

On the formal description of Quadrilateral and Quadrangle Centers. A first start for a Quadri Catalogue.

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Abstract.

In this paper a formal description is given of points occurring in a construction defined by 4 points and/or 4 lines.

The notion of a Quadrigon is introduced. With this notion it is possible to define Centers and related objects depending on 4 points and/or 4 lines in an integrated and differentiated way.

Several tools and examples are given. With these tools the road is free for a structured Catalogue of Quadri-Centers and Quadri-Objects. A first start is given.

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8 JUSTIFICATION

References

1. INTRODUCTION

There are noticeable differences between triangles and quadrilaterals / quadrangles.

- With a triangle there are 3 points and 3 line segments. When we consider a quadrangle as a complex of 4 points then we have to deal with 4 points and 6 possible lines.
- When we consider a quadrilateral as a complex of 4 lines then we have to deal with 4 lines and 6 possible intersection points.
- A triangle can be constructed knowing 3 elements of the triangle. However a quadrilateral cannot be constructed analogously with 4 elements of this figure.

This is quite different from working with a triangle.

Algebraically there are also differences.

- Often trilinear or barycentric coordinates are used in the triangle environment because a point can be related to the 3 side lengths or the 3 angles of a triangle. But in the according situation 4 side lengths or 4 angles are not sufficient for defining a quadrilateral.
- The 3 vertices of a triangle are often $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$. Introducing a 4th point raises the question how to classify this new point.

In this paper we will classify 3 types of construction with 4 points and/or 4 lines.

Most of the time only points or lines per type of construction have relationships with each other.

Each type has its own unique definition for Centers. This will solve several problems and relates to existing practices in Triangle Geometry.

Homogeneous coordinate triples of points and coefficient triples of lines as well as equations of conics will be given according to the developed system.

No synthetic proofs will be given. However with the developed algebraic system it is relatively easy to prove all the described properties.

Calculations are done with Mathematica software using the wonderful collection of formulas and routines in Baricentricas [10] of Francisco Javier García Capitán. Special thanks to him for his valuable advice about the Mathematica software.

Because the intermediate results of algebraic calculations often are very lengthy only the final result is given. Many of the calculated results are checked in a fixed reference system. But nevertheless errors or typos may occur.

References to sources of literature and internet are made where I found information about points, lines and conics. The remaining points, lines and conics I discovered myself in the process of writing this paper. However, several of them may be known and available at other places.

Special thanks also for Peter Moses who helped me with the use of the English language. Very helpful because the English language is not my mother tongue.

Special thanks also for Bernard Gibert who was willing to inspect the “Quadri-cubics” and categorize them.

Special thanks also for Eckart Schmidt who painstakingly worked through 174 pages of the first concept of this paper and contributed with lots of useful additions. He convinced me to use also coordinates relating to Diagonal Triangles (DT) of Quadrangle and Quadrilateral. Most of the DT-coordinates and DT-expressions came from his hand. We had a very nice exchange of ideas alternately in German and English.

2. PRELIMINARIES

In this paper we deal with figures configured with 4 lines and/or 4 points. We will call them Quadri-figures.

There is quite a lot of difference in the use of the words Quadrilateral and Quadrangle. But because of further comprehension it is important to be very precise in making correct definitions.

We use this nomenclature.

Quadrilateral

A quadrilateral is a plane figure consisting of 4 lines no three of which are concurrent. In case of multiple use the prefix and abbreviation **QL** will be used.

Quadrangle

A quadrangle is a plane figure consisting of 4 points no three of which are collinear. In case of multiple use the prefix and abbreviation **QA** will be used.

Also this notion is introduced:

Quadrignon

A Quadrignon is a polygon of 4 points (no 3 of which are collinear) and 4 lines spanned between the 4 points in such a way that each line connects only 2 of the 4 points and each point is the intersection of just 2 of the 4 lines.

In case of multiple use the prefix and abbreviation **QG** will be used.

It is important that a quadrilateral consists of just 4 random lines without intersection points and without any other condition or any order.

In the same way a quadrangle consists of just 4 random points without connecting lines and without any other condition or any order.

However a quadrignon consists of 4 random points with 4 connecting lines in cyclic order.

It also can be seen as a system of 4 random lines with 4 intersection points in cyclic order. Now the order of points and lines are important.

3. DEFINITION OF QUADRI-STRUCTURES

Quadrignon

There is a difference between a quadrilateral and a quadrangle.

Nevertheless this differentiation is not sufficient to describe all constructible points in an environment of 4 points and/or 4 lines. That's why in this chapter a new notion will be introduced "the quadrignon".

A *Quadrignon* is a polygon of 4 points (no 3 of which are collinear) and 4 lines spanned between the 4 points in such a way that each line connects only 2 of the 4 points and each point is the intersection of just 2 of the 4 lines.

There is no limitation for being convex.

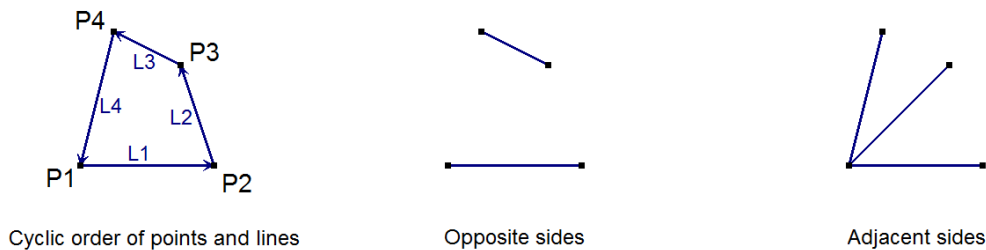
Why this new concept?

1. It is symmetric and according to the duality principle. Just like a triangle is symmetric and according to the duality principle.
2. It appeals to what we usually have in mind with a quadrilateral: 4 points and 4 lines in a certain order.
3. It can be extended in 2 equivalent and symmetric ways: more lines and more points.
4. Some constructions of points cannot be uniquely defined in a quadrangle or quadrilateral environment but it can be done in a quadrigon environment.

A quadrigon is the most specific notion. It determines the cyclic order of lines and points. Unlike the quadrilateral and quadrangle, lines and points now have equal status. A quadrigon has minimal degrees of freedom. It can be seen as the intersection of a quadrilateral and a quadrangle.

$$\text{Quadrigon} = \text{Quadrilateral} \cap \text{Quadrangle}$$

One consequence is that notions like adjacent sides (or points) and opposite sides (or points) become important (that's why the Van Aubel Points are quadrigon points because in their construction the notion of opposite sides is used).

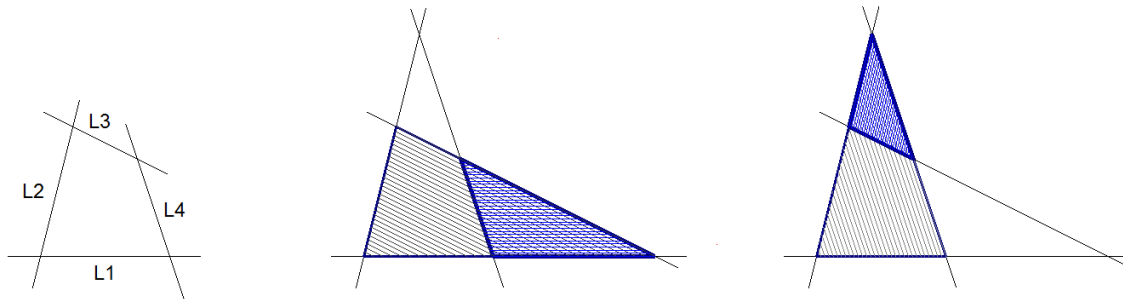


Properties in a Quadrigon

Quadrilateral

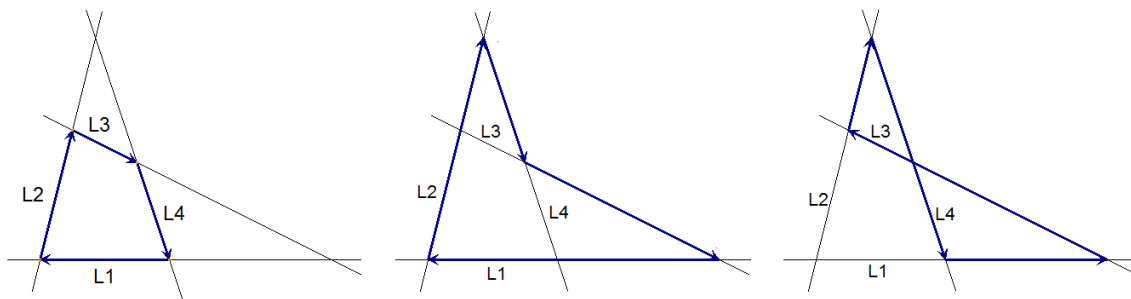
A quadrilateral is a plane figure *consisting of 4 lines* no three of which are concurrent. This figure can be partitioned in several ways.

- a. 4 times 1 line. This case is evident.
- b. 6 possible combinations of 2 lines. This gives us 6 "intersection points".
- c. 4 possible combinations of 3 lines. This gives us 4 "component triangles".
In case of a convex quadrilateral there are 2 pairs of component triangles for which pairwise the area of the inscribed part of the quadrilateral = the subtraction of the triangles per pair.



4 Component Triangles in a Quadrilateral

d. 3 possible cyclic combinations of 4 lines. This gives us 3 “quadrilaterons”.



3 Component Quadrilaterons in a Quadrilateral

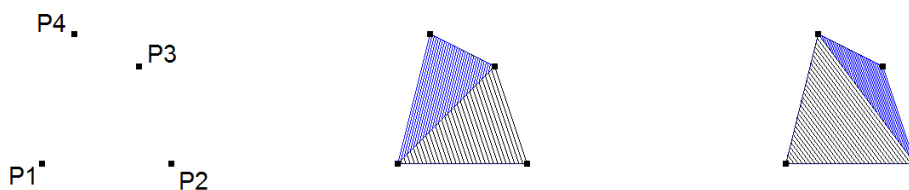
There is no more basic partitioning in a Quadrilateral.

Quadrangle

A quadrangle is a plane figure *consisting of 4 points* no three of which are collinear. This figure can be partitioned in several ways.

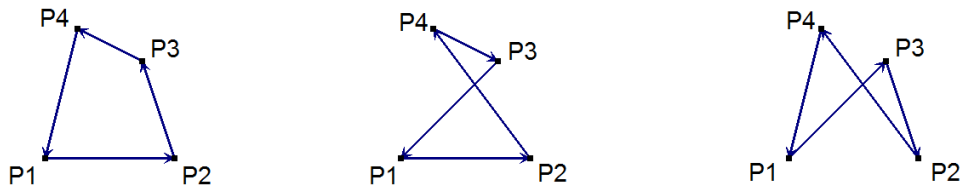
- 4 times 1 point. This case is evident.
- 6 possible combinations of 2 points. This gives us 6 “connecting lines”.
- 4 possible combinations of 3 points. This gives us 4 “component triangles”.

In case of a convex quadrilateral there are 2 pairs of component triangles for which pairwise the area of the inscribed part of the quadrilateral = sum of triangles per pair.



4 Component Triangles in a Quadrangle

d. 3 possible cyclic combinations of 4 points. This gives us 3 “quadrilaterals”.



3 Component Quadrilaterals in a Quadrangle

There is no more basic partitioning in a Quadrangle.

Note that the partitioning of a quadrilateral and a quadrangle leads visually to different “component triangles” and different “quadrilaterals”.

The connections between Quadrangles, Quadrilaterals, Triangles and Quadrilaterals can best be shown in a Data Set Diagram (as used in Computer Science):

Data Set Diagram



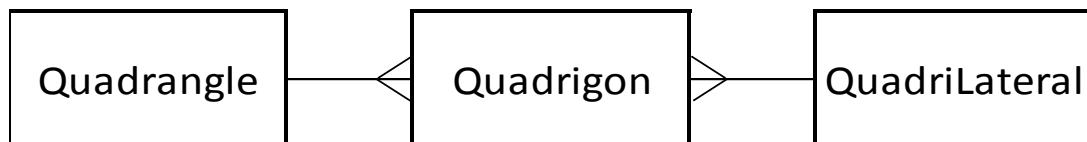
Explanation:

1 Quadrangle has 4 consisting Triangles

A Triangle can be seen as part of a Quadrangle

1 QuadriLateral has 4 consisting Triangles

A Triangle can be seen as part of a QuadriLateral



Explanation:

1 Quadrangle has 3 consisting Quadrigons

A Quadrigon can be seen as part of a Quadrangle

1 QuadriLateral has 3 consisting Quadrigons

A Quadrigon can be seen as part of a QuadriLateral

4. DEFINITION OF QUADRI-CENTERS

Levels of construction

When dealing with Quadri-figures the notions of a Point, Line, Triangle, Quadrilateral, Quadrangle and a Quadrigon represent levels of construction. As explained in last chapter these are all possible levels of construction in a Quadri-figure.

When objects are constructed on the level of a Point, Line or Triangle this construction level usually is evident. However when items are constructed on the level of a Quadrilateral, Quadrangle or Quadrigon it is often unclear to which level this object belongs. Still it is important to know at which level the object is constructed, because each level requires a different approach. Discerning the right level makes it possible to describe and calculate these objects.

Determining the type of Quadri-figure

Making constructions in the Quadri-environment it often is not clear how the dependencies are. Is there a dependency on 4 points, 4 lines or both?

Here are some tools to discover the type of Quadri-figure that is the base of a construction.

When a target point is constructed based on 4 other points in a *quadrangle*, then 4 component triangles and 3 component quadrigons can be discerned. When the same construction method is performed using another component triangle or another component quadrigon then two things can happen:

- The construction point doesn't concur with the target point.
In this case the target point is not a quadrangle point.
- The construction point concurs with the target point.
In this case the target point is probably a quadrangle point.
By performing the same method for all other component triangles or quadrigons (whichever construction level is chosen) and verifying equality with the target point it can be deduced that the target point is indeed a quadrangle point.

When a target point is constructed based on 4 lines in a *quadrilateral*, then 4 component triangles and 3 component quadrigons can be discerned. When the same construction method is performed using another component triangle or another component quadrigon then two things can happen:

- The construction point doesn't concur with the target point.
In this case the target point is not a quadrilateral point.
- The construction point concurs with the target point.
In this case the target point is probably a quadrilateral point.
By performing the same method for all other component triangles or quadrigons (whichever construction level is chosen) and verifying equality with the target point it can be deduced that the target point is indeed a quadrilateral point.

When a target point depending on 4 points or 4 lines is neither a quadrangle point nor a quadrilateral point it should be a *quadrignon point*.

Definition of 3 types of Quadri Centers

This leads us to the conditions for a Quadrangle Center, Quadrilateral Center and a Quadrignon Center.

A *Quadrangle Center* should be a construction point based on 4 points in a quadrangle. When the same construction method is performed using other component triangles or quadrignons then the same point should come out.

A *Quadrilateral Center* should be a construction point based on 4 lines in a quadrangle. When the same construction method is performed using other component triangles or quadrignons then the same point should come out.

A *Quadrignon Center* should be a construction point based on 4 lines or 4 points in a quadrignon not being a Quadrangle Center or a Quadrilateral Center.

Defined in a formal way:

A *Quadrangle Center* QA is:

**a point related to 4 random points P1, P2, P3, P4 (no 3 of which are collinear),
when QA can be described as some transformation of P_i wrt triangle $P_j.P_k.P_l$
and is the same point for all permutations of $(i,j,k,l) \in (1,2,3,4)$, or
when QA can be described as some transformation of P_i wrt quadrignon $P_i.P_j.P_k.P_l$
and is the same point for all permutations of $(i,j,k,l) \in (1,2,3,4)$.**

A *Quadrilateral Center* QL is:

**a point related to 4 random lines L1, L2, L3, L4 (no 3 of which are concurrent),
when QL can be described as some transformation of L_i wrt triangle $L_j.L_k.L_l$
and is the same point for all permutations of $(i,j,k,l) \in (1,2,3,4)$, or
when QL can be described as some transformation of L_i wrt quadrignon $L_i.L_j.L_k.L_l$
and is the same point for all permutations of $(i,j,k,l) \in (1,2,3,4)$.**

A *Quadrignon Center* QG is:

a point solely related to the 4 points and 4 lines of a quadrignon, not being a Quadrangle Center or a Quadrilateral Center.

Algebraic description of Quadri Centers

Since we work with 4 points and/or 4 lines in a plane we can work with 2 Cartesian coordinates as well as 3 homogeneous coordinates.

Also we need to have in mind that a coordinate system that is suitable for a quadrilateral is not necessarily suitable for a quadrangle, etc.

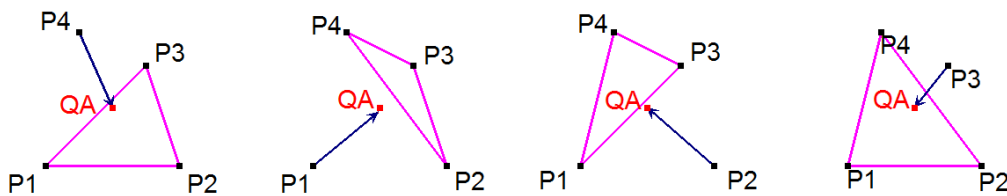
Since we work with “component triangles” and since we concluded that the construction of a center is valid for each component triangle it is worthwhile elaborating the

coordinate systems used in Triangle Geometry. Especially because the science of Triangle Geometry is very well known and documented.

Now the method in a Quadrangle is to identify 3 of the 4 basic points in forming the Reference Triangle. The 4th point is an added point to the Reference Triangle used to perform some construction with.

Again when a point comes out that qualifies as a Center the construction should be working for each choice of reference triangle that is possible in the Quadrangle.

This Quadrangle Center is the result of some transformation on one of the vertices with respect to the triangle formed by the other vertices.



Example:

QA = Inverse of the Isogonal Conjugate of P4 wrt P1.P2.P3

QA = Inverse of the Isogonal Conjugate of P1 wrt P2.P3.P4

QA = Inverse of the Isogonal Conjugate of P2 wrt P3.P4.P1

QA = Inverse of the Isogonal Conjugate of P3 wrt P4.P1.P2

The working method is then:

- choose randomly 3 quadrangle vertices as the vertices of a reference triangle,
- give them homogeneous coordinates (1:0:0), (0:1:0) and (0:0:1),
- identify the remaining point with coordinates (p:q:r).

Every constructed object now can be identified

- as a function of (a,b,c) and (p,q,r) when barycentric triangular coordinates are used and where a, b, c are the side lengths of the reference triangle, or
- as a function of (A,B,C) and (p,q,r) when trilinear triangular coordinates are used and where A, B, C are the angles of the reference triangle.

The *Centroid* and the *Euler-Poncelet Point* in a Quadrangle may serve as example.

As will be shown later the Centroid of the Quadrangle has coordinates:

$$(2p+q+r : p+2q+r : p+q+2r)$$

and the Euler-Poncelet Point:

$$\begin{aligned} & (p(S_B \cdot q - S_C \cdot r)(b^2r(p+q) - c^2q(p+r)) : \\ & \quad q(S_C \cdot r - S_A \cdot p)(c^2p(q+r) - a^2r(q+p)) : \\ & \quad r(S_A \cdot p - S_B \cdot q)(a^2q(r+p) - b^2p(r+q))) \end{aligned}$$

where $S_A = (-a^2 + b^2 + c^2)/2$, etc.

As can be seen the symmetry in construction is reflected in the internal symmetry of the coordinate formulas on the right level.

Identifying Centers on the right level make it possible to algebraically describe relationships between these Centers like collinearity, lying on a conic, etc.

The same consideration can be made with the QL. This results in a system where 3 of the 4 lines will be identified by homogeneous coefficient triples (1:0:0), (0:1:0) and (0:0:1). The 4th line gets coefficient triple (l:m:n).

Note that as 1st coefficient letter “l” is being used in contrast with the QA situation where point coordinates are represented by (p:q:r) with “p” as 1st letter.

This has been done deliberately for direct recognition of Quadri-coordinates.

The letter “l” stands for “line” and tells that QL coordinates have been used. The letter “p” stands for “point” and tells that QA coordinates have been used.

Example of a QL-point is the Miquel Point.

It has QL-coordinates:

$$\{ a^2 m n / (m - n) : b^2 n l / (n - l) : c^2 l m / (l - m) \}.$$

In QA-notation this would be:

$$\{ a^2 (p+q) (p+q+r) : a^2 pq + c^2 qr - b^2 (qr+rp+pq) + 2 S_B q^2 : c^2 (r+q) (p+q+r) \}$$

As can be seen when a QL Center is described in QA-notation there is no longer cyclic symmetry between the 3 coordinates. That is because 3 Quadrigons can be discerned in a Quadrangle each representing a quadrilateral, each having a different Miquel Point. So actually The Miquel Point is dispersed into a triple of points in QA environment.

Nevertheless sometimes it can be useful when interactions are searched between QA-Centers and QL-Centers to calculate a QL-Center in QA-environment or vice versa.

So now we have algebraic descriptions for QA-Centers and QL-Centers.

We also discerned the notion of a Quadrigon and as a matter of fact there are also Quadrigon Centers. We will abbreviate these centers as QG-Centers.

These are centers where the cyclic order of vertices and line segments are used in the construction method.

The algebraic way of describing QA-Centers and QL-Centers are not necessary useful for these QG-Centers. The reason for this is that often the notion of opposite sides/points is used and that gives few connections with component triangles.

I find that a simple Cartesian coordinate system with projective coordinates (i.e. a normal x- and y-coordinate combined with a z-coordinate for indicating infinity points) often gives easiest and symmetric results.

This is not surprising because a Cartesian coordinate system has 4 quadrants in each of which we can pin a vertex even so that the diagonals concur with the origin.

However for the sake of the many relations with quadrangles and quadrilaterals also the QA-and the QL-coordinates are of importance and will be mentioned in this paper.

However since QG-points exist in 3 variants in Quadrangles and Quadrilaterals coordinates will be shown in threefold too. Of course these 3 variants will show a beautiful symmetry too.

This will be shown in Chapter 7.

5. QUADRANGLE OBJECTS

5.0 QUADRANGLES GENERAL INFORMATION

QA/1: Systematics for describing QA-points

In this Encyclopedia of Quadri-Figures 2 coordinate systems are used for Quadrangles:

1. QA-CT-Coordinate system, where 3 arbitrary points of the quadrangle form a Component Triangle (CT). This Component Triangle is defined as Reference Triangle with vertex coordinates (1:0:0), (0:1:0), (0:0:1). The 4th point is defined as (p:q:r).
2. QA-DT-Coordinate system, where the QA-Diagonal Triangle (DT, see QA-Tr1) is defined as the Reference Triangle with vertex coordinates (1:0:0), (0:1:0), (0:0:1). An arbitrary point of the Quadrangle is defined as (p:q:r). The other 3 points now form the Anticevian triangle of Pi wrt the QA-Diagonal Triangle and have vertices (-p : q : r), (p : -q : r), (p : q : -r).

Both coordinate systems can be converted in each other (see QA/6 and QA/7).

Every constructed object now can be identified as:

$$(f(a,b,c, p,q,r) : f(b,c,a, q,r,p) : f(c,a,b, r,p,q))$$

where a,b,c represent the side lengths of the CT- or DT-triangle and

where p,q,r represent the barycentric coordinates wrt the CT- or DT-triangle.

In the description of the points on the following pages only the first of the 3 barycentric coordinates will be shown. The other 2 coordinates can be derived by cyclic rotations:

- $a > b > c > a > \text{etc.}$
- $p > q > r > p > \text{etc.}$

Further the Conway notation has been used in algebraic expressions:

$$S_A = (-a^2 + b^2 + c^2) / 2$$

$$S_B = (+a^2 - b^2 + c^2) / 2$$

$$S_C = (+a^2 + b^2 - c^2) / 2$$

$$S_\omega = (+a^2 + b^2 + c^2) / 2$$

$$S = \sqrt{(S_A S_B + S_B S_C + S_C S_A)} = 2 \Delta$$

Where Δ = area triangle ABC = $\frac{1}{4} \sqrt{((a+b+c) (-a+b+c) (a-b+c) (a+b-c))}$.

Transformed Quadrangles

In the descriptions of Quadrangle Centers often the technique is used of transforming one Quadrangle into another Quadrangle. This is done by performing a Transformation T on P_i wrt triangle $P_j.P_k.P_l$ (for all permutations of $(i,j,k,l) \in (1,2,3,4)$). This produces a new quadrangle $P_1'.P_2'.P_3'.P_4'$.

Consequently the same transformation T can be performed on quadrangle $P_1'.P_2'.P_3'.P_4'$ producing another quadrangle $P_1''.P_2''.P_3''.P_4''$. This quadrangle is called the 2nd generation T -Quadrangle.

It is special that often the Reference Quadrangle and the 2nd generation T -Quadrangle are homothetic. Consequently, this produces a “Center of Homothecy”, in this paper also named “Homothetic Center”.

Another technique of quadrangle transformation is by determining Triangle Centers (see [12]) X_i for 3 points P_j, P_k, P_l (for all permutations $(i,j,k,l) \in (1,2,3,4)$). This produces the (1st) X -Quadrangle.

The same process can be performed on the (1st) X -Quadrangle producing the 2nd X -Quadrangle. This quadrangle is named the 2nd generation X -Quadrangle.

Again often the Reference Quadrangle and the 2nd generation X -Quadrangle are homothetic. Again, this produces a “Center of Homothecy”, in this paper also named “Homothetic Center”.

QA/2: List of QA-Lines

In next list all QA-points are mentioned without prefix "QA-".

All lines in the range QA-P1 - QA-P34 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

These QA-lines are described further with their properties:

- QA-L1: P1, P2, P3, P34
- QA-L2: P2, P4, P6
- QA-L3: P1, P5, P10, P18, P20, P22, P25, P26 (Centroids Line)
- QA-L4: P1, P6, P23
- QA-L5: P10, P11, P12, P13 (QA-DT -Euler Line)
- QA-L6: P1, P15 (Newton-Morley Line)

Other QA-Lines without name but with at least 3 Points on it:

- P1, P4, P7
- P1, P14, P24
- P1, P16, P21
- P1, P32, P33
- P2, P10, P29
- P2, P11, P30
- P3, P20, P29
- P4, P8, P23, P32
- P4, P10, P28
- P5, P17, P19, P21
- P5, P29, P34
- P6, P28, P29
- P10, P16, P19, P31
- P12, P14, P33
- P12, P29, P30
- P20, P21, P31

QA/3: List of parallel QA-Lines

In next list all QA-points are mentioned without prefix "QA-".

All lines in the range QA-P1 - QA-P34 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

These QA-lines are parallel:

- P1.P2 // P22.P29
- P1.P6 // P3.P4 // QA-Cu1-asymptote
- P1.P11 // P3.P30 // P12.P20 // P13.P22 (4th line is midline of 1st and 3rd line)
- P1.P12 // P11.P22
- P1.P13 // P5.P12 // P11.P20
- P1.P16 // P19.P20 // P22.P31 (3rd line is midline of 1st and 2nd line)
- P1.P19 // P16.P22
- P1.P28 // P4.P5
- P1.P29 // P2.P20 // P3.P5 (1st line is midline of 2nd and 3rd line)
- P1.P31 // P5.P17 // P10.P27 // P16.P20 (1st line is midline of 2nd and 4th line)
- P1.P32 // P2.P4 // P7.P8 // P12.P24
- P2.P5 // P3.P20 // P10.P34 d (3rd line, 1st line) = 2*d (3rd line, 2nd line)
- P2.P10 // P25.P34
- P2.P11 // P13.P29
- P2.P12 // P11.P29
- P2.P16 // P3.P21 // P29.P31
- P2.P19 // P16.P29
- P2.P21 // P3.P16
- P2.P23 // P3.P32
- P2.P25 // P3.P26
- P2.P26 // P3.P25
- P3.P10 // P26.P34
- P3.P12 // P20.P30
- P3.P24 // P14.P34
- P3.P30 // P12.P20
- P4.P12 // P13.P28
- P4.P19 // P28.P31
- P4.P20 // P22.P28
- P4.P30 // P6.P11
- P5.P11 // P13.P20
- P5.P16 // P20.P21
- P5.P24 // P13.P32
- P5.P33 // P22.P32
- P10.P14 // P11.P33 // P20.P24
- P10.P24 // P14.P26
- P10.P27 // P16.P20

- P11.P16 // P12.P19 // P13.P31 (3rd line is midline of 1st and 2nd line)
- P11.P19 // P13.P16
- P11.P31 // P12.P16
- P12.P20 // P13.P22
- P16.P25 // P21.P26
- P16.P26 // P21.P25

QA/4: List of perpendicular QA-Lines

In next list all QA-points are mentioned without prefix "QA-".

All lines in the range QA-P1 - QA-P34 have been taken into account.

When lines have more than 2 points, they are defined by the 2 points with lowest serial number.

It is remarkable that there are no point-to-point QA-lines perpendicular !

Apparently because all QA-points (except QA-P12) have been constructed without using perpendicular lines.

The only perpendicular settings in a Quadrangle are:

- QA-Cu7 (QA-Quasi Isogonal Cubic)-asymptote \perp P1.P32 // P2.P4 // P7.P8 // P12.P24.
- 5th point tangent QA-P2 (see QA-L/1) also \perp P1.P32 // P2.P4 // P7.P8 // P12.P24.
- QA-Co2 (QA-Orthogonal Hyperbola) has 2 perpendicular asymptotes.
- QA-Co4 (QA-DT-P3-P12 Orthogonal Hyperbola) has 2 perpendicular asymptotes.
- F1.F2 \perp P4.P12 // P6.P36 // P13.P28,
where F1 and F2 are the foci of the 2 QA-Parabolas (QA-2Co1).
With special property that $P4.P12 = 2 * P6.P36 = 4 * P13.P28$.

When we are looking for perpendicular lines between QA-points and also include QG-points, then there are plenty of perpendicular lines. See QG/4.

QA/5: List of QA-Crosspoints

When 3 lines connecting QA-points concur, the point of concurrence is called a QA-Crosspoint.

In this list all possible non-registered QA-Crosspoints are listed originating from at least 3 connecting lines of QA-points in the range QA-P1 – QA-P34.

QA-points are mentioned without prefix “QA-”.

Lines are defined by the first 2 points on it with lowest serial number.

There are regularly recurring crossing lines with these Crosspoints. This is an indication for the occurrence of Perspective Fields (see QA-PF1).

When the intersection points have fixed ratios of the distances to the defining points on the defining lines, then they are mentioned. There are many of them.

When there are no fixed ratios this is indicated by the remark “x : y”.

For point P on line P1.P2 the ratio $d1 : d2$ means that $d(P,P1) : d(P,P2) = d1 : d2$, where:

- $d1$ is positive when P is positioned wrt P1 at the same side of the line as P2. If not then $d1$ is negative.
- $d2$ is positive when P is positioned wrt P2 at the same side of the line as P1. If not then $d1$ is negative.

– P1.P4 ^ P3.P6 ^ P20.P28	1 : 2 / 2 : 1 / 4 : -1
– P1.P4 ^ P5.P28 ^ P6.P34	1 : 4 / 4 : 1 / 3 : 2
– P1.P8 ^ P2.P23 ^ P4.P33	x : y / x : y / x : y
– P1.P8 ^ P2.P32 ^ P7.P33	x : y / x : y / x : y
– P1.P11 ^ P3.P30 ^ P12.P20 ^ P13.P22	Infinity Point
– P1.P11 ^ P5.P12 ^ P13.P20 ^ P30.P34	-1 : 2 / 1 : 1 / -1 : 2 / 3 : -1
– P1.P12 ^ P5.P13 ^ P24.P32	1 : 4 / 4 : 1 / 4 : 1
– P1.P12 ^ P11.P26 ^ P14.P32	1 : 6 / 9 : -2 / 4 : 3
– P1.P13 ^ P5.P12 ^ P11.P20	Infinity Point
– P1.P13 ^ P11.P22 ^ P12.P20	2 : -1 / 2 : -1 / 1 : 1
– P1.P16 ^ P19.P20 ^ P22.P31	Infinity Point
– P1.P17 ^ P10.P16 ^ P18.P21	x : y / x : y / x : y
– P1.P17 ^ P10.P27 ^ P16.P18	-1 : 4 / x : y / x : y
– P1.P19 ^ P5.P16 ^ P10.P27	-1 : 3 / 2 : 1 / x : y
– P1.P19 ^ P5.P31 ^ P10.P21	1 : 4 / 4 : 1 / 2 : 3
– P1.P28 ^ P5.P6 ^ P12.P24	x : y / x : y / x : y
– P1.P29 ^ P2.P20 ^ P3.P5	Infinity Point
– P1.P29 ^ P3.P10 ^ P20.P34	1 : 1 / 3 : 1 / 3 : 1
– P1.P29 ^ P6.P20 ^ P12.P24	x : y / x : y / x : y
– P1.P31 ^ P5.P17 ^ P10.P27 ^ P16.P20	Infinity Point
– P1.P31 ^ P16.P22 ^ P19.P20	2 : -1 / 2 : -1 / 1 : 1
– P1.P32 ^ P2.P4 ^ P7.P8 ^ P12.P24	Infinity Point
– P1.P32 ^ P3.P6 ^ P4.P34	x : y / 1 : 1 / 3 : 1
– P2.P5 ^ P3.P20 ^ P10.P34	Infinity Point
– P2.P7 ^ P3.P23 ^ P6.P34	x : y / x : y / x : y

- P2.P7 ^ P3.P33 ^ P32.P34	$x:y / 2:3 / 9:-4$
- P2.P7 ^ P6.P32 ^ P23.P33	$x:y / x:y / x:y$
- P2.P8 ^ P3.P4 ^ P7.P34	$x:y / x:y / x:y$
- P2.P8 ^ P4.P33 ^ P7.P23	$x:y / x:y / x:y$
- P2.P16 ^ P3.P21 ^ P29.P31	Infinity Point
- P2.P16 ^ P19.P29 ^ P21.P34	$2:-1 / 2:-1 / 3:-2$
- P2.P20 ^ P3.P10 ^ P22.P29	$1:1 / 3:-1 / -1:2$
- P2.P20 ^ P3.P22 ^ P10.P34	$2:1 / 4:-1 / -1:2$
- P2.P26 ^ P10.P34 ^ P25.P29	$6:-1 / 2:3 / 2:3$
- P2.P32 ^ P3.P6 ^ P23.P34	$x:y / x:y / x:y$
- P3.P5 ^ P4.P20 ^ P12.P24	$x:y / x:y / x:y$
- P4.P11 ^ P6.P30 ^ P12.P28	$2:1 / 1:2 / 4:-1$
- P5.P12 ^ P13.P20 ^ P30.P34	$1:1 / -1:2 / 3:-1$
- P5.P16 ^ P10.P21 ^ P19.P20	$2:-1 / -2:3 / 2:-1$
- P5.P17 ^ P10.P27 ^ P16.P20	Infinity Point
- P5.P29 ^ P12.P24 ^ P20.P28	$x:y / x:y / x:y$
- P5.P31 ^ P10.P27 ^ P19.P20	$4:-1 / x:y / 2:1$
- P10.P14 ^ P11.P33 ^ P20.P24	Infinity Point
- P10.P21 ^ P16.P18 ^ P17.P20	$x:y / x:y / x:y$
- P10.P21 ^ P16.P20 ^ P22.P31	$-1:3 / 1:1 / -1:2$
- P10.P21 ^ P16.P26 ^ P19.P25	$1:1 / 3:-1 / 3:1$
- P10.P21 ^ P19.P26 ^ P25.P31	$2:5 / 6:1 / 4:3$
- P10.P27 ^ P16.P22 ^ P20.P21	$x:y / 4:-1 / 1:2$
- P10.P27 ^ P16.P25 ^ P19.P26	$x:y / 3:2 / 6:1$
- P10.P27 ^ P19.P25 ^ P26.P31	$x:y / 3:-1 / -2:3$
- P10.P33 ^ P11.P14 ^ P26.P32	$2:3 / 3:2 / -4:9$
- P11.P16 ^ P12.P19 ^ P13.P31	Infinity Point
- P11.P31 ^ P12.P19 ^ P13.P16	$2:-1 / 1:1 / -1:2$
- P16.P17 ^ P18.P21 ^ P19.P20	$x:y / x:y / x:y$

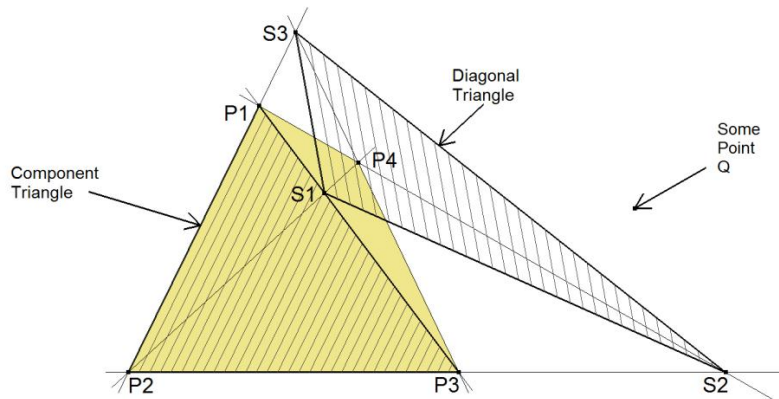
QA/6: QA-Conversion CT -> DT - coordinates

Let P1.P2.P3.P4 be the Reference Quadrangle.

Let P1.P2.P3 be the random Reference Component Triangle en let P4 be the 4th point.

The QA-Diagonal Triangle S1.S2.S3 is the *Cevian Triangle* of P4 wrt P1.P2.P3.

Let Q be some point to be converted from CT- to DT-coordinates.



Let $Q_c (x_c : y_c : z_c)$ be the presentation of Q in barycentric coordinates wrt the Component Triangle. Let $Q_d (x_d : y_d : z_d)$ be the presentation of Q in barycentric coordinates wrt the Diagonal Triangle.

Now $Q_c = x_c.cfc1.P1 + y_c.cfc2.P2 + z_c.cfc3.P3$ wrt the Reference Component Triangle and $Q_d = x_d.cfd1.S1 + y_d.cfd2.S2 + z_d.cfd3.S3$ wrt the Diagonal Triangle, where:

- $(x_c : y_c : z_c)$ are the barycentric coordinates of Q wrt the Component Triangle,
- $(x_d : y_d : z_d)$ are the barycentric coordinates of Q wrt the Diagonal Triangle,
- $cfc1, cfc2, cfc3$ are the Compliance Factors of the Component Triangle,
- $cfd1, cfd2, cfd3$ are the Compliance Factors of the Diagonal Triangle.

Explanation of Compliance Factors can be found at [26b page 40].

Since the Component Triangle is the Reference Triangle, the Compliance Factors of the Component Triangle are all equal 1.

The Compliance Factors of the Diagonal Triangle are:

- $cfd1 = \text{Det} [G_d, S2, S3] / \text{Det} [S1, S2, S3],$
- $cfd2 = \text{Det} [S1, G_d, S3] / \text{Det} [S1, S2, S3],$
- $cfd3 = \text{Det} [S1, S2, G_d] / \text{Det} [S1, S2, S3],$

where G_d = the Centroid of the Diagonal Triangle and “Det” is the abbreviation for “Determinant”.

Calculation gives 2 presentations of the coordinates of Q wrt the Component Triangle:

- $Q_c = (x_c : y_c : z_c),$
- $Q_d = (p (q+r) (p y_d + q y_d + p z_d + r z_d) : q (p+r) (p x_d + q x_d + q z_d + r z_d) : r (p+q)(p x_d + r x_d + q y_d + r y_d))$

Since Qc and Qd present the same point we can now calculate the coordinates of Q wrt the Diagonal Triangle:

- $(x_d : y_d : z_d) = ((q+r)(q r x_c - p r y_c - p q z_c) : (p+r)(-q r x_c + p r y_c - p q z_c) : (p+q)(-q r x_c - p r y_c + p q z_c)).$

However we have to bear in mind that the variables in these coordinates are expressions in (a,b,c) and (p,q,r), which are variables wrt the Component Triangle.

Therefore the CT > DT-conversion of $P(x : y : z)$ consists of 3 consecutive steps:

1. Transform Point $(x, y, z) \rightarrow$

$$\begin{aligned} & ((q+r)(q r x - p r y - p q z) : \\ & (p+r)(-q r x + p r y - p q z) : \\ & (p+q)(-q r x - p r y + p q z)) \end{aligned}$$

2. Replace: $p \rightarrow (-p + q + r)$

$$q \rightarrow (p - q + r)$$

$$r \rightarrow (p + q - r)$$

3. Replace: $a^2 \rightarrow (4 p^2 (SA q^2 + SB q^2 - 2 SA q r + SA r^2 + SC r^2)) / ((p + q - r)^2 (p - q + r)^2)$

$$b^2 \rightarrow (4 q^2 (SA p^2 + SB p^2 - 2 SB p r + SB r^2 + SC r^2)) / ((p - q - r)^2 (p + q - r)^2)$$

$$c^2 \rightarrow (4 r^2 (SA p^2 + SC p^2 - 2 SC p q + SB q^2 + SC q^2)) / ((p - q - r)^2 (p - q + r)^2)$$

QA/7: QA-Conversion DT -> CT - coordinates

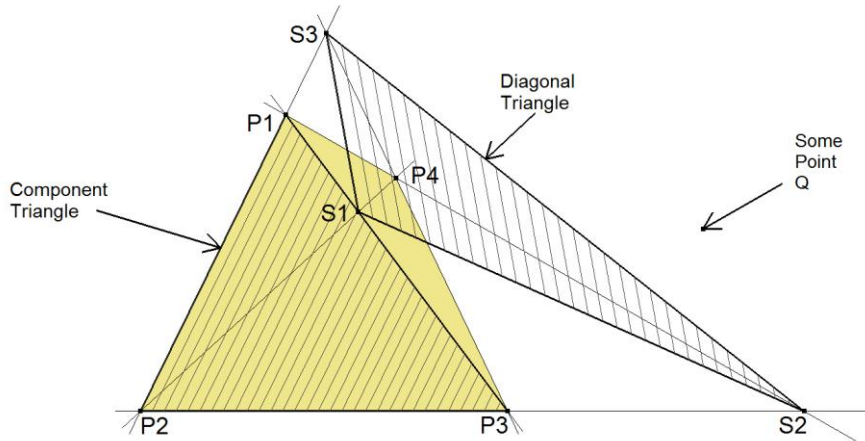
Let $S1.S2.S3$ be the QA-Diagonal Triangle of the Reference Quadrangle $P1.P2.P3.P4$.

Let $S1.S2.S3$ be the Reference Triangle.

Let $P4$ be an arbitrary point of the Quadrangle with coordinates $(p:q:r)$ wrt the DT.

The Component Triangle $P1.P2.P3$ is the *Anticevian Triangle* of $P4$ wrt $S1.S2.S3$.

Let Q be some point to be converted from DT- tot CT-coordinates.



Let $Q_c (x_c : y_c : z_c)$ be the presentation of Q in barycentric coordinates wrt the Component Triangle. Let $Q_d (x_d : y_d : z_d)$ be the presentation of Q in barycentric coordinates wrt the Diagonal Triangle.

Now $Q_d = x_d.cfd1.S1 + y_d.cfd2.S2 + z_d.cfd3.S3$ wrt the Reference Diagonal Triangle

also $Q_c = x_c.cfc1.P1 + y_c.cfc2.P2 + z_c.cfc3.P3$ wrt the Diagonal Triangle,

where:

- $(x_d : y_d : z_d)$ are the barycentric coordinates of Q wrt the Diagonal Triangle,
- $(x_c : y_c : z_c)$ are the barycentric coordinates of Q wrt the Component Triangle,
- $cfc1, cfc2, cfc3$ are the Compliance Factors of the Component Triangle,
- $cfd1, cfd2, cfd3$ are the Compliance Factors of the Diagonal Triangle.

Explanation of Compliance Factors can be found at [26b page 40].

Since the Diagonal Triangle is the Reference Triangle, the Compliance Factors of the Component Triangle are all equal 1.

The Compliance Factors of the Component Triangle are:

- $cfc1 = \text{Det} [G_c, P2, P3] / \text{Det} [P1, P2, P3]$
- $cfc2 = \text{Det} [P1, G_c, P3] / \text{Det} [P1, P2, P3]$
- $cfc3 = \text{Det} [P1, P2, G_c] / \text{Det} [P1, P2, P3]$

where G_c = the Centroid of the Component Triangle and "Det" is abbreviation for "Determinant".

Calculation gives 2 presentations of the coordinates of Q wrt the Diagonal Triangle:

- $Q_d = (x_d : y_d : z_d),$

- $Q_c =$
 $(-p(p^2 x_c - q^2 x_c + 2 q r x_c - r^2 x_c + p^2 y_c - q^2 y_c - 2 p r y_c + r^2 y_c + p^2 z_c - 2 p q z_c + q^2 z_c - r^2 z_c) :$
 $-q(-p^2 x_c + q^2 x_c - 2 q r x_c + r^2 x_c - p^2 y_c + q^2 y_c + 2 p r y_c - r^2 y_c + p^2 z_c - 2 p q z_c + q^2 z_c - r^2 z_c) :$
 $-r(-p^2 x_c + q^2 x_c - 2 q r x_c + r^2 x_c + p^2 y_c - q^2 y_c - 2 p r y_c + r^2 y_c - p^2 z_c + 2 p q z_c - q^2 z_c + r^2 z_c))$

Since Q_c and Q_d present the same point we can now calculate the coordinates of Q wrt the Component Triangle:

- $(x_c : y_c : z_c) =$
 $(p(p - q - r)(r y_d + q z_d) : q(-p + q - r)(r x_d + p z_d) : r(-p - q + r)(q x_d + p y_d)).$

However we have to bear in mind that the variables in these coordinates are expressions in (a,b,c) and (p,q,r) , which are variables wrt the Diagonal Triangle.

Therefore the CT > DT-conversion of $P(x : y : z)$ consists of 3 consecutive steps:

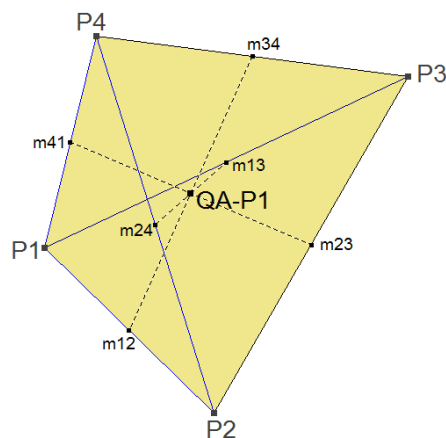
1. Transform Point $(x : y : z) \rightarrow$
 $(p(p - q - r)(r y + q z) :$
 $q(-p + q - r)(r x + p z) :$
 $r(-p - q + r)(q x + p y))$
2. Replace: $p \rightarrow (q + r)$
 $q \rightarrow (p + r)$
 $r \rightarrow (p + q)$
3. Replace: $a^2 \rightarrow (p^2(q - r)^2 SA + q^2(p + r)^2 SB + (p + q)^2 r^2 SC) / ((p + q)^2 (p + r)^2),$
 $b^2 \rightarrow (p^2(q + r)^2 SA + q^2(p - r)^2 SB + (p + q)^2 r^2 SC) / ((p + q)^2 (q + r)^2),$
 $c^2 \rightarrow (p^2(q + r)^2 SA + q^2(p + r)^2 SB + (p - q)^2 r^2 SC) / ((p + r)^2 (q + r)^2).$

5.1 QUADRANGLE CENTERS

QA-P1: QA-Centroid or Quadrangle Centroid

The Centroid of a Quadrangle is actually the center of gravity of a Quadrangle, replacing the points by equal masses.

The usual way to construct it is by connecting midpoints of opposite sides of a chosen component Quadrigon. The two connecting lines as well as the line connecting the midpoints of the diagonals meet at the Centroid.



P_i = Quadrigon Vertices

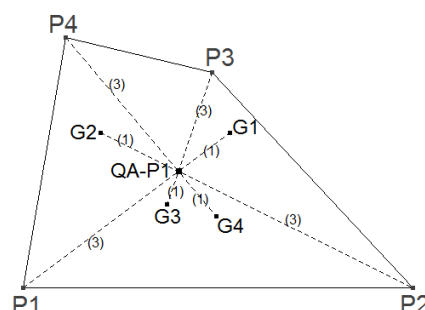
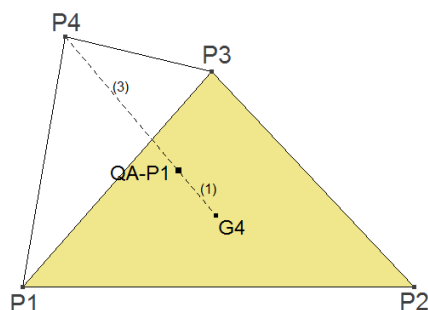
M_{ij} = Midpoint (P_i, P_j)

QA-P1 = Centroid

However there is also another way to construct the Quadrangle Centroid using component triangles. This way of construction makes it clear that it really is a Quadrangle Center because it can be constructed in the same way for all component triangles of the Quadrangle. This picture shows that the Centroid can be constructed by partitioning $G_i.P_i$ in parts 3 : 1.

P_i = Vertices of the Quadrangle
 G_i = Centroids of component Triangles $P_j.P_k.P_i$

The Centroid of a Quadrangle can be constructed by partitioning $P_i.G_i$ in parts (3) : (1) for $i = 1, 2, 3, 4$.



1st CT-Coordinate:

$$2p + q + r$$

1st DT-Coordinate:

$$p^2 (-p^2 + q^2 + r^2)$$

Properties:

- QA-P1 lies on these QA-lines:
 - QA-P2.QA-P3 (1 : 1 => QA-P1 = Midpoint of QA-P2.QA-P3)
 - QA-P4.QA-P7
 - QA-P5.QA-P10 (3 : 1)
 - QA-P6.QA-P23
 - QA-P14.QA-P24 (-1 : 3)
 - QA-P16.QA-P21 (1 : 1 => QA-P1 = Midpoint of QA-P16.QA-P21)
 - QA-P32.QA-P33 (1 : 2)
- QA-P1 lies on these QG-lines:
 - QG-P1.QG-P4 (3 : 1)
 - QG-P2.QG-P12
 - QG-P4.QG-P8 (1 : 1 => QA-P1 = Midpoint of QA-P4.QA-P8)
 - QG-P5.QG-P10 (1 : 1 => QA-P1 = Midpoint of QA-P5.QA-P10)
 - QG-P7.QG-P9 (1 : 1 => QA-P1 = Midpoint of QA-P7.QA-P9)
- QA-P1 = Center of the Nine-point Conic QA-Co1.
- QA-P1 = QA-P10-Ceva conjugate of QA-P16 wrt the QA-Diagonal Triangle.
- QA-P1 = the Involutary Conjugate (see QA-Tf2) of QA-P20.
- QA-P1 = Homothetic Center of the 1st Nine-point Quadrangle and the 1st Midray Quadrangle.
- QA-P1 = the point of tangency of the two congruent tangent circumcircles of the triangles defined by the 3 QA-versions of QG-P7 (1st Quasi Nine-point Center) and the 3 QA-versions of QG-P9 (2nd Quasi Circumcenter).
- QA-P1 = QA-Centroid (QA-P1) of the quadrangle formed by the vertices of the Diagonal Triangle and QA-P5.
- QA-P1 = Gergonne-Steiner (QA-P3) point of the quadrangle formed by the vertices of the Diagonal Triangle and QA-P3.
- QA-P1 lies on the Conic QA-Co5.
- QA-P1 lies on the Cubics QA-Cu2, QA-Cu3, QA-Cu5, QA-Cu6.

QA-P2: Euler-Poncelet Point

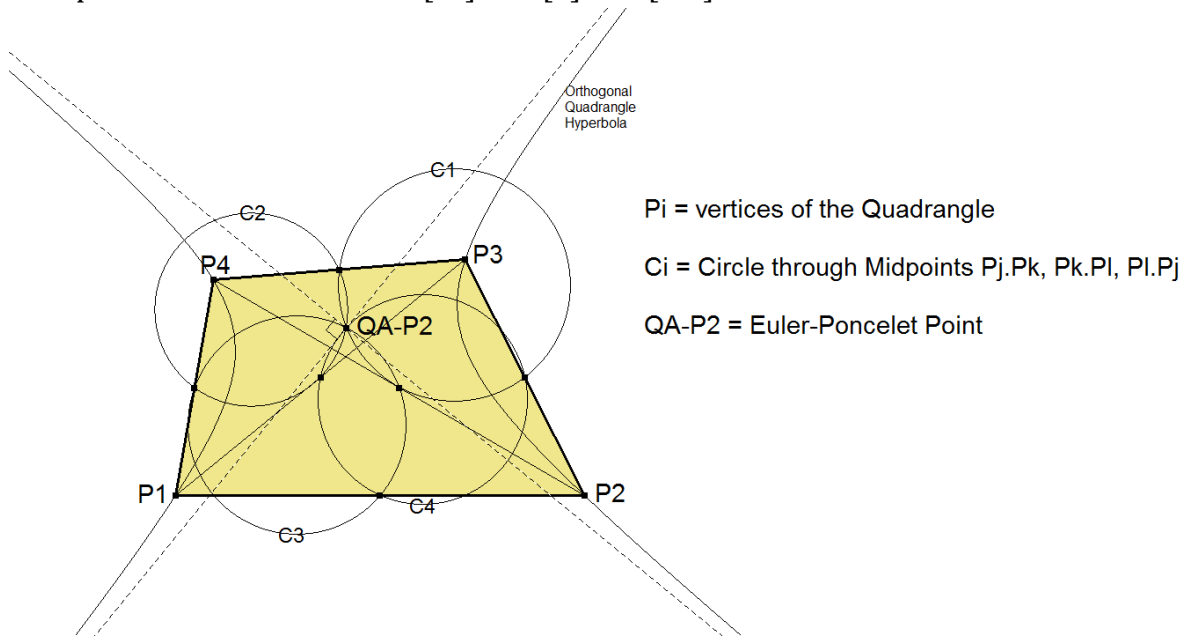
Euler mentioned this point in one of his numerous papers.

In 1821 Brianchon and Poncelet, both captains of artillery, wrote a book [1] in which the orthogonal hyperbola and this point were worked out.

The Euler-Poncelet Point can be defined in different ways.

1. It is the center of the orthogonal hyperbola through P1, P2, P3 and P4.
2. It is the common point of the Nine-point Circles of the triangles Pj.Pk.Pl for all permutations of (j,k,l) ∈ (1,2,3,4).

This point is also described at [2c] and [8] and [15f].



1st CT-Coordinate:

$$p (S_B q - S_C r) (b^2 r (p+q) - c^2 q (p+r))$$

1st DT-Coordinate:

$$1/(b^2 r^2 - c^2 q^2)$$

Properties:

- QA-P2 lies on these QA-lines:
 - QA-P1.QA-P3 (-1 : 2 => QA-P2 = Reflection of QA-P3 in QA-P1)
 - QA-P4.QA-P6 (2 : -1 => QA-P2 = Reflection of QA-P4 in QA-P6)
 - QA-P10.QA-P29 (-2 : 3 => QA-P2 = AntiComplement of QA-P29
wrt QA-DT)
 - QA-P11.QA-P30 (-1 : 2 => QA-P2 = Reflection of QA-P30 in QA-P11)
 - QA-P12.QA-P36 (2 : -1 => QA-P2 = Reflection of QA-P12 in QA-P36)
- QA-P2 is the center of the Orthogonal Hyperbola through P1, P2, P3, P4.
- QA-P2 is the common point of Nine-point Circles Pi.Pj.Pk for all permutations of (i,j,k,l) ∈ (1,2,3,4).

- QA-P2 is the point of concurrence of the four circles determined by the feet of the perpendiculars dropped from each of the four points onto the sides of the triangle formed by the other three. See [13] Nine-point Circle.
- QA-P2 is the Homothetic Center of the Antigonal Quadrangle and the Reference Quadrangle (the Antigonal of a point X is the isogonal conjugate of the inverse in the circumcircle of the isogonal conjugate of X, see [17a]).
- QA-P2 is the Midpoint of the reflections of QA-P4 in $P_i.P_j$ and $P_k.P_l$ (note Eckart Schmidt).
- Let $M = \text{Diagonal Point } P_i.P_j \wedge P_k.P_l$.
Now $M.QA-P4$ and $M.QA-P2$ are symmetric wrt the angle bisector of lines $P_i.P_j$ and $P_k.P_l$ for all permutations of $(i,j,k,l) \in (1,2,3,4)$ [16 page 8].
- QA-P2 is the common point of the circumcircles of the Pedal Triangles $P_i.P_j.P_k$ wrt P_l for all permutations of $(i,j,k,l) \in (1,2,3,4)$. See [8].
- QA-P2 lies on the circumcircle of the QA-Diagonal Triangle.
- QA-P2 lies on the circumcircles of triangles formed by the 3 QA-versions of QG-P1, QG-P6, QG-P10 as well as QG-P14.
- QA-P2 lies on the Nine-point Conic (QA-Co1).
- QA-P2 is concyclic with QA-P7, QA-P8 and QA-P23.

QA-P3: Gergonne-Steiner Point

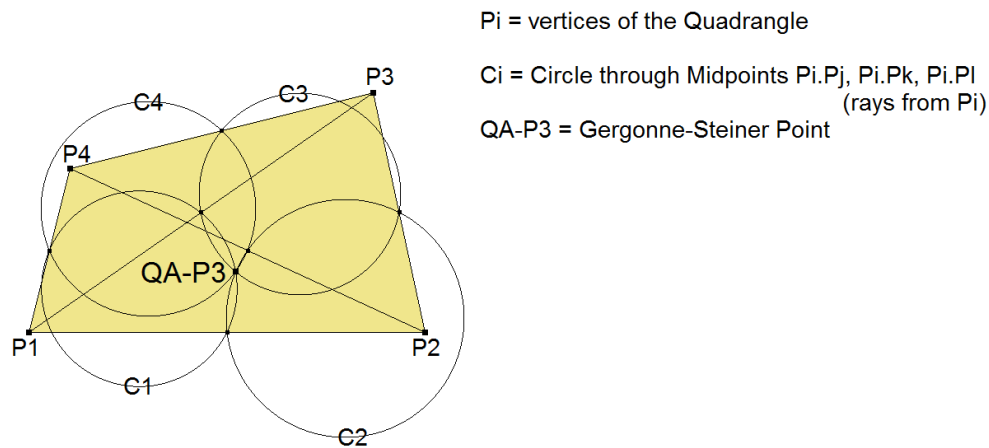
This point is called after Gergonne and Steiner because in the “Annales de Gergonne” the question was posed for the Quadrangle Conic with least eccentricity. Steiner solved this problem. The center of this conic happens to be QA-P3.

It is also the common point of the 4 circles defined by the midpoints of $P_i.P_j$, $P_i.P_k$, $P_i.P_l$ for all permutations of $(i,j,k,l) \in (1,2,3,4)$. This common circle point is described shortly without name in [2c] Jean-Louis Ayme, *Le Point- d'Euler-Poncelet d'un Quadrilatère* on page 10. According to Jean-Louis Ayme this point was mentioned in the writing of Igor Federovitch Sharygin “Problemas de geometria”.

It is strongly related to QA-P2 (the Euler-Poncelet Point) in construction with circles of the same size as well as in position within the Quadrangle.

QA-P3 always appears at the “opposite side” of the reference quadrangle than QA-P2.

The QA-Centroid QA-P1 is their midpoint.



1st CT-Coordinate:

$$\frac{(a^2 (p+q)(p+r) - b^2 p(p+q) - c^2 p(p+r)) * (a^2 q r (2p+q+r) - b^2 p r (q+r) - c^2 p q (q+r))}{1/(-2 a^2 q^2 r^2 + b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2))}$$

1st DT-Coordinate:

$$1/(-2 a^2 q^2 r^2 + b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2))$$

Properties:

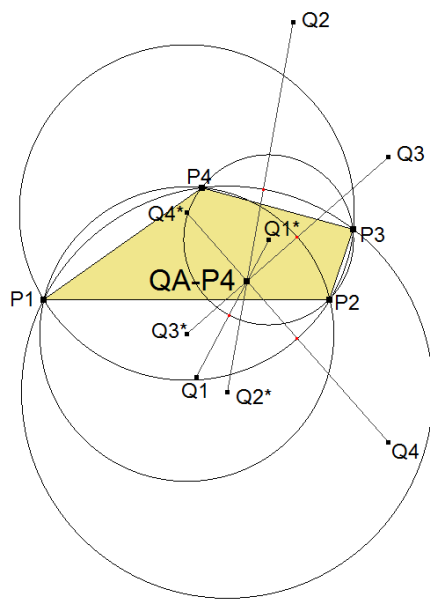
- QA-P3 lies on these QA-lines:
 - QA-P1.QA-P2 $(-1 : 2 \Rightarrow \text{QA-P3} = \text{Reflection of QA-P2 in QA-P1})$
 - QA-P20.QA-P29 $(2 : -1 \Rightarrow \text{QA-P3} = \text{Reflection of QA-P20 in QA-P3})$
 - QA-P22.QA-P35 $(5 : -4)$
- The Reflection of P_i in QA-P3 is a point on the circumcircle of $P_j.P_k.P_l$.
- QA-P3 is the Euler-Poncelet Point (QA-P2) of the 1st Circumcenter Quadrangle (note Eckart Schmidt).
- QA-P3 is the Homothetic Center of the Antigonal Quadrangle and the 1st Centroid Quadrangle (the Antigonal of a point X is the isogonal conjugate of the inverse in the circumcircle of the isogonal conjugate of X, see [17a]).

- QA-P3 is the Perspector of the QA-Diagonal Triangle (QA-Tr1) and the Triangle formed by the Miquel points of the 3 Quadrigons of the Reference Quadrangle (QA-Tr2). See [15] where QA-P3 is called the “Z-Punkt” by Eckart Schmidt.
- QA-P3 is the Isogonal Conjugate of the Complement of QA-P4 wrt the QA-Diagonal Triangle QA-Tr1. See [15f] theorem 25.
- QA-P3 is the Isogonal Conjugate of the AntiComplement of QA-P28 wrt the QA-Diagonal Triangle QA-Tr1.
- QA-P3 is the Isogonal Conjugate of QA-P4 wrt the Miquel Triangle QA-Tr2 [15c page 5].
- QA-P3 lies on the circumcircle of the triangle formed by the 3 QA-versions of the 1st Quasi-Circumcenters (QG-P5).
- QA-P3 lies on the circumcircle of the triangle formed by the 3 QA-versions of the 2nd Quasi-De Longchamps Points (not registered QG-point).
- QA-P3 is the common intersection point of the 3 QA-versions of QL-Ci6, the Dimidium Circle (note Eckart Schmidt).
- QA-P3 lies on the Nine-point Conic (QA-Co1).
- QA-P3 lies on the conic QA-Co4.
- QA-P3 lies on the QA-DT-P4 Cubic (QA-Cu1).

QA-P4: Isogonal Center

This point is mentioned by Clawson [22] as „Isoptic (or Bennett) point”. As shown in the picture below, it is the Inverse of the Isogonal Conjugate of P_i wrt (circumcircle) $P_j P_k P_l$. Many properties of this point are listed by Stärk [16] under the name “Tangentialpunkt” wrt the QA-DT-P4 Conic (QA-Cu1). Other properties also can be found in [15f].

In the discussion of Hyacinthos messages (See [11] #19635, #19649) the point is seen as Homothetic Center. It is called the Isogonal Center because it can be constructed as Homothetic Center of the Reference Quadrangle with the 2nd Isogonal Conjugate Quadrangle and because it has an isogonal conjugate relationship with basic points QA-P2 and QA-P3 (see properties below).



P_i = Vertices

Q_i = Isogonal Conjugates of P_i wrt $P_j P_k P_l$

Q_i^* = Inverse of Q_i wrt circle(P_j, P_k, P_l)

QA-P4 = Isogonal Center

1st CT-Coordinate:

$$a^2 (a^2 qr/p + b^2 r + c^2 q - 2 S_A (p+q+r))$$

1st DT-Coordinate:

$$b^2 c^2 p^4 - a^4 q^2 r^2 + (b^2 - c^2) p^2 (-c^2 q^2 + b^2 r^2)$$

Properties:

- QA-P4 lies on these QA-lines:
 - QA-P1.QA-P7
 - QA-P2.QA-P6 ($2 : -1 \Rightarrow$ QA-P4 = Reflection of QA-P2 in QA-P6)
 - QA-P8.QA-P23
 - QA-P10.QA-P28 ($4 : -3$)
- QA-P4 = Homothetic Center of the 2nd Circumcenter Quadrangle. See [23].
= Homothetic Center of the 2nd Perpendicular Bisector Quadrangle. See[9].
- QA-P4 = Homothetic Center of the 2nd Isogonal Conjugate Quadrangle
- QA-P4 = Homothetic Center of the 1st Circumcenter Quadrangle

and the 1st Isogonal Conjugate Quadrangle. See [11] #19635.

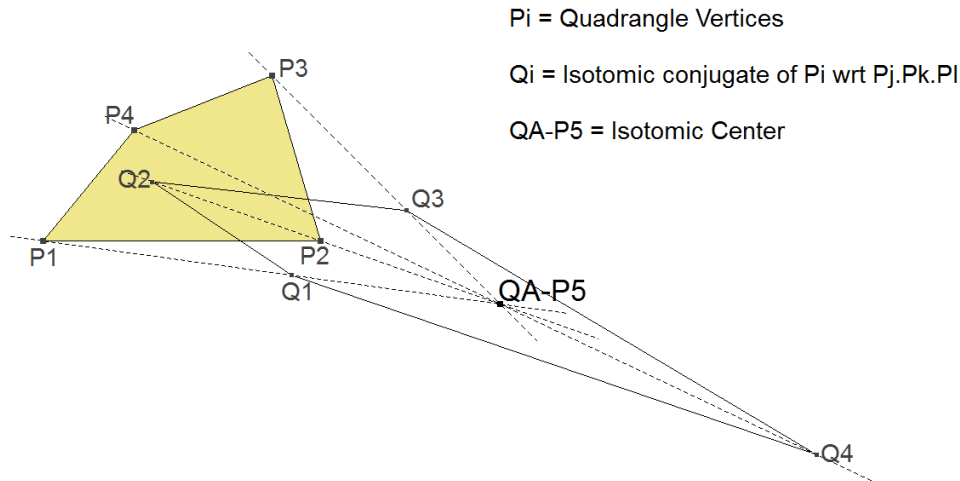
- QA-P4 = Inverse of Isogonal Conjugate of P_i wrt (circumcircle) $P_j.P_k.P_l$. See [18].
- QA-P4 = Isogonal Conjugate of the Reflection of P_i in QA-P2 wrt $P_j.P_k.P_l$. See [18].
- QA-P4 = Isogonal Conjugate of QA-P3 wrt the Miquel Triangle QA-Tr2.
See [15c] page 5.
- QA-P4 = Anticomplement of the Isogonal Conjugate of QA-P3 wrt the QA-Diagonal Triangle QA-Tr1. See [15f] theorem (25).
- QA-P4 lies on the 5th point tangent (see QA-L/1) at QA-P3.
- QA-P4 is also the second intersection point of the circles through the Miquel Point (QL-P1) and 2 opposite vertices of a QA-Quadrigon.
- QA-P4 lies on the QA-DT-P4 Cubic (QA-Cu1).
- Let $M = \text{Diagonal Point } P_i.P_j \wedge P_k.P_l$.
Now $M.QA-P4$ and $M.QA-P2$ are symmetric wrt the angle bisector of lines $P_i.P_j$ and $P_k.P_l$ for all permutations of $(i,j,k,l) \in (1,2,3,4)$. See [16] page 8.
- Let ABCD be a Quadrangle.
The 4 circles ABC, ABD, BCD, CAD can be seen from QA-P4 under the same angle.
That's why this point is also called the Isoptic Point.
This feature is especially the case when 1 of the 4 points lies within the triangle of the other 3 points (see [16] and [22]).
- At Quadrigon-level the Pedal Quadrangle of QA-P4 is a parallelogram with center QA-P6 (Parabola Axes Crosspoint). See [15 f] theorem (23).
- At Quadrigon-level QA-P4 is the Clawson-Schmidt Conjugate of the Diagonal Crosspoint (QG-P1). See [15f] theorem (28).
- The Isogonal Center of the Quadrangle S1.S2.S3.QA-P4 is the Involutary Conjugate (QA-Tf2) of QA-P4 (note Eckart Schmidt).

QA-P5: Isotomic Center

The Isotomic Center is the Perspector of the Reference Quadrangle with the Isotomic Conjugate Quadrangle.

Stated in another way:

The Isotomic Center is the common intersection point of lines $P_i.Q_i$, where $P_i = i^{\text{th}}$ quadrangle vertex, Q_i = Isotomic Conjugate of P_i wrt $P_j.P_k.P_l$ for all permutations of $(i,j,k,l) \in (1,2,3,4)$.



Construction:

QA-P5 is the Reflection of the Anticomplement of QA-P1 (wrt the QA-Diagonal Triangle) in QA-P1.

1st CT-Coordinate:

$$q r (q+r) (2p+q+r)$$

(note that this formula is independent of a,b,c)

1st DT-Coordinate:

$$-4 (p^4 + q^2 r^2) + (p^2 + q^2 + r^2)^2$$

Properties:

- QA-P5 lies on these QA-lines:
 - QA-P1.QA-P10 = QA-L3 (4 : -3)
 - QA-P17.QA-P19
 - QA-P29.QA-P34 (3 : -2)
- QA-P5 lies on these QG-lines:
 - QG-P2.QG-P4 (3 : -2)
- QA-P5 is the Reflection of:
 - QA-P20 in QA-P1
 - QA-P19 in QA-P21.
- QA-P5 is the AntiComplement of QA-P20 wrt the QA-Diagonal Triangle.
- QA-P5.QA-P1 : QA-P1.QA-P10 : QA-P10.QA-P20 = 3 : 1 : 2.

- QA-P5 forms with the vertices of the QA-Diagonal Triangle a quadrangle that shares the same centroid with the Reference Quadrangle.
- QA-P5 is the Involutary Conjugate (see QA-Tf2) of QA-P17.
- QA-P5 lies on the Conic QA-Co4.
- QA-P5 lies on the Cubics QA-Cu2 and QA-Cu4.

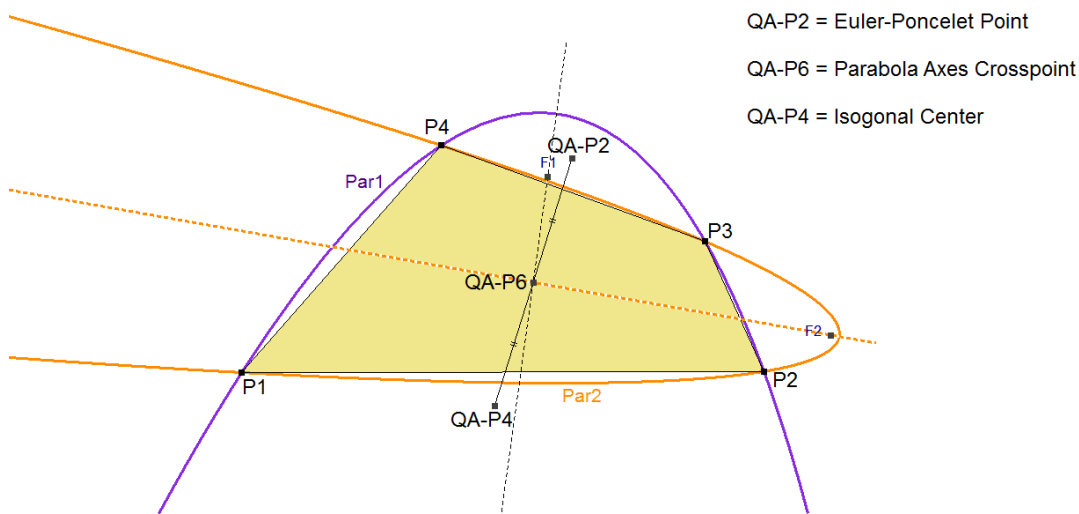
QA-P6: Parabola Axes Crosspoint

The Parabola Axes Crosspoint is the intersection point of the axes of the 2 parabolas that can be constructed through P1, P2, P3, P4. It is also the Midpoint of the Euler-Poncelet Point QA-P2 and the Isogonal Center QA-P4.

Because these parabolas only can be constructed when the Reference Quadrangle is not concave a better definition of this point is: “the Midpoint of the Euler-Poncelet Point and the Isogonal Center”.

Because the property related to the parabolas is much more appealing this point after its primary function.

It also can be reasoned that in a concave quadrangle this point represents the intersection point of the axes of the imaginary parabolas.



1st CT-Coordinate:

$$a^4 q^2 r^2 + c^2 p^2 q^2 S_B + b^2 p^2 r^2 S_C - p q r (a^2 (p + q + r) S_A + 2 (p S^2 - q S_B^2 - r S_C^2))$$

1st DT-Coordinate:

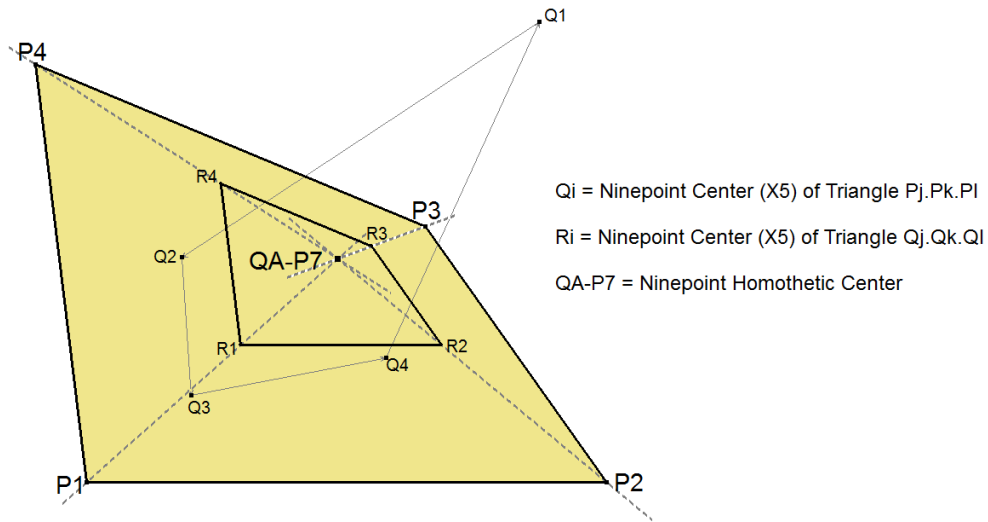
$$p^2 (b^2 S_B r^2 + c^2 S_C q^2 - c^2 b^2 p^2)$$

Properties:

- QA-P6 lies on these QA-lines:
 - QA-P1.QA-P23
 - QA-P2.QA-P4 (1 : 1 => QA-P6 = Midpoint QA-P2.QA-P4)
 - QA-P28.QA-P29 (-1 : 2 => QA-P6 = Reflection of QA-P29 in QA-P28)
- QA-P6 is the Involutory Conjugate (see QA-Tf2) of QA-P30.
- QA-P6 lies on the Simson Line (QA-P6.QA-P36) of QA-P2 occurring on the circumcircle of the QA-Diagonal Triangle.
- For all QA-Quadrilaterals QA-P6 is the center of the Pedal Quadrangle of QA-P4 (Isogonal Center) , which is a parallelogram.
See [15 f] theorem (23).

QA-P7: QA-Nine-point Center Homothetic Center

The QA-Nine-point Homothetic Center (QA-P7) is the homothetic center of the Reference Quadrangle with the Quadrangle composed of four 2nd generation Nine-point Centers (point X(5) in ETC [12]).



1st CT-Coordinate:

$$a^4 q r / p + 3 p S^2 + q S_B^2 + r S_C^2 + (p + q + r) (3 S^2 + 2 S_B S_C)$$

1st DT-Coordinate:

$$-(3 S^2 - S_A^2) p^4 + (3 S^2 + S_A S_B + S_B c^2) p^2 q^2 + (3 S^2 + S_A S_C + S_C b^2) p^2 r^2 - a^4 q^2 r^2$$

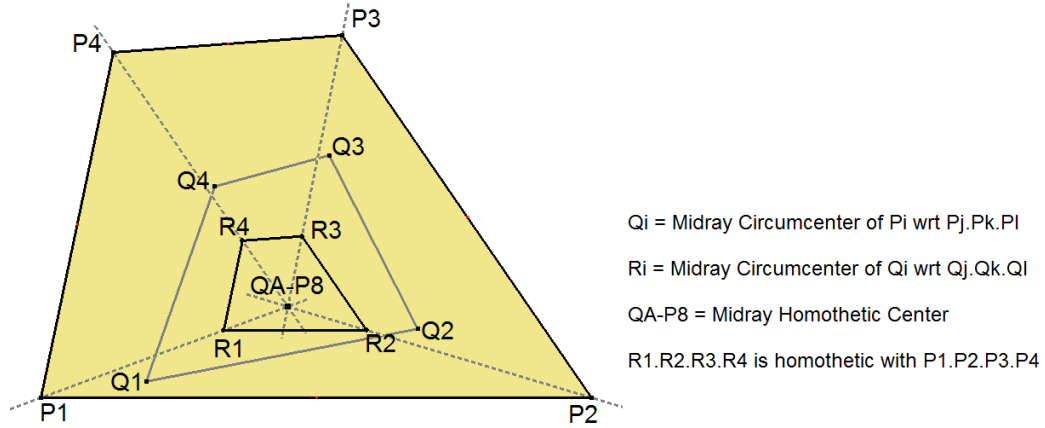
Properties:

- QA-P7 lies on this line:
– QA-P1.QA-P7
- QA-P7.QA-P8 // QA-P2.QA-P4.QA-P6.
- QA-P7 is concyclic with QA-P2, QA-P8 and QA-P23.

QA-P8: Midray Homothetic Center

The Midray Homothetic Center (QA-P8) is the homothetic center of the Reference Quadrangle with the Quadrangle composed of four Midray Circumcenters.

The Midray Circumcenters are the Circumcenters of the triangles $M_{ij}.M_{ik}.M_{il}$ for all permutations of $(i,j,k,l) \in (1,2,3,4)$, where $M_{ij} = \text{Midpoint}(P_i, P_j)$, etc.



1st CT-Coordinate:

$$-a^4 q^2 r^2 + 2 b^2 S_C p^2 r^2 + 2 c^2 S_B p^2 q^2 + 4 a^2 S_A p q r (p+q+r) + p q r (4 S^2 p + a^2 c^2 q + a^2 b^2 r)$$

1st DT-Coordinate:

$$(3 S^2 - S_A^2) p^4 - (3 S^2 + (S_A - 3 c^2) S_B) p^2 q^2 - (3 S^2 + (S_A - 3 b^2) S_C) p^2 r^2 - 3 a^4 q^2 r^2$$

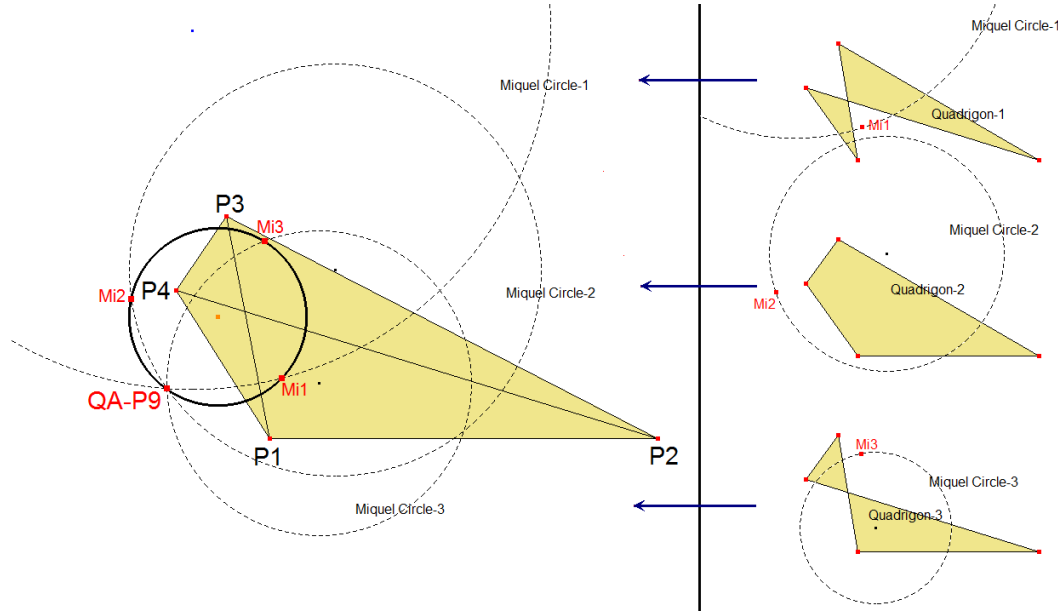
Properties:

- QA-P8 lies on this line:
– QA-P4.QA-P23
- QA-P8 is collinear with QA-P1 (Centroid) and the Reflection of QA-P4 in QA-P2.
- QA-P7.QA-P8 // QA-P2.QA-P4.QA-P6.
- QA-P8 is concyclic with QA-P2, QA-P7 and QA-P23.

QA-P9: QA-Miquel Center

Derived from the famous Miquel Point occurring in the QL-environment, there also is a Miquel Center in the QA-environment.

It is the common point of the 3 Miquel Circles constructed in the 3 Component Quadrilaterals of the Reference Quadrangle.



1st CT-Coordinate:

$$a^2 T_1 T_2 (-a^2 T_3 T_4 / (q+r) + b^2 T_3 T_6 / (p+r) + c^2 T_4 T_5 / (p+q)) - a^2 b^2 c^2 (p+q+r) (-a^2 (p+q+r) T_7 T_8 / (q+r) + T_2 T_5 T_9 / (p+r) + T_1 T_6 T_7 / (p+q))$$

where:

$$\begin{aligned} T_1 &= +a^2 q^2 + b^2 p^2 + 2 S_C p q & T_2 &= +a^2 r^2 + c^2 p^2 + 2 S_B p r \\ T_3 &= +b^2 r + S_A q & T_4 &= +c^2 q + S_A r \\ T_5 &= +b^2 p + S_C q & T_6 &= +c^2 p + S_B r \\ T_7 &= +q (S_C r - S_B q) + p (S_A q + b^2 r) \\ T_8 &= +r (S_B q - S_C r) + p (S_A r + c^2 q) \\ T_9 &= +r (S_0 q + S_A r) + p (-S_C r + c^2 q) \end{aligned}$$

Properties:

- QA-P9 is concyclic with the 3 vertices of the Miquel Triangle (see QA-Tr2).
- QA-P9 is the Reflection of the intersection point of the QA-Cu1 Cubic and its asymptote in the circumcenter of the Miquel Triangle (note Eckart Schmidt).
- The 3 mutual intersection points of the 3 construction circles unequal QA-P9 lie on the Nine-point Conic of the Circumcenter Quadrangle of the Reference Quadrangle.

The Circumcenter Quadrangle is the quadrangle formed by the Circumcenters of the 4 component triangles of the Reference Quadrangle.

- The Reference Quadrangle and the Quadrangle formed by the vertices of the QA-Diagonal Triangle and QA-P4 (Isogonal Center) share the same QA-Miquel Center.

QA-P10: Centroid of the QA-Diagonal Triangle

QA-P10 is the Centroid of the Diagonal Triangle of a Quadrangle.

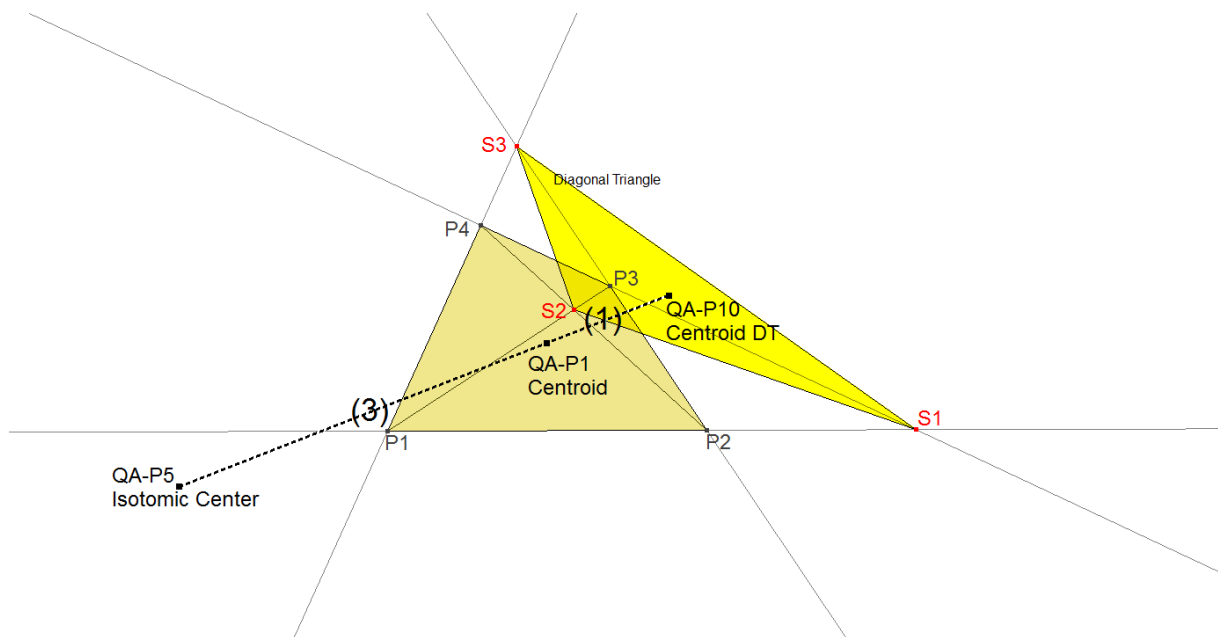
The Diagonal Triangle of a Quadrangle $P1.P2.P3.P4$ is the triangle built from the intersection points $S1 = P1.P2 \wedge P3.P4$, $S2 = P1.P3 \wedge P2.P4$ and $S3 = P1.P4 \wedge P2.P3$.

These points have CT-coordinates: $S1 = (p : q : 0)$, $S2 = (p : 0 : r)$, $S3 = (0 : q : r)$.

Because of the symmetry in $S1, S2, S3$ all Triangle Centers wrt $S1.S2.S3$ as described in [12] Clark Kimberling's Encyclopedia of Triangle Centers also will be Quadrangle Centers. However only those points contributing to the points derived from component Quadrilaterals or component triangles will be described here as Quadrangle Centers.

The Centroid of the Diagonal Triangle does contribute to the points described earlier.

The relation with the Isotomic Center QA-P5 is most special.



1st CT-Coordinate:

$$p (q + r) (2 p + q + r)$$

1st DT-Coordinate:

$$1$$

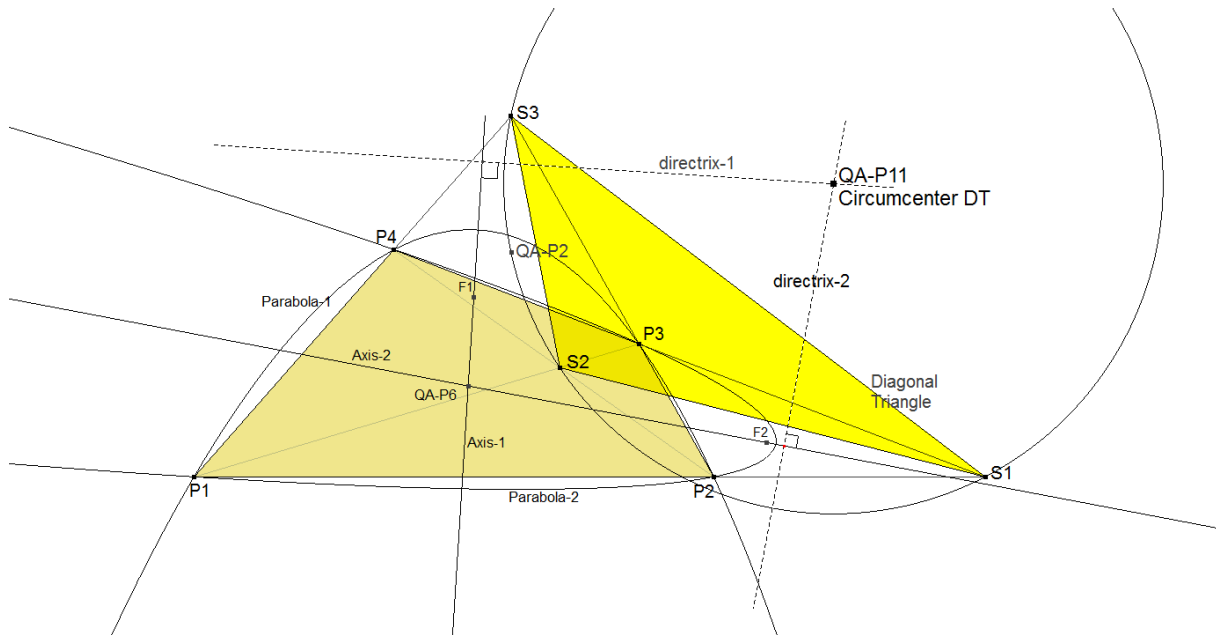
Properties:

- QA-P10 lies on these QA-lines:
 - QA-P1.QA-P5 (4 : -1)
 - QA-P2.QA-P29 (2 : 1 => QA-P29=Complement of QA-P2 wrt QA-DT)
 - QA-P4.QA-P28 (4 : -1)
 - QA-P11.QA-P12 (1 : 2 => QA-P11=Complement of QA-P19 wrt QA-DT)
 - QA-P16.QA-P19 (1 : 2 => QA-P16=Complement of QA-P19 wrt QA-DT)
 - QA-P30.QA-P36 (2 : 1 => QA-P36=Complement of QA-P30 wrt QA-DT)

- QA-P10 lies on this QG-line:
 - QG-P1.QG-P2 (2 : 1 => QG-P2=Complement of QA-P1 wrt QA-DT)
- QA-P10 is the Reflection of QA-P25 in QA-P26.
- QA-P5.QA-P1 : QA-P1.QA-P10 : QA-P10.QA-P20 = 3 : 1 : 2.
 This is the same ratio as used in the construction of the Quadrangle Centroid G using component triangles (when G_i = Centroid $P_j.P_k.P_l$ then $P_i.G : G.G_i = 3 : 1$).
 As a consequence Quadrangle S1.S2.S3.QA-P5 and the Reference Quadrangle P1.P2.P3.P4 share the same Centroid (QA-P1).
- QA-P10 is the Involutary Conjugate (see QA-Tf2) of QA-P16.

QA-P11: Circumcenter of the QA-Diagonal Triangle

QA-P11 is the Circumcenter of the Diagonal Triangle of a Quadrangle.



1st CT-Coordinate:

$$-a^2 q r (2 S_A p^2 q r + T_A) + (S_C T_B p r + S_B T_C p q) + 2 S^2 p^2 q r (q+r) (p+q+r),$$

where:

$$T_A = -a^2 q^2 r^2 + b^2 p^2 r^2 + c^2 p^2 q^2$$

$$T_B = +a^2 q^2 r^2 - b^2 p^2 r^2 + c^2 p^2 q^2$$

$$T_C = +a^2 q^2 r^2 + b^2 p^2 r^2 - c^2 p^2 q^2$$

1st DT-Coordinate:

$$a^2 S_A$$

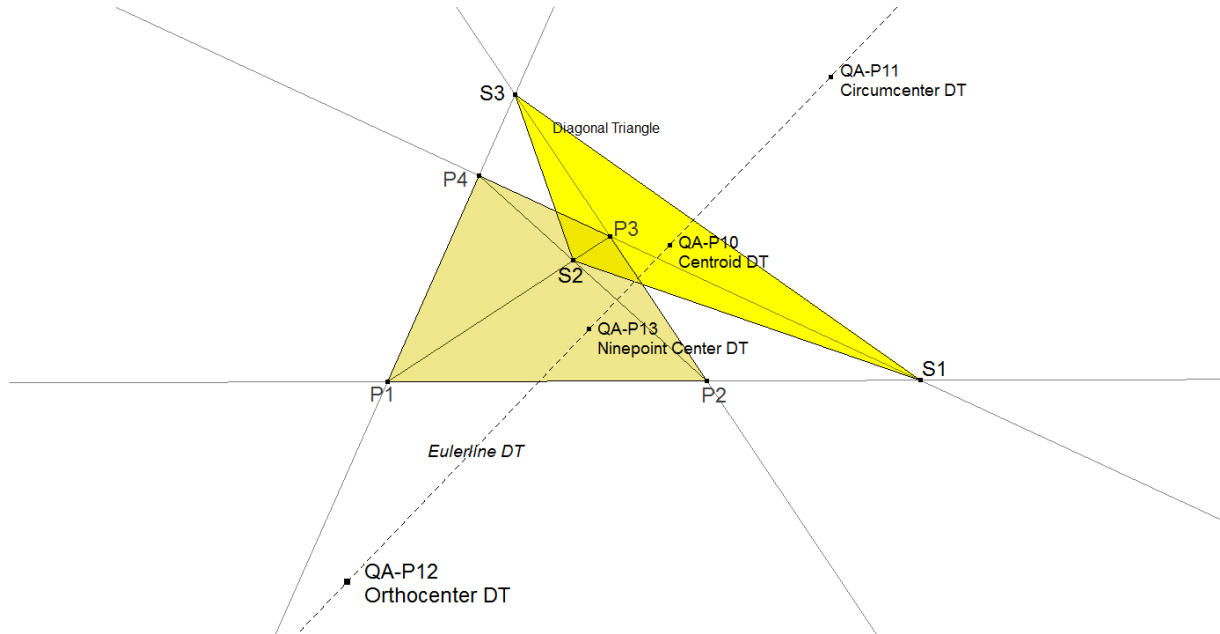
Properties:

- QA-P11 lies on these QA-lines:
 - QA-P2.QA-P30 (1 : 1 => QA-P11=Midpoint QA-P2.QA-P30)
 - QA-P10.QA-P12 (1 : 2 => QA-P11=Complement of QA-P12 wrt QA-DT)
- QA-P11 point is the center of the circumscribed circle through the vertices of the QA-Diagonal Triangle. This circle is interesting because QA-P2 (Euler-Poncelet Point) is situated on it.
- QA-P11 is the intersection point of the directrices of the 2 parabolas of the Reference Quadrangle.
- QA-P11 is collinear with QA-P10 (Centroid DT), QA-P12 (Orthocenter DT), QA-P13 (Nine-point Center DT) on the Euler Line of the QA-Diagonal Triangle.
- QA-P11 is the Gergonne-Steiner point (QA-P3) as well as the Isogonal Center (QA-P4) from the Quadrangle formed by the vertices of the Diagonal Triangle and QA-P2 (Euler-Poncelet Point).

- QA-P11 is concyclic with the Involutary Conjugate (see QA-Tf2) of QA-P11 and the foci of the QA-Parabolas QA-2Co1.

QA-P12: Orthocenter of the QA-Diagonal Triangle

QA-P12 is the Orthocenter of the Diagonal Triangle of a Quadrangle.



1st CT-Coordinate:

$$(2 S_A p^2 q r + T_A) (a^2 q r + S_B p q + S_C p r),$$

where:

$$T_A = -a^2 q^2 r^2 + b^2 p^2 r^2 + c^2 p^2 q^2$$

1st DT-Coordinate:

$$S_B S_C$$

Properties:

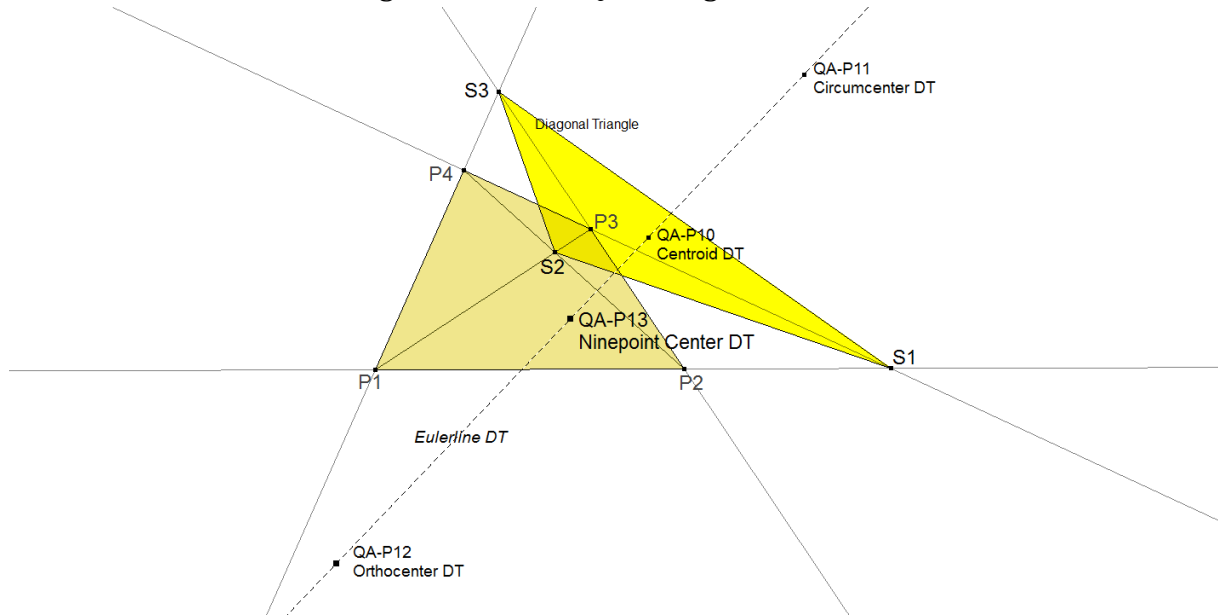
- QA-P12 lies on these QA-lines:
 - QA-P2.QA-P36 (2 : -1 => QA-P12 is Reflection of QA-P2 in QA-P36)
 - QA-P10.QA-P11 (-2 : 3)
 - QA-P14.QA-P33 (-2 : 3)
 - QA-P29.QA-P30 (-1 : 2 => QA-P12 is Reflection of QA-P30 in QA-P29)
- QA-P12 is the Involutary Conjugate (see QA-Tf2) of QA-P23.

QA-P13: Nine-point Center of the QA-Diagonal Triangle

QA-P13 is the Nine-point Center of the Diagonal Triangle of a Quadrangle.

It is also the center of the circumcircle of the Medial Triangle (MT) of the QA-Diagonal Triangle (DT).

The sides of the MT are tangential to both Quadrangle Parabolas.



1st CT-Coordinate:

$$-a^2 q r (2 S_A p^2 q r + T_A) + (S_C T_B p r + S_B T_C p q) - 2 S^2 p^2 q r (q+r) (3p+q+r),$$

where:

$$T_A = -a^2 q^2 r^2 + b^2 p^2 r^2 + c^2 p^2 q^2$$

$$T_B = +a^2 q^2 r^2 - b^2 p^2 r^2 + c^2 p^2 q^2$$

$$T_C = +a^2 q^2 r^2 + b^2 p^2 r^2 - c^2 p^2 q^2$$

1st DT-Coordinate:

$$S^2 + S_B S_C$$

Properties:

- QA-P13 lies on these QA-lines:
 - QA-P10.QA-P11 $(-1 : 3)$
 - QA-P29.QA-P36 $(1 : 1 \Rightarrow \text{QA-P13} = \text{Midpoint QA-P29.QA-P36})$
 - QA-P30.QA-P35 $(5 : -1)$
- QA-P13 = Midpoint of QA-P11.QA-P12.
- QA-P13 = the center of QA-Ci2, the circumcircle of the Medial Triangle (MT) of the QA-Diagonal Triangle (DT).
- QA-P13 = QA-centroid (QA-P1) of the quadrangle formed by the vertices of the Diagonal Triangle and QA-P12.
- QA-P13 = QA-Centroid of Quadrangle QA-P2.QA-P3.QA-P12.QA-P20.

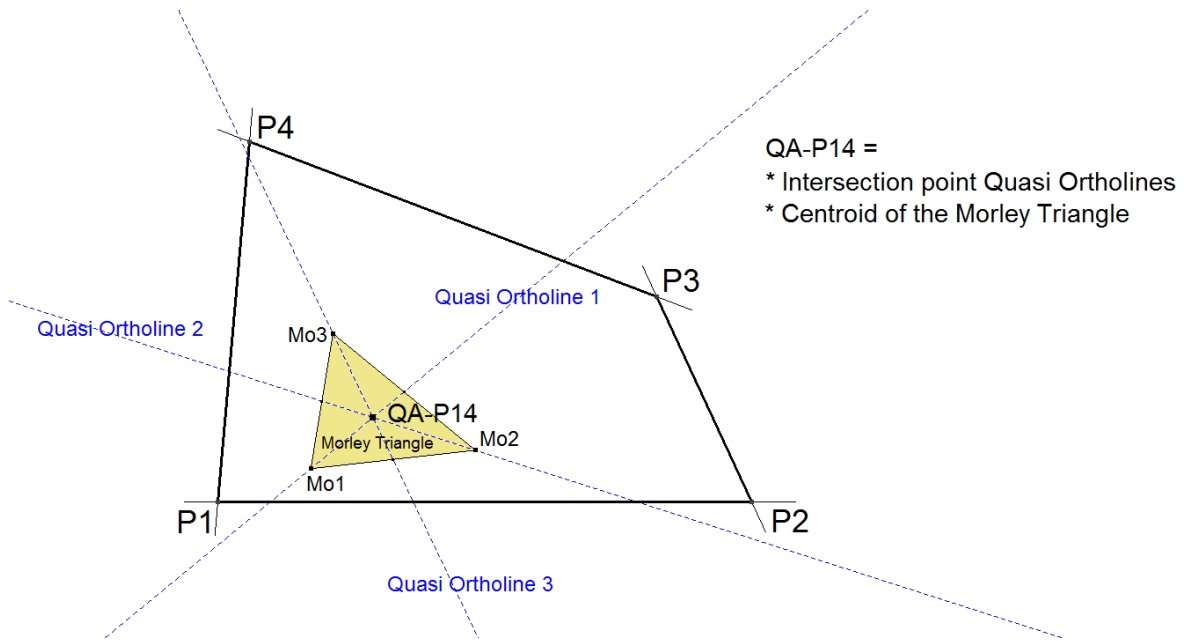
QA-P14: Centroid of the Morley Triangle

The QL-Morley Points (QL-P2) of the 3 Quadrilaterals of the Reference Quadrangle form a triangle Mo1.Mo2.Mo3.

The QL-Quasi Ortholines (see paragraph QL-L6: Quasi Ortholine) of the 3 Quadrilaterals of the Reference Quadrangle pass through Mo1, Mo2, Mo3 and happen to be the medians of the Morley Triangle.

Their common intersection point is the QA-Quasi Ortholine Point.

This point is also the Centroid of the Morley Triangle.



1st CT-coordinate:

$$a^2 S_A T_a - (b^2 S_B + c^2 S_C) T_{bc} + (b^2 S_B - c^2 S_C) p (q - r) (q + r)^2 + 2 a^2 b^2 (p - q) (p + q)^2 r + 2 a^2 c^2 q (p - r) (p + r)^2$$

where:

$$T_a = (3 p^2 q^2 + 3 p q^3 + 2 p^2 q r + 9 p q^2 r + 5 q^3 r + 3 p^2 r^2 + 9 p q r^2 + 6 q^2 r^2 + 3 p r^3 + 5 q r^3)$$

$$T_{bc} = (q + r) (6 p^3 + 9 p^2 q + 9 p^2 r + 4 p q^2 + 4 p r^2 + 3 q^2 r + 3 q r^2 + 10 p q r)$$

1st DT-coordinate:

$$-2 S^2 p^4 - S_B S_C (-p^2 + q^2 + r^2)^2 + 2 (S^2 + 2 S_B c^2) p^2 q^2 + 2 (S^2 + 2 S_C b^2) p^2 r^2 - 4 (a^4 - S_B S_C) q^2 r^2$$

Properties:

- QA-P14 lies on these QA-lines:
 - QA-P1.QA-P24 (1 : 2)
 - QA-P12.QA-P33 (2 : 1)
- QA-P14 divides QA-P12.QA-P33 in line segments with ratio 2:1 (QA-P33 = Complement of QA - P12 wrt the Morley Triangle).
- QA-P14 divides QA-P24.QA-P1 in line segments with ratio 2:1 (QA-P24 = AntiComplement of QA - P12 wrt the Morley Triangle).
- QA-P14 is the Centroid of Triangle QA-P12.QA-P24.QA-P37.

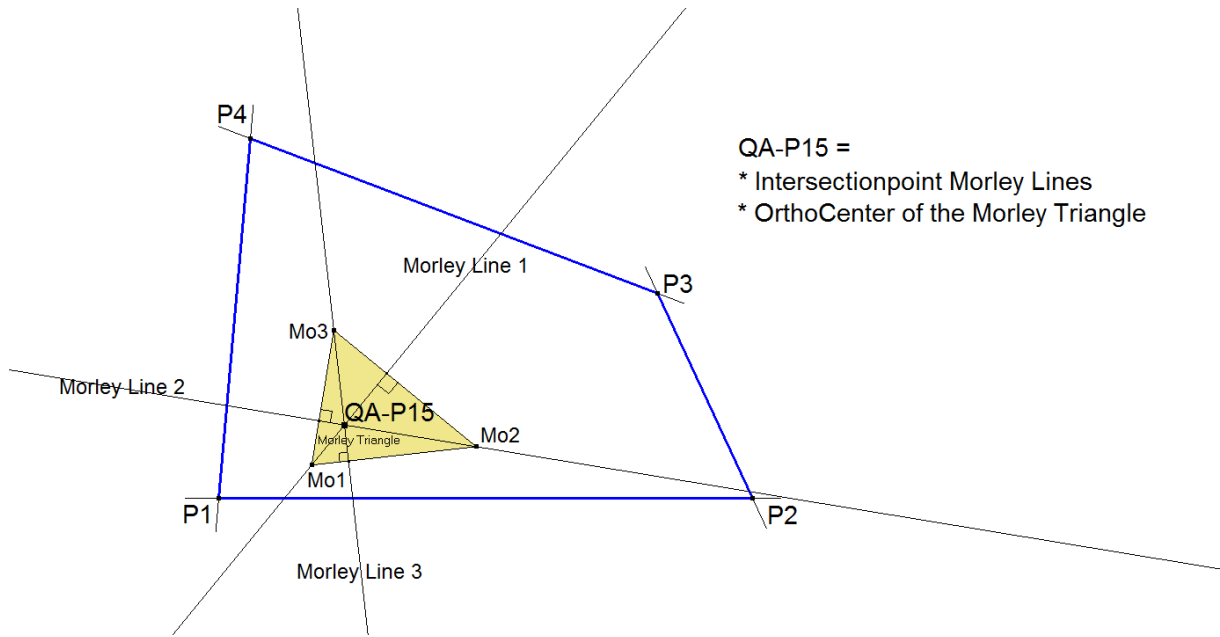
QA-P15: OrthoCenter of the Morley Triangle

The QL-Morley Points (QL-P2) of the 3 Quadrilaterals of the Reference Quadrangle form a triangle Mo1.Mo2.Mo3.

The QL-Morley Lines (QL-L4) of the 3 Quadrilaterals of the Reference Quadrangle pass through Mo1, Mo2, Mo3. So their common intersection point could be called the QA-OrthoPoint.

The QL-Morley Lines also happen to be the altitudes of the Morley Triangle.

So their common intersection point is also the OrthoCenter of the Morley Triangle.



1st CT-coordinate:

$$\begin{aligned}
 & a^4 (p + q) (p + r) (p (q^2 + r^2) + (q + r) (q^2 + q r + r^2)) \\
 & -b^4 (p + q) (q + r) (2 p^3 + r (q + r)^2 + p (q + r) (q + 3 r) + p^2 (3 q + 5 r)) \\
 & -c^4 (p + r) (q + r) (2 p^3 + q (q + r)^2 + p (q + r) (3 q + r) + p^2 (5 q + 3 r)) \\
 & +a^2 b^2 (p + q) (2 p^3 q + q r (q + r)^2 + p r (q + r) (2 q + r) + p^2 (2 q^2 + 5 q r + r^2)) \\
 & +a^2 c^2 (p + r) (2 p^3 r + q r (q + r)^2 + p q (q + r) (2 r + q) + p^2 (2 r^2 + 5 q r + q^2)) \\
 & +b^2 c^2 (q + r) (4 p^4 + 10 p^3 (q + r) + 2 q r (q + r)^2 + p (q + r) (3 q + r) (q + 3 r) + p^2 (9 q^2 + 22 q r + 9 r^2))
 \end{aligned}$$

1st DT-coordinate:

$$\begin{aligned}
 & -2 S_A^2 p^4 (p^2 + q^2 - r^2) (p^2 - q^2 + r^2) - S_B^2 q^2 (3 (p^2 - r^2)^3 - q^2 (p^2 - r^2) (3 p^2 + 5 r^2) \\
 & - q^4 (-p^2 + q^2 + r^2)) - S_C^2 r^2 (3 (p^2 - q^2)^3 - (p^2 - q^2) (3 p^2 + 5 q^2) r^2 - r^4 (-p^2 + q^2 + r^2)) \\
 & + S_B S_C (-8 q^2 r^2 (2 p^4 - p^2 q^2 - p^2 r^2 + 2 q^2 r^2) + (-p^2 + q^2 + r^2)^4) \\
 & -8 S_A p^4 (S_B (p^2 - q^2) q^2 + S_C (p^2 - r^2) r^2) + S^2 p^2 (4 q^2 r^2 (p^2 + q^2 + r^2) + (p^2 - q^2 - r^2)^3)
 \end{aligned}$$

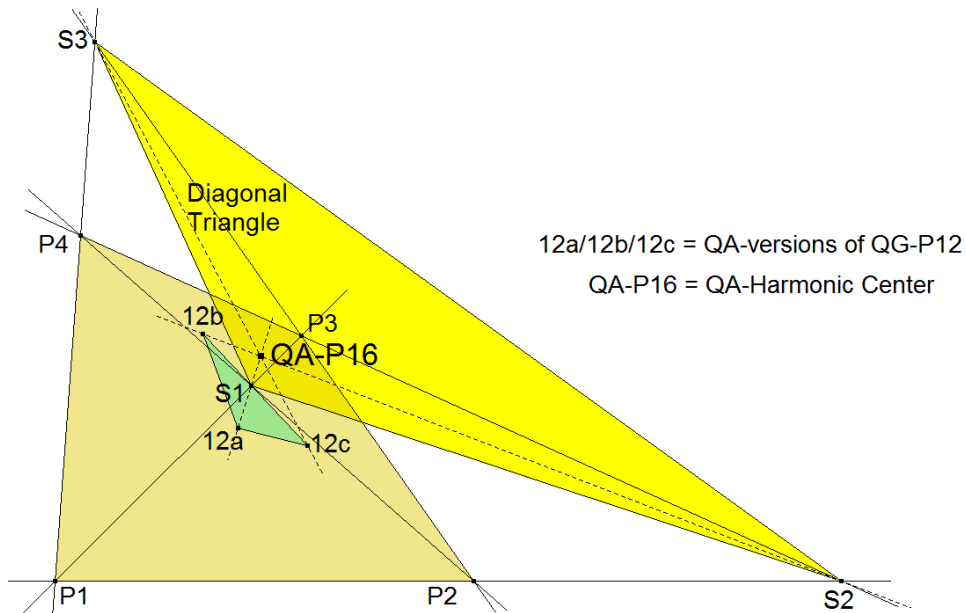
Properties:

- QA-P15 lies on the line QA-L6 (QA-Newton-Morley Line).
- QA-P15 is the circumcenter of the QG-P10 circle in the QA-environment (QG-P10=2nd Quasi Orthocenter) (note Eckart Schmidt).

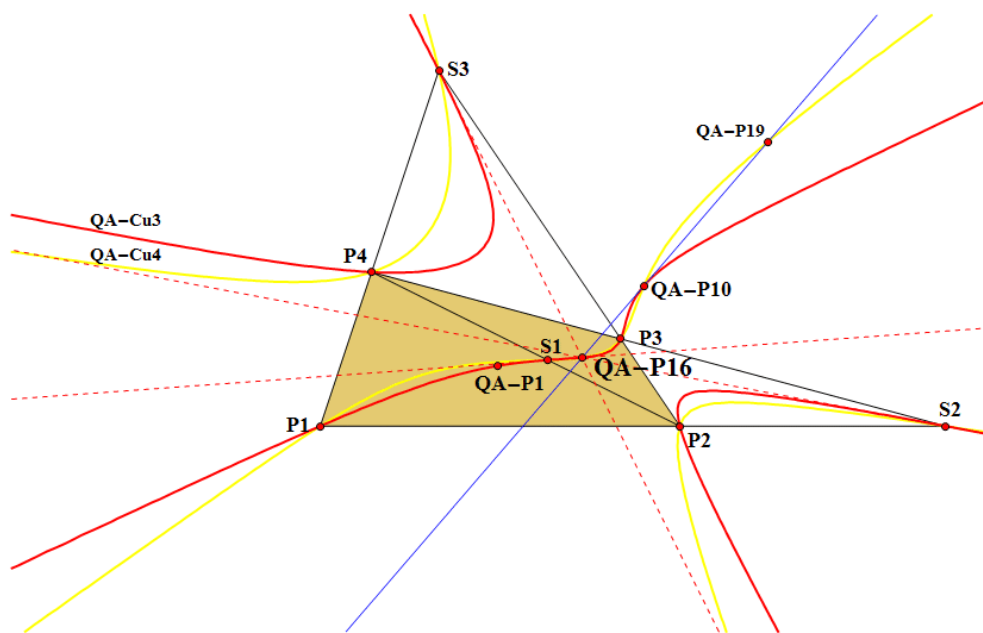
QA-P16: QA-Harmonic Center

The triangle formed by the 3 QA-versions of QG-P12 (Inscribed Harmonic Conic Center) is perspective with the QA-Diagonal Triangle. Their Perspector is QA-P16.

QA-P16 partakes in many QA-Parallelities and with many QA-Crosspoints (see QA/3 and QA/5).



QA-P16 is the perspector of the triangles formed by the 3 QA-versions of QG-P12 (Inscribed Harmonic Conic Center) and the QA-Diagonal Triangle.



QA-P16 is the intersection point of the tangents at the vertices of the QA-Diagonal Triangle and QA-P10 to the QA-DT-P10 Cubic (QA-Cu3).

Construction:

Construct QA-P16 as a complement of the Isotomic Conjugate of the AntiComplement of QA-P1 wrt the QA-Diagonal Triangle

1st CT-coordinate:

$$p(2p + q + r)$$

1st DT-coordinate:

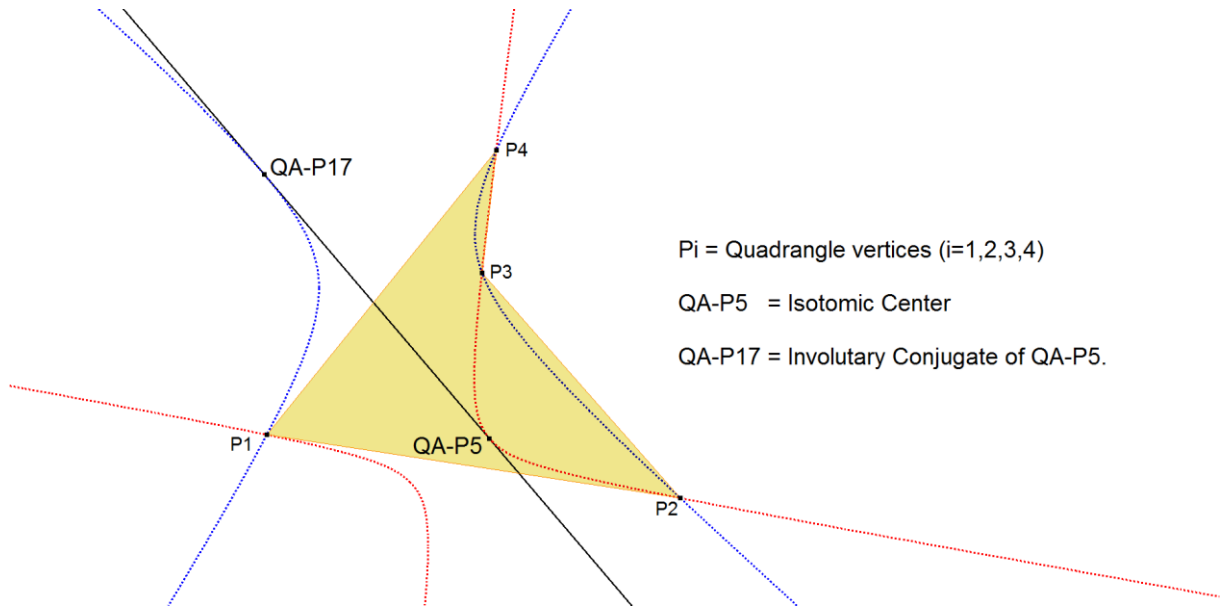
$$p^2$$

Properties:

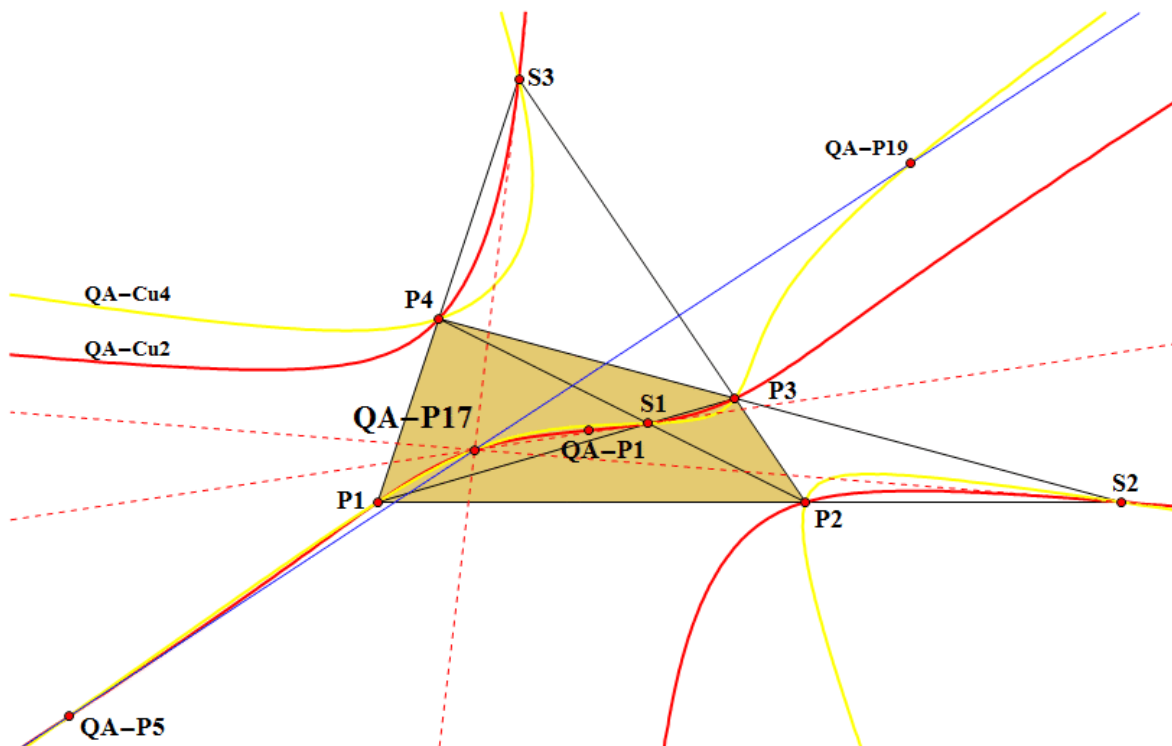
- QA-P16 lies on these QA-lines:
 - QA-P1.QA-P21 $(-1 : 2 \Rightarrow \text{QA-P16} = \text{Reflection QA-P21 in QA-P1})$
 - QA-P10.QA-P19 $(-1 : 3)$
- QA-P16 lies on this QG-line:
 - QG-P1.QG-P12 = QG-L2
- QA-P16 is the Reflection of QA-P19 in QA-P31.
- QA-P16 is the Involutary Conjugate (see QA-Tf2) of QA-P10.
- QA-P16 = QA-P10-Ceva conjugate of QA-P1 wrt the QA-Diagonal Triangle.
- QA-P16 is collinear with QG-P1, QG-P12, QG-P13, QL-P13 on QG-L2.
- QA-P16 is the 4th Perspective Point in the row QG-P13, QG-P12, QG-P1 on line QG-L2 (see [26] Perspective Fields part II).
- QA-P16 lies on the Conic QA-Co5.
- QA-P16 lies on the Cubics QA-Cu3 and QA-Cu4.
- QA-P16 is the pole of the Cubics QA-Cu1 – QA-Cu5 when seen as IsoCubics wrt the QA-Diagonal Triangle and with the Involutary Conjugate as Isoconjugation.
- QA-P16 is the intersection point of the tangents at the vertices of the QA-Diagonal Triangle and QA-P10 to the QA-DT-P10 Cubic (QA-Cu3).
- QA-P16 is the Complement of QA-P19 wrt the QA-Diagonal Triangle.
- QA-P16 is the AntiComplement of QA-P31 wrt the QA-Diagonal Triangle.
- When the Reference Quadrangle is convex, then QA-P16 lies in the overlap of the Quadrangle and its QA-Diagonal Triangle.
- The 3 variants of QG-L2 in a Quadrangle concur in QA-P16.

QA-P17: Involutory Conjugate of QA-P5

QA-P17 is the Involutory Conjugate of QA-P5.



As a consequence the line QA-P5.QA-P17 is the common tangent of the circumscribed conics P1.P2.P3.P4.QA-P5 and P1.P2.P3.P4.QA-P17.



QA-P17 is the intersection point of the tangents at the vertices of the QA-Diagonal Triangle and QA-P5 to the QA-DT-P5 Cubic (QA-Cu2).

1st CT-coordinate:

$$q r (2 p + q + r) (q r (q + r)^2 - p (p + q + r) (q^2 + r^2))$$

1st DT-coordinate:

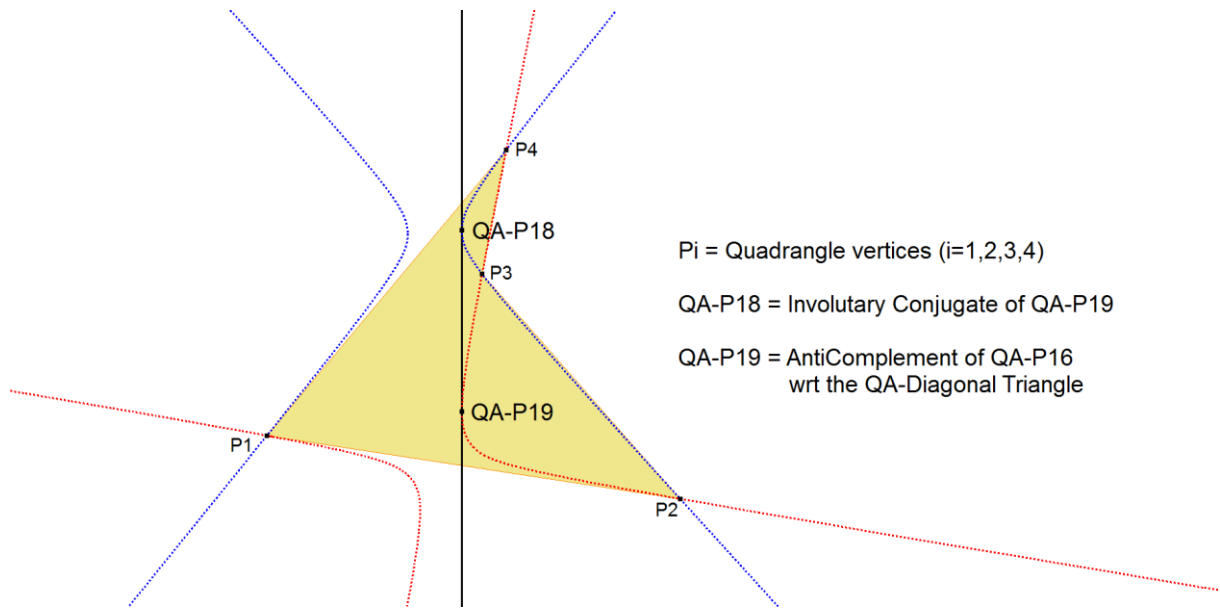
$$p^2 / ((p^4 - (q^2 - r^2)^2) - 2 p^2 (-p^2 + q^2 + r^2))$$

Properties:

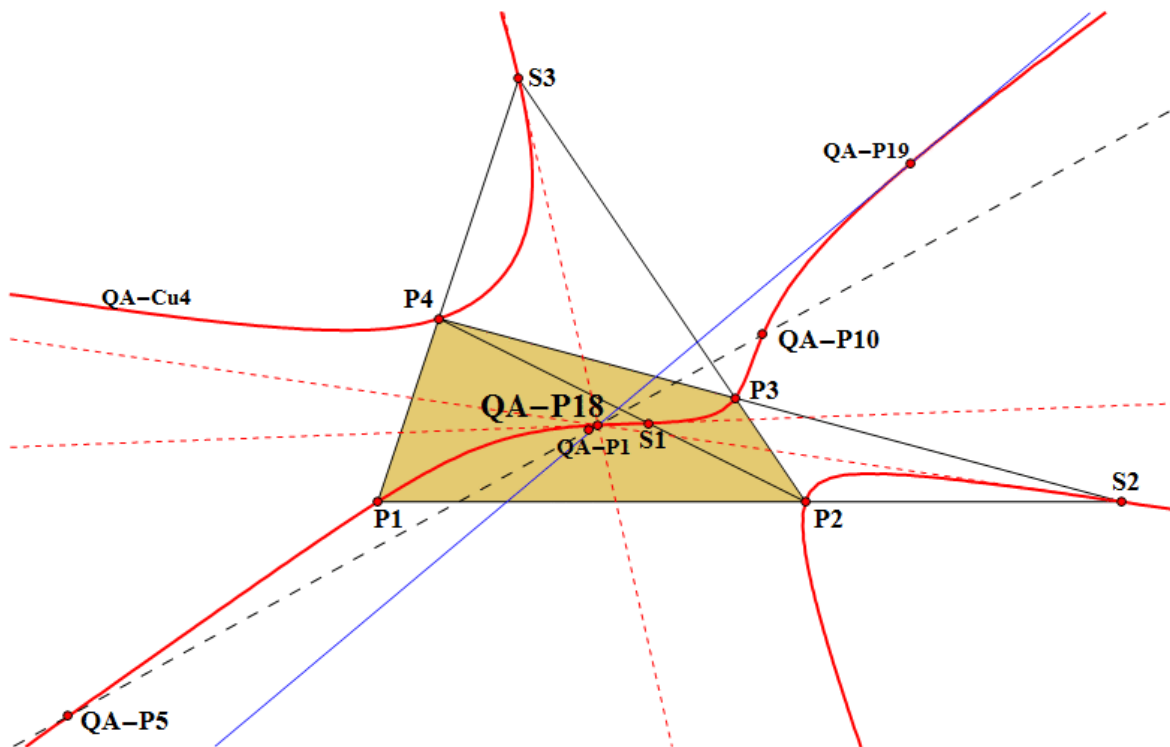
- QA-P17 lies on this QA-line:
 - QA-P5.QA-P19
- QA-P17 is the Involutary Conjugate (see QA-Tf2) of QA-P5.
- QA-P17 lies on the Conic QA-Co5.
- QA-P17 lies on the Cubics QA-Cu2 and QA-Cu4.
- QA-P17 is the intersection point of the tangents at the vertices of the QA-Diagonal Triangle and QA-P5 to the QA-DT-P5 Cubic (QA-Cu2).
- QA-P17 is the Isotomic Center of the Quadrangle formed by the vertices of the QA-Diagonal Triangle and QA-P5 (Isotomic Center).

QA-P18: Involutory Conjugate of QA-P19

QA-P18 is the Involutory Conjugate of QA-P19.



As a consequence the line QA-P18.QA-P19 is the common tangent of the circumscribed conics $P_1.P_2.P_3.P_4.QA-P18$ and $P_1.P_2.P_3.P_4.QA-P19$.



QA-P18 is the intersection point of the tangents at the vertices of the QA-Diagonal Triangle and QA-P19 to the QA-DT-P19 Cubic (QA-Cu4).

1st CT-coordinate:

$$p (q^2 + r^2) (2 p + q + r) (p^2 + p q + p r - q r)$$

1st DT-coordinate:

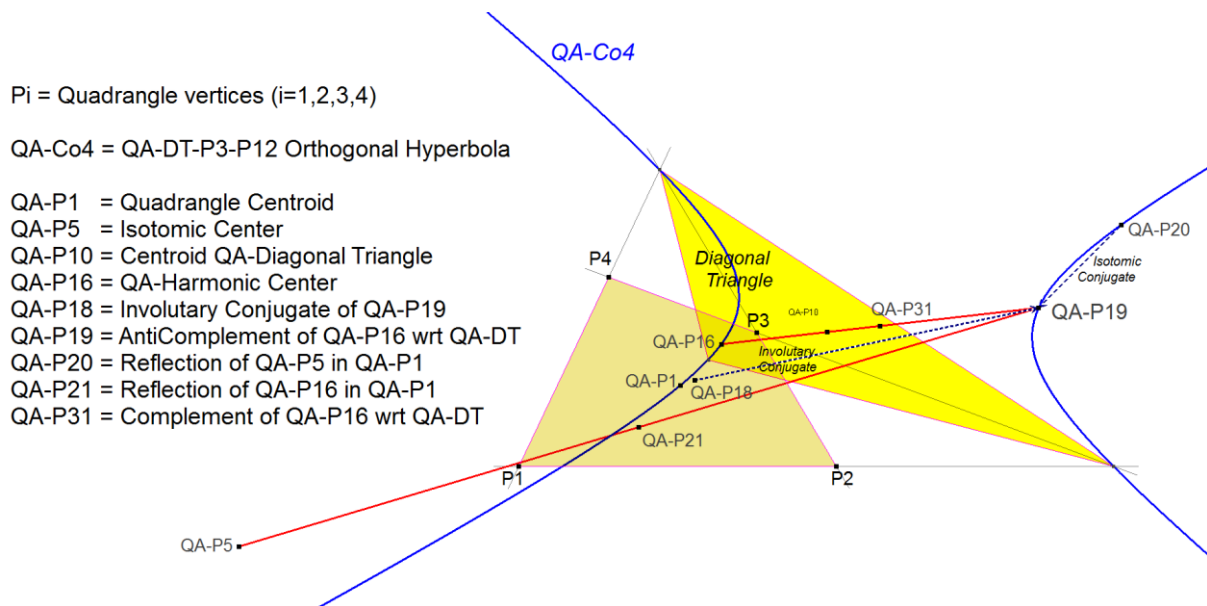
$$p^2 / (-p^2 + q^2 + r^2)$$

Properties:

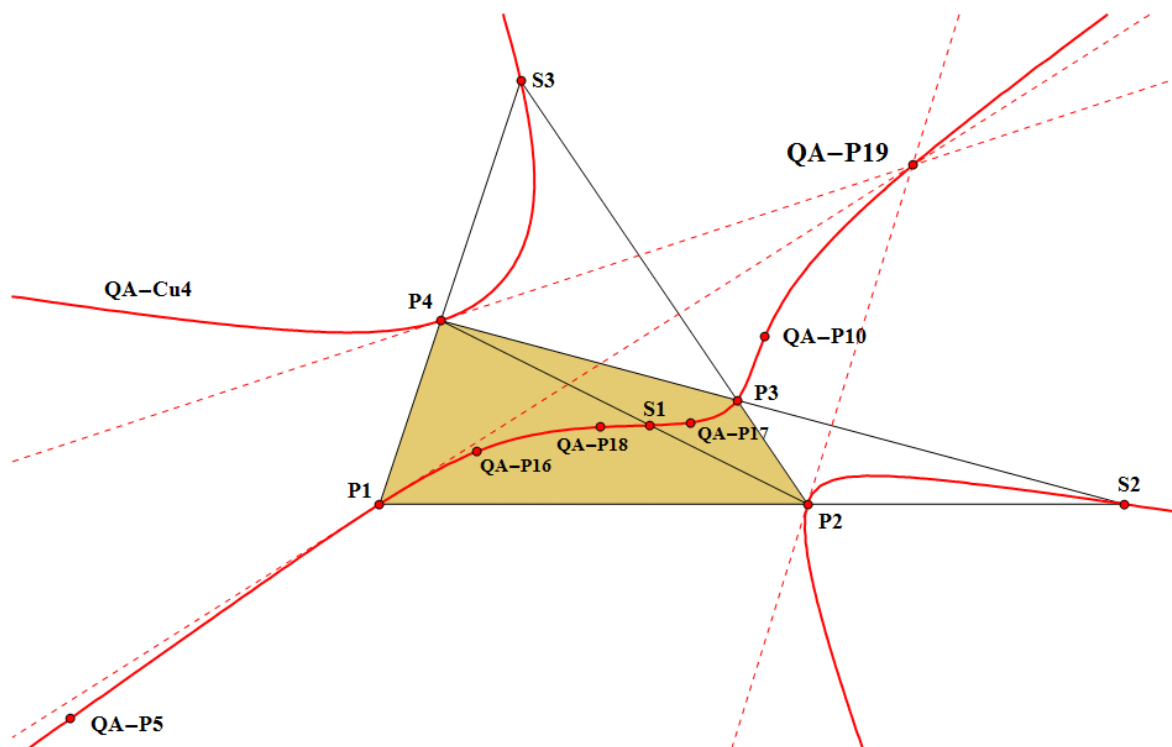
- QA-P18 lies on this QA-line:
 - QA-P1.QA-P5 = QA-L3.
- QA-P18 is the Involuntary Conjugate (see QA-Tf2) of QA-P19.
- QA-P18 lies on the line QA-P1.QA-P5.
- QA-P18 lies on the QA-DT-P19 Cubic (QA-Cu4).
- QA-P18 also lies on the tangent at QA-P19 to the QA-DT-P19 Cubic (QA-Cu4), which is also the tangent at QA-P19 to the Conic (P1,P2,P3,P4,QA-P19), which is also the tangent at QA-P18 to the Conic (P1,P2,P3,P4,QA-P18).

QA-P19: AntiComplement of QA-P16 wrt the QA-Diagonal Triangle

QA-P19 is the AntiComplement of QA-P16 wrt the QA-Diagonal Triangle.



QA-P19 has several properties. See properties below.



QA-P19 is the intersection point of the tangents at the vertices of the Reference Quadrangle to the QA-DT-P19 Cubic (QA-Cu4).

1st CT-coordinate:

$$p (q + r) (q^2 + r^2) (2 p + q + r)$$

1st DT-coordinate:

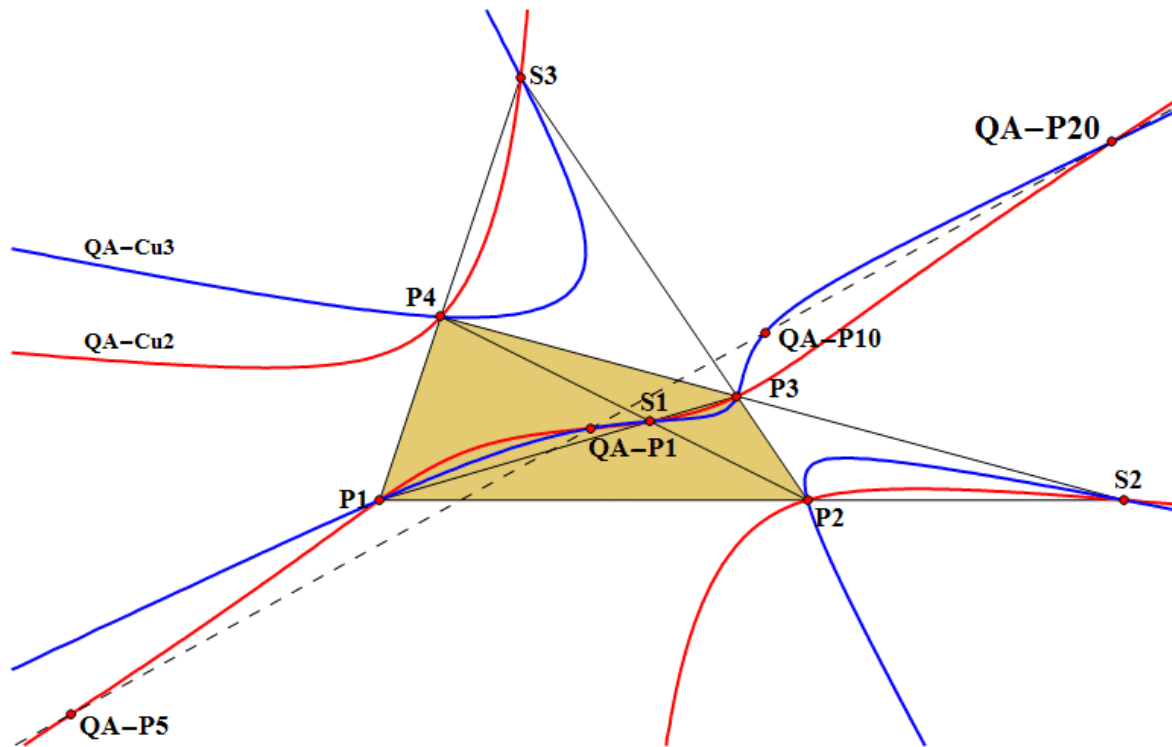
$$-p^2 + q^2 + r^2$$

Properties:

- QA-P19 lies on these QA-lines:
 - QA-P5.QA-P17
 - QA-P10.QA-P16 (3 : -2 => QA-P19 = AntiComplement of QA-P16 wrt QA-DT)
- QA-P19 is the Reflection of QA-P5 in QA-P21.
- QA-P19 is the Reflection of QA-P16 in QA-P31.
- QA-P19 is the Involutary Conjugate (see QA-Tf2) of QA-P18.
- QA-P19 is the Isotomic Conjugate of QA-P20 wrt the QA-Diagonal Triangle.
- QA-P19 is the Anticomplement of QA-P16 wrt the QA-Diagonal Triangle.
- QA-P19 lies on the Conic QA-Co5.
- QA-P19 lies on the Cubic QA-Cu4.

QA-P20: Reflection of QA-P5 in QA-P1

QA-P20 is the Reflection of QA-P5 (Isotomic Center) in QA-P1 (QA-Centroid).



1st CT-coordinate:

$$(q + r) (2 p + q + r) (p^2 + p q + p r - q r)$$

1st DT-coordinate:

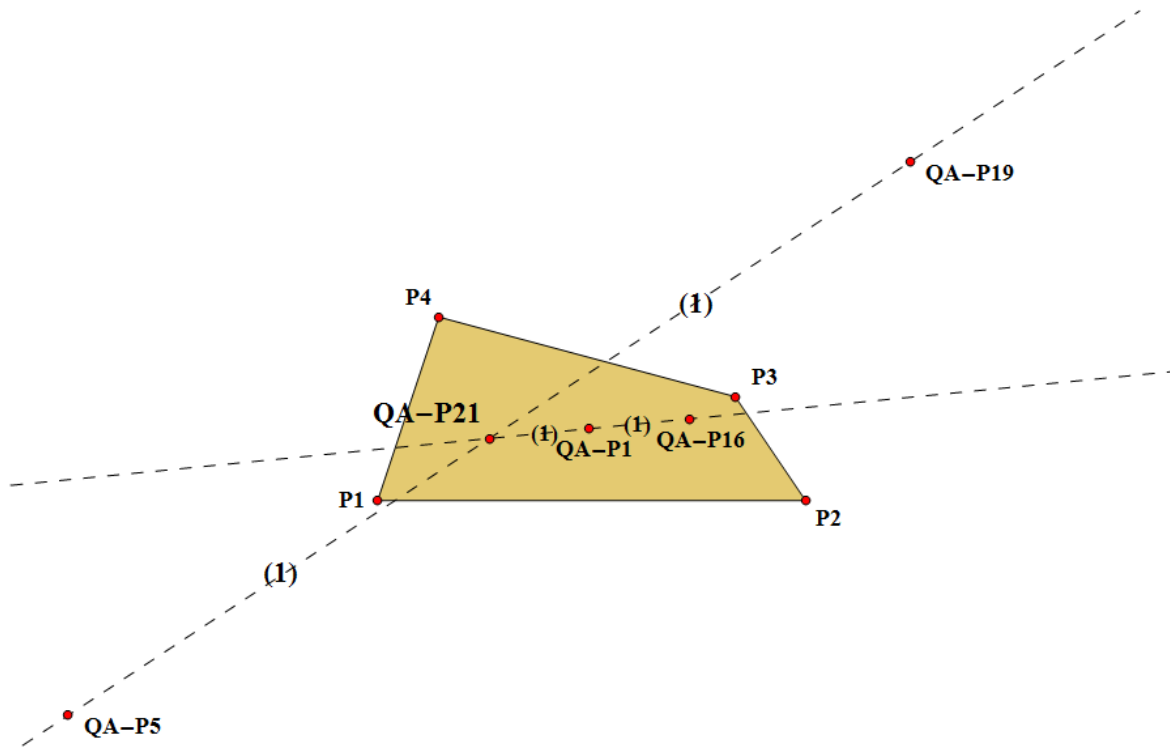
$$1 / (-p^2 + q^2 + r^2)$$

Properties:

- QA-P20 lies on these QA-lines:
 - QA-P1.QA-P5 $(-1 : 2 \Rightarrow \text{QA-P20} = \text{Reflection of QA-P5 in QA-P1})$
 - QA-P3.QA-P29 $(1 : 1 \Rightarrow \text{QA-P20} = \text{Reflection QA-P3 in QA-P29})$
 - QA-P21.QA-P31 $(2 : -1 \Rightarrow \text{QA-P20} = \text{Reflection of QA-P21 in QA-P31})$
 - QA-P34.QA-P35 $(5 : -3)$
- QA-P20 is also the Reflection of:
 - QA-P1 in QA-P22.
- QA-P20 lies on the line QA-P1.QA-P5.QA-P10 in harmonic position:
 $\text{QA-P5.QA-P1} : \text{QA-P1.QA-P10} : \text{QA-P10.QA-P20} = 3 : 1 : 2.$
- QA-P20 is the Involutory Conjugate (see QA-Tf2) of QA-P1.
- QA-P20 is the Isotomic Conjugate of QA-P19 wrt the QA-Diagonal Triangle.
- QA-P20 is the Anticomplement of QA-P1 wrt the QA-Diagonal Triangle.
- QA-P20 lies on the Conics QA-Co4 and QA-Co5..
- QA-P20 lies on the Cubics QA-Cu2, QA-Cu3 and QA-Cu5.

QA-P21: Reflection of QA-P16 in QA-P1

QA-P21 is the Reflection of QA-P16 (QA-Harmonic Center) in QA-P1 (QA-Centroid) as well as the Midpoint of QA-P5 (Isotomic Center) and QA-P19 (AntiComplement of QA-P16 wrt QA-DT).



1st CT-coordinate:

$$(2p + q + r)(q^2 + qr + r^2)$$

1st DT-coordinate:

$$-p^2 (2(-p^4 + q^4 + r^4) + (-p^2 + q^2 + r^2)^2)$$

Properties:

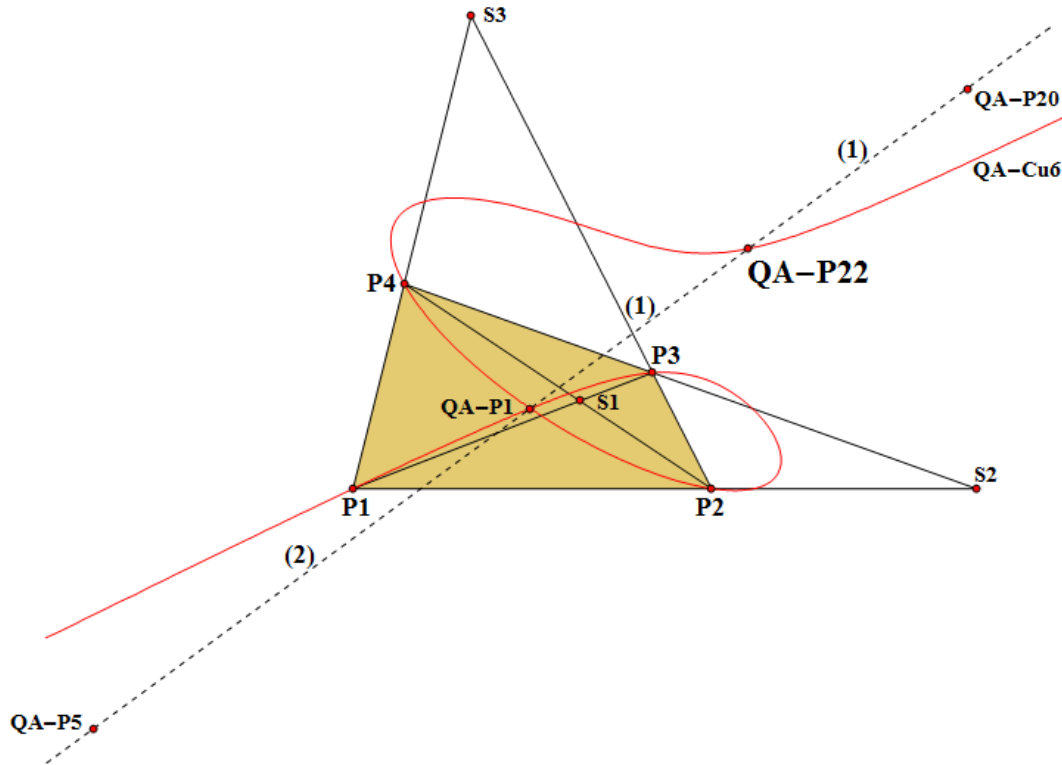
- QA-P21 lies on these QA-lines:
 - QA-P1.QA-P16 $(-1 : 2 \Rightarrow \text{QA-P21 is Reflection QA-P16 in QA-P1})$
 - QA-P5.QA-P17
 - QA-P20.QA-P31 $(2 : -1 \Rightarrow \text{QA-P21 is Reflection QA-P20 in QA-P31})$
- QA-P21 is the Midpoint of QA-P5 and QA-P19
- QA-P21 is the Involutory Conjugate (see QA-Tf2) of QA-P27.

QA-P22: Midpoint QA-P1 and QA-P20

QA-P22 is the Midpoint of QA-P1 (QA-Centroid) and QA-P20 (Reflection of QA-P5 in QA-P1). It is also the Center of the Involution created by the intersection of QA-P1.QA-P5 with the 6 Quadrangle lines (see notes below).

Construction:

Construct the complement of QA-P1 wrt the QA-Diagonal Triangle (DT).



1st CT-coordinate:

$$(2p + q + r)(q + r)(3p^2 + 3pq + 3pr - qr)$$

1st DT-coordinate:

$$p^2(q^2 + r^2) - (q^2 - r^2)^2$$

Properties:

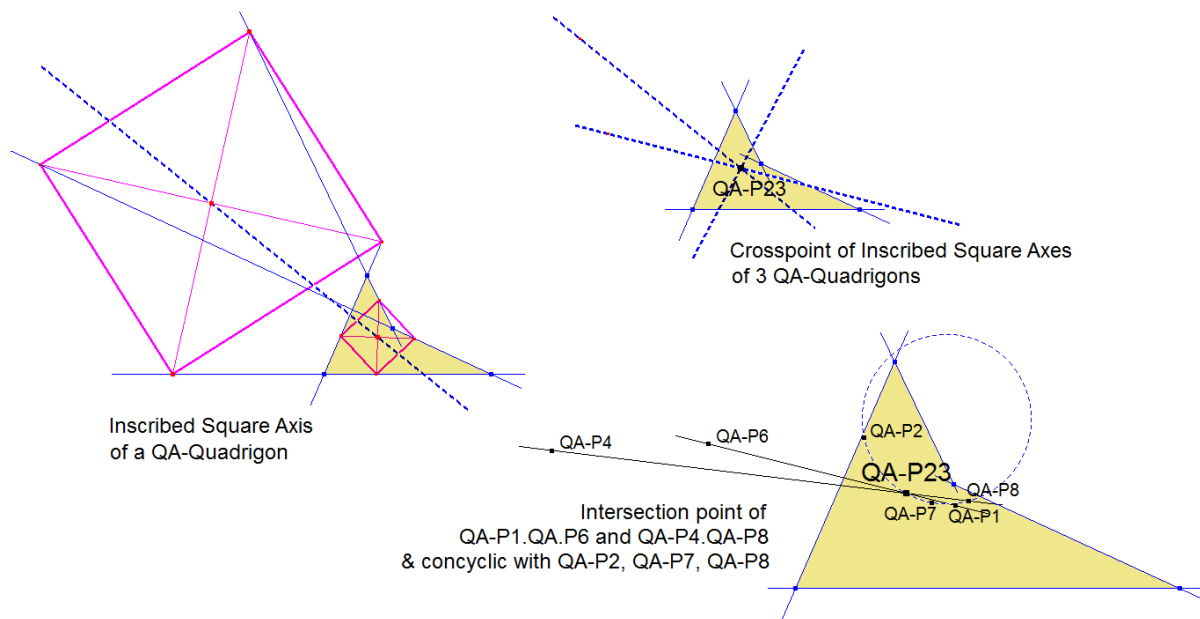
- QA-P22 lies on these QA-lines:
 - QA-P1.QA-P5 $(-1 : 3)$
 - QA-P3.QA-P35 $(5 : -1)$
- QA-P22 is the complement of QA - P1 wrt the QA-Diagonal Triangle..
- QA-P22 is the Center of the Involution on the line QA-L3 = QA-P1.QA-P5. This Line Involution is defined by 2 pairs of points being the intersection points of L with the opposite sides of a component quadrigon of the Reference Quadrangle. The result is the same for all 3 component quadrigons of the Reference Quadrangle. QA-P1 and QA-P20 are the two double points of this Line Involution.
- QA-P22 lies on the cubic QA-Cu6 (QA-P1-InvolutionCenter Cubic).
- QA-P22 = QA-Centroid DT-vertices + QA-P20.
- QA-P22 = QA-Centroid of the Quadrangle QA-P2.QA-P12.QA-P20.QA-P30.

QA-P23: Inscribed Square Axes Crosspoint

Each quadrignon has 2 Inscribed Squares. The centers of these squares are connected by the so-called Inscribed Square Axis. The Inscribed Square Axes of the 3 QA-Quadrignons of the Reference Quadrangle concur in one point. This point is QA-P23. It has very simple coordinates.

Construction:

The construction of these squares is described in [14] and [21].



1st CT-Coordinate:

$$a^2 q r + S_B p q + S_C p r$$

1st DT-Coordinate:

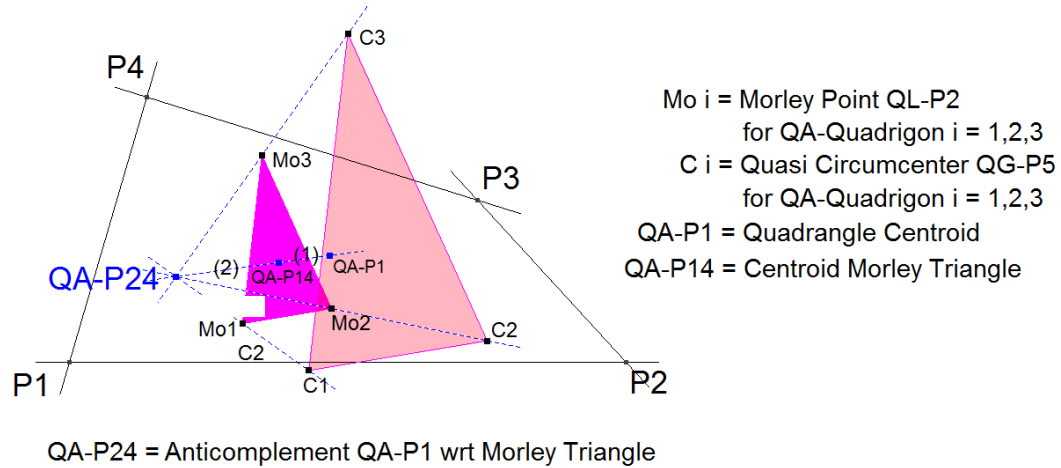
$$p^2 S_A$$

Properties:

- QA-P23 lies on these QA-lines:
 - QA-P1.QA-P6
 - QA-P4.QA-P8
- QA-P23 is the Involutary Conjugate (see QA-Tf2) of QA-P12.
- QA-P23 is concyclic with QA-P2, QA-P7 and QA-P8.

QA-P24: Anticomplement of QA-P1 wrt the Morley Triangle

QA-P24 is the AntiComplement of QA-P1 wrt the Morley Triangle (QA-Tr3).



1st CT-Coordinate:

$$(q T_B + r T_C) ((q + r) T_A + (p + r) T_B + (p + q) T_C) - 2 q r (T_B + T_C)^2 - 4 T_B T_C (r (p + r) + q (p + q))$$

where:

$$T_A = (q + r) S_A \quad T_B = (p + r) S_B \quad T_C = (p + q) S_C$$

1st DT-Coordinate:

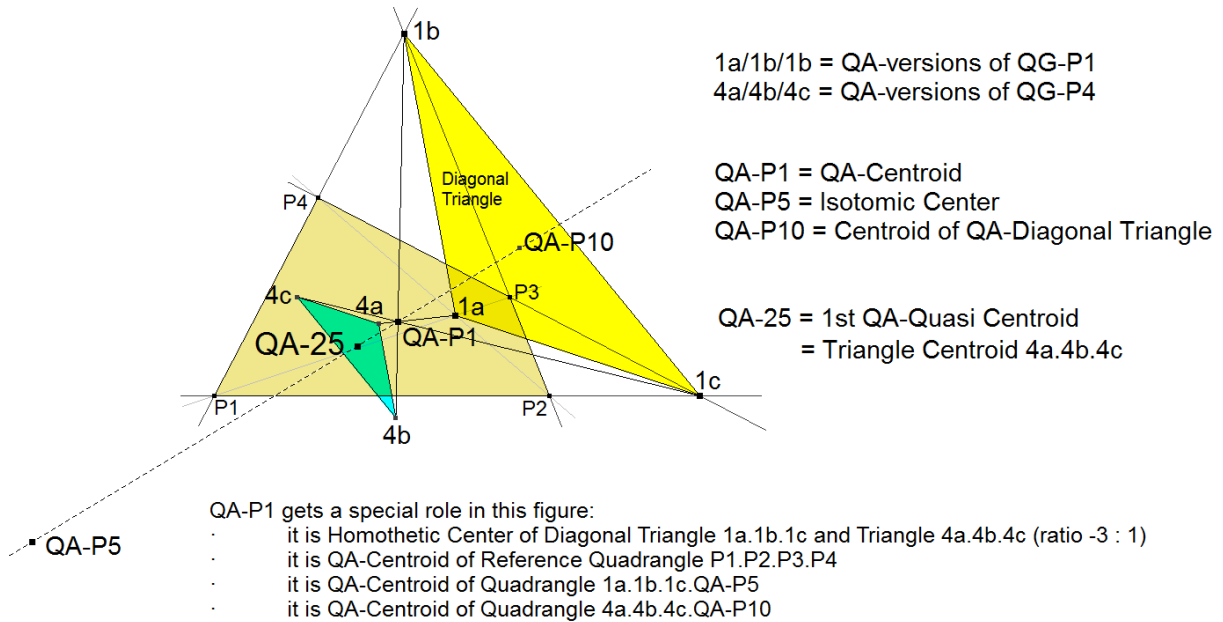
$$(p^2 - (q + r)^2) (p^2 - (q - r)^2) S_B S_C - 4 (-a^4 q^2 r^2 + b^2 S_C p^2 r^2 + c^2 S_B p^2 q^2)$$

Properties:

- QA-P24 lies on this QA-line:
 - QA-P1.QA-P14 (3 : -2 => QA-P24 = AntiCompl. of QA-P1 wrt QA-Tr3)
 - QA-P33.QA-P37 (2 : -1 => QA-P24 = Reflection of QA-P37 in QA-P33)
- QA-P24 is the Homothetic Center of the Morley Triangle and the 1st Quasi Circumcenter (QG-P5) Triangle.

QA-P25: 1st QA-Quasi Centroid

QA-P25 is the Centroid of the triangle formed by the 3 QA-versions of QG-P4 (1st Quasi Centroid), constructed in the QA-Quadrilaterals of a Reference Quadrangle.



1st CT-Coordinate:

$$(q + r) (2 p + q + r) (2 p (p + q + r) + 3 q r)$$

1st DT-Coordinate:

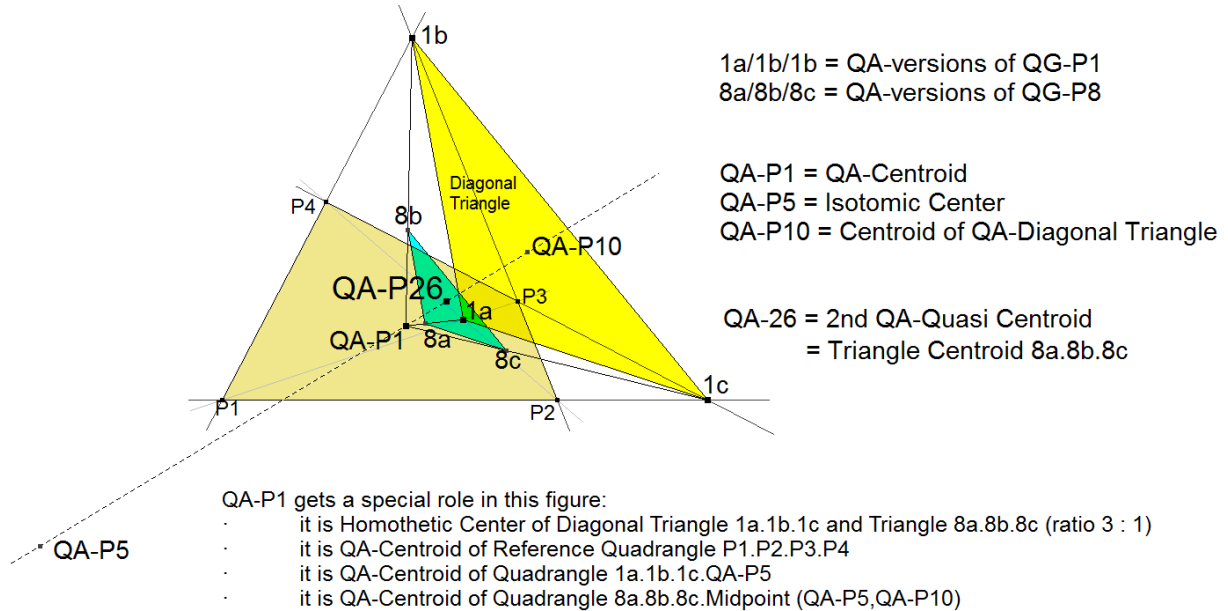
$$(q^2 - r^2)^2 - p^4 + 10 p^2 (-p^2 + q^2 + r^2)$$

Properties:

- QA-P25 lies on QA-L3 (Centroids Line):
– QA-P1.QA-P5 (1 : 8)
- QA-P5.QA-P25 : QA-P25.QA-P1 : QA-P1.QA-P10 = 8 : 1 : 3.
- QA-25 is the Reflection of QA-P26 in QA-P1.

QA-P26: 2nd QA-Quasi Centroid

QA-P26 is the Centroid of the triangle formed by the 3 QA-versions of QG-P8 (2nd Quasi Centroid), constructed in the QA-Quadrilaterals of a Reference Quadrangle.



1st CT-Coordinate:

$$(q + r) (2 p + q + r) (5 p (p + q + r) + 3 q r)$$

1st DT-Coordinate:

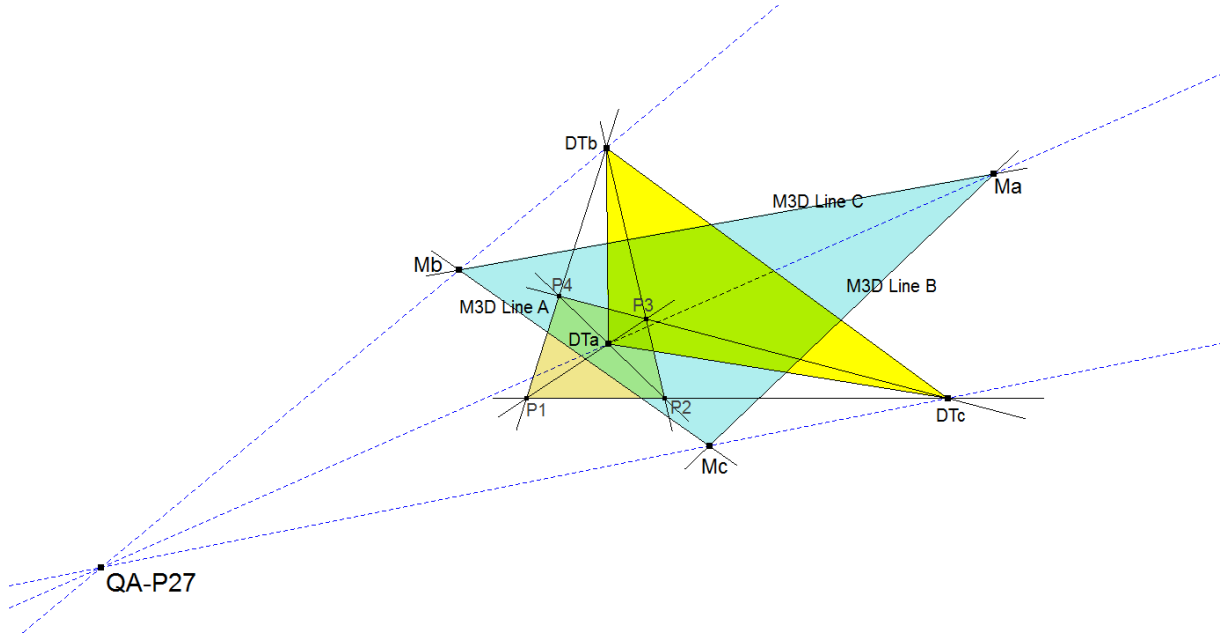
$$(q^2 - r^2)^2 - p^4 - 8 p^2 (-p^2 + q^2 + r^2)$$

Properties:

- QA-P26 lies on QA-L3 (Centroids Line):
– QA-P1.QA-P5 (–1 : 9)
- QA-P5.QA-P1 : QA-P1.QA-P26 : QA-P26.QA-P10 = 9 : 1 : 2.
- QA-26 is the Reflection of QA-P25 in QA-P1.

QA-P27: M3D Center

QA-P27 is the perspector of the QA-Diagonal Triangle and the QA-M3D Triangle.
The QA-M3D Triangle (Ma.Mb.Mc in the figure) is the triangle bounded by the M3D Lines (see paragraph QL-L9) occurring in the 3 QA-Quadrilaterals.



1st CT-Coordinate:

$$(q + r) (2 p + q + r) (q^2 + q r + r^2)$$

$$(-p^4 - 2 p^3 q - 2 p^2 q^2 - p q^3 - 2 p^3 r + p q^2 r + q^3 r - 2 p^2 r^2 + p q r^2 + q^2 r^2 - p r^3 + q r^3)$$

1st DT-Coordinate:

$$1/((p^2 + q^2 + r^2)^2 - 4 ((q^2 + r^2)^2 - q^2 r^2))$$

Area of M3D Triangle (CT-coordinates):

$$2 \Delta (p^2 + p q + q^2 + p r + q r + r^2)^2 / (3 (p + q) (p + r) (q + r) (p + q + r))$$

Properties:

- QA-P27 is the Involutary Conjugate (see QA-Tf2) of QA-P21.

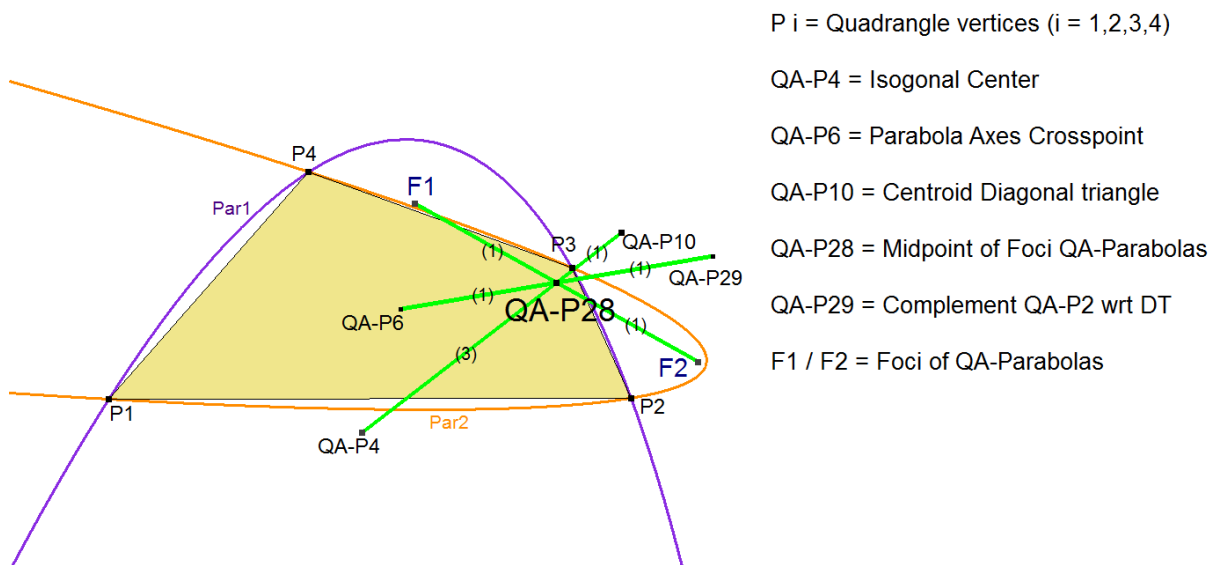
QA-P28: Midpoint of the foci of the QA-Parabolas

QA-P28 is the Midpoint of the foci F1 and F2 of the pair of circumscribed QA-Parabolas (QA-2Co1) .

Because these parabolas only can be constructed when the Reference Quadrangle is not concave a better definition of this point is: “the point on the line QA-P4.QA-P10 such that its distance relation to these points is resp. 3 : 1”.

But because the property related to the parabolas is much more appealing this point has been called after its primary function.

It also can be reasoned that in a concave quadrangle this point represents the Midpoint of the foci of the imaginary parabolas.



1st CT-Coordinate:

$$(q+r) (-a^4 q r (p+q) (p+r) (3 p^2 + 2 p q + 2 p r + q r) - p^2 (q+r) (2 p + q + r) (c^4 p q + b^4 p r + (b^2 - c^2)^2 q r) + p q r (5 p^2 + 3 p q + 3 p r + q r) (a^2 b^2 (p+q) + a^2 c^2 (p+r)))$$

1st DT-Coordinate:

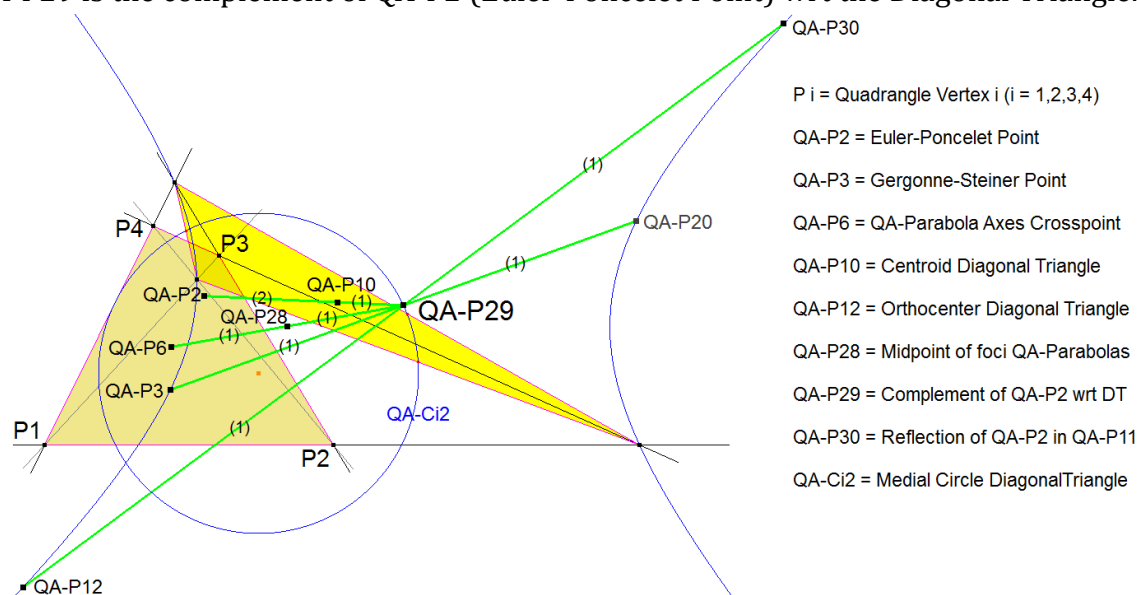
$$-2 b^2 c^2 p^4 - a^2 c^2 q^4 - a^2 b^2 r^4 + c^2 (3 S_C - S_B) p^2 q^2 + b^2 (3 S_B - S_C) p^2 r^2 + a^2 (c^2 + b^2) q^2 r^2$$

Properties:

- QA-P28 lies on these QA-lines:
 - QA-P4.QA-P10 (3 : 1)
 - QA-P6.QA-P29 (1 : 1 => QA-P28 = Midpoint QA-P6.QA-P29)
- QA-P28 = projection of QA-P13 (Nine-point Center DT) on F1.F2
- QA-P28 = Complement of the Isogonal Conjugate of QA-P3 wrt QA-DT
- QA-P28 = QA-Centroid of Quadrangle S1.S2.S3.QA-P4 (Si = vertices QA-DT)
- QA-P28 = QA-Centroid of Quadrangle QA-P2.QA-P3.QA-P4.QA-P20.

QA-P29: Complement of QA-P2 wrt the Diagonal Triangle

QA-P29 is the complement of QA-P2 (Euler-Poncelet Point) wrt the Diagonal Triangle.



1st CT-Coordinate:

$$\begin{aligned}
 & -2a^4 p q r (p+q)(p+r)(q+r)(2p+q+r) \\
 & -b^4 p r (p+q)(q+r)^2 (3p^2 + p q + p r - q r) \\
 & -c^4 p q (p+r)(q+r)^2 (3p^2 + p q + p r - q r) \\
 & -b^2 c^2 p (q+r)^2 (p^3 q + p^2 q^2 + p^3 r - 4p^2 q r - p q^2 r + p^2 r^2 - p q r^2 + 2q^2 r^2) \\
 & + c^2 a^2 p q (p+r)(q+r)(p^2 q + p q^2 + 7p^2 r + 4p q r + q^2 r + 3p r^2 - q r^2) \\
 & + a^2 b^2 p r (p+q)(q+r)(7p^2 q + 3p q^2 + p^2 r + 4p q r - q^2 r + p r^2 + q r^2)
 \end{aligned}$$

1st DT-Coordinate:

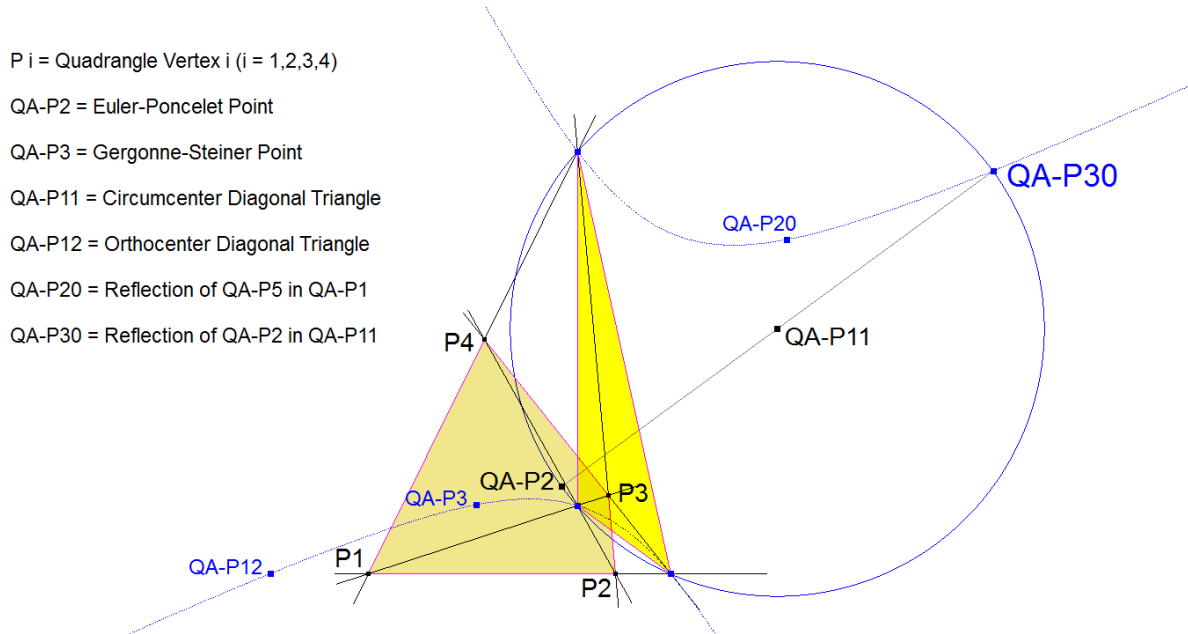
$$(c^2 q^2 - b^2 r^2) [(c^2 - b^2) p^2 + a^2 (q^2 - r^2)]$$

Properties:

- QA-P29 lies on these QA-lines:
 - QA-P2.QA-P10 (3 : -1 \Rightarrow QA-P29 = Complement QA-P2 wrt QA-DT)
 - QA-P3.QA-P20 (1 : 1 \Rightarrow QA-P29 = Midpoint QA-P3.QA-P20)
 - QA-P5.QA-P34 (3 : -1)
 - QA-P6.QA-P28 (2 : -1 \Rightarrow QA-P29 = Reflection of QA-P6 in QA-P28)
 - QA-P12.QA-P30 (1 : 1 \Rightarrow QA-P29 = Midpoint)
 - QA-P13.QA-P36 (-1 : 2 \Rightarrow QA-P29 = Reflection of QA-P36 in QA-P13)
- QA-P29 is the center of the QA-DT-circumscribed orthogonal hyperbola (QA-Co4) additional passing through QA-P3, QA-P12, QA-P20, QA-P30.
- QA-P29 lies on the Medial Circle of the QA-Diagonal Triangle (QA-Ci2).
- QA-P29 lies on the Simson Line of QA-P30 occurring on the circumcircle of the QA-Diagonal Triangle (QA-Ci1).
- QA-P29 = QA-P1 (QA-Centroid) of the Quadrangle QA-P3.QA-P12.QA-P20.QA-P30.
- QA-P29 = QA-P2 (Euler-Poncelet Point) of the quadrangle formed by the vertices of the QA-Diagonal Triangle and QA-P3 of the Reference Quadrangle.

QA-P30: Reflection of QA-P2 in QA-P11

QA-P30 is the Reflection of QA-P2 (Euler-Poncelet Point) in QA-P11 (Circumcenter Diagonal Triangle).



1st CT-Coordinate:

$$\begin{aligned}
 & (2a^4qr(p+q)(p+r) + b^4pr(q+p)(q+r) + c^4pq(r+p)(r+q) \\
 & - b^2c^2p(q+r)(pq+pr+2qr) + a^2b^2p(p+q)(r-3q)r + a^2c^2pq(q-3r)(p+r)) * \\
 & (a^4qr(p+q)(p+r)(pq+pr-2qr) + b^4pr(q+p)(q+r)(pq+2pr-qr) + \\
 & c^4pq(r+p)(r+q)(2pq+pr-qr) + 2b^2c^2pqr(p-q)(p-r)(q+r) - \\
 & 2a^2c^2pqr(p+r)(2pq+pr-qr) - 2a^2b^2pqr(p+q)(pq+2pr-qr))
 \end{aligned}$$

1st DT-Coordinate:

$$1/(c^2b^2p^2 - c^2S_Cq^2 - S_Bb^2r^2)$$

Properties:

- QA-P30 lies on these QA-lines:
 - QA-P2.QA-P11 ($2 : -1 \Rightarrow$ QA-P30 = Reflection of QA-P2 in QA-P11)
 - QA-P10.QA-P36 ($-2 : 3 \Rightarrow$ QA-P30 = Complement of QA-P36)
 - QA-P12.QA-P29 ($2 : -1 \Rightarrow$ QA-P30 = Reflection of QA-P12 in QA-P28)
 - QA-P13.QA-P35 ($5 : -4$)
- QA-P30 is the Involutary Conjugate of QA-P6 (Parabola Axes Crosspoint).
- QA-P30 lies on the circumcircle of the Diagonal Triangle (QA-Ci1).
- QA-P30 lies on the conic QA-Co4.

QA-P31: Complement of QA-P16 wrt the Diagonal Triangle

QA-P31 is the complement of QA-P16 (QA-Harmonic Center) wrt the QA-Diagonal Triangle.

P_i = vertices of Reference Quadrangle ($i = 1, 2, 3, 4$)

QA-P10 = Centroid of QA-Diagonal Triangle

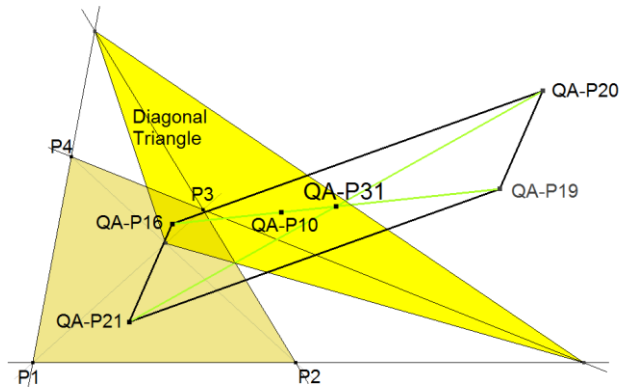
QA-P16 = QA- Harmonic Center

QA-P19 = AntiComplement of QA-P16 wrt QA-DT

QA-P20 = Reflection of QA-P5 in QA-P1

QA-P21 = Reflection of QA-P16 in QA-P1

QA-P31 = Complement of QA-P16 wrt QA-DT



1st CT-Coordinate:

$$p(q+r)(2p+q+r)(p^2+pq+2q^2+pr+qr+2r^2)$$

1st DT-Coordinate:

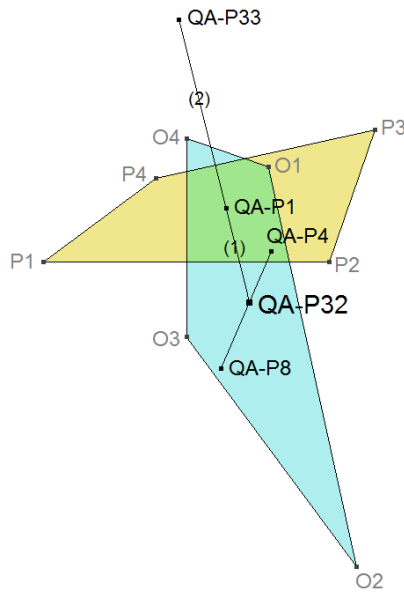
$$q^2 + r^2$$

Properties:

- QA-P31 lies on these QA-lines:
 - QA-P10.QA-P16 ($-1 : 3 \Rightarrow$ QA-P31 = Complement QA-P16 wrt QA-DT)
 - QA-P20.QA-P21 ($1 : 1 \Rightarrow$ QA-P31 = Midpoint QA-P20.QA-P21)
- QA-P31 = Midpoint QA-P16.QA-P19
- QA-P31 is the Center of the parallelogram QA-P16.QA-P21.QA-P19.QA-P20.
- QA-P31 is the QA-Centroid of the Quadrangle formed by the vertices of the QA-Diagonal Triangle and QA-P19.

QA-P32: Centroid of the Circumcenter Quadrangle

QA-P32 is the QA-Centroid of the quadrangle formed by the Circumcenters of the Component Triangles of the Reference Quadrangle.



P_i = Vertices Reference Quadrangle

O_i = Circumcenter Triangle $P_j.P_k.P_l$

QA-P1 = QA-Centroid

QA-P4 = Isogonal Center

QA-P8 = Midray Homothetic Center

QA-P32 = Centroid Circumcenter Quadrangle

QA-P33 = Centroid Orthocenter Quadrangle

1st CT-Coordinate:

$$-a^4 q^2 r^2 + S_B c^2 p^2 q^2 + S_C b^2 p^2 r^2 + 2 S^2 p^2 q r + 3 a^2 S_A p q r (p + q + r)$$

1st DT-Coordinate:

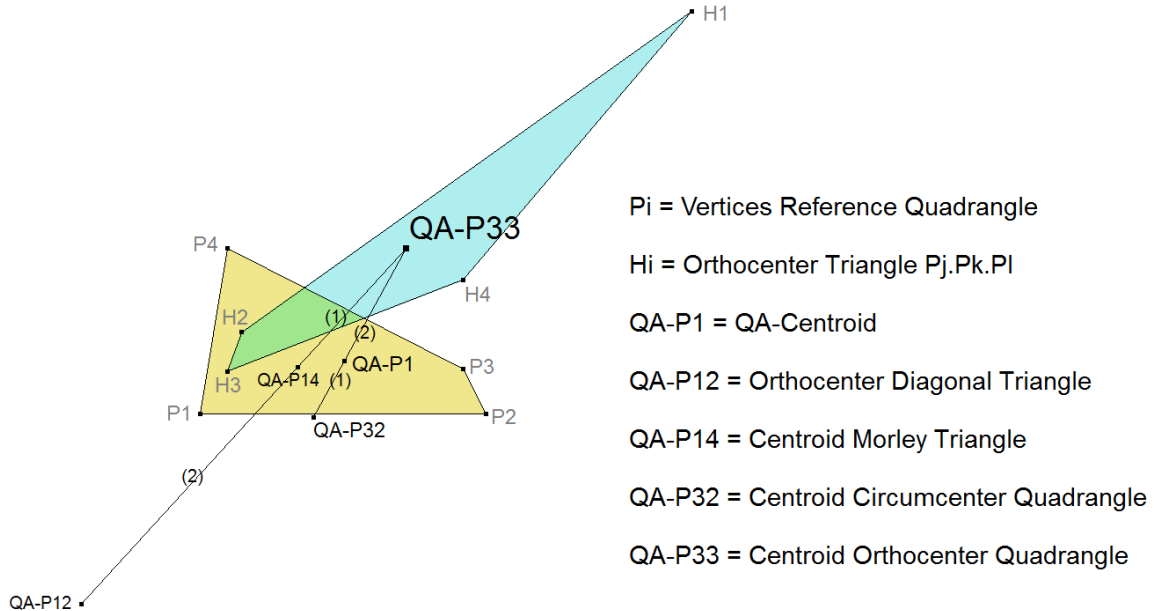
$$a^4 q^2 r^2 - S^2 p^4 + (c^2 S_C - S_B^2) p^2 q^2 + (b^2 S_B - S_C^2) p^2 r^2$$

Properties:

- QA-P32 lies on these QA-lines:
 - QA-P1.QA-P33 $(-2 : 3)$
 - QA-P4.QA-P8 $(1 : 1 \Rightarrow \text{QA-P32} = \text{Midpoint QA-P4.QA-P8})$
- QA-P1.QA-P32.QA-P33 // QA-P2.QA-P4.QA-P6 = QA-L2.

QA-P33: Centroid of the Orthocenter Quadrangle

QA-P33 is the QA-Centroid of the quadrangle formed by the Orthocenters of the Component Triangles of the Reference Quadrangle.



1st CT-Coordinate:

$$+a^4 q^2 r^2 - S_B c^2 p^2 q^2 - S_C b^2 p^2 r^2 + p q r (3 S_B S_C (p + q + r) + S^2 p)$$

1st DT-Coordinate:

$$S^2 p^4 + 2 a^4 q^2 r^2 - (S^2 + 2 S_B c^2) p^2 q^2 - (S^2 + 2 S_C b^2) p^2 r^2$$

Properties:

- QA-P33 lies on these QA-lines:
 - QA-P1.QA-P32 (3 : -2)
 - QA-P12.QA-P14 (3 : -1 => QA-P33 = Complem. of QA-P12 wrt QA-Tr3)
 - QA-P24.QA-P37 (1 : 1 => QA-P33 = Midpoint QA-P24.QA-P37)
- QA-P1.QA-P32.QA-P33 // QA-P2.QA-P4.QA-P6 = QA-L2.

QA-P34: Euler-Poncelet Point of the Centroid Quadrangle

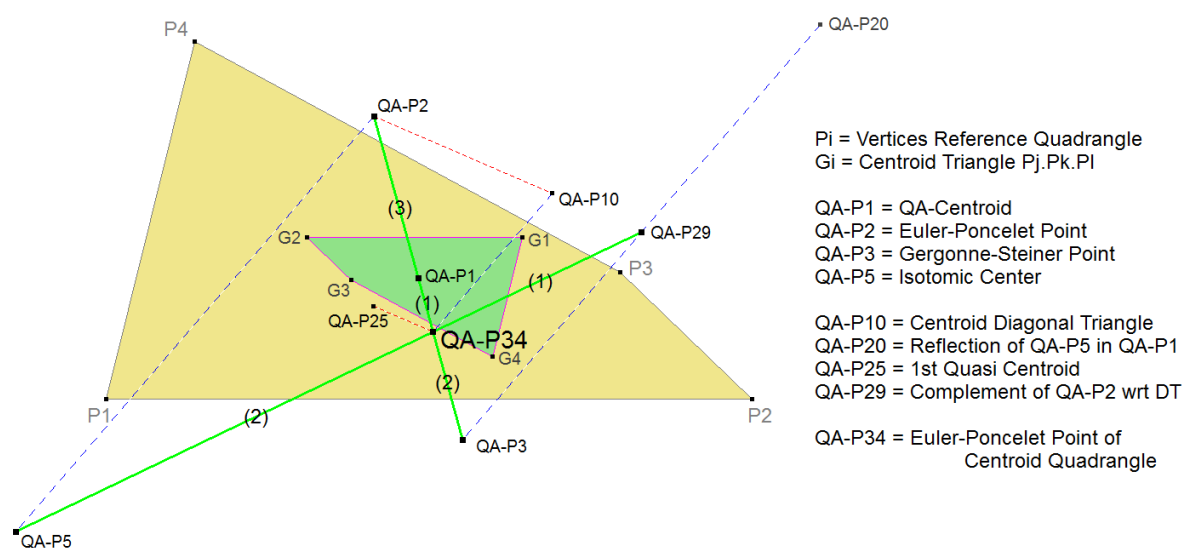
QA-P34 is the Euler-Poncelet Point of the quadrangle formed by the Centroids of the Component Triangles of the Reference Quadrangle.

The Euler-Poncelet point also can be constructed in similar Quadrangles:

- QA-P1 also is Euler-Poncelet Point of the Nine-point Center Quadrangle.
- QA-P2 also is Euler-Poncelet Point of the Orthocenter Quadrangle.
- QA-P3 also is Euler-Poncelet Point of the Circumcenter Quadrangle.

Surprisingly now QA-P1, QA-P2, QA-P3 and QA-P34 have mutual distance ratios similar to the corresponding points in the Triangle Environment on the Euler Line.

This point was contributed by Eckart Schmidt (12/18/2011).



1st CT-Coordinate:

$$\begin{aligned}
 & 2a^4qr(p+q)(p+r)(2p+q+r) \\
 & + b^4p r(p+q)(q+r)(3p+q+r) \\
 & + c^4p q(p+r)(q+r)(3p+q+r) \\
 & + b^2c^2p(q+r)((p+q)(q+r)(r+p) - 3q r(2p+q+r)) \\
 & - a^2b^2p r(p+q)(3q(p+q) + (4q+r)(p+r)) \\
 & - a^2c^2p q(p+r)(3r(p+r) + (4r+q)(p+q))
 \end{aligned}$$

1st DT-Coordinate:

$$\begin{aligned}
 & ((p^2 + q^2 + r^2)^2 - 4(p^4 + q^2r^2))(-b^2p^2 + a^2q^2)(c^2p^2 - a^2r^2) \\
 & + 4p^2(-p^2 + q^2 + r^2)(-c^2q^2 + b^2r^2)((-b^2p^2 + a^2q^2) + (c^2p^2 - a^2r^2))
 \end{aligned}$$

Properties:

- QA-P34 lies on these QA-lines:
 - $QA-P1.QA-P2$ $(-1 : 4)$
 - $QA-P5.QA-P29$ $(2 : 1)$
 - $QA-P20.QA-P35$ $(5 : -2)$
- The distance ratios between points $QA-P3, QA-P34, QA-P1, QA-P2$ are $2 : 1 : 3$.
- $QA-P34.QA-P10 // QA-P2.QA-P5 // QA-P3.QA-P20$.
- $QA-P34.QA-P25 // QA-P2.QA-P10$.

QA-P35: 1st Penta Point

QA-P35 is the Centroid of the Complete Pentangle (system of 5 random points) formed by the points QA-P2, QA-P3, QA-P12, QA-P20, QA-P30. Since there are other Penta Points possible this point is called the 1st Penta Point.

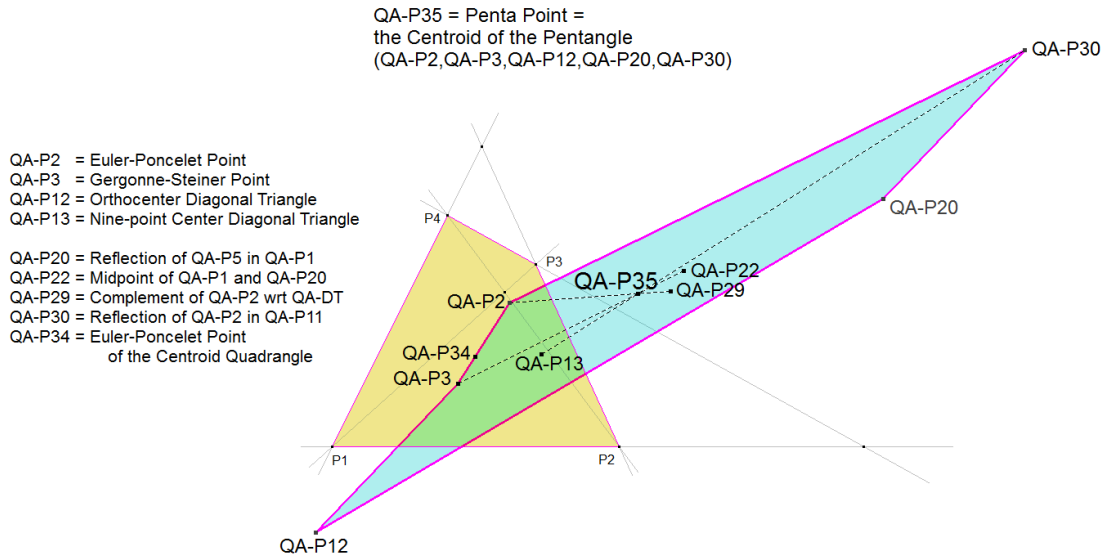
A property from a Pentangle is that its Centroid lies on the lines from each vertex of the Pentangle to the QA-Centroid of the Quadrangle formed by the remainder of its vertices, where the Pentangle Centroid divides these lines in segments with ratio 1 : 4. Since

- QA-P13 = QA-Centroid QA-P2.QA-P12.QA-P20.QA-P3,
- QA-P22 = QA-Centroid QA-P2.QA-P12.QA-P20.QA-P30,
- QA-P29 = QA-Centroid QA-P3.QA-P12.QA-P20.QA-P30,

we know that QA-P35 is dividing the line segments QA-P13.QA-P30, QA-P22.QA-P3 and QA-P29.QA-P2 in segments with ratio 1 : 4.

Also noteworthy in this specific construction is that:

- QA-P13 = QA-Centroid DT-vertices + QA-P12,
- QA-P22 = QA-Centroid DT-vertices + QA-P20,
- QA-P29 = QA-Centroid DT-vertices + Reflection QA-P2 in QA-P29,
- QA-P3.QA-P12 // = QA-P20.QA-P30.



1st CT-Coordinate:

$$\begin{aligned}
 &-4 a^4 q (p+q) r (p+r) (2p+q+r) - b^2 c^2 (q+r) (p^3 q + p^2 q^2 + p^3 r - 12 p^2 q r - 5 p q^2 r + p^2 r^2 - 5 p q r^2 + 2 q^2 r^2) \\
 &-b^4 (p+q) r (q+r) (7 p^2 + 3 p q + 3 p r - q r) + a^2 c^2 q (p+r) (p^2 q + p q^2 + 15 p^2 r + 8 p q r + q^2 r + 7 p r^2 - q r^2) \\
 &-c^4 q (p+r) (q+r) (7 p^2 + 3 p q + 3 p r - q r) + a^2 b^2 (p+q) r (15 p^2 q + 7 p q^2 + p^2 r + 8 p q r - q^2 r + p r^2 + q r^2)
 \end{aligned}$$

1st DT-Coordinate:

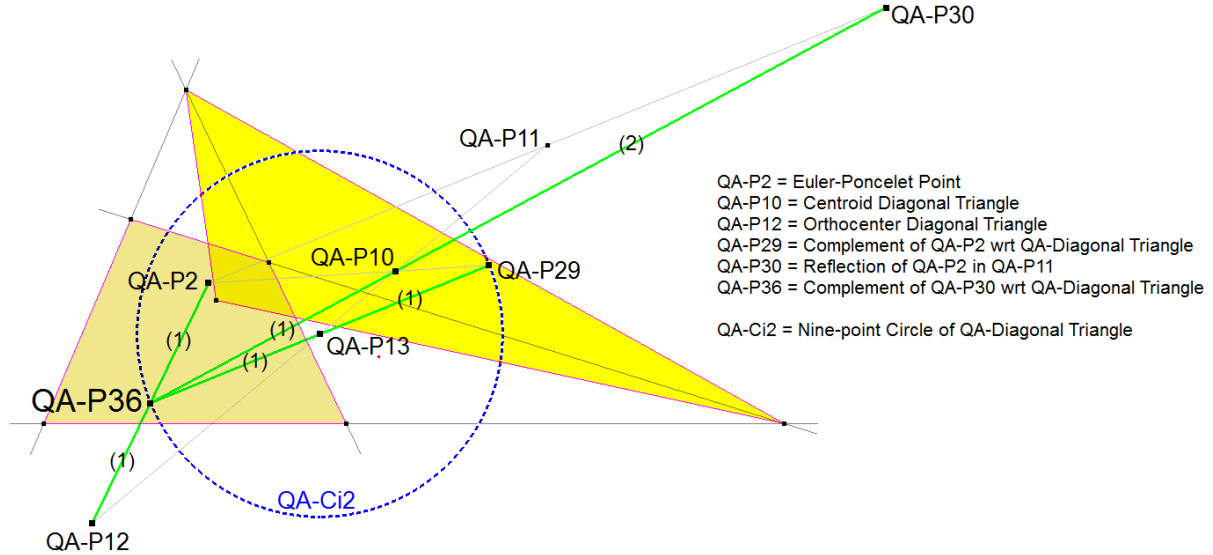
$$\begin{aligned}
 &-a^4 q^2 r^2 - 2 b^4 p^2 r^2 - 2 c^4 p^2 q^2 \\
 &+ a^2 c^2 q^2 (p^2 - 2 q^2 + 2 r^2) + a^2 b^2 r^2 (p^2 + 2 q^2 - 2 r^2) + b^2 c^2 p^2 (-p^2 + 2 q^2 + 2 r^2)
 \end{aligned}$$

Properties:

- QA-P35 lies on these QA-lines:
 - QA-P2.QA-P10 (6 : -1)
 - QA-P3-QA-P22 (4 : 1)
 - QA-P13.QA-P30 (1 : 4)
 - QA-P20.QA-P34 (3 : 2)

QA-P36: Complement of QA-P30 wrt the QA-Diagonal Triangle

QA-P36 is the Complement of QA-P30 (Reflection of QA-P2 wrt the QA-Diagonal Triangle) wrt the Diagonal Triangle.



1st CT-Coordinate:

$$16 \Delta^2 p^2 q r (p+q)(p+r)(q+r)(b^2(p+q)r - c^2 q(p+r))(a^2(-q+r) + (b^2 - c^2)(q+r)) + \\ 2((b^2 - c^2)p(q-r) - a^2(2qr + p(q+r)))(c^4 p q(p+r)(q+r) - c^2 p q r(a^2(p+r) + b^2(q+r)) + \\ (p+q)r(a^4 q(p+r) + b^4 p(q+r) - 2a^2 b^2 p q))(-(c^2 q + b^2 r)p^2(q+r) + a^2 q r(p^2 + q r))$$

1st DT-Coordinate:

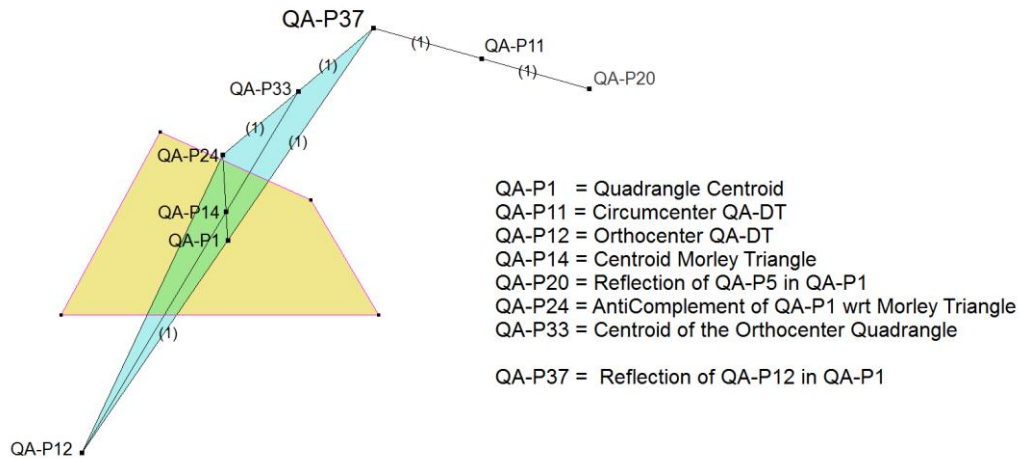
$$(b^2 S_B r^2 + c^2 S_C q^2 - b^2 c^2 p^2)(a^2(S_A p^2 + S_B q^2 + S_C r^2) - 2 S^2 p^2)$$

Properties:

- QA-P36 lies on these QA-lines:
 - QA-P2.QA-P12 (1 : 1 => QA-P36 = Midpoint QA-P2.QA-P12)
 - QA-P10.QA-P30 (-1 : 3 => QA-P36 = Complement of QA-P30)
 - QA-P13.QA-P29 (-1 : 2 => QA-P36 = Reflection of QA-P29 in QA-P13)
- QA-P36 lies on QA-Ci2 (QA-DT-Nine-point Circle).
- QA-P36 is the Center of the circumscribed orthogonal hyperbola of the QA-Diagonal Triangle through QA-P2 and QA-P12.
- QA-P36 lies on the Simson Line (QA-P6.QA-P36) of QA-P2 occurring on the circumcircle of the QA-Diagonal Triangle.
- QA-P36 is QA-P2 (Euler-Poncelet Point) of the Quadrangle formed by the vertices of the QA-Diagonal Triangle and QA-P2.

QA-P37: Reflection of QA-P12 in QA-P1

QA-P37 is the Reflection of QA-P12 (Orthocenter of the QA-Diagonal Triangle)) in QA-P1 (Quadrangle Centroid).



1st CT-Coordinate:

$$S^2 p q r (p + q) (p + r) (q + r) (2 p + q + r) - (p + q + r) (a^2 q r + S_B p q + S_C p r) (2 S_A p^2 q r - a^2 q^2 r^2 + b^2 p^2 r^2 + c^2 p^2 q^2)$$

1st DT-Coordinate:

$$(S^2 + a^2 S_A) p^4 - 2 a^2 S_A p^2 (q^2 + r^2) - S_B S_C (q^2 - r^2)^2$$

Properties:

- QA-P37 lies on these QA-lines:
 - QA-P1.QA-P12 (-1 : 2 = Reflection of QA-P12 in QA-P1)
 - QA-P11.QA-P20 (-1 : 2 = Reflection of QA-P20 in QA-P11)
 - QA-P24.QA-P33 (-1 : 2 = Reflection of QA-P24 in QA-P33)
- QA-P37 forms with QA-P12 and QA-P24 a Triangle that shares the same centroid with the Morley Triangle (QA-Tr3): QA-P14.

5.2 QUADRANGLE LINES

QA-L/1: 5th Point Tangents

General description

5th Point Tangents are the tangents of a conic through the vertices P1, P2, P3, P4 of the Quadrangle at a 5th point unequal to P1, P2, P3, P4.

Let P5 (u : v : w) be the 5th point.

This gives a very simple general formula for the coefficients of the tangent at this point.

1st CT-coefficient:

$$p v w (q w - r v)$$

1st DT-coefficient:

$$u (r^2 v^2 - q^2 w^2)$$

5th Point Tangent at QA-P1 (QA-Centroid)

1st CT-coefficient:

$$p (q - r) / (2p + q + r)$$

1st DT-coefficient:

$$(q^2 - r^2) (-p^2 + q^2 + r^2)$$

Properties:

- This line is QA-L3 and passes through QA-P1 (QA-Centroid), QA-P5 (Isotomic Center) and QA-P10 (Centroid DT).

5th Point Tangent at QA-P3 (Gergonne-Steiner Point)

1st CT-coefficient:

$$p (b^2(p+q)pr - c^2(p+r)pq) / ((a^2 - b^2)pq + (a^2 - c^2)pr + a^2qr - 2S_A p^2)$$

1st DT-coefficient:

$$(q^2 (a^2 r^2 - c^2 p^2) - r^2 (a^2 q^2 - b^2 p^2)) * \\ (b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2) - 2 a^2 q^2 r^2)$$

Properties:

- This line passes through QA-P4 (Isogonal Center).

QA-L1: QA-P1-P2-P3 Line

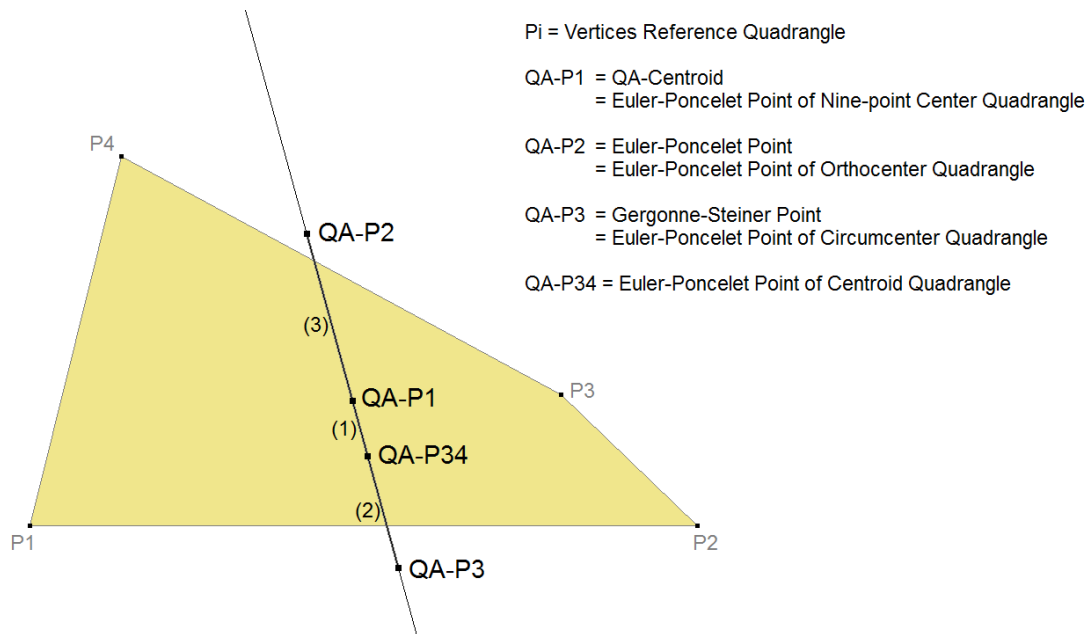
The QA-L1-line is the line through QA-P2 (Euler-Poncelet Point) and QA-P3 (Gergonne-Steiner Point). Both points are constructed in a similar way. They are both common points of circles through midpoints of line segments between Quadrangle points.

QA-P1 (Centroid) also lies on this line and is Midpoint (QA-P2,QA-P3).

Of next 4 points on QA-L1 it also appears that:

- QA-P1 is the Euler-Poncelet Point of the Nine-point Center Quadrangle.
- QA-P2 is the Euler-Poncelet Point of the Orthocenter Quadrangle.
- QA-P3 is the Euler-Poncelet Point of the Circumcenter Quadrangle.
- QA-P34 is the Euler-Poncelet Point of the Centroid Quadrangle.

QA-P1, QA-P2, QA-P3 and QA-P34 have mutual distance ratios similar to their corresponding points in the Triangle Environment on the Euler Line.



1st CT-coefficient:

$$a^4 q (q - r)r/(q+r) + b^4 p r (p+2q+r)/(p+r) - c^4 p q(p+q+2r)/(p+q) - b^2 c^2 p (q - r) - a^2 b^2 r (p+3q) + a^2 c^2 q (p+3r)$$

1st DT-coefficient:

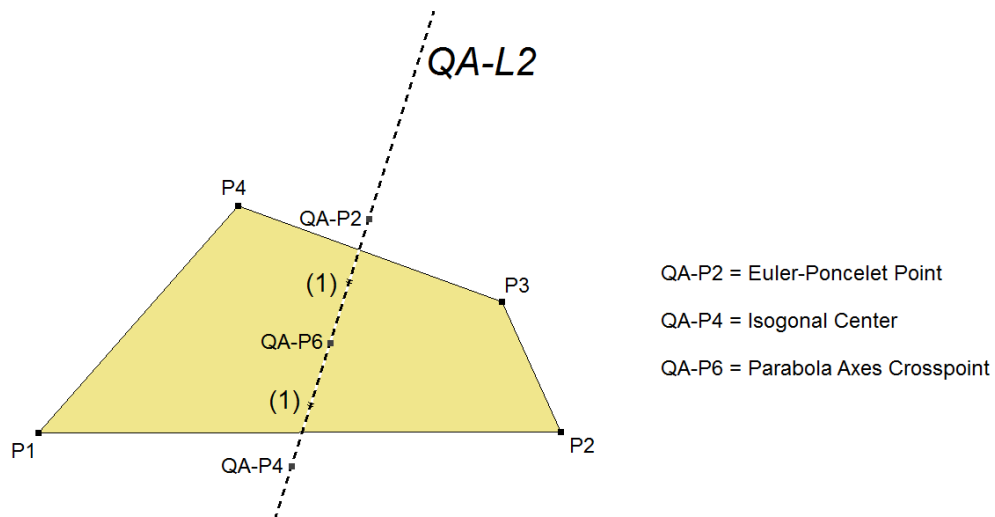
$$p^2 (b^2 r^2 - c^2 q^2) (b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2) - 2 a^2 q^2 r^2)$$

Properties:

- QA-L1 = QA-P1.QA-P2 // QA-P22.QA-P29

QA-L2: QA-P2-P4-P6 Line

The QA-L2-line is the line through QA-P2 (Euler-Poncelet Point) and QA-P4 (Isogonal Center). QA-P6 (Parabola Axes Crosspoint) also lies on this line and is their Midpoint. The expression of the line is not a simple one.



1st CT-coefficient:

$$b^2 p r^2 (2 SA p - 2 SB q) ((b^2 - a^2) p q + b^2 p r - a^2 q r) (b^2 p r + (b^2 - a^2) p q + (b^2 - c^2) q r - 2 SB q^2) - c^2 p q^2 (2 SA p - 2 SC r) ((c^2 - a^2) p r + c^2 p q - a^2 q r) (c^2 p q + (c^2 - a^2) p r + (c^2 - b^2) q r - 2 SC r^2)$$

1st DT-coefficient:

$$(c^2 q^2 - b^2 r^2) * (-a^2 p^2 (c^4 q^4 + b^4 r^4) + (b^4 p^4 + a^4 q^4) r^2 SB + q^2 (c^4 p^4 + a^4 r^4) SC + 2 a^2 p^2 q^2 r^2 (SA^2 - SB SC))$$

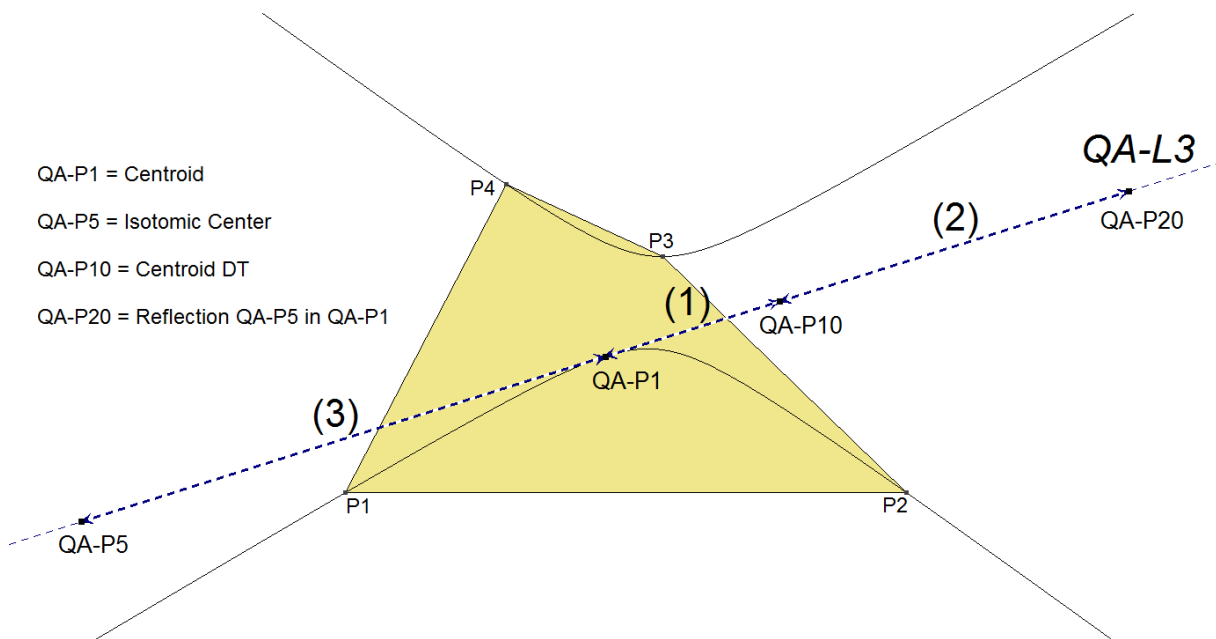
Properties:

- QA-L2 = QA-P2.QA-P4 // QA-P1.QA-P32 // QA-P7.QA-P8 // QA-P12.QA-P24
- QA-L2 and the 5th Point Tangent at QA-P2 are orthogonal. See QA-L/1.
- QA-L2 and the asymptote of QA-Cu7 (QA-Quasi Isogonal Cubic) are perpendicular. See QA-Cu7 and QA/4.

QA-L3: QA-P1-P5-P10 Line or Centroids-Line

The QA-L3-Line is the line through the Centroid of the Quadrangle (QA-P1) and the Centroid of the Diagonal Triangle of the Quadrangle (QA-P10). It also passes through the Isotomic Center (QA-P5) and the 1st and 2nd QA-Quasi Centroids (QA-25 and QA-P26). Interestingly, all these points have coordinates independent of (a,b,c). As a consequence the coefficients of QA-L3 also are independent of (a,b,c).

Because the Centroid of the Quadrangle is on this line and it is divided in segments of fixed ratio this line has the allure of the Euler Line in Triangle environment.



1st CT-coefficient:

$$p (q - r) / (2p + q + r)$$

1st DT-coefficient:

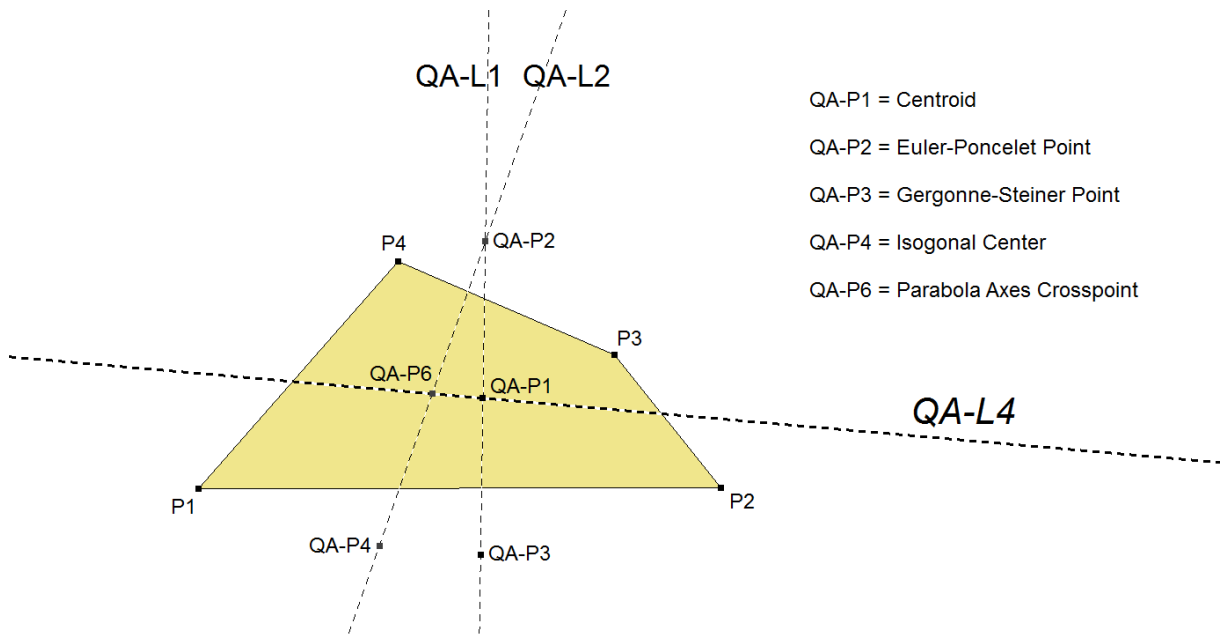
$$(q^2 - r^2) (-p^2 + q^2 + r^2)$$

Properties:

- The major points on this line are: QA-P5, QA-P1, QA-P10, QA-P20 in this order with mutual distance-ratios 3 : 1 : 2 (see comments at QA-P10 and QA-P20).
- Major and minor points on this line are: QA-P5, QA-P25, QA-P1, QA-P26, QA-P10, QA-P22, QA-P20 in this order with mutual distance-ratios: 16 : 2 : 2 : 4 : 3 : 9.
- The barycentric coordinates of QA-L3 are algebraically independent of (a,b,c) just like all known points on this line: QA-P1, QA-P5, QA-P10, QA-P18, QA-P20, QA-P22, QA-P25, QA-P26.
- QA-L3 is the 5th Point tangent at QA-P1 and QA-P20. See QA-L/1.
- QA-P22 is the Center of the QA-Line Involution (see QA-Tf1) on QA-L3.
- QA-Co5 is the Involutary Conjugate (see QA-Tf2) of QA-L3.

QA-L4: QA-P1-P6 Line

The QA-L4-Line is the line through the Midpoint of the segment (QA-P2,QA-P3) on QA-L1 and the Midpoint of the segment (QA-P2,QA-P4) on QA-L2, being QA-P1 (Centroid) and QA-P6 (Parabola Axes Crosspoint).



1st CT-coefficient:

$$a^2 (q - r)/(q + r) - b^2 (p + 3 r)/(p + r) + c^2 (p + 3 q)/(p + q)$$

1st DT-coefficient:

$$q^2 r^2 ((-b^2+c^2) p^2+a^2 (q^2-r^2))$$

1st CT-Coordinate Infinity Point:

$$a^2 - b^2 p/(p + r) - c^2 p/(p + q)$$

1st DT- Coordinate Infinity Point:

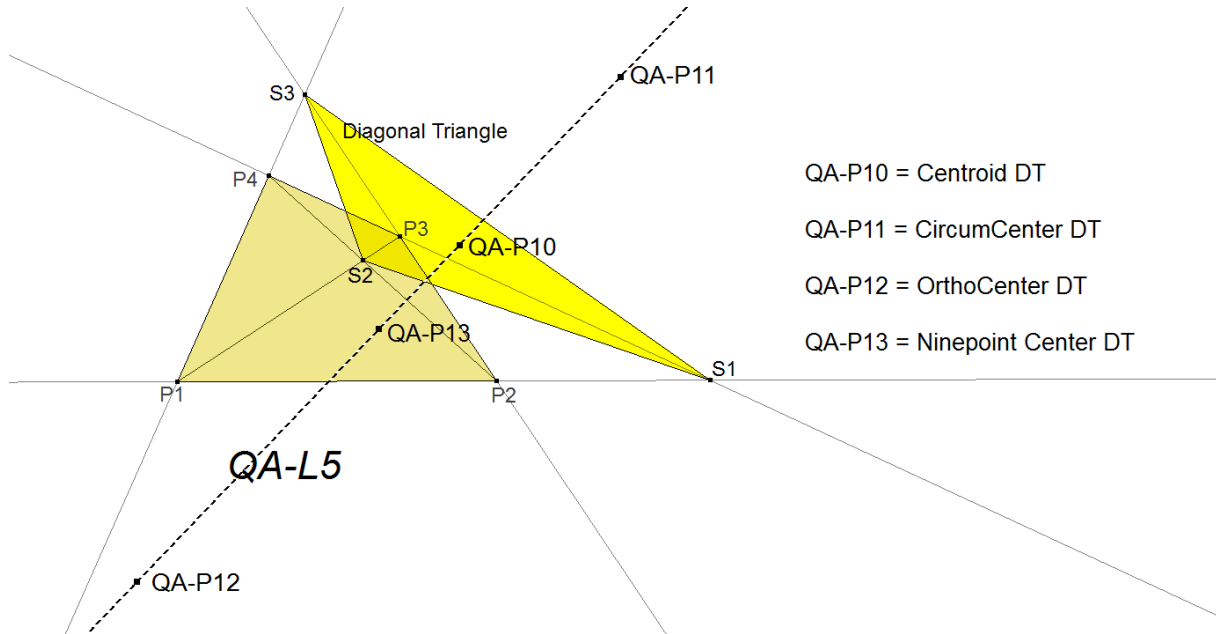
$$p^2 (c^2 q^2 (p^2 - q^2 + r^2) + r^2 (-2 a^2 q^2 + b^2 (p^2 + q^2 - r^2)))$$

Properties:

- QA-L4 // QA-P3.QA-P4.
- QA-L4 // QA-Cu1-asymptote.
- Also QA-P23 (Inscribed Square Axes Crosspoint) lies on QA-L4.
- The two asymptotes of the Orthogonal Quadrangle Hyperbola (QA-Co2) and the two axes of The Gergonne-Steiner Conic (QA-Co3) form a rectangle. QA-P2 and QA-P3 are 2 opposite vertices of this rectangle. The other 2 vertices lie on QA-L4.
- QA-Co4 is the Involutory Conjugate (see QA-Tf2) of QA-L4.

QA-L5: Euler Line of QA-Diagonal Triangle

The QA-L5-Line is the Euler Line of the QA-Diagonal Triangle (DT).



1st CT-coefficient:

$$\begin{aligned}
 & -a^4 q^2 r^2 (p+q)^2 (p+r)^2 (q-r) \\
 & -b^4 p r^2 (p+q)^2 (q+r) (p q^2 + q^2 r - q r^2 + 2 p^2 r + 4 p r^2 + p q r) \\
 & +c^4 p q^2 (p+r)^2 (q+r) (p r^2 + q r^2 - q^2 r + 2 p^2 q + 4 p q^2 + p q r) \\
 & +a^2 b^2 q r^2 (p+q)^2 (3 p^2 r^2 + 3 q^2 r^2 + 2 p^2 q^2 + 2 p^3 r + 3 q r^3 - p r^3 + p^2 q r + 3 p q^2 r) \\
 & -a^2 c^2 q^2 r (p+r)^2 (3 p^2 q^2 + 3 q^2 r^2 + 2 p^2 r^2 + 2 p^3 q + 3 q^3 r - p q^3 + p^2 q r + 3 p q r^2) \\
 & +b^2 c^2 p q r (q^2 - r^2) (2 p^4 - p^2 q^2 - p^2 r^2 + 2 q^2 r^2 + 3 p^3 q + 3 p^3 r + 2 p^2 q r - p q^2 r - p q r^2)
 \end{aligned}$$

1st DT-coefficient:

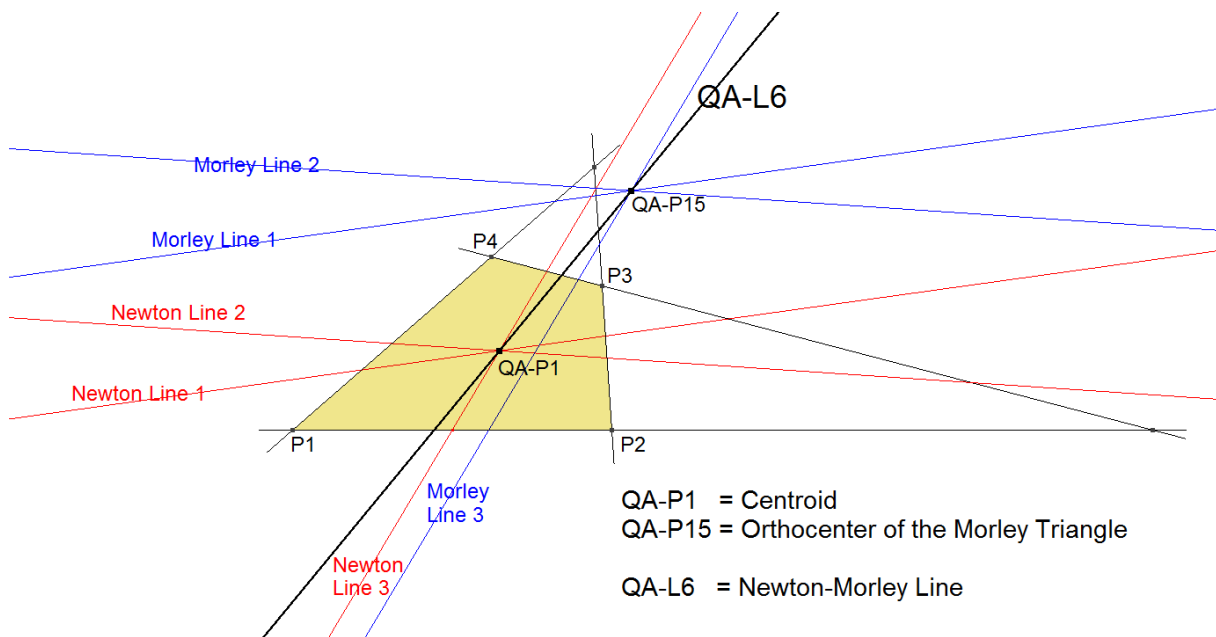
$$SA (SB-SC)$$

Properties:

- QA-L5 passes through QA-P10 (Centroid DT), QA-P11 (Circumcenter DT), QA-P12 (Orthocenter DT) and QA-P13 (Nine-point Center DT).

QA-L6: QA-Newton-Morley Line

Each QL-line parallel to the Newton Line when transferred to the 3 Quadrilaterals of a Quadrangle produces 3 concurrent lines. The locus of the common point is QA-L6. QA-P1 (the Quadrangle Centroid) is a point on this line because the QL-Newton Line is transferred into QA-P1.



1st CT-coefficient:

$$\begin{aligned}
 & -a^4 q r (p + q) (q - r) (p + r) \\
 & -c^4 (p + r) (q + r) (p^3 + 3 p^2 q + 4 p q^2 + 2 q^3 + 2 p^2 r + 3 p q r + 3 q^2 r + p r^2 + q r^2) \\
 & +b^4 (p + q) (q + r) (p^3 + 2 p^2 q + p q^2 + 3 p^2 r + 3 p q r + q^2 r + 4 p r^2 + 3 q r^2 + 2 r^3) \\
 & +b^2 c^2 (q - r) (q + r) (-p^2 q - p q^2 - p^2 r + q^2 r - p r^2 + q r^2) \\
 & +a^2 c^2 (p + r) (p^3 q + 3 p^2 q^2 + 4 p q^3 + 2 q^4 + p^3 r + 7 p^2 q r + 12 p q^2 r + 6 q^3 r + 2 p^2 r^2 \\
 & \quad + 7 p q r^2 + 7 q^2 r^2 + p r^3 + 3 q r^3) \\
 & -a^2 b^2 (p + q) (p^3 q + 2 p^2 q^2 + p q^3 + p^3 r + 7 p^2 q r + 7 p q^2 r + 3 q^3 r + 3 p^2 r^2 + 12 p q r^2 \\
 & \quad + 7 q^2 r^2 + 4 p r^3 + 6 q r^3 + 2 r^4)
 \end{aligned}$$

1st DT-coefficient:

$$\begin{aligned}
 & (-c^2 q^2 + b^2 r^2) \\
 & * (p^2 (p^4 - 3 (q^2 - r^2)^2 + 2 p^2 (q^2 + r^2)) SA + (3 p^4 - (q^2 - r^2)^2 - 2 p^2 (q^2 + r^2)) (q^2 SB + r^2 SC))
 \end{aligned}$$

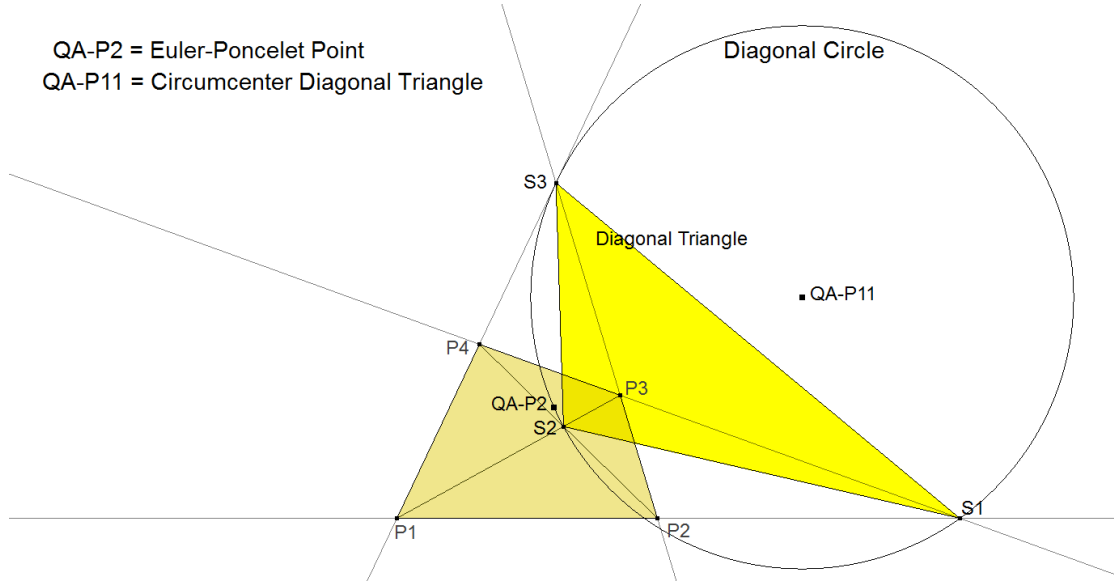
Properties:

- Since the QL-Morley Line is parallel to the QL-Newton Line also the common QA-point of the Morley lines QA-P15 lies on QA-L6.

5.3 QUADRANGLE CONICS

QA-Ci1: QA-Circumcircle Diagonal Triangle

The diagonal circle is the circumcircle of the Diagonal Triangle (DT) of a Quadrangle. The Diagonal Triangle of a Quadrangle is the triangle formed by the intersection points of the 3 distinctive pairs of lines of a Quadrangle.



Equation CT-notation:

$$\begin{aligned} & a^2 q r (p+q) (r+p) (+q r x^2 - p r y^2 - p q z^2 - r (p-q) x y + p (q+r) y z + q (r-p) z x) \\ & + b^2 p r (p+q) (q+r) (-q r x^2 + p r y^2 - p q z^2 + r (p-q) x y - p (q-r) y z + q (r+p) z x) \\ & + c^2 p q (r+p) (q+r) (-q r x^2 - p r y^2 + p q z^2 + r (p+q) x y + p (q-r) y z - q (r-p) z x) = 0 \end{aligned}$$

Equation DT-notation:

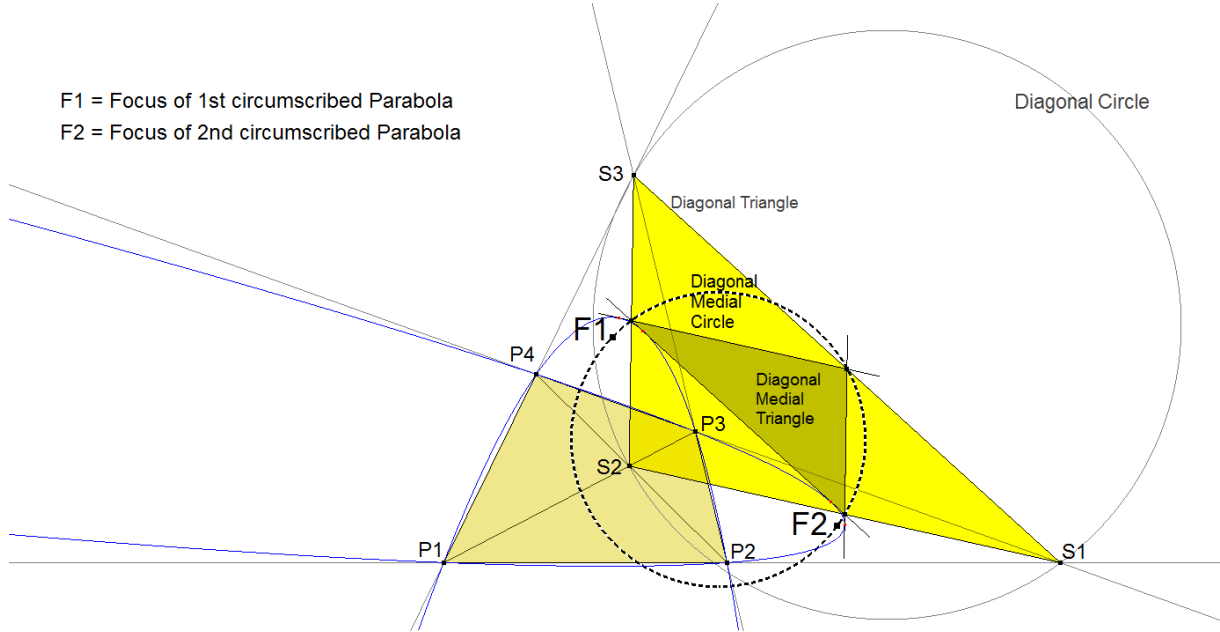
$$a^2 y z + b^2 x z + c^2 x y = 0$$

Properties:

- QA-P11 is the center of the QA-Ci1.
- QA-P2 (Euler-Poncelet Point) and QA-P30 (Reflection of QA-P2 in QA-P11) lie on QA-Ci1.
- QA-Ci1 is the Involutary Conjugate (see QA-Tf2) of the line through QA-P6 perpendicular to QA-L2 (which is the perpendicular bisector of QA-P2.QA-P4).

QA-Ci2: QA-Medial Circle Diagonal Triangle

The Diagonal Medial Circle is the circumcircle of the Medial Triangle (MT) of the Diagonal Triangle (DT) of a Quadrangle.
It is also the Nine-point Circle (or also called Euler Circle) of the QA-Diagonal Triangle.



Equation CT-notation:

$$(x + y + z) (T_{MX} x + T_{MY} y + T_{MZ} z) + 2 T_{XYZ} (a^2 y z + b^2 x z + c^2 x y) = 0$$

where:

$$T_{MX} = qr(a^2 qr(p+q)(r+p) - b^2 pr(p+q)(q+r) - c^2 pq(r+p)(q+r)) - 2 q^2 r^2 (a^2 qr + b^2 rp + c^2 pq)$$

$$T_{MY} = pr(-a^2 qr(p+q)(r+p) + b^2 pr(p+q)(q+r) - c^2 pq(r+p)(q+r)) - 2 r^2 p^2 (a^2 qr + b^2 rp + c^2 pq)$$

$$T_{MZ} = pq(-a^2 qr(p+q)(r+p) - b^2 pr(p+q)(q+r) + c^2 pq(r+p)(q+r)) - 2 p^2 q^2 (a^2 qr + b^2 rp + c^2 pq)$$

$$T_{XYZ} = 2 pqr (p+q) (q+r) (r+p)$$

Equation DT-notation:

$$SA x^2 + SB y^2 + SC z^2 - c^2 x y - b^2 x z - a^2 y z = 0$$

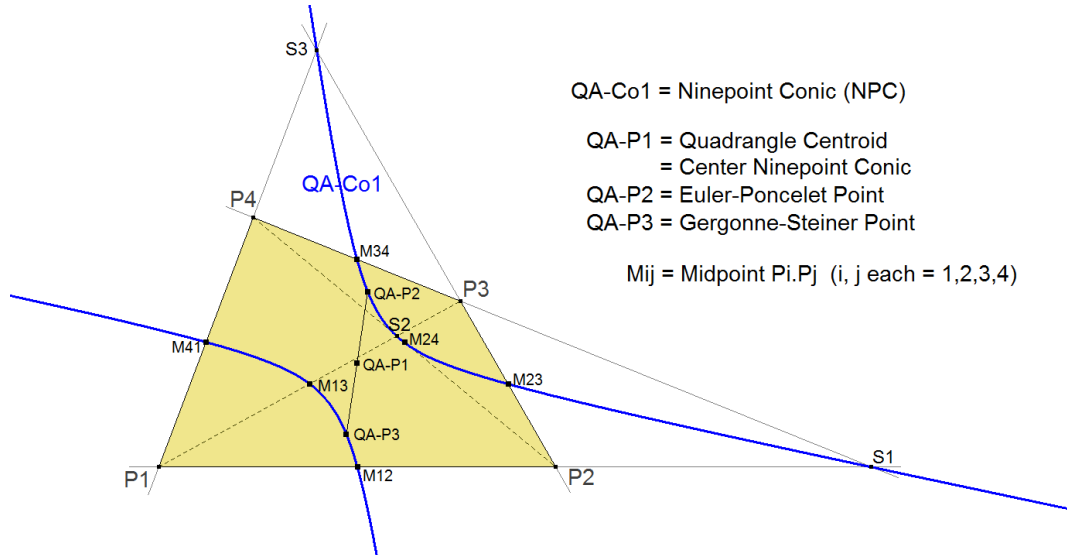
Properties:

- QA-P13 is the center of the Medial Circle.
- QA-P29 and QA-P36 lie on the Medial Circle.
- The foci of both circumscribed Quadrangle Parabolas (QA-2Co1a and QA-2Co1b) lie on the Medial Circle.

QA-Co1: Nine-point Conic

The Nine-point Conic is the conic through the midpoints of all possible line segments connecting the vertices of a Quadrangle. Apart from these 6 midpoints it also passes through the 3 intersection points of all possible pairs of lines connecting the vertices of the Quadrangle. This gives a total of 9 points.

Moreover, there is an analogy with the Nine-point Circle of a Triangle that also passes through the midpoints of all possible line segments connecting its vertices.



Equation CT-notation:

$$q r x^2 + p r y^2 + p q z^2 - r (p + q) x y - q (p + r) x z - p (q + r) y z = 0$$

Equation DT-notation:

$$r^2 x y + q^2 x z + p^2 y z = 0$$

Infinity points CT-notation:

$$\begin{aligned} & (p(q+r) : -pq - \sqrt{(-pqr(p+q+r))} : -pr + \sqrt{(-pqr(p+q+r))}) \\ & (p(q+r) : -pq + \sqrt{(-pqr(p+q+r))} : -pr - \sqrt{(-pqr(p+q+r))}) \end{aligned}$$

Infinity points DT-notation:

$$\begin{aligned} & (2p^2 : -p^2 - q^2 + r^2 - \sqrt{(-4p^2q^2 + (-p^2 - q^2 + r^2)^2)} : -p^2 + q^2 - r^2 + \sqrt{(-4p^2q^2 + (-p^2 - q^2 + r^2)^2)}) \\ & (2p^2 : -p^2 - q^2 + r^2 + \sqrt{(-4p^2q^2 + (-p^2 - q^2 + r^2)^2)} : -p^2 + q^2 - r^2 - \sqrt{(-4p^2q^2 + (-p^2 - q^2 + r^2)^2)}) \end{aligned}$$

These infinity points are equal to the infinity points of the 2 Quadrangle Parabolas.

Properties:

- The center of each circumscribed conic through the vertices of the Reference Quadrangle lies on the Nine-point Conic.
- QA-Co1 is the Involutory Conjugate of the Line at Infinity.
- The asymptotes of the Nine-point Conic are parallel to
 - the axes of the QA-parabolas (provided they are constructible).
- The axes of the Nine-point Conic are parallel to
 - the asymptotes of the QA-Orthogonal Hyperbola as well as
 - the axes of the Gergonne-Steiner Conic.

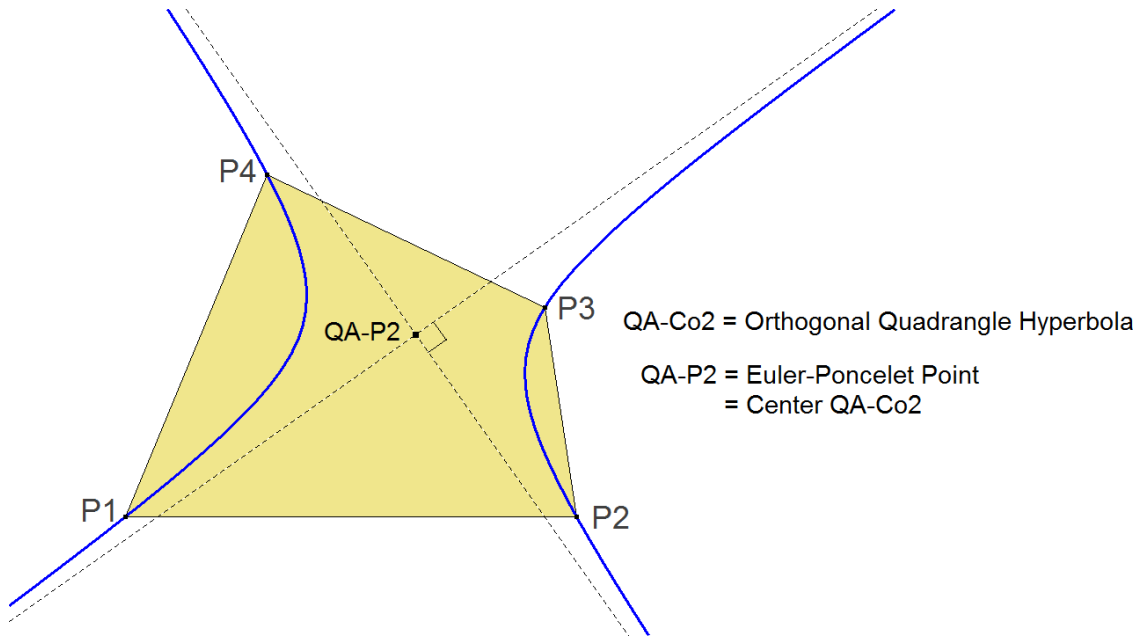
- QA-P1 (Quadri Centroid) is the center of QA-Co1.
- QA-P2 (Euler-Poncelet Point), QA-P3 (Gergonne-Steiner Point) lie on QA-Co1.
- The 3 QA-versions of QG-P13 (Circumscribed Harmonic Conic Center) lie on QA-Co1.
- Let P be a point on the Nine-point Conic in the Quadrigon-environment and let L be a line through P. Let L1, L2, L3, L4 be the lines of the Quadrigon where L1, L3 are opposite sides and L2, L4 are opposite sides.

When P is the Midpoint of the line segment of L between L1 and L3 then is automatically P the Midpoint of the line segment of L between L2 and L4.

QA-Co2: QA-Orthogonal Hyperbola

Since an Orthogonal Hyperbola (OH) is determined by 4 points there is only one Orthogonal Hyperbola defined by the vertices of a quadrangle.

QA-Co2 is the unique Orthogonal Hyperbola defined by the vertices of the reference quadrangle.



Equation CT-notation:

$$(S_A x - S_B y) p q z + (S_B y - S_C z) q r x + (S_C z - S_A x) p r y = 0$$

Equation DT-notation:

$$(b^2 r^2 - c^2 q^2) x^2 + (c^2 p^2 - a^2 r^2) y^2 - (b^2 p^2 - a^2 q^2) z^2 = 0$$

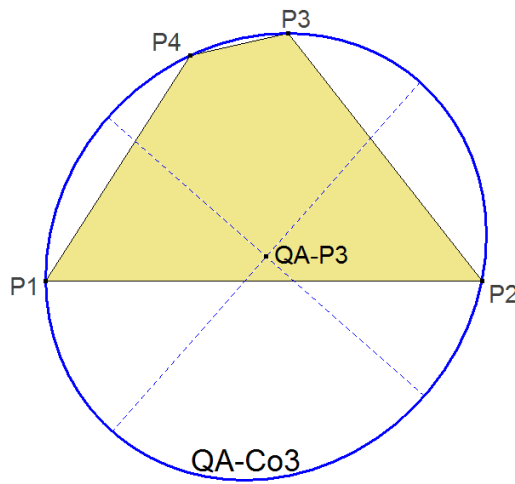
Properties:

- The orthocenters of all QA-Component Triangles lie on this hyperbola.
- QA-P2 (Euler-Poncelet Point) is the center of the QA-Orthogonal Hyperbola.
- The asymptotes of the QA-Orthogonal Hyperbola are parallel to:
 - the axes of the Nine-point Conic (QA-Co1)
 - the axes of the Gergonne-Steiner Conic (QA-Co3).

QA-Co3: Gergonne-Steiner Conic

In a random quadrangle it is generally impossible to circumscribe a circle through all 4 points. But it is always possible to circumscribe a conic (ellipse, parabola or hyperbola) through these 4 points. Technically, the conic with least eccentricity is the conic that deviates least from a circle.

The Gergonne-Steiner Conic is the conic through the vertices of the Reference Quadrangle with least eccentricity. It is the conic with center QA-P3 (Gergonne-Steiner Point). The problem of the Quadrangle conic with least eccentricity was initially posed in the “Annales de Gergonne” and solved by J. Steiner. See [7] page 231.



QA-Co3 = Gergonne-Steiner Conic =
Conic with least eccentricity

QA-P3 = Gergonne-Steiner Point

Equation CT-notation:

$$\begin{aligned} & (a^2 p q r (2p+q+r) - (b^2 r + c^2 q) p^2 (q+r)) y z \\ & + (b^2 p q r (p+2q+r) - (c^2 p + a^2 r) q^2 (p+r)) z x \\ & + (c^2 p q r (p+q+2r) - (a^2 q + b^2 p) r^2 (p+q)) x y = 0 \end{aligned}$$

Equation DT-notation:

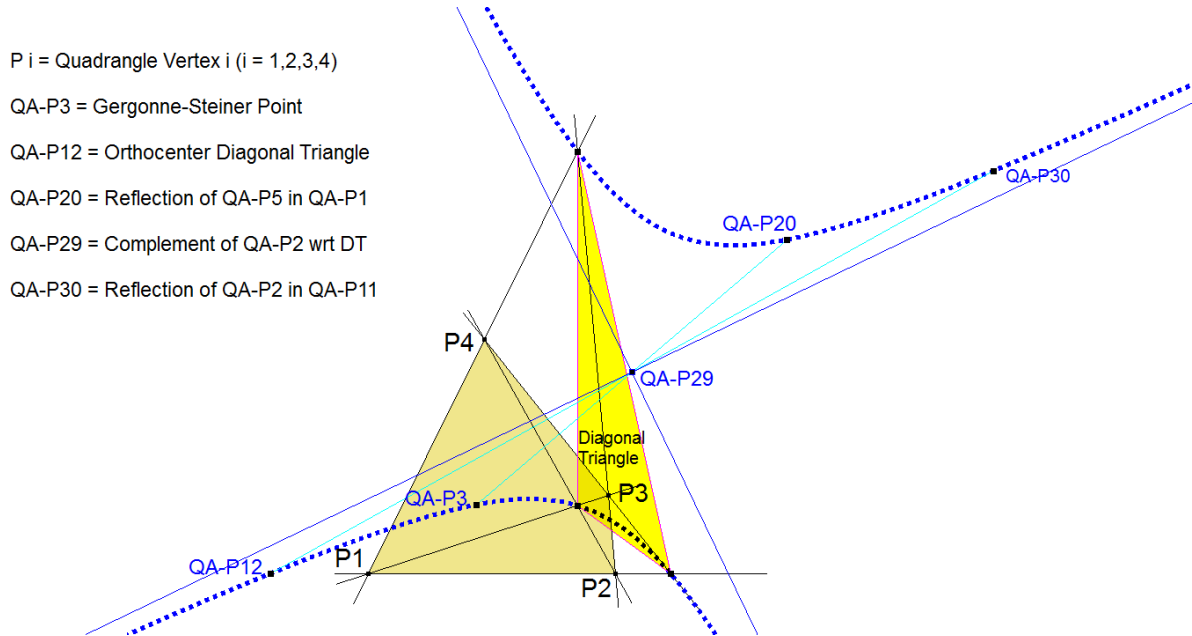
$$\begin{aligned} & (2 a^2 q^2 r^2 - b^2 r^2 (p^2+q^2-r^2) - c^2 q^2 (p^2 - q^2+r^2)) x^2 \\ & + (2 b^2 p^2 r^2 - a^2 r^2 (p^2+q^2-r^2) - c^2 p^2 (-p^2+q^2+r^2)) y^2 \\ & + (2 c^2 p^2 q^2 - a^2 q^2 (p^2-q^2+r^2) - b^2 p^2 (-p^2+q^2+r^2)) z^2 = 0 \end{aligned}$$

Properties:

- Jean-Pierre Ehrmann commented on this conic that when e = eccentricity, then for this conic $e^2 = \sqrt{[2/(k+1)]}$,
where $k = O1P1^*/R1 = O2P2^*/R2 = O3P3^*/R3 = O4P4^*/R4$,
 O_i = circumcenter $P_j P_k P_l$,
 P_i^* = isogonal conjugate of P_i wrt $P_j P_k P_l$,
 R_i = circumradius $P_j P_k P_l$.
(See Hyacinthos [11] message #19970)
- The axes of the Gergonne-Steiner Conic are parallel to:
 - the axes of the QA-Nine-point Conic QA-Co1,
 - the asymptotes of the QA-Orthogonal Hyperbola QA-Co2.

QA-Co4: QA-DT-P3-P12 Orthogonal Hyperbola

QA-Co4 is the circumscribed orthogonal hyperbola of the Diagonal Triangle of the Reference Quadrangle passing through the Gergonne-Steiner Point (QA-P3).



Equation CT-notation:

$$\begin{aligned}
 & (a^2 (q - r) / (q + r) - b^2 (p + 3r) / (p + r) + c^2 (p + 3q) / (p + q)) q r x^2 \\
 & + (a^2 (q + 3r) / (q + r) - b^2 (p - r) / (p + r) - c^2 (3p + q) / (p + q)) p r y^2 \\
 & + (-a^2 (3q + r) / (q + r) + b^2 (3p + r) / (p + r) + c^2 (p - q) / (p + q)) p q z^2 \\
 & - (a^2 (p q + q^2 - p r + 3 q r) / (q + r) - b^2 (p^2 + p q + 3 p r - q r) / (p + r) + c^2 (p - q)) r x y \\
 & - (a^2 (p q - p r - 3 q r - r^2) / (q + r) - b^2 (p - r) + c^2 (p^2 + 3 p q + p r - q r) / (p + q)) q x z \\
 & - (a^2 (q - r) - b^2 (p q - 3 p r - q r - r^2) / (p + r) - c^2 (3 p q + q^2 - p r + q r) / (p + q)) p y z = 0
 \end{aligned}$$

Equation DT-notation:

$$\begin{aligned}
 & ((a^2 - b^2) r^2 - c^2 (p^2 - q^2)) x y \\
 & + ((c^2 - a^2) q^2 - b^2 (r^2 - p^2)) x z \\
 & + ((b^2 - c^2) p^2 - a^2 (q^2 - r^2)) y z = 0
 \end{aligned}$$

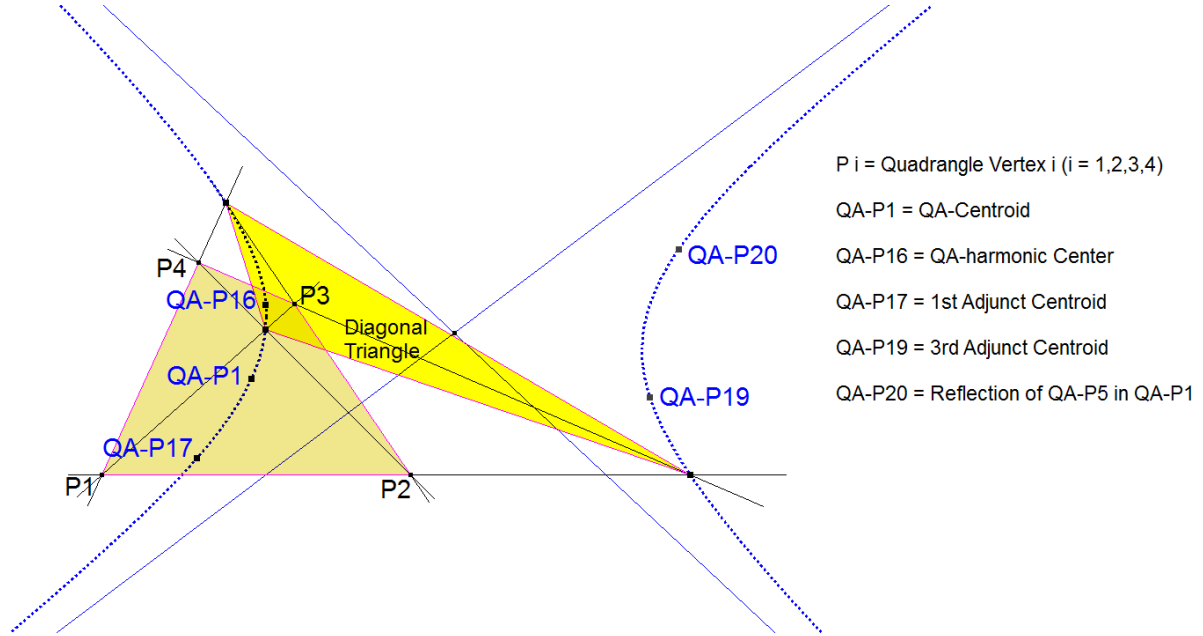
Properties:

- QA-Co4 passes apart from the vertices of the Diagonal Triangle also through QA-P3 (Gergonne-Steiner Point), QA-P12 (Orthocenter QA-Diagonal Triangle), QA-P20 (Reflection of QA-P5 in QA-P1) QA-P30 (Reflection of QA-P2 in QA-P11).
- These intersection points also lie on conic QA-Co4:
 - QA-P1. QA-P2 \wedge QA-P12. QA-P24
 - QA-P1. QA-P5 \wedge QA-P12. QA-P32
 - QA-P1. QA-P12 \wedge QA-P20. QA-P23

- QA-P1. QA-P30 \wedge QA-P6 . QA-P20
 - QA-P2. QA-P11 \wedge QA-P4 . QA-P12
- The center of QA-Co4 is QA-P29 (Complement of QA-P2 wrt the DT).
- QA-Co4 is the Involutary Conjugate of QA-L4 (Line through QA-P1, QA-P6 and QA-P23).

QA-Co5: QA-DT-P1-P16 Conic

QA-Co5 is the circumscribed conic of the Diagonal Triangle of the Reference Quadrangle passing through the QA-Centroid (QA-P1) and the QA-Harmonic Center (QA-P16).



Equation CT-notation:

$$\begin{aligned} & p q r (r - q) (p + 2 q + r) (p + q + 2 r) x^2 \\ & + p q r (p - r) (p + q + 2 r) (2 p + q + r) y^2 \\ & + p q r (q - p) (2 p + q + r) (p + 2 q + r) z^2 \\ & + r (q^2 - p^2) (p + q + 2 r) (-p q + p r + q r + r^2) x y \\ & + q (p^2 - r^2) (p + 2 q + r) (p q + q^2 - p r + q r) x z \\ & + p (r^2 - q^2) (2 p + q + r) (p^2 + p q + p r - q r) y z = 0 \end{aligned}$$

Equation DT-notation:

$$(p^2 - q^2) r^2 (p^2 + q^2 - r^2) x y + q^2 (-p^2 + r^2) (p^2 - q^2 + r^2) x z + p^2 (q^2 - r^2) (-p^2 + q^2 + r^2) y z = 0$$

1st coordinate Conic Center CT-notation:

$$\begin{aligned} & p (q + r) (2 p + q + r) \\ & (3 p^4 q^2 + 6 p^3 q^3 + 3 p^2 q^4 - 4 p^4 q r - 2 p^3 q^2 r - 6 p^2 q^3 r - 8 p q^4 r + 3 p^4 r^2 - 2 p^3 q r^2 + 2 p^2 \\ & q^2 r^2 - 4 p q^3 r^2 + 5 q^4 r^2 + 6 p^3 r^3 - 6 p^2 q r^3 - 4 p q^2 r^3 + 8 q^3 r^3 + 3 p^2 r^4 - 8 p q r^4 + 5 q^2 r^4) \end{aligned}$$

1st coordinate Conic Center DT-notation:

$$(p^2 - q^2 - r^2) (q^2 - r^2)^2$$

Properties:

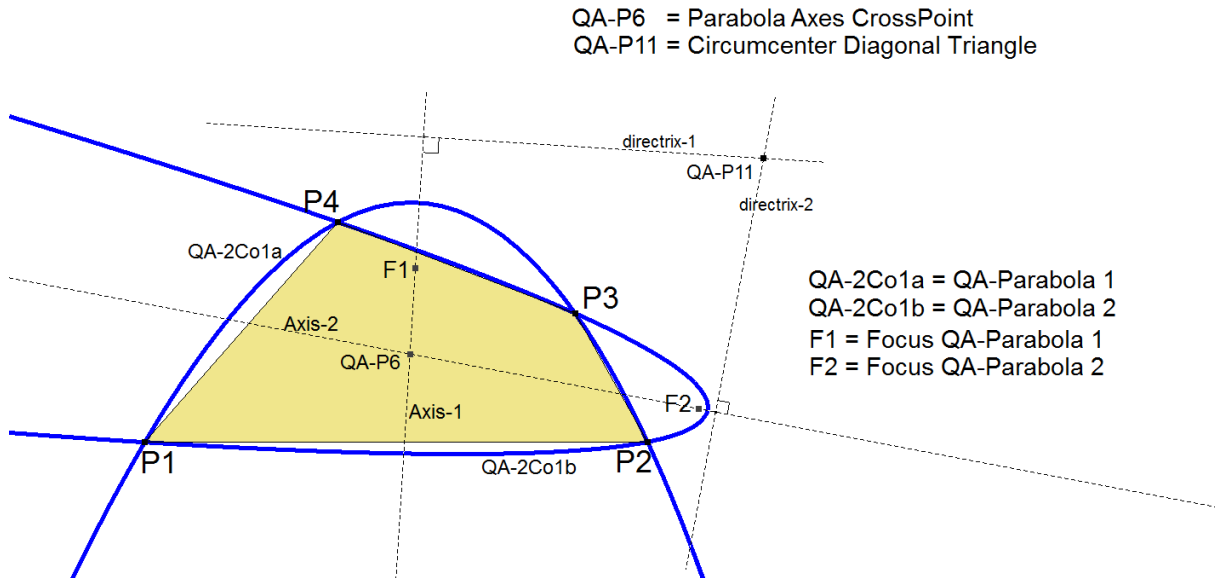
- QA-Co5 passes apart from the vertices of the Diagonal Triangle also through:
QA-P1 (QA-Centroid),
QA-P16 (QA-Harmonic Center),
QA-P17 (Involutory Conjugate of QA-P5),
QA-P19 (AntiComplement of QA-P16 wrt the QA-Diagonal Triangle),

QA-P20 (Reflection of QA-P5 in QA-P1),
as well as the intersection point $QA-P5.QA-P16 \wedge QA-P10.QA-P17$,
as well as the Involutary Conjugates of QA-P1, QA-P5, QA-P10, QA-P18, QA-P20,
QA-P22, QA-P25, QA-P26.

- QA-Co5 is the Involutary Conjugate of QA-L3 (QA-Centroids Line).

QA-2Co1: Pair of Circumscribed QA-Parabolas

Two parabolas can be constructed from 4 points.



There are however some limitations to the positioning of these points for having a real solution of these parabolas.

1. No 3 of the 4 points must be collinear.
2. A Quadrangle consists of 3 Quadrigons.

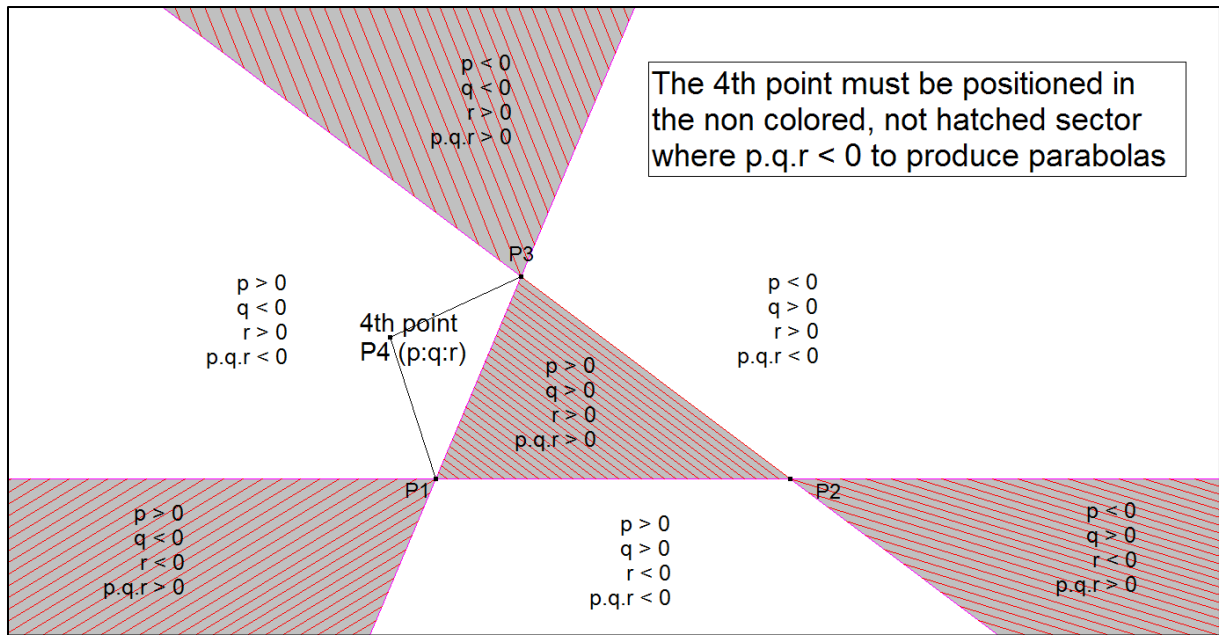
These 3 component quadrigons of a Quadrangle are either

- concave/concave/concave, or
- convex/crossed/crossed.

The first situation does not produce a real solution for circumscribed parabolas. So the best way to describe the positioning of 4 points producing circumscribed parabolas is by stating that these 4 points must not bound a concave form.

This can be confirmed from the formulas.

It contains the element $\sqrt{(-pqr(p+q+r))}$. Now $(p+q+r)$ is always > 0 because barycentric coordinates represent areas and the sum of the 3 areas = area of the reference triangle. So $\sqrt{(-pqr(p+q+r))}$ is only real when $p.q.r$ is negative. That is only when the 4th point in a Quadrigon is a potential mirror point across the sides of the triangle and not across the vertices or inside the triangle as shown in next picture.



The construction of the 2 circumscribed parabolas was described by Newton. This method can be found at [7] as well as [14]. Calculation of the infinity points according to this method delivers these coordinates:

Parabola Equations CT-notation:

$$r (pq-pr+qr+q^2) x y - q (pq-pr-qr-r^2) x z - p (q+r)^2 y z \pm 2 T_c (r x y - q x z) = 0$$

Infinity points CT-notation:

$$(p (q+r) : -p q - T_c : -p r + T_c)$$

$$(p (q+r) : -p q + T_c : -p r - T_c)$$

$$\text{where } T_c = \sqrt{(-p q r (p+q+r))}$$

Parabola Equations DT-notation:

$$(p^2-q^2+r^2) (q^2 x^2-p^2 y^2) + (p^2+q^2-r^2) (r^2 x^2-p^2 z^2) \pm T_d ((q^2-r^2) x^2-p^2 (y^2-z^2)) = 0$$

Infinity points DT-notation:

$$(2 p^2 : -p^2-q^2+r^2 - T_d : -p^2+q^2-r^2 + T_d)$$

$$(2 p^2 : -p^2-q^2+r^2 + T_d : -p^2+q^2-r^2 - T_d)$$

$$\text{where } T_d = \sqrt{((p-q-r) (p+q-r) (p-q+r) (p+q+r))}$$

The Parabola infinity points are equal to the infinity points of the Nine-point Conic.

Properties:

- The sides of the Medial Triangle (of the QA-Diagonal Triangle) are tangential to both circumscribed Quadrangle Parabolas.
- The foci of both circumscribed Quadrangle Parabolas lie on the circumcircle of the Medial Triangle (MT) of the QA-Diagonal Triangle (= Nine-point Circle of the QA-Diagonal Triangle).

- The foci of both circumscribed Quadrangle Parabolas are concyclic with QA-P11 and the involutory conjugate of QA-P11.
- The directrices of both parabolas intersect in QA-P11 (the Circumcenter of the QA-Diagonal Triangle).
- The axes of both parabolas intersect in QA-P6 (the Parabola Axes Crosspoint) which is the Midpoint of QA-P2 (Euler-Poncelet Point) and QA-P4 (Isogonal Center).
- The axes of both parabolas are parallel to the asymptotes of the Nine-point Conic.

5.4 QUADRANGLE CUBICS

QA-Cu/1: Circumscribed QA-Cubics

In this paper 3 types of Quadrangle Cubics are mentioned.

QA-Cubic Type 1

QA-Cubic Type 1 is a cubic that can be constructed as follows:

Let P1, P2, P3, P4 be the vertices of the Reference Quadrangle.

Let V (u:v:w) be a variable point.

Let Lv be some line through V.

Let IC(Lv) be the Involutory Conjugate (QA-Tf2) of Line Lv.

IC(Lv) is a conic since QA-Tf2 is a transformation of the 2nd degree.

The locus of the intersection $IC(Lv) \wedge Lv$ is a QA-Cubic Type 1.

Examples: QA-Cu1 – QA-Cu5, where resp. $V = QA-P4, QA-P5, QA-P10, QA-P19, QA-P1$

The general equation in CT-notation is:

$$q r x^2 (v z - w y) + p r y^2 (w x - u z) + p q z^2 (u y - v x) = 0$$

The general equation in DT-notation is:

$$p^2 y z (v z - w y) + q^2 x z (w x - u z) + r^2 x y (u y - v x) = 0$$

This QA-Cubic Type1 has some interesting general properties:

- The tangents at P1, P2, P3, P4 are concurrent in the Pivot Point V on the cubic.
- The tangents at DT1, DT2, DT3 are concurrent in a point W on the cubic, which is the Involutory Conjugate of V.
- V.W is the only line through V for which Q1 is Double Point itself wrt the created QA-Line Involution, whilst W is the 2nd Double Point on this line.

V.W is tangent in V at the cubic and also at the conic (P1,P2,P3,P4,V)

These cubics can all be seen as “pivotal isocubics” like described by Bernard Gibert [17b]. The reference system here is not a triangle but a quadrangle. The Isoconjugation here is the Involutory Conjugation. Point V is the pivot.

Cubic QA-Cu1 is also a circular cubic because the imaginary circular infinity points lies on this cubic.

QA-Cubic Type 2

QA-Cubic Type 1 is a cubic that can be constructed as follows:

Let P1, P2, P3, P4 be the vertices of the Reference Quadrangle.

Let V (u:v:w) be a variable point.

Let Lv be some line through V.

Let IC(Lv) be the Involutory Conjugate (QA-Tf2) of Line Lv.

IC(Lv) is a conic since QA-Tf2 is a transformation of the 2nd degree.

The locus of the intersection of IC (Lv) \wedge the perpendicular at V to Lv is a QA-Cubic Type-2.

Example: QA-Cu7 (QA-Quasi Isogonal Cubic) with $V = QA-P4$.

Cubic QA-Cu7 is also a circular cubic because the imaginary circular infinity points lies on this cubic.

The general equation in CT-notation is:

$$\begin{aligned} & (-a^2 v w + c^2 v (v + w) + b^2 w (v + w)) (q r x^3 - p r x^2 y - p q x^2 z) \\ & + (-b^2 u w + c^2 u (u + w) + a^2 w (u + w)) (p r y^3 - p q y^2 z - q r x y^2) \\ & + (-c^2 u v + b^2 u (u + v) + a^2 v (u + v)) (p q z^3 - q r x z^2 - p r y z^2) \\ & + (a^2 (q r u^2 - p r u v + q r u v - p r v^2 - p q u w + q r u w + p q v w + p r v w + 2 q r v w - p q w^2) \\ & + b^2 (-q r u^2 + p r u v - q r u v + p r v^2 + p q u w + 2 p r u w + q r u w - p q v w + p r v w - p q w^2) \\ & + c^2 (-q r u^2 + 2 p q u v + p r u v + q r u v - p r v^2 + p q u w - q r u w + p q v w - p r v w + p q w^2)) \\ & x y z = 0 \end{aligned}$$

The general equation in DT-notation is:

$$\begin{aligned} & (-S a u^2 + S b v (u + w) + S c (u + v) w) (r^2 y^2 + q^2 z^2) x \\ & + (S a u (v + w) - S b v^2 + S c (u + v) w) (r^2 x^2 + p^2 z^2) y \\ & + (S a u (v + w) + S b v (u + w) - S c w^2) (q^2 x^2 + p^2 y^2) z \\ & - (r^2 (b^2 u^2 + 2 S c u v + a^2 v^2) + q^2 (c^2 u^2 + 2 S b u w + a^2 w^2) + p^2 (c^2 v^2 + 2 S a v w + b^2 w^2)) x y z = 0 \end{aligned}$$

QA-Cubic Type 3

QA-Cubic Type 3 is a cubic that can be constructed as follows:

Let P1, P2, P3, P4 be the vertices of the Reference Quadrangle.

Let V (u:v:w) be a variable point.

Let Lv be some line through V.

Let IC(Lv) be the Involutary Conjugate (QA-Tf2) of Line Lv.

IC(Lv) is a conic since QA-Tf2 is a transformation of the 2nd degree.

The intersection of IC(Lv) with Lv results in 2 intersection points.

The locus of the Midpoint of these 2 intersection points (which is the Involution Center of the QA-Line Involution on line Lv) produces a QA-Cubic Type 3.

Example: QA-Cu6, where V = QA-P1

The general equation in CT-notation is:

$$\begin{aligned} & r w ((p u + q u + p w) y - (p v + q v + q w) x) x y \\ & + q v ((r v + p w + r w) x - (p u + r u + p v) z) x z \\ & + p u ((q u + q v + r v) z - (r u + q w + r w) y) y z \\ & + (q r u (w - v) + p r v (u - w) + p q w (v - u)) x y z = 0 \end{aligned}$$

The general equation in DT-notation is:

$$\begin{aligned} & p^2 (w y - v z) (-2 u y z + v z (x - y + z) + w y (x + y - z)) \\ & + q^2 (u z - w x) (u z (-x + y + z) - 2 v x z + w x (x + y - z)) \\ & + r^2 (v x - u y) (u y (-x + y + z) + v x (x - y + z) - 2 w x y) = 0 \end{aligned}$$

Properties:

- The vertices of the Reference Quadrangle lie on this cubic.
- The intersection point of the line through V parallel to the side Pi.Pj intersected with the opposite side Pk.Pl is a point on this cubic for all instances of (i,j,k,l) ∈ (1,2,3,4).

Notes from Bernard Gibert (2012, January 22):

Type 1 cubics are $pK(P \times Q, P)$ wrt ABC, in particular :

QACu1 = $pK(P \times igP, P)$, circular cubic see CL035 and SITP 4.2.1

QACu2 = $pK(W1, P)$, central cubic see CL017 and SITP 3.1

QACu3 = $pK(P \times X2P, P)$, where X2P is the centroid of the cevian triangle of P, see CL007

QACu4 = $pK(cP \times ccP \times ct(P^2), P)$, seems complicated...

QACu5 = $pK(W4, P)$, a K^+ , see CL017 and CL049

Type 2 cubics are nK isocubics wrt the cevian triangle of P. See SITP 1.5.3.

it is the locus of M such that P/M lies on the tangent at Q to the circle with center M passing through Q.

QACu7 is obtained with $Q = igP$. It is a focal cubic with focus P/igP , passing through the center S of the rectangular circum-hyperbola through P and P/S which is the infinite real point.

Type 3 cubics are $spK(P, \text{Midpoint } PQ)$ cubics with pole $P \times Q$. See CL055.

These are nodal cubics with node Q.

QACu6 is obtained with $Q = ccP$. It is a K^+ .

notations : c, g, t, i mean complement, isogonal, isotomic, inverse as usual.

SITP = Special Isocubics...

QA-Cu1: QA-DT-P4 Cubic

The QA-DT-P4 Cubic is described by Daniel Baumgartner und Roland Stärk (see [16] page 6). It is more fully described in [15] Eckart Schmidt: “Das Steiner-Dreieck von vier Punkten”. In both papers it is called the “Zirkularkurve”.

If the Reference System is the Reference Quadrangle, this cubic is a pivotal circular isocubic invariant to the Involutory Conjugacy with pivot QA-P4.

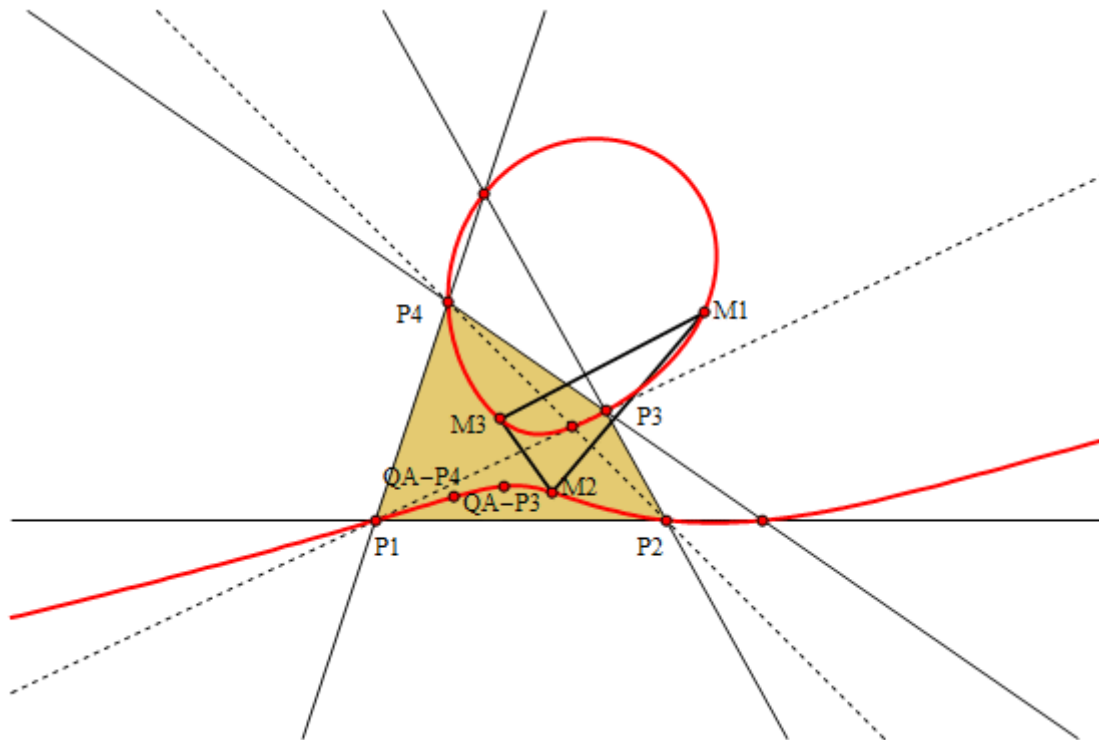
If the Reference System is the Miquel Triangle, this cubic is a pivotal isogonal circular cubic with pivot at the infinity point of the asymptote.

QA-Cu1 is a pK(QA-P16,QA-P4) cubic wrt the QA-Diagonal Triangle in the terminology of Bernard Gibert (see [17b]). (note Eckart Schmidt)

It passes through:

- the vertices of the Reference Quadrangle,
- the vertices of the QA-Diagonal Triangle,
- the vertices of the QA-Miquel Triangle: the Miquel Points of the 3 quadrilaterals of the Reference Quadrangle,
- the incenter and 3 excenters of the Miquel Triangle,
- QA-P3, the Gergonne-Steiner Point,
- QA-P4, the Isogonal Center.

This makes a total of 15 points.



Equation CT-notation:

$$a^2 T_a y z (r y - q z) + b^2 T_b x z (r x - p z) + c^2 T_c x y (q x - p y) = 0$$

where:

$$T_a = a^2 (p + q) (p + r) - b^2 p (p + q) - c^2 p (p + r)$$

$$T_b = b^2 (q + r) (q + p) - c^2 q (q + r) - a^2 q (q + p)$$

$$T_c = c^2 (r + p) (r + q) - a^2 r (r + p) - b^2 r (r + q)$$

Equation DT-notation:

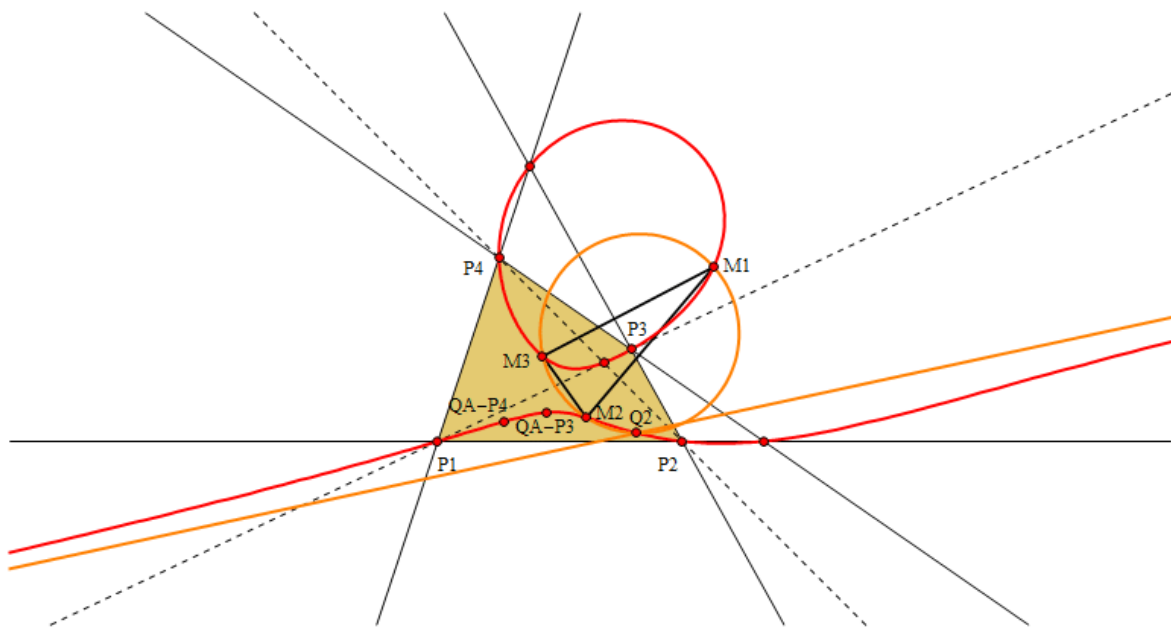
$$\begin{aligned} &+(c^4 p^2 q^2 + b^4 p^2 r^2 - a^4 q^2 r^2 - b^2 c^2 p^2 (-p^2 + q^2 + r^2)) (r^2 y^2 - q^2 z^2) x \\ &+(c^4 p^2 q^2 - b^4 p^2 r^2 + a^4 q^2 r^2 - a^2 c^2 q^2 (p^2 - q^2 + r^2)) (-r^2 x^2 + p^2 z^2) y \\ &+(-c^4 p^2 q^2 + b^4 p^2 r^2 + a^4 q^2 r^2 - a^2 b^2 r^2 (p^2 + q^2 - r^2)) (q^2 x^2 - p^2 y^2) z = 0 \end{aligned}$$

1st coordinate Infinity Point CT-notation:

$$a^2 - b^2 p / (p + r) - c^2 p / (p + q)$$

1st coordinate Infinity Point DT-notation:

$$p^2 (-2 a^2 q^2 r^2 + b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2))$$



Properties (from Eckart Schmidt):

- The tangents at P1, P2, P3, P4 to QA-Cu1 are concurrent in the pivot QA-P4.
- The tangents at S1, S2, S3 and QA-P4 to QA-Cu1 are concurrent in the Involutary Conjugate of QA-P4 on the cubic.
- The tangents at M1, M2, M3 to QA-Cu1 are concurrent on the cubic in the intersection with the asymptote.
- The asymptote of QA-Cu1 // QA-P3.QA-P4 // QA-P1.QA-P6 = QL-L4.
- The vertices of the cevian triangle of QA-P4 wrt the QA-Diagonal Triangle lie on the cubic.
- Triangle M1.M2.M3 is perspective with the QA-Diagonal Triangle with Perspector QA-P3, which is the isogonal conjugate of QA-P4 wrt the Miquel Triangle.
- Every point on the cubic has its isogonal conjugate wrt the Miquel Triangle M1.M2.M3 also on the cubic. All connecting lines of points and its isogonal conjugates (wrt Miquel Triangle) concur in an infinity point Q1 of the asymptote.

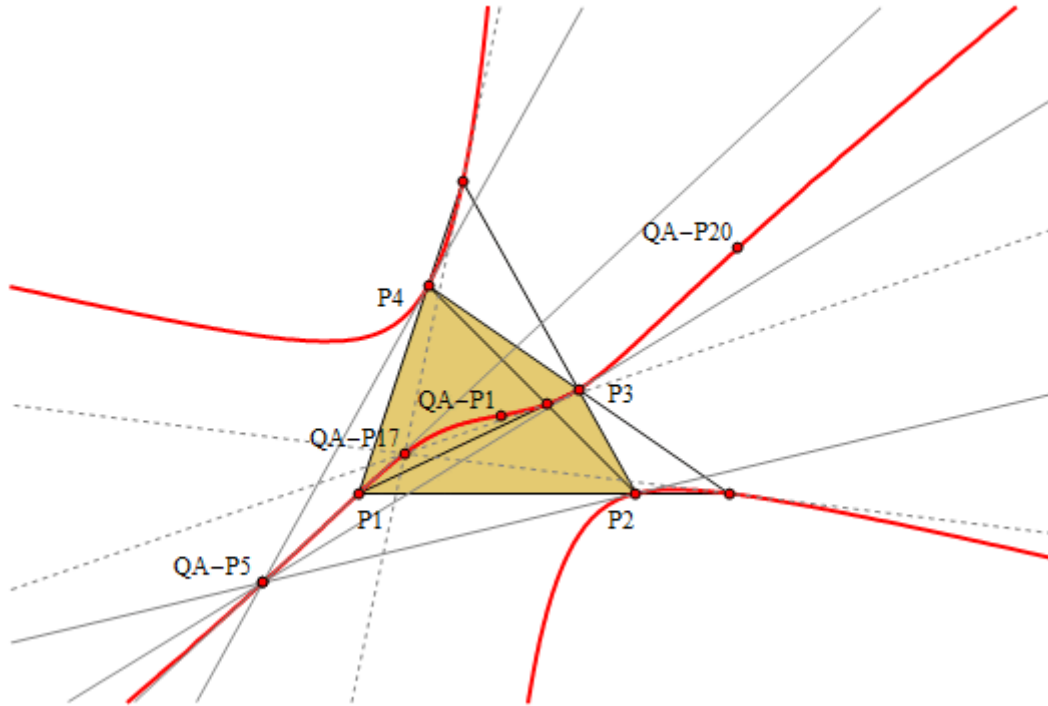
This makes the QA-DT-P4 Cubic also a Pivotal Isogonal Circular Cubic wrt the Miquel Triangle with pivot Q1. See [14] and [17b].

- Q2 is the intersection point of the cubic QA-Cu1 and its asymptote. It is the isogonal conjugate (wrt the Miquel Triangle) of the infinity point of the asymptote of the cubic. Q2 is also the 4th intersection point of the circumcircle of the Miquel Triangle and the cubic. Q2 is also the intersection point of the tangents at M1, M2, M3 to the cubic (see [15c]. Q2 is also the Reflection of QA-P9 in the circumcenter of the Miquel Triangle M1.M2.M3.

QA-Cu2: QA-DT-P5 Cubic

QA-Cu2 is the locus of the Double Points created by the QA-Line Involution of all lines through QA-P5. It is a pivotal isocubic of the Diagonal Triangle, invariant wrt the Involutary Conjugate wrt pivot QA-P5.

QA-Cu2 is a pK(QA-P16,QA-P5) cubic wrt the QA-Diagonal Triangle in the terminology of Bernard Gibert (see [17b]). (note Eckart Schmidt)



Equation CT-notation:

$$\begin{aligned} & (p+q)(p+q+2r)(qx - py)xy \\ & + (p+r)(p+2q+r)(pz - rx)xz \\ & + (q+r)(2p+q+r)(ry - qz)yz = 0 \end{aligned}$$

Equation DT-notation:

$$\begin{aligned} & (4(p^4+q^2r^2) - (p^2+q^2+r^2)^2)(r^2y^2-q^2z^2)x \\ & + (4(q^4+p^2r^2) - (p^2+q^2+r^2)^2)(p^2z^2-r^2x^2)y \\ & + (4(r^4+p^2q^2) - (p^2+q^2+r^2)^2)(q^2x^2-p^2y^2)z = 0 \end{aligned}$$

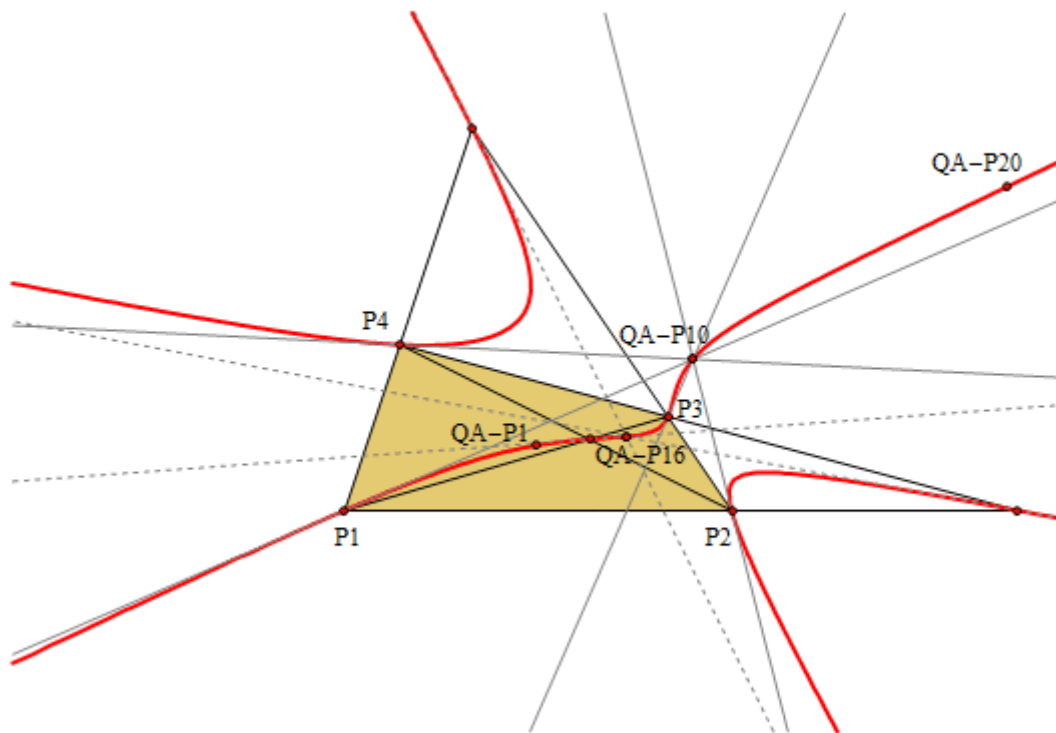
Properties:

- The vertices of the Reference Quadrangle and the QA-Diagonal Triangle lie on this cubic.
- The Involutary Conjugate pairs (QA-P1, QA-P20) and (QA-P5, QA-P17) lie on the cubic.
- The intersection point $QA-P1.QA-P17 \wedge QA-P16.QA-P20$ also lies on QA-Cu2.
- The tangents at P1, P2, P3, P4 meet at QA-P5.
- The tangents at S1, S2, S3 and QA-P5 meet at QA-P17 which is the Involutary Conjugate of QA-P5 on the cubic.
- The 3 asymptotes of QA-Cu2 meet at QA-P1.
- The QA-Cu2 cubic is symmetrical wrt QA-P1 (note Eckart Schmidt).

QA-Cu3: QA-DT-P10 Cubic

QA-Cu3 is the locus of the Double Points created by the QA-Line Involution of all lines through QA-P10. It is a pivotal isocubic of the QA-Diagonal Triangle, invariant wrt the Involutory Conjugate with pivot QA-P10.

QA-Cu3 is a pK(QA-P16,QA-P10) cubic wrt the QA-Diagonal Triangle in the terminology of Bernard Gibert (see [17b]). (note Eckart Schmidt)



Equation CT-notation:

$$\begin{aligned} & r^2 (p+q) (p+q+2r) (q x - p y) x y \\ & + q^2 (p+r) (p+2q+r) (p z - r x) x z \\ & + p^2 (q+r) (2p+q+r) (r y - q z) y z = 0 \end{aligned}$$

Equation DT-notation:

$$r^2 (x-y) x y + p^2 (y-z) y z + q^2 (z-x) z x = 0$$

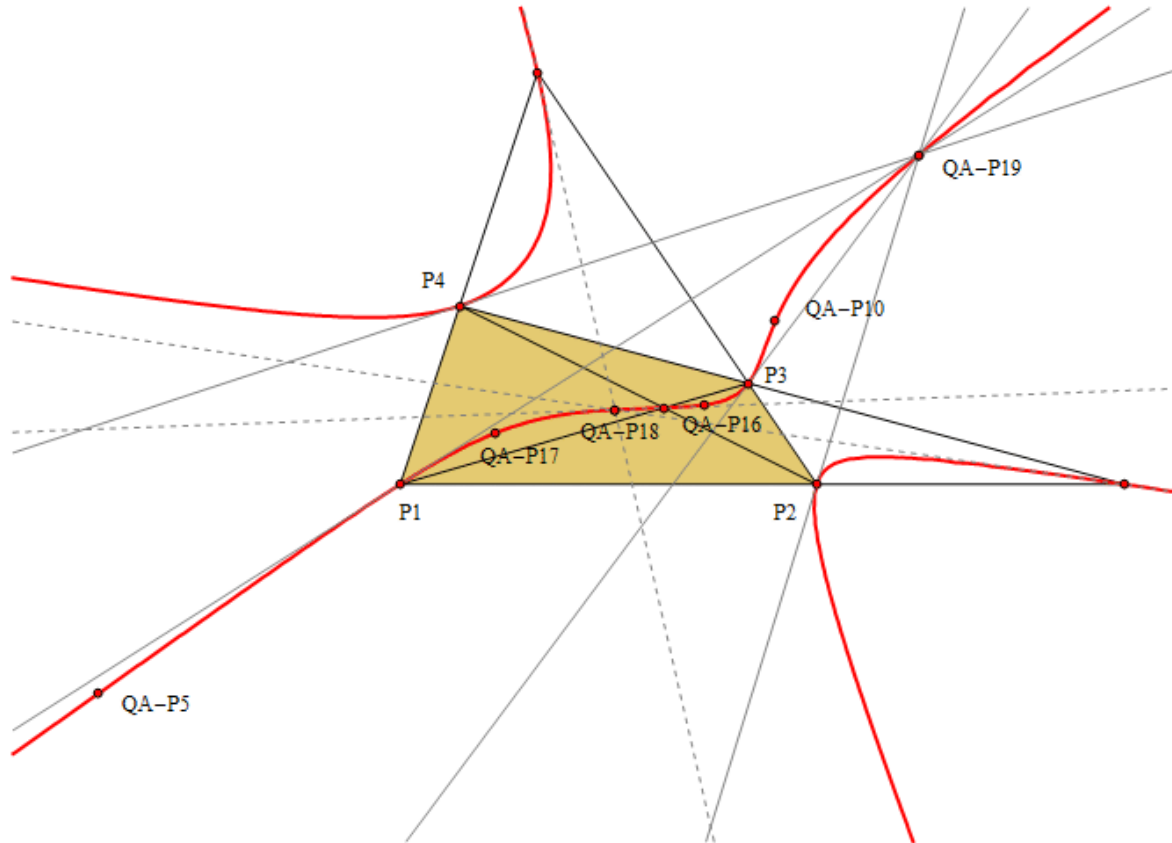
Properties:

- The vertices of the Reference Quadrangle and the QA-Diagonal Triangle lie on this cubic.
- The Midpoints of the sides of the QA-Diagonal Triangle lie on this cubic.
- The Involutory Conjugate pairs (QA-P1, QA-P20) and (QA-P10, QA-P16) lie on the cubic.
- The tangents at P1, P2, P3, P4 meet at QA-P10.
- The tangents at S1, S2, S3 and QA-P10 meet at QA-P16 which is the Involutory Conjugate of QA-P10 on the cubic.

QA-Cu4: QA-DT-P19 Cubic

QA-Cu4 is the locus of the Double Points created by the QA-Line Involution of all lines through QA-P19. It is a pivotal isocubic of the QA-Diagonal Triangle, invariant wrt the Involutary Conjugate with pivot QA-P19.

QA-Cu4 is a pK(QA-P16,QA-P19) cubic wrt the QA-Diagonal Triangle in the terminology of Bernard Gibert (see [17b]). (note Eckart Schmidt)



Equation CT-notation:

$$\begin{aligned} & r^2 (p+q) (p+q+2r) (p^2 + q^2) (q x - p y) x y \\ & + q^2 (p+r) (p+2q+r) (p^2 + r^2) (p z - r x) x z \\ & + p^2 (q+r) (2p+q+r) (q^2 + r^2) (r y - q z) y z = 0 \end{aligned}$$

Equation DT-notation:

$$(-p^2+q^2+r^2)(r^2 y^2-q^2 z^2)x + (p^2-q^2+r^2)(-r^2 x^2+p^2 z^2)y + (p^2+q^2-r^2) (q^2 x^2-p^2 y^2) z = 0$$

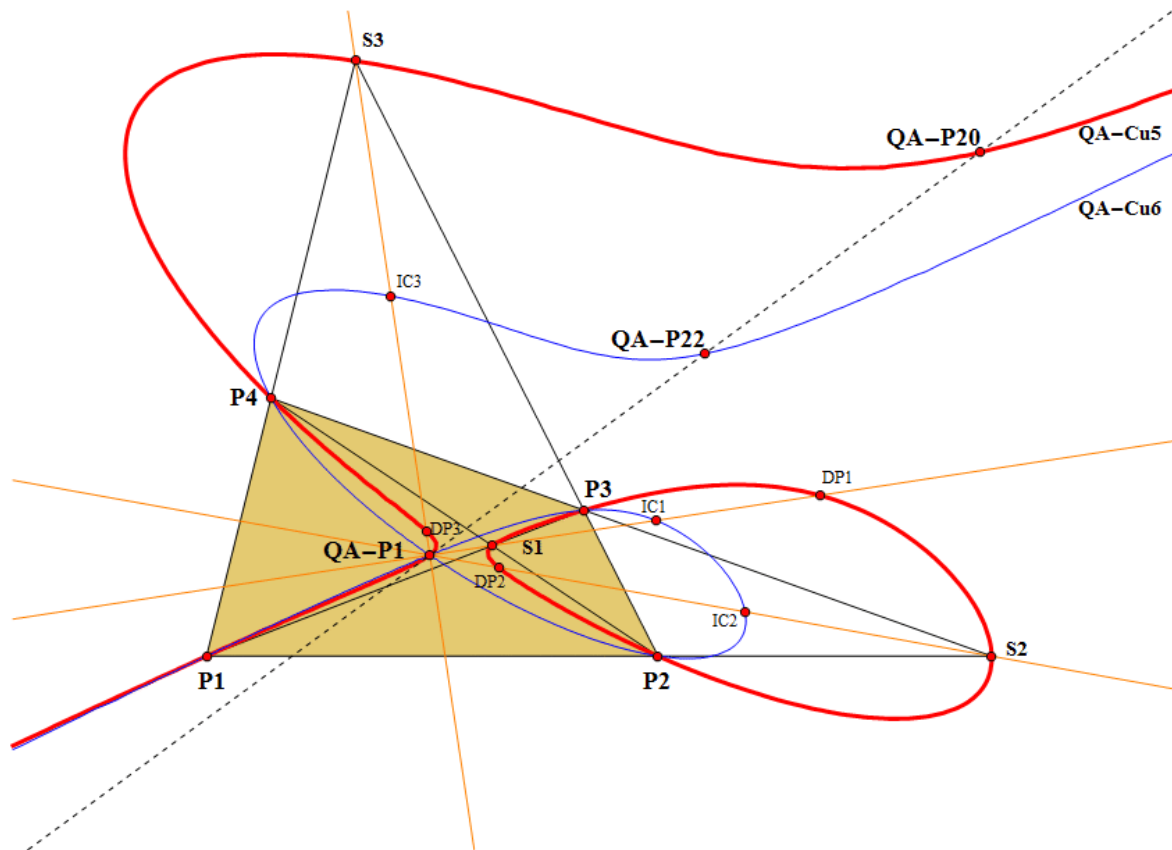
Properties:

- The vertices of the Reference Quadrangle and the QA-Diagonal Triangle lie on this cubic.
- The Involutary Conjugate pairs (QA-P5, QA-P17), (QA-P10, QA-P16) and (QA-P18, QA-P19) lie on the cubic.
- The intersection point $QA-P1.QA-P31 \wedge QA-P16.QA-P18$ also lies on QA-Cu4.
- The tangents at P1, P2, P3, P4 meet at QA-P19.
- The tangents at S1, S2, S3 and QA-P19 meet at QA-P18 which is the Involutary Conjugate of QA-P19 on the cubic.

QA-Cu5: QA-DT-P1 Cubic

QA-Cu5 is the locus of the Double Points created by the QA-Line Involution of all lines through QA-P1. It is a pivotal isocubic of the QA-Diagonal Triangle, invariant wrt the Involutory Conjugate with pivot QA-P1.

QA-Cu5 is a pK(QA-P16,QA-P1) cubic wrt the QA-Diagonal Triangle in the terminology of Bernard Gibert (see [17b]). (note Eckart Schmidt)



Equation CT-notation:

$$p(2p+q+r)(ry-qz)yz + q(p+2q+r)(pz-rx)xz + r(p+q+2r)(qx-py)xy = 0$$

Equation DT-notation:

$$p^2(-p^2+q^2+r^2)(r^2y^2-q^2z^2)x + q^2(-p^2+q^2-r^2)(r^2x^2-p^2z^2)y + r^2(p^2+q^2-r^2)(q^2x^2-p^2y^2)z = 0$$

Properties:

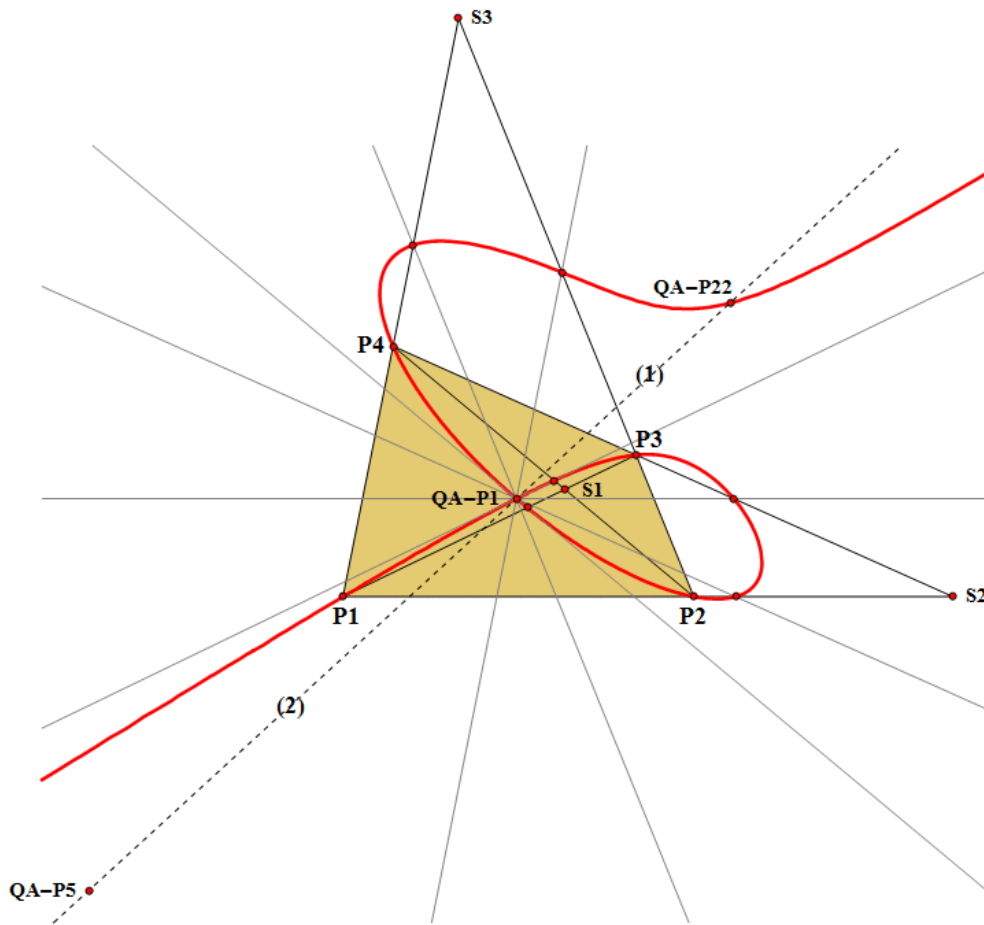
- The vertices of the Reference Quadrangle and the QA-Diagonal Triangle lie on this cubic.
- The Involutory Conjugate pair (QA-P1, QA-P20) lie on the cubic.
- The tangents at P1, P2, P3, P4 meet at QA-P1.
- The tangents at S1, S2, S3 and QA-P1 meet at QA-P20 which is the Involutory Conjugate of QA-P1 on the cubic.

QA-Cu6: QA-P1-Involution Center Cubic

This cubic is the locus of the involution centers of all lines passing through QA-P1 (Quadrangle-Centroid). So it is the locus of Midpoints of the Involutary Conjugates on the cubic QA-Cu5.

The involution is created by the intersection points of the QA-P1-line with the sides of Quadrangle P1.P2.P3.P4 (see chapter QA-Tf1: QA-Line Involution)

QA-P22, the Reflection of the Midpoint (QA-P1,QA-P5) in QA-P1 also lies on the cubic. Similar cubics can be constructed by taking involution centers of lines passing through another point than QA-P1. The other 2 QA-quadrangles deliver the same cubic. This makes the cubic a real QA-Cubic.



Equation CT-notation:

$$\begin{aligned} & r (p+q+2r) ((3q+p)x-(3p+q) y) \times y \\ & + q (p+2q+r) ((3p+r) z-(3r+p) x) \times z \\ & + p (2p+q+r) ((3r+q) y-(3q+r) z) \times y \times z - (p-q)(q-r)(r-p) \times y \times z = 0 \end{aligned}$$

Equation DT-notation:

$$\begin{aligned} & q^2 r^2 (q^2 - r^2) x^3 + q^2 r^2 (2 p^2 (y - z) - (q^2 - r^2) (y + z)) x^2 \\ & + p^2 r^2 (r^2 - p^2) y^3 + p^2 r^2 (2 q^2 (z - x) - (r^2 - p^2) (x + z)) y^2 \\ & + p^2 q^2 (p^2 - q^2) z^3 + p^2 q^2 (2 r^2 (x - y) - (p^2 - q^2) (x + y)) z^2 = 0 \end{aligned}$$

Properties:

- The vertices of the Quadrangle and the Midpoints of the Diagonal triangle and QA-P1, QA-P22 lie on this cubic.
- The intersection point $QA-P1.QA-P6 \wedge QA-P22.QA-P29$ lies on this cubic.
- The intersection points of the lines through QA-P1 parallel to the QA-sidelines with the opposite sidelines all lie on this cubic.

QA-Cu7: DT-Quasi Isogonal Cubic

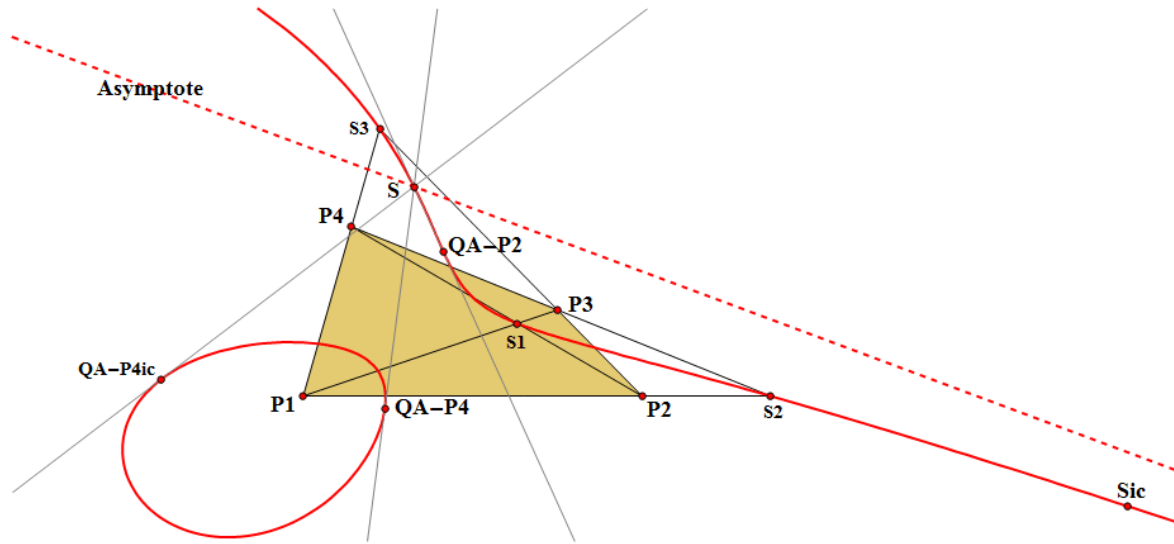
QA-Cu7 is the locus of all points Q for which the Reflection of the line Q.S_i in the angle bisectors of S_i concur, where S_i = ith vertex of the QA-Diagonal Triangle (i = 1,2,3).

The concurring point is also a point of the locus.

With the Reference Quadrangle as reference system this cubic can be seen as a circular isocubic invariant to the Involutory Conjugacy with pivot S_{ic}.

S_{ic} is a point with a long coordinate-formula.

Construction: it is the locus of the intersection pint of the involutory conjugate of some line through QA-P4 and its perpendicular line through QA-P4 (Eckart Schmidt). See QA-Cu/1, QA-Cubic Type 2.



Equation CT-notation:

$$\begin{aligned}
 & - b^2 c^2 p^2 T_x (-q r x + p r y + p q z) x^2 \\
 & - a^2 c^2 q^2 T_y (q r x - p r y + p q z) y^2 \\
 & - a^2 b^2 r^2 T_z (q r x + p r y - p q z) z^2 \\
 & + T_{xyz} x y z
 \end{aligned}$$

where:

$$\begin{aligned}
 T_x &= c^2 q^2 - a^2 q r + b^2 q r + c^2 q r + b^2 r^2, \\
 T_y &= c^2 p^2 + a^2 p r - b^2 p r + c^2 p r + a^2 r^2 \\
 T_z &= b^2 p^2 + a^2 p q + b^2 p q - c^2 p q + a^2 q^2 \\
 T_{xyz} &= b^2 c^2 p^4 + a^2 c^2 q^4 - a^2 (a^2 - b^2 - c^2) q^3 r + a^2 (a^2 + b^2 + c^2) q^2 r^2 - a^2 (a^2 - b^2 - c^2) q r^3 + a^2 \\
 & b^2 r^4 + p^3 (c^2 (a^2 + b^2 - c^2) q + b^2 (a^2 - b^2 + c^2) r) + p^2 (c^2 (a^2 + b^2 + c^2) q^2 + (a^4 - a^2 b^2 - a^2 c^2 + \\
 & 4 b^2 c^2) q r + b^2 (a^2 + b^2 + c^2) r^2) + p (c^2 (a^2 + b^2 - c^2) q^3 + (-a^2 b^2 + b^4 + 4 a^2 c^2 - b^2 c^2) q^2 r + \\
 & (4 a^2 b^2 - a^2 c^2 - b^2 c^2 + c^4) q r^2 + b^2 (a^2 - b^2 + c^2) r^3)
 \end{aligned}$$

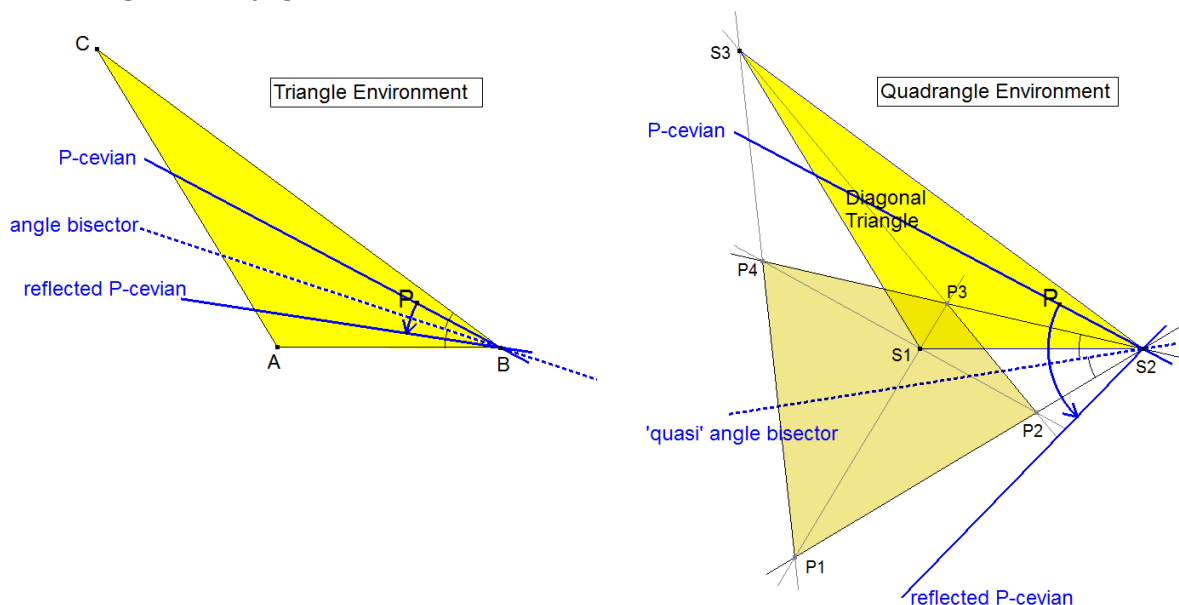
Equation DT-notation:

$$\begin{aligned} & ((b^2 c^2 p^4 + a^4 q^2 r^2) SA + p^2 (c^2 q^2 + b^2 r^2) (S^2 + SB SC)) (r^2 y^2 + q^2 z^2) x \\ & + ((a^2 c^2 q^4 + b^4 p^2 r^2) SB + q^2 (c^2 p^2 + a^2 r^2) (S^2 + SA SC)) (r^2 x^2 + p^2 z^2) y \\ & + ((c^4 p^2 q^2 + a^2 b^2 r^4) SC + (b^2 p^2 + a^2 q^2) r^2 (S^2 + SA SB)) (q^2 x^2 + p^2 y^2) z \\ & + (a^2 b^2 (b^2 p^2 + a^2 q^2) r^4 + a^2 c^2 q^4 (c^2 p^2 + a^2 r^2) + b^2 c^2 p^4 (c^2 q^2 + b^2 r^2) \\ & + 2 p^2 q^2 r^2 (3 SA SB SC + S^2 (SA + SB + SC))) x y z = 0 \end{aligned}$$

Properties:

- The vertices of the QA-Diagonal triangle lie on this cubic.
- QA-P2 (Euler-Poncelet Point) and QA-P4 (Isogonal Center) and their Involutary Conjugates QA-P2ic and QA-P4ic lie on the cubic. QA-P2ic is actually the infinity point of the asymptote of the cubic.
- The Asymptote is perpendicular to QA-L2 = QA-P2.QA-P4.
- The Asymptote // 5th point tangent of QA-P2 (see QA-L/1).
- The Involutary Conjugate of every point on the cubic also lies on the cubic. The cubic is “self-involuntary”.
- The tangents at QA-P2, QA-P4 and QA-P4ic pass through the intersection point S of the asymptote and the cubic.
- QA-Cu7 is the locus of all pairs of involutory conjugated points for which the Thales circle passes through QA-P4 (note Eckart Schmidt).

Quasi Isogonal Conjugate:



An *Isogonal Conjugate* is defined in the environment of a *Triangle*.

To construct an Isogonal Conjugate of a point P, reflections are made of the 3 P-cevians in the corresponding angle bisectors of the vertices of a triangle.

The reflected P-cevians concur in a point P^* . P^* is called the Isogonal Conjugate.

A *Quasi Isogonal Conjugate* is defined in the environment of a *Diagonal Triangle of a Quadrangle*.

To construct a Quasi Isogonal Conjugate of a point P, reflections are made of the 3 P-cevians in a Diagonal Triangle in the corresponding *angle bisectors of the 3 pairs of opposite lines of a quadrangle*.

The reflected cevians only concur in a point P^* when P is situated on the Quasi Isogonal Cubic (QA-Cu7). P^* is called the Quasi Isogonal Conjugate.

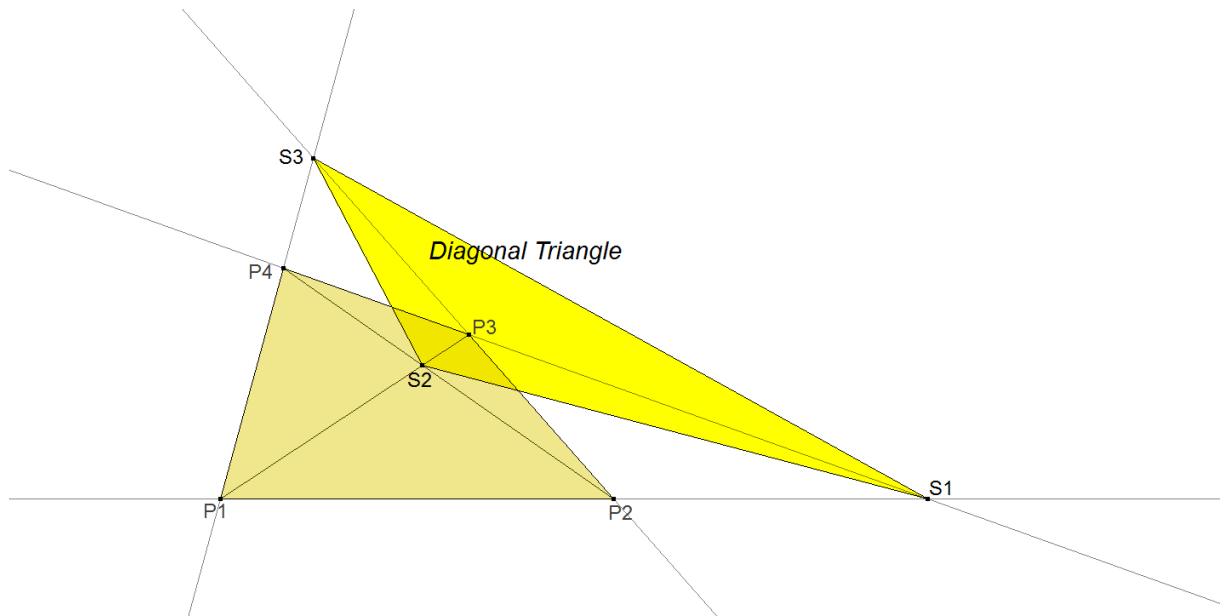
5.5 QUADRANGLE TRIANGLES

QA-Tr1: QA-Diagonal Triangle

Let P_1, P_2, P_3, P_4 be the defining Quadrangle Points.

Let $S_1 = P_1.P_2 \wedge P_3.P_4$, $S_2 = P_1.P_3 \wedge P_2.P_4$ and $S_3 = P_1.P_4 \wedge P_2.P_3$.

Now $S_1.S_2.S_3$ is the QA-Diagonal Triangle of the Reference Quadrangle.



Area QA-Diagonal Triangle in CT-notation:

$$2 p q r \Delta / ((p + q) (p + r) (q + r))$$

Area QA-Diagonal Triangle in DT-notation:

$$2 S$$

Properties:

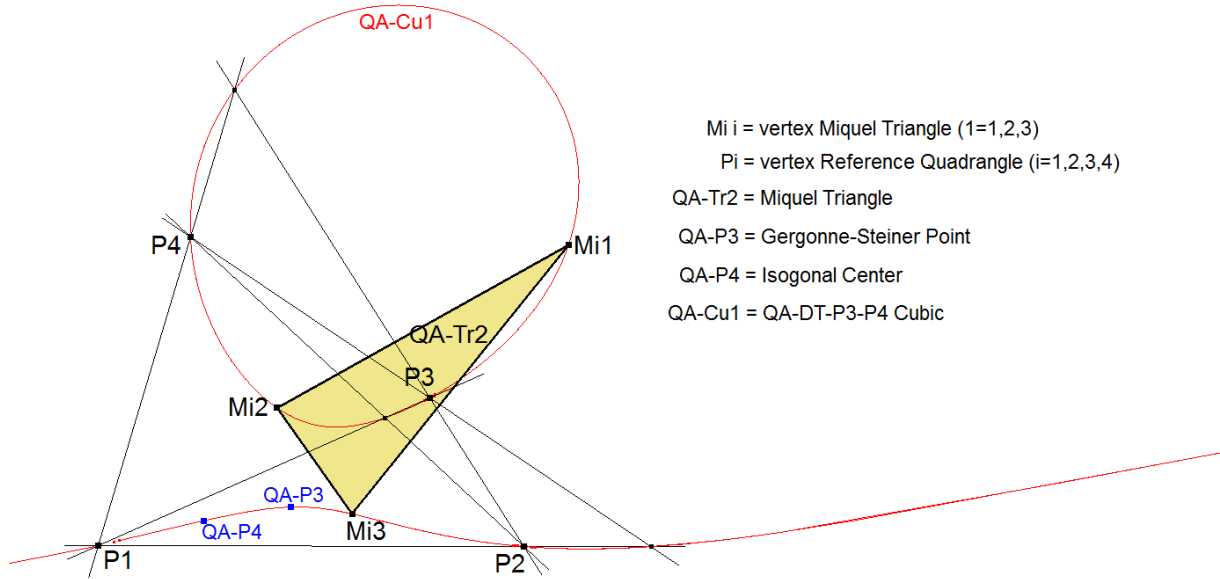
- Every Component Triangle $P_i.P_j.P_k$ is the Anticevian Triangle of the 4th point P_l (for all permutations of $(i,j,k,l) \in (1,2,3,4)$).
- The vertices of the QA-Diagonal Triangle combined with the Isotomic Center form a new quadrangle with the same QA-Centroid as the Reference Quadrangle.
- The Isogonal Center of the Quadrangle $S_1.S_2.S_3.QA-P_4$ is the Involutary Conjugate of QA-P4 (note Eckart Schmidt).

QA-Tr2: Miquel Triangle

The QL-Miquel Points (QL-P1) of the 3 Quadrilaterals of the Reference Quadrangle form a triangle Mi1.Mi2.Mi3.

This Triangle is described in [15c] "Eckart-Schmidt - Das Steiner Dreieck von vier Punkten". In this paper the triangle is called the "Steiner Dreieck".

Special is that the QA-versions of the Steiner Axes (see QL-Tf1) have the role of the internal and external angle bisectors in the Miquel Triangle.



3 QA-versions of Miquel Points in CT-notation:

$$\begin{aligned} & (a^2 (p+q)(p+r) - p(b^2(p+q) + c^2(p+r)) : -b^2(p+q)(p+q+r) : -c^2(p+r)(p+q+r)) \\ & (a^2(p+q)(p+q+r) : a^2 q(p+q) - (-c^2 q + b^2(p+q))(q+r) : c^2(q+r)(p+q+r)) \\ & (a^2(p+r)(p+q+r) : b^2(q+r)(p+q+r) : -c^2(p+r)(q+r) + r(a^2(p+r) + b^2(q+r))) \end{aligned}$$

Area Miquel Triangle in CT-notation:

$$\begin{aligned} & (W_a W_b W_c - (W_a - V_a)(W_b - V_b)(W_c - V_c) \\ & - b^2 c^2 (q+r)^2 V_a - a^2 c^2 (p+r)^2 V_b - a^2 b^2 (p+q)^2 V_c) \Delta / (V_a V_b V_c) \end{aligned}$$

where:

$$\begin{aligned} V_a &= (W_a(2p+q+r) - a^2(p+q)(p+r)) / (p+q+r) \\ V_b &= (W_b(p+2q+r) - b^2(q+r)(q+p)) / (p+q+r) \\ V_c &= (W_c(p+q+2r) - c^2(r+p)(r+q)) / (p+q+r) \\ W_a &= b^2(p+q) + c^2(p+r) \\ W_b &= c^2(q+r) + a^2(q+p) \\ W_c &= a^2(r+p) + b^2(r+q) \end{aligned}$$

3 QA-versions of Miquel Points in DT-notation:

$$\begin{aligned} & (-2 p^2 (p^2 S_a + q^2 S_b + r^2 S_c) : \\ & a^2 q^2 (p^2 - q^2 + r^2) + b^2 p^2 (-p^2 + q^2 + r^2) - 2 c^2 p^2 q^2 : \\ & a^2 r^2 (p^2 + q^2 - r^2) - 2 b^2 p^2 r^2 + c^2 p^2 (-p^2 + q^2 + r^2)) \end{aligned}$$

$$\begin{aligned}
& (a^2 q^2 (p^2 - q^2 + r^2) + b^2 p^2 (-p^2 + q^2 + r^2) - 2 c^2 p^2 q^2 : \\
& \quad -2 q^2 (p^2 Sa + q^2 Sb + r^2 Sc) : \\
& \quad -2 a^2 q^2 r^2 + b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2)) \\
& (a^2 r^2 (p^2 + q^2 - r^2) - 2 b^2 p^2 r^2 + c^2 p^2 (-p^2 + q^2 + r^2) : \\
& \quad -2 a^2 q^2 r^2 + b^2 r^2 (p^2 + q^2 - r^2) + c^2 q^2 (p^2 - q^2 + r^2) : \\
& \quad -2 r^2 (p^2 Sa + q^2 Sb + r^2 Sc))
\end{aligned}$$

Properties:

- QA-Tr2 and QA-Tr1 (QA-Diagonal Triangle) are perspective triangles with Perspector QA-P3 (Gergonne-Steiner Point).
- QA-P3 and QA-P4 are mutually isogonal conjugates wrt the Miquel Triangle.
- The vertices of QA-Tr2 lie on cubic QA-Cu1.
- QA-P9 lies on the circumcircle of the Miquel Triangle.
- The intersection point of the QA-Cu1 cubic and its asymptote lies on the circumcircle of the Miquel Triangle opposite to QA-P9 (note Eckart Schmidt).

QA-Tr3: Morley Triangle

The QL-Morley Points (QL-P2) of the 3 Quadrilaterals of the Reference Quadrangle form a triangle Mo1.Mo2.Mo3.

The QL-Quasi Ortholines (see paragraph QL-L6: Quasi Ortholine) of the 3 Quadrilaterals of the Reference Quadrangle pass through Mo1, Mo2, Mo3 and are the medians of the Morley Triangle.

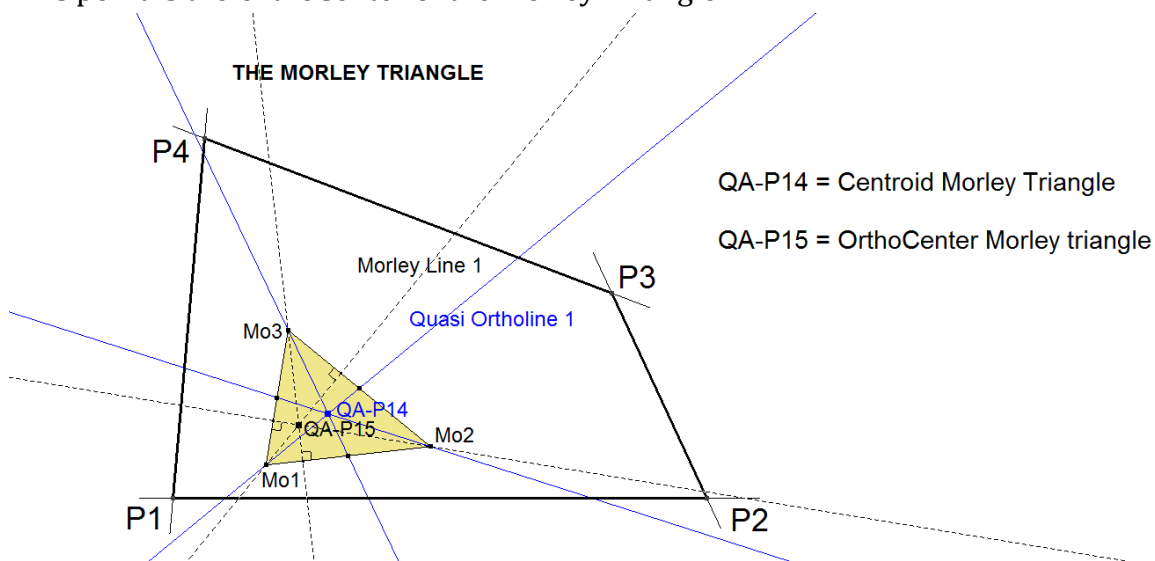
Their intersection point is the QA-Point QA-P14.

This point is the Centroid of the Morley Triangle.

The QL-Morley Lines (QL-L4) of the 3 Quadrilaterals of the Reference Quadrangle pass through Mo1, Mo2, Mo3 and are the altitudes of the Morley Triangle.

Their intersection point is the QA-Point QA-P15.

This point is the OrthoCenter of the Morley Triangle.



Area Morley Triangle in CT-notation:

$$(c^2pq + b^2pr + a^2qr)^2 / (64 (p+q) (p+r) (q+r) (p+q+r) S)$$

Area Morley Triangle in DT-notation:

$$-(SA u^2 + SB v^2 + SC w^2)^2 / (8 S (-u+v+w) (u+v-w) (u-v+w) (u+v+w))$$

Properties:

- The Morley Triangle is the Medial Triangle of the Triangle composed of the points QG-P10 (2nd Quasi Orthocenter) occurring in the 3 QA-Quadrilaterals.
- The Triangle composed of the points QG-P5 (1st Quasi Circumcenter) of the 3 QA-Quadrilaterals is homothetic with the Morley Triangle. Its side lengths are twice the side lengths of the Morley Triangle. Perspector is QA-P24.
- The centroid (QA-P14) of the Morley Triangle plays a role in the construction of complements and anti-complements wrt the Morley triangle:
QA-P24 = AntiComplement of QA-P1 wrt the Morley Triangle
QA-P33 = Complement of QA-P12 wrt the Morley triangle.
- The vertices of the Morley Triangle coincide when the Reference Quadrangle is concyclic.

5.6 QUADRANGLE TRANSFORMATIONS

QA-Tf1: QA-Line Involution

General Information

In mathematics, an Involution, or an involutory function, is a function f that is its own inverse: $f(f(x)) = x$ for all x in the domain of f .

In this paper we will deal with the geometric notion of a “Line Involution”.

It occurs with pairs of points on a line, for whom the product of their distances from a fixed point P is a given constant. P is called the Center of Involution or Involution Center.

Let (A_0, A_1) and (B_0, B_1) be 2 pairs of points for whom the product of their distances from a fixed point O is a given constant c .

According to the definition of Line Involution: $A_0.O * A_1.O = B_0.O * B_1.O = c$.

When the points A_0, A_1 and the constant c are known we can calculate from a variable point B_0 its mate B_1 . When we do the same calculation for B_1 we find its mate B_0 . This makes the transformation an involutory function because $f(f(P)) = P$.

When $B_0=B_1$ we say that $B_0=B_1$ is a Double Point. There are 2 Double Points per Line Involution. Their Midpoint is the Center of Involution O .

Desargues' Involution Theorem

A figure consisting of 4 points and their 6 connecting lines is called a (complete) Quadrangle. The 4 points are the vertices. The 6 connecting lines are the sides of the (complete) Quadrangle. Sides in a (complete) Quadrangle that have no vertices in common are called opposite sides.

Desargues' Involution theorem states that the points of intersection of a line with the three pairs of opposite sides of a complete Quadrangle and a conic section circumscribed about the complete quadrangle form the pairs of an involution (see [7]).

Normally 2 pairs of points describe an involution. Desargues describes in his theorem 4 pairs of points all describing the same involution.

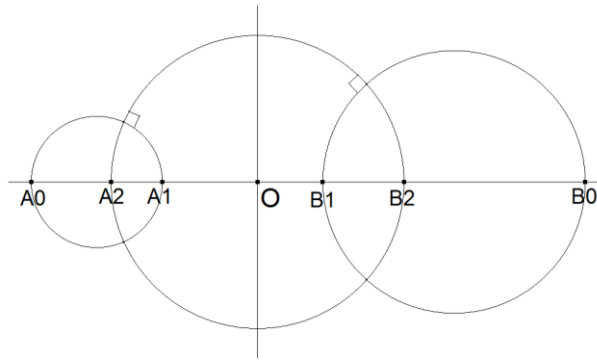
If we restrict ourselves to the 3 pairs of points generated from 3 sets of opposite sides we can deduce that when a line crosses a quadrangle it generates a unique Line Involution with an involution center and 2 double points (when real).

Consequently every line crossing a Quadrangle has an Involution Center and 2 Double Points (real or imaginary).

This line involution is a property of a Quadrangle (and not of a Quadrilateral as might be suspected).

In the QA-environment I call this a QA-Line Involution.

Construction:



For a Line Involution let $(A0, A1)$ and $(B0, B1)$ be two pairs of points and ciA and ciB circles with these diameters.

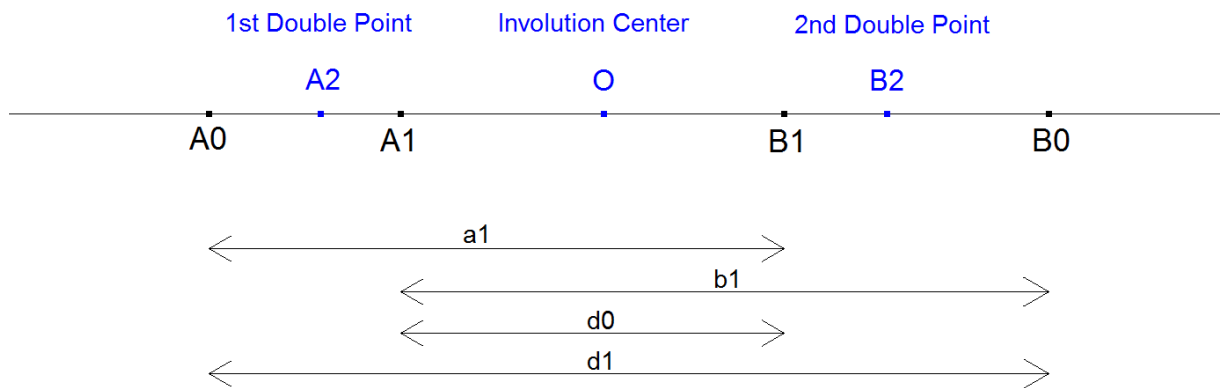
The radical axis of ciA and ciB cuts the line in the Involution Center O .

A circle with center O perpendicular to ciA and ciB cuts the line in the two Double Points $A2, B2$.

Another construction can be found in [19].

Calculations:

Let $(A0, A1)$ and $(B0, B1)$ be 2 pairs of points on a line that create a Line Involution with center O . Then according to the definition: $d(A0, O) * d(A1, O) = d(B0, O) * d(B1, O)$.



Let $A2, B2$ be the corresponding double points.

Now:

$$d(A0, A2) = (a1 * d0 - \sqrt{[d0 * d1 * a1 * b1]}) / (a1 + b1)$$

$$d(A0, B2) = (a1 * d0 + \sqrt{[d0 * d1 * a1 * b1]}) / (a1 + b1)$$

$$d(B0, B2) = (b1 * d0 - \sqrt{[d0 * d1 * a1 * b1]}) / (a1 + b1)$$

$$d(B0, A2) = (b1 * d0 + \sqrt{[d0 * d1 * a1 * b1]}) / (a1 + b1)$$

The center of involution O is given by:

$$d(A0, O) = a1 * d0 / (a1 + b1)$$

$$d(B0, O) = b1 * d0 / (a1 + b1)$$

and $d(A1,O) = d(B1,O) = \sqrt{[d0.d1.a1.b1] / (a1+b1)}$

Let $(A0,A1)$ and $(B0,B1)$ be 2 pairs of points on a line that create a Line Involution with center O. Let $A2$ and $B2$ be the respective Double Points.

When $(A0,A1)$ and $B2$ are known then we can calculate the position of $A2$:

Let $d0 = d(B2,A0)$ and $d1 = d(B2,A1)$

then $e0 = d(A2,A0) = (d0*d0-d0*d2)/(d0+d2),$

and $e1 = d(A2,A1) = (d2*d2-d0*d2)/(d0+d2).$

Properties:

- Let L be a random line and $Q1, Q2$ the Double Points created by the generated QA-Line Involution.

When $Q1$ and $Q2$ are real Double Points then line L is tangent in Qi at the conic $(P1,P2,P3,P4, Qi)$ and also tangent at the QA-DT-cubic with pivot Qi .

QA-Tf2: Involutory Conjugate

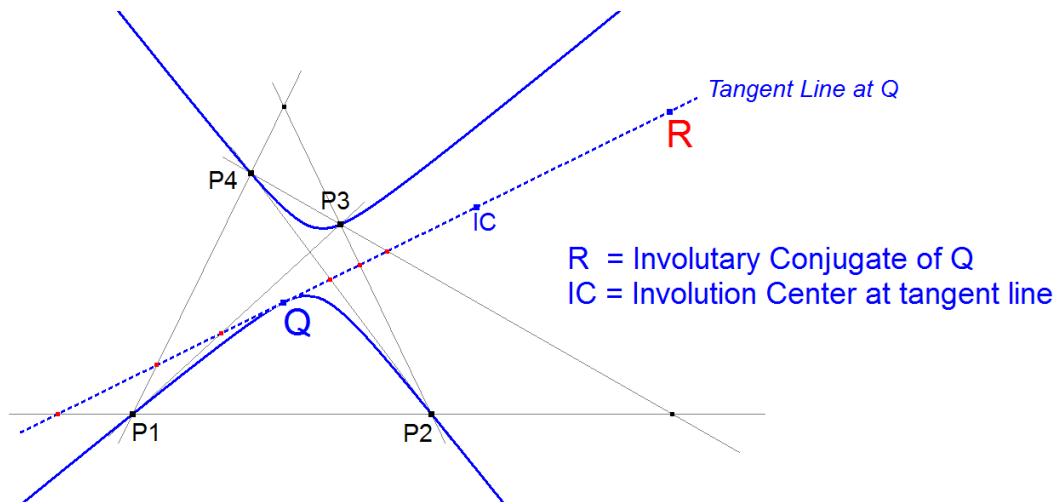
The Involutory Conjugate of a point Q is the 2nd Double Point of the QA-Line Involution (see QA-Tf1) occurring on the tangent at Q to the conic (P_1, P_2, P_3, P_4, Q) .

It is a conjugate because applying two times this transformation ends up in the original point.

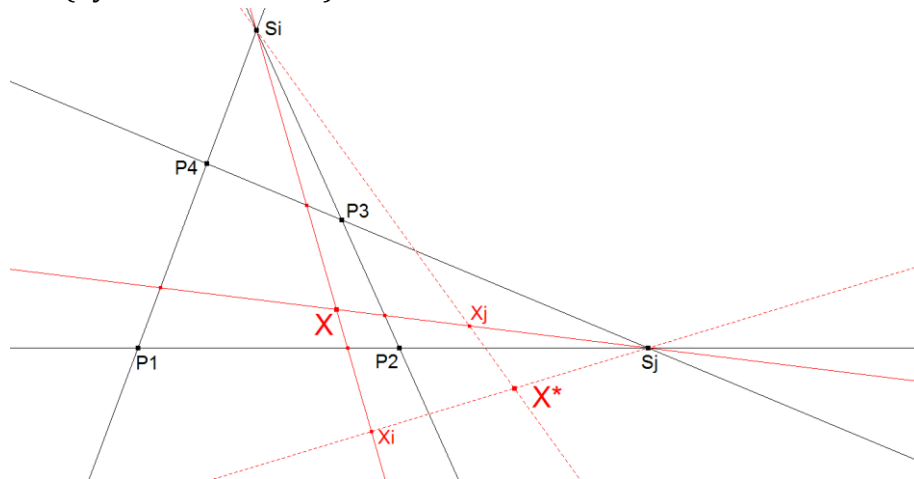
The Involutory Conjugate can also be seen as the Pi-Ceva Conjugate of Q wrt Component Triangle $P_j.P_k.P_l$ (for all permutations of $(i,j,k,l) \in (1,2,3,4)$) (note Bernard Gibert).

Construction 1:

1. Construct the conic through P_1, P_2, P_3, P_4, Q .
2. Construct the tangent L_q at Q to this conic.
3. Construct the Involution Center IC of the created QA-Line Involution for L_q (see [19]).
4. Now $R =$ the Involutory Conjugate of $Q =$ the Reflection of Q in IC .



Construction 2 (by Eckart Schmidt):



1. Choose one of the Component Quadrigons of the Reference Quadrangle.
2. Let S_i and S_j be the intersection points of the opposite sides of the Quadrigon.
3. Connect X (the point to be transformed) with S_i and S_j and construct on these lines the 4th harmonic points X_i and X_j wrt the intersection points with the crossing opposite quadrigon lines.
4. Connecting lines $S_i X_j$ and $S_j X_i$ intersect in X^* , the Involutory Conjugate of X .

Coordinates and Coefficients:

Let $Q(u : v : w)$ be a random point not on one of the connecting lines of the Reference Quadrangle. It is possible to construct a conic through P_1, P_2, P_3, P_4, Q since a conic is defined by 5 points.

The tangent at Q to this conic (P_1, P_2, P_3, P_4, Q) is in CT-notation:

$$(p v w (r v - q w) : q u w (p w - r u) : r u v (q u - p v))$$

The tangent at Q to this conic (P_1, P_2, P_3, P_4, Q) is in DT-notation:

$$(u (q^2 w^2 - r^2 v^2) : v (r^2 u^2 - p^2 w^2) : w (p^2 v^2 - q^2 u^2))$$

This tangent creates an QA-Line Involution where Q is a Double Point.

Its Involution Center in CT-notation is:

$$(u (+q r u^2 - p r u v - p q u w - p q v w - p r v w) : \\ v (-q r u v + p r v^2 - p q u w - q r u w - p q v w) : \\ w (-p r u v - q r u v - q r u w - p r v w + p q w^2))$$

Its Involution Center in DT-notation is:

$$(r^2 u^2 v + q^2 u^2 w + p^2 v w (2 u + v + w) : \\ r^2 u v^2 + p^2 v^2 w + q^2 u w (u + 2 v + w) : \\ q^2 u w^2 + p^2 v w^2 + r^2 u v (u + v + 2 w))$$

The 2nd Double Point on this Q-tangent is the Involutory Conjugate:

in CT-notation:

$$(u (q r u - p r v - p q w) : v (-q r u + p r v - p q w) : w (-q r u - p r v + p q w))$$

in DT-notation:

$$(p^2 v w : q^2 w u : r^2 u v)$$

Examples of Involutary Conjugates:

The following table lists a number of Involutary Conjugated pairs of points.

Point-1	Point-2
QA-P1: QA-Centroid	QA-P20: Reflection of QA-P5 in QA-P1
QA-P5: Isotomic Center	QA-P17: Involutary Conjugate of QA-P5
QA-P6: Parabola Axes Crosspoint	QA-P30: Reflection of QA-P2 in QA-P11
QA-P10: Centroid QA-Diagonal Triangle	QA-P16: QA-Harmonic Center
QA-P12: Orthocenter QA-Diag. Triangle	QA-P23: Inscribed Square Axes Crosspoint
QA-P18: Involutary Conjugate of QA-P19	QA-P19: AntiCompl. of QA-P16 wrt QA-DT
QA-P21: Reflection of QA-P16 in QA-P1	QA-P27: M3D Center

Next table lists a number of Involutary Conjugates of QA-lines and QA-curves.

The Involutary Conjugate transforms lines into circumscribed conics of the QA-Diagonal Triangle. A “5th point tangent” (see QA-L/1) at Q is transformed into a circumscribed conic of the QA-Diagonal Triangle through Q and its Involutary Conjugate.

QA-Line	QA-DT-Conic
Line at Infinity	QA-Co1: Ninepoint Conic
Perpendicular bisector of QA-P2.QA-P4	QA-Ci 1: Circumcircle Diagonal Triangle
QA-L3: QA-Centroids Line	QA-Co5: Circumscribed DT-Conic through vertices Diagonal Triangle and QA-P1, P16, P17, P19, P20
QA-L4: QA-P1.QA-P6 Line	QA-Co4: Orthogonal circumscribed DT-hyperbola through vertices Diagonal Triangle and QA-P3, P12, P20, P30
Line QA-P5.QA-P17.QA-P19.QA-P21	QA-DT-Conic through QA-P5, QA-P17, QA-P18, QA-P27

Next table lists a number of self-involutary cubics.

QA-Curve	comment
QA-Cu1 till QA-Cu5: QA-DT Cubics	self-involutary cubic
QA-Cu7: QA-Quasi Isogonal Cubic	self-involutary cubic

6.0 QUADRILATERAL OBJECTS

6.1 QUADRILATERAL GENERAL INFORMATION

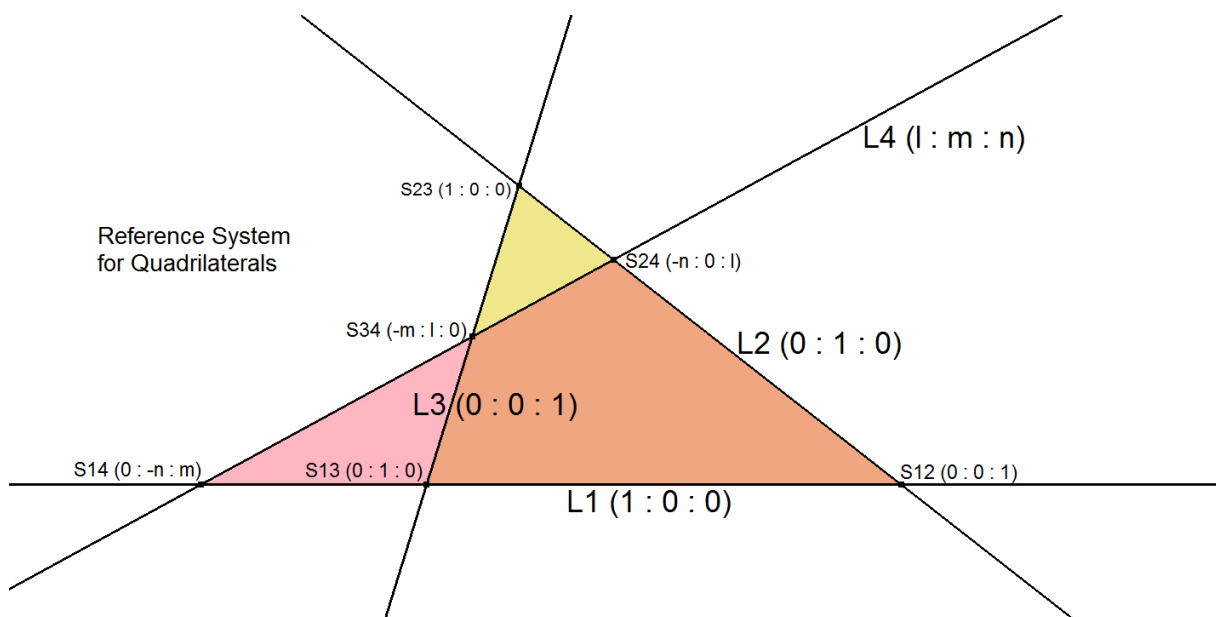
QL/1: Systematics for describing QL-Points

The way to describe points, lines and other curves related to Quadrilaterals is by relating to lines instead of points (as done with the description of Quadrangles).

The big advantage is that all resulting algebraic expressions will be symmetric.

The system used is based on 3 lines of the quadrilateral with barycentric homogeneous coefficients $L1(1:0:0)$, $L2(0:1:0)$ and $L3(0:0:1)$. The 4th line gets coefficients $(l:m:n)$.

Note that the line notation $(l:m:n)$ is used instead of $(p:q:r)$ which is only used for points.



Every constructed object can now be identified as:

$$(f(a,b,c, l,m,n) : f(b,c,a, m,n,l) : f(c,a,b, n,l,m))$$

where a, b, c represent the side lengths of the triangle with vertices $P1 (1:0:0)$, $P2 (0:1:0)$ and $P3 (0:0:1)$.

In the description of the points on the following pages only the first of the 3 barycentric coefficients will be shown. The other 2 coordinates can be derived by cyclic rotations:

- $a > b > c > a > \text{etc.}$
- $l > m > n > l > \text{etc.}$

Further the Conway notation has been used in algebraic expressions:

- $S_A = (-a^2 + b^2 + c^2) / 2$
- $S_B = (+a^2 - b^2 + c^2) / 2$
- $S_C = (+a^2 + b^2 - c^2) / 2$
- $S_\omega = (+a^2 + b^2 + c^2) / 2$

- $S = \sqrt{S_A S_B + S_B S_C + S_C S_A} = 2 \Delta$
 where $\Delta = \text{area triangle ABC} = \frac{1}{4} \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$

QL/2: List of QL-Lines

In next list all QL-points are mentioned without prefix "QL-".

All lines relating to QL-P1 till QL-P26 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

These QL-lines are described further with their properties:

- QL-L1: P5, P7, P12, P20, P22, P23 (Newton Line)
- QL-L2: P2, P7, P9 (Steiner Line)
- QL-L3: P19 (Pedal Line)
- QL-L4: P2, P3 (Morley Line)
- QL-L5: P1, P7, P19 (NSM-Line)
- QL-L6: P2, P10 (Quasi Ortholine)
- QL-L7: P8, P9, P10, P11 (QL-DT-Euler Line)
- QL-L8: P8, P12, P14, P15, P18 (QL-Centroids Line)
- QL-L9: P18, P23 (M3D Line)

Other QL-Lines without name but with at least 3 Points on it are:

- P1, P8, P24
- P1, P10, P16
- P1, P20, P21
- P2, P4, P22
- P2, P6, P12
- P3, P4, P5, P6
- P8, P17, P25
- P13, P17, P24

QL/3: List of parallel QL-Lines

In next list all QL-points are mentioned without prefix “QL-”.

All lines relating to QL-P1 till QL-P26 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

These QL-lines are parallel:

	<u>Railway Watcher (see QL-L/1)</u>
- P1.P7 // P2.P10	P16
- P1.P11 // P9.P16	P10
- P2.P3 // P5.P7	P4
- P2.P5 // P4.P20 // P6.P22	P4+P20, P2+P5, P3
- P2.P20 // P3.P4	
- P7.P21 // P19.P20	P1
- P9.P17 // P11.P25	P10
- P9.P25 // P10.P17	

QL/4: List of perpendicular QL-Lines

In next list all QL-points are mentioned without prefix “QL-”.

All lines relating to QL-P1 till QL-P26 have been taken into account.

When lines have more than 2 points, they are defined by the 2 points with lowest serial number.

These QL-lines are perpendicular:

- P1.P7 \perp P18.P23
- P1.P8 \perp P16.P24
- P1.P25 \perp P16.P17
- P2.P3 \perp P2.P7
- P2.P7 \perp P5.P7
- P2.P10 \perp P18.P23
- P6.P9 \perp P13.P17

QL/5: List of QL-Crosspoints

When 3 lines connecting QL-points concur, the point of concurrence is called a QL-Crosspoint.

In this list all possible non-registered QL-Crosspoints are listed originating from at least 3 connecting lines of QL-points in the range QL-P1 – QL-P26.

QL-points are mentioned without prefix “QL-”.

Lines are defined by 2 points on it with lowest serial number.

There are regularly recurring crossing lines with these Crosspoints. This is an indication for the occurrence of Perspective Fields (see QL-PF1).

When the intersection points have fixed ratios of the distances to the defining points on the defining lines, then they are mentioned. There are many of them.

For point P on line P1.P2 the ratio $d_1 : d_2$ means that $d(P, P_1) : d(P, P_2) = d_1 : d_2$, where:

- d_1 is positive when P is positioned wrt P1 at the same side of the line as P2. If not then d_1 is negative.
- d_2 is positive when P is positioned wrt P2 at the same side of the line as P1. If not then d_1 is negative.

– P2.P5 ^ P3.P20 ^ P4.P12	1 : 1 / 1 : 1 / 3 : -1
– P2.P5 ^ P4.P20 ^ P6.P22	Infinity point
– P5.P8 ^ P14.P22 ^ P18.P20	-3 : 4 / -8 : 9 / -2 : 3
– P5.P14 ^ P8.P22 ^ P18.P20	9 : -4 / 8 : -3 / 2 : 3
– P5.P15 ^ P8.P20 ^ P18.P22	9 : -4 / 2 : 3 / 8 : -3
– P5.P18 ^ P8.P20 ^ P15.P22	4 : -3 / -2 : 3 / -8 : 9
– P9.P12 ^ P10.P14 ^ P11.P18	6 : -1 / 9 : 1 / 3 : 2
– P9.P15 ^ P10.P12 ^ P11.P18	9 : -2 / 6 : 1 / 3 : 4
– P9.P18 ^ P10.P12 ^ P11.P15	3 : 2 / 6 : -1 / 9 : -4
– P9.P18 ^ P10.P14 ^ P11.P12	3 : 1 / 9 : -1 / 3 : -1

QL/6: QL-Conversion CT -> DT-coordinates

Let $L1.L2.L3.L4$ be the Reference Quadrilateral with defining lines $L1, L2, L3, L4$.

Let $La.Lb.Lc$ be the Diagonal Triangle of the Quadrilateral with bounding lines La, Lb, Lc .

The construction of the QL-Diagonal Triangle can be found at QL-Tr1.

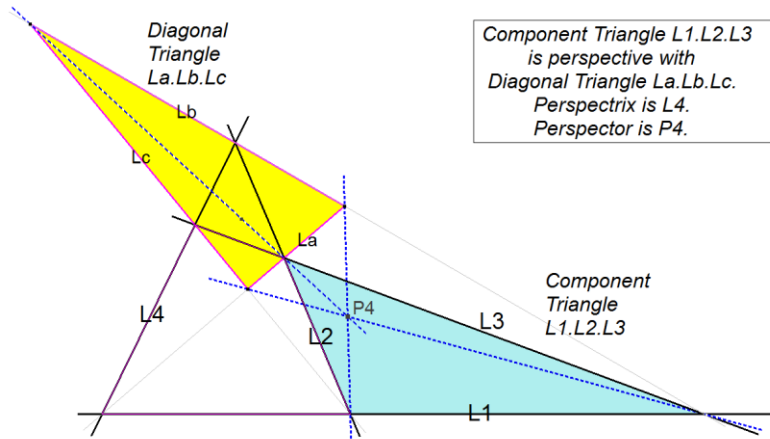
Let $L1.L2.L3$ be the random Reference Component Triangle en let $L4$ be the 4th line.

The QL-Diagonal Triangle $La.Lb.Lc$ is the *AntiCevian Triangle* of $L4$ wrt $L1.L2.L3$, where $L4$ is the perspectrix.

Conversion of a line

Since we deal with lines in a Quadrilateral we first will convert the equation of a line.

Let R be some line to be converted from CT- to DT-coordinates.



Let $R_c (x_c : y_c : z_c)$ be the presentation of R in barycentric coordinates wrt the Component Triangle. Let $R_d (x_d : y_d : z_d)$ be the presentation of R in barycentric coordinates wrt the Diagonal Triangle.

Now $R_c = x_c.cfc1.L1 + y_c.cfc2.L2 + z_c.cfc3.L3$ wrt the Reference Component Triangle and $R_d = x_d.cfd1.La + y_d.cfd2.Lb + z_d.cfd3.Lc$ wrt the Diagonal Triangle, where:

- $(x_c : y_c : z_c)$ are the barycentric coordinates of R wrt the Component Triangle,
- $(x_d : y_d : z_d)$ are the barycentric coordinates of R wrt the Diagonal Triangle,
- $cfc1, cfc2, cfc3$ are the Compliance Factors of the Component Triangle,
- $cfd1, cfd2, cfd3$ are the Compliance Factors of the Diagonal Triangle.

Explanation of Compliance Factors can be found at [See 26b page 40].

Since the Component Triangle is the Reference Triangle, the Compliance Factors of the Component Triangle are all equal 1.

The Compliance Factors of the Diagonal Triangle are:

- $cfd1 = \text{Det} [L1, Lb, Lc] / \text{Det} [La, Lb, Lc],$
- $cfd2 = \text{Det} [La, L1, Lc] / \text{Det} [La, Lb, Lc],$
- $cfd3 = \text{Det} [La, Lb, L1] / \text{Det} [La, Lb, Lc],$

where LI = the line at infinity (1 : 1 : 1) and “Det” is the abbreviation for “Determinant”. Calculation gives 2 presentations of the coordinates of R wrt the Component Triangle:

- $R_c = (x_c : y_c : z_c),$
- $R_d = (l(lm y_d - l n y_d + m n y_d - l m z_d + l n z_d + m n z_d) : -m(-l m x_d - l n x_d + m n x_d + l m z_d - l n z_d - m n z_d) : -n(-l m x_d - l n x_d + m n x_d - l m y_d + l n y_d - m n y_d))$

Since R_c and R_d present the same point we can now calculate the coordinates of R wrt the Diagonal Triangle:

- $(x_d : y_d : z_d) = ((l m - l n - m n)(l m - l n + m n)(-m n x_c + l n y_c + l m z_c) : (l m - l n - m n)(l m + l n - m n)(m n x_c - l n y_c + l m z_c) : (l m + l n - m n)(l m - l n + m n)(-m n x_c - l n y_c + l m z_c))$

However we have to bear in mind that the variables in these coordinates are expressions in (a,b,c) and (l,m,n), which are variables wrt the Component Triangle. So also a conversion of these variables from CT to DT should be done.

Therefore the CT > DT-conversion of line R (x : y : z) consists of 3 consecutive steps:

1. Transform Line (x : y : z) -->

$$((l m - l n - m n)(l m - l n + m n)(-m n x + l n y + l m z) : (l m - l n - m n)(l m + l n - m n)(m n x - l n y + l m z) : (l m + l n - m n)(l m - l n + m n)(-m n x - l n y + l m z))$$
2. Replace $a^2 \rightarrow l^2(m+n)^2(a^2(l+m)(l+n) + (m-n)(b^2(l+m) - c^2(l+n)))$
 $b^2 \rightarrow m^2(l+n)^2(a^2(l+m)(l-n) + (m+n)(b^2(l+m) + c^2(-l+n)))$
 $c^2 \rightarrow n^2(l+m)^2(a^2(l-m)(l+n) + (m+n)(b^2(-l+m) + c^2(l+n)))$
3. Replace $l \rightarrow m n(l+m)(l+n)$
 $m \rightarrow l n(l+m)(m+n)$
 $n \rightarrow l m(l+n)(m+n)$

Conversion of a point

Converting the coordinates of a point in the Quadrilateral-environment is converting the intersection of 2 lines.

This actually only changes the Transform of (x : y : z) and leaves the replacements of variables unchanged:

1. Transform Point (x : y : z) -->

$$((-l m - l n + m n)(m y + n z) : (-l m + l n - m n)(l x + n z) : (l m - l n - m n)(l x + m y))$$
2. Replace $a^2 \rightarrow l^2(m+n)^2(a^2(l+m)(l+n) + (m-n)(b^2(l+m) - c^2(l+n)))$
 $b^2 \rightarrow m^2(l+n)^2(a^2(l+m)(l-n) + (m+n)(b^2(l+m) + c^2(-l+n)))$
 $c^2 \rightarrow n^2(l+m)^2(a^2(l-m)(l+n) + (m+n)(b^2(-l+m) + c^2(l+n)))$
3. Replace $l \rightarrow m n(l+m)(l+n)$
 $m \rightarrow l n(l+m)(m+n)$
 $n \rightarrow l m(l+n)(m+n)$

QL/7: QL-Conversion DT -> CT-coordinates

Let $L1.L2.L3.L4$ be the Reference Quadrilateral with defining lines $L1, L2, L3, L4$.

Let $La.Lb.Lc$ be the Diagonal Triangle of the Quadrilateral with bounding lines La, Lb, Lc .

The construction of the QL-Diagonal Triangle can be found at QL-Tr1.

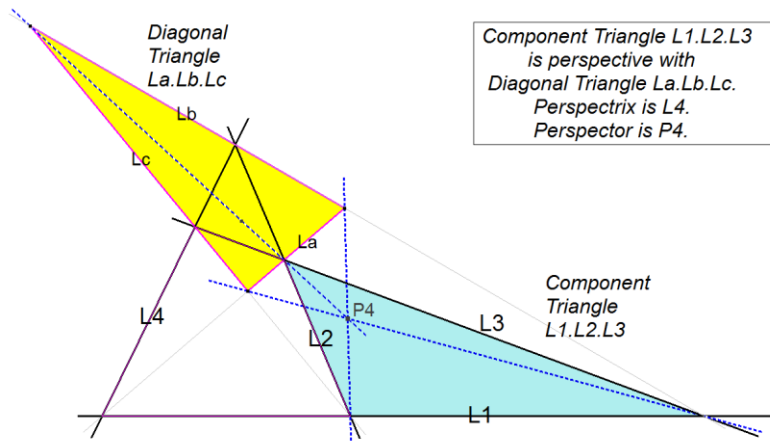
Let $L1.L2.L3$ be the random Reference Component Triangle en let $L4$ be the 4th line.

$L1.L2.L3$ is the *Cevian Triangle* of $L4$ wrt the QL-Diagonal Triangle $La.Lb.Lc$, where $L4$ is the perspectrix.

Conversion of a line

Since we deal with lines in a Quadrilateral we first will convert the equation of a line.

Let R be some line to be converted from DT- to CT-coordinates.



Let $R_c (x_c : y_c : z_c)$ be the presentation of R in barycentric coordinates wrt the Component Triangle. Let $R_d (x_d : y_d : z_d)$ be the presentation of R in barycentric coordinates wrt the Diagonal Triangle.

Now $R_c = x_c.cfc1.L1 + y_c.cfc2.L2 + z_c.cfc3.L3$ wrt the Reference Component Triangle and $R_d = x_d.cfd1.La + y_d.cfd2.Lb + z_d.cfd3.Lc$ wrt the Diagonal Triangle, where:

- $(x_c : y_c : z_c)$ are the barycentric coordinates of R wrt the Component Triangle,
- $(x_d : y_d : z_d)$ are the barycentric coordinates of R wrt the Diagonal Triangle,
- $cfc1, cfc2, cfc3$ are the Compliance Factors of the Component Triangle,
- $cfd1, cfd2, cfd3$ are the Compliance Factors of the Diagonal Triangle.

Explanation of Compliance Factors can be found at [See 26b page 40].

Since the Diagonal Triangle is the Reference Triangle, the Compliance Factors of the Diagonal Triangle are all equal 1.

The Compliance Factors of the Component Triangle are:

- $cfc1 = \text{Det} [L1, L2, L3] / \text{Det} [L1, L2, L3],$
- $cfc2 = \text{Det} [L1, L1, L3] / \text{Det} [L1, L2, L3],$
- $cfc3 = \text{Det} [L1, L2, L1] / \text{Det} [L1, L2, L3],$

where LI = the line at infinity (1 : 1 : 1) and “Det” is the abbreviation for “Determinant”. Calculation gives 2 presentations of the coordinates of R wrt the Component Triangle:

- $R_c = (-l(lmxc + lnxc - lmyc - mnyc - lnzc - mnzc) : -m(-lmxc - lnxc + lmyc + mnyc - lnzc - mnzc) : n(lmxc + lnxc + lmyc + mnyc - lnzc - mnzc))$
- $R_d = (xd : yd : zd),$

Since R_c and R_d present the same point we can now calculate the coordinates of R wrt the Diagonal Triangle:

- $(xc : yc : zc) = ((l+m)(l+n)(nyd + mzd) : (l+m)(m+n)(nxd + lzd) : (l+n)(m+n)(mxd + lzd))$

However we have to bear in mind that the variables in these coordinates are expressions in (a,b,c) and (l,m,n), which are variables wrt the Component Triangle. So also a conversion of these variables from CT to DT should be done.

Therefore the DT > CT-conversion of line R (x : y : z) consists of 3 consecutive steps:

1. Transform Line (x : y : z) -->
 $((l+m)(l+n)(ny + mz) : (l+m)(m+n)(nx + lz) : (l+n)(m+n)(mx + ly))$
2. Replace $a^2 \rightarrow (lm + ln - mn)^2 (b^2 m^2 + a^2 mn - b^2 mn - c^2 mn + c^2 n^2)$
 $b^2 \rightarrow (lm - ln + mn)^2 (a^2 l^2 - a^2 ln + b^2 ln - c^2 ln + c^2 n^2)$
 $c^2 \rightarrow (-lm + ln + mn)^2 (a^2 l^2 - a^2 lm - b^2 lm + c^2 lm + b^2 m^2)$
3. Replace $l \rightarrow (lm - ln - mn)(lm - ln + mn)$
 $m \rightarrow (lm - ln - mn)(lm + ln - mn)$
 $n \rightarrow (-lm - ln + mn)(lm - ln + mn)$

Conversion of a point

Converting the coordinates of a point in the Quadrilateral-environment is converting the intersection of 2 lines.

This actually only changes the Transform of (x : y : z) and leaves the replacements of variables unchanged:

1. Transform Point (x : y : z) -->
 $(-l(m+n)(lp - mq - nr) : -m(l+n)(-lp + mq - nr) : n(l+m)(lp + mq - nr))$
2. Replace $a^2 \rightarrow (lm + ln - mn)^2 (b^2 m^2 + a^2 mn - b^2 mn - c^2 mn + c^2 n^2)$
 $b^2 \rightarrow (lm - ln + mn)^2 (a^2 l^2 - a^2 ln + b^2 ln - c^2 ln + c^2 n^2)$
 $c^2 \rightarrow (-lm + ln + mn)^2 (a^2 l^2 - a^2 lm - b^2 lm + c^2 lm + b^2 m^2)$
3. Replace $l \rightarrow (lm - ln - mn)(lm - ln + mn)$
 $m \rightarrow (lm - ln - mn)(lm + ln - mn)$
 $n \rightarrow (-lm - ln + mn)(lm - ln + mn)$

6.2 QUADRILATERAL CENTERS

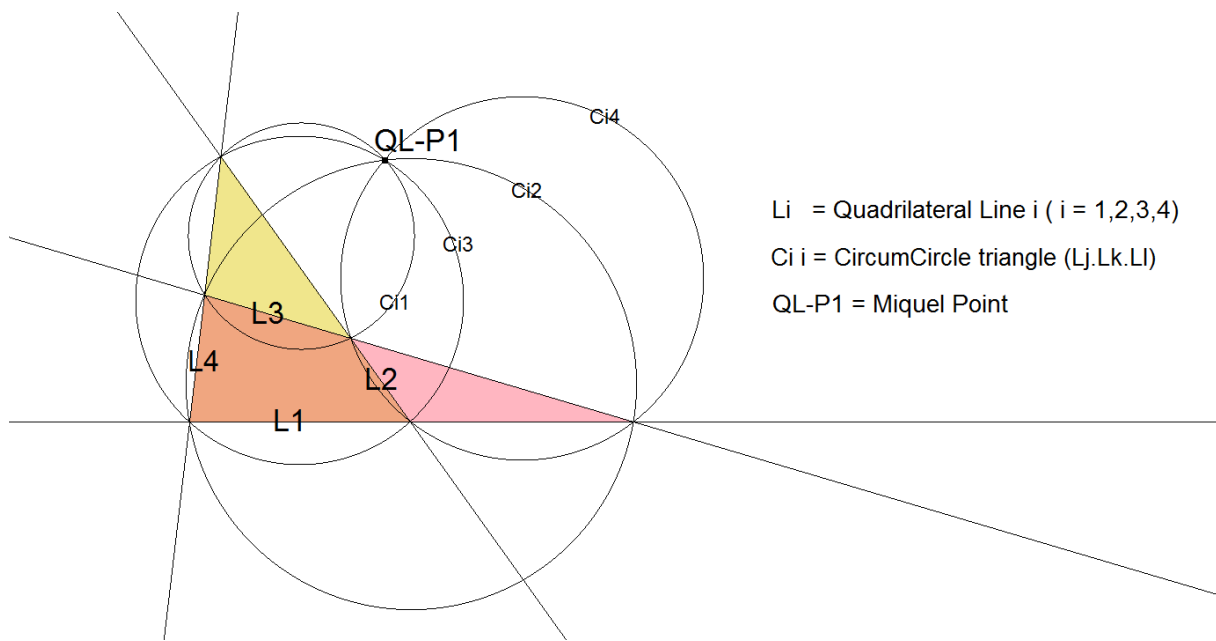
QL-P1: Miquel Point

The Miquel Point is the common point of the circumcircles of the 4 component triangles of the Reference Quadrilateral.

This point is also called the Steiner Point or Clifford Point.

This point is also called the focal point by J.W. Clawson. See [22] page 250.

He describes 134 circles of “more or less interest” through this point.



1st CT-coordinate:

$$a^2 m n / (m - n)$$

1st DT-coordinate:

$$b^2 / (l^2 - n^2) - c^2 / (l^2 - m^2)$$

Properties:

- The Miquel Point QL-P1 lies on these lines:
 - QL-P7.QL-P19 = QL-L5 (2 : -1 => QL-P1 = Reflection of QL-P7 in QL-P19)
 - QL-P8.QL-P24
 - QL-P10.QL-P16 (1 : 1 => QL-P1 = Midpoint of QL-P7 and QL-P19)
 - QL-P20.QL-P21 (-1 : 2 => QL-P1 = Reflection of QL-P21 in QL-P20)
- QL-P1 is the Railway Watcher (see QL-L/1) of:
 - QL-L2 and QL-L3,
 - QL-L1 and the asymptote of QL-Cu1.

- QL-P1 is the focus of the unique inscribed parabola of the Reference Quadrilateral (see [4] page 49).
- QL-P1 and the circumcenters of the 4 component triangles of the Reference Quadrilateral are concyclic on QL-Ci3 (Miquel Circle).
- QL-P1 lies on the 3 coaxal circles:
 - QL-Ci3 (Miquel Circle),
 - QL-Ci5 (Plücker Circle),
 - QL-Ci6 (Dimidium Circle).
- QL-P1 lies on QL-Ci2 (Nine-point Circle of the QL-Diagonal Triangle).
- QL-P1 lies on QL-Cu1 (QL-Quasi Isogonal Cubic).
- QL-P1 lies on QL-Qu1 (QL-Cardioid).
- QL-P1 is the isogonal conjugate of the infinite point of the Newton line with respect to each of the four component triangles (see [4] page 41).
- The symmetric lines of the Steiner Line QL-L2 in the 4 quadrilateral lines coincide at QL-P1.
- QL-P1 relates pairwise to all present line segments in the quadrilateral. It is the center of similarity of line segments $Li^{\wedge}Lk \cdot Li^{\wedge}Ll$ and $Lj^{\wedge}Lk \cdot Lj^{\wedge}Ll$, where $(i,j,k,l) \in (1,2,3,4)$. As a consequence triangles $QL-P1 \cdot Li^{\wedge}Lk \cdot Li^{\wedge}Ll$ and $QL-P1 \cdot Lj^{\wedge}Lk \cdot Lj^{\wedge}Ll$ are similar (see also [9]).
- QL-P1 is the perspector of the QL-Diagonal Triangle and the Triangle formed by the 3 QL-versions of QA-P3 (Gergonne-Steiner Point).
- The Clawson-Schmidt Conjugate (QL-Tf1) is “centered” around QL-P1.
 - Lines through QL-P1 are transformed in other lines through QL-P1.
 - Lines not through QL-P1 are transformed in a circle through QL-P1.
 - Circles with circumcenter QL-P1 are transformed in another circle with circumcenter QL-P1.

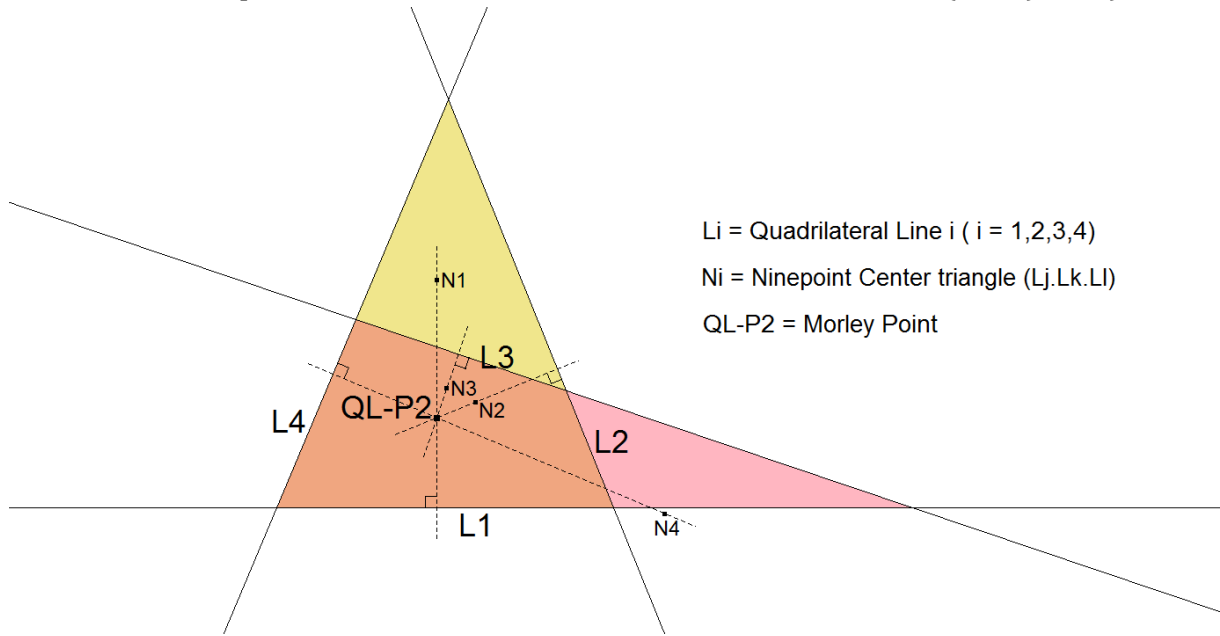
QL-P2: Morley Point

Let N_i be Nine-point Center of triangle $L_j L_k L_l$.

Let L_{pi} be the perpendicular line of N_i at L_i .

Now all perpendicular lines L_{pi} ($i=1,2,3,4$) concur in one point QL-P2.

According to [9] Alexander Bogomolny this point was found by Frank Morley (F. Morley, Orthocentric Properties of the Plane n-Line, Trans Amer Math Soc, 4 (1903) 1-12).



1st CT-coordinate:

$$+a^4 \ l \ (2 \ l - m - n) + b^2 \ c^2 \ (2 \ l^2 - 3 \ l \ m - 3 \ l \ n + 4 \ m \ n) \\ - b^4 \ (l - 2 \ m) \ (l - n) + a^2 \ c^2 \ (l^2 \ m + l^2 \ n - 5 \ l \ m \ n + 2 \ m^2 \ n + l \ n^2) / (m - n) \\ - c^4 \ (l - 2 \ n) \ (l - m) + a^2 \ b^2 \ (l^2 \ m + l^2 \ n - 5 \ l \ m \ n + 2 \ m \ n^2 + l \ m^2) / (n - m)$$

1st DT-coordinate:

$$S_b S_c - (S_b^2 \ m^2 \ (-l^2 + n^2)) / ((l^2 - m^2) \ (m^2 - n^2)) - (S_a \ S_b \ m^2) / (l^2 - m^2) \\ - (S_c^2 \ n^2 \ (l^2 - m^2)) / ((l^2 - n^2) \ (m^2 - n^2)) - (S_a \ S_c \ n^2) / (l^2 - n^2)$$

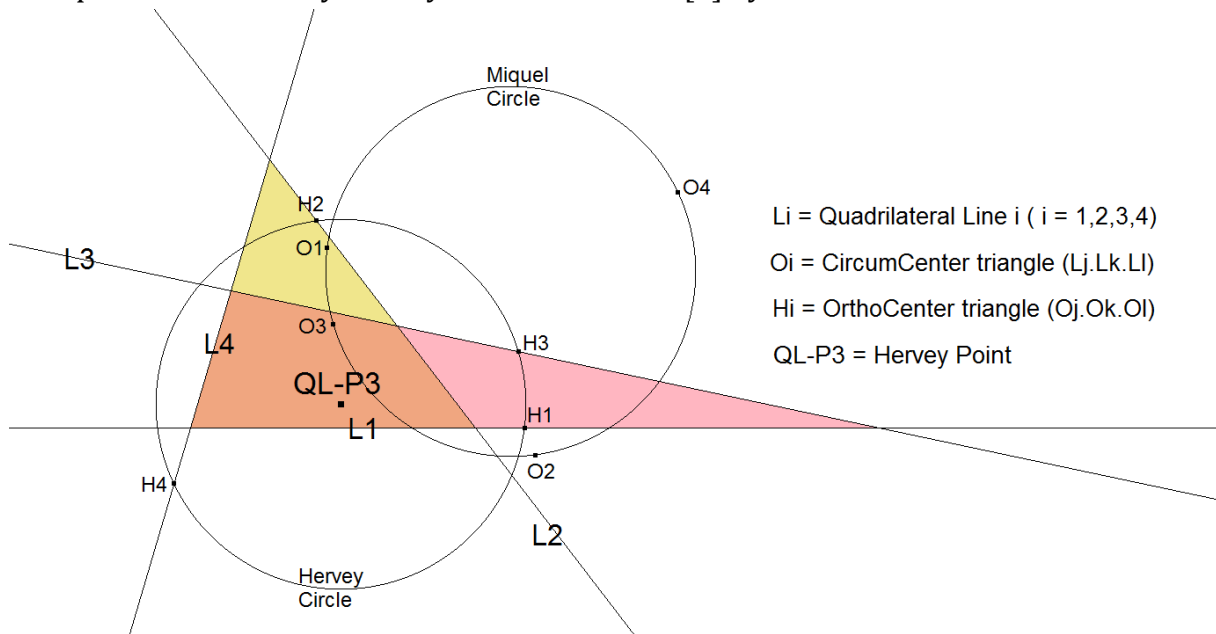
Properties:

- QL-P2 lies on these lines:
 - QL-P4.QL-P22 (2 : -1 => QL-P2 = Reflection of QL-P4 in QL-P22)
 - QL-P6.QL-P12 (3 : -2)
 - QL-P7.QL-P9
- QL-P2.QL-P3 = QL-L4 (Morley Line) // Newton Line QL-L1.
- QL-P2.QL-P10 = QL-L6 (Quasi Ortholine).
- $d(QL-P2, QL-P3) = 3 * d(QL-P5, QL-P12)$. (d = distance)
- $d(QL-P2, QL-L1) = d(OL-P3, QL-L1) = d(OL-P4, QL-L1)$, where QL-L1=Newton Line.
- The Clawson-Schmidt Conjugates of QL-P2 lies on QL-Ci3 (Miquel Circle).
- QL-P2 (Morley Point) lies on the circumcircle of the triangle formed by the 1st Quasi Nine-point Centers (QG-P7) of the 3 QL-Quadrilaterals.

QL-P3: Hervey point

The circumcenters of the 4 component triangles of the Reference Quadrilateral are concyclic (on the Miquel Circle). These circumcenters form a Quadrangle with 4 component triangles whose Orthocenters are also concyclic (on the Hervey Circle) and with circumcenter QL-P3, the Hervey point.

This point was found by Hervey and described in [6] by Alain Levelut.



1st CT-coordinate:

$$a^2 (l - m) (l - n) (b^2 n - c^2 m) + (b^2 - c^2) (m - n) (-b^2 (l - m) n + c^2 m (l - n))$$

1st DT-coordinate:

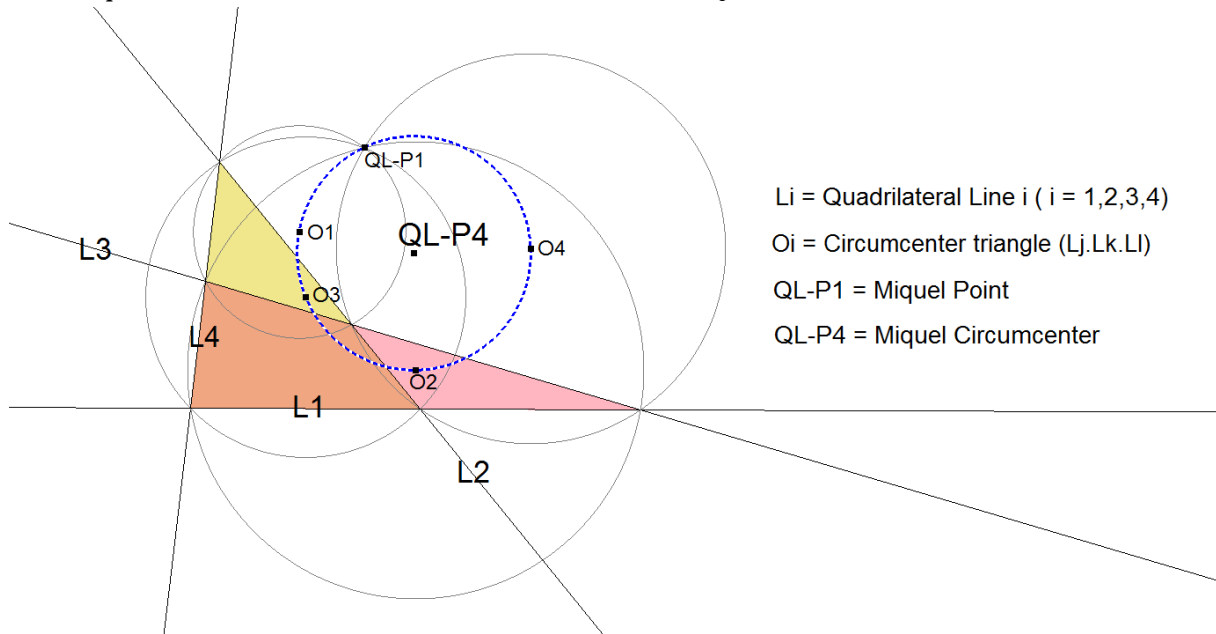
$$a^4 - b^4 (3 m^2 + n^2) / (l^2 - n^2) - c^4 (m^2 + 3 n^2) / (l^2 - m^2) - 4 b^2 c^2 m^2 / (-l^2 + m^2) - 4 b^2 c^2 n^2 / (-l^2 + n^2) + 2 a^2 (b^2 - c^2) (m^2 + n^2) / (m^2 - n^2)$$

Properties:

- QL-P3 lies on this line:
 - QL-P4.QL-P5 (2 : -1 => QL-P3 = Reflection of QL-P4 in QL-P5)
- Distances QL-P4.QL-P6 : QL-P6.QL-P5 : QL-P5.QL-P3 = 1 : 1 : 2.
- QL-P2.QL-P3 = QL-L4 (Morley Line) // Newton Line QL-L1.
- QL-P3 is also the point of concurrence of the four perpendicular bisectors of the segments Oi.Hi of the Euler Lines of the QL-Component Triangles Ti ([6] page 5).
- QL-P3 is the center of the QL-Ci4 Hervey Circle.
- QL-P3 is the Gergonne-Steiner Point (QA-P3) as well as the Isogonal Conjugate (QA-P4) as well as the Midray Homothetic Center (QA-P8) as well as the QA-DT-Orthocenter (QA-P12) from the Orthocenter Quadrangle in the Circumcenter Quadrangle H1.H2.H3.H4. These 4 QA-points concur. H1.H2.H3.H4 is concyclic.
- $d(QL-P3, QL-P2) = 3 * d(QL-P5, QL-P12)$
- $d(QL-P3, QL-L1) = d(OL-P2, QL-L1) = d(OL-P4, QL-L1)$, where d=distance and QL-L1=Newton Line.

QL-P4: Miquel Circumcenter

The circumcenters of the 4 component triangles of the Reference Quadrilateral are concyclic on the Miquel Circle. It is special that QL-P1, the Miquel Point, also resides on the Miquel Circle. The circumcenter of this circle is QL-P4.



1st CT-coordinate:

$$\begin{aligned} & a^2 (a^2 (m - n) (l - m) (n - l) \\ & + b^2 (l - m) (n^2 + l m - 2 m n) \\ & + c^2 (n - l) (m^2 + l n - 2 m n)) \end{aligned}$$

1st DT-coordinate:

$$\begin{aligned} & Sa^2 (m^2 - n^2)^3 - Sb^2 (l^2 - n^2)^2 (3 m^2 + n^2) + Sc^2 (l^2 - m^2)^2 (m^2 + 3 n^2) \\ & + 2(m^2 - n^2) (Sa Sc (l^2 - m^2)^2 + Sa Sb (l^2 - n^2)^2 - S^2 (l^2 (l^2 + m^2 + n^2) - 3 m^2 n^2)) \end{aligned}$$

Properties:

- QL-P4 lies on these lines:
 - QL-P2.QL-P22 (2 : -1 => QL-P4 = Reflection QL-P2 in QL-P22)
 - QL-P3.QL-P5 (2 : -1 => QL-P4 = Reflection QL-P3 in QL-P5)
- QL-P4 is the Reflection of QL-P5 in QL-P6.
- QL-P4 is the Railway Watcher (see QL-L/1) of QL-L1 (Newton Line) and QL-L4 (Morley Line).
- QL-P4 is the Gergonne-Steiner Point (QA-P3) as well as the Isogonal Conjugate (QA-P4) as well as the Midray Homothetic Center (QA-P8) as well as the QA-DT-Orthocenter (QA-P12) from the Circumcenter Quadrangle (see QL-P3). These 4 QA-points concur because the Circumcenter Quadrangle is concyclic.
- The Centroid of the 8 centers of circles as described in rule (9) from Steiner (seen as a system of 8 random points) is QL-P4 (Miquel Circumcenter). See QL-8P1.

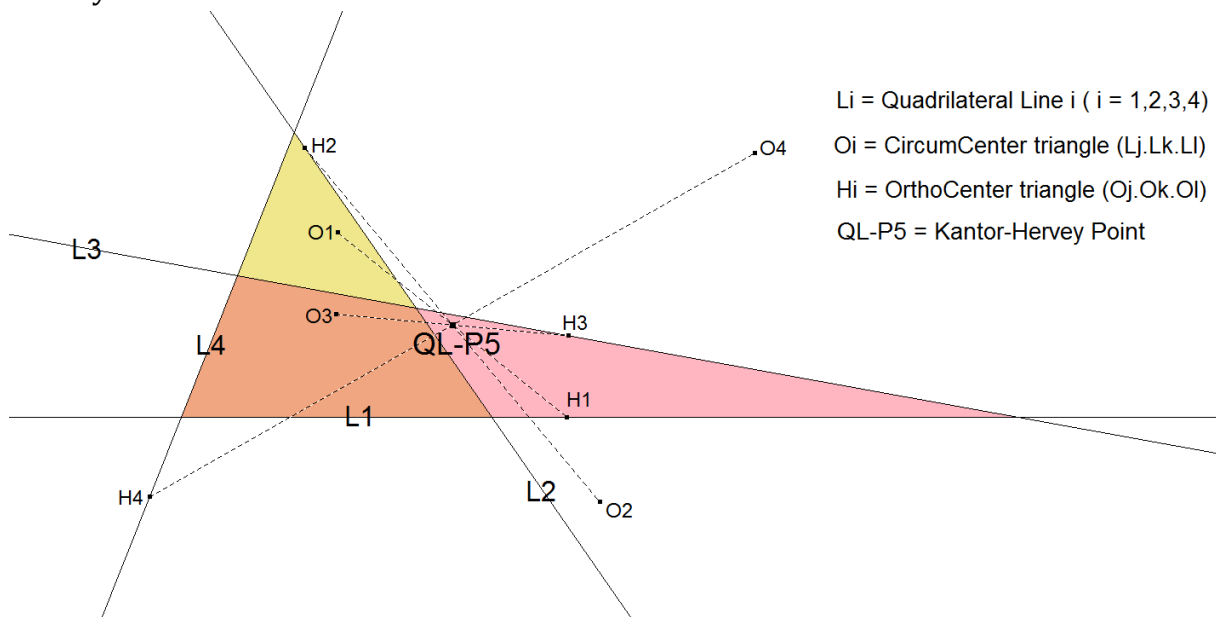
QL-P5: Kantor-Hervey Point

The circumcenters O1, O2, O3, O4 of the 4 component triangles of the Reference Quadrilateral are concyclic (on the Miquel Circle). These circumcenters form a Quadrangle with 4 component triangles whose Orthocenters H1, H2, H3, H4 again are concyclic (on the Hervey Circle with circumcenter QL-P3 the Hervey point).

Now QL-P5 is the common midpoint of Oi.Hi (i=1,2,3,4).

Since Quadrangles O1.O2.O3.O4 and H1.H2.H3.H4 are homothetic it also can be said that QL-P5 is the center of homothety of Quadrangles O1.O2.O3.O4 and H1.H2.H3.H4.

There is a description of this point in [2b] Jean-Louis Ayme "Le point de Kantor-Hervey".



1st CT-coordinate:

$$l (m - n) (a^2 S_A l + b^2 S_B m + c^2 S_C n + 8 \Delta^2 (m n - l m - l n) / l)$$

1st DT-coordinate:

$$-(m^2-n^2) (+Sa^2 (m^2-n^2)^2 + Sb^2 (l^2-n^2)^2 + Sc^2 (l^2-m^2)^2 \\ + 2 (Sb Sc l^4 + Sa Sc m^4 + Sa Sb n^4) + 2 S^2 (l^2 (m^2+n^2) - 3 m^2 n^2))$$

Properties:

- QL-P5 lies on these lines:
 - QL-P7.QL-P12 = QL-L1
 - QL-P3.QL-P4 (1 : 1 => QL-P5 = Midpoint QL-P3 and QL-P4)
- QL-P5 is the Reflection of QL-P4 in QL-P6.
- QL-P5 is the Reflection of QL-P20 in QL-P22.
- The distance ratios between points QL-P20, QL-P22, QL-P12, QL-P5 are 3 : 1 : 2.
- QL-P5 is the shared Euler-Poncelet Point (QA-P2) from the Circumcenter Quadrangle and from the Orthocenter Quadrangle in the Circumcenter Quadrangle (resp. O1.O2.O3.O4 and H1.H2.H3.H4 in picture QL-P3).
- $d(QL-P5, QL-P12) = d(QL-P2, QL-P3) / 3$ (d = distance)

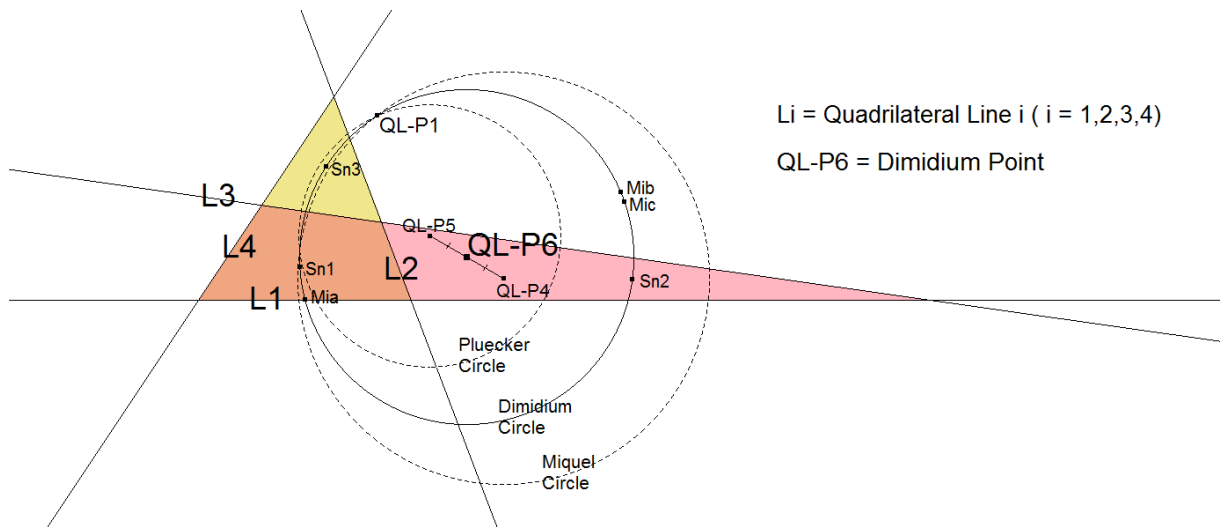
QL-P6: Dimidium Point

The Dimidium Point is the Circumcenter of the Dimidium Circle.

I call this point the Dimidium Point because "Dimidium" is the Latin word for "half".

This is because:

- The Dimidium Point is the **midpoint** of QL-P4 (Miquel Circumcenter) and QL-P5 (Kantor-Hervy Point).
- The intersection points of the Nine-point Conics of the 3 component quadrilaterals of the Reference Quadrilateral have 3 common points: Sn1, Sn2, Sn3. These points lie on the Dimidium Circle (which moreover passes through QL-P1 the Miquel Point). As is known the Conic is a conic through all **midpoints** of all line segments in a quadrangle or quadrigon.
- The Gergonne-Steiner Points Mia, Mib, Mic of the 3 component quadrilaterals of the Reference Quadrilateral also lie on the Dimidium Circle. As is known the Gergonne-Steiner Point (QA-P3) is constructed from the **midpoints** of a ray of line segments.
- The Dimidium Circle (QL-Ci6) lies exactly between (**midway**) the Plücker Circle (QL-Ci5) and the Miquel Circle (QL-Ci3).



1st CT-coordinate:

$$\begin{aligned}
 &+3 a^4 (l-m)(m-n)(n-l) + b^2 c^2 (m-n)(-l m - l n + 2 m n) \\
 &+ b^4 (l-m)(m-n)n + c^2 a^2 (n-l)(-l m + 3 l n - 6 m n + 4 m^2) \\
 &- c^4 m(m-n)(n-l) + a^2 b^2 (l-m)(-l n + 3 l m - 6 m n + 4 n^2)
 \end{aligned}$$

1st DT-coordinate:

$$\begin{aligned}
 &Sb^2 (l^2 - n^2)^2 m^2 - Sc^2 (l^2 - m^2)^2 n^2 \\
 &+ (m^2 - n^2) (Sb Sc l^4 + Sa Sb l^2 n^2 + Sa Sc l^2 m^2 + S^2 (l^2 m^2 + l^2 n^2 - 3 m^2 n^2))
 \end{aligned}$$

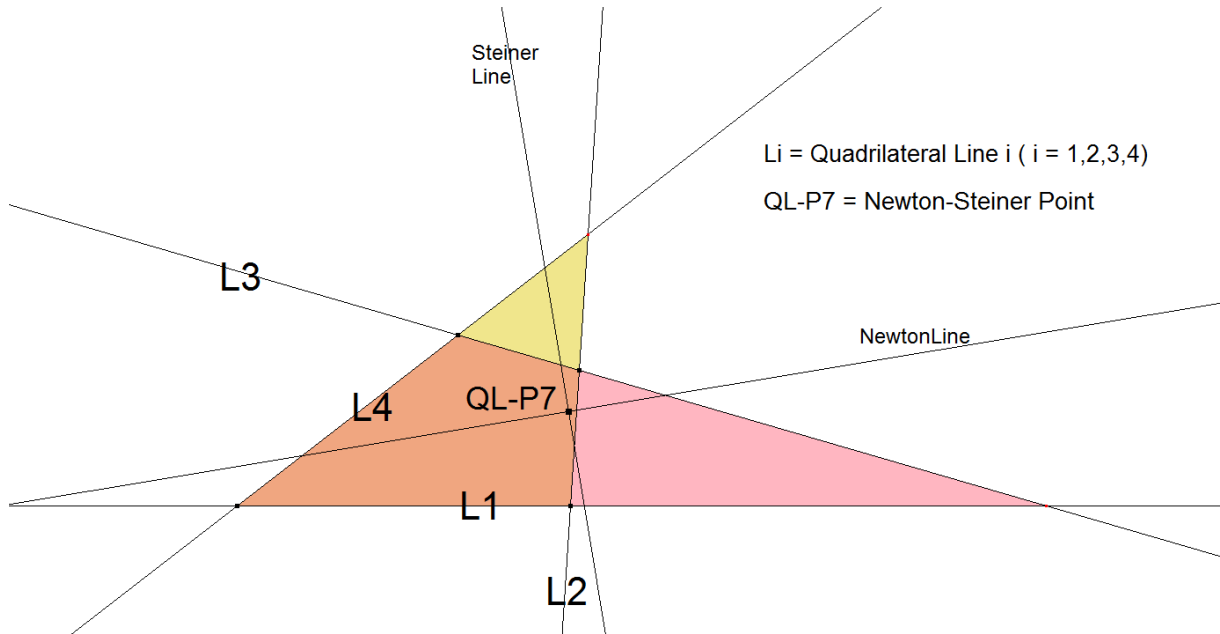
Properties:

- QL-P6 lies on these lines:
 - QL-P2.QL-P12 (3 : 1)
 - QL-P3.QL-P4 (3 : 1)

- QL-P6 = Midpoint of QL-P4 and QL-P5.
- QL-P6 = Circumcenter of QL-Ci6 Dimidium Circle.
- QL-P6 = the Quadrangle Centroid (QA-P1) from the Circumcenter Quadrangle (O1.O2.O3.O4 in picture QL-P3).
- Distances $QL-P6.QL-P12 : QL-P12.QL-P2 = 1 : 2$.

QL-P7: Newton-Steiner Point

The Newton-Steiner Point is the intersection point of the two most prominent Quadrilateral Lines: QL-L1 The Newton Line and QL-L2 The Steiner Line.



1st CT-coordinate:

$$-a^2 m^2 n^2 + S_B l m^2 (3n - l) + S_C l n^2 (3m - l) + a^2 l m n (l - m - n)$$

1st DT-coordinate:

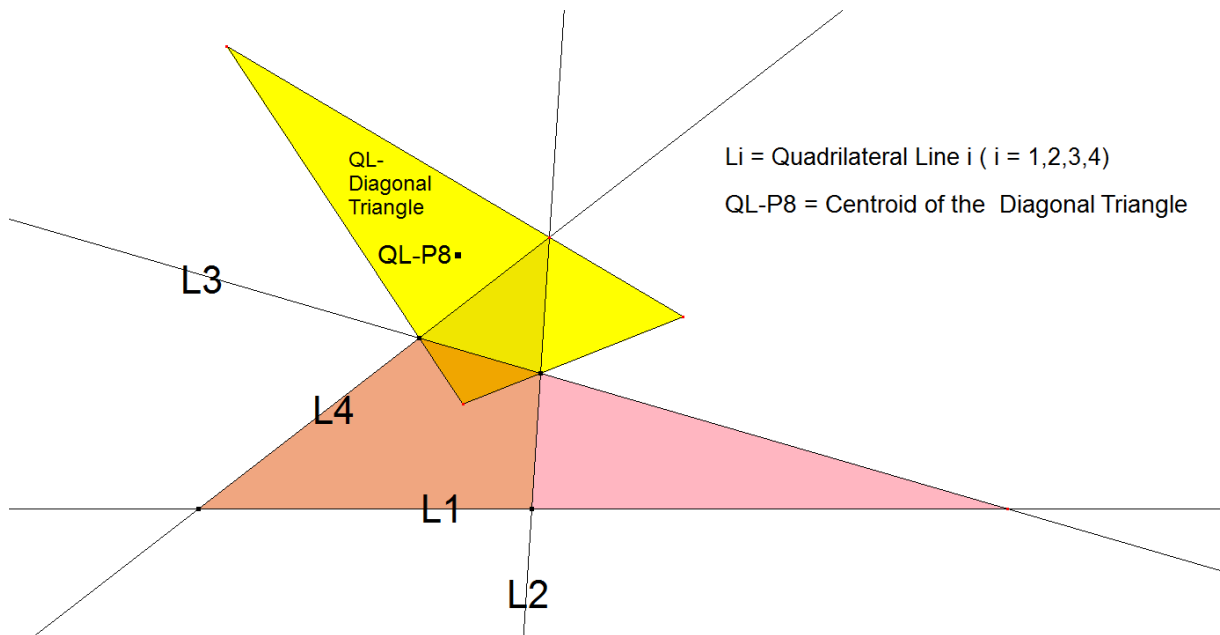
$$a^2 l^2 (m^2 + n^2) - 2 a^2 m^2 n^2 - (m^2 - n^2) (b^2 m^2 - c^2 n^2)$$

Properties:

- QL-P7 lies on these lines:
 - QL-P1.QL-P19=QL-L5 (2 : -1 => QL-P7 = Reflection QL-P1 in QL-P19)
 - QL-P2.QL-P9 =QL-L2
 - QL-P5.QL-P12=QL-L1
- QL-P7.QL-P1 // QL-P2.QL-P10
- QL-P7 is the Reflection of QL-P1 in QL-P19.
- The Clawson-Schmidt Conjugates (QL-Tf1) of QL-P7 lies on QL-Ci3 (Miquel Circle).

QL-P8: Centroid QL-Diagonal Triangle

QL-P8 is the Centroid of the Diagonal Triangle of the Reference Quadrilateral.



1st CT-coordinate:

$$(l m + l n - m n)^2 m n - 4 (l - m) (l - n) m^2 n^2 \quad (\text{note that this formula is independent of } a, b, c)$$

1st DT-coordinate:

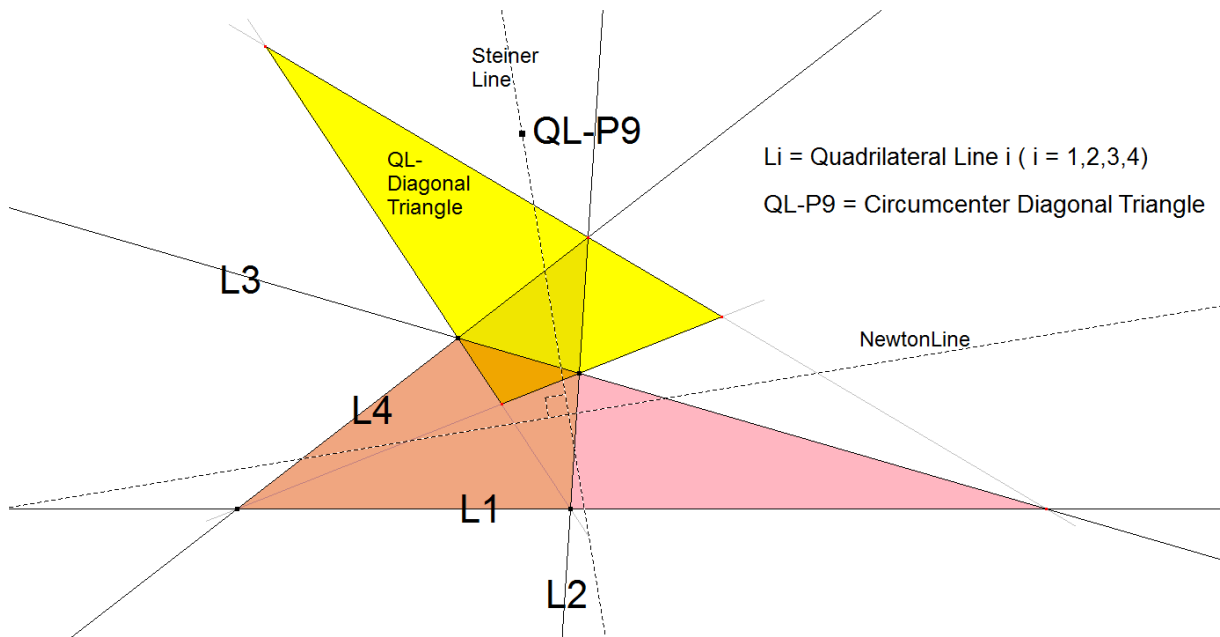
$$1$$

Properties:

- QL-P8 lies on these lines:
 - QL-P1.QL-P24
 - QL-P9.QL-P10 = QL-L7 (1 : 2)
 - QL-P12.QL-P14 = QL-L8 (-3 : 4)
 - QL-P17.QL-P25 (2 : 1)
- The ratio of distances between QL-P9, QL-P8, QL-P11, QL-P10 in this order is 2 : 1 : 3.
- The ratio of distances between QL-P8, QL-P15, QL-P12, QL-P14, QL-P18 in this order is 2 : 1 : 1 : 2.
- $d(\text{QL-P8}, \text{QL-L9}) = 2 * d(\text{QL-P12}, \text{QL-L9}) = 4/3 * d(\text{QL-P19}, \text{QL-L9})$

QL-P9: Circumcenter QL-Diagonal Triangle

QL-P9 is the Circumcenter of the Diagonal Triangle of the Reference Quadrilateral.



1st CT-coordinate:

$$4 m n (-4 \Delta^2 m^2 n^2 + a^2 l^3 (S_C m + S_B n) - l^2 (S_C m + S_B n)^2 + l m n (S_B b^2 m + S_C c^2 n))$$

where:

$$S_A = (-a^2 + b^2 + c^2)/2 \quad S_B = (+a^2 - b^2 + c^2)/2 \quad S_C = (+a^2 + b^2 - c^2)/2$$

$$\Delta = \text{Area} = 1/4 \sqrt{[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]}$$

1st DT-coordinate:

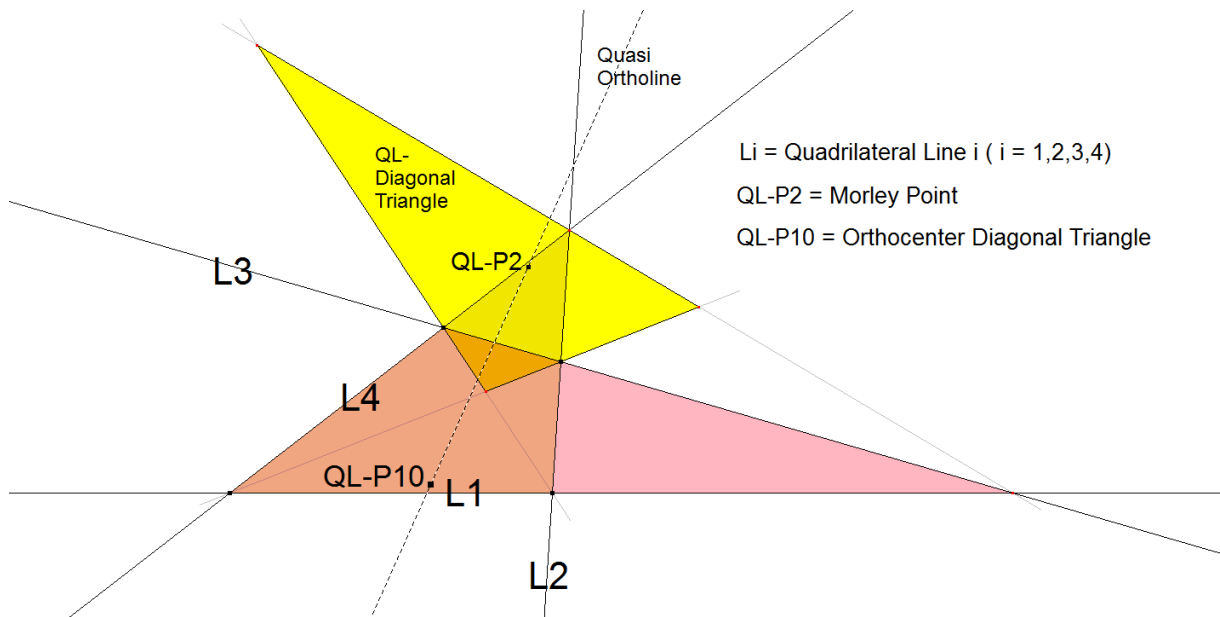
$$a^2 S_a$$

Properties:

- QL-P9 lies on these lines:
 - QL-P2.QL-P7 = QL-L2
 - QL-P8.QL-P10 = QL-L7 (-1 : 3 => QA-P9 = Complement of QA-P10)
- QL-P9 lies together with QL-P8, QL-P10 and QL-P11 on QL-L7, the QL-Euler Line of the Diagonal Triangle. The ratio of distances between QL-P9, QL-P8, QL-P11, QL-P10 in this order is 2 : 1 : 3.
- QL-P9.QL-P16 // QL-P1.QL-P11.
- QL-P9 lies on the circumcircle of the triangle formed by the 3 QL-versions of QG-P9.
- The Clawson-Schmidt Conjugate (QL-Tf1) of QL-P9 lies on QL-Ci3 (Miquel Circle).
- The directrices of QL-Co1 and QL-Co3 meet in QL-P9 (Circumcenter QL-Diagonal Triangle).

QL-P10: Orthocenter QL-Diagonal Triangle

QL-P10 is the Orthocenter of the Diagonal Triangle of the Reference Quadrilateral.



1st CT-coordinate:

$$m n (2 \Delta^2 (l m + l n - m n) (l m + l n + m n) - a^2 l^2 (2 m n S_A + l n S_B + l m S_C) + l m n (n S_A S_B + m S_A S_C) + l^2 (n^2 S_B^2 + m^2 S_C^2))$$

1st DT-coordinate:

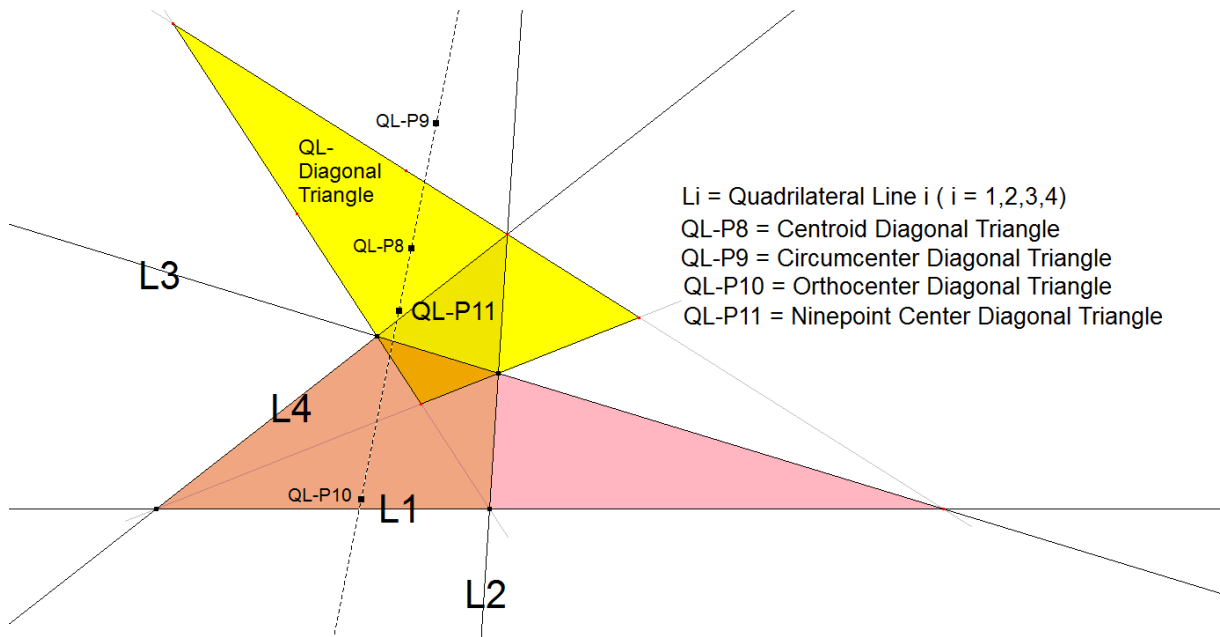
$$1 / S_a$$

Properties:

- QL-P10 lies on these lines:
 - QL-P1.QL-P16 $(-1 : 2 \Rightarrow \text{QL-P10} = \text{Reflection of QL-P16 in QL-P1})$
 - QL-P2.QL-P10 = QL-L6
 - QL-P8.QL-P9 $(-2 : 3)$
- QL-P10 lies together with QL-P8, QL-P9 and QL-P11 on QL-L7, the QL-Euler Line of the Diagonal Triangle. The ratio of distances between QL-P9, QL-P8, QL-P11, QL-P10 in this order is 2 : 1 : 3.
- QL-P10.QL-P2 // QL-P7.QL-P1.
- QL-P10 is a Railway Watcher (see QL-L/1) of lines QL-L9.QL-P16 and line QL-P1.QL-P11.

QL-P11: Nine-point Center QL-Diagonal Triangle

QL-P11 is the Nine-point Center of the Diagonal Triangle of the Reference Quadrilateral.



1st CT-coordinate:

$$a^2 l^2 m n (2 m n S_A + n l S_B + l m S_C - b^2 m^2 - c^2 n^2) - m^2 n^2 (8 \Delta^2 (l m + l n - m n) - b^2 l m S_B - c^2 l n S_C)$$

1st DT-coordinate:

$$S^2 + S_b S_c$$

Properties:

- QL-P11 lies together with QL-P8, QL-P9 and QL-P10 on QL-L7 (QL-Euler Line). The ratio of distances between QL-P9, QL-P8, QL-P11, QL-P10 in this order is 2 : 1 : 3.
- QL-P1.QL-P11 // QL-P16.QL-P9.
- QL-P11 lies on perpendicular bisector F1.F2 (F1, F2 are Foci QL-Co1, QL-Co3).

QL-P12: QL-Centroid or Lateral Centroid

Let G_i be the Triangle Centroid (X2) of Triangle $L_j.L_k.L_l$.

Let TC_i be the Tripolar Centroid ^{*)} of the 3 intersection points of L_j, L_k, L_l with L_i .

Now all lines $G_i.TC_i$ ($i=1,2,3,4$) concur in one point QL-P12, the Lateral Centroid.

It is called the Lateral Centroid because it is constructed of 4 Centroids occurring in the Quadrilateral environment. Just like in the QA-environment, the Centroid in the QL-environment is the Midpoint of 1st and 2nd Quasi Centroids.

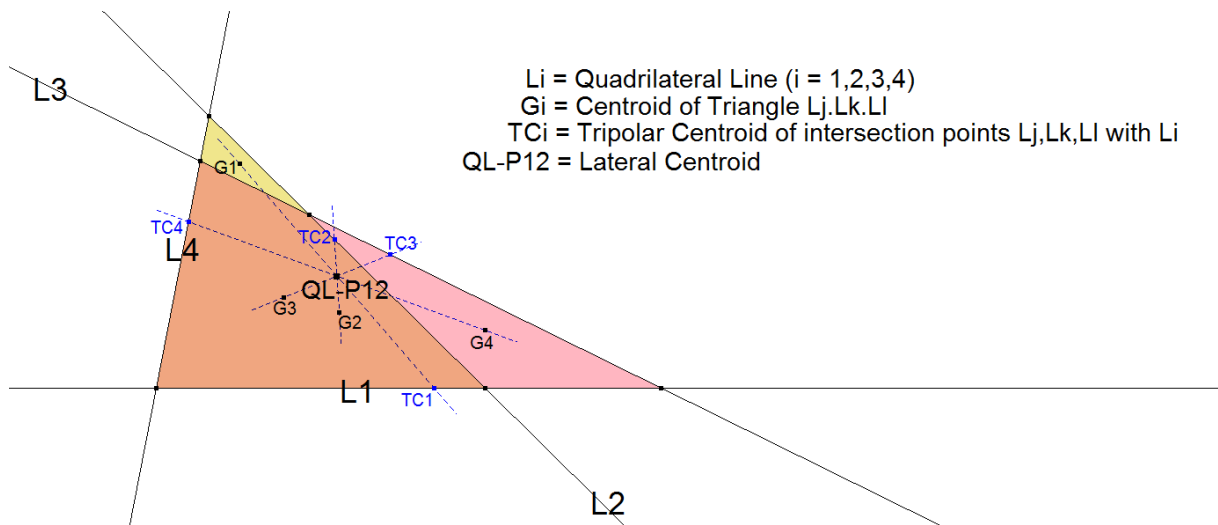
3 other ways of construction:

- QL-P12 is the Tripolar Centroid ^{*)} of the QA-Centroids in the 3 QL-Quadrilaterals.
- Let G_i be the Centroid of component triangle $L_j.L_k.L_l$ and let G_{jkl} be the Centroid of Centroids Triangle $G_j.G_k.G_l$, where $(i,j,k,l) \in (1,2,3,4)$. Lines $G_i.G_{jkl}$ concur in QL-P12.
- QL-P12 is the homothetic Center of the Reference Quadrilateral $L_1.L_2.L_3.L_4$ and the homothetic Quadrilateral formed by the lines parallel to the lines L_1, L_2, L_3, L_4 through the Centroids of the corresponding component triangles.

^{*)} The Tripolar Centroid is the Centroid of a "flat" triangle formed by 3 collinear points.

Construction-method: Suppose P, Q, R are collinear points not on the line at infinity. Let $M = \text{Midpoint}(Q, R)$.

The segment PM has two trisectors. The trisector closer to M is the Tripolar Centroid.



1st CT-coordinate:

$$(m - n) (l (l + m + n) - 3 (l m + l n - m n)) \quad (\text{note that this formula is independent of } a,b,c)$$

1st DT-coordinate:

$$(m^2 - n^2) (m^2 (l^2 - n^2) + n^2 (l^2 - m^2))$$

Properties:

- QL-P12 lies on these lines:
 - QL-P2.QL-P6 (2 : 1)
 - QL-P5.QL-P7 = QL-L1
 - QL-P8. QL-P14 = QL-L8 (3 : 1)

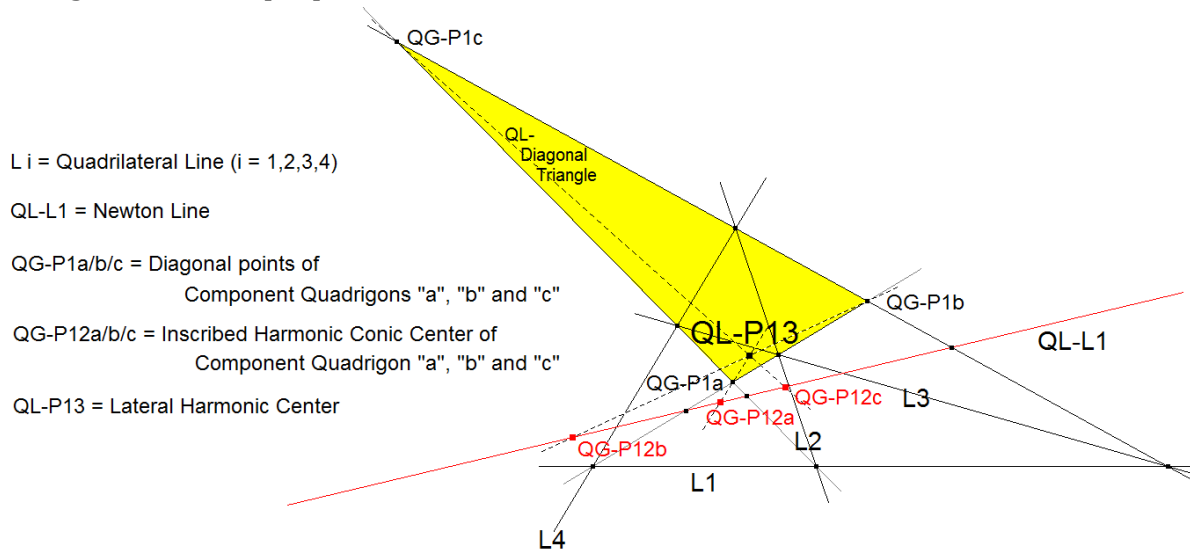
- QL-L12 is the Midpoint of QL-P14 (1st QL-Quasi Centroid) and QL-P15 (2nd QL-Quasi Centroid).
- QL-L12 is the Midpoint of QL-P8 and QL-P18.
- The distance ratios between points QL-P20, QL-P22, QL-P12, QL-P5 are 3 : 1 : 2.
- $d(\text{QL-P6}, \text{QL-P12}) = d(\text{QL-P2}, \text{QL-P12}) / 2$ (d = distance)
- $d(\text{QL-P5}, \text{QL-P12}) = d(\text{QL-P2}, \text{QL-P3}) / 3$
- QL-P12 is also the centroid of the 6 intersection points of the Reference Quadrilateral.
- QL-P12 is the point where the sum of the squares of the distances to the 6 intersection points of the Reference Quadrilateral is minimal.

QL-P13: QL-Harmonic Center or Lateral Harmonic Center

The QL-Harmonic Center or Lateral Harmonic Center is the Perspector of:

- the QL-Diagonal Triangle formed by the 3 QL-versions of QG-P1 (Diagonal Crosspoint), and
- the triangle formed by the 3 QL-versions of QG-P12 (Inscribed Harmonic Conic Center). This happens to be a “flat triangle” because these points are collinear on QL-L1 (Newton Line).

QL-P13 is named harmonic because its construction is based on projective principles leading to harmonic properties.



Another construction:

1. Let QG-P2a, QG-P2b, QG-P2c be the Midpoints of the diagonals of a quadrilateral. These points are collinear at the Newton Line (QL-L1). See QG-P2 and QL-L1.
2. The triangle bounded by the lines QG-P2a.QG-P1a, QG-P2b.QG-P1b, QG-P2c.QG-P1c is perspective with the QL-Diagonal Triangle (QG-P1a.QG-P1b.QG-P1c). Their perspector is QL-P13.

1st CT-coordinate:

$$m \ n \ (3 \ m \ n - l \ m - l \ n) \quad (\text{note that this formula is independent of } a,b,c)$$

1st DT-coordinate:

$$1 / l^2$$

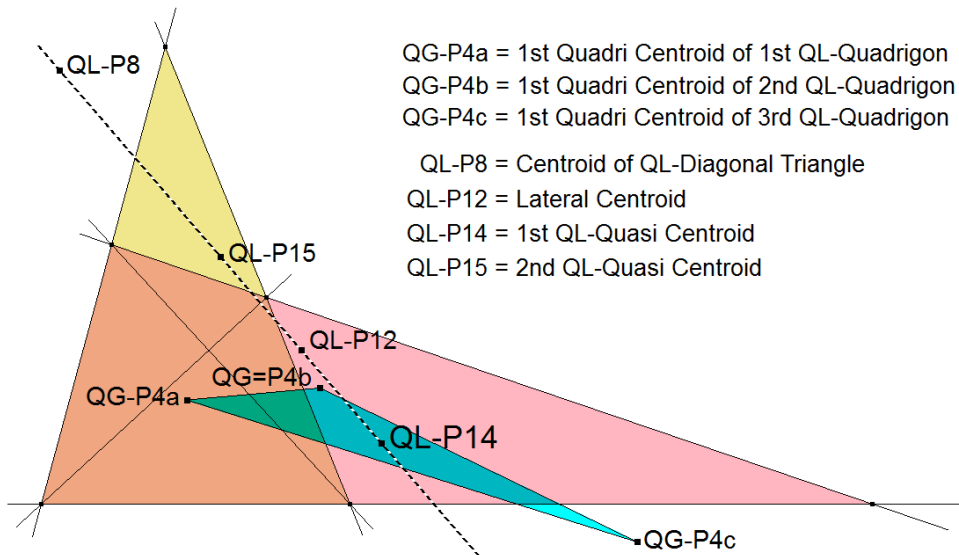
Properties:

- QL-P13 lies on QL-P17.QL-P24.
- QL-P13 is collinear with QA-P16, QG-P1, QG-P12 and QG-P13 (see QG-L2).
- The 3 variants of QG-L2 in a Quadrilateral concur in QL-P13.
- QL-P13 is also the Perspector of the triangles formed by the 3 QL-versions resp. of QG-P1 and QG-P12.
- QL-P13 is also the Perspector of the triangle bounded by the 3 QL-versions of the connecting lines of QG-P1 and QG-P2 and the triangle formed by the 3 QL-versions of QG-P1 (which is the QL-Diagonal Triangle).

QL-P14: 1st QL-Quasi Centroid

QL-P14 is the Centroid of the triangle formed by the three 1st Quasi Centroids (QG-P4) in a Reference Quadrilateral.

The prefix QL- has been added in the name of QL-P14 because there is also a 1st Quasi Centroid in a Quadrangle.



1st CT-coordinate:

$$(m - n) (2 l^5 m^3 - 4 l^4 m^4 - 2 l^5 m^2 n - l^4 m^3 n + 9 l^3 m^4 n - 2 l^5 m n^2 + 10 l^4 m^2 n^2 - 9 l^3 m^3 n^2 - 3 l^2 m^4 n^2 + 2 l^5 n^3 - l^4 m n^3 - 9 l^3 m^2 n^3 + 13 l^2 m^3 n^3 - 5 l m^4 n^3 - 4 l^4 n^4 + 9 l^3 m n^4 - 3 l^2 m^2 n^4 - 5 l m^3 n^4 + 3 m^4 n^4)$$

1st DT-coordinate:

$$(m^2 - n^2) (l^4 + 3 l^2 m^2 + 3 l^2 n^2 - 7 m^2 n^2)$$

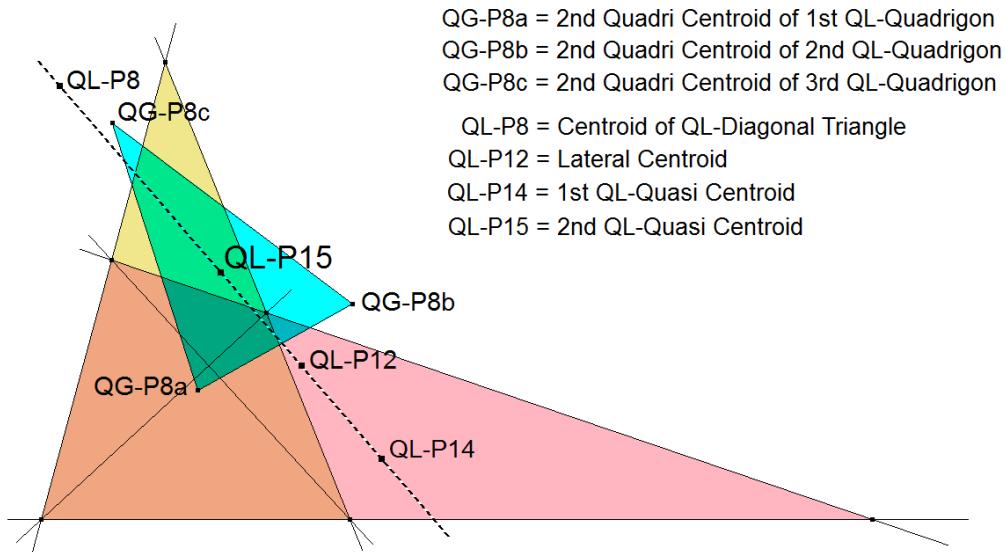
Properties:

- QL-P14 lies on this line:
 – QL-P8.QL-P12 = QL-L8 (4 : -1)
- QL-P14 is the Reflection of QL-P15 (2nd QL-Quasi Centroid) in QL-P12 (Lateral Centroid).

QL-P15: 2nd QL-Quasi Centroid

QL-P15 is the Centroid of the triangle formed by the three 2nd Quasi Centroids (QG-P8) in a Reference Quadrilateral.

The prefix QL- has been added in the name of QL-P15 because there is also a 2nd Quasi Centroid in a Quadrangle.



1st CT-coordinate:

$$(m - n) (l^5 m^3 - 2 l^4 m^4 - l^5 m^2 n - 2 l^4 m^3 n + 6 l^3 m^4 n - l^5 m n^2 + 8 l^4 m^2 n^2 - 9 l^3 m^3 n^2 + l^5 n^3 - 2 l^4 m n^3 - 9 l^3 m^2 n^3 + 20 l^2 m^3 n^3 - 10 l m^4 n^3 - 2 l^4 n^4 + 6 l^3 m n^4 - 10 l m^3 n^4 + 6 m^4 n^4)$$

1st DT-coordinate:

$$(m^2 - n^2) (l^4 - 3 l^2 m^2 + 3 l^2 n^2 + 5 m^2 n^2)$$

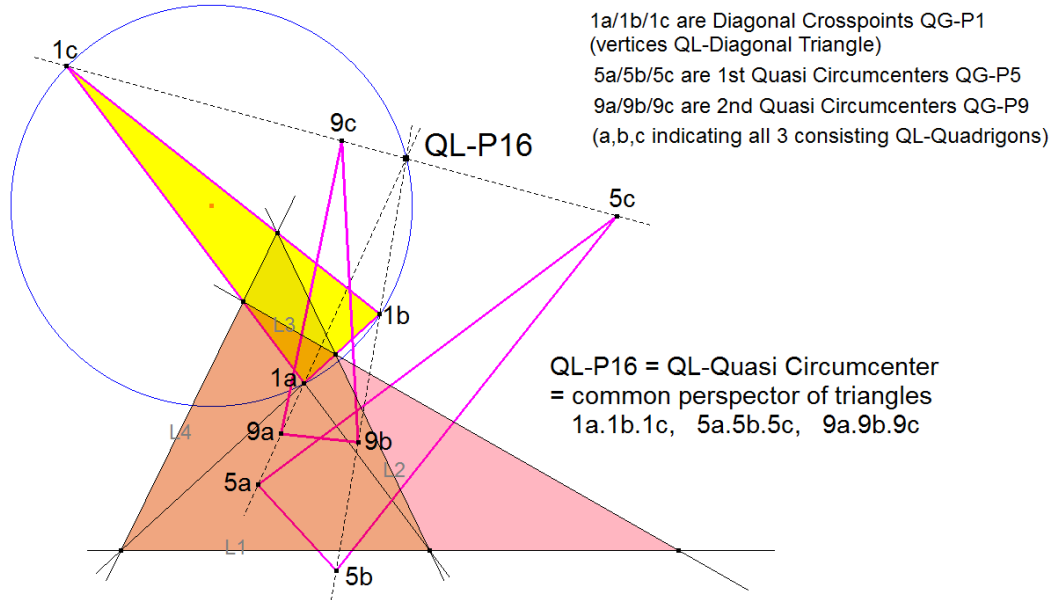
Properties:

- QL-P15 lies on this line:
 – QL-P8.QL-P12=QL-L8 (2 : 1)
- QL-P15 is the Reflection of QL-P14 (1st QL-Quasi Centroid) in QL-P12 (Lateral Centroid).

QL-P16: QL-Quasi Circumcenter

QL-P16 is the perspector of the QL-Diagonal Triangle and the Triangle formed by the points QG-P5 (QG-1st Quasi Circumcenter) of the 3 QL-Quadrilaterals.

QL-P16 is also the perspector of the QL-Diagonal Triangle and the Triangle formed by the points QG-P9 (QG-2nd Quasi Circumcenter) of the 3 QL-Quadrilaterals.



1st CT-coordinate:

$$\begin{aligned}
 & m n (32 a^2 \Delta^2 (l-m)(l-n)(l m-l n+m n)(-l m+l n+m n)(-m n+l m+l n) \\
 & - (a^2 m n (l-m)(l-n) + b^2 l n (m-l)(m-n) + c^2 l m (n-l)(n-m))) \\
 & * (4 a^4 l^2 (l m+l n-m n) - a^4 (l m+l n-m n)^2 - b^4 (l m-l n+m n)^2 - c^4 (-l m+l n+m n)^2 \\
 & + 2 b^2 c^2 (l^2 m^2 - 2 l^2 m n + l^2 n^2 + m^2 n^2) \\
 & + 2 a^2 b^2 (2 l^3 m - 3 l^2 m^2 - 2 l^3 n + 2 l^2 m n + l^2 n^2 - 2 l m n^2 + m^2 n^2) \\
 & + 2 a^2 c^2 (-2 l^3 m + l^2 m^2 + 2 l^3 n + 2 l^2 m n - 2 l m n^2 - 3 l^2 n^2 + m^2 n^2)))
 \end{aligned}$$

1st DT-coordinate:

$$a^2 (S_c l^2 - b^2 m^2 + S_a n^2) (S_b l^2 + S_a m^2 - c^2 n^2)$$

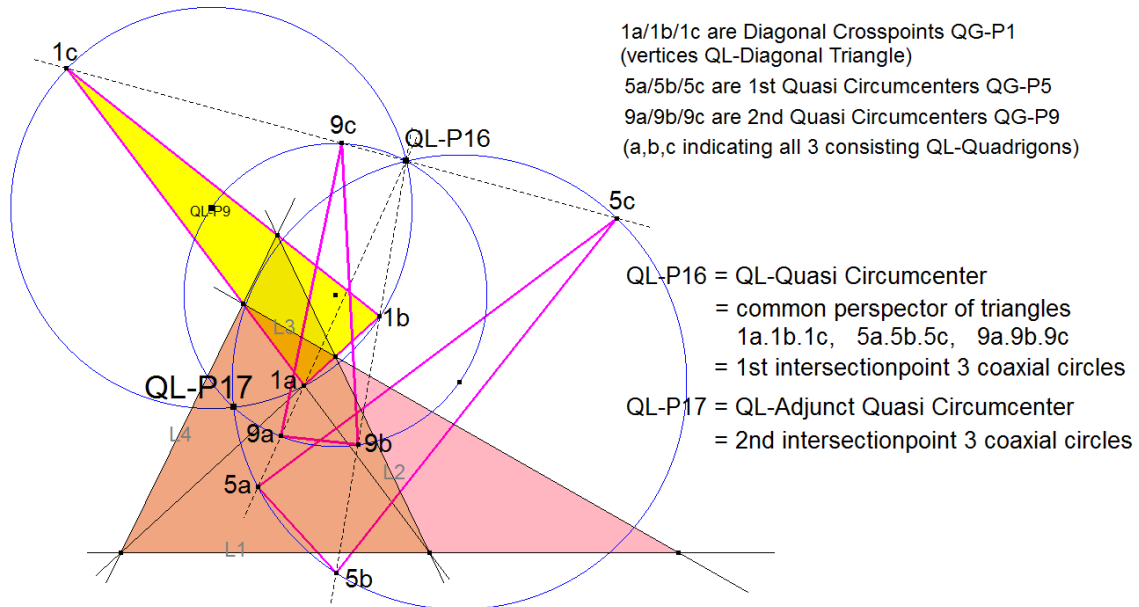
Properties:

- QL-P16 lies on this line:
 - QL-P1.QL-P10 $(-1 : 2 \Rightarrow \text{QL-P16} = \text{Reflection of QL-P10 in QL-P1})$
- QL-P16.QL-P9 // QL-P1.QL-P11.
- QL-P16 lies on the circumcircle of QL-Ci1, the QL-Diagonal Triangle.
- QL-P16 lies on the circumcircles of triangles formed by the 3 QL-versions of QG-P1, QG-P5 as well as QG-P9.
- QL-P16 is Railway Watcher (see QL-L/1) of lines QL-L5 (NSM Line) and QL-L6 (Quasi OrthoLine).

QL-P17: QL-Adjunct Quasi Circumcenter

QL-P17 is the 2nd common intersection point of the coaxial circumcircles of the 3 triangles formed by the 3 QL-versions of resp. QG-P1, QG-P5, QG-P9.

The 1st intersection point of these circles is QL-P16 (QL-Quasi Circumcenter).



1st CT-coordinate:

$$m n (a^2 (l - m) (l - n) (2 l^2 (m - n)^2 - m^2 n^2 + l m n (m + n)) \\ + b^2 (m - n) (m - l) n l (-3 l m + l n + m n) \\ + c^2 (n - m) (n - l) m l (l m - 3 l n + m n)))$$

1st DT-coordinate:

$$a^2 m^2 n^2 / (m^2 - n^2)$$

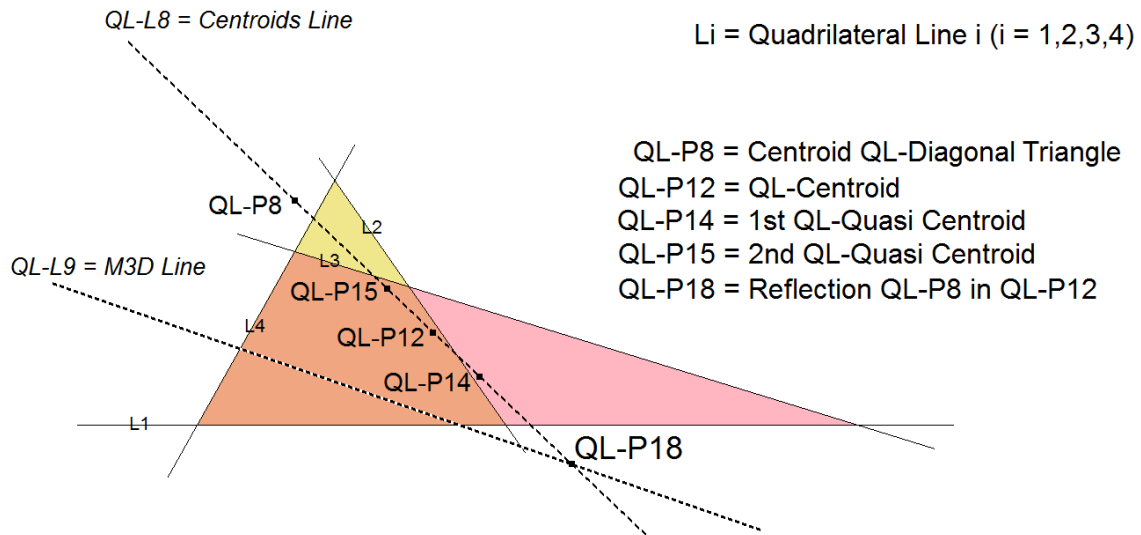
Properties:

- QL-P17 lies on these lines:
 - QL-P8.QL-P25 $(-2 : 3)$
 - QL-P13.QL-P24
- QL-P17 is the AntiComplement of QL-P25 wrt the QL-Diagonal Triangle.
- QL-P25 lies on the polar of QL-P17 wrt the Polar Circle of the QL-Diagonal Triangle (note Eckart Schmidt).
- QL-P17 lies on the circumcircle QL-Ci1 of the QL-Diagonal Triangle.
- QL-P17 lies on the Dimidium Circle QL-Ci6. The second intersection point with the circumcircle of the QL-Diagonal Triangle (QL-Ci1) is QL-P24.

QL-P18: Reflection of QL-P8 in QL-P12

QL-P18 is the Reflection of QL-P8 in QL-P12.

Special about this point is that it lies on QL-L9, the M3D Line.



1st CT-coordinate:

$$l^3 (m - n)^2 (m + n) - 2 l^2 (m^2 - n^2)^2 - 2 m^2 n^2 (m^2 - m n + n^2) + l m n (m + n) (4 m^2 - 7 m n + 4 n^2)$$

1st DT-coordinate:

$$(m^2-n^2) (l^2 (l^2+m^2+n^2) - 3 m^2n^2)$$

Properties:

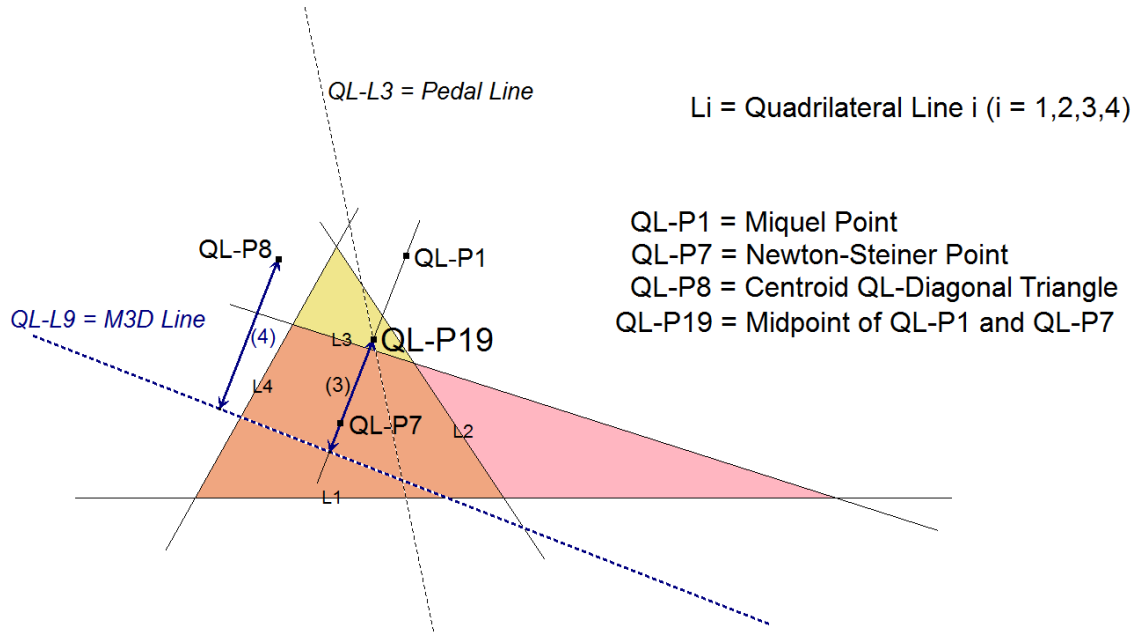
- QL-P18 lies on these lines:
 - QL-P8.QL-P12 = QL-L8 (2 : -1 => P18 = Reflection of QL-P8 in QL-P12)
 - QL-P18.QL-P23 = QL-L9

QL-P19: Midpoint of QL-P1 and QL-P7

QL-P19 is the Midpoint of QL-P1 and QL-P7.

Special about this point is that it lies on QL-L3, the Pedal Line.

Besides that it has a distance relation with the M3D Line (QL-L9).



1st CT-coordinate:

$$(a^2 m n - S_B l m - S_C l n) (3 m n - l m - l n)$$

1st DT-coordinate:

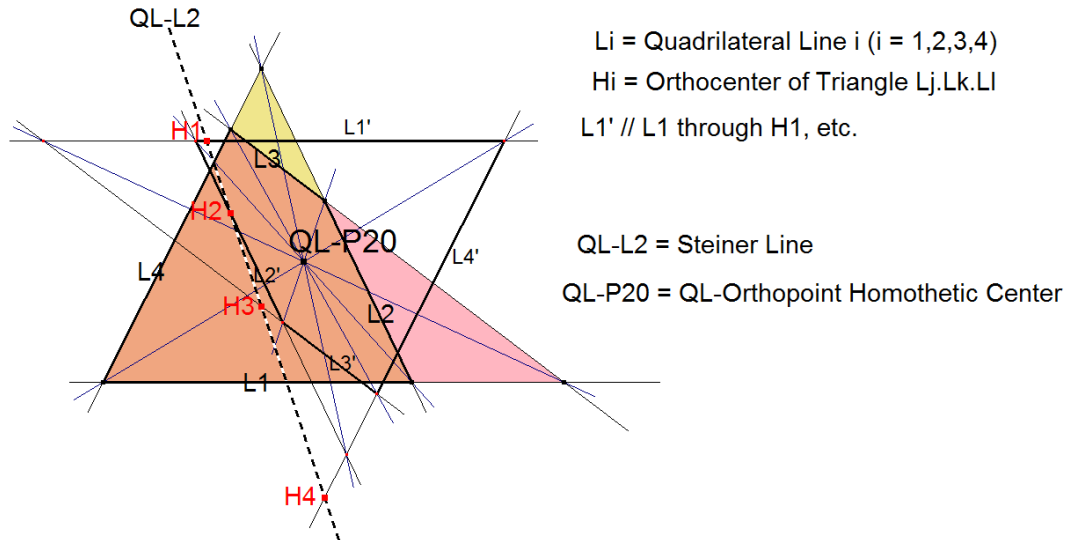
$$S_a (m^2 - n^2)^2 - S_b (l^2 - n^2) n^2 - S_c (l^2 - m^2) m^2$$

Properties:

- QL-P19 lies on this line:
 - QL-P1.QL-P7 = QL-L5 (1 : 1 => QL-P19 = Midpoint of QL-P1, QL-P7)
- QL-P19.QL-P20 // QL-P7.QL-P21
- QL-P19 lies on QL-L3, the Pedal Line.
- QL-P19 is the Quadrangle Centroid of the “flat quadrangle” formed by the 4 projection points of QL-P1 (collinear on QL-L3, the Pedal Line) on the basic lines of the Reference Quadrilateral.
- $d(\text{QL-P19}, \text{QL-L9}) = \frac{3}{4} * d(\text{QL-P8}, \text{QL-L9})$

QL-P20: Orthocenter Homothetic Center

QL-P20 is the homothetic Center of the Reference Quadrilateral $L_1.L_2.L_3.L_4$ and the Quadrilateral formed by the lines parallel to the lines L_1, L_2, L_3, L_4 through the Orthocenters of the corresponding component triangles.



1st Coordinate

$$(m - n) (a^2 S_A m n - b^2 S_B l m - c^2 S_C l n + S_B S_C (l^2 + m n))$$

or

$$(m - n) (S_B S_C l^2 - b^2 S_B l m - c^2 S_C l n + 4 \Delta^2 m n)$$

1st DT-coordinate:

$$(m^2 - n^2) ((-a^2 l^2 + b^2 m^2 + c^2 n^2)^2 - 4 S a^2 m^2 n^2)$$

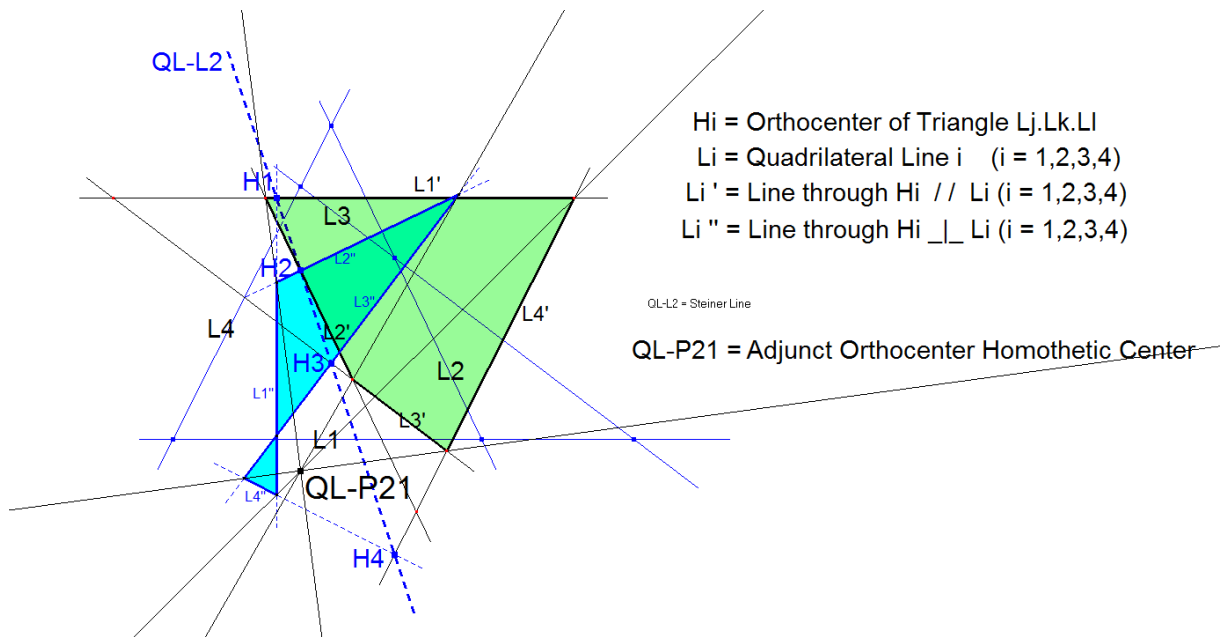
Properties:

- QL-P20 lies on these lines:
 - $QL-P1.QL-P21 = QL-L5$ ($1 : 1 \Rightarrow QL-P20 = \text{Midpoint of } QL-P1, QL-P21$)
 - $QL-P5.QL-P7 = QL-L1$
- $QL-P19.QL-P20 \parallel QL-P7.QL-P21$
- QL-P20 is the Reflection of QL-P5 (Kantor-Hervey Point) in QL-P22 (QL-Nine-point Center Homothetic Center).
- The distance ratios between points QL-P20, QL-P22, QL-P12, QL-P5 are $3 : 1 : 2$.

QL-P21: Adjunct Orthocenter Homothetic Center

QL-P21 is the Perspector of

- the Quadrilateral formed by the lines parallel to the lines L_1, L_2, L_3, L_4 through the Orthocenters of the corresponding component triangles and
- the Quadrilateral formed by the lines *perpendicular* to the lines L_1, L_2, L_3, L_4 through the Orthocenters of the corresponding component triangles.



1st Coordinate

$$\begin{aligned}
 & 2a^2(a^4 + (b^2 - c^2)^2 - 2a^2(b^2 + c^2))(l - m)^2 m(l - n)^2(m - n)n \\
 & + 2(-m + n)(a^2 m n(l - m)(l - n) + b^2 l n(m - l)(m - n) + c^2 l m(n - l)(n - m))(2a^2(b^2 m(l - n) + c^2 n(l - m)) \\
 & + a^4(-l^2 + m n) + (b^2 - c^2)(b^2(l^2 - 2lm + mn) - c^2(l^2 - 2ln + mn)))
 \end{aligned}$$

1st DT-coordinate:

$$\begin{aligned}
 & Tl(Sa^3 Tl^4 + Sb^3 Tm^4 + Sc^3 Tn^4 \\
 & + 2(m^2 + n^2)SaSbScTmTn(Tn - Tm) \\
 & + Sa^2 Tl^2((m^2 + 3n^2)SbTm - (3m^2 + n^2)ScTn) \\
 & + Sc^2 Tn^2(SbTm(Tl + 3Tm) + Sa(3l^2 Tl - 2n^2 Tl - 3m^2 Tn + m^2 Tm)) \\
 & + Sb^2 Tm^2(ScTn(Tl + 3Tn) + Sa(3n^2 Tm + 2m^2 Tl - 3l^2 Tl - n^2 Tn)))
 \end{aligned}$$

where:

$$Tl = m^2 - n^2$$

$$Tm = n^2 - l^2$$

$$Tn = l^2 - m^2$$

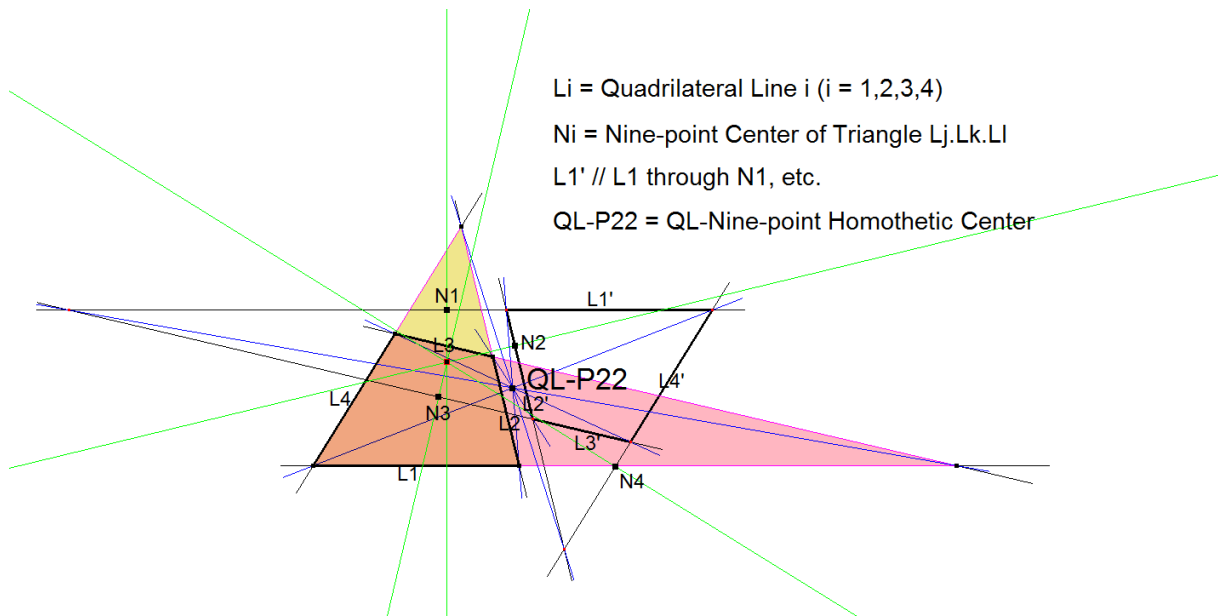
Properties:

- QL-P21 lies on this line:
 - QL-P1.QL-P20 $(-1 : 2 \Rightarrow P21 = \text{Reflection of QL-P1 in QL-P20})$
- QL-P21 lies on the asymptote of the cubic QL-Cu1.

QL-P22: QL-Nine-point Center Homothetic Center

QL-P22 is the homothetic Center of the Reference Quadrilateral $L_1.L_2.L_3.L_4$ and the homothetic Quadrilateral formed by the lines parallel to the lines L_1, L_2, L_3, L_4 through the Nine-point Centers of the corresponding component triangles.

The prefix “QL-” has been added in the name of QL-P22 because there is also a Nine-point Homothetic Center in a Quadrangle.



1st CT-coordinate:

$$(m - n) (a^2 S_A l^2 + b^2 S_B l m + c^2 S_C l n + 8 \Delta^2 (l m + l n - 2 m n - l^2))$$

1st DT-coordinate:

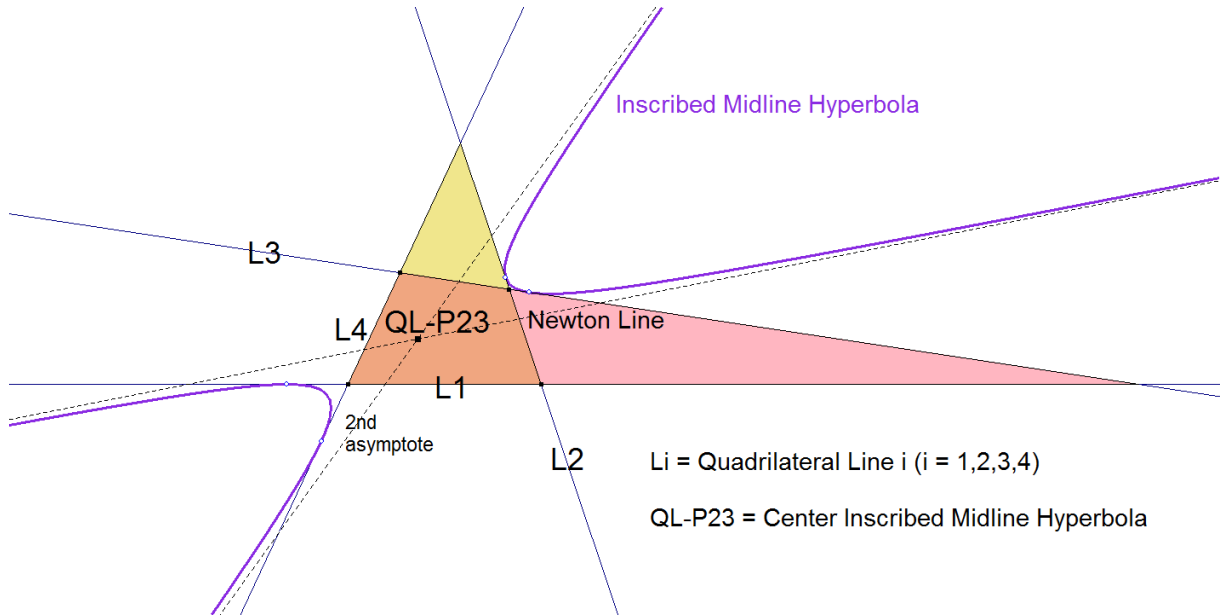
$$(a^2 (l^2 - m^2) - c^2 (m^2 - n^2)) (a^2 (l^2 - n^2) + b^2 (m^2 - n^2))$$

Properties:

- QL-P22 lies on these lines:
 - QL-P2.QL-P4 ($1 : 1 \Rightarrow$ QL-P22 is Midpoint of QL-P2 and QL-P4)
 - QL-P5.QL-P7 = QL-L1
- QL-P22.QL-P6 \parallel QL-P2.QL-P5 \parallel QL-P4.QL-P20
- QL-P22 is the Midpoint of QL-P5 (Kantor-Hervey Point) and QL-P20 (Orthocenter Homothetic Center).
- The distance ratios between points QL-P20, QL-P22, QL-P12, QL-P5 are $3 : 1 : 2$.

QL-P23: Center of the Inscribed Midline Hyperbola

QL-P23 is the center of the Inscribed Midline Hyperbola. This is the conic touching the defining lines of the Reference Quadrilateral and with the Newton Line as asymptote. See Paragraph QL-Co2.



1st CT-coordinate:

$$l(m-n) [(l m + l n + m n)^2 - 4 m n (l^2 + m n)]$$

1st DT-coordinate:

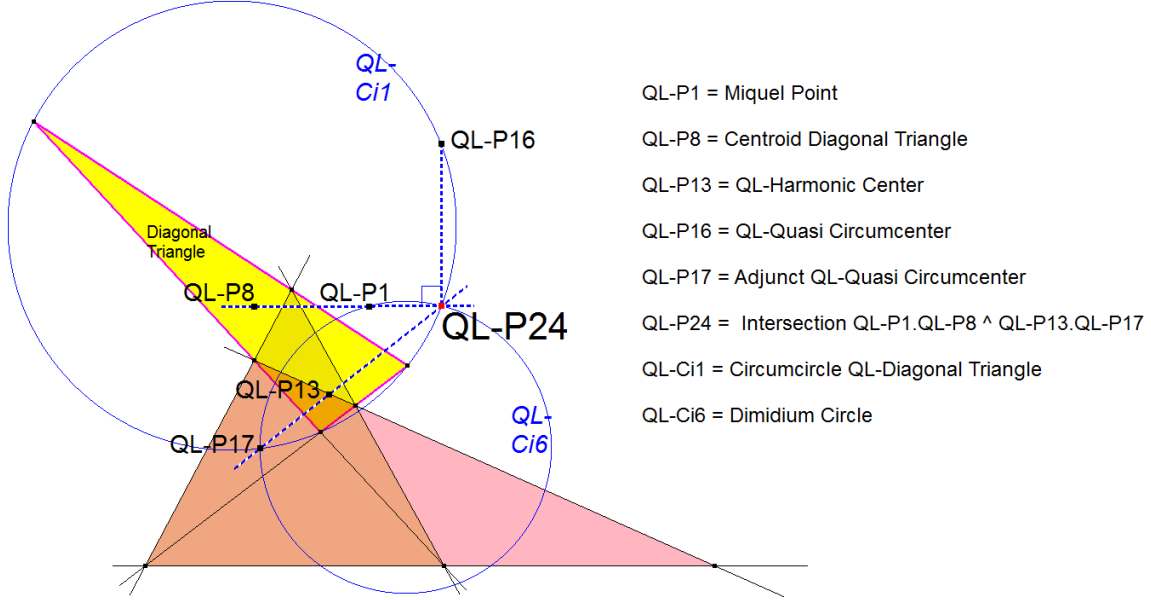
$$m^2 n^2 (m^2 - n^2)$$

Properties:

- QL-P23 lies on QL-L1, the Newton Line.
- QL-P23 lies on QL-L9, the M3D Line.

QL-P24: Intersection QL-P1.QL-P8 ^ QL-P13.QL-P17

QL-P24 is the point of intersection of the lines QL-P1.QL-P8 and QL-P13.QL-P17.
It is also one of the points of intersection of the circles QL-Ci1 (Circumcircle QL-Diagonal Triangle) and QL-Ci6 (Dimidium Circle).



1st CT-coordinate:

$$(l - m) (l - n) m n$$

$$* (T_{mn} (a^4 (4 (l - m) (l - n) (l m - l n - m n) (l m - l n + m n) + (l m + l n - m n))$$

$$+ b^4 (m - n) (+l m - l n + m n) / (l - n)$$

$$+ c^4 (m - n) (-l m + l n + m n) / (m - l) - 2 b^2 c^2 (m - n)^2)$$

$$- 2 a^2 b^2 ((l - m) (m - n) T_{nl} - (l m - l n + m n) T_{lm})$$

$$- 2 a^2 c^2 ((l - n) (m - n) T_{lm} - (l m - l n - m n) T_{nl}))$$

where:

$$T_{mn} = (l^2 m^2 - 2 l^2 m n + 2 l m^2 n + l^2 n^2 + 2 l m n^2 - 3 m^2 n^2)$$

$$T_{lm} = (l^2 m^2 + 2 l^2 m n - 2 l m^2 n - 3 l^2 n^2 + 2 l m n^2 + m^2 n^2)$$

$$T_{nl} = (3 l^2 m^2 - 2 l^2 m n - 2 l m^2 n - l^2 n^2 + 2 l m n^2 - m^2 n^2)$$

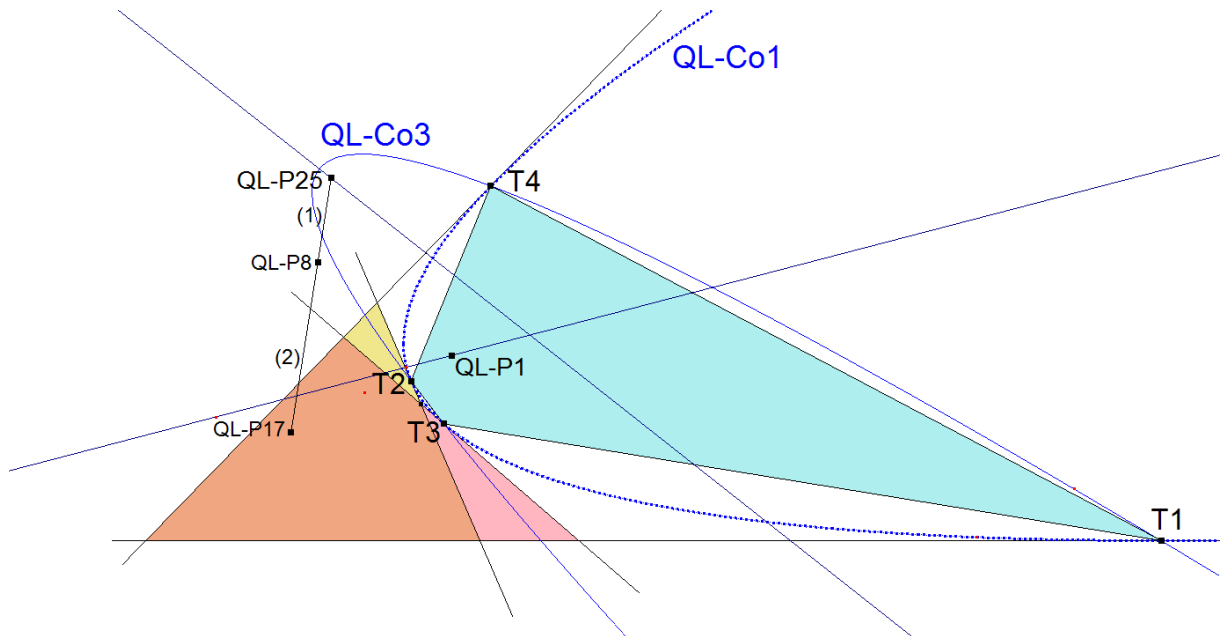
1st DT-coordinate:

$$(a^2 (l^2 - n^2) + b^2 (m^2 - n^2)) (a^2 (l^2 - m^2) + c^2 (-m^2 + n^2))$$

Properties:

- QL-P24 lies on these lines:
 - QL-P1.QL-P8
 - QL-P13.QL-P17
- QL-P24 is the projection of QL-P16 on the line QL-P1.QL-P8.
- QL-P24 is one of the points of intersection of the circles QL-Ci1 (Circumcircle QL-Diagonal Triangle) and QL-Ci6 (Dimidium Circle). The other point of intersection is QL-P17 (QL-Adjunct Quasi Circumcenter).

QL-P25 is the Focus of the 2nd QL-Parabola QL-Co3.
For an explanation of the 2nd QL-Parabola see QL-Co3.



1st CT-coordinate:

$$\begin{aligned} & a^2 m n (l - m) (l - n) (l^5 m^5 - 3 l^5 m^4 n + l^4 m^5 n + 2 l^5 m^3 n^2 - 4 l^4 m^4 n^2 + 2 l^3 m^5 n^2 + 2 l^5 m^2 \\ & n^3 + 6 l^4 m^3 n^3 - 6 l^3 m^4 n^3 - 2 l^2 m^5 n^3 - 3 l^5 m n^4 - 4 l^4 m^2 n^4 - 6 l^3 m^3 n^4 + 16 l^2 m^4 n^4 - 3 l m^5 \\ & n^4 + l^5 n^5 + l^4 m n^5 + 2 l^3 m^2 n^5 - 2 l^2 m^3 n^5 - 3 l m^4 n^5 + m^5 n^5) \\ & - l m n (l - m) (l - n) (m - n) m^2 n^2 (4 b^2 n (l - m) (3 l m - l n - m n)) \\ & - l m n (l - m) (l - n) (m - n) m^2 n^2 (4 c^2 m (l - n) (l m - 3 l n + m n)) \end{aligned}$$

1st DT-coordinate:

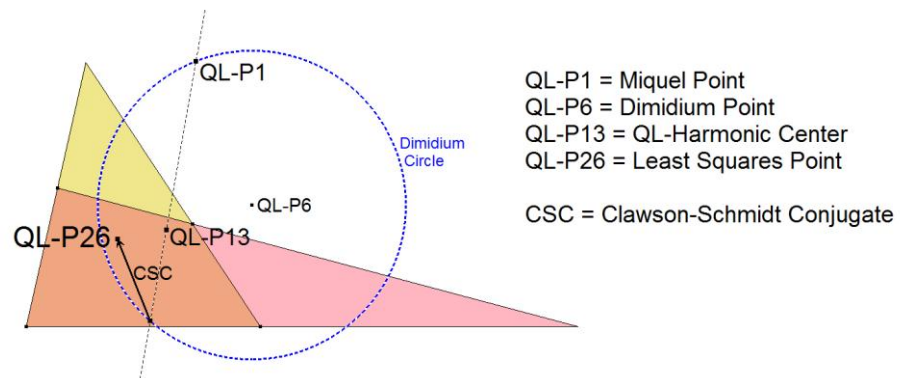
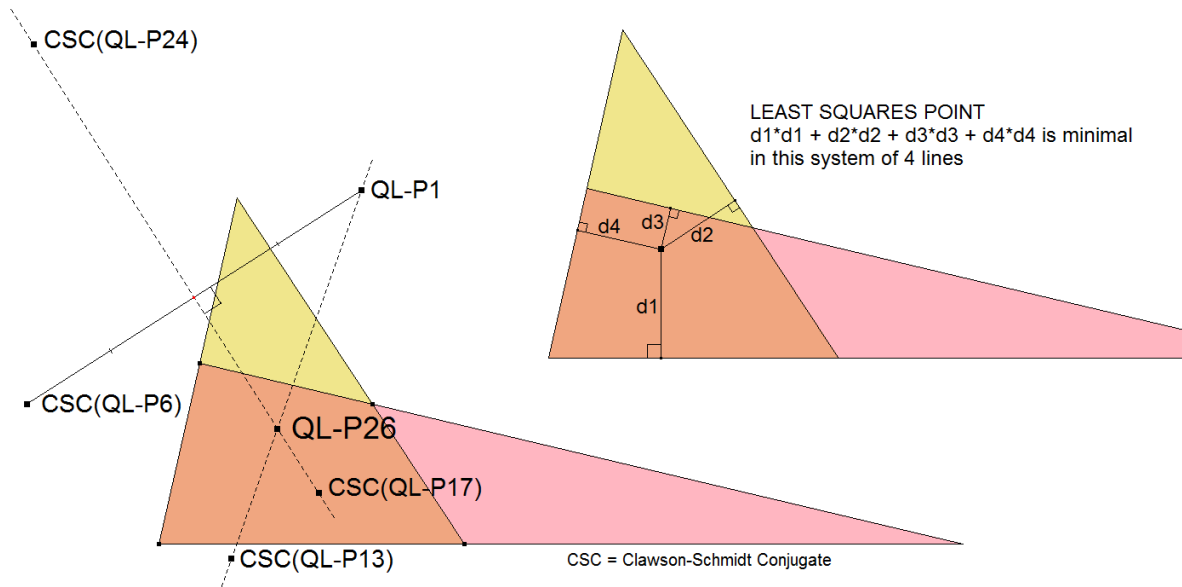
$$l^2 (m^2-n^2) (b^2 n^2 (-l^2+m^2) +c^2 m^2 (l^2-n^2))$$

Properties:

- QL-P25 lies on the line:
 - QL-P8.QL-P17 $(-1 : 3 \Rightarrow \text{QL-P25} = \text{Complement of QL-P17 wrt QL-DT})$
- QL-P25 lies on the Axis of QL-Co3 which is a line \perp QL-P1.QL-P7.
- QL-P25 lies on the QL-Medial Circle QL-Ci2.
- QL-P25 lies on the polar of QL-P17 wrt the Polar Circle of the QL-Diagonal Triangle (note Eckart Schmidt).

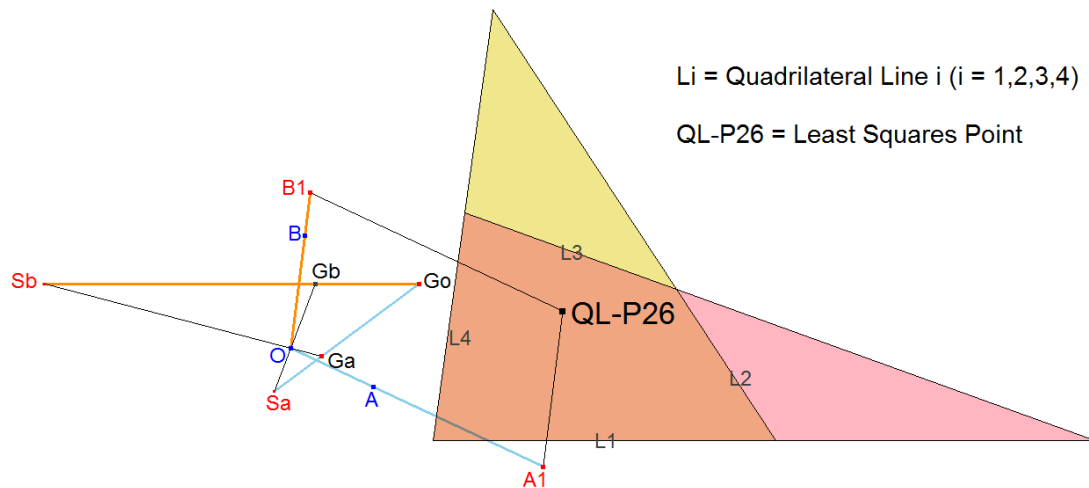
QL-P26: Least Squares Point

QL-P26 is the point in a quadrilateral such that the sum of the squares of the distances to the 4 basic lines is minimal.



QL-P26 is the Clawson-Schmidt Conjugate of the 2nd intersection point of QL-P1.QL-P13 with the Dimidium Circle QL-Ci6.

Construction:



This construction is a modified version of the construction of Coolidge as described in [25].

Let O (origin), A and B be random non-collinear points.

Go = Quadrangle Centroid of the projection points of O on the 4 basic lines of the Reference Quadrilateral.

Ga = Quadrangle Centroid of the projection points of O on the 4 lines through point A parallel to the 4 basic lines of the Reference Quadrilateral.

Gb = Quadrangle Centroid of the projection points of O on the 4 lines through point B parallel to the 4 basic lines of the Reference Quadrilateral.

Let Sa = Ga.Go ^ O.Gb and Sb = Gb.Go ^ O.Ga.

Construct A1 on line O.A such that Sa.Ga : Ga.Go = O.A : A.A1.

Construct B1 on line O.B such that Sb.Gb : Gb.Go = O.B : B.B1.

Construct P such that O.A1.P.B1 is a parallelogram and where O and P are opposite vertices. P is the Least Squares Point QL-P26.

1st CT-coordinate:

$$a^2 (a^2 (l - m) (l - n) + b^2 (2m - n) (m - l) + c^2 (2n - m) (n - l))$$

Least Sum of 4 Square Distances in CT-notation:

$$\begin{aligned} & (-a^4 m^2 n^2 (l - m) (l - n) + b^2 c^2 l^2 (m - n)^2 ((l + m + n) (2l + m + n) - mn) \\ & - b^4 l^2 n^2 (m - l) (m - n) + c^2 a^2 m^2 (n - l)^2 ((l + m + n) (l + 2m + n) - nl) \\ & - c^4 m^2 l^2 (n - l) (n - m) + a^2 b^2 n^2 (l - m)^2 ((l + m + n) (l + m + 2n) - lm)) / \\ & ((l + m + n)^2 (a^2 mn (l - m) (l - n) + b^2 nl (m - l) (m - n) + c^2 lm (n - l) (n - m))) \end{aligned}$$

1st DT-coordinate:

$$\begin{aligned} & m^2 n^2 (a^4 (l^2 - m^2) (l^2 - n^2) - b^4 (m^2 - l^2) (m^2 - n^2) - c^4 (n^2 - l^2) (n^2 - m^2) \\ & + 2 a^2 b^2 (l^2 - m^2) n^2 + 2 a^2 c^2 m^2 (l^2 - n^2)) \end{aligned}$$

Least Sum of 4 Square Distances in DT-notation:

$$\begin{aligned} & (4 S^2 l^2 m^2 n^2 (a^2 l^2 + b^2 m^2 + c^2 n^2)) \\ & / (a^4 (l^2 - m^2) (l^2 - n^2) (l^2 m^2 + l^2 n^2 - m^2 n^2) + 2 b^2 c^2 l^4 (m^2 - n^2)^2 \\ & + b^4 (m^2 - n^2) (m^2 - l^2) (l^2 m^2 - l^2 n^2 + m^2 n^2) + 2 c^2 a^2 m^4 (n^2 - l^2)^2 \\ & + c^4 (n^2 - l^2) (n^2 - m^2) (-l^2 m^2 + l^2 n^2 + m^2 n^2) + 2 a^2 b^2 n^4 (l^2 - m^2)^2) \end{aligned}$$

Properties:

- QL-P26 is the Clawson-Schmidt Conjugate of the 2nd intersection point of the line QL-P1.QL-P13 with the Dimidium Circle QL-Ci6.
- QL-P26 lies on the line QL-P1.CSC(QL-P13).
(CSC = Clawson-Schmidt Conjugate QL-Tf1)
- QL-P26 lies on the line CSC(QL-P17).CSC(QL-P24), which is the perpendicular bisector of QL-P1.CSC(QL-P6)

6.3 QUADRILATERAL LINES

QL-L/1: The Railway Watcher

Within the system of a quadrilateral a strange phenomenon pops up.

Often when a QL-line becomes known there also is a parallel line that requires attention. And moreover there is also a prominent point not on these lines at the same distance from one of the lines as the mutual distance of these lines.

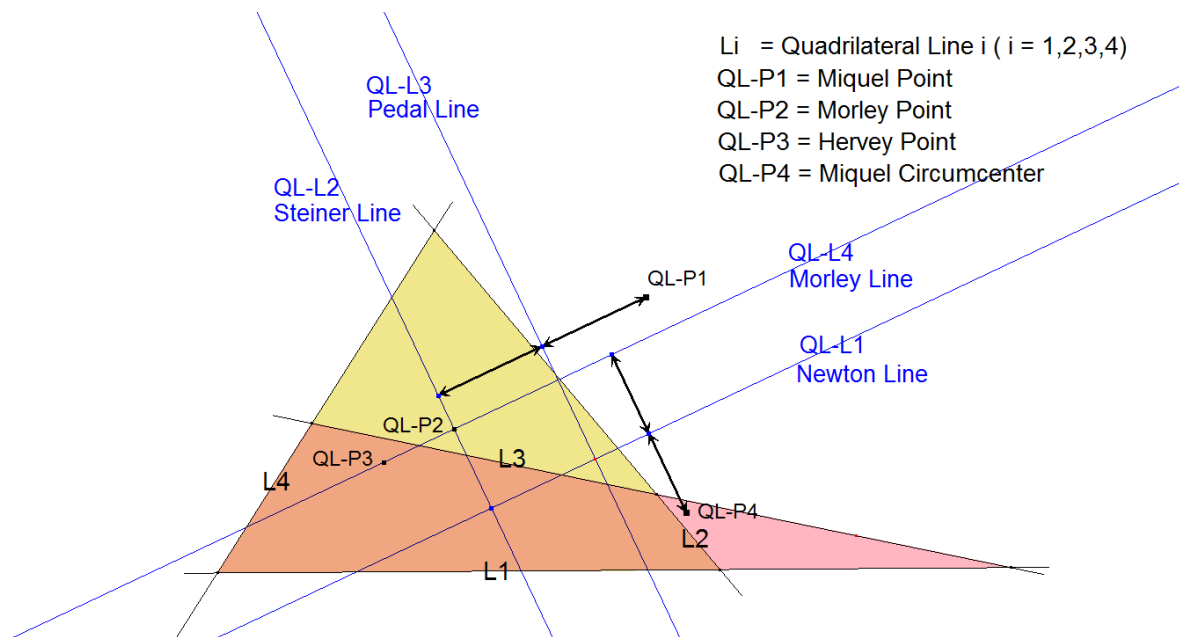
I call this the system of a Railway Watcher because the parallel lines can be represented as a railroad and the point at a minimal distance from the railway as a “railway watcher”.

Examples:

In next picture

- QL-P1 is Railway Watcher of lines QL-L2 (Steiner Line) and QL-L3 (Pedal Line).
- QL-P4 is Railway Watcher of lines QL-L1 (Newton Line) and QL-L4 (Morley Line).

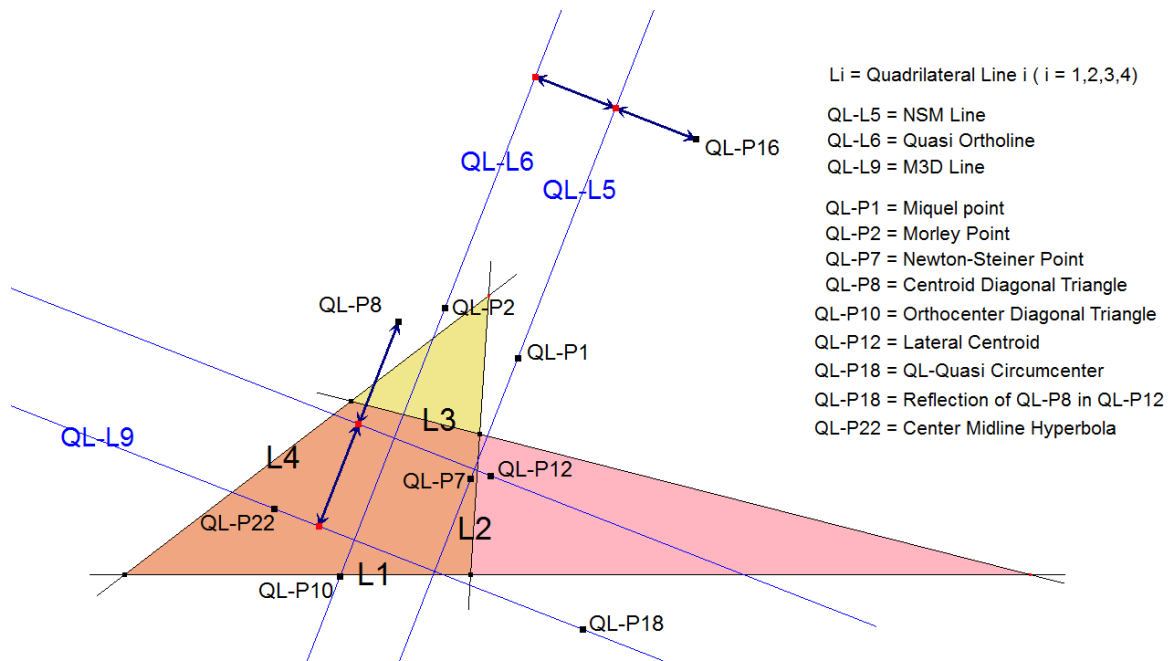
Notable is that both railways cross at right angles.



In next picture

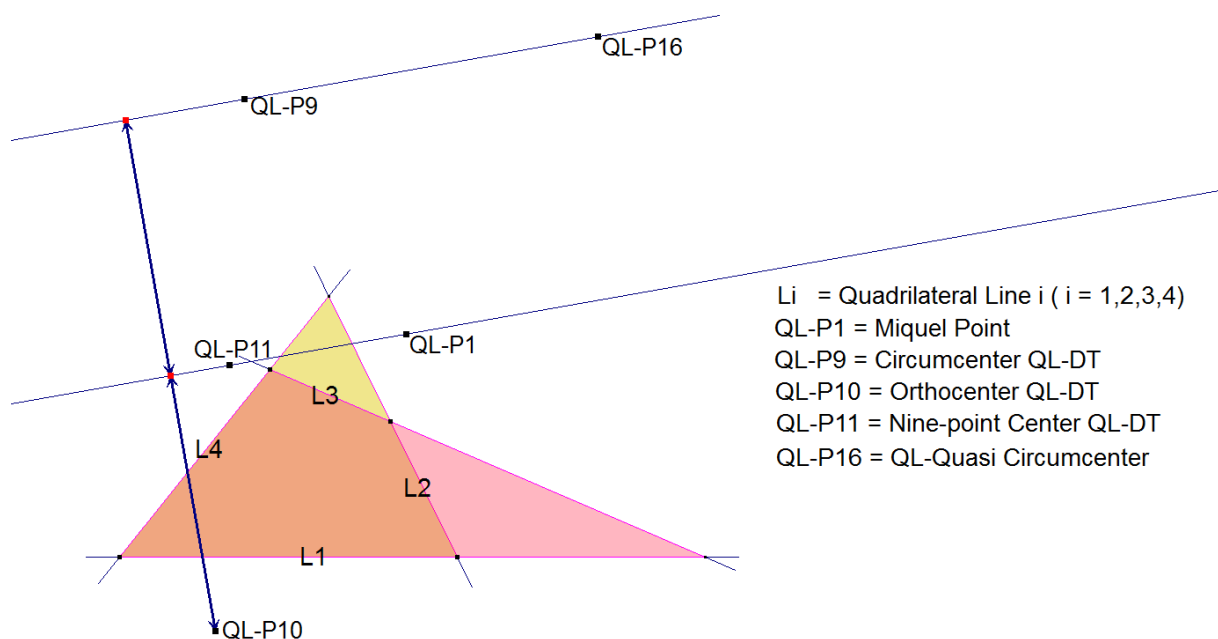
- QL-P8 is Railway Watcher of lines QL-L9 (M3D Line) and line // QL-L9 through QL-P12.
- QL-P16 is Railway Watcher of lines QL-L5 (NSM Line) and QL-L6 (Quasi OrthoLine).

Again both railways cross at right angles.



In next picture

- QL-P10 is Railway Watcher of lines QL-P9.QL-P16 and line QL-P1.QL-P11.



This system of Railway Watchers can be coincidental, but still it is a remarkable item. See QL-Cu1 and QG-Co3 for other Railway Watchers where the Newton Line and an asymptote form a railway.

See also QG-P1 where QG-P1 is the Railway Watcher of the 1st and 2nd Quasi Euler Line.

QL-Cartesian Systems

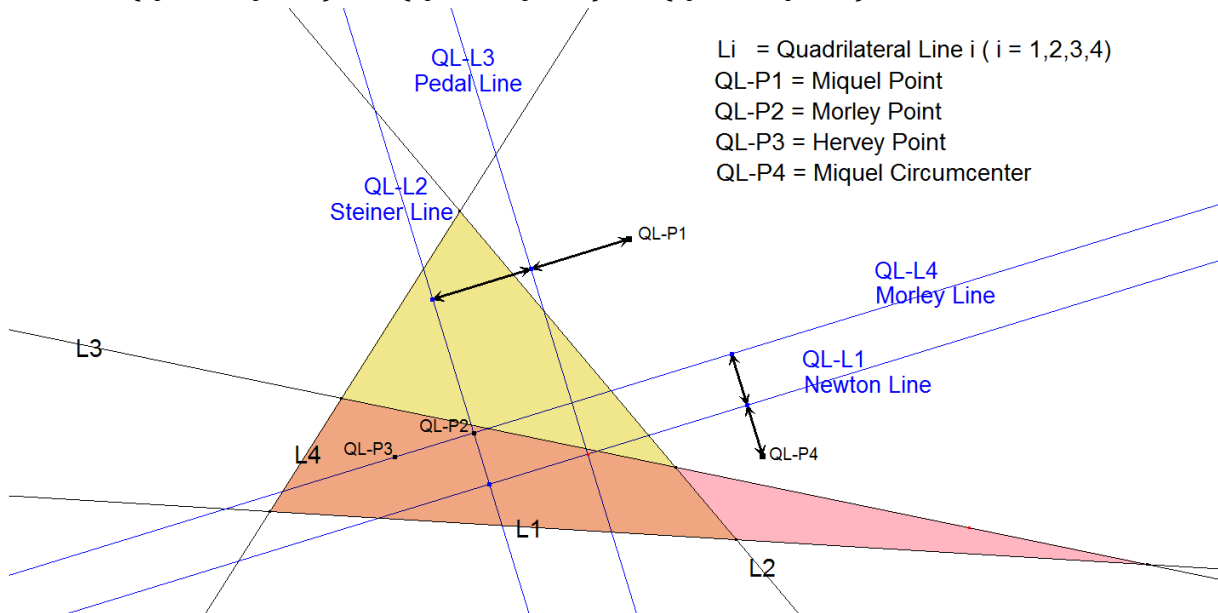
The four most outstanding lines of a quadrilateral are the Newton Line, the Steiner Line, the Pedal Line and the Morley Line.

They form a formidable rectangular construction with a special role for the Miquel Point (QL-P1) and The Miquel Circumcenter (QL-P4).

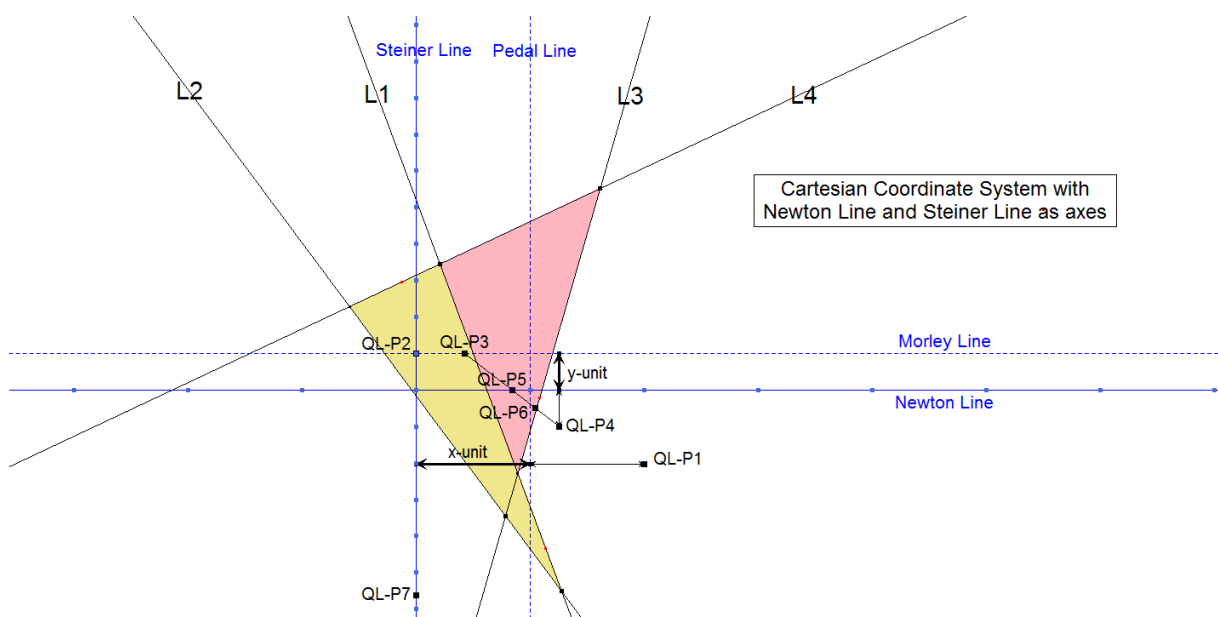
Especially because:

$$d(QL-P1, QL-L3) = d(QL-L2, QL-L3) = d(QL-P2, QL-L3),$$

$$d(QL-P4, QL-L1) = d(QL-L4, QL-L1) = d(QL-P2, QL-L1).$$



Since Steiner Line/Pedal Line and Newton Line/Morley Line are parallel and Steiner line/Newton Line are perpendicular lines, these lines can function as a Cartesian coordinate system with Steiner and Newton Line as axes. Moreover the difference in distances of the parallel lines can function as units at the axes. This would look like this:



Quadrilateral Points

Now all points QL-P1 to QL-P7 have these Cartesian coordinates (relating to CT-coordinates):

	<u>x-coordinate</u>	<u>y-coordinate</u>
P1:	2	$Y_1.Y_2$
P2:	0	+1
P3:	$2X_1 / X_3$	+1
P4:	$(2X_2-2X_1) / (3X_3)$	-1
P5:	$(X_2+2X_1) / (3X_3)$	0
P6:	$X_2 / (2X_3)$	$-1/2$
P7:	0	0

The units used are:

$$X_3 / (8 AL_5^3) \qquad AL_3^2 / (16 ABC AL_5 LM_1^2)$$

where:

$$\begin{aligned} X_1 &= AL_1 \cdot AL_5^2 & Y_1 &= LM_1 / AL_3 \\ X_2 &= ABC \cdot AL_2 \cdot AL_5 & Y_2 &= ABC \cdot AL_6 / AL_5 \\ X_3 &= ABC^2 \cdot LM_0 \cdot LM_1^2 \end{aligned}$$

and:

$$ABC = (-a + b + c) (a - b + c) (a + b - c) (a + b + c)$$

$$LM_0 = l m n$$

$$LM_1 = (l - m) (m - n) (n - l)$$

$$AL_1 = (a^4 + b^2 c^2) (2l - m - n) + (b^4 + c^2 a^2) (2m - n - l) + (c^4 + a^2 b^2) (2n - l - m)$$

$$AL_2 = -a^2 (l - m) (n - l) (3 l m^2 - 8 l m n + m^2 n + 3 l n^2 + m n^2)$$

$$-b^2 (l - m) (m - n) (3 l^2 m + l^2 n - 8 l m n + l n^2 + 3 m n^2)$$

$$-c^2 (n - l) (m - n) (l^2 m + l m^2 + 3 l^2 n - 8 l m n + 3 m^2 n)$$

$$AL_3 = -(l - m) (m - n) (n - l) (a^4 (l m - m n + n l) + b^4 (l m + m n - n l) + c^4 (-l m + m n + n l))$$

$$-(l^2 m^2 + m^2 n^2 + n^2 l^2 - l m n (l + m + n)) (2 b^2 c^2 (m - n) + 2 c^2 a^2 (n - l) + 2 a^2 b^2 (l - m))$$

$$AL_5 = a^2 mn(l-m)(l-n) + b^2 nl(m-l)(m-n) + c^2 lm(n-l)(n-m)$$

$$AL_6 = a^2 mn(l-m)(l-n)(-lm-ln+mn) + b^2 nl(m-l)(m-n)(-lm+ln-mn) + c^2 lm(n-l)(n-m)(lm-ln-mn)$$

Quadrilateral Lines

The Pedal Line has equation $x = 1$.

The Morley Line has equation $y = 1$.

One advantage of this system is that working with circles is easier in a Cartesian coordinate system than in a barycentric/trilinear coordinate system.

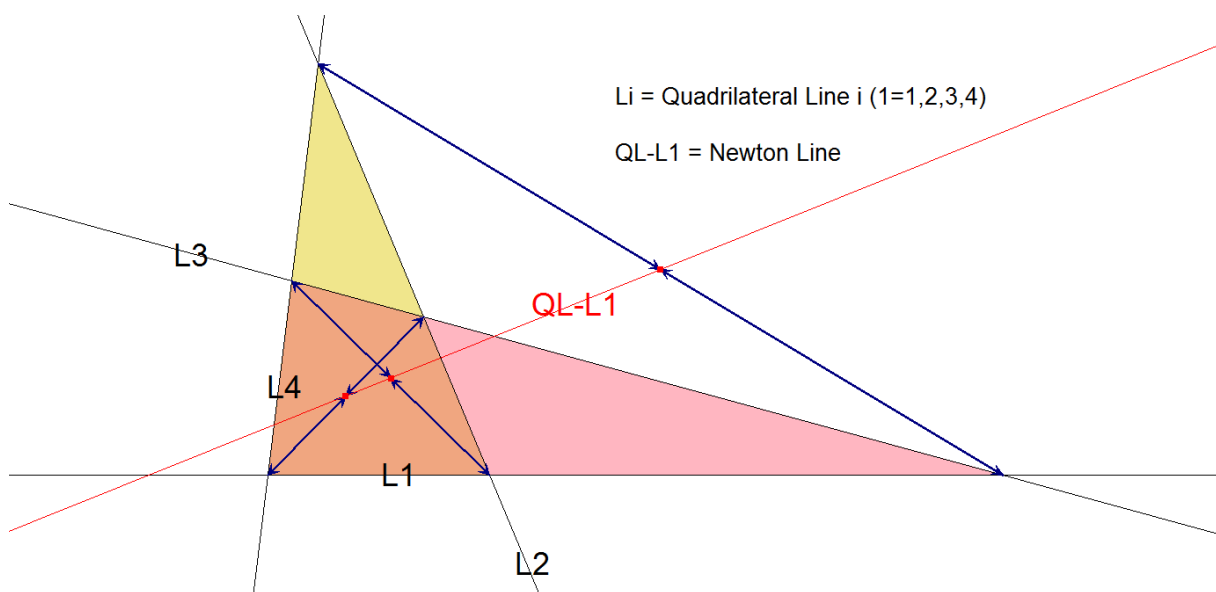
QL-L1: The Newton Line

The intersection point of two lines of a quadrilateral and the intersection point of the other two lines of a quadrilateral are called opposite points and their connective line is called a diagonal. The midpoints of the 3 diagonals of a Quadrilateral are collinear. The line through these 3 midpoints is the Newton Line.

Newton discovered this line because all centers of inscribed conics in a Quadrilateral reside on this line. Gauss discovered that this line of Newton's also passes through the midpoints of the diagonals of the quadrilateral. That's why the Newton Line is also known as Gauss Line or Gauss-Newton Line.

Jean-Louis Ayme wrote a book with 30 proofs about this line at [3].

Also he wrote [2a] about the Newton Line as well as the Steiner Line.



1st CT-coefficient:

$$l m + l n - m n$$

(note that this formula is independent of a,b,c)

1st DT-coefficient:

$$l^2$$

Properties:

- QL-P5, QL-P7, QL-P12, QL-P20, QL-P22, QL-P23 are points on QL-L1.
- Quadrigon point QG-P12 also lies on the Newton Line QL-L1.
- QL-L1 is also the line that connects the Centroids of the 3 component quadrilaterals of the quadrilateral.
- The centers of all inscribed conics of a Quadrilateral lie on QL-L1 (proved by Newton; see also [4] page 49).
- QL-L1 is parallel to QL-L4 (the Morley Line).
- QL-L1 is parallel to the axis of QL-Co1 (the Inscribed Quadrilateral Parabola).

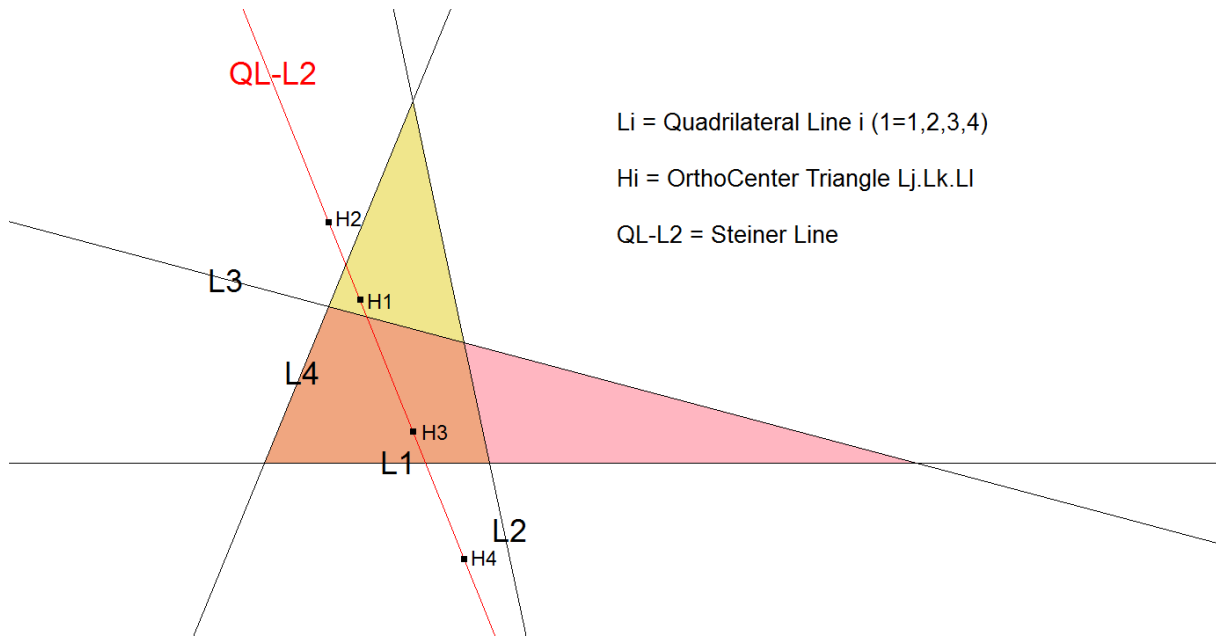
- QL-L1 is perpendicular to QL-L2 (Steiner Line) (known as the Gauss-Bodenmiller theorem) and QL-L3 (Pedal Line).
- QL-L1 is the locus of homothetic centers of L1.L2.L3.L4 with the 4 lines formed by the parallel lines to L1, L2, L3, L4 through Euler Line points Q1, Q2, Q3, Q4 of the Component Triangles, where Qi have proportional ratio wrt OrthoCenter and CircumCenter on respective Euler Line.
- QL-L1 is the Trilinear Polar of QL-P13 (note Eckart Schmidt).
- The asymptote of QL-Cu1 // QL-L1.
- The asymptote of QG-Co3 // QL-L1.

QL-L2: Steiner Line

The Orthocenters of the 4 component triangles of a Quadrilateral are collinear. The line through these 4 Orthocenters is the Steiner Line.

The Steiner Line is also known as Ortholine.

Jean-Louis Ayme wrote [2a] about the Newton Line as well as the Steiner Line.



1st CT-coefficient:

$$S_A \mid (m - n)$$

1st DT-coefficient:

$$b^2 (l^2 - m^2) - c^2 (l^2 - n^2)$$

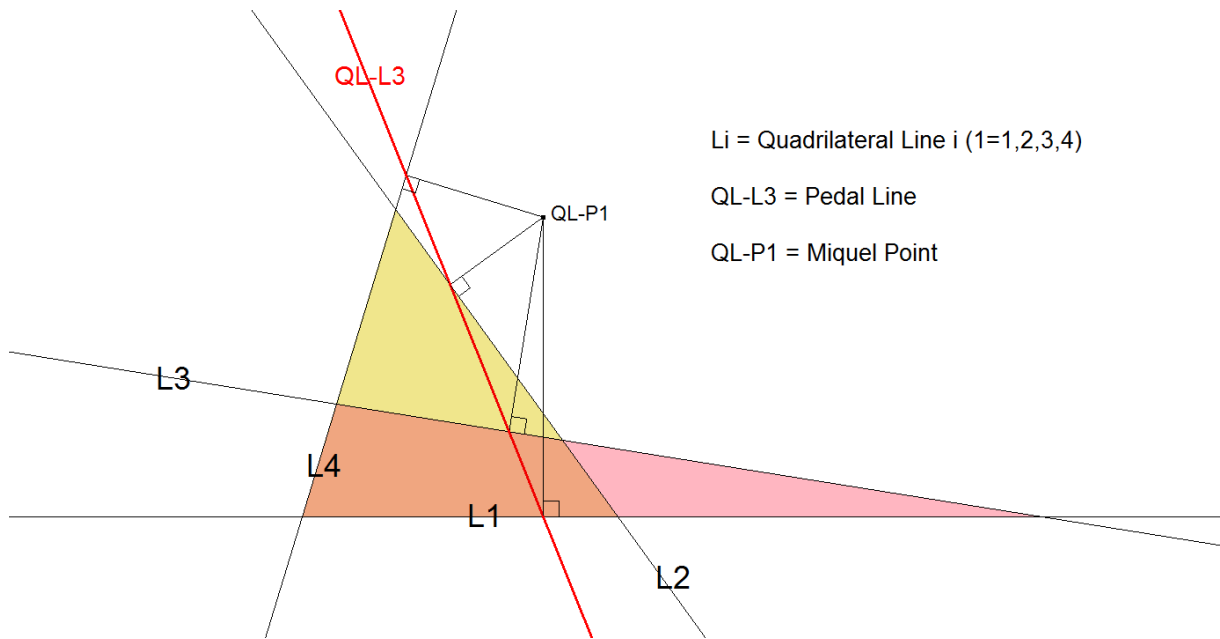
Properties:

- QL-P2 (Morley Point) and QL-2P1a/b (1st/2nd Plücker Point) and QL-P7 (Newton-Steiner Point) and QL-P9 (Circumcenter QL-Diagonal Triangle) lie on QL-L2.
- QL-L2 is parallel to QL-L3 (Pedal Line).
- QL-L2 is perpendicular to QL-L1 (Newton Line) and QL-L4 (Morley Line).
- QL-L2 is the directrix of QL-Co1, the Inscribed Quadrilateral Parabola.
- QL-L2 is the common radical axis of the three circles constructed on the diagonals of the Reference Quadrilateral as diameters.
- QL-L2 is the Clawson-Schmidt Conjugate (QL-Tf1) of QL-Ci3 (Miquel Circle).
- The Orthopole of a sideline of the complete quadrilateral with respect to the triangle bounded by the three other sidelines lies on the Steiner line (see [4] page 42).
The Orthopole of a line L wrt a Triangle = the common intersection point of the three lines perpendicular to the sidelines of the triangle, each passing through the projection of the opposite vertex on line L.

QL-L3: Pedal Line

The perpendicular feet from the Miquel Point (QL-P1) to the four lines lie on the same line R, and the Miquel Point is the only point with this property.

This is statement 3 of Steiner's note on the complete quadrilateral as described on page 1 of [4]. This line R is now according to [4] named the Pedal Line.



1st CT-coefficient:

$$l(m-n) / (a^2 m n - S_B l m - S_C l n)$$

1st DT-coefficient:

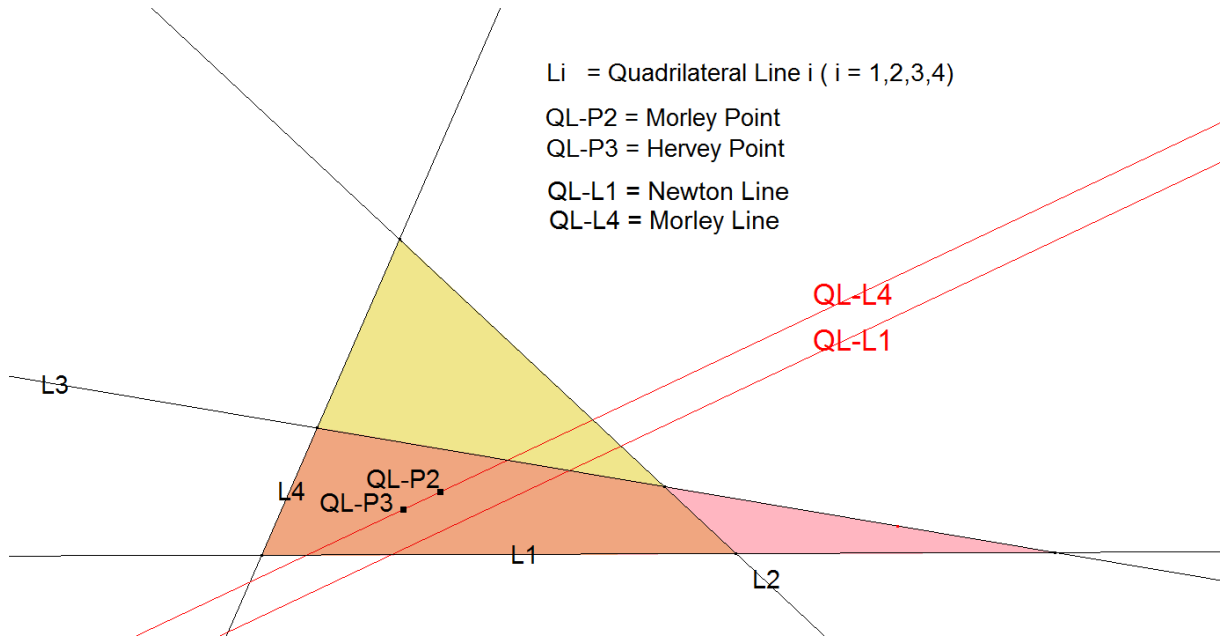
$$(m^2-n^2)^3 SA^2 + (l^2-n^2)^3 SB^2 + (m^2-l^2)^3 SC^2 + 2(m^2-n^2) SA ((l^2-n^2)^2 SB + (l^2-m^2)^2 SC)$$

Properties:

- QL-P19 lies on QL-L3.
- QL-L3 is parallel to QL-L2 (Steiner Line).
- QL-L3 is perpendicular to QL-L1 (Newton Line) and QL-L4 (Morley Line).
- QL-L3 is tangent to the QL-Inscribed Parabola QL-Co1 at the vertex.
- QL-Co1 is the 5th Line Conic (see QL-Co/1) of QL-L3.

QL-L4: Morley Line

The Morley Line is the line through QL-P2 (Morley Point) and QL-P3 (Hervey Point).



1st CT-coefficient:

$$\begin{aligned}
 &+a^4 (l-m) (l-n) (m-n) (lm+ln-mn) \\
 &+b^4 (l-m) (l-n) (m-n) lm \\
 &+c^4 (l-m) (l-n) (m-n) ln \\
 &+b^2 c^2 (n-m) U_0 \\
 &+a^2 c^2 (l-n) (U_0 - U_1) \\
 &+a^2 b^2 (m-l) (U_0 - U_2)
 \end{aligned}$$

where:

$$\begin{aligned}
 U_0 &= l(l^2 m + l^2 n - 4lmn + m^2 n + mn^2) \\
 U_1 &= (l-m)(l+m-2n)(lm+ln-mn) \\
 U_2 &= (l-n)(l+n-2m)(lm+ln-mn)
 \end{aligned}$$

1st DT-coefficient:

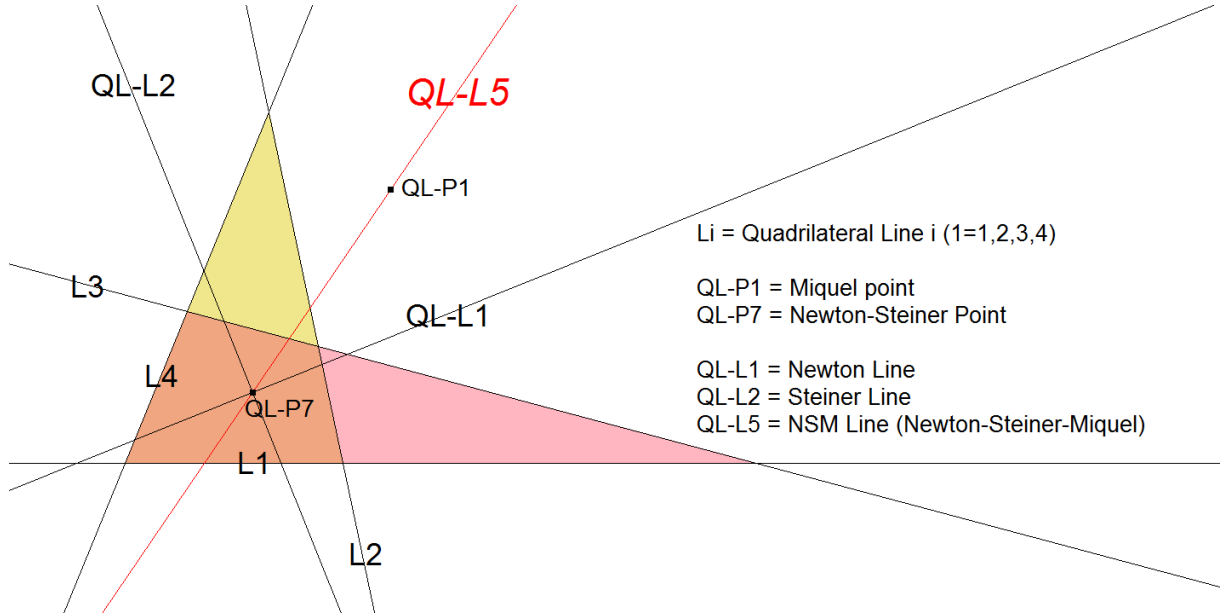
$$\begin{aligned}
 &l^2 (m^2-n^2)^3 SA^2 - m^2 (l^2-n^2)^3 SB^2 + n^2 (l^2-m^2)^3 SC^2 \\
 &+ (l^2-m^2) (l^2-n^2) (m^2-n^2) (2l^2 S^2 - l^2 SB SC - m^2 SC SA - n^2 SA SB)
 \end{aligned}$$

Properties:

- QL-L4 is parallel to QL-L1 (Newton Line).
- QL-L4 is perpendicular to QL-L2 (Steiner Line) and QL-L3 (Pedal Line).
- QL-L4 is parallel to the Trilinear Polar of QL-P24 (note Eckart Schmidt).

QL-L5: NSM Line

The NSM Line is the line through the Newton-Steiner Point (QL-P7) and the Miquel Point (QL-P1)



1st CT-coefficient:

$$l(m-n) \left[(a^2 - b^2) b^2 (l-m)^2 n^2 (3lm - ln - mn) - (a^2 - c^2) c^2 m^2 (l-n)^2 (lm - 3ln + mn) + b^2 c^2 (-2m^3 n^3 + 4lm^2 n^2 (m+n) - 5l^2 mn (m^2 + n^2) + l^3 (m+n) (m^2 + n^2)) \right]$$

1st CT-Coordinate Infinity point:

$$lmn(a^2(2l-m-n) - (b^2 - c^2)(m-n)) - m(l-n)(lm - ln + mn)S_B + n(l-m)(lm - ln - mn)S_C$$

1st DT-coefficient:

$$l^2(l^2 - m^2)(l^2 - n^2)(m^2 - n^2)S^2 + l^2(m^2 - n^2)^3 SA^2 - m^2(-l^2 + n^2)^3 SB^2 - (l^2 - m^2)^3 n^2 SC^2 + (m^2 - n^2)((l^2 + m^2)(l^2 - n^2)^2 SA SB + (l^2 - m^2)^2 (l^2 + n^2) SA SC)$$

1st DT-Coordinate Infinity point:

$$a^2 m^2 n^2 - S_B l^2 m^2 - S_C l^2 n^2$$

Properties:

- QL-P1, QL-P7, QL-P19 lie on QL-L5.
- QL-L5 is parallel to QL-L6 (Quasi Ortholine).
- QL-L9 (M3D Line) is perpendicular to QL-L5.
- QL-L5 // Directrix of the QL-Co3-parabola.

QL-L6: Quasi Ortholine

The Quasi Ortholine originates from the centers from 3 parallelograms that are collinear on the Quasi Ortholine.

These parallelograms are each constructed from the 3 Quadrigons of the Reference Quadrilateral.

Let $P_1.P_2.P_3.P_4$ be a Quadrigon and S be the intersection point of the diagonals:

$$S = P_1.P_3 \wedge P_2.P_4$$

h_{ij} = Orthocenter in Triangle $S.P_i.P_j$ (i and j consecutive nrs in the cycle 1,2,3,4)

Now $h_{12}.h_{23}.h_{34}.h_{41}$ is a parallelogram.

It is remarkable that:

line $h_{41}.h_{12}$ passes through P_1 ,

line $h_{12}.h_{23}$ passes through P_2 ,

line $h_{23}.h_{34}$ passes through P_3 ,

line $h_{34}.h_{41}$ passes through P_4 .

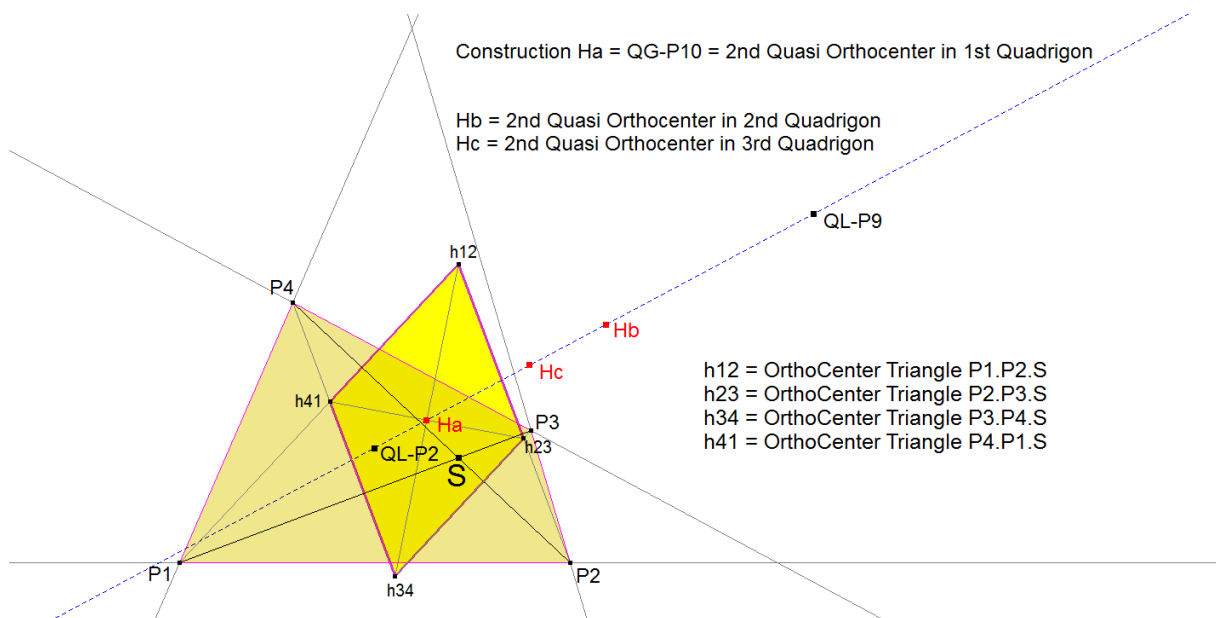
The parallelogram has Center $H_a = QG-P_{10} = 2^{\text{nd}}$ Quasi Orthocenter.

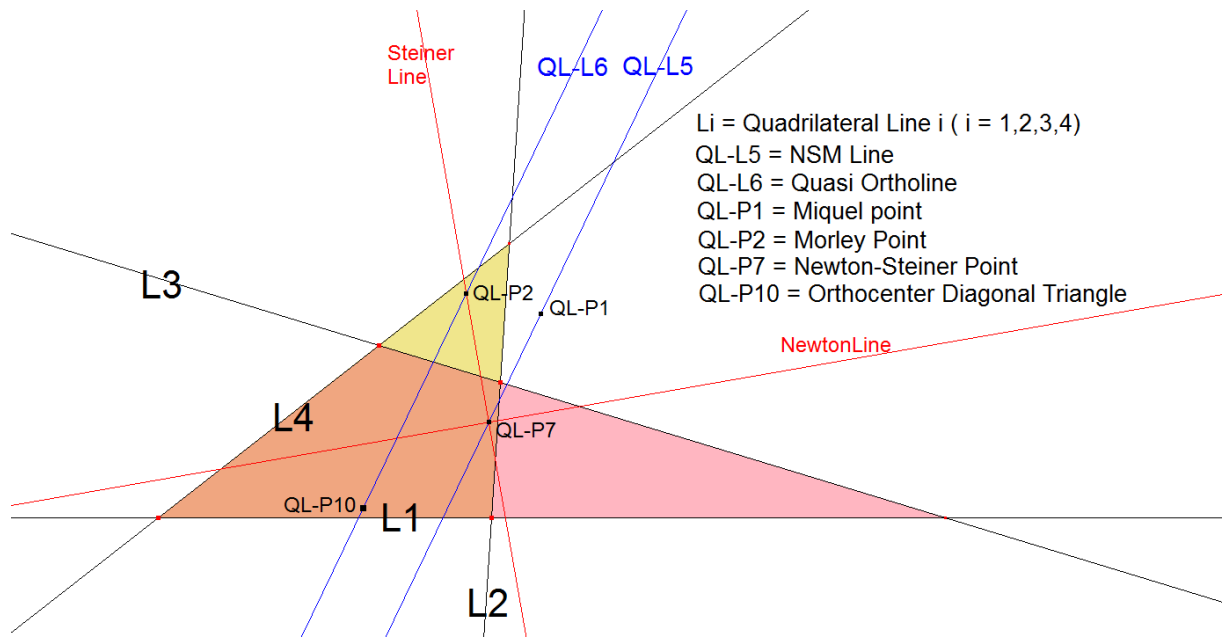
Do this for all 3 Quadrigons in a Quadrilateral.

This gives centers H_a, H_b, H_c . These centers H_a, H_b, H_c are collinear.

The line through H_a, H_b, H_c also passes through $QL-P_2$ (Morley Point) as well as $QL-P_{10}$ (Orthocenter of the QL-Diagonal Triangle) in the Quadrilateral.

The line is called Quasi Ortholine because it passes through the 2^{nd} Quasi Orthocenters of the 3 component Quadrigons of the Reference Quadrangle.





1st CT-coefficient:

$$l m n (b^2 (l - m) - c^2 (l - n)) + l (m - n) (l m + l n - m n) S_A$$

1st CT-Coordinate Infinity point:

$$l m n (a^2 (2 l - m - n) - (b^2 - c^2) (m - n)) - m (l - n) (l m - l n + m n) S_B + n (l - m) (l m - l n - m n) S_C$$

1st CT-Coordinate Infinity point of perpendicular line to QL-L6:

(very simple coordinates)

$$l (m - n) (-3 m n + l m + l n)$$

1st DT-coefficient:

$$l^2 (m^2 - n^2) S_A$$

1st DT-Coordinate Infinity point:

$$m^2 (l^2 - n^2) S_B + n^2 (l^2 - m^2) S_C$$

1st DT-Coordinate Infinity point of perpendicular line to QL-L6:

(very simple coordinates)

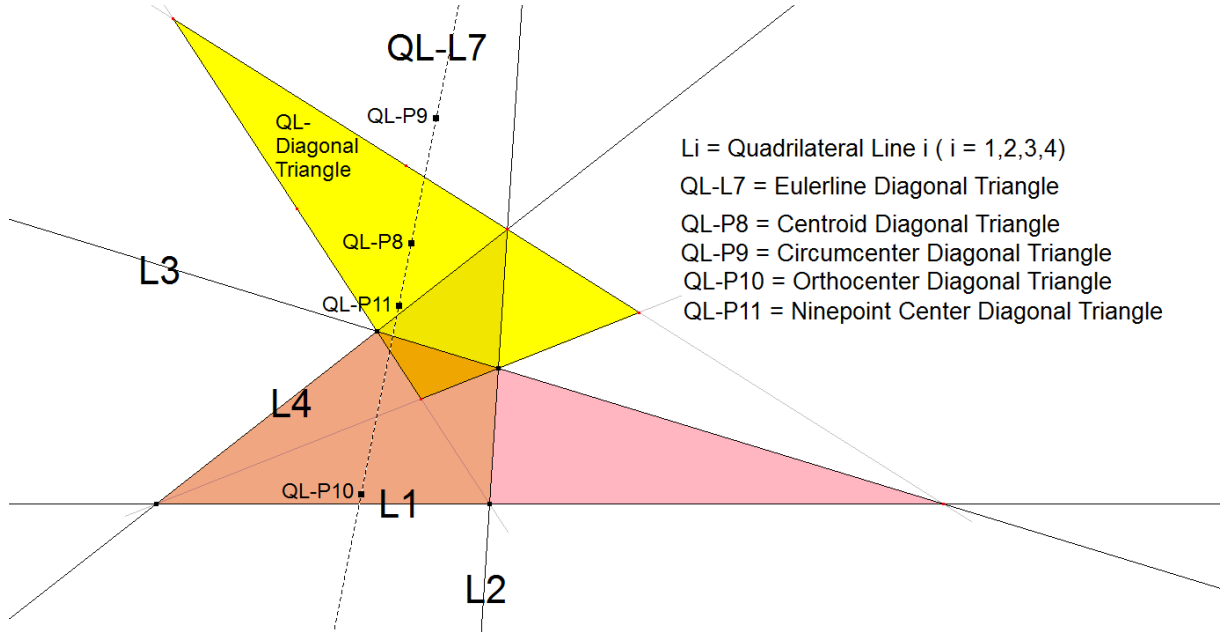
$$l^2 (m^2 - n^2)$$

Properties:

- QL-L6 passes through QL-P2 (Morley point) and QL-P10 (OrthoCenter QL-Diagonal Triangle)
- QL-L6 is parallel to QL-L5 (NSM Line)
- QL-L9 (M3D Line) is perpendicular to QL-L6.
- The circumcenter of the triangle formed by the 3 QL-versions of QG-P13 lies on QL-L6.

QL-L7: Euler Line of QL-Diagonal Triangle

QL-L7 is the Euler Line of the QL-Diagonal Triangle.



1st CT-coefficient:

$$l(m^3 n^3 S_A (n^2 (S_A + S_B) - m^2 (S_A + S_C)) + 3 l^3 (m^2 - n^2) (n S_B + m S_C) (-2 m n S_A + n^2 (S_A + S_B) + m^2 (S_A + S_C)) + l^2 m n (-n^4 (3 S_A^2 + S_A S_B - 2 S_B^2) + 7 m^2 n^2 S_A (S_B - S_C) - 3 m^3 n (2 S_A^2 + S_A (S_B - S_C) + (S_B - S_C) S_C) + m^4 (3 S_A^2 + S_A S_C - 2 S_C^2) + 3 m n^3 (2 S_A^2 + S_A (-S_B + S_C) + S_B (-S_B + S_C))) - l^4 (m - n) (n^3 (S_A (S_B + S_C) + S_B (3 S_B + S_C)) + m^2 n (-S_A (S_B + S_C) + S_C (3 S_B + S_C)) + m n^2 (-S_A (S_B + S_C) + S_B (S_B + 3 S_C)) + m^3 (S_A (S_B + S_C) + S_C (S_B + 3 S_C))) + l m^2 n^2 (n^3 (2 S_A^2 + 3 S_A S_B + S_B^2) - m^3 (2 S_A^2 + 3 S_A S_C + S_C^2) - m n^2 (6 S_A^2 + S_B S_C + S_A (4 S_B + S_C)) + m^2 n (6 S_A^2 + S_B S_C + S_A (S_B + 4 S_C))))$$

1st DT-coefficient:

$$S_A (S_B - S_C)$$

Properties:

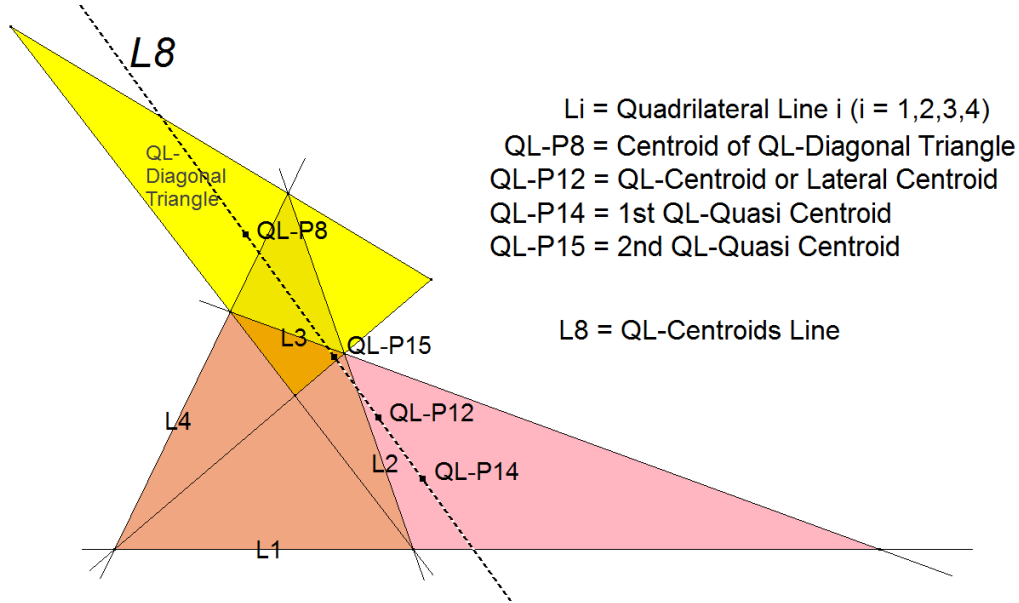
- Quadrilateral points QL-P8 (Centroid DT), QL-P9 (Circumcenter DT), QL-P10 (Orthocenter DT) and QL-P11 (Nine-point Center DT) lie on this line.
- The ratio of distances between QL-P9, QL-P8, QL-P11, QL-P10 in this order is 2 : 1 : 3.

QL-L8: QL-Centroids Line

QL-L8 is the line through 4 Quadrilateral points, all related to Centroids.

These points are:

- QL-P8 = Centroid QL-Diagonal Triangle
- QL-P12 = QL-Centroid or Lateral Centroid
- QL-P14 = 1st QL-Quasi Centroid
- QL-P15 = 2nd QL-Quasi Centroid



1st CT-coefficient:

$$l (6 l^4 m^4 - 3 l^3 m^5 - 10 l^4 m^3 n - 5 l^3 m^4 n + 5 l^2 m^5 n + 8 l^4 m^2 n^2 + 8 l^3 m^3 n^2 - 4 l^2 m^4 n^2 - l m^5 n^2 - 10 l^4 m n^3 + 8 l^3 m^2 n^3 - l m^4 n^3 - m^5 n^3 + 6 l^4 n^4 - 5 l^3 m n^4 - 4 l^2 m^2 n^4 - l m^3 n^4 + 4 m^4 n^4 - 3 l^3 n^5 + 5 l^2 m n^5 - l m^2 n^5 - m^3 n^5)$$

1st DT-coefficient:

$$(l^2 - n^2) (l^2 n^2 - m^4) + (l^2 - m^2) (l^2 m^2 - n^4)$$

Properties:

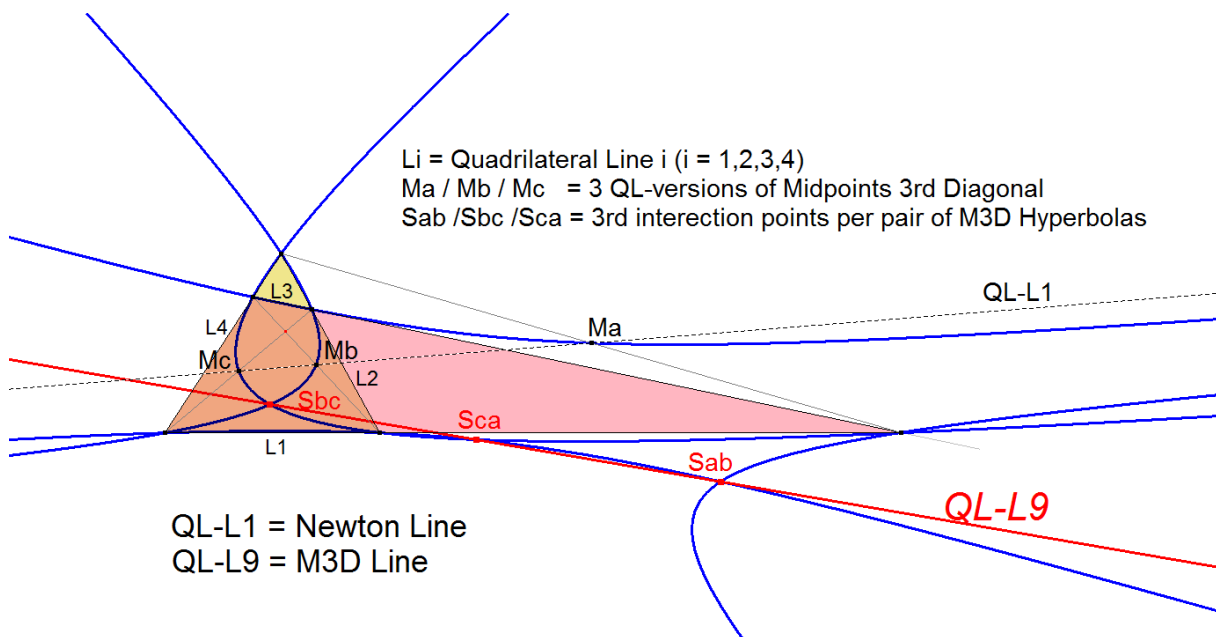
- The ratio of distances between QL-P8, QL-P15, QL-P12, QL-P14 in this order is 2 : 1 : 1.

QL-L9: M3D Line

In the Reference Quadrilateral 3 QL-versions of the M3D Hyperbola QG-Co3 can be constructed. These 3 hyperbolas have pairwise a 3rd intersection point (the 1st and 2nd intersection point are mutual intersection points of L1, L2, L3, L4). These three 3rd intersection points are collinear on a line which is called here the M3D Line. M3D stands for “Midpoint 3rd Diagonal”.

Construction:

QL-L9 is the line through QL-P18 (the Reflection of QL-P8 in QL-P12) perpendicular to QL-L6 (Quasi Ortholine).



1st CT-coefficient:

$$m n (2 l^2 + m n - l m - l n)$$

1st CT-Coordinate Infinity point:

$$l (m - n) (3 m n - l m - l n)$$

1st DT-coefficient:

$$l^2 (m^2 + n^2)$$

1st DT-Coordinate Infinity point:

$$l^2 (m^2 - n^2)$$

Properties:

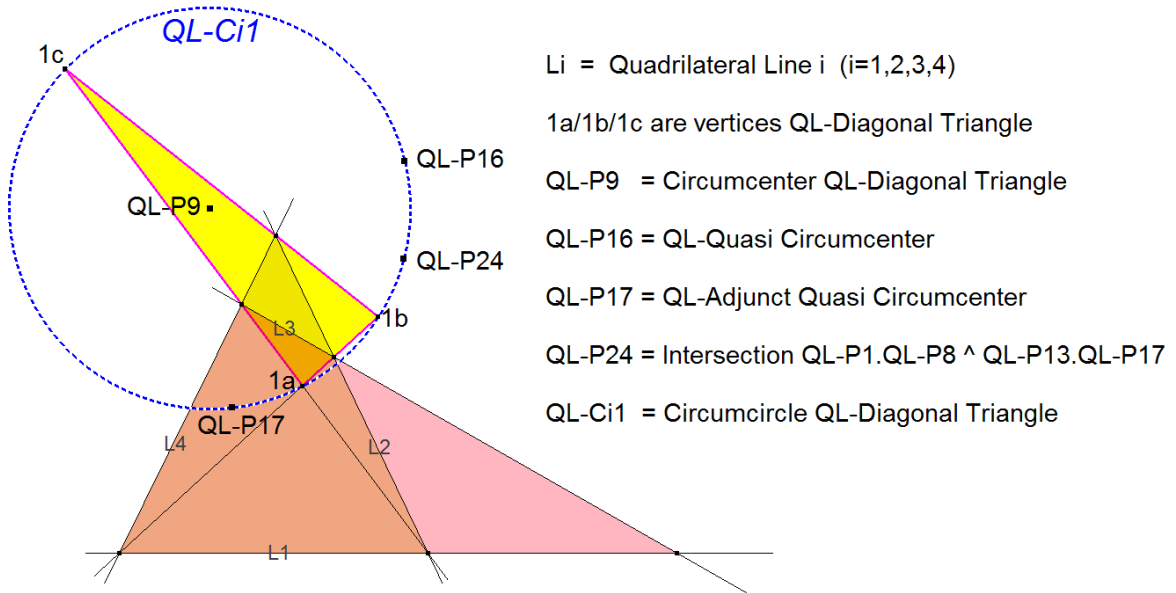
- QL-P18 and QL-P23 lie on this line.
- QL-L9 is perpendicular to the lines QL-L5 (NSM Line) and QL-L6 (Quasi Ortholine).
- The axis of the 2nd QL-Parabola QL-Co3 // QL-L9.
- $d(\text{QL-P8}, \text{QL-L9}) = 2 * d(\text{QL-P12}, \text{QL-L9}) = 4/3 * d(\text{QL-P19}, \text{QL-L9})$

6.4 QUADRILATERAL CONICS

QL-Ci1: QL-Circumcircle Diagonal Triangle

QL-Ci1 is the circumcircle of the QL-Diagonal Triangle (see QL-TR1 and QG-P1).

QL-P9 is its center.



Equation of Circle in CT-notation:

$$\begin{aligned} & l^2 (l m + l n - m n) (b^2 m^2 + c^2 n^2 - 2 S_A m n) x^2 \\ & + m^2 (l m - l n + m n) (c^2 n^2 + a^2 l^2 - 2 S_B l n) y^2 \\ & + n^2 (-l m + l n + m n) (a^2 l^2 + b^2 m^2 - 2 S_C l m) z^2 \\ & + 2 (S_C l^2 m^2 + S_B l^2 n^2 + S_A m^2 n^2) (l m x y + l n x z + m n y z) = 0 \end{aligned}$$

Radius² of Circle in CT-notation:

$$\Delta = \text{Area} = 1/4 \sqrt{[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]}$$

$$\begin{aligned} & (a^2 l^2 - a^2 l m - b^2 l m + c^2 l m + b^2 m^2) \\ & * (a^2 l^2 - a^2 l n + b^2 l n - c^2 l n + c^2 n^2) \\ & * (b^2 m^2 + a^2 m n - b^2 m n - c^2 m n + c^2 n^2) \\ & * l^2 m^2 n^2 / (4 \Delta^2 (l m - l n - m n)^2 (l m + l n - m n)^2 (l m - l n + m n)^2) \end{aligned}$$

Equation of Circle in DT-notation:

$$a^2 y z + b^2 x z + c^2 x y = 0$$

Radius² of Circle in DT-notation:

$$a^2 b^2 c^2 / (4 S^2)$$

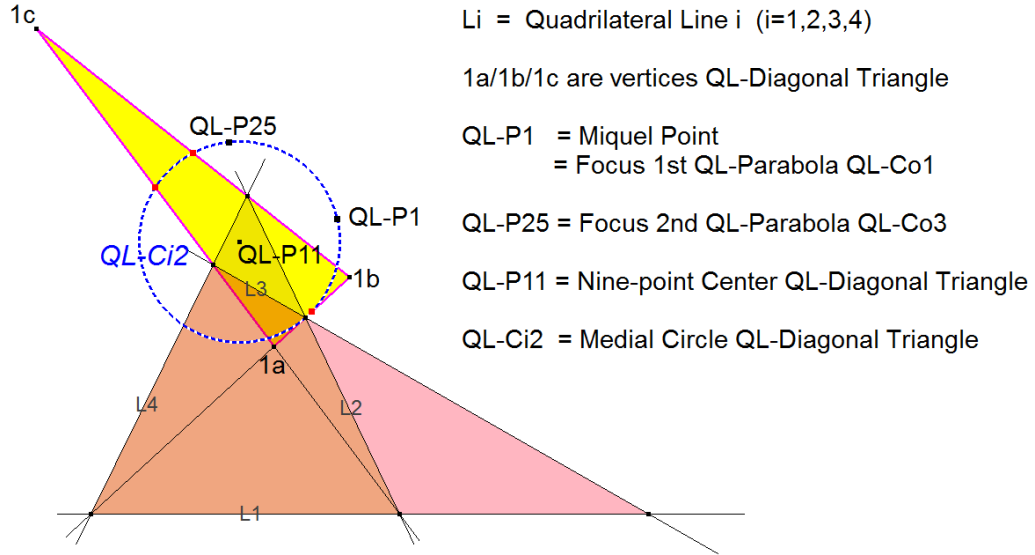
Properties:

- QL-P16 (QL-Quasi Circumcenter) and QL-P17 (QL-Adjunct Quasi Circumcenter) and QL-P24 (Intersection QL-P1.QL-P8 ^ QL-P13.QL-P17) lie on QL-Ci1.
- QL-Ci1 is orthogonal wrt the Plücker Circle (QL-Ci5).

QL-Ci2: QL-Medial Circle Diagonal Triangle

QL-Ci2 is the Medial Circle of the QL-Diagonal Triangle (see paragraph QL-Tr1).

QL-P11 is its center.



Equation in CT-notation:

$$\begin{aligned}
 & l^3 (m - n) (b m - c n) (b m + c n) x^2 \\
 & + m^3 (l - n) (a l - c n) (a l + c n) y^2 \\
 & + n^3 (l - m) (a l - b m) (a l + b m) z^2 \\
 & + l m ((a^2 + b^2 - c^2) l^2 m^2 + (c^2 - b^2) l^2 m n + (c^2 - a^2) l m^2 n - 2 c^2 l m n^2 + c^2 l n^3 + c^2 m n^3) x y \\
 & + l n ((a^2 - b^2 + c^2) l^2 n^2 + (b^2 - a^2) l m n^2 + (b^2 - c^2) l^2 m n - 2 b^2 l m^2 n + b^2 m^3 n + b^2 l m^3) x z \\
 & + m n ((-a^2 + b^2 + c^2) m^2 n^2 + (a^2 - c^2) l m^2 n + (a^2 - b^2) l m n^2 - 2 a^2 l^2 m n + a^2 l^3 m + a^2 l^3 n) y z \\
 & = 0
 \end{aligned}$$

Radius² of Circle in CT-notation:

$$\Delta = \text{Area} = 1/4 \sqrt{[(a + b + c) (-a + b + c) (a - b + c) (a + b - c)]}$$

$$\begin{aligned}
 & (a^2 l^2 - a^2 l m - b^2 l m + c^2 l m + b^2 m^2) \\
 & * (a^2 l^2 - a^2 l n + b^2 l n - c^2 l n + c^2 n^2) \\
 & * (b^2 m^2 + a^2 m n - b^2 m n - c^2 m n + c^2 n^2) \\
 & * l^2 m^2 n^2 / (16 \Delta^2 (l m - l n - m n)^2 (l m + l n - m n)^2 (l m - l n + m n)^2)
 \end{aligned}$$

Equation of Circle in DT-notation:

$$SA x (-x+y+z) + SB y (x-y+z) + SC (x+y-z) z = 0$$

Radius² of Circle in DT-notation:

$$a^2 b^2 c^2 / (16 S^2)$$

Properties:

- QL-P1 = Miquel Point = Focus 1st QL-Parabola lies on QL-Ci2.
- QL-P25 = Focus 2nd QL-Parabola lies on QL-Ci2.

QL-Ci3: Miquel Circle

The first two statements of Steiner about Quadrilaterals (published in 1828) are:
Suppose four lines intersect two by two at six points.

(1) These four lines, taken three by three, form four triangles whose circumcircles pass through the same point F.

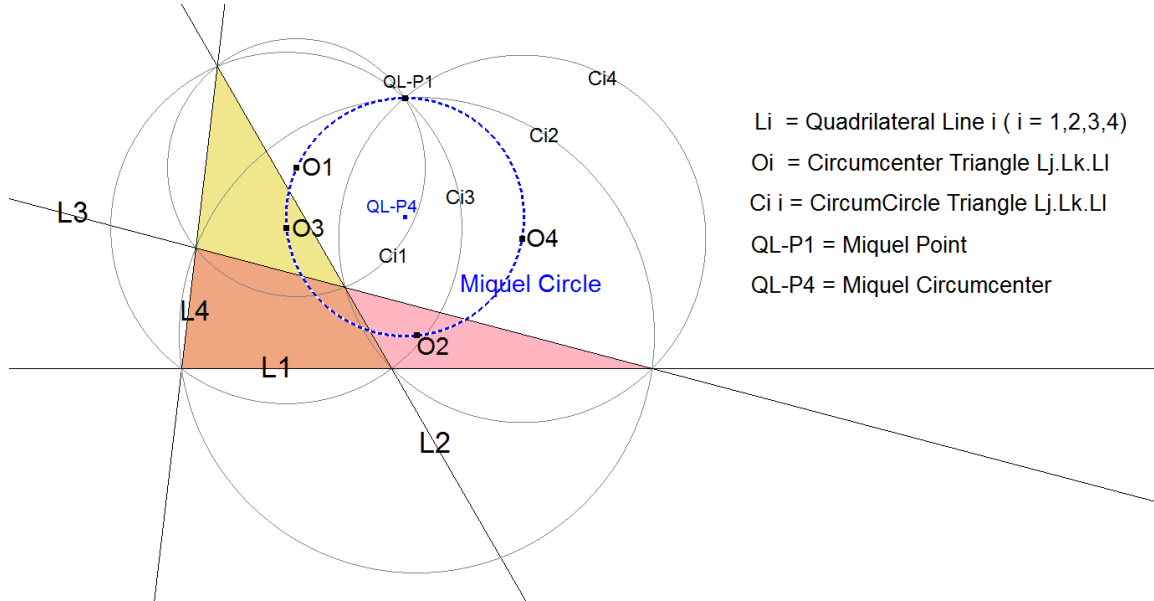
(2) The centers of the four circles (and the point F) lie on the same circle.

(see Jean-Pierre Ehrmann [4] page 35).

This circle is called the circumcentric circle by J.W. Clawson. See [22] page 250.

He appoints 47 points on this circle.

This circle is named here the Miquel Circle. QL-P4 is its center.



Equation in CT-notation:

$$(x + y + z) (b^2 c^2 l (m - n) T_A x + c^2 a^2 m (n - l) T_B y + a^2 b^2 n (l - m) T_C z) - T_{AL} (a^2 y z + b^2 z x + c^2 x y) = 0$$

where:

$$T_A = -2 a^2 l + 2 S_C m + 2 S_B n$$

$$T_B = -2 b^2 m + 2 S_A n + 2 S_C l$$

$$T_C = -2 c^2 n + 2 S_B l + 2 S_A m$$

$$T_{AL} = (a - b - c) (a + b - c) (a - b + c) (a + b + c) (l - m) (l - n) (m - n)$$

Radius² of Circle in CT-notation: $\Delta = \text{Area} = 1/4 \sqrt{[(a + b + c) (-a + b + c) (a - b + c) (a + b - c)]}$

$$(a^2 (l - m) (l - n) + b^2 (m - l) (m - n) + c^2 (n - l) (n - m))$$

$$* (a^2 m n (l - m) (l - n) + b^2 l n (m - l) (m - n) + c^2 l m (n - l) (n - m))$$

$$* a^2 b^2 c^2 / (256 \Delta^4 (l - m)^2 (l - n)^2 (m - n)^2)$$

Properties:

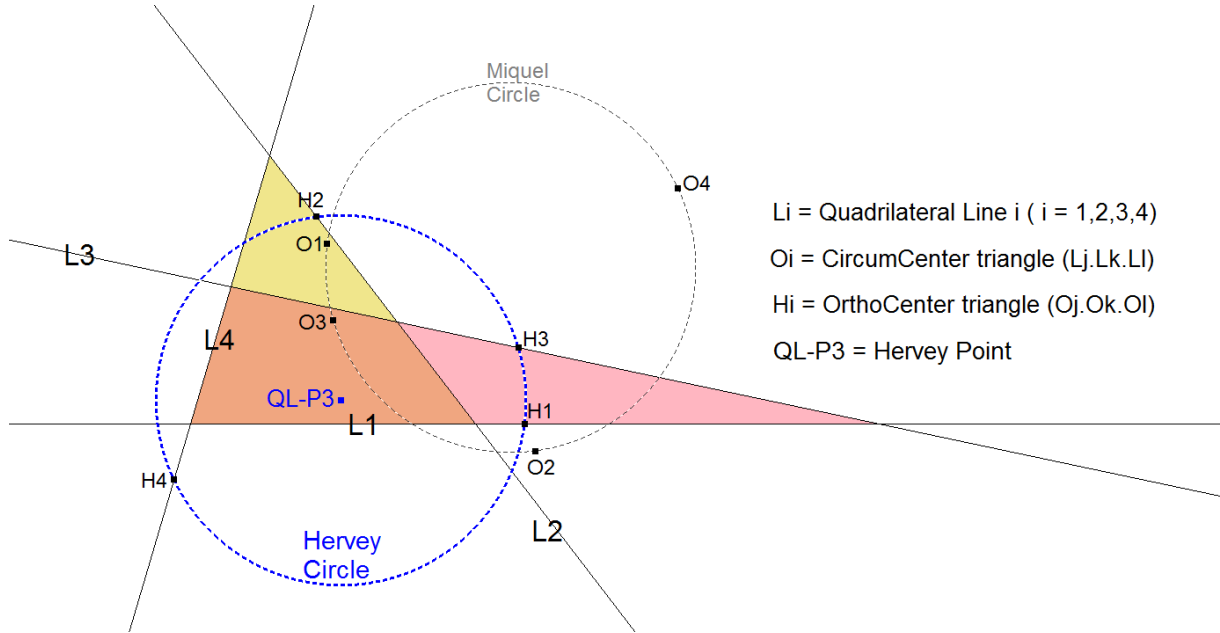
- QL-Ci3 plays a role in QA-P9 (Miquel Center).
- QL-Ci3 has the same radius as the Hervey Circle (QL-Ci4).
- QL-Ci3 is the Reflection of QL-Ci4 in QL-P5.
- QL-Ci3 is the Clawson-Schmidt Conjugate (QL-Tf1) of the Steiner Line (QL-L2).
 As a consequence the Clawson-Schmidt Conjugates of QL-P2, QL-P7 and QL-P9 lie on QL-Ci3.

QL-Ci4: Hervey Circle

Let H_i ($i=1,2,3,4$) be the Orthocenters of the 4 component triangles of the Quadrangle formed by the Circumcenters of the 4 component triangles of the Reference Quadrilateral.

These 4 Orthocenters lie on one of the 4 defining Lines of the Reference Quadrilateral and are concyclic. The circle through these points is the Hervey Circle.

QL-P3 (Hervey Point) is its center.



Equation in CT-notation:

$$2(x + y + z)(A_M A_N S_A x + B_L B_N S_B y + C_M C_L S_C z) - T_{AL}^2(c^2 xy + b^2 xz + a^2 yz) = 0$$

where:

$$A_M = +a^4 m(l-m)(l-n) + b^4 l(l-m)(m-n) + c^4 l(l-n)(m-n) - b^2 c^2(m-n)l(2l-m-n) + a^2 c^2(l-n)(m^2+ln-2mn) - a^2 b^2(l-m)(2lm-ln-mn)$$

$$A_N = -a^4 n(l-m)(l-n) + b^4 l(l-m)(m-n) + c^4 l(l-n)(m-n) - b^2 c^2(m-n)l(2l-m-n) - a^2 c^2(l-n)(l-m-2ln+mn) - a^2 b^2(l-m)(lm-2mn+n^2)$$

$$B_L = -a^4 m(l-m)(l-n) - b^4 l(l-m)(m-n) + c^4 m(l-n)(m-n) + b^2 c^2(m-n)(l^2-2ln+mn) + a^2 c^2(l-n)m(l-2m+n) + a^2 b^2(l-m)(2lm-ln-mn)$$

$$B_N = -a^4 m(l-m)(l-n) + b^4 n(l-m)(m-n) + c^4 m(l-n)(m-n) - b^2 c^2(m-n)(lm+ln-2mn) + a^2 c^2(l-n)m(l-2m+n) + a^2 b^2(l-m)(lm-2ln+n^2)$$

$$C_M = +a^4 n(l-m)(l-n) + b^4 n(l-m)(m-n) + c^4 m(l-n)(m-n) - b^2 c^2(m-n)(lm+ln-2mn) + a^2 c^2(l-n)(2lm-m^2-ln) - a^2 b^2(l-m)n(l+m-2n)$$

$$C_L = +a^4 n(l-m)(l-n) + b^4 n(l-m)(m-n) - c^4 l(l-n)(m-n) + b^2 c^2(m-n)(l^2-2lm+mn) + a^2 c^2(l-n)(lm-2ln+mn) - a^2 b^2(l-m)n(l+m-2n)$$

$$T_{AL} = (a - b - c)(a + b - c)(a - b + c)(a + b + c)(l - m)(l - n)(m - n)$$

Radius² of Circle in CT-notation:

$$\Delta = \text{Area} = 1/4 \sqrt{[(a + b + c)(-a + b + c)(a - b + c)(a + b - c)]}$$

$$(a^2(l-m)(l-n) + b^2(m-l)(m-n) + c^2(n-l)(n-m))$$

$$* (a^2 m n(l-m)(l-n) + b^2 l n(m-l)(m-n) + c^2 l m(n-l)(n-m))$$

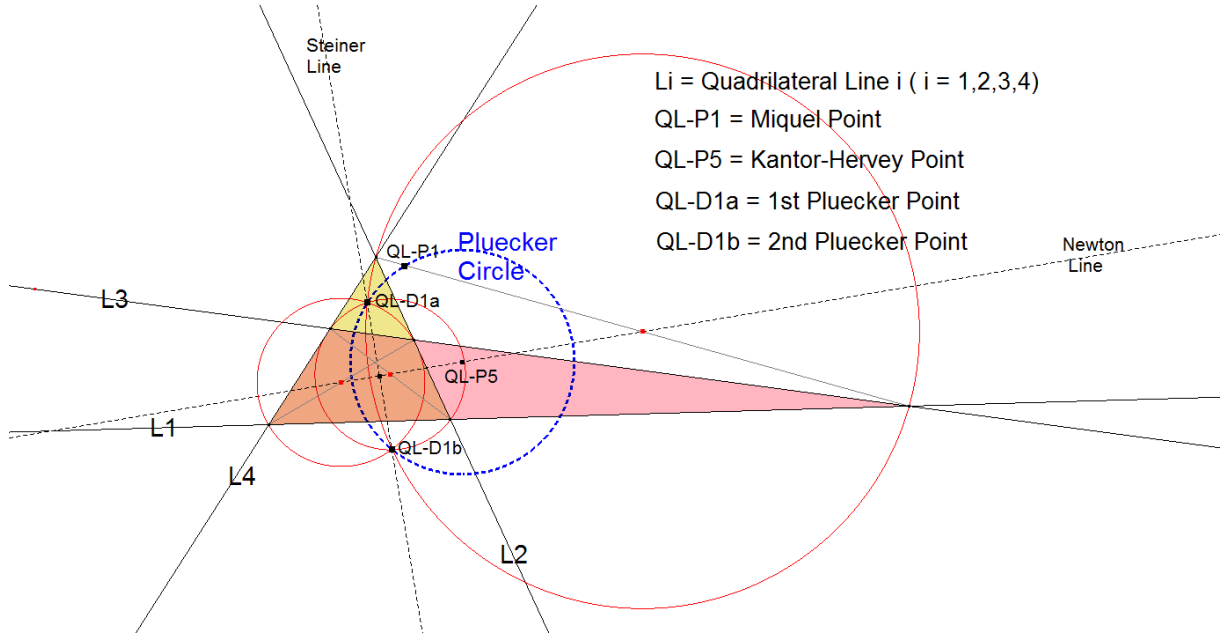
$$* a^2 b^2 c^2 / (256 \Delta^4 (l-m)^2 (l-n)^2 (m-n)^2)$$

Properties:

- The Hervey Circle has the same radius as the Miquel Circle (QL-Ci3).
- The Hervey Circle is the Reflection of QL-Ci3 in QL-P5.

QL-Ci5: Plücker Circle

The Plücker Circle is the circle through the Plücker Points (QL-2P1a and QL-2P1b) and the Miquel Point (QL-P1). Its center is the Kantor-Hervy Point (QL-P5).



Equation in CT-notation:

$$c^2 x y + b^2 x z + a^2 y z + (x + y + z) ((R - T_1/T_4) x + (R - T_2/T_4) y + (R - T_3/T_4) z) = 0$$

where:

$$T_1 = + T_M^2 c^2 (1 - n)^2 + T_N^2 b^2 (1 - m)^2 - 2 T_M T_N S_A (1 - m) (1 - n)$$

$$T_2 = + T_N^2 a^2 (m - 1)^2 + T_L^2 c^2 (m - n)^2 - 2 T_N T_L S_B (m - n) (m - 1)$$

$$T_3 = + T_L^2 b^2 (n - m)^2 + T_M^2 a^2 (n - 1)^2 - 2 T_L T_M S_C (n - 1) (n - m)$$

$$T_4 = 1024 \Delta^4 (1 - m)^2 (m - n)^2 (n - 1)^2$$

$$R = R_1^3 R_2^2 - 8 R_1 R_2 R_3 \Delta^2 - 64 R_1 R_4 \Delta^4 \quad (R = \text{radius}^2 \text{ Plücker Circle})$$

$$R_1 = a^2 m n (1 - m) (1 - n) + b^2 n l (m - 1) (m - n) + c^2 l m (n - 1) (n - m)$$

$$R_2 = a^2 l S_A + b^2 m S_B + c^2 n S_C$$

$$R_3 = S_L (2 S_A (a^2 m^2 n^2 + b^2 l^2 n^2 + c^2 l^2 m^2) + 4 l m n (b^2 S_B n + c^2 S_C m - b^2 c^2 l) - 16 \Delta^2 m^2 n^2) \\ + S_M (2 S_B (a^2 m^2 n^2 + b^2 l^2 n^2 + c^2 l^2 m^2) + 4 l m n (c^2 S_C l + a^2 S_A n - a^2 c^2 m) - 16 \Delta^2 l^2 n^2) \\ + S_N (2 S_C (a^2 m^2 n^2 + b^2 l^2 n^2 + c^2 l^2 m^2) + 4 l m n (a^2 S_A m + b^2 S_B l - a^2 b^2 n) - 16 \Delta^2 l^2 m^2)$$

$$R_4 = -b^2 c^2 S_L^2 - c^2 a^2 S_M^2 - a^2 b^2 S_N^2 + 2 a^2 S_A S_N S_M + 2 b^2 S_B S_N S_L + 2 c^2 S_C S_L S_M$$

and:

$$\Delta = \text{Area} = 1/4 \sqrt{[(a + b + c) (-a + b + c) (a - b + c) (a + b - c)]}$$

$$S_A = (-a^2 + b^2 + c^2) / 2$$

$$S_B = (+a^2 - b^2 + c^2) / 2$$

$$S_C = (+a^2 + b^2 - c^2) / 2$$

$$S_L = l (m - n)^2 (-mn + lm + ln)$$

$$S_M = m (n - 1)^2 (mn + lm - ln)$$

$$S_N = n (1 - m)^2 (mn + lm + ln)$$

$$T_L = -2 l R_2 + 16 \Delta^2 (-mn + lm + ln) \quad T_M = -2 m R_2 + 16 \Delta^2 (mn + lm - ln) \quad T_N = -2 n R_2 + 16 \Delta^2 (mn - lm + ln)$$

Properties:

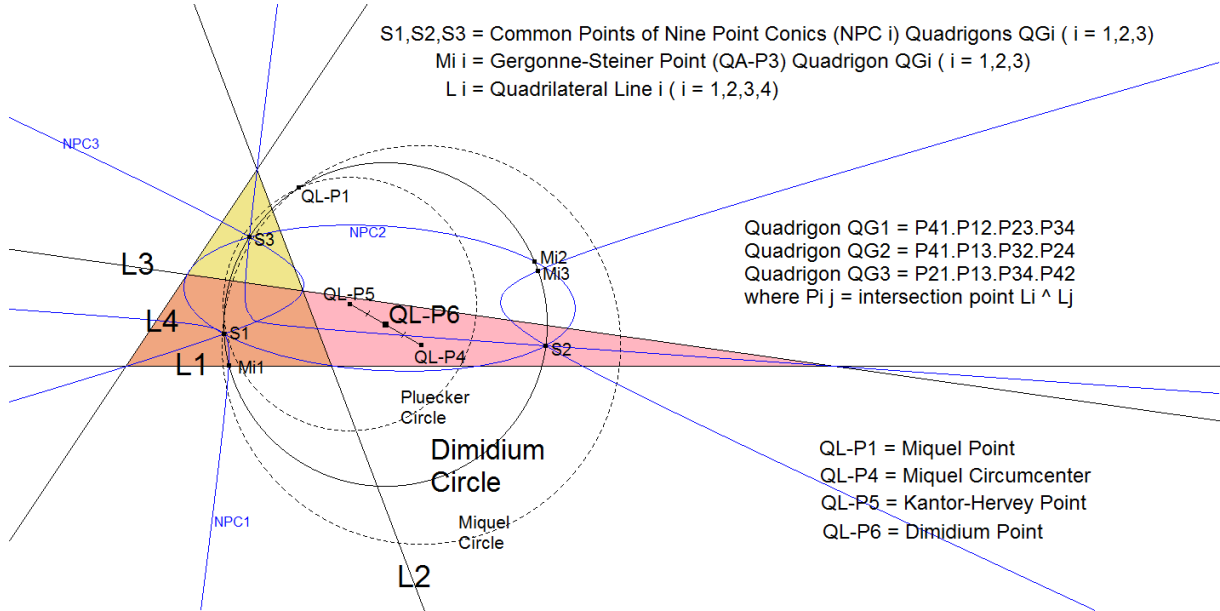
- QL-Ci5 is orthogonal wrt the QL-Diagonal Triangle Circumcircle (QL-Ci1), just like all 3 Plücker Diagonal Circles are.

QL-Ci6: Dimidium Circle

The Dimidium Circle is the circle through the Gergonne-Steiner Points (QA-P3) of the 3 component Quadrilaterals of the Reference Quadrilateral. Its center is QL-P6.

The word "Dimidium" is the Latin word for "half".

Somehow of all Quadrilateral Circles this circle has the simplest algebraic CT-equation.



Equation in CT-notation:

$$\begin{aligned}
 & l(m-n)(b^2(l-m) + c^2(l-n))x^2 \\
 & + m(n-l)(c^2(m-n) + a^2(m-l))y^2 \\
 & + n(l-m)(a^2(n-l) + b^2(n-m))z^2 \\
 & - (m-n)(3a^2(l-m)(l-n) + b^2n(l-m) + c^2m(l-n))yz \\
 & - (n-l)(3b^2(m-l)(m-n) + c^2l(m-n) + a^2n(m-l))zx \\
 & - (l-m)(3c^2(n-m)(n-l) + a^2m(n-l) + b^2l(n-m))xy = 0
 \end{aligned}$$

Equation in DT-notation:

$$\begin{aligned}
 & (m^2 - n^2)(b^2(l^2 - m^2) + c^2(l^2 - n^2))l^2x^2 \\
 & + (m^2 - n^2)(-a^2l^4 + (a^2 - c^2)l^2m^2 + (a^2 - b^2)l^2n^2 + 2SA m^2n^2)yz \\
 & + (n^2 - l^2)(c^2(m^2 - n^2) + a^2(m^2 - l^2))m^2y^2 \\
 & + (n^2 - l^2)(-b^2m^4 + (b^2 - a^2)m^2n^2 + (b^2 - c^2)l^2m^2 + 2SB l^2n^2)xz \\
 & + (l^2 - m^2)(a^2(n^2 - l^2) + b^2(n^2 - m^2))n^2z^2 \\
 & + (l^2 - m^2)(-c^2n^4 + (c^2 - b^2)l^2n^2 + (c^2 - a^2)m^2n^2 + 2SC l^2m^2)xy = 0
 \end{aligned}$$

Properties:

- QL-Ci6 passes through QL-P1 (the Miquel Point).
- QL-Ci6 passes also through QL-P17 (QL-Adjunct Quasi Circumcenter) and QL-P24 (Intersection QL-P1.QL-P8 ^ QL-P13.QL-P17) which are the intersection points of QL-Ci6 with QL-Ci1 (Circumcircle of the QL-Diagonal Triangle).

- The Clawson-Schmidt Conjugate of QL-P26 lies on QL-Ci6. It is the 2nd intersection point of QL-P1.QL-P13 with circle QL-Ci6.
- The center of QL-Ci6 is QL-P6 (the Dimidium Point). This is the Midpoint of QL-P4 (Miquel Circumcenter) and QL-P5 (Kantor-Hervey Point).
- The intersection points of the Nine-point Conics of the 3 component Quadrilaterals of the Reference Quadrilateral have 3 common points: S1, S2, S3. These points lie on the Dimidium Circle.
- The Dimidium Circle (QL-Ci6) lies exactly between the Plücker Circle (QL-Ci5) and the Miquel Circle (QL-Ci3). This is a set of 3 coaxial circles. One of their common points is QL-P1 (Miquel Point).
- The 3 QA-versions of QL-Ci6 meet in QA-P3 (Gergonne-Steiner Point) (note Eckart Schmidt).

QL-Co/1: Inscribed Quadrilateral Conics

5th Line Conics

By adding an extra line to the set of 4 basic lines of a Quadrilateral we get a configuration of 5 lines. Let L5 (u:v:w) be a random 5th line. Five lines do define a unique (inscribed) conic just like 5 points do define a unique (circumscribed) conic.

Equation Conic in CT-notation:

$$T_x^2 x^2 + T_y^2 y^2 + T_z^2 z^2 + 2 T_y T_z y z + 2 T_z T_x x z + 2 T_x T_y x y = 0$$

where:

$$T_x = l u (m w - n v)$$

$$T_y = m v (n u - l w)$$

$$T_z = n w (l v - m u)$$

CT-Coordinates Center:

$$(T_y + T_z : T_x + T_z : T_x + T_y)$$

CT-Coefficients Asymptotes:

$$\begin{aligned} \text{Asy-1 } & (T_x (T_x + T_y) (T_x + T_y + T_z) - (T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_y(T_x + T_y) (T_x + T_y + T_z) - (T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_z(T_x - T_y) (T_x + T_y + T_z) + (T_x + T_y + 2 T_z) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]}) \\ \text{Asy-2 } & (T_x (T_x + T_y) (T_x + T_y + T_z) + (T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_y(T_x + T_y) (T_x + T_y + T_z) + (T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_z(T_x - T_y) (T_x + T_y + T_z) - (T_x + T_y + 2 T_z) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]}) \end{aligned}$$

Equation Conic in DT-notation:

$$T_y T_z x^2 + T_z T_x y^2 + T_x T_y z^2 = 0$$

where:

$$T_x = n^2 v^2 - m^2 w^2 \quad T_y = l^2 w^2 - n^2 u^2 \quad T_z = m^2 u^2 - l^2 v^2$$

DT-Coordinates Center:

$$(T_x : T_y : T_z)$$

DT-Coefficients Asymptotes:

$$\begin{aligned} \text{Asy-1 } & (T_z (T_y^2 + T_x T_y + T_y T_z - \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_z (T_x^2 + T_x T_y + T_x T_z + \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & (T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]}) \\ \text{Asy-2 } & (T_z (T_y^2 + T_x T_y + T_y T_z + \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -T_z (T_x^2 + T_x T_y + T_x T_z) - \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]} : \\ & -(T_x + T_y) \sqrt{[-T_x T_y T_z (T_x + T_y + T_z)]}) \end{aligned}$$

Properties:

- QL-Co1 (Inscribed Parabola) and QL-Co2 (Inscribed Midline Hyperbola) are both examples of 5th Line Conics.
- The Center of a 5th Line Conic is always a point on the Newton Line QL-L1.

Center constructed inscribed Conics

It is also possible to define an inscribed Quadrilateral Conic by the 4 basic lines of the Reference Quadrilateral and the Center of the conic. This appointed center should be a point on the Newton Line (see properties 5th Line Conics).

Let $N = (d : e : f)$ be a point on the Newton Line and be appointed as the center of an inscribed quadrilateral conic.

Equation Conic in CT-notation:

$$\begin{aligned} & (-d+e+f)^2 x^2 - 2 (d+e-f) (d-e+f) y z \\ & + (d+e+f)^2 y^2 - 2 (-d+e+f)(d+e-f) x z \\ & + (d+e-f)^2 z^2 - 2 (-d+e+f)(d-e+f) x y = 0 \end{aligned}$$

4 Points of tangency in CT-notation:

- $(0 : d+e-f : d-e+f)$
- $(d+e-f : 0 : -d+e+f)$
- $(d-e+f : -d+e+f : 0)$
- $((d+e-f)^2 m^2 + 2 (d-e+f) f n^2 + 2 (d+e-f) (d-e+f) m n : (d+e-f)^2 l^2 + 2 (-d+e+f) f n^2 + 2 (-d+e+f) (d+e-f) l n : (d+e-f) ((d-e+f) l^2 + (-d+e+f) m^2))$

Equation Conic in DT-notation:

$$e f x^2 + d f y^2 + d e z^2 = 0$$

4 Points of tangency in DT-notation:

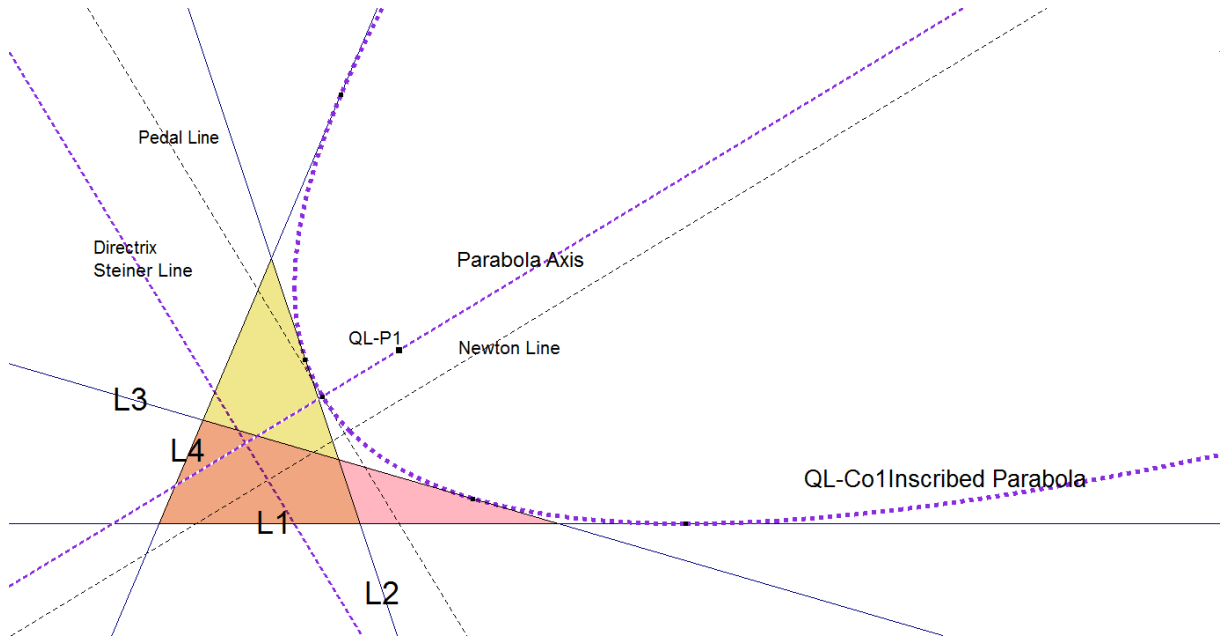
- $(-dl : em : fn)$
- $(dl : -em : fn)$
- $(dl : em : -fn)$
- $(dl : em : fn)$

Properties:

- When $N = QL-L1 \wedge QL-L6$ then one of the axes of the Center Conic coincides with the Newton Line (note Eckart Schmidt).

QL-Co1: Inscribed Parabola

The inscribed Parabola is the unique parabola that can be inscribed within a Quadrilateral.



Equation CT-notation:

$$\begin{aligned} & l^2 (m - n)^2 x^2 - 2 m n (l - m)(n - l) y z \\ & + m^2 (l - n)^2 y^2 - 2 n l (l - m)(m - n) x z \\ & + n^2 (l - m)^2 z^2 - 2 l m (n - l)(m - n) x y = 0 \end{aligned}$$

1st CT-coefficient Axis Parabola:

$$l (m - n) (c l m + b l n - b m n - c m n) (c l m - b l n + b m n - c m n)$$

Infinity point Axis CT-notation:

$$(l (m - n) : m (n - l) : n (l - m))$$

Points of tangency with L1, L2, L3, L4 in CT-notation:

$$\begin{aligned} & (0 : n (l - m) : m (n - l)) \\ & (n (l - m) : 0 : l (m - n)) \\ & (m (n - l) : l (m - n) : 0) \\ & (m n (m - n) : n l (n - l) : l m (l - m)) \end{aligned}$$

Equation DT-notation:

$$x^2 / (m^2 - n^2) + y^2 / (-l^2 + n^2) + z^2 / (l^2 - m^2) = 0$$

1st DT-coefficient Axis Parabola:

$$(l^2 - m^2) (l^2 - n^2) ((l^2 - n^2) SB + (l^2 - m^2) SC)$$

Infinity point Axis DT-notation:

$$(m^2 - n^2 : -l^2 + n^2 : l^2 - m^2)$$

Points of tangency with L1, L2, L3, L4 in DT-notation:

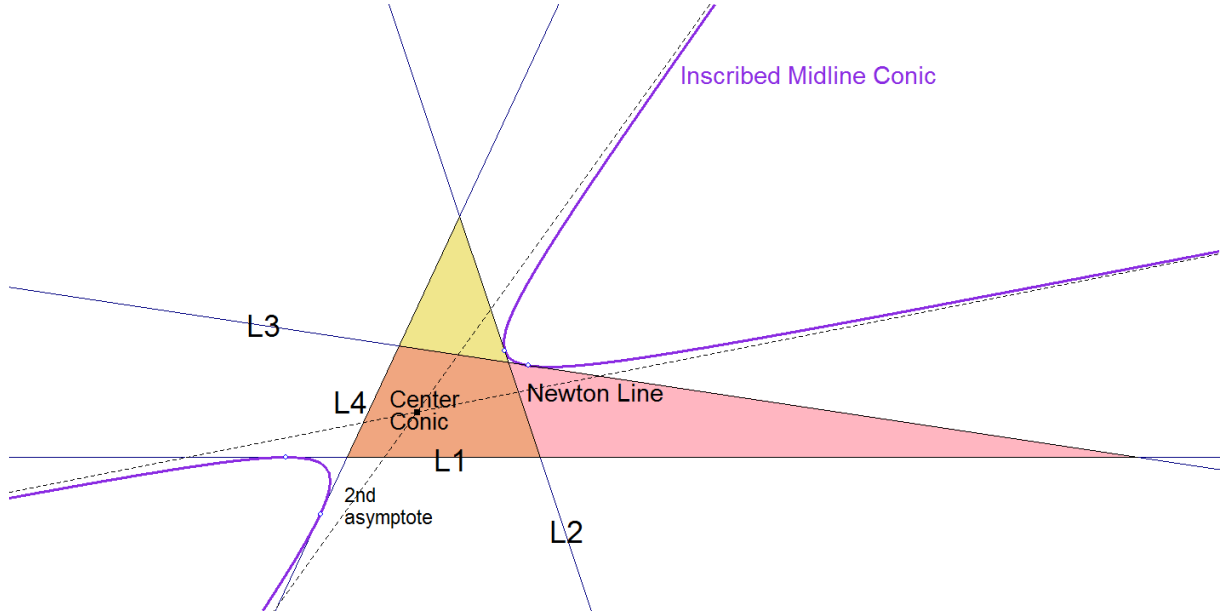
$$\begin{aligned} & (-l (m^2 - n^2) : m (l^2 - n^2) : (l^2 - m^2) n) \\ & (l (m^2 - n^2) : -m (l^2 - n^2) : (l^2 - m^2) n) \\ & (l (m^2 - n^2) : m (l^2 - n^2) : -(l^2 - m^2) n) \\ & (l (m^2 - n^2) : m (l^2 - n^2) : (l^2 - m^2) n) \end{aligned}$$

Properties:

- QL-Co1 is the only parabola inscribed in the quadrilateral (see [4] page 51).
- QL-Co1 is the 5th Line Conic of QL-L3 (see QL-Co/1).
- The Focus is QL-P1 the Miquel Point.
- The Directrix is QL-L2 the Steiner Line.
- The Axis of QL-Co1 is parallel to QL-L1, the Newton Line.
- QL-Co1 is also an inscribed parabola of the QL-Medial Triangle (the medial triangle of the QL-Diagonal Triangle).
- The Nine-point Conics of the 3 quadrigons of a quadrilateral have 3 common points: N1, N2, N3. The Inscribed Parabola of the quadrilateral is also an inscribed parabola of triangle N1.N2.N3.
- Let T1, T2, T3, T4 be the points of tangency of QA-Co1 with the basic lines of the Reference Quadrilateral. Now the QL-Diagonal Triangle of the Reference Quadrilateral and the QA-Diagonal Triangle of the Quadrangle T1.T2.T3.T4 are identical.

QL-Co2: Inscribed Midline Hyperbola

The Inscribed Midline Hyperbola is the conic tangent at the 4 basic lines of the Reference Quadrilateral and the Newton Line (or Midline). It is remarkable that the point of tangency at the Newton Line is at its infinity point. That makes the Newton Line the asymptote of this conic and this makes this conic a hyperbola.



Equation Conic CT-notation:

$$MN^2 x^2 + NL^2 y^2 + LM^2 z^2 - 2 MN NL xy - 2 LM NL yz - 2 LM MN zx = 0$$

Coordinates Center of Conic in CT-notation:

- $(LM + NL : LM + MN : MN + NL)$ (1st presentation)
- $(l(m-n)LL : m(n-l)MM : n(l-m)NN)$ (2nd presentation)

Coefficients 2nd asymptote in CT-notation:

$$(lm + ln - mn)/LL : (lm - ln + mn)/MM : (-lm + ln + mn)/NN$$

where:

$$\begin{aligned} MN &= l(m-n)(lm + ln - mn)^2 \\ NL &= m(n-l)(lm - ln + mn)^2 \\ LM &= n(l-m)(lm - ln - mn)^2 \\ LL &= (lm + ln + mn)^2 - 4mn(l^2 + m^2) \\ MM &= (lm + ln + mn)^2 - 4nl(m^2 + n^2) \\ NN &= (lm + ln + mn)^2 - 4lm(n^2 + l^2) \end{aligned}$$

Equation Conic DT-notation:

$$l^2 x^2 / (m^2 - n^2) + m^2 y^2 / (n^2 - l^2) + n^2 z^2 / (l^2 - m^2) = 0$$

Coordinates Center of Conic in DT-notation:

$$(m^2 n^2 (m^2 - n^2) : n^2 l^2 (n^2 - l^2) : l^2 m^2 (l^2 - m^2))$$

Coefficients 2nd asymptote in DT-notation:

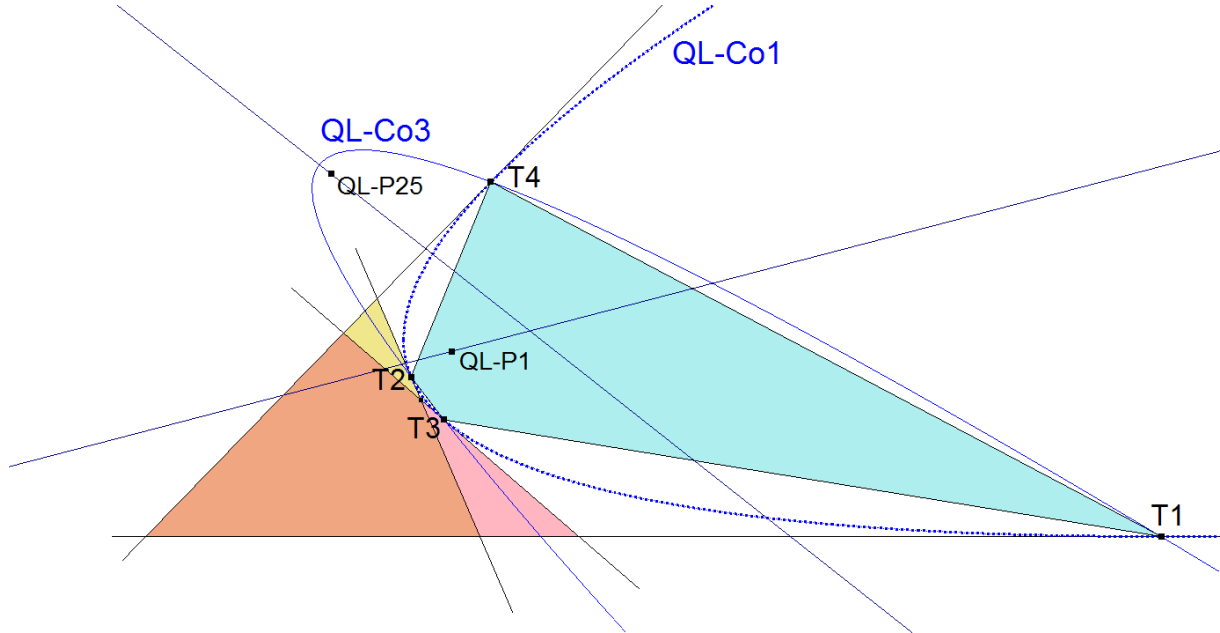
$$(l^2 (-l^2 + m^2 + n^2) : m^2 (l^2 - m^2 + n^2) : n^2 (l^2 + m^2 - n^2))$$

Properties:

- QL-Co2 is the 5th Line Conic of QL-L1 (see QL-Co/1), which is an asymptote.
- The Center of the conic is QL-P23. This is a point on the Newton Line QL-L1, like all centers of inscribed Quadrilateral Conics are.
- Algebraically this conic is independent of (a,b,c) just like the Newton Line.

QL-Co3: 2nd QL-Parabola

The points of tangency of the inscribed QL-Parabola QL-Co1 form a Quadrangle T1.T2.T3.T4. This Quadrangle has 2 circumscribed parabola's where QL-Co1 is one of them. The other circumscribed parabola is QL-Co3.



Equation in CT-notation:

$$l^2 (m - n)^2 ((l m + l n + m n)^2 - 8 l^2 m n) x^2 + 2 m n (l - m) (l - n) (l m + l n - m n)^2 y z \\ + m^2 (l - n)^2 ((l m + l n + m n)^2 - 8 l m^2 n) y^2 + 2 l n (m - l) (m - n) (l m - l n + m n)^2 x z \\ + n^2 (l - m)^2 ((l m + l n + m n)^2 - 8 l m n^2) z^2 + 2 l m (n - l) (n - m) (l m - l n - m n)^2 x y = 0$$

Equation in DT-notation:

$$m^2 n^2 (l^2 - m^2)(l^2 - n^2) x^2 + l^2 n^2 (m^2 - l^2)(m^2 - n^2) y^2 + l^2 m^2 (n^2 - l^2)(n^2 - m^2) z^2 = 0$$

Properties:

- The focus of QL-Co3 is QL-P25 which is the complement of QL-P17 wrt the QL-Diagonal Triangle.
- The axis of the 2nd QL-Parabola // QL-L9 (M3D Line through QL-P18, QL-P23).
- QL-P1.QL-P7 ⊥ Axis QL-Co3, as a consequence QL-P1.QL-P7 // Directrix QL-Co3.
- Directrices QL-Co1 and QL-Co3 meet in QL-P9 (Circumcenter QL-Diagonal Triangle).
- QL-P11 lies on perpendicular bisector F1.F2 (F1, F2 are Foci QL-Co1, QL-Co3).
- The tangents at T1,T2,T3,T4 of QL-Co3 form a tangential quadrilateral.
The QL-DT of this new quadrilateral as well as the QL-DT of the Reference Quadrilateral as well as the QA-DT of T1.T2.T3.T4 are identical.

6.5 QUADRILATERAL CUBICS

QL-Cu1: QL-Quasi Isogonal Cubic

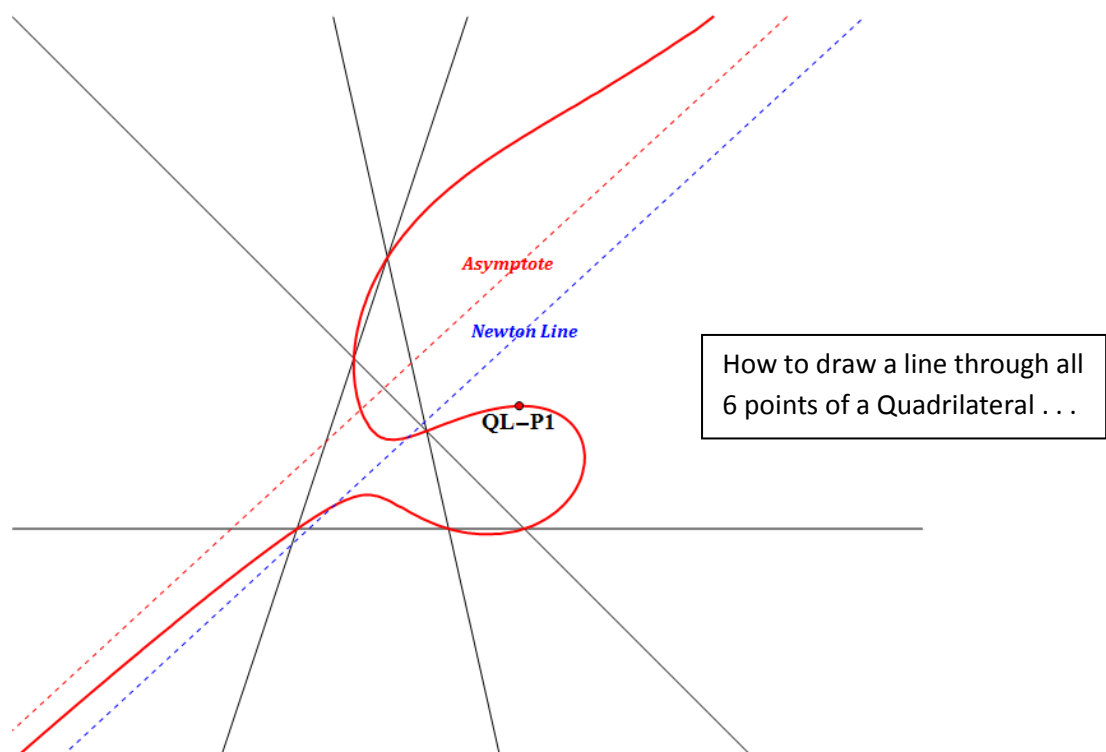
QL-Cu1 has several special characteristics:

1. QL-Cu1 is the locus of all points P for which the Isogonal Conjugates wrt all 4 component triangles coincide in a point on the same cubic. So the cubic is in terms of a quadrilateral QL-self-isogonal.
2. QL-Cu1 is the locus of all points P for which the QG-Quasi Isogonal Conjugates wrt all 3 component quadrilaterals coincide in a point on the same cubic.
3. QL-Cu1 is the locus of the foci of all inscribed conics in the Reference Quadrilateral.
4. QL-Cu1 is the locus of all points for whom the Midpoint of this point and its Clawson-Schmidt Conjugate lie on the Newton Line QL-L1.
5. QL-Cu1 is the locus of all points P for which the feet of the perpendiculars from P to the 4 quadrilateral lines are concyclic (all lie on a circle).
6. QL-Cu1 is the locus of all points P where all perpendiculars from $Li^{\wedge}Lj$ at $Si.Sj$ coincide in a point on the same cubic, where Si = pedal point of P on Li .

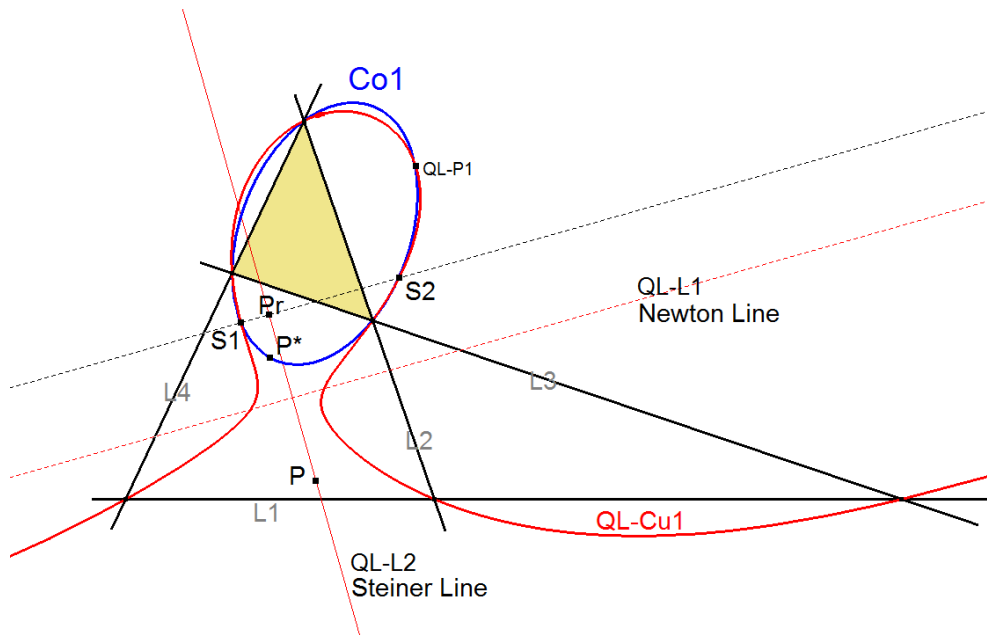
If the Reference System is the Reference Quadrilateral, this cubic is a circular isocubic invariant to the Clawson-Schmidt Conjugate QL-Tf1.

If the Reference System is the Orthic Triangle of the QL-Diagonal Triangle, this cubic is an isogonal circular cubic.

This Quadrilateral Cubic is also described (with other names) by Fred Lang (see [24] page 3) and Eckart Schmidt (see 15e).



Construction:



1. Let P be some point on QL-L2 (Steiner Line).
 2. P* = Isogonal Conjugate of P wrt Triangle L2.L3.L4 (or any other QL-Component Triangle).
 3. Co1= Conic through vertices triangle L2.L3.L4 and QL-P1 and P*.
 4. Pr = P reflected in QL-L1 (Newton Line).
 5. Lr = line through Pr // QL-L1 (now P is railway watcher).
 6. S1 and S2 are intersection points Lr ^ Co1.
- QL-Cu1 is the locus of S1 and S2 with variable P on QL-L2 (Steiner Line).

Equation in CT-notation:

$$a^2 l (m y + n z) y z + b^2 m (n z + l x) x z + c^2 n (l x + m y) x y + 2 (S_A m n + S_B l n + S_C l m) x y z = 0$$

CT-coordinates Infinity Point Asymptote:

$$(l(m - n) : m(n - l) : n(l - m))$$

Equation in DT-notation:

$$l^2 x^2 (S_A x - S_B y - S_C z) - m^2 y^2 (S_A x - S_B y + S_C z) - n^2 z^2 (S_A x + S_B y - S_C z) - (a^2 l^2 + b^2 m^2 + c^2 n^2) x y z = 0$$

DT-coordinates Infinity Point Asymptote:

$$(m^2 - n^2 : n^2 - l^2 : l^2 - m^2)$$

Properties:

- The cubic passes through all 6 intersection points of the 4 basic lines of the Reference Quadrilateral.
- QL-P1 (Miquel Point) lies on the cubic QL-Cu1.
- The vertices of the Orthic Triangle of the QL-Diagonal Triangle lie on QL-Cu1.
- The cubic is circular because it passes through the circular points at infinity.
- The asymptote of QL-Cu1 // QL-L1 = Newton Line.
- Distance Miquel Point (QL-P1) to Asymptote is twice the distance from Miquel Point to Newton Line (Railway Watcher system, see QL-L/1).
- QL-P21 lies on the asymptote of QL-Cu1.
- The Clawson-Schmidt Conjugate (QL-Tf1) of some point P on the cubic is also a point on the cubic. Thus the cubic is invariant under Clawson-Schmidt-conjugation. See [15e].
- The Clawson-Schmidt Conjugate (QL-Tf1) of some point P on QL-Cu1 is identical with the isogonal conjugate of P wrt the Orthic Triangle of the QL-Diagonal Triangle (note Eckart Schmidt).

6.6 QUADRILATERAL QUARTICS

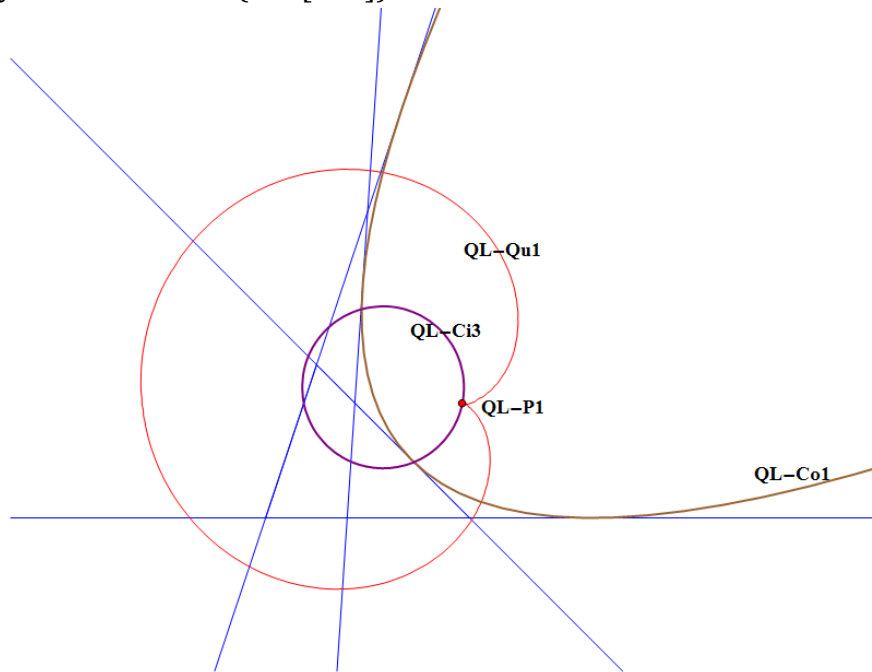
QL-Qu1: QL-Cardioid

QL-Qu1 is the Cardioid which is the envelope of the circles through fixed point QL-P1 (Miquel Point) and with circumcenter on QL-Ci3 (Miquel Circle).

The equation is of the 4th degree, so it is a quartic.

It can also be generated by applying the Clawson-Schmidt Conjugate (QL-Tf1) on each point of the QL-Inscribed Parabola (QL-Co1).

The QL-Cardioid as well as the 33333333 Clawson-Schmidt Conjugate are both described by Eckart Schmidt (see [15d]).



Equation in CT-notation:

$$a^4 (m - n)^2 T_a^2 + b^4 (l - n)^2 T_b^2 + c^4 (l - m)^2 T_c^2 + 2 a^2 b^2 (n - l) (n - m) T_a T_b + 2 b^2 c^2 (l - m) (l - n) T_b T_c + 2 a^2 c^2 (m - l) (m - n) T_c T_a = 0$$

where:

$$T_a = a^2 (l - m) (l - n) y z + c^2 (l - n) y (l x + m y + m z) + b^2 (l - m) z (l x + n y + n z)$$

$$T_b = b^2 (m - l) (m - n) x z + c^2 (m - n) x (l x + m y + l z) + a^2 (m - l) z (n x + m y + n z)$$

$$T_c = c^2 (n - m) (n - l) x y + b^2 (n - m) x (l x + l y + n z) + a^2 (n - l) y (m x + m y + n z)$$

Equation in DT-notation:

$$\begin{aligned} & (l^2 - n^2) (m^2 - n^2) (-2 a^2 b^2 n (m x + l y) z + a^4 m n z (x - y + z) + b^4 l n z (-x + y + z) - \\ & c^4 l m z (x + y + z) - 2 b^2 c^2 l (-l^2 x^2 + (y + z) (m^2 y + l^2 z)) + 2 a^2 c^2 m (-m^2 y^2 + (x + z) (l^2 x + m^2 z)))^2 \\ & + (l^2 - m^2) (l^2 - n^2) (b^4 l n x (x + y - z) + c^4 l m x (x - y + z) - a^4 m n x (x + y + z) - 2 b^2 c^2 l x (n y + m z) \\ & + 2 a^2 b^2 n ((x + y) (n^2 x + m^2 y) - n^2 z^2) - 2 a^2 c^2 m (-m^2 y^2 + (x + z) (m^2 x + n^2 z)))^2 \\ & - (l^2 - m^2) (m^2 - n^2) (a^4 m n y (x + y - z) + c^4 l m y (-x + y + z) - b^4 l n y (x + y + z) \\ & - 2 a^2 c^2 m y (n x + l z) - 2 a^2 b^2 n ((x + y) (l^2 x + n^2 y) - n^2 z^2) + 2 b^2 c^2 l (-l^2 x^2 + (y + z) (l^2 y + n^2 z)))^2 \\ & = 0 \end{aligned}$$

Properties:

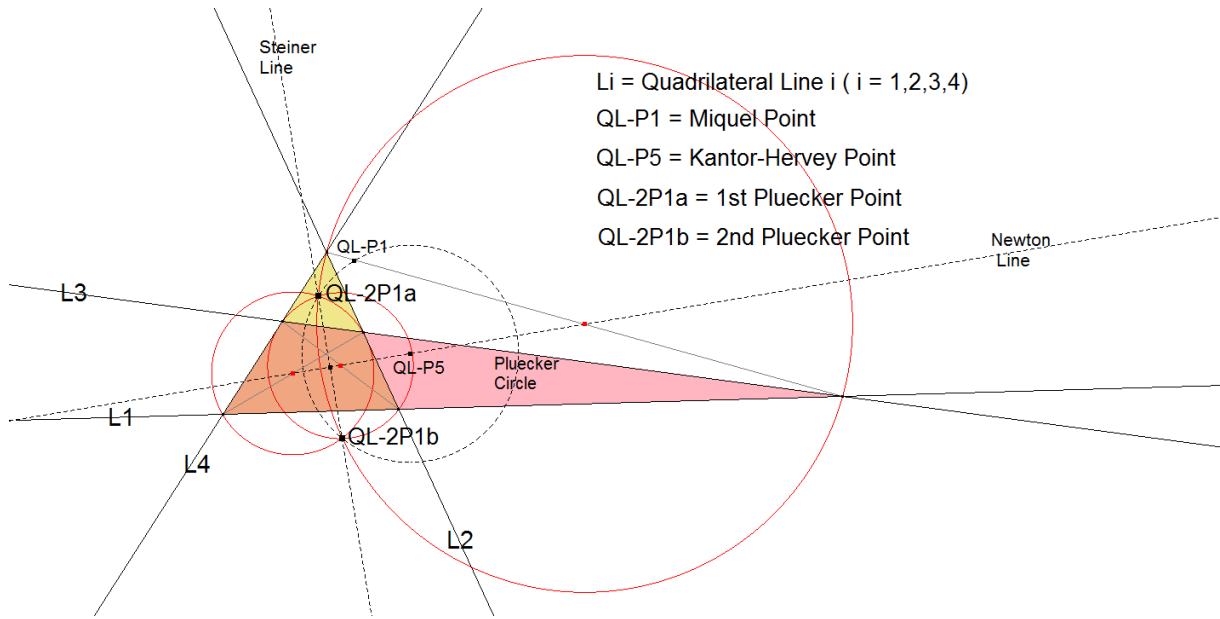
- The cusp of the Cardioid is at QL-P1, the Miquel Point.

6.7 QUADRILATERAL PAIRS/OCTETS/DOZENS OF POINTS/LINES

QL-2P1: Plücker Pair of Points

Plücker proved that the circles having the three diagonals as diameters have two common points which lie on the line joining the four triangles' orthocenters (Wells 1991). That's why these points are called the Plücker Points. See [13] item "Complete Quadrilateral".

These points have added value because the circle with QL-P5 (Kantor-Hervey Point) as center passing through QL-P1 (Miquel Point) also passes through the Plücker Points.



CT-Coordinates:

Different presentations of QL-2P1a: 1st Plücker Point:

$$\text{Plu1a} = \{S_B S_C T_a : S_C (T_n + (l - m) n T_T) : S_B (T_m + (l - n) m T_T)\}$$

$$\text{Plu1b} = \{S_C (T_n + (m - l) n T_T) : S_A S_C T_b : S_A (T_l + (m - n) l T_T)\}$$

$$\text{Plu1c} = \{S_B (T_m + (n - l) m T_T) : S_A (T_l + (n - m) l T_T) : S_A S_B T_c\}$$

Different presentations of QL-2P1b: 2nd Plücker Point:

$$\text{Plu2a} = \{S_B S_C T_a : S_C (T_n - (l - m) n T_T) : S_B (T_m - (l - n) m T_T)\}$$

$$\text{Plu2b} = \{S_C (T_n - (m - l) n T_T) : S_A S_C T_b : S_A (T_l - (m - n) l T_T)\}$$

$$\text{Plu2c} = \{S_B (T_m - (n - l) m T_T) : S_A (T_l + (n - m) l T_T) : S_A S_B T_c\}$$

where:

$$T_a = l m n (-a^2 l + S_B n + S_C m)$$

$$T_b = l m n (+S_A n - b^2 m + S_C l)$$

$$T_c = l m n (+S_A m + S_B l - c^2 n)$$

$$T_l = +2 \Delta^2 (l_2 m^2 + l m n^2 + l^2 n^2 + l m^2 n) - l m n (a^2 S_A l + b^2 S_B m + c^2 S_C n)$$

$$T_m = +2 \Delta^2 (m^2 n^2 + l^2 m n + l^2 m^2 + l m n^2) - l m n (a^2 S_A l + b^2 S_B m + c^2 S_C n)$$

$$T_n = +2 \Delta^2 (l^2 n^2 + l m^2 n + m^2 n^2 + l^2 m n) - l m n (a^2 S_A l + b^2 S_B m + c^2 S_C n)$$

$$T_T = +2 \Delta \sqrt{[\Delta^2 (l m + l n + m n)^2 - l m n (a^2 S_A l + b^2 S_B m + c^2 S_C n)]}$$

$$\Delta = \text{Area} = 1/4 \sqrt{[(a + b + c) (-a + b + c) (a - b + c) (a + b - c)]}$$

$$S_A = (-a^2 + b^2 + c^2) / 2 \quad S_B = (+a^2 - b^2 + c^2) / 2 \quad S_C = (+a^2 + b^2 - c^2) / 2$$

Properties:

- The Plücker Points lie on the Steiner Line (QL-L2).
- The circle with center QL-P5 (Kantor-Hervey Point) and passing through QL-P1 (Miquel Point) also passes through the Plücker Points.

QL-8P1: Steiner Angle Bisector Center Octet

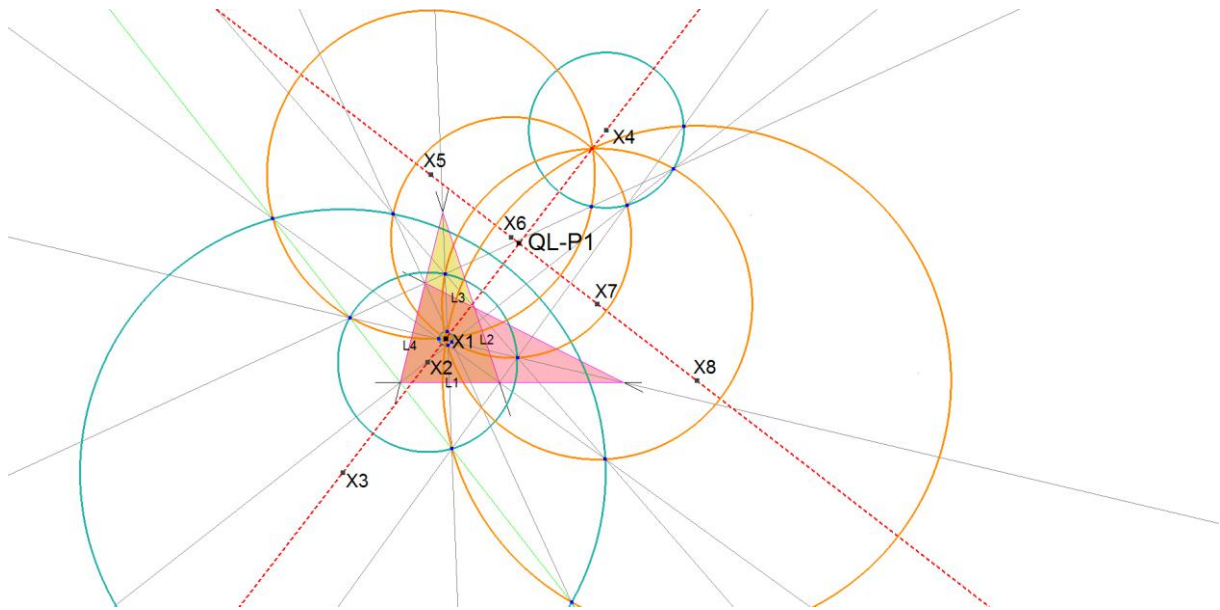
In 1828 Jakob Steiner published in Gergonne's *Annales* 10 rules on the complete quadrilateral. For a complete description see [4].

Rules 8, 9, 10 are:

(8) Each of the four possible triangles has an incircle and three excircles. The centers of these 16 circles lie, four by four, on eight new circles.

(9) These eight new circles form two sets of four, each circle of one set being orthogonal to each circle of the other set. The centers of the circles of each set lie on a same line. These two lines are perpendicular.

(10) Finally, these last two lines intersect at the point F (Miquel Point).



CT-coordinates:

In CT-notation let $L1 = (1 : 0 : 0)$, $L2 = (0 : 1 : 0)$, $L3 = (0 : 0 : 1)$, $L4 = (l : m : n)$.

Let B_{ij} be the bisector (internal or external) between the lines L_i and L_j .

Let T_j be intersection point $B_{ij} \wedge B_{jk}$, where i, j, k are consecutive numbers in the cycle $(1, 2, 3, 4)$.

Let $T1, T2, T3, T4$ be the points that define the Steiner Circles in rule (8).

The coordinates of these points are:

- $T1 = (-a : +b : +c)$
- $T2 = (+b m + c n - W : -b l : -c l)$
- $T3 = (-a m : +a l - c n + W : +c m)$
- $T4 = (+a n : -b n : -a l + b m - W)$

where $W = \sqrt{[a^2(l-m)(l-n) + b^2(m-l)(m-n) + c^2(n-l)(n-m)]}$

Whether $T1, T2, T3, T4$ were created by "internal" or "external" bisectors (which actually is arbitrary in a quadrilateral) is only dependent on the signs in $T1, T2, T3, T4$ of the variables a, b, c, W . So the undetermined status of the bisectors being "internal" or

“external” finds an algebraic solution in a combination of signs of a, b, c and W, as described here:

- X1 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution a -> - a
- X2 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution b -> - b
- X3 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution c -> - c
- X4 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution W -> -W

X1, X2, X3, X4 are collinear

- X5 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution a -> -a, W-> -W
- X6 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution b -> -b, W-> -W
- X7 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with substitution c -> -c, W-> -W
- X8 = Perp. Bisector (T1,T2) ^ Perp. Bisector (T3,T4) with no substitution

X5, X6, X7, X8 are collinear

Finally X1 – X8 end up in an algebraic expression that is not quite simple.

Here are the coordinates of X1 (with the substitution a -> - a):

$$\left(a (a^2 b l^2 - b^2 (a + c) m^2 + c (a^2 - b^2 + 2 a c + b c + c^2) n^2 - (a + b) (a - b + c) n W - (a + c) W^2 + m (b (a + b) (a - b + c) n + 2 b (a + c) W) + l (-a b (a - b + c) m + (-a^2 b + b^3 - a^2 c + b^2 c - 2 a c^2 - b c^2 - c^3) n + a (a - b + c) W) \right) :$$

$$b (a (-b^2 + a c + c^2) l^2 + b^3 m^2 + c (a^2 - b^2 + a c) n^2 - (a + b) (a - b + c) n W + b W^2 + m (b (a + b) (a - b + c) n - 2 b^2 W) + l (b (a - b + c) (b + c) m + (-a^2 b + b^3 - 2 a^2 c - 2 a b c - 2 a c^2 - b c^2) n - (a - b + c) (b + c) W) :$$

$$c (a (a^2 + a b - b^2 + 2 a c + c^2) l^2 - b^2 (a + c) m^2 + b c^2 n^2 + c (a - b + c) n W - (a + c) W^2 + m (-b c (a - b + c) n + 2 b (a + c) W) + l (b (a - b + c) (b + c) m + (-a^3 - a^2 b + a b^2 + b^3 - 2 a^2 c - a c^2 - b c^2) n - (a - b + c) (b + c) W))$$

The coordinates of X2 – X8 are similar.

Properties:

- The Centroid of the 8 centers of circles as described in rule (9) from Steiner (seen as a system of 8 random points) is QL-P4 (Miquel Circumcenter).

QL-12L1: A dozen of Equidistance Lines

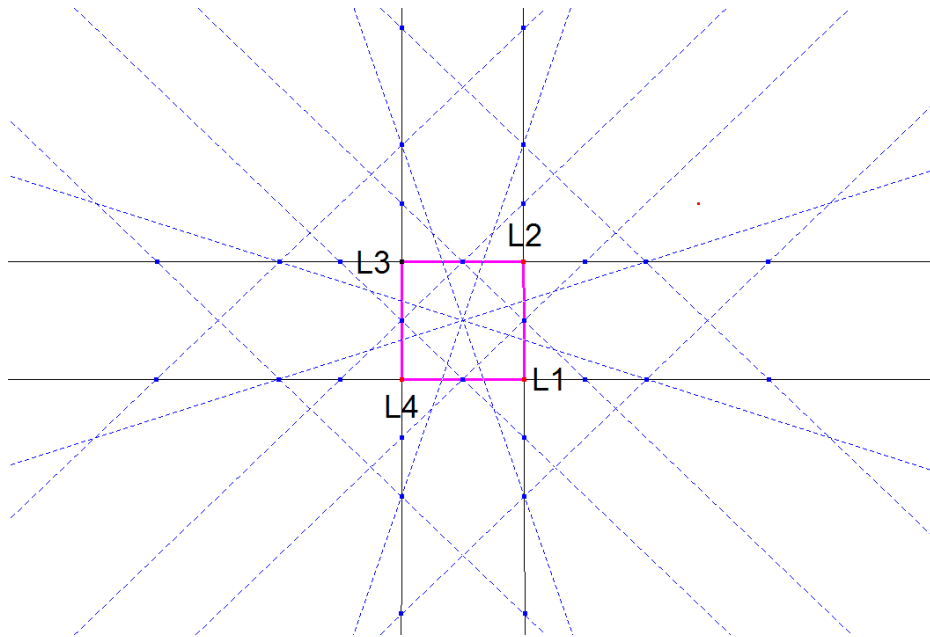
There is the general question of constructing a line in a quadrilateral (a system of 4 lines) such that this line is divided by the 4 lines in 3 equal parts.

This problem is dealt with at [14] Philippe Chevanne, *Mad Maths, Recreational mathematic collection*.

Also a simple synthetic solution is given.

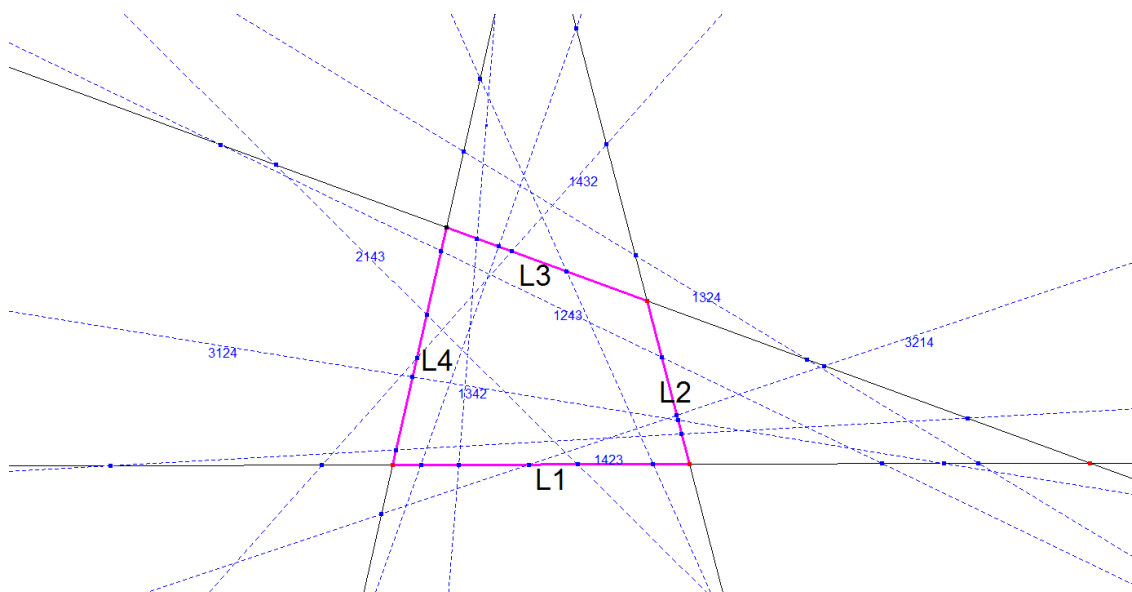
In this paper the 12 algebraically solutions will be given in QL-notation.

The next picture illustrates how these solutions look like in a square-situation.

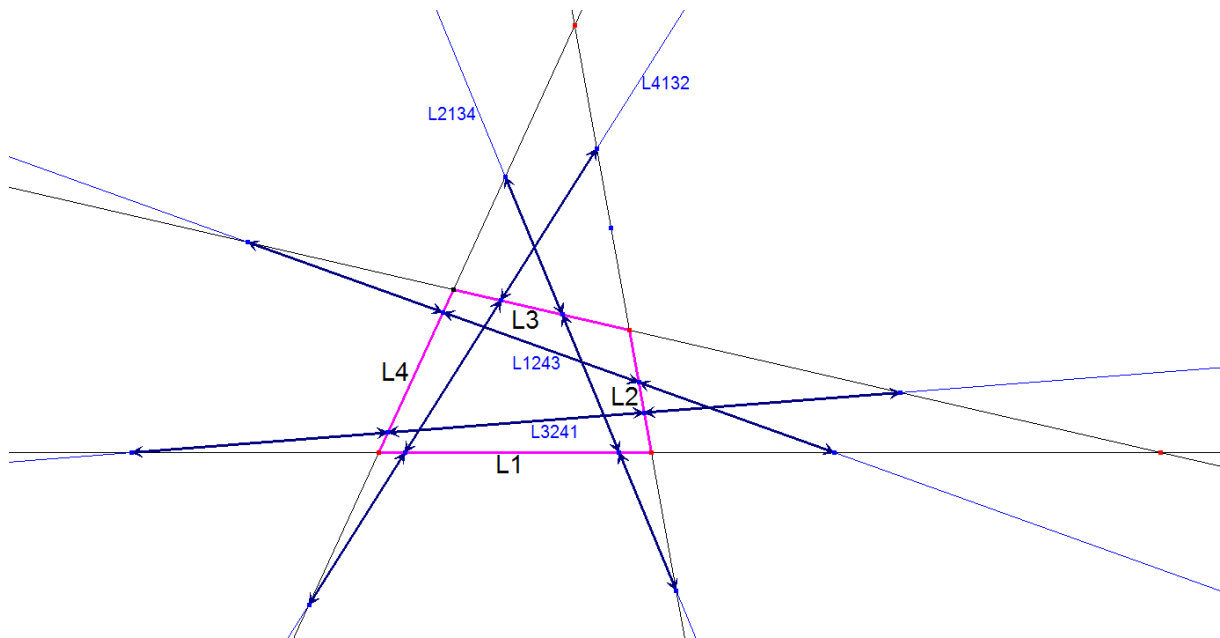


All dotted lines are Equidistance Lines and are divided by the 4 bounding lines of the square in equal parts.

When choosing random basic lines for the Reference Quadrilateral it looks like this:



We can split up the dozen equidistance lines per quadrigon of the quadrilateral.
 In each quadrigon 4 equidistance lines occur where each middle splitted part of the equidistance line is a line segment between 2 opposite lines of the quadrigon.
 In next figure quadrigon L1.L2.L3.L4 is shown with its 4 equidistance lines.



Accordingly each equidistance line can be ascribed to 1 of the 3 component quadrigons:

	Quadrigon <u>L1.L2.L3.L4</u>	Quadrigon <u>L1.L3.L2.L4</u>	Quadrigon <u>L1.L2.L4.L3</u>
Equidistance Line	L4132	L3124	L2143
Equidistance Line	L2134	L4123	L3142
Equidistance Line	L3241	L1342	L1234
Equidistance Line	L1243	L2341	L4231

Note 1:

L4132 indicates the equidistance line intersecting QL-lines in order L4, L1, L3, L2.

Note 2:

The two middle line numbers per equidistance line in the table are opposite lines in the quadrigons in which they occur.

Coefficients of the 12 lines as well as their 12 Midpoints (see also QL-12P1):

The infinity points of the Equidistance Lines (indicating their direction) are important in this setting and have simple coordinates. These coordinates are actually the building blocks of the Equidistance Line itself and its Midpoint as will be shown in next tables.
 Next all mentioned coordinates will be CT-coordinates.

EquiDistance Lines in Quadrigon L1.L2.L3.L4:

	<u>L2134</u>	<u>L4132</u>	<u>L1243</u>	<u>L3241</u>
Infinity point EquiDist.Line	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$
	where:	where:	where:	where:
	$x = 3m - n$	$x = 3m - 2n$	$x = -m - n$	$x = +m - 2n$
	$y = n - 2l$	$y = 2n - l$	$y = +n + 2l$	$y = +2n + l$
	$z = 2l - 3m$	$z = l - 3m$	$z = -2l + m$	$z = -l - m$
EquiDistance Line	$(-2/x : 1/y : 1/z)$	$(1/x : 1/y : -2/z)$	$(2/x : -3/y : 1/z)$	$(1/x : -3/y : 2/z)$
EquiDistance Midpoint	$(x : 3y : -z)$	$(x : -3y : -z)$	$(x : y/3 : -z)$	$(x : -y/3 : -z)$

EquiDistance Lines in Quadrigon L1.L3.L2.L4:

	<u>L3124</u>	<u>L4123</u>	<u>L2341</u>	<u>L1342</u>
Infinity point EquiDist.Line	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$
	where:	where:	where:	where:
	$x = m - 3n$	$x = 2m - 3n$	$x = 2m - n$	$x = -m - n$
	$y = -3n + 2l$	$y = 3n - l$	$y = n + l$	$y = n - 2l$
	$z = -2l + m$	$z = l - 2m$	$z = -l - 2m$	$z = 2l + m$
EquiDistance Line	$(-2/x : 1/y : 1/z)$	$(1/x : -2/y : 1/z)$	$(1/x : 2/y : -3/z)$	$(2/x : 1/y : -3/z)$
EquiDistance Midpoint	$(x : -y : 3z)$	$(x : -y : -3z)$	$(x : -y : -z/3)$	$(x : -y : z/3)$

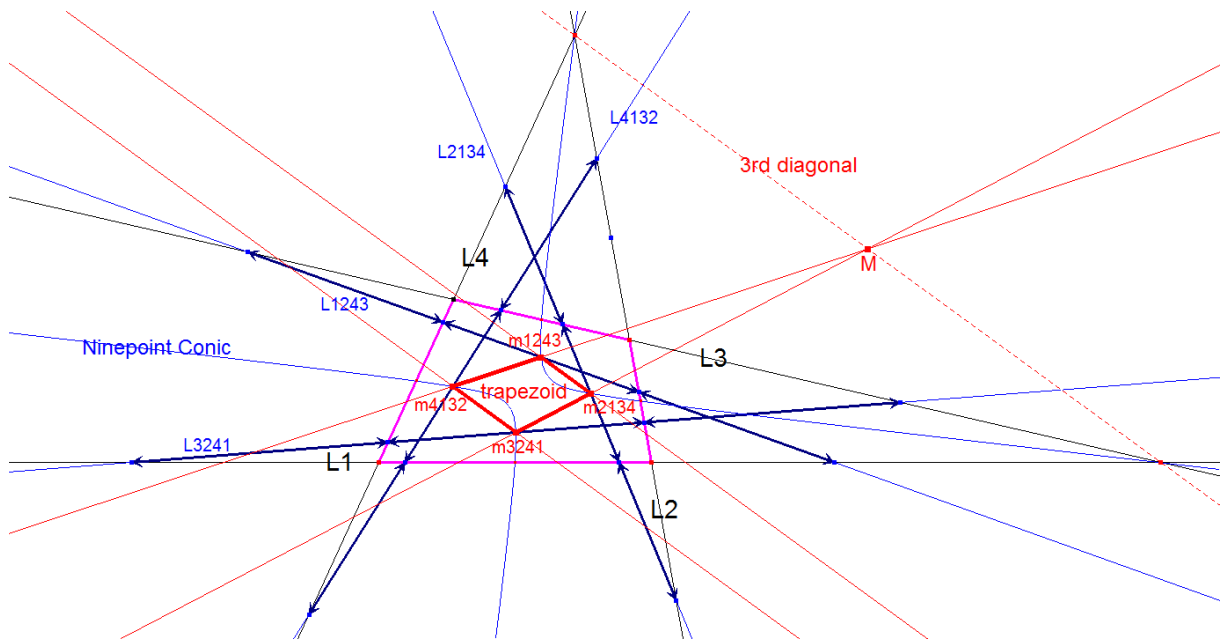
EquiDistance Lines in Quadrigon L1.L2.L4.L3:

	<u>L2143</u>	<u>L3142</u>	<u>L1234</u>	<u>L4231</u>
Infinity point EquiDist.Line	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$	$(x : y : z)$
	where:	where:	where:	where:
	$x = 2m + n$	$x = m + 2n$	$x = 2m - n$	$x = -m - 2n$
	$y = -n - l$	$y = -2n + l$	$y = n - 3l$	$y = 2n - 3l$
	$z = l - 2m$	$z = -l - m$	$z = 3l - 2m$	$z = 3l - m$
EquiDistance Line	$(-3/x : 2/y : 1/z)$	$(-3/x : 1/y : 2/z)$	$(1/x : -2/y : 1/z)$	$(1/x : 1/y : -2/z)$
EquiDistance Midpoint	$(x/3 : y : -z)$	$(-x/3 : y : -z)$	$(3x : y : -z)$	$(-3x : y : -z)$

QL-12P1: A dozen of Equidistance Midpoints

As described by lines QL-12L1 (Equidistance Lines Dozen) there are 12 lines per quadrilateral such that each line is separated by the 4 lines in 3 equal parts. It is interesting that these 3 parts have a midpoint. It is in fact the midpoint of the part in the middle.

Since there are 4 equidistance lines per Quadrilateral of the Reference Quadrilateral there are also 4 Equidistance Midpoints per quadrilateral of the Reference Quadrilateral. The 4 Equidistance Lines per Quadrilateral are those lines where the middle segment of the Equidistance Line is the segment between the opposite lines.



1st CT-coordinate:

The coordinates of the Equidistance Midpoints are described in the paragraph of the Equidistance Lines (QL-12L1).

Properties:

- The 4 Equidistance Midpoints in a QL-Quadrilateral lie on the corresponding Nine-point Conic (see QA-Co1) of the Quadrilateral in question.
- The 4 Equidistance Midpoints in a QL-Quadrilateral form a trapezoid, where $m_{3241}.m_{4132} \parallel m_{2134}.m_{1243} \parallel 3^{\text{rd}} \text{ diagonal of the Quadrilateral}$.
- $m_{3241}.m_{2134} \wedge m_{4132}.m_{1243} = \text{QG-P2 (midpoint } 3^{\text{rd}} \text{ diagonal of the Quadrilateral)}$.
- Let L1, L2, L3, L4 be the lines of a Quadrilateral where L1, L3 are opposite sides and L2, L4 are opposite sides. Let P be a point on the Nine-point Conic of the Quadrilateral and let L be a line through P.

When P is the midpoint of the line segment of L between L1 and L3 then P is automatically the midpoint of the line segment of L between L2 and L4.

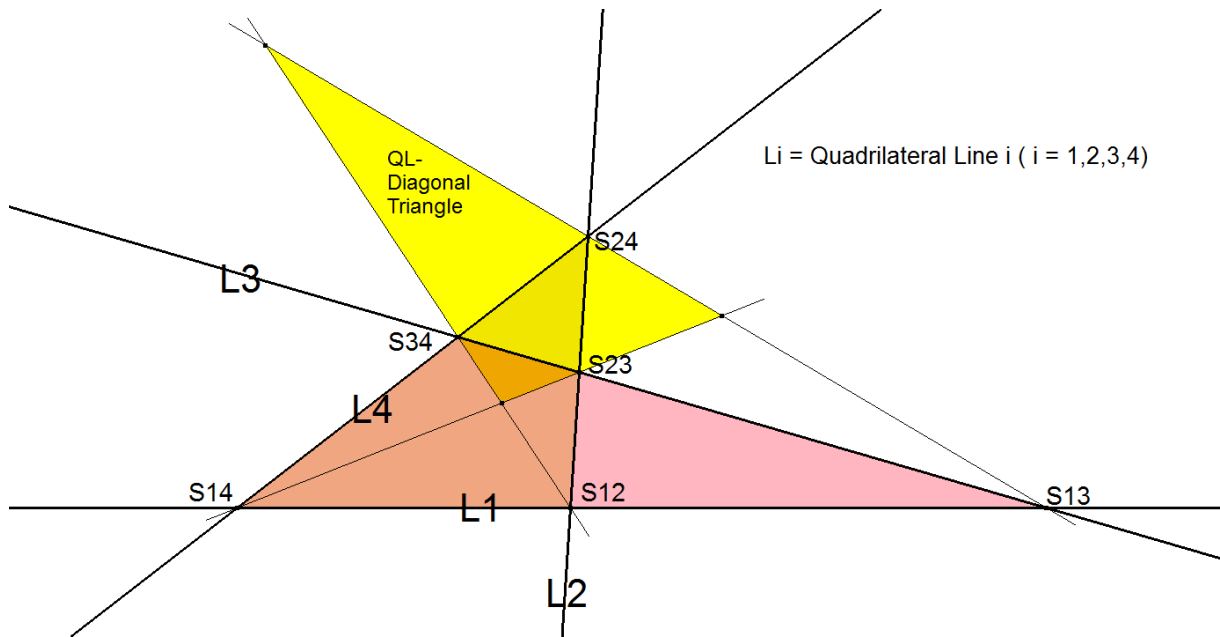
- The Centroid of the 4 Equidistance Midpoints in a QL-Quadrignon lies on the Newton Line (see QL-L1).
- The Centroid of the 12 Equidistance Midpoints of all 3 QL-Quadrignons is a point also on the Newton Line (QL-L1).

6.8 QUADRILATERAL TRIANGLES

QL-Tr1: QL-Diagonal Triangle

The Diagonal Triangle of the Quadrilateral (L1, L2, L3, L4) is the Triangle bounded by lines S12.S34, S13.S24, S14.S23, where Sij = intersection point Li ^ Lj, where i and j are different numbers and $\in (1,2,3,4)$.

The connection of the intersection points of 2 adjacent lines and the intersection point of the remaining 2 lines produces a side of the QL-Diagonal Triangle. By doing this for all 3 possible combinations of 2 sets of 2 adjacent lines the QL-Diagonal Triangle is produced.



Area QL-Diagonal Triangle in CT-notation:

$$\frac{4 l^2 m^2 n^2 \Delta}{(l m - l n - m n) (l m + l n - m n) (l m - l n + m n)}.$$

Radius² Circumcircle QL-Diagonal Triangle in CT-notation:

$$\frac{l^2 m^2 n^2 (a^2 l^2 - a^2 l m - b^2 l m + c^2 l m + b^2 m^2) (a^2 l^2 - a^2 l n + b^2 l n - c^2 l n + c^2 n^2) (b^2 m^2 + a^2 m n - b^2 m n - c^2 m n + c^2 n^2)}{4 \Delta^2 (l m - l n - m n)^2 (l m + l n - m n)^2 (l m - l n + m n)^2}$$

Area QL-Diagonal Triangle in DT-notation:

$$S / 2$$

Radius² Circumcircle QL-Diagonal Triangle in CT-notation:

$$a^2 b^2 c^2 / (4 S^2)$$

Properties:

- The QL-Diagonal Triangle is the AntiCevian Triangle of each of the 4 Component Triangles Li.Lj.Lk, where Ll is the perspectrix and where (i,j,k,l) $\in (1,2,3,4)$.

6.9 QUADRILATERAL TRANSFORMATIONS

QL-Tf1: Clawson-Schmidt Conjugate

The Clawson-Schmidt Conjugate of a point P is a transformation of a point P in such a way that each defining line of the Reference Quadrilateral is transformed in the circumcircle of the triangle formed by the other 3 lines of the Reference Quadrilateral. It is a conjugate because applying two times this transformation ends up in the original point.

Origin of the conjugate

John Wentworth Clawson suggested an inversion transformation wrt some circle with the Miquel point as center (see [22]). It was Eckart Schmidt who gave a very useful form to this inversion by modifying it in this way:

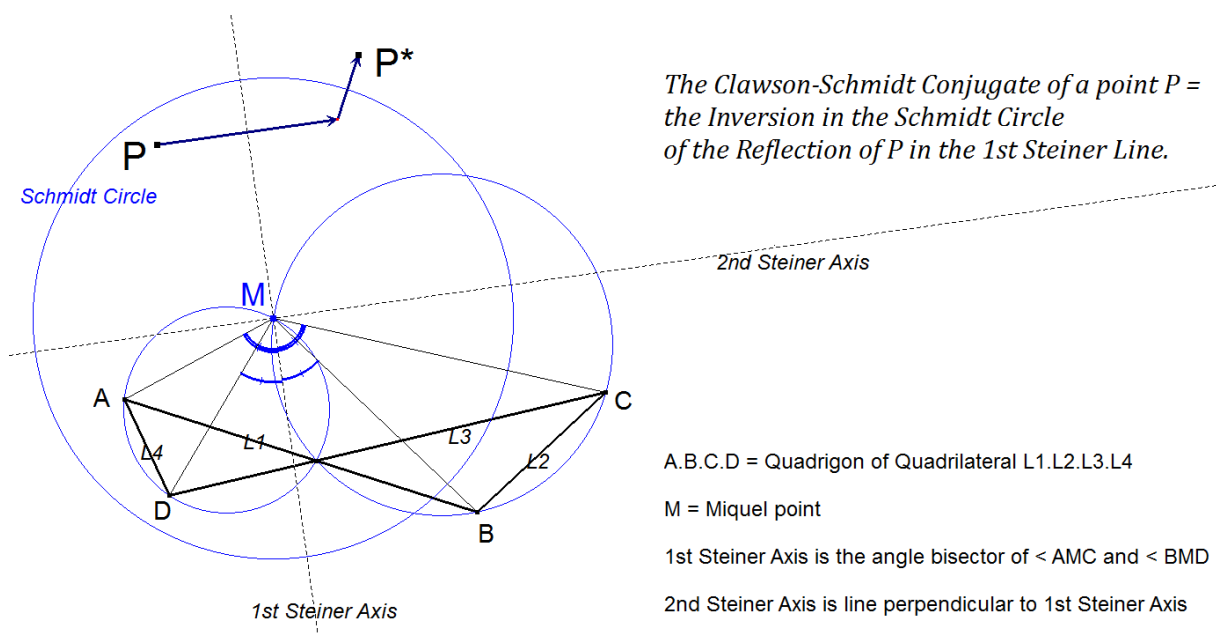
In each quadrigon of a quadrilateral we have 4 points A, B, C, D in this order. Let M be the Miquel Point. Now the Angle Bisectors of $\angle AMC$ and of $\angle BMD$ are the same line. See [15d]. Moreover this line is the same (invariant) for each QL-Quadrigon. Let this line be called the 1st Steiner Axis. Let the 2nd Steiner Axis be the line perpendicular at the 1st Steiner Line in M.

Let the circle with Circumcenter M and with radius $= \sqrt{[MA.MC]} = \sqrt{[MB.MD]}$ be called the Schmidt Circle.

The Clawson-Schmidt Conjugate of a point P is the Inversion wrt the Schmidt Circle of the Reflection in the 1st Steiner Line.

Moreover the reflection and inversion can be performed in reversed order.

The result is a conjugation that transforms each defining line of the Reference Quadrilateral into the circumcircle of the triangle formed by the other 3 lines of the Reference Quadrilateral.



Relationship with Steiner-rules

The 1st and 2nd Steiner-Axes are described in the Steiner rules (8) and (9). That's why they are named after Steiner.

Rules 8, 9, 10 of Steiner about quadrilaterals say (see [4]):

(8) Each of the four possible triangles in a quadrilateral has an incircle and three excircles. The centers of these 16 circles lie, four by four, on eight new circles.

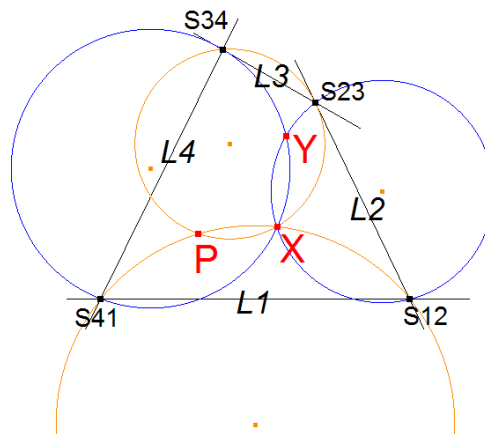
(9) These eight new circles form two sets of four, each circle of one set being orthogonal to each circle of the other set. The centers of the circles of each set lie on a same line. These two lines are perpendicular.

(10) Finally, these last two lines intersect at the point F (Miquel Point).

Both sets of 4 circles as described in (9) are coaxal. One set has an axis with real intersection points of the 4 circles, the other set has an axis with imaginary intersection points of the 4 circles. These axes are called the 1st Steiner-axis (axis with 2 real common intersection points of 4 circles) and the 2nd Steiner-axis (axis with 2 imaginary common intersection points of 4 circles). The common intersection points on the 1st Steiner Axis are the only points in the real plane that are invariant under Clawson-Schmidt Conjugation. See [15d].

Construction:

Eckart Schmidt describes this construction in [15d]:



1. Choose one Quadrigon of the Reference Quadrilateral.
For example $S_{41}.S_{12}.S_{23}.S_{34}$, where $S_{ij} = \text{Intersection } L_i \wedge L_j$.
2. Let P be a random point. Draw a circle $Ci1$ through P and the 2 vertices on a side $L1$ of the Quadrigon. Draw a circle $Ci2$ through P and the 2 vertices on the opposite side $L3$ of the Quadrigon. Let X be the 2nd intersection point of $Ci1$ and $Ci2$.
3. Draw a circle $Ci3$ through X and the 2 vertices on a third side $L2$ of the Quadrigon. Draw a circle $Ci4$ through X and the 2 vertices on the last side $L4$ of the Quadrigon. Let Y be the 2nd intersection point of $Ci3$ and $Ci4$.
4. Y is the Clawson-Schmidt Conjugate.

Coordinates:

Let $Q(u : v : w)$ be a random point not on one of the defining lines of the Reference Quadrilateral.

The Clawson-Schmidt Conjugate in CT-coordinates gives this result:

$$\begin{aligned} & (a^2 m n (a^2 (l-m) (l-n) v w + c^2 (l-n) v (l u+m v+m w) + b^2 (l-m) w (l u+n v+n w)) : \\ & b^2 l n (b^2 (m-l) (m-n) u w + c^2 (m-n) u (l u+m v+l w) + a^2 (m-l) w (n u+m v+n w)) : \\ & c^2 l m (c^2 (n-l) (n-m) u v + b^2 (n-m) u (l u+l v+n w) + a^2 (n-l) v (m u+m v+n w))) \end{aligned}$$

The Clawson-Schmidt Conjugate in DT-coordinates gives this result:

$$\begin{aligned} & (b^4 L N u (u + v - w) + c^4 L M u (u - v + w) - a^4 M N u (u + v + w) - 2 b^2 c^2 L u (N v + M w) \\ & + 2 a^2 b^2 N ((u + v) (N^2 u + M^2 v) - N^2 w^2) - 2 a^2 c^2 M (-M^2 v^2 + (u + w) (M^2 u + N^2 w))) : \\ & a^4 M N v (u + v - w) + c^4 L M v (-u + v + w) - b^4 L N v (u + v + w) - 2 a^2 c^2 M v (N u + L w) \\ & - 2 a^2 b^2 N ((u + v) (L^2 u + N^2 v) - N^2 w^2) + 2 b^2 c^2 L (-L^2 u^2 + (v + w) (L^2 v + N^2 w))) : \\ & - 2 a^2 b^2 N (M u + L v) w + a^4 M N w (u - v + w) + b^4 L N w (-u + v + w) - c^4 L M w (u + v + w) \\ & - 2 b^2 c^2 L (-L^2 u^2 + (v + w) (M^2 v + L^2 w)) + 2 a^2 c^2 M (-M^2 v^2 + (u + w) (L^2 u + M^2 w))) \end{aligned}$$

where:

$$L = m^2 - n^2$$

$$M = n^2 - l^2$$

$$N = l^2 - m^2$$

Examples of Clawson-Schmidt Conjugates:

Point/Line/Curve -1	Line/QA-Curve -2
SPECIFIC	
QL-P1: Miquel Point	Some (undefined) point at infinity
Defining Quadrilateral Lines: L1,L2,L3,L4	Circumcircle of corresponding QL-Component Triangle
Intersection point $L_i \wedge L_j$	Opposite intersection point $L_k \wedge L_l$
QL-L2 Steiner Line	QL-Ci3 Miquel Circle
QL-Co1: Inscribed Parabola	QL-Qi1: Cardioide
GENERAL	
A line not through QL-P1	A circle through QL-P1
A line through QL-P1	A line through QL-P1
A circle with QL-P1 as center	Another circle with QL-P1 as center
INVARIANCIES	
Steiner Bisector Circle-i (i=1-8) (circles in rule 9 of Steiner)	Same Steiner Bisector Circle-i (i=1-8)
Schmidt Circle	Schmidt Circle
Intersection point 1 st Steiner-Axis \wedge Schmidt Circle	Same intersection point 1 st Steiner-Axis \wedge Schmidt Circle (the only invariant points there are)
Intersection point 2 nd Steiner-Axis \wedge Schmidt Circle	Other intersection point 2 nd Steiner-Axis \wedge Schmidt Circle (switching points)
QL-Cu1: Quasi Isogonal Cubic	QL-Cu1: Quasi Isogonal Cubic

Performances in Quadrangles

There are special surprises applying the Clawson-Schmidt Conjugate wrt Quadrangles:

- QL-Tf1 produces at Quadrigon-level using QG-P1 (the Diagonal Crosspoint) a Quadrangle-Point: QA-P4 (Isogonal Center) which is valid for all its QA-Quadrigons. See [15d].
- Consider the 4 points of a QL-Quadrigon as a QA-Quadrigon also defining a Quadrangle. Let M2 and M3 be the Miquel Points of the other two QA-Quadrigons. Now M2 and M3 are mutual Clawson-Schmidt Conjugates wrt the 1st QA-Quadrigon. See [15b] and [15d].
- For a Quadrangle there are three Clawson-Schmidt Conjugates wrt the quadrilaterals

P1P2, P2P3, P3P4, P4P1 (CSCa),

P1P2, P2P4, P4P3, P3P1 (CSCb),

P1P4, P4P2, P2P3, P3P1 (CSCc).

Now the product of two of these conjugates achieves the third one. The product of all three achieves the identity. See [15d]. The QA-DT-P4 Cubic (QA-Cu1) of the quadrangle is invariant under these three transformations.

- Let P be some point on the QA-DT-P4 Cubic (QA-Cu1) and let Pa = CSCa(P), Pb = CSCb(P), Pc = CSCc(P). The tangents at P, Pa, Pb, Pc to the cubic QA-Cu1 coincide in the Isogonal Center (QA-P4) of the Quadrangle P.Pa.Pb.Pc. See [15b].

7. QUADRIGON OBJECTS

QG/1: Systematics for describing QG-Points

A Quadrigon is also called a Tetragon.

It is called a Quadrigon because of the relationship with a Quadrangle and a Quadrilateral.

In the descriptions of EQF these are the definitions of these terms:

- A Quadrangle is a system of 4 random points without conditions. It also often is called a Complete Quadrangle.
- A Quadrilateral is a system of 4 random lines without conditions. It also often is called a Complete Quadrilateral.
- A Quadrigon is a system of 4 consecutive points and 4 consecutive connecting lines. It is also called a Tetragon. There is a cyclic order of points and lines, meaning here that the permutation of 4 points in a certain order equalizes the permutation of 4 points in the reversed order. When the order in a permutation of a cycle equalizes the permutation with reversed order we will name this condition “vv-cyclic” (“vv” stands for “vice versa”).

Within a Quadrangle (4 random points) 3 permutations can be discerned of 4 vv-cyclic consecutive points.

Within a Quadrilateral (4 random lines) 3 permutations can be discerned of 4 vv-cyclic consecutive lines.

Knowing a Quadrigon it can belong to a Quadrangle as well as to a Quadrilateral.

It can be seen as the “intersection” of a Quadrangle and a Quadrilateral.

A Quadrigon occurs where Quadrangle and Quadrilateral meet.

Because in a Quadrigon lines as well as points play a role the algebraic description of a Quadrigon can be done with QA-coordinates (see QA/1) as well as with QL-coordinates (see QL/1). The only disadvantage in both cases is that the 3 coordinates of a QG-point will not be cyclic. However when we observe the coordinates of 3 QG-points in the 3 Quadrigons implied by the 3 vv-cyclic permutations within a Quadrangle/Quadrilateral they can be seen as “tricyclic symmetric points”, meaning that the components of their coordinates rotate as well as are cyclically re-ordered.

There is also another way of algebraic description in a Quadrigon.

I found that a simple Cartesian coordinate system with projective coordinates (i.e. a normal x- and y-coordinate combined with a z-coordinate for indicating infinity points) also gives symmetric results in the x- and y-coordinate.

This is not surprising because a Cartesian coordinate system has 4 quadrants in each of which we can pin a vertex even so that the diagonals concur with the origin.

However because of the relationship with Quadrangles and Quadrilaterals this method will not be used yet in this description.

QG/2: List of QG-Lines

In next list all lines are mentioned with at least one QG-point and additional at least 2 other QG-/QA-/QL-points on it.

All lines with points in the range QG-P1 – QG-P13 and QA-P1 - QA-P36 and QL-P1 - QL-P26 have been taken into account.

QG-P1, QG-P2, QA-P10	
QG-P1, QG-P3, QL-P8	
QG-P1, QG-P4, QG-P8, QA-P1	= QG-L3
QG-P1, QG-P5, QG-P9, QL-P16	
QG-P1, QG-P6, QG-P10	
QG-P1, QG-P7, QG-P11	
QG-P1, QG-P12, QG-P13, QA-P16, QL-P13	= QG-L2
QG-P2, QG-P12, QA-P1, QL-P5, QL-P7, QL-P12, QL-P20, QL-P22, QL-P23	= QL-L1
QG-P2, QG-P13, QG-P14	
QG-P2, QG-P4, QA-P5	
QG-P4, QG-P5, QG-P6, QG-P7	
QG-P5, QG-P10, QA-P1	
QG-P7, QG-P9, QA-P1	// QL-P2.QL-P7
QG-P8, QG-P9, QG-P10, QG-P11	
QG-P10, QL-P2, QL-P10	

QG/3: List of Parallel QG-Lines

In next list all lines with QG-points involved are mentioned that are parallel.

All lines with points in the range QA-P1 - QA-P36 and QG-P1 – QG-P13 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

These lines with purely QG-points are parallel:

QG-P1.QG-P5 // QG-P7.QG-P10
QG-P1.QG-P6 // QG-P7.QG-P9 // QL-P2.QL-P7
QG-P1.QG-P14 // QG-P2.QG-P3
QG-P4.QG-P5 // QG-P8.QG-P9
QG-P4.QG-P9 // QG-P7.QG-P8
QG-P4.QG-P10 // QG-P5.QG-P8

These lines with QG-, QA- and QL-points are parallel:

QG-P1.QG-P2 // QG-P4.QA-P25 // QG-P8.QA-P26
QG-P1.QG-P4 // QG-P2.QA-P22
QG-P1.QG-P5 // QG-P7.QG-P10 // QG-P11.QA-P1
QG-P1.QG-P12 // QG-P2.QA-P31
QG-P1.QA-P2 // QG-P2.QA-P29 // QG-P4.QA-P34
QG-P1.QA-P3 // QG-P7.QA-P15 // QG-P8.QA-P34
QG-P1.QA-P5 // QG-P2.QA-P20 // QG-P4.QA-P10 // QG-P12.QA-P26
QG-P1.QA-P11 // QG-P2.QA-P13
QG-P1.QA-P12 // QG-P2.QA-P11 // QG-P6.QA-P24 // QG-P9.QA-P32
QG-P1.QA-P19 // QG-P2.QA-P16
QG-P1.QA-P20 // QG-P2.QG-P12 // QG-P14.QA-P5 // QG-P8.QA-P10
QG-P1.QA-P24 // QG-P8.QA-P14
QG-P1.QA-P30 // QG-P2.QA-P36
QG-P1.QL-P9 // QG-P3.QL-P11
QG-P1.QL-P10 // QG-P3.QL-P9 // QG-P7.QL-P2
QG-P1.QL-P17 // QG-P3.QL-P25

QG-P2.QG-P4 // QG-P8.QA-P20 // QG-P12.QA-P25
QG-P2.QG-P5 // QG-P6.QA-P5
QG-P2.QG-P7 // QG-P5.QA-P5 // QG-P10.QA-P20
QG-P2.QA-P11 // QG-P6.QA-P24 // QG-P9.QA-P32
QG-P2.QA-P20 // QG-P4.QA-P10 // QG-P12.QA-P26
QG-P2.QA-P26 // QA-P25.QA-P31
QG-P2.QA-P29 // QG-P4.QA-P34

QG-P4.QG-P5 // QG-P8.QG-P9

QG-P4.QG-P9 // QG-P7.QG-P8
QG-P4.QG-P10 // QG-P5.QG-P8
QG-P4.QA-P2 // QG-P8.QA-P3
QG-P4.QA-P3 // QG-P8.QA-P2
QG-P4.QA-P10 // QG-P12.QA-P26
QG-P4.QA-P16 // QG-P8.QA-P21
QG-P4.QA-P20 // QG-P8.QA-P5
QG-P4.QA-P21 // QG-P8.QA-P16
QG-P4.QA-P25 // QG-P8.QA-P26
QG-P4.QA-P26 // QG-P8.QA-P25

QG-P5.QA-P2 // QG-P10.QA-P3
QG-P5.QA-P3 // QG-P10.QA-P2
QG-P5.QA-P5 // QG-P10.QA-P20
QG-P5.QA-P16 // QG-P10.QA-P21
QG-P5.QA-P20 // QG-P10.QA-P5
QG-P5.QA-P21 // QG-P10.QA-P16
QG-P5.QA-P25 // QG-P10.QA-P26
QG-P5.QA-P26 // QG-P10.QA-P25
QG-P5.QL-P17 // QG-P9.QL-P9

QG-P6.QA-P24 // QG-P9.QA-P32

QG-P7 QG-P10 // QG-P11.QA-P1
QG-P7.QA-P2 // QG-P9.QA-P3
QG-P7.QA-P3 // QG-P9.QA-P2
QG-P7.QA-P5 // QG-P9.QA-P20
QG-P7.QA-P15 // QG-P8.QA-P34
QG-P7.QA-P16 // QG-P9.QA-P21
QG-P7.QA-P20 // QG-P9.QA-P5
QG-P7.QA-P21 // QG-P9.QA-P16
QG-P7.QA-P25 // QG-P9.QA-P26
QG-P7.QA-P26 // QG-P9.QA-P25

QG-P8.QA-P10 // QG-P14.QA-P5
QG-P8.QA-P20 // QG-P12.QA-P25

QG-P10.QL-P2 // QL-P1.QL-P7

QG-P12.QL-P5 // QL-P2.QL-P3

QG/4: List of Perpendicular QG-Lines

In next list all lines with QG-points involved are mentioned that are perpendicular.

All lines with points in the range QA-P1 - QA-P36 and QG-P1 – QG-P13 have been taken into account.

When lines have more than 2 points on it, they are defined by the 2 points with lowest serial number.

QG-P1.QG-P3 \perp QG-P5.QA-P32
QG-P1.QG-P5 \perp QG-P3.QL-P1
QG-P1.QG-P6 \perp QG-P2.QG-P12 // QG-P1.QA-P20 // QG-P8.QA-P10 // QG-P14.QA-P5
// QG-P12.QL-P5 // QL-P2.QL-P3 // QL-L1 = Newton Line
QG-P1.QL-P10 \perp QG-P4.QL-P18
QG-P1.QG-P14 \perp QG-P1.QA-P12 // QG-P2.QA-P11 // QG-P6.QA-P24 // QG-P9.QA-P32
QG-P1.QG-P14 \perp QG-P1.QL-P10 // QG-P3.QL-P9 // QG-P7.QL-P2
QG-P1.QL-P17 \perp QG-P5.QL-P17 // QG-P9.QL-P9
QG-P1.QA-P2 \perp QG-P1.QA-P30 // QG-P2.QA-P36
QG-P1.QA-P3 \perp QG-P5.QA-P3 // QG-P10.QA-P2
QG-P2.QA-P12 \perp QG-P2.QG-P3
QG-P1.QA-P20 \perp QG-P7.QG-P9
QG-P1.QA-P30 \perp QG-P2.QA-P29 // QG-P4.QA-P34

QG-P2.QG-P3 \perp QG-P2.QA-P11 // QG-P6.QA-P24 // QG-P9.QA-P32
QG-P2.QG-P12 \perp QG-P7.QG-P9 // QG-P1.QG-P6
QG-P2.QA-P29 \perp QG-P2.QA-P36
QG-P2.QA-P36 \perp QG-P4.QA-P34
QG-P3.QL-P1 \perp QG-P7.QG-P10
QG-P3.QL-P7 \perp QG-P7.QL-P23
QG-P3.QL-P9 \perp QG-P4.QL-P18
QG-P3.QL-P19 \perp QG-P10.QL-P23
QG-P3.QL-P25 \perp QG-P5.QL-P17 // QG-P9.QL-P9

QG-P4.QL-P18 \perp QG-P7.QL-P2

QG-P5.QA-P3 \perp QG-P7.QA-P15 // QG-P8.QA-P34

QG-P7.QG-P9 \perp QG-P8.QA-P10 // QG-P14.QA-P5
QG-P7.QA-P15 \perp QG-P10.QA-P2

QG-P8.QA-P34 \perp QG-P10.QA-P2
QG-P9.QL-P6 \perp QG-P13.QL-P17

QG-P10.QL-P2 \perp QL-P18.QL-P23

QG-P12.QL-P5 \perp QL-P2.QL-P7

QG-P14.QA-P5 \perp QG-P7.QG-P9 // QG-P12.QL-P5 // QL-P2.QL-P3

QG/5: List of QG-Crosspoints

In next list sets of 3 lines are mentioned that concur.

All lines with points in the range QG-P1 – QG-P14, QA-P1 - QA-P21 and QL-P1 - QL-P23 have been taken into account.

Lines are defined by 2 points on it with lowest serial number.

There are regularly recurring crossing lines with these Crosspoints. This is an indication for the occurrence of Perspective Fields (see QG-PF1).

When the intersection points have fixed ratios of the distances to the defining points on the defining lines, then they are mentioned. There are many of them.

When there are no fixed ratios this is indicated by the remark “x : y”.

For point P on line P1.P2 the ratio $d1 : d2$ means that $d(P,P1) : d(P,P2) = d1 : d2$, where:

- $d1$ is positive when P is positioned wrt P1 at the same side of the line as P2. If not then $d1$ is negative.
- $d2$ is positive when P is positioned wrt P2 at the same side of the line as P1. If not then $d1$ is negative.

lines	-----ratios-----	comment
QG-P1.QG-P2 ^ QG-P10.QA-P14 ^ QG-P13.QA-P2	x:y x:y x:y	
QG-P1.QG-P2 ^ QG-P12.QG-P14 ^ QL-P18.QL-P23	x:y x:y x:y	(simple coordinates)
QG-P1.QG-P4 ^ QG-P2.QG-P12 ^ QG-P5.QG-P10	3:1 x:y 1:1	(simple coordinates)
^ QG-P7.QG-P9 ^ QG-P12.QL-P5	1:1 x:y	point on Newton Line
QG-P1.QG-P4 ^ QG-P2.QA-P20 ^ QG-P7.QG-P10	3:-1 -1:2 -1:2	
^ QG-P14.QA-P5 ^ QL-P18.QL-P23	x:y x:y	(simple coordinates)
QG-P1.QG-P4 ^ QG-P5.QG-P11 ^ QG-P6.QG-P9	3:2 4:1 4:1	(simple coordinates)
QG-P1.QG-P5 ^ QG-P7.QG-P10 ^ QG-P11.QA-P1		Infinity Point
QG-P1.QG-P6 ^ QG-P5.QG-P8 ^ QG-P11.QA-P1	1:3 3:-1 2:-1	
QG-P1.QG-P6 ^ QG-P7.QG-P9 ^ QL-P2.QL-P7		Infinity Point (simple coords)
QG-P1.QG-P7 ^ QG-P4.QG-P10 ^ QG-P6.QA-P1	4:1 2:3 1:4	
QG-P1.QG-P7 ^ QG-P5.QG-P10 ^ QG-P6.QG-P9	2:1 2:1 2:1	(Centroid QG-P7/P9/P10) (Centroid QG-P1/P5/P6)
QG-P1.QG-P14 ^ QG-P2.QG-P3 ^ QG-P6.QL-P18		Infinity Point (simple coords)
QG-P1.QA-P5 ^ QG-P2.QA-P1 ^ QG-P8.QA-P20	1:1 2:1 -1:3	point on Newton Line
QG-P1.QA-P5 ^ QG-P2.QA-P20 ^ QG-P4.QA-P10		Infinity Point
QG-P1.QA-P17 ^ QG-P4.QG-P12 ^ QG-P14.QA-P5	x:y -2:3 x:y	
QG-P1.QA-P19 ^ QG-P2.QG-P3 ^ QG-P14.QA-P5	x:y x:y x:y	
QG-P1.QA-P19 ^ QG-P2.QG-P12 ^ QG-P14.QA-P10	x:y x:y 3:1	
QG-P1.QA-P20 ^ QG-P2.QG-P3 ^ QG-P12.QA-P18	x:y x:y x:y	
^ QG-P14.QA-P10	3:1	
QG-P1.QA-P20 ^ QG-P2.QG-P12 ^ QG-P8.QA-P10		
^ QG-P14.QA-P5		Infinity Point Newton Line
QG-P1.QL-P9 ^ QG-P3.QL-P10 ^ QG-P5.QL-P17	2:-1 -1:2 x:y	point on QL-DT-circumcircle
QG-P1.QL-P10 ^ QG-P3.QL-P9 ^ QG-P7.QL-P2		Infinity Point (simple coords)
QG-P1.QL-P12 ^ QG-P2.QG-P4 ^ QG-P3.QL-P18	6:-1 3:2 3:2	

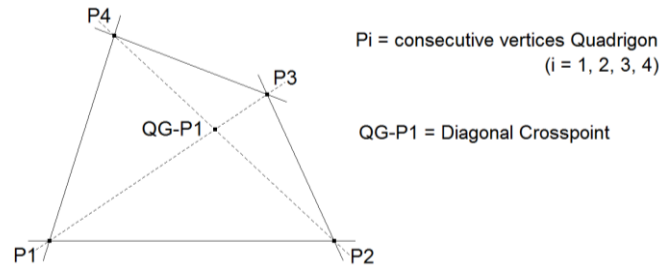
QG-P2.QG-P5	^	QG-P7.QA-P5	^	QG-P9.QA-P10	-1:2	-1:2	3:-2
QG-P2.QG-P6	^	QG-P5.QA-P5	^	QG-P10.QA-P10	-1:2	-1:2	3:-2
QG-P2.QG-P7	^	QG-P5.QA-P5	^	QG-P10.QA-P20			Infinity Point
QG-P3.QG-P7	^	QG-P5.QL-P9	^	QG-P10.QL-P11	x:y	x:y	x:y
QG-P4.QG-P9	^	QG-P5.QG-P11	^	QG-P6.QA-P1	4:3	4:3	8:-1
QG-P6.QG-P9	^	QG-P7.QG-P8	^	QG-P11.QA-P1	3:1	3:1	1:1
QG-P6.QL-P9	^	QG-P7.QL-P2	^	QG-P10.QL-P11	x:y	x:y	x:y

7.1 QUADRIGON POINTS

QG-P1: Diagonal Crosspoint

In a Quadrigon the vertices P_1, P_2, P_3, P_4 have a fixed order.

The Diagonal Crosspoint is the intersection point of the 2 lines connecting opposite points.



Four points can be ordered in 6 ways:

$P_1-P_2-P_3-P_4$

$P_1-P_2-P_4-P_3$

$P_1-P_3-P_2-P_4$

$P_1-P_3-P_4-P_2$ = reversed sequence of $P_1-P_2-P_4-P_3$

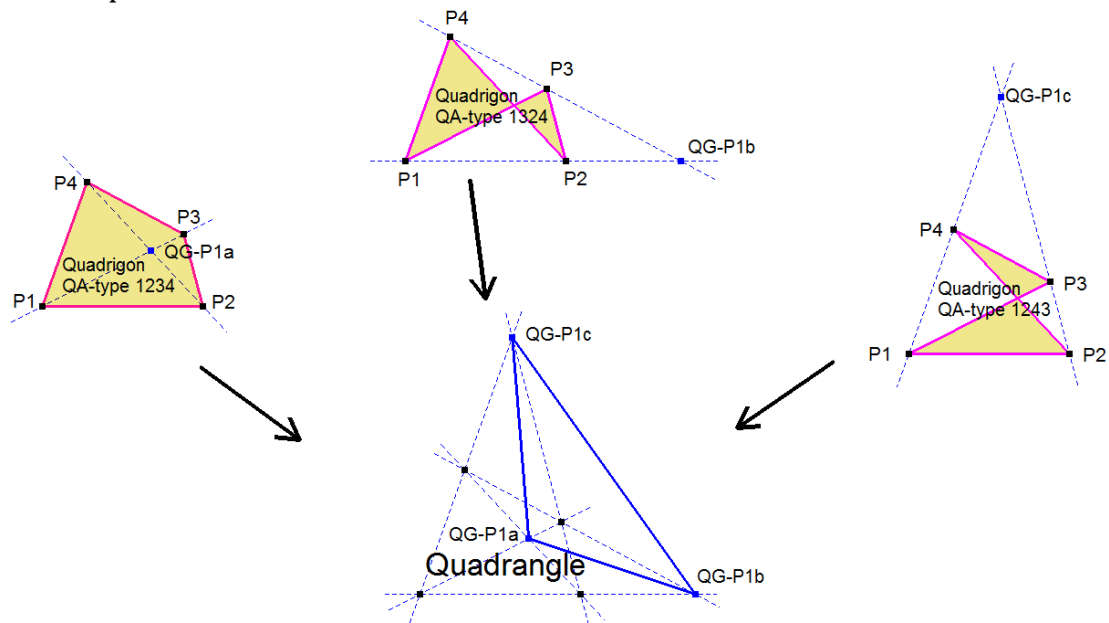
$P_1-P_4-P_2-P_3$ = reversed sequence of $P_1-P_3-P_2-P_4$

$P_1-P_4-P_3-P_2$ = reversed sequence of $P_1-P_2-P_3-P_4$

When we take into account that some sequences are reversed sequences, only 3 types of Quadrigon-orders remain: $P_1-P_2-P_3-P_4$, $P_1-P_2-P_4-P_3$ and $P_1-P_3-P_2-P_4$.

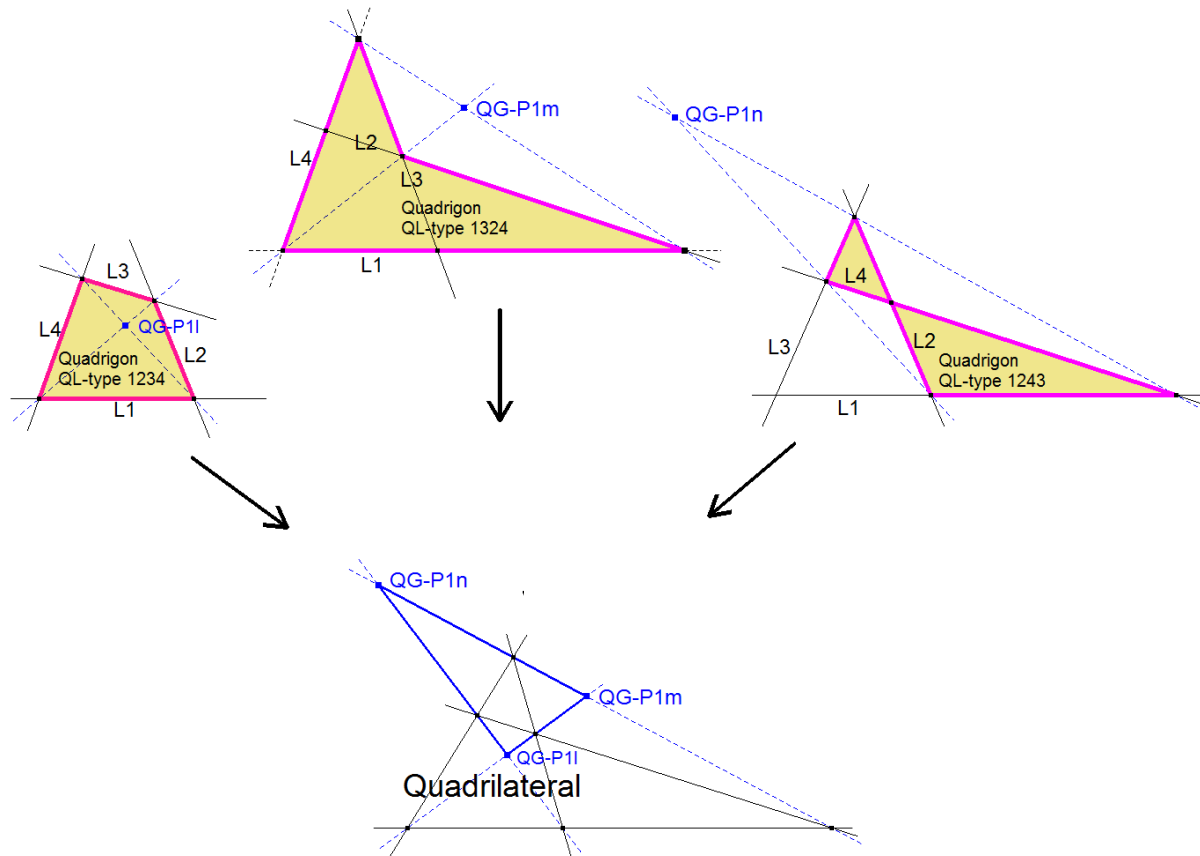
This means that a system of 4 points - also named (complete) quadrangle - consists of 3 quadrigons: $P_1-P_2-P_3-P_4$, $P_1-P_2-P_4-P_3$ and $P_1-P_3-P_2-P_4$.

These 3 quadrigons in a quadrangle produce different Diagonal Crosspoints as can be seen in next picture.



In the same way a system of 4 lines - also named (complete) quadrilateral - consists of 3 quadrigons: L1-L2-L3-L4, L1-L2-L4-L3 and L1-L3-L2-L4.

These 3 quadrigons in a quadrilateral produce different Diagonal Crosspoints as can be seen in next picture.



CT-Coordinates in 3 QA-Quadrigons:

- $(p : q : 0)$, $(0 : q : r)$ and $(p : 0 : r)$

CT-Coordinates in 3 QL-Quadrigons:

- $(-m n : l n : l m)$, $(m n : -l n : l m)$ and $(m n : l n : -l m)$

CT-Area of QG-P1-Triangle in the QA-environment:

- $2 p q r \Delta / ((p + q)(p + r)(q + r))$ = area QA-Diagonal Triangle

CT-Area of QG-P1-Triangle in the QL-environment:

- $4 l^2 m^2 n^2 \Delta / ((-l m + l n + m n)(l m + l n - m n)(l m - l n + m n))$ = area QL-Diagonal Triangle

DT-Coordinates in 3 QA-Quadrigons:

- $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$

DT-Coordinates in 3 QL-Quadrigons:

- $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$

DT-Area of QG-P1-Triangle in the QA-environment:

- $S / 2$

DT-Area of QG-P1-Triangle in the QL-environment:

- $S / 2$

Properties:

- The 3 Diagonal Crosspoints in a Quadrangle form the QA-Diagonal Triangle (see paragraph QA-Tr1: QA-Diagonal Triangle).
- The 3 Diagonal Crosspoints in a Quadrilateral form the QL-Diagonal Triangle (see paragraph QL-Tr1: QL-Diagonal Triangle).
- QG-P1 lies on these lines:
 - QG-P4.QG-P8 (2 : -1 = Reflection of QG-P4 in QG-P8)
 - QG-P5.QG-P9 (2 : -1 = Reflection of QG-P5 in QG-P9)
 - QG-P6.QG-P10 (2 : -1 = Reflection of QG-P6 in QG-P10)
 - QG-P7.QG-P11 (2 : -1 = Reflection of QG-P7 in QG-P11)
 - QG-P12.QG-P13
 - QA-P10.QG-P2 (-2 : 3 = QA-AntiComplement of QG-P2)
 - QL-P8.QG-P3 (-2 : 3 = QL-AntiComplement of QG-P3)
- QG-P1 is the point with minimal sum of distances to the vertices.
- QG-P1 is the Railway Watcher (see QL-L/1) of the 1st and 2nd Quasi Euler Line (note Eckart Schmidt).
- QG-P1 is the Clawson-Schmidt Conjugate (QL-Tf1) of QA-P4 (Quadrangle Isogonal Center).
- QA-P1 (Quadrangle Centroid) lies on QG-P1.QG-P4.QG-P8

QG-P2 Midpoint 3rd QA-Diagonal

We use this terminology.

A “QA-Quadrignon” is a Quadrignon seen as the Component of a Quadrangle.

A “QL-Quadrignon” is a Quadrignon seen as the Component of a Quadrilateral.

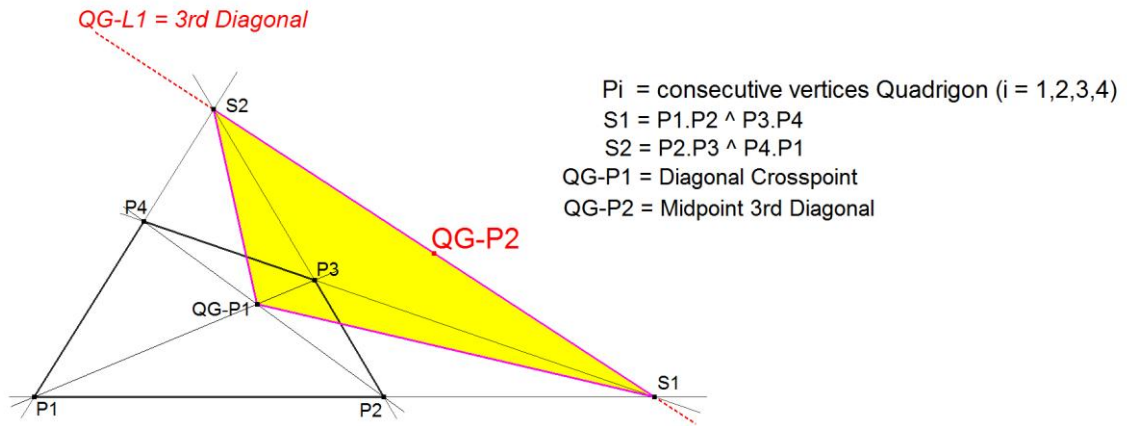
The 3rd Diagonal of a QA-Quadrignon is the same line as the 3rd Diagonal of a QL-Quadrignon. However the Midpoint of a QA-Quadrignon is different from the Midpoint of a QL-Quadrignon.

QG-P2 is the Midpoint of the segment on the 3rd Diagonal of a QA-Quadrignon limited by the intersection points with the 3rd Diagonals of the 2 other Component QA-Quadrignons.

Construction:

Let $S1 = P1.P2 \wedge P3.P4$ and $S2 = P2.P3 \wedge P4.P1$.

QG-P2 = the Midpoint of $S1$ and $S2$.



CT-Coordinates QG-P2 in 3 QA-Quadrignons:

- $(p(2p + q + r) : q(p + r) : r(p + q))$
- $(p(q + r) : q(p + 2q + r) : r(p + q))$
- $(p(q + r) : q(p + r) : r(p + q + 2r))$

CT-Coordinates QG-P2 in 3 QL-Quadrignons:

- $(m - n : -n : m)$
- $(n : n - l : -l)$
- $(-m : l : l - m)$

CT-Area of QG-P2-Triangle in the QA-environment:

(equals $\frac{1}{4}$ x area QL-Diagonal Triangle)

- $pqr \Delta / (2(p + q)(p + r)(q + r))$

CT-Area of QG-P2-Triangle in the QL-environment:

(points are collinear)

- 0

DT-Coordinates QG-P2 in 3 QA-Quadrilaterals:

- $(1 : 0 : 1)$
- $(0 : 1 : 1)$
- $(1 : 1 : 0)$

DT-Coordinates QG-P2 in 3 QL-Quadrilaterals:

- $(n^2 : 0 : -l^2)$
- $(0 : n^2 : -m^2)$
- $(m^2 : -l^2 : 0)$

DT-Area of QG-P2-Triangle in the QA-environment: (equals $\frac{1}{4}$ area QA-Diagonal Triangle)

- $S / 8$

DT-Area of QG-P2-Triangle in the QL-environment: (points are collinear)

- 0

Properties:

- QG-P2 lies on these lines:
 - QG-P1.QA-P10
 - QG-P13.QG-P14
 - QA-P1.QG-P12 = Newton Line
 - QA-P5.QG-P4
 - QG-L1 = 3rd Diagonal of the Quadrilateral
- QG-P2 is the fourth harmonic point of QG-P12 (Inscribed Harmonic Conic Center) on the Newton Line (QL-L1) wrt the midpoints of the diagonals (note Eckart Schmidt).
- The triangle formed by the 3 QA-Versions of QG-P2 is the medial triangle of the QA-Diagonal Triangle.
- The 3 QL-Versions of QG-P2 are 3 points on the Newton Line.

QG-P3 Midpoint 3rd QL-Diagonal

We use this terminology.

A “QA-Quadrignon” is a Quadrignon seen as the Component of a Quadrangle.

A “QL-Quadrignon” is a Quadrignon seen as the Component of a Quadrilateral.

The 3rd Diagonal of a QA-Quadrignon is the same line as the 3rd Diagonal of a QL-Quadrignon. However the Midpoint of a QA-Quadrignon is different from the Midpoint of a QL-Quadrignon.

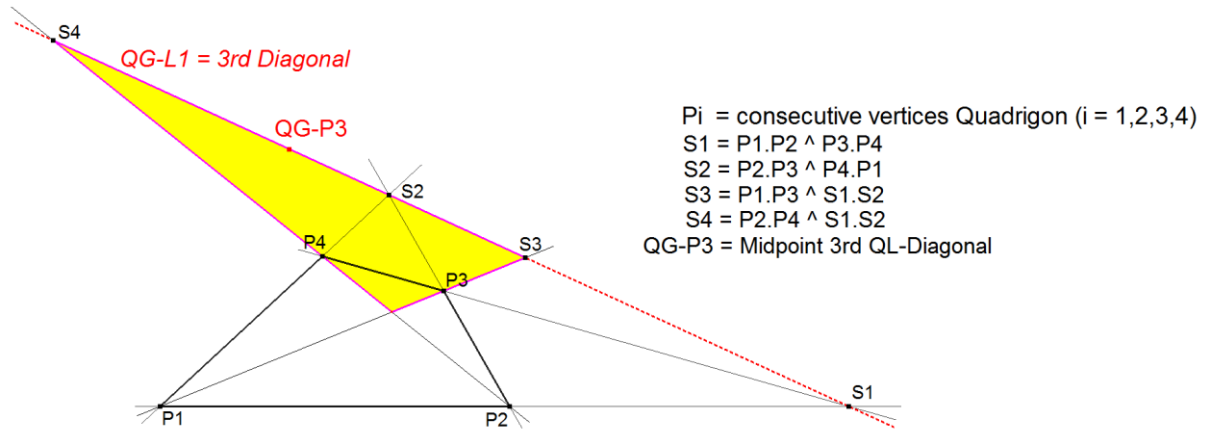
QG-P3 is the Midpoint of the segment on the 3rd Diagonal of a QL-Quadrignon limited by the intersection points with the 3rd Diagonals of the 2 other Component QL-Quadrignons.

Construction:

Let $S1 = P1.P2 \wedge P3.P4$ and $S2 = P2.P3 \wedge P4.P1$. $S1.S2$ is the 3rd Diagonal.

Let $S3 = P1.P3 \wedge S1.S2$ and $S4 = P2.P4 \wedge S1.S2$.

QG-P3 = the Midpoint of $S3$ and $S4$.



CT-Coordinates QG-P3 in 3 QA-Quadrignons:

- $(p(p+r) : -q(q+r) : r(p-q))$
- $(-p(p+q) : q(-p+r) : r(q+r))$
- $(p(q-r) : q(p+q) : -r(p+r))$

CT-Coordinates QG-P3 in 3 QL-Quadrignons:

- $(m^2 n^2 : -n l^2 (n-m) : m l^2 (n-m))$
- $(m^2 (l-n) n : l^2 n^2 : -l m^2 (l-n))$
- $(-m n^2 (m-l) : l(m-l) n^2 : l^2 m^2)$

CT-Area of QG-P3-Triangle in the QA-environment:

(points are collinear)

- 0

CT-Area of QG-P3-Triangle in the QL-environment:

(equals $\frac{1}{4}$ x area QL-Diagonal Triangle)

- $l^2 m^2 n^2 / ((-l m + l n + m n) (l m + l n - m n) (l m - l n + m n))$

DT-Coordinates QG-P3 in 3 QA-Quadrilaterals:

- $(0 : -q^2 : r^2)$
- $(-p^2 : 0 : r^2)$
- $(-p^2 : q^2 : 0)$

DT-Coordinates QG-P3 in 3 QL-Quadrilaterals:

- $(0 : 1 : 1)$
- $(1 : 0 : 1)$
- $(1 : 1 : 0)$

DT-Area of QG-P3-Triangle in the QA-environment: (points are collinear)

- 0

DT-Area of QG-P3-Triangle in the QL-environment: (equals $\frac{1}{4}$ x area QL-Diagonal Triangle)

- $S / 8$

Properties:

- QG-P3 lies on these lines:
 - QG-L1 = 3rd Diagonal
 - QG-P1.QL-P8
- The Triangle formed by the 3 QA-Versions of QG-P3 is the medial triangle of the QL-Diagonal Triangle.
- The 3 QA-Versions of QG-P3 are collinear points.

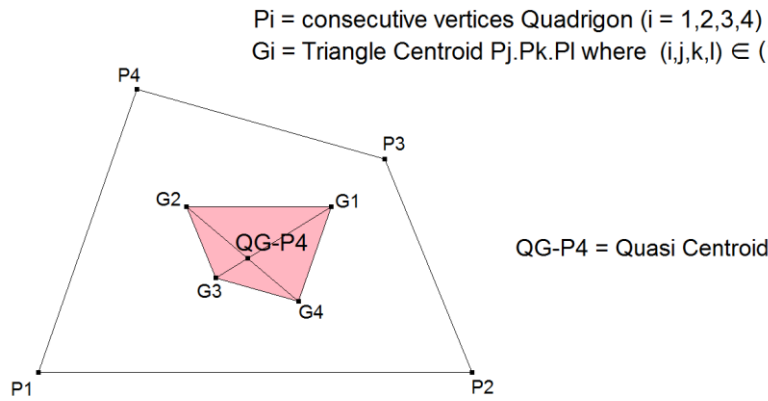
QG-P4: 1st Quasi Centroid

The 1st Quasi Centroid is the Diagonal Crosspoint of the X2-Quadrignon.

The X2-Quadrignon is defined by its vertices being the Triangle Centroids of the component triangles of the Reference Quadrignon.

This point is actually the center of mass of the Quadrignon when it is convex and when its surface is being made of some evenly distributed material.

This point and other 1st Quasi points are described in [5].



CT-Coordinates QG-P4 in 3 QA-Quadrignons:

- $((p+r)^2 + q r : (p+r)(p+2q+r) : (p+r)^2 + p q)$
- $((q+r)(2p+q+r) : (q+r)^2 + p r : (q+r)^2 + p q)$
- $((p+q)^2 + q r : (p+q)^2 + p r : (p+q)(p+q+2r))$

CT-Coordinates QG-P4 in 3 QL-Quadrignons:

- * $((lm+ln+mn)^2 - lmn(l+4m+4n) : (n-l)(lm^2+l^2n+m^2n-2lm(-l+m+n)) : (m-l)(ln^2+l^2m+mn^2)-2ln(-l+m+n))$
- * $((n-m)(l^2m+m^2n+l n-2lm(l-m+n)) : (lm+ln+mn)^2 - lmn(4l+m+4n) : (l-m)(lm^2+mn^2+ln^2-2mn(l-m+n)))$
- * $((m-n)(l^2m+l^2n+mn^2-2ln(l+m-n)) : (l-n)(lm^2+m^2n+ln^2-2mn(l+m-n)) : (lm+ln+mn)^2 - lmn(4l+4m+n))$

CT-Area of QG-P4-Triangle in the QA-environment:

- $2 p q r \Delta / (9 (p+q)(p+r)(q+r))$ (equals $1/9$ * area QA-Diagonal Triangle)

CT-Area of QG-P4-Triangle in the QL-environment:

- $4 l^2 m^2 n^2 \Delta / (9 (-lm+ln+mn)(lm+ln-mn)(lm-ln+mn))$ (equals $1/9$ * area QL-Diagonal Triangle)

DT-Coordinates QG-P4 in 3 QA-Quadrignons:

- $(-4 p^2 (p^2-q^2-r^2) : (p^2-r^2)^2 - 2 q^2 (-p^2+q^2-r^2)- q^4 : -4 r^2 (-p^2-q^2+r^2))$
- $((r^2-q^2)^2 - 2 p^2 (-r^2+p^2-q^2)- p^4 : -4 q^2 (-r^2-p^2+q^2) : -4 r^2 (r^2-p^2-q^2))$
- $(-4 p^2 (-q^2-r^2+p^2) : -4 q^2 (q^2-r^2-p^2) : (q^2-p^2)^2 - 2 r^2 (-q^2+r^2-p^2)- r^4)$

DT-Coordinates QG-P4 in 3 QL-Quadrilaterals:

- $(2 m^2 (m^2 - n^2) : -m^2 (l^2 + m^2 + n^2) + 3 l^2 n^2 : 2 m^2 (-l^2 + m^2))$
- $(-l^2 (l^2 + m^2 + n^2) + 3 n^2 m^2 : 2 l^2 (-n^2 + l^2) : 2 l^2 (l^2 - m^2))$
- $(2 n^2 (-m^2 + n^2) : 2 n^2 (n^2 - l^2) : -n^2 (l^2 + m^2 + n^2) + 3 m^2 l^2)$

DT-Area of QG-P4-Triangle in the QA-environment:

$$S / 18$$

(equals $1/9$ * area QA-Diagonal Triangle)

DT-Area of QG-P4-Triangle in the QL-environment:

$$S / 18$$

(equals $1/9$ * area QL-Diagonal Triangle)

Properties:

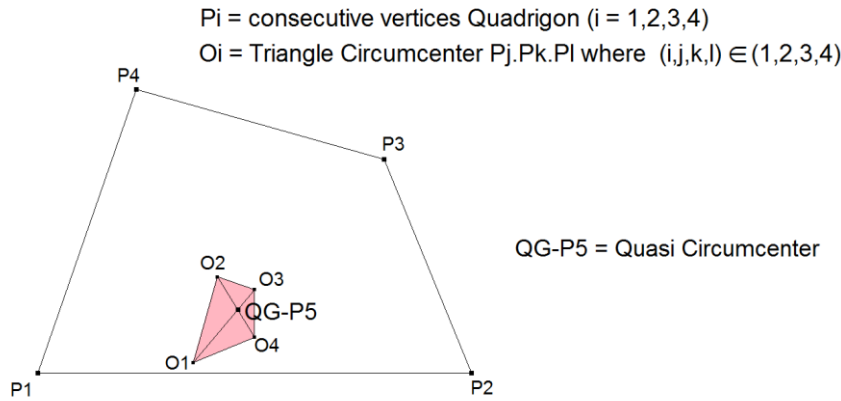
- QG-P4, QG-P5, QG-P6, QG-P7 are collinear on a 1st Quasi Euler Line.
- QG-P4 is the Reflection of QG-P1 in QG-P8.
- QG-P4 is the Reflection of QG-P8 in QA-P1 (QA-Centroid).
- QG-P4 is the Centroid of the Triangle formed by QA-P5 and the two vertices of the QA-Diagonal Triangle unequal QG-P1.
- QA-P25 is the Centroid of the triangle formed by the 3 QA-versions of QG-P4.
- QL-P14 is the Centroid of the triangle formed by the 3 QL-versions of QG-P4.
- The triangle formed by the 3 QA-versions of QG-P4 is homothetic and perspective with the QA-Diagonal Triangle. The side lengths are $1/3$ of the side lengths of the QA-Diagonal triangle. Their Perspector is QL-P1.
- The area of the triangle formed by the 3 QA-versions of QG-P4 equals $1/9$ * the area of the QA-Diagonal Triangle.
- The area of the triangle formed by the 3 QL-versions of QG-P4 also equals $1/9$ * the area of the QL-Diagonal Triangle. However both triangles are not homothetic neither perspective.

QG-P5: 1st Quasi Circumcenter

The 1st Quasi Circumcenter is the Diagonal Crosspoint of the X3-Quadrignon.

The X3-Quadrignon is defined by its vertices being the Triangle Circumcenters of the component triangles of the Reference Quadrignon.

This point and other 1st quasi points are described in [5].



CT-coordinates QG-P5 in 3 QA-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $(a^2 S_A (p^2 + r^2) - (a^2 + c^2) S_C p r - (a^2 - c^2) S_B p q - a^2 (a^2 - c^2) q r + 8 \Delta^2 p r) :$
 $b^2 S_B (p^2 + q r + r^2 + p q) + a^2 b^2 r (p + q) + b^2 c^2 p (q + r) :$
 $c^2 S_C (p^2 + r^2) - (a^2 + c^2) S_A p r + (a^2 - c^2) S_B q r + c^2 (a^2 - c^2) p q + 8 \Delta^2 p r)$

CT-coordinates QG-P5 in 3 QL-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $(b^2 c^2 l m (m - n)^2 - c^4 l m (m - n)^2 + a^4 (l - m) (l m^2 + l n^2 - m n^2) + a^2 (-b^2 (l - m) (l m^2 - 2 l m n + 2 m^2 n + l n^2 - m n^2) + c^2 (-l^2 m^2 + 2 l m^3 + 2 l^2 m n - 2 l m^2 n - l^2 n^2 + l m n^2 - m^2 n^2)) :$
 $-a^4 (l - m) l m n + c^4 l m (m - n) n + b^4 (l - m) (m - n) (l m - l n + m n) - b^2 c^2 (m - n) (l^2 m + l m^2 - l^2 n + l m n - m^2 n) + a^2 (c^2 l m n (l - 2 m + n) - b^2 (l - m) (l m^2 - l m n - m^2 n + l n^2 - m n^2)) :$
 $-a^4 (l - m)^2 m n + c^4 (m - n) (l^2 m - l^2 n - m^2 n) - b^2 c^2 (m - n) (l^2 m - 2 l m^2 - l^2 n + 2 l m n - m^2 n) + a^2 (b^2 (l - m)^2 m n + c^2 (-l^2 m^2 + l^2 m n - 2 l m^2 n + 2 m^3 n - l^2 n^2 + 2 l m n^2 - m^2 n^2)))$

CT-Area of QG-P5-Triangle in the QA-environment: (equals 4x area QA-Morley Triangle)

- $(c^2 p q + b^2 p r + a^2 q r)^2 / (8 \Delta (p + q) (p + r) (q + r) (p + q + r))$

CT-Area of QG-P5-Triangle in the QL-environment:

- $((a^2 m n (l - m) (l - n) (l m - m n + l n) + b^2 n l (m - l) (m - n) (l m + m n - l n) + c^2 l m (n - l) (n - m) (-l m + m n + l n))^2) / (16 \Delta (l - m)^2 (l - n)^2 (m - n)^2 (-l m + l n + m n) (l m + l n - m n) (l m - l n + m n))$

DT-coordinates QG-P5 in 3 QA-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $(2 S_b (S_a p^2 + S_b q^2) (p^2 - r^2) + S_c (p^2 - q^2 - r^2) (S_a p^2 + S_c r^2) + S_b S_c (p^4 - p^2 q^2 - 3 q^2 r^2 - r^4) :$
 $S_a (S_a p^2 + S_b q^2) (p^2 + q^2 - r^2) + S_c (-p^2 + q^2 + r^2) (S_b q^2 + S_c r^2) - S_a S_c q^2 (p^2 - q^2 + r^2) :$
 $-2 S_b (p^2 - r^2) (S_b q^2 + S_c r^2) - S_a (p^2 + q^2 - r^2) (S_a p^2 + S_c r^2) - S_a S_b (p^4 + 3 p^2 q^2 + q^2 r^2 - r^4))$

DT-coordinates QG-P5 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(a^2 m^2 (c^2 n^2 - S_a m^2 - S_b l^2) : m^2 (S_b S_c l^2 + S_a S_c m^2 + S_a S_b n^2) - S^2 l^2 n^2 : c^2 m^2 (a^2 l^2 - S_c m^2 - S_b n^2))$

DT-Area of QG-P5-Triangle in the QA-environment: (equals 4x area QA-Morley Triangle)

- $-(S_a p^2 + S_b q^2 + S_c r^2)^2 / (2 S (-p+q+r) (p+q-r) (p-q+r) (p+q+r))$

DT-Area of QG-P5-Triangle in the QL-environment:

- $(S_c (l^2 - m^2)^2 n^2 + S_b (l^2 - n^2)^2 m^2 + S_a (m^2 - n^2)^2 l^2)^2 / (2 S (l^2 - m^2)^2 (l^2 - n^2)^2 (m^2 - n^2)^2)$

Properties:

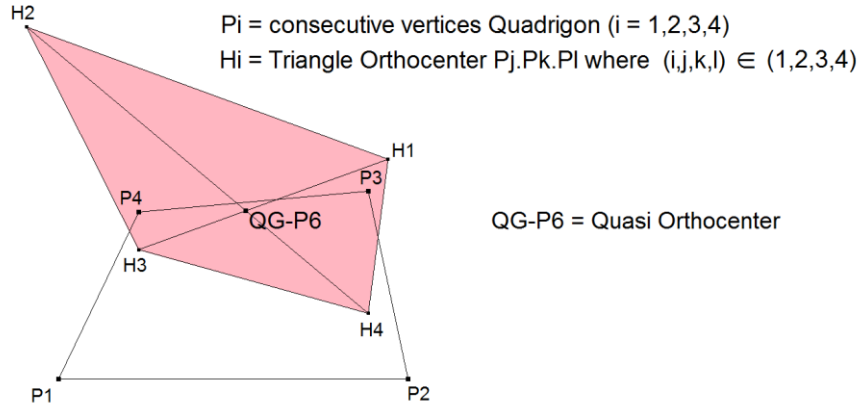
- QG-P4, QG-P5, QG-P6, QG-P7 are collinear on a 1st QG-Quasi Euler Line.
- QG-P5 is the Reflection of QG-P1 in QG-P9 AND THE Reflection of QG-P6 in QG-P7.
- QG-P5 is the intersection point of the perpendicular bisectors of the diagonals.
- QA-P3 (Gergonne-Steiner Point) lies on the circumcircle of the triangle formed by the 1st Quasi Circumcenters of the 3 QA-Quadrilaterals.
- The area of the triangle formed by the 1st Quasi Circumcenters of the 3 QA-Quadrilaterals equals 4x the area of the QA-Morley Triangle (see QA-P24). Both triangles are homothetic with homothetic center QA-P24.
- QL-P16 lies on the QG-P5 circle in the QL-environment and on the QG-P9 circle in the QL-environment. The center of the first circle lies on the second circle.

QG-P6: 1st Quasi Orthocenter

The 1st Quasi Circumcenter is the Diagonal Crosspoint of the X4-Quadrignon.

The X4-Quadrignon is defined by its vertices being the Triangle Orthocenters of the component triangles of the Reference Quadrignon.

This point and other 1st Quasi points are described in [5].



CT-Coordinates QG-P6 in 3 QA-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $$(S_B S_C (p^2 + r^2) + (a^2 - c^2) S_B p q + (a^2 + S_B) S_C q r + S_C (a^2 + c^2) p r,$$

$$S_A S_C (p^2 + r^2) + (a^2 - c^2) S_A p q - (a^2 - c^2) S_C q r - (a^4 + b^4 - 2 a^2 c^2 + c^4) p r/2,$$

$$S_A S_B (p^2 + r^2) - (a^2 - c^2) S_B q r + (c^2 + S_B) S_A p q + S_A (a^2 + c^2) p r)$$

CT-Coordinates QG-P6 in 3 QL-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $$(c^4 (m - n) (l^2 m - l^2 n - m^2 n) - 2 b^2 c^2 (m - n) (l^2 m - l m^2 - l^2 n + l m n - m^2 n) + b^4 (m - n) (l^2 m - 2 l m^2 - l^2 n + 2 l m n - m^2 n) + a^4 (-l^2 m^2 - 2 l^2 m n + 4 l m^2 n - m^3 n - l^2 n^2 + 2 l m n^2 - m^2 n^2) + a^2 (2 b^2 m^3 (l - n) - 2 c^2 m n (2 l m - m^2 - l n)) :$$

$$2 b^2 c^2 m (l m^2 - l m n - m^2 n + l n^2) + b^4 (-l^2 m^2 + 2 l m^3 + 2 l^2 m n - 5 l m^2 n + 2 m^3 n - l^2 n^2 + 2 l m n^2 - m^2 n^2) + c^4 (l^2 m^2 - 2 l^2 m n - l m^2 n + l^2 n^2 + m^2 n^2) + a^4 (l^2 m^2 - l m^2 n + l^2 n^2 - 2 l m n^2 + m^2 n^2) + a^2 (2 b^2 m (-l m^2 + l^2 n - l m n + m^2 n) - 2 c^2 (l^2 m^2 - l^2 m n - l m^2 n + l^2 n^2 - l m n^2 + m^2 n^2)) :$$

$$-2 b^2 c^2 m^3 (l - n) + a^4 (l - m) (l m^2 + l n^2 - m n^2) + b^4 (l - m) (l m^2 - 2 l m n + 2 m^2 n + l n^2 - m n^2) + c^4 (-l^2 m^2 - l m^3 + 2 l^2 m n + 4 l m^2 n - l^2 n^2 - 2 l m n^2 - m^2 n^2) + a^2 (2 c^2 l m (m^2 + l n - 2 m n) - 2 b^2 (l - m) (l m^2 - l m n + m^2 n + l n^2 - m n^2)))$$

CT-Area of QG-P6-Triangle in the QA-environment:

$$2 p q r \Delta / ((p + q) (p + r) (q + r)) \quad (\text{equals area QA-Diagonal Triangle})$$

CT-Area of QG-P6-Triangle in the QL-environment:

$$4 l^2 m^2 n^2 \Delta / ((l m - l n - m n) (l m + l n - m n) (l m - l n + m n)) \quad (\text{equals area QL-Diagonal Triangle})$$

DT-Coordinates QG-P6 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(-2 (c^2 p^2 - a^2 r^2) (Sc (-p^2 + q^2 + r^2) + 2 Sb q^2) : -S^2 (p^2 - q^2 - r^2) (p^2 + q^2 - r^2) - 2 Sa^2 p^2 (p^2 + q^2 - r^2) + 2 Sc^2 r^2 (p^2 - q^2 - r^2) - 4 Sb q^2 (Sa p^2 + Sc r^2) : 2 (Sa (p^2 + q^2 - r^2) + 2 Sb q^2) (c^2 p^2 - a^2 r^2))$

DT-Coordinates QG-P6 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(-2 Sb m^2 (Sb (l^2 - n^2) + Sc (l^2 - m^2)) : -S^2 (m^2 (l^2 + m^2 + n^2) - l^2 n^2) + 2 m^2 (Sb Sc l^2 + Sa Sc m^2 + Sa Sb n^2) : 2 Sb m^2 (Sb (l^2 - n^2) + Sa (m^2 - n^2)))$

DT-Area of QG-P6-Triangle in the QA-environment:

- $S/2$ (equals area QA-Diagonal Triangle)

DT-Area of QG-P6-Triangle in the QL-environment:

- $S/2$ (equals area QL-Diagonal Triangle)

Properties:

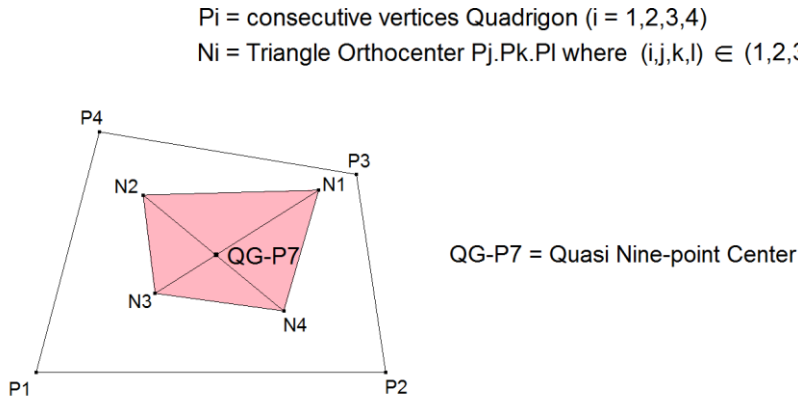
- QG-P4, QG-P5, QG-P6, QG-P7 are collinear on a 1st Quasi Euler Line.
- QA-P2 (Euler-Poncelet Point) lies on the circumcircle of the triangle formed by the 1st Quasi Orthocenters of the 3 QA-Quadrilaterals.
- The area of the triangle formed by the 1st Quasi Orthocenters of the 3 QA-Quadrilaterals equals the area of the QA-Diagonal Triangle.
- The area of the triangle formed by the 1st Quasi Orthocenters of the 3 QL-Quadrilaterals equals the area of the QL-Diagonal Triangle.
- The triangle formed by the 1st Quasi Orthocenters of the 3 QL-Quadrilaterals is perspective with the QL-Diagonal Triangle. The Perspector is the infinity point of QL-L2 (Steiner Line).
- QG-P6 is the Reflection of QG-P1 in QG-P10.
- The line through the 1st and 2nd Quasi Orthocenters (QG-P6 and QG-P10) is perpendicular to the Newton Line (QL-L1).

QG-P7: 1st Quasi Nine-point Center

The 1st Quasi Circumcenter is the Diagonal Crosspoint of the X5-Quadrignon.

The X5-Quadrignon is defined by its vertices being the Triangle Nine-point Centers of the component triangles of the Reference Quadrignon.

This point and other 1st Quasi points are described in [5].



CT-Coordinates QG-P7 in 3 QA-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $$(a^4 q (p+r) - c^4 (p+r) (p+q+r) + b^4 (-p^2 - p r - q r - r^2) + b^2 c^2 (2 p^2 + p q + 3 p r + 2 q r + 2 r^2) + a^2 (c^2 (p+r)^2 + b^2 (p^2 - p q + 3 p r + 2 q r + r^2))) :$$

$$-a^4 (p+r) (p+2 q+r) - c^4 (p+r) (p+2 q+r) + b^4 (-p q - 2 p r - q r) + b^2 c^2 (p^2 + p q + 2 p r + 3 q r + r^2) + a^2 (2 c^2 (p+r) (p+2 q+r) + b^2 (p^2 + 3 p q + 2 p r + q r + r^2)) :$$

$$c^4 q (p+r) - a^4 (p+r) (p+q+r) + b^4 (-p^2 - p q - p r - r^2) + b^2 c^2 (p^2 + 2 p q + 3 p r - q r + r^2) + a^2 (c^2 (p+r)^2 + b^2 (2 p^2 + 2 p q + 3 p r + q r + 2 r^2)))$$

CT-Coordinates QG-P7 in 3 QL-Quadrignons: (only coordinates of 1st Quadrignon point are given)

- $$((-a^4 - b^4 + 2 a^2 c^2 + 2 b^2 c^2 - c^4) m^3 n + (-a^2 b^2 + b^4 - a^2 c^2 - 2 b^2 c^2 + c^4) m^2 n^2 + l^2 ((-a^2 b^2 + b^4 - a^2 c^2 - 2 b^2 c^2 + c^4) m^2 - 2 (a^4 - a^2 b^2 + b^4 - a^2 c^2 - 2 b^2 c^2 + c^4) m n + (-a^2 b^2 + b^4 - a^2 c^2 - 2 b^2 c^2 + c^4) n^2) + l ((-a^4 + 3 a^2 b^2 - 2 b^4 + 2 a^2 c^2 + 3 b^2 c^2 - c^4) m^3 + 2 (2 a^4 - 2 a^2 b^2 + 2 b^4 - 3 a^2 c^2 - 3 b^2 c^2 + c^4) m^2 n + (2 a^2 b^2 - 2 b^4 + 3 a^2 c^2 + 3 b^2 c^2 - c^4) m n^2) :$$

$$b^2 (a^2 + b^2 - c^2) m^3 n + (a^4 - a^2 b^2 - 2 a^2 c^2 - b^2 c^2 + c^4) m^2 n^2 + l^2 ((a^4 - a^2 b^2 - 2 a^2 c^2 - b^2 c^2 + c^4) m^2 + (-a^4 + 3 a^2 b^2 + 3 a^2 c^2 + 2 b^2 c^2 - 2 c^4) m n + (a^4 - a^2 b^2 - 2 a^2 c^2 - b^2 c^2 + c^4) n^2) + l (-b^2 (a^2 - b^2 - c^2) m^3 - 2 b^2 (a^2 + b^2 + c^2) m^2 n + (-2 a^4 + 2 a^2 b^2 + 3 a^2 c^2 + 3 b^2 c^2 - c^4) m n^2) :$$

$$(-a^4 + 3 a^2 b^2 - 2 b^4 + 2 a^2 c^2 + 3 b^2 c^2 - c^4) m^3 n + (a^4 - 2 a^2 b^2 + b^4 - a^2 c^2 - b^2 c^2) m^2 n^2 + l^2 ((a^4 - 2 a^2 b^2 + b^4 - a^2 c^2 - b^2 c^2) m^2 + (-a^4 + 3 a^2 b^2 - 2 b^4 + 3 a^2 c^2 + 2 b^2 c^2) m n + (a^4 - 2 a^2 b^2 + b^4 - a^2 c^2 - b^2 c^2) n^2) + l ((-a^4 + 2 a^2 b^2 - b^4 + 2 a^2 c^2 - c^4) m^3 + 2 (a^4 - 3 a^2 b^2 + 2 b^4 - 3 a^2 c^2 - 2 b^2 c^2 + 2 c^4) m^2 n - 2 (a^4 - 2 a^2 b^2 + b^4 - a^2 c^2 - b^2 c^2 + c^4) m n^2))$$

CT-Area of QG-P7-Triangle in the QA-environment:

$$\frac{(a^4 q r (p+q) (p+r) + b^4 p r (p+q) (q+r) + c^4 p q (p+r) (q+r) - 2 b^2 c^2 p q r (q+r) - 2 a^2 c^2 p q r (p+r) - 2 a^2 b^2 p q r (p+q))}{(32 \Delta (p+q) (p+r) (q+r) (p+q+r))}$$

CT-Area of QG-P7-Triangle in the QL-environment:

- $T_1 T_2 / (64 \Delta (l-m)^2 (l-n)^2 (m-n)^2 (-l m + l n + m n) (l m + l n - m n) (l m - l n + m n))$
 where: $T_1 = a^2 m n (l-m) (l-n) + b^2 l n (m-l) (m-n) + c^2 l m (n-l) (n-m)$
 $T_2 = a^2 m n (l-m) (l-n) (-3 l m + l n + m n) (l m - 3 l n + m n)$
 $+ b^2 l n (m-l) (m-n) (l m + l n - 3 m n) (-3 l m + l n + m n)$
 $+ c^2 l m (n-l) (n-m) (l m + l n - 3 m n) (l m - 3 l n + m n)$

DT-Coordinates QG-P7 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $((-p^2+q^2+r^2)(Sc a^2 r^2+Sb Sa p^2-3 S^2 p^2)+2 Sb q^2 (a^2 r^2-c^2 p^2) :$
 $-(S^2+Sa^2) p^2 (p^2+q^2-r^2)-(S^2+Sc^2) r^2 (-p^2+q^2+r^2)+2 q^2 (-Sa Sb p^2+S^2 q^2-Sb Sc r^2) :$
 $(p^2+q^2-r^2) (-3 S^2 r^2+Sa c^2 p^2+Sb Sc r^2)+2 Sb q^2 (c^2 p^2-a^2 r^2))$

DT-Coordinates QG-P7 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(-m^2 (-Sb a^2 l^2+(S^2+ Sb Sc) m^2-(S^2-Sb^2) n^2) :$
 $Sa a^2 l^2 m^2+Sb b^2 m^4+c^2 Sc m^2 n^2-2 S^2 l^2 n^2 :$
 $-m^2 (-(S^2-Sb^2) l^2+(Sa Sb+S^2) m^2-Sb c^2 n^2))$

CT-Area of QG-P7-Triangle in the QA-environment:

- $-(a^2 b^2 r^4+b^2 c^2 p^4+a^2 c^2 q^4-2 Sa a^2 q^2 r^2-2 Sb b^2 p^2 r^2-2 c^2 Sc p^2 q^2)$
 $/ (8 S (-p+q+r) (p+q-r) (p-q+r) (p+q+r))$

CT-Area of QG-P7-Triangle in the QL-environment:

- $(Sc (l^2-m^2)^2+Sb (l^2-n^2)^2+Sa (m^2-n^2)^2) (Sc n^4 (l^2-m^2)^2+Sb m^4 (l^2-n^2)^2+Sa l^4 (m^2-n^2)^2)$
 $/ (8 S (l^2-m^2)^2 (l^2-n^2)^2 (m^2-n^2)^2)$

Properties:

- QG-P4, QG-P5, QG-P6, QG-P7 are collinear on a 1st Quasi Euler Line.
- QG-P7 is the Reflection of QG-P1 in QG-P11.
- QG-P7 is the Reflection of QG-P9 in the line QG-P2.QG-P12 (which coincides with the Newton Line QL-L1).
- QA-P1 (QA-Centroid) lies on the circumcircle of the triangle formed by the 3 QA-versions of QG-P7.
- QL-P2 (Morley Point) lies on the circumcircle of the triangle formed by the 3 QL-versions of QG-P7.
- The area of the QA-versions of QG-P7 = the area of the QA-versions of QG-P9.
- The area of the QL-versions of QG-P7 = the area of the QL-versions of QG-P9.

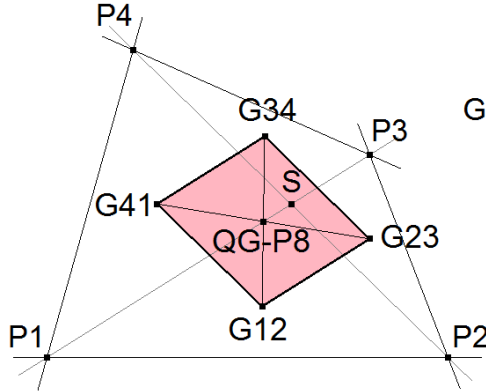
QG-P8 2nd Quasi Centroid

Let P1.P2.P3.P4 be a Quadrigon and let S be its Diagonal Crosspoint QG-P1.

Let G_{i+1} = Circumcenter of Triangle $G_i.G_{i+1}.S$ (i = cyclic sequence 1,2,3,4) .

Now is $G_{12}.G_{23}.G_{34}.G_{41}$ a parallelogram.

QG-P8 is the Center of this parallelogram.



P_i = consecutive Vertices of Quadrigon
($i = 1,2,3,4$)

G_{i+1} = Centroid Triangle $P_i.P_{i+1}.S$
(i = cyclic 1,2,3,4)

S = Diagonal Crosspoint

QG-P8 = 2nd Quasi Centroid

CT-Coordinates QG-P8 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $(-12 p^6 - 9 p^5 q - 63 p^5 r - 39 p^4 q r - 135 p^4 r^2 - 66 p^3 q r^2 - 150 p^3 r^3 - 54 p^2 q r^3 - 90 p^2 r^4 - 21 p q r^4 - 27 p r^5 - 3 q r^5 - 3 r^6 : -3 p^6 - 6 p^5 q - 18 p^5 r - 30 p^4 q r - 45 p^4 r^2 - 60 p^3 q r^2 - 60 p^3 r^3 - 60 p^2 q r^3 - 45 p^2 r^4 - 30 p q r^4 - 18 p r^5 - 6 q r^5 - 3 r^6 : -3 p^6 - 3 p^5 q - 27 p^5 r - 21 p^4 q r - 90 p^4 r^2 - 54 p^3 q r^2 - 150 p^3 r^3 - 66 p^2 q r^3 - 135 p^2 r^4 - 39 p q r^4 - 63 p r^5 - 9 q r^5 - 12 r^6)$

CT-Coordinates QG-P8 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $((m - n) (-l^2 m + 2 l m^2 + l^2 n - 5 l m n + 4 m^2 n) : -l^2 m^2 + 5 l^2 m n - 4 l m^2 n - 4 l^2 n^2 + 5 l m n^2 - m^2 n^2 : -(l - m) (4 l m^2 - 5 l m n + 2 m^2 n + l n^2 - m n^2))$

CT-Area of QG-P8-Triangle in the QA-environment:

- $2 p q r \Delta / (9 (p + q) (p + r) (q + r))$ (equals 1/9 area QA-Diagonal Triangle)

CT-Area of QG-P8-Triangle in the QL-environment: (equals 2/9 area QL-Diagonal Triangle)

- $8 l^2 m^2 n^2 \Delta / (9 (l m - l n - m n) (l m + l n - m n) (l m - l n + m n))$

DT-Coordinates QG-P8 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $(2 p^2 (p^2 - q^2 - r^2) : (p^2 - r^2)^2 - 4 q^2 (p^2 + r^2) + 3 q^4 : 2 r^2 (-p^2 - q^2 + r^2))$

DT-Coordinates QG-P8 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $(m^2 (m^2 - n^2) : -2 (l^2 - m^2)(m^2 - n^2) + (-m^4 + l^2 n^2) : m^2 (-l^2 + m^2))$

DT-Area of QG-P8-Triangle in the QA-environment:

- $S/18$ (equals 1/9 area QA-Diagonal Triangle)

DT-Area of QG-P8-Triangle in the QL-environment: (equals $2/9$ area QL-Diagonal Triangle)

- $S/9$

Properties:

- QG-P8, QG-P9, QG-P10, QG-P11 are collinear on a 2nd Quasi Euler Line (which is a parallel line to the 1st Quasi Euler Line).
- QG-P8 is the Midpoint of QG-P1 and QG-P4.
- QG-P8 is the Reflection of QG-P4 in QA-P1 (QA-Centroid).
- The QG-P8 Triangle in the QA-environment and the QA-Diagonal Triangle are homothetic with Homothetic Center QA-P1.
- The area of the QA-versions of QG-P8 = $1/9$ of the area of the QA-Diagonal Triangle.
- The area of the QL-versions of QG-P8 = $2/9$ of the area of the QL-Diagonal Triangle.

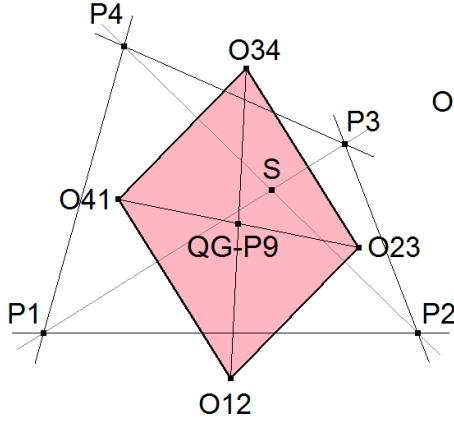
QG-P9 2nd Quasi Circumcenter

Let P1.P2.P3.P4 be a Quadrigon and let S be its Diagonal Crosspoint QG-P1.

Let O_i $i+1$ = Circumcenter of Triangle $O_i.O_{i+1}.S$ (i = cyclic sequence 1,2,3,4) .

Now is $O_{12}.O_{23}.O_{34}.O_{41}$ a parallelogram.

QG-P9 is the Center of this parallelogram.



P_i = consecutive Vertices of Quadrigon
($i = 1,2,3,4$)

O_i $i+1$ = Circumcenter Triangle $P_i.P_{i+1}.S$
(i = cyclic 1,2,3,4)

S = Diagonal Crosspoint

QG-P9 = 2nd Quasi Circumcenter

CT-Coordinates QG-P9 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $(-a^2(a^2 - c^2)qr + a^2(p^2 + r^2)S_A - (a^2 - c^2)pqS_B - (a^2 + c^2)prS_C + 8p(p + q + 2r)\Delta^2,$
 $b^2(c^2pq + a^2pr + c^2pr + a^2qr) + b^2(p^2 + pq + qr + r^2)S_B,$
 $-c^2(-a^2 + c^2)pq - (a^2 + c^2)prS_A + (a^2 - c^2)qrS_B + c^2(p^2 + r^2)S_C + 8r(2p + q + r)\Delta^2)$

CT-Coordinates QG-P9 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $(a^2m(b^2l^2 + c^2l^2 - 2b^2lm + b^2m^2)n - lm(b^2m^2 + a^2n^2 + b^2n^2)S_B + S_A(-a^2(l^2m^2 + l^2n^2 + m^2n^2)$
 $+ 2lm^2nS_B) + 8m(lm^2 - 2lmn + m^2n + 2ln^2 - mn^2)\Delta^2 :$
 $b^2(a^2 + b^2 + c^2)lm^2n - lm(b^2m^2 + a^2n^2 + b^2n^2)S_A - b^2(l^2m^2 + l^2n^2 + m^2n^2)S_B - m(b^2l^2 + c^2l^2 +$
 $b^2m^2)nS_C + 8ln(2lm - 2m^2 - ln + 2mn)\Delta^2 :$
 $c^2lm(b^2m^2 - 2b^2mn + a^2n^2 + b^2n^2) - c^2(l^2m^2 + l^2n^2 + m^2n^2)S_C + S_B(-m(b^2l^2 + c^2l^2 + b^2m^2)n +$
 $2lm^2nS_C) - 8m(l^2m - lm^2 - 2l^2n + 2lmn - m^2n)\Delta^2)$

CT-Area of QG-P9-Triangle in the QA-environment:

- $(a^4qr(p + q)(p + r) + b^4pr(p + q)(q + r) + c^4pq(p + r)(q + r)$
 $- 2b^2c^2pqr(q + r) - 2a^2b^2pqr(p + q) - 2a^2c^2pqr(p + r))$
 $/ (32\Delta(p + q)(p + r)(q + r)(p + q + r))$

CT-Area of QG-P9-Triangle in the QL-environment:

- $T_1T_2 / (64\Delta(l - m)^2(l - n)^2(m - n)^2(-lm + ln + mn)(lm + ln - mn)(lm - ln + mn))$
 where: $T_1 = a^2mn(l - m)(l - n) + b^2ln(m - l)(m - n) + c^2lm(n - l)(n - m)$
 $T_2 = a^2mn(l - m)(l - n)(-3lm + ln + mn)(lm - 3ln + mn)$
 $+ b^2ln(m - l)(m - n)(lm + ln - 3mn)(-3lm + ln + mn)$
 $+ c^2lm(n - l)(n - m)(lm + ln - 3mn)(lm - 3ln + mn)$

DT-Coordinates QG-P9 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrigon point are given)

- $((Sb + a^2)(p^2 - r^2) - Scq^2)(Sa p^2 + Sb q^2 + Sc r^2) - Sb Sc(p - q + r)(p + q - r)(-p + q + r)(p + q + r) :$
 $(Sa p^2 + Sb q^2 + Sc r^2)(Sa(p^2 + q^2 - r^2) + Sc(-p^2 + q^2 + r^2)) - (S^2 + Sa Sc)(-p + q + r)(p + q - r)(p - q + r)(p + q + r) :$
 $-((c^2 + Sb)(p^2 - r^2) + Sa q^2)(Sa p^2 + Sb q^2 + Sc r^2) - Sb Sa(p - q + r)(p + q - r)(-p + q + r)(p + q + r))$

DT-Coordinates QG-P9 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(a^2 m^2 (Sb l^2 + Sa m^2 - c^2 n^2) : - (S^2 + Sb Sc) l^2 m^2 + Sb b^2 m^4 + 2 S^2 l^2 n^2 - (Sa Sb + S^2) m^2 n^2 : + c^2 m^2 (-a^2 l^2 + Sc m^2 + Sb n^2))$

DT-Area of QG-P9-Triangle in the QA-environment:

- $(a^2 b^2 r^4 + b^2 c^2 p^4 + c^2 a^2 q^4 - 2 Sa a^2 q^2 r^2 - 2 Sb b^2 p^2 r^2 - 2 c^2 Sc p^2 q^2) / (8 S (p-q-r) (p+q-r) (p-q+r) (p+q+r))$

DT-Area of QG-P9-Triangle in the QL-environment:

- $((Sc (l^2 - m^2)^2 + Sb (l^2 - n^2)^2 + Sa (m^2 - n^2)^2) (Sc (l^2 - m^2)^2 n^4 + Sb m^4 (l^2 - n^2)^2 + Sa l^4 (m^2 - n^2)^2)) / (8 S (l^2 - m^2)^2 (l^2 - n^2)^2 (m^2 - n^2)^2)$

Properties:

- QG-P8, QG-P9, QG-P10, QG-P11 are collinear on a 2nd Quasi Euler Line (which is a parallel line to the 1st Quasi Euler Line).
- QG-P9 is the Reflection of QG-P7 in QA-P1 (QA-Centroid).
- QG-P9 is the Midpoint of QG-P1 and QG-P5.
- QA-P1 (QA-Centroid) lies on the circumcircle of the triangle formed by the 2nd Quasi Circumcenters of the 3 QA-Quadrilaterals.
- QL-P9 and QL-P16 lie on the circle formed by the 3 points QG-P9 in a Quadrilateral. The center of the similar circle with QG-P5 lies on the QG-P9 circle.
- QG-P9 is the Reflection of QG-P7 in the line QG-P2.QG-P12 (which coincides with the Newton Line QL-L1).
- The area of the QA-versions of QG-P9 = the area of the QA-versions of QG-P7.
- The area of the QL-versions of QG-P9 = the area of the QL-versions of QG-P7.

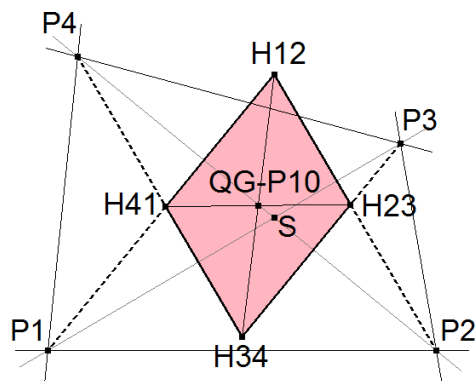
QG-P10 2nd Quasi Orthocenter

Let P_1, P_2, P_3, P_4 be a Quadrilateral and let S be its Diagonal Crosspoint QG-P1.

Let H_{i+1} = Orthocenter of Triangle H_i, H_{i+1}, S (i = cyclic sequence 1,2,3,4).

Now is $H_{12}, H_{23}, H_{34}, H_{41}$ a parallelogram. Special is that each vertex of the Reference Quadrangle lies on a sideline of the parallelogram.

QG-P10 is the Center of this parallelogram.



P_i = consecutive Vertices of Quadrilateral
($i = 1, 2, 3, 4$)

H_{i+1} = Orthocenter Triangle P_i, P_{i+1}, S
(i = cyclic 1,2,3,4)

S = Diagonal Crosspoint

QG-P10 = 2nd Quasi Orthocenter

CT-Coordinates QG-P10 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(-a^2(p^2 + p q + q r + r^2) S_A + (c^2 p q + a^2 p r + c^2 p r + a^2 q r) S_C + 4(2 p^2 + p q + p r + q r + r^2) \Delta^2 : -b^2(a^2 + c^2) p r - (a^2 - c^2) q r S_C + S_A((a^2 - c^2) p q + (p^2 + r^2) S_C) + 8 p r \Delta^2 : S_A(c^2 p q + a^2 p r + c^2 p r + a^2 q r + (p^2 + p q + q r) S_B) - c^2 r^2 S_C + 4 r(p + 2 r) \Delta^2)$

CT-Coordinates QG-P10 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(a^2 m(b^2 l^2 + c^2 l^2 - 2 b^2 l m + b^2 m^2) n - a^2 m^2 n^2 S_A - 2 c^2 l m^2 n S_C + S_B(-l m(b^2 m^2 + a^2 n^2 + b^2 n^2) + l^2(m^2 + n^2) S_C) - 4(l - 2 m) m n(2 l - m + n) \Delta^2 : b^2(a^2 + b^2 + c^2) l m^2 n - b^2 l^2 n^2 S_B - m(b^2 l^2 + c^2 l^2 + b^2 m^2) n S_C + S_A(-l m(b^2 m^2 + a^2 n^2 + b^2 n^2) + m^2(l^2 + n^2) S_C) - 4 l n(l m - 2 l n + m n) \Delta^2 : c^2 l m(b^2 m^2 - 2 b^2 m n + a^2 n^2 + b^2 n^2) - m(b^2 l^2 + c^2 l^2 + b^2 m^2) n S_B + S_A(-2 a^2 l m^2 n + (l^2 + m^2) n^2 S_B) - c^2 l^2 m^2 S_C + 4 l m(2 m - n)(l - m + 2 n) \Delta^2)$

CT-Area of QG-P10-Triangle in the QA-environment: (equals 4 * area QA-Morley Triangle)

- $((a^2 q r + b^2 p r + c^2 p q)^2) / (8 \Delta(p + q)(p + r)(q + r)(p + q + r))$

CT-Area of QG-P10-Triangle in the QL-environment:

- 0 (because vertices are collinear)

DT-Coordinates QG-P10 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(-(S_C(p^2 - q^2 - r^2) - 2 S_B q^2)(-c^2 p^2 + a^2 r^2) : -S_A^2 p^2(p^2 + q^2 - r^2) + S_C^2 r^2(p^2 - q^2 - r^2) - S^2 q^2(p^2 - q^2 + r^2) - 2 S_B q^2(S_A p^2 + S_C r^2) : (S_A(p^2 + q^2 - r^2) + 2 S_B q^2)(c^2 p^2 - a^2 r^2))$

DT-Coordinates QG-P10 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $(S_B m^2(S_B(l^2 - n^2) + S_C(l^2 - m^2)) : S_A a^2 l^2 m^2 + S_C c^2 m^2 n^2 - S_A S_C m^4 - S^2 l^2 n^2 : -S_B m^2(S_B(l^2 - n^2) + S_A(m^2 - n^2)))$

DT-Area of QG-P10-Triangle in the QA-environment: (equals 4 * area QA-Morley Triangle)

- $-(S_a p^2 + S_b q^2 + S_c r^2)^2 / (2S (-p+q+r) (p+q-r) (p-q+r) (p+q+r))$

DT-Area of QG-P10-Triangle in the QL-environment:

- 0 (because vertices are collinear)

Properties:

- QG-P8, QG-P9, QG-P10, QG-P11 are collinear on a 2nd Quasi Euler Line (which is a parallel line to the 1st Quasi Euler Line).
- QG-P10 is the Reflection of QG-P5 in QA-P1 (QA-Centroid).
- QG-P10 is the Midpoint of QG-P1 and QG-P6.
- The three 2nd Quasi Orthocenters in a Quadrilateral are collinear on the line QL-L6 (Quasi Ortholine).
- QA-P2 (Euler-Poncelet Point) lies on the circumcircle of the triangle formed by the three 2nd Quasi Orthocenters in a Quadrangle. QA-P15 is the circumcenter.
- The line through the 1st and 2nd Quasi Orthocenters (QG-P6 and QG-P10) is perpendicular to the Newton Line (QL-L1).

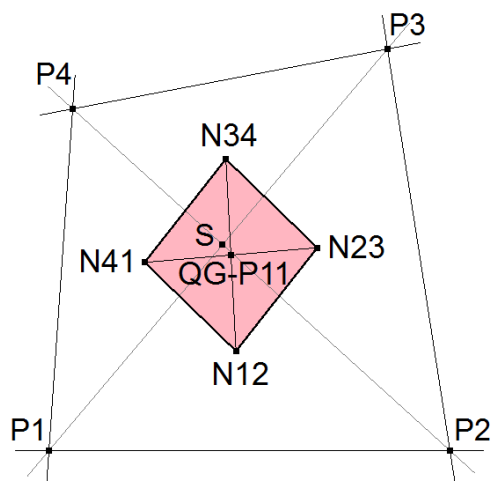
QG-P11 2nd Quasi Nine-point Center

Let $P_1.P_2.P_3.P_4$ be a Quadrigon and let S be its Diagonal Crosspoint QG-P1.

Let N_{i+1} = Nine-point Center of Triangle $G_i.G_{i+1}.S$ (i = cyclic sequence 1,2,3,4) .

Now is $N_{12}.N_{23}.N_{34}.N_{41}$ a parallelogram.

QG-P11 is the Center of this parallelogram.



P_i = consecutive Vertices of Quadrigon
($i = 1,2,3,4$)

N_{i+1} = Nine-point Center Triangle $P_i.P_{i+1}.S$
(i = cyclic 1,2,3,4)

S = Diagonal Crosspoint

QG-P11 = 2nd Quasi Nine-Point Center

CT-Coordinates QG-P11 in 3 QA-Quadrignons: (only coordinates of 1st Quadrigon point are given)

- $$\begin{aligned} &((-24 \Delta^2 + a^2 S_A) p^2 + (-8 \Delta^2 - a^2 (a^2 - c^2)) q r + (-8 \Delta^2 + a^2 S_A) r^2 + p ((-16 \Delta^2 - (a^2 - c^2) S_B) q + (-24 \Delta^2 - \\ &(a^2 + c^2) S_C) r), \\ &(-8 \Delta^2 + b^2 S_B) p^2 + (-16 \Delta^2 + b^2 (a^2 + S_B)) q r + (-8 \Delta^2 + b^2 S_B) r^2 + p ((-8 \Delta^2 - (a^2 - c^2) S_A) q + (-16 \Delta^2 + \\ &b^2 (a^2 + c^2)) r), \\ &(-8 \Delta^2 + c^2 S_C) p^2 + (-16 \Delta^2 + (a^2 - c^2) S_B) q r + (-24 \Delta^2 + c^2 S_C) r^2 + p ((-8 \Delta^2 + (a^2 - c^2) c^2) q + (-24 \Delta^2 - \\ &S_A (a^2 + c^2)) r) \end{aligned}$$

CT-Coordinates QG-P11 in 3 QL-Quadrignons: (only coordinates of 1st Quadrigon point are given)

- $$\begin{aligned} &(2 a^2 m (b^2 l^2 + c^2 l^2 - 2 b^2 l m + b^2 m^2) n - 16 \Delta^2 (-l^2 m^2 + l m^3 + 2 l^2 m n - 6 l m^2 n + 3 m^3 n \\ &- l^2 n^2 + 3 l m n^2 - 3 m^2 n^2) - 2 a^2 (l^2 m^2 + l^2 n^2 + m^2 n^2) S_A - 2 l m (b^2 m^2 + a^2 n^2 + b^2 n^2) S_B \\ &- 4 c^2 l m^2 n S_C : \\ &2 b^2 (a^2 + b^2 + c^2) l m^2 n - 16 \Delta^2 (-l^2 m^2 + 3 l^2 m n - 2 l m^2 n - 3 l^2 n^2 + 3 l m n^2 - m^2 n^2) \\ &- 2 l m (b^2 m^2 + a^2 n^2 + b^2 n^2) S_A - 2 b^2 (l^2 m^2 + l^2 n^2 + m^2 n^2) S_B - 2 m (b^2 l^2 + c^2 l^2 \\ &+ b^2 m^2) n S_C : \\ &-c^2 l (-2 b^2 m^3 + 4 b^2 m^2 n + a^2 l n^2 + b^2 l n^2 - c^2 l n^2 - 2 a^2 m n^2 - 2 b^2 m n^2) - 16 \Delta^2 (-3 l^2 m^2 \\ &+ 3 l m^3 + 3 l^2 m n - 6 l m^2 n + m^3 n - l^2 n^2 + 2 l m n^2 - m^2 n^2) - 4 a^2 l m^2 n S_A - 2 m (b^2 l^2 \\ &+ c^2 l^2 + b^2 m^2) n S_B - 2 c^2 m^2 (l^2 + n^2) S_C \end{aligned}$$

CT-Area of QG-P11-Triangle in the QA-environment:

- $$\begin{aligned} &(a^4 q r (p^2 + p q + p r - 3 q r) + b^4 p r (p q + q^2 - 3 p r + q r) + c^4 p q (-3 p q + p r + q r + r^2) \\ &- 2 b^2 c^2 p q r (4 p + q + r) - 2 a^2 c^2 p q r (p + 4 q + r) - 2 a^2 b^2 p q r (p + q + 4 r)) \\ &/ (128 \Delta (p + q) (p + r) (q + r) (p + q + r)) \end{aligned}$$

CT-Area of QG-P11-Triangle in the QL-environment:

- $$\begin{aligned}
 & (+a^4 m^2 n^2 (l-m)^2 (l-n)^2 (13 l^2 m^2 - 22 l^2 m n - 2 l m^2 n + 13 l^2 n^2 - 2 l m n^2 + m^2 n^2) \\
 & + b^4 l^2 n^2 (l-m)^2 (m-n)^2 (13 l^2 m^2 - 2 l^2 m n - 22 l m^2 n + l^2 n^2 - 2 l m n^2 + 13 m^2 n^2) \\
 & + c^4 l^2 m^2 (l-n)^2 (m-n)^2 (l^2 m^2 - 2 l^2 m n - 2 l m^2 n + 13 l^2 n^2 - 22 l m n^2 + 13 m^2 n^2) \\
 & + 2 b^2 c^2 l^2 m n (l-m) (l-n) (m-n)^2 (l^2 m^2 - 14 l^2 m n + 12 l m^2 n + l^2 n^2 + 12 l m n^2 - 13 m^2 n^2) \\
 & + 2 a^2 c^2 l m^2 n (m-l) (m-n) (l-n)^2 (l^2 m^2 + 12 l^2 m n - 14 l m^2 n - 13 l^2 n^2 + 12 l m n^2 + m^2 n^2) \\
 & + 2 a^2 b^2 l m n^2 (n-l) (m-n) (l-m)^2 (13 l^2 m^2 - 12 l^2 m n - 12 l m^2 n - l^2 n^2 + 14 l m n^2 - m^2 n^2)) \\
 & / (256 \Delta (l-m)^2 (l-n)^2 (m-n)^2 (l m - l n - m n) (l m + l n - m n) (l m - l n + m n))
 \end{aligned}$$
-

DT-Coordinates QG-P11 in 3 QA-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $$\begin{aligned}
 & ((3 S^2 p^2 - Sa Sb p^2 - Sb^2 q^2 - Sc a^2 r^2) (-p^2 + q^2 + r^2) + Sb q^2 ((Sa + c^2) p^2 + Sb q^2 - (a^2 + Sc) r^2) : \\
 & (-p^2 + q^2 + r^2) ((S^2 - Sa^2) p^2 - 4 S^2 q^2 - (S^2 - Sc^2) r^2) + 2 q^2 (5 S^2 r^2 + Sa c^2 p^2 + Sb Sc r^2) : \\
 & (3 S^2 r^2 - Sc Sb r^2 - Sb^2 q^2 - Sa c^2 p^2) (-r^2 + q^2 + p^2) + Sb q^2 ((Sc + a^2) r^2 + Sb q^2 - (c^2 + Sa) p^2)
 \end{aligned}$$

DT-Coordinates QG-P11 in 3 QL-Quadrilaterals: (only coordinates of 1st Quadrilateral point are given)

- $$\begin{aligned}
 & (m^2 (-Sb a^2 l^2 + (S^2 + Sb Sc) m^2 - (S^2 - Sb^2) n^2) : \\
 & (l^2 - m^2) (Sb Sc m^2 + S^2 n^2) - (3 S^2 l^2 - 2 S^2 m^2 + Sa Sb m^2) (m^2 - n^2) : \\
 & -m^2 ((S^2 - Sb^2) l^2 - (Sa Sb + S^2) m^2 + Sb c^2 n^2))
 \end{aligned}$$

DT-Area of QG-P11-Triangle in the QA-environment:

- $$S/32 + 3(Sa p^2 + Sb q^2 + Sc r^2)^2 / (32 S (-p+q+r) (p+q-r) (p-q+r) (p+q+r))$$

DT-Area of QG-P11-Triangle in the QL-environment:

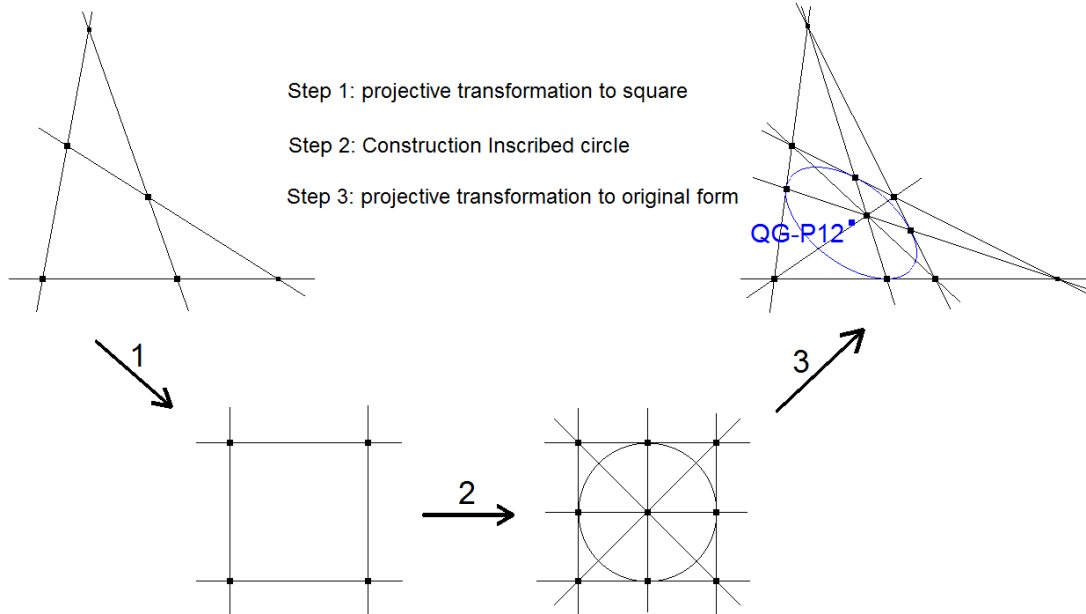
- $$3 S/32 - (Sc (l^2 - m^2)^2 n^2 + Sb m^2 (l^2 - n^2)^2 + Sa l^2 (m^2 - n^2)^2) / (32 S (l^2 - m^2)^2 (l^2 - n^2)^2 (m^2 - n^2)^2)$$

Properties:

- QG-P8, QG-P9, QG-P10, QG-P11 are collinear on a 2nd Quasi Euler Line (which is a parallel line to the 1st Quasi Euler Line).
- QG-P11 is the Midpoint of QG-P1 and QG-P7.
- The Midpoint of QA-P1 and QA-P2 lies on the circumcircle of the triangle formed by the three 2nd Quasi Nine-point Centers in a Quadrangle.

QG-P12: Inscribed Harmonic Conic Center

The Inscribed Harmonic Conic Center is the Center of the Inscribed Harmonic Conic QG-Co1. This conic touches the sidelines of the Quadrigon in their perspective midpoints. See picture below.



CT-Coordinates QG-P12 in 3 QA-Quadrigons:

- $(p(2p+q+r) : q(p-r) : r(p-q))$
- $(p(r-q) : q(r-p) : r(p+q+2r))$
- $(p(q-r) : q(p+2q+r) : r(q-p))$

CT-Coordinates QG-P12 in 3 QL-Quadrigons:

- $(m+n : -2l+n : -2l+m)$
- $(-2m+n : l+n : -2m+l)$
- $(-2n+m : -2n+l : l+m)$

CT-Area of QG-P12-Triangle in the QA-environment:

- $pqr(p+q)(p+r)(q+r) \Delta / ((p^2 + pq + pr - qr)(pq + q^2 - pr + qr)(pq - pr - qr - r^2))$

CT-Area of QG-P12-Triangle in the QL-environment:

- 0 (points are collinear on Newton Line)

DT-Coordinates QG-P12 in 3 QA-Quadrigons:

- $(-p^2 : q^2 : r^2)$
- $(p^2 : -q^2 : r^2)$
- $(p^2 : q^2 : -r^2)$

DT-Coordinates QG-P12 in 3 QL-Quadrilaterals:

- $(-2 m^2 n^2 : l^2 n^2 : l^2 m^2)$
- $(m^2 n^2 : -2 l^2 n^2 : l^2 m^2)$
- $(m^2 n^2 : l^2 n^2 : -2 l^2 m^2)$

DT-Area of QG-P12-Triangle in the QA-environment:

- $(-2 S p^2 q^2 r^2) / ((-p^2+q^2+r^2) (p^2+q^2-r^2) (p^2-q^2+r^2))$

DT-Area of QG-P12-Triangle in the QL-environment:

- 0 (points are collinear on Newton Line)

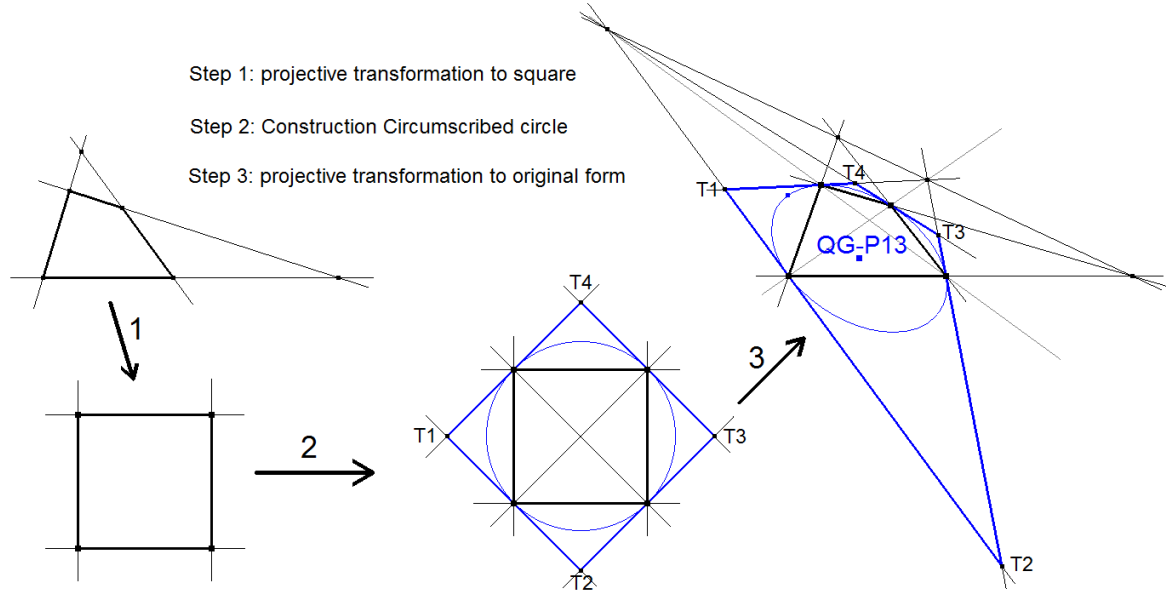
Properties:

- QG-P12 is the fourth harmonic point of QG-P2 (Midpoint 3rd Diagonal) on the Newton Line (QL-L1) wrt the midpoints of the diagonals (note Eckart Schmidt).
- QG-P12 is collinear with QG-P1, QG-P13, QA-P16, QL-P13 on QG-L2.
- The triangle formed by the 3 QA-versions of QG-P12 is perspective with the QA-Diagonal Triangle. Their Perspector is QA-P16 (QA-Harmonic Center).
- The triangle formed by the 3 QL-versions of QG-P12 is perspective with the QL-Diagonal Triangle. Their Perspector is QL-P13 (QL-Harmonic Center).
- QG-P12 is the intersection point of the Trilinear Polars of QL-P13 wrt the QA-Diagonal Triangle and the QL-Diagonal Triangle (note Eckart Schmidt).

QG-P13: Circumscribed Harmonic Conic Center

The Circumscribed Harmonic Conic Center is the Center of the circumscribed Harmonic Conic QG-Co2. This conic touches the sidelines of the projective circumscribed Quadrigon in the vertices of the Reference Quadrigon.

See picture below.



CT-Coordinates QG-P13 in 3 QA-Quadrigons:

- $(2p(2p+q+r) : q(2p+q-r) : r(2p-q+r))$
- $(p(p+2q-r) : 2q(p+2q+r) : r(-p+2q+r))$
- $(p(p-q+2r) : q(-p+q+2r) : 2r(p+q+2r))$

CT-Coordinates QG-P13 in 3 QL-Quadrigons:

- $(mn(lm+ln+mn) : ln(-3lm+ln+mn) : lm(lm-3ln+mn))$
- $(mn(-3lm+ln+mn) : ln(lm+ln+mn) : lm(lm+ln-3mn))$
- $(mn(lm-3ln+mn) : ln(lm+ln-3mn) : lm(lm+ln+mn))$

CT-Area of QG-P13-Triangle in the QA-environment:

- $54 p q r (p+q)(p+r)(q+r) \Delta$
 $/ (((q-r)^2 + 4p(p+q+r))((p-r)^2 + 4q(p+q+r))((p-q)^2 + 4r(p+q+r)))$

CT-Area of QG-P13-Triangle in the QL-environment:

- $16 l^2 m^2 n^2 (lm-ln-mn)(lm+ln-mn)(lm-ln+mn) \Delta$
 $/ ((l^2 m^2 + l^2 n^2 + m^2 n^2 + 2lmn(l+m-3n))(l^2 m^2 + l^2 n^2 + m^2 n^2 + 2lmn(l-3m+n))$
 $(l^2 m^2 + l^2 n^2 + m^2 n^2 + 2lmn(-3l+m+n)))$

DT-Coordinates QG-P13 in 3 QA-Quadrilaterals:

- $(-p^2 : 2q^2 : 2r^2)$
- $(2p^2 : -q^2 : 2r^2)$
- $(2p^2 : 2q^2 : -r^2)$

DT-Coordinates QG-P13 in 3 QL-Quadrilaterals:

- $(-m^2 n^2 : l^2 n^2 : l^2 m^2)$
- $(m^2 n^2 : -l^2 n^2 : l^2 m^2)$
- $(m^2 n^2 : l^2 n^2 : -l^2 m^2)$

DT-Area of QG-P13-Triangle in the QA-environment:

- $-27 S p^2 q^2 r^2 / (2 (-p^2+2q^2+2r^2) (2p^2+2q^2-r^2) (2p^2-q^2+2r^2))$

DT-Area of QG-P13-Triangle in the QL-environment:

- $-2 S l^4 m^4 n^4 / ((-l^2 m^2+l^2 n^2+m^2 n^2) (l^2 m^2+l^2 n^2-m^2 n^2) (l^2 m^2-l^2 n^2+m^2 n^2))$

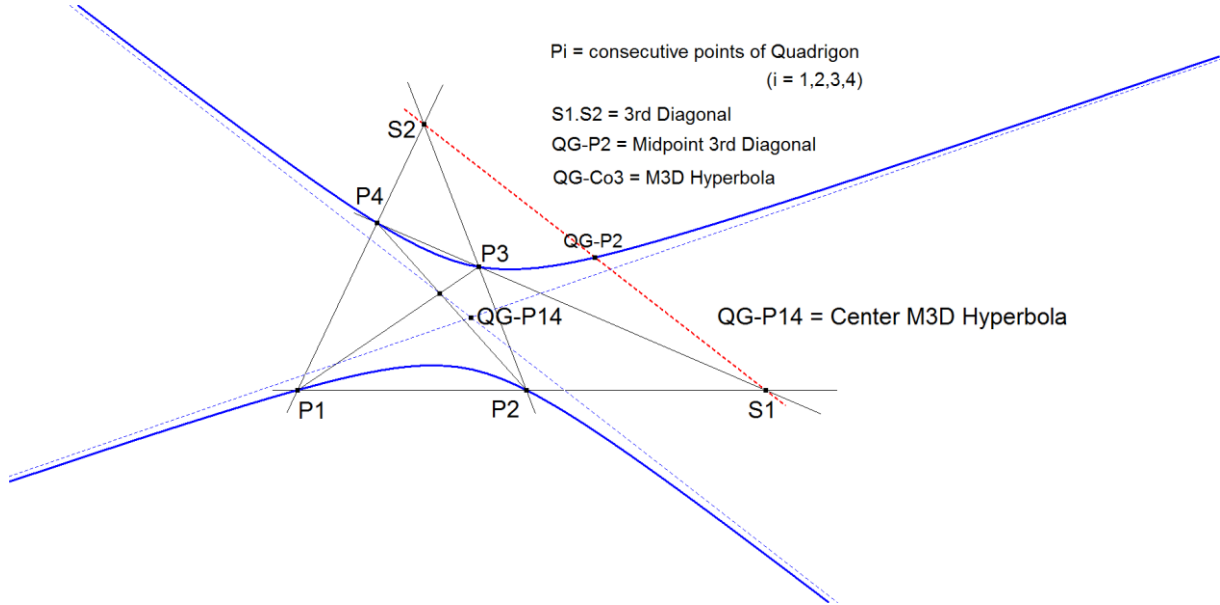
Properties:

- QG-P13 lies on these lines:
 - QG-P1.QG-P12 (=QG-L2)
 - QG-P2.QG-P14
- QG-P13 is the intersection of lines from an intersection of a diagonal with the 3rd diagonal and midpoint of the other diagonal (note Eckart Schmidt).
- The triangle formed by the 3 QA-versions of QG-P13 is perspective with the triangle formed by the 3 QA-versions of QG-P1 (the QA-Diagonal Triangle). Their Perspector is QA-P16 (QA-Harmonic Center).
- The triangle formed by the 3 QL-versions of QG-P13 is perspective with the QL-Diagonal Triangle. Their Perspector is QL-P13 (QL-Harmonic Center).
- The circumcenter of the triangle formed by the 3 QL-versions of QG-P13 is a point on QL-L6 (Quasi Ortholine).
- QG-P13 lies on the Nine-point Conic QA-Co1.
So the vertices of the triangle formed by the 3 QA-versions of QG-P13 lie on the Nine-point Conic QA-Co1.

QG-P14 Center of the M3D Hyperbola

QG-P14 is the center of the M3D Hyperbola (see paragraph QG-Co3).

M3D stands for “Midpoint 3rd Diagonal”.



CT-Coordinates QG-P14 in 3 QA-Quadrigons:

- $(p(q+r)(-pq+pr+3qr+r^2) : q(p-r)^2(p+2q+r) : (p+q)r(p^2+3pq+pr-qr))$
- $(p(q+r)(pq+q^2-pr+3qr) : q(p+r)(p^2+pq+3pr-qr) : (p-q)^2r(p+q+2r))$
- $(p(q-r)^2(2p+q+r) : q(p+r)(-pq+3pr-qr+r^2) : (p+q)r(3pq+q^2-pr+qr))$

CT-Coordinates QG-P14 in 3 QL-Quadrigons:

- $(-ln(-2lm^2+lmn+2m^2n+ln^2-mn^2) : m(l-n)(l^2m-l^2n-2lmn-ln^2+mn^2) : ln(-l^2m+2lm^2+l^2n+lmn-2m^2n))$
- $(lm(lm^2+lmn-m^2n-2ln^2+2mn^2) : -lm(l^2m-l^2n+lmn+2ln^2-2mn^2) : (l-m)n(l^2m+lm^2-l^2n+2lmn-m^2n))$
- $(l(m-n)(lm^2-2lmn-m^2n+ln^2-mn^2) : -mn(-2l^2m+2l^2n+lmn-ln^2+mn^2) : mn(2l^2m-lm^2-2l^2n+lmn+mn^2))$

CT-Area of QG-P14-Triangle in the QA-environment:

- $16pqr(p^2+pq+pr-qr)(pq+q^2-pr+qr)(pq-pr-qr-r^2)\Delta / ((p+q)^3(p+r)^3(q+r)^3)$

CT-Area of QG-P14-Triangle in the QL-environment: (equals 3 x area QL-Diagonal Triangle)

- $12l^2m^2n^2\Delta / ((-lm+ln+mn)(lm+ln-mn)(lm-ln+mn))$

DT-Coordinates QG-P14 in 3 QA-Quadrigons:

- $(-p^2+r^2 : q^2 : p^2-r^2)$
- $(p^2 : -q^2+r^2 : q^2-r^2)$
- $(-p^2+q^2 : p^2-q^2 : r^2)$

DT-Coordinates QG-P14 in 3 QL-Quadrilaterals:

- $(-1 : (l^2 - n^2)/(2 m^2) : 1)$
- $((n^2 - m^2)/(2 l^2) : 1 : -1)$
- $(1 : -1 : (m^2 - l^2)/(2 n^2))$

DT-Area of QG-P14-Triangle in the QA-environment:

- $-(-p^2 + q^2 + r^2)(p^2 - q^2 + r^2)(p^2 + q^2 - r^2) S / (2 p^2 q^2 r^2)$

DT-Area of QG-P14-Triangle in the QL-environment: (equals 3 x area QL-Diagonal Triangle)

- $3 S / 2$

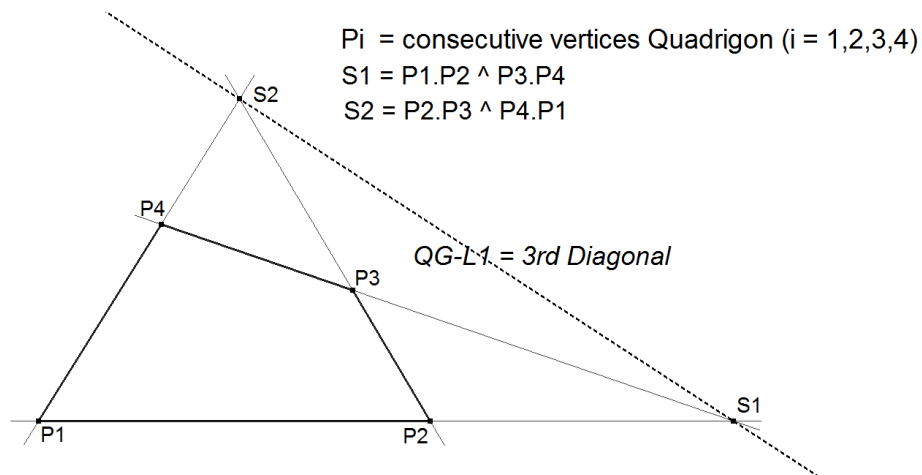
Properties:

- QG-P14 lies on these lines:
 - QG-P2.QG-P13
 - the line through QG-P1 parallel to QG-L1 (3rd Diagonal).
- QA-P2 (Euler-Poncelet Point) lies on the circumcircle of the triangle formed by the points QG-P14 of the 3 QA-Quadrilaterals.
- QG-P14 lies on the Nine-point Conic QA-Co1.

7.2 QUADRIGON LINES

QG-L1: The 3rd diagonal

The 3rd diagonal of a Quadrigon is the connecting line of S1 and S2, where S1 and S2 are the intersection points of the opposite sides of the Reference Quadrigon.



CT-Coefficients QG-L1 in 3 QA-Quadrilaterals:

- $(-q : r : p : p : q)$
- $(q : r : -p : p : q)$
- $(q : r : p : r : -p : q)$

CT-Coefficients QG-L1 in 3 QL-Quadrilaterals:

- $(0 : m : n)$
- $(1 : 0 : n)$
- $(1 : m : 0)$

DT-Coefficients QG-L1 in 3 QA-Quadrilaterals:

- $(0 : 1 : 0)$
- $(1 : 0 : 0)$
- $(0 : 0 : 1)$

DT-Coefficients QG-L1 in 3 QL-Quadrilaterals:

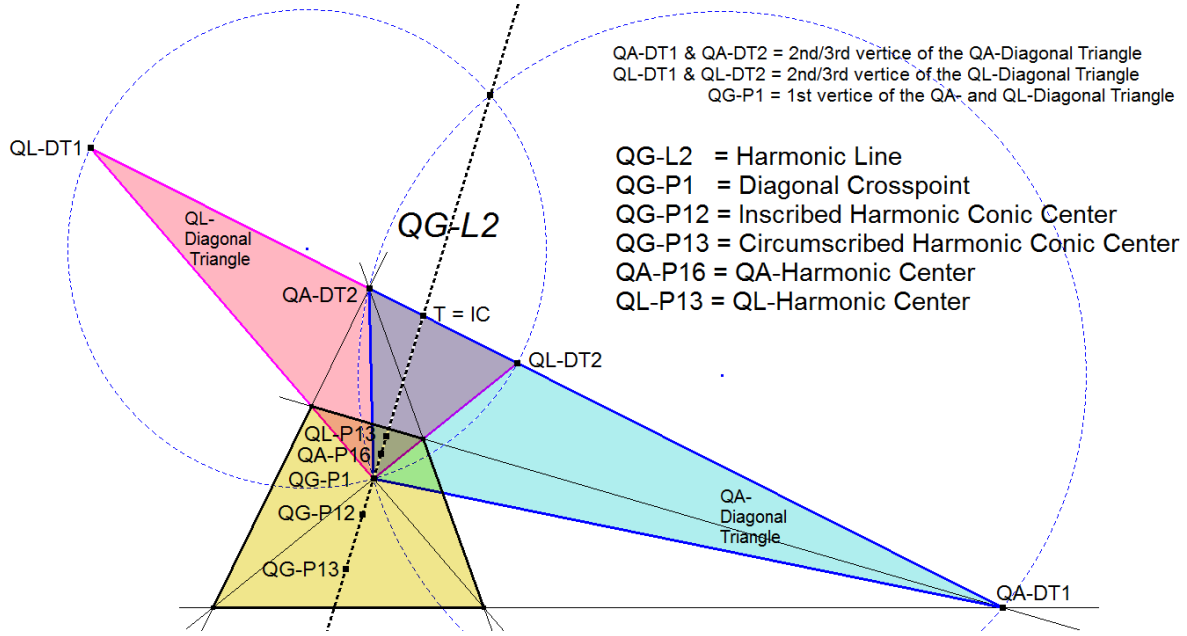
- $(0 : 1 : 0)$
- $(1 : 0 : 0)$
- $(0 : 0 : 1)$

Properties:

- The three 3rd Diagonals in the QA-environment form the QA-Diagonal Triangle.
- The three 3rd Diagonals in the QL-environment form the QL-Diagonal Triangle.

QG-L2: The Harmonic Line

The Harmonic Line is the line through QA-P16 (the harmonic point of a Quadrangle) and QL-P13 (the harmonic point of a Quadrilateral) both meeting in their overlap of a Quadrangle.



CT-Coefficients QG-L2 in 3 QA-Quadrilaterals:

- $(q r (-q + r) : p r (2 p + q + r) : -p q (2 p + q + r))$
- $(q r (p + 2 q + r) : p r (r - p) : -p q (p + 2 q + r))$
- $(q r (p + q + 2 r) : -p r (p + q + 2 r) : p q (-p + q))$

CT-Coefficients QG-L2 in 3 QL-Quadrilaterals:

- $(2 l^2 (m - n) , m (l m - l n + m n) , -n (-l m + l n + m n))$
- $(l (l m + l n - m n) , 2 m^2 (l - n) , -n (-l m + l n + m n))$
- $(l (l m + l n - m n) , -m (l m - l n + m n) , 2 (l - m) n^2)$

DT-Coefficients QG-L2 in 3 QA-Quadrilaterals:

- $(r^2 : 0 : -p^2)$
- $(0 : -r^2 : q^2)$
- $(-q^2 : p^2 : 0)$

DT-Coefficients QG-L2 in 3 QL-Quadrilaterals:

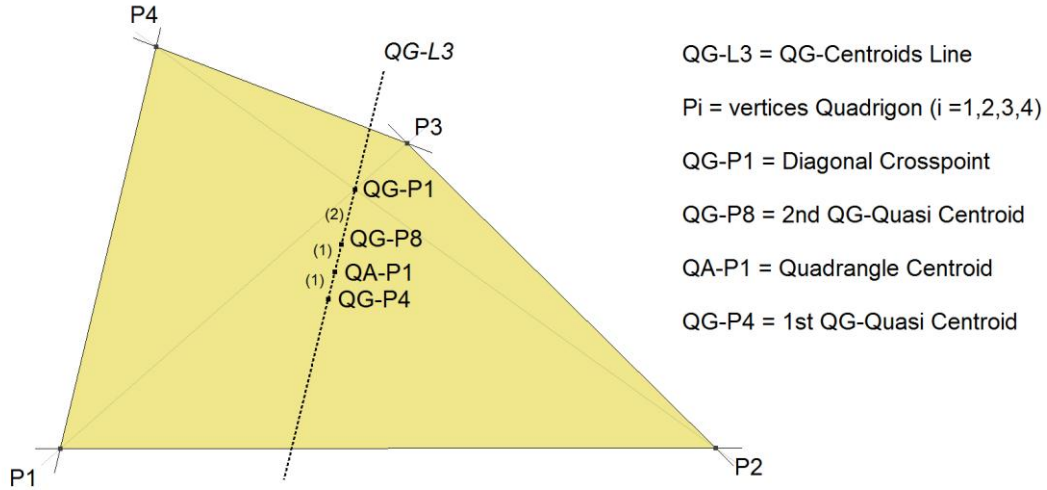
- $(l^2 : 0 : -n^2)$
- $(0 : -m^2 : n^2)$
- $(-l^2 : m^2 : 0)$

Properties:

- QA-P16, QL-P13 are collinear with QG-P1, QG-P12 and QG-P13.
- QG-L2 is the radical axis of the circumcircle of the QA-Diagonal Triangle and the circumcircle of the QL-Diagonal Triangle.
- Let T be the intersection point $QG-L1 \wedge QG-L2$.
Now (QG-P1,T) and (QG-P12,QA-P16) are harmonic conjugated pairs on QG-L2
also (QG-P1,T) and (QG-P13,QL-P13) are harmonic conjugated pairs on QG-L2.
T is also the Involution Center of (QL-DT1,QL-DT2) and (QA-DT1,QA-DT2).
(notes Eckart Schmidt)
- QG-P13, QG-P12, QG-P1, QA-P16, QL-P13 also can be seen as a consecutive perspective row with vanishing point T. See [26b pages 24, 47, 48].
- The 3 variants of QG-L2 in a Quadrangle concur in QA-P16.
- The 3 variants of QG-L2 in a Quadrilateral concur in QL-P13.

QG-L3: The QG-Centroids Line

The QG-Centroids Line is the line connecting the 1st and 2nd QG-Quasi Centroids in a Quadrigon.



CT-Coefficients QG-L3 in 3 QA-Quadrilaterals:

- $(r(p + 2q + r) : (p - r)(p + q + r) : -p(p + 2q + r))$
- $((q - r)(p + q + r) : r(2p + q + r) : -q(2p + q + r))$
- $(q(p + q + 2r) : -p(p + q + 2r) : (p - q)(p + q + r))$

CT-Coefficients QG-L3 in 3 QL-Quadrilaterals:

- $(l(m - n)(lm + ln - mn) : m(l - n)(-lm + 2m^2 + ln - mn) : -(l - m)n(lm - ln - mn))$
- $(l(m - n)(lm + ln - mn) : -m(l - n)(lm - ln + mn) : (-l + m)n(lm - ln - mn + 2n^2))$
- $(-l(m - n)(2l^2 - lm - ln + mn) : -m(l - n)(lm - ln + mn) : -(l - m)n(lm - ln - mn))$

DT-Coefficients QG-L3 in 3 QA-Quadrilaterals:

- $(r^2(-p^2 - q^2 + r^2) : 0 : p^2(-p^2 + q^2 + r^2))$
- $(0 : r^2(p^2 + q^2 - r^2) : q^2(-p^2 + q^2 - r^2))$
- $(q^2(p^2 - q^2 + r^2) : p^2(p^2 - q^2 - r^2) : 0)$

DT-Coefficients QG-L3 in 3 QL-Quadrilaterals:

- $(l^2 - m^2 : 0 : m^2 - n^2)$
- $(n^2 - l^2 : m^2 - n^2 : 0)$
- $(0 : l^2 - m^2 : n^2 - l^2)$

Properties:

- These points lie on QL-L3: QG-P1, QG-P4, QG-P8, QA-P1.
The last three are (quasi-)centroids.
- The distance ratios between points QG-P4, QA-P1, QG-P8, QG-P1 are 1:1:2 in this order.

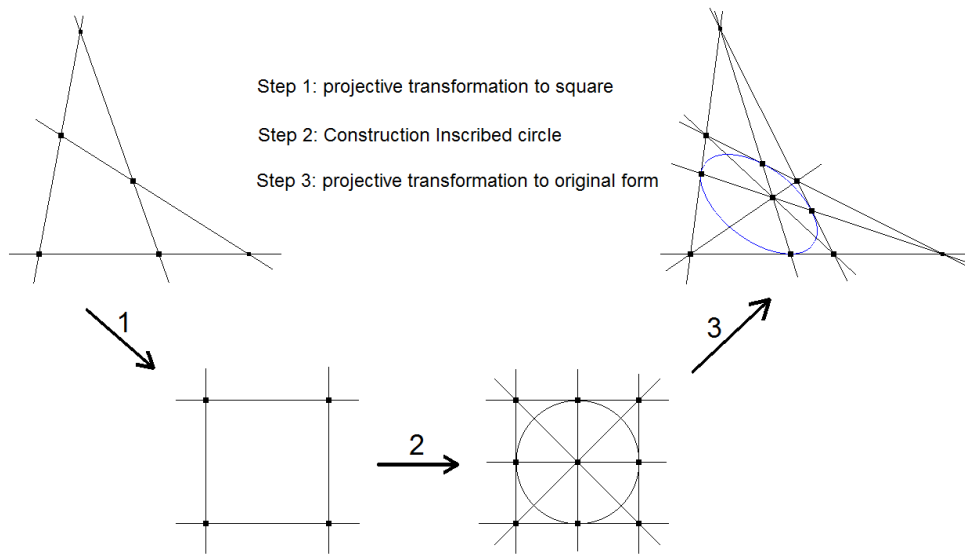
7.3 QUADRIGON CONICS/CUBICS

QG-Co1: Inscribed Harmonic Conic

There is a projective transformation from 4 lines in a square to any other set of 4 lines. See chapter QG-Tf1: QG-Projective Square Transformation.

The Inscribed Harmonic Conic is the projective transformation of the inscribed circle in a square to the Reference Quadrigon. This conic touches the sidelines of the Quadrigon in their perspective midpoints.

See picture below.



Equation QG-Co1 in 3 QA-Quadrigons in CT-notation:

- $q^2 r^2 x^2 + p^2 r^2 y^2 + p^2 q^2 z^2 - 6 p q r^2 x y + 2 p^2 q r y z + 2 p q^2 r x z = 0$
- $q^2 r^2 x^2 + p^2 r^2 y^2 + p^2 q^2 z^2 + 2 p q r^2 x y - 6 p^2 q r y z + 2 p q^2 r x z = 0$
- $q^2 r^2 x^2 + p^2 r^2 y^2 + p^2 q^2 z^2 + 2 p q r^2 x y + 2 p^2 q r y z - 6 p q^2 r x z = 0$

Equation QG-Co1 in 3 QL-Quadrigons in CT-notation:

- $l^2 x^2 + m^2 y^2 + 4 n^2 z^2 - 2 l m x y + 4 m n y z + 4 l n x z = 0$
- $4 l^2 x^2 + m^2 y^2 + n^2 z^2 + 4 l m x y - 2 m n y z + 4 l n x z = 0$
- $l^2 x^2 + 4 m^2 y^2 + n^2 z^2 + 4 l m x y + 4 m n y z - 2 l n x z = 0$

Equation QG-Co1 in 3 QA-Quadrigons in DT-notation:

- $q^2 r^2 x^2 - p^2 r^2 y^2 + p^2 q^2 z^2 = 0$
- $-r^2 q^2 x^2 + r^2 p^2 y^2 + p^2 q^2 z^2 = 0$
- $q^2 r^2 x^2 + r^2 p^2 y^2 - q^2 p^2 z^2 = 0$

Equation QG-Co1 in 3 QL-Quadrigons in DT-notation:

- $2 x^2 l^2 - y^2 m^2 + 2 z^2 n^2 = 0$
- $-x^2 l^2 + 2 y^2 m^2 + 2 z^2 n^2 = 0$

- $2x^2l^2 + 2y^2m^2 - z^2n^2 = 0$

Properties:

- The Center of QG-Co1 is QG-P12.

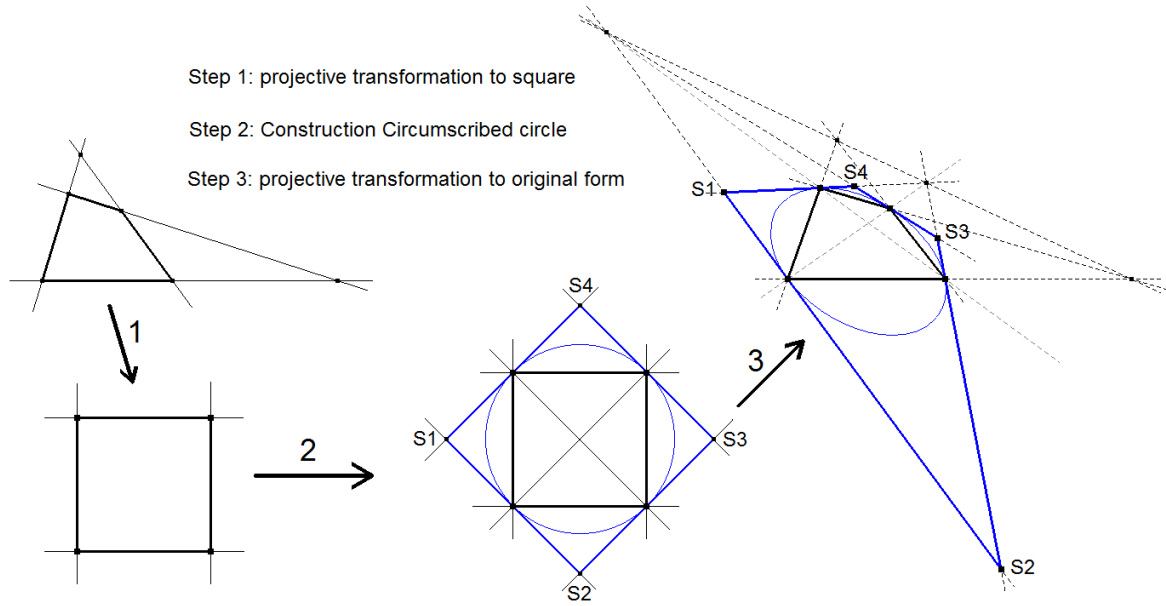
QG-Co2: Circumscribed Harmonic Conic

The Circumscribed Harmonic Conic is the projective transformation of the circumscribed circle of a square to the Reference Quadrigon.

See chapter QG-Tf1: QG-Projective Square Transformation.

This conic touches the sidelines of the projective circumscribed Quadrigon at the vertices of the Reference Quadrigon.

See picture below.



Equation QG-Co2 in 3 QA-Quadrigons in CT-notation:

- $-2 r x y + q x z + p y z = 0$
- $r x y - 2 q x z + p y z = 0$
- $r x y + q x z - 2 p y z = 0$

Equation QG-Co2 in 3 QL-Quadrigons in CT-notation:

- $l^2 x^2 + l m x y + l n x z - m n y z = 0$
- $m^2 y^2 + l m x y - l n x z + m n y z = 0$
- $n^2 z^2 - l m x y + l n x z + m n y z = 0$

Equation QG-Co2 in 3 QA-Quadrigons in DT-notation:

- $q^2 r^2 x^2 - 2 p^2 r^2 y^2 + p^2 q^2 z^2 = 0$
- $-2 q^2 r^2 x^2 + p^2 r^2 y^2 + p^2 q^2 z^2 = 0$
- $q^2 r^2 x^2 + p^2 r^2 y^2 - 2 p^2 q^2 z^2 = 0$

Equation QG-Co2 in 3 QL-Quadrigons in DT-notation:

- $x^2 l^2 - y^2 m^2 + z^2 n^2 = 0$
- $-x^2 l^2 + y^2 m^2 + z^2 n^2 = 0$
- $x^2 l^2 + y^2 m^2 - z^2 n^2 = 0$

Properties:

- The Center of QG-Co2 is QG-P13.

Equation QG-Co3 in 3 QL-Quadrilaterals in DT-notation:

- $(x + z) (x^2 + z^2) - y^2 = 0$
- $(y + z) (y^2 + z^2) - x^2 = 0$
- $(x + y) (x^2 + y^2) - z^2 = 0$

Properties:

- This conic is the locus of Centers of involution of all lines // Newton Line (QL-L1). See paragraph QA-Tf1: Line Involution. This is an example of a combination of QA- and QL-properties in a Quadrilateral (which can be seen as the intersection of a Quadrangle and a Quadrilateral).

8.2 QUADRIGON TRANSFORMATIONS

QG-Tf1: QG-Projective Square Transformation

A projective transformation in a plane is a transformation used in projective geometry.

According to [20] pages 22 and 24:

Definition:

A projective mapping of a projective plane Π onto a projective plane Π' is a one-one mapping of Π onto Π' such that the images of three collinear points are collinear.

A projective mapping of Π onto itself is called a *projective transformation* of Π .

Theorem I.

Given any four points A, B, C, D of a projective plane Π , no three of which are collinear, and four points A', B', C', D' of a projective plane Π' , no three of which are collinear, there exists one and only one projective mapping α of Π onto Π' that takes A, B, C, D into A', B', C', D', respectively.

Square Transformation:

Using this projective transformation it is possible to transform 4 points of a square to any other set of 4 points. Since a square consists of 4 points where the notion of opposite points is relevant it can be seen as a quadrigon. That's why the image of the transformation also will be a quadrigon.

Since projective transformations are invertible, Quadrigons can be uniquely transformed into a Square and a Square can be uniquely transformed back into the Reference Quadrigon.

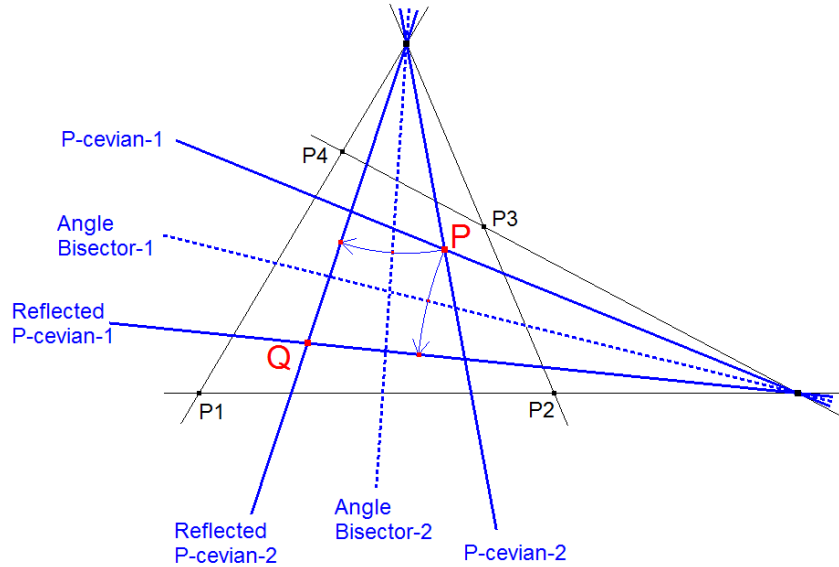
A projective transformation can be used to transform points of the whole plane. This makes it possible to transform the points of a circle circumscribing a square with the same projective transformation that converts the square into a Reference Quadrigon.

More popular:

A projective transformation describes what happens to the perceived positions of observed objects when the point of view of the observer changes. Projective transformations do not preserve sizes or angles but do preserve incidence (the intersection point of 2 lines is after transformation the intersection point of both transformed lines, the connecting line of 2 points is after transformation the connecting line of both transformed points).

QG-Tf2: QG-Quasi Isogonal Conjugate

The QG-Quasi Isogonal Conjugate of a point P is the intersection point of the reflected P-cevians in the two angle bisectors of the 2 pairs of opposite sides of the Quadrigon. This conjugate has been described for points QA-P2 and QA-P4 in [16 page 8].



Let P (u : v : w) be a random point to be transformed, then:

CT-Coordinates QA-Quasi Isogonal Conjugate in 3 QA-Quadrilaterals:

- $(-a^2 (r v - q w) (b^2 p (p + q) r w + a^2 q (p + q) r w - c^2 p q (-q u + p v + r w))) :$
 $a^4 q^2 (p + q) r u w + c^4 p q^2 (q + r) u w - b^2 c^2 p q (p + q - r) (q + r) u w - b^4 p (p + q) r (q + r) u w +$
 $a^2 (-b^2 q (p + q) r (-p + q + r) u w - c^2 q^2 (q r u v - p r v^2 + p q u w + 2 p r u w + q r u w + p q v w)) :$
 $c^2 (q u - p v) (c^2 p q (q + r) u + b^2 p r (q + r) u + a^2 q r (-p u - r v + q w)))$
- $(-a^2 (r v - q w) (c^2 p q (p + r) v + a^2 q r (p + r) v - b^2 p r (-r u + q v + p w))) :$
 $-b^2 (r u - p w) (c^2 p q (q + r) u + b^2 p r (q + r) u - a^2 q r (p u - r v + q w)) :$
 $-a^4 q r^2 (p + r) u v - b^4 p r^2 (q + r) u v + c^4 p q (p + r) (q + r) u v + b^2 c^2 p r (p - q + r) (q + r) u v + a^2$
 $(c^2 q r (p + r) (-p + q + r) u v + b^2 r^2 (2 p q u v + p r u v + q r u v + q r u w + p r v w - p q w^2)))$
- $(-b^4 p^2 (p + q) r v w - c^4 p^2 q (p + r) v w + a^4 q (p + q) r (p + r) v w + b^2 c^2 p^2 (-q r u^2 + p r u v + p q u$
 $w + p q v w + p r v w + 2 q r v w) + a^2 (c^2 p q (p + q - r) (p + r) v w + b^2 p (p + q) r (p - q + r) v w) :$
 $b^2 (r u - p w) (b^2 p (p + q) r w + a^2 q (p + q) r w + c^2 p q (-q u + p v - r w)) :$
 $c^2 (q u - p v) (c^2 p q (p + r) v + a^2 q r (p + r) v + b^2 p r (-r u - q v + p w)))$

CT-Coordinates QL-Quasi Isogonal Conjugate in 3 QL-Quadrilaterals:

- $(a^2 T_b w : b^2 (c^2 n u + a^2 l w) (l u + m v + n w) : c^2 T_b u)$
- $(a^2 T_c v : b^2 T_c u : c^2 (b^2 m u + a^2 l v) (l u + m v + n w))$
- $(-a^2 (c^2 n v + b^2 m w) (l u + m v + n w) : b^2 T_a w : c^2 T_a v)$

where: $T_a = b^2 (l - m) (m - n) u - c^2 (l - n) (m - n) u + a^2 (l m u + l n u - m n u + l m v + l n w)$
 $T_b = a^2 (l - m) (l - n) v + c^2 (l - n) (m - n) v + b^2 (-l m u - l m v + l n v - m n v - m n w)$
 $T_c = a^2 (l - m) (l - n) w - b^2 (l - m) (m - n) w + c^2 (-l n u - m n v + l m w - l n w - m n w)$

DT-Coordinates QA-Quasi Isogonal Conjugate in 3 QA-Quadrilaterals:

- $(-((b^2 p^2 u + a^2 q^2 u + 2 Sc p^2 v)/(2 Sc q^2 u + b^2 p^2 v + a^2 q^2 v)) : 1 : -((2 Sa r^2 v + c^2 q^2 w + b^2 r^2 w)/(c^2 q^2 v + b^2 r^2 v + 2 Sa q^2 w)))$
- $(1 : (-a^2 q^2 v - b^2 p^2 v - 2 Sc q^2 u)/(2 Sc p^2 v + a^2 q^2 u + b^2 p^2 u) : (a^2 r^2 w + c^2 p^2 w + 2 Sb r^2 u)/(-2 Sb p^2 w - a^2 r^2 u - c^2 p^2 u))$
- $((-c^2 p^2 u - a^2 r^2 u - 2 Sb p^2 w)/(2 Sb r^2 u + c^2 p^2 w + a^2 r^2 w) : (c^2 q^2 v + b^2 r^2 v + 2 Sa q^2 w)/(-2 Sa r^2 v - c^2 q^2 w - b^2 r^2 w) : 1)$

DT-Coordinates QL-Quasi Isogonal Conjugate in 3 QL-Quadrilaterals:

- $(2 (Sc a^2 l^2 + Sc b^2 m^2 + (Sa Sb + S^2) n^2) (u^2 l^2 + v^2 m^2 - w^2 n^2) + u v (a^4 l^4 + b^4 m^4 + c^4 n^4 + 2 (S^2 - Sa^2) m^2 n^2 + 2 (S^2 - Sb^2) l^2 n^2 + 2 (S^2 + 3 Sc^2) l^2 m^2) :$
 $4 (Sa u - Sb v + Sc w)^2 l^2 n^2 - (2 Sc u l^2 + a^2 v l^2 + b^2 v m^2 + c^2 v n^2 + 2 Sa w n^2)^2 :$
 $2 ((S^2 + Sb Sc) l^2 + Sa b^2 m^2 + Sa c^2 n^2) (-u^2 l^2 + v^2 m^2 + w^2 n^2) + v w (a^4 l^4 + b^4 m^4 + c^4 n^4 + 2 (S^2 + 3 Sa^2) m^2 n^2 + 2 (S^2 - Sb^2) l^2 n^2 + 2 (S^2 - Sc^2) l^2 m^2))$
- $(4 (Sa u - Sb v - Sc w)^2 m^2 n^2 - (2 Sb w n^2 + c^2 u n^2 + a^2 u l^2 + b^2 u m^2 + 2 Sc v m^2)^2 :$
 $2 ((S^2 + Sa Sb) n^2 + Sc a^2 l^2 + Sc b^2 m^2) (-w^2 n^2 + u^2 l^2 + v^2 m^2) + u v (c^4 n^4 + a^4 l^4 + b^4 m^4 + 2 (S^2 + 3 Sc^2) l^2 m^2 + 2 (S^2 - Sa^2) n^2 m^2 + 2 (S^2 - Sb^2) n^2 l^2) :$
 $2 (Sb c^2 n^2 + Sb a^2 l^2 + (Sc Sa + S^2) m^2) (w^2 n^2 + u^2 l^2 - v^2 m^2) + w u (c^4 n^4 + a^4 l^4 + b^4 m^4 + 2 (S^2 - Sc^2) l^2 m^2 + 2 (S^2 - Sa^2) n^2 m^2 + 2 (S^2 + 3 Sb^2) n^2 l^2))$
- $(2 ((S^2 + Sc Sa) m^2 + Sb c^2 n^2 + Sb a^2 l^2) (-v^2 m^2 + w^2 n^2 + u^2 l^2) + w u (b^4 m^4 + c^4 n^4 + a^4 l^4 + 2 (S^2 + 3 Sb^2) n^2 l^2 + 2 (S^2 - Sc^2) m^2 l^2 + 2 (S^2 - Sa^2) m^2 n^2) :$
 $2 (Sa b^2 m^2 + Sa c^2 n^2 + (Sb Sc + S^2) l^2) (v^2 m^2 + w^2 n^2 - u^2 l^2) + v w (b^4 m^4 + c^4 n^4 + a^4 l^4 + 2 (S^2 - Sb^2) n^2 l^2 + 2 (S^2 - Sc^2) m^2 l^2 + 2 (S^2 + 3 Sa^2) m^2 n^2) :$
 $4 (Sc w - Sa u - Sb v)^2 l^2 m^2 - (2 Sa v m^2 + b^2 w m^2 + c^2 w n^2 + a^2 w l^2 + 2 Sb u l^2)^2)$

Properties:

- In a Quadrangle QA-P2 and QA-P4 are QG-Quasi Isogonal Conjugated wrt the Component Quadrilaterals.
- In the component Quadrilaterals of a Quadrangle the 3 QG-Quasi Isogonal Conjugated points of a point P coincide when P is on the QA-Cu7 Cubic. So QA-Cu7 is self-conjugated wrt QG-Tf2 of the Component Quadrilaterals.
- In the component Quadrilaterals of a Quadrilateral the 3 QG-Quasi Isogonal Conjugated points of a point P coincide when P is on the QL-Cu1 Cubic. So QL-Cu1 is self-conjugated wrt QG-Tf2 of the Component Quadrilaterals.
- QG-Tf2 maps every line (not through a vertex) in a conic through the intersection points of the opposite sides of the Quadrilateral. Moreover this conic divides every side $P_i P_j$ of the Quadrilateral at a point S_{ij} into two parts $P_i S_{ij}$, $S_{ij} P_j$ with ratio $P_i S_{ij} / S_{ij} P_j$. The product of these 4 side ratios is 1 (note Eckart Schmidt).
- For any point at QL-Cu1 the QL-Tf1-transformation point (Clawson-Schmidt Conjugate) equals the QG-Tf2-transformation point (Quasi Isogonal Conjugate). (note Eckart Schmidt)

8. JUSTIFICATION

I mentioned different points and other items in this paper.

When I found this item at some place I mentioned this as a reference.

When no references are mentioned I found this item myself.

Of course it is quite possible that these items were found earlier by someone else.

If this is the case, please let me know, so that I can add this new reference.

Also improvements and other interesting Quadri-items can be send to me, preferably with references.

An interactive version of this information can be found at www.chrisvantienhoven.nl.

Netherlands, June 2012

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