Dynamic Perspective Fields

The hidden structure in Triangle Geometry

Part 2 – Algebraic Approach and Theory

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Structure of a Perspective Field
Perspective Coordinates

There is a most special application for Perspective Fields.
Because points are generated structurally in Perspective Fields this also can be seen in the formulas of the points.
All points in the field we described earlier (defined by X13, X2, X15, X6) have the same algebraic structure in the formulas of their barycentric coordinates:

$$f(a,b,c) = i\, a^2\, SA + j\, SB\, SC + k\, a^2 \Delta$$

Where:
Δ = Area of Reference Triangle
a,b,c are lenghts of sides Reference Triangle
$SA = (-a^2 + b^2 + c^2)\ /\ 2$ $SB = (a^2 - b^2 + c^2)\ /\ 2$ $SC = (a^2 + b^2 - c^2)\ /\ 2$

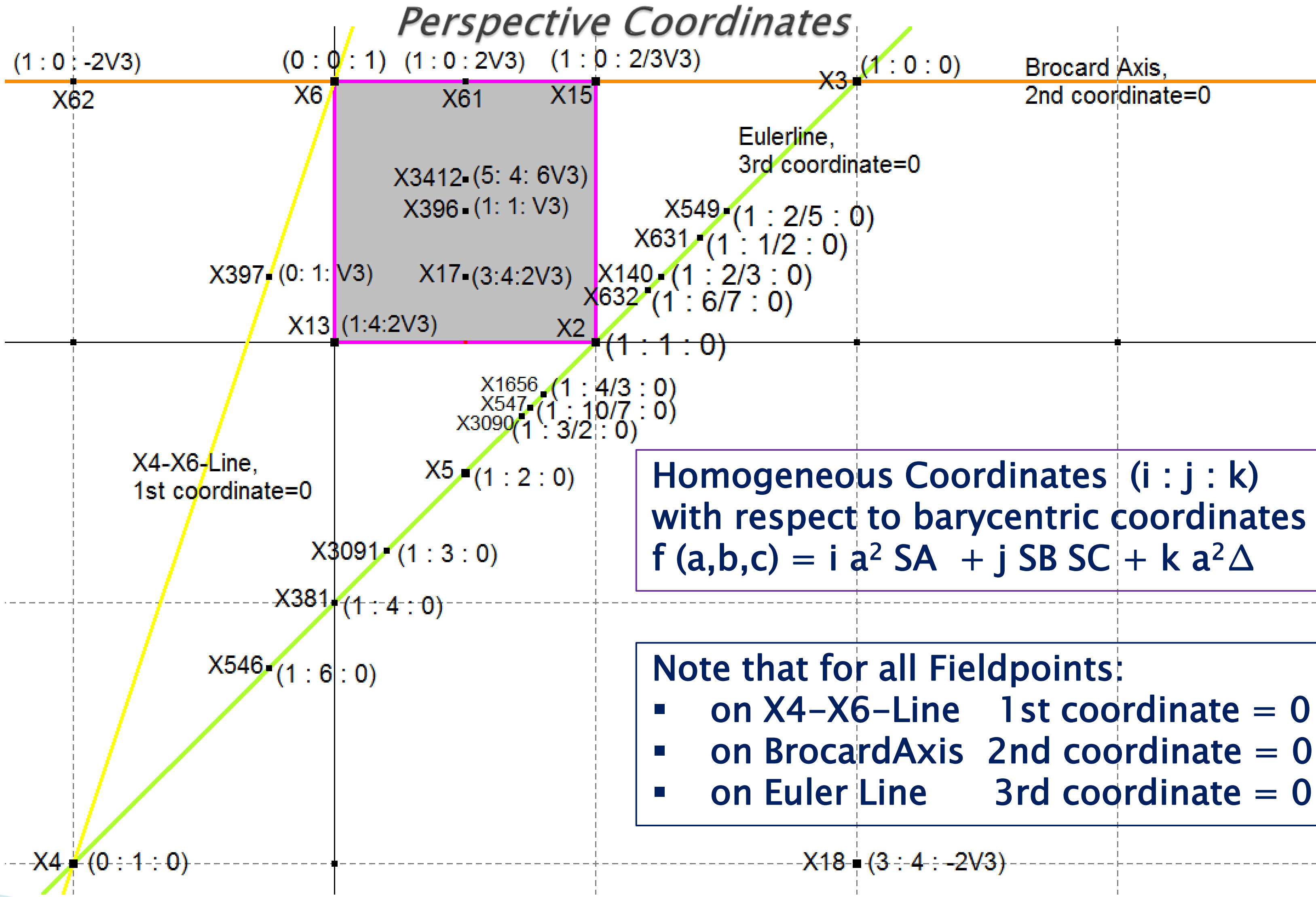
i, j, k are variables uniquely defining the points in a Perspective Field

So every point in the field can be defined as a ***triple (i : j : k)***.
Because only the ratios of variables i, j, k are of importance the triple can be considered as alternate homogeneous coordinates for fieldpoints.
The importance of this is that a fieldpoint now is:

independant of sides and angles of the referencetriangle
and
uniquely defined just by 3 numbers.

Look at next picture to see what structure comes out.

Structure of a Perspective Field



Homogeneous Coordinates (i : j : k)
with respect to barycentric coordinates
f (a,b,c) = i a² SA + j SB SC + k a² Δ

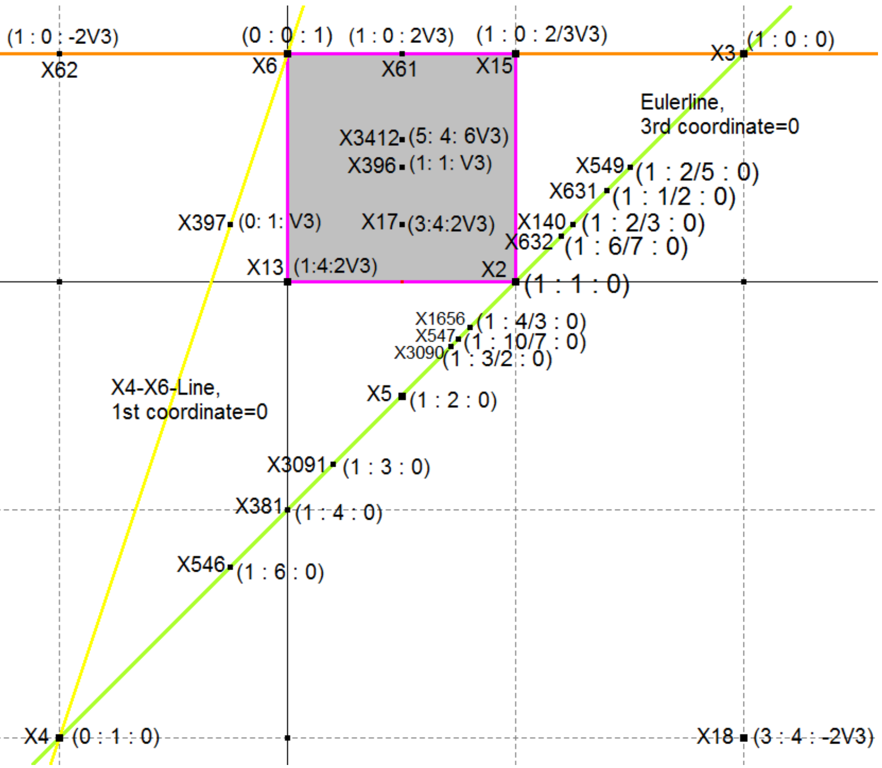
Note that for all Fieldpoints:

- **on X4–X6–Line 1st coordinate = 0**
- **on BrocardAxis 2nd coordinate = 0**
- **on Euler Line 3rd coordinate = 0**

Note that this formula-structure is only valid for Fieldpoints.
Non-fieldpoints will have a different formula-structure.

Structure of a Perspective Field

Analyses Perspective Coordinates



Point of departure:

- Field with Homogeneous Coordinates (i : j : k)
  refering to barycentric coordinates
  f (a,b,c) = i a^2 SA + j SB SC + k $a^2\Delta$

Characteristics:

- X3 = (1 : 0 : 0) because f(a,b,c) = a^2 SA
  X4 = (0 : 1 : 0) because f(a,b,c) = SB SC
  X6 = (0 : 0 : 1) because f(a,b,c) = a^2 Δ

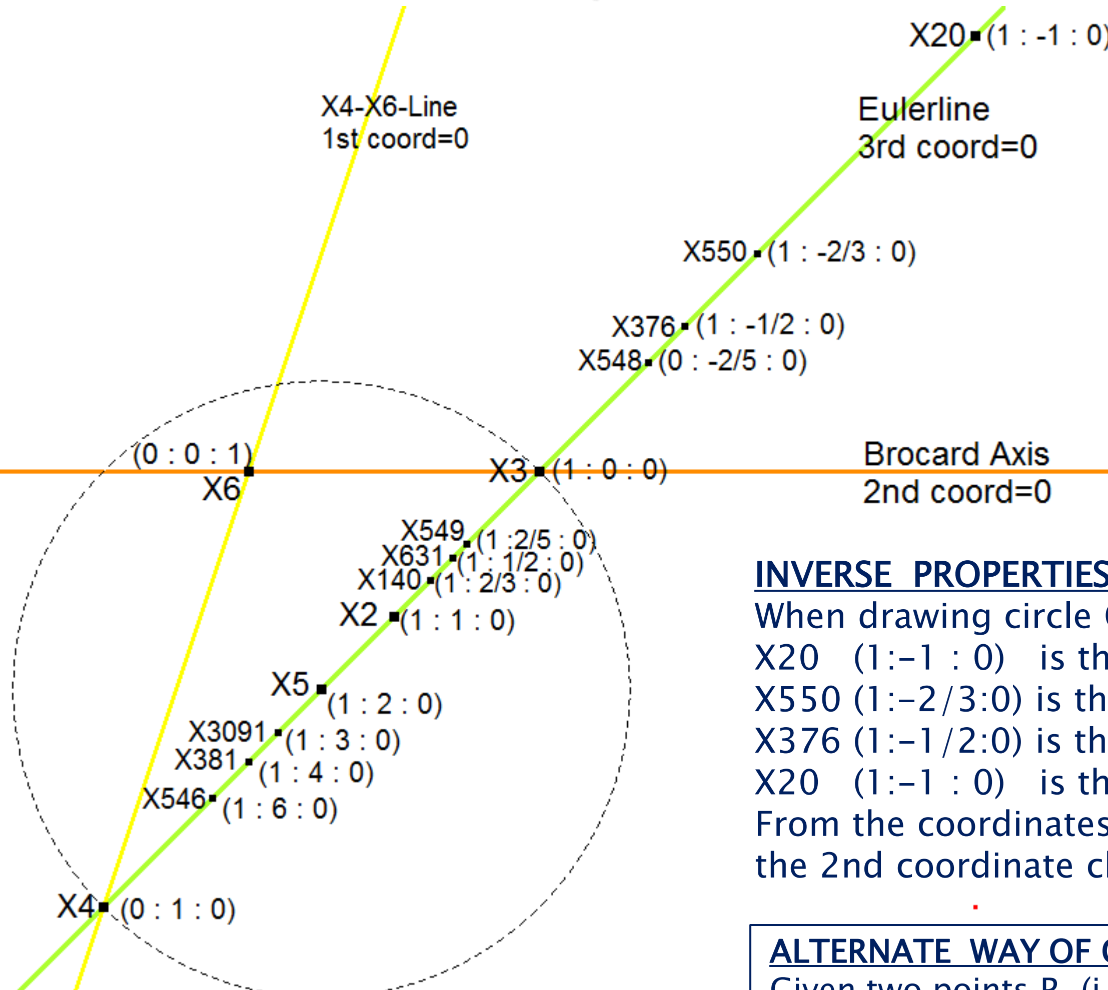
- on X4–X6–Line 1st coordinate = i = 0
  on X3–X6–Line 2nd coordinate = j = 0
  on X3–X4–Line 3rd coordinate = k = 0

- All points on line through X3 have identical ratio j : k
  All points on line through X4 have identical ratio i : k
  All points on line through X6 have identical ratio i : j

- Because X3, X4 and X6 are elements of structure in the Perspective Field I will name them the *referencepoints* wrt the Perspective Field in question. The coordinates will be named accordingly *Perspective Coordinates wrt X3, X4 and X6.*
- You could say that all points in the Perspective Field before us can be seen as *a weighted average of X3, X4 and X6*. The weighting factors are respectively i, j and k, defined as being the Perspective Coordinates.

Structure of a Perspective Field

Perspective Coordinates – scale on lines



Let us analyze the scale on the Eulerline.
For each fieldpoint on the Eulerline the 2nd coordinate varies in the range X3–X4 from 0 to infinite.
Outside this range values are negative.
The scale is not linear, not logarithmic, neither exponential.
*The scale is a **PERSPECTIVE SCALE !***
(will be defined later)

INVERSE PROPERTIES

When drawing circle C with X3.X4 as diameter these inverses occur:
X20 (1:–1 : 0) is the inverse from X2 (1:+1:0) wrt circle C.
X550 (1:–2/3:0) is the inverse from X140 (1:+2/3:0) wrt circle C.
X376 (1:–1/2:0) is the inverse from X631 (1:+1/2:0) wrt circle C.
X20 (1:–1 : 0) is the inverse from X2 (1:+1:0) wrt circle C.
From the coordinates it can be seen that the sign of the value of the 2nd coordinate changes by inverting the point wrt circle C.

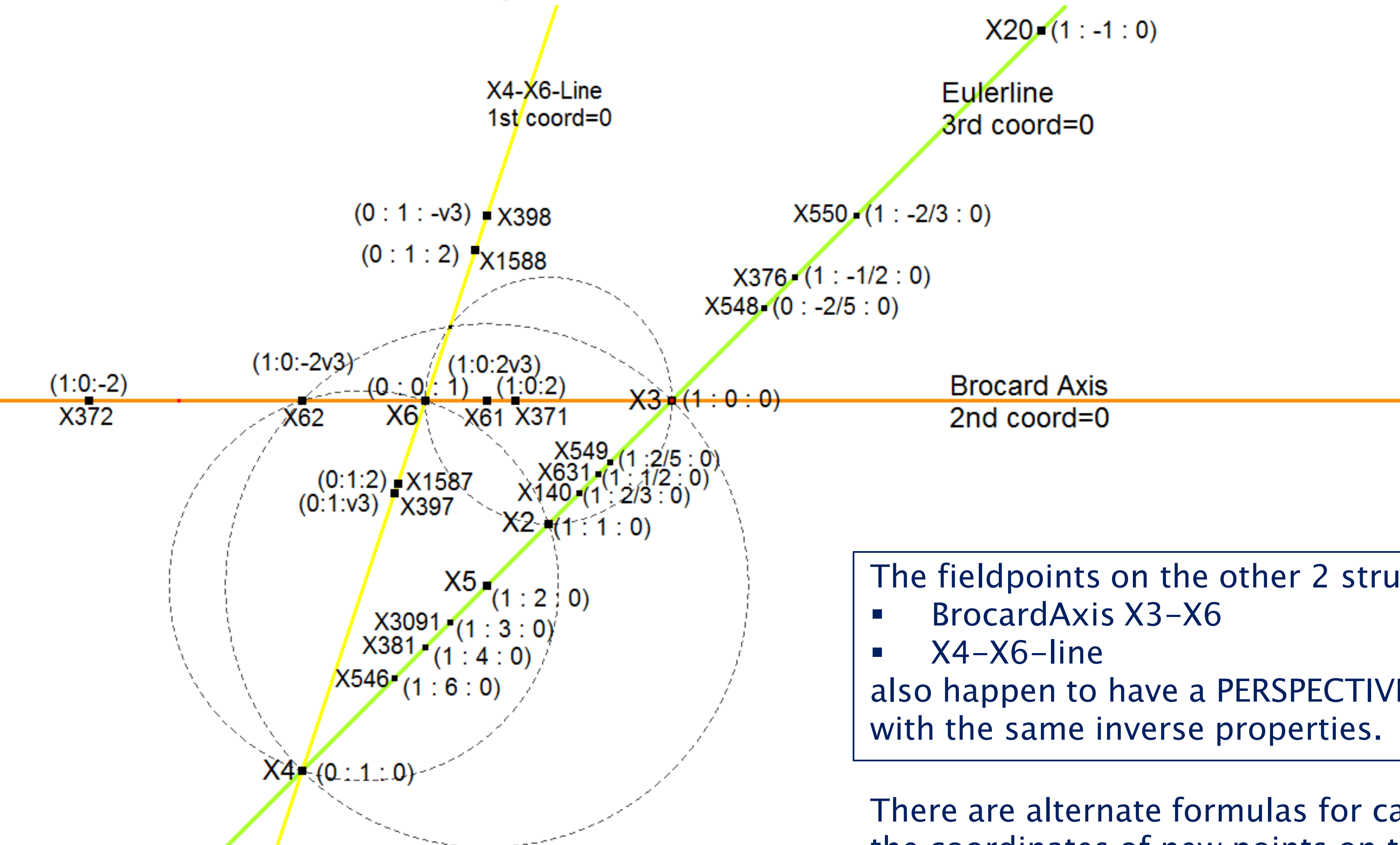
ALTERNATE WAY OF CALCULATING COORDINATES ON EULERLINE

Given two points P_1 $(i_1 : j_1 : 0)$ and P_2 $(i_2 : j_2 : 0)$ on the Eulerline.
The coordinates of P_3 $(i_3 : j_3 : 0)$ such that $P_1.P_3 : P_2.P_3 = m : n$ are

$$i_3 = 2(m+n).i_1.i_2 + (n.i_1.j_2+m.i_2.j_1)$$
$$j_3 = (m+n).j_1.j_2 + 2(m.i_1.j_2+ n.i_2.j_1)$$

Structure of a Perspective Field

Perspective Coordinates – scale on lines



The fieldpoints on the other 2 structural lines:

- BrocardAxis X3-X6
- X4-X6-line

also happen to have a PERSPECTIVE SCALE with the same inverse properties.

There are alternate formulas for calculating the coordinates of new points on these lines but they are not as simple as the one for the Eulerline.

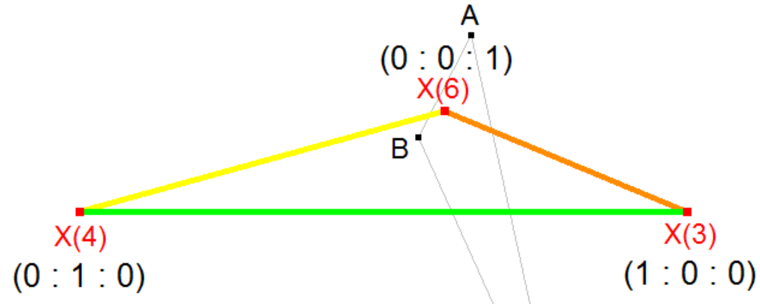
Structure of a Perspective Field

Perspective Coordinates – presentation

The vertices X(3), X(4) and X(6) can be represented in different ways.
For example
(0 :   1   : 0)
= (0 : INF : 0)
= (0 : INF : 1)
= (1 : INF : 0)
represent all the same vertice X(4).

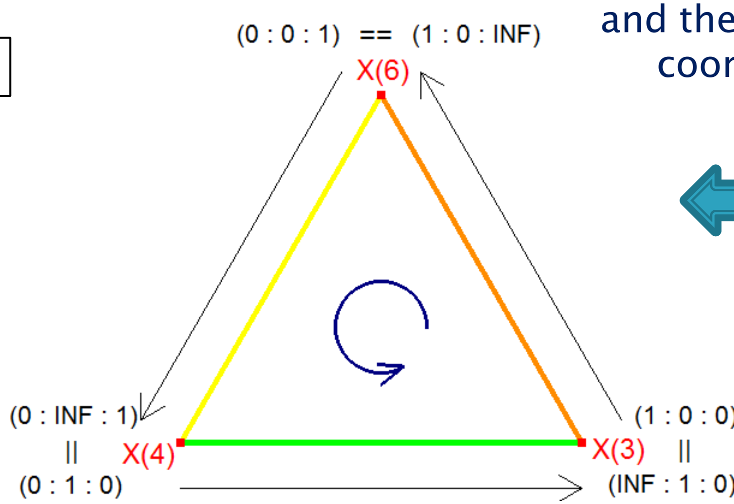
Coordinates (i : j : 0) can be represented in 2 different ways:
(1 : j/i : 0)
= (i/j : 1 : 0).
This leads to different diagrams.

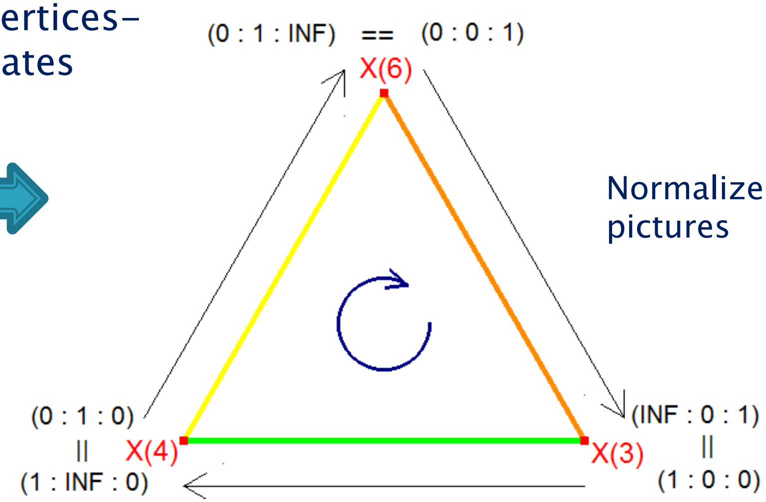
INF=infinite



Actual picture

different presentations of X3-X4-X6-triangle and their vertices-coordinates





Normalized pictures

Structure of a Perspective Field
Perspective Coordinates – normalization

NORMALIZATION TO AN EQUILATERAL TRIANGLE

- The Perspective Network is defined by 4 points.
  To get hold on the Perspective Network we normalized the 4-point-quadrilateral to a square. All other points were transformed accordingly.

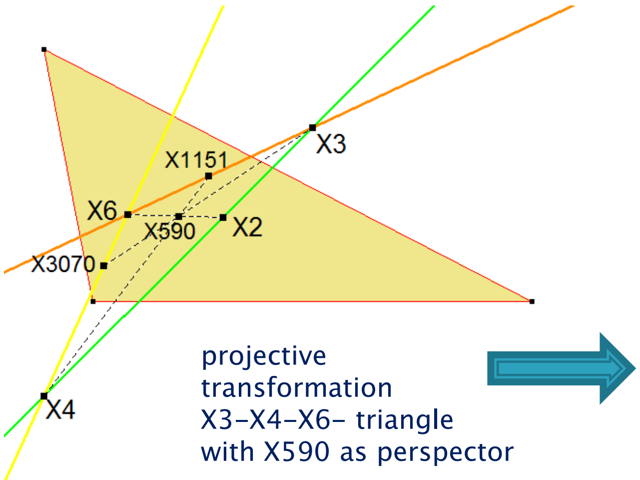
- The FormulaField is describing all points in the Perspective Field with just 3 basic points.
  Now it is obvious to try normalizing these 3 basic points too.
  In this case by *normalizing to an equilateral triangle*.
  Because we want to transform all other networkpoints we need an extra point.
  For that purpose we take the centre of the equilateral triangle after normalization.
  This point also defines the midpoints of the sides of the triangle after normalization.
  That's why we choose this intersectionpoint:
  (1:0:0)-(0:1:1) ^ (0:1:0)-(1:0:1) ^ (0:0:1)-(1:1:0) .

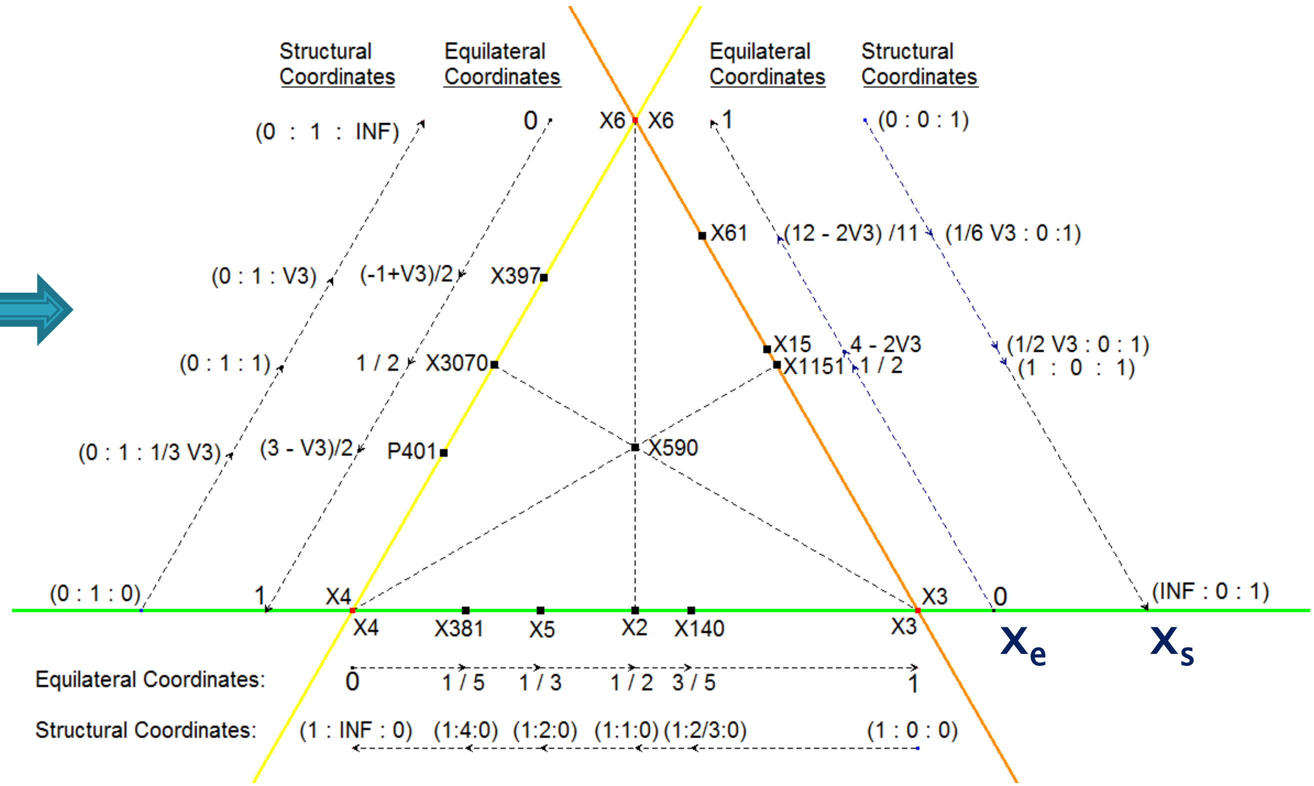
- Doing so a picture shows up that gives a 1:1 relationship between
  - coordinates of the points on the sides of the equilateral triangle, and
  - the weighting factors of the barycentric / trilinear formula-elements.

Look at next picture to see the result >

Structure of a Perspective Field

Perspective Coordinates – determining sidepoints





In essence this diagram explains structure and composition of trilinear and barycentric coordinates.

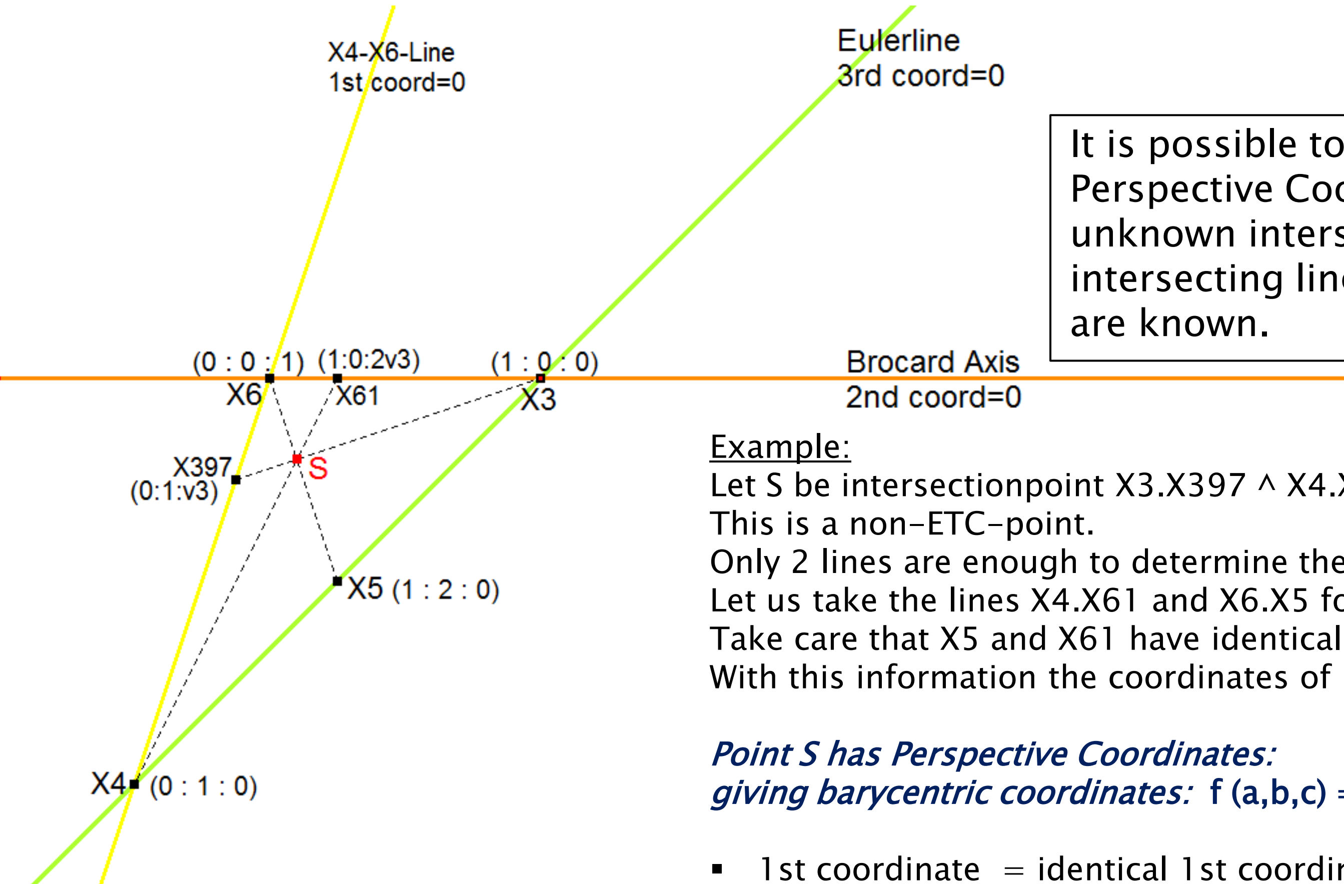
There is a simple relationship between:

- **normalized equilateral coordinate (x_e) = simple distance between vertice and point, and**
- **variable perspective coordinate (x_s) = that one of the perspective coord. triple <> 0 or 1**

$$x_s = (1 - x_e) / x_e \qquad \text{or} \qquad x_e = 1 / (1 + x_s)$$

Structure of a Perspective Field

Perspective Coordinates – determining fieldpoints



It is possible to determine the Perspective Coordinates from an unknown intersectionpoint when the intersecting lines through X3, X4 and X6 are known.

Example:
Let S be intersectionpoint X3.X397 ^ X4.X61 ^ X6.X5.
This is a non-ETC-point.
Only 2 lines are enough to determine the Perspective Coordinates.
Let us take the lines X4.X61 and X6.X5 for determining the new point S.
Take care that X5 and X61 have identical first coordinate = 1.
With this information the coordinates of new point S can be determined.

Point S has Perspective Coordinates: **$(1 : 2 : 2\sqrt{3})$**
giving barycentric coordinates: **$f(a,b,c) = 1\, a^2\, SA + 2\, SB\, SC + 2\sqrt{3}\, a^2 \Delta$**

- 1st coordinate = identical 1st coordinate X5/X61 = 1
- 2nd coordinate = 2nd coordinate from X5
- 3rd coordinate = 3rd coordinate from X61

Immediately it shows that:

- ratios 1st /2nd coordinates from S and X5 are identical.
- ratios 1st /3rd coordinates from S and X61 are identical.
- ratios 2nd/3rd coordinates from S and X397 are identical.

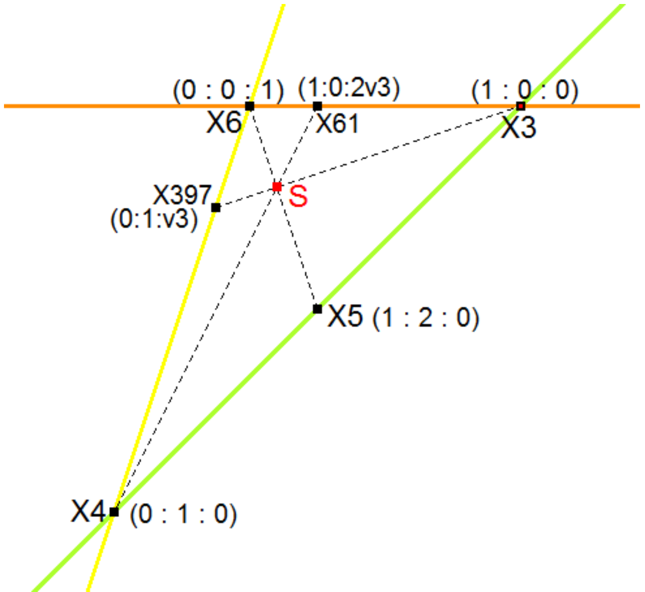
Point S is automatically a fieldpoint.

Structure of a Perspective Field

Perspective Coordinates – determining fieldpoints

It is also possible to determine the Perspective Coordinates from an unknown intersectionpoint

by using the crossdifference !



In Triangle Geometry different types of homogeneous coordinates can be used. To mention:

- ***Projective coordinates.*** These coordinates form a triple, where the first 2 coordinates are actually cartesian coordinates (the x- and y-coordinate) and the 3rd coordinate is usually 1 (except for infinitypoints, then the 3rd coordinate is 0).
- ***Barycentric coordinates.*** These are used in triangle environment and are a triple based on the area the point in question forms with the sidelines of the triangle.
- ***Trilinear coordinates.*** These are also used in triangle environment and are a triple based on the distance of the point in question to the sidelines.
- ***Perspective coordinates*** as being introduced here are a triple figuring in a dynamic triangle environment (vertices are dependant on other variables) and form a weighted average with respect to the vertices.

Strangely enough all these pointcoordinates use the same formulas (based on crossdifferences) for calculating a line or intersectionpoint.

Example Intersectionpoint with Perspective Coordinates

Calculation

S = CD [CD [X3, X397] , CD [X5, X6]]

$= CD [CD [(1:0:0) , (0:1:\sqrt{3})] , CD [(1:2:0) , (0:0:1)]]$

$= CD [(0:-\sqrt{3}:1) , (2:-1:0)]$

$= (1: 2 : 2\sqrt{3})$

CD = CrossDifference

$CD [(i_1 : j_1 : k_1) , (i_2 : j_2 : k_2)] = (j_1.k_2 - j_2.k_1 : k_1.i_2 - k_2.i_1 : i_1.j_2 - i_2.j_1)$

Structure of a Perspective Field

Perspective Coordinates – determining fieldpoints

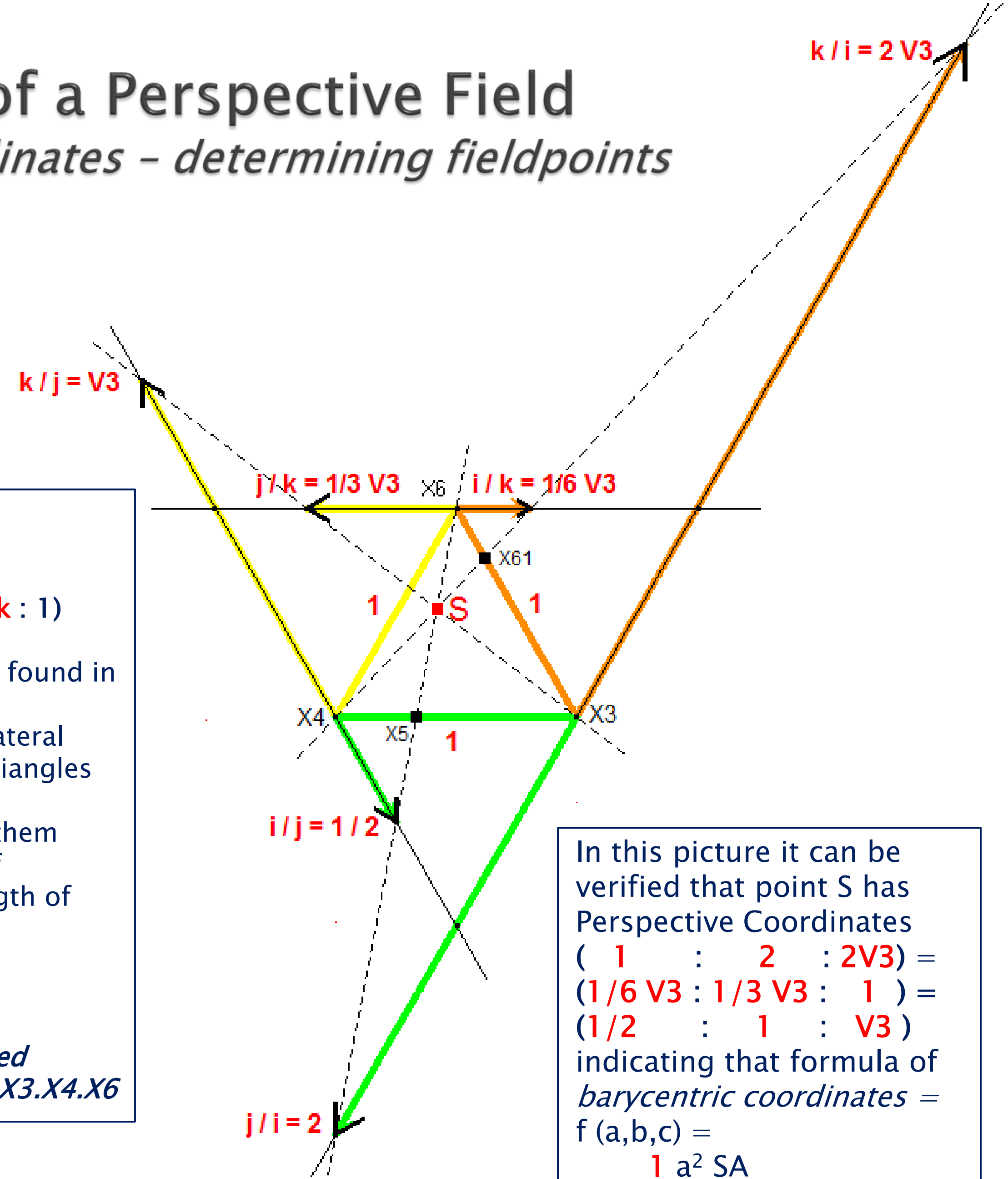
It is also possible to determine the Perspective Coordinates from any point just by measuring surrounding linesegments in the normalized equilateral triangle.

Let S be defined by Perspective Coordinates (i : j : k).

Equivalents for these coordinates are:
(i : j : k) = (1 : j/i : k/i) = (i/j : 1 : k/j) = (i/k : j/k : 1)

- These ratios j/i, k/i, i/j, k/j, i/k and j/k can be found in the picture of the normalized equilateral triangle.
- It is assumed that all sides of the normalized equilateral triangle X3.X4.X6 and its surrounding equilateral triangles have sidelength 1.
- By drawing the cevians of point S and intersecting them with the *sides of the anticomplementary triangle* of X3.X4.X6 linesegments appear with exactly the length of the equivalents of the fieldcoordinates of S.

In this way it is possible
- *to verify Perspective Coordinates, or*
- *to determine the location of points in the normalized equilateral triangle of X3.X4.X6*



In this picture it can be verified that point S has Perspective Coordinates
(1 : 2 : 2V3) =
(1 /6 V3 : 1 /3 V3 : 1) =
(1 /2 : 1 : V3)
indicating that formula of *barycentric coordinates* =
f (a,b,c) =
1 a^2 SA
+ 2 SB SC
+ 2V3 $a^2\Delta$

Structure of a Perspective Field

Perspective Coordinates – types of coordinates

So now we have a new system of coordinates for points in the Triangleplane.

We knew:

- *Trilinear and Barycentric Coordinates* consisting of cyclic formulas based on sides a,b,c and/or angles A,B,C.

We found:

- *Normalized Perspective Square Coordinates* consisting of 2 numbers in a cartesian coordinatesystem and in a setting of 4 predefined points (the perspectivity window) being transformed to the vertices of a square.

At last we found:

- *Perspective coordinates* consisting of 3 numbers in a setting of 3 predefined basic points.
- *Normalized Perspective Equilateral Coordinates* consisting of 3 numbers in the setting of 3 predefined basic points as vertices of an equilateral triangle.

Look at next page for a scheme

Structure of a Perspective Field

Perspective Coordinates – types of coordinates

	Formulas or Numbers	Reference System	Coordinate System	Absolute or Homogeneous Coordinates
Trilinear & Barycentric Coordinates	3 cyclic formulas wrt a,b,c / A,B,C	ABC	Triangular	Homogeneous
Normalized Perspective Square Coordinates	2 numbers	4 fixed points wrt ABC	Cartesian	Absolute
Perspective Coordinates	3 numbers	3 basic points wrt ABC	Triangular	Homogeneous
Normalized Perspective Equilateral Coordinates	3 numbers	3 basic points wrt ABC	Triangular	Absolute

Structure of a Perspective Field
Perspective Coordinates

The implications of these Perspective Coordinates are huge.

- A simple set of 3 numbers is enough for defining a point in a Perspective Field.

- Anticipating the fact that there are many more perspective fields it makes clear that there probably will be
  a finite number of formula-elements that will be the basis for merely numerical coordinates of any point in the Triangle Plane !

Perspective Fields – Theory

So far we investigated the phenomenon of a Perspective Network and a Perspective Field.
I now will close with the theory of these phenomena.

<u>A Perspective Network</u> =

> The system of all possible lines and intersectionpoints
> coming forth from 4 points in a plane,
> no three of which are collinear.

<u>A Perspective Field</u> =

> A system of lines and points that are invariant
> after a projective normalization transformation,
> meaning the projective transformation of 4 points in a plane,
> (no three of which are collinear) to a square.

Perspective Fields – Theory
“a set of points”

- A Perspective Field also can be seen as *a set of points*.
  Just like a line or a circle are they all sets of points.

- The only difference with a line or a circle is that the points in a Rerspective Field are *dispersed*. They are scattered all over the plane.

- The similarity is that in both examples we deal with an infinite number of distinct points.

- A Perspective Field is ***a set of points*** defined by 4 basic points.
  In this way ***it fills the definitiongap of points*** :
  1 point defines a point,
  2 points define a line,
  3 points define a circle,
  4 points define a **PERSPECTIVE FIELD,**
  5 points define a conic.

Perspective Fields – Theory

Compared with other Sets of Points the Perspective Field is quite competitive.

SETS OF POINTS and the numbers of ETC-points they contain

LINES	# ETC-points
EulerLine	**282**
Line at Infinity	**229**
Brocard Axis	**182**
X2-X6 Line	75
X1-X2 Line	73

CIRCLES	# ETC-points
CircumCircle	**258**
InCircle	39
NinePoint Circle	37
Other Circles	< 10

PERSPECTIVE FIELDS *)	**# ETC-points**
X2-X3-X6-X13 Field	**101**
X1-X2-X3-X11 Field	**55**
X1-X2-X3-X355 Field	**54**
X2-X3-X69-X489 Field	**52**
X1-X3-X6-X202 Field	**51**
X2-X3-X6-X542 Field	**49**

HYPERBOLAS	# ETC-points
ABC-X2-X4 Hyperbola (Kiepert)	63
ABC-X1-X4 Hyperbola (Feuerbach)	51
ABC-X1-X2 Hyperbola	46
ABC-X2-X6 Hyperbola	43
ABC-X1-X6 Hyperbola	42
ABC-X3-X4 Hyperbola (Jerabek)	41
ABC-X1-X3 Hyperbola	38

PARABOLAS	# ETC-points
ABC-X476-X685 Parabola	6

ELLIPSES	# ETC-points
Steiner CircumEllipse	23
InEllipse of X1-Anticevian Triangle	22
Steiner InEllipse	9

*) Identification of Perspective Field by 4 centers with lowest ETC-nrs where no 3 centers are collinear.

So far only 4 sets of points were known with more than 100 ETC-points*.
Now there is another one:
the **X2-X3-X6-X13-Perspective Field.**

* ETC-points in the range X(1)-X(3514)

Slight differences in numbers may occur because of precision errors.

Perspective Fields – Theory

A Perspective Network can also be defined in another way.

A Perspective Network =
the system of all possible lines and intersectionpoints,
coming forth from the combination of 2 Perspective Lines,
crossing through some Scaling Point.

Where:

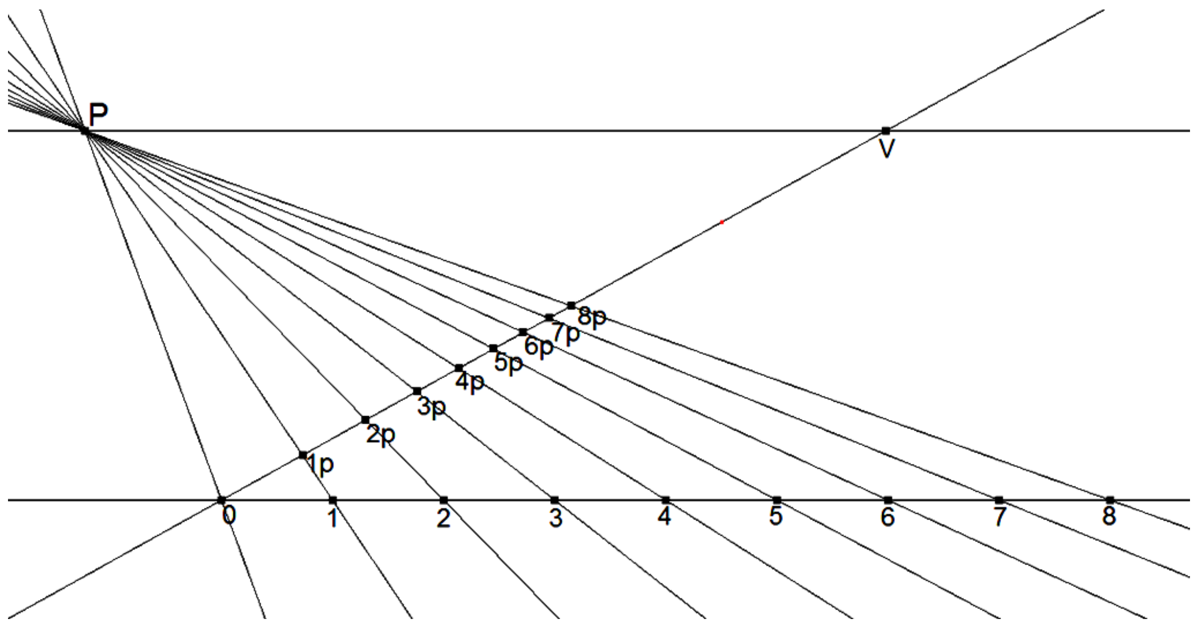
A Perspective Line =
a line with a perspective scale on it.

A Scaling Point =
a measurepoint on a line that is part of the scale.

Here you see 2 lines with a perspective scale on it.
The scale is defined by the sleepers.
This combination of 2 crossing lines (they cross at the vanishing point) with a perspective scale on it make a perfect Perspective Network indeed.
Not all possible lines and points are visible here.
But the “frame” of the network is visible.

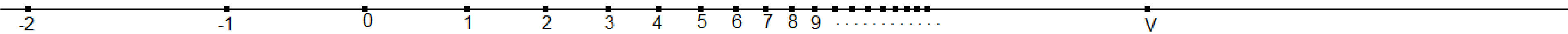
Perspective Fields – Theory
Perspective Scales

<u>Definition</u>
A Perspective Scale can be defined as the projection of some Linear Scale on a crossing line.
In this picture P is a Projection Point (also called Center of Projection).
Points 0, 1, 2, 3, etc. are points of the linear scale with identical mutual distances.
They are projected on some crossing line through 0.
Projected points are 0, 1p, 2p, 3p, etc.
V = intersectionpoint of the ProjectionLine and the line through P parallel to the line with the Linear Scale. V is defined as the Vanishing Point.



Perspective Fields – Theory
Perspective Scales

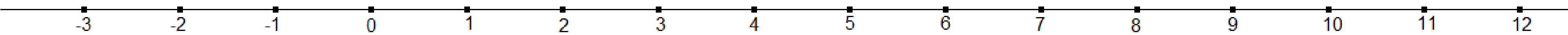
example of a perspective scale:



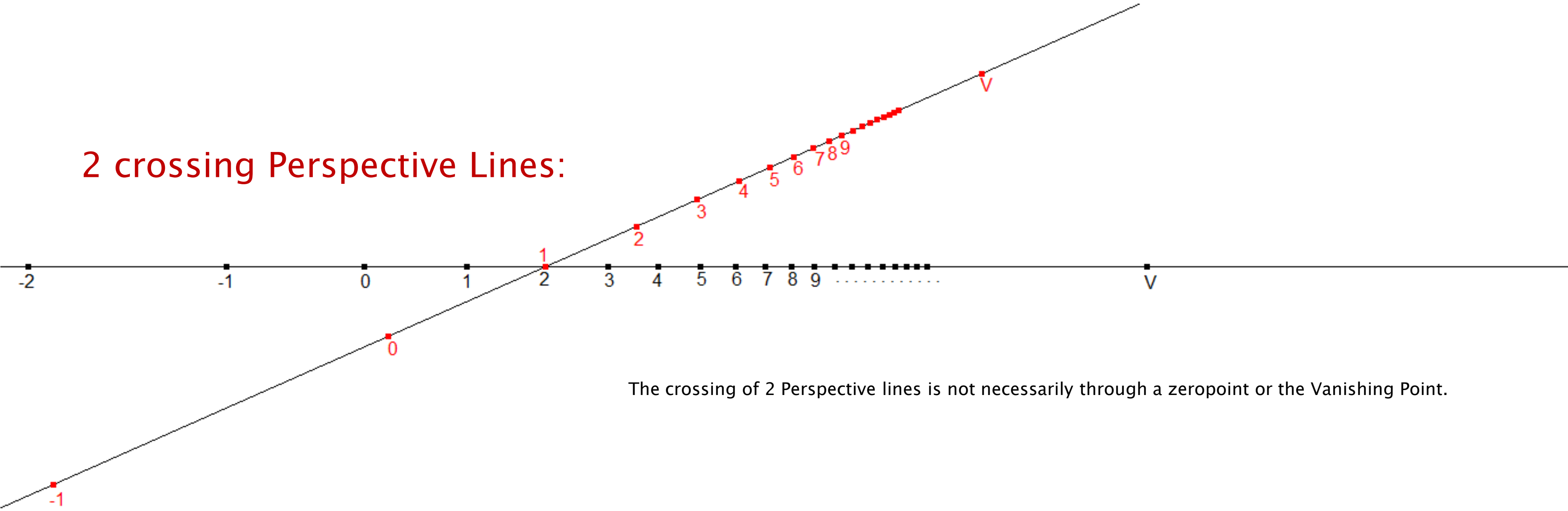
V = Vanishing Point

example of a linear scale:

(a linear scale actually is a normalized perspective scale)

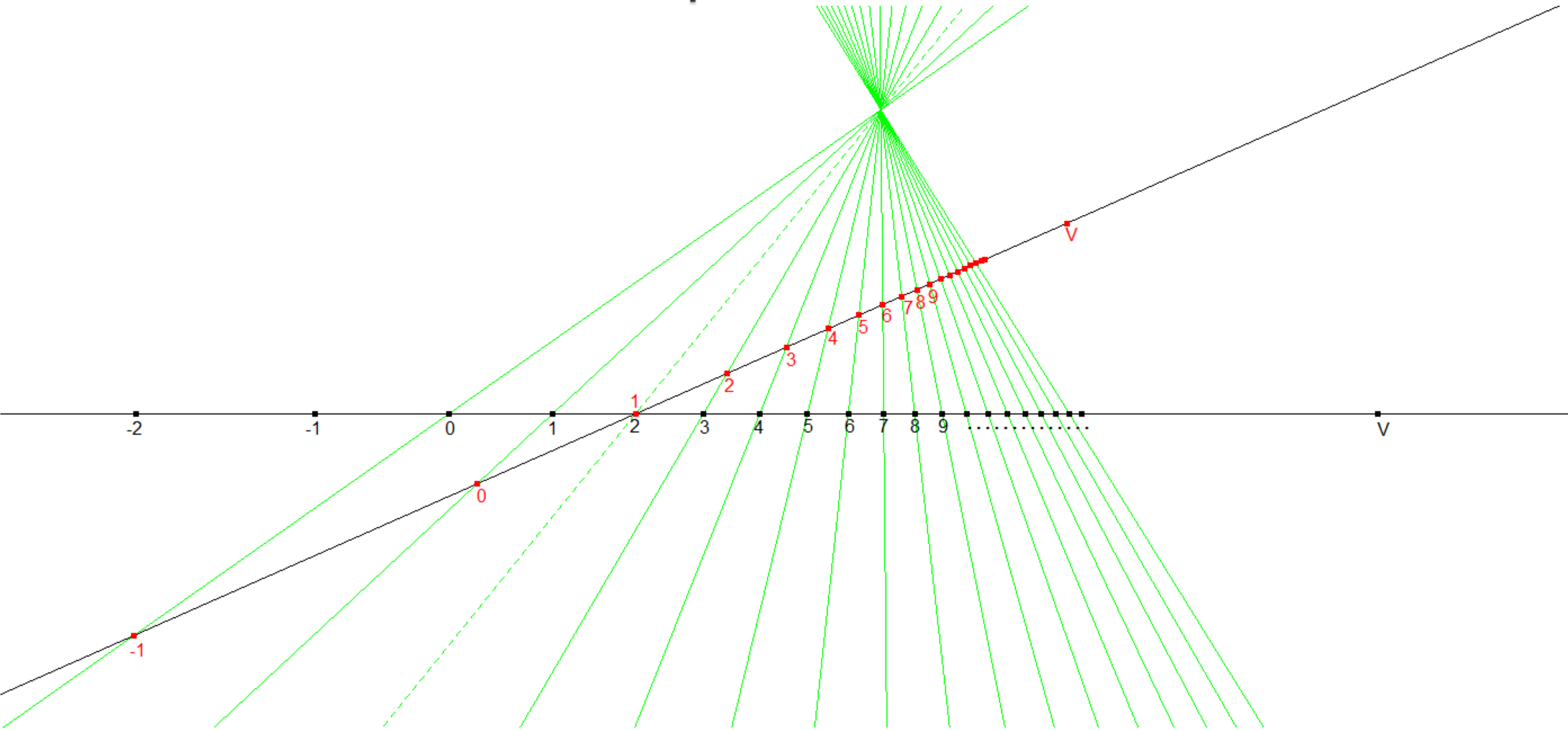


2 crossing Perspective Lines:



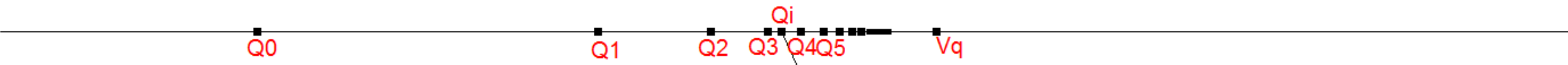
The crossing of 2 Perspective lines is not necessarily through a zeropoint or the Vanishing Point.

Perspective Fields – Theory
Perspective Scales



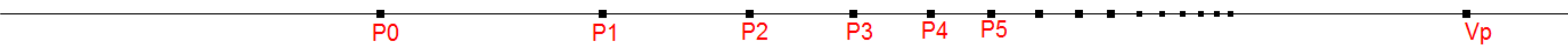
This picture shows *how 2 different perspective scales correspond to each other*. When lines are drawn between corresponding sequential scaling points *(counting from the crossing point)* all connecting lines will be concurrent.

Perspective Fields – Theory
Perspective Scales

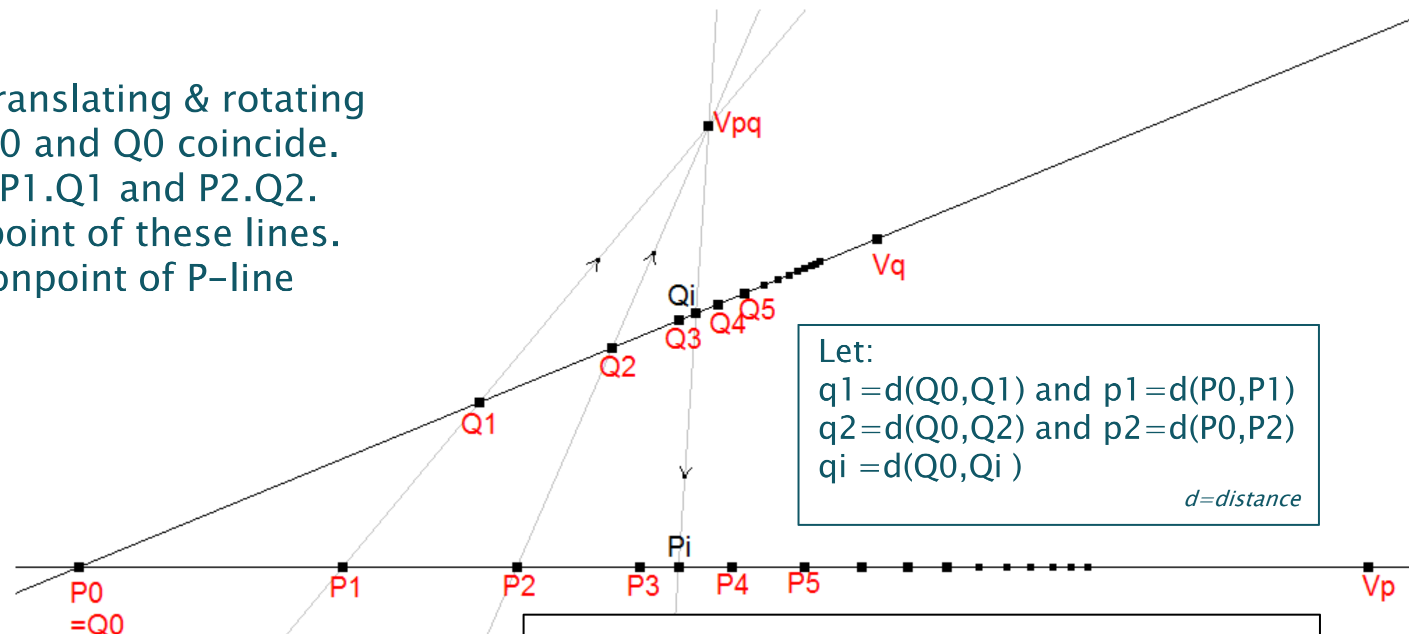


Given:
Q-line with perspective scale Q0, Q1, Q2, Q3, etc.
P-line with perspective scale P0, P1, Q2, P3, etc.
Random point Qi placed in Q-scale.
Question:
Find corresponding point Pi in P-scale.

?



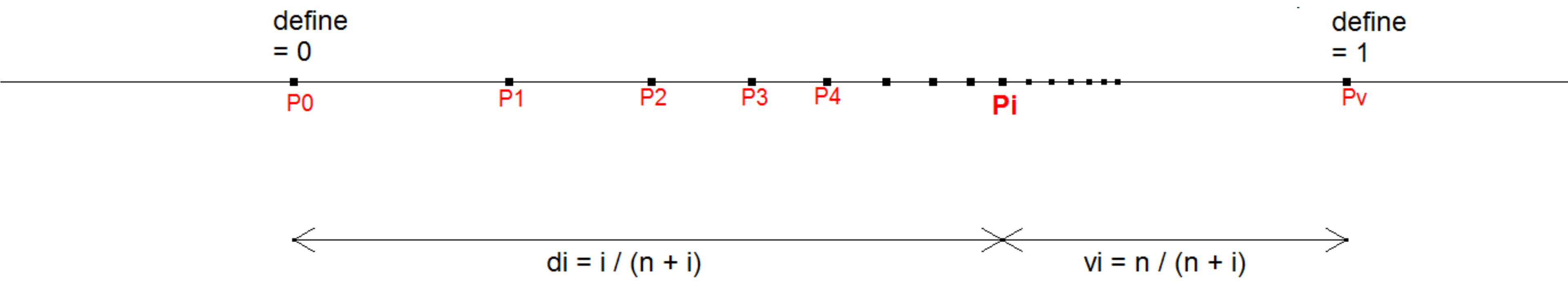
The solution is by translating & rotating the Q-line so that P0 and Q0 coincide.
Then drawing lines P1.Q1 and P2.Q2.
Vpq is intersectionpoint of these lines.
Now Pi = intersectionpoint of P-line and line Vpq.Qi.



Let:
q1=d(Q0,Q1) and p1=d(P0,P1)
q2=d(Q0,Q2) and p2=d(P0,P2)
qi =d(Q0,Qi)
d=distance

Then: $$pi = \frac{p1p2\,(q1 - q2)qi}{p1q2\,(q1 - qi) - p2q1(q2 - qi)}$$

Perspective Fields – Theory
Perspective Scales



- **Basically all Perspective Scales are similar.**

Just like all linear rows are similar.

- What makes a linear scale distinctive is the definition of the unit.

There is also a perspective-norm n that defines a perspective row.

- Every consequetive Perspective Row can be interpreted as follows.

Define some Startingpoint P0 and define Vanishingpoint Pv.
Define the distance (P0,Pv) = 1.
Now di = distance (Pi,Pv) = i / (n + i)
and vi = distance (P0,Pi) = n / (n + i)
where n can be any positive real number
i = row 0,1,2,3,4,5,6,7,……….

- The perspective-norm n is different per perspective scale.

It can be calculated by solving d1=n/(n+1) supposing that d1 is known.

- When distance (P0,Pv) = dv instead of 1 then di and vi can be multiplied by dv.

As an example when n = 1:

d0 = 0	v0 = 1
d1 = 1/2	v1 = 1/2
d2 = 2/3	v2 = 1/3
d3 = 3/4	v3 = 1/4
d4 = 4/5	v4 = 1/5
etc.	

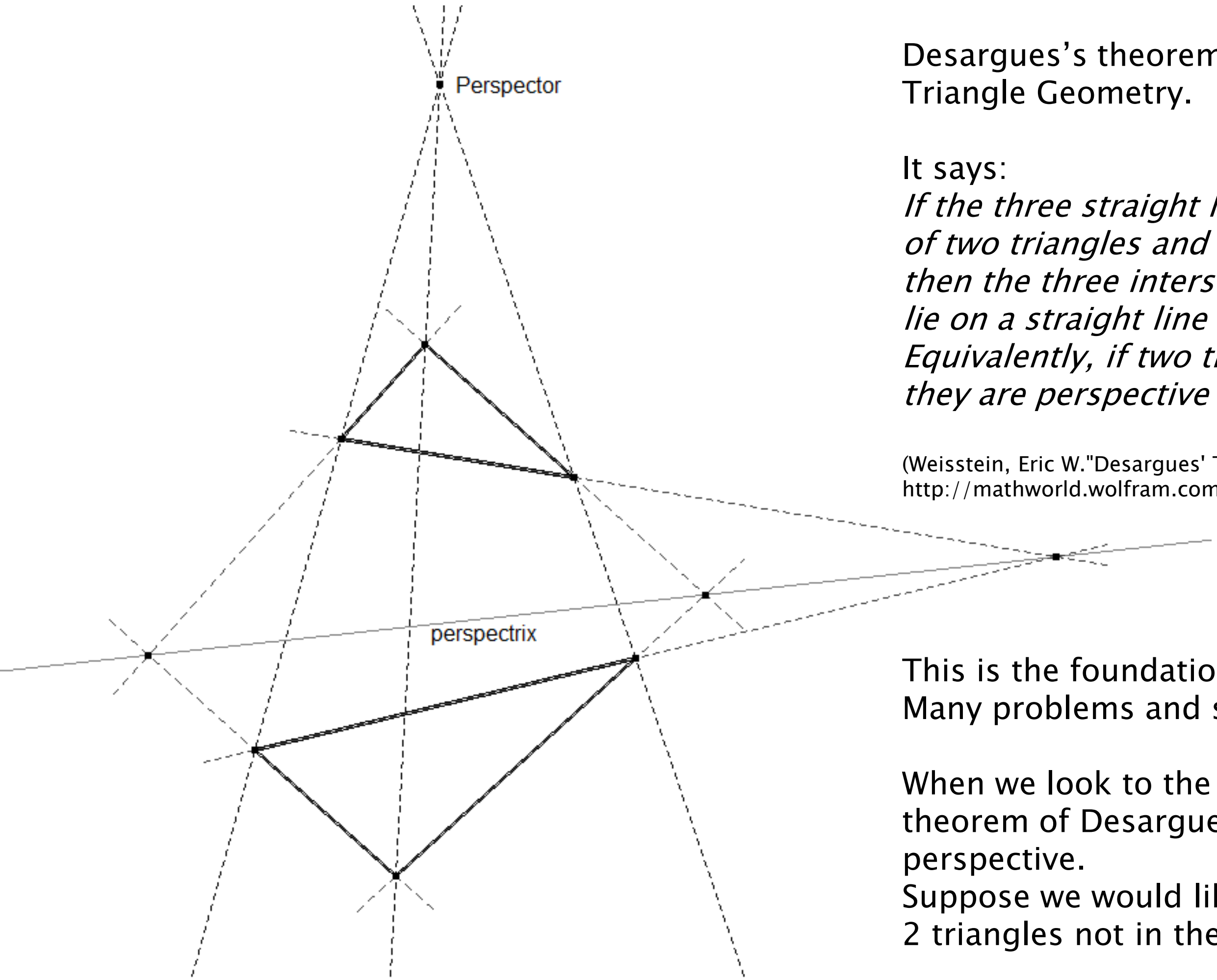
As an example when n = 2:

d0 = 0	v0 = 1
d1 = 1/3	v1 = 2/3
d2 = 2/4	v2 = 2/4
d3 = 3/5	v3 = 2/5
d4 = 4/6	v4 = 2/6
etc.	

n also can be a real number.

It is special to see that the well known row 1, $^{1}/_{2}$, $^{1}/_{3}$, $^{1}/_{4}$, $^{1}/_{5}$, ...
is also an example of a perspective row !

Perspective Fields – Theory
Perspective Centroids



Desargues's theorem is one of the most cited theorems in Triangle Geometry.

It says:
If the three straight lines joining the corresponding vertices of two triangles and all meet in a point (the perspector), then the three intersections of pairs of corresponding sides lie on a straight line (the perspectrix).
Equivalently, if two triangles are perspective from a point, they are perspective from a line.

(Weisstein, Eric W."Desargues' Theorem." From *MathWorld*-- A Wolfram Web Resource. http://mathworld.wolfram.com/DesarguesTheorem.html)

This is the foundation of the classic theory of perspectivity. Many problems and solutions are based on this principle.

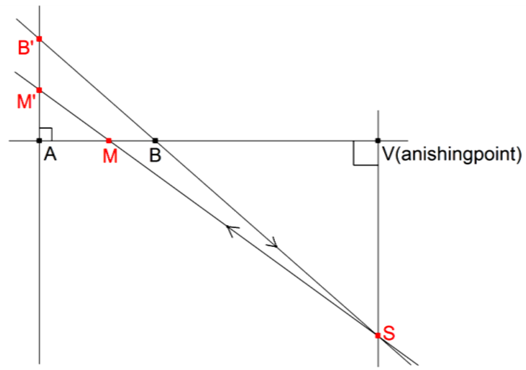
When we look to the picture at the left (presenting the theorem of Desargues) we can imagine seeing 2 triangles in perspective.
Suppose we would like to construct the centroids of these 2 triangles not in the conventional way, but in perspective !

How would this look like ???

A Perspective Centroid is a Centroid as seen under perspective projection

Perspective Fields – Theory
Perspective Centroids

The way to solve this is by constructing the perspective midpoints on all linesegments of the sides of the 2 perspective triangles.

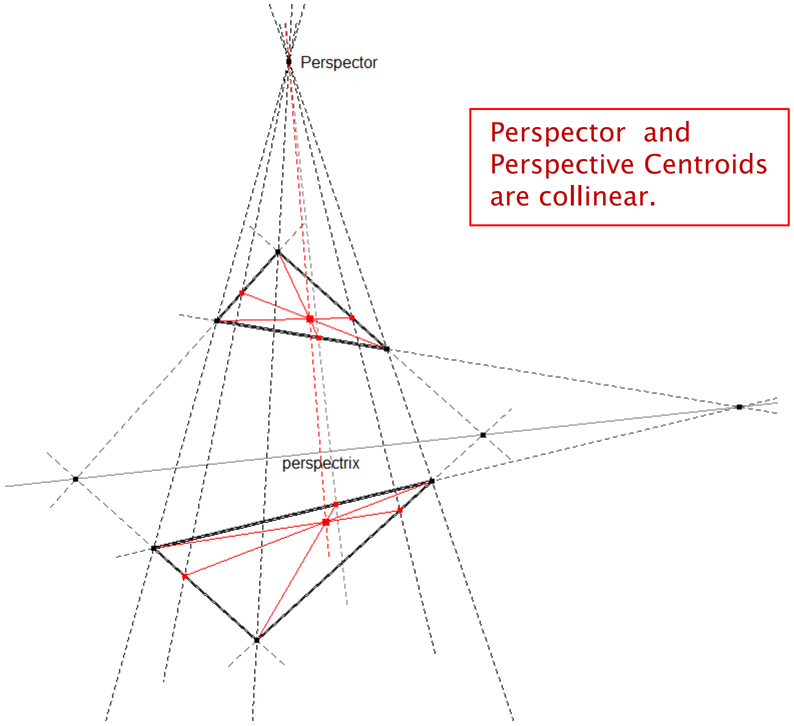


<u>Construction Perspective Midpoint</u>

Let A and B be the points to construct the perspective midpoint from.
Let V be the Vanishingpoint on the perspectrix.
Now draw a line perpendicular at AB through A.
And draw a line perpendicular at AB through V.

Choose some point B' at this perpendicular line.
Determine midpoint M' on linesegment AB'.
Let S be the intersectionpoint BB' ^ perpendicular(V).
Now is M = intersectionpoint SM' ^AB.
This is the perspective midpoint.

When applying this to the triangles of Desargues it looks like this:



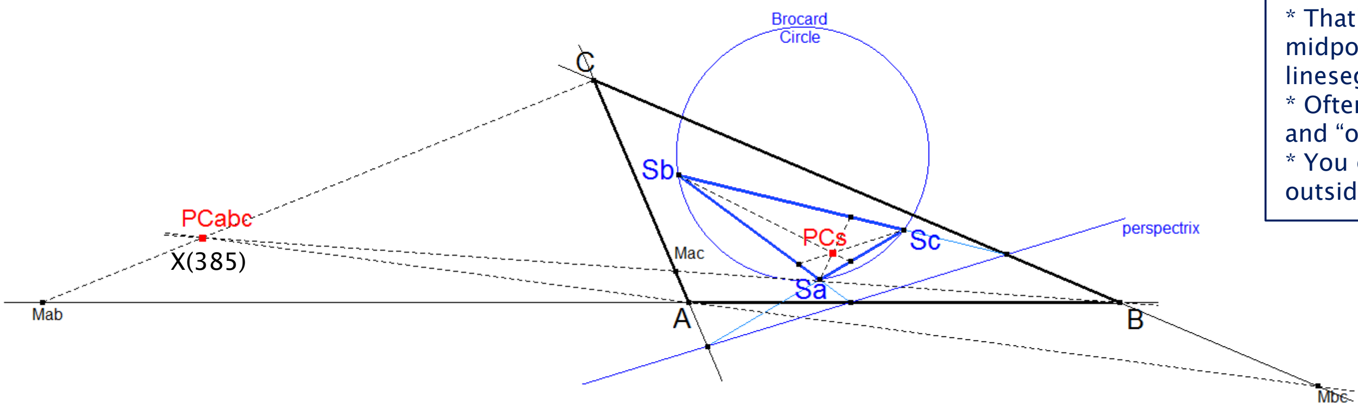
Perspector and Perspective Centroids are collinear.

So when we encounter 2 perspective triangles, not only Perspector and Perspectrix are within reach but also 2 Perspective Centroids ! ! !

<u>A Perspective Centroid is a Centroid as seen under perspective projection</u>

Perspective Fields – Theory
Perspective Centroids

The Perspective Centroids of ABC and the 1st Brocard Triangle (BT1) show how special these centroids are. They are very natural in the perspectivity-structure where they exist.
The Perspective Centroid of ABC wrt BT1 happens to be X(385).
The Perspective Centroid of BT1 wrt ABC is a non-ETC-point with barycentrics $(a^4 + 2\ b^2c^2)$.
It is the intersectionpoint of X(2).X(99) with X(32).X(76). So it is strongly Brocard-related.



* In this picture it can be seen that the Perspective Centroid of ABC wrt BT1 lies outside of ABC, which is a bit awkward for a centroid.
* That's because the perspective midpoints Mab, Mbc lie outside the linesegments AB and BC as well.
* Often in perspectivity "inside" and "outside" can change position.
* You could say that inside or outside is not an item at all.

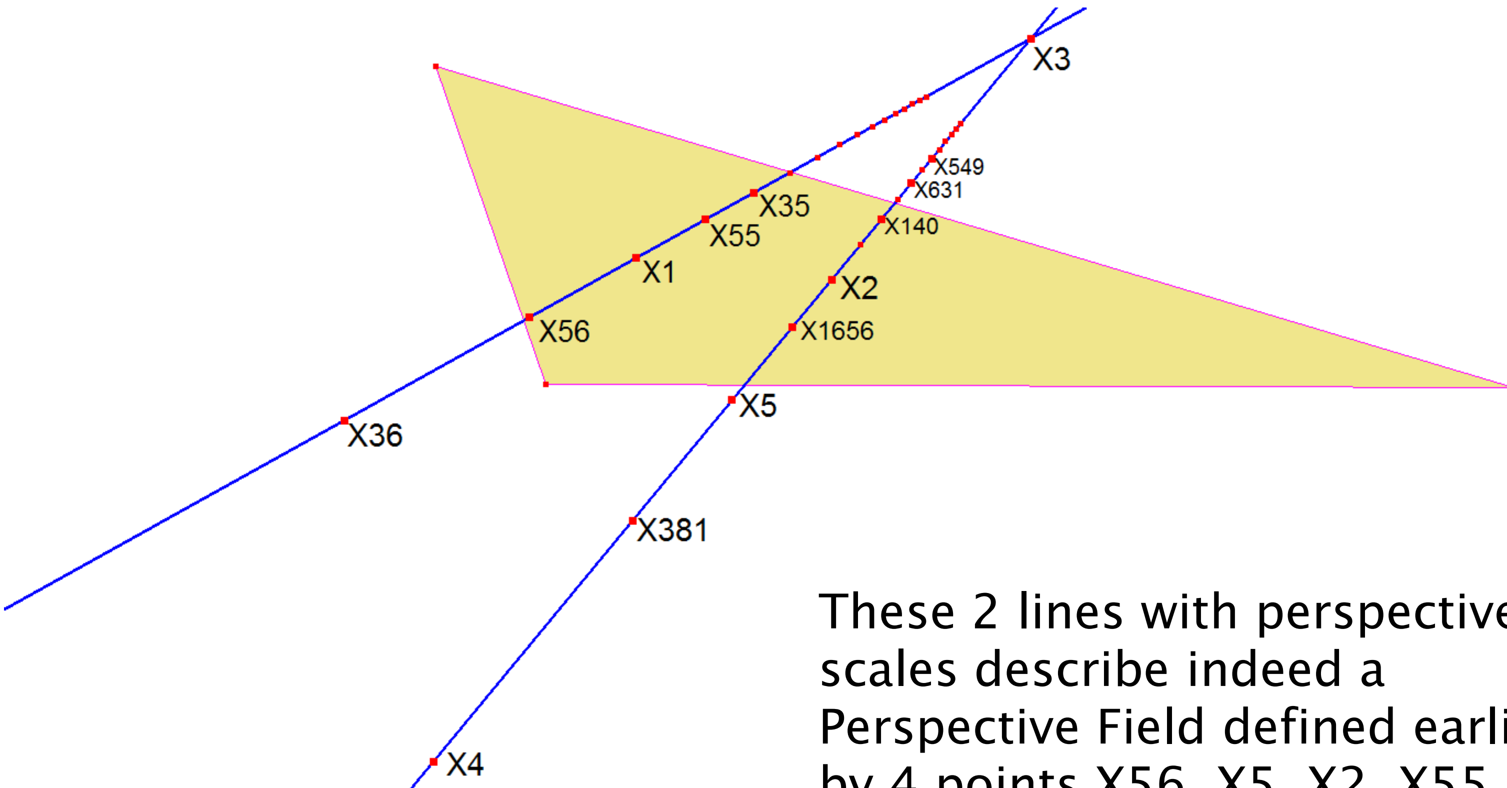
The Perspective Centroids of ABC and the 4 Brocard Triangles are:

	wrt ABC	wrt Brocard Triangle
Brocard Triangle 1	X(385)	$a^4 + 2\ b^2c^2$
Brocard Triangle 2	X(524)	X(2)
Brocard Triangle 3	X(385)	$2\ a^4 + b^2c^2$
Brocard Triangle 4	X(468)	$(a^2-2b^2-2c^2)$ SB SC

A Perspective Centroid is a Centroid as seen under perspective projection

Perspective Fields – Theory

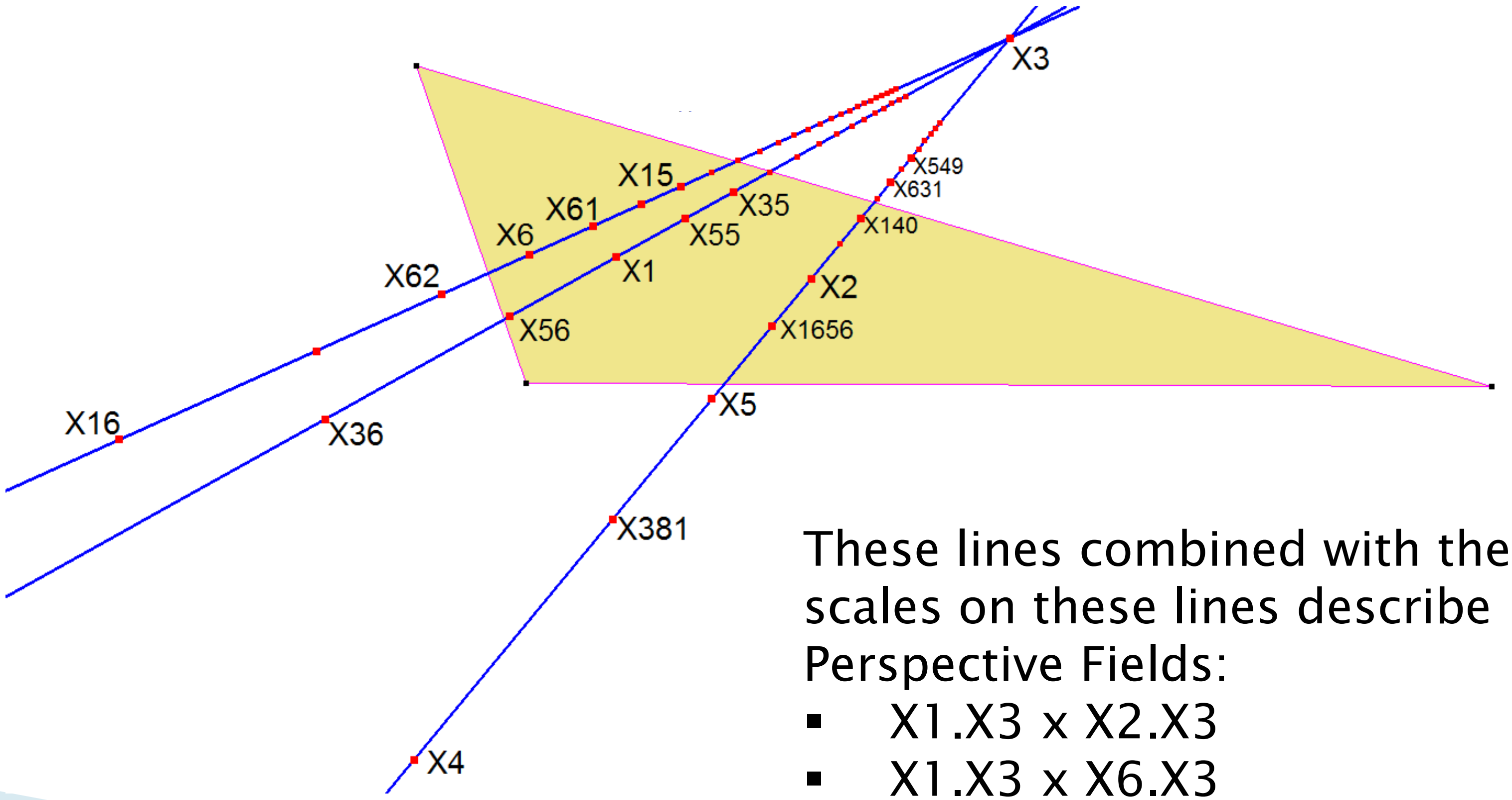
Here is a nice example of a combination of 2 Perspective Scales in the Triangle Plane.
You see 2 crossing Lines (X1.X3 and X2.X3) with some known points distributed in a perspective scale meeting at Scaling Point X3.
In this case X3 is the Vanishing Point of both scales.



These 2 lines with perspective scales describe indeed a Perspective Field defined earlier by 4 points X56, X5, X2, X55.

Perspective Fields – Theory

Here the addition of another Perspective Scale vanishing at X3. You see 3 crossing Lines (X1.X3, X2.X3 and X6.X3) with points distributed in a perspective scale meeting at Scaling Point X3.



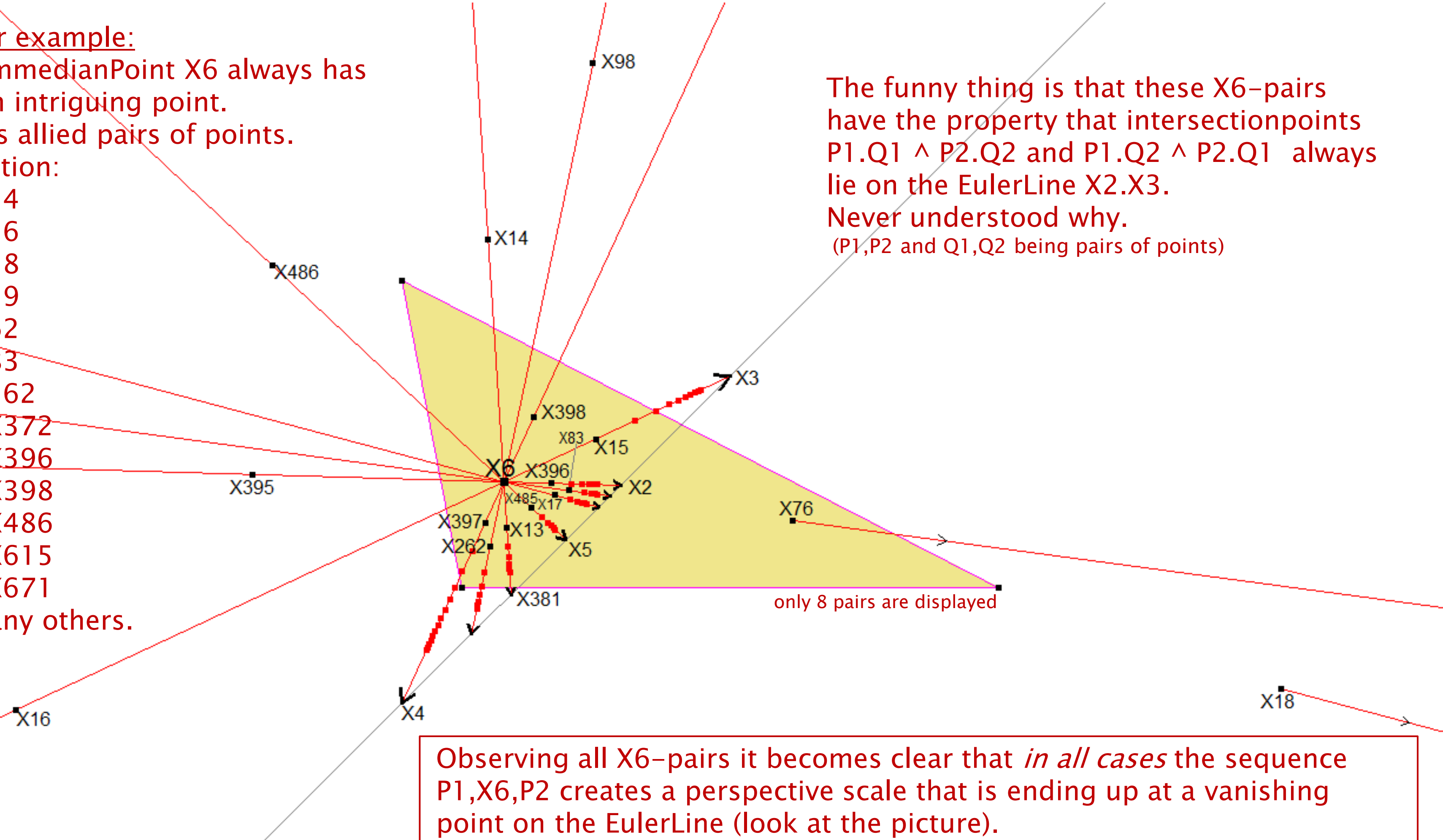
These lines combined with the scales on these lines describe 3 Perspective Fields:

- X1.X3 x X2.X3
- X1.X3 x X6.X3
- X2.X3 x X6.X3

Perspective Fields – Theory

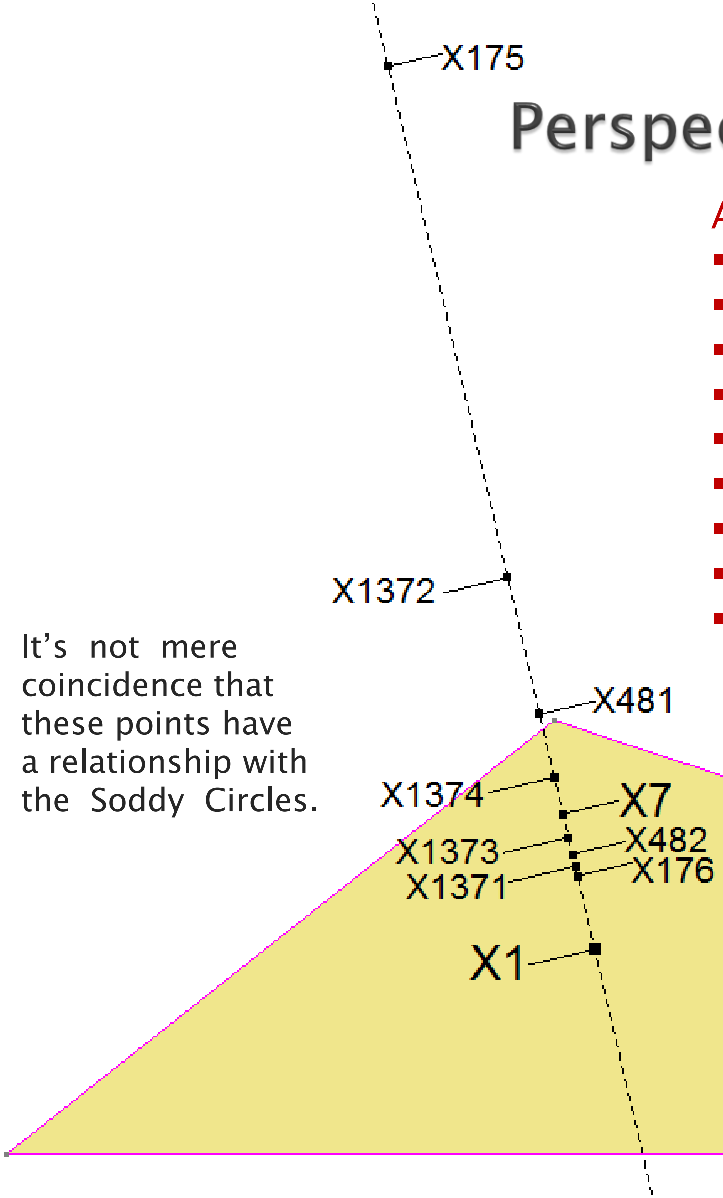
<u>Another example:</u>
The SymmedianPoint X6 always has been an intriguing point.
It liaises allied pairs of points.
To mention:
X13-X14
X15-X16
X17-X18
X32-X39
X61-X62
X76-X83
X98-X262
X371-X372
X395-X396
X397-X398
X485-X486
X590-X615
X598-X671
And many others.

The funny thing is that these X6-pairs have the property that intersectionpoints P1.Q1 ^ P2.Q2 and P1.Q2 ^ P2.Q1 always lie on the EulerLine X2.X3.
Never understood why.
(P1,P2 and Q1,Q2 being pairs of points)



Observing all X6-pairs it becomes clear that *in all cases* the sequence P1,X6,P2 creates a perspective scale that is ending up at a vanishing point on the EulerLine (look at the picture).
Generally spoken 2 crossing perspective scales always create a vanishing line (or horizon line or perspectrix, connecting both vanishing points).
Now when connecting corresponding scalingpoints they concur in a point at the same vanishing line. This explains above mentioned property.

Perspective Fields – Theory



Another nice example of a perspective scale is the sequence:

- X175 Isoperimetric Point
- X1372 2nd Rigby Point
- X481 1st Eppstein Point
- X1374 2nd Griffiths Point
- X7 Gergonne Point
- X1373 1st Griffiths Point
- X482 2nd Eppstein Point
- X1371 1st Rigby Point
- X176 Equal Detour Point

It's not mere coincidence that these points have a relationship with the Soddy Circles.

These points form a perfect *consequetive row of perspective points*.
The vanishingpoint of this row is X1 (Incenter).

It's just one example.

The Triangle Plane is full of perspective rows . . . . .

Perspective Fields – Algebraic Theory

Important is:

- 2 Points determine a Line.
  3 Points on a Line determine a Perspective Scale.

- 3 Points determine a Plane.
  (these 3 points not being collinear)
  4 Points in a Plane determine a Perspective Field.
  (of which no 3 points are collinear)

Perspective Fields – Algebraic Theory

- Every scalingpoint on a line with a Perspective Scale on it can algebraically be described as the weighted average of 2 other randomly chosen scalingpoints on this line.

- Every fieldpoint participating in a Perspective Field can be algebraically described as the weighted average of 3 other randomly chosen points (not on one line) from this field.

- Because of practical reasons it is better only to denote points as referencepoints when they have simple formulas.
  So it becomes much easier to identify the structural coordinates.

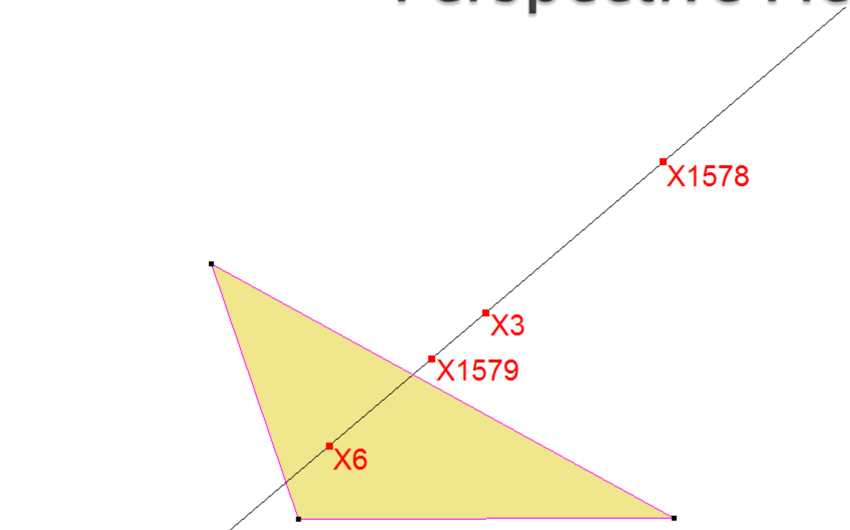
Perspective Fields – Algebraic Theory

- A Perspective Scale best can be defined by choosing these points:
  * ZeroPoint (Origin) $= 0$
  * UnityPoint $= 1$
  * VanishingPoint $= \infty$

- However when certain points are known to occur on a Perspective Scale there is no fixed role for these points. Every point can take the role of Origin, UnityPoint or VanishingPoint.
  Just like the choice of an Origin and a UnityPoint on a Linear Scale is arbitrary.

- Nonetheless when looking to the distribution of points often the best candidate for VanishingPoint can be distinguished by observation. Still this is an arbitrary choice. Next an Origin and UnityPoint will be chosen. Points will be chosen with simple algebraic expressions.

Perspective Fields – Algebraic Theory



		Trilinear coordinates
X(3)	= CircumCenter	cos(A)
X(6)	= Symmedian Point	sin(A)
X(1578)	= Point Alterf I	sin(A) + sec(B) sec(C)
X(1579)	= Point Alterf II	sin(A) – sec(B) sec(C)

Example: These points are 4 scalingpoints within the same Perspective Scale.

At first sight X(1578) can not be seen as a weighted average of X(3) and X(6).
However X(3) can be presented in another way by multiplying cos(A) with tricyclic independant factor: sec(A) sec(B) sec(C). This makes X(3) = sec(B) sec(C).

Now X(1578) and X(1579) can be described as the weighted average of X(3) and X(6):

X(1578) = X(6) + X(3) = sin(A) + sec(B) sec(C) (weights are +1 and +1)

X(1579) = X(6) – X(3) = sin(A) – sec(B) sec(C). (weights are +1 and –1)

But also X(3) and X(6) can be described as the weighted average of X(1578) and X(1579):

X(3) = X(1578)/2 – X(1579)/2 (weights are +1/2 and –1/2)

X(6) = X(1578)/2 + X(1579)/2. (weights are +1/2 and +1/2)

This example shows how a scalingpoint algebraically can be described as the weighted average of 2 other randomly chosen scalingpoints.

Perspective Fields – Algebraic Theory
Compliance Factor

- *In order to calculate a weighted average it is necessary to formulate the formula-elements in such a way that all elements comply to each other.*

- So the presentation of the formulas of the stucturepoints should be scale- or field-compliant.

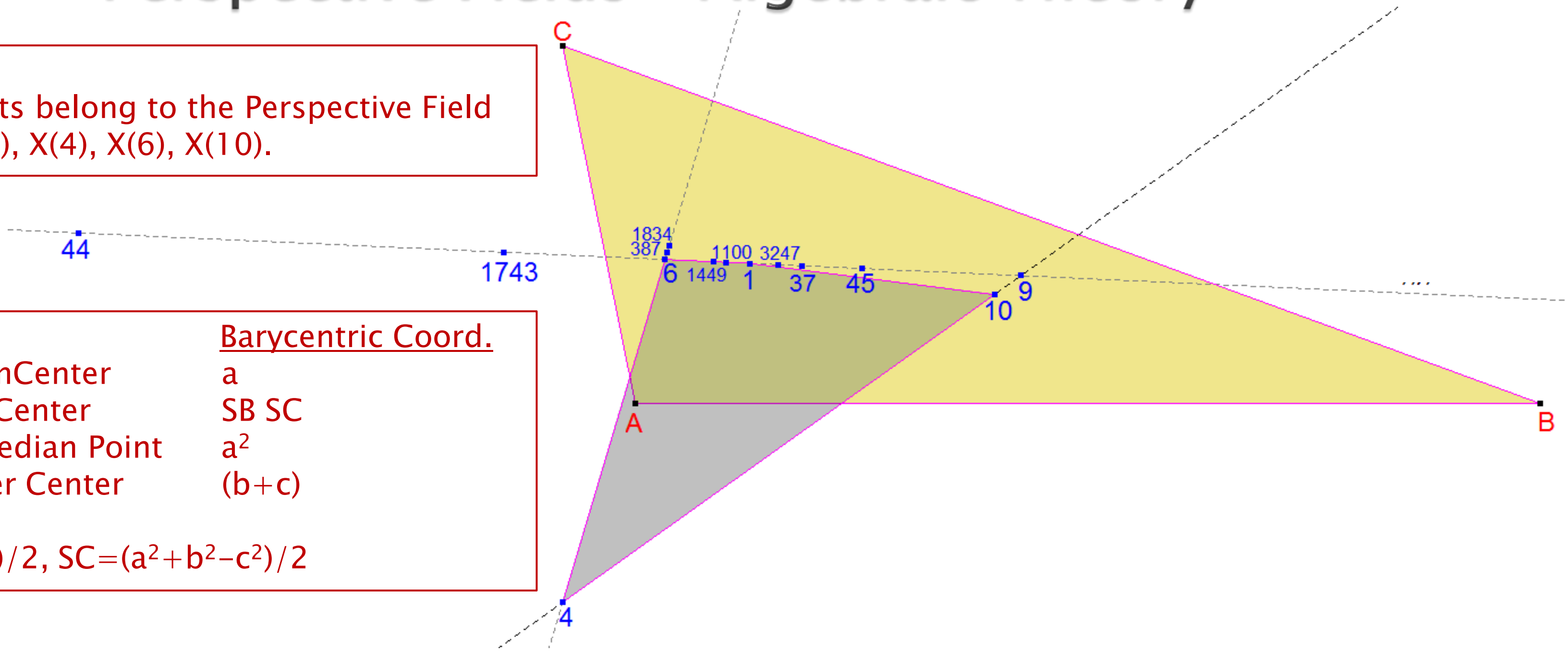
- This can be done by multiplying coordinate formula-elements from Reference Points with a standard ComplianceFactor.
  This ComplianceFactor should also be a formula-element, *but must be TriCyclic Symmetric* (like a+b+c and not a+b-c) in order not to disturb the homogeneity of the coordinates of the referencepoints.

- This ComplianceFactor can not be chosen but is imposed by the Perspective Scale or Field in progress.

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory

Example:
These 14 points belong to the Perspective Field defined by X(1), X(4), X(6), X(10).



	Barycentric Coord.
X(1) = CircumCenter	a
X(4) = OrthoCenter	SB SC
X(6) = Symmedian Point	a^2
X(10) = Spieker Center	(b+c)

SB= $(a^2-b^2+c^2)/2$, SC=$(a^2+b^2-c^2)/2$

Every fieldpoint participating *in a Perspective Field* can be algebraically described as the weighted average of 3 other randomly chosen points.
So it should be possible describing X(10) algebraically as the weighted average of X(1), X(4) and X(6).
This doesn't look naturally. The solution is in finding the right ComplianceFactors for X(1), X(4) and X(6).
In this case the ComplianceFactors for X(1), X(4) and X(6) are $(a+b+c)^3$, 1, $(a+b+c)^2$.
And now **X(10) = 1.X(1) + 4.X(4) – 2.X(6),** because:

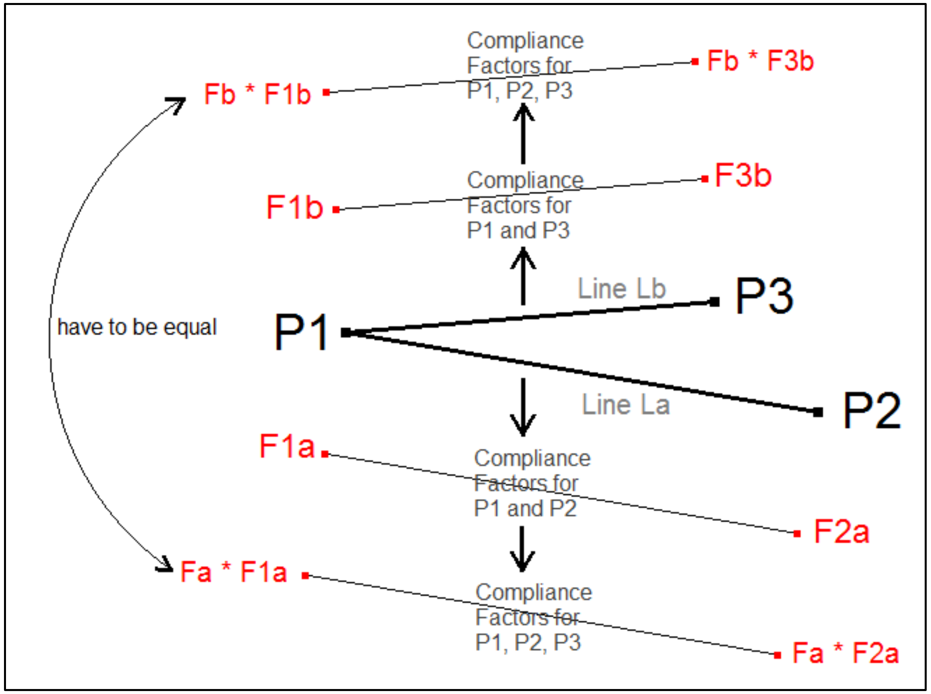
$$1.(a+b+c)^3.a + 4.1.\ SB.SC - 2.(a+b+c)^2.a^2 = -(b+c)\,(a^3+b^3+c^3-a^2 b-a b^2-b^2 c-b c^2-a^2 c-a c^2-2 a b c)$$

The last part of the outcome can be skipped because it is *Tricyclic Symmetric*, so the result is (b+c) indeed.

- Note that the products of ComplianceFactor and Barycentric Coordinate for X(1),X(4),X(6) are of the same degree !
- All other points can be described using these same ComplianceFactors.

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Compliance Factor



Determinining Compliance Factors in a Perspective Field

1. Choose amongst all points in the Field those points with the simplest formulas for Barycentric or Trilinear coordinates. These points P1, P2, P3 will get the role of Reference Points in the field.

2. Choose 2 lines La and Lb from the 3 possible connecting lines P1.P2, P2.P3, P3.P1.
   The coordinates of at least 3 points on these lines should be known.
   Let's say that we choose La=P1.P2 and Lb =P1.P3.

3. Determine the Compliance Factors F1a and F2a of P1 and P2 on line La.
   Determine the Compliance Factors F1b and F3b of P1 and P3 on line Lb.
   (method will be described at next page)

4. Since we now have 2 Compliance Factors F1a and F1b for referencepoint P1 we have to make them comply.

5. Determine the LCM (Least Common Multiple)-factor of factors F1a and F1b on formula level.
   Now for F1a and F1b minimal factors Fa and Fb can be found so that Fa * F1a = Fb * F1b = LCM.

6. Finally the Compliance Factors for the Perspective Field with respect to P1, P2, P3 are:

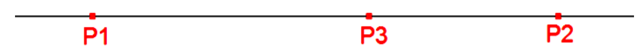
CF1 = F1a * Fa = F1b * Fb
CF2 = F2a * Fa
CF3 = F3b * Fb

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Compliance Factor

Determinining Compliance Factors of 2 Reference Points on a line with a Perspective Scale

Let P_1 and P_2 be the 2 Reference Points on a line and P_3 a third point on this line.



<u>Trilinear *actual* coordinates</u> of P_1, P_2 and P_3

$P_1\ (x_{1t}, y_{1t}, z_{1t})$ $S_{1t} = a\,x_{1t} + b\,y_{1t} + c\,z_{1t}$ $D_1 = d(P_1, P_3)$
$P_2\ (x_{2t}, y_{2t}, z_{2t})$ $S_{2t} = a\,x_{2t} + b\,y_{2t} + c\,z_{2t}$ $D_2 = d(P_2, P_3)$
$P_3\ (x_{3t}, y_{3t}, z_{3t})$ (a, b, c are sidelengths Ref.Triangle) (d=distance)

where: 1)

$$d\left(P_1(x_{1t}:y_{1t}:z_{1t}), P_2(x_{2t}:y_{2t}:z_{2t})\right) = \frac{1}{2\,\sigma}\sqrt{\sum_{cyclic}\left[-a\,b\,c\,\{a(y_{1t}-y_{2t})(z_{1t}-z_{2t})\}\right]}$$

$\sigma = Area$

Now the Compliance Factors of P_1 and P_2 are:
$CF_1 = D_2 * S_{2t}$ $CF_2 = D_1 * S_{1t}$

<u>Barycentric coordinates</u> of P_1, P_2 and P_3

$P_1\ (x_{1b}, y_{1b}, z_{1b})$ $S_{1b} = x_{1b} + y_{1b} + z_{1b}$ $D_1 = d(P_1, P_3)$
$P_2\ (x_{2b}, y_{2b}, z_{2b})$ $S_{2b} = x_{2b} + y_{2b} + z_{2b}$ $D_2 = d(P_2, P_3)$
$P_3\ (x_{3b}, y_{3b}, z_{3b})$ (d=distance)

where: 2)

$$d\left(P_1(x_{1b}:y_{1b}:z_{1b}), P_2(x_{2b}:y_{2b}:z_{2b})\right) = \sqrt{\frac{\sum_{Cyclic}\left[SA\,\{x_{1b}(y_{2b}+z_{2b}) - x_{2b}(y_{1b}+z_{1b})\}^2\right]}{(x_{1b}+y_{1b}+z_{1b})^2(x_{2b}+y_{2b}+z_{2b})^2}}$$

$SA = (-a^2 + b^2 + c^2)/2$

Now the Compliance Factors of P_1 and P_2 are:
$CF_1 = D_2 * S_{2b}$ $CF_2 = D_1 * S_{1b}$

These factors can be cleaned up by:
- dividing by common formulafactors (in fact by the greatest common divisor (GCD) on formulalevel)
- eventually replacing denominators by tricyclic complementary numerators

1) Triangle Centers and Central Triangles by Clark Kimberling, page 31
2) Introduction to the Geometry of the Triangle by Paul Yiu, page 88

<u>ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.</u>

Perspective Fields – Algebraic Theory
Compliance Factor

There is another and much simpler way to determine the Compliance Factors.
Let R1, R2, R3 be 3 Reference Points defining a Perspective Field.
Let P be another point not on the lines R1.R2, R1.R3 or R2.R3 that's part of the Perspective Field. From all points barycentric or trilinear coordinates are known.
Now the 3 Compliance Factors are:

CF1 = Det (P ,R2,R3) / Det (R1,R2,R3)
CF2 = Det (R1,P ,R3) / Det (R1,R2,R3)
CF3 = Det (R1,R2,P) / Det (R1,R2,R3).

Where Det = Determinant:

$$\mathrm{Det}\,(R1, R2, R3) = \mathrm{Det} \begin{vmatrix} R1_x & R1_y & R1_z \\ R2_x & R2_y & R2_z \\ R3_x & R3_y & R3_z \end{vmatrix}$$

$$= R1_x.(R2_y.R3_z - R2_z.R3_y) + R2_x.\,(R3_y.R1_z - R3_z.R1_y) + R3_x.\,(R1_y.R2_z - R1_z.R2_y)$$

$R1_x$, $R1_y$, $R1_z$ are barycentric or trilinear coordinates of R1, etc.

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Area Factor

When Barycentric Coordinates are used:

$$\text{Area}\ (R1,R2,R3) = \text{Det}\ (R1,R2,R3)\ .\ \Delta\ /\ (\text{Tot}(R1).\text{Tot}(R2).\text{Tot}(R3)).$$

When Trilinear Coordinates are used:

$$\text{Area}\ (R1,R2,R3) = \text{Det}\ (R1,R2,R3)\ .\ (a\ b\ c\ /\ (8\ \Delta^2))$$

Det (R1,R2,R3) = Determinant (R1, R2, R3)

Tot (R1) = $R1_x + R1_y + R1_z$

a, b, c = Sidelengths of Reference Triangle

Δ = Area of Reference Triangle

Now for every fieldpoint P in a Triangle formed of fieldpoints this formula is valid:

$$\textbf{Areas}\ (P.R2.R3 : R1.P.R3 : R1.R2.P) = i.\text{Tot}(R1) : j.\text{Tot}(R2) : k.\text{Tot}(R3)$$

(i : j : k) are the Perspective Coordinates of point P

Tot (R1) = $R1_x + R1_y + R1_z$

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Compliance Factor

Recapitulating:

The Compliance Factor is needed to determine the presentation of the coordinate-formulas of a referencepoint in a Perspective Field.
Often we use a standard formula for the trilinear or barycentric presentation of a point.
For example the barycentric coordinates of X(6) = Symmedian Point often are presented by: $(a^2 : b^2 : c^2)$.
But $(a^3\,b\,c \quad : a\,b^3\,c \quad : a\,b\,c^3)$
or $(a^2\,(a+b+c) : b^2\,(a+b+c) : c^2\,(a+b+c))$ also present the same point.
This is because multiplication with a Tricyclic Symmetric Factor does not change homogeneous coordinates.
However when we are producing new points by calculating the weighted average of the coordinates of 3 points then the presentation of a point is vital, because the extra factor affects the weighted sum of the coordinates.
The introduced Compliance Factor takes care of this vital extra factor.

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Compliance Factor

Example:

There are several Perspective Fields with X(1), X(2) and X(3) participating. Because in a Perspective Field every 3 points (not on one line) can be chosen to function as referencepoints all these fields can have the same referencepoints whilst being different.
The Perspective Field X1-X2-X3-X11 has 55 participating points that all can be seen as the weighted average of X(1), X(2) and X(3).

The standard barycentrics are:		The Compliance Factors are:	
X(1)	a	a b c	
X(2)	1	Δ^2	(Δ = Area Ref.Triangle)
X(3)	a^2 SA	1	(SA = $(-a^2 + b^2 + c^2) / 2$)

So all points in this field can be seen as the weighted average of X(1), X(2) and X(3), *however only in this presentation* :

X(1) a^2 b c
X(2) Δ^2
X(3) a^2 SA

Or stated in another way: X(PF-point) = $i \,.\, a^2 b c + j \,.\, \Delta^2 + k \,.\, a^2$ SA,
where i, j, k are real numbers !

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
Compliance Factor

Here are some different Perspective Fields all with X(1), X(2) and X(3) as chosen ReferencePoints and their Compliance Factors in this field.

	X(1)	X(2)	X(3)
Barycentrics	a	1	SB SC
			(SB = $(a^2 - b^2 + c^2) / 2$)
			(SC = $(a^2 + b^2 - c^2) / 2$)

	------------*Compliance Factors*--------------			# ETC-points
Fielddefinition	X(1)	X(2)	X(3)	participating
X1-X2-X3-X7	(-a+b+c) (a-b+c) (a+b-c) + 4 a b c	Δ^2	1	40
X1-X2-X3-X9	(-a+b+c) (a-b+c) (a+b-c) + 4 a b c	a b c (a+b+c)	1	18
X1-X2-X3-X11	a b c	Δ^2	1	55
X1-X2-X3-X355	(-a+b+c) (a-b+c) (a+b-c)	Δ^2	1	54
X1-X2-X3-X1770	(-a+b+c) (a-b+c) (a+b-c) + 2 a b c	Δ^2	1	44

(Δ = Area Ref.Triangle)

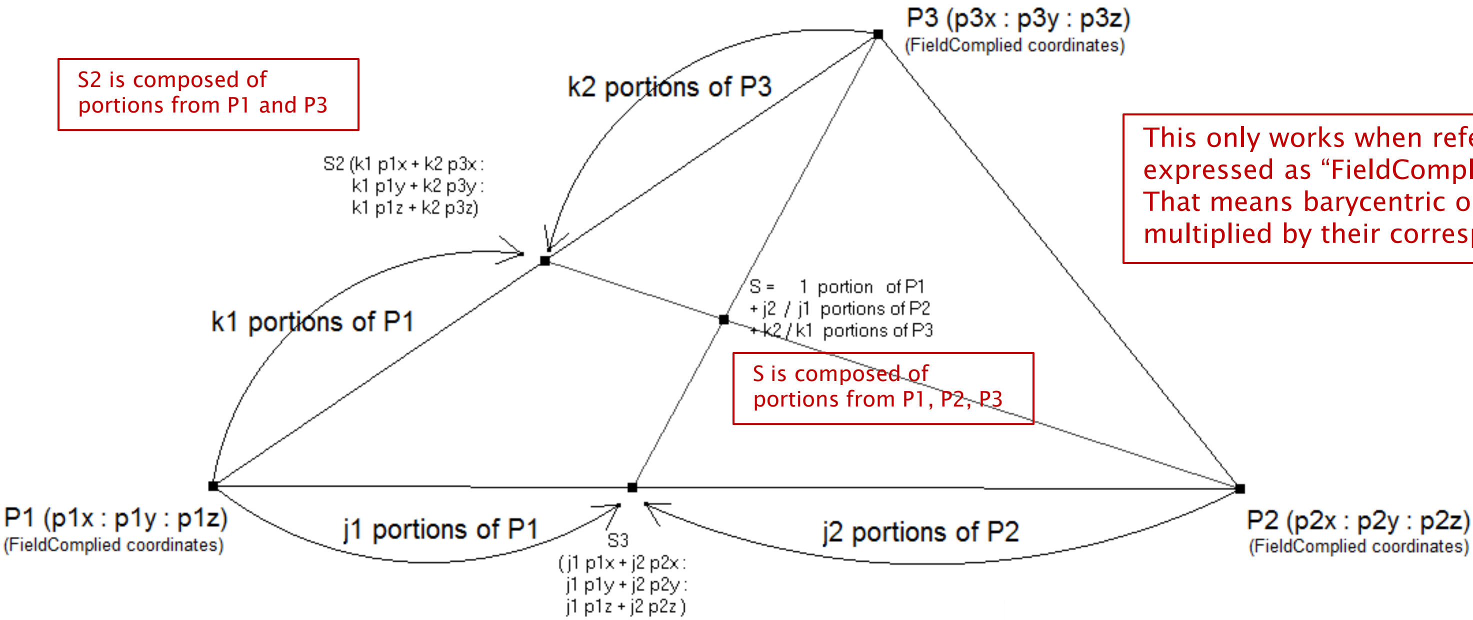
Note:
1. That all Compliance Factors are Tricyclic Symmetric.
2. That for each Perspective Field the product of Barycentric coordinate and Compliance Factor of X(1), X(2) and X(3) are of the same degree. This makes the presentation of X(1), X(2) and X(3) mutually compatible and ready for use in a weighted average.

ComplianceFactors are required for making points algebraic compatible within a Perspective Scale or Field.

Perspective Fields – Algebraic Theory
FieldComplied Coordinates

The unity and interconnection of Fieldcoordinates in a nutshell.

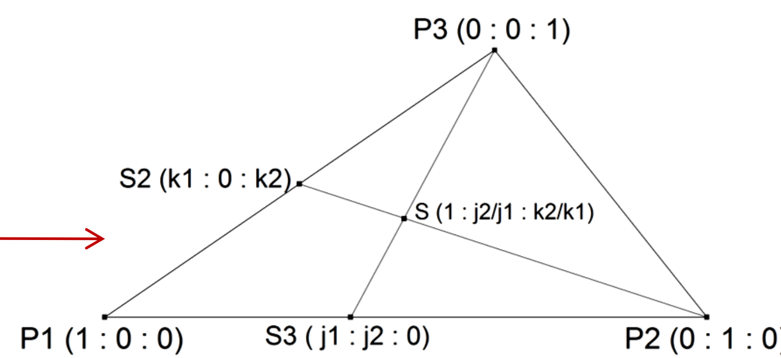
S2 is composed of portions from P1 and P3



This only works when referencepoints P1, P2, P3 are expressed as “FieldComplied Coordinates”.
That means barycentric or trilinear coordinates multiplied by their corresponding Compliance Factor.

S3 is composed of portions from P1 and P2

The same picture with Perspective Coordinates



FieldComplied Coordinates are Barycentric or Trilinear coordinates multiplied by their corresponding ComplianceFactor.

Perspective Fields
The notion of a Weighted Average

The notion of a weighted average has an important role in the algebraic theory of Perspective Fields. Because this theory states that any point in a Perspective Field can be seen as the Weighted Average of any 3 other points (not collinear) in this very Perspective Field provided that the presentation of these 3 points are complied by a compliance factor.

It seems to me that the notion of a Weighted Average (WA) can be used in other formulas using trilinear / barycentric coordinates.
A nice example is the **trilinear distance-formula of 2 points**.

Let $P_1=(x_1, y_1, z_1)$ and $P_2=(x_2, y_2, z_2)$, where $x_1, y_1, z_1, x_2, y_2, z_2$ are *actual* trilinear coordinates.

Now $$\frac{d(P_1,P_2)^2}{2} = \frac{a^2\,SA\,(x_1-x_2)^2 + b^2\,SB\,(y_1-y_2)^2 + c^2\,SC\,(z_1-z_2)^2}{a^2\,SA + b^2\,SB + c^2\,SC}$$ (can be derived from formulas mentioned 5 pages earlier)

It easily can be seen that this is the weighted average of $(x_1-x_2)^2$, $(y_1-y_2)^2$, $(z_1-z_2)^2$
with weigths (a^2 SA, b^2 SB, c^2 SC).
These weights happen to be the barycentric coordinates of X(3)=CircumCenter.

The Cartesian variant with weights (1, 1) gives a similar result:

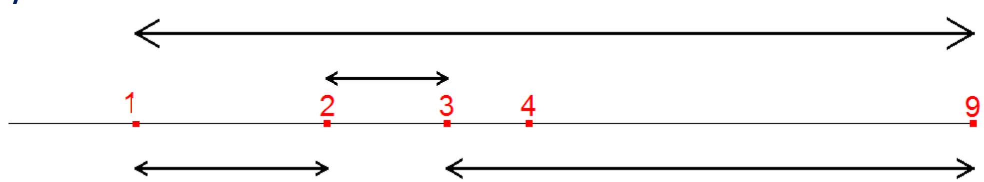
$$\frac{d(P_1,P_2)^2}{2} = \frac{1\,(x_1-x_2)^2 + 1\,(y_1-y_2)^2}{1+1}$$

Perspective Fields
Relationship Harmonic Conjugates

There is a clear relationship between perspective points and harmonic conjugated points.
The definition of harmonic conjugates is (Encyclopedia of Triangle Centers – Clark Kimberling):

> Suppose W, X, Y, Z are collinear points.
> Then W and X are *harmonic conjugates* (of each other) with respect to Y and Z if
> $|WY| / |WZ| = |XY| / |XZ|$.

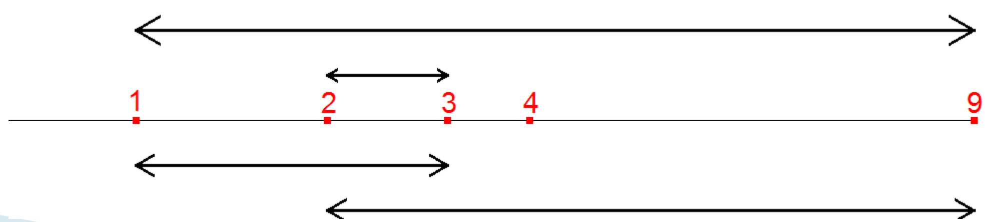
As a consequence $|XY| * |WZ| = |WY| * |XZ|$ or in next figure: 19 * 23 = 12 * 39
In next figure 1,2,3,4 are consequetive perspective points and point 9 is the vanishing point.
It gives special ratios that are valid *for a row of perspective points as well as for harmonic conjugated points*:



$$\frac{19 * 23}{12 * 39} = 1$$

This is in accordance with the definition of Harmonic Conjugation.

Points 1,2,3 can be chosen as ***any consequetive row on some perspective scale***.
Point 9 should be the vanishing point.



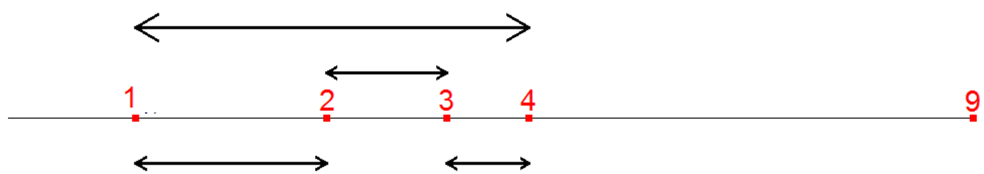
$$\frac{19 * 23}{13 * 29} = \frac{1}{2}$$

This is also valid In case of Harmonic conjugation.

Perspective Fields
Relationship Harmonic Conjugates

When taking as 4th point:
not the vanishing point in a perspective row,
but the 4th sequential point in a perspective row,
then the ratios change as illustrated in below figure.

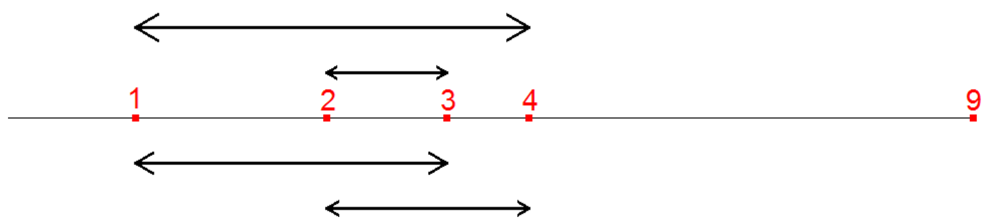
This new relationship could be named a ***semi-harmonic conjugation or perspective conjugation***.



$$\frac{14 * 23}{12 * 34} = 3$$

This is valid for consequetive perspective rows.

Points 1,2,3, 4 can be chosen as any consequetive row on some perspective scale.
Point 9 is the vanishing point (not in use here).



$$\frac{14 * 23}{13 * 24} = \frac{3}{4}$$

This is also valid for consequetive perspective rows.

Perspective Fields
Relationship Harmonic Conjugates

Situations of Harmonic Conjugates

As seen from former pictures 3 consequetive perspective points and the vanishingpoint of this row are harmonic conjugated. Harmonic conjugates (also) can be found in these situations.

SITUATIONS OF HARMONIC CONJUGATES (H1,H2) with respect to (H3,H4)	**(H1,H2)**	**(H3,H4)**
3 Consequetive points P1,P2,P3 on a perspective scale including vanishingpoint VP	P1,P3	P2,VP
Centers of 2 circles M1,M2 + Insimili-/Exsimili-Centers Si,Se	M1,M2	Si,Se
2 Diameterpoints on circle S1,S2 + point P at diameter + its inverse Pi	S1,S2	P,Pi
2 Diameterpoints on circle S1,S2 + 2 points P1,P2 on S1.S2 at orthogonal intersecting circle	S1,S2	P1,P2
Point P in or outside triangle ABC wrt Cevian Triangle A'B'C' and Anticevian Triangle A''B''C''	P,A''	A,A'
Vertices A,B and intersectionpoints Internal- and External AngleBisector from C with AB	A,B	Bi-c,Be-c
Intersectionpoints AB,BC,AC,BD of a quadrilateral with 3rd diagonal	Sab,Sbc	Sac,Sbd

As a matter of fact situations with 4 harmonic conjugated points configure the smallest perspective rows that can be discerned. They define and fit in some Perspective Scale.

Perspective Fields
Quantitative Aspects

Ok, all these Perspective Scales and Perspective Fields look very interesting, but to what extent does it show up in the Triangle Field?

Some questions first:

1. How many Perspective Scales do occur on the different possible lines through X(1), X(2), X(3), X(4), X(5), X(6) . The different lines being constructed by connecting X(1), etc. with all other ETC-points in the range X(1)-X(3514) ?
2. Which percentage of all ETC-points do participate in any Perspective Scale ?

3. How many Perspective Fields do occur on the different possible lines through X(1), the Incenter and X(2), the Centroid (using all ETC-points) ?
4. Which percentage of all ETC-points do participate in any Perspective Field ?

After your own estimate look at next slide.

Perspective Fields
Quantitative Aspects

Let us first define an Active Perspective Scale by *any row of at least 4 points on a line* that can all be scaled in a Perspective Scale defined by any 3 of these points.

Now the answers:

1. How many Active Perspective Scales do occur on the different possible lines through X(1), X(2), X(3), X(4), X(5), X(6) . The different lines being constructed by connecting X(1), etc. with all ETC-points (in the range X(1)-X(3514) ?

- There are 415 different scales going through X(1), the Incenter.
- There are 338 different scales going through X(2), the Centroid.
- There are 265 different scales going through X(3), the CircumCenter.
- There are 281 different scales going through X(4), the OrthoCenter.
- There are 101 different scales going through X(5), the Nine-point Center.
- There are 277 different scales going through X(6), the SymmedainPoint.

This is a tremendous number of Active Perspective Scales !

Slight differences in numbers may occur because of precision errors.

Perspective Fields
Quantitative Aspects

2. Which percentage of all ETC-points do participate in any Active Perspective Scale ?

I calculated with a program all Perspective Scales for ETC-points X(1) – X(125).
That's only 3.6 % of the ETC-database.
These scales drag quite a lot of other ETC-points in their net.
The result is that for only 3.6 % of all ETC-points

1847 of the 3514 ETC-points
are participating in some Active Perspective Scale
combined with a point in the range X(1) – X(125).

I think it can be expected that at least 90% of all ETC-points will be participating in some Active Perspective Scale.

Slight differences in numbers may occur because of precision errors.

Perspective Fields
Quantitative Aspects

3. How many Perspective Fields do occur combined with X(1) and/or X(2), using all ETC-points ?

I did do some calculating with computerprograms.
This calculation did not cover all points and was far from complete.
I crossed couples of Active Perspective Scales with at least 5 points per Scale.
I only registered when more points popped up than already occurred in the Scales.
I removed the doubles and ended up with a list of more than 2.000 different Perspective Fields.
X(1) = Incenter participated in 1.091 of these Fields.
X(2) = Centroid participated in 994 of these Fields.

It came out very clearly that the Incenter and the Centroid are the very important centers in the Triangle Plane.

Slight differences in numbers may occur because of precision errors.

Perspective Fields
Quantitative Aspects

4. Which percentage of all ETC-points do participate in any Perspective Field ?

In this tryout of over 2.000 Perspective Fields 1587 of 3514 ETC-points were involved in these Fields. That is 45.2 %.
However there are many more possible Perspective Fields.
I suspect about *90% of all ETC-points to be involved in some Perspective Field*.

Slight differences in numbers may occur because of precision errors.

Perspective Fields
Quantitative Aspects

My conclusion is:

the Triangle plane
is saturated with perspectivity

Dynamic Perspective Fields – Summary

Synthetic – 1

- We found a new structure in the Triangle Plane named a Perspective Field.
  It is based on a 4-point projective network.

- Most of all known points in the Triangle Plane are participating in some Perspective Field.

- Points in a Perspective Field often pop up at the same issues.

- It is ***a set of points*** defined by 4 basic points.
  In this way ***it fills the definitiongap of points*** :
  1 point defines a point,
  2 points define a line,
  3 points define a circle,
  4 points define a **PERSPECTIVE FIELD**,
  5 points define a conic.

Dynamic Perspective Fields – Summary

Synthetic – 2

- A Perspective Field can be normalized to a Cartesian Coordinatesystem.
- The Normalized Field is invariant and is full of new internal regularities.
- A Perspective Field can also be defined by 2 lines with a perspective scale on it, each crossing in a scalingpoint.
- There is a clear relationship between perspective points on a line and harmonic conjugated points.
- The concept of “semi-harmonic conjugate” or “perspective conjugate” is introduced.

Dynamic Perspective Fields – Summary

Algebraic – 1

- Formulas of trilinear and barycentric coordinates have the same structure for all points occurring on a *Perspective Scale*.
  Formulas of trilinear and barycentric coordinates have the same structure for all points occurring in a *Perspective Field*.

- The concept of *Perspective Coordinates* is introduced.
  Perspective Coordinates of points on a Perspective Scale are a set of 2 numbers relating to 2 major points from this Perspective Scale.
  Perspective Coordinates of points on a Perspective Field are a set of 3 numbers relating to 3 major points from this Perspective Field.

- Not only the fieldpoints on a line within a Perspective Field are literally scalingpoints on a scalingline, also the ratio of the 2 numbers of the Perspective coordinates are according a Perspective Scale.
  This is the main feature explaining the structure of trilinear and barycentric coordinates.

Dynamic Perspective Fields – Summary

Algebraic – 2

- Formulas of points on a line with a Perspective Scale or any line in a Perspective Field can be derived by calculating a weighted average of the formulas of 2 other participating points.

- Formulas of points in a Perspective Field can be derived by calculating a weighted average of the formulas of 3 other participating points.

- The concept of a ComplianceFactor is introduced.

- The concept of FieldComplied Coordinates is introduced.

Dynamic Perspective Fields – Summary

What further can be explored – 1

- A Perspective Network is a Network generated from 4 points.
  It still is uncertain how this network developes per generation.

- A Normalized Perspective Field is full of regularities.
  For example equilateral triangles pop up. Why are they occurring so many in Normalized Fields? Which other regularities do occur?

- A Normalized Perspective Field can be seen as the projection of a plane that cuts the Triangle Plane with some perspectrix as intersectionline.
  I expect a tremendous organization in this 3-dimensional spectacle.

Dynamic Perspective Fields – Summary

What further can be explored – 2

- Is it possible to assign per ETC-point a Perspective Field in which it occurs most? It will give a perfect background of the point in question, because when you know the Perspective Field in which it participates you know the other points it naturally is “doing business with”.

- I developed a list of over 2.000 Perspective Fields.
  I am pretty sure that the most important fields are included in this list.
  For the most important fields it is interesting to determine their major fieldpoints (the triple of points not on one line with the simplest formulas) and their compliance factors.
  Now it will be possible to relate an ETC-point with 3 major fieldpoints and so defining its coordinates with respect to the coordinates of these 3 major fieldpoints.

Dynamic Perspective Fields – Summary

What further can be explored – 3

- Perspective Fields in the Triangle Plane are depending algebraically on 3 major fieldpoints.
  These 3 points are depending on the properties of the 3 vertices of the Reference Triangle. This dependancy makes the Field “dynamic”.
  This Perspective Field in Triangle environment is not static but it is dynamic.
  What happens when we work with other configurations depending on more than 3 variables or other variables ?
  Do Dynamic Perspective Fields occur and how do they behave ?

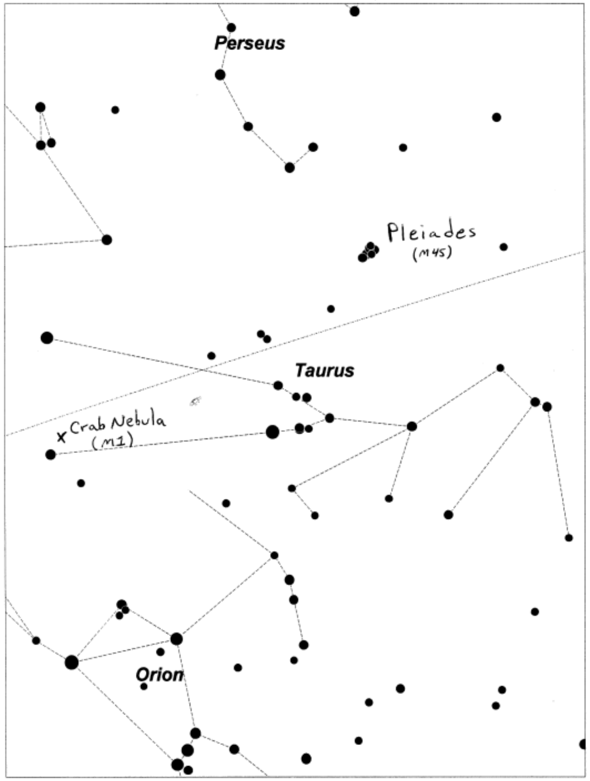
Dynamic Perspective Fields – Summary

Practical Applications- 1

Dynamic Perspective Fields are not just related to the Triangle Field.

The essence is that they are based on a 4-point-defined network or can be seen as the crossing of 2 lines with a perspective scale. Besides that they are flexible in such a way that *they can adopt to external factors (they are dynamic)*. That's why I expect that they are applicable in different disciplines of science. To mention:

(look at next page)



Dynamic Perspective Fields – Summary

Practical Applications– 2

What applications can be thought of in real life?

- For example in *Astronomy* different clusters of stars organized in Dynamic Perspective Fields?
- For example in *Quantum Mechanics* different particles behaving in patterns of a Perspective Field?
- Dynamic Structures in Spacetime Continuum? Gravitational Fields?
- Or whatever comes up ……….

Perspective Fields
– Summary

- *The final word has not be spoken yet ………………*

Chris van Tienhoven
April 2010