

Journal of the Quadri Figures Group 2013

Digital Edition

Chris van Tienhoven et al.

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Messages #1 - #393

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1 Introduction

This journal is a compilation of messages from the

Quadri Figures Group (QFG)

a forum where mathematicians and geometry enthusiasts exchanged ideas on the properties of **quadrilaterals**, **polygons**, and related geometric structures. The discussions covered a wide range of topics, from classical geometric theorems to new discoveries and insights.

The Quadri Figures Group was active from 2013 until November 2019. During these years, the forum developed into a vibrant community and a valuable resource for exploring both well-established results and novel perspectives in geometry. In 2018 and 2019, problems began to arise with Yahoo Groups, the platform that handled the email distribution. Many attachments failed to arrive. In this journal, an effort has been made to recover and include as many of these attachments as possible.

When Yahoo Groups ended its activities in November 2019, the mathematical spirit of QFG did not disappear. Instead, the discussions continued and expanded within the **Quadri- and Poly-Geometry Group (QPG)**, available at <https://groups.io/g/Quadri-and-Poly-Geometry>. QPG took over the baton from QFG, broadening the scope from quadrilaterals to include polygons, poly-figures, and higher-degree curves. Together, the two forums form a continuous line of geometric exploration. An interactive backup of the former Quadri Figures Group is available at <https://groups.io/g/Quadri-Figures-Group>.

This journal was compiled retroactively in 2026 and preserves the annual record of all incoming messages from the Quadri Figures Group. It is available in **PDF format** and includes a **table of contents** that organizes all messages by subject. Navigation is made easy through **hyperlinks** embedded in the message numbers, allowing readers to move quickly between related discussions or return to the table of contents for further reference.

Many of the topics discussed here are closely related to the Encyclopedia of Poly Geometry, available at <https://www.chrisvantienhoven.nl/>, which aims to systematically classify and analyze geometric structures. By collecting the forum messages of the Quadri Figures Group, this journal serves both as a **historical archive** and as a **source of inspiration** for further research in the fascinating world of geometry.

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2 Authors

This section presents an alphabetical overview of the authors who contributed messages to this volume of the Journal.

- Ángel Montesdeoca
- Antreas Hatzipolakis
- Bernard Gibert
- Bernard Keizer
- Chris van Tienhoven
- Dao Thanh Oai
- Eckart Schmidt
- Eisso J. Atzema
- Emmanuel Tsukerman
- Jean-Louis Ayme
- Randy Hutson
- Seiichi Kirikami
- Systems Manager

2.1 Author Index

This section provides an index of all authors who contributed messages to this volume of the Journal.

Each entry lists the author's name, their identifier, and the message numbers associated with their contributions. The list below shows the authors along with the numbers of related messages. Click on a number to go to the corresponding page.

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2.2 Author Information

This section presents background information on the contributing authors. Short biographical notes, areas of interest, and selected publications are included to provide context for their contributions to the Journal. These profiles offer readers an opportunity to become acquainted with the individual behind the names and to appreciate the diverse mathematical backgrounds represented in this volume. Author information is included only insofar as it has been provided or was available.

Ángel Montesdeoca Delgado (1949–2024)

Location

Canary Islands, Spain.

Year of Birth / Generation

1949–2024.

Short Biography

Ángel Montesdeoca Delgado was a highly respected Spanish geometer and former teacher at the Universidad de La Laguna, Canary Islands. He was widely admired for his deep knowledge of projective geometry, his extensive contributions to triangle geometry, and the remarkable clarity and beauty of his mathematical website. Ángel's website, admired for both its content and its elegant layout, remains a testament to his mathematical vision and aesthetic sense. Ángel was known not only for his expertise but also for his kindness, generosity, and willingness to help others. Many members of the geometry community recall his thoughtful explanations, his patient guidance, and his warm, friendly nature. His passing in May 2024 was felt as a profound loss by colleagues, students, and friends around the world. He is remembered with gratitude, respect, and affection by the global geometry community.

Themes and Interests

- Projective geometry
- Triangle geometry
- Geometric constructions and classical configurations
- Mathematical exposition and elegant presentation of results
- Community support, explanation, and mentoring

Selected Publications and Contributions

Ángel produced a large number of geometric results, many of which were shared through his website and through contributions to online geometry communities. His work is frequently cited for its depth, originality, and clarity. A memorial reflection by Francisco Javier García Capitán can be found at: www.garciacapitan.blogspot.com/2025/04/angel-montesdeoca-delgado-1949-2024-un.html

Community Tributes

Members of the geometry community remembered Ángel with great affection:

- His “deep knowledge of Projective Geometry” and “great amount of geometrical results” (F. J. García Capitán).
- His generosity, kindness, and helpful explanations (E. Suppa).
- His profound expertise in triangle geometry (A. P. Hatzipolakis).
- His clarity, thoughtfulness, and warm personality (C. van Tienhoven).

- His influence on many geometers at the beginning of their journey (C. E. Lozada).

Additional Remarks

- Website: <https://amontes.webs.ull.es/>

Antreas P. Hatzipolakis

Location

Lives in Greece.

Year of Birth / Generation

1952.

Short Biography

Antreas P. Hatzipolakis studied mathematics at Athens University. He is the founder of several influential geometry-focused email groups, including *Hyacinthos*, *Anopolis*, and *Euclid*, as well as various Facebook groups dedicated to classical and triangle geometry. For many years, he introduced new problem areas through his email groups, inspiring others to explore, investigate, and solve them. His work has played a significant role in shaping the collaborative culture of modern online geometry communities.

Themes and Interests

- Classical Euclidean geometry
- Triangle geometry
- Problem creation and problem solving

Selected Publications

- Antreas P. Hatzipolakis, Floor van Lamoen, Barry Wolk, and Paul Yiu, *Concurrency of Four Euler Lines*. Forum Geometricorum, Volume 1 (2001), 59–68.
- Antreas P. Hatzipolakis and Paul Yiu, *Reflections in Triangle Geometry*. Forum Geometricorum, Volume 9 (2009), 301–348.

Additional Remarks

Website: <http://www.anthrakitis.blogspot.com/>

Chris van Tienhoven

Global Location

Living in the Netherlands.

Year of Birth

1950.

Short Biography

Chris van Tienhoven graduated in mathematics from Leiden University and has built a career as an entrepreneur working across information technology and graphic design. He also remained active in geometry. Central to his work is a lifelong habit of reducing complexity into simplicity and creating clear, durable structures. He values order, coherence, and long-term vision—principles. All of this eventually led to the creation of the Encyclopedia of Poly Geometry.

Themes, Interests, and Relevant Publications

- Lifelong interest in geometry, beginning in secondary school, with a special fascination for Van Aubel's Theorem.
- Developed the notion of Perspective Fields.
- Initiator of the systematic development and documentation of Quadri Geometry, later expanded into Poly Geometry.
- Founder of the online communities *Quadri Figures Group* and *Quadri and Poly Geometry Group*.
- Editor and compiler of the Annual Journals that collect and preserve the discussions and discoveries of these groups.
- Founder of the Encyclopedia of Poly Geometry (where all entries without external references originate from his own work).

Selected Publications

- Chris van Tienhoven, Dario Pellegrinetti, *Quadrignon Geometry: Circumscribed Squares and Van Aubel Points*. Journal of Geometry and Graphics, Vol. 25, No. 1, 2021.

Other Remarks

Website: www.chrisvantienhoven.nl

Biography: www.chrisvantienhoven.nl/header/biography/

Eckart Schmidt

Location

Living in Germany.

Year of Birth / Generation

1939.

Short Biography

Eckart Schmidt is a former teacher of mathematics and physics at a full-time secondary school, with a long-standing interest in geometry. His work spans several decades and includes numerous contributions to geometric constructions, classical geometry, and the study of n -gons and their transformations.

Themes and Interests

- Geometric constructions using CABRI

Selected Publications

- F. Bachmann & E. Schmidt: *n Ecke*. B.I. Hochschultaschenbuch 471/471a, Mannheim/Wien/Zürich, 1970.
- E. Schmidt: *Abbildungen und Klassen von n Ecken*. MNU XXV (1972), pp. 146–150ff.
- E. Schmidt: *Affin reguläre n Ecke und ihre regulären Komponenten*. MNU XXXIX (1986), pp. 193–198ff.
- E. Schmidt: *Mittelsenkrechtenvierecke eines Vierecks*. PM 2/44 (2002), pp. 84–88ff.
- E. Schmidt: *Circumcenters of Residual Triangles*. Forum Geometricorum 3 (2003), 125–134.
- J. Kühl & E. Schmidt: *Husumer Rechenhandschriften und Paul Halkes Mathematisches Sinnen Confect*. Mitteilungen der Mathematischen Gesellschaft in Hamburg XXIII/2 (2004), 111–156.
- E. Schmidt: *Geradenkonstellationen*. MNU 60/1 (2007), 28–29.
- E. Schmidt: *Billardvierecke eines Sehnenvierecks*. MNU 63/5 (2010), 267–269.
- Additional contributions on geometric constructions (see Themen and EQF-notes).

Additional Remarks

- Co-founder of the Encyclopedia of Poly Geometry and one of the principal contributors to QPG.
- Website: www.eckartschmidt.de

Seiichi Kirikami (1949?–2023)

Location

Japan.

Year of Birth / Generation

Exact year unknown; passed away on 11 December 2023.

Short Biography

In daily life Seiichi worked as a mechanical engineer in the Thermal Power Division of Hitachi, Ltd. In his free time he enriched the geometry community with original ideas, elegant constructions, and generous participation in many collaborative discussions. From 2013 to 2018, Seiichi contributed intensively to the development of Quadri- and Poly-Geometry within the Quadri-Figures Group. His insights, constructions, and discussions were instrumental in the group's formative years, and his contributions helped define several of the key geometric notions that emerged in those years. Later on he contributed extensively to other groups such as Anopolis, Hyacinthos, Romantics of Geometry, ADGEOM, and the Encyclopedia of Triangle Centers (ETC), where many of his ideas became foundational. His work was characterized by simplicity, depth, and a unique ability to see geometric structures from unexpected angles. He was also known for his humility, kindness, and willingness to help others — qualities remembered fondly by colleagues and friends.

Themes and Interests

- Euclidean and projective geometry
- Triangle geometry and classical configurations
- Geometric problem creation and exploration
- OEIS contributions and combinatorial structures
- Applied mathematics, including epidemiological modelling
- Engineering and turbine-related innovations (patents)

Selected Contributions

Seiichi Kirikami's geometric ideas inspired many theorems, conjectures, and new terminology. Among the most notable:

- The *Kirikami six-circles configuration*, which led to the Hatzipolakis–Moses Theorem.
- His prompting led to the introduction of the term *Cyclologic*, now established in triangle-geometry terminology.
- He suggested naming the line through $X(5)$ perpendicular to the Euler line the *Hatzipolakis axis*.
- Numerous contributions to ETC, Hyacinthos, and other geometry forums.
- 29 OEIS entries associated with his name.

- Several patents in turbine-engine technology.
- Publications in epidemiology, including COVID-related infection-spread modelling.

Community Tributes

Colleagues remembered Seiichi with deep affection:

- “Geometry is the poorer of his death.” — A. P. Hatzipolakis
- “He was a great expert in projective and triangle geometry, and a very kind and helpful person.” — E. Suppa
- “He had a way of seeing things from a different angle and presenting them simply.” — C. van Tienhoven
- “He inspired many of my problems.” — A. Altıntaş
- “He wrote many interesting and innovative messages in ADGEOM.” — F. J. García Capitán
- “He helped me in my beginnings with wise and generous advice.” — C. E. Lozada
- “He contributed to OEIS and had important work in epidemiology.” — P. Moses

Additional Remarks

Seiichi Kirikami is remembered as a gentle, insightful, and generous geometer whose ideas continue to inspire new discoveries. His legacy lives on in the many theorems, concepts, and geometric structures that bear his influence.

3 Subjects

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4.2 Messages

Message: #1

Date: 17/5/2013 1:12:01

From: Chris

Subject: Welcome

COPY FROM FORMER GOOGLE GROUP

Dear Friends,

The Group on Quadri-Figures has started.

Messages can be placed at: <https://groups.google.com/forum/#!newtopic/quadri-figures>.

Files on figures, etc. can be attached.

Now all our messages will be retained for the future and can be looked up easily.

Moreover new members easily can join our group.

Let's go on sharing our findings, questions and answers in quadri-geometry!

Best regards,

Chris van Tienhoven

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Message: #2

Date: 17/5/2013 1:15:27

From: Chris

Subject: EQF-Notes

COPY FROM FORMER GOOGLE GROUP

Dear friends,

since 2011 I have followed up the development of Chris' Encyclopedia of Quadri-Figures.

Up to now there are many interesting questions.

Since jan. 2013 I have gathered some remarks in "EQF-Notes".

If you are interested, have a look on my homepage:

[html://eckartschmidt.de](http://eckartschmidt.de) .

Eckart

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Message: #3
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #3 is not available in Yahoo groups.

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Message: #4
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #4 is not available in Yahoo groups.

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Message: #5
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #5 is not available in Yahoo groups.

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Message: #6

Date: 17/5/2013 1:39:08

From: Chris

Subject: a LSD point of a quadrilateral and its Newton line

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami

(April 1, 2013)

Given a quadrilateral $P_1P_2P_3P_4$ and its Newton line, the point which has the property of least square distance (LSD) is given by the following barycentric coordinates.

1st DT: $a^2(2a^4(1-m)^2(1-n)^2mn+b^4(1-m)^2n(m-n)(5lm-3ln+mn)+c^4(1-n)^2m(n-m)(5ln-3lm+nm)-a^2b^2(1-m)^2(1-n)n(3l(m-n)+m(4m-n))-a^2c^2(1-n)^2(1-m)m(3l(n-m)+n(4n-m))-2b^2c^2(1-m)(1-n)(m-n)^2(3l(m+n)-mn))$.

I obtained the above result with the conic contour method, which Mr. van Tienhoven showed me examples. Before computation, I have confirmed that in case of a quadrilateral, the method gives QL-P26 (LSD point) in EQF.

Best regards,

Seiichi Kirikami.

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Message: #7

Date: 17/5/2013 1:40:44

From: Chris van Tienhoven

Subject: a LSD point of a quadrilateral and its Newton line

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami

(April 1, 2013)

1st DT should be replaced by 1st CT.

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Message: #8
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #8 is not available in Yahoo groups.

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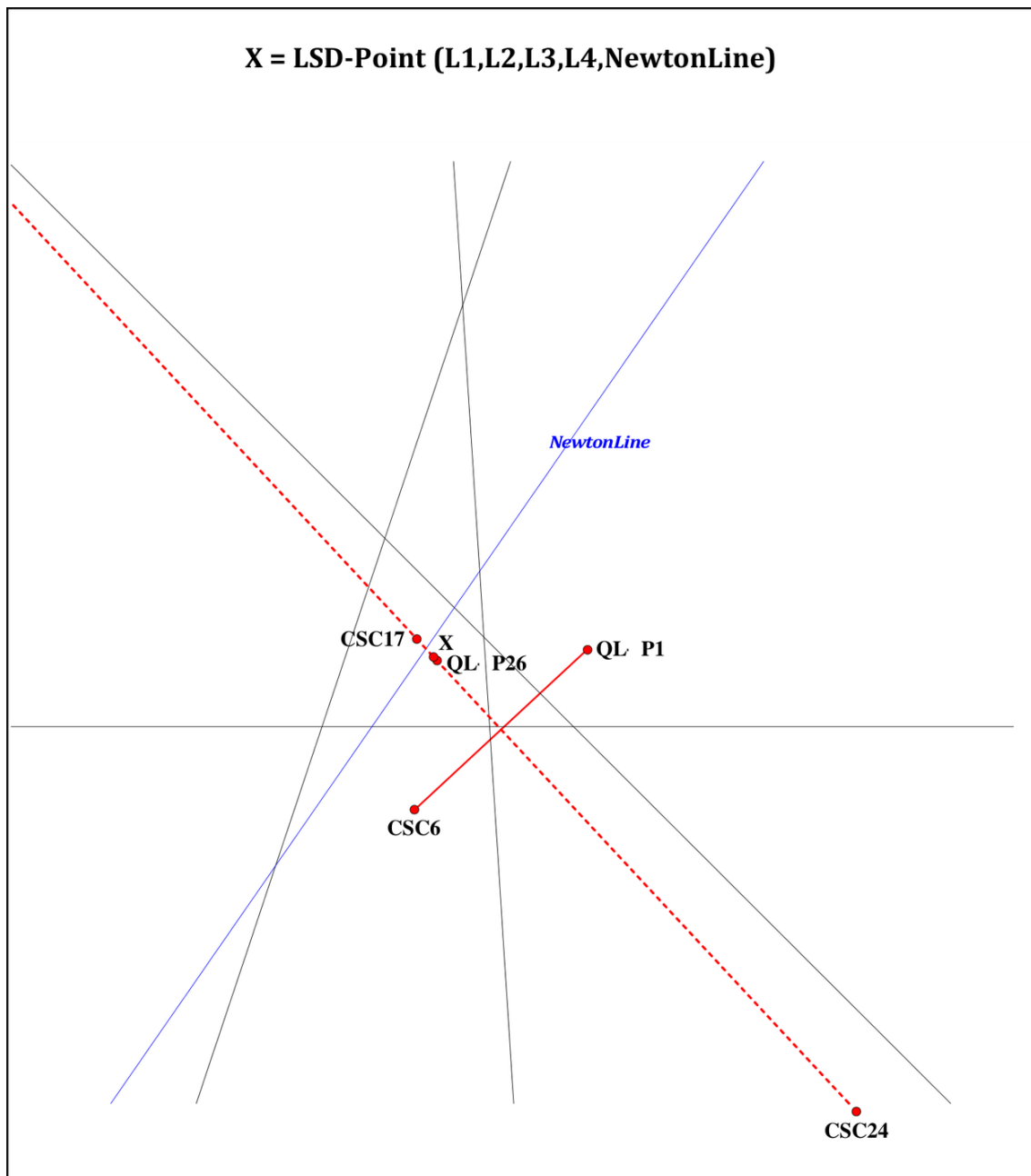
Message: #9
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #9 is not available in Yahoo groups.

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Message: #10
Date: 17/5/2013 2:20:34
From: Chris van Tienhoven
Subject: a LSD point of a quadrilateral and its Newton line

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven
(April 2, 2013)
Dear Seiichi,
Very nice!
This point $LSD(L1, L2, L3, L4, NewtonLine)$ lies on the perpendicular bisector of $QL-P26.CSC(QL-P6)$, which is this line $QL-P26.CSC(QL-P17).CSC(QL-P24)$.
 CSC stands for Clawson-SchmidtConjugate($QL-Tf1$).
You can see these lines in the 1st picture of $QL-P26$ in EQF.
There are no other concurrencies with known lines or curves.
See attached PDF-file.
Best regards,
Chris



LSD-L1L2L3L4NL.pdf

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Message: #11
Date: 17/5/2013 2:36:00
From: Chris
Subject: New names QL-P3 and QL-P5

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven
(April 4, 2013)

Dear friends,

I renamed the points QL-P3 and QL-P5 in EQF.

Thanks to a message from Bernard Keizer it came out that in the document "More theorems on the Complete Quadrilateral" from J.W. Clawson these QL-points:

QL-P1, QL-P2, QL-P3, QL-P4, QL-P5, QL-P12, QL-P20 and QL-P22 were mentioned.

Also it came out that the name Kantor-Hervey Point belonged to QL-P3 and not to QL-P5.

So now

* QL-P3 = Kantor-Hervey Point, and

* QL-P5 = Clawson Center.

This all gave several additions and modifications in EQF.

See "Recently Added" in EQF.

Best regards,

Chris

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Message: #12
Date: 17/5/2013 6:04:35
From: Chris van Tienhoven
Subject: EQF-Note 2013-04-05

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt
(April 5, 2013)

Dear friends,

if you are interested in Involuntary Conjugates (QA-Tf2) of infinity points, there is a paper in the attachement.

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Involutory Conjugate of Infinity Points

In EQF there are some QG-points Involutory Conjugates of infinity points of QG-lines (see tables for QA-Tf2). Here corresponding relationships are tested in the QA-environment.

The Involutory Conjugate QA-Tf2 is an isoconjugation wrt the QA-Diagonal Triangle QA-Tr1 with fixpoints in the vertices of the reference quadrangle. QA-Tf2 maps lines into circumscribed conics of QA-Tr1, especially the line at infinity in the Nine-point Conic QA-Co1. QA-Co1 is the locus of centers of circumscribed conics of the reference quadrangle. But in EQF there are only four circumscribed conics. Here the Involutory Conjugates of infinity points of QA-lines will be discussed wrt their collinearity with other QA-points. – Reference triangle for barycentric coordinates will be the QA-Diagonal Triangle QA-Tr1.

1. Circumscribed Conics of a Quadrangle

The Involutory Conjugate

$$(x : y : z) \rightarrow (p^2 yz : q^2 zx : r^2 xy)$$

of infinity points are points on the Nine-point Conic QA-Co1 with the equation

$$r^2 xy + q^2 zx + p^2 yz = 0$$

and the center

$$QA-P1 = (p^2(-p^2 + q^2 + r^2) : q^2(p^2 - q^2 + r^2) : r^2(p^2 + q^2 - r^2)).$$

If we consider a line $L(e, f, g)$, the Involutory Conjugate of its infinity point

$$\left(\frac{p^2}{f-g} : \frac{q^2}{g-e} : \frac{r^2}{e-f} \right)$$

will be a point of QA-Co1 and a center of a circumscribed conic of the quadrangle. Up to now there are four circumscribed conics of a quadrangle in EQF:

QA-Co2 QA-Orthogonal Hyperbola:

The center QA-P2 is the Involutory Conjugate of infinity points of lines orthogonal to QA-L2.

QA-Co3 Gergonne-Steiner Conic:
The center *QA-P3* is the Involutary Conjugate
of the infinity point of *QA-L4*.

QA-2Co1a,b Pair of Circumscribed QA-Parabolas:
The axes of the *QA-Parabolas* are parallel to the asymptotes
of *QA-Co1*; their infinity points are Involutary Conjugates.

There are two further circumscribed conics of a quadrangle (see
my EQF-Note 2013-01-18):

Circumscribed conic through *QA-P5* and *QA-P10*:
The center, reflected in *QA-P1*, is the Involutary Conjugate
of the infinity point of *QA-P1.QA-P16*.

Circumscribed conic through *QA-P1* and *QA-P16*:
The center, reflected in *QA-P1*, is the Involutary Conjugate
of the infinity point of *QA-L3*.

2. Special Involutary Conjugates of Infinity Points

Considering *QA-lines*, the Involutary Conjugates of their infinity
points on *QA-Co1* are sometimes collinear with *QA-P2* and
another *QA-point*. For example: The Involutary Conjugate of the
infinity point of *QA-P4.QA-P8* is the second intersection point
of *QA-P2.QA-P7* and *QA-Co1*.

<i>infinity point of line ...</i>	<i>Involutary Conjugate</i>
<i>QA-L4</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P3</i>
<i>QA-P1.QA-P11</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P37</i>
<i>QA-P1.QA-P28</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P15</i>
<i>QA-L2</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P23</i>
<i>QA-P2.QA-P23</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P6</i>
<i>QA-P4.QA-P8</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P7</i>
<i>QA-P4.QA-P12</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P11</i>
<i>QA-P10.QA-P14</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P14</i>
<i>QA-P11.QA-P23</i>	<i>QA-Co1</i> ^ <i>QA-P2.QA-P36</i>

If the Involutary Conjugate of an infinity point is reflected in
QA-P1 – always a point on *QA-Co1* – there are corresponding
properties wrt *QA-P3*.

<i>infinity point of line ...</i>	<i>Involutary Conjugate reflected in QA-P1</i>
<i>QA-L1</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P4</i>
<i>QA-L2</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P32</i>
<i>QA-L4</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P2</i>
<i>QA-P1.QA-P4</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P8</i>
<i>QA-P1.QA-P11</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P12</i>
<i>QA-P1.QA-P12</i>	<i>QA-Co1</i> ^ <i>QA-P3.QA-P30</i>

In QG -environment there are some further results: As mentioned above the first three are to be found in EQF under $QA-Tf2$.

<i>infinity point of line ...</i>	<i>Involuntary Conjugate</i>
$QG-P1.QG-P2$	$QG-P13$
$QG-P1.QG-P3$	$QG-P14$
$QL-L1$	$QG-P15$
$QG-L2$	$QA-Co1 \wedge QG-P1.QG-P2$
$QG-L3$	<i>refl. of $QG-P14$ in $QA-P1$</i>

3. Infinity Points of orthogonal Lines

All orthogonal hyperbolas circumscribed $QA-Tr1$ contain the orthocenter $QA-P12$. The Involuntary Conjugate of $QA-P12$ is the Inscribed Square Axes Crosspoint $QA-P23$. Therefore $QA-P23$ can be considered as the common point of all $QA-Tf2$ -images of orthogonal hyperbolas circumscribed $QA-Tr1$.

For orthogonal lines the Involuntary Conjugates of their infinity points lie on $QA-Co1$ with a chord through $QA-P23$.

Remark: Consider chords of $QA-Co1$ through $QA-P23$. The Thales-circle over a chord intersects $QA-Co1$ in $QA-P2$ and a further point X . The orthogonal hyperbola circumscribed $QA-Tr1$ through X cuts the chord in points of an isocubic wrt $QA-Tr1$ for pivot $QA-P23$ and isoconjugation $QA-Tf2$.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-04-05.pdf

Message: #13

Date: 17/5/2013 6:10:04

From: Chris

Subject: a LSD point made from 6 pedal points in QA- environment

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami (April 5,2013)

Dear friends,

Given a quadrangle P1P2P3P4, the following point P has the property of least square distance (LSD) from 6 its pedal points P_{dij} on P_iP_j.

1st CT: $a^2(a^6q^2r^2(p+q)(p+r)+a^4(b^2r^2q(p+q)(-2p^2+2pq+r(2r+q))+c^2q^2r(p+r)(-2p^2+2pr+q(2q+r)))+a^2(b^4p(p+q)r^2(4pq-q^2+3pr+3r^2)+c^4p(p+r)q^2(4pr-r^2+3pq+3q^2)+2b^2c^2pqr(-2p^3-q^3-r^3+3q^2r+3qr^2+p^2(q+r)+2p(q+r)^2))-c^6p^2q^2(p+r)(3q+2r)-b^6p^2r^2(p+q)(2q+3r)+b^2c^4p^2q(3pq(2q+3r)+p^2(6q+4r)+r(7q^2+2qr-4r^2))+b^4c^2p^2r(3pr(2r+3q)+p^2(6r+4q)+q(7r^2+2qr-4q^2)))$

Best regards,

Seiichi Kirikami.

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Message: #14

Date: 17/5/2013 6:11:33

From: Chris van Tienhoven

Subject: a LSD point made from 6 pedal points in QA- environment

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven (April 5,2013)

Dear Seiichi,

I calculated this Quadrangle-point before.

It has pretty long coordinates and it is not lying on known lines or curves.

So I don't estimate it as a very special point.

Chris

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Message: #15

Date: 17/5/2013 6:13:37

From: Chris

Subject: a LSD point made from 6 pedal points in QA- environment

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami

(April 5, 2013)

Dear Chris,

I see. I hope that it will be something in future.

Best regards,

Seiichi Kirikami.

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Message: #16

Date: 17/5/2013 6:17:08

From: Chris

Subject: New properties for QA-P23 and QA-Co1???

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt

(April 5, 2013)

Dear friends,

for a triangle an isoconjugation transforms lines into
circumconics

and QA-Tf2 is an isoconjugation wrt QA-DT:

The Involuntary Conjugates of lines through QA-P23 are the
circumscribed orthogonal hyperbolas of QA-DT.

The Involuntary Conjugates of tangents at QA-Co1 are the
circumscribed parabolas of QA-DT.

Best regards Eckart

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Message: #17
Date: 17/5/2013 6:18:15
From: Chris
Subject: QA-P1 as a LSD point

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami
(April 3, 2013)
Dear friends,
Given a quadrangle $P_1P_2P_3P_4$ and its midpoints M_{ij} of P_i and P_j ,
QA-P1 has the property of least square distance(LSD) from P_1 ,
 P_2 , P_3 and P_4 . And it also has the property of least square
distance(LSD) from M_{12} , M_{13} , M_{14} , M_{23} , M_{24} and M_{34} .
LSD of the former:
$$\text{cyclic sum } \{a^2(2q^2+qr+2r^2+3pq+3pr+p^2)/4/(p+q+r)^2\}.$$

LSD of the latter:
half of the above value.
Best regards,
Seiichi Kirikami.

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Message: #18
Date: 17/5/2013 6:21:12
From: Chris
Subject: QA-P1 as a LSD point

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven
(April 6, 2013)
Dear Seiichi,
I added this property now at QA-P1 in EQF.
Chris

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Message: #19
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #19 is not available in Yahoo groups.

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Message: #20
Date: 17/5/2013 6:23:33
From: Chris
Subject: QA-P1 as a LSD point

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami
(April 7, 2013)
Dear Chris,
Thanks a lot!
Best regards,
Seiichi Kirikami.

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Message: #21
Date: 17/5/2013 8:44:46
From: Chris van Tienhoven
Subject: EQF-Note 2013-04-08 Common point of five Miquel Circles

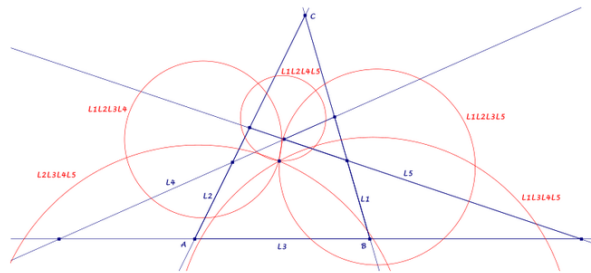
COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt
(April 8, 2013)
Dear friends,
five lines give five quadrilaterals. Their Miquel circles have a
common point.
I don't know, whether this is published anywhere.
In the attachment there are some examples in the
QL-environment.
Best regards Eckart

EQF-Note 2013-04-08

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Common Point of five Miquel Circles

Five lines give five quadr. Their Miquel circles have a common point. In this paper there are some examples for a quadrilateral and a further line. – The results are CABRI-controlled.

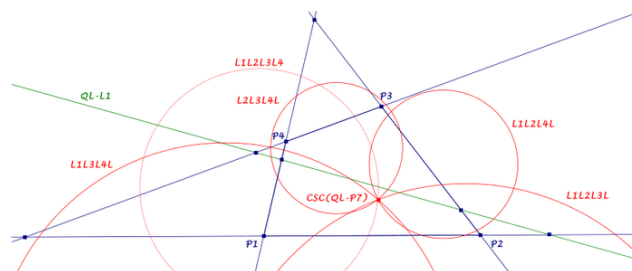


The Miquel circles ($QL-Ci3$) of the five quadrilaterals of five lines have a common point.

A calculation with barycentric coordinates is possible, taking three of the lines for a reference triangle. But the results can't be handled.

We consider now a quadrilateral $L_1L_2L_3L_4$ and a further line L . Then the Miquel circles of the quadrilaterals

$L_1L_2L_3L$, $L_1L_2L_4L$, $L_1L_3L_4L$, $L_2L_3L_4L$ and $L_1L_2L_3L_4$ have a common point, here named MC -point of a line wrt a quadrilateral. The Clawson-Schmidt Conjugate (CSC) of the MC -points lie on $QL-L2$ (CSC image of $QL-Ci3$).



2013-04-08.pdf

Examples:

QL-L1

MC-point is *CSC* of *QL-P7* (see above).

QL-L2,3

MC-point is *QL-P1*.

Lines orthogonal to *QL-L1*

MC-point is *QL-P1*.

QL-L4

MC-point is *CSC* of the reflection of *QL-P2* in *QL-P7*.

Line through *QL-P4*, parallel *QL-L1*

MC-point is *CSC* of *QL-P2*.

Lines parallel to *QL-L1*

MC-point is *CSC* of the reflection of the intersection of the line with *QL-L2* in *QL-P7*.

QG-P1.QG-P14

MC-point is the second intersection of *QL-Ci3* with the *CSC* image *QG-P1.QG-P14*.

Limit cases:

Lines parallel to the quadrangle sides

MC-point is *CSC* of the intersection of the sideline and *QL-L2*.

Diagonals of the quadrilateral

MC-point is *CSC* of the intersection of the diagonal and *QL-L2*.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-04-08.pdf

Message: #22

Date: 17/5/2013 8:49:30

From: Chris van Tienhoven

Subject: EQF-Note 2013-04-08 Common point of five Miquel Circles [1 Attac

Dear Eckart,

The points/lines in a 5L-configuration are very interesting indeed.

Last years I found this article:

<http://www.cut-the-knot.org/triangle/Morley/CenterCircle.shtml>

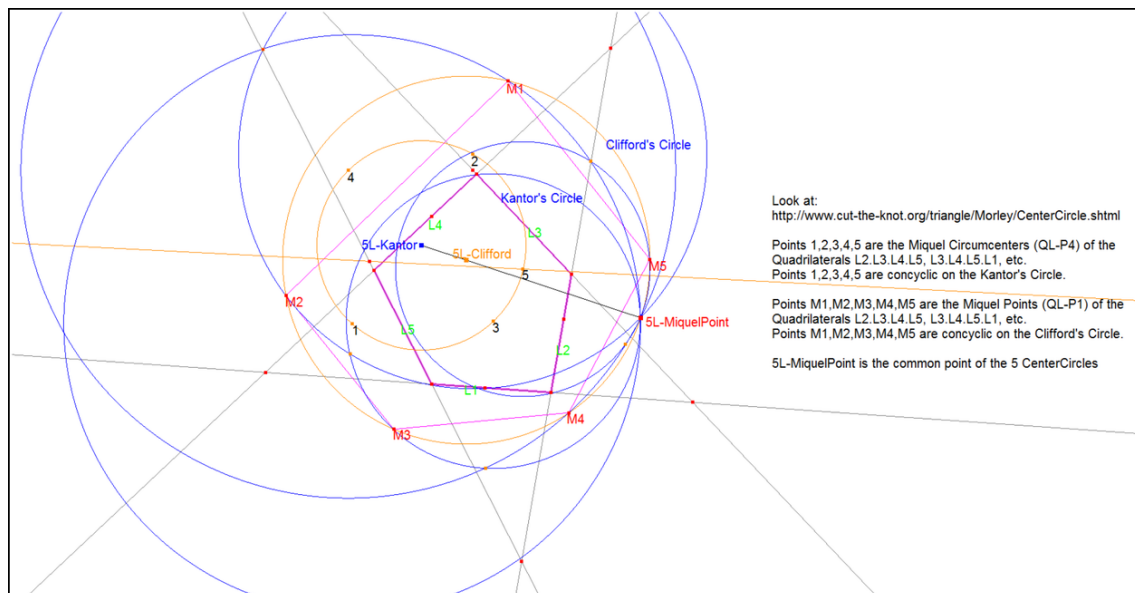
Also I produced attached Cabri-figure converted to PDF-file.

So I think the 5L-Miquel Points is already known.

However your applications with known QL-Lines is very interesting!

Best regards,

Chris



5L-MiquelPoint-01.pdf

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Message: #23

Date: 17/5/2013 8:49:32

From: Chris

Subject: a LSD point of a quadrilateral which is on Newton line

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami

(April 9, 2013)

Dear friends,

Given a quadrilateral $L_1L_2L_3L_4$ and Newton line, the following point on Newton line has the property of least squares distance (LSD) from L_1 , L_2 , L_3 and L_4 .

1st CT: $a^2(b^4(1-m)^2(m-n)n(l(m-n)+mn)+c^4(1-n)^2(n-m)m(l(n-m)+mn)-a^2(1-m)(1-n)(b^2(1-m)n(l(m-n)+mn)+c^2(1-n)m(l(n-m)+mn)+2b^2c^2(1-m)(m-n)(n-l)(l(m+n)-mn))$.

QL-P26, a LSD point of a quadrilateral and Newton line, a LSD point of a quadrilateral which is on Newton line are colinear.

Best regards,

Seiichi Kirikami.

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Message: #24

Date: 17/5/2013 8:51:22

From: Chris van Tienhoven

Subject: a LSD point of a quadrilateral which is on Newton line

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven (April 9, 2013)

Dear Seiichi,

I do not quite understand.

You describe the point as "has the property of least squares distance (LSD) from L_1 , L_2 , L_3 and L_4 ".

This is the point QL-P26.

Do I interpret it wrong?

Chris

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Message: #25

Date: 17/5/2013 8:53:37

From: Chris

Subject: a LSD point of a quadrilateral which is on Newton line

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami (April 10, 2013)

Dear Chris,

I added the condition that a LSD point was on Newton line. So I applied Lagrange's multiplier method to this problem. If you need Mathematica-file, I will send you it.

Best regards,

Seiichi Kirikami

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Message: #26

Date: 17/5/2013 8:56:53

From: Chris van Tienhoven

Subject: Tangential quadrilateral well known properties?

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven (April 10, 2013)

Dear Friends,

This mail for support came from:

Mosca Sebastiano

Maybe we can help him.

Chris

Mosca Sebastiano:

ABCD is a circumscribed quadrilateral,

PQRS the points of tangency with the inscribed circle center O

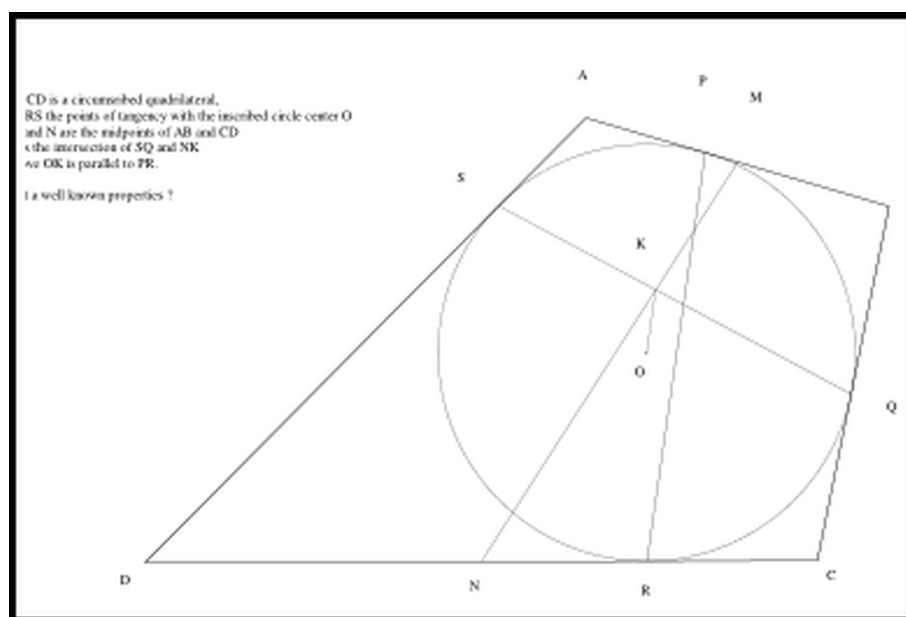
M and N are the midpoints of AB and CD

K is the intersection of SQ and NK

Prove OK is parallel to PR.

Is it a well known properties?

Can yo give me some reference?



quadrilateral properties.pdf

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Message: #27

Date: 17/5/2013 8:58:38

From: Chris van Tienhoven

Subject: Tangential quadrilateral well known properties?

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt
(April 11, 2013)

Dear Chris,

only a short remark to Mosca's question:

Using polar coordinates $P(\cos P, \sin P)$, $Q(\cos Q, \sin Q)$, ...
the lines have the equations

$$\text{OK: } y = -\cot((P+R)/2) x$$

$$\text{PR: } y = -\cot((P+R)/2) x + \cos((P-R)/2) / \sin((P+R)/2)$$

Perhaps the angle relations in the equations lead to a
synthetic proof.

Remarkable that the property doesn't depend on the chord QS.

By the way: In Mosca's message the definition of K must be:

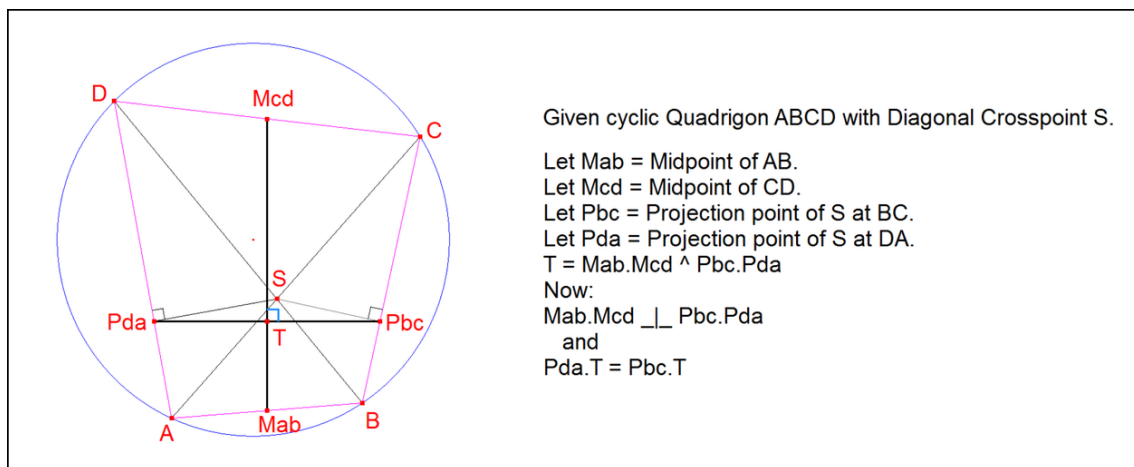
K is the intersection of SQ and NM.

Best regards Eckart

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Message: #28
Date: 17/5/2013 9:08:06
From: Chris van Tienhoven
Subject: Cyclic Quadrilateral

COPY FROM FORMER GOOGLE GROUP from Chris van Tienhoven
(April 12, 2013)
Dear friends,
Once I found this configuration for a Cyclic Quadrilateral.
I don't remember from where.
Given a Cyclic Quadrilateral (actually a Quadrigon) ABCD with
Diagonal Crosspoint S.
Let Mab = Midpoint of AB.
Let Mcd = Midpoint of CD.
Let Pbc = Projection point of S at BC.
Let Pda = Projection point of S at DA.
Let $T = \text{Mab.Mcd} \wedge \text{Pbc.Pda}$
Now:
 $\text{Mab.Mcd} \perp \text{Pbc.Pda}$
 $\text{Pda.T} = \text{Pbc.T}$
See attachment. Is there a synthetic proof for this?
Best regards, Chris



CyclicQuadrilateral.pdf

Message: #29
Date: 17/5/2013 9:11:21
From: Chris
Subject: Cyclic Quadrilateral

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt
(April 12, 2013)
Dear Chris,
in my papers I found two references.
Bundeswettbewerb Mathematik 2003, 2. Runde
Praxis der Mathematik 3/46. Jg. 2004 , S.135
The first one can be found per Google.
Best regards Eckart

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Message: #30
Date: 17/5/2013 9:13:15
From: Chris van Tienhoven
Subject: EQF-Note 2013-04-12: QL-P5, QL-P12, QL-P20, QL-P22

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt
(April 12, 2013)
Dear friends,
for quadrilaterals with the same Inscribed parabola QL-Co1 and
the same QL-Diagonal Triangle QL-Tr1
the loci for the points QL-P5, QL-P12, QL-P20, QL-P22 are lines
with a common point.
There is a paper in the attachement. Perhaps someone can find
further properties of this interesting point.
Best regards Eckart

EQF-Note 2013-04-12

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

QL-P5, QL-P12, QL-P20, QL-P22

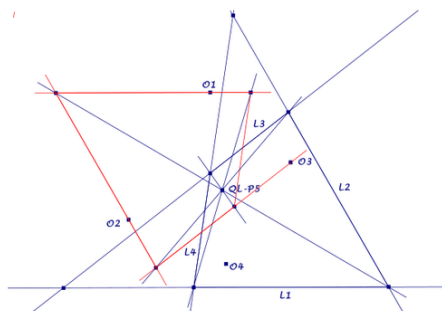
For quadrilaterals with the same Inscribed Parabola QL-Co1 and the same QL-Diagonal Triangle QL-Tr1 the loci for the points above are lines with a common point. This is the reason for the following research. – Reference triangle for barycentric calculation is QL-Tr1.

The Points

The four points above lie collinear on the Newton line $QL-L1$ with rational distance ratios:

$$P5.P12 : P12.P22 : P22.P20 = 2 : 1 : 3.$$

These are the ratios of the Euler line. Background: Take the Euler lines of the triangle components, choose on each Euler line the same special point X and draw a parallel to the corresponding sideline of the quadrilateral, then the constructed quadrilateral is homothetic to the reference quadrilateral with center Y .



X	Y	
circumcenter	QL-P5	(see figure)
centroid	QL-P12	(see EQF)
nine-point center	QL-P22	(see EQF)
orthocenter	QL-P20	(see EQF)

The Lines

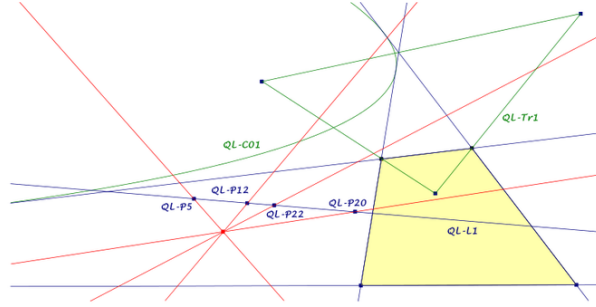
We consider the pencil of quadrilaterals with the same QL -Diagonal Triangle $QL-Tr1$ and the same Inscribed Parabola $QL-Co1$. The inscribed parabola has the equation:

$$\frac{x^2}{m^2-n^2} + \frac{y^2}{n^2-l^2} + \frac{z^2}{l^2-m^2} = 0.$$

For a shorter description the point at infinity of the parabola shall be

$$(u:v:w) = (m^2-n^2:n^2-l^2:l^2-m^2) \text{ with } u+v+w=0.$$

With $l = \sqrt{m^2+w}$ and $n = \sqrt{m^2-u}$ we have a parameter m for the coefficients of the defining lines of the quadrilateral.



The loci of the considered points are lines with the equations:

$$\textbf{QL-P5:} \quad \sum_{cycl} (a^4vw + b^4wu + c^4uv + 4a^2S_Avw)x = 0$$

This line through $QL-P5$ is orthogonal to $QL-P1, QL-P9$.

$$\textbf{QL-P12:} \quad \sum_{cycl} vwx = 0$$

This line is the polar of $QL-P8$ wrt the parabola.

$$\textbf{QL-P20:} \quad \sum_{cycl} (b^2w + c^2v)^2x = 0$$

Construction of this line: a parallel to $QL-L1$ through $QL-P10$ ($QL-DT$ -orthocenter), intersection with the parabola, tangent at the parabola, a parallel through $QL-P20$.

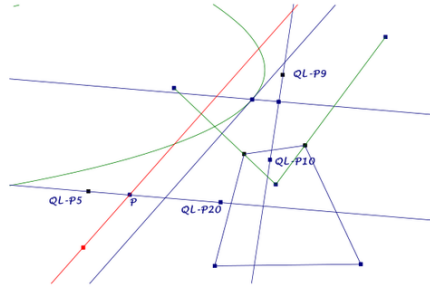
$$\textbf{QL-P22:} \quad \sum_{cycl} (4S^2vw + (-S_Au + S_Bv + S_Cw)^2)x = 0$$

Construction of this line: a parallel to $QL-L1$ through $QL-P11$ ($QL-DT$ -nine-point center), intersection with the parabola, tangent at the parabola, a parallel through $QL-P22$.

Within the meaning of the $X-Y$ -list above the last two constructions can be generalized:

The locus of a point P on $QL-L1$, dividing $QL-P5, QL-P20$ in the ratio κ , can be constructed as follows: a

parallel to $QL-L1$ through a point, dividing $QL-P9.QL-P10$ in the same ratio κ , intersection with the parabola, tangent at the parabola, a parallel through P .



Equation of this line:

$$\sum_{cycl} (-a^4vw + b^4wu + c^4uv + 2a^2(b^2 + c^2)vw + 2(c^2v + b^2w)^2\kappa)x = 0$$

All these lines have a common point.

For quadrilaterals with the same inscribed parabola and the same QL -diagonal triangle the loci for the points $QL-P5$, $QL-P12$, $QL-P20$, $QL-P22$ are lines with a common point.

This common point has the coordinates:

$$\begin{aligned} & (u(v(a^2w + c^2u)^2 - w(a^2v + b^2u)^2) \\ & : (v(w(b^2u + a^2v)^2 - u(b^2w + c^2v)^2) \\ & : (w(u(c^2v + a^2w)^2 - v(c^2u + a^2w)^2)). \end{aligned}$$

The polar of this point wrt the inscribed parabola is a line through $QL-P8$. The direction of the polar is orthogonal to a line through $QL-P10$ and a point, which is the intersection of $QL-L2$ and a perpendicular line to $QL-L7$ through $QL-P3$.

– Unfortunately no further properties can be given. –

A special Case

Taking three lines for a reference triangle and a fourth line parallel to the Euler line of the reference triangle, then we get a quadrilateral, where the four considered points coincide.

For quadrilaterals with sidelines parallel to the Euler lines of their triangle components holds $QL-P5 = QL-P12 = QL-P20 = QL-P22$.

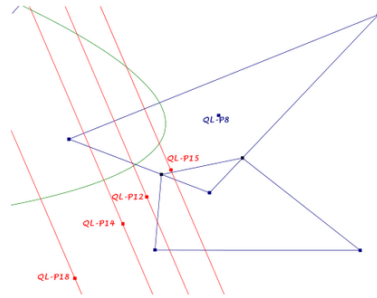
There are four quadrilaterals of this type among those with the same inscribed parabola and the same QL -diagonal triangle, taking lines

$$\begin{aligned}
&(\pm\sqrt{3c^4v^2+2(3S_A^2+S^2)vw+3b^4w^2}, \\
&\quad \pm\sqrt{3a^4w^2+2(3S_B^2+S^2)wu+3c^4u^2}, \\
&\quad \pm\sqrt{3b^4u^2+2(3S_C^2+S^2)uv+3a^4v^2}).
\end{aligned}$$

for defining the quadrilateral.

Final Remark

Among the quadrilaterals of the considered pencil there are four further collinear points, whose loci are lines with a common point. The common point is a point at infinity, so the lines are parallel. The first one is already mentioned.



QL-P12: $\sum_{cycl} vwx = 0$

This line is the polar of *QL-P8* wrt the parabola.
Polar distance from *QL-P8*:

$$d = \frac{S(uv + vw + wu)}{3\sqrt{S_A u^2(v-w)^2 + S_B v^2(w-u)^2 + S_C w^2(u-v)^2}}$$

QL-P14: $\sum_{cycl} (u^2 - 10vw)x = 0$

Parallel line with distance $4/3d$ from *QL-P8*.

QL-P15: $\sum_{cycl} (u^2 + 8vw)x = 0$

Parallel line with distance $2/3d$ from *QL-P8*.

QL-P18: $\sum_{cycl} (v-w)^2 x = 0$

Parallel line with distance $2d$ from *QL-P8*.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-04-12.pdf

Message: #31

Date: 17/5/2013 9:17:06

From: Chris van Tienhoven

Subject: EQF-Note 2013-04-15: Involutary Conjugate of the Newton Line

COPY FROM FORMER GOOGLE GROUP from Eckart Schmidt

(April 15, 2013)

Dear friends,

there is a conic for quadrilaterals in the attachment
with interesting properties in the QL-environment.

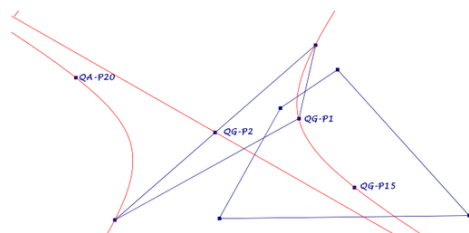
Best regards Eckart

EQF-Note 2013-04-15

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Involutive Conjugate of the Newton Line

For a quadrigon a QA-DT circumscribed conic with center QG-P2 through QG-P15 will be discussed. – Reference triangle for barycentric coordinates is QA-DT.



The Conic in QG-environment

In the QG-environment we consider the Newton Line

$$QL-L1: \quad q^2x + (p^2 - r^2)y - q^2z = 0.$$

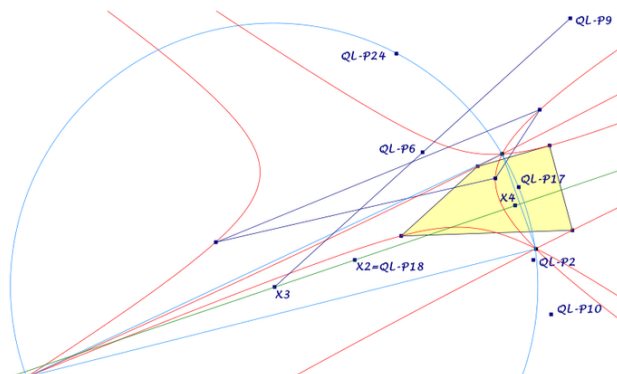
Its Involutionary Conjugate (QA-Tf2) is a QA-DT circumscribed conic with the equation

$$-r^2xy + (p^2 - r^2)zx + p^2yz = 0.$$

- The center of the conic is QG-P2.
- The conic contains the points
QG-P15 (image of the point at infinity of QL-L1),
QA-P20 (image of QA-P1),
reflection of QG-P1 in QG-P2 (image of QG-P12),
the reflections of QG-P15, QA-P20 in QG-P2.
- The asymptotes are parallel to the legs of the QL-diagonal triangle. Their points at infinity are the images of the midpoints of the diagonals of the quadrigon.
- The tangent in QG-P1 is QG-L2,
the tangent in QG-P15 is QG-P12. QG-P15,
the pole of QG-L3 is QG-P12.
The tangents in the endpoints of the 3rd diagonal are parallel QG-P1. QG-P3.
- The conic is the locus of QG-P15 for all quadrigons with the same QA- and QL-diagonal triangle.
- The conic divides the sidelines in ratios with product 1.

The Conic in QL -environment

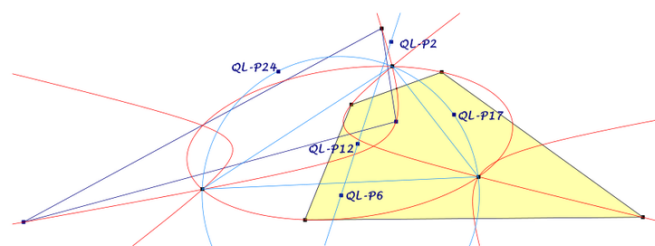
For a quadrangle there are three conics of the new type. Common points are the vertices of $QA-DT$ and $QA-P20$. For a quadrilateral there are also three conics but with three common points. The calculation needs solutions of equations with degree 3. So the following interesting properties are only Cabri controlled.



- The three conics have three common points.
- The centroid $X2$ of this triangle is $QL-P18$ (reflection of $QL-P8$ in $QL-P12$).
- The orthocenter $X4$ of this triangle is the reflection of $QL-P10$ in $QL-P2$.
- The circumcenter $X3$ of this triangle is the reflection of $QL-P9$ in $QL-P6$.
- The circumcircle of this triangle contains $QL-P17$ and $QL-P24$.
- The Simson line of $QL-P17$ wrt this triangle is a parallel to $QL-L6$ half the distance to $QL-P17$.

Comparison with the Nine-point Conic $QA-Co1$

We can compare these properties with those of another conic. Taking the three Nine-Point Conics $QA-Co1$ for a quadrilateral, we get also three common points (see $QL-P6$ in EQF).



The following list shows centroid, circumcenter and orthocenter of the QL -diagonal triangle $QL-DT$, the triangle wrt $QA-CoI$ and the triangle wrt $QG-Cox$.

	$QL-DT$	$QA-CoI-\Delta$	$QG-Cox-\Delta$
<i>centroid</i>	$QL-P8$	$QL-P12$	$QL-P18$
<i>circumcenter</i>	$QL-P9$	$QL-P6$	<i>see above</i>
<i>orthocenter</i>	$QL-P10$	$QL-P2$	<i>see above</i>

Reflecting one of the $QL-DT$ -points in the corresponding point of $QA-CoI-\Delta$ we get the corresponding point of $QG-Cox-\Delta$. The circumcircles of the three triangles contain $QL-P17$ and $QL-P24$.

Eckart Schmidt
<http://eckartschmidt.de>
 eckart_schmidt@t-online.de

2013-04-15.pdf

Message: #32
Date: 17/5/2013 9:21:20
From: Chris van Tienhoven
Subject: Cardioïd(s) tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP from Bernard Keizer
(April 17, 2013)

Dear friends,

This first contribution is rather a question.

I would like to know how it is possible to find the cardioïd(s)
tangent to 4 lines.

Any synthetic or analytical approach or bibliographic reference
would be welcome ...

You will find in the attached file a short introduction to the
subject.

Best regards

Bernard

Cardioïds tangent to 4 lines

(Background for this note is Chris van Tienhoven : Encyclopedia of Quadrifigures)

- For a 4-lines or complete quadrilateral QL1, there is one parabola tangent to the 4 lines (QL-Co1), the focus being the Miquel point M (QL-P1), the directrix the Steiner Line (QL-L2) and the axis being parallel to the Newton Line QL-L1. There is also one deltoïd tangent to the 4 lines, the center being the Kantor-Hervey point H (QL-P3) and the inner circle the Hervey circle (QL-Ci4). If I call θ_i the angles between the Steiner Line and the lines L_i , the angle of the 3 axes of the deltoïd and the axis of the parabola is $\frac{1}{3} \sum \theta_i$ (to the modulus $\pi/3$). Last, there is a cardioïd tangent to the 4 circles C_i circumscribed to the 4 reference triangles T_i of the QL (QL-Qu1), the center being the Miquel circumcenter O (QL-P4), the cusp the Miquel point M (QL-P1) and the inner circle the Miquel circle (QL-Ci3). The parabola and the cardioïd are Clawson-Schmidt conjugates and the angle between the axis of the cardioïd and the axis of the parabola is $\sum \theta_i$. I suppose these are more or less well-known properties.
- I consider now the 4 tangent lines to the cardioïd, which are also tangents to the circles C_i ; these 4 tangents to the cardioïd form a new complete quadrilateral QL2. The contact points with the cardioïd E_i are the reflections of the cusp M in the tangents to the Miquel circle in the points O_i , centers of the 4 reference circles; we have ME_i parallel to OO_i . If r is the radius of the Miquel and the Hervey circles, the cardioïd has a bitangent perpendicular to it's axis in a point T at a distance $3r/2$ from it's center O. Let's K_i be the intersections between this bitangent and each of the 4 tangent lines to the cardioïd. The main property of the cardioïd is that K_iO trisects the angle TK_iE_i .
- Conversely, for a given cardioïd (with center O and cusp M and therefore bitangent) and 4 tangent lines forming the QL2, it's easy to find the points K_i as intersections of the tangents and the bitangent, which gives the O_i as reflections of the cusp M in the segments K_iO . The reflections of M in O_jO_k are the vertices of the QL1.
- The problem is much more difficult if the cardioïd isn't given and we have only the 4 lines. If I believe Frank Morley, there are 8 cardioïds tangent to 4 lines not directed, but only one tangent to the 4 lines to which the deltoïd assigns a definite direction (as the cardioïd is a curve of direction).
- Morley proved also in his Morley's Miracle that the locus of the centers of cardioïds inscribed in a triangle is made of 3 sets of 3 parallel lines, forming equilateral triangles, the direction of the lines being the mean direction of the 3 sides of the triangle to the modulus $\pi/3$ (see American Journal of Mathematics, vol 51, n°3 : Extensions of Clifford's Chain-Theorem by F. Morley). For the QL, the centers of the cardioïds tangent to the 4 lines must therefore belong to the 4 groups of the 3 sets of 3 parallel lines for each of the 4 reference triangles of the QL ...

The question is precisely how to find the cardioïd(s) tangent to 4 given lines.

Synthetic or analytical approach and bibliographic references would be most welcome.

Bernard Keizer
(bernard.keizer@wanadoo.fr)

Cardio_ds tangent to 4 lines.doc

Message: #33
Date: 17/5/2013 9:24:20
From: Chris van Tienhoven
Subject: Cardioïd(s) tangent to 4 lines [1 Attac

Dear Bernard,
in addition to your paper "Cardioids tangent to 4 lines"
there is a further property described in the attachement.
Best regards Eckart

21.04.2013

Dear Bernard,

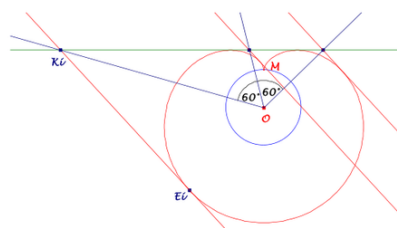
in addition to your paper **Cardioids tangent to 4 lines:**

There is the following property for cardioids in

H. Schmidt: Ausgewählte höhere Kurven.

Kesselringsche Verlagsbuchhandlung Wiesbaden, 1949, S.122.

„Die Tripel paralleler Tangenten der Kardioiden schneiden die Doppeltangente in Tripeln von Punkten, die vom Mittelpunkt des Grundkreises aus unter dem Winkel 60° erscheinen.“



Not to be found there is a further property:

The sum of the orientated distances of these parallel tangents from the center O of the basic circle is zero.

We can start with a quadrilateral, for example described by the vertices of a quadrilateral component in cartesian coordinates:

$$A(a, \frac{1}{a}), \quad B(b, \frac{1}{b}), \quad C(c, \frac{1}{c}), \quad D(d, \frac{1}{d}) .$$

The cardioid can be described by the center of the base circle $O(u, v)$ and the intersection $T(s, t)$ of the axis and the bitangent.

The four sidelines AB, BC, CD, DA are tangent to the cardioid. Their parallel tangents can easily be calculated. The sums 0 of their orientated distances to Z lead to four equations for u, v, s, t .

So far so good ...

These four equations (see beyond) with rational coefficients are up to degree 3 in every variable, but perhaps there are possible interpretations.

Best regards Eckart

Cardio_ds .doc

$$\begin{aligned}
& a^3 + 3a^2b + 3ab^2 + b^3 - 3as^2 - 3bs^2 - 3a^3b^2s^2 - 3a^2b^3s^2 + 2s^3 + \\
& 2a^2b^2s^3 + 2abs^2t + 2a^3b^3s^2t - 3at^2 - 3bt^2 - 3a^3b^2t^2 - \\
& 3a^2b^3t^2 + 2st^2 + 2a^2b^2st^2 + 2abt^3 + 2a^3b^3t^3 - 3a^2u - 6abu - \\
& 3b^2u + 6asu + 6bsu + 6a^3b^2su + 6a^2b^3su - 3s^2u - 3a^2b^2s^2u - \\
& 4abstu - 4a^3b^3stu + t^2u + a^2b^2t^2u - 3a^3b^2u^2 - 3a^2b^3u^2 + \\
& 2abtu^2 + 2a^3b^3tu^2 + a^2b^2u^3 - 3a^3bv - 6a^2b^2v - 3ab^3v + \\
& abs^2v + a^3b^3s^2v + 6atv + 6btv + 6a^3b^2tv + 6a^2b^3tv - \\
& 4stv - 4a^2b^2stv - 3abt^2v - 3a^3b^3t^2v + 6a^2buv + 6ab^2uv - \\
& 2absuv - 2a^3b^3suv - 2tuv - 2a^2b^2tuv - 2abu^2v + a^3b^3u^2v - \\
& 3av^2 - 3bv^2 + 2sv^2 + 2a^2b^2sv^2 + uv^2 - 2a^2b^2uv^2 + abv^3 \S 0
\end{aligned}$$

$$\begin{aligned}
& b^3 + 3b^2c + 3bc^2 + c^3 - 3bs^2 - 3cs^2 - 3b^3c^2s^2 - 3b^2c^3s^2 + 2s^3 + \\
& 2b^2c^2s^3 + 2bcs^2t + 2b^3c^3s^2t - 3bt^2 - 3ct^2 - 3b^3c^2t^2 - \\
& 3b^2c^3t^2 + 2st^2 + 2b^2c^2st^2 + 2bct^3 + 2b^3c^3t^3 - 3b^2u - 6bcu - \\
& 3c^2u + 6bsu + 6csu + 6b^3c^2su + 6b^2c^3su - 3s^2u - 3b^2c^2s^2u - \\
& 4bcstu - 4b^3c^3stu + t^2u + b^2c^2t^2u - 3b^3c^2u^2 - 3b^2c^3u^2 + \\
& 2bctu^2 + 2b^3c^3tu^2 + b^2c^2u^3 - 3b^3cv - 6b^2c^2v - 3bc^3v + \\
& bcs^2v + b^3c^3s^2v + 6btv + 6ctv + 6b^3c^2tv + 6b^2c^3tv - \\
& 4stv - 4b^2c^2stv - 3bct^2v - 3b^3c^3t^2v + 6b^2cuv + 6bc^2uv - \\
& 2bcsuv - 2b^3c^3suv - 2tuv - 2b^2c^2tuv - 2bcu^2v + b^3c^3u^2v - \\
& 3bv^2 - 3cv^2 + 2sv^2 + 2b^2c^2sv^2 + uv^2 - 2b^2c^2uv^2 + bcv^3 \S 0
\end{aligned}$$

$$\begin{aligned}
& c^3 + 3c^2d + 3cd^2 + d^3 - 3cs^2 - 3ds^2 - 3c^3d^2s^2 - 3c^2d^3s^2 + 2s^3 + \\
& 2c^2d^2s^3 + 2cds^2t + 2c^3d^3s^2t - 3ct^2 - 3dt^2 - 3c^3d^2t^2 - \\
& 3c^2d^3t^2 + 2st^2 + 2c^2d^2st^2 + 2cdt^3 + 2c^3d^3t^3 - 3c^2u - 6cdu - \\
& 3d^2u + 6csu + 6dsu + 6c^3d^2su + 6c^2d^3su - 3s^2u - 3c^2d^2s^2u - \\
& 4cdstu - 4c^3d^3stu + t^2u + c^2d^2t^2u - 3c^3d^2u^2 - 3c^2d^3u^2 + \\
& 2cdtu^2 + 2c^3d^3tu^2 + c^2d^2u^3 - 3c^3dv - 6c^2d^2v - 3cd^3v + \\
& cds^2v + c^3d^3s^2v + 6ctv + 6d^2tv + 6c^3d^2tv + 6c^2d^3tv - \\
& 4stv - 4c^2d^2stv - 3cdt^2v - 3c^3d^3t^2v + 6c^2duv + 6cd^2uv - \\
& 2cdsuv - 2c^3d^3suv - 2tuv - 2c^2d^2tuv - 2cdu^2v + c^3d^3u^2v - \\
& 3cv^2 - 3dv^2 + 2sv^2 + 2c^2d^2sv^2 + uv^2 - 2c^2d^2uv^2 + cdv^3 \S 0
\end{aligned}$$

$$\begin{aligned}
& a^3 + 3a^2d + 3ad^2 + d^3 - 3as^2 - 3ds^2 - 3a^3d^2s^2 - 3a^2d^3s^2 + 2s^3 + \\
& 2a^2d^2s^3 + 2ads^2t + 2a^3d^3s^2t - 3at^2 - 3dt^2 - 3a^3d^2t^2 - \\
& 3a^2d^3t^2 + 2st^2 + 2a^2d^2st^2 + 2adt^3 + 2a^3d^3t^3 - 3a^2u - 6adu - \\
& 3d^2u + 6asu + 6dsu + 6a^3d^2su + 6a^2d^3su - 3s^2u - 3a^2d^2s^2u - \\
& 4adstu - 4a^3d^3stu + t^2u + a^2d^2t^2u - 3a^3d^2u^2 - 3a^2d^3u^2 + \\
& 2adtu^2 + 2a^3d^3tu^2 + a^2d^2u^3 - 3a^3dv - 6a^2d^2v - 3ad^3v + \\
& ads^2v + a^3d^3s^2v + 6atv + 6d^2tv + 6a^3d^2tv + 6a^2d^3tv - \\
& 4stv - 4a^2d^2stv - 3adt^2v - 3a^3d^3t^2v + 6a^2duv + 6ad^2uv - \\
& 2adsuv - 2a^3d^3suv - 2tuv - 2a^2d^2tuv - 2adu^2v + a^3d^3u^2v - \\
& 3av^2 - 3dv^2 + 2sv^2 + 2a^2d^2sv^2 + uv^2 - 2a^2d^2uv^2 + adv^3 \S 0
\end{aligned}$$

Cardio_ds.doc

Message: #34

Date: 17/5/2013 9:29:14

From: Chris van Tienhoven

Subject: Cardioïd(s) tangent to 4 lines [1 Attac

Dear Friends,

I already had an earlier discussion with Bernard about these Cardioids.

I like the approach of Eckart. However the 4 equations he found seem pretty tough to solve.

I had another approach.

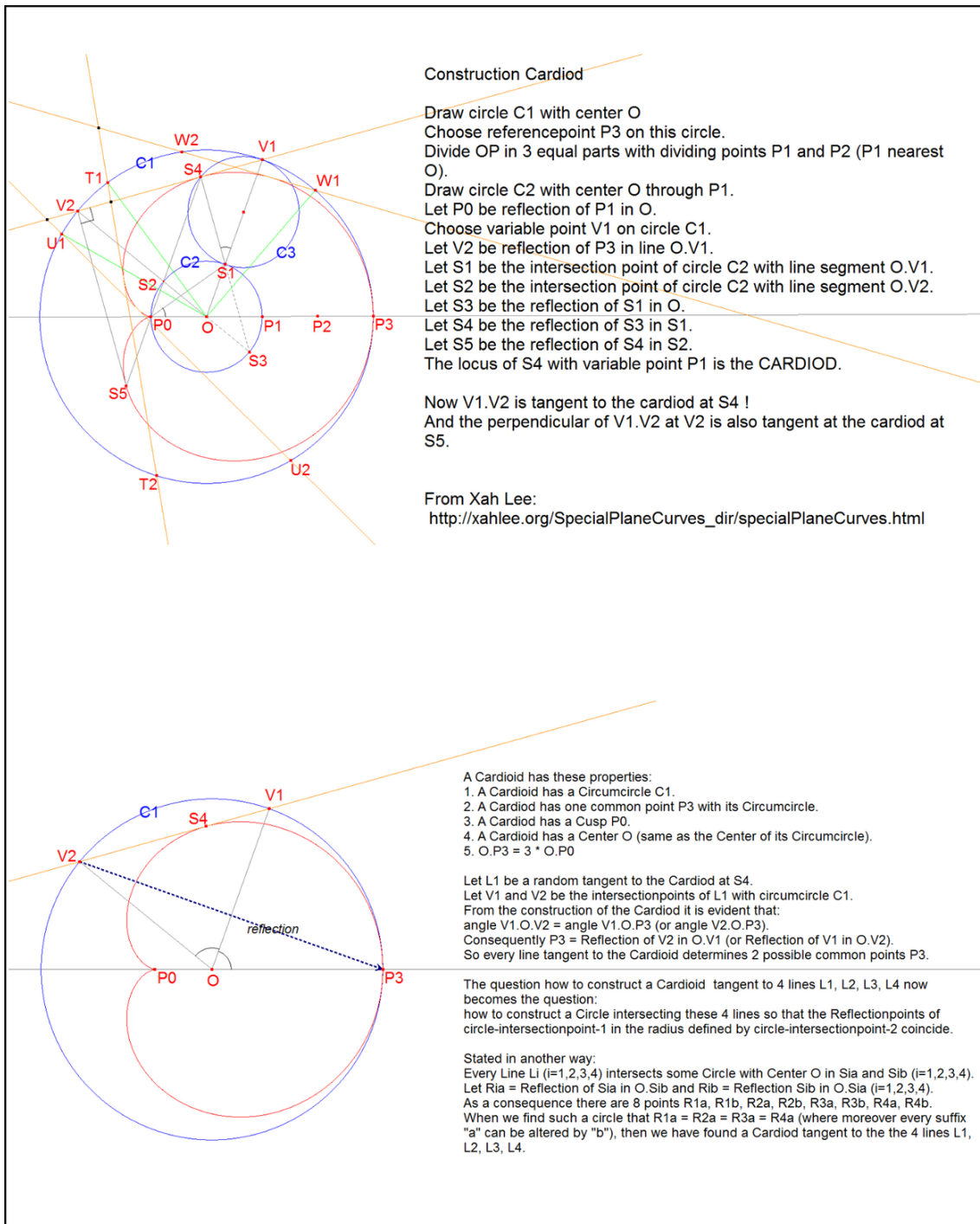
But again the algebraic way as well as the synthetic way of solving its simplification shown in attached file didn't pay off.

Maybe others can use clues in this approach.

See attachment.

Best regards,

Chris van Tienhoven



Cardiod-Construction-02-03.pdf

Message: #35
Date: 17/6/2013 2:37:36
From: Chris
Subject: A classification of QA-points in EQF

COPY FROM FORMER GOOGLE GROUP from Seiichi Kirikami
(April 24, 2013)

Dear friends,

[1] An idea of classification of QA-points:

Given a triangle ABC and a point $P\{p, q, r\}$, cevian points are $D\{0, q, r\}$, $E\{p, 0, r\}$ and $F\{p, q, 0\}$. Naturally AD, BE and CF are concurrent.

Likewise, given a triangle ABC, a point $P\{p, q, r\}$ and QA-point $P_x\{f(p, q, r), g(p, q, r), h(p, q, r)\}$, 3 points are determined as $D\{f(0, q, r), g(0, q, r), h(0, q, r)\}$, $E\{f(p, 0, r), g(p, 0, r), h(p, 0, r)\}$ and $F\{f(p, q, 0), g(p, q, 0), h(p, q, 0)\}$.

According to the colinear determinant of AD, BE and CF, QA-point P_x is classified as follows.

[2] Classification

(1) vertex type: D, E and F degenerate to A, B and C respectively. Examples, QA-P4, P5, P7, P17, P41.

(2) Decomposition type: the colinear determinant consists of the simple product of $p, q, r, (p+q), (q+r), (r+p)$. Examples, QA-P1, P10, P16, P18 to P22, P25 to P28, P38.

(3) Orthocentral type: D, E and F are on AH, BH and CH, where H means the orthocenter of ABC. Examples, QA-P2, P6, P8, P11 to P13, P23, P24, P30, P32, P33, P36, P37, P39.

(4) Cubic type: the main portion of the colinear determinant is cubic. Examples, QA-P3, P29, P34, P35.

(5) Other higher curve type: the main portion of the colinear determinant is higher than cubic. Examples, QA-P9, P14, P15, P31, P40.

Best regards,

Seiichi Kirikami

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Message: #36
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #36 is not available in Yahoo groups.

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Message: #37
Date: 17/6/2013 3:11:01
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(April 25, 2013)

Dear friends,

before starting for holidays a "short" message:

I have realized my concept for calculation cardioids tangent to 4 lines:

The centers of these cardioids lie on cubics.

You find the equations for cartesian and barycentric coordinates in the attachment as Mathematica file.

It seems, that no QL-point is on the cubic.

Best regards Eckart

■ Cardioids tangent to 4 lines

**The centers of cardioids tangent to for lines
lie on cubics with the following equations :**

Cartesian coordinates : $A(a; \frac{1}{a})$, $B(b; \frac{1}{b})$, $C(c; \frac{1}{c})$, $D(d; \frac{1}{d})$

with lines AB, BC, CD, DA

$$\begin{aligned} \text{cub}[\{x_ , y_ , 1\}] := & -a^2 + b^2 - a^4 b^2 + a^2 b^4 - 4ac + a^4 bc - a^3 b^2 c + 3a^2 b^3 c + ab^4 c - c^2 + a^3 b c^2 + \\ & a^2 b^2 c^2 + 3ab^3 c^2 + b^4 c^2 - a^4 b^4 c^2 + a^2 b c^3 - ab^2 c^3 + a^4 b^3 c^3 - a^3 b^4 c^3 + abc^4 - b^2 c^4 + a^3 b^3 c^4 - \\ & a^2 b^4 c^4 + 4bd - a^4 bd - 3a^3 b^2 d + a^2 b^3 d - ab^4 d + a^4 cd + 2a^3 bcd - 2ab^3 cd - b^4 cd + a^3 c^2 d + \\ & 2a^2 b c^2 d + b^3 c^2 d - a^4 b^3 c^2 d - 3a^3 b^4 c^2 d + a^2 c^3 d + 2ab c^3 d - 3b^2 c^3 d + a^4 b^2 c^3 d + 2a^3 b^3 c^3 d - \\ & 3a^2 b^4 c^3 d + a^4 c^4 d - b c^4 d + a^3 b^2 c^4 d - a^2 b^3 c^4 d + d^2 - a^4 d^2 - 3a^3 b d^2 - a^2 b^2 d^2 - ab^3 d^2 + \\ & a^4 b^4 d^2 - a^3 c^2 d^2 - 2ab^2 c^2 d^2 - b^3 c^2 d^2 + 3a^4 b^3 c^2 d^2 + a^3 b^4 c^2 d^2 + a^2 c^2 d^2 - b^2 c^2 d^2 + a^4 b^2 c^2 d^2 - \\ & a^2 b^4 c^2 d^2 - a^3 c^2 d^2 - 3b c^3 d^2 + a^4 b c^3 d^2 + 2a^3 b^2 c^3 d^2 + ab^4 c^3 d^2 - c^4 d^2 + a^3 b c^4 d^2 + a^2 b^2 c^4 d^2 + \\ & 3ab^3 c^4 d^2 + b^4 c^4 d^2 - a^4 b^4 c^4 d^2 + a^2 b d^3 - ab^2 d^3 + a^4 b^3 d^3 - a^3 b^4 d^3 + 3a^2 c d^3 - 2ab c d^3 - \\ & b^2 c d^3 + 3a^4 b^2 c d^3 - 2a^3 b^3 c d^3 - a^2 b^4 c d^3 + 3a^2 c^2 d^3 + b c^2 d^3 - a^4 b c^2 d^3 - 2a^2 b^3 c^2 d^3 - \\ & ab^4 c^2 d^3 + a^4 c^3 d^3 + 2a^3 b c^3 d^3 - 2ab^3 c^3 d^3 - b^4 c^3 d^3 + a^3 c^4 d^3 - a^2 b c^4 d^3 + 3ab^2 c^4 d^3 + \\ & b^3 c^4 d^3 - 4a^4 b^3 c^4 d^3 + a^2 d^4 - ab d^4 + a^4 b^2 d^4 - a^3 b^3 d^4 + a c d^4 - b c d^4 + a^3 b^2 c d^4 - a^2 b^3 c d^4 + \\ & c^2 d^4 - a^4 c^2 d^4 - 3a^3 b c^2 d^4 - a^2 b^2 c^2 d^4 - ab^3 c^2 d^4 + a^4 b^4 c^2 d^4 - a^3 c^3 d^4 - 3a^2 b c^3 d^4 + \\ & ab^2 c^3 d^4 - b^3 c^3 d^4 + 4a^4 b^4 c^3 d^4 - a^2 c^4 d^4 + b^2 c^4 d^4 - a^4 b^2 c^4 d^4 + a^2 b^4 c^4 d^4 + 3ax - 3bx + \\ & 3a^3 b^2 x - 3a^2 b^3 x + 3cx - 3a^3 b c x - 3a^2 b^2 c x - 3ab^3 c x - 3a^2 b c^2 x - 3ab^2 c^2 x - 3b^3 c^2 x + \\ & 3a^3 b^4 c^2 x - 3ab^3 c^3 x + 3a^2 b^4 c^3 x - 3a^3 b^3 c^3 x + 3a^2 b^4 c^3 x - 3dx + 3a^3 b d x + 3a^2 b^2 d x + \\ & 3ab^3 d x - 3a^3 c d x - 3a^2 b c d x + 3ab^2 c d x + 3b^3 c d x - 3a^2 c^2 d x - 3ab c^2 d x + 3b^2 c^2 d x + \\ & 3a^3 b^3 c^2 d x + 6a^2 b^4 c^2 d x - 3a^3 c^3 d x + 3b c^3 d x - 3a^3 b^2 c^3 d x + 3a^2 b^3 c^3 d x + 3a^3 d^2 x + \\ & 3a^2 b d^2 x + 3ab^2 d^2 x - 3a^4 b^3 d^2 x - 3a^2 c d^2 x + 3ab c d^2 x + 3b^2 c d^2 x - 6a^4 b^2 c d^2 x - \\ & 3a^3 b^3 c d^2 x - 3a^2 c^2 d^2 x + 3b c^2 d^2 x - 3a^3 b^2 c^2 d^2 x + 3a^2 b^3 c^2 d^2 x + 3c^3 d^2 x - 3a^3 b c^3 d^2 x - \\ & 3a^2 b^2 c^3 d^2 x + 3a^3 b^3 d^3 x - 3a c d^3 x + 3b c d^3 x - 3a^3 b^2 c d^3 x + 3a^2 b^3 c d^3 x - 3c^2 d^3 x + \\ & 3a^3 b c^2 d^3 x + 3a^2 b^2 c^2 d^3 x + 3ab^3 c^2 d^3 x - 3a^3 c^3 d^3 x + 3a^2 b c^3 d^3 x - 3ab^2 c^3 d^3 x + \\ & 3b^3 c^3 d^3 x + 6a^4 b^3 c^3 d^3 x - 6a^3 b^4 c^3 d^3 x - 3b^2 c^4 d^3 x + 3a^4 b^2 c^4 d^3 x + 6a^3 b^3 c^4 d^3 x + \\ & 3a^3 b^4 c^4 d^3 x + 6a^2 b^2 c^4 d^3 x - 3a^3 b^4 c^2 d^4 x + 3a^2 c^3 d^4 x - 6a^3 b^3 c^3 d^4 x - 3a^2 b^4 c^3 d^4 x + \\ & 3a^2 b c x^2 + 3ab^2 c x^2 + 3ab c^2 x^2 - 3a^4 b^4 c^2 x^2 - 3a^2 b d x^2 - 3ab^2 d x^2 + 3a^2 c d x^2 - 3b^2 c d x^2 + \\ & 3a^2 c^2 d x^2 - 3b c^2 d x^2 - 6a^2 b^3 c^2 d x^2 - 3ab^2 d^2 x^2 + 3a^4 b^2 d^2 x^2 + 3a^2 c d^2 x^2 - 3b c d^2 x^2 + \\ & 6a^3 b^2 c d^2 x^2 + 6a^2 b^3 c^2 d^2 x^2 - 3a^4 b^3 c^3 d^2 x^2 + 3a^3 b^4 c^3 d^2 x^2 + 3b^2 c^4 d^2 x^2 - 3a^3 b^3 c^4 d^2 x^2 - \\ & 6a^2 b c^2 d^3 x^2 - 3a^4 b^3 c^2 d^3 x^2 + 3a^3 b^4 c^2 d^3 x^2 - 3a^4 b^2 c^3 d^3 x^2 + 3a^2 b^4 c^3 d^3 x^2 - 3a^3 b^2 c^4 d^3 x^2 - \\ & 3a^2 b^3 c^4 d^3 x^2 - 3a^2 c^2 d^4 x^2 + 3a^3 b^3 c^2 d^4 x^2 + 3a^2 b^2 c^3 d^4 x^2 + 3a^2 b^3 c^3 d^4 x^2 - ab c x^3 + \\ & a^2 b^3 c^2 x^3 + ab d x^3 - a c d x^3 + b c d x^3 + a^2 b^2 c^2 d x^3 - a^3 b^2 d^2 x^3 - a^2 b^2 c d^2 x^3 + a^2 b c^2 d^2 x^3 - \\ & ab^2 c^2 d^2 x^3 + a^4 b^3 c^2 d^2 x^3 - a^3 b^4 c^2 d^2 x^3 - b^2 c^3 d^2 x^3 + a^3 b^3 c^3 d^2 x^3 - a^2 b^4 c^3 d^2 x^3 + a^2 b^3 c^4 d^2 x^3 + \\ & a^2 c^2 d^3 x^3 + a^4 b^2 c^2 d^3 x^3 - a^3 b^3 c^2 d^3 x^3 + a^3 b^2 c^3 d^3 x^3 - a^2 b^3 c^3 d^3 x^3 + a^2 b^2 c^4 d^3 x^3 - \\ & a^3 b^2 c^4 d^3 x^3 - a^2 b^2 c^3 d^4 x^3 + 3a^2 c y + 6ab c y - 3a^2 b^4 c y + 3a^2 c^2 y - 6a^2 b^3 c^2 y - 3ab^4 c^2 y - \\ & 6ab d y - 3b^2 d y + 3a^4 b^2 d y + 6a c d y - 6b c d y - 3a^4 b c d y + 3a^3 b^2 c d y - 3a^2 b^3 c d y + \\ & 3ab^4 c d y - 3a^3 b c^2 d y - 3a^2 b^2 c^2 d y - 3ab^3 c^2 d y + 3a^4 b^4 c^2 d y - 3a^2 b c^3 d y + 3ab^2 c^3 d y - \\ & 3a^4 b^3 c^3 d y + 3a^3 b^4 c^3 d y - 3ab c^4 d y + 3b^2 c^4 d y - 3a^3 b^3 c^4 d y + 3a^2 b^4 c^4 d y - 3b d^2 y + \\ & 3a^4 b d^2 y + 6a^3 b^2 d^2 y + 3a^3 b c d^2 y + 3a^2 b^2 c d^2 y + 3ab^3 c d^2 y - 3a^4 b^4 c d^2 y - 3a^2 b c^2 d^2 y + \\ & 3ab^2 c^2 d^2 y - 3a^4 b^3 c^2 d^2 y + 3a^3 b^4 c^2 d^2 y + 3ab c^3 d^2 y + 6b^2 c^3 d^2 y - 3a^4 b^2 c^3 d^2 y - \\ & 3a^3 b^3 c^3 d^2 y + 3a^2 b^4 c^3 d^2 y + 3b c^4 d^2 y - 3a^3 b^2 c^4 d^2 y - 3a^2 b^3 c^4 d^2 y - 3ab^4 c^4 d^2 y - \\ & 3a^2 b c d^3 y + 3ab^2 c d^3 y - 3a^4 b^3 c d^3 y + 3a^3 b^4 c d^3 y - 6a^2 c^2 d^3 y - 3ab c^2 d^3 y - 3a^4 b^2 c^2 d^3 y + \\ & 3a^3 b^3 c^2 d^3 y + 3a^2 b^4 c^2 d^3 y - 3a^4 b c^3 d^3 y - 3a^3 b^2 c^3 d^3 y + 3a^2 b^3 c^3 d^3 y + 3ab^4 c^3 d^3 y - \\ & 3a^3 b c^4 d^3 y - 3a^2 b^2 c^4 d^3 y - 3ab^3 c^4 d^3 y + 3a^4 b^4 c^4 d^3 y - 3a^2 c d^4 y + 3ab c d^4 y - 3a^4 b^2 c d^4 y + \\ & 3a^3 b^3 c d^4 y - 3a^2 b^4 c d^4 y + 3a^4 b c^2 d^4 y + 3a^3 b^2 c^2 d^4 y + 3a^2 b^3 c^2 d^4 y + 3ab^4 c^2 d^4 y + \\ & 3a^2 b^2 c^3 d^4 y + 3ab^3 c^3 d^4 y - 3a^4 b^4 c^3 d^4 y + 3a^2 b c^4 d^4 y - 3ab^2 c^4 d^4 y + 3a^4 b^3 c^4 d^4 y - \\ & 3a^3 b^4 c^4 d^4 y - 6a c x y + 6a^2 b^3 c x y + 6a^2 b^2 c^2 x y + 6ab^3 c^2 x y + 6b d x y - 6a^4 b^3 c d x y + \\ & 6a^3 b c d x y - 6ab^3 c d x y + 6a^2 b c^2 d x y - 6a^3 b^4 c^2 d x y + 6ab c^3 d x y - 6b^2 c^3 d x y + \\ & 6a^3 b^3 c^3 d x y - 6a^2 b^4 c^3 d x y - 6a^3 b^2 d^2 x y - 6a^2 b^2 d^2 x y - 6ab^2 c d^2 x y + 6a^4 b^3 c d^2 x y + \end{aligned}$$

$$\begin{aligned}
& 6a^2c^2d^2xy - 6b^2c^2d^2xy + 6a^4b^2c^2d^2xy - 6a^2b^4c^2d^2xy - 6b^4c^3d^2xy + 6a^3b^2c^3d^2xy + \\
& 6a^2b^2c^4d^2xy + 6ab^3c^4d^2xy + 6a^2c^3d^3xy - 6abc^3d^3xy + 6a^4b^2c^3d^3xy - 6a^3b^3c^3d^3xy + \\
& 6a^2c^3d^3xy - 6a^2b^3c^3d^3xy + 6a^3b^3c^3d^3xy - 6ab^3c^3d^3xy + 6ab^2c^4d^3xy - 6a^4b^3c^4d^3xy - \\
& 6a^3b^2c^4d^3xy - 6a^2b^2c^4d^3xy - 6a^2b^3c^4d^3xy + 6a^3b^4c^3d^4xy - 3a^2b^2c^2x^2y - 3ab^2c^2x^2y + \\
& 3a^2b^2d^2x^2y - 3a^2bcd^2x^2y + 3ab^2cd^2x^2y - 3abc^2d^2x^2y + 3b^2c^2d^2x^2y + 3a^2b^4c^2d^2x^2y + \\
& 3a^2b^2c^2d^2x^2y - 3a^2c^2d^2x^2y + 3ab^2c^2d^2x^2y - 3a^4b^2c^2d^2x^2y - 3a^2c^2d^2x^2y + 3b^2c^2d^2x^2y - \\
& 3a^3b^2c^2d^2x^2y + 3a^2b^3c^2d^2x^2y - 3a^2b^2c^3d^2x^2y - 3ab^2c^4d^2x^2y + 3a^2b^2c^2d^3x^2y + \\
& 3a^4b^3c^3d^3x^2y - 3a^3b^4c^3d^3x^2y + 3a^3b^3c^4d^3x^2y + 3a^2b^2c^2d^4x^2y - 3a^3b^3c^3d^4x^2y - \\
& 3a^2b^2c^2d^4x^2y - 3ab^2c^2d^4x^2y + 3a^2b^4c^2d^4x^2y + 3a^2b^2d^2y^2 + 3ab^2d^2y^2 - 3a^2cd^2y^2 + \\
& 3b^2cd^2y^2 - 3a^2cd^2y^2 + 3b^2cd^2y^2 + 6a^2b^3c^2d^2y^2 + 3ab^2d^2y^2 - 3a^4b^2d^2y^2 - 3a^2cd^2y^2 + \\
& 3b^2cd^2y^2 - 6a^3b^2cd^2y^2 - 6ab^2c^3d^2y^2 + 3a^4b^3c^3d^2y^2 - 3a^3b^4c^3d^2y^2 - 3b^2c^4d^2y^2 + \\
& 3a^3b^3c^4d^2y^2 + 6a^2b^2c^3d^3y^2 + 3a^4b^3c^3d^3y^2 - 3a^3b^4c^3d^3y^2 + 3a^2b^2c^3d^3y^2 - 3a^2b^4c^3d^3y^2 + \\
& 3a^3b^2c^4d^3y^2 + 3a^2b^3c^4d^3y^2 + 3a^2c^2d^4y^2 - 3a^3b^3c^2d^4y^2 - 3a^3b^2c^3d^4y^2 - 3a^2b^3c^3d^4y^2 + \\
& 3abc^2x^2y^2 - 3a^2b^3c^2x^2y^2 - 3ab^2cd^2x^2y^2 + 3ac^2dx^2y^2 - 3bcd^2x^2y^2 - 3a^2b^2c^2dx^2y^2 + \\
& 3a^3b^2d^2x^2y^2 + 3a^2b^2cd^2x^2y^2 - 3a^2b^2c^2d^2x^2y^2 + 3ab^2c^2d^2x^2y^2 - 3a^4b^3c^2d^2x^2y^2 + \\
& 3a^3b^4c^2d^2x^2y^2 + 3b^2c^3d^2x^2y^2 - 3a^3b^3c^3d^2x^2y^2 + 3a^2b^4c^3d^2x^2y^2 - 3a^2b^3c^4d^2x^2y^2 - \\
& 3a^2c^2d^3x^2y^2 - 3a^4b^2c^2d^3x^2y^2 + 3a^3b^3c^2d^3x^2y^2 - 3a^3b^2c^3d^3x^2y^2 + 3a^2b^3c^3d^3x^2y^2 - \\
& 3a^2b^2c^4d^3x^2y^2 + 3a^3b^2c^2d^4x^2y^2 + 3a^2b^2c^3d^4x^2y^2 + a^2b^2c^2y^3 - a^2b^2d^2y^3 + a^2cd^2y^3 - \\
& ab^2cd^2y^3 - ab^2cd^2y^3 + ab^2cd^2y^3 - b^2c^2d^2y^3 - a^2b^4c^2d^2y^3 - a^2b^2d^2y^3 + a^2cd^2y^3 - ab^2cd^2y^3 + \\
& a^4b^2c^2d^2y^3 + a^2cd^2y^3 - b^2c^2d^2y^3 + a^3b^2c^2d^2y^3 - a^2b^3c^2d^2y^3 + a^2b^2c^3d^2y^3 + ab^2c^4d^2y^3 - \\
& a^2b^2c^2d^3y^3 - a^4b^3c^3d^3y^3 + a^3b^4c^3d^3y^3 - a^2b^3c^4d^3y^3 - a^2b^2c^4d^3y^3 + a^3b^3c^3d^4y^3
\end{aligned}$$

Barycentric coordinates : Reference triangle QL - Diagonal Triangle,
generating line : $lx + my + nz = 0$

$$\begin{aligned}
\text{cube}[\{x_, y_, z_ \}] := & 1^2 m^4 Sa^3 x^3 - 2 1^2 m^2 n^2 Sa^3 x^3 + 1^2 n^4 Sa^3 x^3 + 2 1^4 m^2 Sa^2 Sb x^3 + 2 1^4 n^2 Sa^2 Sb x^3 + \\
& 2 1^2 m^2 n^2 Sa^2 Sb x^3 + 2 1^2 n^4 Sa^2 Sb x^3 + 1^6 Sa Sb^2 x^3 + 6 1^4 n^2 Sa Sb^2 x^3 + 1^2 n^4 Sa Sb^2 x^3 + \\
& 2 1^4 m^2 Sa^2 Sc x^3 + 2 1^2 m^4 Sa^2 Sc x^3 + 2 1^4 n^2 Sa^2 Sc x^3 + 2 1^2 m^2 n^2 Sa^2 Sc x^3 + 2 1^6 Sa Sb Sc x^3 + \\
& 6 1^4 m^2 Sa Sb Sc x^3 + 6 1^4 n^2 Sa Sb Sc x^3 + 2 1^2 m^2 n^2 Sa Sb Sc x^3 + 4 1^6 Sb^2 Sc x^3 + 4 1^4 n^2 Sb^2 Sc x^3 + \\
& 1^6 Sa Sc^2 x^3 + 6 1^4 m^2 Sa Sc^2 x^3 + 1^2 m^4 Sa Sc^2 x^3 + 4 1^6 Sb Sc^2 x^3 + 4 1^4 m^2 Sb Sc^2 x^3 + 3 1^2 m^4 Sa^2 Sb x^2 y + \\
& 18 1^2 m^2 n^2 Sa^2 Sb x^2 y + 3 1^2 n^4 Sa^2 Sb x^2 y + 6 1^4 m^2 Sa Sb^2 x^2 y + 6 1^4 n^2 Sa Sb^2 x^2 y + \\
& 6 1^2 m^2 n^2 Sa Sb^2 x^2 y + 6 1^2 n^4 Sa Sb^2 x^2 y + 3 1^6 Sb^3 x^2 y - 6 1^4 n^2 Sb^3 x^2 y + 3 1^2 n^4 Sb^3 x^2 y + \\
& 12 1^2 m^4 Sa^2 Sc x^2 y + 12 1^2 m^2 n^2 Sa^2 Sc x^2 y + 18 1^4 m^2 Sa Sb Sc x^2 y + 6 1^2 m^4 Sa Sb Sc x^2 y + \\
& 6 1^4 n^2 Sa Sb Sc x^2 y + 18 1^2 m^2 n^2 Sa Sb Sc x^2 y + 6 1^6 Sb^2 Sc x^2 y + 6 1^4 m^2 Sb^2 Sc x^2 y + \\
& 6 1^4 n^2 Sb^2 Sc x^2 y + 6 1^2 m^2 n^2 Sb^2 Sc x^2 y + 12 1^4 m^2 Sa Sc^2 x^2 y + 12 1^2 m^4 Sa Sc^2 x^2 y + 3 1^6 Sb Sc^2 x^2 y + \\
& 18 1^4 m^2 Sb Sc^2 x^2 y + 3 1^2 m^4 Sb Sc^2 x^2 y + 3 m^6 Sa^3 x y^2 - 6 m^4 n^2 Sa^3 x y^2 + 3 m^2 n^4 Sa^3 x y^2 + \\
& 6 1^2 m^4 Sa^2 Sb x y^2 + 6 1^2 m^2 n^2 Sa^2 Sb x y^2 + 6 m^4 n^2 Sa^2 Sb x y^2 + 6 m^2 n^4 Sa^2 Sb x y^2 + 3 1^4 m^2 Sa Sb^2 x y^2 + \\
& 18 1^2 m^2 n^2 Sa Sb^2 x y^2 + 3 m^2 n^4 Sa Sb^2 x y^2 + 6 1^2 m^4 Sa^2 Sc x y^2 + 6 m^6 Sa^2 Sc x y^2 + 6 1^2 m^2 n^2 Sa^2 Sc x y^2 + \\
& 6 m^4 n^2 Sa^2 Sc x y^2 + 6 1^4 m^2 Sa Sb Sc x y^2 + 18 1^2 m^2 n^2 Sa Sb Sc x y^2 + 18 1^2 m^2 n^2 Sa Sb Sc x y^2 + \\
& 6 m^4 n^2 Sa Sb Sc x y^2 + 12 1^4 m^2 Sb^2 Sc x y^2 + 12 1^2 m^2 n^2 Sb^2 Sc x y^2 + 3 1^4 m^2 Sa Sc^2 x y^2 + \\
& 18 1^2 m^4 Sa Sc^2 x y^2 + 3 m^6 Sa Sc^2 x y^2 + 12 1^4 m^2 Sb Sc^2 x y^2 + 12 1^2 m^4 Sb Sc^2 x y^2 + m^6 Sa^2 Sb y^3 + \\
& 6 m^4 n^2 Sa^2 Sb y^3 + m^2 n^4 Sa^2 Sb y^3 + 2 1^2 m^4 Sa Sb^2 y^3 + 2 1^2 m^2 n^2 Sa Sb^2 y^3 + 2 m^4 n^2 Sa Sb^2 y^3 + \\
& 2 m^2 n^4 Sa Sb^2 y^3 + 1^4 m^2 Sb^3 y^3 - 2 1^2 m^2 n^2 Sb^3 y^3 + m^2 n^4 Sb^3 y^3 + 4 m^6 Sa^2 Sc y^3 + 4 m^4 n^2 Sa^2 Sc y^3 + \\
& 6 1^2 m^4 Sa Sb Sc y^3 + 2 m^6 Sa Sb Sc y^3 + 2 1^2 m^2 n^2 Sa Sb Sc y^3 + 6 m^4 n^2 Sa Sb Sc y^3 + 2 1^4 m^2 Sb^2 Sc y^3 + \\
& 2 1^2 m^4 Sb^2 Sc y^3 + 2 1^2 m^2 n^2 Sb^2 Sc y^3 + 2 m^4 n^2 Sb^2 Sc y^3 + 4 1^2 m^4 Sa Sc^2 y^3 + 4 m^6 Sa Sc^2 y^3 + \\
& 1^4 m^2 Sb Sc^2 y^3 + 6 1^2 m^4 Sb Sc^2 y^3 + m^6 Sb Sc^2 y^3 + 12 1^2 m^2 n^2 Sa^2 Sb x^2 z + 12 1^2 n^4 Sa^2 Sb x^2 z + \\
& 12 1^4 n^2 Sa Sb^2 x^2 z + 12 1^2 n^4 Sa Sb^2 x^2 z + 3 1^2 m^4 Sa^2 Sc x^2 z + 18 1^2 m^2 n^2 Sa^2 Sc x^2 z + \\
& 3 1^2 n^4 Sa^2 Sc x^2 z + 6 1^4 m^2 Sa Sb Sc x^2 z + 18 1^4 n^2 Sa Sb Sc x^2 z + 18 1^2 m^2 n^2 Sa Sb Sc x^2 z + \\
& 6 1^2 n^4 Sa Sb Sc x^2 z + 3 1^6 Sb^2 Sc x^2 z + 18 1^4 n^2 Sb^2 Sc x^2 z + 3 1^2 n^4 Sb^2 Sc x^2 z + 6 1^4 m^2 Sa Sc^2 x^2 z + \\
& 6 1^2 m^4 Sa Sc^2 x^2 z + 6 1^4 n^2 Sa Sc^2 x^2 z + 6 1^2 m^2 n^2 Sa Sc^2 x^2 z + 6 1^6 Sb Sc^2 x^2 z + 6 1^4 m^2 Sb Sc^2 x^2 z + \\
& 6 1^4 n^2 Sb Sc^2 x^2 z + 6 1^2 m^2 n^2 Sb Sc^2 x^2 z + 3 1^6 Sc^3 x^2 z - 6 1^4 m^2 Sc^3 x^2 z + 3 1^2 m^4 Sc^3 x^2 z + \\
& 3 m^6 Sa^3 x y z - 3 m^4 n^2 Sa^3 x y z - 3 m^2 n^4 Sa^3 x y z + 3 n^6 Sa^3 x y z + 9 1^2 m^4 Sa^2 Sb x y z + \\
& 6 1^2 m^2 n^2 Sa^2 Sb x y z + 9 m^4 n^2 Sa^2 Sb x y z + 9 1^2 n^4 Sa^2 Sb x y z + 6 m^2 n^4 Sa^2 Sb x y z + \\
& 9 n^6 Sa^2 Sb x y z + 9 1^4 m^2 Sa Sb^2 x y z + 9 1^4 n^2 Sa Sb^2 x y z + 6 1^2 m^2 n^2 Sa Sb^2 x y z + \\
& 6 1^2 n^4 Sa Sb^2 x y z + 9 m^2 n^4 Sa Sb^2 x y z + 9 n^6 Sa Sb^2 x y z + 3 1^6 Sb^3 x y z - 3 1^4 n^2 Sb^3 x y z - \\
& 3 1^2 n^4 Sb^3 x y z + 3 n^6 Sb^3 x y z + 9 1^2 m^4 Sa^2 Sc x y z + 9 m^6 Sa^2 Sc x y z + 6 1^2 m^2 n^2 Sa^2 Sc x y z +
\end{aligned}$$

$$\begin{aligned}
& 6m^4n^2Sa^2Scxyz + 9l^2n^4Sa^2Scxyz + 9m^2n^4Sa^2Scxyz + 18l^4m^2SaSbScxyz + \\
& 18l^2m^4SaSbScxyz + 18l^4n^2SaSbScxyz - 12l^2m^2n^2SaSbScxyz + 18m^4n^2SaSbScxyz + \\
& 18l^2n^4SaSbScxyz + 18m^2n^4SaSbScxyz + 9l^6Sb^2Scxyz + 9l^4m^2Sb^2Scxyz + \\
& 6l^4n^2Sb^2Scxyz + 6l^2m^2n^2Sb^2Scxyz + 9l^2n^4Sb^2Scxyz + 9m^2n^4Sb^2Scxyz + \\
& 9l^4m^2SaSc^2xyz + 6l^2m^4SaSc^2xyz + 9m^6SaSc^2xyz + 9l^4n^2SaSc^2xyz + 6l^2m^2n^2SaSc^2xyz + \\
& 9m^4n^2SaSc^2xyz + 9l^6SbSc^2xyz + 6l^4m^2SbSc^2xyz + 9l^2m^4SbSc^2xyz + 9l^4n^2SbSc^2xyz + \\
& 6l^2m^2n^2SbSc^2xyz + 9m^4n^2SbSc^2xyz + 3l^6Sc^3xyz - 3l^4m^2Sc^3xyz - 3l^2m^4Sc^3xyz + \\
& 3m^6Sc^3xyz + 12m^4n^2Sa^2Sb^2yz^2 + 12m^2n^4Sa^2Sb^2yz^2 + 12l^2m^2n^2SaSb^2yz^2 + \\
& 12m^2n^4SaSb^2yz^2 + 3m^6Sa^2Sc^2yz^2 + 18m^4n^2Sa^2Sc^2yz^2 + 3m^2n^4Sa^2Sc^2yz^2 + 6l^2m^4SaSbSc^2yz^2 + \\
& 18l^2m^2n^2SaSbSc^2yz^2 + 18m^4n^2SaSbSc^2yz^2 + 6m^2n^4SaSbSc^2yz^2 + 3l^4m^2Sb^2Sc^2yz^2 + \\
& 18l^2m^2n^2Sb^2Sc^2yz^2 + 3m^2n^4Sb^2Sc^2yz^2 + 6l^2m^4SaSc^2yz^2 + 6m^6SaSc^2yz^2 + 6l^2m^2n^2SaSc^2yz^2 + \\
& 6m^4n^2SaSc^2yz^2 + 6l^4m^2SbSc^2yz^2 + 6l^2m^4SbSc^2yz^2 + 6l^2m^2n^2SbSc^2yz^2 + \\
& 6m^4n^2SbSc^2yz^2 + 3l^4m^2Sc^3yz^2 - 6l^2m^4Sc^3yz^2 + 3m^6Sc^3yz^2 + 3m^4n^2Sa^3xz^2 - \\
& 6m^2n^4Sa^3xz^2 + 3n^6Sa^3xz^2 + 6l^2m^2n^2Sa^2Sb^2xz^2 + 6l^2n^4Sa^2Sb^2xz^2 + 6m^2n^4Sa^2Sb^2xz^2 + \\
& 6n^6Sa^2Sb^2xz^2 + 3l^4n^2SaSb^2xz^2 + 18l^2n^4SaSb^2xz^2 + 3n^6SaSb^2xz^2 + 6l^2m^2n^2Sa^2Sc^2xz^2 + \\
& 6m^4n^2Sa^2Sc^2xz^2 + 6l^2n^4Sa^2Sc^2xz^2 + 6m^2n^4Sa^2Sc^2xz^2 + 6l^4n^2SaSbSc^2xz^2 + \\
& 18l^2m^2n^2SaSbSc^2xz^2 + 18l^2n^4SaSbSc^2xz^2 + 6m^2n^4SaSbSc^2xz^2 + 12l^4n^2Sb^2Sc^2xz^2 + \\
& 12l^2n^4Sb^2Sc^2xz^2 + 3l^4n^2SaSc^2xz^2 + 18l^2m^2n^2SaSc^2xz^2 + 3m^4n^2SaSc^2xz^2 + \\
& 12l^4n^2SbSc^2xz^2 + 12l^2m^2n^2SbSc^2xz^2 + 3m^4n^2Sa^2Sb^2yz^2 + 18m^2n^4Sa^2Sb^2yz^2 + \\
& 3n^6Sa^2Sb^2yz^2 + 6l^2m^2n^2SaSb^2yz^2 + 6l^2n^4SaSb^2yz^2 + 6m^2n^4SaSb^2yz^2 + 6n^6SaSb^2yz^2 + \\
& 3l^4n^2Sb^3yz^2 - 6l^2n^4Sb^3yz^2 + 3n^6Sb^3yz^2 + 12m^4n^2Sa^2Sc^2yz^2 + 12m^2n^4Sa^2Sc^2yz^2 + \\
& 18l^2m^2n^2SaSbSc^2yz^2 + 6m^4n^2SaSbSc^2yz^2 + 6l^2n^4SaSbSc^2yz^2 + 18m^2n^4SaSbSc^2yz^2 + \\
& 6l^4n^2Sb^2Sc^2yz^2 + 6l^2m^2n^2Sb^2Sc^2yz^2 + 6l^2n^4Sb^2Sc^2yz^2 + 6m^2n^4Sb^2Sc^2yz^2 + \\
& 12l^2m^2n^2SaSc^2yz^2 + 12m^4n^2SaSc^2yz^2 + 3l^4n^2SbSc^2yz^2 + 18l^2m^2n^2SbSc^2yz^2 + \\
& 3m^4n^2SbSc^2yz^2 + 4m^2n^4Sa^2Sb^2z^3 + 4n^6Sa^2Sb^2z^3 + 4l^2n^4SaSb^2z^3 + 4n^6SaSb^2z^3 + \\
& m^4n^2Sa^2Sc^2z^3 + 6m^2n^4Sa^2Sc^2z^3 + n^6Sa^2Sc^2z^3 + 2l^2m^2n^2SaSbSc^2z^3 + 6l^2n^4SaSbSc^2z^3 + \\
& 6m^2n^4SaSbSc^2z^3 + 2n^6SaSbSc^2z^3 + l^4n^2Sb^2Sc^2z^3 + 6l^2n^4Sb^2Sc^2z^3 + n^6Sb^2Sc^2z^3 + \\
& 2l^2m^2n^2SaSc^2z^3 + 2m^4n^2SaSc^2z^3 + 2l^2n^4SaSc^2z^3 + 2m^2n^4SaSc^2z^3 + 2l^4n^2SbSc^2z^3 + \\
& 2l^2m^2n^2SbSc^2z^3 + 2l^2n^4SbSc^2z^3 + 2m^2n^4SbSc^2z^3 + l^4n^2Sc^3z^3 - 2l^2m^2n^2Sc^3z^3 + m^4n^2Sc^3z^3
\end{aligned}$$

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Message: #38
Date: 17/6/2013 3:14:30
From: Chris
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(April 26, 2013)

Dear Eckart,

This looks like a very interesting result.

Could you share with us (after your holiday) how you came to this result?

Do you have a CT-equation of this cubic?

Then I can try to draw the cubic.

Best regards,

Chris

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Message: #39

Date: 17/6/2013 3:21:56

From: Chris van Tienhoven

Subject: Varignon points (temporarily named)

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami
(April 22, 2013)

Dear friends, Welcome! Bernard. I am sorry that this is not about Cardioid here.

[1] Varignon points: Given a quadrangle $P_1P_2P_3P_4$, the midpoints M_{ij} of P_iP_j and the midpoints T_{ikj} of $M_{ij}M_{jk}$, then 4 circles through the following 3 points have a common point.

- (1) $T_{134}M_{13}T_{243}$, $T_{243}M_{24}T_{132}$, $T_{132}M_{13}T_{241}$, $T_{241}M_{24}T_{134}$.
- (2) $T_{134}M_{24}T_{243}$, $T_{243}M_{13}T_{132}$, $T_{132}M_{24}T_{241}$, $T_{241}M_{24}T_{134}$.
- (3) $T_{123}M_{12}T_{342}$, $T_{342}M_{34}T_{124}$, $T_{124}M_{12}T_{341}$, $T_{341}M_{34}T_{123}$.
- (4) $T_{123}M_{34}T_{342}$, $T_{342}M_{12}T_{124}$, $T_{124}M_{34}T_{341}$, $T_{341}M_{12}T_{123}$.
- (5) $T_{234}M_{23}T_{142}$, $T_{142}M_{14}T_{231}$, $T_{231}M_{23}T_{143}$, $T_{143}M_{14}T_{234}$.
- (6) $T_{234}M_{14}T_{142}$, $T_{142}M_{23}T_{231}$, $T_{231}M_{14}T_{143}$, $T_{143}M_{23}T_{234}$.

[2] Example

(1) Case(1) in CT- environment:

Given a quadrangle $P_1P_2P_3$, the midpoints M_{12} , M_{23} , M_{34} , M_{14} , M_{13} and M_{24} , the midpoints T_{132} , T_{134} , T_{241} and T_{243} of $M_{12}M_{23}$, $M_{14}M_{34}$, $M_{12}M_{14}$ and $M_{23}M_{34}$, then 4 circles through $T_{134}M_{13}T_{243}$, $T_{243}M_{24}T_{132}$, $T_{132}M_{13}T_{241}$ and $T_{241}M_{24}M_{134}$ have a common point. See the attached pdf. See Mathematica-file for the coordinates. My idea in Mathematica is as follows. We know that M_{13} is the common point of the circles through $T_{134}M_{13}T_{243}$ and $T_{132}M_{13}T_{241}$. We compute the centers of these circles. The other intersection is given as the reflection point of M_{13} across the line through the centers of these circles.

Similarly we compute the other intersection as the reflection point of M_{24} . The former other intersection coincides with the latter other one. Since the coordinates are too complicated to be written here, the above Mathematica-file is attached.

(2) Case(1) in DT- environment: Change $pt1\{0,1,0\}$, $pt2\{0,0,1\}$, $pt3\{1,0,0\}$ and $pt4\{p,q,r\}$ to $\{p,-q,r\}$, $\{p,q,-r\}$, $\{-p,q,r\}$ and $\{p,q,r\}$.

(3) Case(2) in CT and DT- environment: Interchange M_{13} and M_{24} . Quadrilateral other cases (3), (4), (5) and (6) are similarly computed.

P. S. I received the email from Chris that he received my message personally, but there was not my message in the mailing list of the group. So I post it again. If you receive the 2nd message of the same contents personally, I wish that you do not mind erasing it.

Best regards, Seiichi Kirkami.

Message: #40
Date: 17/6/2013 3:26:09
From: Chris
Subject: Varignon points (temporarily named)

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven (May 12, 2013)

Dear Seiichi,
I checked your calculations and they look alright to me.
There should be 6 Varignon Points. Is that correct?
What can be done more with these 6 points?
Are they cyclic?
What is their Centroid?
Best regards,
Chris

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Message: #41
Date: 17/6/2013 3:28:42
From: Chris
Subject: New property of QA-Tf2 (Involuntary Conjugate)

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 13, 2013)

Dear friends,
the six conics through
... two vertices of the quadrangle,
... two diagonal intersections (not collinear with the vertices)
... and a point P
have a second common point,
which is the QA-Tf2 image of P.
Best regards Eckart

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Message: #42
Date: 17/6/2013 3:31:24
From: Chris
Subject: 3 LSD-points on Newton line in QL-environment

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami
(May 5, 2013)

Dear friends,

Given a quadrilateral $L_1L_2L_3L_4$ and their intersections s_{12} , s_{13} , s_{14} , s_{23} , s_{24} and s_{34} , the LSD point P_1 from s_{14} , s_{12} , s_{23} , s_{34} , the LSD P_2 point from s_{14} , s_{13} , s_{23} , s_{24} and the LSD point P_3 from s_{12} , s_{13} , s_{24} , s_{34} are on Newton line of the quadrilateral.
Best regards,
Seiichi Kirikami

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Message: #43
Date: 17/6/2013 3:33:12
From: Chris
Subject: 3 LSD-points on Newton line in QL-environment

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(May 12, 2013)

Dear Seiichi,

This result is not quite unexpected.

The 3 LSD-points (points with Least Sum of Squared Distances to the 4 designated points) are the LSD-points of the 3 Component Quadrilaterals of the Reference Quadrilateral.

It is known that the LSD-point of 4 points is the Centroid QA-P1.

It is also known that the Centroids of the 3 Component Quadrilaterals lie on the Newton Line.

So the 3 LSD-points of your 3 triangles coincide with the 3 Centroids of the 3 QL-Component Quadrilaterals and lie on the Newton Line.

Best regards,
Chris

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Message: #44

Date: 17/6/2013 3:34:48

From: Chris

Subject: 3 LSD-points on Newton line in QL-environment

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami

(May 13, 2013)

Dear Chris,

Thank you for your information. After I computed LSD point to 6 points of the quadrilateral, I found that it was QL-P12 on Newton line. Then I knew what you said below.

I found a weighted LSD problem in a book written in Japanese. If I can formulate it for EQF, I will post it.

Best regards,

Seiichi Kirikami.

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Message: #45

Date: 17/6/2013 3:41:04

From: Chris van Tienhoven

Subject: Conic poles of a quadrangle wrt its circumcubics

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 15, 2014)

Dear friends,

consider a quadrangle, a circumcubic and the pencil of circumscribed conics with further intersections P and Q with the cubic.

The lines PQ have a common point on the cubic, called "conic pole" of the quadrangle.

(H.M.Cundy and C.F.Parry. Some cubic curves associated with the triangle. Journal of Geometry 53 (1995), p. 45).

Examples:

QA-Cu1: QA-Tf2 of QA-P4

QA-Cu2: QA-P17

QA-Cu3: QA-P16

QA-Cu4: QA-P18

QA-Cu5: QA-P20

QA-Cu6: QA-P22.

The first five cubics are pivotal isocubics of QA-DT wrt the isoconjugation QA-Tf2.

For these cubics the conic pole of the quadrangle is the common point of the tangents in the vertices of QA-DT.

With this property the points are already mentioned in EQF.

New will be the conic pole QA-P22 for a quadrangle and its circumcubic QA-Cu6.

Best regards Eckart

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Message: #46

Date: 17/6/2013 3:42:37

From: Chris

Subject: Conic poles of a quadrangle wrt its circumcubics

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 16, 2014)

Dear friends,

sorry, I have forgotten the following property:

For a pivotal circumscribed isocubic of a quadrangle (wrt QA-DT and QA-Tf2) the conic pole is the QA-Tf2 image of the pivot.

Best regards Eckart

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Message: #47

Date: 17/6/2013 3:45:10

From: Chris van Tienhoven

Subject: What is the necessary and sufficient condition of quadrifigure funct

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami

(may 17, 2013)

Dear friends,

After a while when I knew EQF, I wondered what the necessary and sufficient condition of quadrifigure function would be. As it seems to me that there is no literature about this theme, I post this subject as a question.

An example of the necessary condition of quadrangle: If the quadrangle is a square, $a=1$, $b=1$, $c=\sqrt{2}$ and $p=1$, $q=1$, $r=-1$, then the quadrangle point has the absolute coordinates

$\{1/2, 1/2, 0\}$.

If we denote the quadrangle function by

$\{f(a,b,c; p,q,r), g(a,b,c; p,q,r), h(a,b,c; p,q,r)\}$,

then the necessary condition is $h(1,1,\sqrt{2}; 1,1,-1)=0$.

So $h(p,q,r)=p+q+2r$ (QA-P1) can be a quadrangle function because $h(1,1,-1)=0$. $h(p,q,r)=p+q+3r$ is not a quadrangle function. I

would like to know the form of quadrifigure function, especially, for the sufficient condition.

Best regards, Seiichi Kirikami.

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Message: #48

Date: 17/6/2013 3:47:41

From: Chris van Tienhoven

Subject: EQF-Note 2013-05-17: A Cubic, a Circle and a Line wrt a Quadrilatera

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 17, 2013)

Dear friends,

there is a short essay about a new line for a quadrilateral in the attachment.

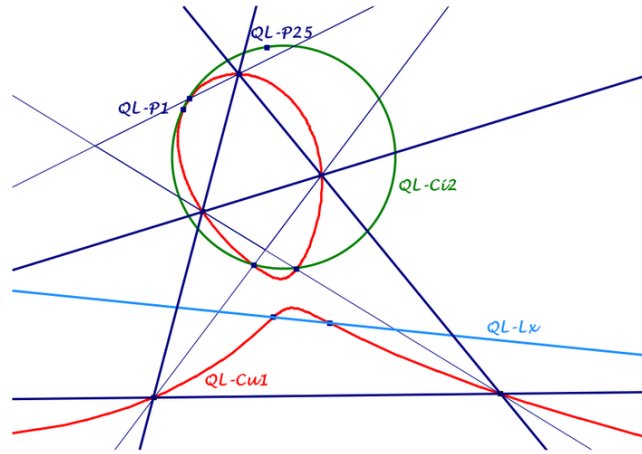
Best regards Eckart

EQF-Note 2013-05-17

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Cubic, a Circle and a Line wrt a Quadrilateral

The cubic will be QL-Cu1, which contains the foci of inscribed conics, the circle will be QL-Ci2, the nine-point circle of the QL-diagonal triangle, and the line will be the Clawson-Schmidt Conjugate of the circle QL-Ci2. This line is not mentioned in EQF up to now. – Reference triangle for barycentric coordinates is QL-DT.

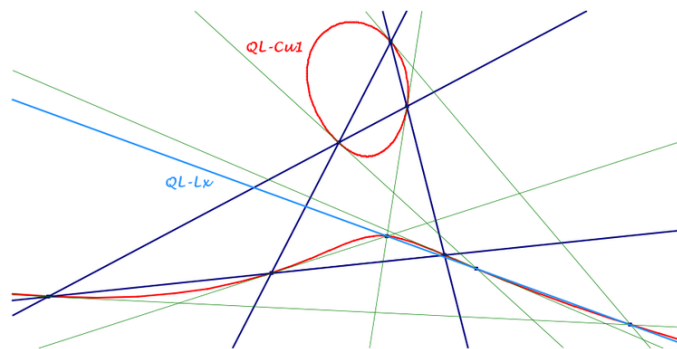


We start with the cubic *QL-Cu1*. The vertices of the orthic triangle of *QL-DT* lie on this cubic as well as on the nine-point circle *QL-Ci2* of *QL-DT*. The Clawson-Schmidt Conjugate *QL-Tf1* of *QL-Ci2* is a line *QL-Lx*, which cuts the cubic in the *QL-Tf1* images of the vertices of the orthic triangle. Other points of *QL-Lx* are the *QL-Tf1* images of the Miquel Point *QL-P1* and *QL-P25*.

The equation of the line *QL-Lx*:

$$\sum_{cycl} (l^4 a^2 (S^2 + 3S_B S_C) + m^4 b^4 S_A + n^4 c^4 S_A + 2m^2 n^2 S_A (S^2 - S_A^2) + 2l^2 m^2 (a^2 b^2 S_A + 2S^2 S_C) + 2l^2 n^2 (a^2 c^2 S_A + 2S^2 S_B)) x = 0$$

But there is a further connection to the cubic $QL-Cu1$: The tangents in opposite points of the quadrilateral intersect on the cubic in three collinear points on $QL-Lx$ (which are the $QL-Tf1$ images of the vertices of the orthic triangle).



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2013-05-17.pdf

Message: #49

Date: 17/6/2013 3:50:51

From: Chris

Subject: EQF-Note 2013-05-17: A Cubic, a Circle and a Line wrt a Quadrila

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 17, 2013)

Dear friends,

sorry, just found out: The new QL-line is the connection of the three collinear QG-P18 for the quadrigon components, which lie on the cubic QL-Cu1.

Best regards Eckart

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Message: #50

Date: 17/6/2013 3:54:09

From: Chris van Tienhoven

Subject: a quadrangle identity derived from a tetrahedron identity

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami

(May 24, 2013)

Dear friends,

The following may be well known, but I post it because it relates a tetrahedron to a quadrangle.

Given a tetrahedron $P_1P_2P_3P_4$ and the midpoints of its edges

P_1P_2 , P_2P_3 , P_3P_1 , P_1P_4 , P_2P_4 , P_3P_4 are denoted by m_{12} , m_{23} , m_{31} , m_{14} , m_{24} , m_{34} respectively.

Then $P_1P_2^2 + P_2P_3^2 + P_3P_1^2 + P_1P_4^2 + P_2P_4^2 + P_3P_4^2$
 $= 4 \cdot (m_{12}m_{34}^2 + m_{23}m_{14}^2 + m_{31}m_{24}^2).$

If we compress the tetrahedron to the quadrangle, then we obtain an identity of quadrangle.

Best regards,

Seiichi Kirikami.

Show trimmed content

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Message: #51

Date: 17/6/2013 3:57:24

From: Chris

Subject: a quadrangle identity derived from a tetrahedron identity

COPY FROM FORMER GOOGLE GROUP by Seiichi Kirikami

(May 27, 2014)

I add 2 other relations.

(1) QA-environment: Given a quadrangle $P_1P_2P_3P_4$, a point (QA-P1) and their midpoints M_{12} , M_{23} , M_{31} , M_{14} , M_{24} , M_{34} , SD(square distances) from P to P_i and M_{ij} has the relation:

$$\begin{aligned} & PP_1^2 + PP_2^2 + PP_3^2 + PP_4^2 \\ &= 2 * (PM_{12}^2 + PM_{23}^2 + PM_{31}^2 + PM_{14}^2 + PM_{24}^2 + PM_{34}^2). \end{aligned}$$

(2) QL-environment: Given a quadrilateral $L_1L_2L_3L_4$, a point (QL-P26) and pedal points P_i on L_i , SD(square distances) from P to P_i and of P_iP_j has the relation:

$$\begin{aligned} & P_1P_2^2 + P_2P_3^2 + P_3P_1^2 + P_1P_4^2 + P_2P_4^2 + P_3P_4^2 \\ &= 4 * (PP_1^2 + PP_2^2 + PP_3^2 + PP_4^2). \end{aligned}$$

Case(1) is from my previous post. I learned Case(2) from Eckart.

Best regards,

Seiichi Kirikami.

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Message: #52
Date: 17/6/2013 4:06:03
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(April 26, 2013)

Dear friends,

Chris has asked for a CT-equation for the cubic for the centers of the cardioids tangent to 4 lines.

It was a great surprise: The equation is very simple, see in the Mathematica file in the attachment.

Further: The asymptotes cut under an angle of 60° !

If Chris can draw the cubic, he can say, whether the asymptotes intersect in one point.

Perhaps this point lies on the cubic.

Best regards Eckart

Cardioids tangent to 4 lines

**The centers of cardioids tangent to four lines
lie on cubics with the following equations :**

Barycentric coordinates : Reference triangle

L1 (1, 0, 0), L2 (0, 1, 0), L3 (0, 0, 1),
fourth line L4 (1, m, n)

$$\begin{aligned} \text{card}[\{x_, y_, z_\}] := & b^2 c^2 l (-b^2 (-1+m) (m-n) - c^2 (-1+n) (-m+n) + a^2 (1m+1n-mn)) x^3 + \\ & a^2 c^2 m (-a^2 (1-m) (1-n) - c^2 (-1+n) (-m+n) + b^2 (1m-1n+mn)) y^3 + \\ & a^2 b^2 n (-a^2 (1-m) (1-n) - b^2 (-1+m) (m-n) + c^2 (-1m+1n+mn)) z^3 + \\ & 3 a^2 b^2 c^2 (1x+my) (1x+nz) (my+nz) \end{aligned}$$

The asymptotes intersect with angles of 60° .

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Card-nb.pdf

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Message: #53

Date: 17/6/2013 4:09:40

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(April 26, 2013)

Dear Eckart,

Wonderful this equation in CT-format!

It is not surprising it is more simple in CT-format, because the problem is about 4 lines and there's no relationship with the Diagonal Triangle.

Attached is the picture of the Cardioids Cubic.

The 3 asymptotes concur in an unknown point.

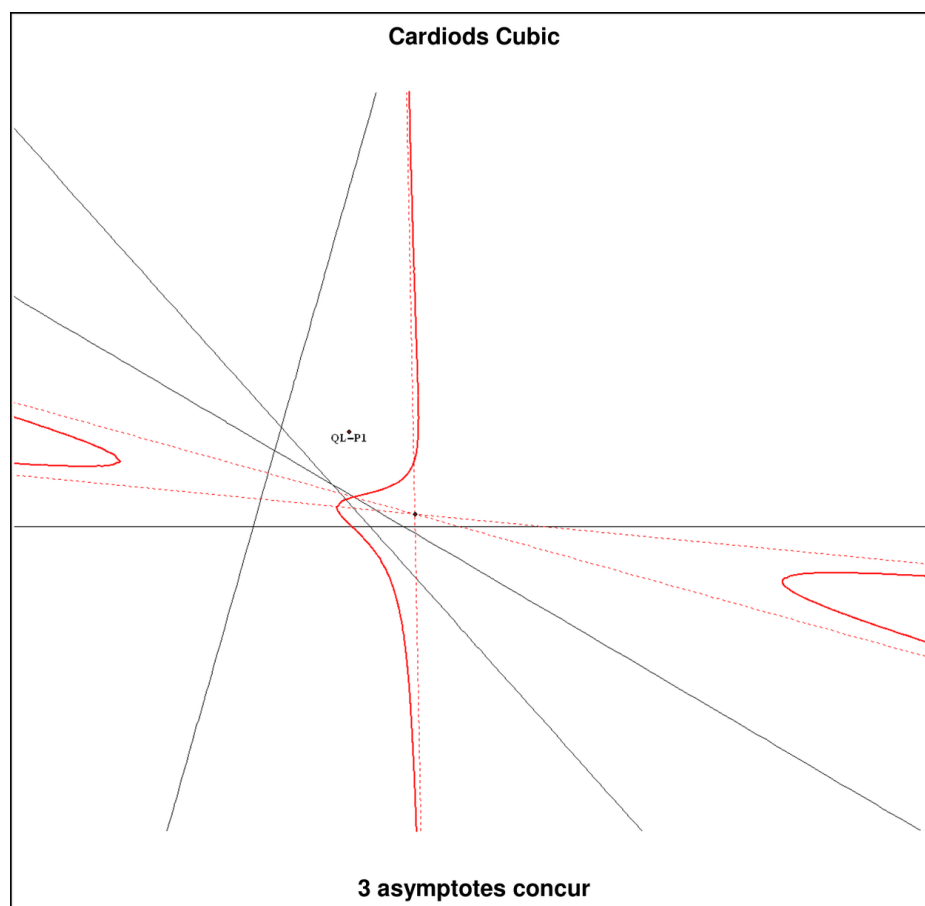
I did NOT see the asymptotes cutting under an angle of 60° .

This far I didn't manage to calculate the coordinates of this interesting point.

Maybe someone else can?

Can you please tell me how you discovered the cubic and what exactly is the relationship with the 8 Cardioids?

Chris



Cardioids Cubic.pdf

Message: #54
Date: 17/6/2013 4:12:35
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

Dear Eckart and Chris,
Thank you for your efforts!
I wasn't able to read Eckart's attached file and I discovered the drawing of the cubic with surprise!
In fact, I join Chris in his questions: where does this cubic come from and what's the link to the 8 searched cardioids?
I'm sure there are 8 cardioids tangent to 4 lines and only but one for oriented lines.
May be the answer to a simple question could help: if I have a cardioid and 4 tangent lines, is it possible to find the other 7 cardioids tangent to the same lines?
As all cardioids are similar to one another, we need for each cardioid 4 parameters, for exemple coordinates of center and cusp, or center and dopple tangent, or coordinates of the center of similarity, angle and ratio of the similarity with another known cardioid ...
And again, the 8 centers belong to 4 groups of 3×3 lines (one group for each reference triangle); is it possible to calculate the equation of those lines and find the centers as intersection of the 4 groups (meaning a maximum of 81 possible points ...)?
Anyhow, it's not the end of the question: if we had the centers, we still would have to find then the corresponding cusps in order to draw the cardioids tangent to the 4 lines; it's another question, for which the answer could be useful.
In the same way, the cusps belong to 4 groups of 3×3 cubics (one for each reference triangle) ...
Best regards
Bernard

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Message: #55

Date: 17/6/2013 4:18:11

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 7, 2013)

Dear friends,

... back from holiday: You ask, how I got the equation of the cubic for the centers of the cardioids tangent to four lines. I have described the way already in my first message wrt this theme. The following file was damaged, so you find an update in the attachment with a possibility to calculate the cusp for a given center.

Best regards Eckart

PS. Dear Bernard, can you give me a short description of the deltoid mentioned in your paper?

Thank you Eckart

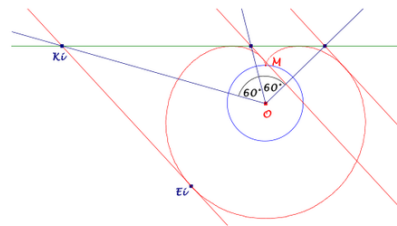
Cardioids tangent to 4 lines

- A cubic for the centers and a formula for the cusps -

There are two properties for tangents of cardioids:

„Die Tripel paralleler Tangenten der Kardioiden schneiden die Doppeltangente in Tripeln von Punkten, die vom Mittelpunkt des Grundkreises aus unter dem Winkel 60° erscheinen.“

H. Schmidt: Ausgewählte höhere Kurven.
Kesselringsche Verlagsbuchhandlung Wiesbaden, 1949, S.122.



Not to be found there is a further property:

For the three parallel tangents of a cardioid the sum of orientated distances from the center O is zero.

Let O be the center of the cardioid, M its cusp, t its bitangent and l_0, l_1, l_2 parallel tangents.

... the angle property gives the intersections of the tangents l_0, l_1, l_2 with the bitangent t .

... the distance property gives an equation for further interpretations.

The necessary calculations are very hard and the terms extreme extensive! Using barycentric coordinates the reference triangle will be the triangle of three sidelines of the quadrilateral with a fourth line $l(l, m, n)$. The balance of orientated distances can be done with the intersections of the parallels with a sideline of the reference triangle.

The three equations for the sidelines of the reference triangle lead to a calculation of the cusp M for a center O . This is also valid for inscribed cardioïds of a triangle.

**A cardioïd tangent to three lines with center $O(u:v:w)$
has the cusp $M(b^2c^2u^3 : c^2a^2v^3 : a^2b^2w^3)$.**

The four equations for the sidelines of a quadrilateral lead to a cubic, locus for the centers of inscribed cardioïds.

**The centers of cardioïds tangent to four lines lie on a cubic
with the equation**

$$\sum_{cycl} b^2c^2l(-a^2(lm + ln - mn) + b^2(m - n)(m - l) + c^2(n - l)(n - m))x^3 \\ = 3a^2b^2c^2(lx + my)(my + nz)(nz + lx) .$$

The asymptotes of this cubic intersect with angles of 60° .

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-05-07 .pdf

Message: #56
Date: 17/6/2013 4:19:40
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 9, 2013)
Dear friends,
in addition to my last message:
The asymptotes of the cubic for the centers of cardioids tangent
to 4 lines
are parallel to the axes of the deltoid tangent to 4 lines (see
Bernard's message).
Best regards Eckart

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Message: #57
Date: 17/6/2013 4:24:51
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(May 12, 2013)
Dear Eckart, dear friends,

What a wonderful cubic !!!
What a wonderful result that the asymptotes of the cubic are
parallel to the axes of the Deltoid !!!
I also have a nice result: the asymptotes of the cubic concur
in the Miquel Point QL-P1!
Let me tell how I proceeded using Eckart's results:

1. When we have in a triangle a Cardioid with center $(u: v: w)$
tangent to all sidelines,
then the cusp has coordinates $(b^2c^2u^3: a^2c^2v^3: a^2b^2w^3)$.
Not every point however in the Triangle Plane can function as
the center of a tangent Cardioid.
Still there are infinite point in the Triangle Plane that can
function as the center of a tangent Cardioid.
2. To diminish the number of points with possibility to function
as the center of a tangent Cardioid we introduce a 4th line that
also has to be tangent to the Cardioid. Now we deal with a
Quadrilateral instead of a Triangle.

3. The Quadrilateral has 4 Component Triangles (CT). Given a center P , then for each Component Triangle the calculation of the cusp has to deliver the same point. I did draw this in Cabri and indeed when 2 CT-cusps concur, automatically all 4 CT-cusps concur.

4. When we equalize the calculated coordinates of the cusps in 2 Component Triangles with a given point P , then your Cubic rolls out.

5. It is important to realize that this cubic only show the points $P(u: v: w)$ for which the transformed points $(b^2c^2u^3: a^2c^2v^3: a^2b^2w^3)$ for the 4 Component Triangles concur. Not each point on the cubic is the center of a tangent Cardioid. However when there is a center of a tangent Cardioid then it should lie on this cubic.

6. Enclosed you will find a new picture of the Cubic and indeed it looks like the asymptotes concur in a point and when I am not wrong this point is the Miquel Point $QL-P1$!!!
I checked it in several numerical examples and the coordinates of $QL-P1$ and the Asymptote Crosspoint are exactly the same.

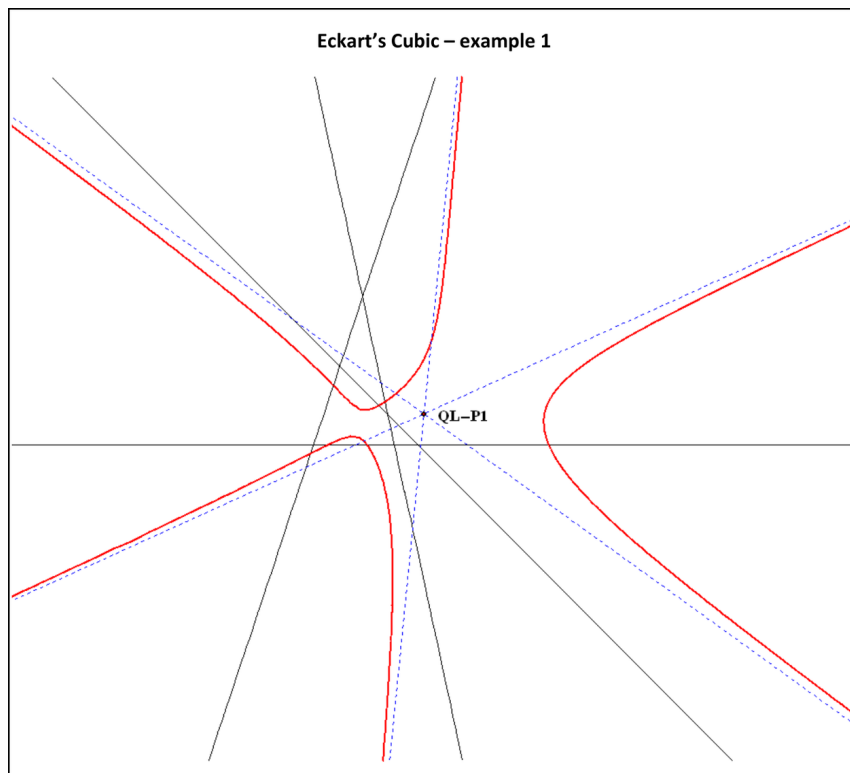
7. Moreover the picture also shows -like Eckart told- the asymptotes cross under angles of 60° degrees.

8. Just like Eckart I already earlier had calculated the Deltoid however again the equations of the axes are very long. Maybe I can try to include the Deltoid in the same picture as Eckarts cubic later.

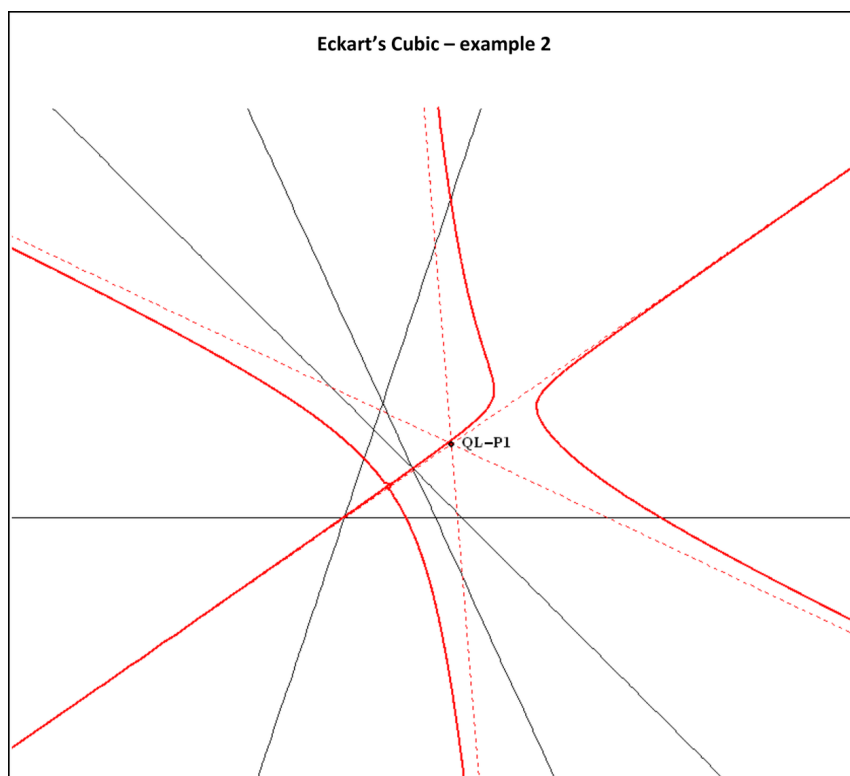
Eckart, some questions:

- Could you tell me how you calculated the cusp coordinates $(b^2c^2u^3: a^2c^2v^3: a^2b^2w^3)$?
- Did you find simple equations of the asymptotes or the infinity points of the asymptotes? I found the equations however far from simple.
- How did you prove the asymptotes crossing under angles of 60° degrees?
- How did you calculate the Deltoid and its axes?

Best regards,
Chris



EckartsCubic.pdf



EckartsCubic.pdf

Message: #58

Date: 17/6/2013 4:27:21

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt

(May 12, 2013)

Dear Chris, dear friends,

thank you very much Chris for the new picture of the cubic for the centers of cardioids tangent to four lines.

It will be a nice result, that the asymptotes intersect in QL-P1!

To your questions:

The coordinates for the cusps are a consequence of the four equations mentioned in my last message, describing the bitangent with O and M .

If you take the infinity point $(1, f, -1-f)$, you get with the cubic an equation for f ,

which leads to the crossing angles of 60° .

The properties of the deltoid I only have proved with Cabri, using the description of Bernard.

The real reason for my message is, there will be 27 cardioids tangent to four lines

. The centers of the inscribed cardioids of the reference triangle lie on 3 sets of 3 parallel lines.

Each line has 3 points of intersection with the cubic for the centers,

so there are 27 possible centers.

I have calculated an example (see the Mathematica file in the attachment), drawing most of the cardioids with Cabri

(using the calculated center and cusp) and it seems alright! The four groups of three sets of three parallel lines

for the centers of the cardioids can be found.

Best regards Eckart

Centers of Cardioids Tangent to 4 Lines, an Example (12.05.2013)

Reference triangle and fourth line:

$a:=6;b:=5;c:=4;l:=-1;m:=-2;n:=2$

For the reference triangle there are three sets of three parallel lines for the centers of inscribed cardioids, intersecting with angles of 60° (see Bernard's message).

For a quadrilateral the centers of inscribed cardioids lie on a cubic (see my last messages).

Each line cuts the cubic in three points, so there are $9 \times 3 = 27$ centers for inscribed cubics.

Centers of cardioids:

01:={-59.78173625283743°,42.165047287904294°,1};

02:={-0.203511113184554°, -0.332074102406412°,1};

03:={0.060606252681044546°, -0.48627193182838463°,1}

04:={0.46707804440590894°, -0.7759758006335196°,1};

05:={1.101895148780922°,1.071079720158494°,1};

06:={1.4669799267249983°,0.8246623879358206°,1};

07:={1.576555916219293°,0.6206153569597764°,1};

08:={3.1347898750709837°, -2.2810514619917557°,1}

09:={6.800318699532346°,7.196195725014429°,1}

010:={-118.11486997562572°,0°,1}

011:={-16.291106305040064°, -52.702072298551705°,1}

012:={-5.630704830920701°,3.990640240900943°,1}

013:={-2.295603569683266°, -0.04916986488667343°,1}

014:={-2.0265143243867794`, -0.0026242615925480264`, 1}

015:={-1.7369505064696762`, 1.496911448196238`, 1}

016:={-1.4284127080083318`, 0.03475928403102694`, 1}

017:={-1.1406814741361457`, 1.2731330086853312`, 1}

018:={-0.9699454275148954`, 1.2321324378106324`, 1}

019:={-0.5496842851246039`, -0.1657206649926386`, 1}

020:={0.3951953482797242`, -3.308535439200701`, 1}

021:={0.5219728461181129`, 1.158069062394834`, 1}

022:={0.6657709472051169`, -0.962307875435281`, 1}

023:={0.8156503060159306`, -2.077717398028317`, 1}

024:={1.6346115236594532`, -0.5806351452160409`, 1}

025:={2.0879758424326775`, -1.4692625981628826`, 1}

026:={2.340553042869985`, -1.683799644726599`, 1}

027:={22.888718892371514`, -3252.4707475677624`, 1}

Cusps of cardioids

M1:={-94956.14051740553`, 47977.51491845804`, 1};

M2:={-0.003746121607258228`, -0.023436081318746842`, 1};

M3:={0.0000989395152867481`, -0.07358979293768209`, 1}

M4:={0.045288281516972485`, -0.29903671073929927`, 1};

M5:={0.5946183329443193`, 0.7864033653062752`, 1};

M6:={1.4031044294890394`, 0.35892898907724097`, 1};

M7:={1.7415887784074824`, 0.15298453388442712`, 1};

M8:={13.691240144445958`, -7.596004686931817`, 1}

M9:={139.76720536496666`, 238.5002698914323`, 1}

M10:={-732371.1205364267`, 0., 1}

M11 := {-1921.6275421056544`, -93683.48787269817`, 1}
M12 := {-79.34247922520326`, 40.673140482795155`, 1}
M13 := {-5.376605303659266`, -0.00007608114126702285`, 1}
M14 := {-3.6988482617096423`, -1.1566483617190364`*^-8, 1}
M15 := {-2.3290553665460507`, 2.146684910110426`, 1}
M16 := {-1.295324553091914`, 0.000026877720962941166`, 1}
M17 := {-0.6596455644224091`, 1.3206912754310147`, 1}
M18 := {-0.40556398529447424`, 1.1971598622027408`, 1}
M19 := {-0.07381717919302891`, -0.002912795372660838`, 1}
M20 := {0.02743171461076124`, -23.178607790321074`, 1}
M21 := {0.06320642318203891`, 0.9939930618404149`, 1}
M22 := {0.1311571589284251`, -0.5703245862452914`, 1}
M23 := {0.2411734479114258`, -5.740363617722843`, 1}
M24 := {1.9411591827455632`, -0.12528236207403975`, 1}
M25 := {4.045702005879118`, -2.0299168273264825`, 1}
M26 := {5.698662609732382`, -3.055281389233865`, 1}
M27 := {5329.44442871325`, -2.2020144862990997`*^10, 1}

Four groups of three sets of three parallel lines (not completely):

group 1: set1: koll[O1,O5,O6] // koll[O12,O13,O20] ...
 set2: koll[O2,O3,O8] ...
 set3: koll[O4,O7,O9] ...

group 2: set1: koll[O4,O20,O21] // koll[O2,O17,O23] ...
 set2: koll[O7,O13,O14] // koll[O3,O19,O26] ...
 set3: koll[O1,O18,O24] // koll[O8,O24,...] ...

group3: set1: koll[O2,O13,O24] // koll[O4,O16,O25] ...
 set2: koll[O5,O15,O21] // koll[O7,O17,O18] // koll[O3,O12,O22]
 set3: koll[O8,O20,...] // koll[O9,O26,...] ...

group4: set1: koll[O21,O24,O26] // koll[O6,O7,O8] ...
 set2: koll[O1,O3,O4] // koll[O2,O22,...] ...
 set3: koll[O2,O5,O9] // koll[O18,O19,O20] ...

Eckart Schmidt
<http://eckartschmidt.de>

2013-05-13-nb.pdf

Message: #59
Date: 17/6/2013 4:27:49
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(May 12, 2013)
Dear Eckart,

I am not quite surprised.
I already had calculated in a triangle the locus of possible centers of cardioids that were tangent to the sides of the triangle.
Two loci came out, one of the 11th degree and one of the 13th degree!
Pretty complicated. But in Mathematica we have the possibility to draw these functions.
When I drew them I saw overlapping structures.
I also saw several lines under angles of 60 degrees but also complicated curved lines.
So the functions probably could have been factorized in lines and another part.
However even Mathematica could not deal with these huge equations.
That's why I am not quite sure yet if your 27 solutions are the only solutions.
But maybe you have good reasoning for your 27 centers.
Let me know please.
These 27 points can be divided in 9 points per quadrigon?

Best regards,
Chris

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Message: #60
Date: 17/6/2013 4:34:30
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(May 23, 2013)

Dear friends,

Thank you for the beautiful information about the amazing
QL-Cardioids.

It is a pleasure developing it together.

Although most of it comes from Eckart.

I am trying right now to find the equation of a cardioid in
barycentrics.

I already managed to find an equation for a very specific case
in reasonable format (4th degree).

Eckart: can you give me more information about the 27 lines?

How you found them? Do you have CT-coefficients? If so, could
you send them to me in a Mathematica-notebook-file (even when
they are huge)?

Tomorrow morning I leave for 2 days then I will be back for half
a day and leave again for 5 days.

So I can't react during these days.

As soon as I find time I will try to make a picture of the
Cardioids.

Best regards,

Chris

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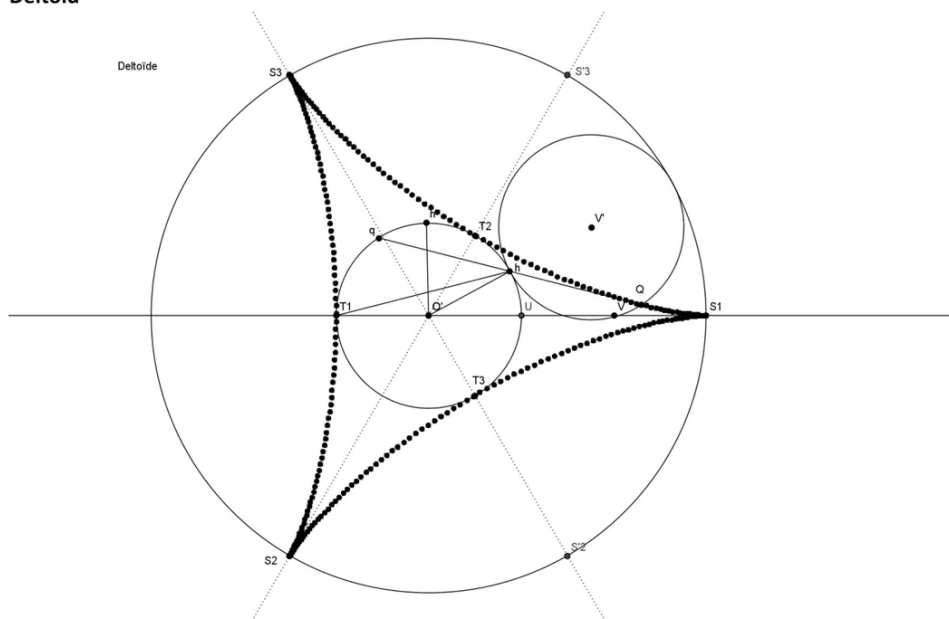
Message: #61
Date: 17/6/2013 4:35:16
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Bernard Keizer
(May 21, 2013)
Dear friends,

I had a lot of problems in the last days, technical (I change my computer in the Dordogne and had great difficulties to get back all my files) as well as familar ...
It seems, you have worked hard during this time!
I'm still not able to read Eckart's attached files ...
I was intrigued by the barycentric coordinates of the cusp, I have found an explanation (see attached file).
I have tried to put in the same file all I know about deltoïd (to answer Eckart's question) and cardioïd.
I'm now totally convinced with the 27 cardioïds of Eckart, it's simply great ! The mistake was that a line to a cardioïd has not 2, but 3 possible directions ($m^\circ 3\pi$).
I'm really impatient now to see some drawings and I hope you will be able to show the possible cardioïds.
I still have a question about what Morley and Bogomolny mean by the unique cardioïd tangent to the 4 directed lines, the direction being given by the deltoïd; may be, it will be obvious with your calculations and drawings ...

Best regards
Bernard

Deltoid



qQ is one tangent to the deltoid : $T_1q = -2 T_1h$ (this goes for T_2 and T_3) : h is the primary point, q the secondary point and Q the symmetric of q in h is the contact point ; the tangent is the symmetric of T_1h in the perpendicular to the axis in h and the angle of the tangent with the axis is $\pi - \theta$.

If $\theta = OT_1h$, $UOh = T_1hq = 2\theta$, $Ohq = Oqh = 3\theta$ and $T_1Oq = 4\theta$.

The distance from O to the tangent is $r \cos 3\theta$.

If we consider 4 lines L_i tangent to the deltoid, the QL formed by the 4 primary points h_i is cyclic by construction ; we have $h_iOh_j = -2(D_i, D_j)$ and $h_ih_kh_j = h_ih_lh_j = -(D_i, D_j)$.

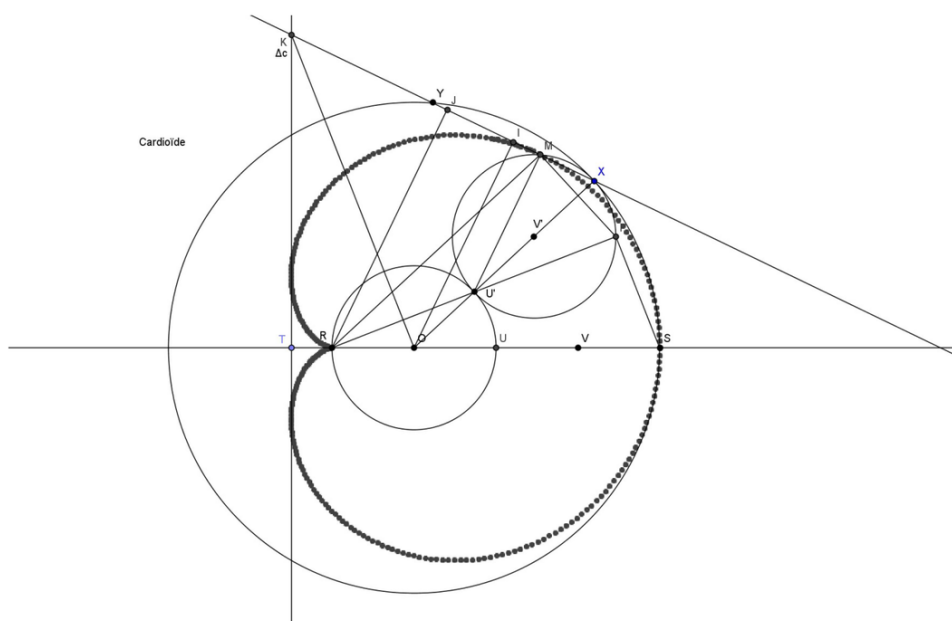
To find the center O , we have to inscribe this QL in the 4 lines ; each triangle $h_jh_kh_l$ is inversely similar to the reference triangle T_i built by the lines D_j , D_k and D_l or directly similar to its medial triangle. For each triangle T_i , all the triangles $h_jh_kh_l$ are similar, the center of similarity being the center of the circumscribed circle to the triangle T_i and the locus of the centers of the circumscribed circles to the triangles $h_jh_kh_l$ is the perpendicular bisector of the segment joining the center of similarity and its isogonal conjugate (the orthocenter of the triangle).

The center of the inscribed circle in the deltoid is therefore on the 4 perpendicular bisectors of the Euler segments ; that's the Kantor-Hervey point (theorem of Hervey) and the inscribed circle of the deltoid is in fact the Hervey circle ...

All that is well known, we have 4 tangents L_i and the symmetric of the axis of the parabola tangent to the 4 lines in the Newton Line (asymptote of the van Rees cubic) is a 5th tangent to the deltoid.

The important thing here is to note that there is only but one way to inscribe a cyclic QL of a given shape on 4 lines.

Cardioid



XY is a tangent to the cardioid, X is the primary point, Y, the symmetric of S in OX is the secondary point and M, at the third of XY next to X, is the contact point. O is the center of the cardioid, R its cusp and S its apse ; the angle between the tangent and the axis is $\pi/2 + 3\theta$.

If the barycentric coordinates of the center are (u,v,w) , the barycentric coordinates of the cusp are $(u^3b^2c^2, v^3a^2c^2, w^3a^2b^2)$; if we deal with the trilinear coordinates used by Clark Kimberling for the triangle, the coordinates of the center are $(u/a, v/b, w/c)$ and those of the cusp $(u^3/a^3, v^3/b^3, w^3/c^3)$.

If $\angle TKO = \theta$, $\angle SRU' = \theta$, $\angle SOX = 2\theta$ and $\angle SOI = \angle SRJ = 3\theta$.

$OX = 3r$ and the distance from O to the tangent is $OI = 3r \cos\theta$; $RU' = 2r \cos\theta$, $RS = 4r$, $RM = 4r \cos^2\theta$ and the distance from R to the tangent is $RJ = 4r \cos^3\theta$, which explains the calculation ...

If we consider 4 lines L_i tangent to the cardioid, the QL formed by the 4 primary points X_i is always cyclic, but we have this time $X_i O X_j = 2/3 (D_i D_j)$ and $h_i h_k h_j = h_i h_l h_j = 1/3 (D_i D_j)$.

The question is now to determine the number of QL from different shapes to inscribe in the 4 lines : it seems in fact that there are 27 possible QL (and again one solution corresponding to one QL). Each angle is modulo $\pi/3$ and we have to choose independently 3 of the 4 angles (for example (Oh_1, Oh_2) , (Oh_2, Oh_3) and (Oh_3, Oh_4)).

For one given triangle, the locus of the centers of the inscribed cardioids is one set of 3×3 lines (the mean direction of the 3 sides of the triangle modulo $\pi/3$). The locus of the cusps is one set of 3×3 cubics. For the 4 lines, the centers are at the intersection of 4 sets (considering the intersection of 2 sets, the maximum is $9 \times 9 = 81$). There are in fact 27 points, 3 on each line of each set and each on 4 lines (one of each set) and all the 27 points belong to Eckart's cubic !

Answer to Chris-Eckart.docx

Message: #62
Date: 17/6/2013 4:36:10
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Bernard Keizer
(May 24, 2013)

Dear Eckart,

I would like to share your enthousiasm and I hoped to see at last those figures I have dreamed of so long!

But unfortunately, as I already told, I'm not able to open your files (I suppose you use a program, for which I have no licence ...).

Could you send it again in another file (I use Geogebra) or Adobe Reader?

By the way, where have you been teaching in Germany? (I spent 4 years in Bonn between 1976 and 1980 in the french embassy ...).

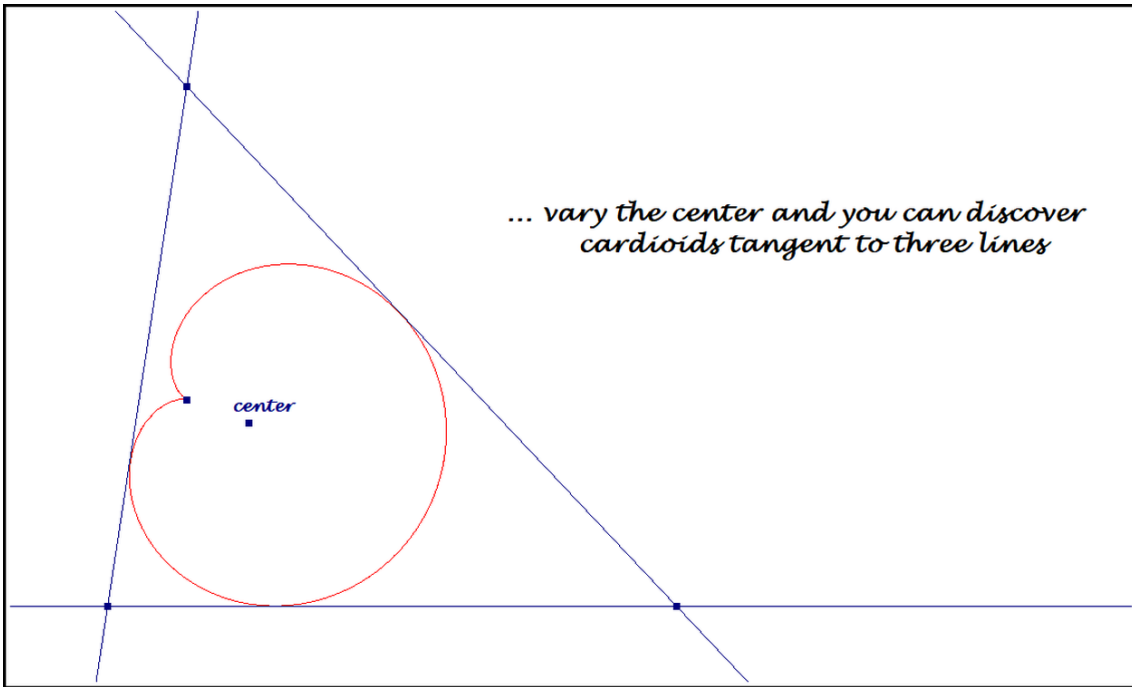
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Message: #63
Date: 17/6/2013 4:36:44
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

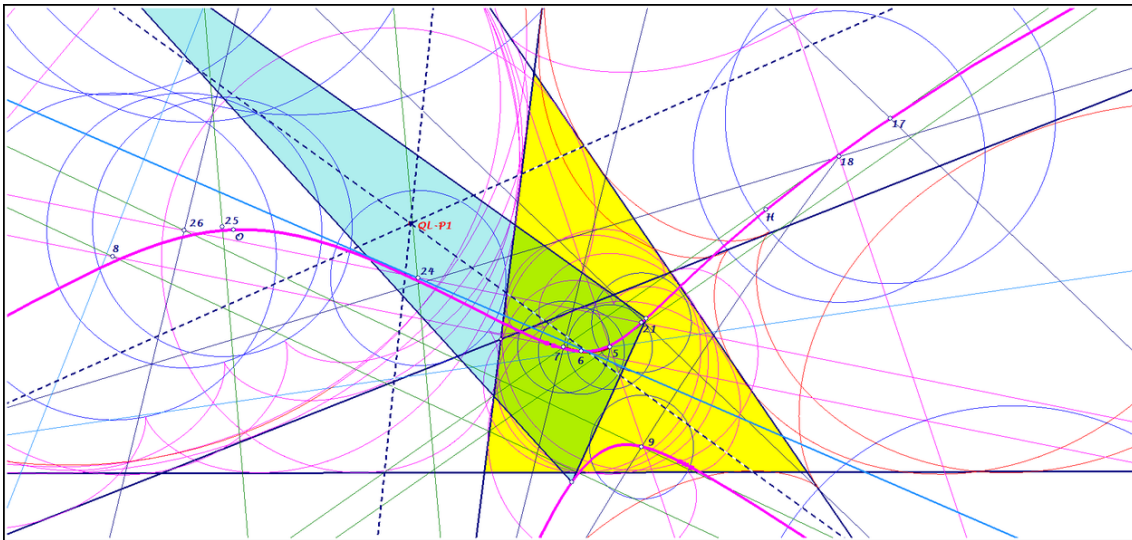
COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 24, 2013)

Dear friends,
thanks to Bernard for the description of the deltoid.
I also do not understand Morley's remark wrt directed lines.
For me there is up to now no sign for a special cardioid.
There are new properties for my cubic of the centers:
The asymptotes cut the cubic in collinear points on a parallel
of the Newton Line.
I have proved this for several numeral examples. This is a
property of the McCay Cubic
(see B. Gibert's catalogue). Maybe it holds for all $pK60+$, I
don't know.
I am positive about: The cubic is a McCay Cubic.

The McCay Cubic is an isogonal pivotal isocubic with pivot in
the circumcenter of the reference triangle.
The problem is the reference triangle for my cubic: The only
property I know is, that the centroid is the Miquel Point.
... What about this reference triangle? ...
You can find all these properties in a Cabri file in the
attachment.
It is a drawing to my calculated example, not constructed, the
calculated points drawn by hand!!!
So it is somewhat unexact and confuse, only a view for
reflecting.
But it contains about 20 cardioids with their centers, the 3
sets of parallels for the reference triangle,
the asymptotes and the cubic, drawn as McCay Cubic of a tested
reference triangle.
... just for fun: There is a further Cabri file for discovering
cardioids tangent to 3 lines.
Best regards
Eckart



cuspid-fig.png



cardioids-fig.png

Message: #64
Date: 17/6/2013 4:38:03
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 25, 2013)

Dear friends,

Chris asked for equations to calculate the centers and cusps of cardioids tangent to 4 lines.

There is a Mathematica file in the attachment.

In addition to the speculation, that my cubic for the centers is a McCay Cubic:

The asymptotes of the cubic are parallel to the axes of the deltoid,

that is the envelope of the Simson lines of the reference triangle of the McCay cubic.

Best regards Eckart

Basic equations for the centers of cardioids tangent to 4 lines

For a cardioid with center $(u:v:w)$ and bitangent (e,f,g)
a line (λ,μ,ν) is tangent, if the following Term is zero:

$$\begin{aligned}
 T[(\lambda, \mu, \nu)] := & \\
 & (-f^4 Sa^2 u^3 \lambda^3 + 4 f^3 g Sa^2 u^3 \lambda^3 - 6 f^2 g^2 Sa^2 u^3 \lambda^3 + 4 f g^3 Sa^2 u^3 \lambda^3 - g^4 Sa^2 u^3 \lambda^3 + e^2 f^2 Sa Sb u^3 \lambda^3 - \\
 & 2 e^2 f g Sa Sb u^3 \lambda^3 + 4 e f^2 g Sa Sb u^3 \lambda^3 + e^2 g^2 Sa Sb u^3 \lambda^3 - 8 e f g^2 Sa Sb u^3 \lambda^3 - \\
 & 2 f^2 g^2 Sa Sb u^3 \lambda^3 + 4 e g^3 Sa Sb u^3 \lambda^3 + 4 f g^3 Sa Sb u^3 \lambda^3 - 2 g^4 Sa Sb u^3 \lambda^3 - 3 e^2 g^2 Sb^2 u^3 \lambda^3 + \\
 & 4 e g^3 Sb^2 u^3 \lambda^3 - g^4 Sb^2 u^3 \lambda^3 + e^2 f^2 Sa Sc u^3 \lambda^3 + 4 e f^3 Sa Sc u^3 \lambda^3 - 2 f^4 Sa Sc u^3 \lambda^3 - \\
 & 2 e^2 f g Sa Sc u^3 \lambda^3 - 8 e f^2 g Sa Sc u^3 \lambda^3 + 4 f^3 g Sa Sc u^3 \lambda^3 + e^2 g^2 Sa Sc u^3 \lambda^3 + \\
 & 4 e f g^2 Sa Sc u^3 \lambda^3 - 2 f^2 g^2 Sa Sc u^3 \lambda^3 + e^2 f^2 Sb Sc u^3 \lambda^3 - 8 e^2 f g Sb Sc u^3 \lambda^3 + \\
 & 4 e f^2 g Sb Sc u^3 \lambda^3 + e^2 g^2 Sb Sc u^3 \lambda^3 + 4 e f g^2 Sb Sc u^3 \lambda^3 - 2 f^2 g^2 Sb Sc u^3 \lambda^3 - \\
 & 3 e^2 f^2 Sc^2 u^3 \lambda^3 + 4 e f^3 Sc^2 u^3 \lambda^3 - f^4 Sc^2 u^3 \lambda^3 + 6 e f^3 Sa Sb u^2 v \lambda^3 - 12 e f^2 g Sa Sb u^2 v \lambda^3 + \\
 & 6 e f g^2 Sa Sb u^2 v \lambda^3 - 6 e^2 f g Sb^2 u^2 v \lambda^3 + 6 e f g^2 Sb^2 u^2 v \lambda^3 + 6 e f^3 Sa Sc u^2 v \lambda^3 - \\
 & 12 e f^2 g Sa Sc u^2 v \lambda^3 + 6 e f g^2 Sa Sc u^2 v \lambda^3 - 6 e^2 f^2 Sb Sc u^2 v \lambda^3 + 6 e f^3 Sb Sc u^2 v \lambda^3 - \\
 & 6 e^2 f g Sb Sc u^2 v \lambda^3 + 6 e f g^2 Sb Sc u^2 v \lambda^3 - 6 e^2 f^2 Sc^2 u^2 v \lambda^3 + 6 e f^3 Sc^2 u^2 v \lambda^3 + \\
 & 3 f^4 Sa Sb u^2 v^2 \lambda^3 - 6 f^3 g Sa Sb u^2 v^2 \lambda^3 + 3 f^2 g^2 Sa Sb u^2 v^2 \lambda^3 - 3 e^2 f^2 Sb^2 u^2 v^2 \lambda^3 + \\
 & 3 f^2 g^2 Sb^2 u^2 v^2 \lambda^3 + 3 f^4 Sa Sc u^2 v^2 \lambda^3 - 6 f^3 g Sa Sc u^2 v^2 \lambda^3 + 3 f^2 g^2 Sa Sc u^2 v^2 \lambda^3 - \\
 & 6 e^2 f^2 Sb Sc u^2 v^2 \lambda^3 + 3 f^4 Sb Sc u^2 v^2 \lambda^3 + 3 f^2 g^2 Sb Sc u^2 v^2 \lambda^3 - 3 e^2 f^2 Sc^2 u^2 v^2 \lambda^3 + \\
 & 3 f^4 Sc^2 u^2 v^2 \lambda^3 - 2 e f^3 Sb^2 v^3 \lambda^3 + 2 f^3 g Sb^2 v^3 \lambda^3 - 4 e f^3 Sb Sc v^3 \lambda^3 + 2 f^4 Sb Sc v^3 \lambda^3 + \\
 & 2 f^3 g Sb Sc v^3 \lambda^3 - 2 e f^3 Sc^2 v^3 \lambda^3 + 2 f^4 Sc^2 v^3 \lambda^3 + 6 e f^2 g Sa Sb u^2 w \lambda^3 - 12 e f g^2 Sa Sb u^2 w \lambda^3 + \\
 & 6 e g^3 Sa Sb u^2 w \lambda^3 - 6 e^2 g^2 Sb^2 u^2 w \lambda^3 + 6 e g^3 Sb^2 u^2 w \lambda^3 + 6 e f^2 g Sa Sc u^2 w \lambda^3 - \\
 & 12 e f g^2 Sa Sc u^2 w \lambda^3 + 6 e g^3 Sa Sc u^2 w \lambda^3 - 6 e^2 f g Sb Sc u^2 w \lambda^3 + 6 e f^2 g Sb Sc u^2 w \lambda^3 - \\
 & 6 e^2 g^2 Sb Sc u^2 w \lambda^3 + 6 e g^3 Sb Sc u^2 w \lambda^3 - 6 e^2 f g Sc^2 u^2 w \lambda^3 + 6 e f^2 g Sc^2 u^2 w \lambda^3 + \\
 & 6 f^3 g Sa Sb u v w \lambda^3 - 12 f^2 g^2 Sa Sb u v w \lambda^3 + 6 f g^3 Sa Sb u v w \lambda^3 - 6 e^2 f g Sb^2 u v w \lambda^3 + \\
 & 6 f g^3 Sb^2 u v w \lambda^3 + 6 f^3 g Sa Sc u v w \lambda^3 - 12 f^2 g^2 Sa Sc u v w \lambda^3 + 6 f g^3 Sa Sc u v w \lambda^3 - \\
 & 12 e f^2 g Sb Sc u v w \lambda^3 + 6 f^3 g Sb Sc u v w \lambda^3 + 6 f g^3 Sb Sc u v w \lambda^3 - 6 e^2 f g Sc^2 u v w \lambda^3 + \\
 & 6 f^3 g Sc^2 u v w \lambda^3 - 6 e f^2 g Sb^2 v^2 w \lambda^3 + 6 f^2 g^2 Sb^2 v^2 w \lambda^3 - 12 e f^2 g Sb Sc v^2 w \lambda^3 + \\
 & 6 f^3 g Sb Sc v^2 w \lambda^3 + 6 f^2 g^2 Sb Sc v^2 w \lambda^3 - 6 e f^2 g Sc^2 v^2 w \lambda^3 + 6 f^3 g Sc^2 v^2 w \lambda^3 + \\
 & 3 f^2 g^2 Sa Sb u w^2 \lambda^3 - 6 f g^3 Sa Sb u w^2 \lambda^3 + 3 g^4 Sa Sb u w^2 \lambda^3 - 3 e^2 g^2 Sb^2 u w^2 \lambda^3 + \\
 & 3 g^4 Sb^2 u w^2 \lambda^3 + 3 f^2 g^2 Sa Sc u w^2 \lambda^3 - 6 f g^3 Sa Sc u w^2 \lambda^3 + 3 g^4 Sa Sc u w^2 \lambda^3 - \\
 & 6 e^2 g^2 Sb Sc u w^2 \lambda^3 + 3 f^2 g^2 Sb Sc u w^2 \lambda^3 + 3 g^4 Sb Sc u w^2 \lambda^3 - 3 e^2 g^2 Sc^2 u w^2 \lambda^3 + \\
 & 3 f^2 g^2 Sc^2 u w^2 \lambda^3 - 6 e f g^2 Sb^2 v w^2 \lambda^3 + 6 f g^3 Sb^2 v w^2 \lambda^3 - 12 e f g^2 Sb Sc v w^2 \lambda^3 + \\
 & 6 f^2 g^2 Sb Sc v w^2 \lambda^3 + 6 f g^3 Sb Sc v w^2 \lambda^3 - 6 e f g^2 Sc^2 v w^2 \lambda^3 + 6 f^2 g^2 Sc^2 v w^2 \lambda^3 - \\
 & 2 e g^3 Sb^2 w^3 \lambda^3 + 2 g^4 Sb^2 w^3 \lambda^3 - 4 e g^3 Sb Sc w^3 \lambda^3 + 2 f g^3 Sb Sc w^3 \lambda^3 + 2 g^4 Sb Sc w^3 \lambda^3 - \\
 & 2 e g^3 Sc^2 w^3 \lambda^3 + 2 f g^3 Sc^2 w^3 \lambda^3 - 2 e^3 f Sa Sb u^3 \lambda^2 \mu + 2 e^3 g Sa Sb u^3 \lambda^2 \mu - 2 e^3 f Sa Sc u^3 \lambda^2 \mu - \\
 & 6 e^2 f^2 Sa Sc u^3 \lambda^2 \mu + 2 e^3 g Sa Sc u^3 \lambda^2 \mu + 12 e^2 f g Sa Sc u^3 \lambda^2 \mu - 6 e^2 g^2 Sa Sc u^3 \lambda^2 \mu - \\
 & 2 e^3 f Sb Sc u^3 \lambda^2 \mu + 8 e^3 g Sb Sc u^3 \lambda^2 \mu - 6 e^2 g^2 Sb Sc u^3 \lambda^2 \mu + 6 e^3 f Sc^2 u^3 \lambda^2 \mu - \\
 & 6 e^2 f^2 Sc^2 u^3 \lambda^2 \mu - 3 f^4 Sa^2 u^2 v \lambda^2 \mu + 12 f^3 g Sa^2 u^2 v \lambda^2 \mu - 18 f^2 g^2 Sa^2 u^2 v \lambda^2 \mu + \\
 & 12 f g^3 Sa^2 u^2 v \lambda^2 \mu - 3 g^4 Sa^2 u^2 v \lambda^2 \mu - 9 e^2 f^2 Sa Sb u^2 v \lambda^2 \mu + 12 e^2 f g Sa Sb u^2 v \lambda^2 \mu + \\
 & 12 e f^2 g Sa Sb u^2 v \lambda^2 \mu - 3 e^2 g^2 Sa Sb u^2 v \lambda^2 \mu - 24 e f g^2 Sa Sb u^2 v \lambda^2 \mu - 6 f^2 g^2 Sa Sb u^2 v \lambda^2 \mu + \\
 & 12 e g^3 Sa Sb u^2 v \lambda^2 \mu + 12 f g^3 Sa Sb u^2 v \lambda^2 \mu - 6 g^4 Sa Sb u^2 v \lambda^2 \mu + 6 e^3 g Sb^2 u^2 v \lambda^2 \mu - \\
 & 15 e^2 g^2 Sb^2 u^2 v \lambda^2 \mu + 12 e g^3 Sb^2 u^2 v \lambda^2 \mu - 3 g^4 Sb^2 u^2 v \lambda^2 \mu - 9 e^2 f^2 Sa Sc u^2 v \lambda^2 \mu - \\
 & 6 f^4 Sa Sc u^2 v \lambda^2 \mu + 12 e f g Sa Sc u^2 v \lambda^2 \mu + 12 f^3 g Sa Sc u^2 v \lambda^2 \mu - 3 e^2 g^2 Sa Sc u^2 v \lambda^2 \mu - \\
 & 6 f^2 g^2 Sa Sc u^2 v \lambda^2 \mu + 12 e^3 f Sb Sc u^2 v \lambda^2 \mu - 9 e^2 f^2 Sb Sc u^2 v \lambda^2 \mu + 6 e^3 g Sb Sc u^2 v \lambda^2 \mu - \\
 & 12 e^2 f g Sb Sc u^2 v \lambda^2 \mu + 12 e f^2 g Sb Sc u^2 v \lambda^2 \mu - 3 e^2 g^2 Sb Sc u^2 v \lambda^2 \mu - 6 f^2 g^2 Sb Sc u^2 v \lambda^2 \mu + \\
 & 12 e^3 f Sc^2 u^2 v \lambda^2 \mu - 9 e^2 f^2 Sc^2 u^2 v \lambda^2 \mu - 3 f^4 Sc^2 u^2 v \lambda^2 \mu - 6 e f^2 g Sa Sb u v^2 \lambda^2 \mu + \\
 & 6 e f g^2 Sa Sb u v^2 \lambda^2 \mu + 6 e^3 f Sb^2 u v^2 \lambda^2 \mu - 12 e^2 f g Sb^2 u v^2 \lambda^2 \mu + 6 e f g^2 Sb^2 u v^2 \lambda^2 \mu -
 \end{aligned}$$

$$\begin{aligned}
& 6e^2f^2gSbScv^2w\lambda\mu^2 + 24efg^2SbScv^2w\lambda\mu^2 + 6f^2g^2SbScv^2w\lambda\mu^2 - 12fg^3SbScv^2w\lambda\mu^2 - \\
& 12e^2fgSc^2v^2w\lambda\mu^2 + 6e^2fgSc^2v^2w\lambda\mu^2 + 6f^3gSc^2v^2w\lambda\mu^2 + 3f^2g^2Sa^2u^2w\lambda\mu^2 - \\
& 6fg^3Sa^2u^2w\lambda\mu^2 + 3g^4Sa^2u^2w\lambda\mu^2 - 3e^2g^2SaSbu^2w\lambda\mu^2 + 3g^4SaSbu^2w\lambda\mu^2 - \\
& 3e^2g^2SaScu^2w\lambda\mu^2 + 12efg^2SaScu^2w\lambda\mu^2 + 6f^2g^2SaScu^2w\lambda\mu^2 - 12eg^3SaScu^2w\lambda\mu^2 - \\
& 6fg^3SaScu^2w\lambda\mu^2 + 3g^4SaScu^2w\lambda\mu^2 - 3e^2g^2SbScv^2w\lambda\mu^2 + 3g^4SbScv^2w\lambda\mu^2 - \\
& 15e^2g^2Sc^2u^2w\lambda\mu^2 + 12efg^2Sc^2u^2w\lambda\mu^2 + 3f^2g^2Sc^2u^2w\lambda\mu^2 - 6efg^2SaSbv^2w\lambda\mu^2 + \\
& 6fg^3SaSbv^2w\lambda\mu^2 - 6efg^2SaScv^2w\lambda\mu^2 + 12f^2g^2SaScv^2w\lambda\mu^2 - 6g^4SaScv^2w\lambda\mu^2 - \\
& 6e^2g^2SbScv^2w\lambda\mu^2 - 6efg^2SbScv^2w\lambda\mu^2 + 12eg^3SbScv^2w\lambda\mu^2 + 6fg^3SbScv^2w\lambda\mu^2 - \\
& 6g^4SbScv^2w\lambda\mu^2 - 6e^2g^2Sc^2v^2w\lambda\mu^2 - 6efg^2Sc^2v^2w\lambda\mu^2 + 12f^2g^2Sc^2v^2w\lambda\mu^2 - \\
& 2eg^3SaSbw^3\lambda\mu^2 + 2g^4SaSbw^3\lambda\mu^2 - 2eg^3SaScw^3\lambda\mu^2 + 6fg^3SaScw^3\lambda\mu^2 - \\
& 4g^4SaScw^3\lambda\mu^2 - 2eg^3SbScw^3\lambda\mu^2 + 2g^4SbScw^3\lambda\mu^2 - 6eg^3Sc^2w^3\lambda\mu^2 + 6fg^3Sc^2w^3\lambda\mu^2 - \\
& 2e^3fSa^2u^3\mu^3 + 2e^3gSa^2u^3\mu^3 + 2e^4SaScu^3\mu^3 - 4e^3fSaScu^3\mu^3 + 2e^3gSaScu^3\mu^3 + \\
& 2e^4Sc^2u^3\mu^3 - 2e^3fSc^2u^3\mu^3 - 3e^2f^2Sa^2u^2v\mu^3 + 3e^2g^2Sa^2u^2v\mu^3 + 3e^4SaSb^2u^2v\mu^3 - \\
& 6e^3gSaSb^2u^2v\mu^3 + 3e^2g^2SaSb^2u^2v\mu^3 + 3e^4SaSc^2u^2v\mu^3 - 6e^2f^2SaSc^2u^2v\mu^3 + \\
& 3e^2g^2SaSc^2u^2v\mu^3 + 3e^4SbSc^2u^2v\mu^3 - 6e^3gSbSc^2u^2v\mu^3 + 3e^2g^2SbSc^2u^2v\mu^3 + \\
& 3e^4Sc^2u^2v\mu^3 - 3e^2f^2Sc^2u^2v\mu^3 - 6ef^2gSa^2u^2v\mu^3 + 6efg^2Sa^2u^2v\mu^3 + \\
& 6e^3fSaSbu^2v\mu^3 - 12e^2fgSaSbu^2v\mu^3 + 6efg^2SaSbu^2v\mu^3 + 6e^3fSaScu^2v\mu^3 - \\
& 6e^2f^2SaScu^2v\mu^3 - 6ef^2gSaScu^2v\mu^3 + 6efg^2SaScu^2v\mu^3 + 6e^3fSbScu^2v\mu^3 - \\
& 12e^2fgSbScu^2v\mu^3 + 6efg^2SbScu^2v\mu^3 + 6e^3fSc^2u^2v\mu^3 - 6e^2f^2Sc^2u^2v\mu^3 - \\
& 3f^2g^2Sa^2v^3\mu^3 + 4fg^3Sa^2v^3\mu^3 - g^4Sa^2v^3\mu^3 + e^2f^2SaSbv^3\mu^3 + 4e^2fgSaSbv^3\mu^3 - \\
& 2ef^2gSaSbv^3\mu^3 - 2e^2g^2SaSbv^3\mu^3 - 8efg^2SaSbv^3\mu^3 + f^2g^2SaSbv^3\mu^3 + \\
& 4eg^3SaSbv^3\mu^3 + 4fg^3SaSbv^3\mu^3 - 2g^4SaSbv^3\mu^3 - e^4Sb^2v^3\mu^3 + 4e^3gSb^2v^3\mu^3 - \\
& 6e^2g^2Sb^2v^3\mu^3 + 4eg^3Sb^2v^3\mu^3 - g^4Sb^2v^3\mu^3 + e^2f^2SaScv^3\mu^3 + 4e^2fgSaScv^3\mu^3 - \\
& 8ef^2gSaScv^3\mu^3 - 2e^2g^2SaScv^3\mu^3 + 4efg^2SaScv^3\mu^3 + f^2g^2SaScv^3\mu^3 - \\
& 2e^4SbScv^3\mu^3 + 4e^3fSbScv^3\mu^3 + e^2f^2SbScv^3\mu^3 + 4e^3gSbScv^3\mu^3 - 8e^2fgSbScv^3\mu^3 - \\
& 2ef^2gSbScv^3\mu^3 - 2e^2g^2SbScv^3\mu^3 + 4efg^2SbScv^3\mu^3 + f^2g^2SbScv^3\mu^3 - \\
& e^4Sc^2v^3\mu^3 + 4e^3fSc^2v^3\mu^3 - 3e^2f^2Sc^2v^3\mu^3 - 6ef^2gSa^2u^2w\mu^3 + 6e^2g^2Sa^2u^2w\mu^3 + \\
& 6e^3gSaSc^2u^2w\mu^3 - 12e^2fgSaSc^2u^2w\mu^3 + 6e^2g^2SaSc^2u^2w\mu^3 + 6e^3gSc^2u^2w\mu^3 - \\
& 6e^2f^2gSc^2u^2w\mu^3 - 6ef^2gSa^2u^2w\mu^3 + 6eg^3Sa^2u^2w\mu^3 + 6e^3gSaSbu^2w\mu^3 - \\
& 12e^2g^2SaSbu^2w\mu^3 + 6eg^3SaSbu^2w\mu^3 + 6e^3gSaScu^2w\mu^3 - 12ef^2gSaScu^2w\mu^3 + \\
& 6eg^3SaScu^2w\mu^3 + 6e^3gSbScu^2w\mu^3 - 12e^2g^2SbScu^2w\mu^3 + 6eg^3SbScu^2w\mu^3 + \\
& 6e^3gSc^2u^2w\mu^3 - 6ef^2gSc^2u^2w\mu^3 - 6f^2g^2Sa^2v^2w\mu^3 + 6fg^3Sa^2v^2w\mu^3 + \\
& 6e^2fgSaSbv^2w\mu^3 - 12efg^2SaSbv^2w\mu^3 + 6fg^3SaSbv^2w\mu^3 + 6e^2fgSaScv^2w\mu^3 - \\
& 6ef^2gSaScv^2w\mu^3 - 6f^2g^2SaScv^2w\mu^3 + 6fg^3SaScv^2w\mu^3 + 6e^2fgSbScv^2w\mu^3 - \\
& 12efg^2SbScv^2w\mu^3 + 6fg^3SbScv^2w\mu^3 + 6e^2fgSc^2v^2w\mu^3 - 6ef^2gSc^2v^2w\mu^3 - \\
& 6efg^2Sa^2u^2w\mu^3 + 6eg^3Sa^2u^2w\mu^3 + 6e^2g^2SaScu^2w\mu^3 - 12efg^2SaScu^2w\mu^3 + \\
& 6eg^3SaScu^2w\mu^3 + 6e^2g^2Sc^2u^2w\mu^3 - 6efg^2Sc^2u^2w\mu^3 - 3f^2g^2Sa^2v^2w\mu^3 + 3g^4Sa^2v^2w\mu^3 + \\
& 3e^2g^2SaSbv^2w\mu^3 - 6eg^3SaSbv^2w\mu^3 + 3g^4SaSbv^2w\mu^3 + 3e^2g^2SaScv^2w\mu^3 - \\
& 6eg^3SbScv^2w\mu^3 + 3g^4SbScv^2w\mu^3 + 3e^2g^2SbScv^2w\mu^3 - 3f^2g^2Sc^2v^2w\mu^3 - 2fg^3Sa^2w^3\mu^3 + 2g^4Sa^2w^3\mu^3 + \\
& 2eg^3SaScw^3\mu^3 - 4fg^3SaScw^3\mu^3 + 2g^4SaScw^3\mu^3 + 2eg^3Sc^2w^3\mu^3 - 2fg^3Sc^2w^3\mu^3 + \\
& 2e^3fSaSbu^3\lambda^2v - 6e^2f^2SaSbu^3\lambda^2v - 2e^3gSaSbu^3\lambda^2v + 12efgSaSbu^3\lambda^2v - \\
& 6e^2g^2SaSbu^3\lambda^2v + 6e^3gSb^2u^3\lambda^2v - 6e^2g^2Sb^2u^3\lambda^2v + 2e^3fSaScu^3\lambda^2v - \\
& 2e^3gSaScu^3\lambda^2v + 8e^3fSbScu^3\lambda^2v - 6e^2f^2SbScu^3\lambda^2v - 2e^3gSbScu^3\lambda^2v + \\
& 6e^2f^2SaSbu^2v\lambda^2v - 12ef^3SaSbu^2v\lambda^2v - 6e^2fgSaSbu^2v\lambda^2v + 24ef^2gSaSbu^2v\lambda^2v - \\
& 12efg^2SaSbu^2v\lambda^2v + 6e^3fSb^2u^2v\lambda^2v + 6e^2fgSb^2u^2v\lambda^2v - 12efg^2Sb^2u^2v\lambda^2v + \\
& 6e^2f^2SaScu^2v\lambda^2v - 6e^2fgSaScu^2v\lambda^2v + 6e^3fSbScu^2v\lambda^2v + 12e^2f^2SbScu^2v\lambda^2v - \\
& 12ef^3SbScu^2v\lambda^2v - 6e^2fgSbScu^2v\lambda^2v + 6ef^3SaSbu^2v\lambda^2v - 6f^4SaSbu^2v\lambda^2v - \\
& 6ef^2gSaSbu^2v\lambda^2v + 12f^3gSaSbu^2v\lambda^2v - 6f^2g^2SaSbu^2v\lambda^2v + 12e^2f^2Sb^2u^2v\lambda^2v - \\
& 6e^2fgSb^2u^2v\lambda^2v - 6f^4SbScu^2v\lambda^2v - 6ef^3SaScu^2v\lambda^2v - 6ef^2gSaScu^2v\lambda^2v + \\
& 12e^2f^2SbScu^2v\lambda^2v - 6f^4SbScu^2v\lambda^2v - 6ef^3gSbScu^2v\lambda^2v + 2f^4SaSbv^3\lambda^2v - \\
& 2f^3gSaSbv^3\lambda^2v + 6ef^3Sb^2v^3\lambda^2v - 6f^3gSb^2v^3\lambda^2v + 2f^4SaScv^3\lambda^2v - \\
& 2f^3gSaScv^3\lambda^2v + 6ef^3SbScv^3\lambda^2v - 4f^4SbScv^3\lambda^2v - 2f^3gSbScv^3\lambda^2v - \\
& 3f^4Sa^2u^2w\lambda^2v + 12f^3gSa^2u^2w\lambda^2v - 18f^2g^2Sa^2u^2w\lambda^2v + 12fg^3Sa^2u^2w\lambda^2v - \\
& 3g^4Sa^2u^2w\lambda^2v - 3e^2f^2SaSbu^2w\lambda^2v + 12e^2fgSaSbu^2w\lambda^2v - 9e^2g^2SaSbu^2w\lambda^2v - \\
& 6f^2g^2SaSbu^2w\lambda^2v + 12fg^3SaSbu^2w\lambda^2v - 6g^4SaSbu^2w\lambda^2v + 12e^3gSb^2u^2w\lambda^2v - \\
& 9e^2g^2Sb^2u^2w\lambda^2v - 3g^4Sb^2u^2w\lambda^2v - 3e^2f^2SaScu^2w\lambda^2v + 12ef^3SaScu^2w\lambda^2v - \\
& 6f^4SaScu^2w\lambda^2v + 12e^2fgSaScu^2w\lambda^2v - 24ef^2gSaScu^2w\lambda^2v + 12f^3gSaScu^2w\lambda^2v -
\end{aligned}$$

$$\begin{aligned}
& 9e^2g^2SaScu^2w\lambda^2v+12efg^2SaScu^2w\lambda^2v-6f^2g^2SaScu^2w\lambda^2v+6e^3fSbScu^2w\lambda^2v- \\
& 3e^2f^2SbScu^2w\lambda^2v+12e^3fSbScu^2w\lambda^2v-12e^2fgSbScu^2w\lambda^2v-9e^2g^2SbScu^2w\lambda^2v+ \\
& 12efg^2SbScu^2w\lambda^2v-6f^2g^2SbScu^2w\lambda^2v+6e^3fSc^2u^2w\lambda^2v-15e^2f^2Sc^2u^2w\lambda^2v+ \\
& 12ef^2Sc^2u^2w\lambda^2v-3f^4Sc^2u^2w\lambda^2v+6ef^3SaSbuvw\lambda^2v-12f^3gSaSbuvw\lambda^2v- \\
& 6efg^2SaSbuvw\lambda^2v+24f^2g^2SaSbuvw\lambda^2v-12fg^3SaSbuvw\lambda^2v+6e^3fSb^2uvw\lambda^2v+ \\
& 12e^2fgSb^2uvw\lambda^2v-6efg^2Sb^2uvw\lambda^2v-12fg^3Sb^2uvw\lambda^2v+6ef^3SaScuvw\lambda^2v- \\
& 6efg^2SaScuvw\lambda^2v+12e^3fSbScuvw\lambda^2v-12e^2f^2SbScuvw\lambda^2v+6ef^3SbScuvw\lambda^2v+ \\
& 12e^2fgSbScuvw\lambda^2v-12f^3gSbScuvw\lambda^2v-6efg^2SbScuvw\lambda^2v+6e^3fSc^2uvw\lambda^2v- \\
& 12e^2f^2Sc^2uvw\lambda^2v+6ef^3Sc^2uvw\lambda^2v+3f^4SaSbv^2w\lambda^2v-3f^2g^2SaSbv^2w\lambda^2v+ \\
& 3e^2f^2Sb^2v^2w\lambda^2v+12ef^2gSb^2v^2w\lambda^2v-15f^2g^2Sb^2v^2w\lambda^2v+3f^4SaScv^2w\lambda^2v- \\
& 3f^2g^2SaScv^2w\lambda^2v+6ef^2SbScv^2w\lambda^2v-6ef^3SbScv^2w\lambda^2v+3f^4SbScv^2w\lambda^2v+ \\
& 12ef^2gSbScv^2w\lambda^2v-12f^3gSbScv^2w\lambda^2v-3f^2g^2SbScv^2w\lambda^2v+3e^2f^2Sc^2v^2w\lambda^2v- \\
& 6ef^3Sc^2v^2w\lambda^2v+3f^4Sc^2v^2w\lambda^2v+6ef^2gSaSbuw^2\lambda^2v-6efg^2SaSbuw^2\lambda^2v- \\
& 6f^2g^2SaSbuw^2\lambda^2v+12fg^3SaSbuw^2\lambda^2v-6g^4SaSbuw^2\lambda^2v+6e^3gSb^2uw^2\lambda^2v- \\
& 6g^4Sb^2uw^2\lambda^2v+6ef^2gSaScuw^2\lambda^2v-6efg^2SaScuw^2\lambda^2v+12e^3gSbScuw^2\lambda^2v- \\
& 12ef^2gSbScuw^2\lambda^2v+6ef^2gSbScuw^2\lambda^2v-6f^2g^2SbScuw^2\lambda^2v+6e^3gSc^2uw^2\lambda^2v- \\
& 12ef^2fgSc^2uw^2\lambda^2v+6ef^2gSc^2uw^2\lambda^2v+6f^3gSaSbv^2w\lambda^2v-6f^2g^2SaSbv^2w\lambda^2v+ \\
& 6e^2fgSb^2v^2w\lambda^2v+6efg^2Sb^2v^2w\lambda^2v-12fg^3Sb^2v^2w\lambda^2v+6f^3gSaScv^2w\lambda^2v- \\
& 6f^2g^2SaScv^2w\lambda^2v+12ef^2fgSbScv^2w\lambda^2v-12ef^2gSbScv^2w\lambda^2v+6f^3gSbScv^2w\lambda^2v+ \\
& 6efg^2SbScv^2w\lambda^2v-12f^2g^2SbScv^2w\lambda^2v+6e^2fgSc^2v^2w\lambda^2v-12ef^2gSc^2v^2w\lambda^2v+ \\
& 6f^3gSc^2v^2w\lambda^2v+3f^2g^2SaSbw^3\lambda^2v-4fg^3SaSbw^3\lambda^2v+g^4SaSbw^3\lambda^2v+ \\
& 3e^2g^2Sb^2w^3\lambda^2v-3g^4Sb^2w^3\lambda^2v+3f^2g^2SaScw^3\lambda^2v-4fg^3SaScw^3\lambda^2v+ \\
& g^4SaScw^3\lambda^2v+6ef^2gSbScw^3\lambda^2v-6efg^2SbScw^3\lambda^2v+3f^2g^2SbScw^3\lambda^2v- \\
& 4fg^3SbScw^3\lambda^2v+g^4SbScw^3\lambda^2v+3e^2g^2Sc^2w^3\lambda^2v-6efg^2Sc^2w^3\lambda^2v+ \\
& 3f^2g^2Sc^2w^3\lambda^2v-6e^2f^2Sa^2u^3\lambda\mu v+12efgSa^2u^3\lambda\mu v-6e^2g^2Sa^2u^3\lambda\mu v- \\
& 2e^4SaSbu^3\lambda\mu v+4e^3fSaSbu^3\lambda\mu v+4e^3gSaSbu^3\lambda\mu v-6e^2g^2SaSbu^3\lambda\mu v- \\
& 2e^4SaScu^3\lambda\mu v+4e^3fSaScu^3\lambda\mu v-6e^2f^2SaScu^3\lambda\mu v+4e^3gSaScu^3\lambda\mu v- \\
& 8e^4SbScu^3\lambda\mu v+4e^3fSbScu^3\lambda\mu v+4e^3gSbScu^3\lambda\mu v-12ef^3Sa^2u^2v\lambda\mu v+ \\
& 24ef^2gSa^2u^2v\lambda\mu v-12efg^2Sa^2u^2v\lambda\mu v+6ef^2SaSbu^2v\lambda\mu v+12ef^2fgSaSbu^2v\lambda\mu v- \\
& 6e^2g^2SaSbu^2v\lambda\mu v-12efg^2SaSbu^2v\lambda\mu v-6e^4Sb^2u^2v\lambda\mu v+12e^3gSb^2u^2v\lambda\mu v- \\
& 6e^2g^2Sb^2u^2v\lambda\mu v-12ef^3SaScu^2v\lambda\mu v+12ef^2fgSaScu^2v\lambda\mu v-6e^4SbScu^2v\lambda\mu v- \\
& 12ef^3fSbScu^2v\lambda\mu v+6e^2f^2SbScu^2v\lambda\mu v+12ef^2fgSbScu^2v\lambda\mu v-6f^4Sa^2u^2v\lambda\mu v- \\
& 12ef^2gSa^2u^2v\lambda\mu v-6f^2g^2Sa^2u^2v\lambda\mu v+6ef^2SaSbu^2v\lambda\mu v+12ef^2gSaSbu^2v\lambda\mu v- \\
& 12efg^2SaSbu^2v\lambda\mu v-6f^2g^2SaSbu^2v\lambda\mu v-12e^3fSb^2u^2v\lambda\mu v+24ef^2fgSb^2u^2v\lambda\mu v- \\
& 12ef^2fgSb^2u^2v\lambda\mu v+6e^2f^2SaScu^2v\lambda\mu v-12ef^3SaScu^2v\lambda\mu v-6f^4SaScu^2v\lambda\mu v+ \\
& 12ef^2gSaScu^2v\lambda\mu v-12ef^3fSbScu^2v\lambda\mu v+12ef^2fgSbScu^2v\lambda\mu v+ \\
& 4ef^3SaSbv^3\lambda\mu v-2f^4SaSbv^3\lambda\mu v+4f^3gSaSbv^3\lambda\mu v-6f^2g^2SaSbv^3\lambda\mu v- \\
& 6e^2f^2Sb^2v^3\lambda\mu v+12ef^2gSb^2v^3\lambda\mu v-6f^2g^2Sb^2v^3\lambda\mu v+4ef^3SaScv^3\lambda\mu v- \\
& 8f^4SaScv^3\lambda\mu v+4f^3gSaScv^3\lambda\mu v-6e^2f^2SbScv^3\lambda\mu v+4ef^3SbScv^3\lambda\mu v- \\
& 2f^4SbScv^3\lambda\mu v+4f^3gSbScv^3\lambda\mu v-12ef^2gSa^2u^2w\lambda\mu v+24efg^2Sa^2u^2w\lambda\mu v- \\
& 12eg^3Sa^2u^2w\lambda\mu v+12ef^2fgSaSbu^2w\lambda\mu v-12eg^3SaSbu^2w\lambda\mu v-6e^2f^2SaScu^2w\lambda\mu v+ \\
& 12ef^2fgSaScu^2w\lambda\mu v-12ef^2gSaScu^2w\lambda\mu v+6e^2g^2SaScu^2w\lambda\mu v- \\
& 6e^4SbScu^2w\lambda\mu v-12e^3gSbScu^2w\lambda\mu v+12ef^2fgSbScu^2w\lambda\mu v+6e^2g^2SbScu^2w\lambda\mu v- \\
& 6e^4Sc^2u^2w\lambda\mu v+12e^3fSc^2u^2w\lambda\mu v-6e^2f^2Sc^2u^2w\lambda\mu v-6f^4Sa^2uvw\lambda\mu v+ \\
& 12f^3gSa^2uvw\lambda\mu v-12f^2g^2Sa^2uvw\lambda\mu v+12fg^3Sa^2uvw\lambda\mu v-6g^4Sa^2uvw\lambda\mu v- \\
& 12ef^2fSaSbuvw\lambda\mu v+36ef^2fgSaSbuvw\lambda\mu v+36ef^2gSaSbuvw\lambda\mu v- \\
& 12e^2g^2SaSbuvw\lambda\mu v-48efg^2SaSbuvw\lambda\mu v-12f^2g^2SaSbuvw\lambda\mu v+ \\
& 12eg^3SaSbuvw\lambda\mu v+12fg^3SaSbuvw\lambda\mu v-12g^4SaSbuvw\lambda\mu v-6e^4Sb^2uvw\lambda\mu v+ \\
& 12e^3gSb^2uvw\lambda\mu v-12e^2g^2Sb^2uvw\lambda\mu v+12eg^3Sb^2uvw\lambda\mu v-6g^4Sb^2uvw\lambda\mu v- \\
& 12e^2f^2SaScuvw\lambda\mu v+12ef^3SaScuvw\lambda\mu v-12f^4SaScuvw\lambda\mu v+ \\
& 36ef^2fgSaScuvw\lambda\mu v-48ef^2gSaScuvw\lambda\mu v+12f^3gSaScuvw\lambda\mu v- \\
& 12e^2g^2SaScuvw\lambda\mu v+36efg^2SaScuvw\lambda\mu v-12f^2g^2SaScuvw\lambda\mu v- \\
& 12e^4SbScuvw\lambda\mu v+12e^3fSbScuvw\lambda\mu v-12ef^2SbScuvw\lambda\mu v+ \\
& 12e^3gSbScuvw\lambda\mu v-48ef^2fgSbScuvw\lambda\mu v+36ef^2gSbScuvw\lambda\mu v- \\
& 12ef^2g^2SbScuvw\lambda\mu v+36efg^2SbScuvw\lambda\mu v-12f^2g^2SbScuvw\lambda\mu v- \\
& 6e^4Sc^2uvw\lambda\mu v+12e^3fSc^2uvw\lambda\mu v-12ef^2Sc^2uvw\lambda\mu v+12ef^3Sc^2uvw\lambda\mu v- \\
& 6f^4Sc^2uvw\lambda\mu v+12ef^2gSaSbv^2w\lambda\mu v-12fg^3SaSbv^2w\lambda\mu v-12ef^2fgSb^2v^2w\lambda\mu v+ \\
& 24efg^2Sb^2v^2w\lambda\mu v-12fg^3Sb^2v^2w\lambda\mu v-6f^4SaScv^2w\lambda\mu v+12ef^2gSaScv^2w\lambda\mu v-
\end{aligned}$$

$$\begin{aligned}
& 12 f^3 g S a S c v^2 w \lambda \mu \nu + 6 f^2 g^2 S a S c v^2 w \lambda \mu \nu - 6 e^2 f^2 S b S c v^2 w \lambda \mu \nu - 12 e^2 f g S b S c v^2 w \lambda \mu \nu + \\
& 12 e^2 f^2 g S b S c v^2 w \lambda \mu \nu + 6 f^2 g^2 S b S c v^2 w \lambda \mu \nu - 6 e^2 f^2 S c^2 v^2 w \lambda \mu \nu + 12 e f^3 S c^2 v^2 w \lambda \mu \nu - \\
& 6 f^4 S c^2 v^2 w \lambda \mu \nu - 6 f^2 g^2 S a^2 u w^2 \lambda \mu \nu + 12 f g^3 S a^2 u w^2 \lambda \mu \nu - 6 g^4 S a^2 u w^2 \lambda \mu \nu + \\
& 6 e^2 g^2 S a S b u w^2 \lambda \mu \nu + 12 e f g^2 S a S b u w^2 \lambda \mu \nu - 12 e g^3 S a S b u w^2 \lambda \mu \nu - 6 g^4 S a S b u w^2 \lambda \mu \nu - \\
& 12 e f^2 g S a S c u w^2 \lambda \mu \nu + 6 e^2 g^2 S a S c u w^2 \lambda \mu \nu + 12 e f g^2 S a S c u w^2 \lambda \mu \nu - \\
& 6 f^2 g^2 S a S c u w^2 \lambda \mu \nu - 12 e^3 g S b S c u w^2 \lambda \mu \nu + 12 e f g^2 S b S c u w^2 \lambda \mu \nu - 12 e^3 g S c^2 u w^2 \lambda \mu \nu + \\
& 24 e^2 f g S c^2 u w^2 \lambda \mu \nu - 12 e f^2 g S c^2 u w^2 \lambda \mu \nu + 12 e f g^2 S a S b v w^2 \lambda \mu \nu + 6 f^2 g^2 S a S b v w^2 \lambda \mu \nu - \\
& 12 f g^3 S a S b v w^2 \lambda \mu \nu - 6 g^4 S a S b v w^2 \lambda \mu \nu - 6 e^2 g^2 S b^2 v w^2 \lambda \mu \nu + 12 e g^3 S b^2 v w^2 \lambda \mu \nu - \\
& 6 g^4 S b^2 v w^2 \lambda \mu \nu - 12 f^3 g S a S c v w^2 \lambda \mu \nu + 12 e f g^2 S a S c v w^2 \lambda \mu \nu - 12 e^2 f g S b S c v w^2 \lambda \mu \nu - \\
& 6 e^2 g^2 S b S c v w^2 \lambda \mu \nu + 12 e f g^2 S b S c v w^2 \lambda \mu \nu + 6 f^2 g^2 S b S c v w^2 \lambda \mu \nu - 12 e^2 f g S c^2 v w^2 \lambda \mu \nu + \\
& 24 e f^2 g S c^2 v w^2 \lambda \mu \nu - 12 f^3 g S c^2 v w^2 \lambda \mu \nu + 4 e g^3 S a S b w^3 \lambda \mu \nu + 4 f g^3 S a S b w^3 \lambda \mu \nu - \\
& 8 g^4 S a S b w^3 \lambda \mu \nu - 6 f^2 g^2 S a S c w^3 \lambda \mu \nu + 4 e g^3 S a S c w^3 \lambda \mu \nu + 4 f g^3 S a S c w^3 \lambda \mu \nu - \\
& 2 g^4 S a S c w^3 \lambda \mu \nu - 6 e^2 g^2 S b S c w^3 \lambda \mu \nu + 4 e g^3 S b S c w^3 \lambda \mu \nu + 4 f g^3 S b S c w^3 \lambda \mu \nu - \\
& 2 g^4 S b S c w^3 \lambda \mu \nu - 6 e^2 g^2 S c^2 w^3 \lambda \mu \nu + 12 e f g^2 S c^2 w^3 \lambda \mu \nu - 6 f^2 g^2 S c^2 w^3 \lambda \mu \nu + \\
& 6 e^3 f S a^2 u^3 \mu^2 \nu - 6 e^3 g S a^2 u^3 \mu^2 \nu + 2 e^4 S a S b u^3 \mu^2 \nu - 2 e^3 g S a S b u^3 \mu^2 \nu - 4 e^4 S a S c u^3 \mu^2 \nu + \\
& 6 e^3 f S a S c u^3 \mu^2 \nu - 2 e^3 g S a S c u^3 \mu^2 \nu + 2 e^4 S b S c u^3 \mu^2 \nu - 2 e^3 g S b S c u^3 \mu^2 \nu + \\
& 12 e^2 f^2 S a^2 u^2 v \mu^2 \nu - 6 e^2 f g S a^2 u^2 v \mu^2 \nu - 6 e^2 g^2 S a^2 u^2 v \mu^2 \nu - 6 e^4 S a S b u^2 v \mu^2 \nu + \\
& 6 e^3 f S a S b u^2 v \mu^2 \nu + 12 e^3 g S a S b u^2 v \mu^2 \nu - 6 e^2 f g S a S b u^2 v \mu^2 \nu - 6 e^2 g^2 S a S b u^2 v \mu^2 \nu - \\
& 6 e^4 S a S c u^2 v \mu^2 \nu + 12 e^2 f^2 S a S c u^2 v \mu^2 \nu - 6 e^2 f g S a S c u^2 v \mu^2 \nu + 6 e^3 f S b S c u^2 v \mu^2 \nu - \\
& 6 e^2 f g S b S c u^2 v \mu^2 \nu + 6 e^3 S a^2 u v^2 \mu^2 \nu + 6 e f^2 g S a^2 u v^2 \mu^2 \nu - 12 e f^2 g S a^2 u v^2 \mu^2 \nu - \\
& 12 e^2 f g S a S b u v^2 \mu^2 \nu + 6 e^2 f^2 S a S b u v^2 \mu^2 \nu + 24 e^2 f g S a S b u v^2 \mu^2 \nu - 6 e f^2 g S a S b u v^2 \mu^2 \nu - \\
& 12 e f g^2 S a S b u v^2 \mu^2 \nu - 12 e^3 f S a S c u v^2 \mu^2 \nu + 12 e^2 f^2 S a S c u v^2 \mu^2 \nu + 6 e f^3 S a S c u v^2 \mu^2 \nu - \\
& 6 e f^2 g S a S c u v^2 \mu^2 \nu + 6 e^2 f^2 S b S c u v^2 \mu^2 \nu - 6 e f^2 g S b S c u v^2 \mu^2 \nu + 6 f^3 g S a^2 v^3 \mu^2 \nu - \\
& 6 f^2 g^2 S a^2 v^3 \mu^2 \nu - 6 e^2 f^2 S a S b v^3 \mu^2 \nu + 2 e f^3 S a S b v^3 \mu^2 \nu + 12 e f^2 g S a S b v^3 \mu^2 \nu - \\
& 2 f^3 g S a S b v^3 \mu^2 \nu - 6 e^2 g^2 S a S b v^3 \mu^2 \nu - 6 e^2 f^2 S a S c v^3 \mu^2 \nu + 8 e f^3 S a S c v^3 \mu^2 \nu - \\
& 2 f^3 g S a S c v^3 \mu^2 \nu + 2 e f^3 S b S c v^3 \mu^2 \nu - 2 f^3 g S b S c v^3 \mu^2 \nu + 3 e^2 f^2 S a^2 u^2 w \mu^2 \nu + \\
& 12 e^2 f g S a^2 u^2 w \mu^2 \nu - 15 e^2 g^2 S a^2 u^2 w \mu^2 \nu + 3 e^4 S a S b u^2 w \mu^2 \nu - 3 e^2 g^2 S a S b u^2 w \mu^2 \nu + \\
& 3 e^4 S a S c u^2 w \mu^2 \nu - 6 e^3 f S a S c u^2 w \mu^2 \nu + 6 e^2 f^2 S a S c u^2 w \mu^2 \nu - 12 e^3 g S a S c u^2 w \mu^2 \nu + \\
& 12 e^2 f g S a S c u^2 w \mu^2 \nu - 3 e^2 g^2 S a S c u^2 w \mu^2 \nu + 3 e^4 S b S c u^2 w \mu^2 \nu - 3 e^2 g^2 S b S c u^2 w \mu^2 \nu + \\
& 3 e^4 S c^2 u^2 w \mu^2 \nu - 6 e^3 f S c^2 u^2 w \mu^2 \nu + 3 e^2 f^2 S c^2 u^2 w \mu^2 \nu + 6 e f^3 S a^2 u v w \mu^2 \nu + \\
& 12 e f^2 g S a^2 u v w \mu^2 \nu - 6 e f g^2 S a^2 u v w \mu^2 \nu - 12 e g^3 S a^2 u v w \mu^2 \nu + 6 e^3 f S a S b u v w \mu^2 \nu - \\
& 12 e^3 g S a S b u v w \mu^2 \nu + 24 e^2 f^2 g S a S b u v w \mu^2 \nu - 6 e f g^2 S a S b u v w \mu^2 \nu - 12 e g^3 S a S b u v w \mu^2 \nu + \\
& 6 e^3 f S a S c u v w \mu^2 \nu - 12 e^2 f^2 S a S c u v w \mu^2 \nu + 12 e f^3 S a S c u v w \mu^2 \nu - 12 e^3 g S a S c u v w \mu^2 \nu + \\
& 12 e f^2 g S a S c u v w \mu^2 \nu - 6 e f g^2 S a S c u v w \mu^2 \nu + 6 e^3 f S b S c u v w \mu^2 \nu - 6 e f g^2 S b S c u v w \mu^2 \nu + \\
& 6 e^3 f S c^2 u v w \mu^2 \nu - 12 e^2 f^2 S c^2 u v w \mu^2 \nu + 6 e f^3 S c^2 u v w \mu^2 \nu + 12 f^3 g S a^2 v^2 w \mu^2 \nu - \\
& 9 f^2 g^2 S a^2 v^2 w \mu^2 \nu - 3 g^4 S a^2 v^2 w \mu^2 \nu - 3 e^2 f^2 S a S b v^2 w \mu^2 \nu + 12 e f^2 g S a S b v^2 w \mu^2 \nu - \\
& 6 e^2 g^2 S a S b v^2 w \mu^2 \nu - 9 f^2 g^2 S a S b v^2 w \mu^2 \nu + 12 e g^3 S a S b v^2 w \mu^2 \nu - 6 g^4 S a S b v^2 w \mu^2 \nu - \\
& 3 e^4 S b^2 v^2 w \mu^2 \nu + 12 e^3 g S b^2 v^2 w \mu^2 \nu - 18 e^2 g^2 S b^2 v^2 w \mu^2 \nu + 12 e g^3 S b^2 v^2 w \mu^2 \nu - \\
& 3 g^4 S b^2 v^2 w \mu^2 \nu - 3 e^2 f^2 S a S c v^2 w \mu^2 \nu + 6 e f^3 S a S c v^2 w \mu^2 \nu - 12 e f^2 g S a S c v^2 w \mu^2 \nu + \\
& 12 f^3 g S a S c v^2 w \mu^2 \nu - 6 e^2 g^2 S a S c v^2 w \mu^2 \nu + 12 e f g^2 S a S c v^2 w \mu^2 \nu - 9 f^2 g^2 S a S c v^2 w \mu^2 \nu - \\
& 6 e^4 S b S c v^2 w \mu^2 \nu + 12 e^3 f S b S c v^2 w \mu^2 \nu - 3 e^2 f^2 S b S c v^2 w \mu^2 \nu + 12 e^3 g S b S c v^2 w \mu^2 \nu - \\
& 24 e^2 f g S b S c v^2 w \mu^2 \nu + 12 e f^2 g S b S c v^2 w \mu^2 \nu - 6 e^2 g^2 S b S c v^2 w \mu^2 \nu + 12 e f g^2 S b S c v^2 w \mu^2 \nu - \\
& 9 f^2 g^2 S b S c v^2 w \mu^2 \nu - 3 e^4 S c^2 v^2 w \mu^2 \nu + 12 e^3 f S c^2 v^2 w \mu^2 \nu - 15 e^2 f^2 S c^2 v^2 w \mu^2 \nu + \\
& 6 e f^3 S c^2 v^2 w \mu^2 \nu + 6 e f^2 g S a^2 u w^2 \mu^2 \nu + 6 e f g^2 S a^2 u w^2 \mu^2 \nu - 12 e g^3 S a^2 u w^2 \mu^2 \nu + \\
& 6 e^3 g S a S b u w^2 \mu^2 \nu - 6 e^2 g^2 S a S b u w^2 \mu^2 \nu + 6 e^3 g S a S c u w^2 \mu^2 \nu - 12 e^2 f g S a S c u w^2 \mu^2 \nu + \\
& 12 e f^2 g S a S c u w^2 \mu^2 \nu - 12 e^2 g^2 S a S c u w^2 \mu^2 \nu + 6 e f g^2 S a S c u w^2 \mu^2 \nu + 6 e^3 g S b S c u w^2 \mu^2 \nu - \\
& 6 e^2 g^2 S b S c u w^2 \mu^2 \nu + 6 e^3 g S c^2 u w^2 \mu^2 \nu - 12 e^2 f g S c^2 u w^2 \mu^2 \nu + 6 e f^2 g S c^2 u w^2 \mu^2 \nu + \\
& 6 f^3 g S a^2 v w^2 \mu^2 \nu - 6 g^4 S a^2 v w^2 \mu^2 \nu + 6 e^2 f g S a S b v w^2 \mu^2 \nu - 6 e^2 g^2 S a S b v w^2 \mu^2 \nu - \\
& 6 e f g^2 S a S b v w^2 \mu^2 \nu + 12 e g^3 S a S b v w^2 \mu^2 \nu - 6 g^4 S a S b v w^2 \mu^2 \nu + 6 e^2 f g S a S c v w^2 \mu^2 \nu - \\
& 12 e f^2 g S a S c v w^2 \mu^2 \nu + 12 f^3 g S a S c v w^2 \mu^2 \nu - 6 e^2 g^2 S a S c v w^2 \mu^2 \nu + 6 e^2 f g S b S c v w^2 \mu^2 \nu - \\
& 6 e f g^2 S b S c v w^2 \mu^2 \nu + 6 e^2 f g S c^2 v w^2 \mu^2 \nu - 12 e f^2 g S c^2 v w^2 \mu^2 \nu + 6 f^3 g S c^2 v w^2 \mu^2 \nu + \\
& 3 f^2 g^2 S a^2 w^3 \mu^2 \nu - 3 g^4 S a^2 w^3 \mu^2 \nu + 3 e^2 g^2 S a S b w^3 \mu^2 \nu - 4 e g^3 S a S b w^3 \mu^2 \nu + \\
& g^4 S a S b w^3 \mu^2 \nu + 3 e^2 g^2 S a S c w^3 \mu^2 \nu - 6 e f g^2 S a S c w^3 \mu^2 \nu + 6 f^2 g^2 S a S c w^3 \mu^2 \nu - \\
& 4 e g^3 S a S c w^3 \mu^2 \nu + g^4 S a S c w^3 \mu^2 \nu + 3 e^2 g^2 S b S c w^3 \mu^2 \nu - 4 e g^3 S b S c w^3 \mu^2 \nu + \\
& g^4 S b S c w^3 \mu^2 \nu + 3 e^2 g^2 S c^2 w^3 \mu^2 \nu - 6 e f g^2 S c^2 w^3 \mu^2 \nu + 3 f^2 g^2 S c^2 w^3 \mu^2 \nu + 3 e^2 f^2 S a^2 u^3 \lambda v^2 - \\
& 6 e^2 f g S a^2 u^3 \lambda v^2 + 3 e^2 g^2 S a^2 u^3 \lambda v^2 + e^4 S a S b u^3 \lambda v^2 - 4 e^3 f S a S b u^3 \lambda v^2 + \\
& 3 e^2 f^2 S a S b u^3 \lambda v^2 - 6 e^2 f g S a S b u^3 \lambda v^2 + 6 e^2 g^2 S a S b u^3 \lambda v^2 - 3 e^4 S b^2 u^3 \lambda v^2 + \\
& 3 e^2 g^2 S b^2 u^3 \lambda v^2 + e^4 S a S c u^3 \lambda v^2 - 4 e^3 f S a S c u^3 \lambda v^2 + 3 e^2 f^2 S a S c u^3 \lambda v^2 +
\end{aligned}$$

$$\begin{aligned}
& e^4 \text{Sb Sc u}^3 \lambda v^2 - 4 e^3 f \text{Sb Sc u}^3 \lambda v^2 + 3 e^2 f^2 \text{Sb Sc u}^3 \lambda v^2 + 6 e f^3 \text{Sa}^2 u^2 v \lambda v^2 - \\
& 12 e f^2 g \text{Sa}^2 u^2 v \lambda v^2 + 6 e f g^2 \text{Sa}^2 u^2 v \lambda v^2 - 12 e^2 f^2 \text{Sa Sb u}^2 v \lambda v^2 + 6 e f^3 \text{Sa Sb u}^2 v \lambda v^2 + \\
& 6 e^2 f g \text{Sa Sb u}^2 v \lambda v^2 - 12 e f^2 g \text{Sa Sb u}^2 v \lambda v^2 + 12 e f g^2 \text{Sa Sb u}^2 v \lambda v^2 - 12 e^3 f \text{Sb}^2 u^2 v \lambda v^2 + \\
& 6 e^2 f g \text{Sb}^2 u^2 v \lambda v^2 + 6 e f g^2 \text{Sb}^2 u^2 v \lambda v^2 - 6 e^2 f^2 \text{Sa Sc u}^2 v \lambda v^2 + 6 e f^3 \text{Sa Sc u}^2 v \lambda v^2 - \\
& 6 e^2 f^2 \text{Sb Sc u}^2 v \lambda v^2 + 6 e f^3 \text{Sb Sc u}^2 v \lambda v^2 + 3 f^4 \text{Sa}^2 u v^2 \lambda v^2 - 6 f^3 g \text{Sa}^2 u v^2 \lambda v^2 + \\
& 3 f^2 g^2 \text{Sa}^2 u v^2 \lambda v^2 - 3 e^2 f^2 \text{Sa Sb u} v^2 \lambda v^2 - 12 e f^3 \text{Sa Sb u} v^2 \lambda v^2 + 3 f^4 \text{Sa Sb u} v^2 \lambda v^2 + \\
& 12 e f^2 g \text{Sa Sb u} v^2 \lambda v^2 - 6 f^3 g \text{Sa Sb u} v^2 \lambda v^2 + 6 f^2 g^2 \text{Sa Sb u} v^2 \lambda v^2 - 15 e^2 f^2 \text{Sb}^2 u v^2 \lambda v^2 + \\
& 12 e f^2 g \text{Sb}^2 u v^2 \lambda v^2 + 3 f^2 g^2 \text{Sb}^2 u v^2 \lambda v^2 - 3 e^2 f^2 \text{Sa Sc u} v^2 \lambda v^2 + 3 f^4 \text{Sa Sc u} v^2 \lambda v^2 - \\
& 3 e^2 f^2 \text{Sb Sc u} v^2 \lambda v^2 + 3 f^4 \text{Sb Sc u} v^2 \lambda v^2 - 2 e f^3 \text{Sa Sb v}^3 \lambda v^2 - 4 f^4 \text{Sa Sb v}^3 \lambda v^2 + \\
& 6 f^3 g \text{Sa Sb v}^3 \lambda v^2 - 6 e f^3 \text{Sb}^2 v^3 \lambda v^2 + 6 f^3 g \text{Sb}^2 v^3 \lambda v^2 - 2 e f^3 \text{Sa Sc v}^3 \lambda v^2 + \\
& 2 f^4 \text{Sa Sc v}^3 \lambda v^2 - 2 e f^3 \text{Sb Sc v}^3 \lambda v^2 + 2 f^4 \text{Sb Sc v}^3 \lambda v^2 + 6 e f^2 g \text{Sa}^2 u^2 w \lambda v^2 - \\
& 12 e f g^2 \text{Sa}^2 u^2 w \lambda v^2 + 6 e g^3 \text{Sa}^2 u^2 w \lambda v^2 - 6 e^2 f^2 \text{Sa Sb u}^2 w \lambda v^2 + 6 e f^2 g \text{Sa Sb u}^2 w \lambda v^2 - \\
& 12 e f g^2 \text{Sa Sb u}^2 w \lambda v^2 + 12 e g^3 \text{Sa Sb u}^2 w \lambda v^2 - 6 e^4 \text{Sb}^2 u^2 w \lambda v^2 + 6 e g^3 \text{Sb}^2 u^2 w \lambda v^2 - \\
& 6 e^2 f g \text{Sa Sc u}^2 w \lambda v^2 + 6 e f^2 g \text{Sa Sc u}^2 w \lambda v^2 - 6 e^4 \text{Sb Sc u}^2 w \lambda v^2 + 12 e^3 f \text{Sb Sc u}^2 w \lambda v^2 - \\
& 6 e^2 f^2 \text{Sb Sc u}^2 w \lambda v^2 - 6 e^2 f g \text{Sb Sc u}^2 w \lambda v^2 + 6 e f^2 g \text{Sb Sc u}^2 w \lambda v^2 + 6 f^3 g \text{Sa}^2 u v w \lambda v^2 - \\
& 12 e f^2 g^2 \text{Sa}^2 u v w \lambda v^2 + 6 f g^3 \text{Sa}^2 u v w \lambda v^2 - 12 e f^3 \text{Sa Sb u} v w \lambda v^2 - 6 e^2 f g \text{Sa Sb u} v w \lambda v^2 + \\
& 6 f^3 g \text{Sa Sb u} v w \lambda v^2 + 12 e f g^2 \text{Sa Sb u} v w \lambda v^2 - 12 f^2 g^2 \text{Sa Sb u} v w \lambda v^2 + \\
& 12 f g^3 \text{Sa Sb u} v w \lambda v^2 - 12 e^3 f \text{Sb}^2 u v w \lambda v^2 - 6 e^2 f g \text{Sb}^2 u v w \lambda v^2 + 12 e f g^2 \text{Sb}^2 u v w \lambda v^2 + \\
& 6 f g^3 \text{Sb}^2 u v w \lambda v^2 - 6 e^2 f g \text{Sa Sc u} v w \lambda v^2 + 6 f^3 g \text{Sa Sc u} v w \lambda v^2 - 12 e^3 f \text{Sb Sc u} v w \lambda v^2 + \\
& 24 e f^2 g^2 \text{Sb Sc u} v w \lambda v^2 - 12 e f^3 \text{Sb Sc u} v w \lambda v^2 - 6 e^2 f g \text{Sb Sc u} v w \lambda v^2 + 6 f^3 g \text{Sb Sc u} v w \lambda v^2 - \\
& 6 f^4 \text{Sa Sb v}^2 w \lambda v^2 - 6 e f^2 g \text{Sa Sb v}^2 w \lambda v^2 + 12 f^2 g^2 \text{Sa Sb v}^2 w \lambda v^2 - 6 e^2 f^2 \text{Sb}^2 v^2 w \lambda v^2 - \\
& 6 e f^2 g \text{Sb}^2 v^2 w \lambda v^2 + 12 f^2 g^2 \text{Sb}^2 v^2 w \lambda v^2 - 6 e f^2 g \text{Sa Sc v}^2 w \lambda v^2 + 6 f^3 g \text{Sa Sc v}^2 w \lambda v^2 - \\
& 6 e^2 f^2 \text{Sb Sc v}^2 w \lambda v^2 + 12 e f^3 \text{Sb Sc v}^2 w \lambda v^2 - 6 f^4 \text{Sb Sc v}^2 w \lambda v^2 - 6 e f^2 g \text{Sb Sc v}^2 w \lambda v^2 + \\
& 6 f^3 g \text{Sb Sc v}^2 w \lambda v^2 - 3 f^4 \text{Sa}^2 u w^2 \lambda v^2 + 12 f^3 g \text{Sa}^2 u w^2 \lambda v^2 - 15 f^2 g^2 \text{Sa}^2 u w^2 \lambda v^2 + \\
& 6 f g^3 \text{Sa}^2 u w^2 \lambda v^2 - 6 e^2 f^2 \text{Sa Sb u} w^2 \lambda v^2 + 12 e f^2 g \text{Sa Sb u} w^2 \lambda v^2 - 9 e^2 g^2 \text{Sa Sb u} w^2 \lambda v^2 - \\
& 12 e f g^2 \text{Sa Sb u} w^2 \lambda v^2 - 3 f^2 g^2 \text{Sa Sb u} w^2 \lambda v^2 + 12 e g^3 \text{Sa Sb u} w^2 \lambda v^2 + 6 f g^3 \text{Sa Sb u} w^2 \lambda v^2 - \\
& 3 e^4 \text{Sb}^2 u w^2 \lambda v^2 - 9 e^2 g^2 \text{Sb}^2 u w^2 \lambda v^2 + 12 e g^3 \text{Sb}^2 u w^2 \lambda v^2 - 6 e^2 f^2 \text{Sa Sc u} w^2 \lambda v^2 + \\
& 12 e f^3 \text{Sa Sc u} w^2 \lambda v^2 - 6 f^4 \text{Sa Sc u} w^2 \lambda v^2 + 12 e f^2 g \text{Sa Sc u} w^2 \lambda v^2 - 24 e f^2 g \text{Sa Sc u} w^2 \lambda v^2 + \\
& 12 f^2 g^2 \text{Sa Sc u} w^2 \lambda v^2 - 9 e^2 g^2 \text{Sa Sc u} w^2 \lambda v^2 + 12 e f g^2 \text{Sa Sc u} w^2 \lambda v^2 - 3 f^2 g^2 \text{Sa Sc u} w^2 \lambda v^2 - \\
& 6 e^4 \text{Sb Sc u} w^2 \lambda v^2 + 12 e^3 f \text{Sb Sc u} w^2 \lambda v^2 - 6 e^2 f^2 \text{Sb Sc u} w^2 \lambda v^2 - 9 e^2 g^2 \text{Sb Sc u} w^2 \lambda v^2 + \\
& 12 e f g^2 \text{Sb Sc u} w^2 \lambda v^2 - 3 f^2 g^2 \text{Sb Sc u} w^2 \lambda v^2 - 3 e^4 \text{Sc}^2 u w^2 \lambda v^2 + 12 e^3 f \text{Sc}^2 u w^2 \lambda v^2 - \\
& 18 e^2 f^2 \text{Sc}^2 u w^2 \lambda v^2 + 12 e f^3 \text{Sc}^2 u w^2 \lambda v^2 - 3 f^4 \text{Sc}^2 u w^2 \lambda v^2 - 12 f^3 g \text{Sa Sb v} w^2 \lambda v^2 - \\
& 6 e f g^2 \text{Sa Sb v} w^2 \lambda v^2 + 12 f^2 g^2 \text{Sa Sb v} w^2 \lambda v^2 + 6 f g^3 \text{Sa Sb v} w^2 \lambda v^2 - 12 e^2 f g \text{Sb}^2 v w^2 \lambda v^2 + \\
& 6 e f g^2 \text{Sb}^2 v w^2 \lambda v^2 + 6 f g^3 \text{Sb}^2 v w^2 \lambda v^2 - 6 e f g^2 \text{Sa Sc v} w^2 \lambda v^2 + 6 f^2 g^2 \text{Sa Sc v} w^2 \lambda v^2 - \\
& 12 e^2 f g \text{Sb Sc v} w^2 \lambda v^2 + 24 e f^2 g \text{Sb Sc v} w^2 \lambda v^2 - 12 f^3 g \text{Sb Sc v} w^2 \lambda v^2 - 6 e f g^2 \text{Sb Sc v} w^2 \lambda v^2 + \\
& 6 f^2 g^2 \text{Sb Sc v} w^2 \lambda v^2 - 6 f^2 g^2 \text{Sa Sb w}^3 \lambda v^2 - 2 e g^3 \text{Sa Sb w}^3 \lambda v^2 + 8 f g^3 \text{Sa Sb w}^3 \lambda v^2 - \\
& 6 e^2 g^2 \text{Sb}^2 w^3 \lambda v^2 + 6 e g^3 \text{Sb}^2 w^3 \lambda v^2 - 2 e g^3 \text{Sa Sc w}^3 \lambda v^2 + 2 f g^3 \text{Sa Sc w}^3 \lambda v^2 - \\
& 6 e^2 g^2 \text{Sb Sc w}^3 \lambda v^2 + 12 e f g^2 \text{Sb Sc w}^3 \lambda v^2 - 6 f^2 g^2 \text{Sb Sc w}^3 \lambda v^2 - 2 e g^3 \text{Sb Sc w}^3 \lambda v^2 + \\
& 2 f g^3 \text{Sb Sc w}^3 \lambda v^2 - 6 e^3 f \text{Sa}^2 u^3 \mu v^2 + 6 e^3 g \text{Sa}^2 u^3 \mu v^2 - 4 e^4 \text{Sa Sb u}^3 \mu v^2 - \\
& 2 e^3 f \text{Sa Sb u}^3 \mu v^2 + 6 e^3 g \text{Sa Sb u}^3 \mu v^2 + 2 e^4 \text{Sa Sc u}^3 \mu v^2 - 2 e^3 f \text{Sa Sc u}^3 \mu v^2 + \\
& 2 e^4 \text{Sb Sc u}^3 \mu v^2 - 2 e^3 f \text{Sb Sc u}^3 \mu v^2 - 15 e^2 f^2 \text{Sa}^2 u^2 v \mu v^2 + 12 e^2 f g \text{Sa}^2 u^2 v \mu v^2 + \\
& 3 e^2 g^2 \text{Sa}^2 u^2 v \mu v^2 + 3 e^4 \text{Sa Sb u}^2 v \mu v^2 - 12 e^3 f \text{Sa Sb u}^2 v \mu v^2 - 3 e^2 f^2 \text{Sa Sb u}^2 v \mu v^2 - \\
& 6 e^3 g \text{Sa Sb u}^2 v \mu v^2 + 12 e^2 f g \text{Sa Sb u}^2 v \mu v^2 + 6 e^2 g^2 \text{Sa Sb u}^2 v \mu v^2 + 3 e^4 \text{Sb}^2 u^2 v \mu v^2 - \\
& 6 e^3 g \text{Sb}^2 u^2 v \mu v^2 + 3 e^2 g^2 \text{Sb}^2 u^2 v \mu v^2 + 3 e^4 \text{Sa Sc u}^2 v \mu v^2 - 3 e^2 f^2 \text{Sa Sc u}^2 v \mu v^2 + \\
& 3 e^4 \text{Sb Sc u}^2 v \mu v^2 - 3 e^2 f^2 \text{Sb Sc u}^2 v \mu v^2 - 12 e f^3 \text{Sa}^2 u v^2 \mu v^2 + 6 e f^2 g \text{Sa}^2 u v^2 \mu v^2 + \\
& 6 e f g^2 \text{Sa}^2 u v^2 \mu v^2 + 6 e^3 f \text{Sa Sb u} v^2 \mu v^2 - 12 e^2 f^2 \text{Sa Sb u} v^2 \mu v^2 - 12 e^2 f g \text{Sa Sb u} v^2 \mu v^2 + \\
& 6 e f^2 g \text{Sa Sb u} v^2 \mu v^2 + 12 e f g^2 \text{Sa Sb u} v^2 \mu v^2 + 6 e^3 f \text{Sb}^2 u v^2 \mu v^2 - 12 e^2 f g \text{Sb}^2 u v^2 \mu v^2 + \\
& 6 e f g^2 \text{Sb}^2 u v^2 \mu v^2 + 6 e^3 f \text{Sa Sc u} v^2 \mu v^2 - 6 e^2 f^2 \text{Sa Sc u} v^2 \mu v^2 + 6 e^3 f \text{Sb Sc u} v^2 \mu v^2 - \\
& 6 e^2 f^2 \text{Sb Sc u} v^2 \mu v^2 - 3 f^4 \text{Sa}^2 v^3 \mu v^2 + 3 f^2 g^2 \text{Sa}^2 v^3 \mu v^2 + 3 e^2 f^2 \text{Sa Sb v}^3 \mu v^2 - \\
& 4 e f^3 \text{Sa Sb v}^3 \mu v^2 + f^4 \text{Sa Sb v}^3 \mu v^2 - 6 e f^2 g \text{Sa Sb v}^3 \mu v^2 + 6 f^2 g^2 \text{Sa Sb v}^3 \mu v^2 + \\
& 3 e^2 f^2 \text{Sb}^2 v^3 \mu v^2 - 6 e f^2 g \text{Sb}^2 v^3 \mu v^2 + 3 f^2 g^2 \text{Sb}^2 v^3 \mu v^2 + 3 e^2 f^2 \text{Sa Sc v}^3 \mu v^2 - \\
& 4 e f^3 \text{Sa Sc v}^3 \mu v^2 + f^4 \text{Sa Sc v}^3 \mu v^2 + 3 e^2 f^2 \text{Sb Sc v}^3 \mu v^2 - 4 e f^3 \text{Sb Sc v}^3 \mu v^2 + \\
& f^4 \text{Sb Sc v}^3 \mu v^2 - 6 e^2 f^2 \text{Sa}^2 u^2 w \mu v^2 - 6 e^2 f g \text{Sa}^2 u^2 w \mu v^2 + 12 e^2 g^2 \text{Sa}^2 u^2 w \mu v^2 - \\
& 6 e^4 \text{Sa Sb u}^2 w \mu v^2 - 6 e^2 f g \text{Sa Sb u}^2 w \mu v^2 + 12 e^2 g^2 \text{Sa Sb u}^2 w \mu v^2 - 6 e^4 \text{Sa Sc u}^2 w \mu v^2 + \\
& 12 e^3 f \text{Sa Sc u}^2 w \mu v^2 - 6 e^2 f^2 \text{Sa Sc u}^2 w \mu v^2 + 6 e^3 g \text{Sa Sc u}^2 w \mu v^2 - 6 e^2 f g \text{Sa Sc u}^2 w \mu v^2 + \\
& 6 e^3 g \text{Sb Sc u}^2 w \mu v^2 - 6 e^2 f g \text{Sb Sc u}^2 w \mu v^2 - 12 e f^3 \text{Sa}^2 u v w \mu v^2 - 6 e f^2 g \text{Sa}^2 u v w \mu v^2 + \\
& 12 e f g^2 \text{Sa}^2 u v w \mu v^2 + 6 e g^3 \text{Sa}^2 u v w \mu v^2 - 12 e^3 f \text{Sa Sb u} v w \mu v^2 + 6 e^3 g \text{Sa Sb u} v w \mu v^2 - \\
& 6 e f^2 g \text{Sa Sb u} v w \mu v^2 - 12 e^2 g^2 \text{Sa Sb u} v w \mu v^2 + 12 e f g^2 \text{Sa Sb u} v w \mu v^2 +
\end{aligned}$$

$$\begin{aligned}
& 12 e g^3 S a S b u v w \mu v^2 + 6 e^3 g S b^2 u v w \mu v^2 - 12 e^2 g^2 S b^2 u v w \mu v^2 + 6 e g^3 S b^2 u v w \mu v^2 - \\
& 12 e^3 f S a S c u v w \mu v^2 + 24 e^2 f^2 S a S c u v w \mu v^2 - 12 e f^3 S a S c u v w \mu v^2 + 6 e^3 g S a S c u v w \mu v^2 - \\
& 6 e f^2 g S a S c u v w \mu v^2 + 6 e^3 g S b S c u v w \mu v^2 - 6 e f^2 g S b S c u v w \mu v^2 - 6 f^4 S a^2 v^2 w \mu v^2 + \\
& 6 f g^3 S a^2 v^2 w \mu v^2 - 6 e^2 f^2 S a S b v^2 w \mu v^2 + 6 e^2 f g S a S b v^2 w \mu v^2 - 12 e f g^2 S a S b v^2 w \mu v^2 + \\
& 12 f g^3 S a S b v^2 w \mu v^2 + 6 e^2 f g S b^2 v^2 w \mu v^2 - 12 e f g^2 S b^2 v^2 w \mu v^2 + 6 f g^3 S b^2 v^2 w \mu v^2 - \\
& 6 e^2 f^2 S a S c v^2 w \mu v^2 + 12 e f^3 S a S c v^2 w \mu v^2 - 6 f^4 S a S c v^2 w \mu v^2 + 6 e^2 f g S a S c v^2 w \mu v^2 - \\
& 6 e f^2 g S a S c v^2 w \mu v^2 + 6 e^2 f g S b S c v^2 w \mu v^2 - 6 e f^2 g S b S c v^2 w \mu v^2 - 12 e f^2 g S a^2 u w^2 \mu v^2 + \\
& 6 e f g^2 S a^2 u w^2 \mu v^2 + 6 e^3 g S a^2 u w^2 \mu v^2 - 12 e^3 g S a S b u w^2 \mu v^2 + 12 e^2 g^2 S a S b u w^2 \mu v^2 - \\
& 6 e f g^2 S a S b u w^2 \mu v^2 + 6 e^3 g S a S b u w^2 \mu v^2 - 12 e^3 g S a S c u w^2 \mu v^2 + 24 e^2 f g S a S c u w^2 \mu v^2 - \\
& 12 e f^2 g S a S c u w^2 \mu v^2 + 6 e^2 g^2 S a S c u w^2 \mu v^2 - 6 e f g^2 S a S c u w^2 \mu v^2 + 6 e^2 g^2 S b S c u w^2 \mu v^2 - \\
& 6 e f g^2 S b S c u w^2 \mu v^2 - 3 f^4 S a^2 v w^2 \mu v^2 - 9 f^2 g^2 S a^2 v w^2 \mu v^2 + 12 f g^3 S a^2 v w^2 \mu v^2 - \\
& 6 e^2 f^2 S a S b v w^2 \mu v^2 + 12 e f^2 g S a S b v w^2 \mu v^2 - 3 e^2 g^2 S a S b v w^2 \mu v^2 - 12 e f g^2 S a S b v w^2 \mu v^2 - \\
& 9 f^2 g^2 S a S b v w^2 \mu v^2 + 6 e^3 g S a S b v w^2 \mu v^2 + 12 f g^3 S a S b v w^2 \mu v^2 - 3 e^4 S b^2 v w^2 \mu v^2 + \\
& 12 e^3 g S b^2 v w^2 \mu v^2 - 15 e^2 g^2 S b^2 v w^2 \mu v^2 + 6 e g^3 S b^2 v w^2 \mu v^2 - 6 e^2 f^2 S a S c v w^2 \mu v^2 + \\
& 12 e f^3 S a S c v w^2 \mu v^2 - 6 f^4 S a S c v w^2 \mu v^2 - 3 e^2 g^2 S a S c v w^2 \mu v^2 + 12 e f g^2 S a S c v w^2 \mu v^2 - \\
& 9 f^2 g^2 S a S c v w^2 \mu v^2 - 6 e^4 S b S c v w^2 \mu v^2 + 12 e^3 f S b S c v w^2 \mu v^2 - 6 e^2 f^2 S b S c v w^2 \mu v^2 + \\
& 12 e^3 g S b S c v w^2 \mu v^2 - 24 e^2 f g S b S c v w^2 \mu v^2 + 12 e f^2 g S b S c v w^2 \mu v^2 - 3 e^2 g^2 S b S c v w^2 \mu v^2 + \\
& 12 e f g^2 S b S c v w^2 \mu v^2 - 9 f^2 g^2 S b S c v w^2 \mu v^2 - 3 e^4 S c^2 v w^2 \mu v^2 + 12 e^3 f S c^2 v w^2 \mu v^2 - \\
& 18 e^2 f^2 S c^2 v w^2 \mu v^2 + 12 e f^3 S c^2 v w^2 \mu v^2 - 3 f^4 S c^2 v w^2 \mu v^2 - 6 f^2 g^2 S a^2 w^3 \mu v^2 + \\
& 6 f g^3 S a^2 w^3 \mu v^2 - 6 e^2 g^2 S a S b w^3 \mu v^2 + 8 e g^3 S a S b w^3 \mu v^2 - 2 f g^3 S a S b w^3 \mu v^2 - \\
& 6 e^2 f^2 S a S c w^3 \mu v^2 + 12 e f g^2 S a S c w^3 \mu v^2 - 6 f^2 g^2 S a S c w^3 \mu v^2 + 2 e g^3 S a S c w^3 \mu v^2 - \\
& 2 f g^3 S a S c w^3 \mu v^2 + 2 e g^3 S b S c w^3 \mu v^2 - 2 f g^3 S b S c w^3 \mu v^2 + 2 e^3 f S a^2 u^3 v^3 - 2 e^3 g S a^2 u^3 v^3 + \\
& 2 e^4 S a S b u^3 v^3 + 2 e^3 f S a S b u^3 v^3 - 4 e^3 g S a S b u^3 v^3 + 2 e^4 S b^2 u^3 v^3 - 2 e^3 g S b^2 u^3 v^3 + \\
& 6 e^2 f^2 S a^2 u^2 v^3 - 6 e^2 f g S a^2 u^2 v^3 + 6 e^3 f S a S b u^2 v^3 + 6 e^2 f^2 S a S b u^2 v^3 - \\
& 12 e^3 f g S a S b u^2 v^3 + 6 e^3 f S b^2 u^2 v^3 - 6 e^2 f g S b^2 u^2 v^3 + 6 e f^3 S a^2 u v^3 - \\
& 6 e f^2 g S a^2 u v^3 + 6 e^2 f^2 S a S b u v^3 + 6 e f^3 S a S b u v^3 - 12 e f^2 g S a S b u v^3 + \\
& 6 e^2 f^2 S b^2 u v^3 - 6 e f^2 g S b^2 u v^3 + 2 f^4 S a^2 v^3 v^3 - 2 f^3 g S a^2 v^3 v^3 + 2 e f^3 S a S b v^3 v^3 + \\
& 2 f^4 S a S b v^3 v^3 - 4 f^3 g S a S b v^3 v^3 + 2 e f^3 S b^2 v^3 v^3 - 2 f^3 g S b^2 v^3 v^3 + 3 e^2 f^2 S a^2 u^2 w v^3 - \\
& 3 e^2 g^2 S a^2 u^2 w v^3 + 3 e^4 S a S b u^2 w v^3 + 3 e^2 f^2 S a S b u^2 w v^3 - 6 e^2 g^2 S a S b u^2 w v^3 + \\
& 3 e^4 S b^2 u^2 w v^3 - 3 e^2 g^2 S b^2 u^2 w v^3 + 3 e^4 S a S c u^2 w v^3 - 6 e^3 f S a S c u^2 w v^3 + 3 e^2 f^2 S a S c u^2 w v^3 + \\
& 3 e^4 S b S c u^2 w v^3 - 6 e^3 f S b S c u^2 w v^3 + 3 e^2 f^2 S b S c u^2 w v^3 + 6 e f^3 S a^2 u v w v^3 - \\
& 6 e f g^2 S a^2 u v w v^3 + 6 e^3 f S a S b u v w v^3 + 6 e f^3 S a S b u v w v^3 - 12 e f g^2 S a S b u v w v^3 + \\
& 6 e^3 f S b^2 u v w v^3 - 6 e f g^2 S b^2 u v w v^3 + 6 e^3 f S a S c u v w v^3 - 12 e^2 f^2 S a S c u v w v^3 + \\
& 6 e f^3 S a S c u v w v^3 + 6 e^3 f S b S c u v w v^3 - 12 e^2 f^2 S b S c u v w v^3 + 6 e f^3 S b S c u v w v^3 + \\
& 3 f^4 S a^2 v^2 w v^3 - 3 f^2 g^2 S a^2 v^2 w v^3 + 3 e^2 f^2 S a S b v^2 w v^3 + 3 f^4 S a S b v^2 w v^3 - \\
& 6 f^2 g^2 S a S b v^2 w v^3 + 3 e^2 f^2 S b^2 v^2 w v^3 - 3 f^2 g^2 S b^2 v^2 w v^3 + 3 e^2 f^2 S a S c v^2 w v^3 - \\
& 6 e f^3 S a S c v^2 w v^3 + 3 f^4 S a S c v^2 w v^3 + 3 e^2 f^2 S b S c v^2 w v^3 - 6 e f^3 S b S c v^2 w v^3 + \\
& 3 f^4 S b S c v^2 w v^3 + 6 e f^2 g S a^2 u w^2 v^3 - 6 e f g^2 S a^2 u w^2 v^3 + 6 e^3 g S a S b u w^2 v^3 + \\
& 6 e f^2 g S a S b u w^2 v^3 - 6 e^2 g^2 S a S b u w^2 v^3 - 6 e f g^2 S a S b u w^2 v^3 + 6 e^3 g S b^2 u w^2 v^3 - \\
& 6 e^2 g^2 S b^2 u w^2 v^3 + 6 e^3 g S a S c u w^2 v^3 - 12 e^2 f g S a S c u w^2 v^3 + 6 e f^2 g S a S c u w^2 v^3 + \\
& 6 e^3 g S b S c u w^2 v^3 - 12 e^2 f g S b S c u w^2 v^3 + 6 e f^2 g S b S c u w^2 v^3 + 6 f^3 g S a^2 v w^2 v^3 - \\
& 6 f^2 g^2 S a^2 v w^2 v^3 + 6 e^2 f g S a S b v w^2 v^3 + 6 f^3 g S a S b v w^2 v^3 - 6 e f g^2 S a S b v w^2 v^3 - \\
& 6 f^2 g^2 S a S b v w^2 v^3 + 6 e^2 f g S b^2 v w^2 v^3 - 6 e f g^2 S b^2 v w^2 v^3 + 6 e^2 f g S a S c v w^2 v^3 - \\
& 12 e f^2 g S a S c v w^2 v^3 + 6 f^3 g S a S c v w^2 v^3 + 6 e^2 f g S b S c v w^2 v^3 - 12 e f^2 g S b S c v w^2 v^3 + \\
& 6 f^3 g S b S c v w^2 v^3 - f^4 S a^2 w^3 v^3 + 4 f^3 g S a^2 w^3 v^3 - 3 f^2 g^2 S a^2 w^3 v^3 - 2 e^2 f^2 S a S b w^3 v^3 + \\
& 4 e^2 f g S a S b w^3 v^3 + 4 e f^2 g S a S b w^3 v^3 + e^2 g^2 S a S b w^3 v^3 - 8 e f g^2 S a S b w^3 v^3 + \\
& f^2 g^2 S a S b w^3 v^3 - e^4 S b^2 w^3 v^3 + 4 e^3 g S b^2 w^3 v^3 - 3 e^2 g^2 S b^2 w^3 v^3 - 2 e^2 f^2 S a S c w^3 v^3 + \\
& 4 e f^3 S a S c w^3 v^3 - 2 f^4 S a S c w^3 v^3 + 4 e^2 f g S a S c w^3 v^3 - 8 e f^2 g S a S c w^3 v^3 + \\
& 4 f^3 g S a S c w^3 v^3 + e^2 g^2 S a S c w^3 v^3 - 2 e f g^2 S a S c w^3 v^3 + f^2 g^2 S a S c w^3 v^3 - 2 e^4 S b S c w^3 v^3 + \\
& 4 e^3 f S b S c w^3 v^3 - 2 e^2 f^2 S b S c w^3 v^3 + 4 e^3 g S b S c w^3 v^3 - 8 e^2 f g S b S c w^3 v^3 + \\
& 4 e f^2 g S b S c w^3 v^3 + e^2 g^2 S b S c w^3 v^3 - 2 e f g^2 S b S c w^3 v^3 + f^2 g^2 S b S c w^3 v^3 - \\
& e^4 S c^2 w^3 v^3 + 4 e^3 f S c^2 w^3 v^3 - 6 e^2 f^2 S c^2 w^3 v^3 + 4 e f^3 S c^2 w^3 v^3 - f^4 S c^2 w^3 v^3)
\end{aligned}$$

For the 4 lines of a quadrilateral $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, (l,m,n)
this gives 4 equations.

Eliminating e, f, g there remain two equations

for the centers (if the following terms are zero).

```
T1[{u_, v_, w_}] :=
(a^4 b^5 c^6 u^9 - 2 a^2 b^8 c^6 u^9 + b^10 c^6 u^9 - 2 a^2 b^6 c^8 u^9 - 2 b^8 c^8 u^9 + b^6 c^10 u^9 + 27 a^2 b^6 c^8 u^7 v^2 +
3 a^6 b^4 c^6 u^6 v^3 - 6 a^4 b^6 c^6 u^6 v^3 + 3 a^2 b^8 c^6 u^6 v^3 - 6 a^4 b^4 c^8 u^6 v^3 + 21 a^2 b^6 c^8 u^6 v^3 +
3 a^2 b^4 c^10 u^6 v^3 - 54 a^4 b^4 c^8 u^5 v^4 - 54 a^4 b^4 c^8 u^4 v^5 + 3 a^8 b^2 c^6 u^3 v^6 - 6 a^6 b^4 c^6 u^3 v^6 +
3 a^4 b^6 c^6 u^3 v^6 + 21 a^6 b^2 c^8 u^3 v^6 - 6 a^4 b^4 c^8 u^3 v^6 + 3 a^4 b^2 c^10 u^3 v^6 + 27 a^6 b^2 c^8 u^2 v^7 +
a^10 c^6 v^9 - 2 a^8 b^2 c^6 v^9 + a^6 b^4 c^6 v^9 - 2 a^8 c^8 v^9 - 2 a^6 b^2 c^8 v^9 + a^6 c^10 v^9 - 27 a^4 b^6 c^6 u^7 v w +
27 a^2 b^8 c^6 u^7 v w + 27 a^2 b^6 c^8 u^7 v w - 27 a^4 b^6 c^6 u^6 v^2 w + 27 a^2 b^8 c^6 u^6 v^2 w + 54 a^2 b^6 c^8 u^6 v^2 w +
27 a^6 b^4 c^6 u^5 v^3 w - 27 a^4 b^6 c^6 u^5 v^3 w - 27 a^4 b^4 c^8 u^5 v^3 w - 108 a^4 b^4 c^8 u^4 v^4 w -
27 a^6 b^4 c^6 u^3 v^5 w + 27 a^4 b^6 c^6 u^3 v^5 w - 27 a^4 b^4 c^8 u^3 v^5 w + 27 a^8 b^2 c^6 u^2 v^6 w - 27 a^6 b^4 c^6 u^2 v^6 w +
54 a^6 b^2 c^8 u^2 v^6 w + 27 a^8 b^2 c^6 u v^7 w - 27 a^6 b^4 c^6 u v^7 w + 27 a^6 b^2 c^8 u v^7 w + 27 a^2 b^8 c^6 u^7 w^2 -
27 a^4 b^6 c^6 u^6 v w^2 + 54 a^2 b^8 c^6 u^6 v w^2 + 27 a^2 b^6 c^8 u^6 v w^2 + 54 a^6 b^4 c^6 u^4 v^3 w^2 -
54 a^4 b^4 c^8 u^4 v^3 w^2 + 54 a^4 b^6 c^6 u^3 v^4 w^2 - 54 a^4 b^4 c^8 u^3 v^4 w^2 + 54 a^8 b^2 c^6 u v^6 w^2 -
27 a^6 b^4 c^6 u v^6 w^2 + 27 a^6 b^2 c^8 u v^6 w^2 + 3 a^6 b^6 c^4 u^6 w^3 - 6 a^4 b^8 c^4 u^6 w^3 +
3 a^2 b^10 c^4 u^6 w^3 - 6 a^4 b^6 c^6 u^6 w^3 + 21 a^2 b^8 c^6 u^6 w^3 + 3 a^2 b^6 c^8 u^6 w^3 + 27 a^6 b^6 c^4 u^5 v w^3 -
27 a^4 b^8 c^4 u^5 v w^3 - 27 a^4 b^6 c^6 u^5 v w^3 + 54 a^6 b^6 c^4 u^4 v^2 w^3 - 54 a^4 b^8 c^4 u^4 v^2 w^3 -
21 a^4 b^4 c^8 u^3 v^3 w^3 + 42 a^6 b^6 c^4 u^3 v^3 w^3 - 21 a^4 b^8 c^4 u^3 v^3 w^3 + 42 a^6 b^4 c^6 u^3 v^3 w^3 +
42 a^4 b^6 c^6 u^3 v^3 w^3 - 21 a^4 b^4 c^8 u^3 v^3 w^3 - 54 a^8 b^4 c^4 u^2 v^4 w^3 + 54 a^6 b^6 c^4 u^2 v^4 w^3 -
27 a^6 b^4 c^6 u v^5 w^3 + 27 a^6 b^6 c^4 u v^5 w^3 - 27 a^6 b^4 c^6 u v^5 w^3 + 3 a^10 b^2 c^4 v^6 w^3 - 6 a^8 b^4 c^4 v^6 w^3 +
3 a^6 b^6 c^4 v^6 w^3 + 21 a^8 b^2 c^6 v^6 w^3 - 6 a^6 b^4 c^6 v^6 w^3 + 3 a^6 b^2 c^8 v^6 w^3 - 54 a^4 b^8 c^4 u^5 w^4 -
108 a^4 b^8 c^4 u^4 v w^4 - 54 a^4 b^8 c^4 u^3 v^2 w^4 + 54 a^4 b^6 c^6 u^3 v^2 w^4 - 54 a^8 b^4 c^4 u^2 v^3 w^4 +
54 a^6 b^4 c^6 u^2 v^3 w^4 - 108 a^8 b^4 c^4 u v^4 w^4 - 54 a^8 b^4 c^4 v^5 w^4 - 54 a^8 b^8 c^4 u^4 w^5 - 27 a^6 b^6 c^4 u^3 v w^5 -
27 a^4 b^8 c^4 u^3 v w^5 + 27 a^4 b^6 c^6 u^3 v w^5 - 27 a^8 b^4 c^4 u v^3 w^5 - 27 a^6 b^6 c^4 u v^3 w^5 +
27 a^6 b^4 c^6 u v^3 w^5 - 54 a^8 b^4 c^4 v^4 w^5 + 3 a^8 b^6 c^2 u^3 w^6 + 21 a^6 b^8 c^2 u^3 w^6 + 3 a^4 b^10 c^2 u^3 w^6 -
6 a^6 b^6 c^4 u^3 w^6 - 6 a^4 b^8 c^4 u^3 w^6 + 3 a^4 b^6 c^6 u^3 w^6 + 27 a^8 b^6 c^2 u^2 v w^6 + 54 a^6 b^8 c^2 u^2 v w^6 -
27 a^6 b^6 c^4 u^2 v w^6 + 54 a^8 b^6 c^2 u v^2 w^6 + 27 a^6 b^8 c^2 u v^2 w^6 - 27 a^6 b^6 c^4 u v^2 w^6 +
3 a^10 b^4 c^2 v^3 w^6 + 21 a^8 b^6 c^2 v^3 w^6 + 3 a^6 b^8 c^2 v^3 w^6 - 6 a^8 b^4 c^4 v^3 w^6 - 6 a^6 b^6 c^4 v^3 w^6 +
3 a^6 b^4 c^6 v^3 w^6 + 27 a^6 b^8 c^2 u^2 w^7 + 27 a^8 b^6 c^2 u v w^7 + 27 a^6 b^8 c^2 u v w^7 - 27 a^6 b^6 c^4 u v w^7 +
27 a^8 b^6 c^2 v^2 w^7 + a^10 b^6 w^9 - 2 a^8 b^8 w^9 + a^6 b^10 w^9 - 2 a^8 b^6 c^2 w^9 - 2 a^6 b^8 c^2 w^9 + a^6 b^6 c^4 w^9)
```

```
T2[{u_, v_, w_}] := b^2 c^2 l (-b^2 (-1+m) (m-n) - c^2 (-1+n) (-m+n) + a^2 (1 m+1 n-m n)) u^3
+a^2 c^2 m (-a^2 (1-m) (1-n) - c^2 (-1+n) (-m+n) + b^2 (1 m-1 n+m n)) v^3
+a^2 b^2 n (-a^2 (1-m) (1-n) - b^2 (-1+m) (m-n) + c^2 (-1 m+1 n+m n)) w^3
+3 a^2 b^2 c^2 (1 u+m v) (1 u+n w) (m v+n w)
```

The first term includes the 9 lines of the 3 sets of 3 parallels for the reference triangle. All points on these lines are centers. The second Term gives a cubic for the centers wrt a quadrilateral. The intersections of the 9 lines with the cubic are centers of cardioids tangent to 4 lines.

With this two equations you can calculate the centers for a numeral example:

```
a := 6; b := 5; c := 4; l := -1; m := -2; n := 2
```



```

w := 1; Solve[{T1[{u, v, w}] == 0, T2[{u, v, w}] == 0}, {u, v}] // N
{{v -> 42.1651, u -> -59.7817}, {v -> -0.332074, u -> -0.203511}, {v -> -0.486272, u -> 0.0606063},
 {v -> -0.775976, u -> 0.467078}, {v -> 1.07108, u -> 1.1019}, {v -> 0.824662, u -> 1.46698},
 {v -> 0.620615, u -> 1.57656}, {v -> -2.28105, u -> 3.13479}, {v -> 7.1962, u -> 6.80032},
 {v -> 0., u -> -118.115}, {v -> -52.7021, u -> -16.2911}, {v -> 3.99064, u -> -5.6307},
 {v -> -0.0491699, u -> -2.2956}, {v -> -0.00262426, u -> -2.02651}, {v -> 1.49691, u -> -1.73695},
 {v -> 0.0347593, u -> -1.42841}, {v -> 1.27313, u -> -1.14068}, {v -> 1.23213, u -> -0.969945},
 {v -> -0.165721, u -> -0.549684}, {v -> -3.30854, u -> 0.395195}, {v -> 1.15807, u -> 0.521973},
 {v -> -0.962308, u -> 0.665771}, {v -> -2.07772, u -> 0.81565}, {v -> -0.580635, u -> 1.63461},
 {v -> -1.46926, u -> 2.08798}, {v -> -1.6838, u -> 2.34055}, {v -> -3252.47, u -> 22.8887}}

Clear[a, b, c, u, v, w]

```

The cusps corresponding to a center (u:v:w) can be calculated as follows:

```

cusp[{u_, v_, w_}] := {b^2 c^2 u^3, a^2 c^2 v^3, a^2 b^2 w^3}

```

```

Eckart Schmidt
eckart_schmidt@t-online.de
http://eckartschmidt.de

```

Message: #65
Date: 17/6/2013 4:46:39
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven (May 26, 2013)

Dear Friends,

Tonight I will be flying to Malta with my wife for a 5-day-holiday.

Before I go want to thank you for all further information.

Also I want to share with you what I found about Cardioids tangent to all sides of a triangle:

The intersection points of nearest trisectors to the sides of ABC

(which are the vertices of the 1st and 3rd Morley Triangle) are centers of Cardioids that are bitangent to one of the sidelines of ABC.

The Cardioid is double tangent (bitangent) to the sideline where the trisectors are nearest to.

This property is valid for internal as well as external trisectors.

In beside picture only Cardioids bitangent to one of the sides of the triangle are shown.

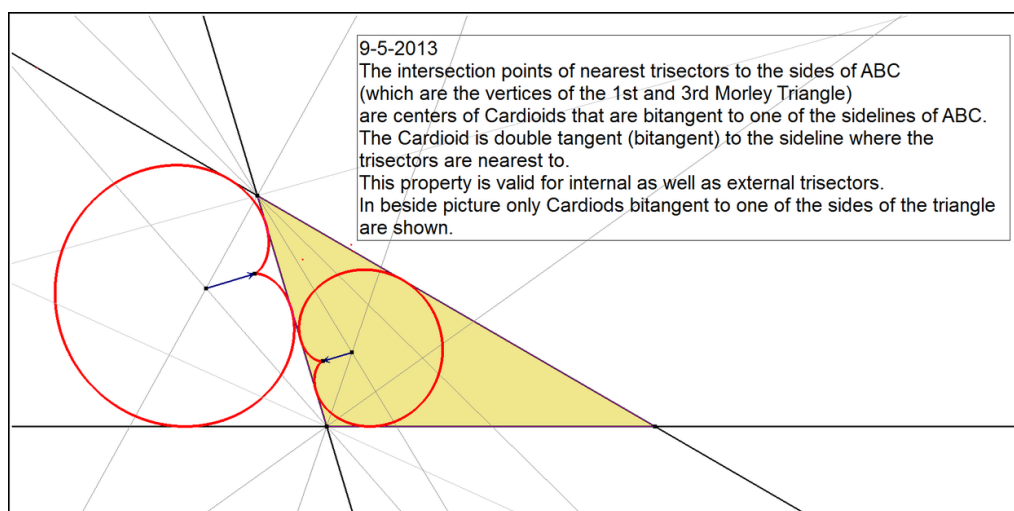
I made a picture from this property . See attachment in different formats..

Does anyone of you know if this property is known? References?

When I come back I want to investigate further Cardioids tangent to all sides of a Quadrilateral and make pictures.

See you next week,

Chris



QL-Qu98-Cardioids-12-Tr.png

Message: #66

Date: 17/6/2013 5:16:42

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven
(June 7, 2013)

Dear friends,

I managed to make a picture of all possible 27 Cardioids
inscribed in a Quadrilateral.

I made a presentation of the process of constructing the
inscribed Cardioids.

See attachment.

Next step remains calculating the coordinates of the 27 Cardioid
centers.

Best regards,

Chris van Tienhoven

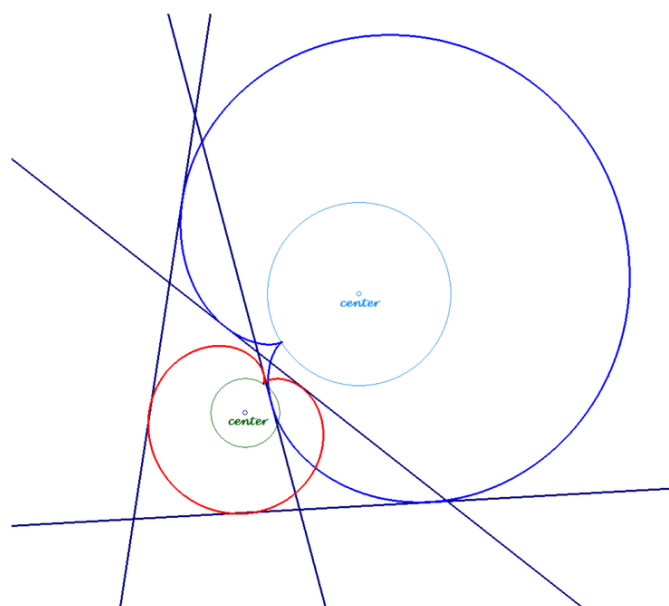
Morley's Miracle

Chris van Tienhoven

June 7, 2013

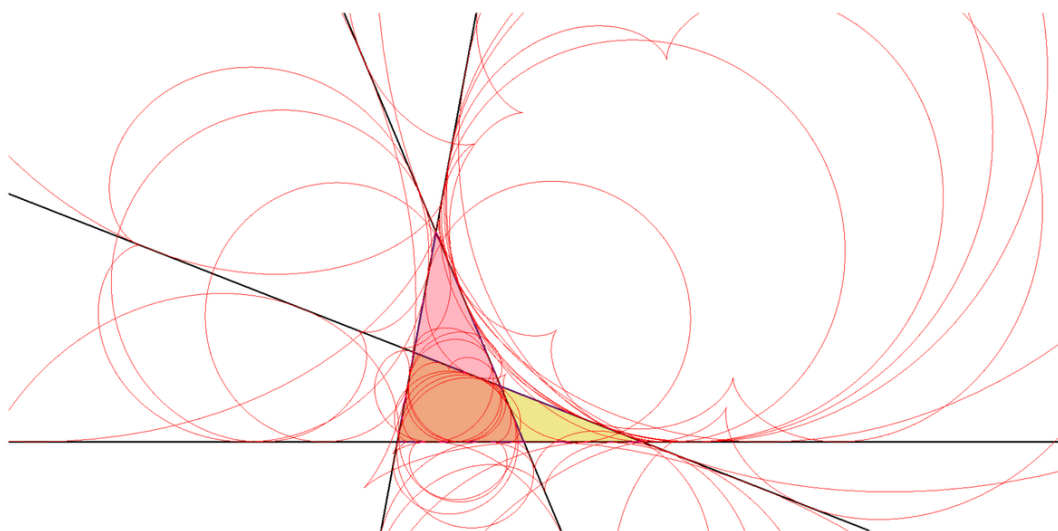
QL-Qu98-MorleyCardioids-01.pdf

2 examples of inscribed Cardioids in a Quadrilateral



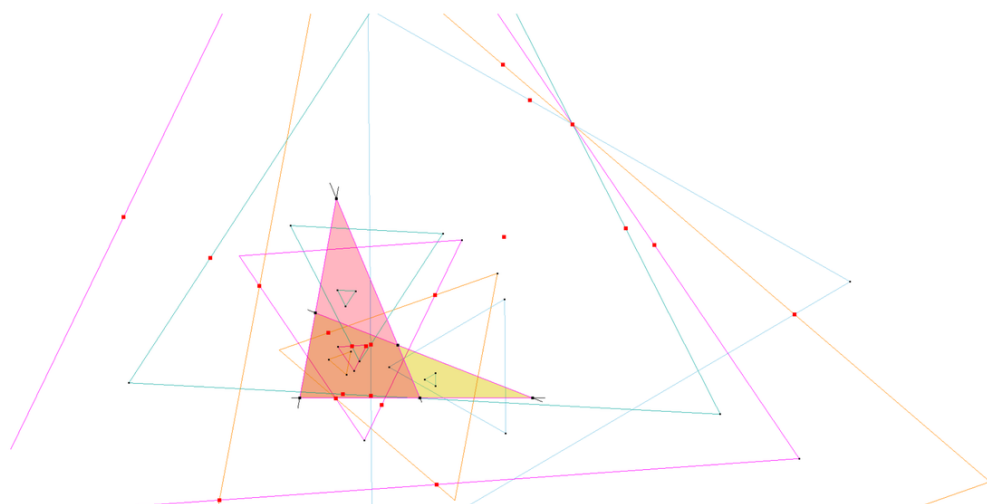
QL-Qu98-MorleyCardioids-01.pdf

According to Morley there are 27 inscribed Cardioids in a Quadrilateral,
and indeed there are:



QL-Qu98-MorleyCardioids-01.pdf

4 Times 3 equilateral triangles in a Quadrilateral are needed to construct them.

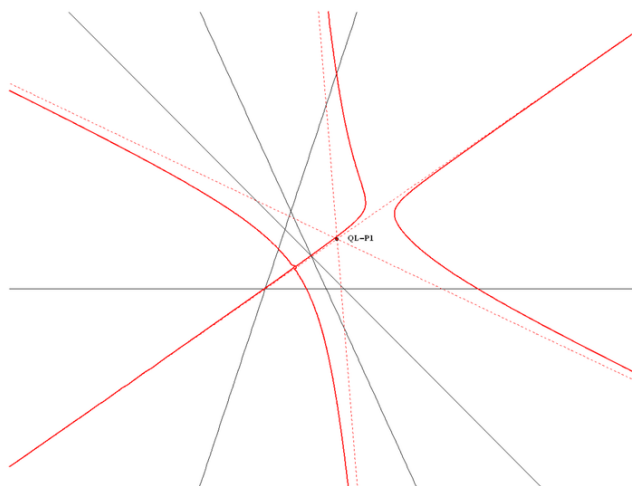


The 27 red intersections of the (extended) sides of the 4x3 equilateral triangles are the centers of the inscribed Cardioids.
Each center is the intersection point of 4 sidelines.

QL-Qu98-MorleyCardioids-01.pdf

The 27 Cardioid Centers lie on Eckart's Cubic

(found by Eckart Schmidt, May 2013)



The asymptotes are concurrent in the Miquel Point (QL-P1)
and cross each other at 60 degrees

QL-Qu98-MorleyCardioids-01.pdf

Message: #67

Date: 17/6/2013 5:16:43

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Chris van Tienhoven

(June 5, 2013)

Dear friends,

After a wonderful holiday at Malta I couldn't wait for further exploring the Cardioids.

However I am very busy with all kinds of things and they all seem to be very important.

Anyway I took my time to read the paper of Frank Morley:

Extensions of Clifford's Chain-Theorem.

At page 470 Morley says quite below:

"the $n(n-1)^{(n-2)}$ axes of the component $(n-1)$ -lines meet in the $(n-1)^{(n-1)}$ incenters,

there being n axes on a point and $(n-1)$ points on an axis."

When I substitute $n = 4$, then it says:

"the $4 \cdot 9$ axes of the component 3-lines (triangles) meet in 27 incenters,

there being 4 axes on a point and 3 points on an axis."

This is exactly what Eckart found out.

Stated in my own words:

In a Quadrilateral we have 4 Component Triangles. Each Component Triangle has its own set of $3 \cdot 3$ parallel axes.

Morley's Miracle is that these 4 sets of $3 \cdot 3$ axes intersect in exactly 27 points. Each Axis contains 3 of these points.

Eckart found out that these points lie on a Eckart's cubic.

It's hard to understand how it can be, but it happens to be.

I myself started to calculate these 27 points and I think I just need some time to succeed.

Besides this Morley also wrote:

" $n(n-2)^{(n-2)}$ incenters are on the $(n-2)^{(n-1)}$ circles, there being n points on each circle, $(n-2)$ circles on each point.

These $(n-2)$ circles cut at the angle $\pi/(n-2)$.

The $(n-3)^{(n-1)}$ cardioids arise from the $n(n-3)^{(n-2)}$ circles; the circles are osculants of the cardioids, each having n osculants $C3$'s;

And each $C2$ osculating $(n-3)$ $C3$'s."

When I substitute $n = 4$ and rephrase a bit, then it says:

" $2 \cdot 4$ incenters are on the 8 circles, there being 4 points on each circle, 2 circles on each point.

These 2 circles cut at the angle $\pi/2$. (this is Seiners theorem, rules 8 and 9! See QL-8P1.)

The 1 cardioid arising from the $4 \cdot 1$ circles; the circles are inscribed in the single cardioid;

And each circle osculating (is inscribed) in the single cardioid. (This is QL-Qu1) !"

Bernard, you already mentioned the property of the single Cardioid that Morley mentioned! Thanks, it made me think.
Eckart, I wish you a beautiful and inspired holiday!
Best regards,
Chris

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Message: #68
Date: 17/6/2013 5:16:44
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(June 5, 2013)
Dear friends,
before 10 days holiday a last message. Thanks to Bernard for his summary with new questions.
But calculations and constructions wrt the center cubic are difficult because of the trisection of angles in the background. At most numeral examples can be tested.
But I am sure of the following properties for the reference triangle of the McCay Cubic:
... centroid is the Miquel Point QL-P1
... Brocard axis $X3.X6=OK$ is the Newton Line QL-L1
... axes of the deltoid of the Simson lines are parallel to the axes of the deltoid of the quadrilateral
... a parallel to the Newton Line with $2/3$ distance to QL-P1 cuts the cubic on its asymptotes
There are so many properties for the McCay Cubic in B. Gibert's catalogue perhaps with further relations to the problem.
Best regards Eckart
P.S. For Bernard: The 27 centers make no Morley set for the reference triangle of the McCay Cubic.

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Message: #69
Date: 17/6/2013 5:16:47
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 27, 2013)

Dear friends,

Chris has opened a new sight of the geometry of the centers of inscribed cardioids of a triangle - thanks!

There are further properties wrt this question in the attachment.

Best regards Eckart

27.05.2013

**Inscribed cardioids of a triangle
bitangent to one side**

Morley proved, that the centers of inscribed cardioids of a triangle lie on 3 sets of 3 parallel lines, forming equilateral triangles (see Bernard's first message). The sidelines of the first and third Morley triangle belong to these sets. Every point on these lines can be a center of an inscribed cardioid.

Here we examine those inscribed cardioids, which are bitangent to one side: Chris (message 26.05) has worked out, that the vertices of the first and third Morley triangle belong to these centers. For one side of the triangle – further on AB – there will be nine centers. For a numeral example the centers $(u:v:w)$ can be calculated with the following equations (see my message 25.05.)

$$\begin{aligned} -c^4u^3 + 3c^2a^2uw^2 + 2a^2S_bw^3 &= 0, \\ -c^4v^3 + 3b^2c^2vw^2 + 2b^2S_av^3 &= 0. \end{aligned}$$

Now the geometry of these centers will be described: Let $A'B'C'$ be the first Morley triangle and $A''B''C''$ the third Morley triangle. Further points will be:

$$\begin{aligned} A^* &= AC' \cap A'B', \quad B^* = BC' \cap A'B', \\ A^{**} &= AC'' \cap B'C', \quad B^{**} = BC'' \cap A'C', \\ U &= AB^* \cap BA^*, \quad V = AA^* \cap BB^{**}, \quad W = BB^* \cap AA^{**}. \end{aligned}$$

The centers of the inscribed cardioids bitangent to AB are:

$$\boxed{C', C'', A^*, B^*, A^{**}, B^{**}, U, V, W}.$$

Collinear on the sides of the first Morley triangle:

$$\boxed{A', B', A^*, B^* - B', C', A^{**} - C', A', B^{**}}$$

Collinear on one side of the third Morley triangle:

$$\boxed{A'', B'', V, W}$$

Further collinear points:

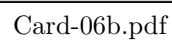
$$\boxed{A, B^*, B^{**}, U - B, A^*, A^{**}, U}$$

$$\boxed{A, A^*, V - B, B^*, W - A, A^{**}, C'', W - B, B^{**}, C'', V}$$

The 3 sets of 3 parallels for all centers are:

$$\boxed{A'C', A''C'', UW - B'C', B^*C^*, UV - A'B', A''B'', A^{**}B^{**}}$$

Card-06 .pdf



114

Message: #70
Date: 17/6/2013 5:16:49
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Bernard Keizer
(May 27, 2013)

Dear friends,

Thank you again for all the new discoveries

For Chris: in my first questionning paper, I wrote " The main property of the cardioïd is that the line Ki0 trisects the angle between the tangent and the bitangent ".

That's exactly how Morley discovered his Morley's theorem (known often as Morley's miracle: see Bogomolny ...)

For a given triangle, there are exactly 27 centers of cardioïds bitangent to one of the 3 sides: they are the vertices of 27 equilateral triangles (of which the first and the third Morley's triangles) and those 27 points are on $3 \times 3 = 9$ lines , whose direction is the mean direction ($m^\circ \pi/3$) of the 3 sides..

That's what we could call the Morley set for this triangle.

Back to the QL: the centers of the cardioïds tangent to the 4 lines belong to the 4 Morley's sets of the 4 reference triangles of the QL.

The discovery (for me, at last) is what Eckart proved: there are 27 points intersection of the 4 sets (each point on one line of each set and each of the $4 \times 9 = 36$ lines having 3 points) and those 27 points belong to the McKay cubic of a new triangle!

For Eckart: how do you find that there is a deltoïd envelopping the Simson lines of that triangle? If this is true, the triangle is a main triangle of the deltoïd and the deltoïd is the Steiner deltoïd (tangent to the the sides and the altitudes) of this triangle and the Euler point of the triangle is the center of the inscribed circle of the deltoïd. As you have the centroïd as the point QL-P1 and the circumcenter and the orthocenter on the cubic, I wonder where all those points are (you say the circumcenter, which is the pivot of the cubic is on the line QL-L1 ...)

My last question is on those 27 points: did I understand correctly that they are making a Morley set for the last triangle? In that case, the direction of the lines is again the mean direction of the 3 sides and therefore parallel to the 3 axes of the Steiner deltoïd of the last triangle (what's the link between this new deltoïd and the deltoïd tangent to the 4 lines?).

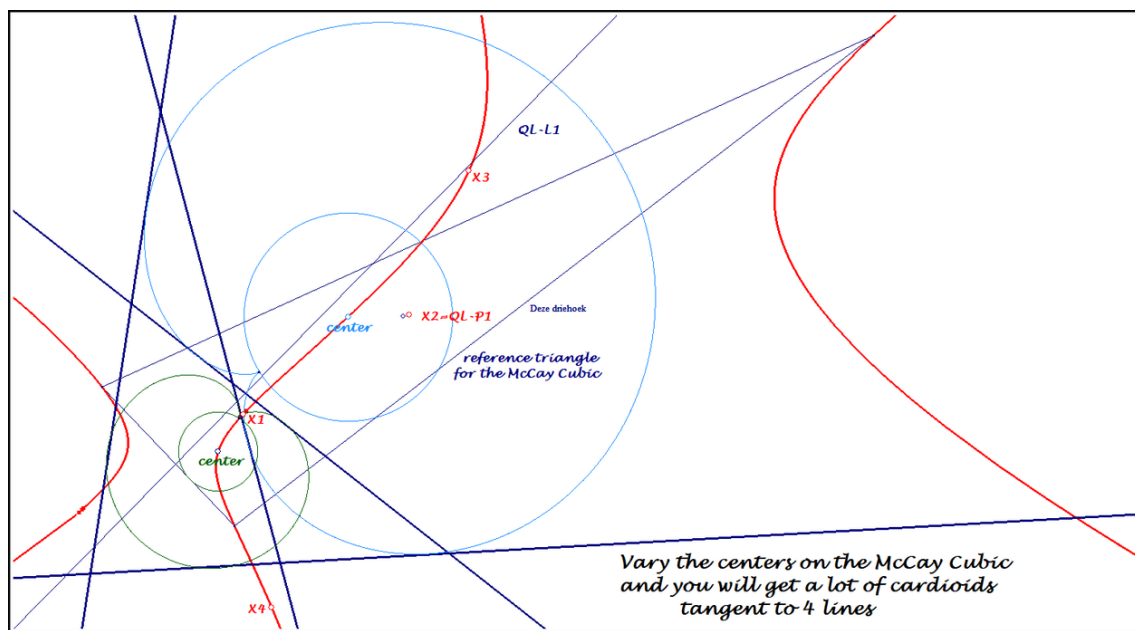
Best regards

Bernard

Message: #71
Date: 17/6/2013 5:18:30
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(May 26, 2013)

Dear friends,
there is a Cabri file in the attachment for a quadrilateral and its inscribed cardioids.
I have drawn (free hand, not constructed) a reference triangle for the McCay Cubic.
Vary the shown centers on the cubic and you can find most of the 27 inscribed cardioids.
The reference triangle for the McCay Cubic has centroid in QL-P1 and circumcenter on QL-L1.
The axes of the deltoid of its Simson lines are parallel to the asymptotes of the cubic.
Best regards Eckart



McCayCubic-fig.png

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Message: #72

Date: 17/6/2013 5:18:55

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Eckart Schmidt
(may 28, 2013)

Dear friends,

there are problems with the exactness in my Cabri file (26.05),
when reloaded (???)

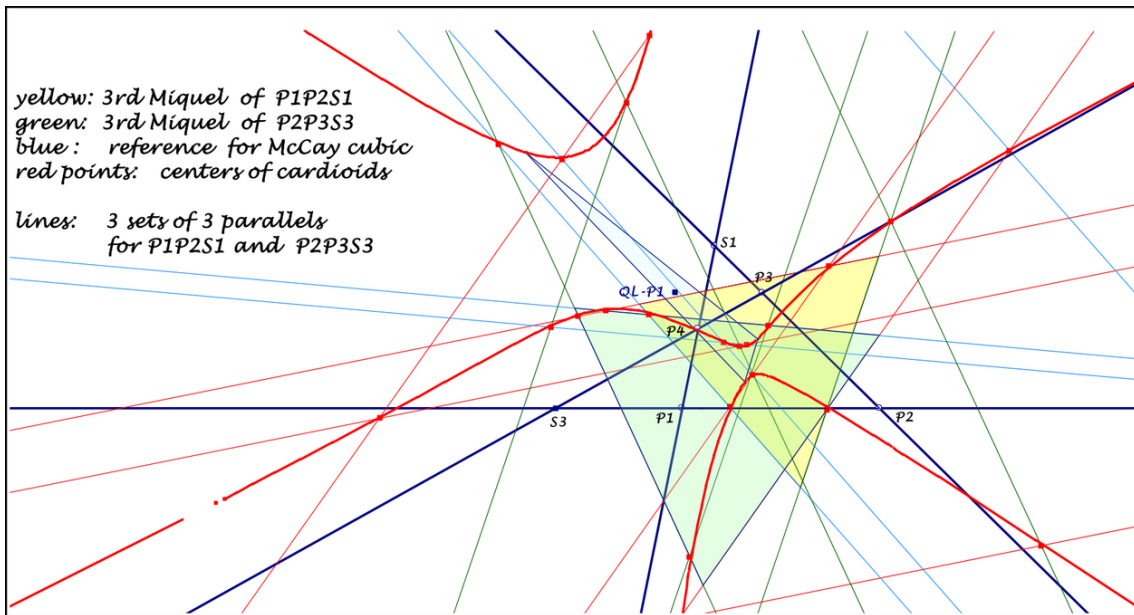
So there is a new drawing of the centers for cardioids tangent
to four lines in the attachment.

For two triangles of the quadrilateral the 3 sets of 3 parallels
for the centers are drawn

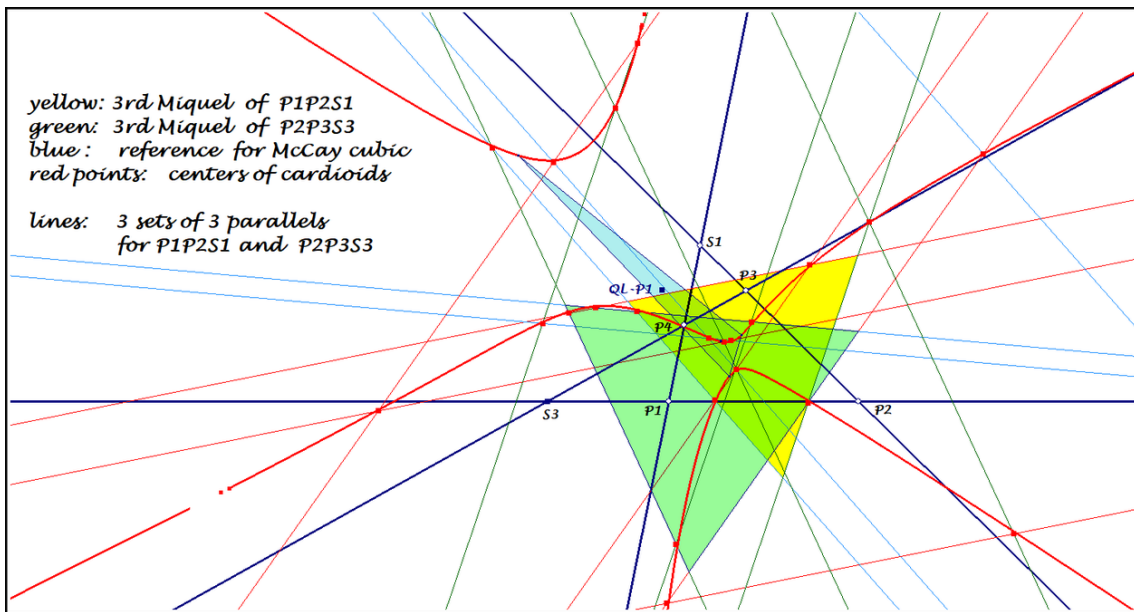
and you can study the intersections, which are centers, with the
macro in my Cabri file (24.05).

The McCay cubic is also shown.

Best regards Eckart



2013-05-27.pdf



13-05-28-fig.png

Message: #73

Date: 17/6/2013 5:23:19

From: Chris van Tienhoven

Subject: Cardioids tangent to 4 lines

COPY FROM FORMER GOOGLE GROUP by Bernard Keizer

(June 8, 2013)

Dear friends,

I'm really happy that the question of the cardioids tangent to 4 lines has raised so much attention from your side.

First, I'll give you all the references I found on the subject: 4 articles by Alex Bogomolny in Cut the knot: Doodlings and Miracles, nov 98; Morley's Pursuit of Incidence, dec 98; Lines, Circles and Beyond jan 99; On Motivation and Understanding, fev 99.

On page 1 of the 4th text, I found the 8 cardioids that touch 4 lines, that's the origin of my initial mistake, but I suppose the 8 cardioids were tangent to 4 lines themselves tangent to a hypocycloid with 5 cusps, as explained on page 4 of the 3rd text.

4 articles and a book by Frank Morley: On the metric geometry of the plane n-line, 1900; Orthocentric properties of the plane n-line, 1902; On reflexive geometry, 1907; Extensions of Clifford's Chain-Theorem, 1929 (followed by another interesting article on the same matter by Paul Smith Wagner) and last, but not least Inversive Geometry, 1929.

Encouraged by your discoveries, I tried again to draw more exactly my 4 Morley sets and I found also the 27 points (as Eckart says, it's really hard to draw it precisely, because of the trisection of all angles!). But of course, as I use Geogebra, my drawings are less beautiful than yours ...

I made an interesting observation, which is visible on the last drawing from Chris: each of the 36 lines (or extended sides of equilateral triangles) belonging to one set cuts each of the 3 other sets in 3 points, which are in 3 different directions (in other words, 2 points on the same line are never on 2 parallels). That explains why there are only 27 points and not 81 as intersections of 2 sets of 9 lines each; maybe it's obvious, but I don't see why ...

I still have my questions on the reference triangle of the MacCay cubic: I read the article on the cubic K03 in Bernard Gibert, but I don't see the link between the 4 reference triangles of the QL and this 5th triangle; if t_i is the angle between the ortholine and the line Li of the QL, the angle between the axis of the parabola and the axis of the cardioid tangent to the 4 circles is Sum of t_i and the angle between the axis of the parabola and the axes of the deltoïd are $1/3$ Sum of t_i m° $\pi/3$...

My last curiosity is the question of the directed lines: the circle is a curve of direction and there are 4 circles tangent to 3 lines (centers are incenter and excenters), but only one tangent to 3 directed lines; the same goes for the cardioïd, which is also a curve of direction and I wonder if you could see a regularity after drawing all your 27 cardioïds!
That's all for today
Best regards
Bernard

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Message: #74
Date: 20/6/2013 3:37:58
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear friends,
I hope being successful in using the new Quadri-Figures-Group.
In the attachment (editorial note: the attachment is omitted because of very large size) there is the equation of cardioids tangent to 3 or 4 lines, when the center is given.
Perhaps someone can simplify this very extensive term!
Best regards Eckart

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Message: #75
Date: 20/6/2013 9:41:57
From: Chris
Subject: New points QL-P28 and QL-P29

Dear friends,
Welcome to the new Yahoo Group!
Eckart has been on his holiday and we had a Group removal!
In the meantime Antreas and me had a discussion at his Anopolis Group site. About several interesting Quadrilateral topics.
See message #392 "Is this Miquel point interesting?" and following messages at:
<http://tech.groups.yahoo.com/group/Anopolis/> †)
As a result two new interesting QL-points QL-P28 and QL-P29 were born.
Seiichi found as a consequence 2 marvelous follow-up properties of these points.
I leave it to him to mention them at the Quadri Group.
So far this time.
Best regards, Chris

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†) Editorial note: The referenced message(s) can be found as listed in [EPG-References \[33\]](#).

Message: #76
Date: 21/6/2013 1:07:23
From: Antreas
Subject: New points QL-P28 and QL-P29

Dear Chris,
Very nice!!!!!!!!!!!!!!!!!!!!!!
I think there should be a dual problem: which similar lines (Euler lines, Brocard axes, in general X_iX_j -lines) of the component triangles of a quadrangle do envelope a circle? And how about to return to Triangle Geometry?
Let ABC be a triangle and L a line.
For $P = X_{186}$ or X_{265} , the P-points of the triangles ABC and the triangles bounded by (AB,BC,L) , (BC,CA,L) , (CA,AB,L) are concyclic.
Which are the centers (with respect ABC) of the circles for special lines $L =$ Euler Line, Brocard axis, etc
Antreas

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Message: #77
Date: 21/6/2013 2:20:57
From: jeanlouisayme
Subject: With the Euler-Poncelet's point

Dear Geometers,
I propose this problem coming from Brianchon and Poncelet for which I haven't a synthetic proof (avoid conic...)
1. ABCD a quadrilateral
2. P_o the Euler-Poncelet's point of ABCD,
3. E,F,G the points of intersection of AC and BD, AD and BC, AB and CD
4. (O) the circumcircle of the triangle XYZ.
Prove that (O) goes through P_o .
Chris knows my problem. An elementary proof is welcome.
Sincerely
Jean-Louis

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Message: #78
Date: 21/6/2013 6:38:49
From: Antreas
Subject: New points QL-P28 and QL-P29

Dear Chris

Also, with a good drawing program, is possible I think to draw the locus of the centers of the circles as the Euler line (for example) moves around the circumcenter O or the orthocenter H (the most important points) of ABC .

Antreas

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Message: #79
Date: 22/6/2013 8:00:27
From: Chris
Subject: New points QL-P28 and QL-P29

Dear Antreas,

When I transcribe your proposal I get this construction:

1. Let ABC be reference Triangle with bounding lines a, b, c .
2. Let d be a line through O .
3. Construct the P-points QL-P28 (based on X186), QL-P29 (based on X265) or QL-P4 (based on X3) of the Quadrilateral $abcd$.
4. Determine the locus of the P-point with variable line d .

I found these results in Cabri:

The locus of QL-P4 is a cubic through $X(3)$.

The locus of QL-P28 is a quartic.

The locus of QL-P29 is a cubic through $X(4)$?

When changing d to a variable line through $X(4)$ the results are:

The locus of QL-P4 is a cubic.

The locus of QL-P28 is a quartic.

The locus of QL-P29 is a cubic.

Of course also loci could be determined with other points than QL-P4, QL-P28 and QL-P29.

This is a nice example of combining Quadri-Geometry into Triangle Geometry.

Best regards,

Chris

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Message: #80

Date: 24/6/2013 4:59:57

From: emmanueltsukerman

Subject: Circumcenter of Mass and generalized Euler line

Dear friends,

For your interest, I wanted to bring to your attention the preprint "Circumcenter of Mass and generalized Euler line" which you may find at <http://arxiv.org/abs/1301.0496> .

The preprint contains generalizations of QG-L4 (Quasi Euler line) and the points on it (e.g., QG-P5 Quasi Circumcenter, QG-P6 Quasi Orthocenter, QG-P7 Quasi Nine-point Center) to n -polygons and, more generally, simplicial polytopes in Euclidean, spherical and hyperbolic geometries.

From the abstract:

- We define and study a variant of the center of mass of a polygon and, more generally, of a simplicial polytope which we call the Circumcenter of Mass (CCM). The Circumcenter of Mass is an affine combination of the circumcenters of the simplices in a triangulation of a polytope, weighted by their volumes.
- Our motivation comes from the study of completely integrable discrete dynamical systems, where the CCM is an invariant of the discrete bicycle (Darboux) transformation and of recuttings of polygons.
- We show that the CCM satisfies an analog of Archimedes' Lemma, a familiar property of the center of mass.
- We define and study a generalized Euler line associated to any simplicial polytope, extending the previously studied Euler line associated to the quadrilateral. We show that the generalized Euler line for polygons consists of all centers satisfying natural continuity and homogeneity assumptions and Archimedes' Lemma.
- Finally, we show that CCM can also be defined in the spherical and hyperbolic settings.

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Message: #81
Date: 24/6/2013 1:00:56
From: Antreas
Subject: CONICS IN C. QUADRILATERAL

An old problem on conics was:
How many conics are tangent to five given conics?
See here the history of the problem and a special solution:
<http://www.mat.ucm.es/serv/revmat/vol10-2/vol10-2h.pdf>
Now, in a complete quadrilateral, we have a special case
of 5 concurrent conics (circles):
the circumcircles of the 4 component triangles and the Miquel
circle passing through the 4 circumcenters.
How many conics are tangent to these 5 conics (circles)
and which ones?
On the other hand, five points determine a conic.
Which is the conic passing through the centers of the
above 4+1 circumcircles?
APH

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Message: #82

Date: 25/6/2013 2:22:55

From: Seiichi Kirikami

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P4

Dear friends,

Chris found QL-P28 and QL-P29 and their circles, which are determined by 4 concyclic points. Then I studied them graphically, expecting that there might be such chains as center-circle or Clifford's like the case of QL-P4 and its circle.

"Cut The Knot! Morley's Pursuit of Incidence by Alex Bogomolny, Dec. 1998" has description about center-circle and Clifford's chains.

I first describe the case of QL-P4 in comparison. Then QL-P28 and P29. The following notation is applied to these cases. Given 5 lines 1, 2, 3, 4, 5, the intersection of 1,2 is denoted by 12. Other intersections similarly denoted. $X(i)$ of triangle {12, 12, 23} is denoted by 123. Other points similarly denoted. The points 123, 124, 341, 234 are on a circle with its center denoted by 1234. Other centers similarly denoted. $i=3, 186$ and 265 . See the attached files(gsp or MS word).

(1) 5 line configuration of QL-P4 with the centers of Miquel circles denoted by 1234 and others:

These circles concur in point 12345(red). Their centers are on a circle with its center 12345(green).

(2) 5 line configurations of QL-P28 and P29 with the centers of their circles denoted by 1234 and others:

These circles concur in point 12345(red). Their centers are not on a circle. I also confirmed that 6 line configurations of QL-P28 and P29 do not produce any concurrency of point 12345 and others.

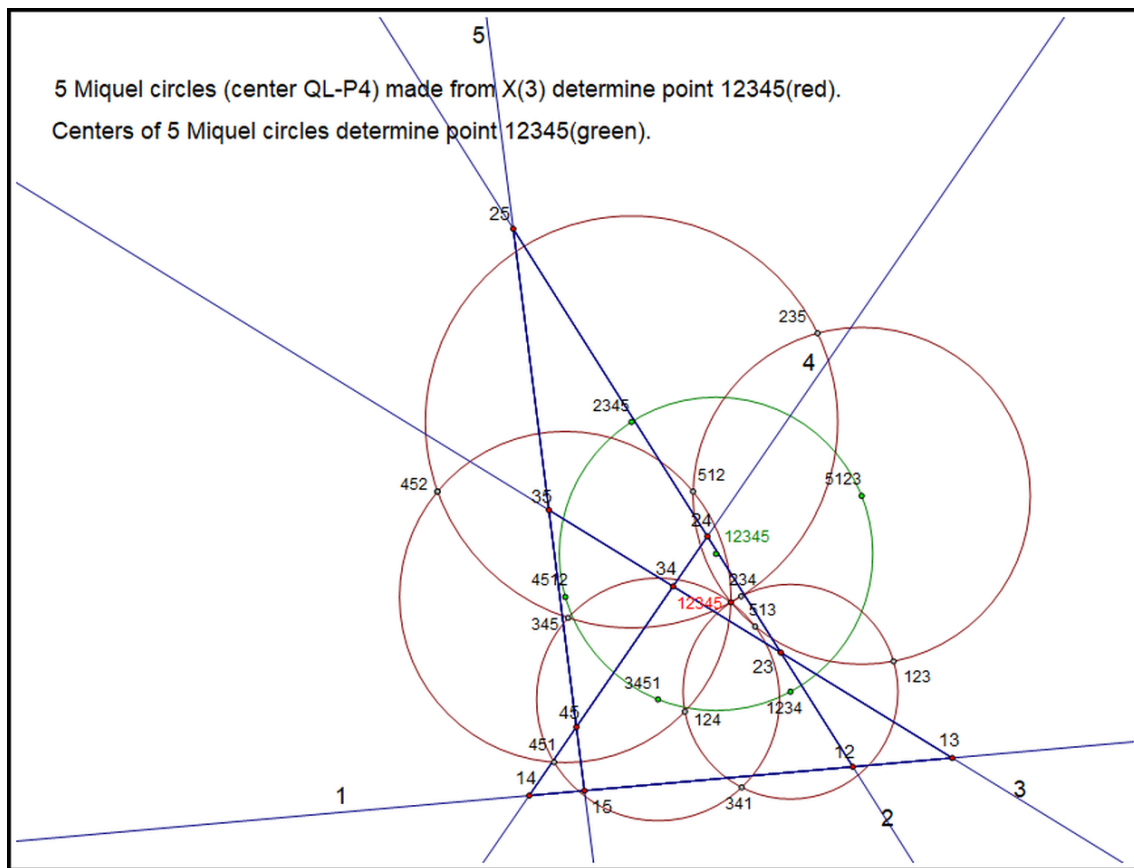
My conclusion is as follows.

(1) 5 line configurations of QL-P28 and P29 do not have the same properties as QL-P4, but they rather determine pentalateral points.

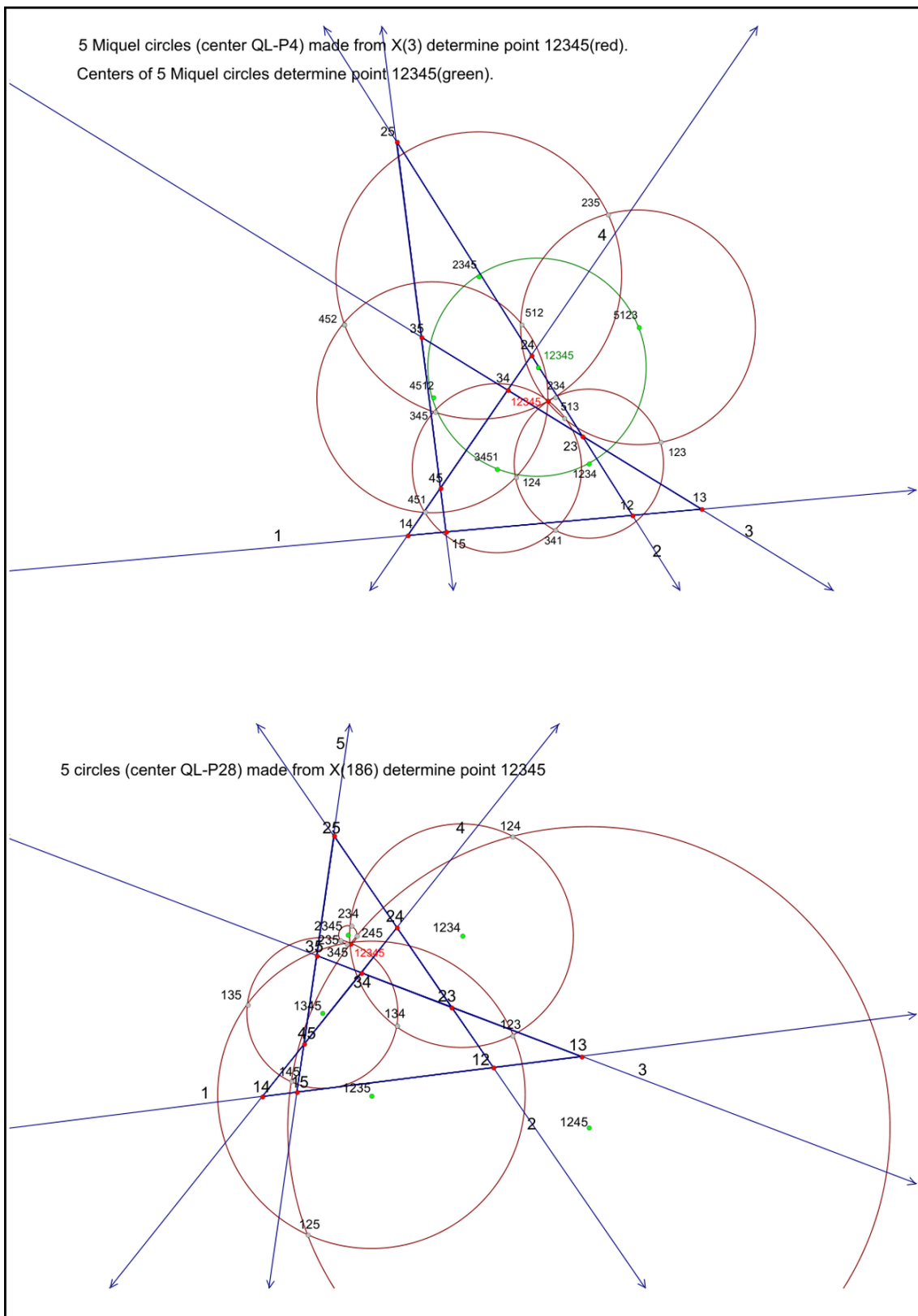
(2) Center- circle and Clifford's chains with QL-P4 are rare phenomena.

Best regards,

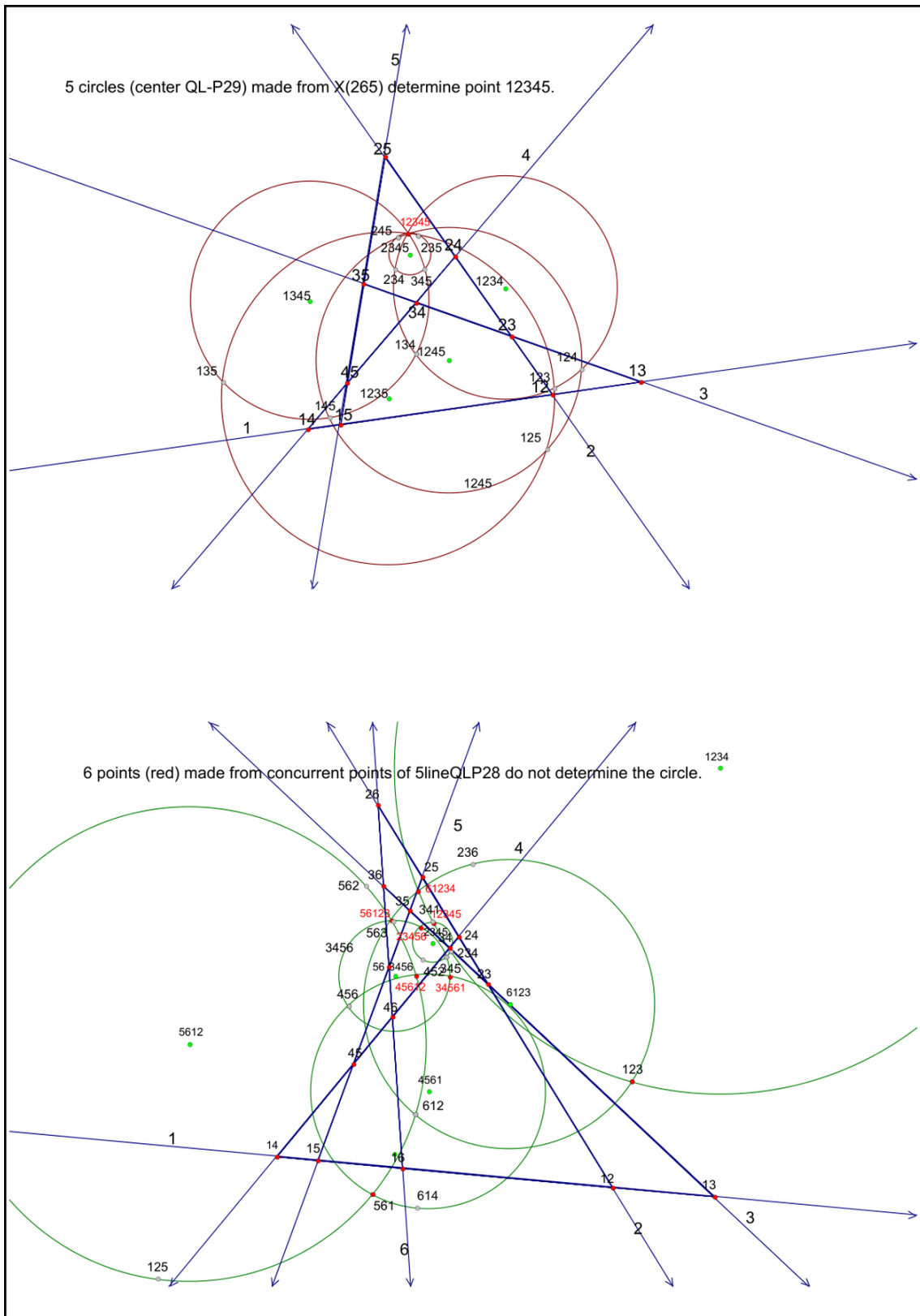
Seiichi.



QLP4P28P29-gsp.png



QLP4P28P29.docx



QLP4P28P29.docx

Message: #83

Date: 25/6/2013 3:29:14

From: Quadri-Figures-Group@yahoogroups.com

Subject: No Subject

Hello,

This email message is a notification to let you know that a file has been uploaded to the Files area of the Quadri-Figures-Group group.

File: /Equilateral triangle on angle bisectors.pdf

Uploaded by: yeumtrondoitb85 < yeumtrondoitb85@yahoo.com >

Description:

You can access this file at the URL:

<http://groups.yahoo.com/group/Quadri-Figures-Group/files/Equilateral%20triangle%20on%20angle%20bisectors.pdf>

To learn more about file sharing for your group, please visit:

<http://help.yahoo.com/l/us/yahoo/groups/original/members/web/index.html>

Regards,

yeumtrondoitb85 < yeumtrondoitb85@yahoo.com >

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Message: #84

Date: 25/6/2013 3:30:14

From: yeumtrondoitb85

Subject: D point and secon circle and ten equilateral triangles

Dear group!

I'm sorry. I'm not mathematician so I introduce some problems at:

<http://tech.groups.yahoo.com/group/Quadri-Figures-Group/files/>

I wish you proved it. Thanks very much

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Message: #85

Date: 25/6/2013 10:15:06

From: Chris

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Seiichi,

When I say it in my own words:

* When we draw the points $X(186)$ for all 4 Component Triangles in a Quadrilateral (4L or system of 4 random lines) then these points are concyclic on a circle. Let's call it $C186$. Its center is QL-P28.

* When we draw the circles $C186$ for all 5 Component Quadrilaterals in a Pentalateral (5L or system of 5 random lines) then these 5 circles all pass through a common point. The same is valid for $X(265)$. It is most remarkable indeed.

Congratulations! You discovered 2 brandnew 5L-points (actually centers)!

One based on $X(186)$ and one based on $X(265)$.

And these 5L-points are very rare!

Best regards,

Chris

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Message: #86

Date: 26/6/2013 8:47:29

From: eckart_schmidt@t-online.de

Subject: RE. Cardioids tangent to 4 lines

Dear friends,

sorry, I have to cancel my conjecture, that the cubic for the centers of cardioids tangent to 4 lines generally is a McCay Cubic.

Bernard Gibert has given me a counter-example: The cubic ...

"... is generally not a McCay cubic for some triangle since it can be a nodal cubic, try $l:m:n = X190$ for example."

Best regards Eckart

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Message: #87

Date: 26/6/2013 12:00:41

From: seiichikiri

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Chris, dear friends,
Thank you very much for your kind comment. I highly appreciate your ability of Mathematica.
Congratulations for the anniversary of EQF! Though it was several days earlier, I think.
Best regards,
Seiichi.

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Message: #88

Date: 26/6/2013 9:29:59

From: Antreas Hatzipolakis

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Friends
Is there an evidence whether there are more points with that concyclicity property?
I would bet there are infinitely many (that is, there is a locus where they are lying on).
Probably some day Chris will check the rest 1000+ points in ETC.... :-)
APH

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Message: #89

Date: 28/6/2013 4:22:04

From: Chris

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Antreas, dear friends,

[APH]

>

> > Is there an evidence whether there are more points with that
> concyclicity property?

> > Probably some day Chris will check the rest 1000+ points in
ETC....:-)

I checked all ETC-points in the range X(1)-X(5372) and only
X(3), X(186) and X(265) give 4 concyclic points in the Component
Triangles of a Quadrilateral.

[APH]

>

> > I would bet there are infinitely many (that is, there is a
locus where

> they are lying on).

That's a very interesting question.

I am not so sure there will be a locus of points having this
property.

If so then there won't be an evidence that there aren't more of
these loci, because the locus probably would be dependent of
some construction method, and why wouldn't be there another
construction method also giving the concyclic-property?

It is a bit of speculating as long as we don't have a (general)
method to determine a locus.

Somebody with an idea?

Best regards,

Chris

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Message: #90

Date: 28/6/2013 9:10:19

From: Chris

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Seiichi and Friends,

[SK]

>> Congratulations for the anniversary of EQF!

You are right Seiichi.

At June 15, 2012 I introduced the "Encyclopedia of Quadri-Figures (EQF)" at the internet.

Since then lots of beautiful comments and additions came to me.

I had a hard time digesting it all.

The volume of EQF increased with 30% !!

And now we started up a Discussion Group related to Quadrilaterals and n-gons.

I am glad how it worked out and hope we have lots of fun together exploring this beautiful part of Geometry.

Best regards,

Chris van Tienhoven

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Message: #91

Date: 28/6/2013 11:34:37

From: Antreas Hatzipolakis

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Chris

Although I do not know how to determine a locus in 4-lateral/gon geometry,

I think there should be a method, since such loci exist!

An example, where the locus is the entire plane, is Orthopolar circles

(from a discussion in Hyacinthos. I have the conjecture here:

<http://anthrakitis.blogspot.gr/2013/03/orthopolar-circles-quadrilateral.html>)

By the way, could someone provide a proof (synthetic or not)?

APH

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Message: #92

Date: 28/6/2013 12:59:32

From: Seiichi Kirikami

Subject: Graphical observation: X(186) and X(265) in cyclic QA-environment

Dear friends,

This message is the addition of the message #82.

Given a cyclic quadrangle $P_1P_2P_3P_4$, let $X(186)$ of triangle $P_iP_jP_k$ be Q_1 . Then Q_1, Q_2, Q_3 and Q_4 are concyclic.

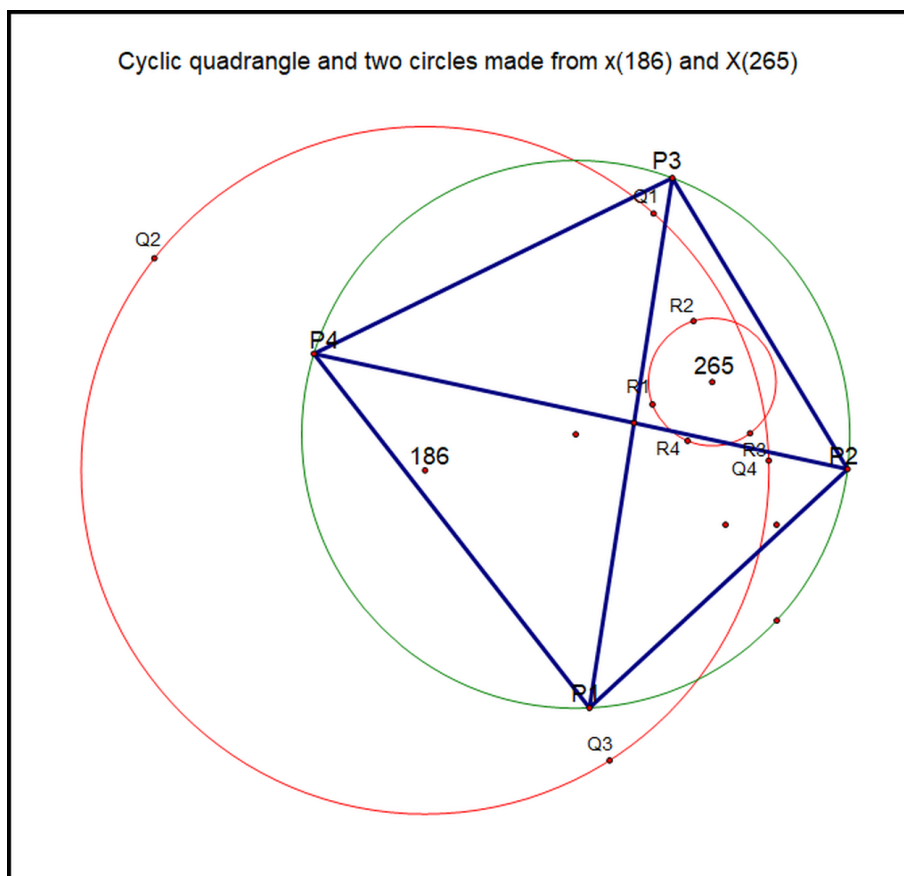
Given a cyclic quadrangle $P_1P_2P_3P_4$, let $X(265)$ of triangle $P_iP_jP_k$ be R_1 . Then R_1, R_2, R_3 and R_4 are concyclic.

See the attached files (gsp or MS word).

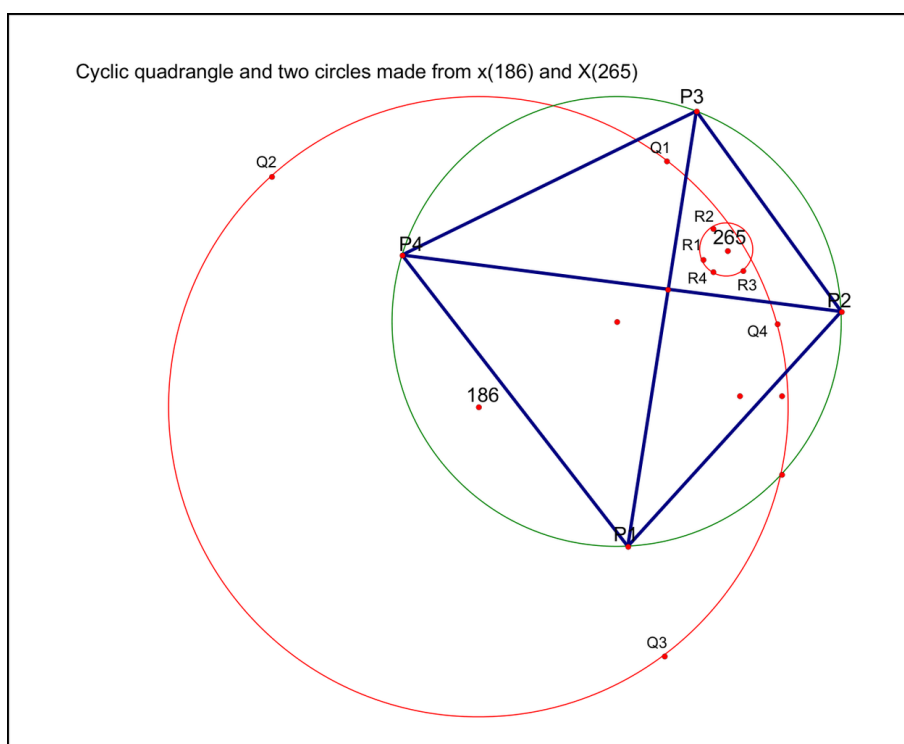
This property is easier for me to understand $X(186)$ and $X(265)$.

Best regards,

Seiichi.



CyclicQAX(186)X(265)-gsp.png



CyclicQAX(186)X(265).docx

Message: #93

Date: 28/6/2013 2:45:31

From: Seiichi Kirikami

Subject: Graphical observation: QL-P28 and QL-P29 in comparison with QL-P

Dear Chris, dear friends,

You are the driver and conductor of the train of EQF! I see that your work is very, very hard. No one except for you can endure it. I think that, including me, many people are grateful to you. Thanks a lot!

Best regards,

Seiichi.

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Message: #94

Date: 29/6/2013 10:42:37

From: eckart_schmidt@t-online.de

Subject: Cardioids tangent to 4 lines

Dear friends,

after Bernard Gibert had given a counter-example I have to modify my conjecture for the cubic of the centers of cardioids tangent to 4 lines. I risk to sum up my new sight:

* The cubic is a $K60+$. The three asymptotes are parallel to the axes of the deltoid of the quadrilateral and concur in the Miquel Point QL-P1.

* The asymptotes cut the cubic collinear on a line parallel QL-L1 in $2/3$ distance to QL-P1.

The Newton Line QL-L1 can have three intersections with the cubic:

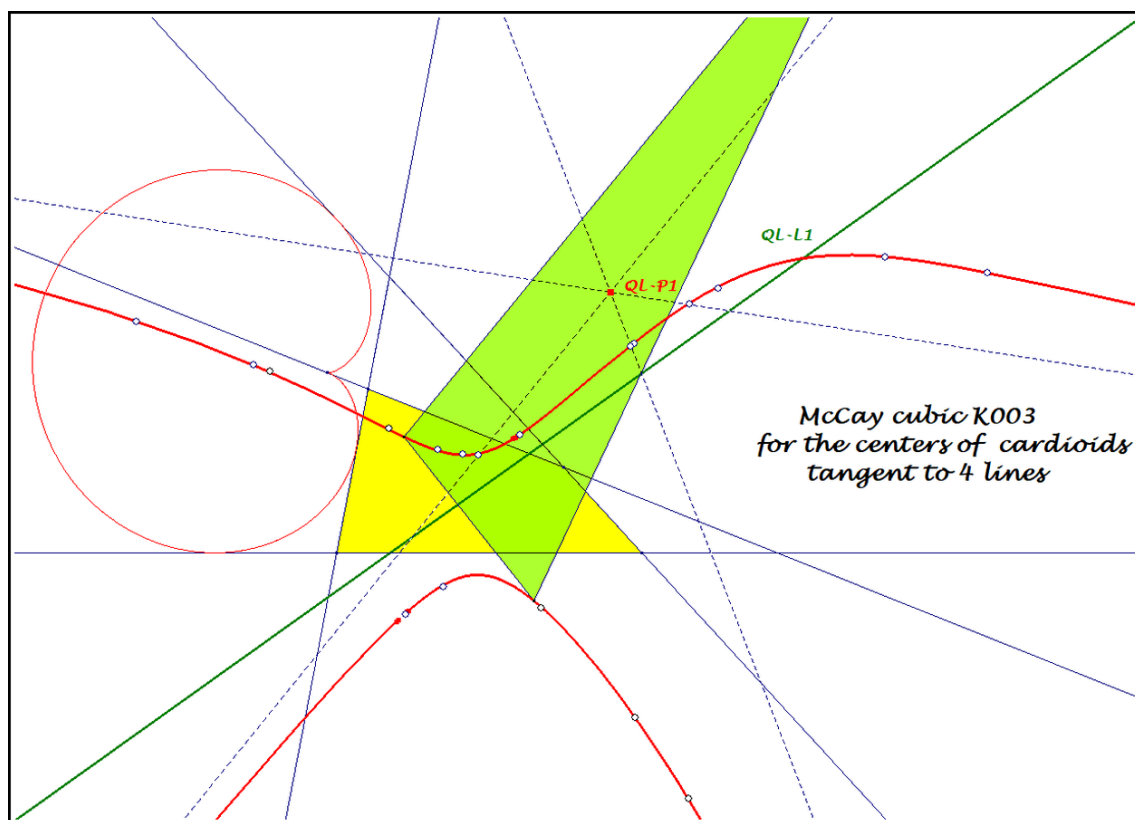
* If there is only one real intersection, the cubic will be a McCay cubic $K003$. The reference triangle has its centroid in the Miquel point QL-P1 and its circumcenter is the point of intersection (QL-L1=X3.X6).

* If there are one and a double counted real intersection, the cubic will be a Musselman cubic $K028$. In this case the quadrilateral is tangent to a circle with center QL-P10, which is the nodal point of the cubic. The reference triangle for $K028$ has its orthocenter in QL-P10 and the midpoint of the orthocenter and the centroid (X381) is the Miquel point QL-P1.

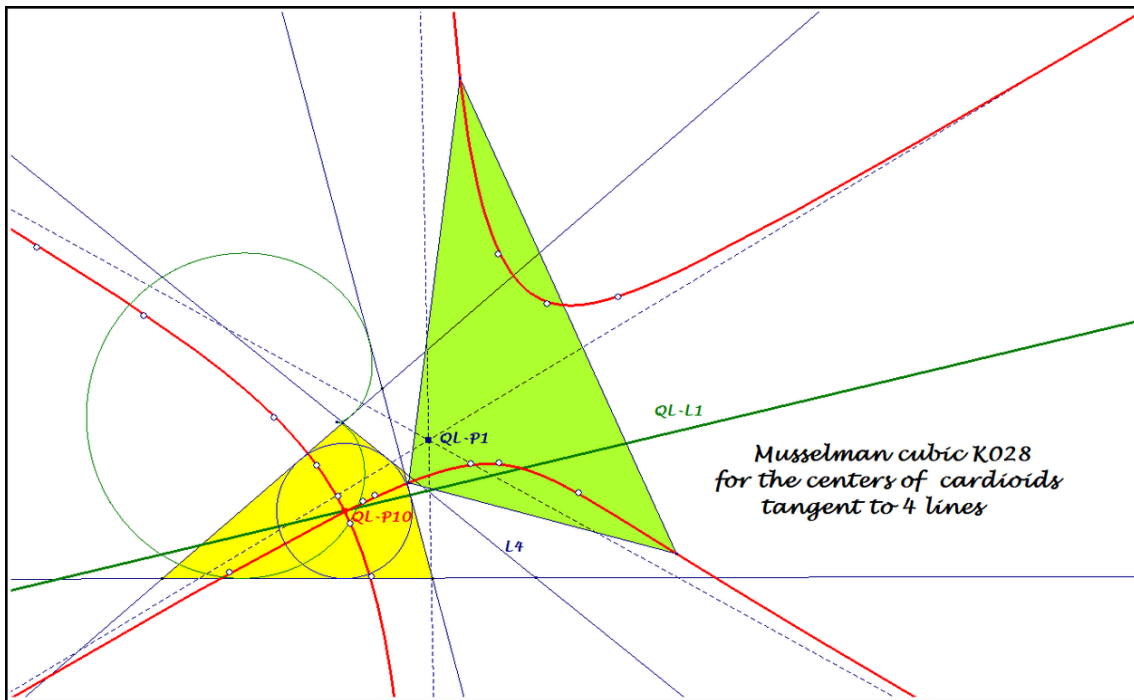
* If there are three real intersections, the cubic will be a Kjp $K024$. The reference triangle has its centroid in the Miquel Point QL-P1 and the points of intersection are the centers of the Apollonion circles.

I hope, there are no basic faultes. You find Cabri files in the attachment.

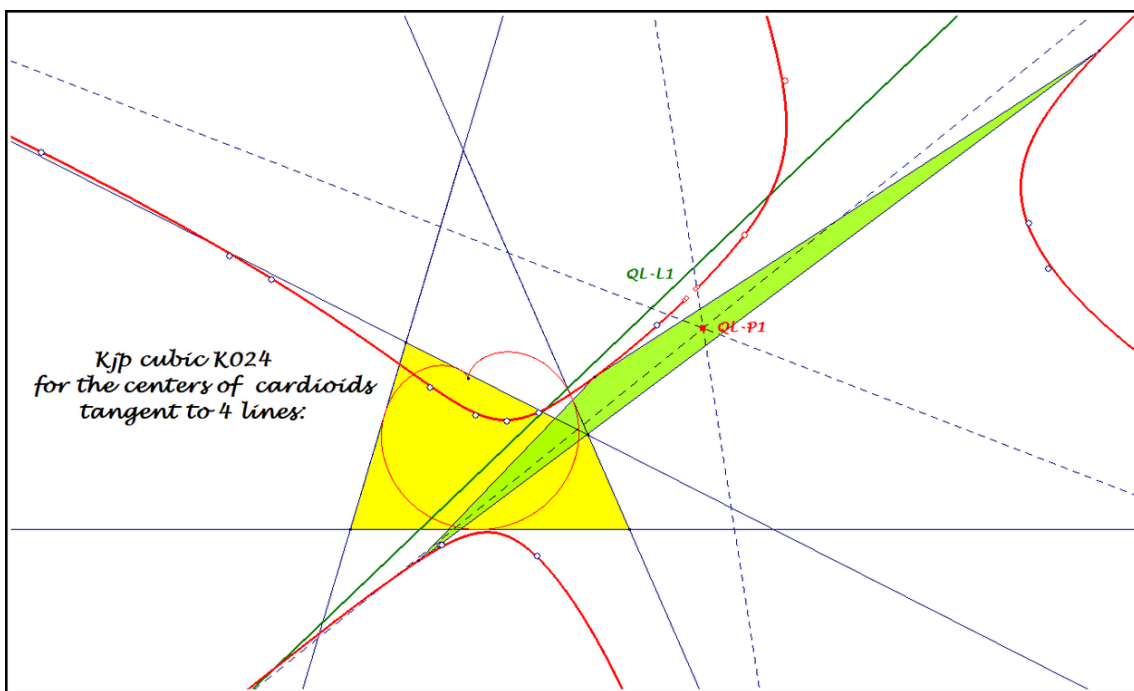
Best regards Eckart



13-06-29-a-fig.png



13-06-29-b-fig.png



13-06-29-c-fig.png

Message: #95
Date: 29/6/2013 9:11:28
From: Chris
Subject: Cardioids tangent to 4 lines

Dear Eckart,
Very nice results!
I know this takes a lot of time to resolve.
However I wonder,
1. How a QL-Cubic can belong to different Triangle Cubic-types dependant on circumstances. Or is it that actually these 3 Triangle Cubic-types belong to the same family?
2. Is there a standard Reference Triangle for these 3 types of Triangle cubcis? What is the relation of this/these Reference Triangle(s)?
Best regards,
Chris

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Message: #96
Date: 29/6/2013 9:34:42
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Chris,
I wonder too and have the same questions, for my knowledge of cubics is bounded.
Corresponding with Bernard Gibert, he announced connections of the cubics in the next week.
A further question:
What is the geometrical background for having one or three real intersections between QL-L1 and the cubic?
Best regards Eckart

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Message: #97
Date: 01/7/2013 10:06:35
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris and Eckart,

>> [Chris] What is the geometrical background for having one or three real
>> intersections between QL-L1 and the cubic?

I have more or less achieved the study of these cubics although a couple of details are still not fully completed.

I can answer the question above:

the Newton line QL-L1 is the orthic line of the hessian $H(P)$ of the Eckart cubic $E(P)$.

the number of real intersections depends of the position of the line $L(P)$ $lx+my+nz=0$ wrt to the 4 in/excircles of ABC.

1. if $L(P)$ cuts an even number of circles then $H(P)$ is bipartite and $E(P)$ meets QL-L1 thrice.

2. if $L(P)$ cuts an odd number of circles then $H(P)$ is unipartite and $E(P)$ meets QL-L1 once.

3. if $L(P)$ is tangent to one circle then $H(P)$ and $E(P)$ are nodal with node the center of the circle.

4. if $L(P)$ is tangent to two circles then $H(P)$ and $E(P)$ must decompose into a line and a circle for $H(P)$, a hyperbola with excentricity 2 for $E(P)$.

I have a lot of additional informations and I wonder whether it could be worth a paper...

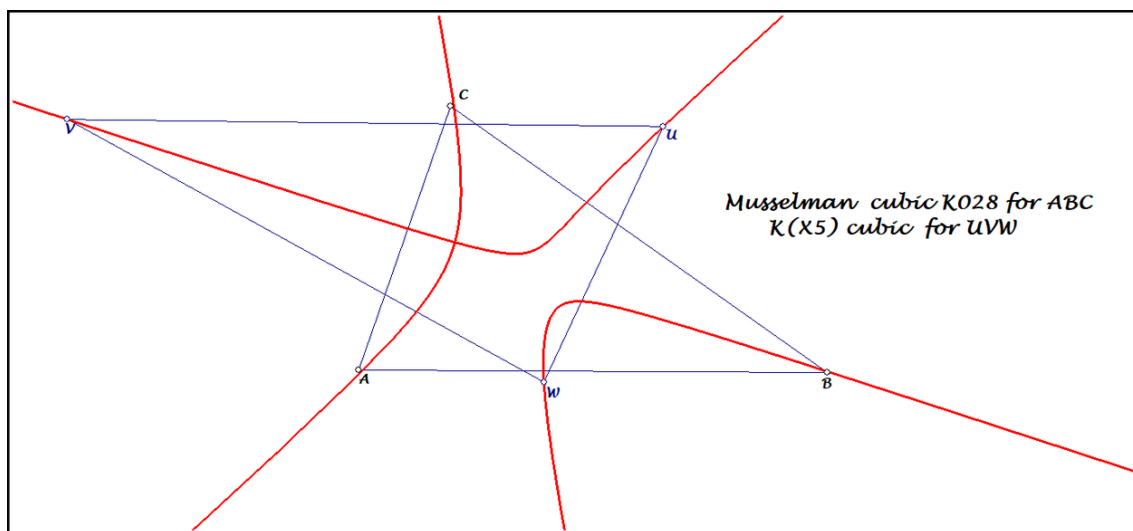
Best regards

Bernard

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Message: #98
Date: 01/7/2013 10:36:24
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Chris, dear friends,
perhaps there is a common sight of the three types (McCay K003, Musselman K028, Kjp K024) of my cubic:
In "Special Isocubics ..." from Bernard Gibert there is a "McCay-Kjp pencil" in table 22 with the property:
"The locus of M whose pedal circle cuts the nine-point circle under a given angle ... is a circum-cubic which is always a K60+..."
It is mentioned, that this pencil contains two nodal cubics.
Perhaps it is possible for my cubic to find reference triangles, so that this nodal cubics are the Musselman cubics K028 in the case of quadrilaterals tangent to a circle. I can't test it, for I have no construction for these cubics.
Unexpected: I found, that on a Musselman cubic K028 there is a reference triangle, for which the cubic is a K(X5).
This cubic is described by Bernard Gibert in Forum Geometricorum, Volume 2 (2002), p.55.
There is a Cabri file in the attachment.
Best regards Eckart



13-06-30-a-fig.png

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Message: #99

Date: 01/7/2013 2:49:55

From: eckart_schmidt@t-online.de

Subject: Cardioids tangent to 4 lines

Dear Chris, dear Bernard Gibert,

thanks to Bernard for the answer on my question, I searched without success.

In addition to my last message:

On a Musselman cubic K028 there are reference triangles, so that K028 belongs to the McCay-Kjp-pencil (table 22):

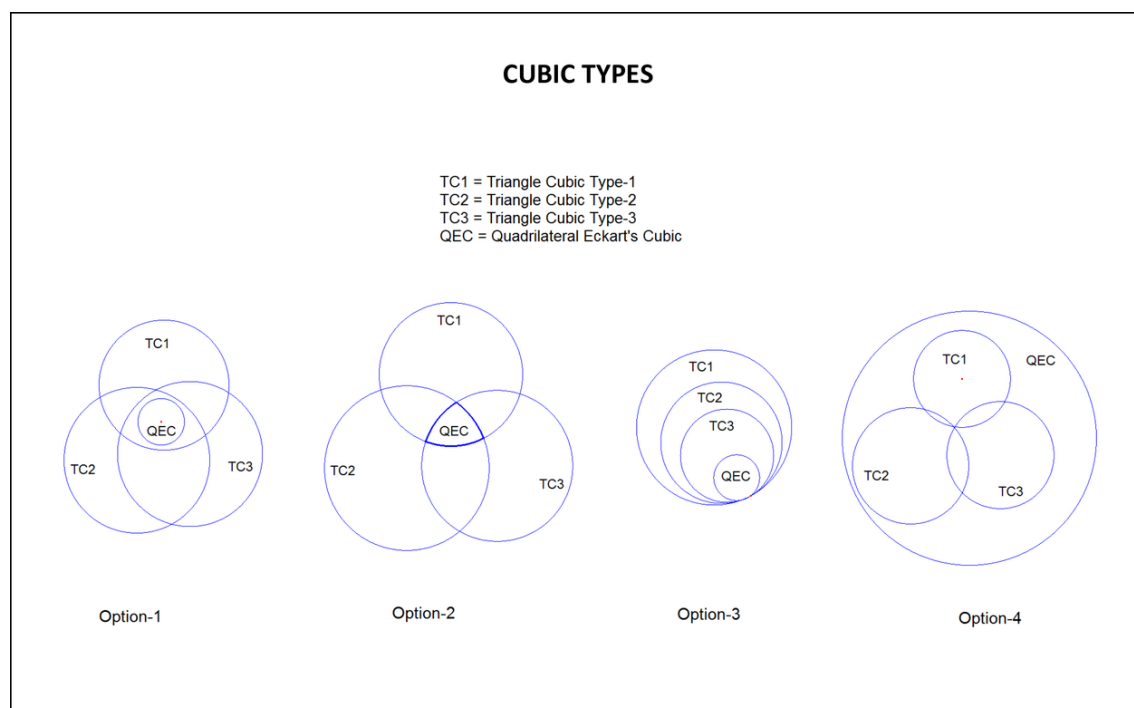
Construction for a given K028 and its reference triangle:

- * Take the midpoint $X_2.X_4$ as centroid G of a new reference triangle ABC .
 - * Take the nodal point X_4 as a focus F for the inscribed Steiner ellipse of ABC .
 - * Choose a point A on the cubic with its homothetic Ma under $h(G, -1/2)$.
 - * Construct the Steiner ellipse with center G , Focus F and point Ma .
 - * The tangents from A to the ellipse and the tangent in Ma give a triangle ABC on the cubic.
 - * For ABC as reference triangle the cubic belongs to the McCay-Kjp pencil.
- So the three types for my cubic belong to the same pencil.
Best regards Eckart
-

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Message: #100
Date: 01/7/2013 3:10:20
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

Dear Bernard, Dear Eckart,
Thanks for all information.
Bernard I hope it is worth a paper -like you suggested- for clarity and for reference in EQF.
One question remains for my own clarity.
A QL-Cubic is a unit in itself. How can it be 3 types in the Triangle environment?
If so, when we consider a cubic to belong to a set of similar cubics, how do these sets of cubics, type 1 (McCay), type 2 (Musselman), type3 (Kjp) relate to Eckart's (type of) cubic?
I made a picture with set diagrams of 4 options. See attachment.
Does one option apply to this situation or is there a better set diagram to be made?
Best regards,
Chris



CUBIC TYPES.pdf

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Message: #101
Date: 01/7/2013 3:33:53
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris,

>> A QL-Cubic is a unit in itself. How can it be 3 types in the Triangle
>> environment?

All depends of the reference triangle you choose.
A nK in one triangle can be a pK in another triangle.
Actually, any non-singular cubic is, in infinitely many ways, a pK wrt another triangle but this latter triangle is not always real.
It will be if you can draw four real tangents to the cubic from one of its points.
In particular, if the cubic is circular with an oval, this is always possible with a point on the oval.
See this paper for more information:
<http://bernard.gibert.pagesperso-orange.fr/files/bitripk.html>
Best regards
Bernard

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Message: #102
Date: 01/7/2013 5:58:03
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Correction:

>>
>> this is always possible with a point NOT
>> on the oval.
>>

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Message: #103
Date: 02/7/2013 9:24:41
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends
can it be, that the Hessian of my cubic is the QL-Quasi Isogonal Cubic QL-Cu1, containing the foci of all inscribed conics of the quadrilateral? Then Bernards criterion for intersections of QL-L1 and my cubic can be formulated wether QL-Cu1 is bipartite, unipartite or nodal.
Best regards Eckart

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Message: #104
Date: 2020-02-20
From: Systems Manager
Subject: Deleted Messages

Message number #104 is not available in Yahoo groups.

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Message: #105
Date: 03/7/2013 3:57:42
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear friends,
meanwhile I know, how to get the Hessian of my cubic. I think I'm right:
The Hessian of my cubic is QL-Cu1 (QL-Quasi Isogonal Cubic).
By the way, for a quadrilateral tangent to a circle the cubic QL-Cu1 is a strophoid of the line QL-L1, pole QL-P1 and fixed point QL-P10 (see "A book of curves" by E.H.Lockwood). Let S1, S2, S3 be the further intersections of the circle round QL-P10 through QL-P1 with the strophoid, then the lines QL-P1.Si are the asymptotes of my cubic.
Best regards Eckart

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Message: #106

Date: 04/7/2013 11:14:27

From: eckart_schmidt@t-online.de

Subject: Cardioids tangent to 4 lines

Dear friends,
please excuse, that my last message appears twice. I am not perfect in computing. Perhaps Chris can cut one message. There is a further criterium for the types of my cubic: Consider the conic C tangent to the four lines of the quadrilateral with center $QL-L1^{\wedge}QL-L6$; one axis is $QL-L1$ (see EQF, $QL-Co/1$).
If the conic C is a circle (center $QL-P10$), then ...
 $QL-Cu1$ is nodal,
 $QL-Cu1$ is the strophoid of $QL-L1$ with pole $QL-P1$ and fixed point $QL-P10$,
my cubic has a nodal point ($QL-P10$) and a further real intersection with $QL-L1$,
my cubic can be considered as a Musselman cubic $K028$ or a $K(X5)$ cubic or a cubic of the McCay-Kjp pencil (depending on corresponding reference triangles).
If the principal axis of the conic C is $QL-L1$, then
 $QL-Cu1$ is unipartite,
my cubic has one real intersection with $QL-L1$,
my cubic is a McCay cubic (for a corresponding reference triangle).
If the principal axis of the conic C is orthogonal $QL-L1$, then
 $QL-Cu1$ is bipartite,
 $QL-Cu1$ is a strophoid of $QL-L1$ with pole $QL-P1$ and fixed points in the foci of C ,
my cubic has three distinct real intersections with $QL-L1$,
my cubic is a Kjp cubic $K024$ (for a corresponding reference triangle).
If $QL-L1=QL-L6$, the quadrilateral is tangent to two circles (see Bernard's message), then ...
 $QL-Cu1$ decomposes into $QL-L1$ and a circle C_i with center $QL-P1$ and the intersections of the four lines not on $QL-L1$, my cubic decomposes into $QL-L1$ and a hyperbola with eccentricity 2 (see Bernards message) centered in $QL-P1$ with main circle C_i .
By the way: In the first and third case $QL-Cu1$ can be described as a strophoid in the definition of E. H. Lockwood: A Book of Curves (p.135). In the second case $QL-Cu1$ is a modified strophoid in the following sense: The line will be $QL-L1$ again, also the pole $QL-P1$, but the circles centered at $QL-L1$ have not a fixed point, they have to be orthogonal to the Thales circle wrt the foci of C . Best regards Eckart

Message: #107

Date: 05/7/2013 8:05:27

From: Chris

Subject: New transformation QA-Tf3 and related new point QA-P42

Dear Friends,

Thanks to Antreas' message #91, I once again looked at his orthopole circle at <http://anthrakitis.blogspot.gr/2013/03/orthopolar-circles-quadrilateral.html>

Already when we discussed this feature at Hyacinthos I found it most remarkable.

Summarized:

Let O_1, O_2, O_3, O_4 be the circumcenters of the Component Triangles of Reference Quadrangle $P_1.P_2.P_3.P_4$.

Let P be some random point.

Let OP_i be the Orthopole of line $P.P_i$ wrt Component Triangle $P_j.P_k.P_l$, where $(i,j,k,l) \in (1,2,3,4)$. The 4 Orthopoles OP_1, OP_2, OP_3, OP_4 are concyclic.

The circle through OP_1, OP_2, OP_3, OP_4 also passes through QA-P2, the Euler-Poncelet Point.

The center of this circle appears to be a linear transformation of starting point P . I called it QA-Tf3. It has very interesting features.

One of them is that the consecutive points generated from P diverge oscillating from another interesting fixed point: QA-P42.

Best regards,

Chris

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Message: #108
Date: 06/7/2013 2:51:28
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris and Eckart,

>>

>> I know this takes a lot of time to resolve.

>>

Indeed, and a very big margin as well!...

I know how to draw the reference triangle for a non-singular Eckart cubic when it meets the orthic line of its hessian at one real point (giving a McCay cubic) or at three real points (giving a Kjp cubic).

The method is difficult and involves all the heavy artillery related to general cubic curves: hessian, prehessians, poloconics, etc.

Naturally, these constructions are not possible with ruler and compass.

I will post a figure with the Kjp configuration.

Best regards

Bernard

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Message: #109
Date: 06/7/2013 7:21:20
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris and Eckart,

here are two figures with K003 and K024 types of Eckart's cubics.

Best regards

Bernard

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Message: #110
Date: 07/7/2013 8:44:26
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear Eckart,
Thanks for all your effort!
It is a tough, challenging and wonderful cubic we are dealing with
Bernard, you said you are using all the *heavy artillery* related to general cubic curves.
I think you act completely in the style of your famous countrymen Poncelet and Brianchon, who were both *soldiers in artillery* in Napoleon's army.
About the possible Reference Triangle I wonder if there is only one or possibly more of them.
What are the geometric properties of such triangle?
Since the basis of Eckart's Cubic is a Quadrilateral, being a system of 4 random lines, it is not unreasonable to think of a Reference Triangle, being a system of 3 lines (instead of 3 points) with an extra random line to complete the system.
Best regards,
Chris

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Message: #111
Date: 07/7/2013 11:44:04
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris,
>>
>> About the possible Reference Triangle I wonder
>> if there is only one or possibly more of them.
>>

when the Eckart cubic is elliptic (= not singular = without node), there is only one such triangle.
I'm still struggling with nodal cubics which have only one prehessian and the geometry is quite different.
Best regards
Bernard

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Message: #112
Date: 08/7/2013 9:57:10
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends,
I am very glad, that my conjectures seem right. Gratulation to Bernard for constructing the reference triangles for the McCay and Kjp cubic. But up to now I cannot reproduce the construction.

A remark to the nodal case:

There are infinite reference triangles, so that the cubic belongs to the McCay-Kjp-pencil. Here the construction of an example, if the axes of the deltoid of the quadrilateral are known (see also #99):

- * The parallels to the deltoid-axes through QL-P1 cut QL-L1 in three points. Their homothetics wrt $h(QL-P1, 2/3)$ are points of the cubic.

- * Take one of these points as vertice A of the reference triangle.

- * Let QL-P1 be the centroid of the reference triangle.

- * The homothetic of A wrt $h(QL-P1, -1/2)$ is the middle Ma of BC.

- * Construct the (Steiner) ellipse with center QL-P1, focus QL-P10 (nodal point) and point Ma.

- * The tangents from A to the ellipse and the tangent in Ma give a triangle ABC on the cubic.

- * For ABC as reference triangle the cubic belongs to the McCay-Kjp pencil.

Best regards Eckart

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Message: #113
Date: 09/7/2013 11:55:39
From: bernard.keizer
Subject: Cardioids tangent to 4 lines

Dear Eckart, dear Chris and dear Bernard

Although I'm not a specialist in cubics, I had several times visited Bernard's site and I'm glad to see he joined his efforts to help us to understand the wonderful cubic of the 27 centers of cardioids tangent to 4 lines discovered by Eckart. I must confess I was a little lost with your previous messages, but this one is more precise and I think I begin to see the possible construction of the reference triangle of the MacCay cubic ... We are now far from the initial Morley sets of the 4 reference triangles! I noticed that each line from one set cuts a second set in 9 points (as there are 3×3 parallels = 9 lines in each set), but only three of them are centers of cardioids tangent to the 4 lines, one point in each group of parallels. The other points are centers of 2 different cardioids, one tangent to 3 lines and the other also to 3 lines, but 2 lines only are common to both.

And by drawing cardioids, I was wondering what the locus of the 27 cusps could be (also a cubic?)

Best regards
Bernard

I begun to read other messages from the group, I saw the group is expanding, I noticed in particular that Jean-Louis Aymé had joined and I am happy of that because he is the one who advised me to contact Chris. (I haven't found yet a synthetic proof to his problem with the Euler-Poncelet point belonging to the circle circumscribed to the diagonal triangle of a quadrangle)

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Message: #114
Date: 10/7/2013 2:59:52
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends,
it seems accepted, that my cubic belongs to the McCay-Kjp pencil (see table 22). These cubics are defined (wrt a reference triangle) by an angle "theta" , which is the angle of intersection of the nine-point circle and the pedal circles of points on the cubic. For the McCay cubic "theta" is 0° , for a Kjp cubic "theta" is 90° . Here is a general construction of these cubics, knowing the angle "theta". I have not found this construction on table 22, it is a generalisation of the construction for the Kjp cubic:

Let P be a point on the circumcircle of the reference triangle ABC,

- ... let L(P) be the parallel at O to the Simson line of P;
- ... L(P) cuts the circumcircle in two points,
- ... consider the rectangular hyperbola
 through these two points and B, C,
- ... consider the line AP rotated by $90^\circ + \text{"theta"}$ at A,
- ... the intersections of this line and the rectangular hyperbola
 will be points of the cubic.

Best regards Eckart

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Message: #115
Date: 10/7/2013 8:28:50
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Eckart,

>

>> it seems accepted,

>> that my cubic belongs to the McCay-Kjp pencil

>>

I am more and more convinced that this is not true.

K003 and K024 are two very special cases of apolar isogonal cubics with same radial center G and their hessians are focal cubics with focus G.

Conversely, if you consider a pencil of focal isogonal nKs with same focus F on the circumcircle, the (real) prehessians of these cubics do not generally form a pencil!

So, the problem is not that simple and certainly much more challenging than expected!

We need time and reflexion, and above all, PROOFS!

Best regards

Bernard

PS: are you aware of the paper by Sister Mary Gervase on cardioids?

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Message: #116
Date: 11/7/2013 12:43:27
From: Seiichi Kirikami
Subject: van Aubel points

Dear friends,

(1) Preparation of van Aubel quadrangle.

Given a quadrangle $P_1P_2P_3P_4$, we denote the midpoint of the segment P_iP_j by M_{ij} or M_{ji} . We erect the vertical segment of half length of its side at M_{ij} in both direction of it. We denote the endpoint of the vertical segment by T_{ijR} or T_{ijL} , if we proceed from P_i to P_j and see the endpoint at the right or left direction respectively. So T_{ijR} and T_{jiL} indicate the same endpoint. van Aubel quadrangles can be constructed indefinitely.

(2) 4 circles through the following 3 points are concurrent.

(2.1) 1st cases: See van Aubel points (Aubel-SK-01a)

- (a) $T_{12RM31}T_{23R}$, $T_{23RM24}T_{34R}$, $T_{34RM31}T_{41R}$, $T_{41RM24}T_{12R}$
- (b) $T_{12LM31}T_{23L}$, $T_{23LM24}T_{34L}$, $T_{34LM31}T_{41L}$, $T_{41LM24}T_{12L}$
- (c) $T_{12RM14}T_{24R}$, $T_{24RM23}T_{43R}$, $T_{43RM14}T_{31R}$, $T_{31RM23}T_{12R}$
- (d) $T_{12LM14}T_{24L}$, $T_{24LM23}T_{43L}$, $T_{43LM14}T_{31L}$, $T_{31LM23}T_{12L}$
- (e) $T_{23RM12}T_{31R}$, $T_{31RM34}T_{14R}$, $T_{14RM12}T_{42R}$, $T_{42RM34}T_{23R}$
- (f) $T_{23LM12}T_{31L}$, $T_{31LM34}T_{14L}$, $T_{14LM12}T_{42L}$, $T_{42LM34}T_{23L}$

(2.2) 2nd cases: See van Aubel points (Aubel-SK-02a)

- (a) $T_{12RM24}T_{23R}$, $T_{23RM31}T_{34R}$, $T_{34RM24}T_{41R}$, $T_{41RM31}T_{12R}$
- (b) $T_{12LM24}T_{23L}$, $T_{23LM31}T_{34L}$, $T_{34LM24}T_{41L}$, $T_{41LM31}T_{12L}$
- (c) $T_{12RM23}T_{24R}$, $T_{24RM14}T_{43R}$, $T_{43RM23}T_{31R}$, $T_{31RM14}T_{12R}$
- (d) $T_{12LM23}T_{24L}$, $T_{24LM14}T_{43L}$, $T_{43LM23}T_{31L}$, $T_{31LM14}T_{12L}$
- (e) $T_{23RM34}T_{31R}$, $T_{31RM12}T_{14R}$, $T_{14RM34}T_{42R}$, $T_{42RM12}T_{23R}$
- (f) $T_{23LM34}T_{31L}$, $T_{31LM12}T_{14L}$, $T_{14LM34}T_{42L}$, $T_{42LM12}T_{23L}$

(2.3) 3rd cases: See van Aubel points (Aubel-SK-03a)

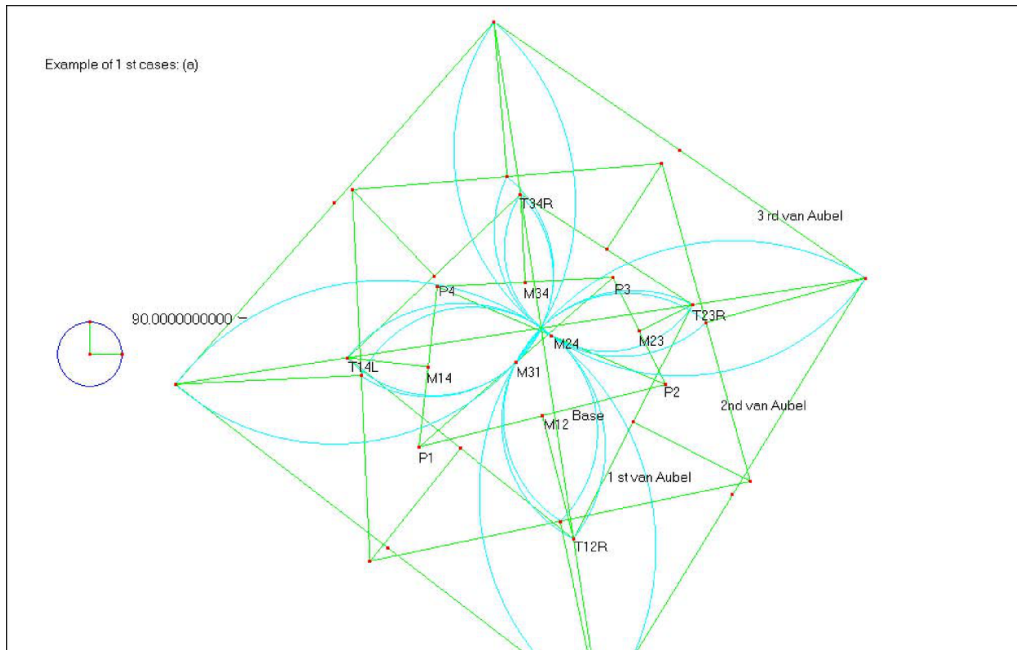
- (a) $S_{124M31}S_{123}$, $S_{123M24}S_{234}$, $S_{234M31}S_{134}$, $S_{134M24}S_{124}$ (outer)
- (b) $S_{124M31}S_{123}$, $S_{123M24}S_{234}$, $S_{234M31}S_{134}$, $S_{134M24}S_{124}$ (inner)
- (c) $S_{123M14}S_{124}$, $S_{124M23}S_{234}$, $S_{234M14}S_{134}$, $S_{134M23}S_{123}$ (outer)
- (d) $S_{123M14}S_{124}$, $S_{124M23}S_{234}$, $S_{234M14}S_{134}$, $S_{134M23}S_{123}$ (inner)
- (e) $S_{234M12}S_{123}$, $S_{123M34}S_{134}$, $S_{134M12}S_{124}$, $S_{124M34}S_{234}$ (outer)
- (f) $S_{234M12}S_{123}$, $S_{123M34}S_{134}$, $S_{134M12}S_{124}$, $S_{124M34}S_{234}$ (inner)

Note: S_{ijk} from (a) to (f) are different points.

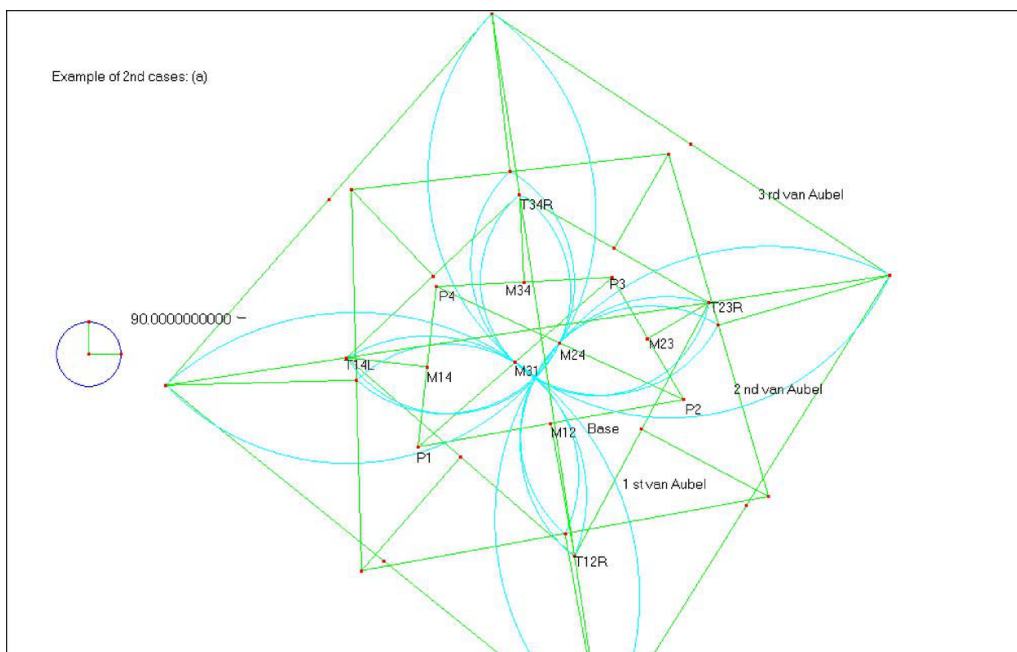
(2.4) 4th cases: See van Aubel points (Aubel-SK-04a)

- (a) $S_{124M24}S_{123}$, $S_{123M31}S_{234}$, $S_{234M24}S_{134}$, $S_{134M31}S_{124}$ (outer)
- (b) $S_{124M24}S_{123}$, $S_{123M31}S_{234}$, $S_{234M24}S_{134}$, $S_{134M31}S_{124}$ (inner)
- (c) $S_{123M23}S_{124}$, $S_{124M14}S_{234}$, $S_{234M23}S_{134}$, $S_{134M14}S_{123}$ (outer)
- (d) $S_{123M23}S_{124}$, $S_{124M14}S_{234}$, $S_{234M23}S_{134}$, $S_{134M14}S_{123}$ (inner)
- (e) $S_{234M34}S_{123}$, $S_{123M12}S_{134}$, $S_{134M34}S_{124}$, $S_{124M12}S_{234}$ (outer)
- (f) $S_{234M34}S_{123}$, $S_{123M12}S_{134}$, $S_{134M34}S_{124}$, $S_{124M12}S_{234}$ (inner)

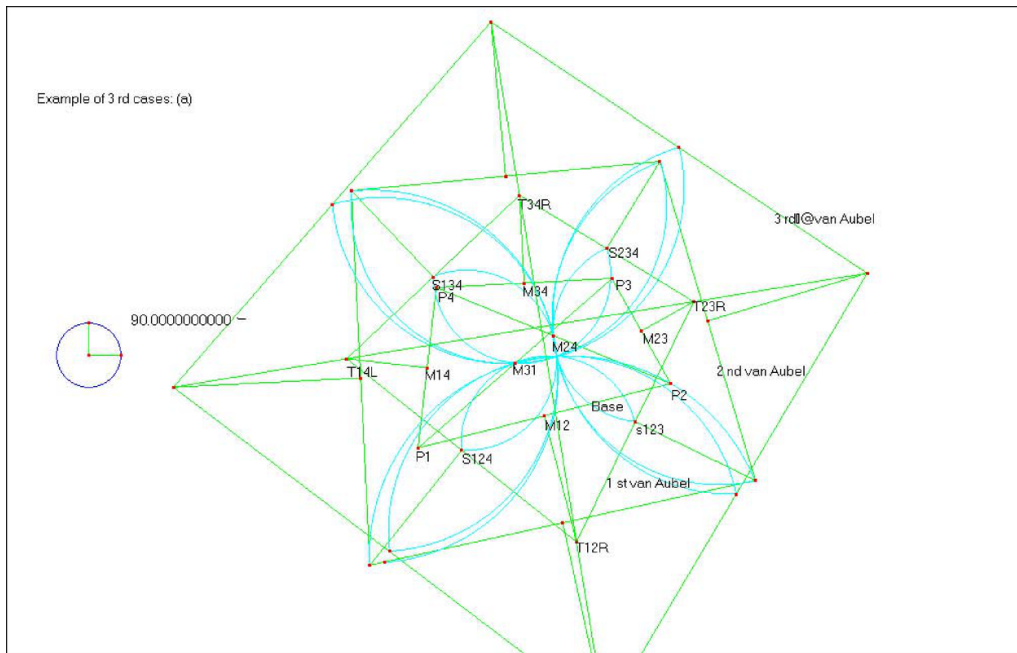
Note: Sijk from (a) to (f) are different points.
 N th quadrangle has concurrent points, which coincide with those of <N th quadrangles.
 Now I have no Mathematica evidence of concurrency.
 Sorry in advance if the above description were false.
 Seiichi.



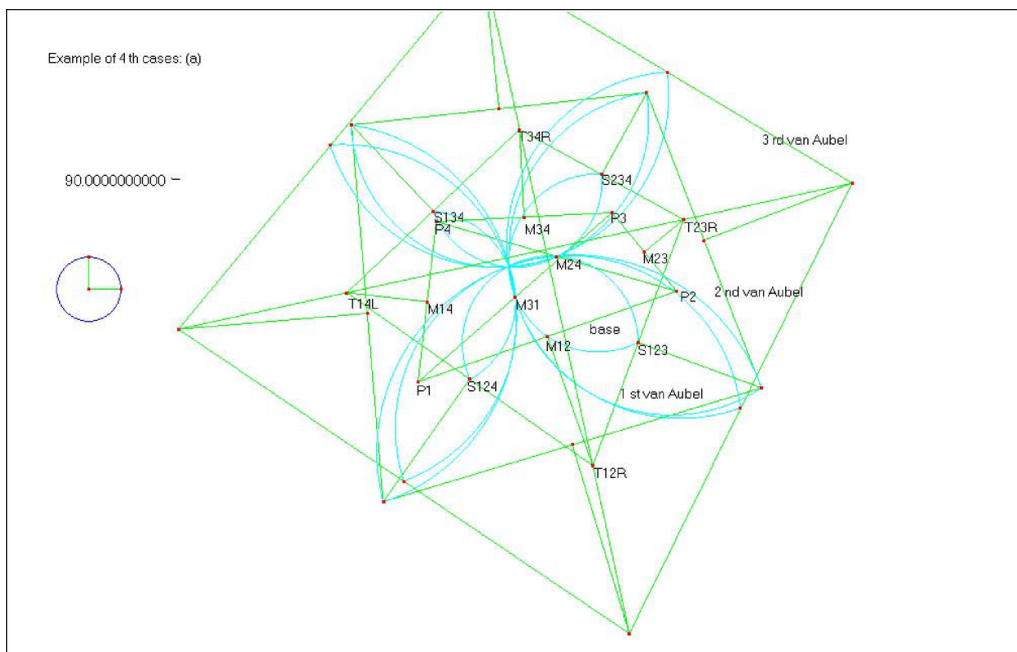
van Aubel points.pdf



van Aubel points.pdf



van Aubel points.pdf



van Aubel points.pdf

Message: #117
Date: 14/7/2013 3:02:23
From: Seiichi Kirikami
Subject: A quadrangle point

Dear friends,

[1] Given a quadrangle $P_1P_2P_3P_4$, let H_i be the orthocenters of $P_jP_kP_l$ and D_i be the isogonal of H_i wrt $P_1P_2P_3$. D_1, D_2, D_3 and D_4 are colinear.

See Anopolis message #537.

See the attached files (gsp or MSword).

[2] Given a quadrangle $P_1P_2P_3P_4$, let H_i be the orthocenters of $P_jP_kP_l$. The isagonals of $P_jP_kP_l$ determine L_i . L_1, L_2, L_3 and L_4 concur in a point Q .

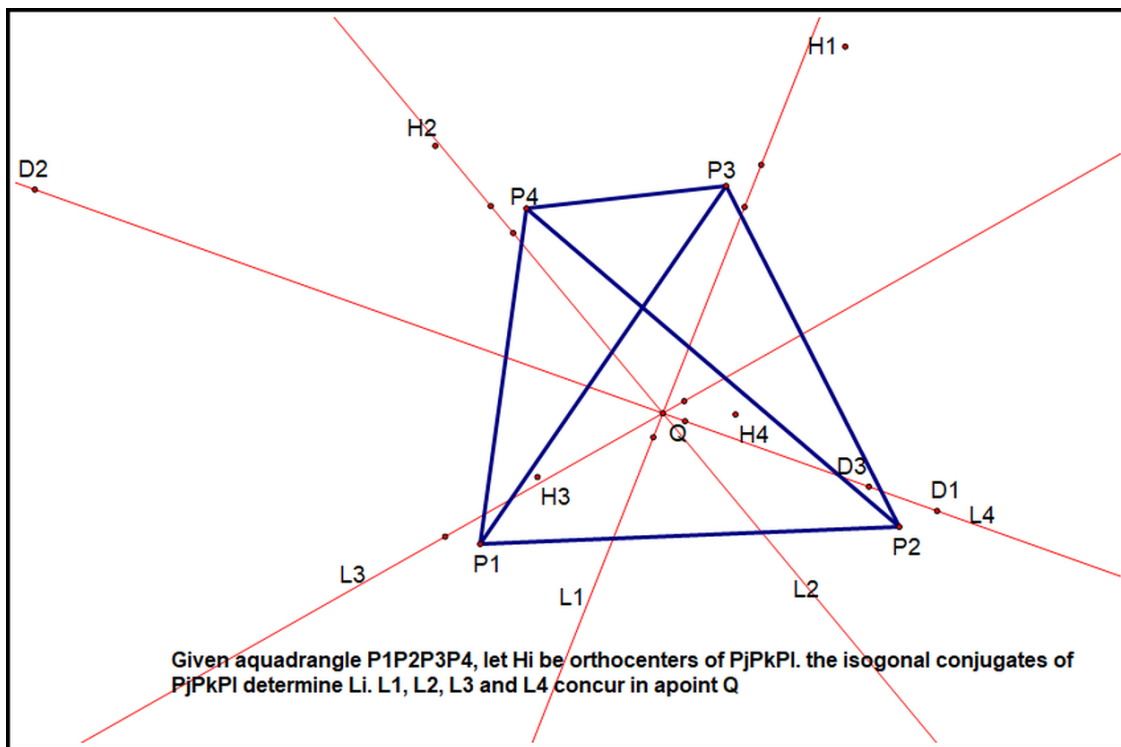
See the attached files (gsp or MSword).

Sorry in advance if the above were wrong.

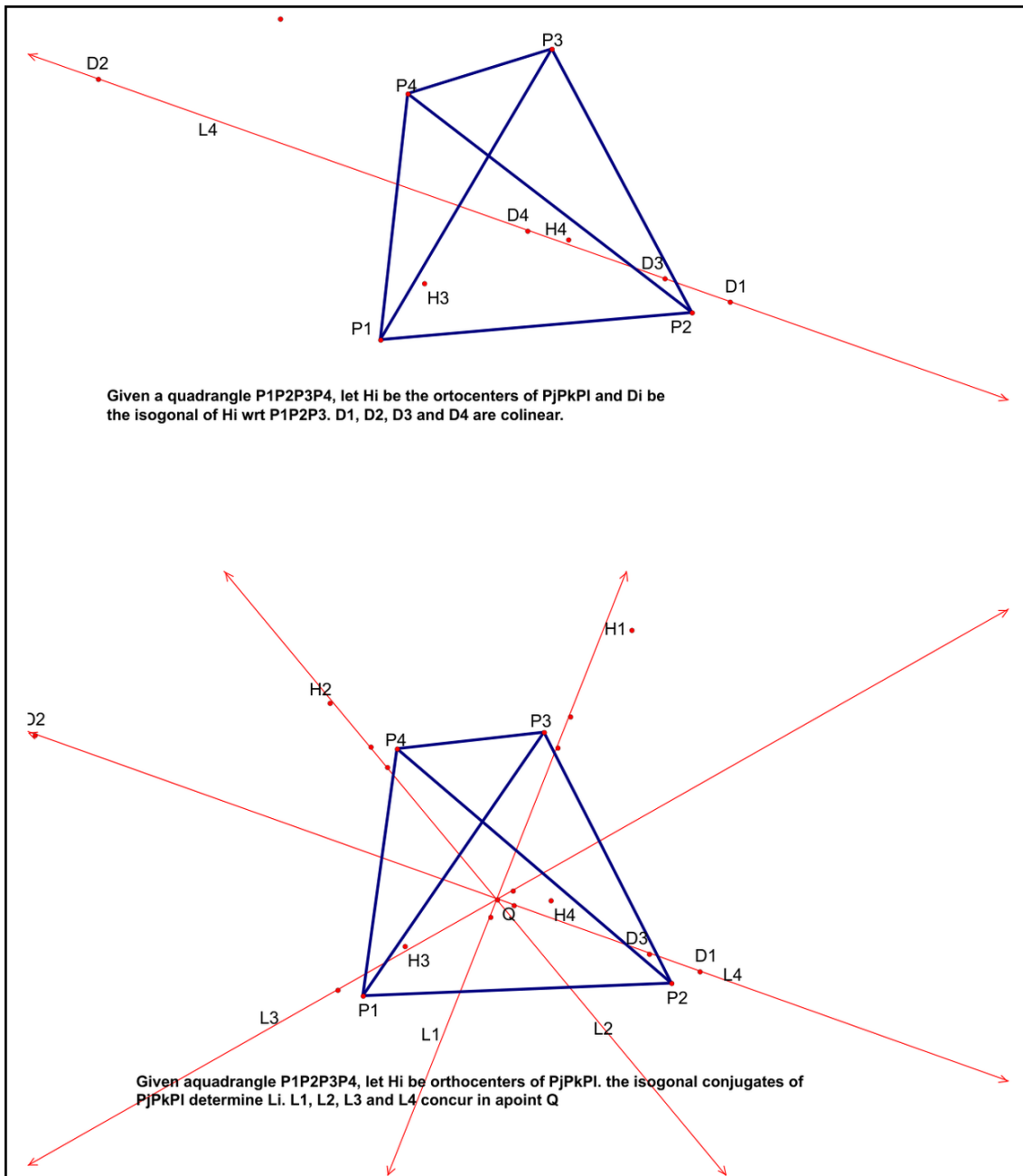
Sorry Chris that I can not send the coordinates. Now I am learning copy and run program of "ConjugadoIsogonal" by Professor GarciaCapitan.

Is there a point in EQF which may coincide the above one?

Best regards,
Seiichi.



QuadranglePoint-gsp.png



QuadranglePoint.docx

Message: #118
Date: 14/7/2013 8:21:36
From: seiichikiri
Subject: A quadrangle point

Dear friends,
This point is QA-P4(isogonal center), because I confirmed numerically that, for example, $\angle P_1P_3P_2 + \angle P_1P_4P_2 = \angle P_1QP_2$.
Best regards,
Seiichi.

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Message: #119
Date: 14/7/2013 5:35:21
From: Chris
Subject: A quadrangle point

Dear Seiichi,
This is a most remarkable construction of QA-P4.
I never saw this type of construction before.
Thanks to an observation of Antreas and your implementation in a Quadrangle. Congratulations!
I tried some other Triangle points with the same construction, however without success.
By the way QA-P4 surprises me with its many ways of beautiful construction all the time.
I mentioned your new construction as a new property in EQF.
See:
<http://www.chrisvantienhoven.nl/quadrangle-objects/15-mathematics/quadrangle-objects/artikelen-qa/25-qa-p4.html>
Best regards,
Chris

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Message: #120
Date: 15/7/2013 10:47:11
From: Antreas
Subject: A quadrangle point

Dear Seiichi and Chris
Seiichi has noticed that it seems collinearity appears also for isotomic conjugates.
See:
<http://tech.groups.yahoo.com/group/Anopolis/message/544>
If it is true: Are, in a quadrangle, the four lines concurrent as well?
APH

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Message: #121
Date: 15/7/2013 12:42:38
From: seiichikiri
Subject: A quadrangle point

Dear Antreas,
I tried the isotomic case. But there was no colinearity. I would also like to post something interesting in Anopolis.
Dear Chris,
Thank you very much for the inclusion in EQF. When I drew the figure, it gave me some kind of dejavu feeling and it took some time to find that it was QA-P4. But as you say, it is a beautiful point.
Best regards,
Seiichi.

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Message: #122
Date: 15/7/2013 7:37:06
From: emmanueltsukerman
Subject: A quadrangle point

Hello Friends,
I hope I don't disappoint anyone if I point out a reference that considers the QA-P4 construction mentioned: "Central Points of the Complete Quadrangle" by Benedetto Scimemi.
You may also find related content in
<http://forumgeom.fau.edu/FG2012volume12/FG201214.pdf>
involving QA-P4, isogonal conjugation and collinearity, but this time via the circumcenters.
Best Regards,
Emmanuel

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Message: #123
Date: 16/7/2013 11:48:41
From: seiichikiri
Subject: A quadrangle point

Dear Emmanuel, dear friends,
Thank you very much, Emmanuel, for your information. Could I post a problem which I noticed when I knew QA-P4? There may be literatures about it or it may be already in EQF or it may not exist.
Given a quadrangle $P_1P_2P_3P_4$ and a point P , P has the property:
 $\angle P_1P_4P + \angle PP_3P_2 = \angle P_1PP_2$.
 $\angle P_2P_1P + \angle PP_4P_3 = \angle P_2PP_3$.
 $\angle P_3P_2P + \angle PP_1P_4 = \angle P_3PP_4$.
 $\angle P_4P_3P + \angle PP_2P_1 = \angle P_4PP_1$.
In case of QA-P4, the 1st relation is written:
 $\angle P_1P_4P_2 + \angle P_1P_3P_2 = \angle P_1PP_2$, as shown in Corollary 10 of FG201214.
whether it exists or not, it remains a problem for me.
Is the Simson line in Fig.18 of FG201214 different from QL-L3(pedal line) in EQF?
Best regards,
Seiichi.

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Message: #124
Date: 17/7/2013 9:10:59
From: eckart_schmidt@t-online.de
Subject: A quadrangle point

Dear Seiichi, dear friends,
I had a short look on your discussion about the new property of QA-P4 (but i am still blocked with cardioids).
There is a synthetic proof for the last angle property of QA-P4.
Jean Louis Ayme asked for a proof and I posted the letter in the attachment.
Best regards Eckart.

01.03.2013

Dear Mister Ayme,

Chris van Tienhoven told me, that you need a synthetic proof for the following property of $T=QA-P4$:

$$(*) \quad \angle ACB + \angle ADB = \angle ATB$$

for all permutations of the vertices of the quadrangle $ABCD$. I try to describe a possibility, but my English isn't the best ... et mon Francais plus mauvais.

Perhaps you are interested in the following information:

The property (*) is discussed in

Praxis der Mathematik in der Schule 1/44, Jg 2002, 19-27

(see the scan of the beginning).

Ein merkwürdiger Punkt des Vierecks

Roland Stärk und Daniel Baumgartner

Gegeben sind in der Ebene vier Punkte A, B, C, D . Man zeige, dass es einen Punkt T gibt, der die folgenden Winkelbedingungen (orientierte Winkel modulo 180°) erfüllt:

$$\angle ATB = \angle ADB + \angle ACB, \quad \angle BTC = \angle BAC + \angle BDC,$$

$$\angle CTD = \angle CBD + \angle CAD, \quad \angle DTA = \angle DCA + \angle DBA.$$

In chapter 2 Stärk gives an analytical proof with special cartesian coordinates for the following property (see scan), which is the definition of $QA-P4$ in EQF .

Satz: (Fig. 5): Der zur Ecke A des Vierecks $ABCD$ bezüglich des Dreiecks BCD isogonalkonjugierte Punkt A^* und der Punkt T des Vierecks sind invers bezüglich des Umkreises des Dreiecks BCD .

Now my idea for a synthetic proof, that $QA-P4$ of a quadrangle $ABCD$ has the property (*). $QA-P4=T$ shall be defined as the invers of the isogonal conjugate of a vertex D wrt the residue triangle ABC .

Angles:

$$\angle BAD = \alpha, \quad \angle CBA = \beta, \quad \angle ACB = \varphi_1, \quad \angle ADB = \varphi_2.$$

Ayme-TPkt.pdf

These angles define the quadrangle.

Points:

M midpoint of the circumcircle of ABC ,

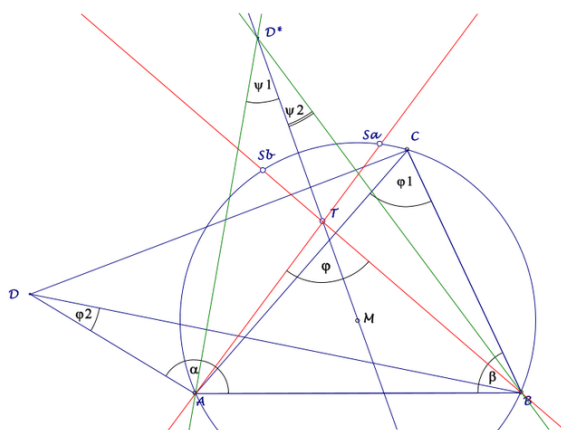
D^* isogonal conjugate of D wrt ABC ,

$T = QA-P4$ reflection of D^* in the circumcircle of ABC .

Angles:

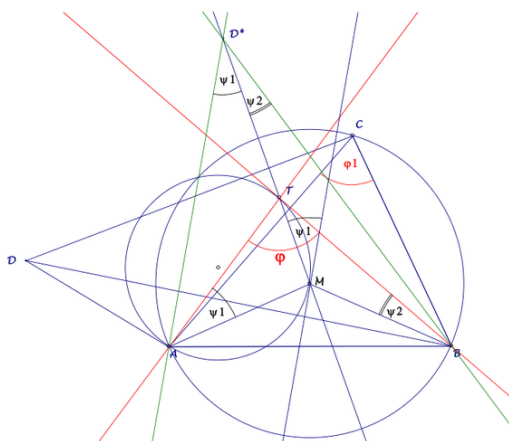
$$\angle ATB = \varphi, \quad \angle AD^*T = \psi_1, \quad \angle TD^*B = \psi_2.$$

To prove: $\varphi = \varphi_1 + \varphi_2$.



1. The reflection in a circle is angle-true. Reflecting AD^* in the circumcircle of ABC , we get a circumcircle of ATM with a tangent in M parallel AD^* . The angle ψ_1 becomes chord-tangent-angle of MT and equals the peripheral-angle:

$$\angle MAT = \psi_1 \quad \text{and in the same way:} \quad \angle TBM = \psi_2.$$



Angle balance in $ATBM$ gives with $\angle AMB = 2\varphi_1$

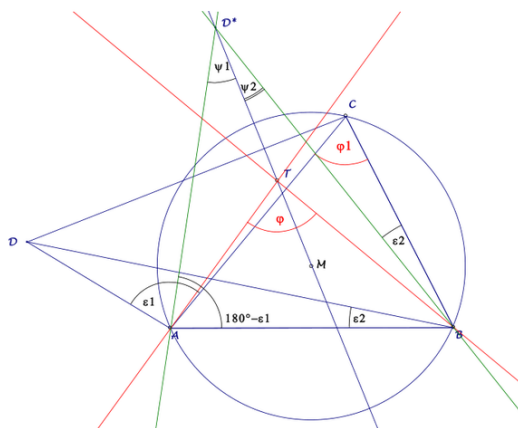
$$(1) \quad \varphi = 2\varphi_1 - \psi_1 - \psi_2 .$$

2. The isogonal conjugate wrt ABC gives

$$\angle CAD = \angle D^*AB = \varepsilon_1, \quad \angle DBA = \angle CBD^* = \varepsilon_2$$

with the angle balance in ABD^*

$$(2) \quad \psi_1 + \psi_2 = \varepsilon_1 + \varepsilon_2 - \beta .$$



3. Angle balance in ABD :

$$(3) \quad \varepsilon_2 = 180^\circ - \alpha - \varphi_2 .$$

Angle balance in ABC :

$$(4) \quad \varepsilon_1 = -180^\circ + \alpha + \beta + \varphi_1 .$$

From (1), (2), (3), (4) follows directly:

$$\varphi = \varphi_1 + \varphi_2 \quad \blacksquare$$

Yours sincerely
Eckart Schmidt

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Message: #125
Date: 17/7/2013 12:13:18
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends,
up to now I cannot reproduce Bernard's construction of reference triangles for my cubic. Here is my own way, how to find the reference triangles in the McCay or Kjp case:

- # X is the intersection of QL-L1 and QL-L6.
- # Co1 is the conic with center X and tangent to the lines of the quadrilateral.
- # Ci1 is the Thales circle for the foci of Co1.
- # O is a variable point on the secondary axis of Co1.
- # Ci2 is the circle round O perpendicular Ci1.
- # Point L is the reflection of X in Ci2.
- # Point A is a variable point on Ci2, the corresponding triangle ABC has circumcircle Ci2 and centroid QL-P1.
- # Ci3 is a circle as locus of the Lemoine points L' of ABC.
- # Chose O on QL-L1, that Ci3 contains L.
- # Chose A on Ci2, that L' = L.
- # If the secondary axis of Co1 is QL-L1, my cubic is the McCay cubic.

If the secondary axis of Co1 is perpendicular QL-L1, my cubic is the Kjp cubic.

This method degenerates for a quadrilateral tangent to 4 lines: Co1 is a circle, Ci1 is the point X = QL-P10, the secondary axis can be QL-L1 (or a perpendicular line). With the order next to last the reference triangle degenerates collinear and my cubic is the limit McCay (or Kjp) cubic..

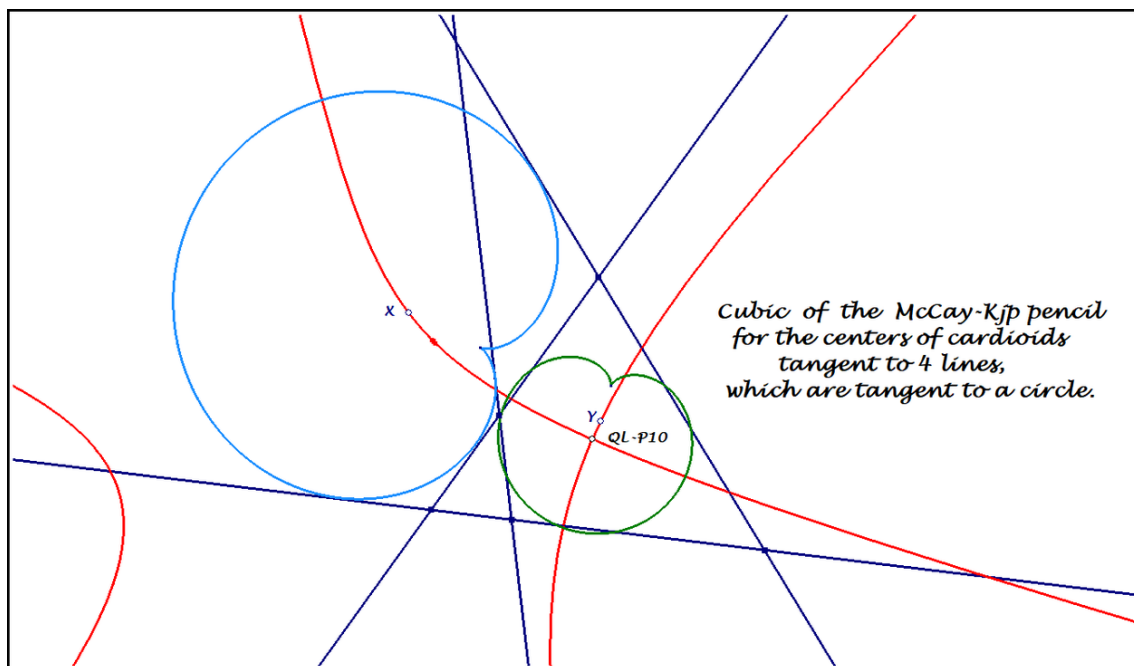
Bernard has doubts, that for a quadrilateral tangent to 4 lines my cubic belongs to the McCay-Kjp pencil.

Here is a construction, knowing the deltoid axes of the quadrilateral:

- # Consider parallels to the deltoid axes through QL-P1 and their intersections with a line parallel to QL-L1 in $\frac{2}{3}$ distance from QL-P1. These points lie on the cubic.
- # Chose one of these points as A and construct a special reference triangle:
 - * Take QL-P1 as centroid G of the reference triangle ABC.
 - * Take QL-P10 as a focus F for the inscribed Steiner ellipse of ABC.
 - * Let Ma be the homothetic of A under $h(G, -1/2)$.
 - * Construct the Steiner ellipse with center G, Focus F and point Ma.
 - * The tangents from A to the ellipse and the tangent in Ma

- give a special reference triangle ABC.
- # Take another point of the cubic (see above) and construct its pedal circle wrt ABC. Let "theta" be the angle of intersection of this pedal circle and the nine-point circle.
 - # Construct the cubic of the McCay-Kjp pencil with the angle "theta" (see message #114)
 - * Let P be a point on the circumcircle of the reference triangle ABC,
 - * ... let $L(P)$ be the parallel at O to the Simson line of P;
 - * ... $L(P)$ cuts the circumcircle in two points,
 - * ... consider the rectangular hyperbola through these two points and B, C,
 - * ... consider the line AP rotated by $90^\circ + \text{"theta"}$ at A,
 - * ... the intersections of this line and the rectangular hyperbola will be points of the cubic.

In the attachment there is a Cabri file with an quadrilateral tangent to 4 lines and my cubic constructed in the described way. You can test the variable centers on the cubic for cardioids tangent to 4 lines (the cubic is constructed in two parts, therefore the centers X and Y). I hope, there are no basic mistakes.
Best regards Eckart



McCay-Kjp-fig.png

Message: #126

Date: 17/7/2013 1:46:26

From: Seiichi Kirikami

Subject: 4 concurrent circles points similar to Euler-poncelet and Varignon p

Dear Emmanuel, dear friends,

Your paper FG201214 led me to the following 4 concurrent circles points. See the attached files (gsp or MSword).

[1] Euler-Poncelet like point:

Given a quadrangle $P_1P_2P_3P_4$, $QA-P_4$ and pedal N_{ij} from $QA-P_4$ on P_iP_j , the circles through $N_{12}N_{14}N_{24}$, $N_{23}N_{24}N_{34}$, $N_{12}N_{13}N_{23}$ and $N_{13}N_{14}N_{34}$ concur in a point Q .

[2] Gergonne-Steiner like point:

Given a quadrangle $P_1P_2P_3P_4$, $QA-P_4$ and pedal N_{ij} from $QA-P_4$ on P_iP_j , the circles through $N_{12}N_{13}N_{14}$, $N_{13}N_{23}N_{34}$, $N_{14}N_{24}N_{34}$ and $N_{12}N_{24}N_{23}$ concur in a point $QA-P_4$. This is not interesting.

[3] Varignon like points(1)

Similar to Varignon points. See the message#39(Varignon Points).

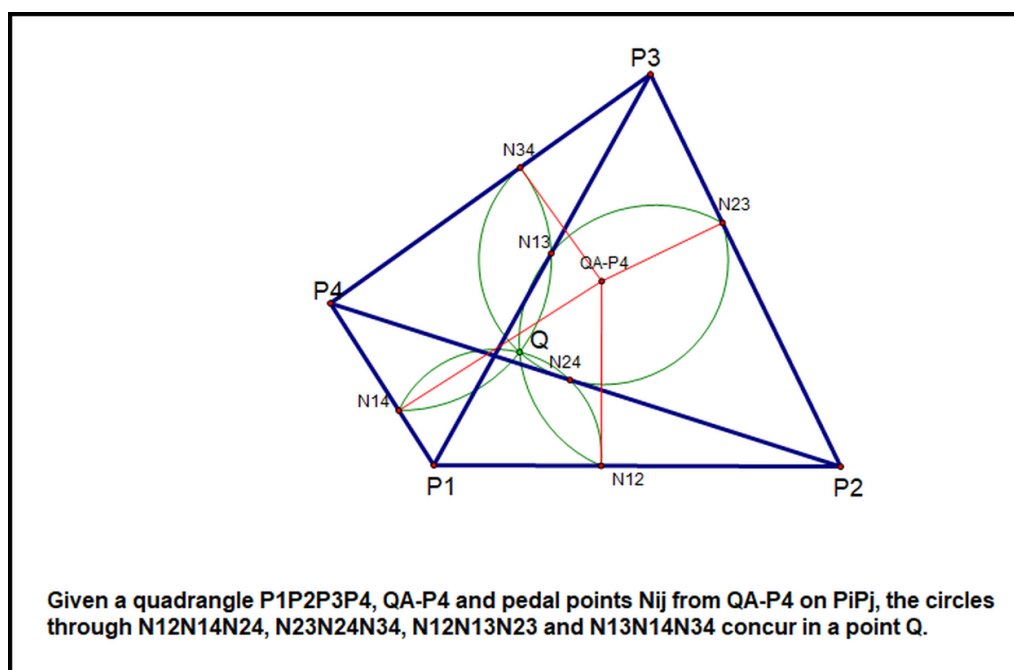
[4] Varignon like points(2)

Similar to Varignon points. See the message#39(Varignon Points).

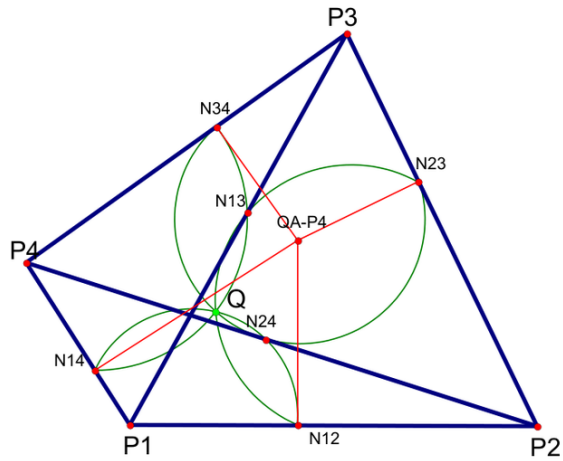
I will make the coordinates of [1].

Best regards,

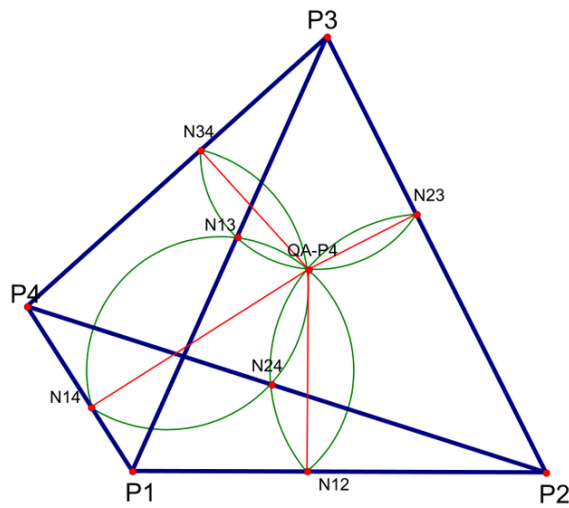
Seiichi.



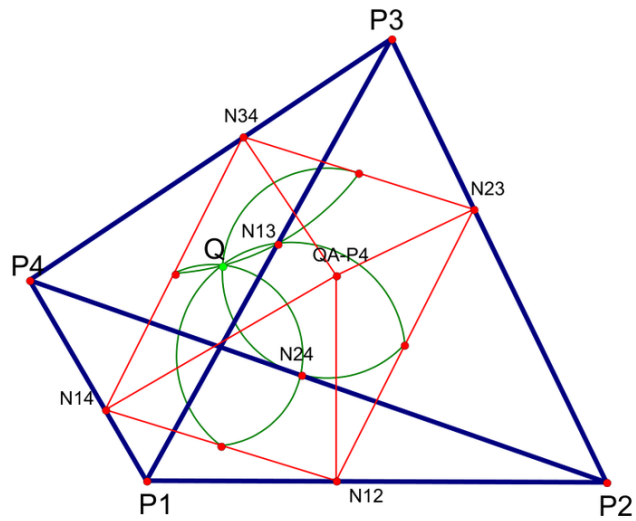
4ConcurrentCircles-gsp.png



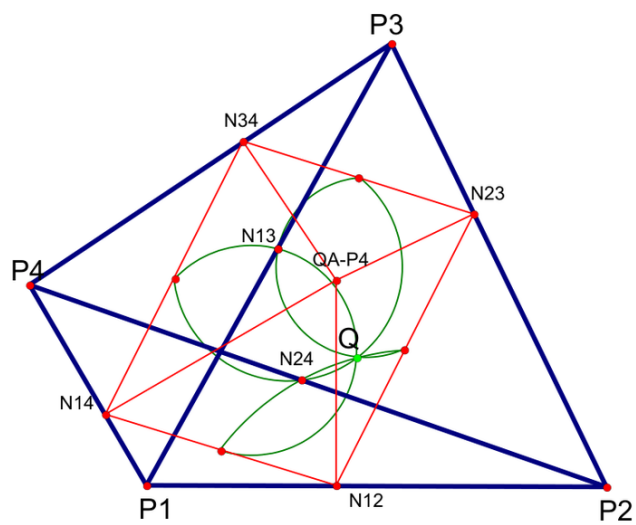
Given a quadrangle $P_1P_2P_3P_4$, $QA-P_4$ and pedal points N_{ij} from $QA-P_4$ on P_iP_j , the circles through $N_{12}N_{14}N_{24}$, $N_{23}N_{24}N_{34}$, $N_{12}N_{13}N_{23}$ and $N_{13}N_{14}N_{34}$ concur in a point Q .



Given a quadrangle $P_1P_2P_3P_4$, $QA-P_4$ and pedal points N_{ij} from $QA-P_4$ on P_iP_j , the circles through $N_{12}N_{13}N_{14}$, $N_{13}N_{23}N_{34}$, $N_{14}N_{24}N_{34}$ and $N_{12}N_{24}N_{23}$ concur in a point $QA-P_4$.



See message #39 (Varignon points)



See message #39 (Varignon points)

4ConcurrentCircles.docx

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Message: #127

Date: 18/7/2013 8:14:57

From: Chris

Subject: 4 concurrent circles points similar to Euler-poncelet and Varignon

Dear Seiichi,

Interesting!

These points are made of restricted permutations of points.

Restated:

Let N_{ij} be the feet of the perpendiculars from QA-P4 to $P_i.P_j$.

[1] The 4 versions of circles through N_{ij} , N_{jk} , N_{ki} (for all permutations $(i,j,k) \in (1,2,3,4)$) have one common point, which is QA-P2.

[2] The 4 versions of circles through N_{ij} , N_{ik} , N_{il} (for all permutations $(i,j,k,l) \in (1,2,3,4)$) have one common point, which is QA-P4.

[3] and [4] are actually Quadrignon points. However their midpoint is Quadrangle point QA-P6.

Best regards,

Chris

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Message: #128

Date: 18/7/2013 10:01:02

From: eckart_schmidt@t-online.de

Subject: Isogonal conjugate quadrigon of a point

Dear friends,

Emmanuel led me to the paper FG201214, I had a short look and found on the last pages the following definition:

* "Let P be a point on the plane of ABCD. Let l_A, l_B, l_C, l_D be the reflections of the lines AP, BP, CP, DP in the bisectors of
* Definition. Let $PA = l_A \wedge l_B$, $PB = l_B \wedge l_C$, $PC = l_C \wedge l_D$, $PD = l_D \wedge l_A$. The quadrilateral BA.PB.PC.PD will be called the isogonal conjugate of P wrt to ABCD ..."

Wrt Chris classification, this is a definition for quadrigons. I have tested the question, for which points the isogonal conjugate quadrigon degenerates into a four times counted point. Here the results:

* If the isogonal conjugate quadrigon of P degenerates, the four counted point Q is the QG-Tf2 image of P, or the isogonal conjugated of P wrt QG-Tr3.

* The points P, for which the isogonal conjugate quadrigon degenerates, and the four counted point Q lie on a cubic with the equation

* $2 SA z (q^2 x^2 - p^2 y^2) + 2 SC x (r^2 y^2 - q^2 z^2) + (c^2 q^2 + b^2 r^2) x^2 y - (b^2 p^2 + a^2 q^2) y z^2 + (a^2 r^2 - c^2 p^2) y^3 = 0$ (using the diagonal triangle QA-DT as reference triangle for barycentric coordinates).

* This cubic is the QL-Quasi Isogonal Cubic QL-Cu1.

* The points P and Q are foci of inscribed conics of the corresponding quadrilateral.

* The three quadrigons of a quadrilateral have the same points, for which the isogogonal conjugate quadrigon degenerates.

Best regards Eckart

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Message: #129

Date: 18/7/2013 1:20:45

From: Seiichi Kirikami

Subject: 4 concurrent circles points similar to Euler-poncelet and Varignon

Dear Chris,

Thank you very much for your identification. I confirmed QA-P2.

What I found was not Euler Poncelet like point, but Euler Poncelet point itself. I could not help laughing.

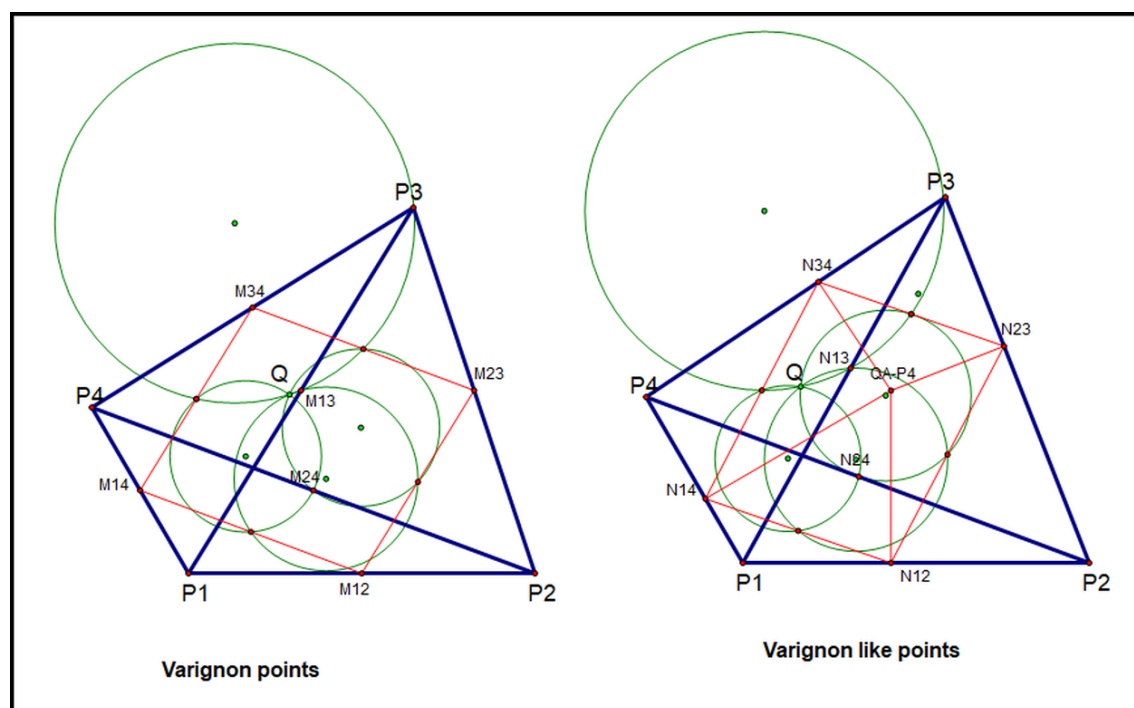
As for [3] and [4], I could not understand "However, their midpoint is Quadrangle point QA-P6".

I attached the file. I hope your detailed explanation.

Eckart's results about FG201214: Splendid!

Best regards,

Seiichi.



4ConcurrentCirclesR1-gsp.png

Message: #130

Date: 18/7/2013 3:32:03

From: Chris

Subject: 4 concurrent circles points similar to Euler-poncelet and Varignon

Dear Seiichi,

The Varignon-like points deal with the midpoints of the sides of the Varignon Parallelogram. It is good to realize that the sides of the Varignon Parallelogram are parallel to 2 of the 3 diagonals of a Quadrangle. In a Quadrignon we also work with 2 diagonals. That's why the construction of the Varignon-like points are Quadrignon points.

You work actually in the Quadrignon P1.P2.P3.P4 (order is significant).

There are also Quadrignons P1.P2.P4.P3 and P1.P3.P2.P4.

When you construct the same duo of Varignon-like points for Quadrignons P1.P2.P4.P3 and P1.P3.P2.P4 then you will find other points.

However the connecting lines of the 3 duo's intersect in one point, which is QA-P6.

Best regards,

Chris

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Message: #131

Date: 18/7/2013 6:54:46

From: emmanueltsukerman

Subject: A quadrangle point

Dear Seiichi,

Thank you for your question. That's correct - the Simson line in Fig.18 of FG201214 is QL-L3(pedal line) in EQF.

Best Regards,

Emmanuel

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Message: #132
Date: 19/7/2013 9:59:28
From: Chris
Subject: Finding point X on QA-Cu1 Cubic

Dear friends,
Given random point P and Quadrangle P1.P2.P3.P4.
Let R_{ij} = Reflection of P in $P_i.P_j$, where i,j can have values 1,2,3,4.
The locus where lines $R_{12}.R_{34}$, $R_{13}.R_{24}$, $R_{14}.R_{23}$ are concurrent is the cubic QA - Cu1.
See: http://www.chrisvantienhoven.nl/other-quadrangle-objects/15_mathematics/quadrangle-objects/artikelen-qa/83-qa-cu1.html
Let's call this concurrency point the QA-Reflect Image.
Example:
QA-P2 is the QA-Reflect-Image of QA-P4.
I observed that QA-P38 is the QA-Reflect-Image of some point X on QA-Cu1.
Anyone who can help me finding the coordinates of this point X?
Best regards,
Chris

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Message: #133
Date: 20/7/2013 11:20:07
From: eckart_schmidt@t-online.de
Subject: Finding Point X on QA-Cu1 Cubic

Dear Chris, dear friends,
the "QA-Reflect Image" is only defined for points on QA-Cu1. The connection of a point on QA-Cu1 and its image is parallel to QA-P2.QA-P4 or QA-P11.QA-P38. So the searched point is the intersection of QA-P11.QA-P38 with the cubic.
First DT-coordinate:
$$p^2 (b^2 r^4 (b^4 p^2 - a^4 q^2) + c^2 q^4 (c^4 p^2 - a^4 r^2) + b^2 c^2 p^2 q^2 r^2 (2a^2 - b^2 - c^2))$$

Best regards Eckart

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Message: #134
Date: 20/7/2013 12:21:24
From: seiichikiri
Subject: A quadrangle point

Dear Emmanuel, dear friends,
Concerning another Simson line, I would like to mention Eckart's
Probe(Themen 2005, 05-1, Simson-Gerade eines Kreisvierecks).
<http://eckartschmidt.de/>
Best regards,
Seiichi.

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Message: #135
Date: 20/7/2013 1:49:26
From: Chris
Subject: Finding Point X on QA-Cu1 Cubic

Dear Eckart, dear friends,

Thanks Eckart!

I used the same trick as you did, however I was not able to find simple coordinates as an intersection of a cubic and a line. I wonder how you did.

Now that the coordinates are known it is so much simpler to find other properties of this point.

I found it lies on:

- * The line QA-P11.QA-P38,
- * The line through QA-P41 parallel to QA-P3.QA-P4,
- * The perpendicular bisector of QA-P2.QA-P4,
- * Cubic QA-Cu1.

The perpendicular bisector of QA-P2.QA-P4 is an old acquaintance of ours. We (Eckart, Seiichi and me) discussed it before in mails before this Quadri-Figures-Group existed.

I quote:

*** Mail 8-1-2013 from 13:22 Seiichi Kirikami

"Montesdeoca-Hutson point (QA-P38) is determined by the diagonal lines of the hexagon $s_{12}s_{14}s_{13}s_{34}s_{23}s_{24}$, which I call MH hexagon.

The intersections of the lines $s_{12}.s_{14}*s_{23}.s_{34}$, $s_{13}.s_{14}*s_{23}.s_{24}$ and $s_{13}.s_{34}*s_{12}.s_{24}$ determine a Pascal line."

*** Mail 9-1-2013 12:25 from Eckart Schmidt

"Further properties of the MH Pascal line:

- ... polar of QA-P38 wrt QA-Ci1
- ... QA-Tf2 image of QA-Ci1
- ... perpendicular bisector of QA-P2.QA-P4

The line is already mentioned under QA-Ci1, where the last two properties are to be found."

I also will mention now some other properties related to the QA-Reflect transformation, that only is valid for points on the QA-Cu1 cubic.

1. The line through 2 Involutary Conjugated points on QA-Cu1 passes through QA-P4.
2. The line through a point on QA-Cu1 and its QA-Reflect image is parallel to QA-P2.QA-P4.
3. The QA-Reflect-images of a point on QA-Cu1 and its Involutary Conjugate pass through the Reflection of QA-P4 in QA-P2.

4. The line through a point on QA-Cu1 and the QA-Reflect image of its Involutary Conjugate pass through QA-P2.

It looks like that QA-P2 and QA-P4 are pretty dominant for this transformation.

Knowing these properties I wonder if it is possible to construct the reverse of the QA-Reflect transformation in a simple way.

Any Suggestions?

Best regards,
Chris

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Message: #136
Date: 21/7/2013 3:09:07
From: Chris
Subject: A quadrangle point

Dear Emmanuel, dear friends,
I looked through the paper of Benedetto Scimemi.
Thanks Emmanuel for the reference.
I found all kinds of interesting properties regarding QA-P1 (Centroid), QA-P2 (Euler-Poncelet Point/ nine-Circle Point) and QA-P4 (Isogonal Center/ Isoptic Point), but not the one we were discussing.
So as yet I consider it to be revealed by Antreas and Seiichi.
Best regards,
Chris

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Message: #137
Date: 22/7/2013 7:15:46
From: emmanueltsukerman
Subject: A quadrangle point

Dear Chris,
That's excellent! I'm glad you found the reference interesting
and I'm glad my memory served me wrong here.
Best Regards,
Emmanuel

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Message: #138
Date: 22/7/2013 11:56:48
From: Chris
Subject: What is the necessary and sufficient condition of quadrifigure f

Dear Seiichi,
I studied your original thoughts about the necessary and
sufficient condition of quadrifigure function.
When I see it right your premise is that when a Quadrangle point
is placed into a Square it should be in the center of the
square.
And -I checked it- that is true for all points QA-P1 till
QA-P42.
However I still wonder if we can say this true for all
Quadrangle points.
There is an equivalence in a triangle.
When we place triangle points in an equilateral triangle almost
all triangle points end up in the center.
However not all of them.
You can see this when you make a triangle almost equilateral and
see where the points "fly".
See: http://sites.paideiaschool.org/steve_sigur/resources/all%20points%20web/all-points.html
Especially Yao Liu's observations (at 30% from the beginning of
the page).
How do you see this?
Best regards,
Chris

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Message: #139

Date: 23/7/2013 3:34:39

From: seiichikiri

Subject: What is the necessary and sufficient condition of quadrifigure f

Dear Chris,

Thank you very much for your pointing out the complexity of the center of almost equilateral triangle. I vaguely remember the case that as the triangle becomes almost equilateral, the center $\{p, q, r\}$ limits to $\{0, 0, 0\}$. I think that in some cases the points fly somewhere out of the center. I must correct the contents of message#47. What I proposed is not the necessary condition. It is one of many methods to check the possibility of a quadrangle point.

Best regards,

Seiichi.

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Message: #140

Date: 23/7/2013 4:29:17

From: seiichikiri

Subject: a property of QA-P2

Dear friends,

I learned the following property from Mr. Peter J. C. Moses.

Given a quadrangle $P_1P_2P_3P_4$ and the antigonal image Q_i of P_i wrt P_jPkPl , QA-P2 is the midpoint of P_iQ_i .

I changed the property of a triangle to that of a quadrangle.

Best regards,

Seiichi.

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Message: #141

Date: 24/7/2013 9:10:37

From: eckart_schmidt@t-online.de

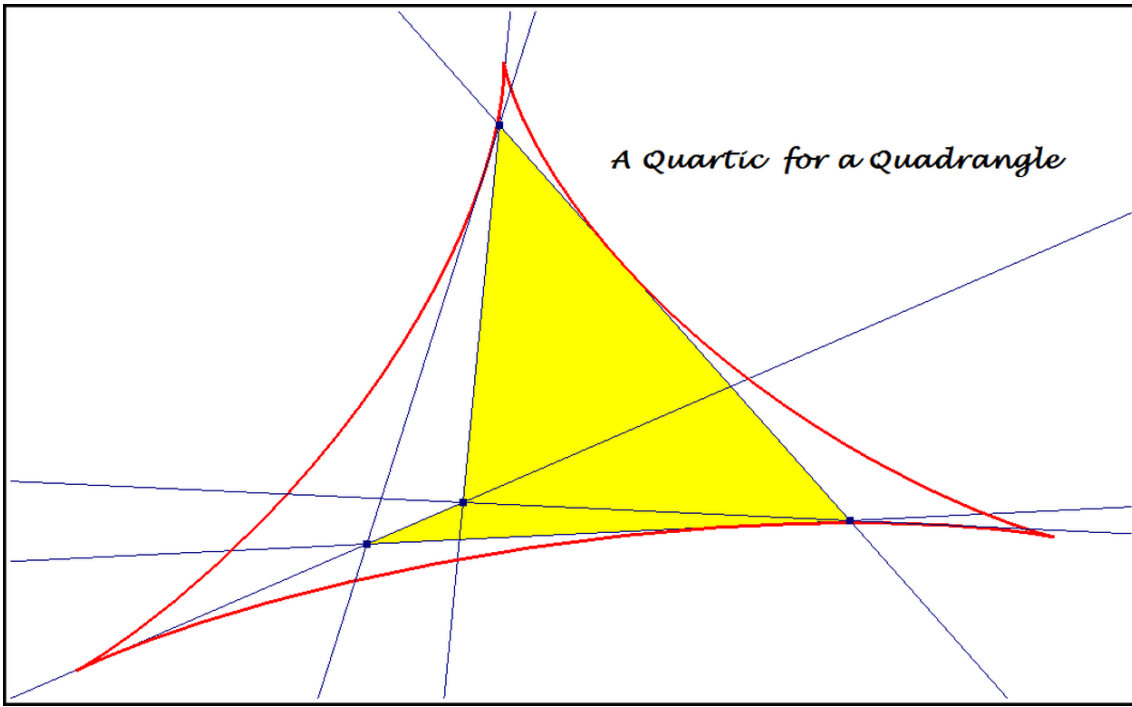
Subject: A Quartic for a Quadrangle

Dear friends,

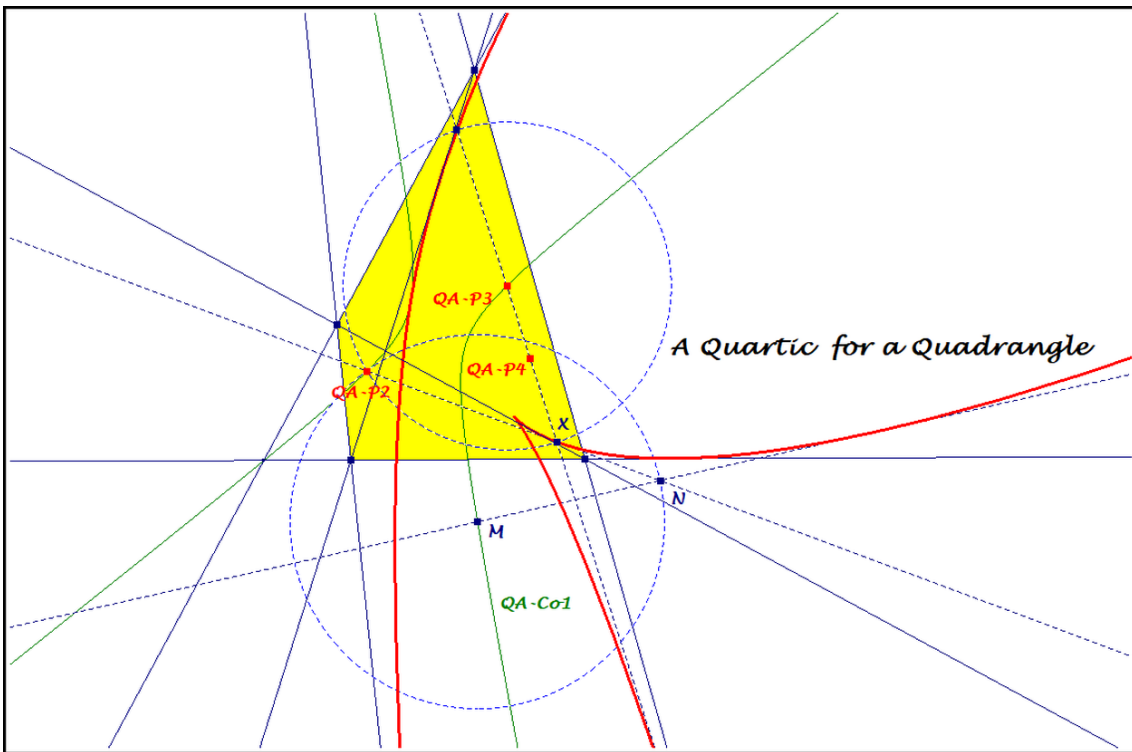
I have found a quartic tangent to the six lines of a quadrangle. Geometrical background, equation and construction are described in a Mathematica file, two examples are to be found in Cabri files in the attachment.

Not the points of the quartic, but rather the tangents are interesting.

Best regards Eckart



quartic02-fig.png



quartic01-fig.png

A Quartic for a Quadrangle

Not difficult to prove:

Let P be a point in the plane of the quadrilateral L1L2L3L4 (with the intersections Sij).

Let Lij be the reflection of PSij in the bisector of $\angle L_i L_j$.

The points L12^L34, L23^L41, L13^L24 coincide for points P on QL-Cu1 in the QL-Tf1 image of P.

More interesting, changing points and lines:

Let L be a line in the plane of the quadrangle P1P2P3P4 (with connecting lines Lij).

Let Sij be the reflections of L^Lij in the midpoint of PiPj.

The lines S12.S34, S23.S41, S13.S24 will be the same for lines L(e,f,g) (DT-coefficients) with

$$\frac{e p^2}{f-g} + \frac{f q^2}{g-e} + \frac{g r^2}{e-f} = 0$$

If a line belongs to this pencil, the three counted image line belongs also to the pencil.

Example1: QA-P3.QA-P4 and QA-P2.QA-Tf2(QA-P2).

Example 2: Parallel to QG-P1.QG-P2 through QG-P13 and

$$\{ (q^2 - 4r^2) (3p^2 - q^2 + r^2), 2(p^2 - r^2)^2, (q^2 - 4p^2) (3r^2 + p^2 - q^2) \}$$

Through every point there are up to three lines of the pencil.

For any direction (u:v:w) with $u + v + w = 0$ there is only one line of the pencil:

$$L_{uvw} := \{ u(q^2 w^2 - r^2 v^2), v(r^2 u^2 - p^2 w^2), w(p^2 v^2 - q^2 u^2) \}$$

The lines envelope a quartic with the equation:

$$\begin{aligned} \text{quad}[x_, y_, z_] &:= 4q^2(q^2 - r^2)^2 r^2 x^4 + 4p^2(p^2 - r^2)^2 r^2 y^4 + 4p^2(p^2 - q^2)^2 q^2 z^4 \\ &+ 4q^2 r^2 (3p^2 q^2 + 5p^2 r^2 + (q^2 - r^2)^2) x^3 y \\ &+ 4q^2 r^2 (3p^2 r^2 + 5p^2 q^2 + (q^2 - r^2)^2) x^3 z \\ &+ 4p^2 r^2 (3p^2 q^2 + 5q^2 r^2 + (p^2 - r^2)^2) x y^3 \\ &+ 4p^2 r^2 (3q^2 r^2 + 5p^2 q^2 + (p^2 - r^2)^2) y^3 z \\ &+ 4p^2 q^2 (3p^2 r^2 + 5q^2 r^2 + (p^2 - q^2)^2) x z^3 \\ &+ 4p^2 q^2 (3q^2 r^2 + 5p^2 r^2 + (p^2 - q^2)^2) y z^3 \\ &+ p^2(p^6 - 2p^4 q^2 + p^2 q^4 - 2p^4 r^2 + 38p^2 q^2 r^2 + 12q^4 r^2 + p^2 r^4 + 12q^2 r^4) y^2 z^2 \\ &+ q^2(p^4 q^2 - 2p^2 q^4 + q^6 + 12p^4 r^2 + 38p^2 q^2 r^2 - 2q^4 r^2 + 12p^2 r^4 + q^2 r^4) x^2 z^2 \\ &+ r^2(12p^4 q^2 + 12p^2 q^4 + p^4 r^2 + 38p^2 q^2 r^2 + q^4 r^2 - 2p^2 r^4 - 2q^2 r^4 + r^6) x^2 y^2 \\ &+ 2q^2 r^2 (11p^4 + 24p^2 q^2 + q^4 + 24p^2 r^2 - 2q^2 r^2 + r^4) x^2 y z \\ &+ 2p^2 r^2 (p^4 + 24p^2 q^2 + 11q^4 - 2p^2 r^2 + 24q^2 r^2 + r^4) x y^2 z \\ &+ 2p^2 q^2 (p^4 - 2p^2 q^2 + q^4 + 24p^2 r^2 + 24q^2 r^2 + 11r^4) x y z^2 \end{aligned}$$

The point of tangency of the line L_{uvw} will be:

$$\{ q^2 r^2 u^4 - p^4 v^2 w^2 + p^2 u(v-w)(r^2 v^2 - q^2 w^2), \\ p^2 r^2 v^4 - q^4 u^2 w^2 + q^2 v(u-w)(r^2 u^2 - p^2 w^2), p^2 q^2 w^4 - r^4 u^2 v^2 + r^2 w(u-v)(q^2 u^2 - p^2 v^2) \}$$

The other two points of intersection with the quartic have midpoint:

$$\{ p^2 v w, q^2 u w, r^2 u v \}$$

This is the QA-Tf2 image of the point at infinity of L_{uvw} .

That means that the further intersections of the tangents to the quartic have midpoint on the

QA-DT circumconic, which is the QA-Tf2 image of the line at infinity.

The quartic is tangent to the lines Lij.

The points of tangency Tij divide Pi.Pj in the following ratios:

$$\begin{aligned}\tau_{12} &:= \frac{-p+q+r}{p-q+r}; \tau_{23} := \frac{p-q+r}{p+q-r}; \tau_{34} := \frac{p+q-r}{p+q+r}; \tau_{41} := \frac{p+q+r}{-p+q+r}; \\ \tau_{13} &:= \frac{-p+q+r}{p+q-r}; \\ \tau_{24} &:= \frac{p-q+r}{p+q+r};\end{aligned}$$

The lines T12.T34, T23.T41, T13.T24 have a common point:

$$\begin{aligned}\{ & p(p^6 - 2p^5q - p^4q^2 + 4p^3q^3 - p^2q^4 - 2p^5r + q^6 - 2p^5r - 6p^4qr - \\ & 4p^3q^2r + 4p^2q^3r + 6p^4r + 2q^5r - p^4r^2 - 4p^3qr^2 + 10p^2q^2r^2 - 4p^3q^3r^2 - q^4r^2 + \\ & 4p^3r^3 + 4p^2q^3r^3 - 4p^2q^2r^3 - 4q^3r^3 - p^2r^4 + 6p^4qr^4 - q^2r^4 - 2p^5r^5 + 2q^5r^5 + r^6), \\ & q(p^6 - 2p^5q - p^4q^2 + 4p^3q^3 - p^2q^4 - 2p^5r + q^6 + 2p^5r + 6p^4qr + 4p^3q^2r - 4p^2q^3r - \\ & 6p^4r - 2q^5r - p^4r^2 - 4p^3qr^2 + 10p^2q^2r^2 - 4p^3q^3r^2 - q^4r^2 - 4p^3r^3 - \\ & 4p^2q^3r^3 + 4p^2q^2r^3 + 4q^3r^3 - p^2r^4 + 6p^4qr^4 - q^2r^4 + 2p^5r^5 - 2q^5r^5 + r^6), \\ & r(p^6 + 2p^5q - p^4q^2 - 4p^3q^3 - p^2q^4 + 2p^5r + q^6 - 2p^5r + 6p^4qr - 4p^3q^2r - 4p^2q^3r + \\ & 6p^4r - 2q^5r - p^4r^2 + 4p^3qr^2 + 10p^2q^2r^2 + 4p^3q^3r^2 - q^4r^2 + 4p^3r^3 - \\ & 4p^2q^3r^3 - 4p^2q^2r^3 + 4q^3r^3 - p^2r^4 - 6p^4qr^4 - q^2r^4 - 2p^5r^5 - 2q^5r^5 + r^6) \}\end{aligned}$$

It seems, that no QA-point lies on this quartic!

Not the points, but rather the tangents are interesting!

Finally a simple construction of the quartic:

Consider the line QA-P2.X, when X is one of the intersections of QA-P3.QA-P4 and the circle round QA-P3 through QA-P2 (X is a point of the quartic).

Let M be a variable point on QA-Co1 as center of a circle through QA-P2.

This circle has a second intersection N with QA-P2.X.

The lines MN envelope the quartic.

Eckart Schmidt
eckart_schmidt@t-online.de
<http://eckartschmidt.de>

Message: #142
Date: 24/7/2013 12:55:44
From: Chris
Subject: A Quartic for a Quadrangle

Dear Eckart,
Very nice!
What are the coordinates of:
* the Cusp
* the point of tangency Quartic and NinePointConic?
Best regards,
Chris

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Message: #143
Date: 25/7/2013 3:36:20
From: seiichikiri
Subject: 4 concurrent circles points similar to Euler-poncelet and Varignon

Dear Chris,
I see quadrilaterals. I confirmed the following.
[1] Varignon parallelogram: the lines through 3 duos determined by concurrent circles concur in QA-P1.
[2] Varignon-like parallelogram: the lines through 3 duos determined by concurrent circles concur in QA-P6.
Incidentally I hope that Chris will give a proper name to Varignon-like parallelogram on the basis of literatures, as it is inconvenient to always call it Varignon-like.
Best regards,
Seiichi.

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Message: #144

Date: 25/7/2013 7:39:03

From: Chris

Subject: 4 concurrent circles points similar to Euler-poncelet and Varign

Dear Seiichi,

I think that the predicate "Varignon-like" is very understandable.

I don't know of better terminology in literature with the same meaning.

Maybe anyone else has a suggestion?

Best regards,

Chris

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Message: #145

Date: 28/7/2013 1:08:34

From: seiichikiri

Subject: 4 concurrent circles points similar to Euler-poncelet and Varign

Dear Chris,

I think that you should give a proper name on the basis of 1st appearance in the literatures.

For example, someone will find the same Varignon-like after 10 years and it will be named something. Then there will be possibility of negotiation of names.

Other naming: Varignon parallelograms of the 1st kind and 2nd kind.

Best regards,

Seiichi.

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Message: #146
Date: 29/7/2013 7:32:55
From: seiichikiri
Subject: A property of QA-P4

Dear friends,
Given a quadrangle $P_1P_2P_3P_4$ and the antigonal image A_i of P_i wrt P_jPkPl , the isogonal conjugates of A_i wrt P_jPkPl coincide with each other, QA-P4.
Best regards,
Seiichi.

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Message: #147
Date: 29/7/2013 9:06:53
From: Chris
Subject: A property of QA-P4

Dear Seiichi,
This is not quite surprising, because:
The Antigonal of a point P is the isogonal conjugate of the inverse in the circumcircle of the isogonal conjugate of QA-vertex P_i ($i=1,2,3,4$).
So Antigonal = $g.i.g.P_i$
When you take the isogonal conjugate of the antigonal this is the result: $X = g.g.i.g.P_i$
Since 2 times Isogonal Conjugate is Identity $X = i.g.P$.
In other words $X = QA-P4 =$ the Inverse of the Isogonal Conjugate of P_i .
This property is already known.
Best regards,
Chris
notations: g = isogonal, i = inverse

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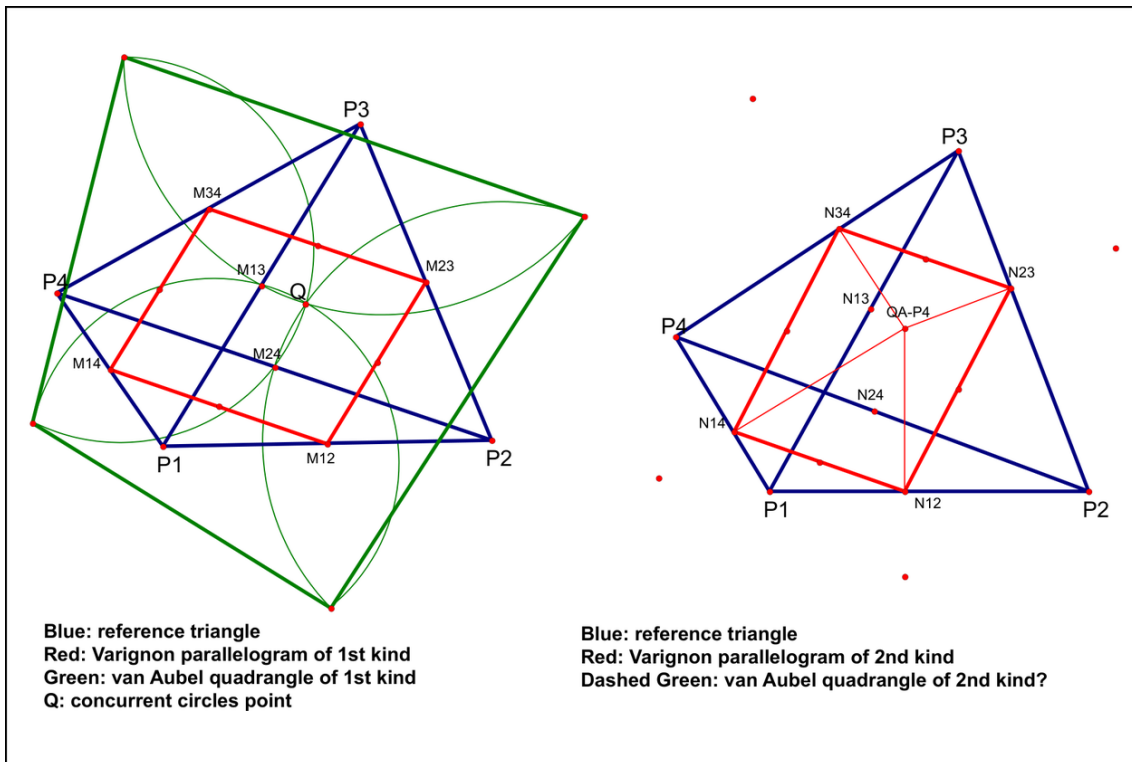
Message: #148
Date: 29/7/2013 10:18:25
From: seiichikiri
Subject: A property of QA-P4

Dear Chris,
I see. When I saw the description of QA-P4 precisely, what I found is not beyond it.
My experience: antigonal image produces something, but syngonal image does not produce anything.
Best regards,
Seiichi.

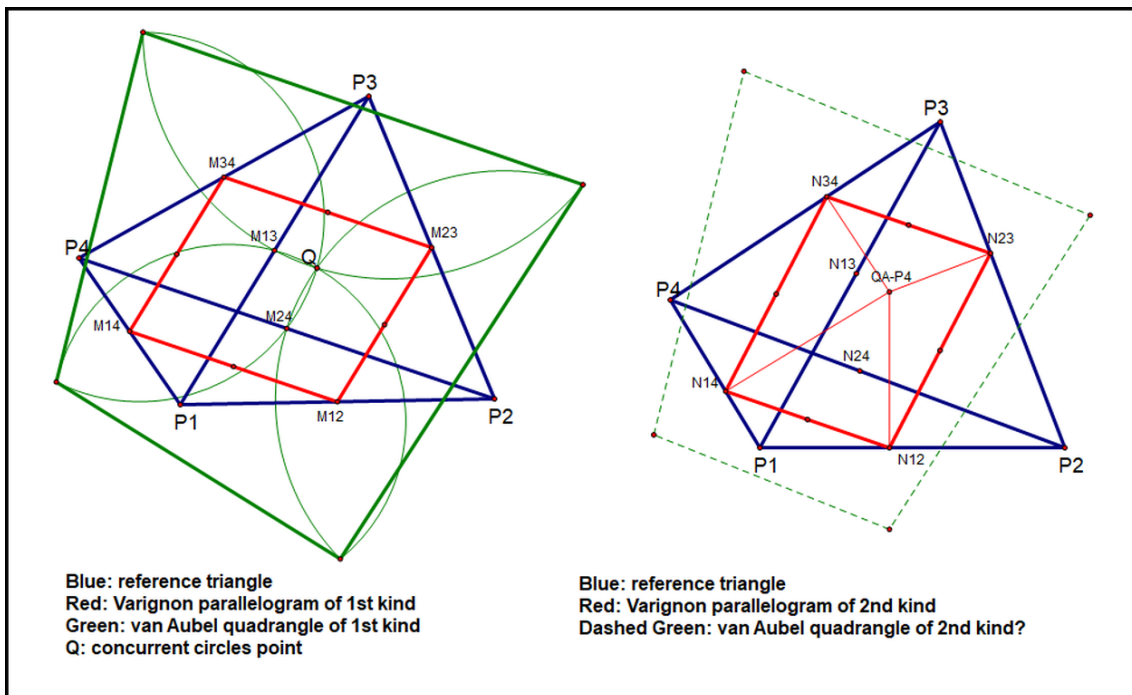
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Message: #149
Date: 30/7/2013 12:20:02
From: Seiichi Kirikami
Subject: Are there van Aubel quadrangle and points of 2nd kind?

Dear friends,
I would like to propose a problem about van Aubel quadrangle and points of 2nd kind.
Sorry in advance if there were no such things as van Aubel quadrangle and points of 2nd kind.
[1] van Aubel quadrangle of 1st kind: See Matheworld "van Aubel's theorem". For example, the centers of the squares placed outwardly on each side of a quadrangle determine its van Aubel quadrangle of 1st kind.
[2] van Aubel points of 1st kind: These points accompany van Aubel quadrangle of 1st kind. See the message #116.
[3] Varignon parallelogram of 1st kind: See Matheworld "Varignon parallelogram".
[4] Varignon parallelogram of 2nd kind: See the description of QA-P4 in EQF.
They([3] and [4]) coincide with each other in a cyclic quadrangle when vertices are the same.
[5] I think that van Aubel quadrangle and points of 2nd kind accompany Varignon parallelogram of 2nd kind.
See the attached files(gsp or MSword).
Best regards,
Seiichi.



vanAubel2nd.docx



van Aubel 2nd-gsp.png

Message: #150

Date: 30/7/2013 12:43:08

From: seiichikiri

Subject: What is the necessary and sufficient condition of quadrifigure f

Dear Chris, dear friends,

I saw the figure of Yao Liu's almost equilateral triangle.

My observation is as follows.

- (1) the triangle centers which are not constrained on a circle or other figure limit to the center of the equilateral triangle, $(1,1,1)$.
- (2) the triangle centers which are constrained on a circle or other figure are on the same circle or other figure.

Examples of the circumcircle, $X(107)$, $X(162)$; NP circle, $X(123)$, $X(134)$; the circumcircle of the anticomplementary triangle, $X(147)$, $X(148)$.

In these limiting cases, the coordinates are indefinite, $(0,0,0)$. If they are finite other than $(1,1,1)$, it is not a triangle center.

My generalization is as follows.

- (1) the quadrangle point which is not constrained limits to the center of the square. $f(-1,1,1,\sqrt{2},1,1)=0$.
 - (2) If the quadrangle point is constrained on a figure, its coordinates are indefinite, $(0,0,0)$. If they have finite values other than $(0,1/2,1/2)$, it is not a quadrangle point.
- In any case, $f(-1,1,1,\sqrt{2},1,1) = 0$.

Best regards,
Seiichi.

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Message: #151

Date: 31/7/2013 10:54:43

From: eckart_schmidt@t-online.de

Subject: Modified Involutary Conjugate QA-Tf2 for Quadrigons

Dear friends,

Involutary Conjugate (*) is an isoconjugation for the QA-Diagonal Triangle with fixed points in the vertices of the quadrangle. Here for quadrigons a modified transformation (^) will be considered, which swaps opposite vertices. There are some interesting aspects in the QL-environment. - Reference triangles for barycentric coordinates are QA-DT or QL-DT.

The Modified Involutary Conjugate (shorted MIC) $X^\wedge$ is the fourth harmonic point of the Involutary Conjugate X^* wrt QG-P1 and the intersection of $X^*.QG-P1$ with QG-L1:

$$(x:y:z) \rightarrow (p^2 y : z : -q^2 z x : r^2 x y).$$

Some MIC-images in QG-environment:

QG-P12 $^\wedge$: QA-P10

QG-P13 $^\wedge$: midpoint of QG-P1.QG-P2

QG-P14 $^\wedge$

QG-P15 $^\wedge$: intersection of QG-P2.QA-P16

and a QG-L1-parallel half the distance to QG-P1

QG-P16 $^\wedge$: fourth harmonic point of QL-P1 on the QG-P1-traversal

In QA-environment a point P has three MIC-images, which are the vertices of the QA-DT-anticevian triangle of the Involutary Conjugate P^* .

In QL-environment the three MIC -images of a point $P(u:v:w)$ are collinear on a line $L(P)$ with simple coefficients

$$L(P) = (l^2 u, m^2 v, n^2 w)$$

The line $L(P)$ is the image of the trilinear polar of P wrt the "isoconjugation for lines" (see my homepage, 12.3).

For an intersection P of two lines of the quadrilateral $L(P)$ is the connection of the opposite point with the noncollinear vertex of QL-DT.

$L(QL-P8)$ is QL-L1,

for all points on QA-P8.QA-P13 $L(P)$ is a parallel to QL-L1,
and for all points on QL-L1 the line $L(P)$ contains QL-P8.

The last property can be generalized: For points P on a line $L(e,f,g)$

the lines $L(P)$ have a common point

$$P(L) = (m^2 n^2 e : n^2 l^2 f : l^2 m^2 g),$$

which can be interpreted as follows:

Taking the QL-DT -isoconjugation with fixed points in the trilinear poles of the sidelines of the quadrilateral, then $P(L)$ is the image of the trilinear pole of L .

For these QL-transformations $L(P)$ (points into lines) and $P(L)$ (lines into points) holds $P(L(P))=P$ and $L(P(L))=L$.

Best regards Eckart

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Message: #152

Date: 31/7/2013 11:23:37

From: eckart_schmidt@t-online.de

Subject: Modified Involutary Conjugate QA-Tf2 for Quadrigons

Dear friends,
excuse the outward appearance of my last message and the missing MIC-image of QG-P14:

QG-P14 $\wedge$: midpoint of QG-P1.QG-P3

Best regards Eckart

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Message: #153

Date: 31/7/2013 11:48:59

From: seiichikiri

Subject: QA-P4 as an image of a triangle and a point

Dear friends,
The following image D coincides with QA-P4 in EQF. See Anopolis message #724.

[APH&AM]

Let ABC be a triangle, P a point and H_a, H_b, H_c the orthocenters of PBC, PCA, PAB respectively. Da, Db, Dc = the isogonal conjugates of H_a, H_b, H_c (wrt the triangle ABC).

The circumcircles of $DaBC, DbCA, DCAB$ are concurrent.

If $P=(u, v, w)$, the circumcircles of $DaBC, DbCA, DcAB$ are concurrent in $D=(a^2v*w(a^2(u+v)(u+w)-u(b^2(u+v)+c^2(u+w))));$.

Best regards,

Seiichi.

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Message: #154
Date: 31/7/2013 4:16:28
From: Chris
Subject: QA-P4 as an image of a triangle and a point

Dear Seiichi,
Nice method for constructing QA-P4.
I suppose you also noticed that D , D_a , D_b , D_c and $X(3)$ are collinear.
Best regards,
Chris

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Message: #155
Date: 01/8/2013 11:39:09
From: eckart_schmidt@t-online.de
Subject: Circular QG-cubic

Dear friends,
for triangles the locus of points, whose pedal triangle is a cevian triangle is the Darboux cubic. In the attachment there is an analogon for quadrilaterals. I think, the geometrical background is interesting, but there are rarely relationships to QG-geometry. Perhaps someone can find further properties.
Best regards Eckart

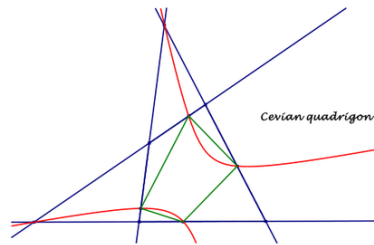
A Pivotal Isogonal Circular QG-Cubic

For a triangle the locus of points, whose pedal triangles are cevian triangles is the Darboux cubic. Defining cevian quadrilaterons in a corresponding way there is a pivotal isogonal circular cubic for quadrilaterons. – The following information is a summary of my paper 12.2 on my homepage.

Definition: A cevian quadrilateral is an inscribed quadrilateral, dividing the sides in ratios with product 1.

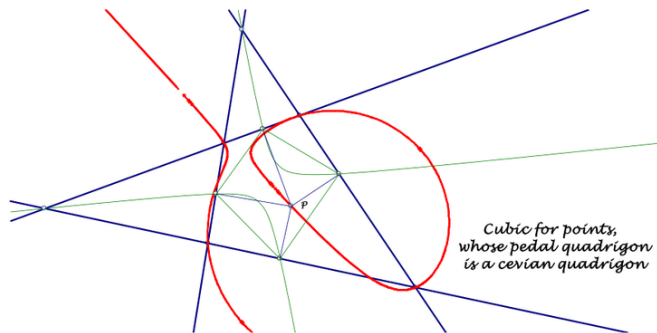
For example inscribed conics of a quadrilateral give cevian quadrilaterons.

Property: The vertices of a cevian quadrilateral and the intersections of opposite sides of the reference quadrilateral lie on a conic.



With this property we get easily the equation for the points, whose pedal quadrilateral is a cevian quadrilateral. But the equation is a bit extensive.

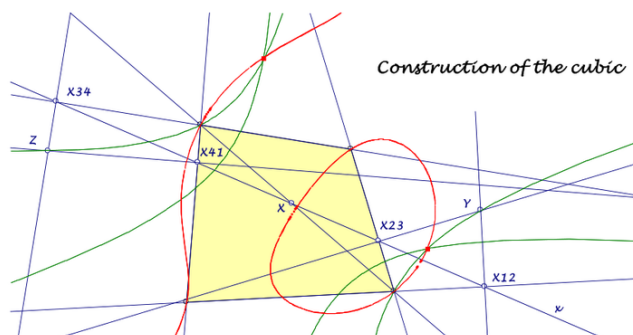
Theorem: The locus of points, whose pedal quadrilateral is a cevian quadrilateral, is a circular cubic, containing the Miquel point with an asymptote parallel to the bisector of the intersections of opposite sides.



A reference triangle can be constructed, so that the cubic is a pivotal isogonal circular cubic.

Finally a possible construction for the cubic:

Let $P_1P_2P_3P_4$ be the quadrignon. Consider points X on a diagonal and lines x through X , which cut the sidelines in X_{12} , X_{23} , X_{34} , X_{41} . Let Y be the intersection of perpendiculars in X_{12} and X_{23} to the corresponding sidelines, let Z be the intersection of perpendiculars in X_{34} and X_{41} to the corresponding sidelines. The loci of Y and Z changing the lines x through X are conics, the intersections of these conics are points of the cubic.



Eckart Schmidt
eckart_schmidt@t-online.de
<http://eckartschmidt.de>

QG-cubic.pdf

Message: #156

Date: 01/8/2013 1:41:21

From: seiichikiri

Subject: An inequality of the bimedial segments of a quadrangle

Dear friends,

Given a quadrangle $P_1P_2P_3P_4$ and the midpoints M_{ij} on P_iP_j , the sum of the lengths of the bimedial segments equals or less than the square root of the square sum of the sidelengths of the quadrangle multiplied by $\text{Sqrt}(3)/2$.

$M_{12}M_{34}+M_{13}M_{24}+M_{14}M_{23}$

$\leq \text{Sqrt}(3)/2 * \text{Sqrt}(P_1P_2^2+P_2P_3^2+P_3P_4^2+P_4P_1^2+P_1P_3^2+P_2P_4^2)$.

A simple application of Cauchy Schwarts inequality gives the result.

Best regards,

Seiichi.

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Message: #157

Date: 03/8/2013 7:53:41

From: seiichikiri

Subject: An inequality of the bimedial segments of a quadrangle

Dear friends,

Numerical experiment of a linear two sided inequality of a quadrangle:

$(P_1P_2+P_2P_3+P_3P_4+P_4P_1+P_1P_3+P_2P_4)/4$

$\leq M_{12}M_{34}+M_{13}M_{24}+M_{14}M_{23} \leq (P_1P_2+P_2P_3+P_3P_4+P_4P_1+P_1P_3+P_2P_4)/2$.

Best regards,

Seiichi.

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Message: #158
Date: 03/8/2013 8:55:20
From: Chris
Subject: New items in EQF

Dear friends,
I placed several new items in EQF.
See Recently added:
<http://www.chrisvantienhoven.nl/index.php/recently-added.html>
Especially:
* A QA-Moebius Conjugate QA-Tf4 at
<http://www.chrisvantienhoven.nl/other-quadrangle-objects/index.php/15-mathematics/encyclopedia-of-quadri-figures/quadrangle-objects/artikelen-qa/230-qa-tf4.html>
* A QL-Line Isoconjugate QL-Tf2 at
<http://chrisvantienhoven.nl/other-quadrilateral-objects/index.php/17-mathematics/encyclopedia-of-quadri-figures/quadrilateral-objects/artikelen-ql/231-ql-tf2.html>
* New properties QL-Cardioid QL-Qu1 at
<http://www.chrisvantienhoven.nl/other-quadrilateral-objects/17-mathematics/quadrilateral-objects/artikelen-ql/157-ql-qu1.html>
Best regards,
Chris

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Message: #159
Date: 03/8/2013 11:53:28
From: Bernard Gibert
Subject: Cardioids and Eckart Cubics

Dear Chris, Eckart and friends,
I've written a draft and put it on my web site. Although it's far from being completed, feel free to comment and correct possible mistakes.
<http://bernard.gibert.pagesperso-orange.fr/files/Resources/eckart.pdf>
Best regards
Bernard

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Message: #160
Date: 03/8/2013 3:32:56
From: eckart_schmidt@t-online.de
Subject: Circle for a quadrigon

Dear friends,
here the short geometry of a circle for a quadrigon: QG-Cix
round QG-P9 through QG-P1
* Points on the circle: QG-P1, QG-P5, QA-P3, QL-P17 and the
vertices of QA-Tr2 unequal QL-P1
* QG-Cix is the QL-Tf1 image of a circle through QA-P4, QG-P16
and the vertices of QA-Tr2 unequal QL-P1
* QG-Cix is the QA-Tf4 image of QL-Ci3 (Miquel Circle)
Best regards Eckart

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Message: #161
Date: 06/8/2013 11:42:16
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends,

thanks to Bernard for the reference to your paper. You describe our previous efforts on a higher level. Up to now I miss an explanation for the construction of the reference triangles for the cubic (message #109). I try it once more, to describe my "construction". The description in message #125 contains unfortunately a mix-up of the axes of Co1!

Here is a better version of my own way, how to find the reference triangles for my cubic, if the quadrilateral is not tangent to a circle:

- # X is the intersection of QL-L1 (Newton line) and QL-L6.
- # Co1 is the conic with center X and tangent to the lines of the quadrilateral.
- # Ci1 is the Thales circle for the foci of Co1.
- # O is a variable point on the axis of Co1.
- # Ci2 is the circle round O perpendicular Ci1.
- # Point K is the reflection of X in Ci2.
- # Let $G = QL-P1$ (Miquel point) and $H = hG, -2(O)$ and $Y = hG, -2(K)$.
- # Consider the rectangular hyperbola through O, H, K, Y with center Z.
- # Chose O on the axis of Co1 so that the reflection of H in Z lies on the circle Ci2.
- # Then the further intersections of the hyperbola and Ci2 give the reference triangle.
- # If the axis of Co1 is QL-L1, the cubic is the McCay cubic. If the axis of Co1 is perpendicular QL-L1, the cubic is the Kjp cubic.

Remark: G, O, H, K are finally X2, X3, X4, X6 of the reference triangle.

Best regards Eckart

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Message: #162
Date: 06/8/2013 11:47:23
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Eckart,

>> thanks to Bernard for the reference to your paper.
>> You describe our previous efforts on a higher level.
>> Up to now I miss an explanation for the construction of
>> the reference triangles for the cubic (message #109).

It's on its way! Please leave me some more time since it's difficult, at least for me.

Best regards
Bernard

PS: I will probably post an updated version of my draft paper this evening.

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Message: #163
Date: 07/8/2013 2:18:38
From: seiichikiri
Subject: What is the necessary and sufficient condition of quadrifigure f

Dear friends,

The following Q is an example which satisfies $f(-1,1,1;\sqrt{2},1,1)=0$, but is not a quadrangle point.

$Q = (u((v+w)(w+u-v)(u+v-w)+2uvw) : :)$
from Advanced Plane Geometry message #443 (AMM11615) †).
Best regards,
Seiichi.

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†) Editorial note: The referenced message(s) can be found as listed in [EPG-References \[50\]](#).

Message: #164

Date: 08/8/2013 6:27:33

From: bernard.keizer

Subject: Diagonal quadrilaterals and conics through the S-points of the Dimid

Dear Chris, dear Eckart and dear friends of the Quadri-group
When I came to the quadri-group, I was not only interested in cardioïds, but in plenty of other things; among them are the S-points of the Dimidium circle and I was and am still intrigued by these 3 points.

I've tried to make a link with what I call the first and second diagonal triangles and the first and second diagonal quadrilaterals of the QL. I hope it could interest you and, may be, lead to new definitions or new points.

Best regards

Bernard Keizer

I put the attached file in the files, I hope it will work ...

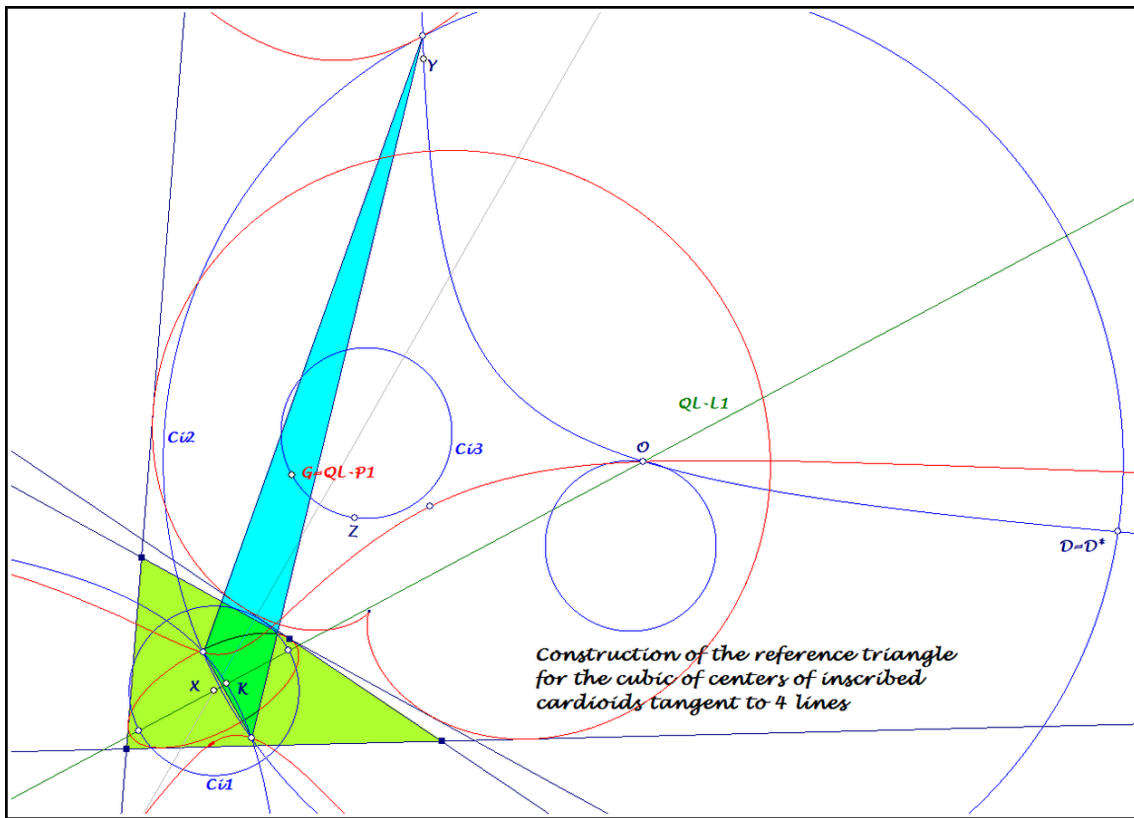
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Message: #165
Date: 09/8/2013 9:30:44
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard, dear Chris, dear friends,
I have updated my "construction" for the reference triangles of my cubic, if the quadrilateral is not tangent to 4 lines: It needs only an intersection point of a cubic and a quartic -which can be marked by Cabri- to construct the reference triangles.

- # X is the intersection of QL-L1 (Newton line) and QL-L6.
- # Co1 is the conic with center X and tangent to the lines of the quadrilateral.
- # Ci1 is the Thales circle for the foci of Co1.
- # O is a variable point on the axis of Co1.
- # Ci2 is the circle round O perpendicular Ci1.
- # Point K is the inverse of X wrt Ci2.
- # Let G = QL-P1 (Miquel point) and H = hG,-2(O) and Y = hG,-2(K).
- # The intersections of GO and Ci2 give the Hessian QL-Cu1 of the cubic.
- # Consider the rectangular hyperbola through O, H, K, Y with center Z.
- # Let D be the reflection of H in Z.
- # Changing O on the axis of Co1,
 - ... the locus of Z is a circle Ci3 (through QL-P1),
 - ... the locus of D is a cubic (through the foci of Co1),
 - ... the locus of the intersections of the hyperbola and Ci2 is a quartic (through the foci of Co1).
- # Let D* be the further intersection of the cubic and the quartic.
- # Reflecting D* in points Z of Ci3 you get circle Ci4.
- # The homothetic circle hG-1/2(Ci4) cuts the Newton line in two points.
- # Let O be finally that intersection point with D=D*.
- # Then the intersections of the hyperbola and Ci2 -unequal D- give the reference triangle ABC.
- # G, O, H, K and Y are finally X2, X3, X4, X6 and X69 of the reference triangle.
- # The rectangular hyperbola is the polar conic of H.
- # If the axis of Co1 is QL-L1, the cubic is the McCay cubic. If the axis of Co1 is perpendicular QL-L1, the cubic is the Kjp cubic.

I hope there are no basic mistakes. The Cabri file in the attachment can be an orientation. Perhaps Bernard can lighten the background for the circles and curves.
Best regards Eckart



cardMcCay-fig.png

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Message: #166
Date: 10/8/2013 12:10:56
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

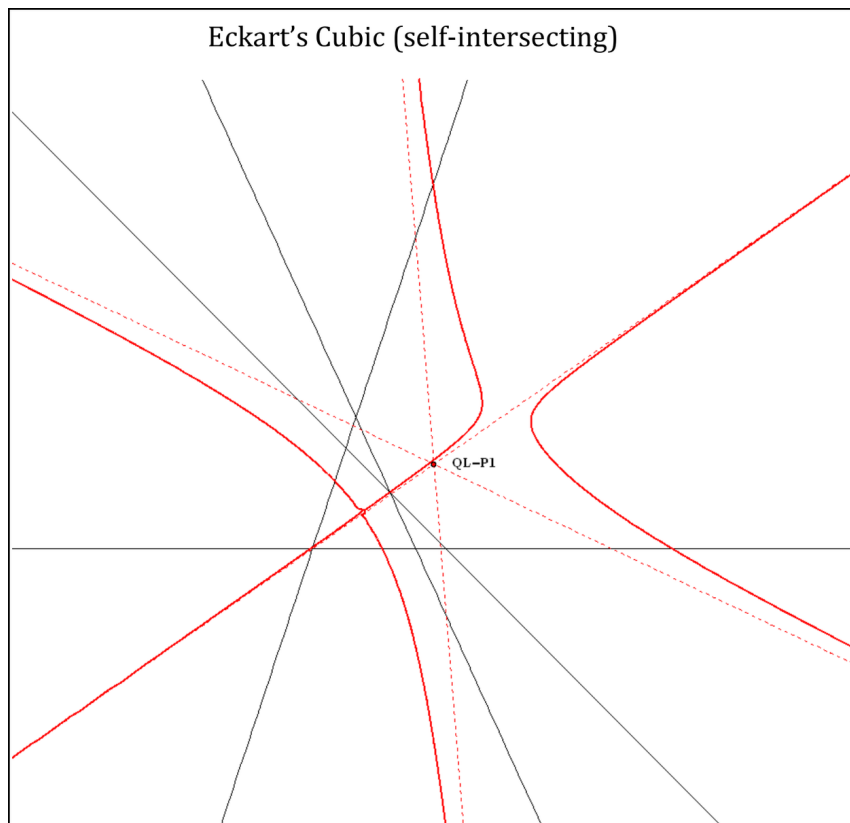
Dear friends,
sorry, it is better, to cut out the following remarks in my last message:

The intersections of G_0 and Ci_2
give the Hessian $QL-Cu_1$ of the cubic.
The rectangular hyperbola is the polar conic of H .
They are only valid, if the axis of Co_1 is $QL-L_1$.
(If the axis of Co_1 is perpendicular to $QL-L_1$, circles with
midpoint M on $QL-L_1$ through the foci of Co_1 cut MG in points of
the Hessian of the cubic.)
Best regards Eckart

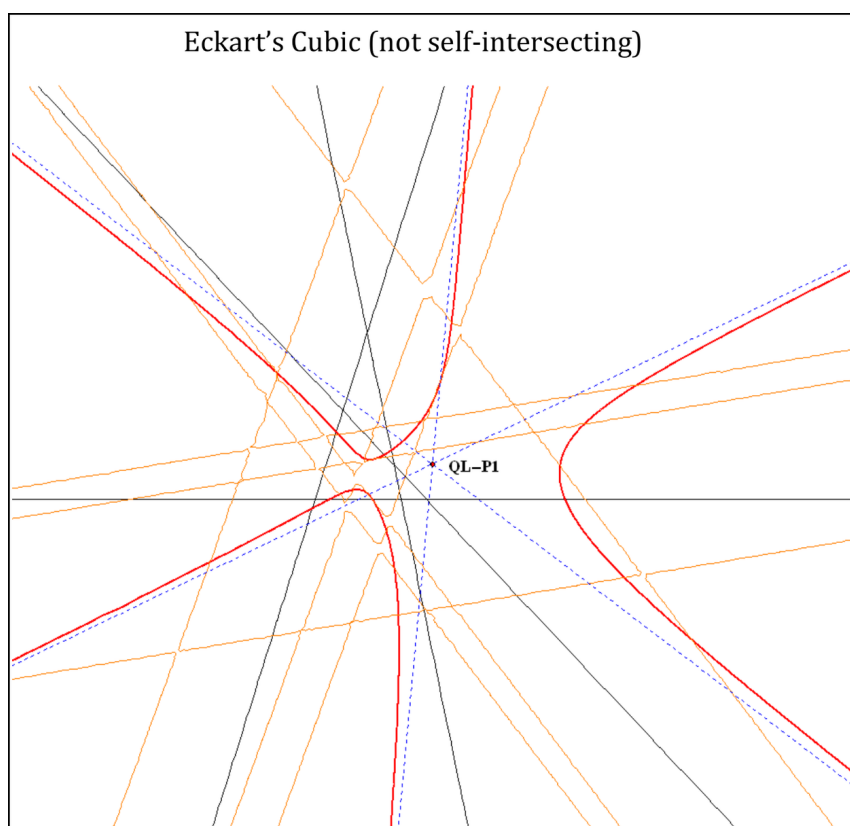
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Message: #167
Date: 10/8/2013 7:31:10
From: Chris van Tienhoven
Subject: Cardioids tangent to 4 lines

Dear Eckart, dear friends,
Sometimes Eckart's Cubic is self-intersecting.
Often this indicates a special point, even when it is not
self-intersecting.
Is already known which point the self-intersecting point of
Eckart's cubic is?
What are the coordinates of this point?
Best regards,
Chris
See attachment.



EckartsCubic2.pdf



EckartsCubic2.pdf

Message: #168
Date: 10/8/2013 8:07:49
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Chris,

>> Sometimes Eckart's Cubic is self-intersecting.
>> Often this indicates a special point, even when it is not
>> self-intersecting.
>> Is already known which point the self-intersecting point
>> of Eckart's cubic is?
>> What are the coordinates of this point?

This happens when the quadrilateral circumscribes a circle.
This is detailed in the draft paper available on my web site.

Best regards
Bernard

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Message: #169
Date: 11/8/2013 10:57:59
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard,
trying to construct the nodal $E(L)$, I succeeded in the
alternative construction (p.12), but I failed in the first
construction (p.11).
In my construction wrt 6. the points P_1 , P_2 , P_3 are not the
vertices of an equilateral triangle and their focus line is not
 $E(L)$.
Best regards Eckart

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Message: #170
Date: 11/8/2013 11:50:14
From: Bernard Gibert
Subject: Cardioids tangent to 4 lines

Dear Eckart,

>> trying to construct the nodal $E(L)$,
>> I succeeded in the alternative, construction (p.12),
>> but I failed in the first construction (p.11).

it's only a typo in item 5: the center of (Ho) is the midpoint of X_1Go and not X_1Ho .

sorry!

thank you for checking my paper

please feel free to comment and criticize

Best regards

Bernard

PS: item 1 is also incoherent: forget the circumcircle of ABC...

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Message: #171
Date: 14/8/2013 8:12:31
From: Bernard Gibert
Subject: interesting paper?

Dear friends,

here's a paper by Jules Marchand I cannot find on the internet.
It could be of interest...

Marchand, J. Géométrie du quadrilatère complet. (French) Bull. Soc. Vaudoise Sci. nat. 59, 251-270. Published: 1937

Best regards

Bernard

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Message: #172

Date: 14/8/2013 9:59:30

From: Antreas Hatzipolakis

Subject: interesting paper?

> On Wed, Aug 14, 2013 at 9:12 AM, Bernard Gibert <
bg42@orange.fr >
> wrote:
>> Dear friends,
>> here's a paper by Jules Marchand
>> I cannot find on the internet. It could be of interest...
>> Marchand, J. Géométrie du quadrilatère complet. (French)
>> Bull. Soc. Vaudoise Sci. nat. 59, 251-270. Published: 1937
>> Best regards
>> Bernard
>> (
>> http://groups.yahoo.com/group/Quadri-Figures-Group/post;_ylc=X3oDMTJwdDc1M3FxBF9TAzk3MzU5NzE0BGdycElkAzg3Njg2NjExBGdycHNwSWQD_MTcwNTA4MzM4NgRtc2dJZAMxNzEEc2VjA2Z0cgRzbGsDcnBseQRzdGltZQMxMzc2_NDYwNzUx?act=reply&messageNum=171
>>)
Review in German:
<http://www.zentralblatt-math.org/zmath/en/search/?q=an:02523907&type=pdf&format=complete>

aph

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Message: #173
Date: 14/8/2013 11:24:07
From: Antreas Hatzipolakis
Subject: QUADRILATERAL Bibliography

----- Forwarded message -----

From: Antreas <anopolis72@gmail.com >
Date: Thu, Aug 15, 2013 at 12:06 AM
Subject: [EGML] QUADRILATERAL Bibliography
To: Anopolis@yahoogroups.com

Thebault, V.

Sur le quadrilatere complet. (French)

C. R. Acad. Sci., Paris 217, 97-99 (1943).

<http://gallica.bnf.fr/ark:/12148/bpt6k31698/f97.image>

<http://gallica.bnf.fr/ark:/12148/bpt6k31698/f98.image>

<http://gallica.bnf.fr/ark:/12148/bpt6k31698/f99.image>

Thebault, V.

Sur la géométrie du quadrilatere complet. (French)

C. R. Acad. Sci., Paris 218, 97-99 (1944).

<http://gallica.bnf.fr/ark:/12148/bpt6k3170r/f97.image>

<http://gallica.bnf.fr/ark:/12148/bpt6k3170r/f98.image>

<http://gallica.bnf.fr/ark:/12148/bpt6k3170r/f99.image>

aph

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Message: #174
Date: 18/8/2013 11:10:35
From: Antreas Hatzipolakis
Subject: QUADRILATERAL NEWTON LINE (A medial line)

----- Forwarded message -----

From: *Antreas < anopolis72@gmail.com >
Date: Mon, Aug 19, 2013 at 12:07 AM
Subject: [EGML] QUADRILATERAL NEWTON LINE (Re: A medial line)
To: Anopolis@yahoogroups.com

[APH]:

> > Let ABC be a triangle and A'B'C' the medial triangle.
> > A line L intersects BC,CA,AB at P1,P2,P3, resp.
> > Let M1, M2, M3 be the midpoints of P2P3, P3P1, P1P2, resp.
> > and A*, B*, C* the midpoints of A'M1, B'M2, C'M3, resp.
> > The points A*,B*,C* are collinear.
> > The line A*B*C* is the Newton line
> > of the complete quadrilateral
> > (AB, BC, CA, L) ie the line passing through the midpoints
> > of AP1, BP2, CP3.

Let (a,b,c,d) be a complete quadrilateral.

The A*,B*,C* points, as described above, coincide
for the four triangles having as L-line the fourth line
ie the triangles

(b,c,d, L=a), (c,d,a, L=b), (d,a,b, L=c), (a,b,c, L=d).

So, for a complete quadrilateral we have three new (?)
points on the Newton line.

At first glance, I did not find them in Chris' page

<http://www.chrisvantienhoven.nl/other-quadrilateral-objects/17-mathematics/quadrilateral-objects/artikelen-ql/137-ql-11.html>

Sorry if they are listed.

APH

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Message: #175
Date: 19/8/2013 9:20:48
From: Chris
Subject: QUADRILATERAL Bibliography

Dear friends,

In the paper of V. Thebault "Sur le quadrilatere complet" page 98 "Dans les quatre triangles formés par quatre droites d'un plan, prises trois par trois, les perpendiculaires élevées au milieu de la distance du centre du cercle circonscrit a l'orthocentre par un même point"

The described point is QL-P3, the Kantor-Hervey Point.

The circle described right after is QL-Ci5, The Plücker circle.

At page 97 there was a special description "Les extrémités . . ."

Transcribed in English:

The ends of the diameters (perpendicular to the corresponding line Li) of the circumcircles of the component triangles $Lj.Lk.Ll$ lie on two perpendicular axes through the Miquel Point F .

Each of these axes is the angle bisector of $A.F.A'$, etc.

where A' , B' , C' are the intersection points of 1st Component Triangle ABC .

This construction as well as the axes were new to me.

These axes are other lines than the perpendicular axes from Steiner (described in QL-8P1 and QL-Tf1).

Best regards,
Chris

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Message: #176

Date: 19/8/2013 9:59:52

From: Chris

Subject: QUADRILATERAL NEWTON LINE (A medial line)

Dear Antreas,

The Newton Line is defined as the line through the 3 midpoints of the diagonals of a Quadrilateral.

Let's name these 3 midpoints Mid1, Mid2, Mid3.

The points you mentioned are related points. They are:

- * Midpoint(Mid1, Mid2)

- * Midpoint(Mid2, Mid3)

- * Midpoint(Mid3, Mid1)

I wasn't familiar with these points.

Very nice!

The Tripolar Centroid of Mid1, Mid2, Mid3 is QL-P12.

The Tripolar Centroid of your points is also QL-P12.

Best regards,

Chris

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Message: #177

Date: 20/8/2013 8:42:18

From: eckart_schmidt@t-online.de

Subject: Cardioids tangent to 4 lines

Dear Bernard, dear friends,

there are some EQF-elements in the constructions in Bernard's paper, perhaps of interest:

- * The point "Omega" on page 13 is $QL-L1 \wedge QL-L6$.

- * The circle "Gamma" on page 16 is the Schmidt Circle (see EQF QL-Tf1).

- * The conjugation "Phi"(M) on page 15 is the Clawson-Schmidt Conjugate QL-Tf1 in EQF.

- * The points F1, F2 on page 16 are the fixed points of QL-Tf1.

- * The lines L1, L2 are the angle bisectors of the lines from the Miquel point to two opposite points of the quadrilateral.

Best regards Eckart

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Message: #178
Date: 20/8/2013 10:05:28
From: Chris
Subject: QUADRILATERAL Bibliography

Dear friends,

I made a mistake with the construction of V. Thebault with 2 axes through the Miquel Point.
They still are identical with The Steiner Axes as described in QL-8P1 (Steiner Octet) and QL-Tf1 (Clawson-Schmidt Conjugate).

Best regards,
Chris

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Message: #179
Date: 21/8/2013 9:35:35
From: Chris
Subject: Two points on the Newton Line

Dear friends,
There are exactly 2 points on the Newton Line that are each other's Isogonal Conjugate wrt all 4 QL-Component Triangles.
These points lie on QL-Cu1.
Their midpoint is the intersection point of the Newton Line and the Quasi Ortholine: $QL-L1 \wedge QL-L6$.
Best regards,
Chris

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Message: #180
Date: 21/8/2013 11:09:12
From: Antreas Hatzipolakis
Subject: Two points on the Newton Line

Dear Chris

A quiz:
Which points are the antigonal conjugates of the center of the Miquel circle
wrt the four component triangles of a complete quadrilateral?

APH

> On Wed, Aug 21, 2013 at 10:35 PM, Chris <van10hoven@gmail.com>
> wrote:
>> Dear friends,
>> There are exactly 2 points on the Newton Line that are each
>> other's Isogonal Conjugate wrt all 4 QL-Component Triangles.
>> These points lie on QL-Cu1.
>> Their midpoint is the intersection point of the Newton Line
>> and the Quasi Ortholine: $QL-L1^{QL-L6}$.
>> Best regards,
>> Chris

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Message: #181
Date: 21/8/2013 11:11:37
From: Antreas Hatzipolakis
Subject: Two points on the Newton Line

> On Thu, Aug 22, 2013 at 12:09 AM,
> Antreas Hatzipolakis <anopolis72@gmail.com>
> wrote:
>> Dear Chris
>> A quiz:
>> Which points are the antigonal conjugates
>> of the center of the Miquel circle

Sorry: Miquel point

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Message: #182
Date: 22/8/2013 8:14:03
From: Chris
Subject: Two points on the Newton Line

Dear Antreas,

The Antigonal Conjugate of some point on the circumcircle of a triangle coincides with its Orthocenter.
The Miquel Point of a Quadrilateral is the common point of the circumcircles of its 4 QL-Component Triangles.
That's why the Antigonal Conjugate of the Miquel Point wrt the 4 QL-Component Triangles coincide with the Orthocenters of the 4 QL-Component Triangles.
As already known these 4 Orthocenters are collinear on the Steiner Line QL-L2.

Best regards,
Chris

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Message: #183
Date: 22/8/2013 11:25:42
From: Antreas
Subject: Two points on the Newton Line

Dear Chris,

So we have a circle (of infinity radius, Steiner line) where are lying the four antigonal conj. points of a point (Miquel point) wrt the component triangles.

Question:

Which other points have the same property?
Either one point or four points respective to component triangles.

To make it clear:

1. Let P be a point on the plane of a c. 4lateral.
Let P_a, P_b, P_c, P_d be the antigonal conjugates of P wrt the four component triangles.
For which points P the points P_a, P_b, P_c, P_d are concyclic?
2. Let P be a point with h. coordinates $(x:y:z)$ wrt some reference triangle..
Let P_a, P_b, P_c, P_d be the points with h. coordinates $(x:y:z)$ wrt the four component triangles of a c. 4lateral.

For which P 's the P_a, P_b, P_c, P_d are concyclic?
I guess the general problem is quite complicated.
Probably by some program one could investigate the points P (centers $X(i)$) listed in ETC.

APH

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Message: #184
Date: 22/8/2013 11:32:25
From: Antreas
Subject: Two points on the Newton Line

--- In Quadri-Figures-Group@yahoogroups.com , "Antreas" wrote:

> > Dear Chris,
> > So we have a circle (of infinity radius, Steiner line) where
> > are lying the four antigonal conj. points of a point
> > (Miquel point) wrt the component triangles.
> > Question:
> > Which other points have the same property?
> > Either one point or four points respective
> > to component triangles.
> > To make it clear:
> > 1. Let P be a point on the plane of a c. 4lateral.
> > Let P_a, P_b, P_c, P_d be the antigonal conjugates of
> > P wrt the four component triangles.
> > For which points P the points P_a, P_b, P_c, P_d are concyclic?
> > 2. Let P be a point with h. coordinates $(x:y:z)$ wrt some
> > reference triangle..
> > Let P_a, P_b, P_c, P_d be the points with h. coordinates
> > $(x:y:z)$ wrt the four component triangles of a c. 4lateral.

Let Q_a, Q_b, Q_c, Q_d be the antigonal conjugates of P_a, P_b, P_c, P_d
wrt the four component triangles.

> > For which P 's the P_a, P_b, P_c, P_d are concyclic?

Read instead:

For which P 's the Q_a, Q_b, Q_c, Q_d are concyclic.

Sorry!
aph

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Message: #185
Date: 22/8/2013 4:31:23
From: bernard.keizer
Subject: Two points on the Newton Line

Dear Chris,

These 2 points are each other's Clawson-Schmidt conjugate.
The Clawson-Schmidt conjugate of the Newton Line is a circle through the 2 points and QL-P1.

One property of the curve QL-Cu1 is that 2 CS-conjugate points on the curve have the same tangential point (point where the tangent to the curve recuts the curve) and that the CS-conjugate of this tangential is in line with the 2 points.

The tangential point of the 2 points on the Newton Line is QL-P1 and the conjugate of QL-P1 is the infinity point of the Newton Line.

The tangential point of QL-P1 and this infinity point of the Newton Line is the point where QL-Cu1 cuts its asymptote and is on the line QL-P1 QL-P4 (axis of the mono-cardioid); the CS-conjugate of this point is on the axis of the parabola (parallel to the Newton Line).

The 2 points of the curve on the Newton Line are the 2 characteristic points of 2 orthogonal sets of circles (the centers of one set being on the Newton Line and the centers of the other set on the perpendicular bisector of the 2 points). The first set is composed of circles of which a diameter is through QL-P1 and the second is composed of circles such as the cord through the 2 contact points of the 2 tangents from QL-P1 to the circle is through the tangential of QL-P1.

If the first set is with the 2 points as basis-points (and the other with the 2 points as Poncelet points), the curve QL-Cu1 is made of 2 parts and the 2 points are on the circle having as diameter QL-P1 and its tangential.

In the reverse case (2 points as Poncelet points of the first set and basis-points of the second), the curve QL-Cu1 is in one piece.

In this case, it's easy to draw the curve as the locus of the contact points of the tangents from QL-P1 to the circles having their center on the perpendicular bisector of the 2 points.

Best regards
Bernard

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Message: #186
Date: 22/8/2013 7:23:53
From: Bernard Gibert
Subject: Two points on the Newton Line

Dear Chris and Bernard K.,

>> There are exactly 2 points on the Newton Line that are each
>> other's Isogonal Conjugate wrt all 4 QL-Component Triangles.
>> These points lie on QL-Cu1.
>> Their midpoint is the intersection point of the Newton Line
>> and the Quasi Ortholine: $QL-L1 \wedge QL-L6$.

All this stuff is directly related with focal cubics.
This is explained and detailed in my draft paper
<http://bernard.gibert.pagesperso-orange.fr/files/cardioids.html>
and also in
<http://bernard.gibert.pagesperso-orange.fr/files/isocubics.html>

Best regards
Bernard

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Message: #187
Date: 22/8/2013 10:24:20
From: Chris
Subject: Two points on the Newton Line

Dear Bernard G. and Bernard K.,
Thanks for all information!
I also calculated that the line through the Miquel Point (QL-P1)
and the Center of the Miquel Circle (QL-P4) is tangent to the
QL-Cu1-cubic.
Best regards,
Chris

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Message: #188
Date: 23/8/2013 8:52:06
From: eckart_schmidt@t-online.de
Subject: Two points on the Newton line

Dear Bernard K., dear friends,
it seems, that Bernard's (K) construction of QL-Cu1 has the same background as the "strophoid"-construction in my message #106.
A short summary (two possibilities, see brackets):
Let F1, F2 be the foci of an inscribed conic
with center $QL-L1 \wedge QL-L6$
... and C1 the circle with diameter F1F2.
... If the axis of the inscribed conic is QL-L1
(orthogonal QL-L1),
... let M be a variable point on QL-L1
... as midpoint of a circle C2 perpendicular C1
(through F1, F2),
... then the lines M.QL-P1 cut C2 in points of QL-Cu1.
Best regards Eckart

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Message: #189
Date: 23/8/2013 11:26:22
From: eckart_schmidt@t-online.de
Subject: Cardioids tangent to 4 lines

Dear Bernard (G), dear friends,
in addition to Bernard's paper "Inscribed Cardioids ...":
For a quadrilateral tangent to a circle the cubic $E(L)$ can be considered as Musselman cubic $K028$ (see my message #94).
Every point A on $E(L)$ can easily be completed to a reference triangle ABC for $K028$ with

$X_4 = QL-P10$,
 $X_2 =$ reflection of X_4 in $QL-P1$,
 $X_3 =$ reflection of $QL-P1$ in X_2 .

Well known points are for example:

... the common midpoint X_3 of the circumcircle,
... the intersection of $QL-L1$

with the perpendicular bisector of $X_3.X_4$,

... the homothetic $h(QL-P1, -2)$ of the last point.

The reference triangle degenerates for the intersections of $QL-Cu1$ and the circle with diameter $X_2.X_3$.

If one vertice is $QL-P10$, there are two rectangular reference triangles.

In addition I cite my message #99:

On a Musselman cubic $K028$ there are reference triangles, so that $K028$ belongs to the McCay-Kjp-pencil (table 22):

Construction for a given $K028$ and its reference triangle:

- * Take the midpoint $X_2.X_4$ as centroid G of a new reference triangle ABC .
- * Take the nodal point X_4 as a focus F for the inscribed Steiner ellipse of ABC .
- * Choose a point A on the cubic with its homothetic Ma under $h(G, -1/2)$.
- * Construct the Steiner ellipse with center G , Focus F and point Ma .
- * The tangents from A to the ellipse and the tangent in Ma give a triangle ABC on the cubic.
- * For ABC as reference triangle the cubic belongs to the McCay-Kjp pencil.

Best regards Eckart

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Message: #190
Date: 24/8/2013 8:17:09
From: Seiichi Kirikami
Subject: An area problem in QG- environment

Dear friends,

The following is what I found as an optimum problem in a Japanese book and I changed a bit.

(1) Quadrigon P1P2P3P4: P1(0,1,0), P2(0,0,1), P3(1,0,0), P4(p,q,r).

Given a quadrigon P1P2P3P4 with angle A_i of vertex P_i , a point P and its pedals Q_{ij} on P_iP_j , the differential area of P1P2P3P4-2*Q1Q2Q3Q4 has the expression:

$$\sin[2A_1]PP_1^2 + \sin[2A_2]PP_2^2 + \sin[2A_3]PP_3^2 + \sin[2A_4]PP_4^2$$

If area (P1P2P3P4)=2*area(Q1Q2Q3Q4),

we have a conic and its center as follows.

$$x_1 = a^2(q(SAq+b^2r) \\ (2SBpr+c^2p^2+a^2r^2)+b^2pr(r(SAp+SBq-SCr)-c^2pq)):$$

$$y_1 = b^2(p(SBp+a^2r) \\ (2SAqr+b^2r^2+c^2q^2)+a^2qr(r(SAp+SBq-SCr)-c^2pq)):$$

$$z_1 = SC (2SAqr+b^2r^2+c^2q^2) \\ (2SBpr+c^2p^2+a^2r^2)+a^2b^2r^2(r(SAp+SBq-SCr)-c^2pq)$$

(2) Quadrigon P1P2P4P3: P3(1,0,0), P1(0,1,0), P2(0,0,1), P4(p,q,r).

$$x_2 = y_1/. \{a \rightarrow c, b \rightarrow a, c \rightarrow b, p \rightarrow r, q \rightarrow p, r \rightarrow q\}.$$

$$y_2 = z_1/. \{a \rightarrow c, b \rightarrow a, c \rightarrow b, p \rightarrow r, q \rightarrow p, r \rightarrow q\}.$$

$$z_2 = x_1/. \{a \rightarrow c, b \rightarrow a, c \rightarrow b, p \rightarrow r, q \rightarrow p, r \rightarrow q\}.$$

(3) Quadrigon P1P3P2P4: P2(0,0,1), P3(1,0,0), P1(0,1,0), P4(p,q,r).

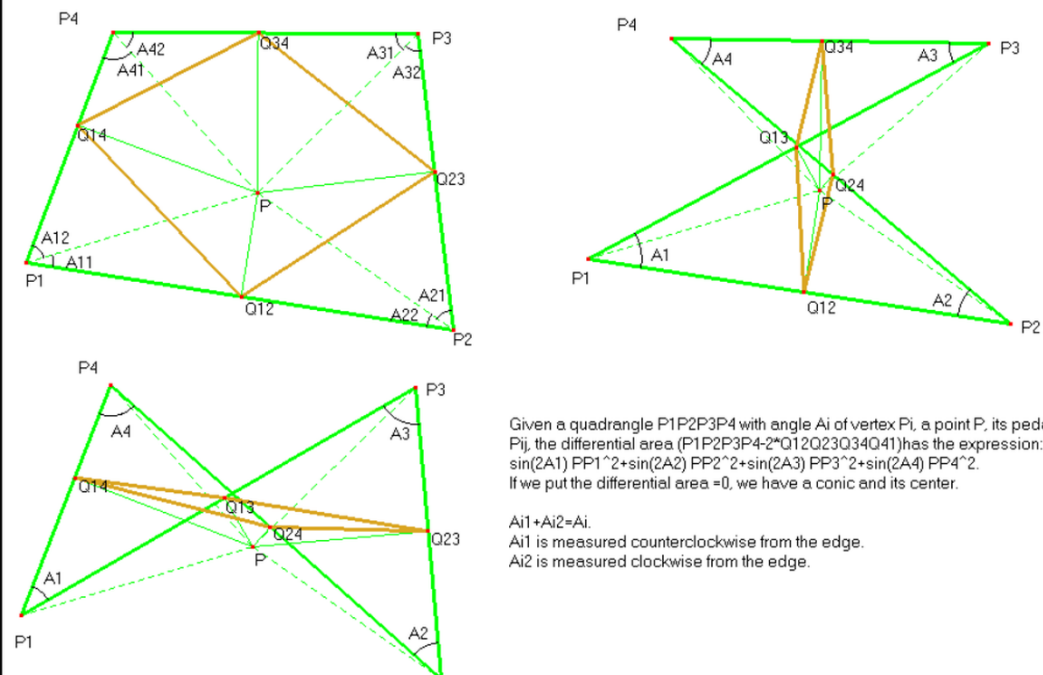
$$x_3 = z_1/. \{a \rightarrow b, b \rightarrow c, c \rightarrow a, p \rightarrow q, q \rightarrow r, r \rightarrow p\}.$$

$$y_3 = x_1/. \{a \rightarrow b, b \rightarrow c, c \rightarrow a, p \rightarrow q, q \rightarrow r, r \rightarrow p\}.$$

$$z_3 = y_1/. \{a \rightarrow b, b \rightarrow c, c \rightarrow a, p \rightarrow q, q \rightarrow r, r \rightarrow p\}.$$

Best regards,
Seiichi.

Configurations of a quadrigon in a quadrigon.



quadrigon in a quadrigon.docx

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Message: #191

Date: 24/8/2013 12:40:02

From: Seiichi Kirikami

Subject: The sufficient conditions of a quadrangle point

Dear friends,

The sufficient conditions of a quadrangle point are as follows.

A quadrangle point is denoted by $P_1P_2P_3P_4$ such that

$P_3=\{1,0,0\}$, $P_1=\{0,1,0\}$, $P_2=\{0,0,1\}$, $P_4=\{p, q, r\}$,
where $p+q+r=1$, based on the triangle system $P_1P_2P_3$.

$$P_1P_2^2 = a^2,$$

$$P_2P_3^2 = b^2,$$

$$P_1P_3^2 = c^2,$$

$$P_1P_4^2 = a^2r^2+c^2p^2+2SBrp,$$

$$P_2P_4^2 = b^2p^2+a^2q^2+2SCpq,$$

$$P_3P_4^2 = c^2q^2+b^2r^2+2SAqr,$$

A point $\{x,y,z\}$ is a quadrangle one if the absolute area PP_jP_{j+1} of the triangle system $P_{j-1}P_jP_{j+1}$ equals the absolute area PP_jP_{j+1} of the triangle system $P_jP_{j+1}P_{j+2}$ and the sum of the absolute area PP_jP_{j+1} equals the absolute area of $P_1P_2P_3P_4$.

If we suppose a quadrangle function $\{f(a^2, b^2, c^2; p, q, r);;\}$, the above conditions are as follows.

(1) $f(b^2, c^2, a^2; q, r, p)/\text{cyclic sum } f(b^2, c^2, a^2; q, r, p) =$
 $qf(b^2, c^2q^2+b^2r^2+2SAqr, b^2p^2+a^2q^2+2SCpq;$
 $1, -r, -p)/\text{cyclic sum } f(b^2, c^2q^2+b^2r^2+2SAqr,$
 $b^2p^2+a^2q^2+2SCpq; 1, -r, -p).$

(2) $qf(c^2q^2+b^2r^2+2SAqr, b^2p^2+a^2q^2+2SCpq, b^2; -r, -p,$
 $1)/\text{cyclic sum } f(c^2q^2+b^2r^2+2SAqr, b^2p^2+a^2q^2+2SCpq, b^2;$
 $-r, -p, 1) = -rf(c^2q^2+b^2r^2+2SAqr, a^2r^2+c^2p^2+2SBrp, c^2;$
 $q, p, -1)/\text{cyclic sum } f(c^2q^2+b^2r^2+2SAqr, a^2r^2+c^2p^2+2SBrp,$
 $c^2; q, p, -1).$

(3) $-rf(a^2r^2+c^2p^2+2SBrp, c^2, c^2q^2+b^2r^2+2SAqr; p, -1,$
 $q)/\text{cyclic sum } f(a^2r^2+c^2p^2+2SBrp, c^2, c^2q^2+b^2r^2+2SAqr;$
 $p, -1, q) = pf(a^2r^2+c^2p^2+2SBrp, a^2, b^2p^2+a^2q^2+2SCpq;$
 $-r, 1, -q)/\text{cyclic sum } f(a^2r^2+c^2p^2+2SBrp, a^2,$
 $b^2p^2+a^2q^2+2SCpq; -r, 1, -q).$

(4) $pf(a^2, b^2p^2+a^2q^2+2SCpq, a^2r^2+c^2p^2+2SBrp; 1, -q,$
 $-r)/\text{cyclic sum } f(a^2, b^2p^2+a^2q^2+2SCpq, a^2r^2+c^2p^2+2SBrp;$
 $1, -q, -r) = f(a^2, b^2, c^2; p, q, r)/\text{cyclic sum } f(a^2, b^2,$
 $c^2; p, q, r).$

(5) $1 - r =$ the sum of the right handed sides of (1), (2), (3) and (4).

If the quadrangle function is independent of the sides a^2 , b^2 and c^2 , then the following formulae hold.

$$(6) \quad f(q, r, p) / \text{cyclic sum } f(q, r, p) \\ = qf(1, -r, -p) / \text{cyclic sum } f(1, -r, -p).$$

$$(7) \quad qf(-r, -p, 1) / \text{cyclic sum } f(-r, -p, 1) \\ = -rf(q, p, -1) / \text{cyclic sum } f(q, p, -1).$$

$$(8) \quad -rf(p, -1, q) / \text{cyclic sum } f(p, -1, q) \\ = pf(-r, 1, -q) / \text{cyclic sum } f(-r, 1, -q).$$

$$(9) \quad pf(1, -q, -r) / \text{cyclic sum } f(1, -q, -r) \\ = f(p, q, r) / \text{cyclic sum } f(p, q, r).$$

(10) $1-r$ = the sum of the right handed sides of (6), (7), (8) and (9).

A result which I obtained from the above conditions:

If the quadrangle function is independent of the sides, there is no other linear quadrangle point except for QA-P1($x = 2p+q+r$;).

Best regards,
Seiichi.

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Message: #192

Date: 25/8/2013 11:43:39

From: eckart_schmidt@t-online.de

Subject: Cardioids tangent to 4 lines

Dear Bernard (G),

In my message #99 I already used your "construction of groups of pivots" for the nodal case to find reference triangles, so that $E(L)$ belongs to the McCay-Kjp-pencil. This holds also in general: The triangles $P_1P_2P_3$ constructed as in 7 of your paper are reference triangles so that $E(L)$ belongs to the McCay-Kjp-pencil.

Best regards Eckart

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Message: #193

Date: 25/8/2013 6:31:16

From: Chris van Tienhoven

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear friends,

Here are some remarks joining the remarks of Bernard Keizer.

Conjugates on QL-Cu1

The Clawson-Schmidt Conjugate is very important on the cubic QL-Cu1.

It is very special that this conjugate has the same performance for points on QL-Cu1 as

the Isogonal Conjugate of P wrt the Orthic Triangle of the QL-Diagonal Triangle (note Eckart Schmidt).

the Quasi Isogonal Conjugate (QG-Tf2) of P wrt any Quadrignon of the Reference Quadrilateral (note Eckart Schmidt).

Easy construction method of a CS-conjugated point of any point on QL-Cu1

Any point X on QL-Cu1 can be seen as the intersection point of the line L_n parallel to the Newton line and the connecting line $L_p = QL-P1.X$.

Now $CSC(X)$ can be constructed as the intersection point of $L_{a'}$ and $L_{b'}$, where:

$L_{n'} = L_n$ Reflected in the Newton Line

$L_{p'} = L_p$ reflected in the 1st Steiner Axis (see QL-Tf1 in EQF)

The 1st Steiner Axis easily can be constructed as:

1. an internal angle bisector of $S_{ij}.QL-P1.S_{kl}$, where $S_{ij} = L_i \wedge L_j$, $S_{kl} = L_k \wedge L_l$ for all $(i,j,k,l) \in (1,2,3,4)$, or
2. the internal angle bisector of $QL2P2a.QL-P1.QL-2P2b$, where $QL-2P2a$ and $QL-2P2b$ are the two points on the Newton Line discussed before.

As a consequence, when X_1 and X_2 are CS-conjugated points, then the line through X_1 parallel to $QL-L1$ and the line $X_2.QL-P1$ will cross in another point on QL-Cu1. X_1 and X_2 can be interchanged here.

Two new interesting points on QL-Cu1

* S = the intersection point of the QL-Cu1 and its asymptote. It can be constructed as the intersection of the asymptote and line $QL-P1.QL-P4$.

* T = the intersection point of the QL-Cu1 and the line through $QL-P1$ parallel to the Newton Line $QL-L1$. It can be constructed as the perpendicular bisector of $QL-2P2a/b$ and the line // $QL-L1$ through $QL-P1$.

* S and T are each other's Clawson-Schmidt Conjugate

* and so the intersection point W of their tangents at QL-Cu1 also lies at QL-Cu1.

Tangents at QL-Cu1

* The tangent at QL-Cu1 in QL-P1 = QL-P1.QL-P4.

This line is also tangent in QL-P1 at the circle (QL-P1,QL-2P2a,QL-2P2b),

which is the Clawson-Schmidt Conjugate of the Newton Line QL-L1

* The tangent at QL-Cu1 in T is S.T reflected in the perpendicular bisector of line segment QL-2P2a.QL-2P2b.

* How to construct the tangent in S ?????

Open questions:

I think and I feel there should be a simple construction method for constructing tangents at QL-Cu1, but how??

CS-conjugated pairs of points on QL-Cu1 lie at different sides of the Newton line at equal distances. There is a maximal distance from the Newton Line. What are the coordinates of the two points (on both sides of the Newton Line) on QL-Cu1 with maximal distance from the Newton Line?

Chris

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Message: #194

Date: 25/8/2013 9:22:48

From: eckart_schmidt@t-online.de

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear Chris,

on my homepage I have described QL-Cu1 under 11.1. So I can answer your first question:

Construction of the tangent in a point P on QL-Cu1:

* Let P* be the isogonal conjugate of P

* ... and Co1 the isogonal conjugate of PP*
wrt the triangle L1,L2,L3

* ... and Co2 the isogonal conjugate of PP*
wrt the triangle L2,L3,L4.

* Let Q be the fourth intersection of Co1 and Co 2.

* Then PQ is the tangent in P wrt QL-Cu1.

Best regards Eckart

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Message: #195
Date: 26/8/2013 11:58:55
From: Antreas Hatzipolakis
Subject: QUADRANGLE POINT? (12 MIDPOINTS)

--- In Anopolis@yahooogroups.com , "Antreas" wrote:

>> Let ABC be a triangle, P a point and A'B'C'
>> the cevian triangle of P.
>> Denote:
>> Ab, Ac = the midpoints of AB', AC', resp.
>> Bc, Ba = the midpoints of BC', BA', resp.
>> Ca, Cb = the midpoints of CA', CB', resp.
>> A1,B1,C1 = the midpoints of AA',BB',CC', resp.
>> A2,B2,C2 = the midpoints of BcCb, CaAc, AbBa, resp.
>> For which P's the triangles A1B1C1, A2B2C2 are perspective?
>> For all?

Conjecture 1: For all P's the triangles are perspective.
Now, probably a new point in quadrangle (or a new construction
of an old point),
if my conjectures are true!!! :-)
Let ABCD be a quadrangle.
Denote:
Da, Db, Dc, Dd the above described perspectors
wrt (BCD), (CDA), (DAB), (ABC), resp.

Conjecture 2:
The Lines ADa, BDb, CDc, DDd are concurrent.

Antreas

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Message: #196
Date: 27/8/2013 1:37:20
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

> On Mon, Aug 26, 2013 at 11:43 PM,
> Antreas <anopolis72@gmail.com>
> wrote:
>> Let ABC be a triangle, P a point and A'B'C'
>> the cevian triangle of P.
>> Denote:
>> Ab, Ac = the midpoints of AB', AC', resp.
>> Bc, Ba = the midpoints of BC', BA', resp.
>> Ca, Cb = the midpoints of CA', CB', resp.
>> A1,B1,C1 = the midpoints of AA',BB',CC', resp.
>> A2,B2,C2 = the midpoints of BcCb, CaAc, AbBa, resp.
>> For which P's the triangles A1B1C1, A2B2C2 are perspective?
>> For all?

We can also take a line L intersecting BC, CA,AB at A',B',C',
resp.
The triangles A1B1C1, A2B2C2 are perspective on the Newton line
of the complete quadrilateral (AB,BC,CA, L)
If we consider a complete quadrilateral (a,b,c,d) we get 4
points on the Newton line of the quadrilateral,
corresponding to triangles (b,c,d, L=a), (c,d,a, L=b), (d,a,b,
L=c), (a,b,c, L=d).
But it's now too late here (in Greece) to draw a figure! :-)
Antreas

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Message: #197
Date: 27/8/2013 8:40:22
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

[APH]

>> Let ABC be a triangle, P a point and A'B'C'
>> the cevian triangle of P.
>> Denote:
>> Ab, Ac = the midpoints of AB', AC', resp.
>> Bc, Ba = the midpoints of BC', BA', resp.
>> Ca, Cb = the midpoints of CA', CB', resp.
>> A1,B1,C1 = the midpoints of AA',BB',CC', resp.
>> A2,B2,C2 = the midpoints of BcCb, CaAc, AbBa, resp.
>> For which P's the triangles A1B1C1, A2B2C2 are perspective?
>> For all?
>> Antreas

[César Lozada]:
A1B1C1, A2B2C2 are perspective because A2B2C2 is the medial
triangle of A1B1C1 !!!
So, their perspector is the centroid of A1,B1,C1.

Dear Cesar,

Very good!!!
So we need 3 only midpoints!!
"Quadranglization": Let ABCD be a quadrangle.

Denote:
 $A' = AD \cap BC$, $B' = BD \cap CA$, $C' = AC \cap AB$
 $M1, M2, M3$ = the midpoints of AA' , BB' , CC'
 Gd = the centroid of $M1M2M3$
Similarly Ga, Gb, Gc for the triangles BCD, CDA, DAB
The lines AGa, BGb, CGc, DGd are concurrent (??)

APH

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Message: #198
Date: 27/8/2013 8:44:57
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

Sorry for the typo:

>> $A' = AD \cap BC$, $B' = BD \cap CA$, $C' = AC \cap AB$

Read:

$C' = CD \cap AB$

APH

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Message: #199
Date: 27/8/2013 9:25:55
From: eckart_schmidt@t-online.de
Subject: 12 Midpoints

Dear Andreas,
the intersection of the lines in your last message is a point on QA-L3, dividing QA-P1.QA-P10 with ratio 3:4.
Best regards Eckart

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Message: #200
Date: 27/8/2013 12:24:00
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

[APH]:

Equivalently:

Let ABC be a triangle, D a point and A'B'C' the cevian triangle of D. (ABCD is a quadrangle)

Denote:

M_1, M_2, M_3 = the midpoints of AA', BB', CC' , resp.

Ma, Mb, Mc = the midpoints of DA', DB', DC' , resp.

G_a, G_b, G_c, G_d = the centroids of BCD, CDA, DAB, ABC, resp.

The lines AG_a, BG_b, CG_c, DG_d are concurrent.

Antreas

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Message: #201
Date: 27/8/2013 12:29:57
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

[APH]:
Equivalently:
Let ABC be a triangle, D a point and A'B'C' the cevian triangle of D. (ABCD is a quadrangle)
Denote:
M1,M2,M3 = the midpoints of AA',BB',CC', resp.
Ma,Mb,Mc = the midpoints of DA', DB', DC', resp.
Ga,Gb,Gc,Gd = the centroids of MaM2M3,M1MbM3,M1M2Mc,M1M2M3, resp.
The lines AGa, BGb, CGc, DGd are concurrent.
Antreas

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Message: #202
Date: 27/8/2013 7:34:34
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

[APH]

Let $(1,2,3,4,5,6)$ be a hexagon.

Consider the set $\{A,B,C\}$, subset of $\{1,2,3,4,5,6\}$, and the complementary $\{A',B',C'\}$.

Denote:

Bc, Ba = the midpoints of BC', BA' , resp.

Ca, Cb = the midpoints of CA', AB' , resp.

Ab, Ac = the midpoints of AB', AC' , resp.

Ma, Mb, Mc = the midpoints of $BcCb, CaAc, AbBa$, resp.

$M1, M2, M3$ = the midpoints of AA', BB', CC' , resp.

The lines $MaM1, MbM2, McM3$ are concurrent at a point P . (Proof?)

The point P is the same for all sets $\{A,B,C\}$, subsets of $\{1,2,3,4,5,6\}$

So we have a point in a sexangle.

Is this point known?

Note: Two points may coincide and then we have a point P in a quintangle.

Antreas

P.S. Notice in the terminology:

The prefices quadra-, quinta-, sexa - septa- etc come from Latin,

while the prefices tetra-, penta-, hexa, hepta- etc from Greek.

John H. Conway and I had worked on the names of the regular polygons and polyhedra

<http://mathforum.org/dr.math/faq/faq.polygon.names.html>

APH

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Message: #203
Date: 27/8/2013 10:31:05
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

The point P is the centroid of $M_1M_2M_3$ [Cesar Lozada, Anopolis #899]

So a simpler form is:

Let $(1,2,3,4,5,6)$ be a sexangle.

Let $\{A,B,C\}$ be a subset of the set $\{1,2,3,4,5,6\}$

and $\{A',B',C'\}$ the complementary of $\{A,B,C\}$ with respect $\{1,2,3,4,5,6\}$.

Let M_1, M_2, M_3 be the midpoints of AA', BB', CC' , resp.

The centroid of $M_1M_2M_3$ is a fixed point, ie it is independent from the choice of $\{A,B,C\}$.

Antreas

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Message: #204
Date: 27/8/2013 11:24:48
From: Antreas Hatzipolakis
Subject: 12 MIDPOINTS

[APH:

>> The point P is the centroid of $M_1M_2M_3$
>> [Cesar Lozada, Anopolis #899]
>> So a simpler form is:
>> Let $(1,2,3,4,5,6)$ be a sexangle.
>> Let $\{A,B,C\}$ be a subset of the set $\{1,2,3,4,5,6\}$
>> and $\{A',B',C'\}$ the complementary of $\{A,B,C\}$
>> with respect $\{1,2,3,4,5,6\}$.
>> Let M_1, M_2, M_3 be the midpoints of AA', BB', CC' , resp.
>> The centroid of $M_1M_2M_3$ is a fixed point, ie
>> it is independent from the choice of $\{A,B,C\}$.

For octangle:

Let 12345678 be an octangle.

Let $\{A,B,C,D\}$ be a subset of the set $X =: \{1,2,3,4,5,6,7,8\}$
and $\{A',B',C',D'\}$ the complementary of $\{A,B,C,D\}$ with respect X .

Let M_1, M_2, M_3, M_4 be the midpoints of AA', BB', CC', DD' , resp.

The centroid of the quadrangle $M_1M_2M_3M_4$ is a fixed point, ie
independent from the choice of $\{A,B,C,D\}$

Antreas

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Message: #205

Date: 28/8/2013 2:32:49

From: eckart_schmidt@t-online.de

Subject: Two points on the Newton line QL-Cu1 revisited

Dear friends,

just for fun:

Let F_1, F_2 be the foci of the inscribed conic of a quadrilateral with center $QL-L1 \wedge QL-L6$.

... Consider the circumscribed conics of the triangle components through F_1, F_2 .

... These conics have a common point in $QL-P1$.

Let F_1, F_2 be the foci of an inscribed conic of a quadrilateral.

... Consider the circumscribed conics of the triangle components through F_1, F_2 .

... These conics have a common point X on $QL-Cu1$.

... X is the isogonal conjugate of the third intersection of F_1F_2 and $QL-Cu1$.

... F_1, F_2, X and $QL-P1$ are concyclic.

Let F_1, F_2 be a $QL-Tf1$ pair .

... Consider the circumscribed conics of the triangle components through F_1, F_2 .

... These conics have a common point X .

... F_1, F_2, X and $QL-P1$ are concyclic.

Let L be a line.

... Consider the isogonal conjugates of L wrt the triangle components.

... The common points of these conics lie on $QL-Cu1$.

... Consider a line pencil and you have a construction for $QL-Cu1$.

Best regards Eckart

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Message: #206
Date: 28/8/2013 3:13:12
From: Angel
Subject: A Quadrangle Centers

Dear Members of the group Quadri-Figures-Group

(Adopting the notation of Chris van Tienhoven in
<http://www.chrisvantienhoven.nl/>)

Let P_1, P_2, P_3, P_4 be the defining Quadrangle Points.
Let $S_1 = P_1.P_2 \setminus P_3.P_4$, $S_2 = P_1.P_3 \setminus P_2.P_4$ and $S_3 = P_1.P_4 \setminus P_2.P_3$.

Now $S_1 S_2 S_3$ is the QA-Diagonal Triangle of the Reference Quadrangle.
QA-DT-Coordinate system, where the QA-Diagonal Triangle is defined as the Reference Triangle with barycentric vertex coordinates $S_1 = (1:0:0)$, $S_2 = (0:1:0)$, $S_3 = (0:0:1)$.
An arbitrary point of the Quadrangle is defined as $P_4 = (p:q:r)$.

The other 3 points now form the Anticevian triangle of P_i wrt the QA-Diagonal Triangle and have vertices $P_1 = (-p:q:r)$, $P_2 = (p:-q:r)$, $P_3 = (p:q:-r)$.
Let $V = (u:v:w)$ be a point and C_i is the center of the conic through points S_1, S_2, S_3, P_i and V ($i=1,2,3,4$).
The lines $P_i C_i$ ($i=1,2,3,4$) are concurrent if and only if $P = (x:y:z)$ lies on the circumconic that passes through the vertices of the QA-Diagonal Triangle of perspector the barycentric square of $P_4 = (p,q,r)$, together with the quartic that passes through the $P_1, P_2, P_3, P_4, S_1, S_2, S_3, \text{QA-P10} (= \text{Centroid of the QA-Diagonal Triangle}), \dots$, of equation

(DT-Coordinate):
$$p^2(q^2-r^2)y^2z^2 + q^2(r^2-p^2)z^2x^2 + r^2(p^2-q^2)x^2y^2 = 0.$$

Particular case:
If $V = \text{QA-P10}$ (Centroid of the QA-Diagonal Triangle)
The lines $P_i C_i$ ($i=1,2,3,4$) intersect at Quadrangle Centers of
1st DT-Coordinate: $p^2(q^2+r^2)$

Best regards
Angel Montesdeoca

Message: #207
Date: 28/8/2013 6:20:12
From: Chris
Subject: A Quadrangle Centers

Dear Angel,

Very nice construction and very nice point.
The point you found is collinear with QA-P1, QA-P16, QA-P21 and also the QA-Involution Center (QA-Tf1) of the line QA-P1.QA-P16.

I will study the involved conic and quartic later.
Best regards,
Chris

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Message: #208
Date: 28/8/2013 11:54:27
From: Chris
Subject: A Quadrangle Centers

Dear Angel,

Some extra properties:
The conic is the Ninepoint Conic QL-Co1.
The Quartic also passes through QA-P20.
There are four known QA-points X that are on these curves for which the lines P_iC_i ($i=1,2,3,4$) are concurrent in some point Y.
When $X = \text{QA-P2}$ then $Y = \text{QA-P1.QA-P6} \wedge \text{QA-P11.QA-P38}$, which is the Involution Center of the line QA-P1.QA-P6.
When $X = \text{QA-P3}$ then $Y = \text{QA-P1.QA-Tf2(QA-P2)} \wedge \text{QA-P22.QA-P29}$
When $X = \text{QA-P10}$ then $Y =$ the Involution Center of the line QA-P1.QA-P16.
When $X = \text{QA-P20}$ then $Y = \text{QA-P22}$, which is the Involution Center of the line QA-P1.QA-P5.
All these points Y lie on the cubic QA-Cu6 because they are all Involution Centers of some line through QA-P1.
I do not quite see why. Do you?

Best regards,
Chris

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Message: #209
Date: 29/8/2013 10:31:16
From: eckart_schmidt@t-online.de
Subject: A quadrangle center

Dear Angel, dear Chris,
interesting new curves! My sight: (
Let X be points for which the lines P_iC_i are concurrent in
points Y.)
One locus for X is the QA-DT circumconic QA-Co1, which is the
image of the line at infinity under the isoconjugation QA-Tf1
(see Chris' message). So it is easy to study the locus for the
corresponding Y points.
It is a cubic with a nodal point in QA-P1 and the equation
$$\begin{aligned} & q^2r^2x^2(q^2-r^2)(-x+y+z) \\ & + r^2p^2y^2(r^2-p^2)(x-y+z) \\ & + p^2q^2z^2(p^2-q^2)(x+y-z) \\ & + 2p^2q^2r^2(-y)(y-z)(z-x) = 0 \end{aligned}$$

The second locus for X is a quartic. It is the isotomic
conjugate of a conic with the equation
$$p^2x^2(q^2-r^2)+q^2y^2(r^2-p^2)+r^2z^2(p^2-q^2) = 0$$

This conic contains with a point also the vertices of the
anticevian triangle. So it is easy to construct the quartic:
Take QA-P10 and QA-P20 on the quartic (see Chris' message) and
their isotomic images QA-P10 and QA-P19 and for QA-P19 the
vertices of its anticevian triangle, then the conic can be
drawn. Its isotomic conjugate is the quartic. The quartic
contains the vertices of the anticomplementary triangle. The
locus for the Y points is the same as above.
So far, there will be more properties.
Best regards Eckart

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Message: #210
Date: 29/8/2013 11:53:40
From: eckart_schmidt@t-online.de
Subject: A quadrangle center

Dear Angel, dear Chris,
sorry there is a typo in the first equation of my last message.
Chris already mentioned this cubic as QA-Cu6.
Best regards Eckart

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Message: #211

Date: 30/8/2013 6:57:41

From: seiichikiri

Subject: The sufficient conditions of a quadrangle point

Dear friends,

About a quadratic quadrangle function independent of the sidelengths of the reference triangle.

The conditions without the restriction $p+q+r=1$ are as follows:

- (6) $(p+q+r)f(q,r,p) / \text{cyclic sum } f(p,q,r)$
 $= qf(p+q+r, -r, -p) / \text{cyclic sum } f(p+q+r, -r, -p).$
- (7) $qf(-r, -p, p+q+r) / \text{cyclic sum } f(-r, -p, p+q+r)$
 $= -rf(q, p, -p-q-r) / \text{cyclic sum } f(q, p, -p-q-r).$
- (8) $-rf(p, -p-q-r, q) / \text{cyclic sum } f(p, -p-q-r, q)$
 $= pf(-r, p+q+r, -q) / \text{cyclic sum } f(-r, p+q+r, -q).$
- (9) $pf(p+q+r, -q, -r) / \text{cyclic sum } f(p+q+r, -q, -r)$
 $= (p+q+r)f(p, q, r) / \text{cyclic sum } f(p, q, r).$
- (10) $p+q = \text{the sum of the right handed sides}$
 $\text{of (6), (7), (8) and (9).}$

From (9), $f(p,q,r)$ must be $p(hp+kq+lr)$, because the denominators do not have factors p or $p+q+r$ owing to their symmetry of variables. Cyclic sum $f(p,q,r)$ and $f(p+q+r, -q, -r)$ must equal each other.

Then we have $QA-P16 = \{p(2p+q+r) : : \}$.

It seems to me that there is no other quadratic quadrangle function except for QA-P16 when it is independent of the sidelengths of the reference triangle.

Best regards,
Seiichi.

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Message: #212
Date: 01/9/2013 10:37:50
From: Antreas Hatzipolakis
Subject: Conjecture - Concyclic points

----- Forwarded message -----

From: Antreas <anopolis72@gmail.com>
Date: Sun, Sep 1, 2013 at 11:35 AM
Subject: [EGML] Re: Conjecture - Concyclic points
To: Anopolis@yahoogroups.com
[César Lozada]:
>> Let P be a point in the plane of ABC. The perpendicular
bisector
>> of [BC] cuts the circumcircle of PBC at A', with A' and A in
>> the same side w/r to (BC). Build B' and C' cyclically.
>> Conjectures:
>> 1) The points A', B', C' and P are concyclic.
>> 2) If Z(P) is the center of the circle(P, A', B', C') then:
>> 2.1) if P is on the circumcircle of ABC then Z(P)=O
>> 2.2) Z(O) = O, Z(H) = X(355), Z(X(13))=G=X(2),
>> Z (X(14))= X(628) and maybe others

Dear César,

This is very nice and very good for quadrangle loci!

Now, let ABCD be a quadrangle and P a point.

Denote:

aZ(P) = the center Z(P) for the triangle BCD and point P

bZ(P) = the center Z(P) for the triangle CDA and point P

cZ(P) = the center Z(P) for the triangle DAB and point P

dZ(P) = the center Z(P) for the triangle ABC and point P

For which points P we have:

aZ(P)A, bZ(P)B, cZ(P)C, dZ(P)D are concurrent?

Also, for which P's the four centers are concyclic?

Antreas

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Message: #213
Date: 01/9/2013 11:38:55
From: eckart_schmidt@t-online.de
Subject: Six-point QL-cubic wrt a point

Dear friends,
here is an QL-analogon to QA-cubics of type 1 (see QA-Cu/1).
These QA-cubics (e.g. QA-Cu1,2,3,4,5) are pivotal isocubics wrt
QA-Tf2 and can easily be constructed in the following way: For a
pivot P these cubics are the locus of the tangential points of P
wrt circumscribed conics of the quadrangle (Is this anywhere
described?). This construction can be generalized for
quadrilaterals:

For a given point P the cubic is the locus of the tangential
points of P wrt inscribed conics of the quadrilateral.
Another construction: Lines through P cut their QL-Tf2 image in
points of the cubic.

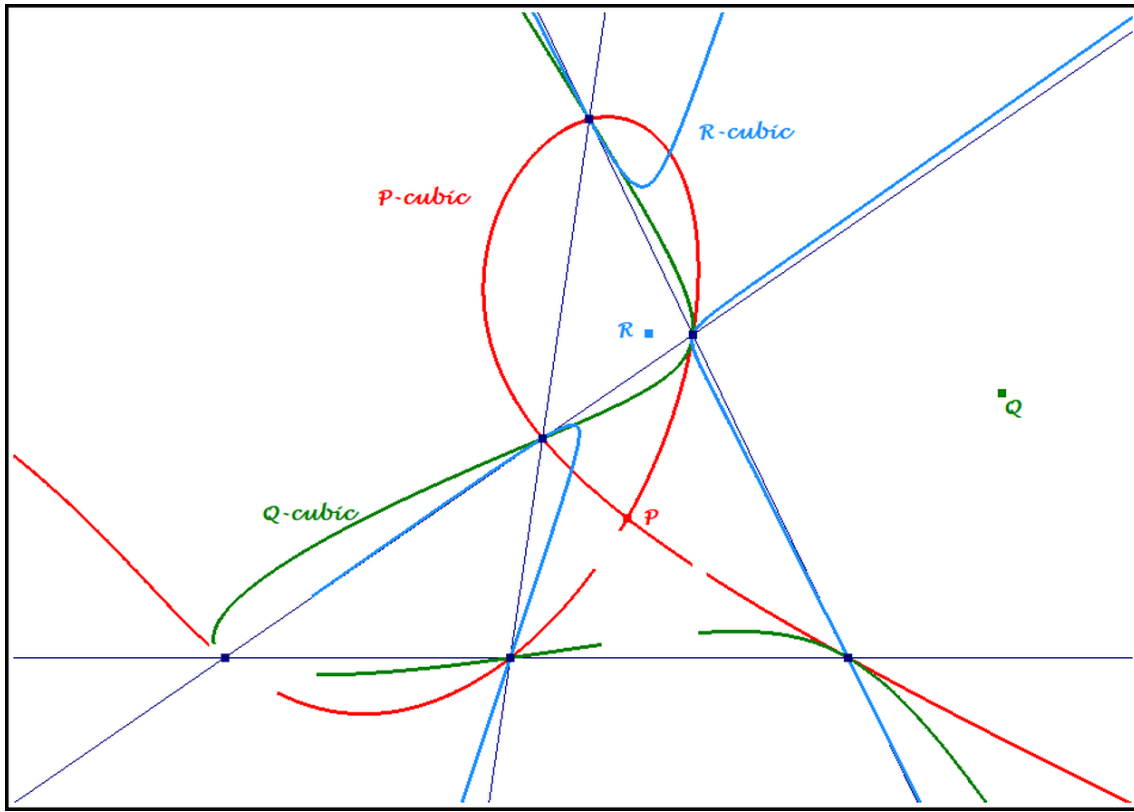
Properties:

1. The cubic contains the six points of the quadrilateral.
2. For $P(u:v:w)$ (CT-notation) the cubic has the equation
$$\begin{aligned} & m n x (w^2 y^2 + v^2 z^2) \\ & + n l y (u^2 z^2 + w^2 x^2) \\ & + l m z (v^2 x^2 + u^2 y^2) \\ & + 2 (l m u v + n l w u + m n v w) x y z = 0 \end{aligned}$$
3. Tangents to the cubic in opposite points
of the quadrilateral intersect on the cubic.
4. The three intersection points are collinear.
5. The equation of this line is
$$m n v^2 w^2 (m v + n w) (2 l n + m v + n w) x + \dots = 0$$
6. Tangents to the cubic in the tangential points of P wrt an
inscribed conic intersect on the cubic.

Observation: The cubic has a nodal point in P, if P lies in the
"convex heart" of the quadrilateral. Changing the region about a
line, the cubic will lose its nodal character; changing the
region about a point, the cubic will hold its nodal character...

Remark: Using QL-points for P, I found no relationships to other
QL-points. Perhaps Chris will find some.
There is a Cabri file in the attachment showing three examples.

Best regards Eckart



13-08-31-a-fig.png

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Message: #214
Date: 01/9/2013 12:01:12
From: Bernard Gibert
Subject: Six-point QL-cubic wrt a point

Dear Eckart,

>> 2. For $P(u:v:w)$ (CT-notation) the cubic has the equation
>> $m n x (w^2 y^2 + v^2 z^2) + n l y (u^2 z^2 + w^2 x^2)$
>> $+ l m z (v^2 x^2 + u^2 y^2)$
>> $+ 2 (l m u v + n l w u + m n v w) x y z = 0$
There's a typo in your equation since this cubic is a cK with
node P and root the tripole of L.
See SITP §8 (and also §1.4.1)

Best regards
Bernard

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Message: #215
Date: 01/9/2013 4:07:04
From: eckart_schmidt@t-online.de
Subject: Six-points QL-cubic wrt a point

Dear Bernard,

thanks for correction. The right equation of the cubic must be:

$$\begin{aligned} & m n x (w^2 y^2 + v^2 z^2) \\ & + n l y (u^2 z^2 + w^2 x^2) \\ & + l m z (v^2 x^2 + u^2 y^2) \\ & - 2(l m u v + n l w u + m n v w) x y z = 0 \end{aligned}$$

Thanks also for a classification as a conico-pivotal isocubic (cK). So the cubic is invariant under an isoconjugation with fixed point P.

The pivotal conic of the cubic (see SITP, §8.1) is an inscribed conic of QL-DT as envelope of the QL-Tf2 images of lines through P.

Best regards Eckart

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Message: #216
Date: 02/9/2013 2:00:45
From: amontes
Subject: A Quadrangle Center

Dear Chris

Response to post [Quadri-Figures-Group] #208.

An additional Quadrangle Center:

(Eckart Schmidt

<http://groups.yahoo.com/neo/groups/Quadri-Figures-Group/conversations/messages/209>)

The quartic that passes through the $P_1, P_2, P_3, P_4, S_1, S_2, S_3, QA-P10$,

$QA-P20$ is the isotomic conjugate of a conic with the equation $p^2x^2(q^2-r^2)+q^2y^2(r^2-p^2)+r^2z^2(p^2-q^2)=0$.

This conic contains with a point also the vertices of the anticevian triangle: $QA-P10$ and $QA-P19$ and the vertices of its anticevian triangles.

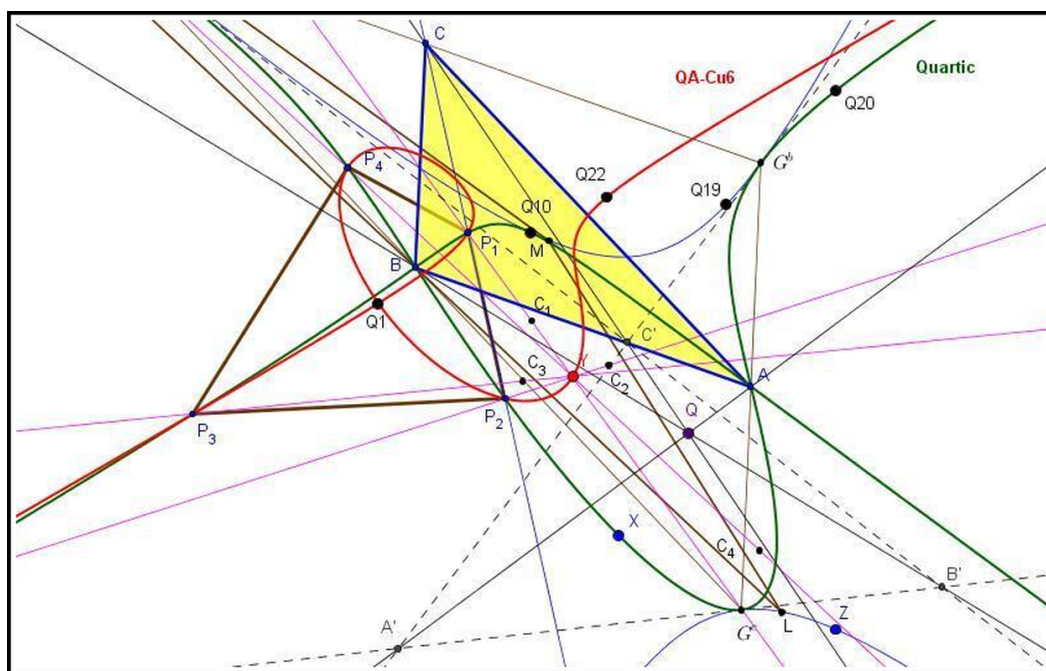
Thus, the quartic passes through the vertices of the triangle antimedial of the diagonal triangle.

The conic and the quartic are tangent in the vertices of the antimedial triangle.

The three tangents bound a triangle $A'B'C'$ perspective with the diagonal triangle, its perspector Q has 1st DT-Coordinate: $1 / (p^2(q^2-r^2))$

Best regards

Angel Montesdeoca



QAPnuevoQuartic.jpg

Message: #217
Date: 05/9/2013 4:08:02
From: eckart_schmidt@t-online.de
Subject: Two points on the Newton Line

Dear Chris,
the discussed points are now QL-2P2a,b in EQF. Here is a further property, I haven't found there:

- # These points are QL-Tf1 images as well as QG-Tf2 images.
- # These points are not only foci of an inscribed conic of the quadrilateral but also foci of inscribed conics for the triangles QG-Tr3 of the corresponding quadrigon components.
- # These QG-conics touch QG-L1 in QG-P17.
- # If there are intersections of the three QG-conics for a quadrilateral, they are orthogonal.

Best regards Eckart

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Message: #218
Date: 07/9/2013 1:15:52
From: seiichikiri
Subject: A parametric expression of QA-L3

Dear friends,
While I examined QA-points independent of the sidelengths of the reference triangle with the conditions of message #211, I noticed a parametric expression of QA-L3 as follows.

$QA-L3 = \{(q+r)(2p+q+r)(mp(p+q+r)+nqr);; \}$.

- (1) $m=n=1$ gives QA-P1.
- (2) $m=0, n=1$ gives QA-P5.
- (3) $m=1, n=0$ gives QA-P10.
- (4) $m=-1, n=1$ gives QA-P20.
- (5) $m=3, n=-1$ gives QA-P22.
- (6) $m=2, n=3$ gives QA-P25.
- (7) $m=5, n=3$ gives QA-P26.

Best regards,
Seiichi.

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Message: #219
Date: 08/9/2013 6:27:38
From: seiichikiri
Subject: A parametric expression of QA-L3

Dear friends,

Addition: QAP31 extended.

QAP31extended = {p(q+r)(2p+q+r)(mp^2+mpq+mpr+mqr+nq^2+nr^2);; }.

Best regards,
Seiichi.

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Message: #220
Date: 08/9/2013 9:48:07
From: Chris
Subject: Six-point QL-cubic wrt a point

Dear Eckart,

I found these points / asymptotes:

- * Asymptotes QL-P5, QL-P7, QL-P12, QL-P20, QL-P22, QL-P23-cubics equal or parallel to Newton Line:
- * Asymptotes QL-P23-cubic equal or parallel to 2nd asymptote QL-Co2
- * QL-P8-cubic passes through the 3 QL-versions of QG-P3.

Best regards,
Chris

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Message: #221
Date: 09/9/2013 9:35:05
From: eckart_schmidt@t-online.de
Subject: Six-point QL-cubic wrt a point

Dear Chris,
in addition to your last message:
For all points P on the Newton line there is an asymptote for
the cubic equal or parallel the Newton Line.
Best regards Eckart

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Message: #222
Date: 09/9/2013 8:50:54
From: bernard.keizer
Subject: Extension to QL-8P1

Reference for this is F. Loud 1900 Sundry metric theorems concerning n lines in a plane
Considering the 8 circles described in the 8th point of Steiner, the first points of intersection are the $4 \times 4 = 16$ in- and excenters of the reference triangles. The second points of intersection of the same circles are 16 other points belonging to the same circles. But they belong to 12 other circles too, 4 on each circle and 3 circles on each point; these 12 circles are all through the point QL-P1 and form 6 pairs of orthogonal circles, the second points of intersection of the pairs being the 6 vertices of the QL. The centers of the 12 circles are 2 on each perpendicular bisector of the segments joining QL-P1 to the 6 vertices, which are the sides and diagonals of the inscribed quadrilateral formed by the circumcenters of the reference triangles. As there are 3 circles on each point and all circles are through QL-P1, there are 16 alignments of 3 centers on the perpendicular bisectors of the segments joining QL-P1 to the 16 points.

Best regards
Bernard Keizer

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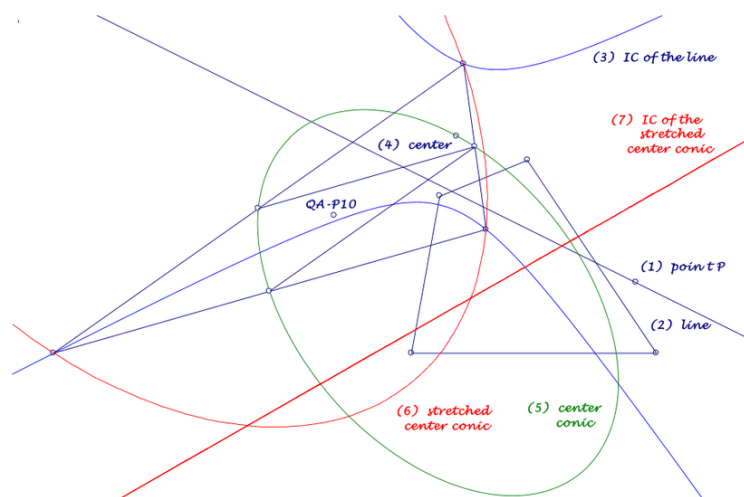
Message: #223
Date: 10/9/2013 3:07:26
From: eckart_schmidt@t-online.de
Subject: Circumscribed conics of the QA-DT Medial Triangle

Dear friends,
a special construction leads from an arbitrary point P to a circumscribed conic of the QA-DT medial triangle. Stretching this conic to a circumscribed conic of QA-DT it is the Involuntary Conjugate of a line L . There are a lot of EQF relationships, see the attachment.
Best regards Eckart

Background for these notes is:
 Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Circumscribed Conics of the *QA-DT* Medial Triangle

A special construction leads from an arbitrary point P to a circumscribed conic of the *QA-DT* medial triangle. Stretching this conic to a circumscribed conic of *QA-DT* it is the Involutory Conjugate of a line L . Here several examples are listed without right to completeness. – Reference triangle for barycentric coordinates is *QA-DT*.



Construction

- Line pencil of an arbitrary point $P(u:v:w)$.
- Involutory conjugates of the lines are *QA-DT* circumscribed conics.
- Locus of the centers is a circumscribed conic of the medial triangle of *QA-DT*:

$$q^2 r^2 u x^2 + r^2 p^2 v y^2 + p^2 q^2 w z^2 - p^2 (q^2 w + r^2 v) y z - q^2 (r^2 u + p^2 w) z x - r^2 (p^2 v + q^2 u) x y = 0$$

with center

$$(2p^2 v w + q^2 w u + r^2 u v : p^2 v w + 2q^2 w u + r^2 u v : p^2 v w + q^2 w u + 2r^2 u v) .$$

- The anticomplement of the center conic is a circumscribed conic of $QA-DT$:

$$p^2(q^2w + r^2v)yz + q^2(r^2u + p^2w)zx + r^2(p^2v + q^2u)xy = 0$$

with center

$$(u(q^2w + r^2v) : v(r^2u + p^2w) : w(p^2v + q^2u)).$$

- The Involutary Conjugate of the stretched center conic is a line:

$$(q^2w + r^2v)x + (r^2u + p^2w)y + (p^2v + q^2u)z = 0.$$

Examples

$P = QA-P1$

The center conic contains $QA-P29$ and has center $QA-P22$.

The stretched center conic is the Nine-point Conic $QA-Col$.

The final line is the line at infinity.

$P = QA-P2$

The center conic and the stretched center conic are parabolas with axis orthogonal to $QA-L2$.

The final line contains $QA-P2$ and is orthogonal to $QA-P2.QA-P23$.

$P = QA-P3$

The center conic and the stretched center conic are parabolas with axis parallel to $QA-L4$.

The final line contains $QA-P3$ and is orthogonal to $QA-P3.QA-P32$.

$P = QA-P6$

The center conic is an orthogonal hyperbola through $QA-P11$ and $QA-P29$ with center in the midpoint of $QA-P11.QA-P29$.

The stretched center conic is an orthogonal hyperbola through $QA-P12$ and has the center $QA-P36$.

The final line contains $QA-P23$ and is orthogonal to $QA-P1.QA-P32$.

$P = QA-P10$

The center conic has a center in the midpoint of $QA-P16.QA-P31$.

The center of the stretched center conic is $QA-P31$.

The final line is the $QA-DT$ trilinear polar of the isotomic conjugate of $QA-P31$ (orthogonal to $QA-P12.QA-P32$).

P = QA-P16

The center conic is the *QA-DT* inscribed Steiner ellipse with center *QA-P10*.

The stretched center conic is the *QA-DT* circumscribed Steiner ellipse with center *QA-P10*.

The final line is the *QA-DT* trilinear polar of *QA-P16*.

QA-P17

The center of the center conic is *QA-P1*.

The center of the stretched center conic is *QA-P20*.

The final line is a *QA-DT* trilinear polar of a point on *QA-P1.QA-P17...*

QA-P18

The center of the center conic is *QA-P31*.

The center of the stretched center conic is *QA-P16*.

The final line is the trilinear polar of *QA-P20*.

QA-P19

The center of the center conic divides *QA-P10.QA-P18* with ratio *1:3*.

The center of the stretched center conic divides *QA-P10.QA-P18* with ratio *-1:3*.

QA-P20

The center of the center conic is the midpoint of *QA-P1.QA-P22*.

The center of the stretched center conic is *QA-P22*.

The final line is orthogonal to *QA-P12.QA-32...*

QA-P23

The center conic is the circumcircle *QA-Ci2* of the *QA-DT* medial triangle.

The stretched center conic is the circumcircle *QA-Ci1* of *QA-DT*.

The final line is the connection of *QA-P6* and the reflection of *QA-P38* in *QA-Ci1* (orthogonal to *QA-P1.QA-P32* and *QA-P11.QA-P38*). It is the *QA-DT* trilinear polar of a point which is the isotomic conjugate of the isogonal conjugate of *QA-P16*:

$$a^2q^2r^2x + b^2r^2p^2y + c^2p^2q^2z = 0 .$$

QA-P27

The center of the center conic is the midpoint of $QA-P1.QA-P31$ (or divides $QA-P10.QA-P21$ with ratio $1:3$).
The center of the stretched center conic divides $QA-P10.QA-P21$ with ratio $-1:3$.

QA-P30

The center of the center conic divides $QA-P10.QA-P6$ with ratio $1:3$.
The center of the stretched center conic divides $QA-P10.QA-P6$ with ratio $-1:3$.

QA-P36

The final line is orthogonal to $QA-P12.QA-P29$.

In the QG -environment the center conic always contains $QG-P2$ and the stretched center conic always contains $QG-P1$ (further not mentioned). The parallel to $QG-L1$ half the distance to $QG-P1$ shall be denoted as QG -mid-parallel.

QG-P1

The construction degenerates.

QG-P2

The center of the center conic is the midpoint of $QG-P1.QG-P2$. The center conic contains $QG-P1$ and the intersection of $QG-L1$ and $QG-L2$.
The stretched center conic has center $QG-P2$.
The final line is a parallel to $QG-L1$ through $QG-P12$.

QG-P3

The center conic degenerates into two lines: $QG-P1.QG-P2$ and QG -mid-parallel.

QG-P4

The center of the stretched center conic lies on $QG-P2.QG-P12$. It is the reflection of the intersection of $QG-P2.QG-P12$ and the QG -mid-parallel in $QA-P1$.
The final line is parallel to $QG-P1.QG-P3$ through the reflection of $QG-P1$ in $QG-P15$.

QG-P12

The Involutary Conjugate of all lines through *QG-P12* are *QA-DT* circumscribed conics with center *QG-P2*. So the construction degenerates.

QG-P13

The center conic is a parabola with an axis parallel to *QG-P1.QG-P2* and *QG-L1* tangent in *QG-P2*.

The stretched center conic is a parabola with an axis parallel to *QG-P1.QG-P2* and *QG-P1.QG-P14* tangent in *QG-P1*.

The final line is *QG-P3.QG-P13*.

QG-P14

The center conic is a parabola through *QG-P3* with an axis parallel to *QG-P1.QG-P3*.

The stretched center conic is a parabola through the *QA-DT*-anticomplement of *QG-P3* with an axis parallel to *QG-P1.QG-P3*.

The final line contains *QG-P14*.

QG-P15

The center conic is a parabola with an axis parallel to *QG-P2.QG-P12*.

The stretched center conic is a parabola through *QG-P14* with an axis parallel to *QG-P2.QG-P12*.

The final line is parallel to *QG-P1.QG-P3* through *QG-P15*.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Message: #224
Date: 11/9/2013 2:03:45
From: eckart_schmidt@t-online.de
Subject: Simple QL-Cu1 construction

Dear friends,
there is a simple QL-Cu1 construction:
lines L through QL-P1
lines L' = QL-Tf1(L)
 = reflections of L in the 1st Steiner axis (see QL-Tf1)
conics Co = QG-Tf2(L)
intersections of L' and Co give points on QL-Cu1.

Background is the following geometry:
Let P be an arbitrary point and L a line through P.
QL-Tf1(L) is a circle, QG-Tf2(L) is a conic.
There are 4 intersections of the circle and the conic.
3 points lie on QL-Cu1, one point on the line
QL-Tf1(P).QG-Tf2(P).
This line cuts QL-Cu1 in 3 points: The QL-Tf2 images of these
points lie concyclic with QL-P1 and P.

Best regards Eckart

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Message: #225
Date: 12/9/2013 8:55:14
From: eckart_schmidt@t-online.de
Subject: Similar construction for QA-Cu1

Dear friends,
consider a quadrigon component of a quadrangle with lines L through QG-P1.
The line QA-Tf2(L) and the circle QL-Tf1(L) intersect on QA-Cu1.
Best regards Eckart

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Message: #226
Date: 13/9/2013 8:58:44
From: Chris
Subject: Similar construction for QA-Cu1

Dear Eckart,
1. This again shows how a Quadrilateral and Quadrangle meet in a Quadrigon. Very nice!
2. Your method also gives a new construction method of the cubic QA-Cu1: the intersection $QA-Tf2(L) \wedge QL-Tf1(L)$, where L = random line through QG-P1 (intersection point diagonals Quadrigon).
Best regards,
Chris

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Message: #227

Date: 14/9/2013 8:51:16

From: seiichikiri

Subject: Algebraic quadrangle points independent of the sidelengths of the re

Dear friends,

I confirmed that quadrangle points independent of the sidelengths of the reference triangle satisfied five conditions of message#211 and derived "algebraic quadrangle points from them. "Algebraic quadrangle point" means one which does not have its corresponding geometry now:

[1] Quadrangle points ordered according to their polynomial degree.

- (1) QAP1: $x=2p+q+r$.
- (2) QAP16: $x=p(2p+q+r)$.
- (3) QAP10: $x=p(q+r)(2p+q+r)$.
- (4) QAP21: $x=(2p+q+r)(q^2+qr+r^2)$.
- (5) QAP5: $x=qr(q+r)(2p+q+r)$.
- (6) QAP20: $x=(q+r)(2p+q+r)(-p^2-pq-pr+qr)$.
- (7) QAP22: $x=(q+r)(2p+q+r)(-3p^2-3pq-3pr+qr)$.
- (8) QAP25: $x=(q+r)(2p+q+r)(2p^2+2pq+2pr+3qr)$.
- (9) QAP26: $x=(q+r)(2p+q+r)(5p^2+5pq+5pr+3qr)$.
- (10) QAP19: $x=p(2p+q+r)(q^3+q^2r+qr^2+r^3)$.
- (11) QAP31: $x=p(q+r)(2p+q+r)(p^2+pq+pr+qr+2q^2+2r^2)$.
- (12) QAP18: $x=p(2p+q+r)(p^2+pq+pr-qr)(q^2+r^2)$.
- (13) QAP17: $x=qr(2p+q+r)$
 $(-p^2q^2-pq^3-pq^2r+q^3r-p^2r^2-pqr^2+2q^2r^2-pr^3+qr^3)$.
- (14) QAP27: $x=(q+r)(2p+q+r)(q^2+qr+r^2)$
 $(-p^4-2p^3q-2p^2q^2-pq^3-2p^3r+pq^2r+q^3r-2p^2r^2+pqr^2+q^2r^2-pr^3+qr^3)$.

[2] Algebraic quadrangle points

- (1) $x=p(q+r)(2p+q+r)(p^2+pq+pr+qr)$.
- (2) $x=p(q+r)(2p+q+r)(q^2+r^2)$.

quadrangle points may contain more than an algebraic point.

Best regards,

Seiichi.

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Message: #228

Date: 14/9/2013 3:02:57

From: Chris

Subject: Algebraic quadrangle points independent of the sidelengths of th

Dear Seiichi,

Interesting way of finding new QA-points!

In Triangle Geometry it is relatively easy to make up new points.

Just find a new function $f(a,b,c)$ and there you are . . .

In Quadrangle Geometry there are several conditions to fulfill when you have a new function $f(a,b,c,p,q,r)$.

Like you pointed out the most important is that substitution in the x-coordinate with

$\{a \rightarrow \sqrt{2}, b \rightarrow 1, c \rightarrow 1, p \rightarrow -1, q \rightarrow 1, r \rightarrow 1\}$
should give zero.

This implies that every function with factor $(2p+q+r)$ makes a QA-point.

About your two new points:

(1) $x=p(q+r)(2p+q+r)(p^2+pq+pr+qr)$.

(2) $x=p(q+r)(2p+q+r)(q^2+r^2)$.

I checked nr (1) and there are no linear connections with other QA-points, neither does this point lie on known conics or cubics.

Nr (2) happens to be QA-P19.

Best regards,
Chris

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Message: #229

Date: 15/9/2013 2:17:49

From: seiichikiri

Subject: Algebraic quadrangle points independent of the sidelengths of th

Dear Chris, dear friends,

Thank you very much Chris for your comments.

I have a plan to make the conditions for quadrilateral points. I wonder why there are no linear or quadratic points (independent of the sidelengths). They are 3^n degree polynomials.

Best regards,

Seiichi.

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Message: #230

Date: 16/9/2013 4:42:06

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Chris,

I've just discovered the new properties recently added to QL-Cu1;

I must apologise, I should visit the site EQF more often ...

I wonder if you could a few more properties and may be other interesting points.

Dear Eckart, dear Bernard G.,

As I know I'm dealing with experts, I thank you in advance to check these propositions and if possible to answer my 3 final questions.

Best regards

Bernard K.

A) Properties

- 2 CL-S conjugate points are isogonal wrt any triangle inscribed in the curve QL-Cu1 ; this is a generalisation of the remark concerning the orthic triangle of the diagonal triangle. This is true in particular for the triangle QL-P1QL-2P2aQL-2P2b.
- The point where the tangents to QL-Cu1 in 2 CL-S conjugates points intersect is the tangential of the 2 points.
- Any circle through QL-P1 cuts QL-Cu1 in 3 other points : the CL-S conjugates of the 3 points are on a line and the tangentials are on a line too ; the converse is true.
- In particular, any circle through 2 CL-S conjugate points and their tangential is through QL-P1 and the line through the 2 CL-S conjugate points is the CL-S transform of the circle and cuts QL-Cu1 in a 3rd point, which is the CL-S conjugate of the tangential (the vertices of the orthic triangle of the diagonal triangle are the CL-S conjugates of the tangentials of the 3 pairs of CL-S conjugate points forming the vertices of the QL).
- Any line through QL-P1 cuts QL-Cu1 in 2 points and the middle of the segment joining the 2 points is on the Newton Line (even if the points are not CL-S conjugates).
- Any line through the point S (which is the tangential of QL-P1) cuts the curve QL-Cu1 in 2 points on equal distance from QL-P1.

B) New points

- After S and T, you could add the points U and V, the points where the circle of inversion of the CL-S transformation or Schmidt circle cuts the curve QL-Cu1 ; U and V are CL-S conjugates, symmetric of one another wrt the 1st Steiner axis and are on the line perpendicular to the 1st Steiner axis trough the point S. T is the tangential of U and V and T, U, V and QL-P1 are concyclic. QL-2P2a, QL-2P2b, U and V are concyclic, the center of the circle being the intersection between the 1st Steiner axis and the perpendicular bisector of QL-2P2a and QL-2P2b.
- UQL-2P2a and VQL-2P2b intersect in a point A and VQL-2P2a and UQL-2P2b intersect in a point A' ; A and A' are CL-S conjugates on the curve QL-Cu1 and have for tangential the infinity point of the Newton Line (meaning that the tangents to QL-Cu1 in A and A' are parallel to the Newton Line).
- Now we can have 2 pairs of points B and B' and C and C' : B and B' are CL-S conjugates , lie on the bisector of QL-P1QL-2P2aQL-2P2b, have QL-2P2a for middle and QL-2P2b for tangential and the circle through B, B' and QL-P1 is through QL-2P2b (the center of this circle is the intersection between the perpendicular bisector of QL-P1QL-2P2b and the circle circumscribed to QL-P1, QL-2P2a and QL-2P2b) ; the same construction goes for C and C' by inverting the points QL-2P2a and QL-2P2b. A, B and C' are on a line, A, C and B' and A', B' and C' are on lines too and this QL formed by the 6 points (the only one with 2 tangents to QL-Cu1 parallel to the Newton Line) could be named the main QL of QL-Cu1.

C) Three questions

- The curve QL-Cu1 has 3 inflexion points (points where the curve cuts it's tangent) ; these points are therefore their own tangential. From each of the 3 points one can draw 3 other tangents to the curve. The 3 inflexion points are on a line and for each of them, the 3 contact points with the 3 other tangents are on a line. There are totally 12 points on 4 different lines. How is it possible to find the points and lines for a given QL-Cu1 curve ?

- The points S (where the curve cuts its asymptote) and its CL-S conjugate T have the same tangential (4th intersection of the curve and the circle through S, T and QL-P1). Does this point have particular properties ?
- In general, the main QL described above has no other properties than to have 2 tangents to its van Rees circular focal cubic parallel to the Newton Line. What condition must the points QL-P1, QL-2P2a and QL-2P2b observe if we want the diagonals BB' and CC' be orthogonal to the sides AC and AB ? In this case, we have a triangle and the 4th line as the line through the feet of 2 altitudes of the triangle ...

Properties of QL-Cu1.docx

Message: #231
Date: 16/9/2013 11:23:48
From: Chris
Subject: A Quadrangle Center

Dear Angel,
Nice point Q.
I checked this point in relationship with other QA-objects.
However I couldn't find any collinearity with existing
QA-points,
neither does it lie on any known QA-Curve.
Best regards,
Chris

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Message: #232
Date: 17/9/2013 12:38:50
From: eckart_schmidt@t-online.de
Subject: Precisions concerning the curve QL-Cu1

Dear Bernard,

I am just studying your interesting paper about QL-Cu1.

First remarks:

- # If the curve is bipartite, the points T, U, V, ...
are not real.
- # If the curve is unipartite, the "main QL of QL-Cu1"
has the same QL-P1, QL-2P2a,b, QL-L1 and QL-Cu1
as the reference quadrilateral (perhaps more analogies).
- # QL-Cu1 is an isogonal circular isocubic wrt QL-P1
QL-2P2a QL-2P2b (I think up to now not mentioned).
- # If the curve is bipartite,
there is a pivot in the point at infinity of QL-L1
(see my homepage 11.1).

So far, best regards Eckart

PS. My homepage: <http://eckartschmidt.de>

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Message: #233

Date: 18/9/2013 12:39:08

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard,

there is a mistake in my last message: If QL-Cu1 is bipartite, point T exists as CL-S conjugate of S (not as tangential of U, V, as mentioned in your paper).

If QL-Cu1 is unipartite, the "main QL of QL-Cu1" can be shortened described as the quadrilateral for the points, where QL-Cu1 cuts the following lines:

outer bisector QL-2P2a QL-P1 QL-2P2b and
inner bisectors of QL-P1 QL-2P2a QL-2P2b and
QL-2P2a QL-2P2b QL-P1 (not equal QL-P1, QL-2P2a, QL-2P2b).

If QL-Cu1 is bipartite, these bisectors cut the cubic in the centers of the in- and excircles of QL-2P2a QL-P1 QL-2P2b.

If QL-Cu1 is bipartite every point P1 on the cubic can be completed by P2, P3, P4 on the cubic, so that the quadrigon P1P2P3P4 has the same QL-P1, QL-2P2a,b, QL-L1, QL-Cu1 as the reference quadrilateral (in analogy to the "main QL" in the unipartite case) in the following way:

For every triangle there is a transformation $t(P)$ wrt a reference vertice, composed of a reflection in the angle bisector at the reference vertice and an inversion wrt a circle round the reference vertice, so that the other two vertices change.

Consider now the triangle QL-2P2a QL-P1 QL-2P2b and the images $t(P1)$ for the reference vertices QL-2P2a, QL-P1, QL-2P2b as P2, P3, P4.

Background: The triangle QL-2P2a QL-P1 QL-2P2b is the Miquel Triangle QA-Tr2 of the quadrangle P1P2P3P4.

Best regards Eckart

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Message: #234

Date: 19/9/2013 10:19:25

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard,

in addition to my last message:

The last construction of the quadrigon P1P2P3P4 on QL-Cu1 also holds for the unipartite case.

Wrt question 2 in your paper:

Let X be the tangential of S and T on the cubic.

Point X is the CL-S conjugate

of the third intersection R of ST with QL-Cu1.

(This point is the second intersection of ST with the circle through T, QL-2P2a,b.)

Point X is the isogonal conjugate of R

wrt QL-2P2a QL-P1 QL-2P2b.

Point X is the second intersection of the circles

S, T, QL-P1 and S, QL-2Pa, QL-2P2b.

(First circle already mentioned in your paper.)

The tangent in S to QL-Cu1 is also tangent in S

to the circle through R, S, QL-P1.

The tangent in T to QL-Cu1 is also tangent in T

to the circle through R, T, QL-P1.

The tangent in S to QL-Cu1 is the isogonal conjugate

wrt QL-2P2a QL-P1 QL-2P2b of the conic

through R, T, QL-P1, QL-2P2a, QL-2P2b.

The tangent in T to QL-Cu1 is the isogonal conjugate

wrt QL-2P2a QL-P1 QL-2P2b

of the conic through R, S, QL-P1, QL-2P2a, QL-2P2b.

Best regards Eckart

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Message: #235

Date: 19/9/2013 7:07:01

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Eckart,

Thanks a lot for your commentars and explanations!

For the part A) Properties in my paper, I suppose you didn't find errors ... I suggest to add at the end of the first one your development about the transformation $t(P)$ (reflexion + inversion ...), which in fact holds for the unipartite case too. For the second part B) New points, I suggest to add to the points S , T , U and V the tangential W of S and T with some of the properties you pointed out (I personally prefer the intersection of the circles S , T and $QL-P1$ and S , $QL-2P2a$ and $QL-2P2b$).

Those developments were meant for EQF, but it belongs to Chris to decide about that ...

There is a mistake at the end of this part B: this main QL is obviously not the only one with 2 tangents to $QL-Cu1$ parallel to the Newton Line, but the only one to have 2 middles of diagonals in the points $QL-2P2a$ and $QL-2P2b$.

The part C) three questions was more personal: you answered marvellously the second one (point W), but not the first (inflexion points).

I would like to precise the last question. There are a lot of QL's having the points A and A' as vertices, $QL-P1$ as Miquel point and $QL-2P2a$ and $QL-2P2b$ as intersection points of $QL-Cu1$ with the Newton Line (any circle through $QL-P1$ and A gives 2 other points B and C on the curve ...). Two are particularly interesting: the first one is what I called the main QL, with the 2 points $QL-2P2a$ and $QL-2P2b$ as middles of 2 diagonals; the second one has the 2 diagonals perpendicular to the sides (the orthocenter of the triangle ABC is on the bisector of the angle $QL-2P2aQL-P1QL-2P2b$) and we could call it orthogonal QL. I'm curious to know in which conditions these 2 QL-s coincide. To be honest, I started with a triangle ABC and the altitudes BB' and CC' and have the orthogonal QL as the sides of the triangle ABC and the line through the feet B' and C' as the 4th line, which cuts BC in A' ; I soon discovered that the points $QL-2P2a$ and $2P2b$ weren't the middles of the diagonals and that it was possible to form the main QL of the same $QL-Cu1$ with same Miquel point and Newton Line ...

Thanks again for your attention

Best regards

Bernard

Message: #236

Date: 20/9/2013 4:54:25

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard,

thanks for your extensive answer. Some remarks:

If QL-Cu1 is unipartite, your "main QL" is a special one, for B, B', C, C' lie on a circle. The midpoint is the center of the QL-P1 corresponding excircle of QL-2P2a QL-P1 QL-2Pb. This holds also for the further constructions in your last message. The midpoint of the circle for your "orthogonal QL" seems to be the midpoint of BC.

If QL-Cu1 is bipartite and you define a "main QL" analogue with the intersections of the cubic with the outer bisector at QL-P1 and the inner bisectors at QL-2P2a,b wrt QL-2P2a QL-P1 QL-2P2b, then the points of the "main QL" are QL-2P2a,b and the in- and excenters of QL-2P2a QL-P1 QL-2P2b. A very special QL. Sorry, but I cannot find an analogue for the "orthogonal QL".

I prefer the following sight:

Consider quadrigons P1P2P3P4 on QL-Cu1 of a reference quadrilateral, constructed with the reference triangle QL-2P2a QL-P1 QL-2P2b

as described in message #233.

- # Opposite points of P1P2P3P4 as quadrilateral are CL-S conjugate on the cubic.
- # QL-Cu1 is an isogonal isocubic wrt the reference triangle.
- # P1P2P3P4 as quadrilateral has the same QL-Cu1 (QL-L1, QL-P1, QL-2P2a,b) as the reference quadrilateral.
- # If QL-Cu1 is bipartite,
 - ... QL-Cu1 is a pivotal isogonal circular cubic wrt the reference triangle with pivot in the point at infinity of QL-L1,
 - ... P1, P2, P3, P4 have the same tangential,
 - ... the reference triangle is the Miquel Triangle QA-Tr2 of the quadrangle P1P2P3P4,
 - ... QL-Cu1 of the reference quadrilateral is QA-Cu1 of P1P2P3P4 as quadrangle.
- # If QL-Cu1 is unipartite, P1P2P3P4 are special quadrigons:
 - ... opposite vertices of the quadrigon P1P2P3P4 are isogonal conjugate,
 - ... the diagonals are parallel,

... $QA-P4 = QL-P1$,
... intersections of opposite sides are $QL-2P2a,b$.

I hope, there are no basic mistakes!

Best regards Eckart

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Message: #237

Date: 21/9/2013 1:22:05

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard, dear Chris,

I think it's remarkable, that QL-Cu1 is isogonal invariant wrt the reference triangle QL-2P2a QL-P1 QL-2P2b = QL-Trx.

Let $\wedge$ denote the Clawson-Schmidt conjugate and $*$ the isogonal conjugate wrt QL-Trx.

If X is a point on QL-Cu1, then holds $X^* = X^\wedge$.

If X is not a point on QL-Cu1,
the line $X^\wedge X^*$ contains QL-P1.

For a line L the image $L^\wedge$ is a circle through QL-P1
and L^* is a conic through QL-P1,
... both have 4 common points on QL-Cu1
(one of them QL-P1),
... the Simson line of QL-P1 wrt the further three common
points is perpendicular QL-L1.

For a side line L_i the image L_i^* is a common circumconic
of the triangles QL-Trx and $L_j L_k L_l$.

$X^\wedge X^* = X^* X^\wedge$ and $X.X^\wedge X^* = X.X^* X^\wedge$ contains QL-P1.

For X and $X^\wedge = X^*$ on QL-Cu1
... the circumconics of the triangle components through these
two points have a further common point Y on the cubic,
... $X, Y, X^\wedge = X^*$, and QL-P1 lie concyclic,
... $Y^\wedge = Y^*$ is the third intersection of $X^* X^\wedge$ with QL-Cu1.

For X not on QL-Cu1 ($X^\wedge \neq X^*$)
... the circumconics of the triangle components
through X and $X^\wedge$ have a further common point Y ,
... $X, X^\wedge, Y$ and QL-P1 are concyclic,
 $X, X^\wedge, Y^\wedge$ are collinear,
... the circumconics of the triangle components
through X and X^* have a further common point Z ,
... X, X^*, Z^* are collinear.

There will be more properties.

Best regards Eckart

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Message: #238

Date: 21/9/2013 2:14:25

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard, dear Chris,

sorry, the properties in my last message are only valid in the unipatite case of QL-Cu1.

In the bipartite case there are significant corrections!

I will describe it later.

But QL-Cu1 is also in the bipartite case isogonal invariant wrt QL-2P2a QL-P1 QL-2P2b.

Excuse, Eckart

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Message: #239

Date: 22/9/2013 12:07:43

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard, dear Chris,

I repeat: It is remarkable, that QL-Cu1 is isogonal invariant wrt the reference triangle QL-2P2a QL-P1 QL-2P2b = QL-Trx.

The unipartite case of QL-Cu1 (see also my last message):
Let $\wedge$ denote the Clawson-Schmidt conjugate and $*$ the isogonal conjugate wrt QL-Trx.

- # If X is a point on QL-Cu1, then holds $X^* = X^\wedge$.
- # If X is not a point on QL-Cu1,
the line $X^\wedge X^*$ contains QL-P1.
- # For a line L the image $L^\wedge$ is a circle through QL-P1
and L^* is a conic through QL-P1,
... both have 4 common points on QL-Cu1 (one of them QL-P1),
... the Simson line of QL-P1
wrt the further three common points is perpendicular
QL-L1.
- # For a side line L_i the image L_i^* is a common circumconic
of the triangles QL-Trx and $L_j L_k L_l$.
- # $X^\wedge X^* = X^* X^\wedge$ collinear with X , QL-P1.
- # For X and $X^\wedge X^*$ on QL-Cu1
... the circumconics of the triangle components through these
two points have a further common point Y on the cubic,
... X , Y , $X^\wedge X^*$, and QL-P1 lie concyclic,
... $Y^\wedge Y^*$ is the third intersection of $X^* X^\wedge$ with QL-Cu1.
- # For X not on QL-Cu1 ($X^\wedge \neq X^*$)
... the circumconics of the triangle components
through X and $X^\wedge$ have a further common point Y ,
... X , $X^\wedge$, Y and QL-P1 are concyclic, X , $X^\wedge$, $Y^\wedge$ are collinear,
... the circumconics of the triangle components
through X and X^* have a further common point Z ,
... X , X^* , Z^* are collinear.

The bipartite case of QL-Cu1:

- # The line $X^\wedge X^*$ contains QL-P1.
- # For a line L the image $L^\wedge$ is a circle through QL-P1
and L^* is a conic through QL-P1,
... the Simson line of QL-P1 wrt the further three
common points of $L^\wedge$ and L^* is parallel QL-L1.
- # For a side line L_i the image L_i^* is a circumconic of QL-Trx.
Further intersections of these conics lie on QL-Cu1
as third intersections of the the lines through QL-P1
and a point of the quadrilateral.

```

#  $X^* = X^\wedge$  collinear with  $X$ ,  $QL-P1$ .
# For  $X$ ,  $X^\wedge$ ,  $X^*$  on  $QL-Cu1$ 
... the circumconics of the triangle components
    through  $X$  and  $X^\wedge$  have a further common point  $Y$  on the
    cubic,
...  $X$ ,  $Y$ ,  $X^\wedge$ , and  $QL-P1$  lie concyclic,
... the circumconics of the triangle components
    through  $X$  and  $X^*$  have a further common point
    in  $QL-P1$  on the cubic,
# For  $X$ ,  $X^\wedge$ ,  $X^*$  not on  $QL-Cu1$ 
... the circumconics of the triangle components
    through  $X$  and  $X^\wedge$  have a further common point  $Y$ .
...  $X$ ,  $Y$ ,  $X^\wedge$  and  $QL-P1$  are concyclic,
... the circumconics of the triangle components
    through  $X$  and  $X^*$  have a further common point  $Z$ 
... Apparently there are more interesting properties
    in the unipartite case.

```

Best regards Eckart

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Message: #240

Date: 23/9/2013 3:57:26

From: seiichikiri

Subject: colinearity and cyclic sums of quadrangle points

Dear friends,

I observed the following about colinearity and cyclic sums of quadrangle points.

Cyclic sum means $f(p,q,r)+f(q,r,p)+f(r,p,q)$.

[1] the following points are colinear as QA-L3. Cyclic sums:

- (1) QAP1: $4(p+q+r)$.
- (2) QAP10: $3(p+q)(q+r)(r+p)$.
- (3) QAP5: $(p+q)(q+r)(r+p)(p+q+r)$
- (4) QAP20: $-2(p+q)(q+r)(r+p)(p+q+r)$.
- (5) QAP22: $-8(p+q)(q+r)(r+p)(p+q+r)$.
- (6) QAP25: $9(p+q)(q+r)(r+p)(p+q+r)$.
- (7) QAP26: $18(p+q)(q+r)(r+p)(p+q+r)$.

[2] the following points are colinear. Cyclic Sums:

- (1) QAP10: $3(p+q)(q+r)(r+p)$.
- (2) QAP16: $2(p^2+q^2+r^2+qr+rp+pq)$.
- (3) QAP19: $(p+q)(q+r)(r+p)(p^2+q^2+r^2+qr+rp+pq)$.
- (4) QAP31: $4(p+q)(q+r)(r+p)(p^2+q^2+r^2+qr+rp+pq)$.

We can think that colinear points are derived from 2 points of the simplest cyclic sums.

Best regards,
Seiichi.

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Message: #241

Date: 23/9/2013 5:08:12

From: yeuemtrondoitb85

Subject: A generalization Brahmagupta's Theorem

Dear all Member!

Let a quadrilateral ABCD,
let E point and $\text{Angle}(\text{DEA}) + \text{Angle}(\text{CEB}) = 180^\circ$ and $\text{DE} \cdot \text{EB} = \text{EC} \cdot \text{EA}$.
F is midpoint of DC.
EF meet AB at G.
F, I are midpoints of CD, AB respectively.
K, L, M are circumcircle of (EDA), (EIG), (ECB)

Prove that

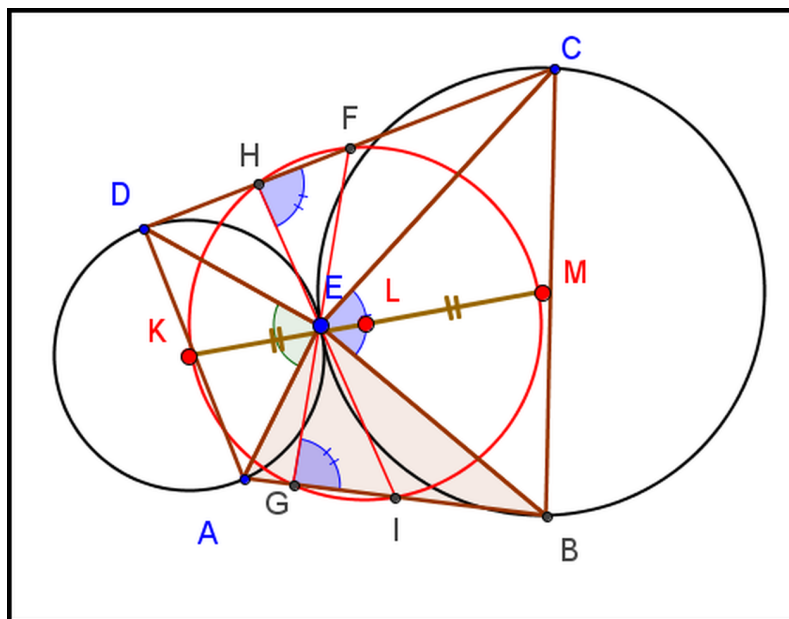
1- $\text{Angle DEA} = \text{Angle AGE}$

2- L is midpoints of KM

I'm not arithmetician, I want you proved and publish it.

Best regard

Dao Thanh Oai



A generalization Brahmagupta's Theorem.png

Message: #242

Date: 23/9/2013 5:10:10

From: yeuemtrondoitb85

Subject: A generalization Brahmagupta's Theorem

Dear all Member!

Let a quadrilateral ABCD,

let E point and

angle DEA + angle CEB = 180 deg and

$DE * EB = EC * EA$.

F is midpoint of DC.

EF meet AB at G. F, I are midpoints of CD, AB respectively.

K, L, M are circumcircle of (EDA), (EIG), (ECB)

Prove that

1- angle DEA = angle AGE

2- L is midpoints of KM

I'm not arithmetician, I want you proved and publish it.

Best regard

Dao Thanh Oai

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Message: #243

Date: 23/9/2013 10:20:30

From: Chris van Tienhoven

Subject: A generalization Brahmagupta's Theorem

Dear Dao Thanh Oai,

Your generalization of Brahmagupta's Theorem is very interesting.

I checked the locus of points with the property

$$\angle DEA + \angle CEB = 180 \text{ degree.}$$

Surprisingly it is the cubic QL-Cu1.

Note that this is a property belonging to a Quadrilateral (system of 4 random lines) and not to a Quadrangle (system of 4 random points).

When we define a Reference Quadrilateral as a system of 4 random line, then we can discern 3×2 sets of 2 opposite line segments.

For each 2 sets of 2 opposite line segments (e.g. AB/CD, AC/BD in your example) we have the rule that the sum of the angles formed from E to this line segments are $180 \bmod 180$, where the angles can be negative depending on orientation.

The property for points E that:

* $DE \cdot EB = EC \cdot EA$ is valid for some selected points on QL-Cu1.

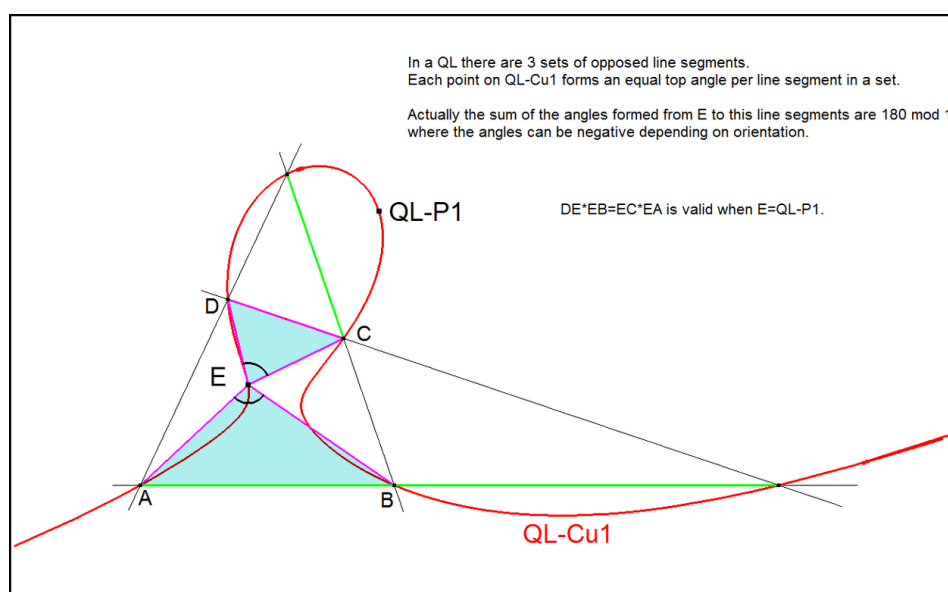
The Miquel Point QL-P1 is one of these points!

The property for points E that:

* $\angle BGE = \angle CHE = \angle CEB$ is valid for another selection of points on QL-Cu1.

Best regards,

Chris van Tienhoven



QL-Cu1-BrahmaguptaTheorem-01.png

Message: #244

Date: 23/9/2013 10:49:25

From: eckart_schmidt@t-online.de

Subject: A generalization Brahmaguptas Theorem

Dear mister Dao Thanh Oai, dear Chris,

only a short remark: Points E of a quadrigon with the angle property $\angle DEA + \angle BEC = 180^\circ$ lie on a section of QL-Cu1. This will give a new property of QL-Cu1 wrt a quadrigon as locus for points X with $\angle P_1XP_2 + \angle P_3XP_4 = 0^\circ$ or 180° .

Best regards Eckart

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Message: #245

Date: 23/9/2013 9:27:34

From: Chris van Tienhoven

Subject: A generalization Brahmagupta's Theorem

Dear friends,

There are 2 different cubics with the property that the sum of the angles formed from E to the opposite QL-line segments are $180 \bmod 180$.

I checked the cubics with the property that $\cos[E_1]^2 - \cos[E_2]^2 = 0$.

The first one is the well-known cubic QL-Cu1.

The 2nd one is a new cubic QL-Cux with CT-equation:

$$-c^2 l n x^2 y - c^2 m n x y^2 + l (b^2 m + a^2 n - b^2 n - c^2 n) x^2 z + m (a^2 l + b^2 l - c^2 l + a^2 n - b^2 n - c^2 n) x y z + a^2 l m y^2 z + (a^2 l + b^2 l - c^2 l - b^2 m) n x z^2 + a^2 l n y z^2$$

Somehow the equation is not quite symmetric although it is valid for all 3 QL-Quadrilaterals.

But it passes through the Plücker Pair of points QL-2P1a and QL-2P1b and also through QG-P1 (this looks like a separate point).

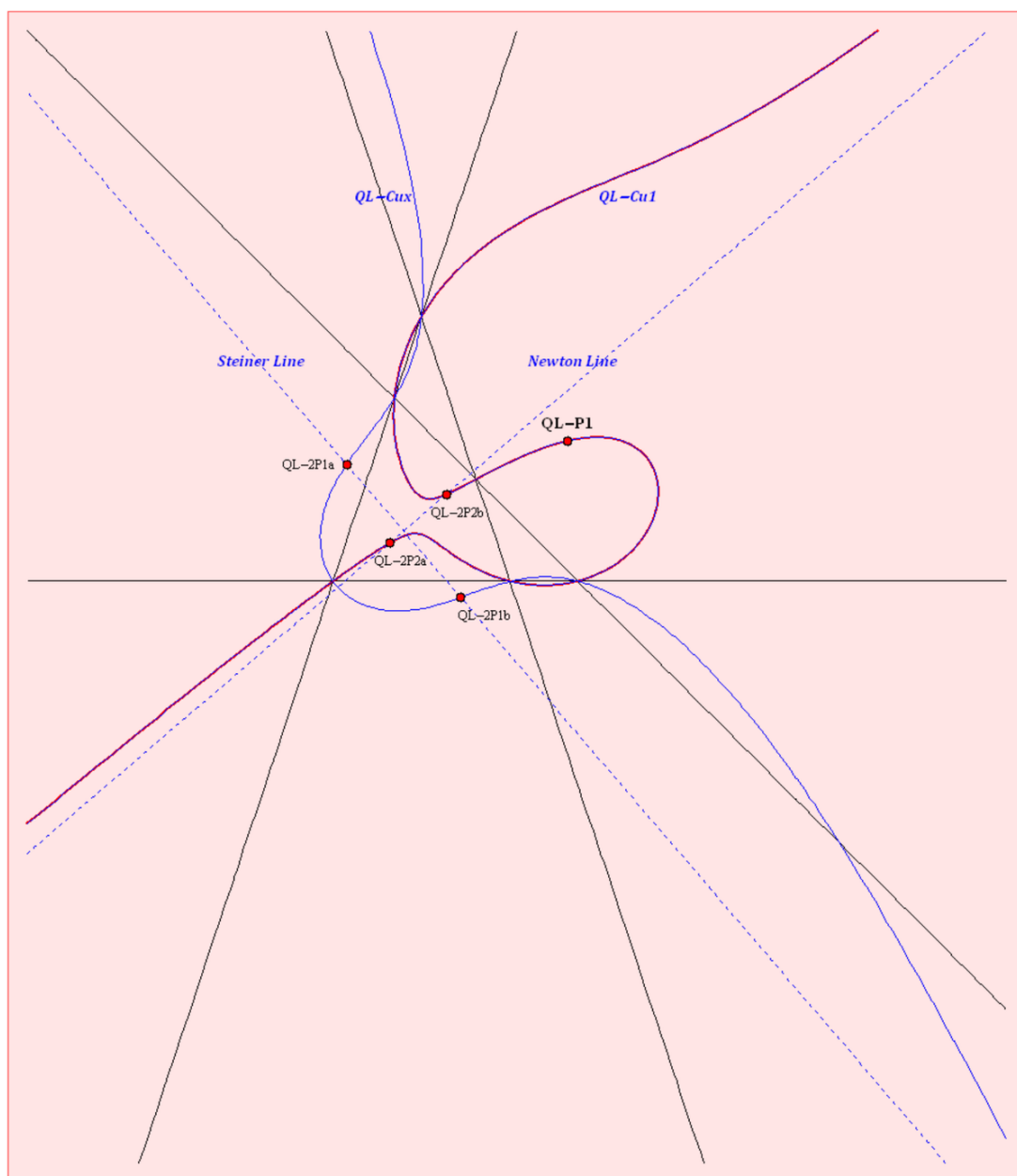
The Plücker Pair of points QL-2P1a and QL-2P1b are easy to verify that the formed angles are "complementary".

See attachment.

Best regards,

Chris

2 Cubics with points that form "complementary angles" with QL-line segments



QL-Cux-01.pdf

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Message: #246
Date: 23/9/2013 9:48:55
From: Eisso J. Atzema
Subject: interesting paper?

Dear Bernard,

In 1937, Jules Marchand also had a paper with the exact same title in the following less obscure publication by the University of Lausanne (where Marchand was a professor at the time):

Recueil de travaux publiés à l'occasion du quatrième centenaire de la fondation de l'Université juin MCMXXXVIII. Faculté des Sciences (Lausanne, 1937)

I bet this is the same article. There are several libraries in France that have a copy of the Recueil (Paris, Toulouse, Montpellier, Dijon) (see Sudoc.fr). I can see if I can order a copy of Marchand's paper through Interlibrary Loan as there are at least six libraries in the US that own the Recueil.

best,
Eisso

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Message: #247

Date: 24/9/2013 8:55:37

From: Dao Thanh Oai

Subject: A generalization Brahmagupta's Theorem

Dear Chris van Tienhoven

Let a quadrilateral ABCD,
let E point and $\angle DEA + \angle CEB = 180^\circ$ deg and $DE \cdot EB = EC \cdot EA$.
F is midpoint of DC.
EF meet AB at G.
F, I are midpoints of CD, AB respectively.
K, L, M are circumcircle of (EDA), (FIG), (ECB)

Prove that
1- $\angle DEA = \angle AGE$
2-L is midpoints of KM

I'm not mathematicians.
I want you publish it.
Because I don't know QL-cu1.
Its is yours

Best regard
Dao Thanh Oai

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Message: #248

Date: 25/9/2013 9:26:11

From: yeuemtrondoitb85

Subject: May be a new property of cyclic quadrilateral?

Dear all Member!

I have a question to you for this problem:

Let a cyclic quadrilateral ABCD and G is circumcircle of (ABCD).

BC meets AD at F.

CD meets BA at E.

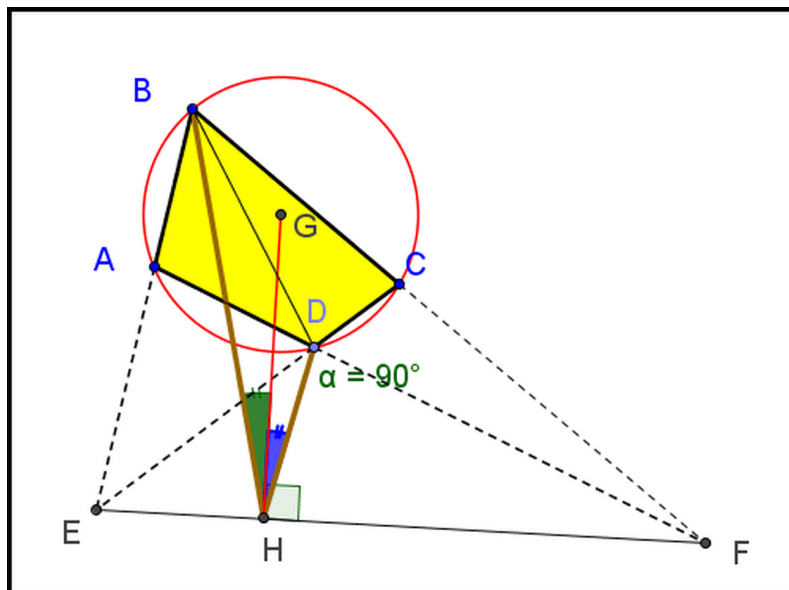
H is on EF and GH perpendicular with EF.

Prove that HG is angle bisector of angle BHD

Maybe a new property of cyclic quadrilateral?

Best regards

Dao Thanh Oai



May be a new property of cyclic quadrilateral.png

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Message: #249

Date: 25/9/2013 11:46:58

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Eckart,

1) In the unipartite case, for the main QL in fact, B , B' , C and C' are concyclic and the circumcenter is on 3 remarkable circles: through QL-P1, B , B' and QL-2P2b, through QL-P1, C , C' and QL-2P2a and through QL-2P2a, QL-2P2b, U and V ; on this last circle, it's the opposite of the incenter of the triangle QL-P1QL-2P2aQL-2P2b and, as you say, it's one of the excenters of the same triangle. For the orthogonal QL, this point is the middle of BC , but I haven't found yet in which conditions these 2 QL's coincide ...

2) If I'm not wrong, in the bipartite case, the points QL-2P2a and QL-2P2b aren't real ...

3) Do you know a way to draw the inflexion points of QL-Cu1 in the unipartite case?

Best regards
Bernard

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Message: #250

Date: 26/9/2013 1:35:55

From: Antreas Hatzipolakis

Subject: REFERENCE

Dear Chris

The paper in your reference:

[16] Daniel Baumgartner, Roland Stärk, Ein merkwürdiger Punkt des Vierecks, available at

<http://mathematicus.ch/geometrie/empdv.pdf>
is no longer available.

Daniel Baumgartner's web site is at:

<http://www.geometria.ch/>

and the paper is available at:

<http://www.geometria.ch/geometrie/empdv.pdf>

Antreas

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Message: #251

Date: 26/9/2013 11:22:24

From: bernard.keizer

Subject: May be a new property of cyclic quadrilateral?

Dear Dao Thanh Oai,

For a cyclic quadrilateral, the circumcenter is the orthocenter of the diagonal triangle.

From the vertices of the orthic triangle of this diagonal triangle, you see the sides AB and CD under the same angle and these vertices belong to the cubic QA-Cu1 ...

You may find demos and developments in Cyclic Quadrilaterals - The Big Picture Yufei Zhao

Best regards

Bernard Keizer

>

> ---In Quadri-Figures-Group@yahoogroups.com, wrote:

> Dear all Member!

> I have a question to you for this problem:

> Let a cyclic quadrilateral ABCD and G is circumcircle of

> (ABCD). BC meets AD at F. CD meets BA at E.

> H is on EF and GH perpendicular with EF.

> Prove that HG is angle bisector of angle BHD

> May be a new property of cyclic quadrilateral?

> Best regards

> Dao Thanh Oai

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Message: #252

Date: 26/9/2013 12:42:50

From: seiichikiri

Subject: colinearity and cyclic sums of quadrangle points

Dear friends,

Addition (an algebraic QA point derived from QAP17)

(1) QAP17: $\{qr(2p+q+r)(-p^2q^2-pq^3-pq^2r+q^3r-p^2r^2-pqr^2+2q^2r^2-pr^3+qr^3);; \}$.

Cyclic sum =

$-(p+q+r)(p^2+p(q+r)-qr)(q^2+q(r+p)-rp)(r^2+r(p+q)-pq)$.

(2) QAPX: $\{p(2p+q+r)(p^2(q-r)^2+p(q-r)^2(q+r)-qr(q^2+r^2))\}$.

Cyclic sum = $-3(p^2+p(q+r)-qr)(q^2+q(r+p)-rp)(r^2+r(p+q)-pq)$.

The factor $(p+q+r)$ of the cyclic sum of QAP17 are eliminated.

See the attached Mathematica file.

Best regards,

Seiichi.

>

> ---In quadri-figures-group@yahoogroups.com, wrote:

> Dear friends,

> I observed the following about colinearity and cyclic sums of quadrangle

> points.

> Cyclic sum means $f(p,q,r)+f(q,r,p)+f(r,p,q)$.

> [1] the following points are colinear as QA-L3. Cyclic sums:

> (1) QAP1: $4(p+q+r)$.

> (2) QAP10: $3(p+q)(q+r)(r+p)$.

> (3) QAP5: $(p+q)(q+r)(r+p)(p+q+r)$

> (4) QAP20: $-2(p+q)(q+r)(r+p)(p+q+r)$.

> (5) QAP22: $-8(p+q)(q+r)(r+p)(p+q+r)$.

> (6) QAP25: $9(p+q)(q+r)(r+p)(p+q+r)$.

> (7) QAP26: $18(p+q)(q+r)(r+p)(p+q+r)$.

> [2] the following points are colinear. Cyclic Sums:

> (1) QAP10: $3(p+q)(q+r)(r+p)$.

> (2) QAP16: $2(p^2+q^2+r^2+qr+rp+pq)$.

> (3) QAP19: $(p+q)(q+r)(r+p)(p^2+q^2+r^2+qr+rp+pq)$.

> (4) QAP31: $4(p+q)(q+r)(r+p)(p^2+q^2+r^2+qr+rp+pq)$.

> We can think that colinear points are derived from 2 points of the simplest cyclic sums.

> Best regards,

> Seiichi.

```

(* confirmation of 5 conditions *)
(* QAP17 *)
f1[1] = q r (2 p + q + r)
      (-p^2 q^2 - p q^3 - p q^2 r + q^3 r - p^2 r^2 - p q r^2 + 2 q^2 r^2 - p r^3 + q r^3);
f1[2] = f1[1] /. {p -> q, q -> r, r -> p};
f1[3] = f1[2] /. {p -> q, q -> r, r -> p};
F1 = Sum[f1[i], {i, 3}] // Factor // Simplify
f2[1] = f1[1] /. {p -> p + q + r, q -> -q, r -> -r};
f2[2] = f1[2] /. {p -> p + q + r, q -> -q, r -> -r};
f2[3] = f1[3] /. {p -> p + q + r, q -> -q, r -> -r};
F2 = Sum[f2[i], {i, 3}] // Factor // Simplify
f3[1] = f1[1] /. {p -> p + q + r, q -> -r, r -> -p};
f3[2] = f1[2] /. {p -> p + q + r, q -> -r, r -> -p};
f3[3] = f1[3] /. {p -> p + q + r, q -> -r, r -> -p};
F3 = Sum[f3[i], {i, 3}] // Factor // Simplify
f4[1] = f1[1] /. {p -> q, q -> r, r -> p};
f4[2] = f1[2] /. {p -> q, q -> r, r -> p};
f4[3] = f1[3] /. {p -> q, q -> r, r -> p};
F4 = Sum[f4[i], {i, 3}] // Factor // Simplify
f5[1] = f1[1] /. {p -> q, q -> p, r -> -p - q - r};
f5[2] = f1[2] /. {p -> q, q -> p, r -> -p - q - r};
f5[3] = f1[3] /. {p -> q, q -> p, r -> -p - q - r};
F5 = Sum[f5[i], {i, 3}] // Factor // Simplify
f6[1] = f1[1] /. {p -> -r, q -> -p, r -> p + q + r};
f6[2] = f1[2] /. {p -> -r, q -> -p, r -> p + q + r};
f6[3] = f1[3] /. {p -> -r, q -> -p, r -> p + q + r};
F6 = Sum[f6[i], {i, 3}] // Factor // Simplify
f7[1] = f1[1] /. {p -> -r, q -> p + q + r, r -> -q};
f7[2] = f1[2] /. {p -> -r, q -> p + q + r, r -> -q};
f7[3] = f1[3] /. {p -> -r, q -> p + q + r, r -> -q};
F7 = Sum[f7[i], {i, 3}] // Factor // Simplify
f8[1] = f1[1] /. {p -> p, q -> -p - q - r, r -> q};
f8[2] = f1[2] /. {p -> p, q -> -p - q - r, r -> q};
f8[3] = f1[3] /. {p -> p, q -> -p - q - r, r -> q};
F8 = Sum[f8[i], {i, 3}] // Factor // Simplify
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Simplify

(p + q + r) (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))
p (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))
q (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))
(p + q + r) (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))
-r (p^2 - q r + p (q + r)) (p (-q + r) - q (q + r)) (p (-q + r) + r (q + r))
q (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))

```

```

p (p^2 - q r + p (q + r)) (p (q - r) + q (q + r)) (p (q - r) - r (q + r))
-r (p^2 - q r + p (q + r)) (p (-q + r) - q (q + r)) (p (-q + r) + r (q + r))

0
0
0
0
0

(* QAPX *)
f1[1] = p (2 p + q + r) (p^2 (q - r)^2 + p (q - r)^2 (q + r) - q r (q^2 + r^2));
f1[2] = f1[1] /. {p -> q, q -> r, r -> p};
f1[3] = f1[2] /. {p -> q, q -> r, r -> p};
F1 = Sum[f1[i], {i, 3}] // Factor // Simplify
f2[1] = f1[1] /. {p -> p + q + r, q -> -q, r -> -r};
f2[2] = f1[2] /. {p -> p + q + r, q -> -q, r -> -r};
f2[3] = f1[3] /. {p -> p + q + r, q -> -q, r -> -r};
F2 = Sum[f2[i], {i, 3}] // Factor // Simplify
f3[1] = f1[1] /. {p -> p + q + r, q -> -r, r -> -p};
f3[2] = f1[2] /. {p -> p + q + r, q -> -r, r -> -p};
f3[3] = f1[3] /. {p -> p + q + r, q -> -r, r -> -p};
F3 = Sum[f3[i], {i, 3}] // Factor // Simplify
f4[1] = f1[1] /. {p -> q, q -> r, r -> p};
f4[2] = f1[2] /. {p -> q, q -> r, r -> p};
f4[3] = f1[3] /. {p -> q, q -> r, r -> p};
F4 = Sum[f4[i], {i, 3}] // Factor // Simplify
f5[1] = f1[1] /. {p -> q, q -> p, r -> -p - q - r};
f5[2] = f1[2] /. {p -> q, q -> p, r -> -p - q - r};
f5[3] = f1[3] /. {p -> q, q -> p, r -> -p - q - r};
F5 = Sum[f5[i], {i, 3}] // Factor // Simplify
f6[1] = f1[1] /. {p -> -r, q -> -p, r -> p + q + r};
f6[2] = f1[2] /. {p -> -r, q -> -p, r -> p + q + r};
f6[3] = f1[3] /. {p -> -r, q -> -p, r -> p + q + r};
F6 = Sum[f6[i], {i, 3}] // Factor // Simplify
f7[1] = f1[1] /. {p -> -r, q -> p + q + r, r -> -q};
f7[2] = f1[2] /. {p -> -r, q -> p + q + r, r -> -q};
f7[3] = f1[3] /. {p -> -r, q -> p + q + r, r -> -q};
F7 = Sum[f7[i], {i, 3}] // Factor // Simplify
f8[1] = f1[1] /. {p -> p, q -> -p - q - r, r -> q};
f8[2] = f1[2] /. {p -> p, q -> -p - q - r, r -> q};
f8[3] = f1[3] /. {p -> p, q -> -p - q - r, r -> q};
F8 = Sum[f8[i], {i, 3}] // Factor // Simplify
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Simplify

```

$$\begin{aligned}
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 3 \left(p^2 - q r + p (q + r) \right) \left(p (q - r) + q (q + r) \right) \left(p (q - r) - r (q + r) \right) \\
& 0 \\
& 0 \\
& 0 \\
& 0 \\
& 0
\end{aligned}$$

algebraic QA-Point-nb.pdf

Message: #253

Date: 26/9/2013 10:07:58

From: Chris

Subject: colinearity and cyclic sums of quadrangle points

Dear Seiichi,

This point QAPX:

$\{p(2p+q+r)(p^2(q-r)^2+p(q-r)^2(q+r)-q\ r(q^2+r^2))$

is:

$QA-P1.QA-P17 \wedge QA-P10.QA-P16 \wedge QA-P18.QA-P21$

Accidentally this point is already mentioned in QA/5:

List of QA-Crosspoints in EQF.

Best regards,

Chris

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Message: #254

Date: 27/9/2013 9:05:43

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard, dear Chris,

back from some days holiday in Denmark, I have only an explanation wrt Bernard's second question:

Only now I have seen, that Chris' definition of QL-2P2a,b is not my use of this points. If we take the first line of the definition in EQF, Bernard is right: These points are not real in the bipartite case.

Up to now I interpreted QL-2P2a,b in the sense of the fourth line as foci of an inscribed conic with center in the intersection of QL-L1 and QL-L6. In this interpretation you have to read my messages.

In the bipartite case these foci lie on a perpendicular line wrt QL-L1 through the intersection of QL-L1 and QL-L6.

I would appreciate, if Chris changes the definition.

These two foci F1, F2 are always real and of significance for simple "strophoid" constructions for QL-Cu1 (see earlier messages):

If F1, F2 are points on QL-L1 (QL-Cu1 unipartite), the intersections of of circles round points M on QL-L1 orthogonal to the Thales circle about F1F2 cut the line M.QL-P1 in points of QL-Cu1.

If F1, F2 are not points on QL-L1 (QL-Cu1 bipartite), the intersections of of circles round points M on QL-L1 through F1, F2 cut the line M.QL-P1 in points of QL-Cu1.

Best regards Eckart

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Message: #255

Date: 27/9/2013 1:57:31

From: Chris van Tienhoven

Subject: A generalization Brahmagupta's Theorem

Dear Dao,

Unfortunately I am not able to solve your problem synthetically yet.

Probably when I could I also would find a construction method. When I would have a construction method I probably would be able to solve this problem synthetically.

The point(s) E you describe have these properties:
they form supplementary angles with opposite line segments of a quadrilateral.

$EA * EC = EB * ED$, where A,B,C,D, are concyclic.

Circumcenters K,M,L of resp. triangles EDA, EBC and FGHI are collinear and $LM=LK$.

Condition 1 tells us that point E lies on cubic QL-Cu1.

Condition 2 tells us that point E lies on another cubic.

Condition 3 tells us that point E should lie on a curve of 5th degree.

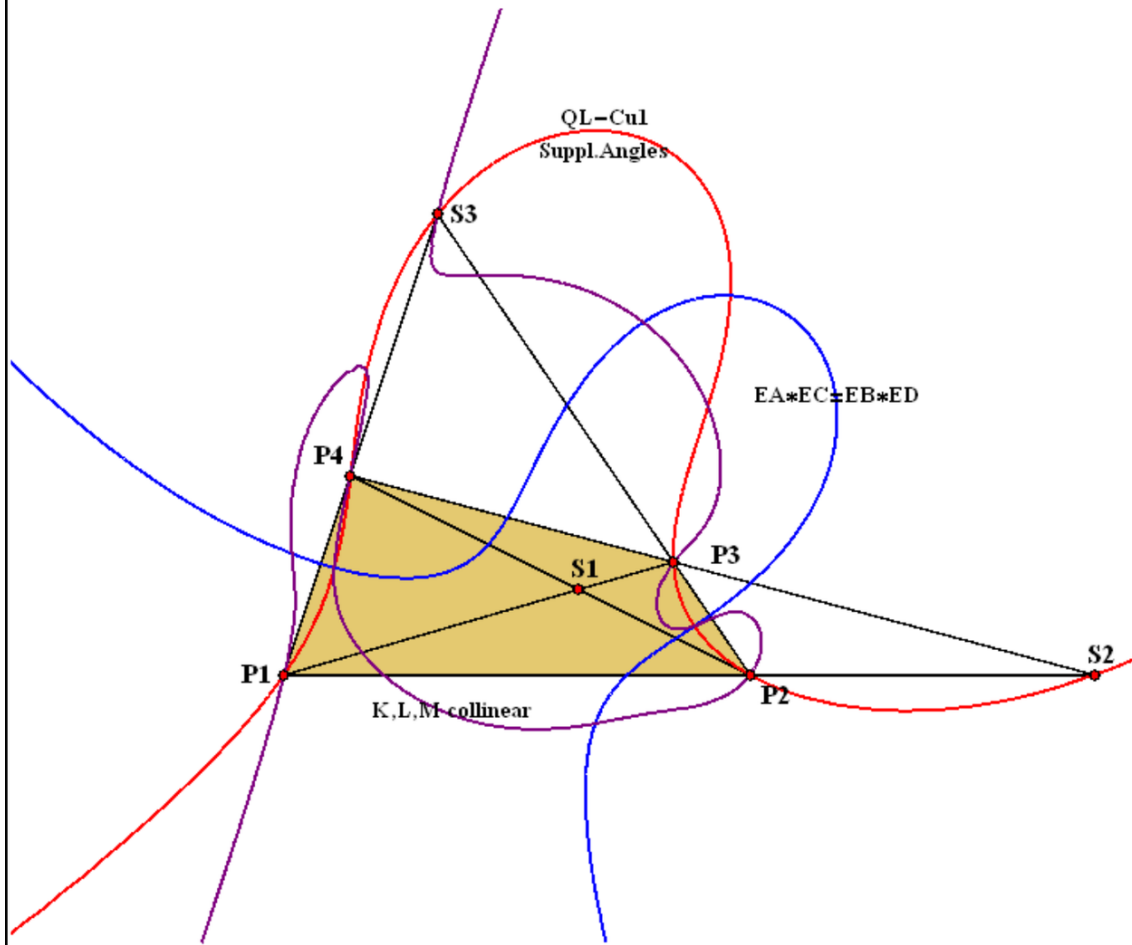
See attached picture for these curves and how they coincide in 2 points.

The question is how to construct these 2 (or more) points with these conditions.

Algebraically I can't find a way either to solve this problem. Anyone else with a good idea?

Best regards,
Chris

A generalization of Brahmagupta's Theorem



A generalization of Brahmagupta's Theorem.pdf

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Message: #256

Date: 27/9/2013 2:02:07

From: Chris van Tienhoven

Subject: A generalization Brahmagupta's Theorem

Dear Friends,

I explored the new cubic a bit further.

I already told that the equation wasn't symmetric.

Now it appears that it is a sextet of QL-cubics.

They can be constructed in a similar way as QL-Cu1.

Construction:

Let L_1, L_2, L_3, L_4 be the 4 defining lines of a Reference Quadrilateral.

Define a pair of opposite QL-line segments = the line segments on L_i, L_j intersected by L_k, L_l , where (i, j, k, l) are elements of $(1, 2, 3, 4)$.

There are 6 pairs of opposite QL-line segments in a Quadrilateral (2 pairs per QL-Quadrilateral).

Erect isosceles triangles at opposite pair of QL-line segments with same top-angle.

Define V-circle = circle centered at the toppoint of this triangle and going through the endpoints of the QL-line segment.

Construct a V-circle-1 at one QL-line segment.

Construct similar V-circles-3 and 4 at both sides of the opposite QL-line segment.

The locus of V-circle-1 intersected with V-circles-3 and 4 will produce two different cubics. One cubic will be QL-Cu1.

The other one is one version of a sextet of cubics QL-6Cu1a.

All 6 cubics are making equal/supplementary angles with the opposite QL-line segments.

They all pass through 5 of 6 vertices of the Reference Quadrilateral.

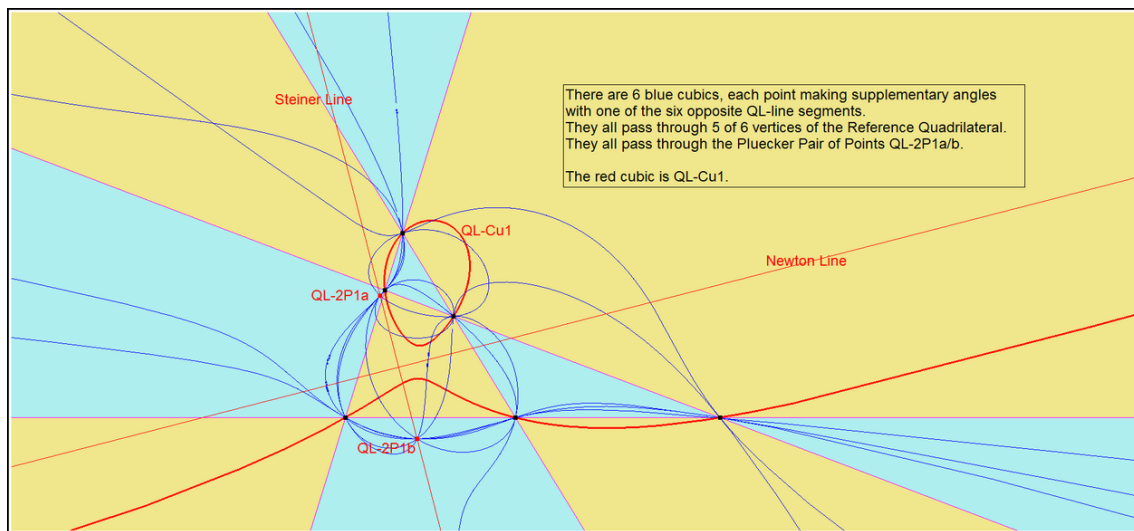
Through each vertex of the Reference Quadrilateral are passing 5 versions of QL-6Cu1.

They all pass through the Pluecker Pair of Points QL-2P1a/b.

See attachment.

Best regards,

Chris



QL-Cu1-Quasi Isogonal Cubic-60-extraCubics.png

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Message: #257

Date: 27/9/2013 3:32:22

From: Dao Thanh Oai

Subject: A generalization Brahmagupta's Theorem

Thank you Chris van Tienhoven very very very much!

You can construct converse for this problem.

Step 1-Construct a triangle ABC

Step 2-Construct a triangle ABD

Step 3-Construct C' and $\angle CAC' = 180^\circ - \angle BAD$

Step 4-Construct a line through AC'

Step 5-Construct a circle with center A and radii $AD \cdot AC / AB$

Step 6-The circle meet AC' at E

Step 7-Rename A by E, B by A, C by B, E by C

Note ABCD don't need cyclic

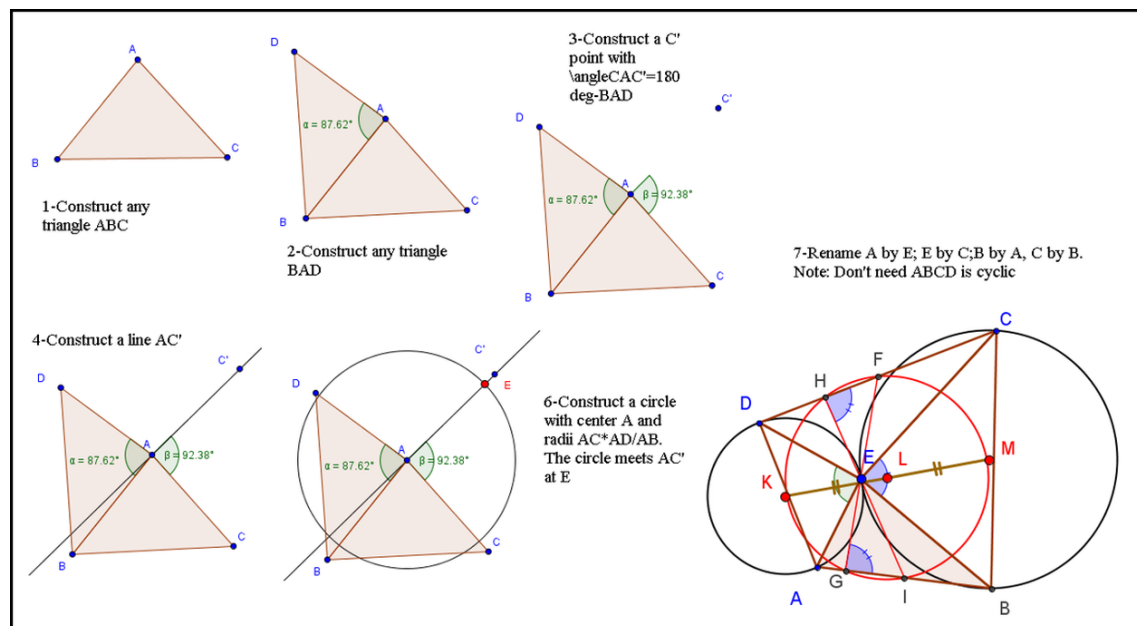
You see it at:

http://data.webdien.com/photo/up/873493f895fd58900b3698809df41a4_d.bmp

or Figure attachment.

Best regard

Dao Thanh Oai



Converse construct A generalization Brahmagupta's Theorem-bmp.png

Message: #258

Date: 27/9/2013 4:29:05

From: eckart_schmidt@t-online.de

Subject: A generalisation Brahmagupta's Theorem

Dear Dao Thanh Oai, dear Chris,
there is synthetical proof for the first property in the
attachment.

Best regards Eckart

A generalization Brahmagupta's Theorem

Quadri-Figures Group message #241 (22.09.2013)

Dear Chris van Tienhoven

Let a quadrilateral $ABCD$, let E point and $\angle DEA + \angle CEB = 180^\circ$ deg and $DE \cdot EB = EC \cdot EA$. F is midpoint of DC . EF meet AB at G . F, I are midpoints of CD, AB respectively. K, L, M are circumcircle of $(EDA), (FIG), (ECB)$

Prove that

1- $\angle DEA = \angle AGE$

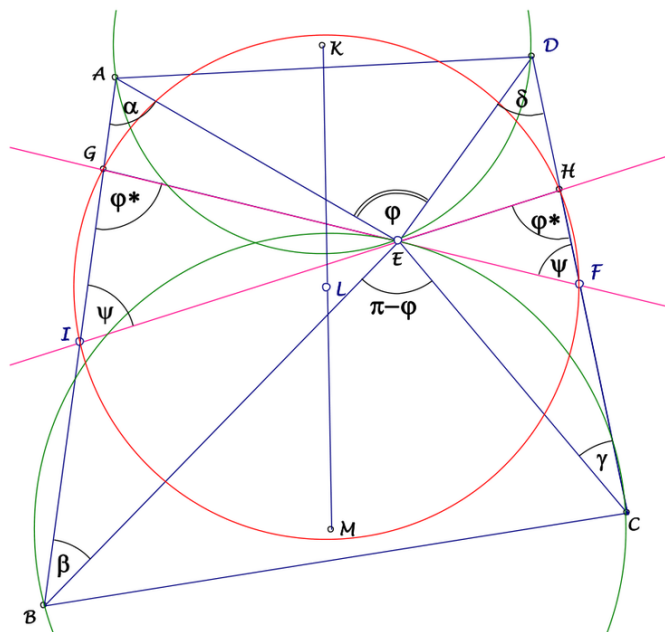
2- L is midpoints of KM

I'm not mathematicians. I want you publish it.

Best regard

Dao Thanh Oai

A synthetical prove of the first property:



Brahmagupta.pdf

1. $EA \cdot EC = EB \cdot ED \Leftrightarrow \sin \alpha \cdot \sin \gamma = \sin \beta \cdot \sin \delta$ (sine rule)
2. $\varphi + \varphi^* = \alpha + \delta + \psi = 180^\circ - \beta - \gamma + \psi$ (angle balance for $AGFD$ and $BGFC$)
3. $\frac{\sin(\alpha + \psi)}{\sin \alpha} = \frac{AI}{EI}$ and $\frac{\sin(\psi - \beta)}{\sin \beta} = \frac{BI}{EI}$ (sine rule)
4. ... give with $AI = BI$ $\frac{\sin(\alpha + \psi)}{\sin(\psi - \beta)} = \frac{\sin \alpha}{\sin \beta} = \frac{\sin \delta}{\sin \gamma}$
5. Using 2.: $\frac{\sin(\varphi + \varphi^* - \delta)}{-\sin(\varphi + \varphi^* + \gamma)} = \frac{\sin \delta}{\sin \gamma}$
... gives $\sin(\gamma + \delta) \cdot \sin(\varphi + \varphi^*) = 0$
6. For $\sin(\gamma + \delta) \neq 0$ the sum $\varphi + \varphi^*$ must be 180° .

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Message: #259

Date: 27/9/2013 7:08:11

From: Chris van Tienhoven

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear Eckart,

Beautiful construction of a tangent to QL-Cu1!

I used your construction for a construction of the tangential of two conjugated points at QL-Cu1 without using conics.

So you don't have to construct the 4th intersection point of two conics.

The tangential of a point at QL-Cu1 is the intersection point of the tangent of this point and the tangent of its conjugate at QL-Cu1.

This point also lies on QL-Cu1.

Construction Tangential Q of a point P on QL-Cu1

Let P^* be the Clawson-Schmidt Conjugate of P.

Let S_{ij} be a random intersection point of $L_i \wedge L_j$ (i,j are different numbers from set {1,2,3,4})

Let S_{ik}^* be the Isogonal Conjugate of S_{ik} wrt triangle $S_{ij}.P.P^*$.

Let S_{il}^* be the Isogonal Conjugate of S_{il} wrt triangle $S_{ij}.P.P^*$.

Let S_{jk}^* be the Isogonal Conjugate of S_{jk} wrt triangle $S_{ij}.P.P^*$.

Let S_{jl}^* be the Isogonal Conjugate of S_{jl} wrt triangle $S_{ij}.P.P^*$.

Now Q = the isogonal Conjugate of intersection point $S_{ik}^*.S_{il}^* \wedge S_{jk}^*.S_{jl}^*$ wrt triangle $S_{ij}.P.P^*$.

Q is the Tangential of P as well as P^* .

PQ is the tangent in P at QL-Cu1.

P^*Q is the tangent in P^* at QL-Cu1.

Best regards,

Chris

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Message: #260

Date: 28/9/2013 6:17:11

From: Chris

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear friends,

Here is a simple method to construct the tangents at the 6 vertices S_{ij} to the cubic QL-Cu1.

The tangents at S_{ij} to QL-Cu1 are the reflection of the diagonal $S_{ij}.Sk1$ in the (internal or external) bisector of lines L_i, L_j .
 i, j are different numbers from the set $(1, 2, 3, 4)$

Best regards,
Chris

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Message: #261

Date: 28/9/2013 6:44:33

From: Chris

Subject: A generalization Brahmagupta's Theorem

Dear Dao, dear friends,

Thanks for the converse construction method.

This made it possible to me to verify that the SuperConjugate of point E has exactly the same properties as E itself:

- * supplementary angles with QL-line segments
- * $EA \cdot EC = EB \cdot ED$
- * K, L, M are collinear

SuperConjugate =

- * Clawson-Schmidt Conjugate, as well as
- * Isogonal Conjugate wrt the Orthic Triangle of the QL-Diagonal Triangle of ABCD, as well as
- * Isogonal Conjugate wrt any Component Triangle of the Quadrilateral defined by ABCD

It would be wonderful to find a simple construction of these two special points from basic points A, B, C & D and to find their coordinates.

Like you observed ABCD doesn't have to be cyclic at all.

It looks like this is a special property at the level of a Quadrilateral.

See:

<http://www.chrisvantienhoven.nl/index.php/quadrilateral-objects.html>

Best regards,
Chris

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Message: #262
Date: 28/9/2013 4:40:47
From: eckart_schmidt@t-online.de
Subject: A generalization Brahmagupta's Theorem

Dear Chris,
for a quadrigon the two "SuperConjugate" points E and E' on $QL-L1$ lie symmetric to $QA-P1$, that means E and E' are the foci of an inscribed conic with center in the centroid of the quadrigon.
Let the lines $E.QL-P1$ and $E'.QL-P1$ cut the cubic in further points F and F' , and the lines $E.E'$ and $F.F'$ cut the cubic in E'' and F'' , then then E, E'', F', F'' and E', E'', F, F'' lie concyclic .
Best regards Eckart

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Message: #263
Date: 29/9/2013 11:01:31
From: eckart_schmidt@t-online.de
Subject: Two points on the Newton Line $QL-Cu1$ revisited

Dear Chris,

... & $QL-Cu1$ revisited:
I think, there is a further property for $QL-Cu1$:
 $QL-Cu1$ is the locus of points P ,
for which the second intersection of the circles through P ,
 $L1^{\wedge}L2$, $L3^{\wedge}L4$ and P , $L2^{\wedge}L3$, $L4^{\wedge}L1$ are Clawson-Schmidt Conjugates.

Best regards Eckart

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Message: #264
Date: 29/9/2013 11:14:20
From: Chris
Subject: Two points on the Newton Line QL-Cu1 revisited

Dear Eckart and friends,

I think I will have to rewrite the EQF-page of QL-Cu1.
There are so many additions!
There remains this open question:
What are the coordinates of the inflexion points?
Bernard G, do you got a solution for this?

Best regards,
Chris

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Message: #265
Date: 29/9/2013 11:21:26
From: Bernard Gibert
Subject: Two points on the Newton Line QL-Cu1 revisited

Dear Chris,

>> There remains this open question:
>> What are the coordinates of the inflexion points?
>> Bernard G, do you got a solution for this?

when the class is 6 (non singular cubic) there are 9 inflexion points and 3 are real, hence no chance to find them!

Best regards
Bernard

PS: if you rewrite the QL-Cu1 page, it would be good to speak of focus and its polar conic...

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Message: #266

Date: 30/9/2013 10:19:57

From: Chris

Subject: A generalization Brahmagupta's Theorem

Dear Dao, Eckart and friends,

Thanks Eckart.

This actually defines the Brahmagupta's points E and E' in a Quadrigon (4-gon).

They can be constructed now like this:

1. Given 4 consecutive points P1,P2,P3,P4 connected by 4 lines L12, L23, L34, L41.
2. Construct the Newton Line (QL-L1) of the Quadrilateral defined by L12,L23,L34,L41.
3. Construct the Quadrangle Centroid (QA-P1) of the Quadrangle defined by P1, P2, P3, P4. This point lies on the Newton Line,
4. Construct the inscribed Conic within the the Quadrilateral defined by lines L12,L23,L34,L41.
5. Construct the foci of this conic. They are Brahmagupta's points E and E'.

Regarding 4., this an easy way to construct an inscribed conic in a Quadrilateral knowing its Center (per definition on the Newton Line).

1. Determine Centerpoint Ce on the Newton Line of Quadrilateral L1.L2.L3.L4.
2. Let Ca be the Anticomplement of Ce wrt some QL-Component Triangle L1.L2.L3.
3. Let Pe be the Isotomic Conjugate of Ca wrt QL-Component Triangle L1.L2.L3.
Pe is the Perspector of the inconic of QL-Component Triangle L1.L2.L3.
4. The traces of Pe on L1.L2.L3 lie on the inscribed conic we are looking for.
5. Also the Reflections of the traces in Ce lie on the conic we are looking for.
6. With these 6 points we can construct the conic.

The question stays open: what are the coordinates of the Brahmagupta's points?

I already had a try, but it is not easy to calculate the foci of a given conic.

Anyone can help?

Best regards,

Chris

Message: #267

Date: 30/9/2013 1:36:22

From: eckart_schmidt@t-online.de

Subject: A generalization Brahmagupta's Theorem

Dear Chris,

The coordinates of the points E are horrible (see attachment).

Best regards Eckart

A generalization Brahmagupta's Theorem

CT - Coordinates for points E

Let

$$z := 1$$

and

$$y := - \left(a^2 1n + a^2 mn - 2a^2 n^2 + 2a^2 1mx - c^2 1mx - a^2 1nx + c^2 1nx - a^2 mnx + 2c^2 mnx - 2c^2 n^2 x + c^2 1mx^2 - c^2 1nx^2 \right) / \left(2a^2 1m - a^2 1n - a^2 mn - c^2 1mx + c^2 1nx + 2c^2 mnx - 2c^2 n^2 x \right)$$

and x one of the following values

$$\left\{ \left\{ x := \begin{aligned} & - \left(2a^2 1^2 m^2 + 2b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 mn - 3b^2 1^2 mn + c^2 1^2 mn + a^2 1^2 m^2 n - 5b^2 1^2 m^2 n + c^2 1^2 m^2 n \right. \right. \\ & \quad \left. \left. + b^2 1^2 n^2 - 5a^2 1mn^2 + 7b^2 1mn^2 + c^2 1mn^2 - a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 1n^3 - 2b^2 1n^3 - 2c^2 1n^3 \right. \right. \\ & \quad \left. \left. + 2a^2 mn^3 - 2b^2 mn^3 + 2a^2 mn^3 - 2b^2 mn^3 - 6c^2 mn^3 + 4c^2 n^4 \right) / \right. \\ & \quad \left(4c^2 1(1+m-2n)(m-n)n \right) - \frac{1}{2} \sqrt{\left(\frac{2a^2}{c^2} + (2a^2 1^2 m^2 + 2b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 mn - \right.} \\ & \quad \quad 3b^2 1^2 mn + c^2 1^2 mn + a^2 1^2 m^2 n - 5b^2 1^2 m^2 n + c^2 1^2 m^2 n + b^2 1^2 n^2 - 5a^2 1mn^2 + \\ & \quad \quad 7b^2 1mn^2 + c^2 1mn^2 - a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 1n^3 - 2b^2 1n^3 - 2c^2 1n^3 + \\ & \quad \quad \left. 2a^2 mn^3 - 2b^2 mn^3 - 6c^2 mn^3 + 4c^2 n^4 \right)^2 / \left(4c^4 1^2 (1+m-2n)^2 (m-n)^2 n^2 \right) - \\ & \quad \left(4a^4 1^2 m^2 - 4a^2 b^2 1^2 m^2 - 3a^2 c^2 1^2 m^2 - b^2 c^2 1^2 m^2 + c^4 1^2 m^2 - 4a^4 1^2 mn + \right. \\ & \quad \quad 4a^2 b^2 1^2 mn + 4a^2 c^2 1^2 mn + 2b^2 c^2 1^2 mn - 2c^4 1^2 mn - 4a^4 1^2 m^2 n + 4a^2 b^2 1^2 m^2 n + \\ & \quad \quad 10a^2 c^2 1^2 m^2 n + 4b^2 c^2 1^2 m^2 n - 4c^4 1^2 m^2 n + a^4 1^2 n^2 - a^2 b^2 1^2 n^2 - b^2 c^2 1^2 n^2 + \\ & \quad \quad c^4 1^2 n^2 + 2a^4 1mn^2 - 2a^2 b^2 1mn^2 - 12a^2 c^2 1mn^2 - 8b^2 c^2 1mn^2 + 8c^4 1mn^2 + \\ & \quad \quad a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3a^2 c^2 m^2 n^2 - 4b^2 c^2 m^2 n^2 + 4c^4 m^2 n^2 + 4b^2 c^2 1n^3 - 4c^4 1n^3 + \\ & \quad \quad \left. 8b^2 c^2 mn^3 - 8c^4 mn^3 + 4a^2 c^2 n^4 - 4b^2 c^2 n^4 + 4c^4 n^4 \right) / \left(c^4 1(1+m-2n)(m-n)n \right) \Bigg) - \\ & \quad \frac{1}{2} \sqrt{\left(-\frac{2a^2}{c^2} + (2a^2 1^2 m^2 + 2b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 mn - 3b^2 1^2 mn + c^2 1^2 mn + \right.} \\ & \quad \quad a^2 1^2 m^2 n - 5b^2 1^2 m^2 n + c^2 1^2 m^2 n + b^2 1^2 n^2 - 5a^2 1mn^2 + 7b^2 1mn^2 + c^2 1mn^2 - \\ & \quad \quad a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 1n^3 - 2b^2 1n^3 - 2c^2 1n^3 + 2a^2 mn^3 - \\ & \quad \quad \left. 2b^2 mn^3 - 6c^2 mn^3 + 4c^2 n^4 \right)^2 / \left(2c^4 1^2 (1+m-2n)^2 (m-n)^2 n^2 \right) - \\ & \quad \left(4a^4 1^2 m^2 - 4a^2 b^2 1^2 m^2 - 3a^2 c^2 1^2 m^2 - b^2 c^2 1^2 m^2 + c^4 1^2 m^2 - 4a^4 1^2 mn + \right. \\ & \quad \quad 4a^2 b^2 1^2 mn + 4a^2 c^2 1^2 mn + 2b^2 c^2 1^2 mn - 2c^4 1^2 mn - 4a^4 1^2 m^2 n + 4a^2 b^2 1^2 m^2 n + \\ & \quad \quad 10a^2 c^2 1^2 m^2 n + 4b^2 c^2 1^2 m^2 n - 4c^4 1^2 m^2 n + a^4 1^2 n^2 - a^2 b^2 1^2 n^2 - b^2 c^2 1^2 n^2 + \\ & \quad \quad c^4 1^2 n^2 + 2a^4 1mn^2 - 2a^2 b^2 1mn^2 - 12a^2 c^2 1mn^2 - 8b^2 c^2 1mn^2 + 8c^4 1mn^2 + \\ & \quad \quad a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3a^2 c^2 m^2 n^2 - 4b^2 c^2 m^2 n^2 + 4c^4 m^2 n^2 + 4b^2 c^2 1n^3 - 4c^4 1n^3 + \\ & \quad \quad \left. 8b^2 c^2 mn^3 - 8c^4 mn^3 + 4a^2 c^2 n^4 - 4b^2 c^2 n^4 + 4c^4 n^4 \right) / \left(c^4 1(1+m-2n)(m-n)n \right) \Bigg) - \\ & \quad \left(- \left(2a^2 1^2 m^2 + 2b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 mn - 3b^2 1^2 mn + c^2 1^2 mn + a^2 1^2 m^2 n - \right. \right. \\ & \quad \quad \left. \left. 5b^2 1^2 m^2 n + c^2 1^2 m^2 n + b^2 1^2 n^2 - 5a^2 1mn^2 + 7b^2 1mn^2 + c^2 1mn^2 - \right. \right. \\ & \quad \quad \left. \left. a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 1n^3 - 2b^2 1n^3 - 2c^2 1n^3 + 2a^2 mn^3 - \right. \right. \\ & \quad \quad \left. \left. 2b^2 mn^3 - 6c^2 mn^3 + 4c^2 n^4 \right)^3 / \left(c^6 1^3 (1+m-2n)^3 (m-n)^3 n^3 \right) - \right. \\ & \quad \left. \left(8 \left(2a^4 1^2 m^2 + 2a^2 b^2 1^2 m^2 - a^2 c^2 1^2 m^2 - a^4 1^2 mn - 3a^2 b^2 1^2 mn + a^2 c^2 1^2 mn + \right. \right. \right. \end{aligned} \right\} \right\}$$

$$\begin{aligned}
& \frac{1}{2} \sqrt{\left(\frac{2a^2}{c^2} + (2a^2 l^2 m^2 + 2b^2 l^2 m^2 - c^2 l^2 m^2 - a^2 l^2 m n - 3b^2 l^2 m n + c^2 l^2 m n + \right. \\
& \quad a^2 l m^2 n - 5b^2 l m^2 n + c^2 l m^2 n + b^2 l^2 n^2 - 5a^2 l m n^2 + 7b^2 l m n^2 + c^2 l m n^2 - \\
& \quad a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 l n^3 - 2b^2 l n^3 - 2c^2 l n^3 + 2a^2 m n^3 - \\
& \quad \left. 2b^2 m n^3 - 6c^2 m n^3 + 4c^2 n^4) \right)^2 / (4c^4 l^2 (1+m-2n)^2 (m-n)^2 n^2) - \\
& (4a^4 l^2 m^2 - 4a^2 b^2 l^2 m^2 - 3a^2 c^2 l^2 m^2 - b^2 c^2 l^2 m^2 + c^4 l^2 m^2 - 4a^4 l^2 m n + \\
& \quad 4a^2 b^2 l^2 m n + 4a^2 c^2 l^2 m n + 2b^2 c^2 l^2 m n - 2c^4 l^2 m n - 4a^4 l m^2 n + \\
& \quad 4a^2 b^2 l m^2 n + 10a^2 c^2 l m^2 n + 4b^2 c^2 l m^2 n - 4c^4 l m^2 n + a^4 l^2 n^2 - \\
& \quad a^2 b^2 l^2 n^2 - b^2 c^2 l^2 n^2 + c^4 l^2 n^2 + 2a^4 l m n^2 - 2a^2 b^2 l m n^2 - 12a^2 c^2 l m n^2 - \\
& \quad 8b^2 c^2 l m n^2 + 8c^4 l m n^2 + a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3a^2 c^2 m^2 n^2 - \\
& \quad 4b^2 c^2 m^2 n^2 + 4c^4 m^2 n^2 + 4b^2 c^2 l n^3 - 4c^4 l n^3 + 8b^2 c^2 m n^3 - 8c^4 m n^3 + \\
& \quad \left. 4a^2 c^2 n^4 - 4b^2 c^2 n^4 + 4c^4 n^4) / (c^4 l (1+m-2n) (m-n) n) \right) - \\
& \frac{1}{2} \sqrt{\left(-\frac{2a^2}{c^2} + (2a^2 l^2 m^2 + 2b^2 l^2 m^2 - c^2 l^2 m^2 - a^2 l^2 m n - 3b^2 l^2 m n + c^2 l^2 m n + \right. \\
& \quad a^2 l m^2 n - 5b^2 l m^2 n + c^2 l m^2 n + b^2 l^2 n^2 - 5a^2 l m n^2 + 7b^2 l m n^2 + c^2 l m n^2 - \\
& \quad a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 l n^3 - 2b^2 l n^3 - 2c^2 l n^3 + 2a^2 m n^3 - \\
& \quad \left. 2b^2 m n^3 - 6c^2 m n^3 + 4c^2 n^4) \right)^2 / (2c^4 l^2 (1+m-2n)^2 (m-n)^2 n^2) - \\
& (4a^4 l^2 m^2 - 4a^2 b^2 l^2 m^2 - 3a^2 c^2 l^2 m^2 - b^2 c^2 l^2 m^2 + c^4 l^2 m^2 - 4a^4 l^2 m n + \\
& \quad 4a^2 b^2 l^2 m n + 4a^2 c^2 l^2 m n + 2b^2 c^2 l^2 m n - 2c^4 l^2 m n - 4a^4 l m^2 n + 4a^2 b^2 l m^2 n + \\
& \quad 10a^2 c^2 l m^2 n + 4b^2 c^2 l m^2 n - 4c^4 l m^2 n + a^4 l^2 n^2 - a^2 b^2 l^2 n^2 - b^2 c^2 l^2 n^2 + \\
& \quad c^4 l^2 n^2 + 2a^4 l m n^2 - 2a^2 b^2 l m n^2 - 12a^2 c^2 l m n^2 - 8b^2 c^2 l m n^2 + 8c^4 l m n^2 + \\
& \quad a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3a^2 c^2 m^2 n^2 - 4b^2 c^2 m^2 n^2 + 4c^4 m^2 n^2 + 4b^2 c^2 l n^3 - 4c^4 l n^3 + \\
& \quad 8b^2 c^2 m n^3 - 8c^4 m n^3 + 4a^2 c^2 n^4 - 4b^2 c^2 n^4 + 4c^4 n^4) / (c^4 l (1+m-2n) (m-n) n) + \\
& (- (2a^2 l^2 m^2 + 2b^2 l^2 m^2 - c^2 l^2 m^2 - a^2 l^2 m n - 3b^2 l^2 m n + c^2 l^2 m n + a^2 l m^2 n - \\
& \quad 5b^2 l m^2 n + c^2 l m^2 n + b^2 l^2 n^2 - 5a^2 l m n^2 + 7b^2 l m n^2 + c^2 l m n^2 - \\
& \quad a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 l n^3 - 2b^2 l n^3 - 2c^2 l n^3 + 2a^2 m n^3 - \\
& \quad \left. 2b^2 m n^3 - 6c^2 m n^3 + 4c^2 n^4) \right)^3 / (c^6 l^3 (1+m-2n)^3 (m-n)^3 n^3) - \\
& (8 (2a^4 l^2 m^2 + 2a^2 b^2 l^2 m^2 - a^2 c^2 l^2 m^2 - a^4 l^2 m n - 3a^2 b^2 l^2 m n + a^2 c^2 l^2 m n + \\
& \quad a^4 l m^2 n - 5a^2 b^2 l m^2 n + a^2 c^2 l m^2 n + a^2 b^2 l^2 n^2 - 5a^4 l m n^2 + 7a^2 b^2 l m n^2 + \\
& \quad a^2 c^2 l m n^2 - a^4 m^2 n^2 + 2a^2 b^2 m^2 n^2 + 2a^2 c^2 m^2 n^2 + 2a^4 l n^3 - 2a^2 b^2 l n^3 - \\
& \quad \left. 2a^2 c^2 l n^3 + 2a^4 m n^3 - 2a^2 b^2 m n^3 - 6a^2 c^2 m n^3 + 4a^2 c^2 n^4) \right) / \\
& (c^4 l (1+m-2n) (m-n) n) + (4 (2a^2 l^2 m^2 + 2b^2 l^2 m^2 - c^2 l^2 m^2 - a^2 l^2 m n - \\
& \quad 3b^2 l^2 m n + c^2 l^2 m n + a^2 l m^2 n - 5b^2 l m^2 n + c^2 l m^2 n + b^2 l^2 n^2 - 5a^2 l m n^2 + \\
& \quad 7b^2 l m n^2 + c^2 l m n^2 - a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 l n^3 - 2b^2 l n^3 - \\
& \quad \left. 2c^2 l n^3 + 2a^2 m n^3 - 2b^2 m n^3 - 6c^2 m n^3 + 4c^2 n^4) (4a^4 l^2 m^2 - 4a^2 b^2 l^2 m^2 - \right. \\
& \quad 3a^2 c^2 l^2 m^2 - b^2 c^2 l^2 m^2 + c^4 l^2 m^2 - 4a^4 l^2 m n + 4a^2 b^2 l^2 m n + 4a^2 c^2 l^2 m n + \\
& \quad 2b^2 c^2 l^2 m n - 2c^4 l^2 m n - 4a^4 l m^2 n + 4a^2 b^2 l m^2 n + 10a^2 c^2 l m^2 n + \\
& \quad 4b^2 c^2 l m^2 n - 4c^4 l m^2 n + a^4 l^2 n^2 - a^2 b^2 l^2 n^2 - b^2 c^2 l^2 n^2 + c^4 l^2 n^2 + 2a^4 l m n^2 - \\
& \quad \left. 2a^2 b^2 l m n^2 - 12a^2 c^2 l m n^2 - 8b^2 c^2 l m n^2 + 8c^4 l m n^2 + a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - \right. \\
& \quad \left. 3a^2 c^2 m^2 n^2 - 4b^2 c^2 m^2 n^2 + 4c^4 m^2 n^2 + 4b^2 c^2 l n^3 - 4c^4 l n^3 + 8b^2 c^2 m n^3 - \right. \\
& \quad \left. 8c^4 m n^3 + 4a^2 c^2 n^4 - 4b^2 c^2 n^4 + 4c^4 n^4) \right) / (c^6 l^2 (1+m-2n)^2 (m-n)^2 n^2) \Big) / \\
& \left(4 \sqrt{\left(\frac{2a^2}{c^2} + (2a^2 l^2 m^2 + 2b^2 l^2 m^2 - c^2 l^2 m^2 - a^2 l^2 m n - 3b^2 l^2 m n + c^2 l^2 m n + \right. \right. \\
& \quad a^2 l m^2 n - 5b^2 l m^2 n + c^2 l m^2 n + b^2 l^2 n^2 - 5a^2 l m n^2 + 7b^2 l m n^2 + c^2 l m n^2 - \\
& \quad \left. \left. a^2 m^2 n^2 + 2b^2 m^2 n^2 + 2c^2 m^2 n^2 + 2a^2 l n^3 - 2b^2 l n^3 - 2c^2 l n^3 + 2a^2 m n^3 - \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \left(\frac{2b^2mn^3 - 6c^2mn^3 + 4c^2n^4}{c^4} \right)^2 / \left(4c^4l^2(1+m-2n)^2(m-n)^2n^2 \right) - \right. \\
& \left(4a^4l^2m^2 - 4a^2b^2l^2m^2 - 3a^2c^2l^2m^2 - b^2c^2l^2m^2 + c^4l^2m^2 - 4a^4l^2mn + \right. \\
& 4a^2b^2l^2mn + 4a^2c^2l^2mn + 2b^2c^2l^2mn - 2c^4l^2mn - 4a^4l^2m^2n + \\
& 4a^2b^2l^2m^2n + 10a^2c^2l^2m^2n + 4b^2c^2l^2m^2n - 4c^4l^2m^2n + a^4l^2n^2 - \\
& a^2b^2l^2n^2 - b^2c^2l^2n^2 + c^4l^2n^2 + 2a^4l^2mn^2 - 2a^2b^2l^2mn^2 - 12a^2c^2l^2mn^2 - \\
& 8b^2c^2l^2mn^2 + 8c^4l^2mn^2 + a^4m^2n^2 - a^2b^2m^2n^2 - 3a^2c^2m^2n^2 - \\
& 4b^2c^2m^2n^2 + 4c^4m^2n^2 + 4b^2c^2l^2n^3 - 4c^4l^2n^3 + 8b^2c^2mn^3 - 8c^4mn^3 + \\
& \left. \left. \left. 4a^2c^2n^4 - 4b^2c^2n^4 + 4c^4n^4 \right) / \left(c^4l(1+m-2n)(m-n)n \right) \right) \right) \Bigg\}, \\
\{x := & - \left(2a^2l^2m^2 + 2b^2l^2m^2 - c^2l^2m^2 - a^2l^2mn - 3b^2l^2mn + c^2l^2mn + a^2l^2m^2n - \right. \\
& 5b^2l^2m^2n + c^2l^2m^2n + b^2l^2n^2 - \\
& 5a^2l^2mn^2 + 7b^2l^2mn^2 + c^2l^2mn^2 - \\
& a^2m^2n^2 + 2b^2m^2n^2 + 2c^2m^2n^2 + \\
& 2a^2l^2n^3 - 2b^2l^2n^3 - 2c^2l^2n^3 + \\
& 2a^2mn^3 - 2b^2mn^3 - 6c^2mn^3 + \\
& \left. \left. \left. 4c^2n^4 \right) / \left(4c^2l(1+m-2n)(m-n)n \right) + \right. \right. \\
& \frac{1}{2} \sqrt{\left(\frac{2a^2}{c^2} + \left(2a^2l^2m^2 + 2b^2l^2m^2 - c^2l^2m^2 - a^2l^2mn - 3b^2l^2mn + c^2l^2mn + \right. \right. \\
& a^2l^2m^2n - 5b^2l^2m^2n + c^2l^2m^2n + b^2l^2n^2 - 5a^2l^2mn^2 + 7b^2l^2mn^2 + c^2l^2mn^2 - \\
& a^2m^2n^2 + 2b^2m^2n^2 + 2c^2m^2n^2 + 2a^2l^2n^3 - 2b^2l^2n^3 - 2c^2l^2n^3 + 2a^2mn^3 - \\
& \left. \left. \left. 2b^2mn^3 - 6c^2mn^3 + 4c^2n^4 \right) \right)^2 / \left(4c^4l^2(1+m-2n)^2(m-n)^2n^2 \right) - \right. \\
& \left(4a^4l^2m^2 - 4a^2b^2l^2m^2 - 3a^2c^2l^2m^2 - b^2c^2l^2m^2 + c^4l^2m^2 - 4a^4l^2mn + \right. \\
& 4a^2b^2l^2mn + 4a^2c^2l^2mn + 2b^2c^2l^2mn - 2c^4l^2mn - 4a^4l^2m^2n + \\
& 4a^2b^2l^2m^2n + 10a^2c^2l^2m^2n + 4b^2c^2l^2m^2n - 4c^4l^2m^2n + a^4l^2n^2 - \\
& a^2b^2l^2n^2 - b^2c^2l^2n^2 + c^4l^2n^2 + 2a^4l^2mn^2 - 2a^2b^2l^2mn^2 - 12a^2c^2l^2mn^2 - \\
& 8b^2c^2l^2mn^2 + 8c^4l^2mn^2 + a^4m^2n^2 - a^2b^2m^2n^2 - 3a^2c^2m^2n^2 - \\
& 4b^2c^2m^2n^2 + 4c^4m^2n^2 + 4b^2c^2l^2n^3 - 4c^4l^2n^3 + 8b^2c^2mn^3 - 8c^4mn^3 + \\
& \left. \left. \left. 4a^2c^2n^4 - 4b^2c^2n^4 + 4c^4n^4 \right) / \left(c^4l(1+m-2n)(m-n)n \right) \right) + \right. \\
& \frac{1}{2} \sqrt{\left(- \frac{2a^2}{c^2} + \left(2a^2l^2m^2 + 2b^2l^2m^2 - c^2l^2m^2 - a^2l^2mn - 3b^2l^2mn + c^2l^2mn + \right. \right. \\
& a^2l^2m^2n - 5b^2l^2m^2n + c^2l^2m^2n + b^2l^2n^2 - 5a^2l^2mn^2 + 7b^2l^2mn^2 + c^2l^2mn^2 - \\
& a^2m^2n^2 + 2b^2m^2n^2 + 2c^2m^2n^2 + 2a^2l^2n^3 - 2b^2l^2n^3 - 2c^2l^2n^3 + 2a^2mn^3 - \\
& \left. \left. \left. 2b^2mn^3 - 6c^2mn^3 + 4c^2n^4 \right) \right)^2 / \left(2c^4l^2(1+m-2n)^2(m-n)^2n^2 \right) - \right. \\
& \left(4a^4l^2m^2 - 4a^2b^2l^2m^2 - 3a^2c^2l^2m^2 - b^2c^2l^2m^2 + c^4l^2m^2 - 4a^4l^2mn + \right. \\
& 4a^2b^2l^2mn + 4a^2c^2l^2mn + 2b^2c^2l^2mn - 2c^4l^2mn - 4a^4l^2m^2n + \\
& 4a^2b^2l^2m^2n + 10a^2c^2l^2m^2n + 4b^2c^2l^2m^2n - 4c^4l^2m^2n + a^4l^2n^2 - \\
& a^2b^2l^2n^2 - b^2c^2l^2n^2 + c^4l^2n^2 + 2a^4l^2mn^2 - 2a^2b^2l^2mn^2 - 12a^2c^2l^2mn^2 - \\
& 8b^2c^2l^2mn^2 + 8c^4l^2mn^2 + a^4m^2n^2 - a^2b^2m^2n^2 - 3a^2c^2m^2n^2 - \\
& 4b^2c^2m^2n^2 + 4c^4m^2n^2 + 4b^2c^2l^2n^3 - 4c^4l^2n^3 + 8b^2c^2mn^3 - 8c^4mn^3 + \\
& \left. \left. \left. 4a^2c^2n^4 - 4b^2c^2n^4 + 4c^4n^4 \right) / \left(c^4l(1+m-2n)(m-n)n \right) + \right. \right. \\
& - \left(2a^2l^2m^2 + 2b^2l^2m^2 - c^2l^2m^2 - a^2l^2mn - 3b^2l^2mn + c^2l^2mn + a^2l^2m^2n - \right. \\
& 5b^2l^2m^2n + c^2l^2m^2n + b^2l^2n^2 - 5a^2l^2mn^2 + 7b^2l^2mn^2 + c^2l^2mn^2 - \\
& a^2m^2n^2 + 2b^2m^2n^2 + 2c^2m^2n^2 + 2a^2l^2n^3 - 2b^2l^2n^3 - 2c^2l^2n^3 + 2a^2mn^3 - \\
& \left. \left. \left. 2b^2mn^3 - 6c^2mn^3 + 4c^2n^4 \right) \right)^3 / \left(c^6l^3(1+m-2n)^3(m-n)^3n^3 \right) - \right. \\
& \left(8 \left(2a^4l^2m^2 + 2a^2b^2l^2m^2 - a^2c^2l^2m^2 - a^4l^2mn - 3a^2b^2l^2mn + a^2c^2l^2mn + \right. \right. \\
& \left. \left. \left. a^4l^2m^2n - 5a^2b^2l^2m^2n + a^2c^2l^2m^2n + a^2b^2l^2n^2 - 5a^4l^2mn^2 + 7a^2b^2l^2mn^2 + \right. \right.
\end{aligned}$$

$$\left. \begin{aligned} & \left(\frac{a^2 c^2 1 m n^2 - a^4 m^2 n^2 + 2 a^2 b^2 m^2 n^2 + 2 a^2 c^2 m^2 n^2 + 2 a^4 1 n^3 - 2 a^2 b^2 1 n^3 - 2 a^2 c^2 1 n^3 + 2 a^4 m n^3 - 2 a^2 b^2 m n^3 - 6 a^2 c^2 m n^3 + 4 a^2 c^2 n^4}{(c^4 1 (1+m-2n) (m-n) n) + (4 (2 a^2 1^2 m^2 + 2 b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 m n - 3 b^2 1^2 m n + c^2 1^2 m n + a^2 1 m^2 n - 5 b^2 1 m^2 n + c^2 1 m^2 n + b^2 1^2 n^2 - 5 a^2 1 m n^2 + 7 b^2 1 m n^2 + c^2 1 m n^2 - a^2 m^2 n^2 + 2 b^2 m^2 n^2 + 2 c^2 m^2 n^2 + 2 a^2 1 n^3 - 2 b^2 1 n^3 - 2 c^2 1 n^3 + 2 a^2 m^2 n^2 - 2 b^2 m^2 n^2 - 6 c^2 m^2 n^2 + 4 c^2 n^4) (4 a^4 1^2 m^2 - 4 a^2 b^2 1^2 m^2 - 3 a^2 c^2 1^2 m^2 - b^2 c^2 1^2 m^2 + c^4 1^2 m^2 - 4 a^4 1^2 m n + 4 a^2 b^2 1^2 m n + 4 a^2 c^2 1^2 m n + 2 b^2 c^2 1^2 m n - 2 c^4 1^2 m n - 4 a^4 1 m^2 n + 4 a^2 b^2 1 m^2 n + 10 a^2 c^2 1 m^2 n + 4 b^2 c^2 1 m^2 n - 4 c^4 1 m^2 n + a^4 1^2 n^2 - a^2 b^2 1^2 n^2 - b^2 c^2 1^2 n^2 + c^4 1^2 n^2 + 2 a^4 1 m n^2 - 2 a^2 b^2 1 m n^2 - 12 a^2 c^2 1 m n^2 - 8 b^2 c^2 1 m n^2 + 8 c^4 1 m n^2 + a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3 a^2 c^2 m^2 n^2 - 4 b^2 c^2 m^2 n^2 + 4 c^4 m^2 n^2 + 4 b^2 c^2 1 n^3 - 4 c^4 1 n^3 + 8 b^2 c^2 m n^3 - 8 c^4 m n^3 + 4 a^2 c^2 n^4 - 4 b^2 c^2 n^4 + 4 c^4 n^4) / (c^6 1^2 (1+m-2n)^2 (m-n)^2 n^2)} \right) \Bigg/ \\ & \left(4 \sqrt{\left(\frac{2 a^2}{c^2} + (2 a^2 1^2 m^2 + 2 b^2 1^2 m^2 - c^2 1^2 m^2 - a^2 1^2 m n - 3 b^2 1^2 m n + c^2 1^2 m n + a^2 1 m^2 n - 5 b^2 1 m^2 n + c^2 1 m^2 n + b^2 1^2 n^2 - 5 a^2 1 m n^2 + 7 b^2 1 m n^2 + c^2 1 m n^2 - a^2 m^2 n^2 + 2 b^2 m^2 n^2 + 2 c^2 m^2 n^2 + 2 a^2 1 n^3 - 2 b^2 1 n^3 - 2 c^2 1 n^3 + 2 a^2 m^2 n^2 - 2 b^2 m^2 n^2 - 6 c^2 m^2 n^2 + 4 c^2 n^4) \wedge 2 / (4 c^4 1^2 (1+m-2n)^2 (m-n)^2 n^2) - (4 a^4 1^2 m^2 - 4 a^2 b^2 1^2 m^2 - 3 a^2 c^2 1^2 m^2 - b^2 c^2 1^2 m^2 + c^4 1^2 m^2 - 4 a^4 1^2 m n + 4 a^2 b^2 1^2 m n + 4 a^2 c^2 1^2 m n + 2 b^2 c^2 1^2 m n - 2 c^4 1^2 m n - 4 a^4 1 m^2 n + 4 a^2 b^2 1 m^2 n + 10 a^2 c^2 1 m^2 n + 4 b^2 c^2 1 m^2 n - 4 c^4 1 m^2 n + a^4 1^2 n^2 - a^2 b^2 1^2 n^2 - b^2 c^2 1^2 n^2 + c^4 1^2 n^2 + 2 a^4 1 m n^2 - 2 a^2 b^2 1 m n^2 - 12 a^2 c^2 1 m n^2 - 8 b^2 c^2 1 m n^2 + 8 c^4 1 m n^2 + a^4 m^2 n^2 - a^2 b^2 m^2 n^2 - 3 a^2 c^2 m^2 n^2 - 4 b^2 c^2 m^2 n^2 + 4 c^4 m^2 n^2 + 4 b^2 c^2 1 n^3 - 4 c^4 1 n^3 + 8 b^2 c^2 m n^3 - 8 c^4 m n^3 + 4 a^2 c^2 n^4 - 4 b^2 c^2 n^4 + 4 c^4 n^4) / (c^4 1 (1+m-2n) (m-n) n) \right) \Bigg) \Bigg\} \end{aligned} \right.$$

... then two points are real,

that are the searched E-points, two points are not real.

Eckart Schmidt

<http://eckartschmidt.de>

eckart_schmidt@t-online.de

Message: #268

Date: 30/9/2013 2:38:55

From: Dao Thanh Oai

Subject: A generalization Brahmagupta's Theorem

Dear Chris van Tienhoven and Eckart!

Thank to you very much! (I want chat many word but my english is not good)

You can see reference: Ellipse with two foci tangent pedal circle and tangent three side of a triangle at here: <http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47&t=554936>. Maybe giving to you your idear about ellipse tangent four sides quadrilateral.

Best regard

Dao Thanh Oai

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Message: #269

Date: 30/9/2013 3:02:46

From: Chris van Tienhoven

Subject: Precisions concerning the curve QL-Cu1

Dear Eckart,

To quote Shakespeare: "There is something special in Denmark"
(statement a bit adapted by me).

QL-Cu1 has real brain teasing aspects.

You are right, the foci of the inscribed conic with center
QL-L1^QL-L6 can be on QL-L1 as well as the perpendicular line
wrt QL-L1 through QL-L1^QL-L6.

My analyses is as follows.

The foci of a conic -when it is an ellipse- always reside on the
main axis.

The main axis of the inscribed conic with center QL-L1^QL-L6
obviously can coincide with QL-L1 as well as its perpendicular
line through QL-L1^QL-L6 (depending on the shape of the
Reference Quadrilateral).

That's why the described foci can reside on alternate lines.

I am wondering if the following is true: An ellipse has 4 foci,
2 of them real, 2 of them imaginary. Maybe others can confirm or
correct this.

(just like 2 circles always have 4 intersection points, 2 of
them real or imaginary and 2 of them always being the imaginary
intersection points at infinity)

This would also explain why it is so difficult to define a
simple method for constructing the foci of an ellipse unless you
know which axis is the main axis.

When my conjecture is right, then the foci on QL-L1 as well as
the foci on the perpendicular line through QL-L1^QL-L6 are both
valid, although per pair alternately real and imaginary.

My conclusion is anyway that they are not identical points
because they reside on different lines and have distinct
coordinates.

That's why I am reluctant to give them a corporate name.

But thanks to your correct comment I will make a remark about
the special character of QL-2P2 in EQF regarding being focus or
not.

If you have another opinion please let me know. I am open
minded.

Best regards,
Chris

Message: #270

Date: 30/9/2013 5:13:41

From: bernard.keizer

Subject: Tangential quadrilateral well known properties?

Dear Chris, dear Eckart, dear friends

May be you remember this problem with ABCD circumscribed quadrilateral, forming the tangential quadrilateral of a cyclic quadrilateral PQRS in circle center O, M, N, M' and N' middles of the sides of the first one.

We have to prove that MN cuts QS in a point K on a parallel through O to PR (and the same with M'N' cutting PR in a point K' on a parallel trough O to QS).

I first tried the idea of Eckart with the polar coordinates of P ($\cos P$, $\sin P$), but it's rather boring ...

The well known property asked by Chris is that the tangential QL and the cyclic QA have the same diagonal triangle EFG self polar wrt the circle (AC, BD, PR and QS cut in a point G and the inverse of G wrt the circle is the Miquel point of the inscribed QL). O is on the Newton Line of the tangential QL.

In fact, the property is true for a conic and is mentioned in EQF for the quadrangle formed by the 4 contact points of the inscribed parabola QL-Co1 tangent to 4 lines forming it's tangential QL.

So, OKGK' is a parallelogram ...

Despite all my efforts, I'm not able to find a synthetic proof of this property and I didn't find any reference ...

Could someone help me?

Best regards

Bernard Keizer

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Message: #271

Date: 01/10/2013 9:02:53

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

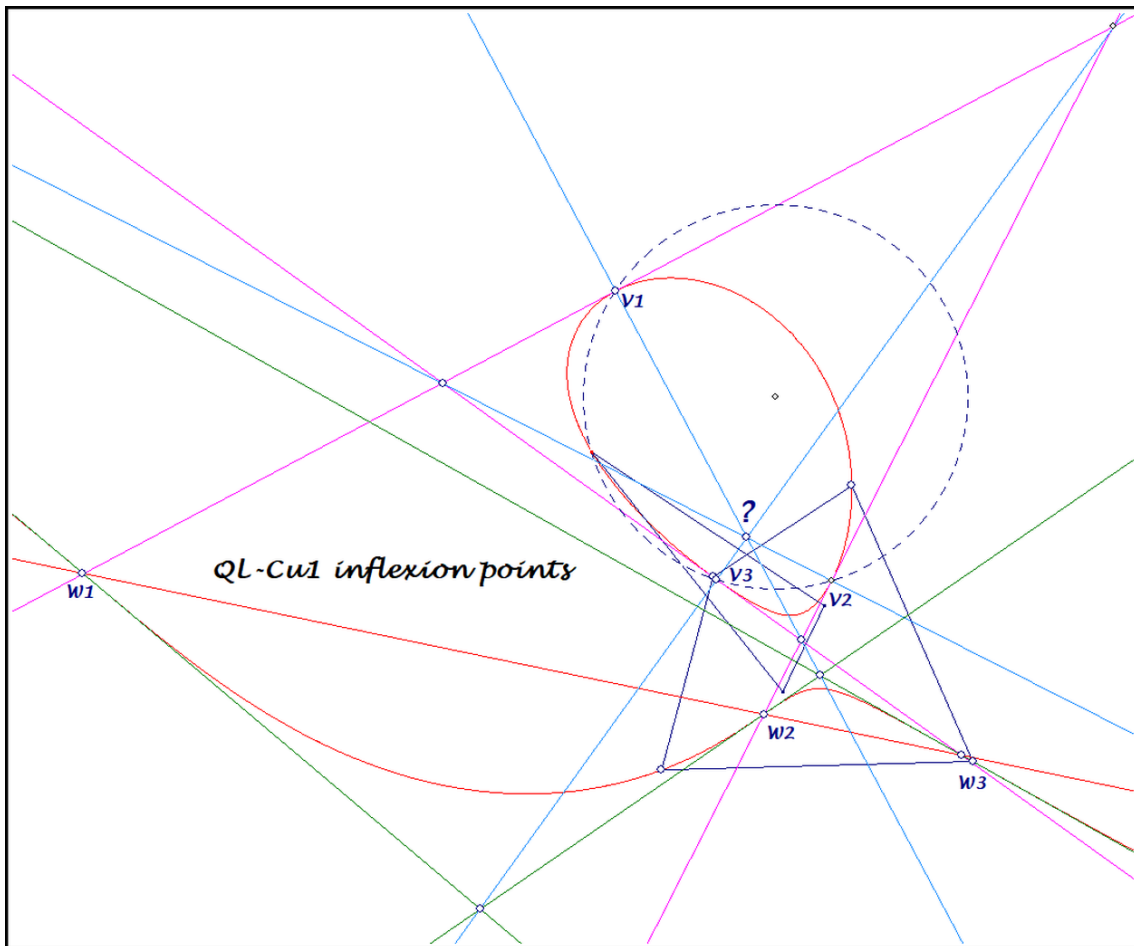
Dear Chris,

I don't know, whether I have understood your last message. Even if it is possible, to consider 4 foci for a conic, there will be no common sight of the uni- and bipartite case of QL-Cu1. For example wrt the "strophoid"-construction: On the one hand we need circles centered on QL-L1 through the foci of the inscribed conic with center $QL-L1 \wedge QL-L6$, on the other hand we need circles orthogonal to the Thales circle about these foci. I would prefer to define QL-2P2a,b as the addressed foci, for QL-Cu1 is isogonal invariant wrt the triangle with the vertices QL-P1 and these foci (independent whether the foci are on QL-L1 or not).

Another point: Even if Bernard gives no chance in calculation, some remarks wrt the inflexion points of QL-Cu1 (see attachment):

- ... QL-Cu1 and QL-Cu2 have the same inflexion points (see Bernard's paper 8.1).
 - ... The three inflexion points W_i lie collinear on a line L ,
 - ... The Clawson-Schmidt Conjugate (QL-Tr1) of L is a circle through QL-P1 and the isogonal conjugate of L (see above) is a conic through QL-P1. The three intersections V_i unequal QL-P1 are points on QL-Cu1.
 - ... The tangentials of V_i wrt QL-Cu1 are their QL-Tr1 images, which are the inflexion points W_i . So the inflexion points are the tangentials of its QL-Tf1 image.
 - ... The Simson line of QL-P1 wrt the V-triangle is perpendicular QL-L1.
 - ... The normal at V_i contains the intersection of the tangents in V_j, V_k .
 - ... The normal at V_i contains the intersection of the tangents in W_j, W_k .
 - ... The normals at V_i, V_j, V_k have a common point.
- What about this point???

Best regards
Eckart



13-09-29-e-fig.png

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Message: #272

Date: 02/10/2013 10:27:08

From: eckart_schmidt@t-online.de

Subject: Tangential quadrilateral well known properties?

Dear Bernard,

there is even not a synthetic proof, but a
Mathematica-calculation in the attachment wrt your problem for
an inscribed conic.

Best regards Eckart

Dear Bernard,

wrt your message "Re: Tangential quadrilateral well known properties?":

Dear Chris, dear Eckart, dear friends

May be you remember this problem with ABCD circumscribed quadrilateral, forming the tangential quadrilateral of a cyclic quadrilateral PQRS in circle center O, M, N, M' and N' middles of the sides of the first one. We have to prove that MN cuts QS in a point K on a parallel through O to PR (and the same with M'N' cutting PR in a point K' on a parallel through O to QS).

I first tried the idea of Eckart with the polar coordinates of P (cosP, sinP), but it's rather boring...

...

In fact, the property is true for a conic and is mentioned in EQF for the quadrangle formed by the 4 contact points of the inscribed parabola QL-Co1 tangent to 4 lines forming its tangential QL.

So, OKGK' is a parallelogram...

Despite all my efforts, I'm not able to find a synthetic proof of this property and I didn't find any reference...

Could someone help me?

Best regards

Bernard Keizer

... here is not a synthetic proof, but a calculation in CT-coordinates for an inscribed conic:

The quadrigon:

$$P1 := \{1, 0, 0\}; P2 := \{0, 1, 0\}; P3 := \{0, -n, m\}; P4 := \{n, 0, -1\}$$

Let Z be the center of an inscribed conic on the Newton line of a quadrigon with parameter κ

$$Z := \left\{ \kappa, \frac{1m - 1n - mn - (1m + 1n - mn)\kappa}{1m - 1n + mn}, 1 \right\}$$

... then the Brianchon point T of the conic wrt the reference triangle is the isotomic conjugate of the anticomplement of Z:

$$T := \{mn(n+1\kappa)(-m-1\kappa+m\kappa), 1n(m+1\kappa-m\kappa)(-m+n+m\kappa), -1m(n+1\kappa)(-m+n+m\kappa)\}$$

This gives the contact points

$$P := \{-m(n+1\kappa), 1(-m+n+m\kappa), 0\}$$

$$S := \{n(-m-1\kappa+m\kappa), 0, -1(-m+n+m\kappa)\}$$

$$Q := \{0, -n(-m-1\kappa+m\kappa), -m(n+1\kappa)\}$$

Analogue you get the Brianchon point T' of the conic wrt the reference triangle P2 P3 P1P2^P3P4

$$T' := \{mn(n+1\kappa)(-m+n+m\kappa), -1n(m^2-mn+n^2 - (-1m+2m^2-1n-mn)\kappa + (1^2-1m+m^2)\kappa^2), 1m(n+1\kappa)(m+1\kappa-m\kappa)\}$$

This gives the contact point

$$R := \{-mn(-m+n+mx), 1n(n+1x), -1m(m+1x-mx)\}$$

The intersection K of QS and MN is

$$K := \{-n(-21m(m-n) - (1-m)(1m-1n-mn)x), \\ -n(-1^2m-1m^2+1^2n+m^2n - (1-m)(1m-1n-mn)x), -(21m-1n-mn)(1m-1n+mn)\}$$

The points at infinity of PR and OK are the same

$$\{2mn-2n^2+(1m-1n-mn)x, 1m-1n-mn+2n^2-(1m-1n-mn)x, -1m+1n-mn\}$$

So PR and OK are parallel.

Best regards Eckart

Keizer-01-nb.pdf

Message: #273

Date: 03/10/2013 9:28:42

From: eckart_schmidt@t-online.de

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear friends,

sorry there is something wrong in my message #263. Correct is:

For every point P in the QL-environment the second intersections of the circles through P , $L_1^{\wedge}L_2$, $L_3^{\wedge}L_4$ and P , $L_2^{\wedge}L_3$, $L_4^{\wedge}L_1$ are Clawson-Schmidt Conjugates (QL-Tf1).

If P is a point on QL-Cu1 and P' its QL-Tf1 image and Q , Q' the intersections of the circles, then these points on QL-Cu1 give a quadrigon $PQP'Q'$ with the same point QL-P1, the same line QL-L1 and the same cubic QL-Cu1 as the reference quadrilateral.

Correcting my message the following must hold:

QL-Cu1 is the locus of points P , for which the second intersections of the circles through P , $L_1^{\wedge}L_2$, $L_3^{\wedge}L_4$ and P , $L_2^{\wedge}L_3$, $L_4^{\wedge}L_1$ are

- ... isogonal conjugate wrt a QL triangle component,
- ... or isogonal conjugate wrt the triangle with the vertices QL-P1 and the foci of the inscribed conic with center QL-L1^QL-L6,
- ... or isogonal conjugate wrt the orthic triangle of QL Diagonal Triangle,
- ... or QG-Quasi Isogonal Conjugate (QG-Tf2) wrt a quadrigon component.

Best regards Eckart

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Message: #274

Date: 03/10/2013 10:42:14

From: eckart_schmidt@t-online.de

Subject: A generalisation Brahmagupta's Theorem

Dear Dao Thanh Oai, dear Chris,

I have completed my synthetical proof, see attachment.

Best regards Eckart

A generalization Brahmagupta's Theorem

Quadri-Figures Group message #241 (22.09.2013)

Dear Chris van Tienhoven

Let a quadrilateral $ABCD$, let E point and $\angle DEA + \angle CEB = 180^\circ$ deg and $DE \cdot EB = EC \cdot EA$. F is midpoint of DC . EF meet AB at G . F, I are midpoints of CD, AB respectively. K, L, M are circumcircle of $(EDA), (FIG), (ECB)$

Prove that

1- $\angle DEA = \angle AGE$

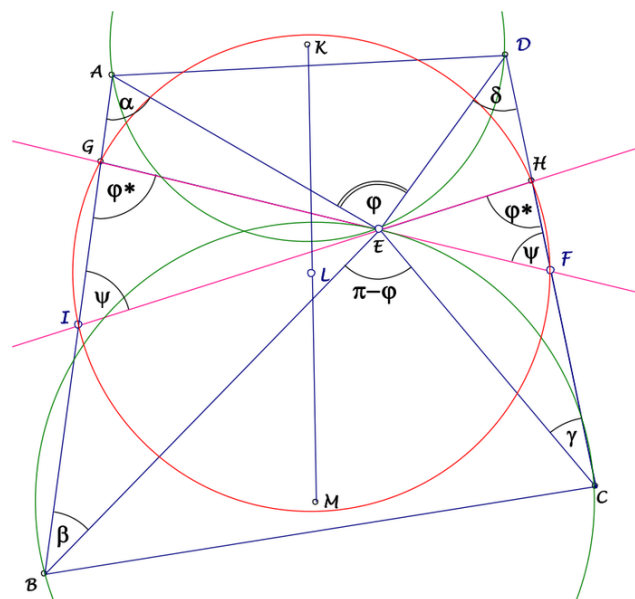
2- L is midpoints of KM

I'm not mathematicians. I want you publish it.

Best regard

Dao Thanh Oai

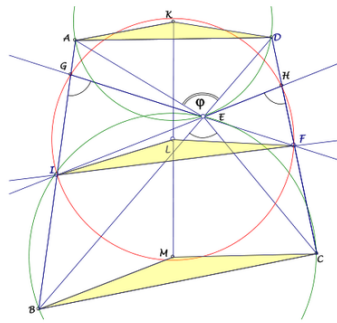
A synthetical prove of the first property:



Brahmagupta-274.pdf

1. $EA \cdot EC = EB \cdot ED \Leftrightarrow \sin \alpha \cdot \sin \gamma = \sin \beta \cdot \sin \delta$ (sine rule)
2. $\varphi + \varphi^* = \alpha + \delta + \psi = 180^\circ - \beta - \gamma + \psi$ (angle balance for $AGFD$ and $BGFC$)
3. $\frac{\sin(\alpha + \psi)}{\sin \alpha} = \frac{AI}{EI}$ and $\frac{\sin(\psi - \beta)}{\sin \beta} = \frac{BI}{EI}$ (sine rule)
4. ... give with $AI = BI$ $\frac{\sin(\alpha + \psi)}{\sin(\psi - \beta)} = \frac{\sin \alpha}{\sin \beta} = \frac{\sin \delta}{\sin \gamma}$
5. Using 2.: $\frac{\sin(\varphi + \varphi^* - \delta)}{-\sin(\varphi + \varphi^* + \gamma)} = \frac{\sin \delta}{\sin \gamma}$
... gives $\sin(\gamma + \delta) \cdot \sin(\varphi + \varphi^*) = 0$
6. For $\sin(\gamma + \delta) \neq 0$ the sum $\varphi + \varphi^*$ must be 180° .

Prove of the second property, using the first property:



The isosceles triangles ADK , IFL , BCM are similar, for central angles are twice the peripheral angles.
The fact, that I is midpoint of AB and F is midpoint of DC implies of course for similar triangles ADK , IFL , BCM , that L is midpoint of KM .

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Brahmagupta-274.pdf

Message: #275

Date: 03/10/2013 11:32:20

From: bernard.keizer

Subject: Tangential quadrilateral well known properties?

Dear Eckart,

I spent hours to try to get a free version of Mathematica, but I didn't succeed!

Can't you send me your file in pdf or Adobe Reader?

Did you also find that the circumcenter O was on the Newton Line of the tangential quadrilateral and that KK' was orthogonal to this Newton Line?

Best regards

Bernard

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Message: #276

Date: 03/10/2013 12:54:33

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Chris, dear Eckart

My understanding of the van Rees focal circular cubic and the 2 points QL-2P2a,b is following:

- * there are 2 ways of drawing the curve: the first one is to determinate the contact points with the tangencies from QL-P1 to a first set of circles and the second one is to determinate the extremities of the diameters through QL-P1 of a second set of circles

- * these 2 sets are orthogonal and have the same 2 points QL-2P2a,b as reference points, basis points for one and Poncelet points for the other

- * if the points are basis points for the first one, they are on the Newton Line, the centers of the circles of this set are on the perpendicular bisector of the segment joining the 2 points, the curve is unicursal and the Newton Line cuts the curve in these 2 points

- * if the points are basis points for the second one, they are on a perpendicular to the Newton Line, the centers of the circles of this set are on the Newton Line (one of those circles through the 2 points is for example the circle with diameter the points QL-P1 and S), the curve is bicursal and the Newton Line doesn't cut the curve

Thanks to Eckart for the inflexion points, but I couldn't open this file either ...

Best regards
Bernard

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Message: #277

Date: 03/10/2013 6:16:42

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Eckart, dear Chris

It's a real pleasure to work this way with you!

Thanks to Eckart, I was able to see your figure.

If we call O the center of the circle through the V_i and QL-P1, I the common point? of the normals in the V_i and K_i the intersection points of the tangents in V_j and V_k , I is the incenter and the K_i the excenters of the V -triangle and the line of the W_i is the orthic axis, perpendicular to OI and the points $V_iV_jW_k$ are on a line perpendicular to OK_i .

The middles of the segments V_iW_i are on the Newton Line QL-L1, which explains that the Simson- and Steiner-Line of QL-P1 wrt the V -triangle is perpendicular to this line.

Last point: the circle through V_i , W_i and QL-P1 is tangent to the curve QL-Cu1 in W_i

All this should bring new ideas about an easy way to construct those points and lines for a given QL-Cu1 ...

Best regards

Bernard

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Message: #278

Date: 03/10/2013 11:30:38

From: Chris

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear friends,

I made some typos in the use of indices i, j, k, l wrt the construction of the Tangential at QL-Cu1.

Here is the right formulation:

Construction Tangential Q of a point P on QL-Cu1

Let P^* be the CSC of P.

Let S_{ij} be a random intersection point of $L_i \wedge L_j$ (i, j are different numbers from set $(1, 2, 3, 4)$)

Let S_{ki}^* be the Isogonal Conjugate of S_{ki} wrt triangle $S_{ij}.P.P^*$.

Let S_{kj}^* be the Isogonal Conjugate of S_{kj} wrt triangle $S_{ij}.P.P^*$.

Let S_{li}^* be the Isogonal Conjugate of S_{li} wrt triangle $S_{ij}.P.P^*$.

Let S_{lj}^* be the Isogonal Conjugate of S_{lj} wrt triangle $S_{ij}.P.P^*$.

Now Q = the isogonal Conjugate of intersection point $S_{ki}^*.S_{kj}^* \wedge S_{li}^*.S_{lj}^*$ wrt triangle $S_{ij}.P.P^*$.

Q is the Tangential of P as well as P^* .

PQ is tangent in P at QL-Cu1. P^*Q is tangent in P^* at QL-Cu1.

Best regards,

Chris

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Message: #279

Date: 04/10/2013 9:29:44

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear friends,

There is a light mistake in my message, for which I apologise.

Please read as following:

... and the points $V_i V_j W_k$ are on a line. Let T_i be the intersection between $V_i K_i$ and $V_j V_k$; then $T_j T_k$ is perpendicular to $T_j T_k$.

The rest seems to hold ...

For Eckart: did you try your drawing with a unicursal curve, with Q1-P1 and Newton Line?

Best regards

Bernard

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Message: #280

Date: 04/10/2013 12:12:14

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard, dear Chris,

Bernard's understanding of QL-2P2a,b has the same background as my understanding. I hope, Chris can change the definition.

If Chris rewrites QL-Cu1, perhaps a remark, that QL-Cu1 is anallamatic (that means invariant under an inversion): see attachment.

Last not least: For a quadrilateral tangent to a circle QL-Cu1 is a strophoid and for a parallelogram the foci of inscribed conics lie on the QA-Orthogonal Hyperbola QA-Co2 (see

<http://eckartschmidt.de/Focus.pdf> and

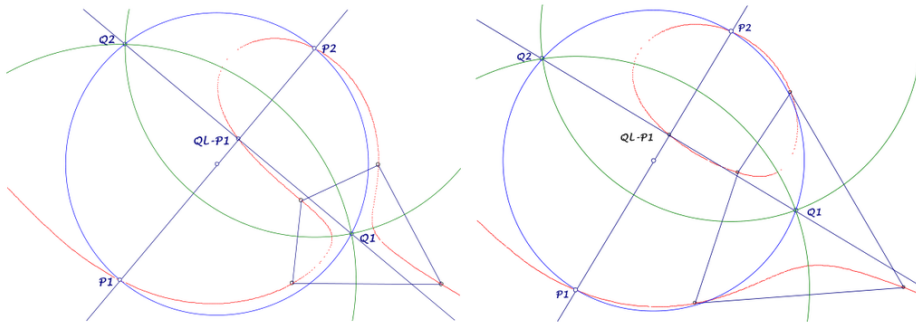
<http://eckartschmidt.de/Strtv.pdf>))

Best regards Eckart

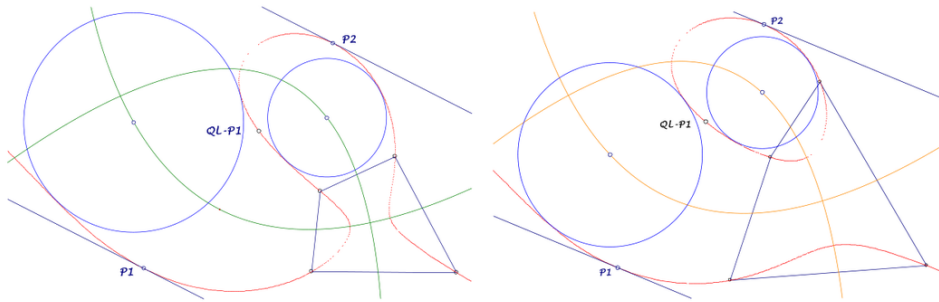
QL-Cu1 as anallagmatic cubic

A curve is anallagmatic, if it is invariant under an inversion. For *QL-Cu1* there are two inversions:

Let Q_1 and Q_2 be the fixed points of *QL-TfI* and P_1, P_2 the intersections of *QL-Cu1* with the perpendicular bisector of Q_1Q_2 . The points P_1, P_2, Q_1, Q_2 lie concyclic with center on *QL-L1*. Then *QL-Cu1* is invariant under reflections in circles round P_1 or P_2 through Q_1, Q_2 (see <http://eckartschmidt.de/Focus.pdf>).



An allagmatic curve is also envelope of two sets of circles: The centers of these circles lie on two parabolas with focus *QL-P1* and directrix in the tangent at P_1 or P_2 parallel *QL-L1*.



Eckart Schmidt

eckart_schmidt@t-online.de

<http://eckartschmidt.de>

QL-Cu1-an.pdf

Message: #281

Date: 04/10/2013 4:03:13

From: eckart_schmidt@t-online.de

Subject: Precisions concerning the curve QL-Cu1

Dear Bernard,

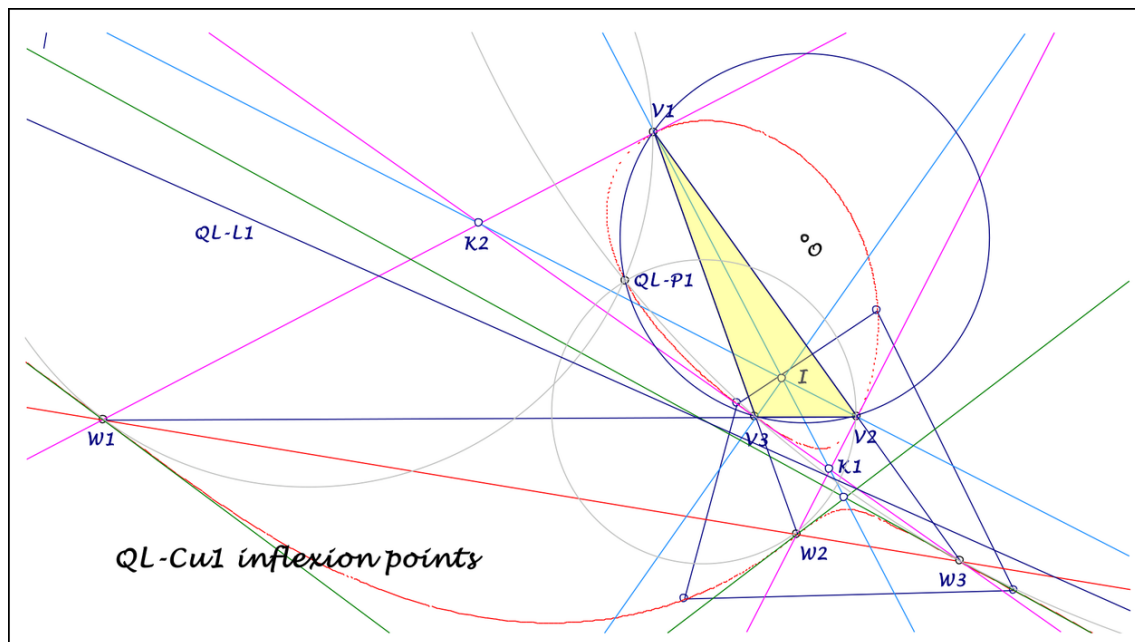
whow, good additional properties for the QL-Cu1 inflexion points constellation!

There are drawings for the unicursal and the bicursal case of QL-Cu1 in pdf-format (see attachment).

Your last correction I haven't understood: "... then TjTk is perpendicular TjTk."

But I disbelieve, that there will be real construction. I think Bernard Gibert will be right: "... no chance", for the inflexion points of QL-Cu1 are also inflexion points of QL-Cu2!

Best regards Eckart



Keizer-06.pdf

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Message: #282
Date: 04/10/2013 5:16:29
From: bernard.keizer
Subject: Precisions concerning the curve QL-Cu1

Dear Eckart,
Thanks for your 2 figures!
Sorry again, it was TjTk is perpendicular to OKi ...Best regards
Bernard

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Message: #283
Date: 06/10/2013 10:55:55
From: eckart_schmidt@t-online.de
Subject: A pair of QG-Cubics

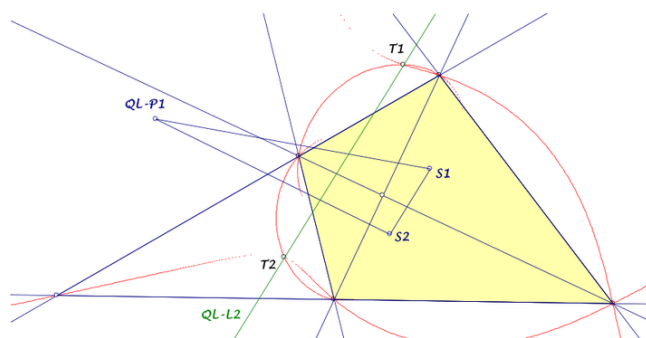
Dear friends,
the loci for points, which have the same angle of view (mod 180°) for the diagonals of a quadrigon, are two cubics (see attachment).
Best regards Eckart

EQF-Note 2013-10-06

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Pair of QG-Cubics

The loci for points, which have the same angle of view (mod 180°) for the diagonals of a quadrigon, are two cubics. – Reference triangle for barycentric coordinates is QA-Diagonal Triangle QA-Tr1.



Constructing about and under the diagonals of a quadrigon the circles for the same peripheral angle, there are four points of intersection, which have as loci two cubics with the equations

$$\begin{aligned} & 2S_A y(r^2 x^2 - p^2 z^2) - 2S_B x(r^2 y^2 - q^2 z^2) \\ & + (c^2 q^2 + b^2 r^2) x^2 z - (c^2 p^2 + a^2 r^2) y^2 z - (b^2 p^2 - a^2 q^2) z^3 = 0, \\ & 2S_B y(q^2 x^2 - p^2 y^2) - 2S_C y(r^2 x^2 - p^2 z^2) \\ & + (b^2 p^2 + a^2 q^2) x z^2 - (c^2 p^2 + a^2 r^2) x y^2 - (b^2 r^2 - c^2 q^2) x^3 = 0. \end{aligned}$$

The points at infinity are

$$\begin{aligned} & (p^2 - q^2 + r^2 : -p^2 + q^2 + r^2 : -2r^2) \\ & (-2p^2 : p^2 + q^2 - r^2 : p^2 - q^2 + r^2). \end{aligned}$$

From the points of these cubics the diagonals of the quadrigon are seen under angles mod 180° .

Properties:

- Both cubics are circumcubics of the quadrigon.
- Both cubics contain the diagonal crosspoint $QG-P1$.
- Each cubic contains one of the intersections $QG-2P2a,b$ of opposite sides.
- Further intersections of the two cubics are the intersections $T1, T2$ of the Thales circles about the diagonals.
- The line $T1T2$ is $QL-L2$.
- Each of the cubics contains one of the vertices $S1, S2$ of the Miquel triangle $QA-Tr2$ unequal $QL-P1$.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-10-06.pdf

Message: #284
Date: 06/10/2013 2:24:31
From: Chris van Tienhoven
Subject: A pair of QG-Cubics

Dear Eckart,

It looks like your cubics are the same as the ones I described in QFG Message #256.

I described 6 cubics for a Quadrilateral, meaning 2 cubics per QL-Quadrignon.

I also described the way how to construct them.

New for me is that one point of the Miquel Triangle lie on these cubics.

The points T1 and T2 are the Pluecker pair of points QL-2P1a/b.

Best regards,
Chris

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Message: #285
Date: 07/10/2013 4:11:36
From: eckart_schmidt@t-online.de
Subject: QL-DT Circumscribed Conics

Dear friends,
in EQF there are no QL-DT circumscribed conics (without the circumcircle). In the attachment you find some examples.
Best regards Eckart

EQF-Note 2013-10-07

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

QL-DT Circumscribed Conics

For a triangle circumscribed conics are isoconjugates of lines. In the QA-environment circumscribed conics of QA-DT are the Involutory Conjugates (QA-Tf2) of lines, for this transformation is an isoconjugation wrt QA-DT with fixed points in the vertices of the quadrangle. Examples: QA-Co1, QA-Co4, QA-Co5. In the QL-environment no circumscribed conics of QL-DT (without the circumcircle) are mentioned in EQF. Here are given some examples. – Reference triangle for barycentric coordinates is QL-DT.

QL-Isoconjugation

Wrt QL-DT there is an isoconjugation with fixed points in the trilinear poles of the side lines of the quadrilateral:

$$(-mn:nl:lm), (mn:-nl:lm), (mn:nl:-lm), (mn:nl:lm).$$

This QL-isoconjugation (for points) is the mapping

$$(x:y:z) \rightarrow (m^2n^2yz:n^2l^2zx:l^2m^2xy)$$

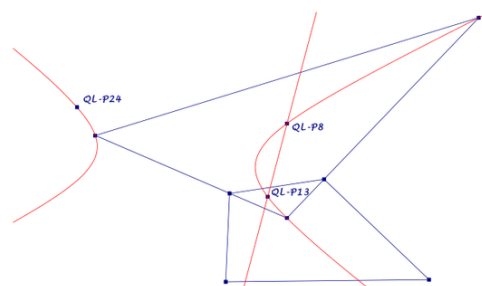
with the special example QL-P8 $\rightarrow$ QL-P13.

Lines will be mapped in QL-circumscribed conics, but there are only a few interesting examples.

QL-L1 The QL-DT circumscribed conic is the Steiner ellipse.

QL-L7,8 The QL-DT circumscribed conics contain QL-P13.

QL-P8, QL-P13 The QL-DT circumscribed conic contains beside QL-P8 and QL-P13 further QL-P24.



2013-10-07.pdf

Another Way to *QL-DT* Circumscribed Conics

Wrt *QL-DT* there is an isoconjugation for lines (*QL-Tf2*). Fixed lines are the side lines of the quadrilateral. This *QL*-isoconjugation for lines is the mapping

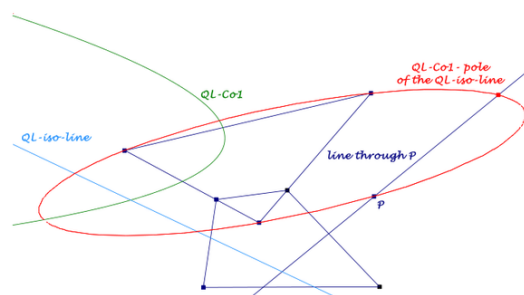
$$(e, f, g) \rightarrow (l^2 fg, m^2 ge, n^2 ef).$$

You get the image line taking the trilinear pole, its *QL*-isoconjugate (for points) and then the trilinear polar.

Construction of a special *QL-DT* circumscribed conic through an arbitrary point $P(u:v:w)$:

Consider the line pencil of P and the images of the lines wrt the *QL*-isoconjugation for lines. The poles wrt the Inscribed Parabola *QL-Co1* are points on the lines again. The locus of these points is a *QL-DT* circumscribed conic with the equation

$$(l^2 - m^2)n^2w xy + (m^2 - n^2)l^2u yz + (n^2 - l^2)m^2v zx = 0.$$



Some Examples:

$P = QL-P2$ The *QL-DT* circumscribed conic is an orthogonal hyperbola through *QL-P2* and *QL-P10*.

$P = QL-P7$ The *QL-DT* circumscribed conic contains *QL-P7* and the point at infinity of *QL-L1* (one asymptote parallel to *QL-L1*).

$P = QL-P10$ The *QL-DT* circumscribed conic is an orthogonal hyperbola through *QL-P10*.

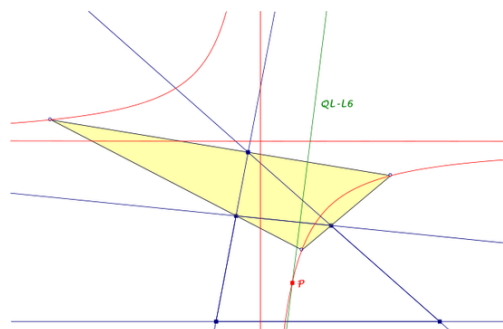
$P = QL-P13$ The *QL-DT* circumscribed conic contains *QL-P8*, *QL-P13*, *QL-P24* (see example above).

$P = QL-P17$ The *QL-DT* circumscribed conic is the circumcircle *QL-Ci1*.

$P = QL-P23$ The *QL-DT* circumscribed conic is a parabola with an axis parallel *QL-L1*.

All points on the line *QL-L6* give *QL-DT* circumscribed orthogonal hyperbolas.

Example: The point at infinity gives an orthogonal hyperbola through $QL-P10$ with asymptotes parallel to $QL-L6$ and $QL-L9$.

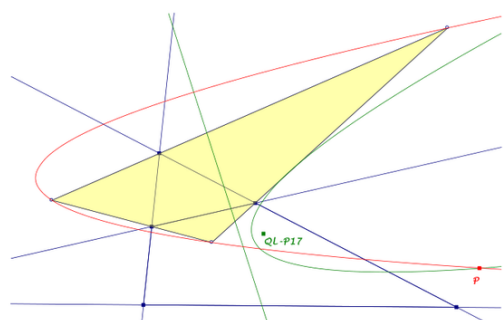


All points on the parabola with focus $QL-P17$ and directrix $QL-L6$ give QL -circumscribed parabolas.

The reference parabola has the equation

$$\sum_{cycl} (l^4 (m^2 - n^2)^2 x^2 - 2(l^2 - m^2)(l^2 - n^2)m^2 n^2 yz) = 0.$$

It is an inscribed conic of $QL-DT$. $QL-L1$ is tangent to this parabola in $QL-P23$. Mapping the parallels to $QL-L1$ by the QL -isoconjugation for lines (see above), the envelope gives this parabola.



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2013-10-07.pdf

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Message: #286
Date: 09/10/2013 4:34:45
From: eckart_schmidt@t-online.de
Subject: Some "splitter"

Dear Chris,

here are some "splitter", found on my desk: perhaps unimportant, perhaps already mentioned:

- * QG-P12 is a point on the QG-circumconic through QA-P1.
 - * QL-P18 is the centroid of the three collinear points QG-P15 on QL-L9.
 - * QG-Co2: QL-P1 lies on the polar of QG-P16 (and invers).
 - * The polar of QG-P12 is parallel QG-L1 (QG-P1 railway watcher).
 - * The tangents in the intersections with QG-L2 are parallel QG-L1.
 - * The tangents in the intersections with QG-P13.QG-P15 are perpendicular QL-L2.
 - * The polar of QG-P3 is QG-P1.QG-P2.
 - * The 3 QL-versions of QG-Co2 touch in the 6 QL-points.
 - * The Miquel points QL-L1 of tangential quadrilaterals wrt circumconics of a quadrangle lie on QA-Ci2.
 - * Center of QA-Co5: Let X be the QA-DT isotomic conjugate of QA-P16, let Y be the point at infinity of the trilinear polar of X wrt QA-DT, let Z be the isoconjugate of QA-P20 wrt a QA-DT isoconjugation with fixed point Y, then Z is the center of QA-Co5.
 - * Center of QA-Co5 is a point on a conic through the side midpoints of QA-DT with center QA-P22 (this conic contains QA-P29).
 - * A new cubic: QA-DT-P23 Cubic (containing QA-P12): Consider the QA-DT circumscribed orthogonal hyperbolas and the intersections with their QA-Tf2 image lines. This leads to a cubic with pivot QA-P23 invariant wrt the isoconjugation QA-Tf2.
- I hope there are no basic errors!

Best regards Eckart

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Message: #287
Date: 10/10/2013 9:47:23
From: Chris van Tienhoven
Subject: Some "splitter"

Dear Eckart,

Nice Splitters!

Some remarks:

* QG-P12 is a point on the QG-circumconic through QA-P1.
This conic is even a QA-circumconic. It passes through the 3
QA-versions of QG-P12 and also through QA-P16.
Its tangent at QA-P1 is the Centroids line QA-L3.
Its tangent at QA-P16 is the line through
QA-P10, QA-P16, QA-P19, QA-P31.

* QG-Co2: The tangents in the intersections with QG-L2 are
parallel QG-L1.

* The tangents in the intersections with QG-P13, QG-P15 are
perpendicular QL-L2.

I cannot confirm this with Cabri.

Regarding the properties wrt QG-Co2, are there similar
properties wrt QG-Co1?

* Center of QA-Co5: Let X be the QA-DT isotomic conjugate of
QA-P16, let Y be the point at infinity of the trilinear polar of
X wrt QA-DT, let Z be the isoconjugate of QA-P20 wrt a QA-DT
isoconjugation with fixed point Y, then Z is the center of
QA-Co5.

I do not understand the part: "Let Z be the Isoconjugate of
QA-P20 wrt"

Best regards,
Chris

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Message: #288
Date: 10/10/2013 2:16:14
From: eckart_schmidt@t-online.de
Subject: Some "splitter"

Dear Chris,

I think, these properties of QG-Co2 are right (the first one holds also for QG-Co1):

- * QG-Co2: The tangents in the intersections with QG-L2 are parallel QG-L1.
- * The tangents in the intersections with QG-P13.QG-P15 are perpendicular QL-L2.

An isoconjugation wrt a reference triangle
and fixed point $(u:v:w)$

is a transformation $(x:y:z) \rightarrow (u^2 y z : v^2 z x : w^2 x y)$.

Best regards Eckart

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Message: #289
Date: 11/10/2013 9:22:09
From: yeuemtrondoitb85
Subject: A generalization Brahmagupta's Theorem

Dear Mr Eckart!

I'm sorry, I don't know your proof about: "A generalization Brahmagupta's Theorem"

Best regard
Dao Thanh Oai

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Message: #290
Date: 11/10/2013 10:05:00
From: Dao Thanh Oai
Subject: A generalization Brahmagupta's Theorem

Dear Eckart!

I didn't understood your proof about: A generalization Brahmagupta theorem!

Best regard
Dao Thanh Oai

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Message: #291
Date: 11/10/2013 8:31:05
From: eckart_schmidt@t-online.de
Subject: A generalization Brahmagupta's Theorem

Dear Mr. Dao Thanh Oai,

for a better understanding of my proof in message #274 Chris has proposed, to add the following phrases:

2. All angles in the constellation can be calculated without EAD and ADE with sum $180^\circ - \phi$ as well as ECB and CBE with sum ϕ .

Using that the sum of all angles in AGFD and BGFC are 360 degrees it follows that:

4. Combining 1 and 3 gives:

When $AI = BI$, then.

5. Combining 2 & 4 and using the formula $\sin(a+b) = \sin(a) \cdot \cos(b) + \cos(a) \cdot \sin(b)$ gives:

I hope, this will be helpful.

Best regards Eckart

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Message: #292
Date: 11/10/2013 8:45:27
From: Dao Thanh Oai
Subject: A generalization Brahmagupta's Theorem

Thank you Eckart!
I didn't notice $AI = BI$
Best regard
Dao Thanh Oai

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Message: #293
Date: 13/10/2013 10:24:52
From: eckart_schmidt@t-online.de
Subject: A Curious QA-Transformation

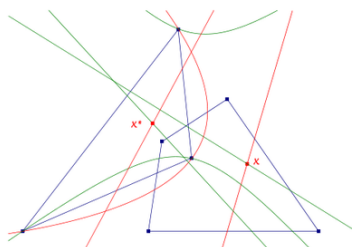
Dear friends,
there is a new type of QA-transformation: not an involution,
but holding fixed the vertices of a quadrangle as well as the
six connection lines. If you are interested, see attachment.
Best regards Eckart

EQF-Note 2013-10-13

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Curious QA-Transformation

This is a new type of QA-transformation: not an involution but holding fixed the vertices of the quadrangle as well as the six connection lines. Further properties are discussed. – Reference triangle for barycentric coordinates is the QA-Diagonal Triangle QA-DT.



Definition QA-Tfx:

Let X be a point, consider lines through X and their Involutory Conjugates (QA-Tf2), which are QA-DT - circumconics. The polars of X wrt these conics have a common point X^* .

For $X = (x : y : z)$ the image point is

$$X^* = (x(-q^2r^2x^2 + r^2p^2y^2 + p^2q^2z^2) : y(q^2r^2x^2 - r^2p^2y^2 + p^2q^2z^2) : z(q^2r^2x^2 + r^2p^2y^2 - p^2q^2z^2)).$$

Properties:

1. QA-Tfx is not an involution.
2. The vertices P_i of the quadrangle as well as the vertices S_i of QA-DT are fixed points.
3. The lines P_iP_j of the quadrangle as well as the sidelines S_iS_j of QA-DT are invariant wrt QA-Tfx.
4. All points on the trilinear polar of P_i wrt QA-DT have the image P_i .
5. All points X, X^* on a sideline S_iS_j of QA-DT divide S_iS_j harmonic.

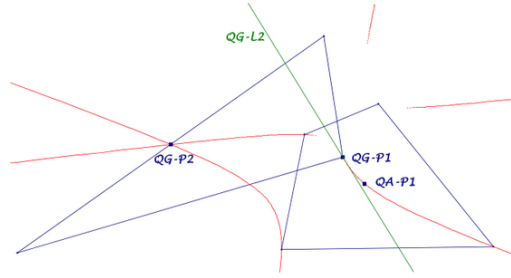
6. Circumscribed conics of the quadrangle are invariant under $QA-Tfx$.

7. $QA-Tfx$ mappings of lines unequal P_iP_j and S_iS_j are circumcubics of the quadrangle. For a line with the coefficients e, f, g The cubic has the equation

$$eq^2r^2(f^2q^2 - g^2r^2)x^3 + fr^2p^2(g^2r^2 - e^2p^2)y^3 + gp^2q^2(e^2p^2 - f^2q^2)z^3 \\ + fq^2r^2(2e^2p^2 - f^2q^2 + g^2r^2)x^2y + ep^2r^2(e^2p^2 - 2f^2q^2 - g^2r^2)xy^2 \\ + gq^2r^2(-2e^2p^2 - f^2q^2 + g^2r^2)x^2z + ep^2q^2(e^2p^2 - f^2q^2 - 2g^2r^2)xz^2 \\ + gp^2r^2(e^2p^2 + 2f^2q^2 - g^2r^2)y^2z + fp^2q^2(-e^2p^2 + f^2q^2 - 2g^2r^2)yz^2 = 0$$

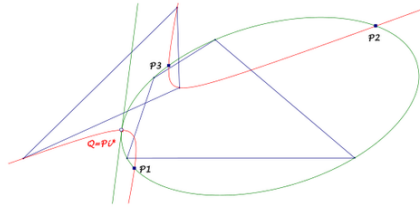
Example $QA-Tfx(QG-L2)$ with equation:

$$(p^2 - r^2)(p^2z - r^2x)y^2 - q^2(z - x)^2(p^2z + r^2x) = 0$$



This circumscribed cubic of the quadrangle contains $QG-P1$ as inflexion point, $QG-P2$ as nodal point and $QA-P1$.

8. For a point Q on a circumconic of a quadrangle there are three points P_i with $QA-Tfx$ -image Q . All four points lie on a circumconic of $QA-DT$, which is the $QA-Tf2$ image of the tangent at Q to the QA -circumconic.



9. For a line L unequal P_iP_j and S_iS_j there is a cubic, whose $QA-Tfx$ -image is the line L .

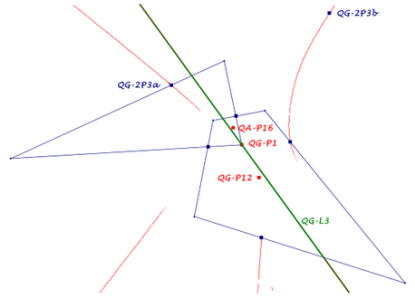
If the line has the coefficients e, f, g , the equation of the cubic is

$$-eq^2r^2x^3 - fr^2p^2y^3 - gp^2q^2z^3 + fq^2r^2x^2y + er^2p^2xy^2 \\ + gp^2r^2y^2z + fp^2q^2yz^2 + ep^2q^2xz^2 + fp^2q^2yz^2 = 0.$$

The cubic can be constructed in the following way: Let X be a point of the line L and Co_1 the QA -circumconic through X . The $QA-Tf2$ -image of the tangent in X at Co_1 is a $QA-DT$ -circumconic Co_2 . The intersections of Co_1 and Co_2 are points of the cubic.

Example: QG -Cubic with QA - Tfx -image QG - $L3$ and equation

$$p^2(-p^2+q^2+r^2)(q^2r^2x^2+r^2p^2y^2-p^2q^2z^2)z + r^2(p^2+q^2-r^2)(-q^2r^2x^2+r^2p^2y^2+p^2q^2z^2)x=0$$



This cubic contains QG - $P1$ as inflexion point, QG - $2P3a,b$, QA - $P16$ and QG - $P12$. Further points on the cubic are the intersections of the lines QG - $P1$, QG - $2P2a,b$ with the sidelines of the quadrilateral.

Final remark

There are analogous relationships in the QL -environment, defining the following transformation for lines:

Definition QL - Tfx :

Let L be a line, consider points X on L and for the lines through X the QL -Line Isoconjugates (QL - $Tf2$), which envelope QL - DT -inconics. The poles of L wrt these conics are collinear on L^* .

Eckart Schmidt
<http://eckartschmidt.de>
 eckart_schmidt@t-online.de

Message: #294

Date: 13/10/2013 1:57:02

From: Dao Thanh Oai

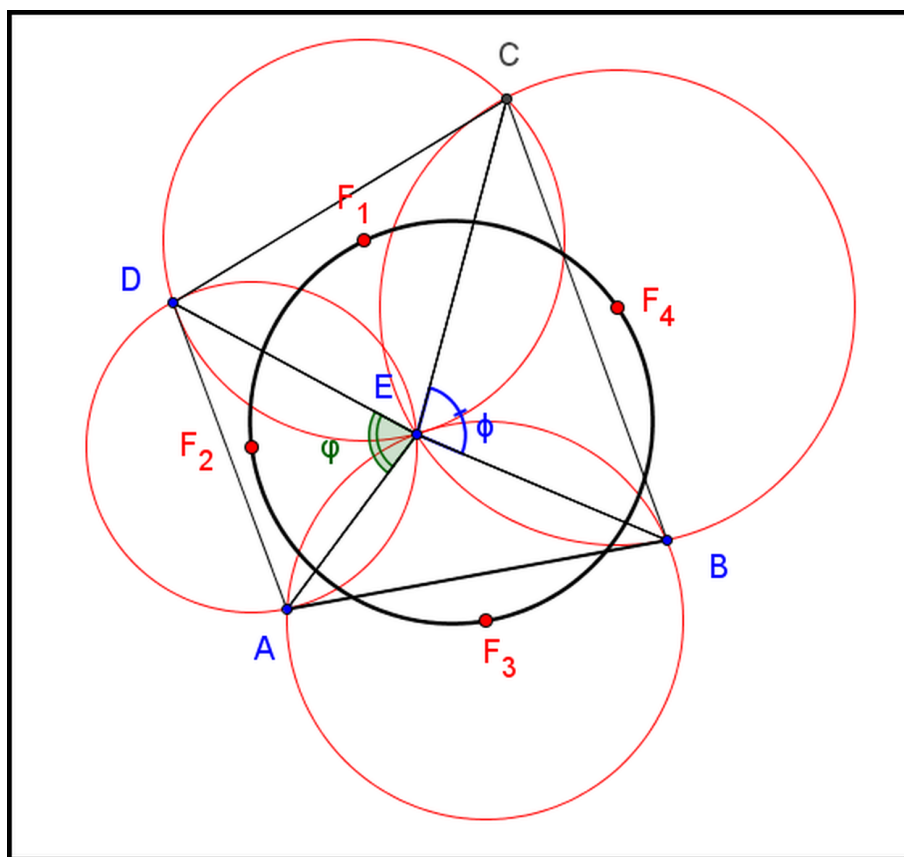
Subject: A generalisation Brahmagupta's Theorem

Dear Eckart and Chris!

I discovered one problem: Centre of four circle AEB,BEC,CED,DEA are cyclic.

Best regard

Dao Thanh Oai



Centre of four circle are cyclic.png

Message: #295
Date: 13/10/2013 2:44:44
From: Dao Thanh Oai
Subject: A generalisation Brahmagupta's Theorem

Dear Chris and Eckart!
I'm sorry because last property very easily.
Best regard
Dao Thanh Oai

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Message: #296
Date: 13/10/2013 10:54:52
From: Chris
Subject: A Curious QA-Transformation

Dear Eckart,
Very nice QA-transformation!
I found these examples:
Let:
X1= InvolutionCenter QA-P1.QA-P16 (see QA-Tf1)
X2= InvolutionCenter QA-P1.QA-P31 (see QA-Tf1)
X3= QA-DT-ConicPerspector of QA-Co5 (see QA-Co/1)
X4= QA-DT-ConicPerspector of QA-DT-Conic through P5-P17-P18-P27
 (see QA-Co/1)
X5= Infinity point QA-P1.QA-P5
QA-Tfx(QA-P1) lies on QA-P1.X3};
QA-Tfx(QA-P1) lies on QA-P16. QA-P22;
QA-Tfx(QA-P5) lies on QA-P5.X4;
QA-Tfx(QA-P5) lies on QA-P10.X3;
QA-Tfx(QA-P10) lies on QA-P5.X3;
QA-Tfx(QA-P10) lies on QA-P10.X1;
QA-Tfx(QA-P16) = QA-P1;
QA-Tfx(QA-P17) lies on QA-P17.X4;
QA-Tfx(QA-P18) lies on QA-P10.X2
QA-Tfx(QA-P18) lies on QA-P16.QA-P17;
QA-Tfx(QA-P20) lies on QA-P20.X3;
QA-Tfx(QA-P22) lies on X3.X5;
It is special that only points on the the Centroids Line QA-L3
(apart from QA-P16) are transformed such that they lie on lines
with other known points.
Best regards,
Chris

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Message: #297
Date: 13/10/2013 11:24:00
From: Chris
Subject: A Curious QA-Transformation

Dear Eckart,

Also:

- * QA-Tfx(QA-P17) lies on cubic QA-Cu2
- * QA-Tfx(QA-P18) lies on cubic QA-Cu4
- * QA-Tfx(QA-P20) lies on cubic QA-Cu5

Best regards,

Chris

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Message: #298

Date: 15/10/2013 4:18:32

From: Seiichi Kirikami

Subject: Algebraic computation of quadratic, cubic and quartic quadrangle poi

Dear friends,

5 conditions of message #211 are as follows:

(1) $pf(p+q+r, -q, -r)/\text{cyclic sum}$

$f(p+q+r, -q, -r) = (p+q+r)f(p, q, r)/\text{cyclic sum } f(p, q, r).$

(2) $(p+q+r)f(q, r, p)/\text{cyclic sum } f(q, r, p) = qf(p+q+r, -r, -p)/\text{cyclic sum } f(p+q+r, -r, -p).$

(3) $qf(-r, -p, p+q+r)/\text{cyclic sum}$

$f(-r, -p, p+q+r) = -rf(q, p, -p-q-r)/\text{cyclic sum } f(q, p, -p-q-r).$

(4) $-rf(p, -p-q-r, q)/\text{cyclic sum}$

$f(p, -p-q-r, q) = pf(-r, p+q+r, -q)/\text{cyclic sum } f(-r, p+q+r, -q).$

(5) $p+q =$ the sum of the right-handed sides of (1), (2), (3) and (4).

The 1st condition is most important. The other conditions are for confirmation.

The 1st condition is an identity. The arbitrary coordinates of the 4th point $P_4 \{p, q, r\}$ satisfy it.

We choose $P_4 \{1, n, n\}$, $n=2, 3, \dots$. We suppose the factor $(2p+q+r)$ in $f(p, q, r)$.

The following solutions are obtained by Mathematica 8. See the attached files.

[1] Quadratic cases:

Assumption: $f(p, q, r) = (2p+q+r)(ap+b(q+r)).$

Solutions: (1) $\{b \rightarrow 0\}$, (2) $\{b \rightarrow a\}$.

Reductions:

(1) $f(p, q, r) = (2p+q+r)p$, (QAP16).

The other case degenerates to linear case $2p+q+r$, (QAP1).

[2] Cubic cases:

Assumption: $f(p, q, r) = (2p+q+r)(ap^2+b(q^2+r^2)+cp(q+r)+dqr).$

Solutions: (1) $\{b \rightarrow a, c \rightarrow d\}$, (2) $\{c \rightarrow a(a+b)/(a-2b), d \rightarrow -2b\}$, (3) $\{c \rightarrow a, d \rightarrow b\}$, (4) $\{a \rightarrow 0, b \rightarrow 0, c \rightarrow d\}$, (5) $\{a \rightarrow 0, b \rightarrow 0, d \rightarrow 0\}$, (6) $\{b \rightarrow a/2, c \rightarrow a, d \rightarrow a/2\}$.

Reductions:

(3) $f(p, q, r) = (2p+q+r)(ap(p+q+r)+b(q^2+r^2+qr)).$

$a=0$ simplifies $f(p, q, r) = (2p+q+r)(q^2+r^2+qr)$, (QAP21).

$b=0$ simplifies $f(p, q, r) = (2p+q+r)p$, (QAP16).

(5) $f(p, q, r) = (2p+q+r)cp(q+r)$, simplifies to $(2p+q+r)p(q+r)$, (QAP10).

The other cases degenerates to linear or quadratic cases.

[3] Quartic cases:

Assumption: $f(p,q,r)=(2p+q+r)(ap^3+bp^2(q+r)+cp(q^2+r^2)+dpqr+e(q^3+r^3)+fqr(q+r))$.

Solutions: (1) $\{a \rightarrow c-2e, b \rightarrow c-2e, f \rightarrow -e\}$, (2) $\{a \rightarrow e, b \rightarrow c, f \rightarrow c\}$, (3) $\{b \rightarrow a+c-e, d \rightarrow (-a^2+ac-2c^2-2ae+4ce-2e^2)/(-c+e), f \rightarrow (a^2-ac+ae+2ce-2e^2)/a\}$, (4) $\{b \rightarrow (-a^2+ac)/2/e, d \rightarrow (-a^3+a^2c+2a^2e-2ace+2ae^2+4ce^2)/2/(a-e)/e, f \rightarrow -e\}$, (5) $\{a \rightarrow 0, b \rightarrow 0, d \rightarrow -2c, f \rightarrow -e\}$, (6) $\{a \rightarrow c-2e, b \rightarrow c-2e, d \rightarrow -2c, f \rightarrow -e\}$, (7) $\{a \rightarrow 0, b \rightarrow c, d \rightarrow 2c, e \rightarrow 0\}$, (8) $\{a \rightarrow 0, b \rightarrow c, e \rightarrow 0, f \rightarrow c\}$, (9) $\{a \rightarrow 0, d \rightarrow 2(b^2-bc+c^2)/(2b-c), e \rightarrow 0, f \rightarrow 0\}$, (10) $\{a \rightarrow c, b \rightarrow c, e \rightarrow 0, f \rightarrow 0\}$, (11) $\{a \rightarrow c, d \rightarrow b, e \rightarrow 0, f \rightarrow 0\}$, (12) $\{b \rightarrow a+c, d \rightarrow (a^2-ac+c^2)/c, e \rightarrow 0, f \rightarrow a-c\}$, (13) $\{a \rightarrow c/3, b \rightarrow c/3, e \rightarrow c/3, f \rightarrow -c/3\}$, (14) $\{a \rightarrow c/3, b \rightarrow c, e \rightarrow c/3, f \rightarrow c\}$, (15) $\{a \rightarrow 0, b \rightarrow 0, e \rightarrow c/2, f \rightarrow -c/2\}$, (16) $\{a \rightarrow 0, b \rightarrow 0, e \rightarrow c, f \rightarrow 2c^2/d\}$, (17) $\{a \rightarrow -c, b \rightarrow -c, e \rightarrow c, f \rightarrow -c\}$.

Reductions:

(1) With additional condition ($e=0$),
 $f(p,q,r)=(2p+q+r)p(c(p^2+p(q+r)+q^2+r^2)+dqr)$.

(7) $f(p,q,r)=(2p+q+r)(q+r)(cp(p+q+r)+fqr)$.

$c=0$ simplifies $f(p,q,r)=(2p+q+r)(q+r)qr$, (QAP5).

$f=0$ simplifies $f(p,q,r)=(2p+q+r)p(q+r)$, (QAP10).

(16) With additional condition ($c=d$),
 $f(p,q,r)=(2p+q+r)(p(q^2+r^2+qr)+q^3+r^3+2qr(q+r))$.

The other cases degenerates to less than quartic cases or do not satisfy the other conditions.

As for the quintic case, Mathematica 8 did not finish itself in 30 minutes. So I obtained no results.

[4] Summary

(1) quadratic: $f(p,q,r)=(2p+q+r)p$.

(2) cubic: $f(p,q,r)=(2p+q+r)(ap(p+q+r)+b(q^2+r^2+qr))$.
 $f(p,q,r)=(2p+q+r)p(q+r)$.

(3) quartic: $f(p,q,r)=(2p+q+r)p(c(p^2+p(q+r)+q^2+r^2)+dqr)$.

$f(p,q,r)=(2p+q+r)(q+r)(cp(p+q+r)+fqr)$.

$f(p,q,r)=(2p+q+r)(p(q^2+r^2+qr)+q^3+r^3+2qr(q+r))$.

Best regards,
 Seiichi.

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(* Quadratic QA-points *)
f1[1] = (2 p + q + r) (a p + b (q + r));
f1[2] = f1[1] /. {p -> q, q -> r, r -> p};
f1[3] = f1[2] /. {p -> q, q -> r, r -> p};
F1 = Sum[f1[i], {i, 3}] // Simplify
f2[1] = f1[1] /. {p -> p + q + r, q -> -q, r -> -r};
f2[2] = f1[2] /. {p -> p + q + r, q -> -q, r -> -r};
f2[3] = f1[3] /. {p -> p + q + r, q -> -q, r -> -r};
F2 = Sum[f2[i], {i, 3}] // Simplify
(p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 2, r -> 2} // Simplify
(p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 3, r -> 3} // Simplify
2 a (p2 + q2 + q r + r2 + p (q + r)) + 2 b (p2 + q2 + 3 q r + r2 + 3 p (q + r))
2 (b (p2 - q2 - q r - r2 - p (q + r)) + a (p2 + q2 + q r + r2 + p (q + r)))
2016 (a - b) b
15360 (a - b) b

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(* Cubic QA-points *)
f1[1] = (2 p + q + r) (a p^2 + b (q^2 + r^2) + c p (q + r) + d q r);
f1[2] = f1[1] /. {p -> q, q -> r, r -> p};
f1[3] = f1[2] /. {p -> q, q -> r, r -> p};
F1 = Sum[f1[i], {i, 3}] // Simplify
f2[1] = f1[1] /. {p -> p + q + r, q -> -q, r -> -r};
f2[2] = f1[2] /. {p -> p + q + r, q -> -q, r -> -r};
f2[3] = f1[3] /. {p -> p + q + r, q -> -q, r -> -r};
F2 = Sum[f2[i], {i, 3}] // Simplify
poly1 = (p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 2, r -> 2} // Simplify
poly2 = (p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 3, r -> 3} // Simplify
poly3 = (p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 4, r -> 4} // Simplify
poly4 = (p + q + r) f1[1] F2 - p f2[1] F1 /. {p -> 1, q -> 5, r -> 5} // Simplify
(* solutions *)
Solve[poly1 == 0 && poly2 == 0 && poly3 == 0 && poly4 == 0, {a, b, c, d}]
(p + q + 2 r) (d p q + b (p^2 + q^2) + r (c (p + q) + a r)) +
(p + 2 q + r) (a q^2 + d p r + c q (p + r) + b (p^2 + r^2)) +
(2 p + q + r) (a p^2 + d q r + c p (q + r) + b (q^2 + r^2))

(2 p + q + r) (d q r - c (q + r) (p + q + r) + a (p + q + r)^2 + b (q^2 + r^2)) +
(p - r) (a r^2 - c r (p + r) - d q (p + q + r) + b (q^2 + (p + q + r)^2)) +
(p - q) (a q^2 - c q (p + q) - d r (p + q + r) + b (r^2 + (p + q + r)^2))

-1008 (5 a^2 + 8 b^2 + 2 b (9 c - 2 d) + 4 (c - d) d - a (13 b + 5 c + 9 d))
-7680 (7 a^2 + 18 b^2 + b (32 c - 9 d) + 9 (c - d) d - a (25 b + 7 c + 16 d))
-31200 (9 a^2 - a (41 b + 9 c + 25 d) + 2 (16 b^2 + b (25 c - 8 d) + 8 (c - d) d))
-92160 (11 a^2 + 50 b^2 + b (72 c - 25 d) + 25 (c - d) d - a (61 b + 11 c + 36 d))

Solve::svars: 方程式はすべての "solve" 変数に対しては解を与えない可能性があります。 >>
{ {b -> a, c -> d}, {c ->  $\frac{a(a+b)}{a-2b}$ , d -> -2 b}, {c -> a, d -> b},
{a -> 0, b -> 0, c -> d}, {a -> 0, b -> 0, d -> 0}, {b ->  $\frac{a}{2}$ , c -> a, d ->  $\frac{a}{2}$ } }

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(* Confirmation of solutions *)
(* {b→a,c→d} *)
f1[1] = (2 p + q + r) (a p^2 + a (q^2 + r^2) + d p (q + r) + d q r);
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Simplify
0
0
0
0
0
0

```

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(* {c→ $\frac{a(a+b)}{a-2b}$ , d→-2 b} *)
f1[1] = (2 p + q + r) (a p^2 + b (q^2 + r^2) + a (a + b) p (q + r) / (a - 2 b) - 2 b q r);
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
(3 (a - 2 b) (a - b) b (q - r)^2 (q + r) (2 p + q + r)^2 (a p (p + q + r) + b (p^2 - 3 q r + p (q + r)))) /
(2 (-b^2 (2 p^3 + 2 q^3 + q^2 r + q r^2 + 2 r^3 + p^2 (q + r) + p (q^2 - 12 q r + r^2)) +
a b (-p^3 + p^2 (q + r) - (q - r)^2 (q + r) + p (q^2 - 3 q r + r^2)) +
a^2 (p^3 + q^3 + 2 q^2 r + 2 q r^2 + r^3 + 2 p^2 (q + r) + p (2 q^2 + 3 q r + 2 r^2)))
(a^2 p (p^2 + q^2 + q r + r^2 + p (q + r)) - a b (p^3 + 4 p^2 (q + r) + 9 q r (q + r) +
p (4 q^2 + 13 q r + 4 r^2)) - b^2 (2 p^3 + 5 p^2 (q + r) - 9 q r (q + r) + p (5 q^2 - 4 q r + 5 r^2))))

```


$$\begin{aligned}
& - \left(3(a-2b)(a-b)b(p-r)^2(p+r)(p+2q+r)^2(aq(p+q+r)+b(p(q-3r)+q(q+r))) \right) / \\
& \left(2(-b^2(2p^3+2q^3+q^2r+qr^2+2r^3+p^2(q+r)+p(q^2-12qr+r^2))+ \right. \\
& \quad ab(-p^3+p^2(q+r)-(q-r)^2(q+r)+p(q^2-3qr+r^2))+ \\
& \quad a^2(p^3+q^3+2q^2r+2qr^2+r^3+2p^2(q+r)+p(2q^2+3qr+2r^2))) \\
& \quad (a^2q(p^2+q^2+qr+r^2+p(q+r))-b^2(p^2(5q-9r)+p(5q^2-4qr-9r^2)+ \\
& \quad q(2q^2+5qr+5r^2))-ab(q(q+2r)^2+p^2(4q+9r)+p(4q^2+13qr+9r^2))) \Big) \\
& - \left(3(a-2b)(a-b)b(q-r)^2(q+r)(2p+q+r)^2(aqr+b(-3p^2+qr-3p(q+r))) \right) / \\
& \left(2(a^2r(p^2+q^2+qr+r^2+p(q+r))+ \right. \\
& \quad b^2(p^2(9q-5r)+p(9q^2+4qr-5r^2)-r(5q^2+5qr+2r^2))- \\
& \quad ab(r(2q+r)^2+p^2(9q+4r)+p(9q^2+13qr+4r^2))) \\
& \quad (a^2q(p^2+q^2+qr+r^2+p(q+r))-b^2(p^2(5q-9r)+p(5q^2-4qr-9r^2)+ \\
& \quad q(2q^2+5qr+5r^2))-ab(q(q+2r)^2+p^2(4q+9r)+p(4q^2+13qr+9r^2))) \Big) \\
& \left(3(a-2b)(a-b)b(p-r)^2(p+r)(p+2q+r)^2(apr+b(p(-3q+r)-3q(q+r))) \right) / \\
& \left(2(a^2r(p^2+q^2+qr+r^2+p(q+r))+ \right. \\
& \quad b^2(p^2(9q-5r)+p(9q^2+4qr-5r^2)-r(5q^2+5qr+2r^2))- \\
& \quad ab(r(2q+r)^2+p^2(9q+4r)+p(9q^2+13qr+4r^2))) (a^2p(p^2+q^2+qr+r^2+p(q+r))- \\
& \quad ab(p^3+4p^2(q+r)+9qr(q+r)+p(4q^2+13qr+4r^2))- \\
& \quad b^2(2p^3+5p^2(q+r)-9qr(q+r)+p(5q^2-4qr+5r^2))) \Big) \\
& \left(3(a-2b)(a-b)b(p-q)(p+q)(p+q+2r) \right. \\
& \quad (a^5p(p-q)qr(p^2+2pq+q^2+3pr+3qr+2r^2)(p^2+q^2+qr+r^2+p(q+r))^2+ \\
& \quad ab^4(-p^8(9q^2+4qr+39r^2)-p^7(48q^3+51q^2r+116qr^2+201r^3)- \\
& \quad p^6(165q^4+97q^3r-243q^2r^2+461qr^3+336r^4)- \\
& \quad p^5(192q^5+98q^4r-702q^3r^2-1077q^2r^3+390qr^4+171r^5)+ \\
& \quad 3r^2(q+r)^2(3q^6+2q^5r+20q^4r^2+56q^3r^3+16q^2r^4-11qr^5-5r^6)+ \\
& \quad p^4(21q^6+290q^5r+96q^4r^2+873q^3r^3+1624q^2r^4+323qr^5+45r^6)+ \\
& \quad p^3(111q^7+538q^6r-51q^5r^2-2028q^4r^3-819q^3r^4+1041q^2r^5+371qr^6-27r^7)+ \\
& \quad p^2(39q^8+228q^7r+156q^6r^2-1392q^5r^3-2161q^4r^4- \\
& \quad 531q^3r^5+141q^2r^6-227qr^7-141r^8)+pr \\
& \quad (4q^8+53q^7r+20q^6r^2+123q^5r^3+640q^4r^4+553q^3r^5-145q^2r^6-297qr^7-87r^8)) + \\
& \quad a^2b^3(p^8(27q^2-44qr+9r^2)+p^7(90q^3-69q^2r-202qr^2+117r^3)+ \\
& \quad p^6(171q^4-122q^3r-564q^2r^2-124qr^3+387r^4)+ \\
& \quad p^5(225q^5+20q^4r-969q^3r^2-762q^2r^3+276qr^4+594r^5)- \\
& \quad 9r^2(q+r)^2(3q^6+8q^5r+14q^4r^2+11q^3r^3+4q^2r^4+4qr^5+r^6)+ \\
& \quad p^4(180q^6+322q^5r-396q^4r^2-1023q^3r^3-349q^2r^4+355qr^5+459r^6)+ \\
& \quad p^3(45q^7+257q^6r+258q^5r^2+168q^4r^3+189q^3r^4+72q^2r^5-20qr^6+135r^7)+p^2 \\
& \quad (-9q^8+78q^7r+123q^6r^2+249q^5r^3+691q^4r^4+720q^3r^5+117q^2r^6-241qr^7-36r^8)+ \\
& \quad pr(44q^8+67q^7r-11q^6r^2-15q^5r^3+158q^4r^4+218q^3r^5-47q^2r^6-162qr^7-36r^8)) + \\
& \quad a^3b^2(p^8(9q^2+19qr+15r^2)+p^7(24q^3+72q^2r+143qr^2+105r^3)+ \\
& \quad p^6(69q^4+85q^3r+303q^2r^2+566qr^3+258r^4)+ \\
& \quad p^5(114q^5+74q^4r+294q^3r^2+1038q^2r^3+1053qr^4+297r^5)+ \\
& \quad 3r^2(q+r)^2(-3q^6-10q^5r-28q^4r^2-34q^3r^3-8q^2r^4+qr^5+r^6)+ \\
& \quad p^4(57q^6-62q^5r-120q^4r^2+735q^3r^3+1490q^2r^4+946qr^5+171r^6)+ \\
& \quad p^3(-15q^7-202q^6r-669q^5r^2-540q^4r^3+315q^3r^4+762q^2r^5+442qr^6+63r^7)- \\
& \quad \left. \right)
\end{aligned}$$

$$\begin{aligned}
& p^2 (15 q^8 + 129 q^7 r + 534 q^6 r^2 + 1047 q^5 r^3 + 1235 q^4 r^4 + 756 q^3 r^5 + \\
& 21 q^2 r^6 - 161 q r^7 - 39 r^8) - p r (19 q^8 + 152 q^7 r + 449 q^6 r^2 + \\
& 906 q^5 r^3 + 1153 q^4 r^4 + 706 q^3 r^5 + 101 q^2 r^6 - 63 q r^7 - 21 r^8) + \\
& a^4 b (-p^8 (9 q^2 + 4 q r + 3 r^2) - p^7 (39 q^3 + 42 q^2 r + 17 q r^2 + 30 r^3) - \\
& p^6 (75 q^4 + 97 q^3 r + 63 q^2 r^2 + 101 q r^3 + 84 r^4) - \\
& p^5 (75 q^5 + 98 q^4 r + 36 q^3 r^2 + 165 q^2 r^3 + 237 q r^4 + 117 r^5) + \\
& 3 r^2 (q + r)^2 (3 q^6 + 5 q^5 r + 5 q^4 r^2 + 5 q^3 r^3 + 4 q^2 r^4 + 4 q r^5 + r^6) - \\
& 2 p^4 (21 q^6 + 8 q^5 r - 66 q^4 r^2 + 151 q^3 r^4 + 131 q r^5 + 45 r^6) - \\
& p^3 (6 q^7 - 52 q^6 r - 273 q^5 r^2 - 285 q^4 r^3 + 63 q^3 r^4 + 246 q^2 r^5 + 124 q r^6 + 27 r^7) + \\
& p^2 (3 q^8 + 39 q^7 r + 210 q^6 r^2 + 336 q^5 r^3 + 188 q^4 r^4 - 18 q^3 r^5 - 39 q^2 r^6 + 25 q r^7 + 12 r^8) + \\
& p r (4 q^8 + 62 q^7 r + 146 q^6 r^2 + 150 q^5 r^3 + 91 q^4 r^4 + 58 q^3 r^5 + 71 q^2 r^6 + 54 q r^7 + 12 r^8) + \\
& b^5 (-2 p^8 (9 q^2 - 32 q r + 15 r^2) + p^7 (-75 q^3 + 135 q^2 r + 251 q r^2 - 183 r^3) + \\
& p^6 (-192 q^4 + 211 q^3 r + 213 q^2 r^2 + 341 q r^3 - 381 r^4) + \\
& p^5 (-228 q^5 + 56 q^4 r + 69 q^3 r^2 - 825 q^2 r^3 + 639 q r^4 - 351 r^5) + \\
& 3 r^2 (q + r)^2 (6 q^6 + 11 q^5 r + 29 q^4 r^2 + 50 q^3 r^3 + 16 q^2 r^4 - 2 q r^5 - 2 r^6) - \\
& p^4 (60 q^6 + 80 q^5 r - 240 q^4 r^2 + 1245 q^3 r^3 + 1075 q^2 r^4 - 1189 q r^5 + 153 r^6) + \\
& p^3 (57 q^7 + 23 q^6 r + 681 q^5 r^2 + 855 q^4 r^3 - 630 q^3 r^4 + 42 q^2 r^5 + 964 q r^6 - 72 r^7) + \\
& p^2 (30 q^8 - 21 q^7 r + 249 q^6 r^2 + 843 q^5 r^3 + 565 q^4 r^4 - 54 q^3 r^5 + 42 q^2 r^6 + 152 q r^7 - 78 r^8) - \\
& p r (64 q^8 + 233 q^7 r + 575 q^6 r^2 + 933 q^5 r^3 + 775 q^4 r^4 + \\
& 436 q^3 r^5 + 272 q^2 r^6 + 126 q r^7 + 42 r^8) + \dots) / \\
& (2 (-b^2 (2 p^3 + 2 q^3 + q^2 r + q r^2 + 2 r^3 + p^2 (q + r) + p (q^2 - 12 q r + r^2)) + \\
& a b (-p^3 + p^2 (q + r) - (q - r)^2 (q + r) + p (q^2 - 3 q r + r^2)) + \\
& a^2 (p^3 + q^3 + 2 q^2 r + 2 q r^2 + r^3 + 2 p^2 (q + r) + p (2 q^2 + 3 q r + 2 r^2))) \\
& (a^2 r (p^2 + q^2 + q r + r^2 + p (q + r)) + \\
& b^2 (p^2 (9 q - 5 r) + p (9 q^2 + 4 q r - 5 r^2) - r (5 q^2 + 5 q r + 2 r^2)) - \\
& a b (r (2 q + r)^2 + p^2 (9 q + 4 r) + p (9 q^2 + 13 q r + 4 r^2))) \\
& (a^2 p (p^2 + q^2 + q r + r^2 + p (q + r)) - \\
& a b (p^3 + 4 p^2 (q + r) + 9 q r (q + r) + p (4 q^2 + 13 q r + 4 r^2)) - \\
& b^2 (2 p^3 + 5 p^2 (q + r) - 9 q r (q + r) + p (5 q^2 - 4 q r + 5 r^2))) \\
& (a^2 q (p^2 + q^2 + q r + r^2 + p (q + r)) - \\
& b^2 (p^2 (5 q - 9 r) + p (5 q^2 - 4 q r - 9 r^2) + q (2 q^2 + 5 q r + 5 r^2)) - \\
& a b (q (q + 2 r)^2 + p^2 (4 q + 9 r) + p (4 q^2 + 13 q r + 9 r^2)))
\end{aligned}$$

```

(* {c ->  $\frac{a(a+b)}{a-2b}$ , d -> -2 b}, {b -> a} *)
f1[1] = (2 p + q + r) (a p^2 + a (q^2 + r^2) + a (a + a) p (q + r) / (a - 2 a) - 2 a q r);
f1[2] = f1[1] /. {p -> q, q -> r, r -> p};
f1[3] = f1[2] /. {p -> q, q -> r, r -> p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p -> p + q + r, q -> -q, r -> -r};
f2[2] = f1[2] /. {p -> p + q + r, q -> -q, r -> -r};
f2[3] = f1[3] /. {p -> p + q + r, q -> -q, r -> -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p -> p + q + r, q -> -r, r -> -p};
f3[2] = f1[2] /. {p -> p + q + r, q -> -r, r -> -p};
f3[3] = f1[3] /. {p -> p + q + r, q -> -r, r -> -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p -> q, q -> r, r -> p};
f4[2] = f1[2] /. {p -> q, q -> r, r -> p};
f4[3] = f1[3] /. {p -> q, q -> r, r -> p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p -> q, q -> p, r -> -p - q - r};
f5[2] = f1[2] /. {p -> q, q -> p, r -> -p - q - r};
f5[3] = f1[3] /. {p -> q, q -> p, r -> -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p -> -r, q -> -p, r -> p + q + r};
f6[2] = f1[2] /. {p -> -r, q -> -p, r -> p + q + r};
f6[3] = f1[3] /. {p -> -r, q -> -p, r -> p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p -> -r, q -> p + q + r, r -> -q};
f7[2] = f1[2] /. {p -> -r, q -> p + q + r, r -> -q};
f7[3] = f1[3] /. {p -> -r, q -> p + q + r, r -> -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p -> p, q -> -p - q - r, r -> q};
f8[2] = f1[2] /. {p -> p, q -> -p - q - r, r -> q};
f8[3] = f1[3] /. {p -> p, q -> -p - q - r, r -> q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
0
0
0
0
0
0

```

```

(* {c→a,d→b} *)
f1[1] = (2 p + q + r) (a p^2 + b (q^2 + r^2) + a p (q + r) + b q r);
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
0
0
0
0
0
0

```

```

(* {a→0,b→0,c→d} *)
f1[1] = (2 p + q + r) (d p (q + r) + d q r);
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
0
0
0
0
0
0

```

```

(* {a→0,b→0,d→0} *)
f1[1] = (2 p + q + r) (c p (q + r) );
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
0
0
0
0
0
0

```

```

(* {b→ $\frac{a}{2}$ , c→a, d→ $\frac{a}{2}$ } *)
f1[1] = (2 p + q + r) (a p^2 + a / 2 (q^2 + r^2) + a p (q + r) + a / 2 q r);
f1[2] = f1[1] /. {p → q, q → r, r → p};
f1[3] = f1[2] /. {p → q, q → r, r → p};
F1 = Sum[f1[i], {i, 3}];
f2[1] = f1[1] /. {p → p + q + r, q → -q, r → -r};
f2[2] = f1[2] /. {p → p + q + r, q → -q, r → -r};
f2[3] = f1[3] /. {p → p + q + r, q → -q, r → -r};
F2 = Sum[f2[i], {i, 3}];
f3[1] = f1[1] /. {p → p + q + r, q → -r, r → -p};
f3[2] = f1[2] /. {p → p + q + r, q → -r, r → -p};
f3[3] = f1[3] /. {p → p + q + r, q → -r, r → -p};
F3 = Sum[f3[i], {i, 3}];
f4[1] = f1[1] /. {p → q, q → r, r → p};
f4[2] = f1[2] /. {p → q, q → r, r → p};
f4[3] = f1[3] /. {p → q, q → r, r → p};
F4 = Sum[f4[i], {i, 3}];
f5[1] = f1[1] /. {p → q, q → p, r → -p - q - r};
f5[2] = f1[2] /. {p → q, q → p, r → -p - q - r};
f5[3] = f1[3] /. {p → q, q → p, r → -p - q - r};
F5 = Sum[f5[i], {i, 3}];
f6[1] = f1[1] /. {p → -r, q → -p, r → p + q + r};
f6[2] = f1[2] /. {p → -r, q → -p, r → p + q + r};
f6[3] = f1[3] /. {p → -r, q → -p, r → p + q + r};
F6 = Sum[f6[i], {i, 3}];
f7[1] = f1[1] /. {p → -r, q → p + q + r, r → -q};
f7[2] = f1[2] /. {p → -r, q → p + q + r, r → -q};
f7[3] = f1[3] /. {p → -r, q → p + q + r, r → -q};
F7 = Sum[f7[i], {i, 3}];
f8[1] = f1[1] /. {p → p, q → -p - q - r, r → q};
f8[2] = f1[2] /. {p → p, q → -p - q - r, r → q};
f8[3] = f1[3] /. {p → p, q → -p - q - r, r → q};
F8 = Sum[f8[i], {i, 3}];
(p + q + r) f1[1] / F1 - p f2[1] / F2 // Factor // Simplify
q f3[1] / F3 - (p + q + r) f4[1] / F4 // Factor // Simplify
-r f5[1] / F5 - q f6[1] / F6 // Factor // Simplify
p f7[1] / F7 - (-r) f8[1] / F8 // Factor // Simplify
p + q - (p + q + r) f1[1] / F1 - q f3[1] / F3 - (-r) f5[1] / F5 - p f7[1] / F7 // Factor //
Simplify
0
0
0
0
0
0

```

Message: #299

Date: 16/10/2013 1:39:05

From: seiichikiri

Subject: Algebraic computation of quadratic, cubic and quartic quadrangle

Dear friends,

Sorry!

Correction:

The last quartic quadrangle point degenerates to QAP21,
because $(p(q^2+r^2+qr)+q^3+r^3+2qr(q+r)) = (q^2+r^2+qr)(p+q+r)$.

Best regards,

Seiichi.

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Message: #300

Date: 17/10/2013 9:04:04

From: eckart_schmidt@t-online.de

Subject: Tangential quadrilateral well known properties?

Dear Bernard K.,

up to now I haven't found a synthetic prove for the discussed
property of a tangential quadrigon, but "... the evening crowns
the day."

In the attachment there is a clearly laid out calculation with
barycentric coordinates, perhaps of interest.

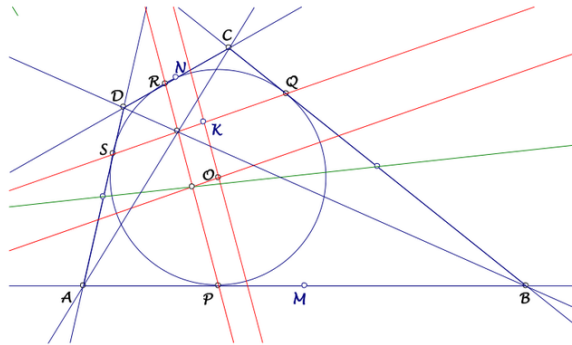
Best regards Eckart

Re: Tangential quadrilateral well known properties?

(QFG message 26)

Mosca Sebastiano:

$ABCD$ is a circumscribed quadrilateral,
 $PQRS$ the points of tangency with the inscribed circle center O ,
 M and N are the midpoints of AB and CD ,
 K is the intersection of SQ and MN
 Prove OK is parallel to PR .



Here is a simple calculation with barycentric coordinates, using the QA -Diagonal Triangle of the quadrigon $PQRS$ as reference triangle:

Let $A_0B_0C_0$ be the reference triangle for barycentric coordinates, obtuse-angled at B_0 .

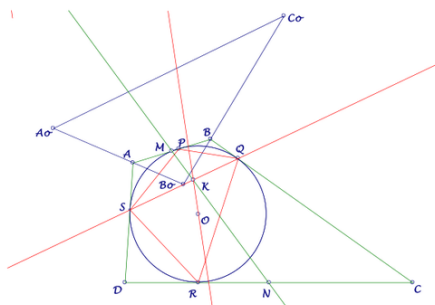
Use the polar circle ([1], p.37) of the reference triangle with the equation

$$S_A x^2 + S_B y^2 + S_C z^2 = 0$$

for a cyclic quadrigon $PQRS$:

$$P(u:v:w), \quad Q(-u:v:w), \quad R(u:-v:w), \quad S(u:v:-w)$$

$$\text{with } v^2 = \frac{S_A u^2 + S_C w^2}{-S_B}.$$



Mosca.pdf

The tangents in P, Q, R, S have the intersections

$$A(S_B v : -S_A u : 0), \quad B(0 : S_C w : -S_B v),$$

$$C(S_B v : S_A u : 0), \quad D(0 : S_C w : S_B v).$$

The midpoints of AB and CD are

$$M(S_B v(S_B v - S_C w) : -S_A S_B uv + 2S_A S_C uw - S_B S_C vw : S_B v(S_B v - S_A u))$$

$$N(S_B v(S_B v + S_C w) : S_A S_B uv + 2S_A S_C uw + S_B S_C vw : S_B v(S_B v + S_A u)).$$

This gives the intersection of QS and MN

$$K(-u : \frac{S_A S_B u^2 + (S^2 + S_A S_C)uw + S_B S_C w^2}{S_B(S_A u - S_C w)} : w).$$

The center of the polar circle is the orthocenter of the reference triangle $O(S_B S_C : S_C S_A : S_A S_B)$.

The common point at infinity of KO and PR is $(u : -u - w : w)$, so KO and PR are parallel.

[1] <http://forumgeom.fau.edu/FG2004volume4/FG200405.pdf>

Eckart Schmidt
<http://eckartschmidt.de>
 eckart_schmidt@t-online.de

Mosca.pdf

Message: #301

Date: 17/10/2013 10:06:15

From: Chris

Subject: Algebraic computation of quadratic, cubic and quartic quadrangle

Dear Seiichi,

Very interesting approach.

Nice to see which QA-points can be deduced using a systematic algebraic approach for composing barycentric coordinates.

It appears that most "simple" points are QA-P1, QA-P10 and QA-P16.

Your quartic case

$f(p,q,r)=(2p+q+r)(p(q^2+r^2+qr)+q^3+r^3+2qr(q+r))$ can be simplified to $f(p,q,r)=(p+q+r)(2p+q+r)(q^2+qr+r^2)$, which is QA-P21.

Not strange that also QA-P21 pops up (Reflection of QA-P16 in QA-P1).

For as far as I can see you only constructed points as a function $f(p,q,r)$ and not as $f(p,q,r,a,b,c)$.

Could be interesting to investigate these kind of points too.

Best regards,

Chris

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Message: #302

Date: 17/10/2013 1:09:07

From: seiichikiri

Subject: Algebraic computation of quadratic, cubic and quartic quadrangle

Dear Chris,

Thank you very much for your comments.

I have very clear aims:

(1) to make an example of an infinite sequence of QA-points.

(2) to find some complex QA-points.

As for the latter aim, the equations which I obtained are the quadratic ones of many variables. At present, I think that no one can say that there are no complex QA-points. So, for my aims, I use a simpler function $f(p, q, r)$. But there are many possibilities that what I said turns out blunders.

Best regards,

Seiichi.

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Message: #303
Date: 19/10/2013 10:40:17
From: bernard.keizer
Subject: Tangential quadrilateral well known properties?

Dear Eckart,

Thank you for your calculations.

Is it possible, with the same referential, to prove following properties:

O is the orthocenter of the diagonal triangle see my message #251.

The QA PQRS and the tangential QL (4 lines are the sides of the QA ABCD) have the same diagonal triangle (see my message #270).

The Miquel point of the inscribed QA is the inverse of B0 wrt the inscribed circle and lies on A0C0

O is on the Newton Line of the tangential QL.

M' and N' being the middles of AD and BC and K' being the intersection of PR and M'N', KK' is orthogonal to this Newton Line (see my message #275).

Best regards

Bernard K.

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Message: #304

Date: 19/10/2013 11:06:28

From: bernard.keizer

Subject: Precisions concerning the curve QL-Cu1

Dear Eckart,

I've just realised that my two points A and A', which have as tangential the infinity point of the Newton Line (and QL-P1 as the CL-S conjugate of this infinity point lies on AA') are precisely the points you called P1 and P2, the centers of the 2 inversions in which QL-Cu1 is invariant (anallagmatic curve). For any QL having A and A' as opposite vertices, the 4 other vertices B, B' C and C' are concyclic and the circumcenter lies on the bisector of the CL-S transformation, orthogonal to AA' in QL-P1.

Among all these QL's is the one I called main QL of the curve QL-Cu1, with the points QL-2Pa and b as middles of BB' and CC' and the circumcenter as excenter of the triangle QL-P1 and QL-2P2a and b, as you told me in a former message.

Among these QL's is also the one I called orthogonal QL, with diagonal orthogonal to the sides; this time, the circumcenter, always on the bisector of the CL-S transformation, is the middle of BC. But there are at least 2 solutions, maybe more ...

I haven't found on which conditions the main is also orthogonal, but it must be a special QL-Cu1, as it isn't true an ordinary one.

Best regards

Bernard

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Message: #305

Date: 19/10/2013 11:24:33

From: bernard.keizer

Subject: La Géométrie du Quadrilatère Complet

Dear Chris, dear Eckart, dear friends of the Quadri-forum,
The technical part took finally more time than I thought, but
now it's done ...

I've just put on the web an article about The Geometry of the
QL.

The link is <http://bernardkeizer.blogspot.fr/>

You may recognise some items already discussed in this forum as
Eckart's cubic of the centers of the 27 cardioïds inscribed in the
QL and many references to EQF.

Thanks in advance for all your remarks and critics.

Best regards

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Message: #306

Date: 20/10/2013 11:52:22

From: eckart_schmidt@t-online.de

Subject: Tangential quadrilateral well known properties?

Dear Bernard,

congratulations on your remarkable paper "La Géométrie du Quadrilatère Complet"! Shame on me, that I had only a look on some chapters.

Your properties wrt the tangential quadrigon can easily be proved with my calculation:

"O is on the Newton Line of the tangential QL"
... evident, for the Newton line is the focus of centers
of inscribed conics.

"The QA PQRS and the tangential QL (4 lines are the sides of the QA ABCD) have the same diagonal triangle (see my message #270)"
... have a look on the coordinates of P, Q, R, S, A, B, C, D.

"O is the orthocenter of the diagonal triangle see my message #251"
... already mentioned in my calculation.

"The Miquel point of the inscribed QA is the inverse of B0 wrt the inscribed circle and lies on A0C0"
... this is a basic property of cyclic quadrigons. The Miquel point is QG-P17 of PQRS.

"M' and N' being the middles of AD and BC and K' being the intersection of PR and M'N',
KK' is orthogonal to this Newton Line (see my message #275)"
... this seems somewhat remarkable! If the circle is replaced by a conic, KK' and the Newton line give conjugated diameters.

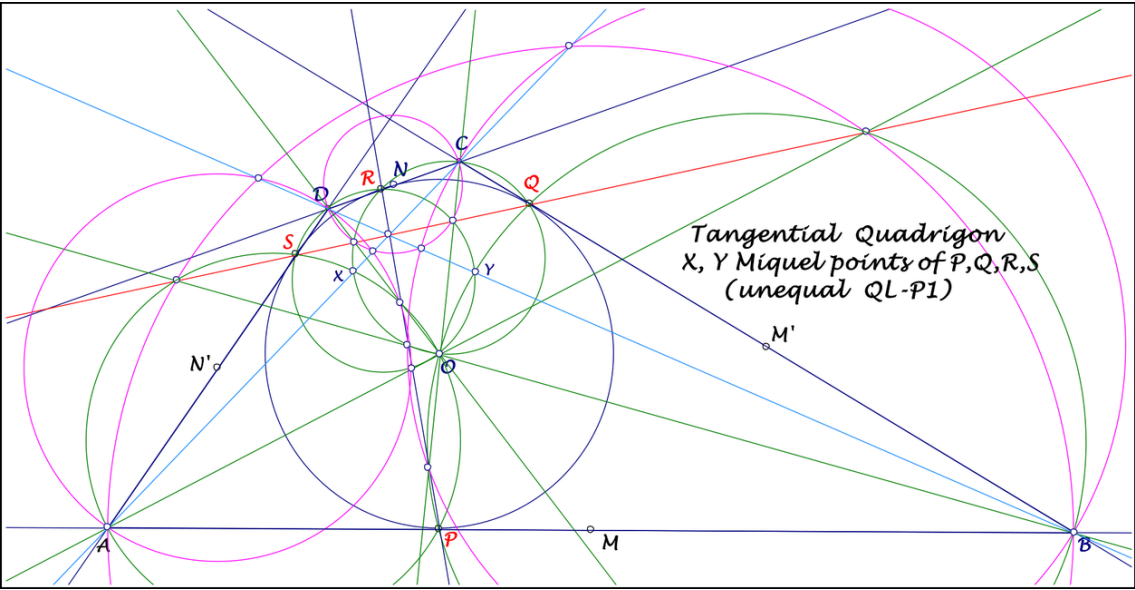
It's a pity, that there are no connections with midpoints!
But there is a glimmer of hope (see attachment): There are a lot of intersections wrt a tangential quadrigon, if you consider the following elements:

Lines: Angle bisectors and diagonals of the quadrigon ABCD and diagonals of the quadrigon PQRS.

Circles: Incircle of ABCD and Thales circles of AB, BC, CD, DA, OA, OB, OC, OD.

There are 15 new intersections of more than two elements. Among these points are the Miquel points of P,Q,R,S (unequal QL-P1 of the quadrigon PQRS). The corresponding angle bisector of the Miquel triangle of P,Q,R,S contains the diagonal crosspoint of ABCD and the center O of the incircle.

Best regards Eckart



Keizer-07.pdf

Message: #307
Date: 21/10/2013 8:33:19
From: seiichikiri
Subject: QA-points at infinity

Dear friends,

[1] QA-point: $f(p,q,r)=(2p+q+r)(q+r)(h(p+q)(p+r)+kqr)$.

Examples:

(1) QA-P25: $(2p+q+r)(q+r)(2p^2+2pq+2pr+3qr)$.

(2) QA-P26: $(2p+q+r)(q+r)(5p^2+5pq+5pr+3qr)$.

Cyclic sum of $f(p,q,r)=(4h+k)(p+q)(q+r)(r+p)(p+q+r)$. If $4h+k=0$, it becomes zero.

A point at infinity: $(2p+q+r)(q+r)(p^2+pq+pr-3qr)$.

[2] QA-point: $f(p,q,r)=(2p+q+r)p(q+r)(h(p+q)(p+r)+k(q^2+r^2))$.

$h=1$ and $k=2$ gives QA-P31.

Cyclic sum of

$f(p,q,r)=(2h+k)(p+q)(q+r)(r+p)(p^2+q^2+r^2+qr+rp+pq)$. If $2h+k=0$, it becomes zero.

A point at infinity: $(2p+q+r)p(q+r)(p^2+pq+pr+qr-2q^2-2r^2)$.

Best regards,
Seiichi.

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Message: #308

Date: 22/10/2013 10:27:44

From: eckart_schmidt@t-online.de

Subject: Tangential quadrilateral well known properties?

Dear Bernard K.,

some remarks perhaps a help for a synthetic prove:

Mosca Sebastiano:

ABCD is a circumscribed quadrilateral,
PQRS the points of tangency with the inscribed circle center O
M and N are the midpoints of AB and CD
K is the intersection of SQ and NM
T is the common diagonal crosspoint of ABCD and PQRS
Prove OK is parallel to PR.

Consider the angles of ABCD

and let the radius of the incircle be 1:

... $QS = 2 \sin(A+B)/2 = \sin(C+D)/2$
... $TQ/TS = (\cos(B/2) \cos(C/2)) / (\cos(D/2) \cos(A/2))$
... $KQ/KS = \cos((B-C)/2) / \cos((D-A)/2)$
... $KT = / \cos((A+D)/2) / \sin((B+D)/2) / = \dots$
... $KO = / \cos((A+B)/2) / \sin((B+D)/2) / = \dots$
... Let A', B' be the intersections of the Thales circle
about AB with QS;
let C', D' be the intersections of the Thales circle
about CD with QS:
 $KA' / KB' = KC' / KD' = K'N' / K'M' = DA / BC =$
 $(\sin(B/2) \sin(C/2)) / (\sin(D/2) \sin(A/2))$

... The midpoints of the concyclic points

P,M,A',B' and R,N,C',D' are collinear with K.

These properties can be calculated with cartesian coordinates as
in message #27 or proved with CABRI.

Best regards Eckart

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Message: #309

Date: 26/10/2013 1:34:36

From: eckart_schmidt@t-online.de

Subject: Tangential quadrilateral well known properties?

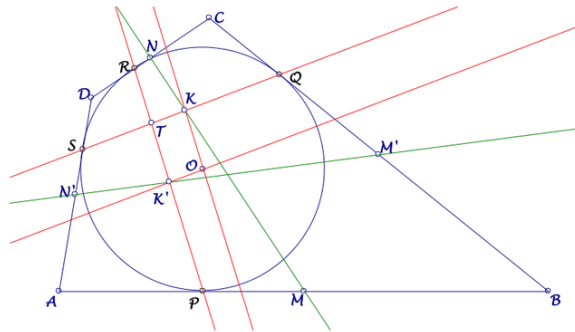
Dear Bernard K., dear Chris,
in the attachment you find a synthetic proof for the discussed
properties. I hope, there are no basic mistakes. Perhaps someone
can make it shorter!
Best regards Eckart

Re: Tangential quadrilateral well known properties?

(QFG message 26)

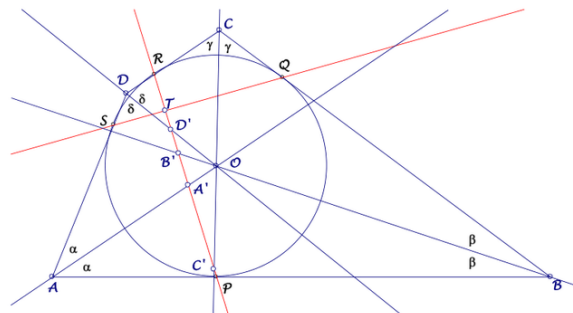
Mosca Sebastiano:

$ABCD$ is a circumscribed quadrilateral,
 $PQRS$ the points of tangency with the inscribed circle center O ,
 M and N are the midpoints of AB and CD ,
 K is the intersection of SQ and MN
 Prove OK is parallel to PR .



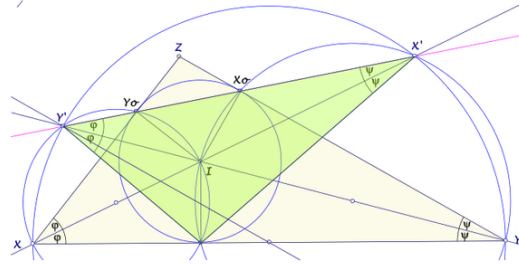
Synthetic Proof

- (1) Let the radius of the incircle be 1 and the angles of $ABCD$ be $2\alpha, 2\beta, 2\gamma, 2\delta$ with $\alpha + \beta + \gamma + \delta = 180^\circ$.
- (2) Side lengths of $ABCD$:
 $AB = AP + PB = \cot \alpha + \cot \beta$,
 $BQ = BQ + QC = \cot \beta + \cot \gamma, \dots$
- (3) Side lengths of $PQRS$:
 $PQ = 2 \cos \beta$, $QR = 2 \cos \gamma, \dots$
- (4) Angles:
 $\angle TPA = 180^\circ - \delta - \alpha$, $\angle TQB = 180^\circ - \alpha - \beta, \dots$
 and $\angle PTQ = \alpha + \gamma$.

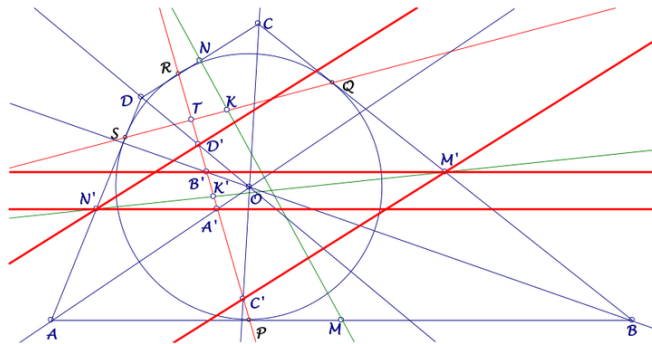


Mosca-synth.pdf

- (5) First distances on PR (law of sines):
 $PT = \frac{2 \cos \alpha \cos \beta}{\sin(\alpha + \gamma)}, RT = \frac{2 \cos \gamma \cos \delta}{\sin(\alpha + \gamma)}, PR = 2 \sin(\beta + \gamma).$
- (6) Further points on PR . These points will be used, to calculate PK', RK' .
 $A' = PR \cap OA, B' = PR \cap OB, \dots$
- (7) Distances of these points on PR :
 In the triangle POA' with
 $OP = 1, \angle A'OP = 90^\circ - \alpha, \angle PA'O = 180^\circ - \delta$
 the law of sines gives $PA' = \frac{\cos \alpha}{\sin \delta}.$
 Analog: $PB' = \frac{\cos \beta}{\sin \gamma}, RC' = \frac{\cos \gamma}{\sin \beta}, RD' = \frac{\cos \delta}{\sin \alpha}.$
- (8) Lemma: In a triangle XYZ with incircle (or excircle), touching the sides in X_o, Y_o, Z_o , the angle bisectors XI / YI cut the line X_oY_o in points X' / Y' , which lie on the Thales circle about XY as well as on the Thales circle about YI / XI . The radii of X' / Y' wrt the Thales circle about XY are parallel to the sidelines XZ / YZ . This can be verified with help of the following drawing (here omitted).



- (9) Projecting these properties in the constellation of the tangential quadrigon $ABCD$, we get the following parallels:
 $M'B' \text{ parallel } N'A' \text{ and } M'C' \text{ parallel } N'D'.$



(10) Intercept theorems give:

$$\frac{K'A'}{K'B'} = \frac{K'N'}{K'M'} = \frac{K'D'}{K'C'}.$$

(11) Further distances on PR : Replace in (10)

$$K'A' = PK' - PA', \quad K'B' = PB' - PK',$$

$$K'C' = RC' - RK', \quad K'D' = RK' - RD'$$

and $PK' + RK' = PR$.

Then we get with (5) and (7):

$$PK' = \frac{\cos(\alpha - \beta)}{\sin(\alpha + \gamma)} \quad \text{and} \quad RK' = \frac{\cos(\gamma - \delta)}{\sin(\alpha + \gamma)}.$$

(12) An interesting consequence of (10) and (11) is:

$$\frac{K'N'}{K'M'} = \frac{DA}{BC}.$$

(13) Now we can calculate $TK' = RK' - RT$ (see (11) and (5));
analog TK :

$$TK' = \frac{\cos(\alpha + \beta)}{\sin(\alpha + \gamma)} \quad \text{and} \quad TK = \frac{\cos(\beta + \gamma)}{\sin(\alpha + \gamma)}.$$

(14) The law of chords give $K'O^2 = 1 - PK' \cdot RK'$ (see (11));
analog KO^2 :

$$K'O^2 = \frac{\cos^2(\beta + \gamma)}{\sin^2(\alpha + \gamma)} \quad \text{and} \quad KO^2 = \frac{\cos^2(\alpha + \beta)}{\sin^2(\alpha + \gamma)}$$

This completes the proof.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Message: #310

Date: 27/10/2013 4:02:06

From: Chris van Tienhoven

Subject: Two points on the Newton Line QL-Cu1 revisited

Dear friends,

All points on QL-Cu1 lie within a stroke bounded by two lines parallel to the Newton Line on equal distances of the Newton Line.

These two bounding lines touch QL-Cu1 in 2 points.

In the last phrase of message #193 I wondered how to identify these points.

I found a method for constructing these points.

They lie on the 2nd Steiner Axis as well as a special circle.

See QL-Tf1 for explanation *Steiner Axes and Schmidt Circle*:

<http://www.chrisvantienhoven.nl/other-quadrilateral-objects/17-mathematics/quadrilateral-objects/artikelen-ql/159-ql-tf1.html>.

See also attachment.

I calculated the CT-coordinates for these points. Unfortunately they are very lengthy.

They are not collinear with other QL-Points (except QL-P1), neither do they occur on other known QL-lines or QL-curves (except one of the Steiner Axes).

Last but not least, here is an extra method for constructing Clawson-Schmidt Conjugates (QL-Tf1) of points P on QL-Cu1.

* Let L_n = line through P parallel to the Newton Line QL-L1.

* Let L_m = line connecting P and the Miquel Point QL-P1

* Let L_n' = L_n reflected in the Newton Line QL-L1.

* Let L_m' = L_m reflected in the 1st Steiner Line.

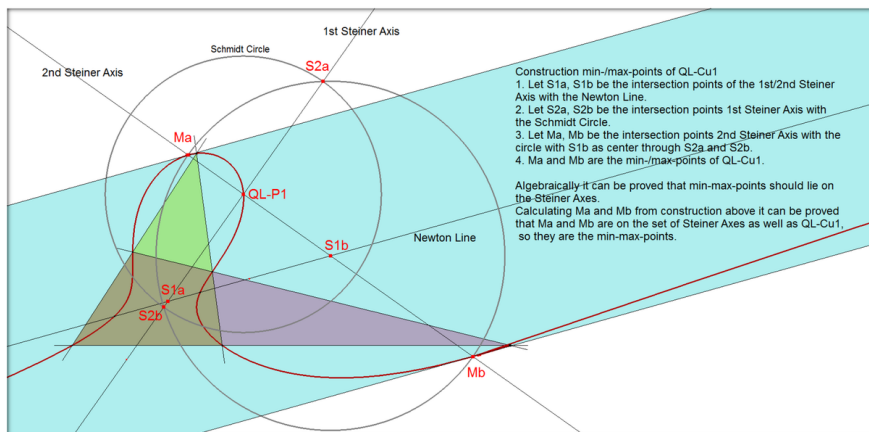
* Now $QL-Tf1(P) = L_n' \wedge L_m'$.

Note: This method is only valid for points on QL-Cu1.

Best regards,

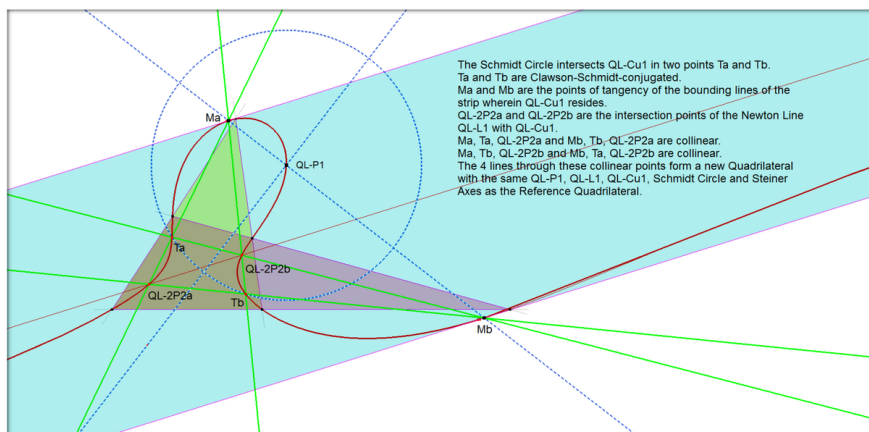
Chris

Construction Min-/Max-Points at Cubic QL-CU1



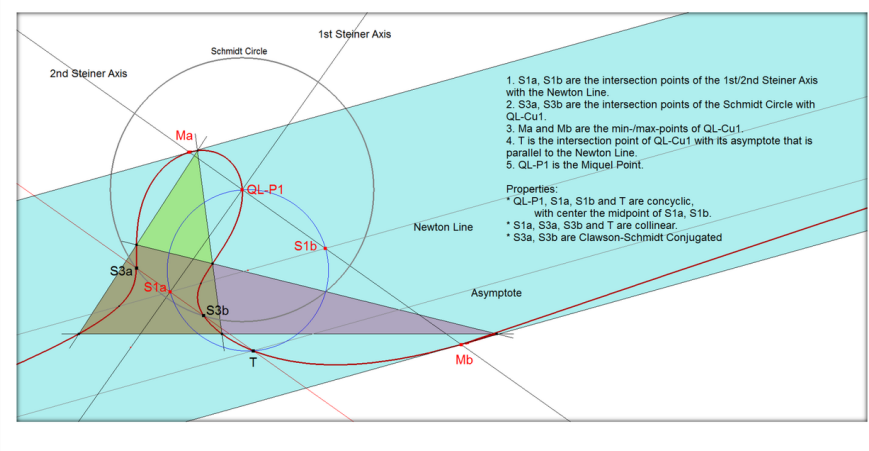
QL-Cu1-MinMax-points.pdf

Related Quadrilateral with identical features



QL-Cu1-MinMax-points.pdf

Related points on Schmidt Circle



QL-Cu1-MinMax-points.pdf

Message: #311

Date: 28/10/2013 3:49:33

From: eckart_schmidt@t-online.de

Subject: Inscribed Parallelograms of a Quadrigon

Dear friends,

I think, it can be of interest: Inscribed parallelograms as rhombi, rectangles, squares. So I have made a summary of an article on my homepage.

Best regards Eckart

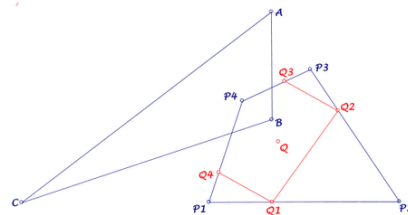
Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Inscribed Parallelograms of a Quadrigon

For any point P in the plane of a quadrigon there is an inscribed parallelogram with center P . The centers of inscribed rhombi and inscribed rectangles lie on special conics. Their intersections give the centers of inscribed squares (see QG-L6 in EQF). This paper is a summary of

<http://eckartschmidt.de/Pgive.pdf>

Reference triangle for barycentric coordinates is $QA-DT$.



Inscribed Parallelograms

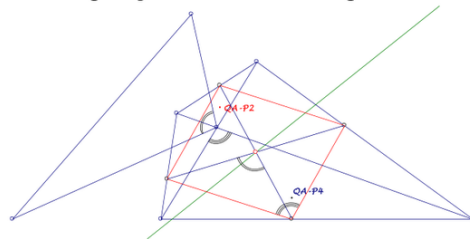
Let $P_1P_2P_3P_4$ be a quadrigon with diagonal triangle ABC as reference triangle for barycentric coordinates:

$$P_1(-p, q, r), \quad P_2(p, -q, r), \quad P_3(p, q, -r), \quad P_4(p, q, r).$$

For a parallelogram center $Q(u : v : w)$ the inscribed parallelogram $Q_1Q_2Q_3Q_4$ can be constructed and calculated as follows: The intersection of P_3P_4 with the reflection of P_1P_2 in Q gives Q_3 , and so on:

$$Q_1(p : -q : \frac{q^2u + p^2v + pqw}{qu - pv}), \quad Q_2(\frac{r^2v + q^2w + qru}{qw - rv} : -q : r)$$

$$Q_3(p : q : \frac{-q^2u - p^2v + pqw}{qu + pv}), \quad Q_4(\frac{-r^2v - q^2w + qru}{qw + rv} : q : r).$$



A special example is mentioned in *EQF*: The pedal quadrigon of the Isogonal Center $QA-P4$ is a parallelogram. This parallelogram is a representative of parallelograms with center on the perpendicular bisector of $QA-P2.QA-P4$. These parallelograms have angles as in the intersection of the diagonals of the quadrigon.

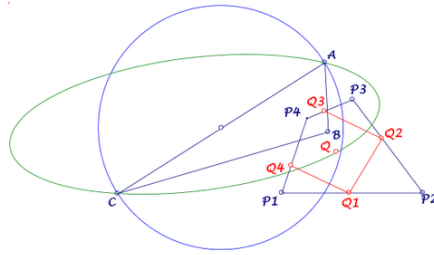
The inscribed parallelogram degenerates for centers on the Nine-point Conic $QA-Co1$.

Inscribed Rhombi

A rhombus is a parallelogram with equal side lengths. The centers of inscribed rhombi lie on a conic with equation

$$q^2(p^2S_Ayz - q^2S_Bzx + r^2S_Cuv) + r^2p^2b^2y^2 = 0.$$

This conic is the Involutory Conjugate $QA-Tf2$ of the Thales circle about the third diagonal AC of the quadrigon.

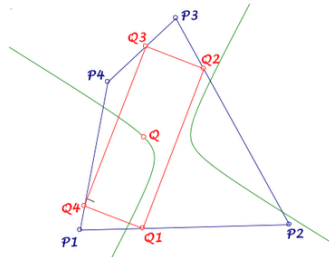


Special examples are two inscribed rhombi with sides parallel to the diagonals; the vertices divide the sides of the quadrigon in the ratio of the diagonal lengths.

Inscribed Rectangles

For rectangles the right angularity gives the equation for the centers:

$$a^2q^4r^2x^2 + b^2p^2r^2(p^2 - r^2)y^2 - c^2p^2q^4z^2 + 2p^2q^2r^2y(S_Cx - S_Az) = 0$$



This conic contains the midpoints of the diagonals, for which the rectangles degenerate collinear. The center of the conic is the midpoint of the Euler-Poncelet Point $QA-P2$ and the Isogonal Center $QA-P4$.

Inscribed Squares

There are two real intersections of the conics for the centers of inscribed rhombi and rectangles:

$$Q^{\pm}(p(S_A p \pm S r : S_B q^2 : r(S_C r \pm S p))).$$

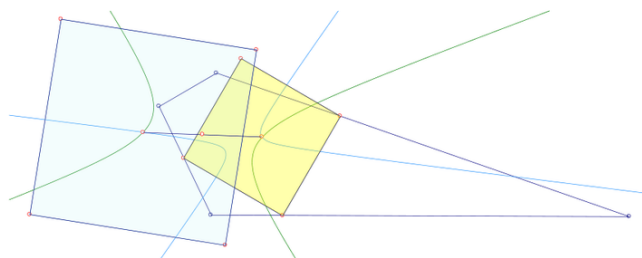
These centers of inscribed squares lie symmetric to the center of the conic for the centers of inscribed rhombi. Their connection is the Inscribed Square Axis $QG-L6$ (see EQF). The vertices of the inscribed squares are:

$$Q_1^{\pm}(p : -q : \frac{S_C r^2 + c^2 p q \pm S r(p + q)}{S_A p - S_B q \pm S r}),$$

$$Q_2^{\pm}(\frac{S_A p^2 + a^2 q r \pm S p(q + r)}{S_C r - S_B q \pm S p} : -q : r),$$

$$Q_3^{\pm}(p : q : \frac{S_C r^2 - c^2 p q \pm S r(p - q)}{S_A p + S_B q \pm S r}),$$

$$Q_4^{\pm}(\frac{S_A p^2 - a^2 q r \pm S p(r - q)}{S_C r + S_B q \pm S p} : q : r).$$



For further properties have a look on my homepage 11.2.

Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

Message: #312
Date: 29/10/2013 9:47:36
From: eckart_schmidt@t-online.de
Subject: Two points on the Newton Line QL-Cu1 revisited

Dear Chris,

if QL-Cu1 is bipartite, there are two further points on QL-Cu1 with tangents parallel and symmetric to the Newton line. These four points lie two by two on the 1st and 2nd Steiner line. They are the in- and excenters of a triangle QL-P1.F1.F2, when F1, F2 are the foci of an inscribed conic with center $QL-L1 \wedge QL-L6$. Background is, that QL-Cu1 in the bipartite case is an isogonal pivotal circular cubic with reference triangle QL-P1.F1.F2.

Best regards Eckart

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Message: #313
Date: 31/10/2013 3:10:46
From: eckart_schmidt@t-online.de
Subject: Involutary Conjugate QA-Tf2

Dear Chris,

what about this construction for QA-Tf2?
Let P be a point and P1, P2, P3, P4 the points of a quadrangle with $P2P3 \wedge P4P1 = Si$ and $P1P2 \wedge P3P4 = Sj$, then the fourth intersection of the conics through P, P1, P3, Si, Sj and P, P2, P4, Si, Sj is QA-Tf2(P).

Best regards Eckart

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Message: #314
Date: 31/10/2013 7:16:50
From: Chris van Tienhoven
Subject: Involutary Conjugate QA-Tf2

Very nice!
Our insight and knowledge is growing every day!
Best regards, Chris

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Message: #315
Date: 02/11/2013 6:15:51
From: Seiichi Kirikami
Subject: Numerically confirmed quadrangle points with a property of angles

Dear friends,

I studied quadrangle points with a property similar to, but a bit different from QA-P4 in EQF.

Given a quadrangle $P_1P_2P_3P_4$ and a point $P\{x, y, z\}$, P has the property:

$$\begin{aligned}\text{angle } P_1PP_2 &= \text{angle } P_1P_4P + \text{angle } PP_3P_2, \\ \text{angle } P_2PP_3 &= \text{angle } P_2P_1P + \text{angle } PP_4P_3, \\ \text{angle } P_3PP_4 &= \text{angle } P_3P_2P + \text{angle } PP_1P_4, \\ \text{angle } P_4PP_1 &= \text{angle } P_4P_3P + \text{angle } PP_2P_1.\end{aligned}$$

In order to determine P , I used the tangent method as shown in p4 of "Ein merkwürdiger Punkt des Vierecks" by Daniel baumgartner and Roland Staerk, Praxis der Mathematik 1/44 Jg, 2002 of message #250 and obtained 4 quartic equations (kr_1, kr_2, kr_3 and kr_4) of x , y and z .

With 2 equations (for example, kr_1 and kr_2) and $x+y+z=1$, I computed the coordinates of P numerically and confirmed that they coincided with each other.

I used the values ($a=6$, $b=9$, $c=13$, $p=3$, $q=23$, $r=-10$) and obtained 4 vertices, 2 real coordinates and 2 complex coordinates as solutions. I confirmed the relations of angles by Cabri in case of real solutions. See the attached file.

Best regards,
Seiichi.

```

(* QA-PX similar to, a bit different from QAP4 *)
Simplificar[lista_] := Simplify[ $\frac{\text{lista}}{\text{Apply}[\text{PolynomialGCD}, \text{lista}]}$ ];
Punto[r_, s_] := Simplificar[Cross[r, s]];
Recta[P_, Q_] := Simplificar[Cross[P, Q]];
DividirRazon[ptP_, ptQ_, m_, n_] := Simplificar[n Tr[ptQ] ptP + m Tr[ptP] ptQ];
Medio[ptP_, ptQ_] := DividirRazon[ptP, ptQ, 1, 1];
PuntoInfinito[{p_, q_, r_}] := {q - r, r - p, p - q};
PuntoInfinitoPerpendicular[{f_, g_, h_}] := PuntoInfinito[{SA f, SB g, SC h}] /.
  {SA →  $\frac{b^2 + c^2 - a^2}{2}$ , SB →  $\frac{c^2 + a^2 - b^2}{2}$ , SC →  $\frac{a^2 + b^2 - c^2}{2}$ };
Perpendiculara[P_, r_] := Recta[P, PuntoInfinitoPerpendicular[PuntoInfinito[r]]];
Perpendic[P_, Q_, R_] := Perpendiculara[P, Recta[Q, R]];
Mediatriz[ptP_, ptQ_] := Perpendic[Medio[ptP, ptQ], ptP, ptQ];
Circuncentro[ptA_, ptB_, ptC_] :=
  Punto[Mediatriz[ptA, ptB], Mediatriz[ptA, ptC]];
Tg[r_, s_] := (S Tr[Cross[r, s]]) / ((r[[2]] - r[[3]]) (s[[2]] - s[[3]]) SA +
  (r[[3]] - r[[1]]) (s[[3]] - s[[1]]) SB + (r[[1]] - r[[2]]) (s[[1]] - s[[2]]) SC) /.
  {S → Sqrt[2 a^2 b^2 + 2 b^2 c^2 + 2 c^2 a^2 - a^4 - b^4 - c^4] / 2,
  SA →  $\frac{b^2 + c^2 - a^2}{2}$ , SB →  $\frac{c^2 + a^2 - b^2}{2}$ , SC →  $\frac{a^2 + b^2 - c^2}{2}$ };

(* 4 vertices *)
pt1 = {0, 1, 0};
pt2 = {0, 0, 1};
pt3 = {1, 0, 0};
pt4 = {p, q, r};
ptX = {x, y, z};

(* lines *)
rt14 = Recta[pt1, pt4]
rt12 = Recta[pt1, pt2]
rt23 = Recta[pt2, pt3]
rt34 = Recta[pt3, pt4]
rt24 = Recta[pt2, pt4]
rt13 = Recta[pt1, pt3]
rt1X = Recta[pt1, ptX]
rt2X = Recta[pt2, ptX]
rt3X = Recta[pt3, ptX]
rt4X = Recta[pt4, ptX]

(* curves *)
kr1 = Numerator[Factor[Tg[rt1X, rt2X] -  $\frac{\text{Tg}[rt14, rt4X] + \text{Tg}[rt3X, rt23]}{1 - \text{Tg}[rt14, rt4X] \times \text{Tg}[rt3X, rt23]}$ ]] /.
  {a → 6, b → 9, c → 13, p → 3, q → 23, r → -10} // Simplify
kr2 = Numerator[Factor[Tg[rt2X, rt3X] -  $\frac{\text{Tg}[rt12, rt1X] + \text{Tg}[rt4X, rt34]}{1 - \text{Tg}[rt12, rt1X] \times \text{Tg}[rt4X, rt34]}$ ]] /.

```



```

{a → 6, b → 9, c → 13, p → 3, q → 23, r → -10} // Simplify
kr3 = Numerator[Factor[Tg[rt3X, rt4X] -  $\frac{Tg[rt23, rt2X] + Tg[rt1X, rt14]}{1 - Tg[rt23, rt2X] \times Tg[rt1X, rt14]}$ ]] /.
{a → 6, b → 9, c → 13, p → 3, q → 23, r → -10} // Simplify
kr4 = Numerator[Factor[Tg[rt4X, rt1X] -  $\frac{Tg[rt34, rt3X] + Tg[rt2X, rt12]}{1 - Tg[rt34, rt3X] \times Tg[rt2X, rt12]}$ ]] /.
{a → 6, b → 9, c → 13, p → 3, q → 23, r → -10} // Simplify
N[Solve[kr1 == 0 && kr2 == 0 && x + y + z == 1, {x, y, z}] // Factor // Simplify, 20]
N[Solve[kr2 == 0 && kr3 == 0 && x + y + z == 1, {x, y, z}] // Factor // Simplify, 20]
N[Solve[kr3 == 0 && kr4 == 0 && x + y + z == 1, {x, y, z}] // Factor // Simplify, 20]
N[Solve[kr4 == 0 && kr1 == 0 && x + y + z == 1, {x, y, z}] // Factor // Simplify, 20]
{r, 0, -p}
{1, 0, 0}
{0, 1, 0}
{0, -r, q}
{-q, p, 0}
{0, 0, 1}
{z, 0, -x}
{-y, x, 0}
{0, -z, y}
{-r y + q z, r x - p z, -q x + p y}
-32  $\sqrt{35}$  (-81 x3 (27 040 y + 13 911 z) + x2 (1 245 361 y2 - 23 733 y z - 154 062 z2) +
108 y z (467 y2 + 3107 y z + 1296 z2) + 3 x (78 923 y3 + 625 661 y2 z + 419 337 y z2 + 79 299 z3))
32  $\sqrt{35}$  (48 281 x3 (169 y + 81 z) -
108 y z (2237 y2 + 634 y z - 4099 z2) + x2 (1 245 361 y2 + 9 851 858 y z + 3 619 161 z2) -
36 x (27 040 y3 + 7493 y2 z - 104 971 y z2 - 29 808 z3))
-32  $\sqrt{35}$  (-81 x3 (27 040 y + 13 911 z) + x2 (1 245 361 y2 - 23 733 y z - 154 062 z2) +
108 y z (467 y2 + 3107 y z + 1296 z2) + 3 x (78 923 y3 + 625 661 y2 z + 419 337 y z2 + 79 299 z3))
32  $\sqrt{35}$  (48 281 x3 (169 y + 81 z) -
108 y z (2237 y2 + 634 y z - 4099 z2) + x2 (1 245 361 y2 + 9 851 858 y z + 3 619 161 z2) -
36 x (27 040 y3 + 7493 y2 z - 104 971 y z2 - 29 808 z3))

```

```

{ {x → 0, y → 0, z → 1.0000000000000000},
  {x → 1.0000000000000000, y → 0, z → 0}, {x → 0, y → 1.0000000000000000, z → 0},
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  {x → -0.75382293859371643960, y → 0.71837161774204917366, z → 1.0354513208516672659},
  {x → 0.35431835598286679878, y → 0.95098681673900111121, z → -0.30530517272186790999},
  {x → -0.19975229130542482041 - 0.21591078091826634944 i,
    y → 0.83467921724052514243 - 0.88436359840345461817 i,
    z → 0.36507307406489967798 + 1.10027437932172096761 i},
  {x → -0.19975229130542482041 + 0.21591078091826634944 i,
    y → 0.83467921724052514243 + 0.88436359840345461817 i,
    z → 0.36507307406489967798 - 1.10027437932172096761 i} }

{ {x → 0, y → 0, z → 1.0000000000000000},
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  {x → 0.1875000000000000, y → 1.4375000000000000, z → -0.6250000000000000},
  {x → -0.75382293859371643960, y → 0.71837161774204917366, z → 1.0354513208516672659},
  {x → 0.35431835598286679878, y → 0.95098681673900111121, z → -0.30530517272186790999},
  {x → -0.19975229130542482041 - 0.21591078091826634944 i,
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    z → 0.36507307406489967798 + 1.10027437932172096761 i},
  {x → -0.19975229130542482041 + 0.21591078091826634944 i,
    y → 0.83467921724052514243 + 0.88436359840345461817 i,
    z → 0.36507307406489967798 - 1.10027437932172096761 i} }

{ {x → 0, y → 0, z → 1.0000000000000000},
  {x → 1.0000000000000000, y → 0, z → 0}, {x → 0, y → 1.0000000000000000, z → 0},
  {x → 0.1875000000000000, y → 1.4375000000000000, z → -0.6250000000000000},
  {x → -0.75382293859371643960, y → 0.71837161774204917366, z → 1.0354513208516672659},
  {x → 0.35431835598286679878, y → 0.95098681673900111121, z → -0.30530517272186790999},
  {x → -0.19975229130542482041 - 0.21591078091826634944 i,
    y → 0.83467921724052514243 - 0.88436359840345461817 i,
    z → 0.36507307406489967798 + 1.10027437932172096761 i},
  {x → -0.19975229130542482041 + 0.21591078091826634944 i,
    y → 0.83467921724052514243 + 0.88436359840345461817 i,
    z → 0.36507307406489967798 - 1.10027437932172096761 i} }

{ {x → 0, y → 0, z → 1.0000000000000000},
  {x → 1.0000000000000000, y → 0, z → 0}, {x → 0, y → 1.0000000000000000, z → 0},
  {x → 0.1875000000000000, y → 1.4375000000000000, z → -0.6250000000000000},
  {x → -0.75382293859371643960, y → 0.71837161774204917366, z → 1.0354513208516672659},
  {x → 0.35431835598286679878, y → 0.95098681673900111121, z → -0.30530517272186790999},
  {x → -0.19975229130542482041 - 0.21591078091826634944 i,
    y → 0.83467921724052514243 - 0.88436359840345461817 i,
    z → 0.36507307406489967798 + 1.10027437932172096761 i},
  {x → -0.19975229130542482041 + 0.21591078091826634944 i,
    y → 0.83467921724052514243 + 0.88436359840345461817 i,
    z → 0.36507307406489967798 - 1.10027437932172096761 i} }

```

Message: #316
Date: 02/11/2013 8:56:56
From: seiichikiri
Subject: Numerically confirmed quadrangle points with a property of angle

Dear friends,
The attached file will open by Mathematica, if you add ".nb" to it.
Best regards,
Seiichi.

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Message: #317
Date: 03/11/2013 9:52:54
From: eckart_schmidt@t-online.de
Subject: Numerically confirmed quadrangle points with a ...

Dear Seiichi,
your discussed points are the fixed points of the
Clawson-Schmidt Conjugate QL-Tf1 for a quadrigon, that are the
intersections of the 1st Steiner axis and the Schmidt circle.
Best regards Eckart

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Message: #318
Date: 03/11/2013 1:28:07
From: seiichikiri
Subject: Numerically confirmed quadrangle points with a ...

Dear Eckart,

I am very glad that you have determined the points, which I
could only compute numerically, such that they are suitable in
EQF.

Best regards,
Seiichi.

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Message: #319
Date: 04/11/2013 9:47:47
From: Eisso J. Atzema
Subject: Question about a Quadrilateral

Colleagues,

I apologize beforehand for triple posting, but I have a somewhat urgent question. While working on a paper, I found a curious 16th century Indian attempt at a derivation for the area formula of a cyclic quadrilateral. The derivation is incorrect, but it is fairly easy to fix it using modern means. The resulting proof is quite interesting, but I just think I have seen it before. I only cannot remember where. Can anyone help me out? The proof goes as follows:

Let ABCD be a (convex) cyclic quadrilateral, with AB of length a , BC of length b and so on. Now, going either clockwise or counter-clockwise, let A' on AB be such that $AA'=(a+c)/2$, B' on BC such that $BB'=(b+c)/2$ and so on. Then, it is straightforward to see that $A'B'C'D'$ is a rectangle with the same area as ABCD. Furthermore, it can be shown that the sides of this rectangle equal $e/f*\sqrt{(s-a)(s-c)}$ and $f/e*\sqrt{(s-b)(s-d)}$, with e is the length of AC and f the length of BD (where I might have switched the role of e and f). Thus, the area of ABCD equals $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

Has anyone seen this proof before? Please provide me with a reference, however vague. Also, basically my verification for the lengths of the rectangle is a tedious computation. Might there be a nicer and shorter way to verify these lengths?

Thanks so much,
Eisso

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Message: #320

Date: 04/11/2013 9:56:13

From: eckart_schmidt@t-online.de

Subject: 3 QG-circumconics for a QL and a fixed point

Dear friends,

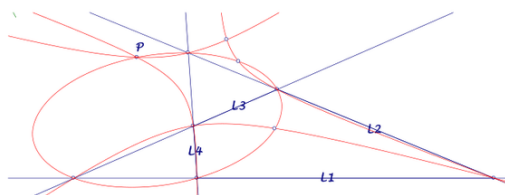
for a quadrilateral with its 3 quadrigon components there are 3 QG-circumconics through a fixed point. These conics have 3 collinear fourth intersections. The geometry of the corresponding line is discussed in the attachment, leading to a new type of QL-cubics.

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Three QG-Circumconics for a Quadrilateral

For the three QG-components of a quadrilateral there are circumconics through a fixed point with three collinear fourth intersections. The corresponding line will be discussed here, leading to a new type of QL-cubics. – Reference triangle for barycentric coordinates is QL-DT.



The three QG-Circumconics and their Intersections

For a quadrilateral with lines

$$L_1(-l, m, n), \quad L_2(l, -m, n), \quad L_3(l, m, -n), \quad L_4(l, m, n)$$

there are the intersections

$$Q_{12}(m:l:0), \quad Q_{23}(0:n:m), \quad Q_{34}(-m:l:0),$$

$$Q_{41}(0:-n:m), \quad Q_{13}(n:0:l), \quad Q_{24}(-n:0:l)$$

and the quadrigon components

$$Q_{12}Q_{23}Q_{34}Q_{41}, \quad Q_{12}Q_{24}Q_{34}Q_{13}, \quad Q_{23}Q_{24}Q_{41}Q_{13}.$$

Let $P(u:v:w)$ be an arbitrary point, then the QG-circumconics through P have the equations

$$(l^2u^2 - m^2v^2 + n^2w^2)zx - (l^2x^2 - m^2y^2 + n^2z^2)wu = 0,$$

$$(l^2u^2 + m^2v^2 - n^2w^2)xy - (l^2x^2 + m^2y^2 - n^2z^2)uv = 0,$$

$$(-l^2u^2 + m^2v^2 + n^2w^2)yz - (-l^2x^2 + m^2y^2 + n^2z^2)vw = 0.$$

The fourth intersections of two of these conics give

$$(m^2uv : l^2u^2 - n^2w^2 : -m^2vw),$$

$$(n^2uw : -n^2vw : l^2u^2 - m^2v^2),$$

$$(n^2w^2 - m^2v^2 : -l^2uv : l^2wu).$$

These intersections are the Involutory Conjugates $QA-Tf2$ of P wrt the QG-components. They are collinear on a line with the coefficients

$$L_p(l^2u(-l^2u^2 + m^2v^2 + n^2w^2), \\ m^2v(l^2u^2 - m^2v^2 + n^2w^2), n^2w(l^2u^2 + m^2v^2 - n^2w^2)).$$

Three Examples

$P = QL-P1$: The coefficients of L_{QL-P1} are extensive. The line is the Clawson-Schmidt Conjugate $QL-Tf1$ of the Dimidium Circle $QL-Ci6$ containing $QL-P26$.

$P = QL-P8$: The line L_{QL-P8} has the coefficients

$$L_{QL-P8}(l^2(-l^2 + m^2 + n^2), m^2(l^2 - m^2 + n^2), n^2(l^2 + m^2 - n^2))$$
and is the second asymptote of $QL-Co2$ containing $QL-P23$.

$P = QL-P13$: The line L_{QL-P13} has the coefficients

$$L_{QL-P13}(l^2m^2 - m^2n^2 + n^2l^2, l^2m^2 + m^2n^2 - n^2l^2, -l^2m^2 + m^2n^2 + n^2l^2)$$
and is a parallel to $QL-L9$ through $QL-P19$.

For points P on L_i the line L_P is L_i again. For the vertices of the diagonal triangle $QL-DT$ the line is the opposite sideline of $QL-DT$. For the midpoints of the sides of $QL-DT$ the line contains the opposite vertex and $QL-P13$.

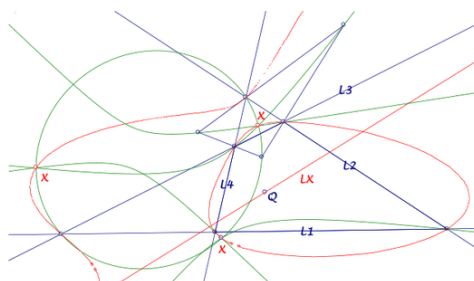
Further Properties

For a line L there are always three points X with $L_X = L$. These points are the common intersections of the conics, which are the $QA-Tf2$ images of L wrt the QG -components.

For all lines through a fixed point $Q(x_o : y_o : z_o)$ these three points lie on a cubic with the equation

$$l^4x_o x^3 + m^4y_o y^3 + n^4z_o z^3 - l^2m^2(x_o y + xy_o)xy - m^2n^2(y_o z + yz_o)yz - n^2l^2(z_o x + zx_o)zx = 0.$$

This cubic is the locus for all points X , whose L_X contains Q , passing through the six points of the quadrilateral.



If Q is a vertex of $QL-DT$, this conic is $QG-Co2$ of the correspondent QG -component.

If $Q = QL-P13$, this cubic has the equation

$$l^2(-x + y + z)x^2 + m^2(x - y + z)y^2 + n^2(x + y - z)z^2 = 0.$$

and is a circumcubic of the medial triangle of $QL-DT$, isotomic invariant wrt this triangle.

A new Type of *QL*-Cubics

The general case of the locus for points X , whose L_X contains a fixed point $Q(x_o : y_o : z_o)$

- ... is a nonpivotal isocubic of type nK containing the points of the quadrilateral.
- The reference triangle $A_oB_oC_o$ for this cubic has its vertices in the third intersections of the cubic with the sidelines of *QL-DT*:

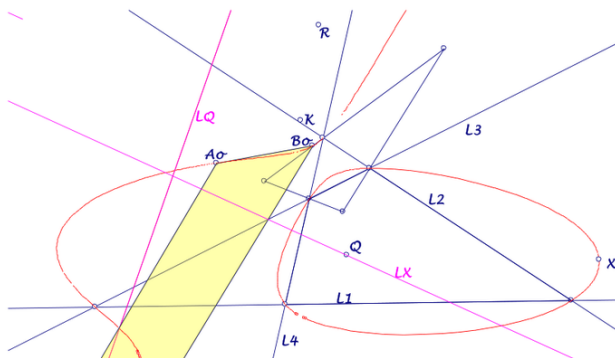
$$A_o(0:n^2z_o:m^2y_o), \quad B_o(n^2z_o:0:l^2x_o), \quad C_o(m^2y_o:l^2x_o:0).$$

$A_oB_oC_o$ is the cevian triangle wrt *QL-DT* of a point

$$K(m^2n^2y_oz_o:l^2n^2z_ox_o:l^2m^2x_oy_o),$$

which is the image of Q wrt a *QL-DT*-isoconjugation with fixed points in the trilinear poles of L_i .

- The isoconjugation wrt $A_oB_oC_o$ for the cubic has fixed point K .
- The root R of the cubic is the trilinear pole of L_Q wrt $A_oB_oC_o$. The coordinates are extensive.



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2013-11-04.pdf

Message: #321

Date: 04/11/2013 9:58:20

From: Eisso J. Atzema

Subject: Question about a Quadrilateral (Minor Correction)

> On Mon, Nov 4, 2013 at 9:47 AM, Eisso J. Atzema <
atzema@math.umaine.edu >
> wrote:
>> Colleagues,
>> I apologize beforehand for triple posting, but I have a
>> somewhat urgent question. While working on a paper,
>> I found a curious 16th century Indian attempt at a derivation
>> for the area formula of a cyclic quadrilateral.
>> The derivation is incorrect, but it is fairly easy to fix it
>> using modern means. The resulting proof is quite interesting,
>> but I just think I have seen it before.
>> I only cannot remember where.
>> Can anyone help me out? The proof goes as follows:
>> Let ABCD be a (convex) cyclic quadrilateral, with AB of
>> length a, BC of length b and so on. Now, let A' on the ray
>> from A to B be such that $AA'=(a+c)/2$, B' on the ray
>> from B to C such that $BB'=(b+c)/2$ and so on.
>> Then, it is straightforward to see that A'B'C'D' is a
>> rectangle with the same area as ABCD. Furthermore,
>> it can be shown that the sides of this rectangle equal
>> $e/f \cdot \sqrt{(s-a)(s-c)}$ and $f/e \cdot \sqrt{(s-b)(s-d)}$,
>> with e is the length of AC and f the length of BD
>> (where I might have switched the role of e and f).
>> Thus, the area of ABCD equals $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.
>> Has anyone seen this proof before? Please provide me with a
>> reference, however vague. Also, basically my verification for
>> the lengths of the rectangle is a tedious computation.
>> Might there be a nicer and shorter way to verify these
>> lengths?
>> Thanks so much,
>> Eisso

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Message: #322
Date: 05/11/2013 12:15:21
From: eckart_schmidt@t-online.de
Subject: Question about a Quadrilateral

Dear Mister Atzema,

indeed no reference, but perhaps a help: see attachment.

Best regards Eckart

Re: Question about a Quadrilateral

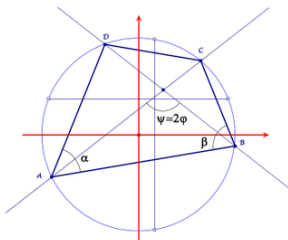
(QFG message 319, 321)

“Let $ABCD$ be a (convex) cyclic quadrilateral, with AB of length a , BC of length b and so on. Now, let A' on the ray from A to B be such that $AA'=(a+c)/2$, B' on the ray from B to C such that $BB'=(b+c)/2$ and so on. Then, it is straightforward to see that $A'B'C'D'$ is a rectangle with the same area as $ABCD$.”

Let $ABCD$ be a cyclic quadrilateral wrt a circle of radius 1. The quadrilateral shall be defined by the following angles:

$$\angle DAB = \alpha, \quad \angle ABC = \beta, \quad \angle ASB = \psi = 2\varphi, \\ (\alpha \leq \beta \leq 90^\circ, \quad S \text{ diagonal crosspoint}).$$

We can choose a cartesian coordinate system, orientated at the midpoints of the arcs (see figure). Advantage: The axes will be parallel to the sides of your rectangle.



The vertices have the coordinates:

$$A(-\cos(\beta - \varphi); -\sin(\beta - \varphi)), \quad B(\cos(\alpha - \varphi); -\sin(\alpha - \varphi), \\ C(-\cos(\beta + \varphi); \sin(\beta + \varphi)), \quad D(\cos(\alpha + \varphi); \sin(\alpha + \varphi))$$

with the following distances

$$a = 2 \cos\left(\frac{\alpha + \beta - \psi}{2}\right), \quad b = 2 \cos\left(\frac{-\alpha + \beta + \psi}{2}\right), \\ c = -2 \cos\left(\frac{\alpha + \beta + \psi}{2}\right), \quad d = 2 \cos\left(\frac{\alpha - \beta + \psi}{2}\right), \\ e = 2 \sin \beta, \quad f = 2 \sin \alpha.$$

The vertices of the discussed rectangle can be calculated:

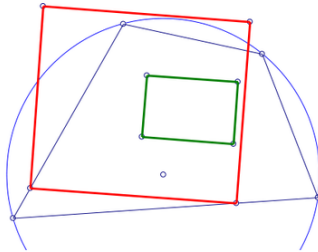
$$A'(-\cos \beta \cos \varphi + \sin \alpha \sin \varphi; -\sin \beta \cos \varphi + \cos \alpha \sin \varphi), \\ B'(-\cos \beta \cos \varphi + \sin \alpha \sin \varphi; \sin \beta \cos \varphi + \cos \alpha \sin \varphi), \\ C'(-\cos \beta \cos \varphi - \sin \alpha \sin \varphi; \sin \beta \cos \varphi + \cos \alpha \sin \varphi), \\ D'(-\cos \beta \cos \varphi - \sin \alpha \sin \varphi; -\sin \beta \cos \varphi + \cos \alpha \sin \varphi).$$

The side lengths of the rectangle are

$$AB = CD = 2 \sin \beta \cos \varphi, \quad BC = DA = 2 \sin \alpha \sin \varphi \\ \text{and the area } 2 \sin \alpha \sin \beta \sin \psi.$$

Additional remark: There is a further rectangle with parallel sides wrt the discussed rectangle, built up of the incenters of the triangles ABC , BCD , CDA , DAB . This rectangle has the side lengths

$$\sin \frac{\alpha + \beta}{2} - \cos \varphi \quad \text{and} \quad \cos \frac{\alpha - \beta}{2} - \sin \varphi.$$



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Message: #323
Date: 08/11/2013 4:00:06
From: eckart_schmidt@t-online.de
Subject: Some "splitter"

Dear Chris, dear friends,

I had a look in "Encyklopädie der Mathematischen Wissenschaften" and got some suggestions, concerning:

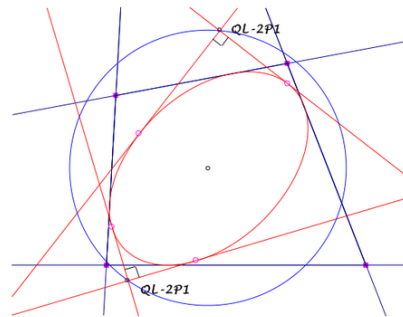
- * further properties of the Plücker Points QL-2P1,
 - * angle bisectors of pairs of tangents wrt QL-inscribed conics
 - * inscribed QL-conics with maximum product of axes.
- ... see attachment.

Best regards Eckart

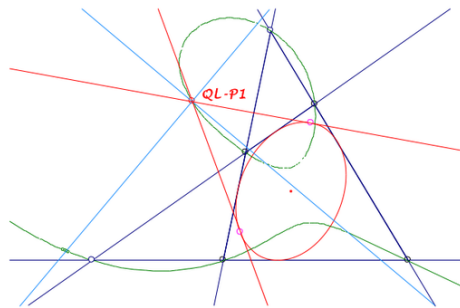
Some „Splitter“

(08.11.2013)

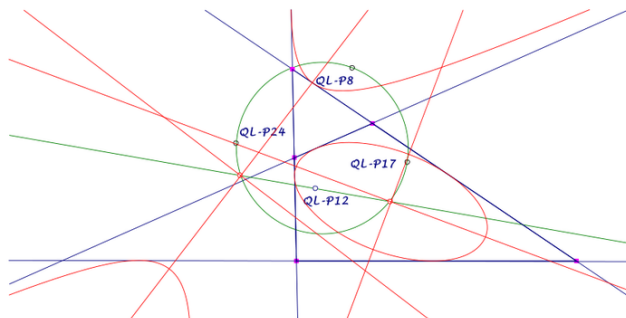
- The Plücker Points $QL-2P1$ (which are the common points of the Thales circles about the QL -diagonals) are the intersections of the directrix circles of the inscribed QL -conics.
For the Plücker Points the tangents to inscribed conics are orthogonal.



- Pairs of tangents from a fixed point on $QL-Cu1$ to QL -inscribed conics have the same angle bisectors. For $QL-P1$ these bisectors are the 1^{st} and 2^{nd} Steiner axis (see $QL-Tf1$).

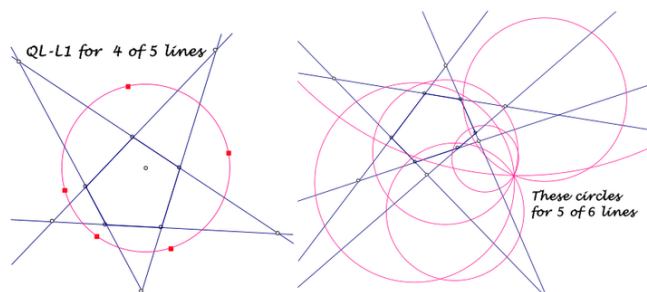


- There are two inscribed QL -conics (an ellipse and a hyperbola) with maximum product of axes. The centers lie symmetric to $QL-P12$ in the intersections of $QL-L1$ and a circle through $QL-P8$, $QL-P17$, $QL-P24$.



- (Special for Chris, extending EQF for $n = 5, 6, \dots$, perhaps well known?):

2 lines give a point (intersection).
 These points for 2 of 3 lines give a circle (circumcircle).
 These circles for 3 of 4 lines give a point (Miquel point).
 These points for 4 of 5 lines give a circle (...).
 These circles for 5 of 6 lines give a point (...) ...



Suggestions out of

Dingeldey, Friedrich,
 Kegelschnitte und Kegelschnittssysteme.
 (Encyklopädie der Mathematischen Wissenschaften,
 Bd.III/2, Heft 1).
 Teubner Leipzig, 1903

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Splitter 13-11-08.pdf

Message: #324

Date: 09/11/2013 10:53:23

From: Seiichi Kirikami

Subject: The perspector of 2 triangles made from 3 QA version Pedal lines and

Dear friends,

Given a quadrangle $P_1P_2P_3P_4$, 2 triangles made from 3 QA version Pedal lines and Steiner lines has a perspector.

Its barycentrics: See the last line of the attached file.

It might be too complicated.

Best regards,

Seiichi.

P. S. If the attachment does not have ".nb", add ".nb" to it.


```

(* Perspector of triangles of Pedal and Steiner lines *)
Simplificar[lista_] := Simplify[
$$\frac{\text{lista}}{\text{Apply}[\text{PolynomialGCD}, \text{lista}]}$$
];
Punto[r_, s_] := Simplificar[Cross[r, s]];
Recta[P_, Q_] := Simplificar[Cross[P, Q]];
DividirRazon[ptP_, ptQ_, m_, n_] := Simplificar[n Tr[ptQ] ptP + m Tr[ptP] ptQ];
Medio[ptP_, ptQ_] := DividirRazon[ptP, ptQ, 1, 1];
PuntoInfinito[{p_, q_, r_}] := {q - r, r - p, p - q};
PuntoInfinitoPerpendicular[{f_, g_, h_}] := PuntoInfinito[{SA f, SB g, SC h}] /.
  {SA  $\rightarrow \frac{b^2 + c^2 - a^2}{2}$ , SB  $\rightarrow \frac{c^2 + a^2 - b^2}{2}$ , SC  $\rightarrow \frac{a^2 + b^2 - c^2}{2}$ };
Perpendicular[P_, r_] := Recta[P, PuntoInfinitoPerpendicular[PuntoInfinito[r]]];
Perpendic[P_, Q_, R_] := Perpendicular[P, Recta[Q, R]];
Mediatriz[ptP_, ptQ_] := Perpendic[Medio[ptP, ptQ], ptP, ptQ];
Circuncentro[{ptA_, ptB_, ptC_}] :=
  Punto[Mediatriz[ptA, ptB], Mediatriz[ptA, ptC]];
Ortocentro[{ptA_, ptB_, ptC_}] :=
  Punto[Perpendic[ptA, ptB, ptC], Perpendic[ptB, ptC, ptA]];
Pie[P_, r_] := Punto[Perpendicular[P, r], r];
SimetriaAxial[P_, r_] := DividirRazon[P, Pie[P, r], 2, -1];
SimetriaAxiala[P_, Q_, R_] := SimetriaAxial[P, Recta[Q, R]];

(* 4 vertices *)
pt1 = {0, 1, 0};
pt2 = {0, 0, 1};
pt3 = {1, 0, 0};
pt4 = {p, q, r};

(* lines and cross points *)
rt12 = Recta[pt1, pt2];
rt13 = Recta[pt1, pt3];
rt14 = Recta[pt1, pt4];
rt23 = Recta[pt2, pt3];
rt24 = Recta[pt2, pt4];
rt34 = Recta[pt3, pt4];
ptS1 = Punto[rt13, rt24];
ptS2 = Punto[rt12, rt34];
ptS3 = Punto[rt23, rt14];

(* circumcenters *)
ct11 = Circuncentro[{pt2, pt3, ptS2}];
ct12 = Circuncentro[{pt3, pt4, ptS3}];
ct21 = Circuncentro[{pt4, pt2, ptS3}];
ct22 = Circuncentro[{pt2, pt3, ptS1}];
ct31 = Circuncentro[{pt3, pt4, ptS1}];
ct32 = Circuncentro[{pt4, pt2, ptS2}];

(* Miquel points *)
ptM1 = SimetriaAxiala[pt3, ct11, ct12];
ptM2 = SimetriaAxiala[pt2, ct21, ct22];

```

```

ptM3 = SimetriaAxiala[pt4, ct31, ct32];

(* perpendicular *)
ppdM11 = Pie[ptM1, rt23];
ppdM12 = Pie[ptM1, rt34];
ppdM21 = Pie[ptM2, rt24];
ppdM22 = Pie[ptM2, rt23];
ppdM31 = Pie[ptM3, rt34];
ppdM32 = Pie[ptM3, rt24];

(* Pedal lines *)
ped1 = Recta[ppdM11, ppdM12];
ped2 = Recta[ppdM21, ppdM22];
ped3 = Recta[ppdM31, ppdM32];

(* Pedal triangle *)
ped12 = Punto[ped1, ped2];
ped23 = Punto[ped2, ped3];
ped31 = Punto[ped3, ped1];

(* Orthocenters *)
ot11 = Ortocentro[{pt2, pt3, ptS2}];
ot12 = Ortocentro[{pt3, pt4, ptS3}];
ot21 = Ortocentro[{pt4, pt2, ptS3}];
ot22 = Ortocentro[{pt2, pt3, ptS1}];
ot31 = Ortocentro[{pt3, pt4, ptS1}];
ot32 = Ortocentro[{pt4, pt2, ptS2}];

(* Steiner lines *)
stn1 = Recta[ot11, ot12];
stn2 = Recta[ot21, ot22];
stn3 = Recta[ot31, ot32];

(* Steiner triangle *)
stn12 = Punto[stn1, stn2];
stn23 = Punto[stn2, stn3];
stn31 = Punto[stn3, stn1];

(* connecting lines *)
rtA = Recta[ped12, stn12];
rtB = Recta[ped23, stn23];
rtC = Recta[ped31, stn31];
Det[{rtA, rtB, rtC}] // Factor // Simplify
Punto[rtA, rtB]

```

PedalSteinerLines-nb.pdf

$$\begin{aligned}
& \{ -a^4 (p+q) r (p+r)^2 - \\
& (q+r) (c^4 (p+r)^2 (2p+2q+r) + b^4 (p+q) (2p^2+4pq+2q^2+5pr+6qr+5r^2) + \\
& b^2 c^2 (4p^3+2q^3+4q^2r+qr^2+r^3+p^2(8q+6r)+3p(2q^2+3qr+r^2))) + \\
& a^2 (c^2 (p+r)^2 (2pq+2q^2+3pr+4qr+r^2) + b^2 (p^3(2q+r)+p^2(6q^2+6qr-r^2) + \\
& p(6q^3+11q^2r+2qr^2-2r^3) + q(2q^3+6q^2r+3qr^2-2r^3))) , \\
& -a^4 (p+r) (p^3+2p^2(q+2r)+qr(3q+5r)+p(q^2+7qr+5r^2)) + \\
& (q+r) (b^4 (p+q) (p-r) (p+q+r) + c^4 (p+r) (5p^2+r(q+r)+p(3q+4r)) + \\
& b^2 c^2 (2p^3+p^2(3q-r)+p(q^2+qr-2r^2)+r(q^2-r^2))) + \\
& a^2 (-c^2 (p^2-r^2) (p^2+3pq+2q^2+4pr+3qr+r^2) + b^2 \\
& (p^4+p^3(q+2r)+p^2(-q^2+qr+r^2)-qr(q^2+3qr+2r^2)-p(q^3+2q^2r+2qr^2+2r^3))) , \\
& a^4 (p+q) (p+r)^2 (p+2(q+r)) + (q+r) \\
& (c^4 p (p+r)^2 + b^4 (p+q) (5p^2+2(q+r)^2+p(6q+5r)) + \\
& b^2 c^2 (2p^3+p^2(-3q+r)-2q(q+r)^2-p(6q^2+5qr+r^2))) + \\
& a^2 (-c^2 (p+r)^2 (p^2+2q(q+r)+p(4q+3r)) + \\
& b^2 (p+q) (p^3+p^2(q+3r)+p(4q^2+9qr+6r^2)+2(q^3+3q^2r+4qr^2+2r^3))) \} \\
& \{ a^4 q (p+q)^2 (p+r) + \\
& (q+r) (b^4 (p+q)^2 (2p+q+2r) + c^4 (p+r) (2p^2+5pq+5q^2+4pr+6qr+2r^2) + \\
& b^2 c^2 (4p^3+q^3+q^2r+4qr^2+2r^3+p^2(6q+8r)+3p(q^2+3qr+2r^2))) - \\
& a^2 (b^2 (p+q)^2 (3pq+q^2+2pr+4qr+2r^2) + c^2 (p^3(q+2r)+p^2(-q^2+6qr+6r^2) + \\
& r(-2q^3+3q^2r+6qr^2+2r^3) + p(-2q^3+2q^2r+11qr^2+6r^3))) , -a^4 (p+q)^2 (p+r) \\
& (p+2(q+r)) - (q+r) (b^4 p (p+q)^2 + c^4 (p+r) (5p^2+2(q+r)^2+p(5q+6r)) + \\
& b^2 c^2 (2p^3+p^2(q-3r)-2r(q+r)^2-p(q^2+5qr+6r^2))) + \\
& a^2 (b^2 (p+q)^2 (p^2+2r(q+r)+p(3q+4r)) - \\
& c^2 (p+r) (p^3+p^2(3q+r)+p(6q^2+9qr+4r^2)+2(2q^3+4q^2r+3qr^2+r^3))) , \\
& a^4 (p+q) (p+r) (p^2+p(4q+r)+q(5q+3r)) - \\
& (q+r) (c^4 (p-q) (p+r) (p+q+r) + b^4 (p+q) (5p^2+q(q+r)+p(4q+3r)) + \\
& b^2 c^2 (2p^3-q^3-p^2(q-3r)+qr^2+p(-2q^2+qr+r^2))) + \\
& a^2 (b^2 (p^2-q^2) (p^2+4pq+q^2+3pr+3qr+2r^2) - \\
& c^2 (p+r) (p^3+2p^2q+p(q^2-qr-r^2)-q(2q^2+3qr+r^2))) \} \\
& \{ a^4 (p+q) (q-r) (p+r) (p+q+r) - (q+r) (b^2 c^2 (q-r) (2p^2+q^2+4qr+r^2+3p(q+r)) - \\
& c^4 (p^2(3q+r)+r(5q^2+4qr+r^2)+p(5q^2+7qr+2r^2)) + \\
& b^4 (p^2(q+3r)+q(q^2+4qr+5r^2)+p(2q^2+7qr+5r^2))) + a^2 \\
& (b^2 (-p^3(q+r)-p^2(q^2+2qr+3r^2)+p(q^3+q^2r-2qr^2-2r^3)+q(q^3+2q^2r+qr^2-2r^3)) + \\
& c^2 (p^3(q+r)+p^2(3q^2+2qr+r^2) + \\
& p(2q^3+2q^2r-qr^2-r^3) - r(-2q^3+q^2r+2qr^2+r^3))) , \\
& -a^4 (p+q) (p+r) (2p^2+4pq+2q^2+6pr+5qr+5r^2) - \\
& (b^2 - c^2) (q+r)^2 (b^2 (p+q) r - c^2 (p+r) (2p+2q+r)) + \\
& a^2 (-c^2 (p+r) (2p^3+4q^3+6q^2r+3qr^2+r^3+p^2(6q+4r)+p(8q^2+9qr+r^2)) + b^2 (2p^4 + \\
& 6p^3(q+r)+qr(q^2-qr-2r^2)+p^2(6q^2+11qr+3r^2)+2p(q^3+3q^2r+qr^2-r^3))) , \\
& a^4 (p+q) (p+r) (2p^2+6pq+5q^2+4pr+5qr+2r^2) + \\
& (b^2 - c^2) (q+r)^2 (-c^2 q (p+r) + b^2 (p+q) (2p+q+2r)) + \\
& a^2 (-c^2 (p+r) (2p^3+p^2(6q+4r)+q(-2q^2-qr+r^2)+p(3q^2+5qr+2r^2)) + \\
& b^2 (p+q) (2p^3+q^3+3q^2r+6qr^2+4r^3+p^2(4q+6r)+p(q^2+9qr+8r^2))) \}
\end{aligned}$$

0

$$\begin{aligned}
& \{ -a^8 (p+q)^2 (p+r)^2 \\
& \quad (p^2 (q^2 + 6qr + r^2) + qr (11q^2 + 26qr + 11r^2) + p (q^3 + 19q^2r + 19qr^2 + r^3)) - 2a^6 (p+q) \\
& \quad (p+r) (c^2 (p+r) (p^4 (3q+r) + p^3 (10q^2 + 7qr - r^2) + p^2 (11q^3 + 4q^2r - 14qr^2 - 5r^3) + \\
& \quad p (4q^4 - 6q^3r - 28q^2r^2 - 21qr^3 - 5r^4) - r (4q^4 + 9q^3r + 10q^2r^2 + 7qr^3 + 2r^4)) + \\
& \quad b^2 (p+q) (p^4 (q+3r) + p^3 (-q^2 + 7qr + 10r^2) + p^2 (-5q^3 - 14q^2r + 4qr^2 + 11r^3) - \\
& \quad p (5q^4 + 21q^3r + 28q^2r^2 + 6qr^3 - 4r^4) - q (2q^4 + 7q^3r + 10q^2r^2 + 9qr^3 + 4r^4)) + \\
& \quad (b^2 - c^2) (q+r)^3 (-c^6 (p+r)^2 (2p^3 + qr (q+r) + p^2 (7q+3r) + p (3q^2 + 6qr + r^2)) + \\
& \quad b^6 (p+q)^2 (2p^3 + qr (q+r) + p^2 (3q+7r) + p (q^2 + 6qr + 3r^2)) + \\
& \quad b^2 c^4 (p+r) (6p^4 + 3p^3 (7q+5r) + p^2 (29q^2 + 43qr + 12r^2) + \\
& \quad 3p (6q^3 + 12q^2r + 9qr^2 + r^3) + q (4q^3 + 10q^2r + 11qr^2 + 5r^3)) - \\
& \quad b^4 c^2 (6p^5 + 21p^4 (q+r) + p^3 (27q^2 + 64qr + 29r^2) + qr (5q^3 + 11q^2r + 10qr^2 + 4r^3) + \\
& \quad p^2 (15q^3 + 70q^2r + 65qr^2 + 18r^3) + p (3q^4 + 32q^3r + 47q^2r^2 + 28qr^3 + 4r^4)) + \\
& \quad a^4 (q+r) (c^4 (p+r)^2 (10p^5 + p^4 (43q+23r) + 7p^3 (9q^2 + 8qr + 3r^2) + \\
& \quad qr (2q^3 + 4q^2r + 3qr^2 + r^3) + p^2 (36q^3 + 41q^2r + 32qr^2 + 7r^3) + \\
& \quad p (6q^4 + 12q^3r + 17q^2r^2 + 8qr^3 - r^4)) + \\
& \quad b^4 (p+q)^2 (10p^5 + p^4 (23q+43r) + 7p^3 (3q^2 + 8qr + 9r^2) + qr (q^3 + 3q^2r + 4qr^2 + 2r^3) + \\
& \quad p^2 (7q^3 + 32q^2r + 41qr^2 + 36r^3) + p (-q^4 + 8q^3r + 17q^2r^2 + 12qr^3 + 6r^4)) + \\
& \quad 2b^2 c^2 (p+q) (p+r) (6p^5 + 2q^5 + 9q^4r + 16q^3r^2 + 16q^2r^3 + 9qr^4 + 2r^5 + \\
& \quad 23p^4 (q+r) + p^3 (31q^2 + 54qr + 31r^2) + p^2 (19q^3 + 51q^2r + 51qr^2 + 19r^3) + \\
& \quad p (7q^4 + 29q^3r + 46q^2r^2 + 29qr^3 + 7r^4)) - \\
& \quad 2a^2 (q+r) (b^6 (p+q)^2 (4p^5 + q^2r (q+r)^2 + 4p^4 (3q+4r) + 2p^3 (8q^2 + 17qr + 9r^2) + \\
& \quad 2pq (q^3 + 6q^2r + 8qr^2 + 3r^3) + p^2 (10q^3 + 33q^2r + 28qr^2 + 5r^3)) + \\
& \quad c^6 (p+r)^2 (4p^5 + q^2r (q+r)^2 + 4p^4 (4q+3r) + 2p^3 (9q^2 + 17qr + 8r^2) + \\
& \quad 2pr (3q^3 + 8q^2r + 6qr^2 + r^3) + p^2 (5q^3 + 28q^2r + 33qr^2 + 10r^3)) - \\
& \quad b^2 c^4 (p+r) (4p^6 + 4p^5 (3q+5r) + 2p^4 (6q^2 + 35qr + 23r^2) + p^3 r (83q^2 + 136qr + 53r^2) - \\
& \quad q (q+r)^2 (2q^3 + q^2r - qr^2 - 3r^3) + p^2 (-11q^4 + 26q^3r + 119q^2r^2 + 110qr^3 + 28r^4) + \\
& \quad p (-9q^5 - 10q^4r + 26q^3r^2 + 57q^2r^3 + 35qr^4 + 5r^5)) - \\
& \quad b^4 c^2 (4p^7 + 12p^6 (2q+r) + 2p^5 (33q^2 + 41qr + 6r^2) + p^4 q (99q^2 + 206qr + 95r^2) + \\
& \quad qr (q+r)^2 (3q^3 + q^2r - qr^2 - 2r^3) + p^3 (81q^4 + 246q^3r + 202q^2r^2 + 26qr^3 - 11r^4) + \\
& \quad p^2 (33q^5 + 145q^4r + 176q^3r^2 + 52q^2r^3 - 21qr^4 - 9r^5) + \\
& \quad p (5q^6 + 38q^5r + 64q^4r^2 + 30q^3r^3 - 13q^2r^4 - 14qr^5 - 2r^6)) , \\
& \quad a^8 (p+q)^2 (p+r)^3 (p^2 (q+r) + p (3q^2 + 6qr + r^2) + q (2q^2 + 7qr + 3r^2)) - \\
& \quad 2 \\
& \quad a^6 \\
& \quad (p+q) \\
& \quad (p+r) \\
& \quad (b^2 (p+q) (p^4 (2q+r) + 2p^3 (5q^2 + 6qr + r^2) + p^2 (16q^3 + 33q^2r + 16qr^2 + r^3) + \\
& \quad 2pq (6q^3 + 17q^2r + 14qr^2 + 3r^3) + q^2 (4q^3 + 16q^2r + 18qr^2 + 5r^3)) + \\
& \quad c^2 (p+r)^2 (4q^4 + 14q^3r + 16q^2r^2 + 9qr^3 + 2r^4 + p^3 (2q+3r) + \\
& \quad p^2 (8q^2 + 17qr + 6r^2) + p (10q^3 + 28q^2r + 20qr^2 + 5r^3)) - \\
& \quad (q+r)^2 (-c^8 (p+r)^3 (p^2 (3q+r) + q (2q^2 + 3qr + r^2) + p (7q^2 + 6qr + r^2)) + \\
& \quad b^8 (p+q)^2 (qr^2 (q+r) + p^3 (q+11r) + pr (6q^2 + 19qr + 11r^2) + p^2 (q^2 + 19qr + 26r^2)) + \\
& \quad 2b^6 c^2 (4p^5 (q-r) + p^4 (15q^2 - 10qr - 9r^2) + p^3 (21q^3 - 2q^2r - 37qr^2 - 10r^3) +
\end{aligned}$$

$$\begin{aligned}
& p^2 (13 q^4 + 11 q^3 r - 42 q^2 r^2 - 31 q r^3 - 7 r^4) + q r (q^4 - q^3 r - 5 q^2 r^2 - 5 q r^3 - 2 r^4) + \\
& p (3 q^5 + 8 q^4 r - 15 q^3 r^2 - 26 q^2 r^3 - 12 q r^4 - 2 r^5) - \\
& b^4 c^4 (2 p^5 (3 q + r) + 6 p^4 (6 q^2 + 3 q r + r^2) + p^3 (63 q^3 + 77 q^2 r + 29 q r^2 + 7 r^3) + \\
& q r (10 q^4 + 23 q^3 r + 21 q^2 r^2 + 7 q r^3 - r^4) + p^2 (43 q^4 + 119 q^3 r + 73 q^2 r^2 + 25 q r^3 + 4 r^4) + \\
& p (10 q^5 + 66 q^4 r + 77 q^3 r^2 + 39 q^2 r^3 + 7 q r^4 + r^5) + \\
& 2 b^2 c^6 (p^4 (5 q^2 + 6 q r + r^2) + p^3 (18 q^3 + 33 q^2 r + 22 q r^2 + 3 r^3) + \\
& 2 q r (2 q^4 + 6 q^3 r + 8 q^2 r^2 + 5 q r^3 + r^4) + p^2 (16 q^4 + 52 q^3 r + 61 q^2 r^2 + 28 q r^3 + 3 r^4) + \\
& p (4 q^5 + 28 q^4 r + 50 q^3 r^2 + 43 q^2 r^3 + 14 q r^4 + r^5)) + \\
& 2 a^2 (q + r) (-c^6 (p + r)^3 (2 p^4 + 2 q (q + r)^2 (2 q + r) + p^3 (9 q + 5 r) + \\
& 2 p^2 (8 q^2 + 10 q r + 3 r^2) + p (14 q^3 + 28 q^2 r + 17 q r^2 + 3 r^3)) + \\
& b^6 (p + q)^2 (2 p^5 - q r (q + r)^2 (3 q + 4 r) + p^4 (5 q + 7 r) + p^3 (5 q^2 + 21 q r + 10 r^2) + \\
& p^2 (q^3 + 14 q^2 r + 28 q r^2 + 9 r^3) - p (q^4 + 7 q^3 r + 4 q^2 r^2 - 6 q r^3 - 4 r^4)) + \\
& b^4 c^2 (p + q) (p + r) (2 p^5 + p^4 (7 q + 9 r) + p^3 (19 q^2 + 29 q r + 16 r^2) + \\
& (q + r)^2 (6 q^3 + 11 q^2 r + 3 q r^2 + 2 r^3) + p^2 (31 q^3 + 51 q^2 r + 46 q r^2 + 16 r^3) + \\
& p (23 q^4 + 54 q^3 r + 51 q^2 r^2 + 29 q r^3 + 9 r^4)) - \\
& b^2 c^4 (p + r) (2 p^6 + p^5 (9 q + 5 r) - 2 p^3 r (13 q^2 + 13 q r + 2 r^2) + p^4 (11 q^2 + 10 q r + 3 r^2) - \\
& q (q + r)^2 (4 q^3 + 12 q^2 r + 18 q r^2 + 5 r^3) - p^2 (12 q^4 + 83 q^3 r + 119 q^2 r^2 + 57 q r^3 + 7 r^4) - \\
& p (12 q^5 + 70 q^4 r + 136 q^3 r^2 + 110 q^2 r^3 + 35 q r^4 + 3 r^5)) + \\
& a^4 (p + r) (-b^4 (p + q)^2 (p^4 (q - r) - p^3 (7 q^2 + 8 q r + 3 r^2) - p^2 (21 q^3 + 32 q^2 r + 17 q r^2 + 4 r^3) - \\
& p (23 q^4 + 56 q^3 r + 41 q^2 r^2 + 12 q r^3 + 2 r^4) - q (10 q^4 + 43 q^3 r + 63 q^2 r^2 + 36 q r^3 + 6 r^4)) + \\
& c^4 (p + r)^2 (p^4 (7 q + 9 r) + 3 p^3 (11 q^2 + 20 q r + 7 r^2) + p^2 (56 q^3 + 135 q^2 r + 94 q r^2 + 21 r^3) + \\
& q (12 q^4 + 42 q^3 r + 56 q^2 r^2 + 33 q r^3 + 7 r^4) + \\
& p (42 q^4 + 128 q^3 r + 135 q^2 r^2 + 60 q r^3 + 9 r^4)) + \\
& 2 b^2 c^2 (p^6 (5 q + 3 r) + p^5 (33 q^2 + 38 q r + 7 r^2) + p^4 (81 q^3 + 145 q^2 r + 64 q r^2 + 4 r^3) + \\
& p^3 (99 q^4 + 246 q^3 r + 176 q^2 r^2 + 30 q r^3 - 3 r^4) + \\
& p^2 (66 q^5 + 206 q^4 r + 202 q^3 r^2 + 52 q^2 r^3 - 13 q r^4 - 5 r^5) + \\
& p (24 q^6 + 82 q^5 r + 95 q^4 r^2 + 26 q^3 r^3 - 21 q^2 r^4 - 14 q r^5 - 2 r^6) + \\
& q (4 q^6 + 12 q^5 r + 12 q^4 r^2 - 11 q^3 r^3 - 9 q r^5 - 2 r^6)) , \\
& a^8 (p + q)^3 (p + r)^2 (p^2 (q + r) + r (3 q^2 + 7 q r + 2 r^2) + p (q^2 + 6 q r + 3 r^2)) - \\
& 2 \\
& a^6 \\
& (p + q) \\
& (p + r) \\
& (b^2 (p + q)^2 (2 q^4 + 9 q^3 r + 16 q^2 r^2 + 14 q r^3 + 4 r^4 + p^3 (3 q + 2 r) + \\
& p^2 (6 q^2 + 17 q r + 8 r^2) + p (5 q^3 + 20 q^2 r + 28 q r^2 + 10 r^3)) + \\
& c^2 (p + r) (p^4 (q + 2 r) + 2 p^3 (q^2 + 6 q r + 5 r^2) + r^2 (5 q^3 + 18 q^2 r + 16 q r^2 + 4 r^3) + \\
& 2 p r (3 q^3 + 14 q^2 r + 17 q r^2 + 6 r^3) + p^2 (q^3 + 16 q^2 r + 33 q r^2 + 16 r^3)) + (q + r)^2 \\
& (-c^8 (p + r)^2 (q^2 r (q + r) + p^3 (11 q + r) + p^2 (26 q^2 + 19 q r + r^2) + p q (11 q^2 + 19 q r + 6 r^2)) + \\
& b^8 (p + q)^3 (p^2 (q + 3 r) + r (q^2 + 3 q r + 2 r^2) + p (q^2 + 6 q r + 7 r^2)) + \\
& 2 b^2 c^6 (4 p^5 (q - r) + p^4 (9 q^2 + 10 q r - 15 r^2) + p^3 (10 q^3 + 37 q^2 r + 2 q r^2 - 21 r^3) + \\
& p^2 (7 q^4 + 31 q^3 r + 42 q^2 r^2 - 11 q r^3 - 13 r^4) + q r (2 q^4 + 5 q^3 r + 5 q^2 r^2 + q r^3 - r^4) + \\
& p (2 q^5 + 12 q^4 r + 26 q^3 r^2 + 15 q^2 r^3 - 8 q r^4 - 3 r^5)) - \\
& 2 b^6 c^2 (p^4 (q^2 + 6 q r + 5 r^2) + p^3 (3 q^3 + 22 q^2 r + 33 q r^2 + 18 r^3) + \\
& 2 q r (q^4 + 5 q^3 r + 8 q^2 r^2 + 6 q r^3 + 2 r^4) + p^2 (3 q^4 + 28 q^3 r + 61 q^2 r^2 + 52 q r^3 + 16 r^4) + \\
& p (q^5 + 14 q^4 r + 43 q^3 r^2 + 50 q^2 r^3 + 28 q r^4 + 4 r^5)) +
\end{aligned}$$

$$\begin{aligned}
& b^4 c^4 (2 p^5 (q+3 r)+6 p^4\left(q^2+3 q r+6 r^2\right)+p^3\left(7 q^3+29 q^2 r+77 q r^2+63 r^3\right)+ \\
& \quad q r\left(-q^4+7 q^3 r+21 q^2 r^2+23 q r^3+10 r^4\right)+p^2\left(4 q^4+25 q^3 r+73 q^2 r^2+119 q r^3+43 r^4\right)+ \\
& \quad p\left(q^5+7 q^4 r+39 q^3 r^2+77 q^2 r^3+66 q r^4+10 r^5\right)))- \\
2 a^2 (q+r)\left(b^6(p+q)^3\left(2 p^4+2 r(q+r)^2(q+2 r)+p^3(5 q+9 r)+\right.\right. \\
& \quad \left.2 p^2\left(3 q^2+10 q r+8 r^2\right)+p\left(3 q^3+17 q^2 r+28 q r^2+14 r^3\right)\right)- \\
c^6(p+r)^2\left(2 p^5-q r(q+r)^2(4 q+3 r)+p^4(7 q+5 r)+p^3\left(10 q^2+21 q r+5 r^2\right)+\right. \\
& \quad \left.p^2\left(9 q^3+28 q^2 r+14 q r^2+r^3\right)+p\left(4 q^4+6 q^3 r-4 q^2 r^2-7 q r^3-r^4\right)\right)- \\
b^2 c^4(p+q)(p+r)\left(2 p^5+p^4(9 q+7 r)+p^3\left(16 q^2+29 q r+19 r^2\right)+\right. \\
& \quad (q+r)^2\left(2 q^3+3 q^2 r+11 q r^2+6 r^3\right)+p^2\left(16 q^3+46 q^2 r+51 q r^2+31 r^3\right)+ \\
& \quad \left.p\left(9 q^4+29 q^3 r+51 q^2 r^2+54 q r^3+23 r^4\right)\right)+ \\
b^4 c^2\left(2 p^7+p^6(7 q+9 r)+p^5\left(8 q^2+19 q r+11 r^2\right)-p^4 q\left(q^2+16 q r+15 r^2\right)-\right. \\
& \quad q r(q+r)^2\left(5 q^3+18 q^2 r+12 q r^2+4 r^3\right)-p^3\left(11 q^4+83 q^3 r+145 q^2 r^2+83 q r^3+12 r^4\right)- \\
& \quad \left.p^2\left(10 q^5+92 q^4 r+229 q^3 r^2+219 q^2 r^3+82 q r^4+12 r^5\right)-\right. \\
& \quad \left.p\left(3 q^6+40 q^5 r+138 q^4 r^2+189 q^3 r^3+116 q^2 r^4+32 q r^5+4 r^6\right)\right))+ \\
a^4(p+q)\left(c^4(p+r)^2\left(p^4(q-r)+p^3\left(3 q^2+8 q r+7 r^2\right)+p^2\left(4 q^3+17 q^2 r+32 q r^2+21 r^3\right)+\right.\right. \\
& \quad \left.r\left(6 q^4+36 q^3 r+63 q^2 r^2+43 q r^3+10 r^4\right)+p\left(2 q^4+12 q^3 r+41 q^2 r^2+56 q r^3+23 r^4\right)\right))+ \\
b^4(p+q)^2\left(p^4(9 q+7 r)+3 p^3\left(7 q^2+20 q r+11 r^2\right)+p^2\left(21 q^3+94 q^2 r+135 q r^2+56 r^3\right)+\right. \\
& \quad \left.r\left(7 q^4+33 q^3 r+56 q^2 r^2+42 q r^3+12 r^4\right)+\right. \\
& \quad \left.p\left(9 q^4+60 q^3 r+135 q^2 r^2+128 q r^3+42 r^4\right)\right))+ \\
2 b^2 c^2\left(p^6(3 q+5 r)+p^5\left(7 q^2+38 q r+33 r^2\right)+p^4\left(4 q^3+64 q^2 r+145 q r^2+81 r^3\right)+\right. \\
& \quad \left.p^3\left(-3 q^4+30 q^3 r+176 q^2 r^2+246 q r^3+99 r^4\right)+p^2\left(-5 q^5-13 q^4 r+52 q^3 r^2+\right.\right. \\
& \quad \left.202 q^2 r^3+206 q r^4+66 r^5\right)+r\left(-2 q^6-9 q^5 r-11 q^4 r^2+12 q^2 r^4+12 q r^5+4 r^6\right)+ \\
& \quad \left.p\left(-2 q^6-14 q^5 r-21 q^4 r^2+26 q^3 r^3+95 q^2 r^4+82 q r^5+24 r^6\right)\right))\}
\end{aligned}$$

Message: #325

Date: 09/11/2013 1:14:48

From: Eisso J. Atzema

Subject: {Disarmed} Question about a Quadrilat

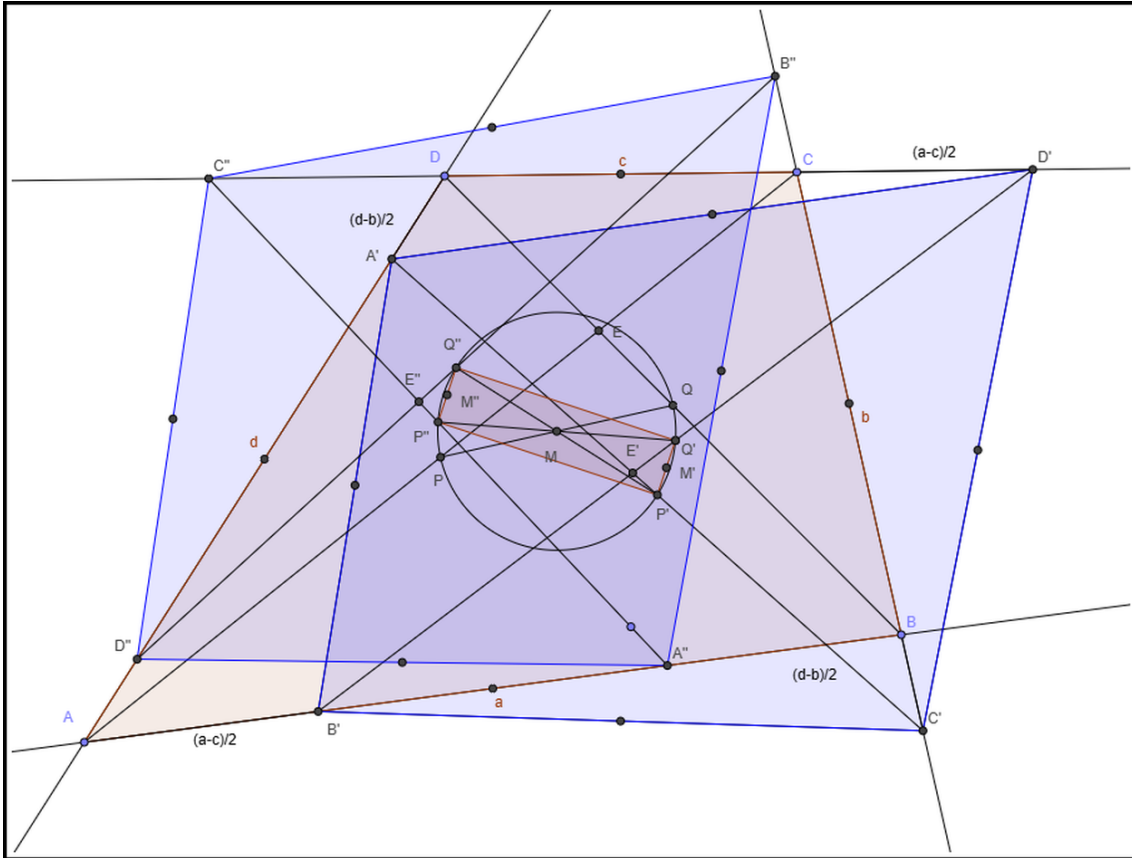
Sehr geehrter Herr Schmidt,

vielen Dank für Ihren Beweis. Der meine ist ungefähr gleich lang (aber weniger trigonometrisch). Mittlerweile habe ich auch noch einige andere Eigenschaften der Konstruktion gefunden:

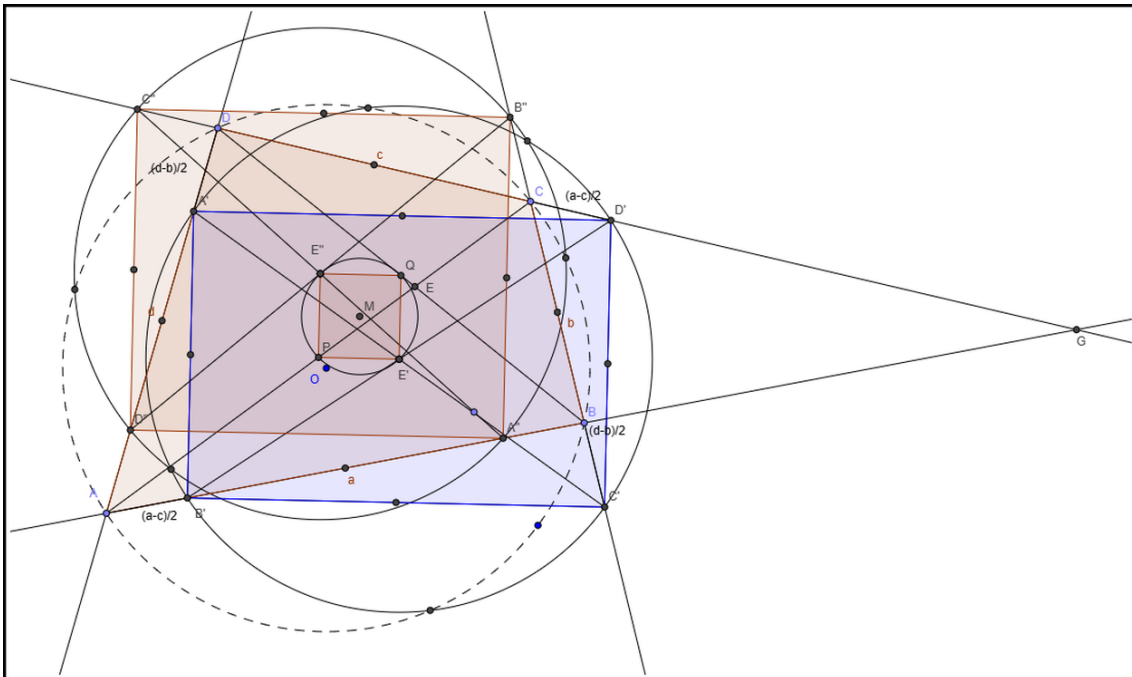
* Nebst dem von mir angedeuteten Rechteck gibt es noch ein zweites, das man bekommt wenn man in die andere Richtung vorgeht: Statt A' auf der Halbgerade AB mit $AA'=(a+c)/2$, nehmen wir einfach B'' auf der Halbgerade BA mit $BB''=(a+c)/2$ usw. Nennen wir die Mittelpunkte von $A'B'C'D'$ und $A''B''C''D''$, E' bzw E'' , so sind E' und E'' Eckpunkten eines Rechteckes deren anderen zwei Eckpunkten die Mittelpunkte P , Q der Diagonalen AC und BD sind und deren Seiten den Seiten von $A'B'C'D'$ und $A''B''C''D''$ parallel sind.

* Falls $ABCD$ ein allgemeines Kreisviereck ist, können $A'B'C'D'$ und $A''B''C''D''$ noch immer konstruiert werden. Natürlich sind $A'B'C'D'$ und $A''B''C''D''$ i.A. keine Rechtecke mehr und auch haben sie eine andere Flächeninhalt als $ABCD$. Einiges kann man aber doch sagen. Auf Grund einer Skizze, sieht es aus als ob P , Q und deren analoge Punkten P' , Q' , P'' , Q'' auf einem Kreis liegen und zwar so dass PQ , $P'Q''$ und $P''Q'$ Diametern sind. Anders gesagt, PQ , $P'Q''$ und $P''Q'$ sind gleichlang und haben den gleichen Mittelpunkt M . Demnach wären auch $P'Q'$ und $P''Q''$ gleichlang (und zwar gleich null falls $ABCD$ ein Kreisviereck ist). Der Punkt M ist auch der Mittelpunkt von $M'M''$, wo M der Mittelpunkt von $P'Q'$, M'' der Mittelpunkt von $P''Q''$ ist. Ich hänge zwei Geogebra Dateien an...

Mit freundlichen Grüßen,
Eisso



ganesh2-ggb.png



ganesh1-ggb.png

Message: #326

Date: 10/11/2013 10:21:41

From: eckart_schmidt@t-online.de

Subject: The perspector of 2 triangles made from 3 QA ...

Dear Seiichi,

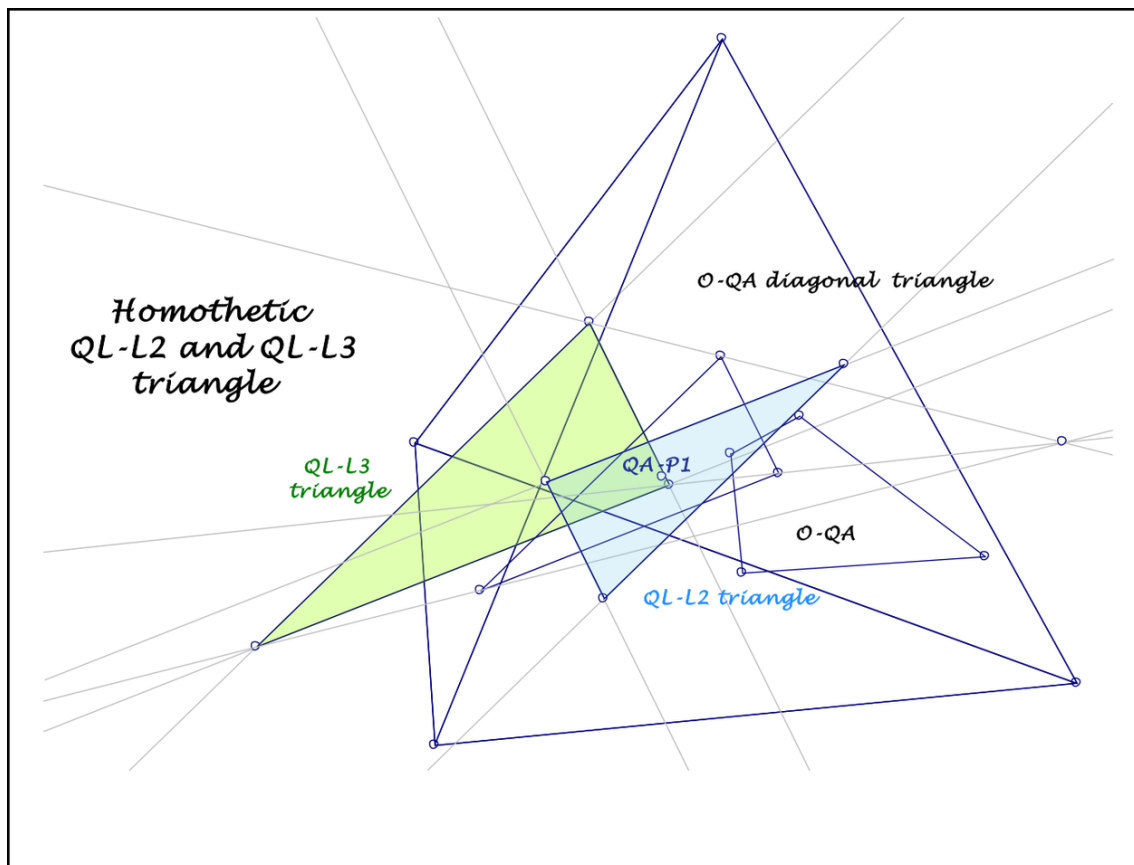
interesting constellation, but extensive coordinates. Some remarks for the geometrical background:

Consider a quadrangle QA and its O-QA (circumcenters of the triangle components) and the diagonal triangle of O-QA:

- * The QL-L2 triangle is the reflection of the O-QA diagonal triangle in the centroid of the reference quadrangle.
- * The QL-L3 triangle is homothetic to the O-QA diagonal triangle.

... see attachment

Best regards Eckart



Seiichi-13-11-10.pdf

Message: #327

Date: 10/11/2013 11:50:40

From: Chris

Subject: The perspector of 2 triangles made from 3 QA version Pedal lines

Dear Seiichi,

Interesting construction.

I found the same CT-coordinates.

Your point QA-Px as well as QA-Tf2[QA-Px], QA-Tf3[QA-Px], QA-Tf4[QA-Px] have no collinearities with known points, neither are they lying on known curves.

Best regards,

Chris

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Message: #328
Date: 10/11/2013 12:35:44
From: Chris
Subject: {Disarmed} Question about a Quadrilateral

Dear Eisso, dear Eckart,

GE: Das Punkt M ist das Schwerpunkt (QA-P1) des Vierecks ABCD.
EN: The point M is the centroid (QA-P1) of the quadrangle ABCD.
NL: Het punt M is het zwaartepunt (QA-P1) van de vierhoek ABCD.

Best regards,
Chris

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Message: #329
Date: 10/11/2013 1:09:28
From: Chris
Subject: Some "splitter"

Dear Eckart,
Very nice properties from an old source!
1. Do you have references to persons who were the "finders" of these properties?
2. About the inscribed conics (3rd point): what is the max. product of axes of an hyperbola?
3. The implications for n-gons I hope to implement them once beside of EQF although I think this can take a lot of time in the future.
Best regards,
Chris

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Message: #330
Date: 11/11/2013 11:12:41
From: seiichikiri
Subject: The perspector of 2 triangles made from 3 QA ...

Dear Eckart,

They have beautiful relations! I think that I have understood them more deeply.
Thanks a lot!

Best regards,
Seiichi.

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Message: #331
Date: 11/11/2013 11:19:38
From: seiichikiri
Subject: The perspector of 2 triangles made from 3 QA version Pedal lines

Dear Chris,
Thank you very much for your computation and confirmation.
I hope that you will appreciate my progress of computation by Mathematica, compared with that one year ago.
Best regards,
Seiichi.

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Message: #332
Date: 11/11/2013 1:03:46
From: eckart_schmidt@t-online.de
Subject: Some "splitter"

Dear Chris,

the background for my "splitter" are pages 106 -108 in
[http://dfg-viewer.de/v2/?set%5Bimage%5D=124&set%5Bmets%5D=http%3A%2F%2Fgdz.sub.uni-goettingen.de%2Fmets_export.php%3FPPN%3DPPN360609856&set%5Bzoom%5D=default&set%5Bdebug%5D=0&set%5Bdouble%5D=0&set%5Bstyle%5D=.](http://dfg-viewer.de/v2/?set%5Bimage%5D=124&set%5Bmets%5D=http%3A%2F%2Fgdz.sub.uni-goettingen.de%2Fmets_export.php%3FPPN%3DPPN360609856&set%5Bzoom%5D=default&set%5Bdebug%5D=0&set%5Bdouble%5D=0&set%5Bstyle%5D=)

... and for the last point page 1010 in
[http://dfg-viewer.de/v2/?set%5Bimage%5D=252&set%5Bmets%5D=http%3A%2F%2Fgdz.sub.uni-goettingen.de%2Fmets_export.php%3FPPN%3DPPN360609767&set%5Bzoom%5D=default&set%5Bdebug%5D=0&set%5Bdouble%5D=0&set%5Bstyle%5D=.](http://dfg-viewer.de/v2/?set%5Bimage%5D=252&set%5Bmets%5D=http%3A%2F%2Fgdz.sub.uni-goettingen.de%2Fmets_export.php%3FPPN%3DPPN360609767&set%5Bzoom%5D=default&set%5Bdebug%5D=0&set%5Bdouble%5D=0&set%5Bstyle%5D=)

I am sure, there are more suggestions.

Best regards Eckart

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Message: #333
Date: 11/11/2013 1:37:36
From: eckart_schmidt@t-online.de
Subject: The perspector of 2 triangles made from ...

Dear Seiichi,

thanks for your message. A further remark:
The QL-L2 triangle is the circumscribed triangle of the Morley Triangle QA-Tr3 with vertices in QG-P10 of the QG-components.

Best regards Eckart

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Message: #334
Date: 11/11/2013 2:25:41
From: seiichikiri
Subject: The perspector of 2 triangles made from ...

Dear Eckart,
Your description of nesting triangles is very interesting for me. As QL-3 triangle is homothetic to O-QA diagonal triangle, I will deduce the homothetic point later. If it had simpler coordinates, I would be very happy.
Best regards,
Seiichi.

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Message: #335
Date: 12/11/2013 2:43:54
From: seiichikiri
Subject: The perspector of 2 triangles made from ...

Dear Eckart,

I obtained the coordinates of the homothetic center or point.
It is also complicated, including $(..a^8(p, q \text{ or } r)^9..)$,
while those of the former case include $(.. a^8(p, q \text{ or } r)^8..)$.

Best regards,
Seiichi.

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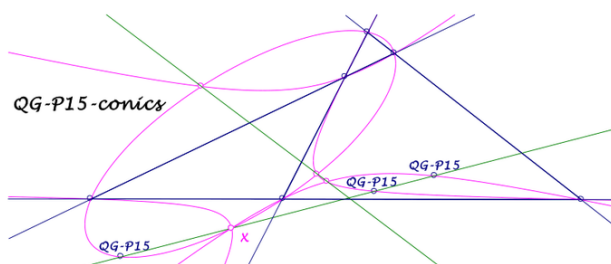
Message: #336
Date: 14/11/2013 9:30:05
From: eckart_schmidt@t-online.de
Subject: 3 QG-circumconics for a QL II

Dear friends,
here is another aspect as in message #320 of three
QG-circumconics for a quadrilateral.
If interested, see attachment.
Best regards Eckart

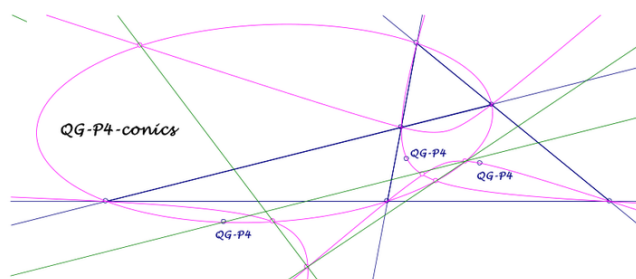
Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Three QG -Circumconics for a Quadrilateral II

Here another aspect is worked out as in EQF-Note 2013-11-04. We consider circumconics of quadrilaterons through a fixed quadrigon point $QG-Px$. For a quadrilateral there are three of these conics. We discuss the cases, that these conics have a common point. – Reference triangle for barycentric coordinates is $QL-DT$.



For quadrilaterons we consider circumconics through a fixed quadrigon point $QG-Px$. For the QG -components of a quadrilateral there are three of these $QG-Px$ -conics, each containing four of the six QL -points. In pairs they have four intersections: two QL -points and two further (not always real) intersections. So there are six further intersections on four lines defining a new quadrilateral (see figure below).



In the following only examples are worked out, where three intersections of the conics coincide (see first figure):

If the three points $QG-Px$ for a quadrilateral are collinear, the three $QG-Px$ -conics have a common point (collinear with the three $QG-Px$) and three further collinear intersections.

QG-P2

The common point of the three *QG-P2*-conics is the point at infinity of the Newton-Line *QL-L1*:

$$X(m^2 - n^2 : n^2 - l^2 : l^2 - m^2).$$

The three further intersections lie on the *M3D* Line *QL-L9*.

QG-P12

The common point of the three *QG-P12*-conics is the *QL*-Harmonic Center *QL-P13*.

The three further intersections lie on a parallel to *QL-L9* through *QL-P19*.

QG-P15 (see first figure)

The common point of the three *QG-P15*-conics is the intersection of *QL-L9* and *QL-P8*, *QL-P13*:

$$X(m^2 n^2 (m^2 n^2 - l^4) : n^2 l^2 (n^2 l^2 - m^4) : l^2 m^2 (l^2 m^2 - n^4)).$$

The three further intersections lie on a line with the equation

$$(m^2 n^2 + l^4)x + (n^2 l^2 + m^4)y + (l^2 m^2 + n^4)z = 0,$$

containing the intersection of *QL-L9* and its *QL-Tf2* image.

QG-P16

The common point of the three *QG-P16*-conics lies collinear with the *QG-P16*-points and *QL-P26* on the *QL-Tf1* image of the Dimidium Circle *QL-Ci6*:

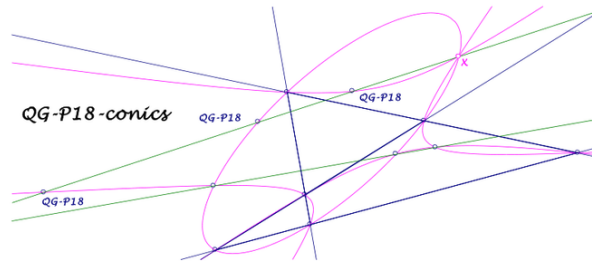
$$X(m^2 n^2 (a^2 M + b^2 L)(a^2 N + c^2 L)(a^2 MN + (m - n)(b^2 n N + c^2 m M)) \\ (a^2 MN + (m + n)(-b^2 n N + c^2 m M)) : \dots)$$

(only the first coordinate with $L = m^2 - n^2, M = n^2 - l^2, N = l^2 - n^2$).

QG-P18

The common point of the three *QG-P18*-conics lies collinear with the *QG-P18*-points on the *QL-Tf1* image of the *QL-DT* Medial Circle *QL-Ci2*:

$$X((m^2 S_C^2 - n^2 S_B^2)(l^2(S_B S_C + S^2) + S_A(m^2 b^2 + n^2 c^2)) \\ : (n^2 S_A^2 - l^2 S_C^2)(m^2(S_A S_C + S^2) + S_B(n^2 c^2 + l^2 a^2)) \\ : (l^2 S_B^2 - m^2 S_A^2)(n^2(S_A S_B + S^2) + S_C(l^2 a^2 + m^2 b^2))).$$



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2013-11-15.pdf

Message: #337

Date: 15/11/2013 3:43:08

From: eckart_schmidt@t-online.de

Subject: Numerically confirmed quadrangle points with a ...

Dear Seiichi,

if you use for the fixed CSC-points the Miquel triangle as reference triangle ABC (with C in the Miquel point), then the barycentric coordinates are

$a(a+b+\sqrt{(a+b)^2-c^2}):b(a+b+\sqrt{(a+b)^2-c^2}): -c^2),$

$a(a+b-\sqrt{(a+b)^2-c^2}):b(a+b-\sqrt{(a+b)^2-c^2}): -c^2).$

Best regards Eckart

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Message: #338

Date: 15/11/2013 4:34:49

From: eckart_schmidt@t-online.de

Subject: Clawson-Schmidt Conjugate QL-Tf1

Dear Chris,

if we take the Miquel triangle as reference triangle ABC (C in the Miquel point), then the Clawson-Schmidt Conjugate QL-Tf1 becomes a very simple form for quadrilaterals:

$(x: y: z)$

$\rightarrow (-a^2 y: -b^2 x: (a^2 y z + b^2 z x + c^2 x y) / (x+y+z)).$

Let $L3 \wedge L4 = P4 = (u: v: w),$

then

$L2 \wedge L3 = P3 = (a^2 w: T: c^2 u),$

$L1 \wedge L2 = P2 = (a^2 v: b^2 u: T),$

$L4 \wedge L1 = P1 = (T: b^2 w: c^2 v)$

with $T = -(a^2 v w + b^2 w u + c^2 u v) / (u+v+w).$

The diagonal crosspoints have not so simple coordinates.

Best regards Eckart

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Message: #339
Date: 15/11/2013 8:50:25
From: Chris van Tienhoven
Subject: Clawson-Schmidt Conjugate QL-Tf1

Dear Eckart,

When the Miquel Triangle is reference triangle, the question arises how to construct a corresponding reference Quadrangle. With your triple transformation:

(u: v: w)

--->((a² w: T: c² u), (a² v: b² u: T), (T: b² w: c² v)),
where $T = -(a^2 v w + b^2 w u + c^2 u v) / (u+v+w)$.

Simplified P1 -> (P2,P3,P4) this becomes clear:

Every point P1 can be the vertex of a Quadrilateral with ABC as Miquel Triangle.

Because P1, P2, P3, P4 represent a Quadrilateral with ABC as Miquel Triangle.

Further when we apply this triple transformation strictly to a triangle, then:

- * The incenter is transformed into the 3 excenters
- * An excenter is transformed into the incenter + other 2 excenters.
- * A triangle vertex is transformed into the other 2 triangle vertices + some undefined point (0:0:0).

This leads us to the solution of how to construct the triple transformation in a triangle.

1. Let P be some point to be triple transformed wrt some triangle ABC.
 2. Construct the incenter I and excenters E1,E2,E3 of ABC. I.E1.E2.E3 forms a Quadrangle (system of 4 points), representing 3 possible Quadrilaterals:
 - Consecutive lines from Quadrigon I.E1.E2.E3
 - Consecutive lines from Quadrigon I.E1.E3.E2
 - Consecutive lines from Quadrigon I.E2.E1.E3
 3. Construct the Clawson-Schmidt Conjugate of P wrt the 3 Quadrilaterals with consecutive lines of resp. I.E1.E2.E3 , I.E1.E3.E2 , I.E2.E1.E3 .
- These 3 points are the triple transformed points of P.

Best regards,
Chris

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Message: #340
Date: 15/11/2013 11:30:57
From: Chris
Subject: Clawson-Schmidt Conjugate QL-Tf1

Dear Eckart,

An extra property of P1 when triple transformed into (P2,P3,P4):

- * P2.P3.P4 is perspective with ABC.
- * The perspector is the isogonal conjugate of P1.

Best regards,
Chris

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Message: #341
Date: 16/11/2013 10:31:54
From: eckart_schmidt@t-online.de
Subject: Clawson-Schmidt Conjugate QL-Tf1

Dear Chris,
you describe a "triple transformation in a triangle". I would describe it in the following way:
For a vertex A of a triangle ABC (incenter I) there is a transformation TA (analogue TB, TC):
... first reflection in AI,
... then reflection in a circle round A with radius $\sqrt{b \cdot c}$.
TA(P), TB(P), TC(P) are the "triple transformation" of P wrt ABC.
Best regards Eckart

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Message: #342
Date: 16/11/2013 11:42:26
From: seiichikiri
Subject: Numerically confirmed quadrangle points with a ...

Dear Eckart,
Thank you very much for your computation of 2 fixed points.
I think that it is very important to obtain the simple solution and you showed its example on the Miquel triangle.
Best regards,
Seiichi

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Message: #343
Date: 17/11/2013 2:21:37
From: eckart_schmidt@t-online.de
Subject: QG-Tr3 a new Reference Triangle

Dear Chris, dear Seiichi,
the not successful efforts to calculate the fixed points of the
Clawson-Schmidt Conjugate QL-Tf1 have led me to QG-Tr3 as new
reference triangle. I think, it is an acceptable way to
calculate the Miquel point, the 1st Steiner line, the Schmidt
circle, the transformation QL-Tf1 and its fixed points. See
Mathematica file in the attachment.
Best regards Eckart

QG-Tr3 a new Reference Triangle

QG-Tr3 shall be the reference triangle ABC for barycentric coordinates:

$$A=QG-2P2a, B=QG-2P2b, C=QG-P18.$$

If we chose a point $P4(u:v:w)$, we get the other vertices of the quadrigon in the following way:

$P2$ is the isogonal conjugate of $P4$ wrt QG-Tr3

$P1$ is the intersection of $A.P4$ and $B.P2$,

$P3$ is the intersection of $A.P2$ and $B.P4$.

$$\begin{aligned} pP1 &:= \{-a^2 w^2, -c^2 uv, -c^2 uw\}; \\ pP2 &:= \{a^2 vw, b^2 uw, c^2 uv\}; \\ pP3 &:= \{c^2 uv, b^2 w^2, c^2 vw\}; \\ pP4 &:= \{u, v, w\} \end{aligned}$$

Then the Miquel point QL-P1 is

$$pMiqu := \{a^2 vB (uA + vB), b^2 uA (uA + vB), -c^2 uvAB\}$$

... with the short cuts

$$A := (c^2 v^2 + 2 SA vw + b^2 w^2); B := (c^2 u^2 + 2 SB uw + a^2 w^2)$$

The 1st Steiner Line has the coefficients

$$gSt1 := \{-bc^2 uA, ac^2 vB, -a b (a - b) (uA + vB)\}$$

The fixed points of the Clawson Schmidt Conjugate are the intersections of the 1st Steiner Line and the Schmidt Circle. The Schmidt Circle has the equation

$$\begin{aligned} krsS[\{x_, y_, z_\}] &:= -c^2 (buA - avB) (buAx^2 - avBy^2) + 2ABc^2 (ab + SC) uvxy \\ &- 2aBv (aA SAu + aB SA v - bc^2 Au) yz - 2bAu (bA SBu + bB SB v - aB c^2 v) xz \\ &- ab (ab (uA + vB)^2 - ABc^2 uv) z^2 \end{aligned}$$

The fixed points of the Clawson Schmidt Conjugate lie also on a circle through $A=QG-2P2a$, $B=QG-2P2b$ and the incenter of QG-Tr3 with the equation

$$\text{krS}[\{x_ , y_ , z_ \}] := a^2 y z + b^2 x z + c^2 x y - a b z (x + y + z)$$

The fixed points of the Clawson Schmidt Conjugate are

$$\text{pfix1} := \left\{ -a + \frac{a \sqrt{vB} \sqrt{c^2 - (a-b)^2}}{(a-b) \sqrt{uA}}, b + \frac{b \sqrt{uA} \sqrt{c^2 - (a-b)^2}}{(a-b) \sqrt{vB}}, \frac{c^2}{(a-b)} \right\}$$

$$\text{pfix2} := \left\{ -a - \frac{a \sqrt{vB} \sqrt{c^2 - (a-b)^2}}{(a-b) \sqrt{uA}}, b - \frac{b \sqrt{uA} \sqrt{c^2 - (a-b)^2}}{(a-b) \sqrt{vB}}, \frac{c^2}{a-b} \right\}$$

Also the Clawson-Schmidt Conjugate QL-Tf1 gets an acceptable form:

$$\begin{aligned} \text{QLTf1}[\{x_ , y_ , z_ \}] := \\ \{ -a^2 B v (-B c^2 v y^2 + A c^2 u x y + (a^2 A u - A b^2 u - 2 B S A v) y z - b^2 (A u + B v) z^2), \\ -b^2 A u (-A c^2 u x^2 + B c^2 v x y + (-a^2 B v + b^2 B v - 2 A S B u) x z - a^2 (A u + B v) z^2), \\ c^2 u v A B (c^2 x y + b^2 x z + a^2 y z) \} \end{aligned}$$

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Chris17-nb.pdf

Message: #344
Date: 17/11/2013 7:47:58
From: Chris
Subject: QG-Tr3 a new Reference Triangle

Dear Eckart,
Very nice approach (change of reference triangle) for dealing with items concerning the Clawson-Schmidt Conjugates (CSC)!
The new QG-circle you mention (through QG-2P2a/b, Incenter of QG-Tr3 and fixed CSC-points looks special. The only special thing I found about this circle is that the 3 QL-versions are coaxial with axis the 1st Steiner Axis and with common points the fixed CSC-points.
Chris

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Message: #345
Date: 18/11/2013 1:30:10
From: eckart_schmidt@t-online.de
Subject: Angle bisectors of a quadrigon

Dear friends,

it is well known, that the intersections of inner / outer angle bisectors of a quadrigon give cyclic quadrigons. But there are rarely connections with EQF.

Some observations with CABRI:

- * The two cyclic quadrigons have the same diagonal lines (angle bisectors of opposite side lines of the reference quadrigon).
- * The midpoints of the circumcircles for the cyclic quadrigons lie on the 1st Steiner axis (see QL-Tf1).
- * The circumcircles of the cyclic quadrigons cut the Schmidt circle (see QL-Tf1) orthogonal.
- * The radical axis of the two circumcircles is the 2nd Steiner axis (see QL-Tf1).
- * The circumcircles of the cyclic quadrigons are invariant wrt the Clawson-Schmidt Conjugate QL-Tf1.
- * The Euler-Poncelet points of the cyclic quadrigons lie on the Newton line QL-L1 of the reference quadrigon.
- * Taking QG-Tr3 as reference triangle (see my message #343), the vertices of the inner cyclic quadrigon can be calculated:
P12(a $-(c \sqrt{B} - b \sqrt{A})$:
b c u $\sqrt{A}$: $c^2 u \sqrt{A}$),

P23(a c v $\sqrt{B}$:
b $-(c \sqrt{A} - u \sqrt{B})$: $c^2 v \sqrt{B}$),

P34((c v $\sqrt{B}$ + c u $\sqrt{A}$:
b w $\sqrt{A}$: c w $\sqrt{A}$),

P41(a w $\sqrt{B}$:
c v $\sqrt{B}$ + (c u $\sqrt{A}$: c w $\sqrt{B}$)).

Not mentioned are general properties of cyclic quadrigons.
Perhaps someone can find further properties.
Best regards Eckart

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Message: #346
Date: 20/11/2013 9:46:34
From: eckart_schmidt@t-online.de
Subject: A new circle for quadrilaterals

Dear Chris,

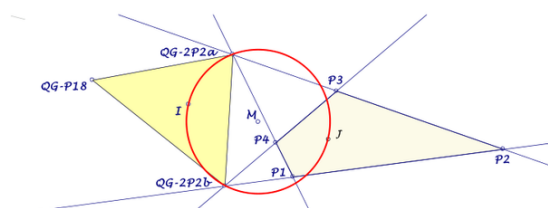
I have collected some properties of the new QG-circle, I mentioned in my message #343.
... see attachment

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A new Circle for Quadrigons

Here a new *QG-circle* will be described connected with the angle bisectors of a quadrigon. – Reference triangle is the Quasi Isogonal Triangle *QG-Tr3*. But most of the properties are only tested with *CABRI*.



Preliminary Remarks

Reference triangle is the Quasi Isogonal Triangle *QG-Tr3* with vertices in the intersections of opposite sides *QG-2P2a,b* and the Quasi Isogonal Crosspoint *QG-P18*. We use the Incenter *I*, the excenter *J* wrt *QG-P18* and the corresponding isogonal conjugate (*QG-Tf2*).

The new *QG-circle* is the Thales circle about *IJ*.

If we chose the reference triangle *ABC* with
 $A = QG - 2P2a, B = QG - 2P2b, C = QG - P18$

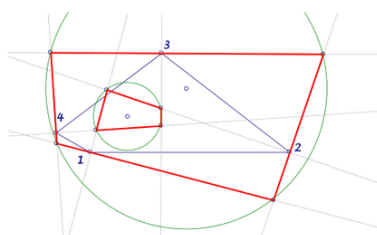
and give P_4 the coordinates u, v, w , then we get

$$P_1(a^2w : c^2uv : c^2uw), \quad P_2(a^2vw : b^2uw : c^2uv),$$

$$P_3(c^2uv : b^2w^2 : c^2vw), \quad P_4(u : v : w)$$

and the equation of the new *QG-circle* is

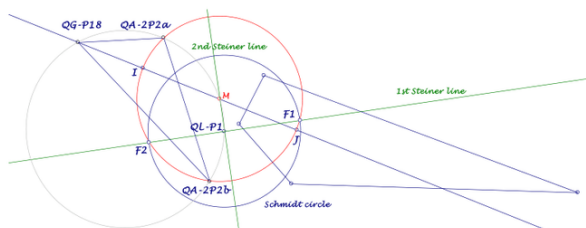
$$a^2yz + b^2zx + c^2xy - abz(x + y + z) = 0.$$



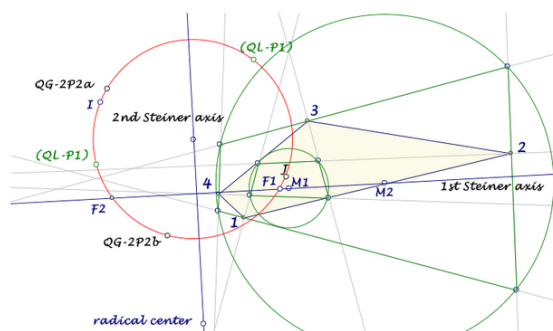
Furthermore we consider the two cyclic quadrigons, constructed with the inner and outer angle bisectors of the quadrigon.

Properties of the new QG -Circle $QG-Cix$

- $QG-Cix$ contains the intersections I and J of the inner and outer angle bisectors of opposite sidelines of the quadrigon.
- $QG-Cix$ contains $QG-2P2a,b$ the intersections of opposite sidelines of the quadrigon.
- $QG-Cix$ contains the fixed points F_1, F_2 of the Clawson-Schmidt Conjugate $QL-TfI$.
- The midpoint M of $QG-Cix$ is the intersection of the 2nd Steiner axis (see $QL-TfI$) and the circumcircle of $QG-Tr3$ (beside $QL-PI$).



- $QG-Cix$ is invariant wrt the Clawson-Schmidt Conjugate $QL-TfI$.
- $QG-Cix$ is invariant wrt the Isogonal Conjugate for $QG-Tr3$ ($QG-Tf2$).
- $QG-Cix$ cuts the circumcircles of the cyclic angle-bisector-quadrilaterals orthogonal.
- $QG-Cix$ contains the Miquel points ($QL-PI$) of the cyclic angle-bisector-quadrilaterals.
- The radical center of $QG-Cix$ and the circumcircles of the angle-bisector-quadrilaterals is the intersection of $QG-LI$ and the 2nd Steiner axis.
- The polar of the radical center wrt $QG-Cix$ is the 1st Steiner axis.



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2013-11-20.pdf

Message: #347
Date: 22/11/2013 4:02:42
From: eckart_schmidt@t-online.de
Subject: Another quadrigon circle

Dear friends,

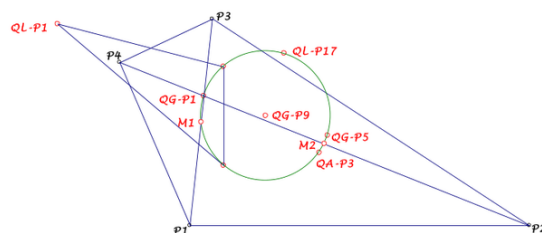
in the attachment there is another quadrigon circle described,
connected with diagonal points.

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

Another Quadrigon Circle

Here another *QG*-circle will be described
connected with diagonal points.



This *QG*-circle is the circumcircle of the Diagonal Crosspoint *QG-P1* and the midpoints of the diagonals.

If we use *QL-Tr1* as reference triangle, this circle has the equation

$$a^2(l^2 - m^2)(n^2x + m^2y + n^2z)z - c^2(m^2 - n^2)(l^2x + m^2y + l^2z)x \\ + b^2(l^2 - m^2)(m^2 - n^2)zx = 0$$

with the midpoint

2nd *QG*-Quasi Circumcenter *QG-P9*

and contains ...

- ... the Diagonal Crosspoint *QG-P1*,
- ... the diagonal midpoints *M1* and *M2*,
- ... the 1st *QG*-Quasi Circumcenter *QG-P5*,
- ... the Gergonne-Steiner Point *QA-P3*,
- ... the *QL*-Adjunct Quasi Circumcenter *QL-P17*
- ... two vertices of the Miquel Triangle unequal *QL-P1*.

Furthermore there is a transformation, consisting of a reflection in the angle bisector at *QG-P1* (wrt *QL-Tr1*) and a reflection in a circle round *QG-P1*, so that the diagonal midpoints *M1* and *M2* change:

$$(x : y : z) \rightarrow (a^2m^4z(x + y + z) \\ : -(m^2n^2 - n^2l^2 + l^2m^2)(a^2z^2 + 2S_Bzx + c^2x^2) - m^4(a^2yz + b^2zx + c^2xy) \\ : c^2m^4x(x + y + z))$$

This transformation gives the new *QG*-circle as image of the Newton Line *QL-L1*.

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Message: #348

Date: 24/11/2013 10:58:13

From: eckart_schmidt@t-online.de

Subject: Numerically confirmed quadrangle points with a ...

Dear Seiichi, dear Chris,
there are further properties for the CSC-fixed points F_1 and F_2
... for a quadrigon:
... P_1, P_3, F_1, F_2 and P_2, P_4, F_1, F_2 are concyclic.
... circumcircles of $P_1P_2F_i$ and $P_3P_4F_i$ or circumcircles
of $P_2P_3F_i$ and $P_4P_1F_i$ are tangent in F_i .
... for a quadrilateral:
... F_1, F_2 points, which are the Isogonal Center (QA-P4)
of their pedal quadrangle.
Best regards Eckart

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Message: #349

Date: 24/11/2013 12:30:17

From: seiichikiri

Subject: Numerically confirmed quadrangle points with a ...

Dear Eckart,
I confirmed the concyclicity and tangency by Cabri pictures. It
is really interesting!!!
Best regards, Seiichi.

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Message: #350

Date: 24/11/2013 2:17:57

From: Chris

Subject: Numerically confirmed quadrangle points with a ...

Dear Eckart, dear Seiichi,
This brings me to this property:
QA-P9 is the Miquel Point of the Antipedal Quadrilateral of
QA-P4.
Best regards, Chris

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Message: #351
Date: 25/11/2013 10:42:50
From: Chris
Subject: Inscribed Harmonic Quadrilateral

Dear Friends,
It is possible to construct an Inscribed Harmonic Quadrilateral in a Quadrangle.

This can be done as follows:

Let P_1, P_2, P_3, P_4 be 4 random points forming our Reference Quadrangle.

Let S_1, S_2, S_3 be the vertices of the Diagonal Triangle.

S_1 =intersection point $P_4.P_1 \wedge P_2.P_3$

S_2 =intersection point $P_4.P_2 \wedge P_1.P_3$

S_3 =intersection point $P_4.P_3 \wedge P_1.P_2$

Let

$M_{12} = P_1.P_2 \wedge S_1.S_2,$

$M_{34} = P_3.P_4 \wedge S_1.S_2,$

$M_{23} = P_2.P_3 \wedge S_2.S_3,$

$M_{41} = P_4.P_1 \wedge S_2.S_3,$

$M_{13} = P_1.P_3 \wedge S_1.S_3,$

$M_{24} = P_2.P_4 \wedge S_1.S_3.$

These 6 points lie on 4 lines:

line $M_{13}.M_{34}.M_{41}$

line $M_{12}.M_{24}.M_{41}$

line $M_{12}.M_{23}.M_{13}$

line $M_{23}.M_{34}.M_{24}$

So these 4 lines actually form a Complete Quadrilateral.

We will call this Quadrilateral the Inscribed Harmonic Quadrilateral IH-QL.

and the Reference Quadrangle we will call Ref-QA.

Surprisingly $IH-QL-DT = Ref-QA-DT$ (DT =Diagonal Triangle).

$IH-QL-P_8 = Ref-QA-P_{10}$ (Centroid Diagonal Triangle)

$IH-QL-P_9 = Ref-QA-P_{11}$ (Circumcenter Diagonal Triangle)

$IH-QL-P_{10} = Ref-QA-P_{12}$ (Orthocenter Diagonal Triangle)

$IH-QL-P_{11} = Ref-QA-P_{13}$ (Nine-point center Diagonal Triangle)

$IH-QL-P_{13} = Ref-QA-P_{16}$ (so the Harmonic Centers Coincide)

$IH-QL-P_{18} = Ref-QG-P_4$ of Reference Quadrigon
(Quadrigon Centroid)

$IH-QL-P_{12} = Ref-QL-P_{12}$ of Reference Quadrigon
(Quadrilateral Centroid)

I don't think I will be surprised with more interesting properties for the Inscribed Harmonic Quadrilateral in a Quadrangle

Best regards,

Chris

Message: #352
Date: 26/11/2013 7:47:45
From: Chris
Subject: Inscribed Harmonic Quadrilateral

Dear friends,
I made some false calculations.
About the coincidences only these are true:
IH-QL-P8 = Ref-QA-P10 (Centroid Diagonal Triangle)
IH-QL-P9 = Ref-QA-P11 (Circumcenter Diagonal Triangle)
IH-QL-P10 = Ref-QA-P12 (Orthocenter Diagonal Triangle)
IH-QL-P11 = Ref-QA-P13 (Nine-point center Diagonal Triangle)
IH-QL-P13 = Ref-QA-P16 (so the Harmonic Centers Coincide)
Best regards,
Chris

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Message: #353
Date: 27/11/2013 12:32:24
From: eckart_schmidt@t-online.de
Subject: Inscribed Harmonic Quadrilateral

Dear Chris,
if you consider an IH-QL for a quadrigon, the points M12, M23, M34, M41 give an IH-QG. The vertices of this IH-QG are the points of tangency for QG-Co1 and divide the sides in ratios with product 1.
Further relationships between IH-QL and Ref-QA can be related wrt the same diagonal triangle. For Example: IH-QL-L7=Ref-QA-L5.
Perhaps new:
IH-QL-L6 contains Ref-QA-P32
IH-QL-Ci2 contains Ref-QA-P29 and Ref-QA-P36 .
Best regards Eckart

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Message: #354
Date: 27/11/2013 1:55:28
From: Chris
Subject: Another quadrigon circle

Dear Eckart,
A small contribution:
The angle bisector in the QL-Diagonal Triangle at QG-P1 is also
angle bisector between lines QG-P1.QG-P6.QG-P10 and
QG-P1.QG-P5.QG-P9.
Best regards,
Chris

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Message: #355
Date: 27/11/2013 8:54:12
From: eckart_schmidt@t-online.de
Subject: QL-Cu1 for quadrigons

Dear Chris,

if you rewrite QL-Cu1 in EQF, here another aspect wrt
quadrigons:
... QL-Cu1 is the locus of points, whose antipedal quadrigon is
cyclic (if not degenerated).
... The antipedal quadrigons degenerate into triangles for
points on the circumcircles of the triangle components.
... A CT-equation of QL-Cu1:
$$\begin{aligned} & a^2 y z (q z - r y) + c^2 x y (p y - q x) \\ & - (b^2 r + 2 q SA) x^2 z + (b^2 p + 2 q SC) x z^2 \\ & + 2(p SA - r SC) x y z = 0 \end{aligned}$$

Best regards Eckart

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Message: #356
Date: 29/11/2013 9:25:27
From: eckart_schmidt@t-online.de
Subject: A curious QG-parabola

Dear friends,
consider a quadrigon, divide the sides in the same ratio and you get an inscribed quadrigon I-QG.
Some properties (EQF-elements of these inscribed quadrigons will be set in parentheses):
... all I-QG have the same centroid (QA-P1)
and the same Newton line (QL-L1) as the reference quadrigon,
... the Miquel points (QL-P1) are collinear on QL-P1.QG-P3,
... (QL-P19) are collinear on a line through QL-P19
perpendicular QG-P5.QG-P10,
... (QG-P3) are collinear on QA-P1.QG-P3.
The NSM Lines (QL-L5) envelope a parabola (tangent to QL-L1 and QL-L5).
What about this parabola?
Best regards Eckart

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Message: #357
Date: 29/11/2013 9:57:15
From: eckart_schmidt@t-online.de
Subject: Some "splitter" III

Dear Chris,
some properties, not found in EQF:
QG-P15: $\text{area}(P_i, P_{i+1}, QG-P15) = \text{area}(P_{i+2}, P_{i+3}, QG-P15)$
QL-Ci1: ... locus of Euler-Poncelet points of QA with vertices
in the boundary points of QL-tangential conics.
QL-P18: ... centroid of the three collinear QG-P15 points
on QL-L9.
QG-P9 : ... QL-P17 lies (with QL-P9 and QL-P16)
on the circle through the 3 QL-versions of QG-P9.
QA-Tr3: ... for a QG: QG-P10 is the 4. parallelogram point
of QA-Tr3.
QA-Tr3: ... for a QL: The three QL versions have parallel
opposite sides of QL-P2 with midpoints on QL-L6.
Best regards Eckart

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Message: #358
Date: 29/11/2013 11:59:42
From: Chris
Subject: Some "splitter" III

Dear Eckart,
I like your splitter!
Regarding your last item (The three QL versions of QA-Tr3 have parallel sides, opposite to QL-P2 with midpoints on QL-L6):
These opposite sides are parallel to the Steiner line QL-L2.
Best regards, Chris

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Message: #359
Date: 30/11/2013 10:52:27
From: eckart_schmidt@t-online.de
Subject: QL-Cu1 for quadrilaterals

Dear Chris,
in addition to my message #355 there is a further property for QL-Cu1 for quadrilaterals:
For quadrilaterals P1P2P3P4 the QL-Quasi Isogonal Cubic QL-Cu1 is the locus of points P, for which the circumcenters of PPiPi+1 are concyclic.
Best regards Eckart

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Message: #360
Date: 01/12/2013 9:45:36
From: eckart_schmidt@t-online.de
Subject: QL-Cu1 for quadrilaterals

Dear Chris,
QL-Cu1 for quadrilaterals stays on interesting:
For quadrilaterals P1P2P3P4 and points P on QL-Cu1 the angle bisectors of
For convex quadrilaterals these angle bisectors are identical for outer points and perpendicular for inner points.
Best regards Eckart

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Message: #361
Date: 01/12/2013 12:03:33
From: Chris van Tienhoven
Subject: Quad Perimeter Problem

Dear friends,

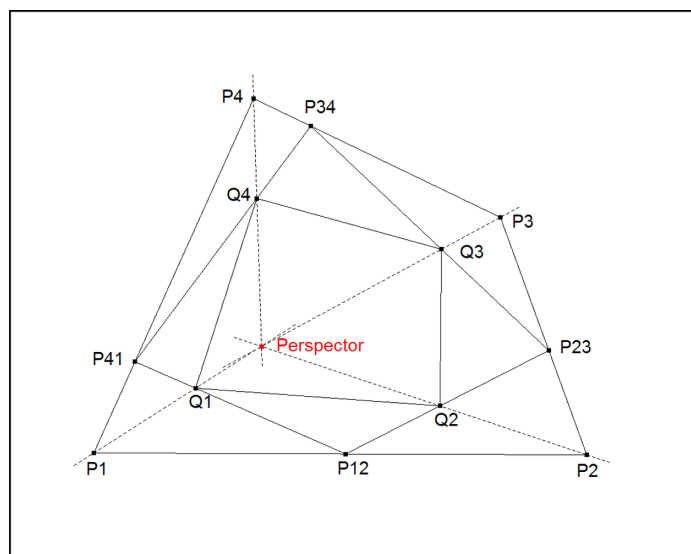
Let $P_1.P_2.P_3.P_4$ be Quadrigon/Tetragon Quad-1 with perimeter peri-1 .
Construct points $P_{12}, P_{23}, P_{34}, P_{41}$ on AB, BC, CD, DA such that:
 $P_1.P_{12} + P_1.P_{41}$
 $= P_2.P_{12} + P_2.P_{23} = P_3.P_{23} + P_3.P_{34} = P_4.P_{34} + P_4.P_{41}$
 $= \text{peri-1} / 4$.
Let $P_{12}.P_{23}.P_{34}.P_{41}$ be Quadrigon/Tetragon Quad-2 with perimeter peri-2 .

Now construct in a similar way an inscribed Quad-3 $Q_1.Q_2.Q_3.Q_4$ such that:
 $P_{12}.Q_1 + P_{12}.Q_2$
 $= P_{23}.Q_2 + P_{23}.Q_3 = P_{34}.Q_3 + P_{34}.Q_4 = P_{41}.Q_4 + P_{41}.Q_1$
 $= \text{peri-2}/4$.

Questions:

1. Is it possible to construct in a general way inscribed Quadrigons Quad-2 and Quad-3 such that Quad-1 and Quad-3 are perspective (see attachment)?
2. Is it possible to construct in a general way inscribed Quad-2 such that $P_{12}.P_{34}$ is perpendicular to $P_{23}.P_{41}$?

Best regards,
Chris



QG-P97-PerimeterPoint-01.png

Message: #362

Date: 05/12/2013 9:34:00

From: eckart_schmidt@t-online.de

Subject: A Quartic for Quadrilaterals

Dear friends,

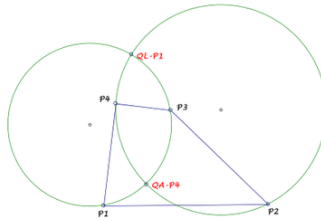
consider for a quadrilateral circles through opposite vertices, the boundary points give an interesting quartic: QG-circumscribed, QL-Tf1 invariant, anallagmatic (see attachment). The constructions are simple, the calculation extensive. But perhaps another reference triangle will give simpler expressions.

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Quartic for Quadrilaterals

The fixed points of a simple quadrilateral transformation give an interesting quartic, anallagmatic and invariant wrt the Clawson-Schmidt Conjugate QL-Tf1. Construction and equation shall be given.



The transformation

Consider for a quadrilateral circles through opposite vertices and a given point P . The second intersection of these circles shall be the image of P .

Let $P_1P_2P_3P_4$ be the quadrilateral, $P_1P_2P_3$ the reference triangle for barycentric coordinates and $P_4(p:q:r)$. For a point $P(x:y:z)$ the image has somewhat extensive coordinates:

$$\left\{ \frac{b^2 (p+q+r) z (rx - pz) + c^2 (p+r) xy + (a^2 p + a^2 r - b^2 p) zy}{a^2 r yz + b^2 r xz + c^2 r (p+q+r) xy - c^2 p qz (x+y+z)}, \frac{b^2 y (x+y+z)}{c^2 xy + b^2 xz + a^2 yz}, \frac{b^2 (p+q+r) x (pz - rx) + a^2 (p+r) zy + (-b^2 r + c^2 r + c^2 p) xy}{b^2 p zx + c^2 p yx + a^2 p (p+q+r) yz - a^2 q r x (x+y+z)} \right\}$$

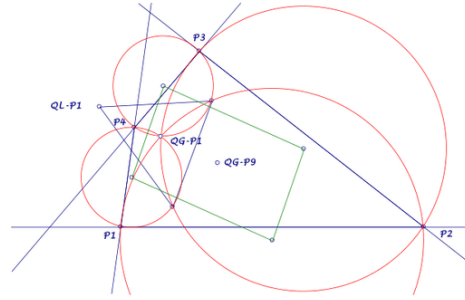
with

$$x := x(q+r) - p(y+z); \quad y := y(r+p) - q(z+x); \quad z := z(p+q) - r(x+y)$$

Some properties:

- Fixed points of the transformation give a quartic (see below).
- The image of the Miquel Point $QL-P1$ is the Isogonal Center $QA-P4$ (see above).

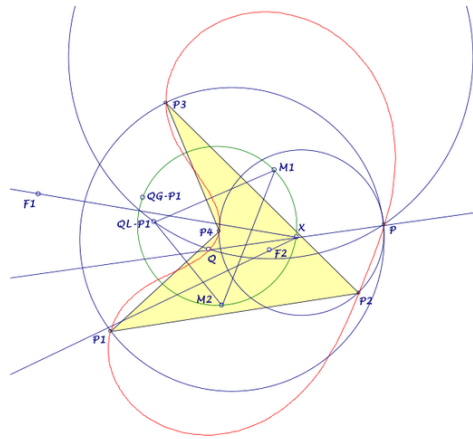
- The images of the intersections of opposite sidelines are vertices M_1 and M_2 (unequal $QL-P1$) of the Miquel Triangle $QA-Tr2$.
- The diagonals are invariant wrt this transformation.
- The images of the sidelines are circles, containing the vertices of the opposite side, the Diagonal Crosspoint $QG-P1$ and a vertex (unequal $QL-P1$) of the Miquel Triangle.
- The midpoints of these circles give a parallelogram with center $QG-P9$ and sidelines perpendicular to the diagonals of the quadrigon.



The Quartic of the Fixed Points

For fixed points P of the transformation the circles through opposite vertices of the quadrigon are tangent in P . These points give a quartic with the equation (X, Y, Z see above):

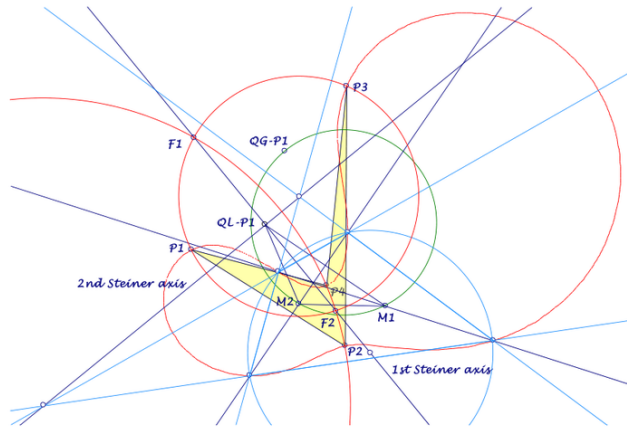
$$\begin{aligned}
 & (p+r) y^2 (a^4 z z + c^4 x x) + a^2 c^2 (p+r) y^2 (x z + z x) \\
 & - b^4 x z (r (x+y) X + p (y+z) Z) \\
 & - a^2 b^2 z (p z y (z+y-x) + r (z y (z-x) + Y (x+z) (y+x))) \\
 & - b^2 c^2 x (r x y (x+y-z) + p (x y (x-z) + Y (x+z) (y+z))) = 0
 \end{aligned}$$



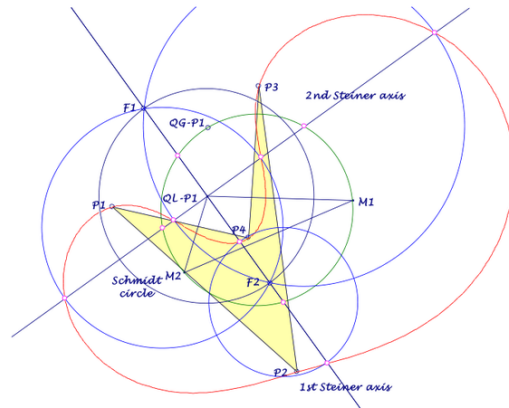
Construction of the quartic: For the construction we need the vertices M_1 and M_2 (unequal $QL-P1$) of the Miquel triangle and the fixed points F_1 and F_2 of the Clawson-Schmidt Conjugate $QL-Tf1$.

1. Circle through $QG-P1$, M_1 , M_2 with a point X
2. L_x angle bisector of $\angle F_1 X F_2$
3. P , Q as intersections of L_x and its image-circle wrt $QL-Tf1$ are points of the quartic.

Properties:



- The quartic is the locus of points, where circles through opposite vertices of the quadrigon are tangent.
- The quartic is a circumquartic of the quadrigon.
- The quartic is invariant wrt $QL-Tf1$.
- For $QL-Tf1$ partners on the quartic the midpoints lie on the circumcircle of M_1 , M_2 , $QG-P1$.
- The angle bisectors of the Miquel triangle at $M_{1,2}$ cut their $QL-Tf1$ image on the quartic in points of a cyclic quadrigon.
- Midpoint of the circumcircle of this cyclic quadrigon is the excenter wrt $QL-P1$ of the Miquel triangle.
- The quartic is anallagmatic, that means: The quartic is invariant wrt a reflection in a circle. There are two those circles with midpoints in the intersections of the opposite sidelines of the cyclic quadrigon (on the 2^{nd} Steiner axis), containing F_1 and F_2 .
- The 1^{st} Steiner axis cuts the circle through M_1 , M_2 , $QG-P1$ in two points. One point is the center of a circle orthogonal to the Schmidt circle, which cuts the 1^{st} Steiner line on the quartic.
- The 2^{nd} Steiner axis cuts the circle through M_1 , M_2 , $QG-P1$ in two points. They are centers of circles through F_1 and F_2 , which cut the 2^{nd} Steiner axis on the quartic.



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2013-12-05.pdf

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Message: #363

Date: 06/12/2013 12:06:57

From: eckart_schmidt@t-online.de

Subject: A Quartic for a Quadrigon

Dear friends,

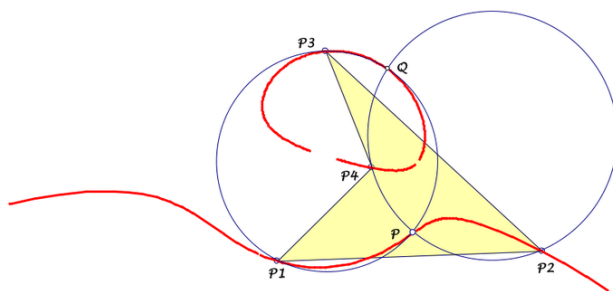
in my last message I have described a quartic for a quadrigon as locus for points, where circles through opposite vertices are tangent. In the attachment is a further quartic as locus of points, where these circles cut perpendicular.

Best regards Eckart

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Second Quartic for Quadrilaterals

For a quadrilateral circles through opposite vertices are tangent in points on a quartic (see EQF-Note 2013-12-05). Here another quartic is described as locus of points, where circles through opposite vertices intersect perpendicular.



Let $P_1P_2P_3P_4$ be a quadrilateral and $P_1P_2P_3$ the reference triangle for barycentric coordinates with $P_4(p:q:r)$. Then the locus for points, where circles through opposite vertices intersect perpendicular, is a quartic with the equation

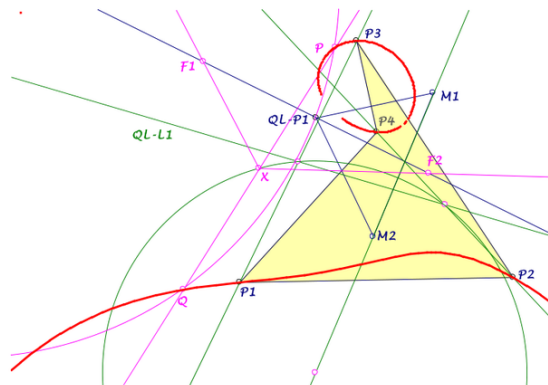
$$\begin{aligned} & (p+r)(a^4 z z y^2 + a^2 c^2 y^2 (x z + x z) + c^4 x x y^2) \\ & - a^2 b^2 z (p z y (-x + y + z) - r (x y z - x y (x + y + z) + x y z)) \\ & - b^2 c^2 x (r x y (x + y - z) - p (x y z - y z (x + y + z) + x y z)) \\ & - b^4 x z (r x (x + y) + p z (y + z)) = 0 \end{aligned}$$

with

$$x := x(q+r) - p(y+z); \quad y := y(r+p) - q(z+x); \quad z := z(p+q) - r(x+y)$$

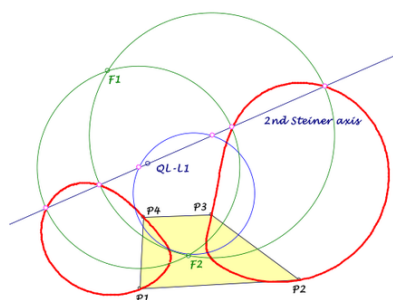
Construction of the quartic: For the construction we need the vertices M_1 and M_2 (unequal $QL-PI$) of the Miquel triangle and the fixed points F_1 and F_2 of the Clawson-Schmidt Conjugate $QL-TfI$.

1. Circle through the intersections of the Newton Line $QL-LI$ and the diagonals with center on M_1M_2 .
2. For points X on this circle consider the angle bisector L_x of $\angle F_1XF_2$.
3. P, Q as intersections of L_x and its image-circle wrt $QL-TfI$ are points of the quartic.

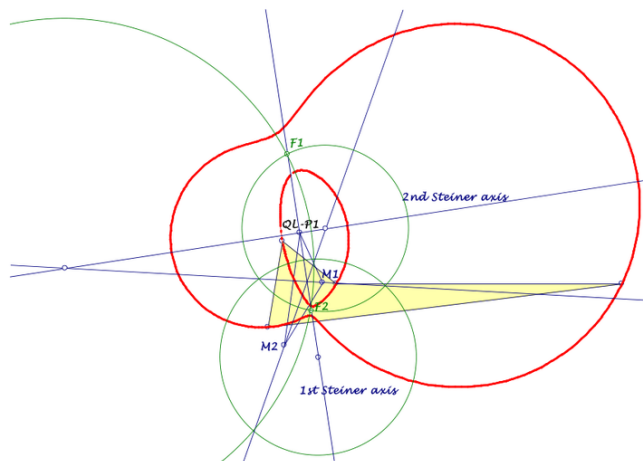


Properties:

- The quartic is the locus of points, where circles through opposite vertices of the quadrigon intersect perpendicular.
- The quartic is a circumquartic of the quadrigon.
- The quartic is invariant wrt $QL-Tf1$.
- The quartic is invariant wrt the transformation described in *EQF-Note 2013-12-05*.
- For $QL-Tf1$ partners on the quartic the midpoints lie on a circle through the intersections of $QL-L1$ and the diagonals, centered on M_1M_2 .
- The 2nd Steiner axis cuts this circle in two points (not always real). Circles round these points through F_1, F_2 cut the 2nd Steiner axis on the quartic.



- The quartic is anallagmatic, that means: The quartic is invariant wrt a reflection in a circle. There are three those circles (not always real): Two circles with midpoints in the intersections of the 2nd Steiner axis with the angle bisectors of the Miquel triangle at M_1 and M_2 , containing F_1 and F_2 , and one circle perpendicular to the Schmidt circle with midpoint in the excenter of the Miquel triangle corresponding $QL-P1$.



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2013-12-06.pdf

Message: #364
Date: 08/12/2013 11:29:16
From: eckart_schmidt@t-online.de
Subject: A Quartic for Quadrilaterals

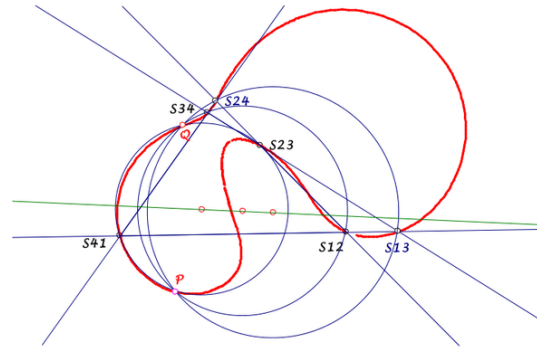
Dear friends,
for a quadrilateral there are three pairs of "opposite" points
and for each pair a circle through a given point. The points,
for which the three circles are coaxal give a quartic (see
attachment).
Best regards Eckart

EQF-Note 2013-12-08

Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
<http://chrisvantienhoven.nl/>

A Quartic for Quadrilaterals

For a quadrilateral there are three pairs of intersections and for each pair a circle through a given point. The points, for which these circles are coaxal give a quartic.



Let L_1, L_2, L_3, L_4 define a quadrilateral, then there are three pairs of intersections:

$$S_{12} = L_1 \cap L_2, S_{34} = L_3 \cap L_4 \quad \text{and} \quad S_{23} = L_2 \cap L_3, S_{41} = L_4 \cap L_1 \\ \text{and} \quad S_{13} = L_1 \cap L_3, S_{24} = L_2 \cap L_4.$$

We take for each pair the circumcircle through a point P and consider the locus of P for collinear centers of these three circles. This curve give a quartic.

If we chose the diagonal triangle $QL-DT$ as reference triangle for barycentric coordinates and give L_4 the coefficients l, m, n , then the quartic has the equation

$$\begin{aligned} & a^2 b^2 (1^2 - m^2) z (1^2 n^2 x x + m^2 n^2 y y - n^4 z z + 2 1^2 m^2 x y (x + y)) \\ & + b^2 c^2 (m^2 - n^2) x (-1^4 x x + 1^2 m^2 y y + 1^2 n^2 z z + 2 m^2 n^2 y z (y + z)) \\ & + a^2 c^2 (n^2 - 1^2) y (1^2 m^2 x x - m^4 y y + m^2 n^2 z z + 2 1^2 n^2 x z (x + z)) \\ & + a^4 (1^2 - m^2) (n^2 - 1^2) y z (-m^2 y (x + y) + n^2 z (x + z)) \\ & + b^4 (1^2 - m^2) (m^2 - n^2) x z (-n^2 z (y + z) + 1^2 x (x + y)) \\ & + c^4 (m^2 - n^2) (n^2 - 1^2) x y (-1^2 x (x + z) + m^2 y (y + z)) = 0 \end{aligned}$$

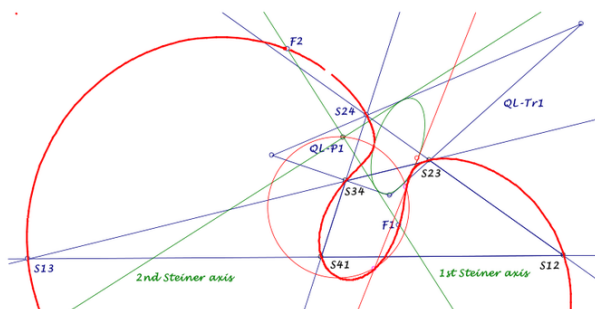
with

$$\begin{aligned} X &:= 2 y z + x (x + y + z); Y := 2 x z + y (x + y + z); \\ Z &:= 2 x y + z (x + y + z) \end{aligned}$$

2013-12-08.pdf

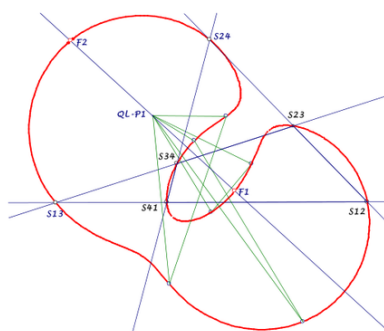
Construction of the quartic: For the construction we need the 1^{st} and 2^{nd} Steiner axis (see *QL-Tr1*):

1. Construct an inscribed conic of the Diagonal Triangle *QL-Tr1* with the 1^{st} and 2^{nd} Steiner axis (and Steiner Line *QL-L2*) as tangents.
2. The intersections of tangents at this conic and their image-circle wrt the Clawson-Schmidt Conjugate *QL-Tr1* will give points of the quartic.



Properties:

- This quartic contains the six intersections of the quadrilateral.
- The quartic contains the vertices M_1, M_2 (unequal *QL-P1*) of the Miquel Triangles *QA-Tr2* of the three quadrigon components.
- The quartic contains the fixed points F_1, F_2 of the Clawson-Schmidt Conjugate *QL-Tr1*.



- The quartic is invariant wrt the Clawson-Schmidt Conjugate *QL-Tr1*.
- The quartic is invariant wrt the transformation described in *EQF-Note 2013-12-05*.

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2013-12-08.pdf

Message: #365
Date: 08/12/2013 1:12:31
From: Chris van Tienhoven
Subject: An excellent book of geometry problems

Dear friends,
See message below from Francisco Javier Garcia Capitan.
He recommends a most special document of
Jose Maria Pedret.
I went through it.
Pages in the document 67-75, 292-334 (pages in PDF-file 69-77,
296-338)
all deal about quadrilaterals!
This is very special!
Best regards,
Chris

Van:
AdvancedPlaneGeometry@yahoogroups.com
[mailto:AdvancedPlaneGeometry@yahoogroups.com] *Namens*
garciacapitan@gmail.com
Verzonden: vrijdag 6 december 2013 8:52
Aan: AdvancedPlaneGeometry@yahoogroups.com
Onderwerp: [ADGEOM] An excellent book of geometry problems
Dear friends, here is an excellent book of geometry problems
written in 1955.
The text it is in Spanish but it contains many figures that can
help.
My friend Jose Maria Pedret bought it in a second hand bookshop
and carefully scanned it. I have the permission from the author
to distribute it.
Enjoy the book!
More information: <http://garciacapitan.99on.com/jsb/>
Direct download: <https://drive.google.com/file/d/0B640DPhcZB-mcXjY4b05ZQ3NPLUE/edit?usp=sharing>

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Message: #366

Date: 08/12/2013 1:21:17

From: Chris van Tienhoven

Subject: problem about circumscribed square of a quadrigon

Dear friends,

Let P_1, P_2, P_3, P_4 be the vertices of a Quadrigon.

Let M_{13} be midpoint $P_1.P_3$.

Let M_{24} be midpoint $P_2.P_4$.

Let M be some point on the circle with diameter $M_{13}.M_{24}$.

Draw lines through P_1 and P_3 parallel to $M.M_{13}$.

Draw lines through P_2 and P_4 parallel to $M.M_{24}$.

These 2+2 parallel lines form a rectangle.

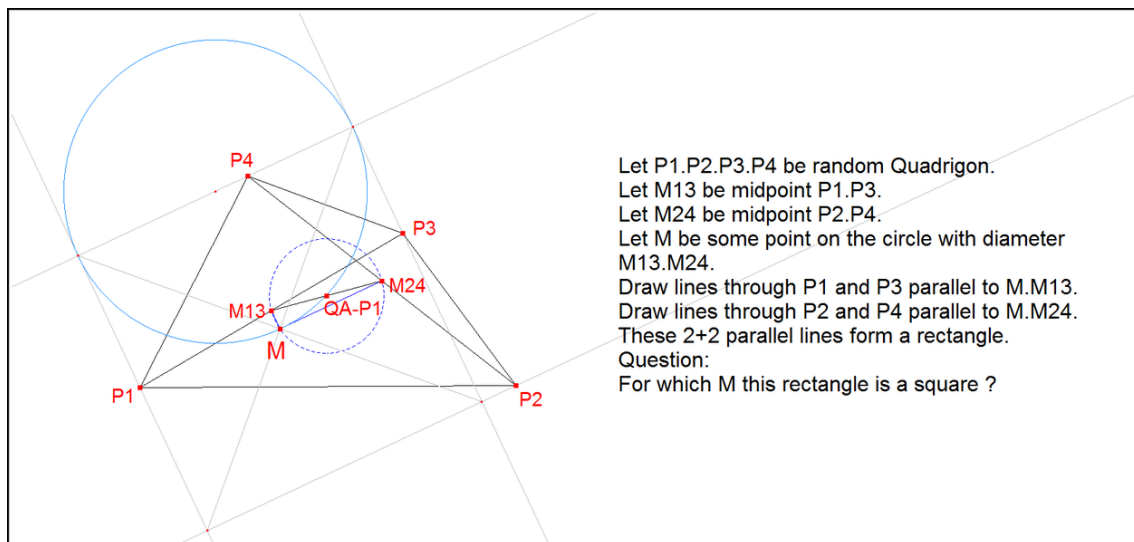
Question:

How to construct and possibly calculate point M such that this rectangle is a square?

Best regards,

Chris

See attachment



QG-Qu-Circumscribed Square-01.png

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Message: #367

Date: 09/12/2013 9:12:27

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

Dear Chris,

here is a construction of the described square (see attachment):

Let $P_1P_2P_3P_4$ be a quadrigon and C_{ij} the Thales circles about P_iP_j with

... P_{12} point on C_{12} ,

$P_{41}=P_1P_{12}\cap C_{41}$ and $P_{23}=P_{12}P_2\cap C_{23}$ (second intersections).

Construct a square about $P_{41}P_{12}$ with P_{412}

as opposite point of P_{12}

... and a square about $P_{12}P_{23}$ with P_{123}

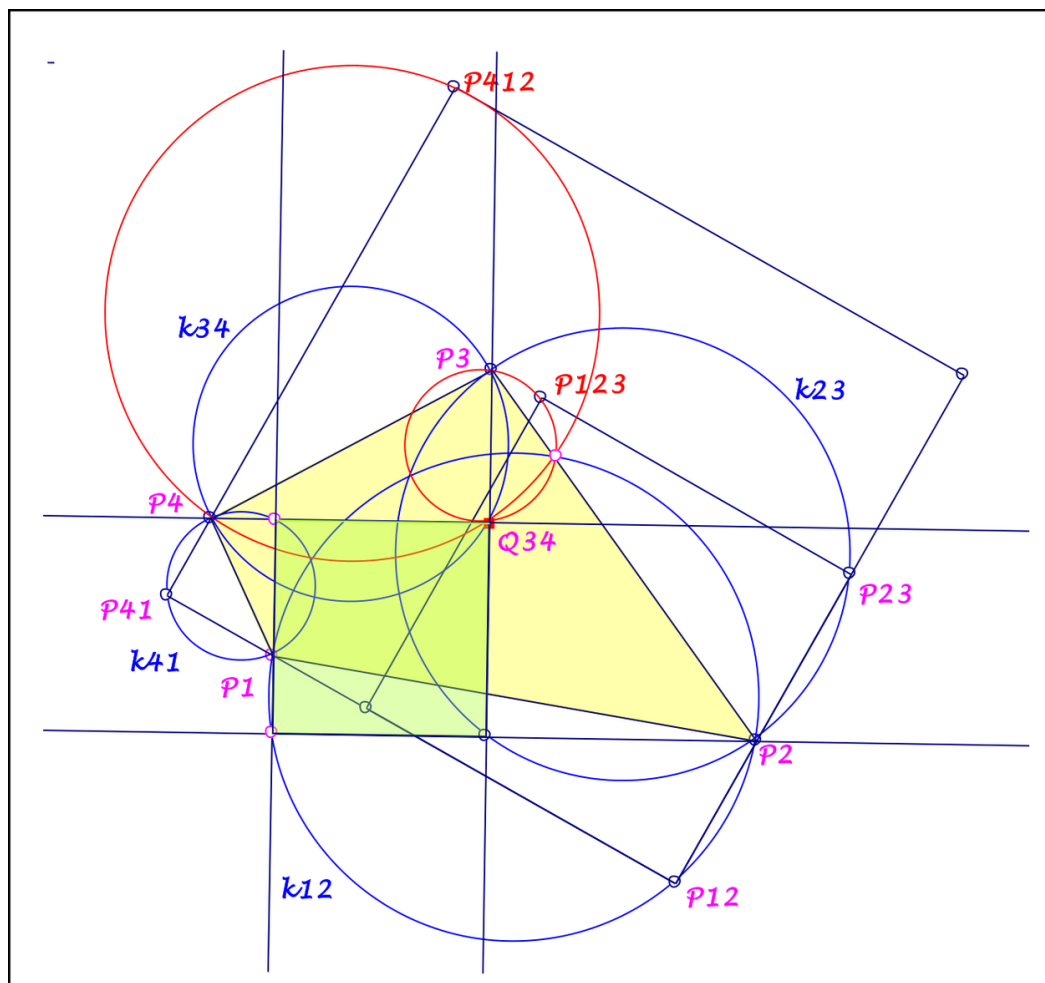
as opposite point of P_{12} .

The loci of these opposite points are circles k_{412} and k_{123} .

One of the intersections of these circles lies on C_{34} .

This point is a vertex Q_{34} of the searched quadrigon ...

Best regards Eckart



Square.pdf

Message: #368

Date: 09/12/2013 5:39:46

From: yeuemtrondoitb85

Subject: Tangential quadrilateral well known properties?

Dear Mr Eckart, Mr. Bernard K, Dear Mr. Chris,

(Denote similar file pdf of Mr Eckart)

ABCD is a circumscribed quadrilateral,

PQRS the points of tangency with the inscribed circle center O,

M and N are the midpoints of AB and CD,

M' and N' are the midpoints of BC and DA

Construct four circle (OAB), (OBC), (OCD), (ODA).

We have some result following:

1-Two tangents of (OAB), (OCD) at O is one line (coincident line), this line meets BC,DA respectively at B₁,D₁. Two tangents of (OBC), (ODA) at O is one line (coincident line), this line meets AB,CD at A₁,C₁

2-A₁B₁C₁D₁ is a parallelogram

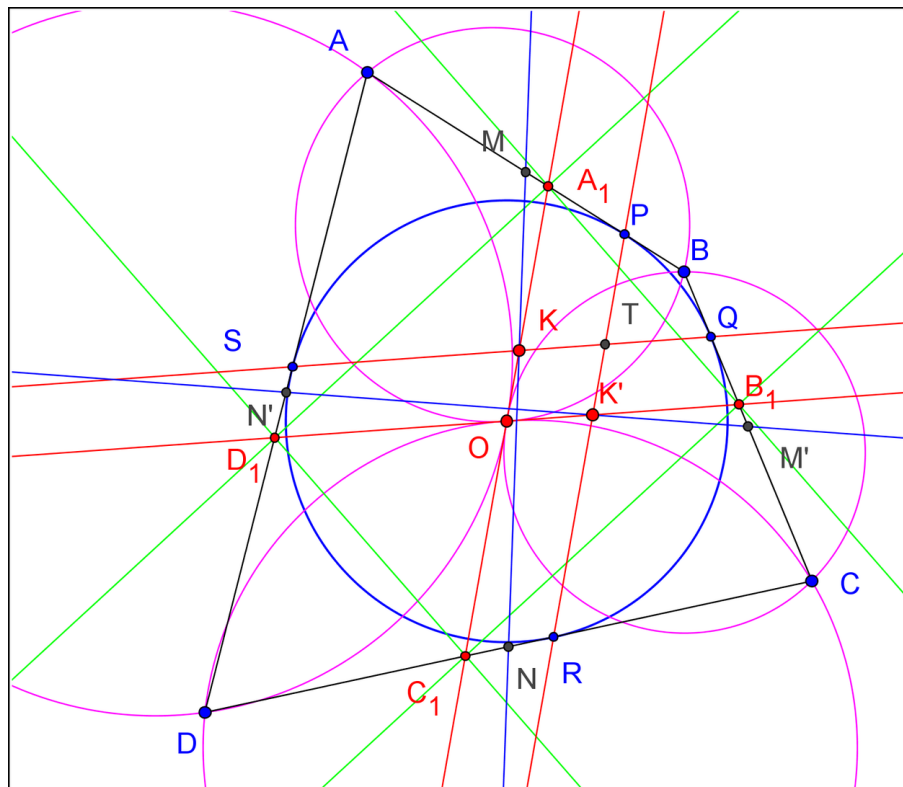
3-QS//B₁D₁ and PR//A₁C₁

4-A₁B₁,MN,QS are concurrent at K; similar B₁D₁, M'N', RP are concurrent at K'

Best regards

Sincerely

Dao Thanh Oai



Tangential quadrilateral.png

Message: #369

Date: 09/12/2013 5:50:23

From: yeuemtrondoitb85

Subject: Tangential quadrilateral well known properties?

Dear Mr Eckart, Mr. Bernard K, Dear Mr. Chris,

(Denote similar file pdf of Mr Eckart)

ABCD is a circumscribed quadrilateral,

PQRS the points of tangency with the inscribed circle center O,

M and N are the midpoints of AB and CD,

M' and N' are the midpoints of BC and DA

Construct four circle (OAB), (OBC), (OCD), (ODA).

We have some result following:

1-Two tangents of (OAB), (OCD) at O is one line (coincident line), this line meets BC, DA respectively at B₁, D₁. Two tangents of (OBC), (ODA) at O is one line (coincident line), this line meets AB, CD at A₁, C₁

2-A₁B₁C₁D₁ is a parallelogram

3-QS//B₁D₁ and PR//A₁C₁

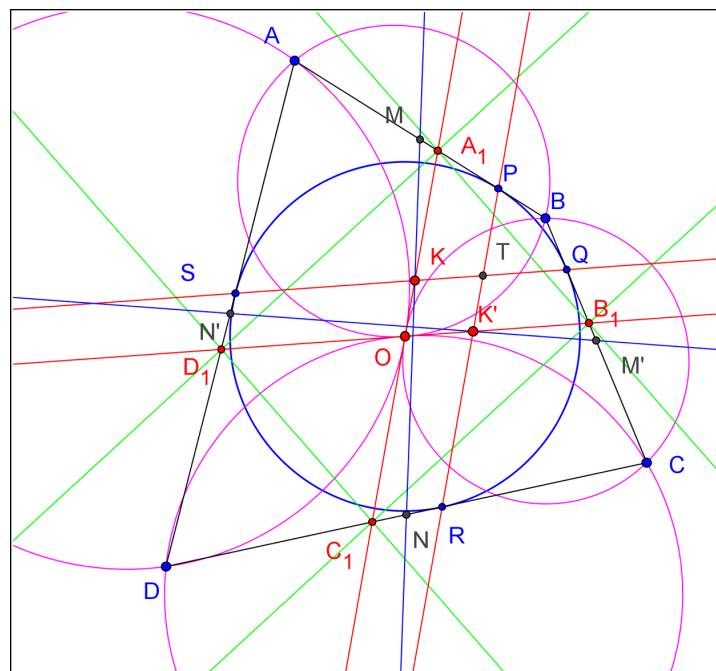
4-A₁B₁, MN, QS are concurrent at K; similar B₁D₁, M'N', RP are concurrent at K'

6-A₁PRC₁; and QB₁D₁S are isosceles trapezoid

Best regards

Sincerely

Dao Thanh Oai



Tangential quadrilateral-369.png

Message: #370

Date: 09/12/2013 11:40:36

From: Chris van Tienhoven

Subject: Quadri-Figures-Group] problem about circumscribed square of a qua

Dear Eckart,

Very nice construction of a circumscribed square around a "quadrilateral"!

I took the liberty to alter your construction a bit. See also attached files.

Construction two circumscribed squares around a quadrigon:
Let $P_1P_2P_3P_4$ be a quadrigon and C_{ij} the Thales circles about P_iP_j . C_{12} is intersected by the perpendicular bisector of P_1P_2 in 2 points.

Let P_{12in} be the inward point (direction QA-Centroid).

Let P_{12out} be the outward point (other direction).

Let S_1 be 2nd intersection point of circles C_{12} and C_{14} .

Let S_2 be 2nd intersection point of circles C_{12} and C_{23} .

Let C_{14in} be the circle through S_1 , P_4 and P_{12in} .

Let C_{23in} be the circle through S_2 , P_3 and P_{12in} .

Let V_{34in} be the 2nd intersection point of C_{14in} and C_{23in} .

V_{34in} = the Vertex of the 1st circumscribed square connecting P_3 and P_4 .

Let C_{14out} be the circle through S_1 , P_4 and P_{12out} .

Let C_{23out} be the circle through S_2 , P_3 and P_{12out} .

Let V_{34out} be the 2nd intersection point of C_{14out} and C_{23out} .

V_{34out} = the Vertex of the 2nd circumscribed square connecting P_3 and P_4 .

V_{34in} and V_{34out} are the constructed vertices of the circumscribed squares.

By connecting these points with P_3 and P_4 the first sidelines of the squares are constructed.

By drawing parallel lines through P_1 and P_2 the other sidelines of the squares are constructed.

Now we have 2 circumscribed squares.

The centers of the squares lie indeed on the Thales Circle of $M_{13}M_{24}$, where M_{13} and M_{24} are the midpoints of the two diagonals of the Quadrigon.

Some other interesting properties:

- * The diagonals of the two circumscribed squares produce 2 other intersection points, also on the Thales Circle of $M_{13}M_{24}$.
- * The connecting line of the two centers of the circumscribed squares passes through this point:

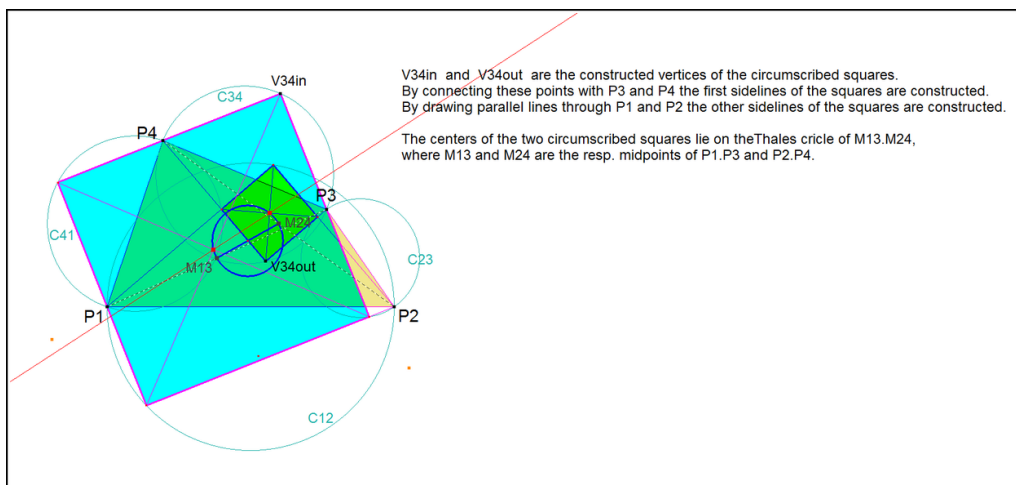
QG-P1.QG-P6 $\wedge$ QG-P5.QG-P8 $\wedge$ QA-P1.QG-P11,
 which is a point on the Nine-point Conic QA-Co1.

See also QG/5: <http://www.chrisvantienhoven.nl/quadrignon-objects/16-mathematics/encyclopedia-of-quadri-figures/quadrignon-objects/artikelen-qg/165-qg-5.html>

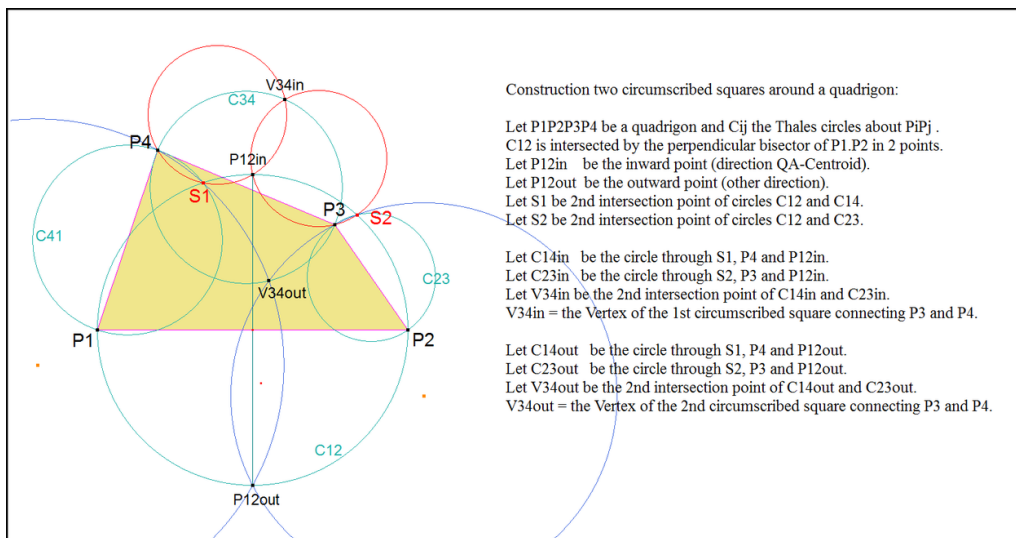
I think there will be more of these intersection points on this circumscribed-square-center line.

* A first check didn't produce concurrencies of the QG-lines of the 3 QA-Quadrignons or the 3 QL-Quadrignons.

Best regards,
 Chris



QG-P9007-CircumscribedSquareCenter-12.png



QG-P9007-CircumscribedSquareCenter-10.png

Message: #371

Date: 10/12/2013 11:16:21

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

Dear Chris,

thank you for the description of the construction for the second circumscribed square. I had supposed it, but wasn't able to find! Reason: The calculation for point M gave two solutions. Now I can give you the coefficients of the circumscribed-square-center line (see attached Mathematica file).

Best regards Eckart

Circumscribed-square-center line (message 370)

Reference triangle for barycentric coordinates:

P1P2P3 with P4:={p,q,r}.

$$\begin{aligned} & \{ (q+r) (2p^2 + 3pq + 2q^2 + 3pr + 4qr + 2r^2) SA^2 \\ & - (p+q) r (p+q+2r) SC^2 \\ & + (p^3 + 2p^2q + 4p^2r + 4pqr + pr^2 - 2qr^2 - 2r^3) SA SB \\ & + r (p^2 - 2pr - 4qr - 3r^2) SB SC \\ & - (p^3 + 4p^2q + 5p^2r + 2q^3 + 4pqr + 3q^2r + 2pr^2 + 2qr^2 + r^3) SA SC, -p (q+r) (2p+q+r) SA^2 \\ & - 2(p+r)^2 (p+2q+r) SB^2 \\ & - (p+q) r (p+q+2r) SC^2 \\ & + (p^3 + 4p^2q + 10p^2r + 4q^3 + 2p^2r + 12pqr + 10q^2r + 3pr^2 + 8qr^2 + 2r^3) SA SB \\ & + (2p^3 + 8p^2q + 10p^2r + 4q^3 + 3p^2r + 12pqr + 10q^2r + 2pr^2 + 4qr^2 + r^3) SB SC \\ & + (3p + 4q + 3r) (p^2 + 2pq + q^2 + pr + 2qr + r^2) SA SC, \\ & -p (q+r) (2p+q+r) SA^2 \\ & + (p+q) (2p^2 + 4pq + 2q^2 + 3pr + 3qr + 2r^2) SC^2 \\ & - p (3p^2 + 4pq + 2pr - r^2) SA SB \\ & - (2p^3 + 2p^2q - p^2r - 4pqr - 4pr^2 - 2qr^2 - r^3) SB SC \\ & - (p^3 + 2p^2q + 3p^2r + 2q^3 + 2p^2r + 4pqr + 5q^2r + 4qr^2 + r^3) SA SC \} \end{aligned}$$

Eckart Schmidt

eckart_schmidt@t-online.de

http://eckartschmidt.de

Chris-13-12-09-nb.pdf

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Message: #372
Date: 12/12/2013 2:29:54
From: eckart_schmidt@t-online.de
Subject: problem about circumscribed square of a quadrigon

Dear Chris,
there is a very simple construction of the circumscribed squares of a quadrigon, using the van Aubel constellation:
Given a quadrigon, place a square outward on each side, and connect the centers of opposite squares. The second intersections of these lines with the Thales circles over the sides are the vertices of an circumscribed square. Placing the squares inside the quadrigon, we get the second circumscribed square.
Best regards Eckart

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Message: #373
Date: 12/12/2013 7:25:24
From: Chris
Subject: problem about circumscribed square of a quadrigon

Dear Eckart,
Remarkably simple indeed!
I already had noticed long before the relation with the Van Aubel configuration. But this simple construction is new to me.
Best regards,
Chris

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Message: #374

Date: 13/12/2013 9:04:36

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

Dear Chris,

there is a further aspect wrt the circumscribed squares of a quadrigon:

Reflecting a random point round the Varignon parallelogram of a quadrigon we get a quadrigon with the same midpoints of the sides as the reference quadrigon. All these quadrigons have circumscribed squares of the same size and parallel sides (see attachments).

Best regards Eckart

Circumscribed-squares of a Quadrigon

Reference triangle for barycentric coordinates:

$P_1P_2P_3$ with $P_4 := \{p, q, r\}$.

Centroids of the circumscribed squares:

(For the second centroid change the sign of S)

$$\begin{aligned} & \{-2p^2S SA - pqS SA - prS SA - 2p^2S SB - 2prS SB + 3p^2S AB + 2pqS AB + \\ & 3prS AB - 2p^2S SC - 4pqS SC - 2q^2S SC - 4prS SC - 3qrS SC - r^2S SC + 3p^2S AC + 3pqS AC + \\ & 4prS AC + 4p^2S BC + 4pqS BC + 5prS BC + 2qrS BC + r^2S BC + prS C^2 + qrS C^2, \\ & (-pS + rS - qSA - rSA + pSC + qSC)(pSA - rSC), -p^2S SA - 3pqS SA - 2q^2S SA - 4prS SA - 4qrS SA - \\ & 2r^2S SA + pqSA^2 + prSA^2 - 2prS SB - 2r^2S SB + p^2S AB + 2pqS AB + 5prS AB + 4qrS AB + 4r^2S AB - \\ & prS SC - qrS SC - 2r^2S SC + 4prS AC + 3qrS AC + 3r^2S AC + 3prS BC + 2qrS BC + 3r^2S BC\} \end{aligned}$$

Coefficients of the circumscribed-square-center line

$$\begin{aligned} & \{(q+r)(2p^2 + 3pq + 2q^2 + 3pr + 4qr + 2r^2)SA^2 \\ & - (p+q)r(p+q+2r)SC^2 \\ & + (p^3 + 2p^2q + 4p^2r + 4pqr + pr^2 - 2qr^2 - 2r^3)SAB \\ & + r(p^2 - 2pr - 4qr - 3r^2)SBC \\ & - (p^3 + 4p^2q + 5pq^2 + 2q^3 + 4pqr + 3q^2r + 2pr^2 + 2qr^2 + r^3)SAC, -p(q+r)(2p+q+r)SA^2 \\ & - 2(p+r)^2(p+2q+r)SB^2 \\ & - (p+q)r(p+q+2r)SC^2 \\ & + (p^3 + 4p^2q + 10pq^2 + 4q^3 + 2p^2r + 12pqr + 10q^2r + 3pr^2 + 8qr^2 + 2r^3)SAB \\ & + (2p^3 + 8p^2q + 10pq^2 + 4q^3 + 3p^2r + 12pqr + 10q^2r + 2pr^2 + 4qr^2 + r^3)SBC \\ & + (3p + 4q + 3r)(p^2 + 2pq + q^2 + pr + 2qr + r^2)SAC, \\ & -p(q+r)(2p+q+r)SA^2 \\ & + (p+q)(2p^2 + 4pq + 2q^2 + 3pr + 3qr + 2r^2)SC^2 \\ & - p(3p^2 + 4pq + 2pr - r^2)SAB \\ & - (2p^3 + 2p^2q - p^2r - 4pqr - 4pr^2 - 2qr^2 - r^3)SBC \\ & - (p^3 + 2p^2q + 3pq^2 + 2q^3 + 2p^2r + 4pqr + 5q^2r + 4qr^2 + r^3)SAC\} \end{aligned}$$

Side-length of a circumscribed square (For the second square change sign of S)

$$(pSA - rSC) / \sqrt{(-2(p+q+r)(pS + rS - pSA - rSC) + (q+r)^2SA + (p+r)^2SB + (p+q)^2SC)}$$

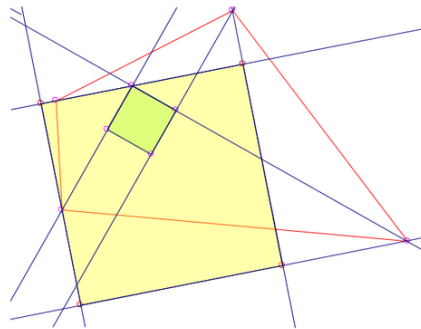
Eckart Schmidt

eckart_schmidt@t-online.de

http://eckartschmidt.de

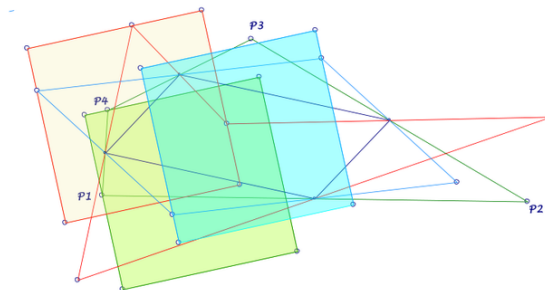
Some remarks wrt circumscribed squares of a quadrigon

For a quadrigon there are two circumscribed squares, see message 366, 370. Possible constructions see message 367, 370, 372. Properties of the circumscribed-square-center line see message 370. Results in calculation see attached Mathematica file.



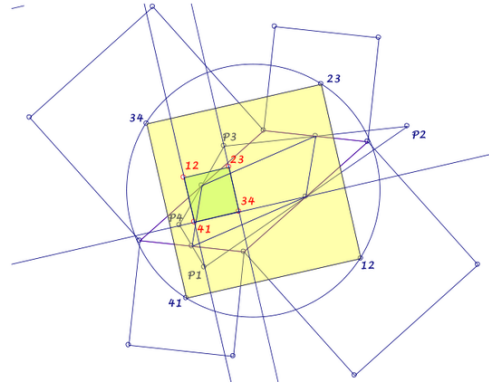
New aspect:

Consider the Varignon parallelogram of the quadrigon. Reflecting a random point P round this parallelogram we get a quadrigon with the same midpoints of the sides as the reference quadrigon. All these quadrigons have circumscribed squares of the same size and parallel sides. (The drawing shows only one square component).

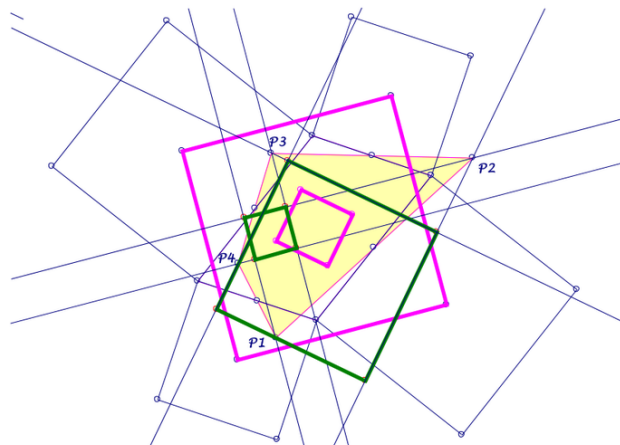


(If the point P runs along a circle round a vertex of the reference quadrigon, the centers of the circumscribed squares run along circles with the same radius round the centers of the circumscribed squares of the reference quadrigon.)

Now we consider the parallelogram with the same Varignon parallelogram as the reference quadrigon and draw squares over / under the sides. The centers of these squares give again a square (Napoleon square). The sides of such a Napoleon square are parallel to those of that circumscribed square of the reference quadrigon with opposite orientation.



The ratio of side-lengths of circumscribed squares of a quadrigon equals the ratio of side-lengths of the Napoleon squares described in the last paragraph. This ratio depends only on the Varignon parallelogram of the reference triangle.



Eckart Schmidt
<http://eckartschmidt.de>
eckart_schmidt@t-online.de

2013-12-13.pdf

Message: #375

Date: 13/12/2013 2:08:47

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

Dear Chris,

the Varignon parallelogram was the background for my examination paper at university. So I have described a further aspect wrt the circumscribed squares of a quadrigon (see attachment).

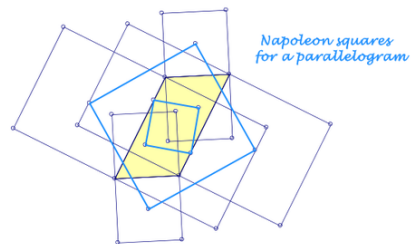
Best regards Eckart

Circumscribed squares of a quadrigon – an additional aspect –

This is a reference to my professor
Prof. Dr. Friedrich Bachmann

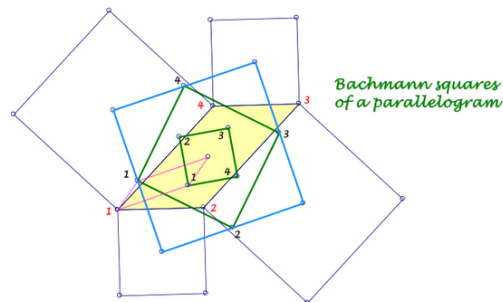
Napoleon squares of a parallelogram

Given a parallelogram, place a square outward / inside of each side, the centers of these squares give two new squares, here called the Napoleon squares (see [1], [2]).



Bachmann squares of a parallelogram

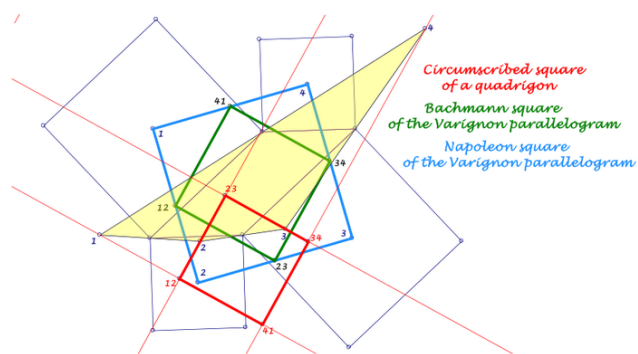
The midpoints of the sides of the Napoleon squares of a parallelogram give two further squares, here called Bachmann squares. The parallelogram is the vectorial sum of these Bachmann squares (see [1], [3]).



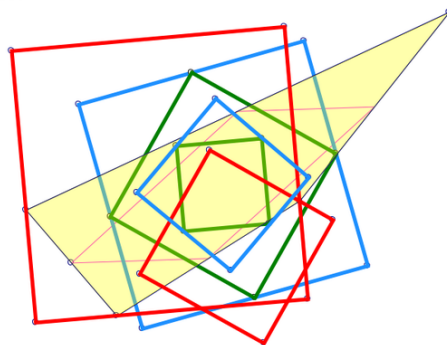
Circumscribed squares of a quadrigon

For a quadrigon there are two circumscribed squares (see QFG message 370).

For all quadrigons with the same Varignon parallelogram these circumscribed squares have the same size. Their sides are parallel to the sides of the Bachmann squares for the Varignon parallelogram.



Finally: There are three pairs of squares, which have the same ratio of size.



References

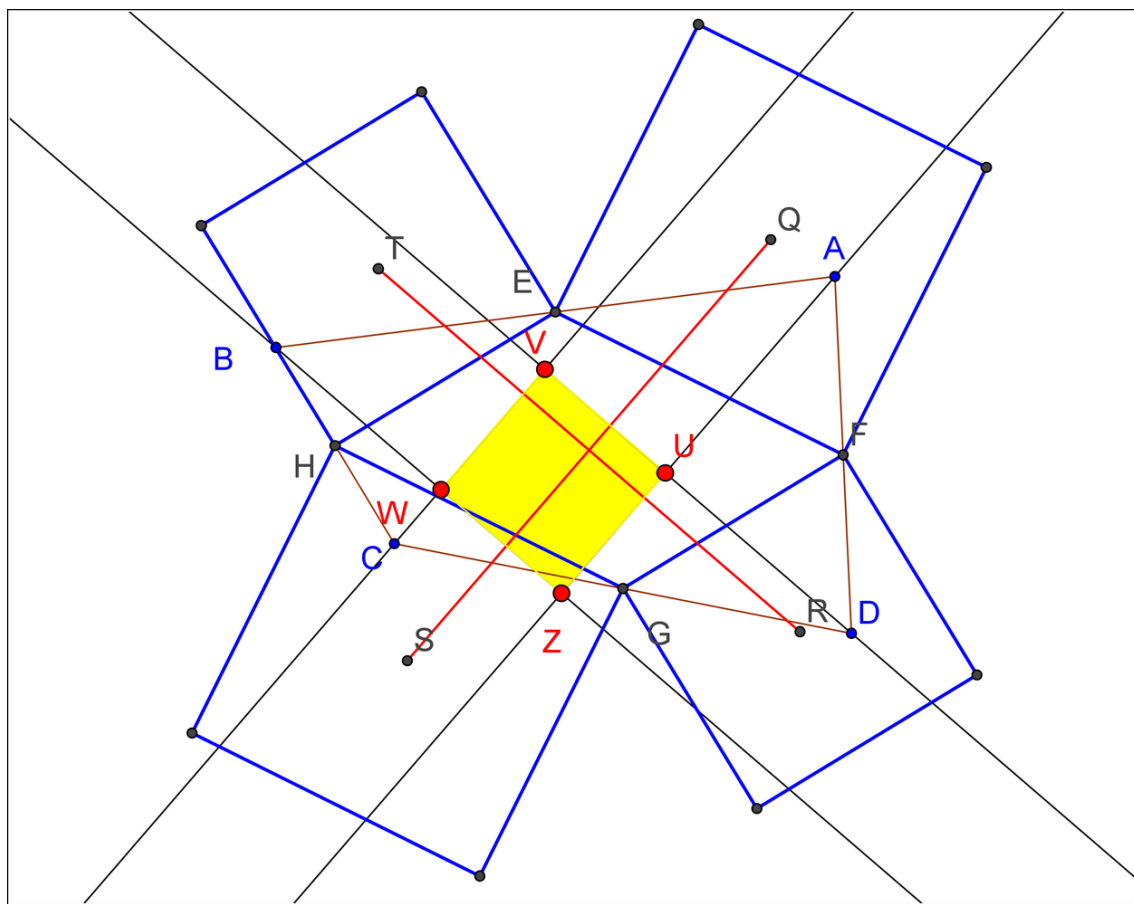
- [1] E. Schmidt: Affin-reguläre n-Ecke und ihre regulären Komponenten. – MNU XXXIX,4.
- [2] My homepage 05-5.
- [3] F. Bachmann / E. Schmidt: n-Ecke.
– B.I - Hochschultaschenbücher 471/471a,
Bibliographisches Institut Mannheim 1970

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 eckart_schmidt@t-online.de

2013-12-14.pdf

Message: #376
Date: 13/12/2013 7:25:01
From: yeuemtrondoitb85
Subject: Problems: Square

Dear Mr Chris and Mr Eckart,
ABCD are a quadrilateral
EFGH are a parallelogram
TQRS is a square
then VWZU are squar
Best regards
Sincerely
Dao Thanh Oai



square.png

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Message: #377
Date: 14/12/2013 8:13:12
From: Chris van Tienhoven
Subject: Problems: Square

Dear Dao, Dear Eckart,
Very nice properties of the circumscribed square of a
quadrilateral.
Thank you!
Best regards,
Chris

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Message: #378

Date: 14/12/2013 9:02:43

From: Chris van Tienhoven

Subject: problem about circumscribed square of a quadrigon

Dear Friends,

I found some old notes of mine about squares related to a Quadrigon and the Van Aubel Configuration.

Here are the notes. See also attached figure.

Notes:

P_{ij} and p_{ij} are erected at $P_i.P_j$ such that $P_3.P_4.P_4.p_3$ is a square.

P_{ij} is erected outwardly and p_{ij} is erected inwardly.

Now $P_{12}.P_{34} \perp P_{23}.P_{41}$ and $P_{12}.P_{34} = P_{23}.P_{41}$

and $p_{12}.p_{34} \perp p_{23}.p_{41}$ and $p_{12}.p_{34} = p_{23}.p_{41}$.

Let $M_j = \text{midpoint}(P_{ij}, P_{jk})$ and $m_j = \text{midpoint}(p_{ij}, p_{jk})$.

$M_1.M_2.M_3.M_4$ and $m_1.m_2.m_3.m_4$ are squares for obvious reasons.

$\text{Area}(P_1.P_2.P_3.P_4) = \text{Area}(M_1.M_2.M_3.M_4) - \text{Area}(m_1.m_2.m_3.m_4)$

$P_1.P_2.P_3.P_4$, $M_1.M_2.M_3.M_4$ and $m_1.m_2.m_3.m_4$ share the same centroid.

Let H_{41} be $\text{OrthoCenter}(P_{12}, P_{23}, P_{34})$,

and H_{12} be $\text{OrthoCenter}(P_{23}, P_{34}, P_{41})$,

and H_{23} be $\text{OrthoCenter}(P_{34}, P_{41}, P_{12})$,

and H_{34} be $\text{OrthoCenter}(P_{41}, P_{12}, P_{23})$.

Now $\text{Area}(H_{12}, H_{23}, H_{34}, H_{41}) = \text{Area}(P_{12}, P_{23}, P_{34}, P_{41}) = 2 * \text{Area}(M_1.M_2.M_3.M_4)$.

$\text{Area}(H_{12}, H_{23}, H_{34}, H_{41}) = \text{Area}(P_{12}, P_{23}, P_{34}, P_{41})$ is general property of a quadrangle.

$\text{Area}(P_{12}, P_{23}, P_{34}, P_{41}) = 2 * \text{Area}(M_1.M_2.M_3.M_4)$ because $M_1.M_2.M_3.M_4$ is Varignon-parallelogram.

The blue squares in the figure are formed by the lines parallel to the diagonals of the brown squares and passing through the vertices of $P_1.P_2.P_3.P_4$.

K_1 and K_2 happen to be their centers.

These squares are circumscribed squares of $P_1.P_2.P_3.P_4$.

Special property:

$\text{Area}[M_1.M_2.M_3.M_4] / \text{Area}[m_1.m_2.m_3.m_4] = \text{Area}[N_1.N_2.N_3.N_4] / \text{Area}[n_1.n_2.n_3.n_4]$

$N_1.N_2.N_3.N_4$ and $n_1.n_2.n_3.n_4$ are the blue squares.

K_1 and K_2 have the same distance d to the Centroid of the Reference Quadrigon.

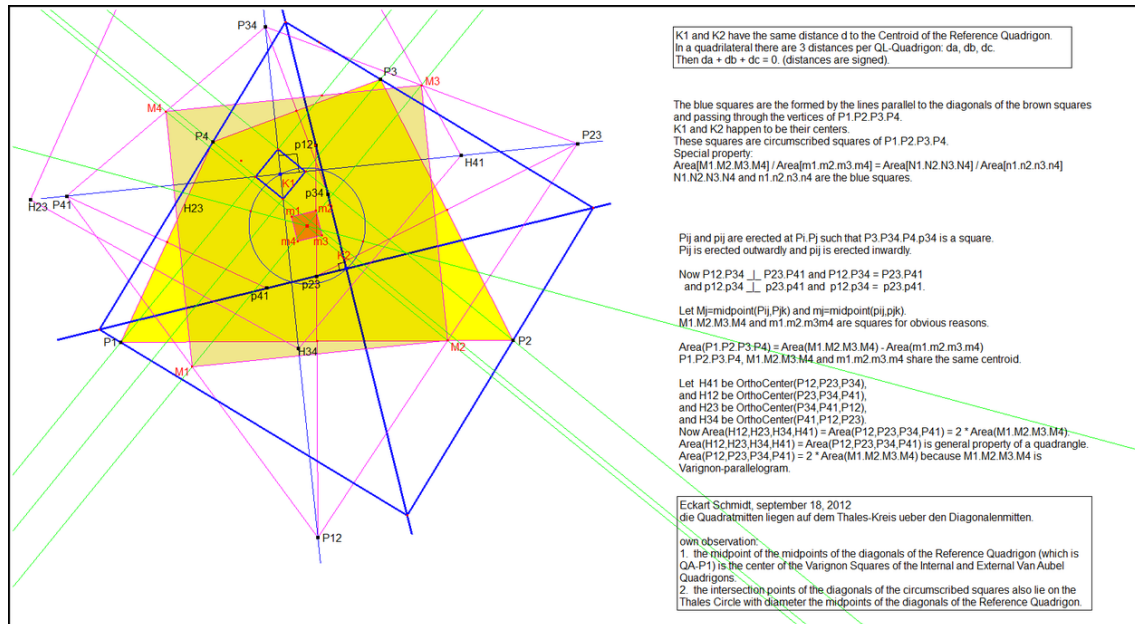
In a quadrilateral there are 3 distances per QL-Quadrigon: d_a , d_b , d_c .

Then $d_a + d_b + d_c = 0$. (distances are signed).

1. the midpoint of the midpoints of the diagonals of the Reference Quadrigon (which is $QA-P_1$) is the center of the Varignon Squares of the Internal and External Van Aubel Quadrigons.

2. the intersection points of the diagonals of the circumscribed squares also lie on the Thales Circle with diameter the midpoints of the diagonals of the Reference Quadrigon.

Best regards,
Chris van Tienhoven



QG-2P99-VanAubelPoints-08a-circumsquares.png

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Message: #379

Date: 15/12/2013 8:15:16

From: yeuemtrondoitb85

Subject: Poncelet point and rectangular hyperbola

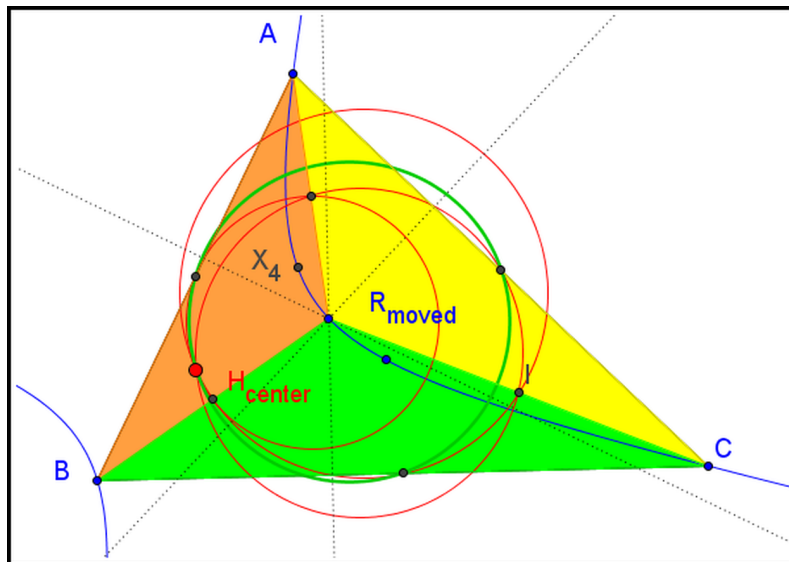
Dear Mr. Chris and Mr. Eckart

Let ABC be a triangle, Let rectangular hyperbola through A, B, C, X_4 (X_4 is the orthocenter of the triangle ABC). Denote D is any point lie on the hyperbola. Then Poncelet point of four points A, B, C, D is center of the rectangular hyperbola. (Show that when D moved on the hyperbola then Poncelet point of $ABCD$ is fixed)

Best regards

Sincerely

Dao Thanh Oai



Poncelet and rectangular hyperbola.png

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Message: #380

Date: 15/12/2013 11:07:51

From: Chris

Subject: Poncelet point and rectangular hyperbola

Dear Dao,

At first sight a remarkable property.

As often occurs at second sight it is evident.

Taking four points on an Orthogonal Hyperbola we have a Quadrangle.

In a Quadrangle only one Orthogonal Hyperbola can be constructed.

So the Orthogonal Hyperbola it is spanned in, is the only one.

The Poncelet Point (QA-P2) of a Quadrangle is the Center of the only Orthogonal Hyperbola of the Quadrangle.

So it is evident that a Quadrangle that is spanned on 4 points of an Orthogonal Hyperbola is a fixed point being the Center of the Orthogonal Hyperbola.

Seen from the Triangle point of view the Orthogonal Hyperbola can be defined by A,B,C,H and a 5th point P5. So the point you describe can be seen as the transformation of random point P5.

It is the Center of the Orthogonal Hyperbola (A,B,C,H,P5).

Best regards,

Chris van Tienhoven

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Message: #381
Date: 15/12/2013 11:49:34
From: eckart_schmidt@t-online.de
Subject: Problems: Square

Dear Dao,
with your simple construction of the circumscribed squares of a quadrigon it is easy to realize, that for all quadrigons with the same Varignon parallelogram these circumscribed squares have parallel corresponding sides and the same size (already mentioned in message #374, #375).
Best regards Eckart

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Message: #382
Date: 15/12/2013 11:59:36
From: yeuemtrondoitb85
Subject: Problems: Square

Dear Mr. Chris, and Mr. Eckart.
Thank to you very much
Best regards
Sincerely
Dao Thanh Oai

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Message: #383
Date: 15/12/2013 12:03:09
From: yeuemtrondoitb85
Subject: Poncelet point and rectangular hyperbola

Dear Mr Chris,
I generalization one theorem at here: Vonk, Jan (2009), "The Feuerbach point and reflections of the Euler line" <http://forumgeom.fau.edu/FG2009volume9/FG2009Volume9.pdf#page=51>, Forum Geometricorum *9*: 47-55
Best regards
Sincerely
Dao Thanh Oai

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Message: #384

Date: 15/12/2013 3:00:59

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

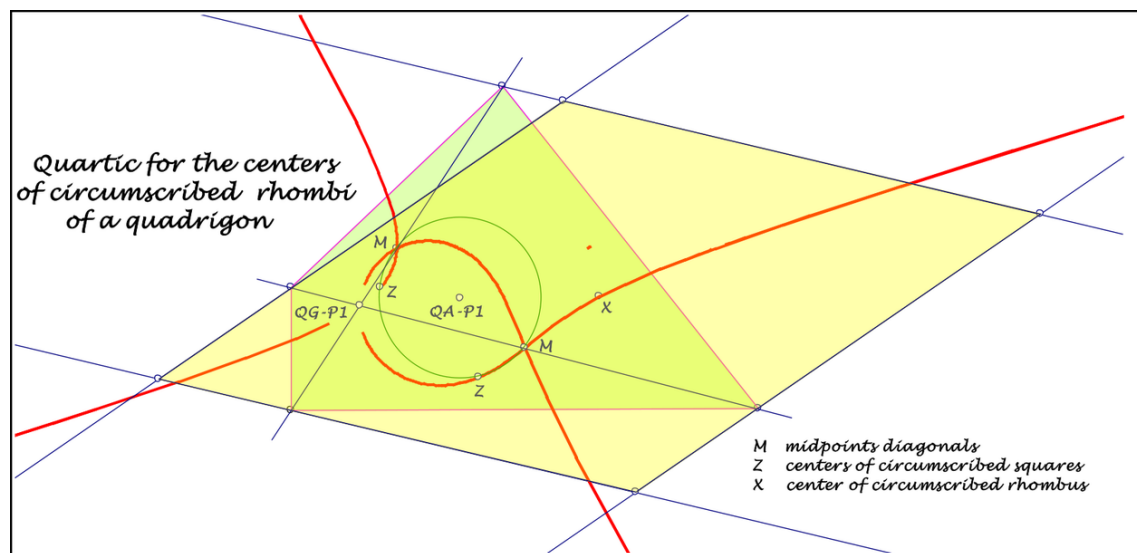
Dear Chris,

in addition to circumscribed squares of a quadrigon:

The centers of circumscribed rectangles lie on the Thales circle about the midpoints of the diagonals.

The centers of circumscribed rhombi lie on a quartic (see attachment). The quartic contains the diagonal crosspoint QG-P1, the midpoints M of the diagonals and the centers Z of the circumscribed squares. Construction and equation can be given.

Best regards Eckart



2013-12-15.pdf

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Message: #385

Date: 15/12/2013 4:17:10

From: Chris

Subject: problem about circumscribed square of a quadrigon

Dear Eckart,
Nice quartic with these familiar points on it.
I tried to construct the circumscribed rhombi.
In my picture I found 4 circumscribed rhombi.
Do you have a construction method?
I am interested in the equation too.
Best regards,
Chris

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Message: #386

Date: 16/12/2013 10:20:50

From: eckart_schmidt@t-online.de

Subject: problem about circumscribed square of a quadrigon

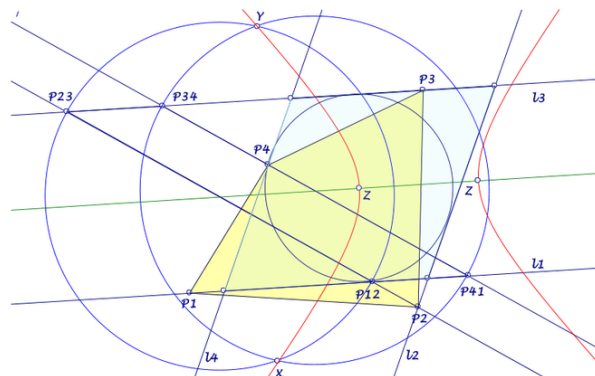
Dear Chris,
attached a description of the construction for the circumscribed rhombi of a quadrigon (I'm sure, there will be a simpler way).
The equation of the quartic is easy to calculate, see attachment.
Best regards Eckart

*Equation of the quartic for the centers
of circumscribed rhombi of a quadrigon
(reference triangle P1P2P3 with P4(p;q;r))*

$$\begin{aligned}
 \text{quad}[\{x_, y_, z_\}] := & -b^2 (r^2 x^4 + p^2 z^4) + (c^2 p^2 + a^2 p r - b^2 p r + c^2 p r + a^2 r^2) y^4 \\
 & + 2 b^2 (p + r) x z (r x^2 + p z^2) - 2 (a^2 - c^2) y (r^2 x^3 - p^2 z^3) - b^2 (p^2 + 4 p r + r^2) x^2 z^2 \\
 & + (p + 2 q + 3 r) (c^2 p + 2 c^2 q - a^2 r + b^2 r) x^2 y^2 \\
 & + (3 p + 2 q + r) (b^2 p - c^2 p + 2 a^2 q + a^2 r) y^2 z^2 \\
 & - 2 (c^2 p^2 + 2 c^2 p q + 2 c^2 p r + a^2 q r - b^2 q r + c^2 q r - b^2 r^2 + c^2 r^2) x y^3 \\
 & - 2 (a^2 p^2 - b^2 p^2 + a^2 p q - b^2 p q + c^2 p q + 2 a^2 p r + 2 a^2 q r + a^2 r^2) y^3 z \\
 & + 2 (a^2 - c^2) x y z (2 p r x + r^2 x - p^2 z - 2 p r z) + \\
 & 2 (a^2 p^2 - b^2 p^2 + 3 a^2 p q - 3 b^2 p q + c^2 p q + 2 a^2 q^2 - 2 b^2 q^2 + 2 c^2 q^2 + \\
 & 3 a^2 p r - 4 b^2 p r + 3 c^2 p r + a^2 q r - 3 b^2 q r + 3 c^2 q r - b^2 r^2 + c^2 r^2) x y^2 z
 \end{aligned}$$

Eckart Schmidt
eckart_schmidt@t-online.de
<http://eckartschmidt.de>

Construction of circumscribed rhombi for a quadrigon



Let $P_1P_2P_3P_4$ be a quadrigon.

- l_1 variable line through P_1 with a variable point P_{12} ,
- l_3 parallel to l_1 through P_3 ,
- $l_2 = P_{12}P_2$,
- l_4 parallel to l_2 through P_4 ,
- $P_{12}P_{23}P_{34}P_{41}$ circumscribed parallelogram.
- Thales circles over $P_{12}P_{23}$ and $P_{34}P_{41}$ with intersections X, Y .
- Changing P_{12} on l_1 the locus of X, Y is a conic.
- $Z_{1,2}$ intersections of the conic and the mid-parallel of l_1 and l_3 , centers of circumscribed rhombi.
- Circle round Z_1 or Z_2 tangent to l_1, l_3 .
- A tangent through P_2 at this circle and a parallel through P_4 give a circumscribed rhombus.
- Changing the line l_1 through P_1 the centers Z_i give the quartic for the centers of circumscribed rhombi.

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2013-12-15-386.pdf

Message: #387
Date: 18/12/2013 5:41:06
From: rhutson2
Subject: conic through reflections

Dear friends,
I have been away from quadri-figures for awhile, so forgive me if this has come up before.
Let $P_1P_2P_3P_4$ be a quadrangle and Q a point. Let Q_{ij} be the reflection of Q in line P_iP_j for all i, j in $\{1, 2, 3, 4\}$, $i < j$
1) What is the locus of Q such that $Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}$ all lie on a common conic? The locus would include $QA-P_4$, but no other listed QA -points.
2) Is there a unique point Q such that the conic is a circle?
3) Is there a unique point Q such that the conic is a rectangular hyperbola?
4) Is there a unique point Q such that Q is also the center of the conic?
If $Q = QA-P_4$, the conic has center at $QA-P_2$ (but is not $QA-Co_2$).
Conjecture for $Q = QA-P_4$: if $P_1P_2P_3P_4$ is concave, the conic is always an ellipse; if convex, it is always a hyperbola, and rectangular if cyclic.
Best regards,
Randy Hutson

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Message: #388
Date: 19/12/2013 10:03:43
From: eckart_schmidt@t-online.de
Subject: conic through reflections

Dear Randy,
interesting questions! Here some remarks to question 1 out of first experiences with Cabri:
* The locus of Q seems to be a cubic...
* ... circumscribed the diagonal triangle $QA-Tr_1$,
* ... circumscribed the Miquel triangle $QA-Tr_2$,
* ... invariant wrt the three versions of $QL-Tf_1$.
Best regards Eckart

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Message: #389
Date: 19/12/2013 3:25:01
From: Chris
Subject: conic through reflections

Dear Randy, Dear Eckart,
The locus of all points for which the reflections in the 6 connecting lines in a Quadrangle are coconic (all laying on a conic) also passes through the Involutory Conjugate of QA-P2 (Euler-Poncelet Point), which is an infinity point. Also the circular points at infinity lie on the locus, so it is circular.
Best regards,
Chris

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Message: #390
Date: 19/12/2013 5:04:12
From: rhutson2
Subject: conic through reflections

Another observation I forgot to mention: when $Q = QA-P4$, the conic also passes through QA-P4 (and, of course, its antipode, the reflection of QA-P4 in QA-P2).
Best regards,
Randy

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Message: #391
Date: 20/12/2013 4:38:01
From: amontes
Subject: conic through reflections

Dear Randy, Dear Eckart, Dear Chris,

[Randy Hutson QFG#387]:

Let $P_1P_2P_3P_4$ be a quadrangle and Q a point. Let Q_{ij} be the reflection

of Q in line P_iP_j for all i, j in $\{1, 2, 3, 4\}$, $i < j$

1.) What is the locus of Q such that $Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}$ all

lie on a common conic? The locus would include $QA-P_4$.

If $Q = QA-P_4$, the conic has center at $QA-P_2$.

[Eckart QFG#388]:

The locus of Q is circumscribed the Miquel triangle $QA-Tr_2$

[Chris van Tienhoven QFG#389]:

The locus of all points for which the reflections in the 6 connecting

lines in a Quadrangle are coconic (all laying on a conic) also passes

through the Involutory Conjugate of $QA-P_2$ (Euler-Poncelet Point), which

is an infinity point.

[Randy Hutson QFG#390]:

Another observation I forgot to mention: when $Q = QA-P_4$, the conic also

passes through $QA-P_4$.

*** The locus of Q is the septic with singularities ("isolated points")

P_1, P_2, P_3 and P_4 . With the properties outlined above by Randy, Eckart and Chris.

Equation of septic, in DT-notation (very complicated, see attached file

Sextic-QFG#387.txt)

Particular cases:

==== If $P_1P_2P_3P_4$ be the quadrangle $IIaIbIc$ (incenter and excenters),

the septic is decomposed into the product of the line at infinity and a

sextic with I, Ia, Ib and Ic only real points ("isolated points"):

$-b^2 c^4 x^4 y^2 - a^2 c^4 x^3 y^3 - b^2 c^4 x^3 y^3 + c^6 x^3 y^3 -$

$a^2 c^4 x^2 y^4 + a^2 b^2 c^2 x^4 y z - b^4 c^2 x^4 y z -$

$$\begin{aligned}
& b^2 c^4 x^4 y z + a^2 b^2 c^2 x^3 y^2 z - b^4 c^2 x^3 y^2 z + \\
& b^2 c^4 x^3 y^2 z - a^4 c^2 x^2 y^3 z + a^2 b^2 c^2 x^2 y^3 z + \\
& a^2 c^4 x^2 y^3 z - a^4 c^2 x y^4 z + a^2 b^2 c^2 x y^4 z - \\
& a^2 c^4 x y^4 z - b^4 c^2 x^4 z^2 + a^2 b^2 c^2 x^3 y z^2 + \\
& b^4 c^2 x^3 y z^2 - b^2 c^4 x^3 y z^2 + 6 a^2 b^2 c^2 x^2 y^2 z^2 + \\
& a^4 c^2 x y^3 z^2 + a^2 b^2 c^2 x y^3 z^2 - a^2 c^4 x y^3 z^2 - \\
& a^4 c^2 y^4 z^2 - a^2 b^4 x^3 z^3 + b^6 x^3 z^3 - b^4 c^2 x^3 z^3 - \\
& z^3 - \\
& a^4 b^2 x^2 y z^3 + a^2 b^4 x^2 y z^3 + a^2 b^2 c^2 x^2 y z^3 + \\
& a^4 b^2 x y^2 z^3 - a^2 b^4 x y^2 z^3 + a^2 b^2 c^2 x y^2 z^3 + \\
& a^6 y^3 z^3 - a^4 b^2 y^3 z^3 - a^4 c^2 y^3 z^3 - a^2 b^4 x^2 z^4 - \\
& z^4 - \\
& a^4 b^2 x y z^4 - a^2 b^4 x y z^4 + a^2 b^2 c^2 x y z^4 - \\
& a^4 b^2 y^2 z^4 = 0.
\end{aligned}$$

=== If P1P2P3P4 be the quadrangle GGaGbGc (the centroid and the vertices

of triangle antimedial), the equation of septic is (in

DT-notation):

$$\begin{aligned}
& a^8 x^6 y - 6 a^6 b^2 x^6 y + 9 a^4 b^4 x^6 y - 2 a^2 b^6 x^6 y \\
& - \\
& 3 a^6 c^2 x^6 y + 12 a^4 b^2 c^2 x^6 y - 6 a^2 b^4 c^2 x^6 y + \\
& 3 a^4 c^4 x^6 y - 6 a^2 b^2 c^4 x^6 y - 2 a^2 c^6 x^6 y + \\
& a^8 x^5 y^2 - 8 a^6 b^2 x^5 y^2 + 15 a^4 b^4 x^5 y^2 + \\
& 4 a^2 b^6 x^5 y^2 - 2 b^8 x^5 y^2 + 4 a^6 c^2 x^5 y^2 + \\
& 3 a^4 b^2 c^2 x^5 y^2 + 6 a^2 b^4 c^2 x^5 y^2 - 2 b^6 c^2 x^5 y^2 - \\
& y^2 - \\
& 9 a^4 c^4 x^5 y^2 + 6 a^2 b^2 c^4 x^5 y^2 + 4 a^2 c^6 x^5 y^2 - \\
& 2 b^2 c^6 x^5 y^2 - 2 c^8 x^5 y^2 - 3 a^8 x^4 y^3 + \\
& 10 a^6 b^2 x^4 y^3 + 18 a^2 b^6 x^4 y^3 - 5 b^8 x^4 y^3 + \\
& 15 a^6 c^2 x^4 y^3 - 12 a^4 b^2 c^2 x^4 y^3 + \\
& 9 a^2 b^4 c^2 x^4 y^3 + 6 b^6 c^2 x^4 y^3 - 6 a^4 c^4 x^4 y^3 + \\
& 6 b^4 c^4 x^4 y^3 - 3 a^2 c^6 x^4 y^3 - 2 b^2 c^6 x^4 y^3 + \\
& 3 c^8 x^4 y^3 - 5 a^8 x^3 y^4 + 18 a^6 b^2 x^3 y^4 + \\
& 10 a^2 b^6 x^3 y^4 - 3 b^8 x^3 y^4 + 6 a^6 c^2 x^3 y^4 + \\
& 9 a^4 b^2 c^2 x^3 y^4 - 12 a^2 b^4 c^2 x^3 y^4 + \\
& 15 b^6 c^2 x^3 y^4 + 6 a^4 c^4 x^3 y^4 - 6 b^4 c^4 x^3 y^4 - \\
& 2 a^2 c^6 x^3 y^4 - 3 b^2 c^6 x^3 y^4 + 3 c^8 x^3 y^4 - \\
& 2 a^8 x^2 y^5 + 4 a^6 b^2 x^2 y^5 + 15 a^4 b^4 x^2 y^5 - \\
& 8 a^2 b^6 x^2 y^5 + b^8 x^2 y^5 - 2 a^6 c^2 x^2 y^5 + \\
& 6 a^4 b^2 c^2 x^2 y^5 + 3 a^2 b^4 c^2 x^2 y^5 + 4 b^6 c^2 x^2 y^5 + \\
& y^5 + \\
& 6 a^2 b^2 c^4 x^2 y^5 - 9 b^4 c^4 x^2 y^5 - 2 a^2 c^6 x^2 y^5 + \\
& 4 b^2 c^6 x^2 y^5 - 2 c^8 x^2 y^5 - 2 a^6 b^2 x y^6 + \\
& 9 a^4 b^4 x y^6 - 6 a^2 b^6 x y^6 + b^8 x y^6 - \\
& 6 a^4 b^2 c^2 x y^6 + 12 a^2 b^4 c^2 x y^6 - 3 b^6 c^2 x y^6 - \\
& 6 a^2 b^2 c^4 x y^6 + 3 b^4 c^4 x y^6 - 2 b^2 c^6 x y^6 + \\
& a^8 x^6 z - 3 a^6 b^2 x^6 z + 3 a^4 b^4 x^6 z - 2 a^2 b^6 x^6 z \\
& -
\end{aligned}$$

$$\begin{aligned}
& 6 a^6 c^2 x^6 z + 12 a^4 b^2 c^2 x^6 z - 6 a^2 b^4 c^2 x^6 z + \\
& 9 a^4 c^4 x^6 z - 6 a^2 b^2 c^4 x^6 z - 2 a^2 c^6 x^6 z + \\
& 2 a^8 x^5 y z - 10 a^6 b^2 x^5 y z + 8 a^2 b^6 x^5 y z - \\
& 4 b^8 x^5 y z - 10 a^6 c^2 x^5 y z + 12 a^4 b^2 c^2 x^5 y z + \\
& 12 a^2 b^4 c^2 x^5 y z - 4 b^6 c^2 x^5 y z + \\
& 12 a^2 b^2 c^4 x^5 y z + 8 a^2 c^6 x^5 y z - 4 b^2 c^6 x^5 y z - \\
& 4 c^8 x^5 y z + 2 a^8 x^4 y^2 z - 16 a^6 b^2 x^4 y^2 z - \\
& 3 a^4 b^4 x^4 y^2 z + 17 a^2 b^6 x^4 y^2 z - 7 b^8 x^4 y^2 z + \\
& 4 a^6 c^2 x^4 y^2 z - 12 a^4 b^2 c^2 x^4 y^2 z + \\
& 18 a^2 b^4 c^2 x^4 y^2 z + 10 b^6 c^2 x^4 y^2 z - \\
& 21 a^4 c^4 x^4 y^2 z + 9 a^2 b^2 c^4 x^4 y^2 z + \\
& 18 b^4 c^4 x^4 y^2 z - 4 a^2 c^6 x^4 y^2 z + 2 b^2 c^6 x^4 y^2 z \\
& + \\
& c^8 x^4 y^2 z - 2 a^8 x^3 y^3 z - 2 a^6 b^2 x^3 y^3 z - \\
& 2 a^2 b^6 x^3 y^3 z - 2 b^8 x^3 y^3 z + 22 a^6 c^2 x^3 y^3 z - \\
& 12 a^4 b^2 c^2 x^3 y^3 z - 12 a^2 b^4 c^2 x^3 y^3 z + \\
& 22 b^6 c^2 x^3 y^3 z - 12 a^2 b^2 c^4 x^3 y^3 z - \\
& 14 a^2 c^6 x^3 y^3 z - 14 b^2 c^6 x^3 y^3 z + 10 c^8 x^3 y^3 z - \\
& 7 a^8 x^2 y^4 z + 17 a^6 b^2 x^2 y^4 z - 3 a^4 b^4 x^2 y^4 z - \\
& 16 a^2 b^6 x^2 y^4 z + 2 b^8 x^2 y^4 z + 10 a^6 c^2 x^2 y^4 z + \\
& 18 a^4 b^2 c^2 x^2 y^4 z - 12 a^2 b^4 c^2 x^2 y^4 z + \\
& 4 b^6 c^2 x^2 y^4 z + 18 a^4 c^4 x^2 y^4 z + \\
& 9 a^2 b^2 c^4 x^2 y^4 z - 21 b^4 c^4 x^2 y^4 z + \\
& 2 a^2 c^6 x^2 y^4 z - 4 b^2 c^6 x^2 y^4 z + c^8 x^2 y^4 z - \\
& 4 a^8 x y^5 z + 8 a^6 b^2 x y^5 z - 10 a^2 b^6 x y^5 z + \\
& 2 b^8 x y^5 z - 4 a^6 c^2 x y^5 z + 12 a^4 b^2 c^2 x y^5 z + \\
& 12 a^2 b^4 c^2 x y^5 z - 10 b^6 c^2 x y^5 z + \\
& 12 a^2 b^2 c^4 x y^5 z - 4 a^2 c^6 x y^5 z + 8 b^2 c^6 x y^5 z - \\
& 4 c^8 x y^5 z - 2 a^6 b^2 y^6 z + 3 a^4 b^4 y^6 z - 3 a^2 b^6 \\
& y^6 z + \\
& b^8 y^6 z - 6 a^4 b^2 c^2 y^6 z + 12 a^2 b^4 c^2 y^6 z - \\
& 6 b^6 c^2 y^6 z - 6 a^2 b^2 c^4 y^6 z + 9 b^4 c^4 y^6 z - \\
& 2 b^2 c^6 y^6 z + a^8 x^5 z^2 + 4 a^6 b^2 x^5 z^2 - \\
& 9 a^4 b^4 x^5 z^2 + 4 a^2 b^6 x^5 z^2 - 2 b^8 x^5 z^2 - \\
& 8 a^6 c^2 x^5 z^2 + 3 a^4 b^2 c^2 x^5 z^2 + 6 a^2 b^4 c^2 x^5 \\
& z^2 - \\
& 2 b^6 c^2 x^5 z^2 + 15 a^4 c^4 x^5 z^2 + 6 a^2 b^2 c^4 x^5 z^2 + \\
& 4 a^2 c^6 x^5 z^2 - 2 b^2 c^6 x^5 z^2 - 2 c^8 x^5 z^2 + \\
& 2 a^8 x^4 y z^2 + 4 a^6 b^2 x^4 y z^2 - 21 a^4 b^4 x^4 y z^2 - \\
& 4 a^2 b^6 x^4 y z^2 + b^8 x^4 y z^2 - 16 a^6 c^2 x^4 y z^2 - \\
& 12 a^4 b^2 c^2 x^4 y z^2 + 9 a^2 b^4 c^2 x^4 y z^2 + \\
& 2 b^6 c^2 x^4 y z^2 - 3 a^4 c^4 x^4 y z^2 + \\
& 18 a^2 b^2 c^4 x^4 y z^2 + 18 b^4 c^4 x^4 y z^2 + \\
& 17 a^2 c^6 x^4 y z^2 + 10 b^2 c^6 x^4 y z^2 - 7 c^8 x^4 y z^2 + \\
& 6 a^8 x^3 y^2 z^2 - 16 a^6 b^2 x^3 y^2 z^2 - \\
& 18 a^4 b^4 x^3 y^2 z^2 - 24 a^2 b^6 x^3 y^2 z^2 + \\
& 8 b^8 x^3 y^2 z^2 - 16 a^6 c^2 x^3 y^2 z^2 - \\
& 30 a^4 b^2 c^2 x^3 y^2 z^2 - 12 a^2 b^4 c^2 x^3 y^2 z^2 - \\
& 4 b^6 c^2 x^3 y^2 z^2 - 18 a^4 c^4 x^3 y^2 z^2 -
\end{aligned}$$

$$\begin{aligned}
& 12 a^2 b^2 c^4 x^3 y^2 z^2 + 12 b^4 c^4 x^3 y^2 z^2 - \\
& 24 a^2 c^6 x^3 y^2 z^2 - 4 b^2 c^6 x^3 y^2 z^2 + 8 c^8 x^3 y^2 z^2 + \\
& 8 a^8 x^2 y^3 z^2 - 24 a^6 b^2 x^2 y^3 z^2 - \\
& 18 a^4 b^4 x^2 y^3 z^2 - 16 a^2 b^6 x^2 y^3 z^2 + \\
& 6 b^8 x^2 y^3 z^2 - 4 a^6 c^2 x^2 y^3 z^2 - \\
& 12 a^4 b^2 c^2 x^2 y^3 z^2 - 30 a^2 b^4 c^2 x^2 y^3 z^2 - \\
& 16 b^6 c^2 x^2 y^3 z^2 + 12 a^4 c^4 x^2 y^3 z^2 - \\
& 12 a^2 b^2 c^4 x^2 y^3 z^2 - 18 b^4 c^4 x^2 y^3 z^2 - \\
& 4 a^2 c^6 x^2 y^3 z^2 - 24 b^2 c^6 x^2 y^3 z^2 + 8 c^8 x^2 y^3 z^2 + \\
& a^8 x y^4 z^2 - 4 a^6 b^2 x y^4 z^2 - 21 a^4 b^4 x y^4 z^2 + \\
& 4 a^2 b^6 x y^4 z^2 + 2 b^8 x y^4 z^2 + 2 a^6 c^2 x y^4 z^2 + \\
& 9 a^4 b^2 c^2 x y^4 z^2 - 12 a^2 b^4 c^2 x y^4 z^2 - \\
& 16 b^6 c^2 x y^4 z^2 + 18 a^4 c^4 x y^4 z^2 + \\
& 18 a^2 b^2 c^4 x y^4 z^2 - 3 b^4 c^4 x y^4 z^2 + \\
& 10 a^2 c^6 x y^4 z^2 + 17 b^2 c^6 x y^4 z^2 - 7 c^8 x y^4 z^2 - \\
& 2 a^8 y^5 z^2 + 4 a^6 b^2 y^5 z^2 - 9 a^4 b^4 y^5 z^2 + \\
& 4 a^2 b^6 y^5 z^2 + b^8 y^5 z^2 - 2 a^6 c^2 y^5 z^2 + \\
& 6 a^4 b^2 c^2 y^5 z^2 + 3 a^2 b^4 c^2 y^5 z^2 - 8 b^6 c^2 y^5 z^2 + \\
& z^2 + \\
& 6 a^2 b^2 c^4 y^5 z^2 + 15 b^4 c^4 y^5 z^2 - 2 a^2 c^6 y^5 z^2 + \\
& 4 b^2 c^6 y^5 z^2 - 2 c^8 y^5 z^2 - 3 a^8 x^4 z^3 + \\
& 15 a^6 b^2 x^4 z^3 - 6 a^4 b^4 x^4 z^3 - 3 a^2 b^6 x^4 z^3 + \\
& 3 b^8 x^4 z^3 + 10 a^6 c^2 x^4 z^3 - 12 a^4 b^2 c^2 x^4 z^3 - \\
& 2 b^6 c^2 x^4 z^3 + 9 a^2 b^2 c^4 x^4 z^3 + 6 b^4 c^4 x^4 z^3 + \\
& 18 a^2 c^6 x^4 z^3 + 6 b^2 c^6 x^4 z^3 - 5 c^8 x^4 z^3 - \\
& 2 a^8 x^3 y z^3 + 22 a^6 b^2 x^3 y z^3 - 14 a^2 b^6 x^3 y z^3 + \\
& 10 b^8 x^3 y z^3 - 2 a^6 c^2 x^3 y z^3 - 12 a^4 b^2 c^2 x^3 y z^3 - \\
& z^3 - \\
& 12 a^2 b^4 c^2 x^3 y z^3 - 14 b^6 c^2 x^3 y z^3 - \\
& 12 a^2 b^2 c^4 x^3 y z^3 - 2 a^2 c^6 x^3 y z^3 + \\
& 22 b^2 c^6 x^3 y z^3 - 2 c^8 x^3 y z^3 + 8 a^8 x^2 y^2 z^3 - \\
& 4 a^6 b^2 x^2 y^2 z^3 + 12 a^4 b^4 x^2 y^2 z^3 - \\
& 4 a^2 b^6 x^2 y^2 z^3 + 8 b^8 x^2 y^2 z^3 - 24 a^6 c^2 x^2 y^2 z^3 - \\
& z^3 - \\
& 12 a^4 b^2 c^2 x^2 y^2 z^3 - 12 a^2 b^4 c^2 x^2 y^2 z^3 - \\
& 24 b^6 c^2 x^2 y^2 z^3 - 18 a^4 c^4 x^2 y^2 z^3 - \\
& 30 a^2 b^2 c^4 x^2 y^2 z^3 - 18 b^4 c^4 x^2 y^2 z^3 - \\
& 16 a^2 c^6 x^2 y^2 z^3 - 16 b^2 c^6 x^2 y^2 z^3 + \\
& 6 c^8 x^2 y^2 z^3 + 10 a^8 x y^3 z^3 - 14 a^6 b^2 x y^3 z^3 + \\
& 22 a^2 b^6 x y^3 z^3 - 2 b^8 x y^3 z^3 - 14 a^6 c^2 x y^3 z^3 - \\
& 12 a^4 b^2 c^2 x y^3 z^3 - 12 a^2 b^4 c^2 x y^3 z^3 - \\
& 2 b^6 c^2 x y^3 z^3 - 12 a^2 b^2 c^4 x y^3 z^3 + \\
& 22 a^2 c^6 x y^3 z^3 - 2 b^2 c^6 x y^3 z^3 - 2 c^8 x y^3 z^3 + \\
& 3 a^8 y^4 z^3 - 3 a^6 b^2 y^4 z^3 - 6 a^4 b^4 y^4 z^3 + \\
& 15 a^2 b^6 y^4 z^3 - 3 b^8 y^4 z^3 - 2 a^6 c^2 y^4 z^3 - \\
& 12 a^2 b^4 c^2 y^4 z^3 + 10 b^6 c^2 y^4 z^3 + 6 a^4 c^4 y^4 z^3 + \\
& +
\end{aligned}$$

$$\begin{aligned}
& 9 a^2 b^2 c^4 y^4 z^3 + 6 a^2 c^6 y^4 z^3 + 18 b^2 c^6 y^4 z^3 - \\
& 5 c^8 y^4 z^3 - 5 a^8 x^3 z^4 + 6 a^6 b^2 x^3 z^4 + \\
& 6 a^4 b^4 x^3 z^4 - 2 a^2 b^6 x^3 z^4 + 3 b^8 x^3 z^4 + \\
& 18 a^6 c^2 x^3 z^4 + 9 a^4 b^2 c^2 x^3 z^4 - 3 b^6 c^2 x^3 z^4 - \\
& 12 a^2 b^2 c^4 x^3 z^4 - 6 b^4 c^4 x^3 z^4 + 10 a^2 c^6 x^3 z^4 \\
& + \\
& 15 b^2 c^6 x^3 z^4 - 3 c^8 x^3 z^4 - 7 a^8 x^2 y z^4 + \\
& 10 a^6 b^2 x^2 y z^4 + 18 a^4 b^4 x^2 y z^4 + 2 a^2 b^6 x^2 y \\
& z^4 + \\
& b^8 x^2 y z^4 + 17 a^6 c^2 x^2 y z^4 + 18 a^4 b^2 c^2 x^2 y z^4 \\
& + \\
& 9 a^2 b^4 c^2 x^2 y z^4 - 4 b^6 c^2 x^2 y z^4 - \\
& 3 a^4 c^4 x^2 y z^4 - 12 a^2 b^2 c^4 x^2 y z^4 - \\
& 21 b^4 c^4 x^2 y z^4 - 16 a^2 c^6 x^2 y z^4 + 4 b^2 c^6 x^2 y \\
& z^4 + \\
& 2 c^8 x^2 y z^4 + a^8 x y^2 z^4 + 2 a^6 b^2 x y^2 z^4 + \\
& 18 a^4 b^4 x y^2 z^4 + 10 a^2 b^6 x y^2 z^4 - 7 b^8 x y^2 z^4 - \\
& 4 a^6 c^2 x y^2 z^4 + 9 a^4 b^2 c^2 x y^2 z^4 + \\
& 18 a^2 b^4 c^2 x y^2 z^4 + 17 b^6 c^2 x y^2 z^4 - \\
& 21 a^4 c^4 x y^2 z^4 - 12 a^2 b^2 c^4 x y^2 z^4 - \\
& 3 b^4 c^4 x y^2 z^4 + 4 a^2 c^6 x y^2 z^4 - 16 b^2 c^6 x y^2 z^4 \\
& + \\
& 2 c^8 x y^2 z^4 + 3 a^8 y^3 z^4 - 2 a^6 b^2 y^3 z^4 + \\
& 6 a^4 b^4 y^3 z^4 + 6 a^2 b^6 y^3 z^4 - 5 b^8 y^3 z^4 - \\
& 3 a^6 c^2 y^3 z^4 + 9 a^2 b^4 c^2 y^3 z^4 + 18 b^6 c^2 y^3 z^4 - \\
& 6 a^4 c^4 y^3 z^4 - 12 a^2 b^2 c^4 y^3 z^4 + 15 a^2 c^6 y^3 z^4 \\
& + \\
& 10 b^2 c^6 y^3 z^4 - 3 c^8 y^3 z^4 - 2 a^8 x^2 z^5 - \\
& 2 a^6 b^2 x^2 z^5 - 2 a^2 b^6 x^2 z^5 - 2 b^8 x^2 z^5 + \\
& 4 a^6 c^2 x^2 z^5 + 6 a^4 b^2 c^2 x^2 z^5 + 6 a^2 b^4 c^2 x^2 \\
& z^5 + \\
& 4 b^6 c^2 x^2 z^5 + 15 a^4 c^4 x^2 z^5 + 3 a^2 b^2 c^4 x^2 z^5 - \\
& 9 b^4 c^4 x^2 z^5 - 8 a^2 c^6 x^2 z^5 + 4 b^2 c^6 x^2 z^5 + \\
& c^8 x^2 z^5 - 4 a^8 x y z^5 - 4 a^6 b^2 x y z^5 - \\
& 4 a^2 b^6 x y z^5 - 4 b^8 x y z^5 + 8 a^6 c^2 x y z^5 + \\
& 12 a^4 b^2 c^2 x y z^5 + 12 a^2 b^4 c^2 x y z^5 + \\
& 8 b^6 c^2 x y z^5 + 12 a^2 b^2 c^4 x y z^5 - 10 a^2 c^6 x y z^5 \\
& - \\
& 10 b^2 c^6 x y z^5 + 2 c^8 x y z^5 - 2 a^8 y^2 z^5 - \\
& 2 a^6 b^2 y^2 z^5 - 2 a^2 b^6 y^2 z^5 - 2 b^8 y^2 z^5 + \\
& 4 a^6 c^2 y^2 z^5 + 6 a^4 b^2 c^2 y^2 z^5 + 6 a^2 b^4 c^2 y^2 \\
& z^5 + \\
& 4 b^6 c^2 y^2 z^5 - 9 a^4 c^4 y^2 z^5 + 3 a^2 b^2 c^4 y^2 z^5 + \\
& 15 b^4 c^4 y^2 z^5 + 4 a^2 c^6 y^2 z^5 - 8 b^2 c^6 y^2 z^5 + \\
& c^8 y^2 z^5 - 2 a^6 c^2 x z^6 - 6 a^4 b^2 c^2 x z^6 - \\
& 6 a^2 b^4 c^2 x z^6 - 2 b^6 c^2 x z^6 + 9 a^4 c^4 x z^6 + \\
& 12 a^2 b^2 c^4 x z^6 + 3 b^4 c^4 x z^6 - 6 a^2 c^6 x z^6 - \\
& 3 b^2 c^6 x z^6 + c^8 x z^6 - 2 a^6 c^2 y z^6 - \\
& 6 a^4 b^2 c^2 y z^6 - 6 a^2 b^4 c^2 y z^6 - 2 b^6 c^2 y z^6 +
\end{aligned}$$

$$3 a^4 c^4 y z^6 + 12 a^2 b^2 c^4 y z^6 + 9 b^4 c^4 y z^6 - 3 a^2 c^6 y z^6 - 6 b^2 c^6 y z^6 + c^8 y z^6 = 0.$$

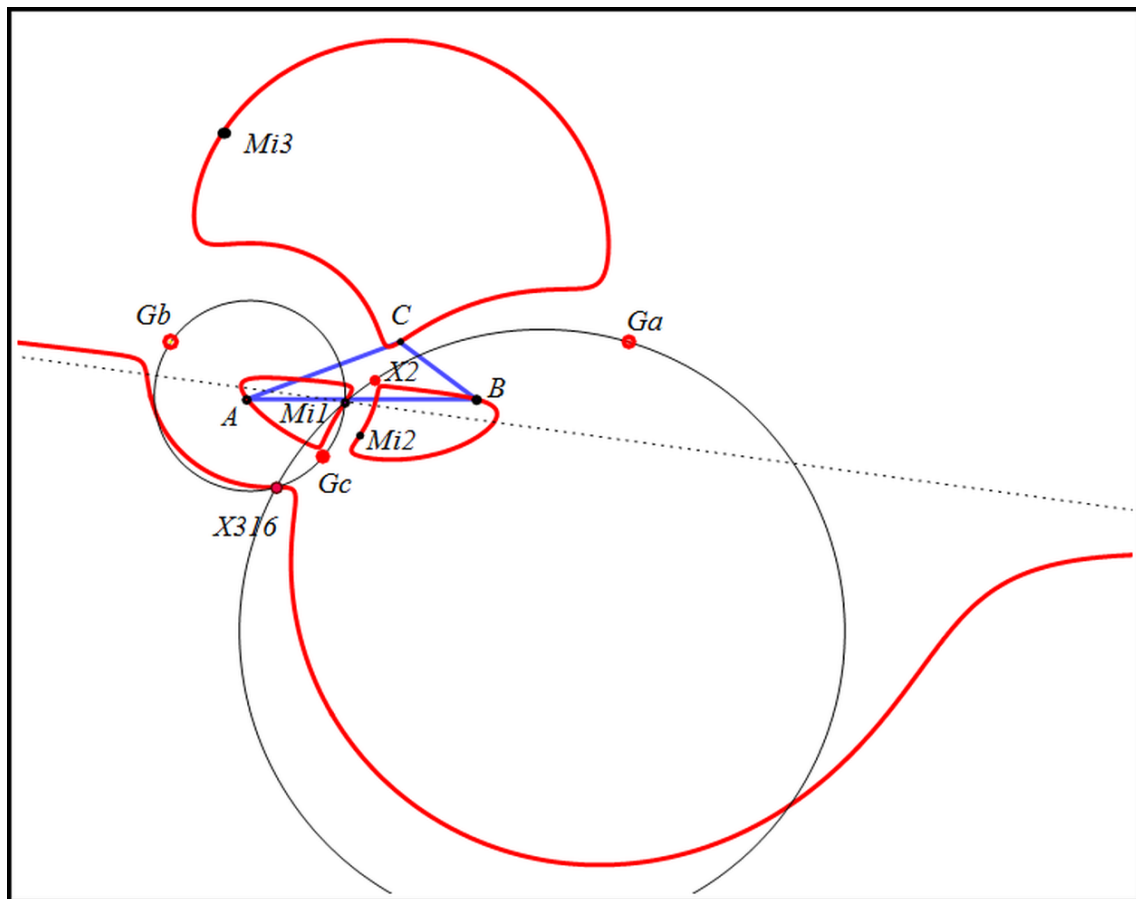
Is circumscribed the Miquel triangle QA-Tr2 =
 $\{ Mi1=(a^2+b^2+c^2 : -a^2-b^2+2c^2 : -a^2+2b^2-c^2) , Mi2, Mi3 \}.$

Passes through X(523)= the Involutory Conjugate of QA-P2=X(99) (Euler-Poncelet Point), which is an infinity point.

If $Q = QA-P4=X(316)=$ DROUSSENT PIVOT, the conic has center at QA-P2=X(99).

(see attached file Quadri-Figures-Group387.png)

Best regards,
 Angel Montesdeca



Quadri-Figures-Group387.png

Message: #392
Date: 23/12/2013 11:48:49
From: eckart_schmidt@t-online.de
Subject: conic through reflections

Dear Randy, Chris, Angel,

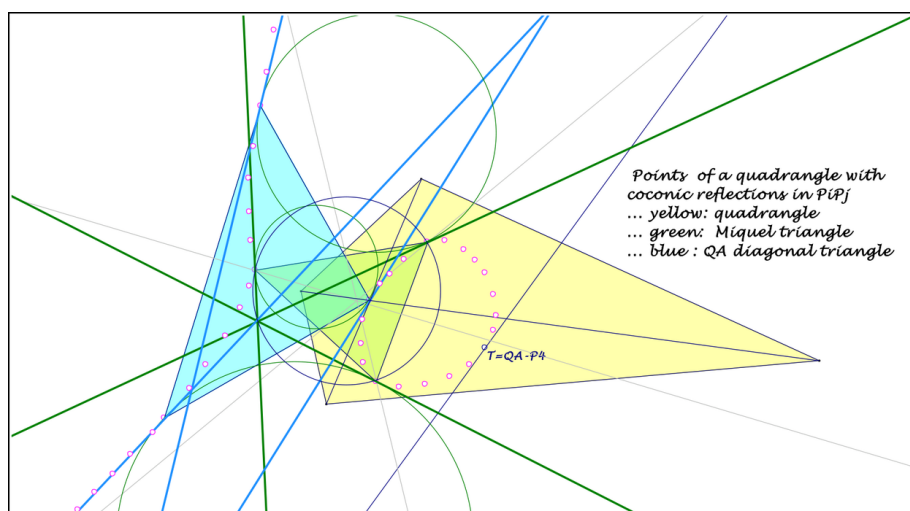
my speculation, that the discussed curve is a cubic was wrong, sorry. But the invariance wrt the three QL-Tf1 versions (CSC) should hold.

I think, the following properties will also be true. For a quadrigon component of the quadrangle let M be the Miquel point, F the point at infinity of perpendicular lines of $QA-L2$ (see Chris' message) and T the Isogonal Center $QA-P4$ (see Randy's message), all points of the curve as already mentioned.

- * The reflection of MF in the corresponding angle bisector of the Miquel triangle is tangent to the curve at M .
- * These tangents have a common point in the isogonal conjugate of F wrt the Miquel triangle (on the circumcircle).
- * TF is tangent to the curve at T .
- * The CSC images of the tangent in T are circles tangent to the curve in a Miquel point and a diagonal crosspoint.
- * The tangents to these circles in the diagonal crosspoints are tangents to the curve.

There is a drawing in the attachment with some "hand-plotted" points of the curve. Perhaps you can understand my first speculation for a cubic, but there are other constellations with three closed curve components and a further curve.

Merry Christmas
Eckart



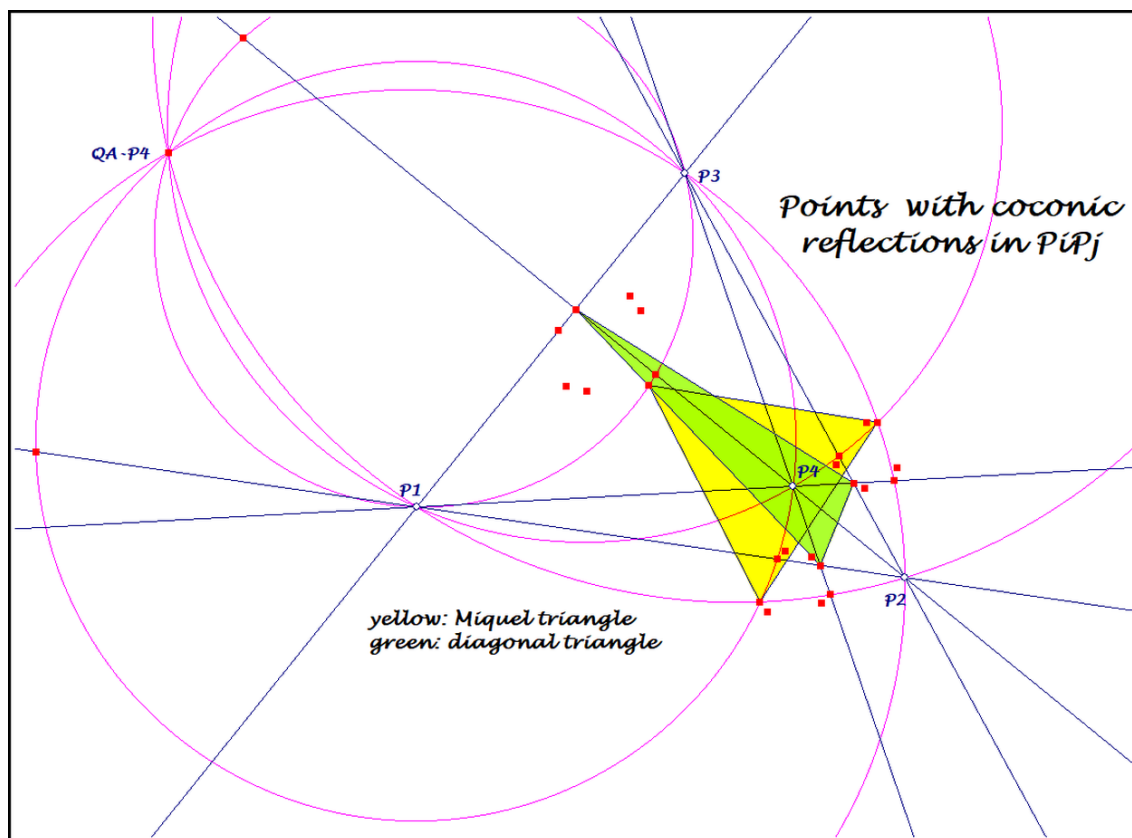
2013-12-23.pdf

Message: #393
Date: 30/12/2013 11:46:08
From: eckart_schmidt@t-online.de
Subject: conic through reflections

Dear friends,

here is a construction for up to 24 special further points with coconic reflections in P_iP_j of a quadrangle $P_1P_2P_3P_4$:
Consider the intersections of P_iP_j and the circumcircle of P_k , P_l and $QA-P_4$. These points and their images wrt the three versions of QL-Tf1 have the discussed property (see attachment).

Best regards and a Happy New Year
Eckart



2013-12-30-fig.png

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6 Colophon

Sources & Contact

Web address (QPG Forum): <https://groups.io/g/Quadri-and-Poly-Geometry>

EPG Encyclopedia (content reference): <https://www.chrisvantienhoven.nl>

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- Volume 4 (2016), messages #1403–#2169
- Volume 5 (2017), messages #2170–#2799
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- Volume 2 (2020), messages #61–#630
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- Volume 4 (2022), messages #1295–#1544
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