Journal of the Quadri- and Poly-Geometry Group 2024

Digital Edition

Chris van Tienhoven et al.

June 10, 2026

Volume 6

(jan. 2024 - dec. 2024)

Messages #2052 - #2559

Contents

1 Introduction

This journal is a compilation of messages from the **Quadri- and Poly-Geometry (QPG)** forum, where mathematicians and geometry enthusiasts exchange ideas on the properties of **quadrilaterals**, **polygons**, and **curves of nth degree**. The discussions cover a wide range of topics, from classical geometric theorems to new discoveries and insights.

The origins of this journal trace back to the Quadri Figures Group (QFG, available at https://groups.io/g/Quadri-Figures-Group), which was active from 2013 until November 2019. In November 2019, the forum transitioned into the Quadri- and Poly-Geometry Group (QPG, available at https://groups.io/g/Quadri-and-Poly-Geometry) forum, which continues to facilitate discussions on quadrilaterals, polygons, and related topics. Over the years, these forums have evolved into valuable resources for exploring both well-established results and novel perspectives in geometry. For both forums, an **annual record of all incoming messages** is compiled in this journal.

This journal is available in **PDF format** and includes a **table of contents** that organizes all messages by subject. Navigation is made easy through **hyperlinks** embedded in the message numbers, allowing users to quickly jump between related discussions or return to the table of contents for further reference.

Many of the topics discussed here are closely related to the Encyclopedia of Poly Geometry, available at https://www.chrisvantienhoven.nl/, which aims to systematically classify and analyze geometric structures. By collecting these forum messages, this journal serves both as a **historical archive** and as a **source of inspiration** for further research in the fascinating world of geometry.

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2 Authors

This section presents an alphabetical overview of the authors who contributed messages to this volume of the Journal.

- Antreas Hatzipolakis
- Bernard Keizer
- Chris van Tienhoven
- Eckart Schmidt
- Francisco Javier García Capitán
- Keita Miyamoto
- M@IMF
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- Tran Quang Hung
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- Seiichi Kirikami (In Memoriam)

2.1 Author Index

This section provides an index of all authors who contributed messages to this volume of the Journal.
Each entry lists the author's name, their identifier, and the message numbers associated with their contributions.
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- **Keita Miyamoto**
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  #2418 #2420 #2469 #2471 #2475 #2535 #2553 #2554 #2558

- **M@IMF**
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  #2493 #2508 #2509 #2517 #2520 #2523 #2527 #2528 #2529 #2532 #2543 #2544 #2546 #2547 #2551 #2552

- **Stanley Rabinowitz**
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- **Tran Quang Hung**
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  #2109 #2113 #2141 #2154 #2158 #2162 #2201 #2203 #2209 #2223 #2231 #2245

- **Seiichi Kirikami**
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  #2534

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2.2 Author Information

This section presents background information on the contributing authors. Short biographical notes, areas of interest, and selected publications are included to provide context for their contributions to the Journal. These profiles offer readers an opportunity to become acquainted with the individual behind the names and to appreciate the diverse mathematical backgrounds represented in this volume. Author information is included only insofar as it has been provided or was available.

Antreas P. Hatzipolakis

Location
Lives in Greece.

Year of Birth / Generation
1952.

Short Biography
Antreas P. Hatzipolakis studied mathematics at Athens University. He is the founder of several influential geometry-focused email groups, including *Hyacinthos*, *Anopolis*, and *Euclid*, as well as various Facebook groups dedicated to classical and triangle geometry. For many years, he introduced new problem areas through his email groups, inspiring others to explore, investigate, and solve them. His work has played a significant role in shaping the collaborative culture of modern online geometry communities.

Themes and Interests

- Classical Euclidean geometry
- Triangle geometry
- Problem creation and problem solving

Selected Publications

- Antreas P. Hatzipolakis, Floor van Lamoen, Barry Wolk, and Paul Yiu, *Concurrency of Four Euler Lines.* Forum Geometricorum, Volume 1 (2001), 59–68.
- Antreas P. Hatzipolakis and Paul Yiu, *Reflections in Triangle Geometry.* Forum Geometricorum, Volume 9 (2009), 301–348.

Additional Remarks

Website: http://www.anthrakitis.blogspot.com/

Chris van Tienhoven

Global Location
Living in the Netherlands.

Year of Birth
1950.

Short Biography
Chris van Tienhoven graduated in mathematics from Leiden University and has built a career as an entrepreneur working across information technology and graphic design. He also remained active in geometry. Central to his work is a lifelong habit of reducing complexity into simplicity and creating clear, durable structures. He values order, coherence, and long-term vision—principles. All of this eventually led to the creation of the Encyclopedia of Poly Geometry.

Themes, Interests, and Relevant Publications

- Lifelong interest in geometry, beginning in secondary school, with a special fascination for Van Aubel's Theorem.
- Developed the notion of Perspective Fields.
- Initiator of the systematic development and documentation of Quadri Geometry, later expanded into Poly Geometry.
- Founder of the online communities *Quadri Figures Group* and *Quadri and Poly Geometry Group*.
- Editor and compiler of the Annual Journals that collect and preserve the discussions and discoveries of these groups.
- Founder of the Encyclopedia of Poly Geometry (where all entries without external references originate from his own work).

Selected Publications

- Chris van Tienhoven, Dario Pellegrinetti, *Quadrigon Geometry: Circumscribed Squares and Van Aubel Points*. Journal of Geometry and Graphics, Vol. 25, No. 1, 2021.

Other Remarks
Website: www.chrisvantienhoven.nl
Biography: www.chrisvantienhoven.nl/header/biography/

Eckart Schmidt

Location
Living in Germany.

Year of Birth / Generation
1939.

Short Biography
Eckart Schmidt is a former teacher of mathematics and physics at a full-time secondary school, with a long-standing interest in geometry. His work spans several decades and includes numerous contributions to geometric constructions, classical geometry, and the study of n-gons and their transformations.

Themes and Interests

- Geometric constructions using CABRI

Selected Publications

- F. Bachmann & E. Schmidt: *n Ecke*. B.I. Hochschultaschenbuch 471/471a, Mannheim/Wien/Zürich, 1970.
- E. Schmidt: *Abbildungen und Klassen von n Ecken*. MNU XXV (1972), pp. 146–150ff.
- E. Schmidt: *Affin reguläre n Ecke und ihre regulären Komponenten*. MNU XXXIX (1986), pp. 193–198ff.
- E. Schmidt: *Mittelsenkrechtenvierecke eines Vierecks*. PM 2/44 (2002), pp. 84–88ff.
- E. Schmidt: *Circumcenters of Residual Triangles*. Forum Geometricorum 3 (2003), 125–134.
- J. Kühl & E. Schmidt: *Husumer Rechenhandschriften und Paul Halckes Mathematisches Sinnen Confect*. Mitteilungen der Mathematischen Gesellschaft in Hamburg XXIII/2 (2004), 111–156.
- E. Schmidt: *Geradenkonstellationen*. MNU 60/1 (2007), 28–29.
- E. Schmidt: *Billardvierecke eines Sehnenvierecks*. MNU 63/5 (2010), 267–269.
- Additional contributions on geometric constructions (see Themen and EQF-notes).

Additional Remarks

- Co-founder of the Encyclopedia of Poly Geometry and one of the principal contributors to QPG.
- Website: www.eckartschmidt.de

Francisco Javier García Capitán

Location
Priego de Córdoba, Andalucía, Spain.

Year of Birth / Generation
1963.

Short Biography
Francisco Javier García Capitán is a mathematician and long-time secondary school teacher with a strong interest in geometry, elementary mathematics, and computational approaches. He is an active explorer of barycentric coordinates and the author of *Baricentricas.nb*. His work bridges classical geometric insight with modern computational tools.

Themes and Interests

- Elementary mathematics
- Geometry
- Mathematica and hobby programming
- Barycentric coordinates

Selected Publications
International Journal of Geometry

- (with Paul Yiu) *Three mutually tangent congruent circles…*, 5 (2016), 15–18.
- *A structure on the circumcircle*, 10 (2021), 71–83.
- *Infinite points and isogonal conjugate*, 12 (2023), 127–134.
- *Isotomic conjugate and parallelism*, 12 (2023), 89–100.

Forum Geometricorum

- *Means as chords*, 8 (2008), 99–101.
- *Trilinear polars of Brocardians*, 9 (2009), 297–300.
- *Collinearity of the first trisection points…*, 11 (2011), 217–221.
- (with Ehrmann & Myakishev) *Construction of circles…*, 11 (2011), 261–268.
- (with Dergiades & Lim) *On six circumcenters…*, 11 (2011), 269–275.
- *Some simple results on cevian quotients*, 13 (2013), 227–231.
- *A simple construction of an inconic*, 14 (2014), 387–388.
- *Lemniscates and a locus…*, 15 (2015), 123–125.
- *Another construction of the Simson lines…*, 15 (2015), 173–176.

- *Locus of centroids of similar inscribed triangles*, 16 (2016), 257–267.
- *A Family of Triangles...*, 18 (2018), 79–82.

Additional Remarks

- Website: www.garciacapitan.epizy.com
- Blog: www.garciacapitan.blogspot.com

Keita Miyamoto

Location
Born and living in Japan.

Year of Birth / Generation
Millennial (approx. 1981–1996).

Short Biography
Keita Miyamoto spent part of his academic training at the Faculty of Science of Kyoto University, after which he continued developing his mathematical interests outside formal education. This independent path has shaped a broad, self-directed approach to geometry. His work reflects a deep curiosity for the foundations and structures of both classical and modern geometric systems.

Themes and Interests

- Euclidean geometry
- Projective geometry
- Spherical geometry
- Geometry in the Lobachevsky plane
- Comparative study of geometric systems

Publications and Contributions

- Explorations and discussions on Euclidean and non-Euclidean geometry.
- Contributions to community exchanges on projective and hyperbolic themes.

M@IMF (Handle)

Location
Japan

Year of Birth / Generation
Generation X (1965 – 1980)

Short Biography
M@IMF is an active contributor to the QPG community and prefers to work anonymously.

Themes and Interests

- Geometry
- Algebraic formulations
- Problem exploration
- Computational experimentation

Stanley Rabinowitz

Location
Living in New Hampshire, USA.

Year of Birth / Generation
1947 (Baby Boomer).

Short Biography
Stanley Rabinowitz is a retired computer programmer with a Ph.D. in Mathematics. Throughout his career he has combined computational thinking with a deep appreciation for classical mathematics, particularly geometry, combinatorics, and number theory. He is the founder and sole proprietor of *MathPro Press*, a small but influential publishing house dedicated to high-quality mathematics problem books, indexes, and reference materials used by educators, problem solvers, and researchers worldwide.

Themes and Interests

- Classical Euclidean geometry
- Problem creation and problem solving
- Combinatorics and number theory
- Mathematical indexing, bibliographic work, and reference compilation
- Computational approaches to mathematical problems

Publications and Contributions
Stanley Rabinowitz enjoys creating elegant and challenging mathematics problems, especially in Euclidean geometry. He is the author of the *Index to Mathematical Problems 1980–1984*, a widely used reference work that reflects his long-standing commitment to organizing and preserving mathematical problem literature. Through MathPro Press, he has contributed to the accessibility of problem-solving resources and supported the broader mathematical community with carefully curated publications.

Selected Publications

- *Algorithmic Manipulation of Fibonacci Identities*, in *Applications of Fibonacci Numbers*, Volume 6, ed. G. E. Bergum et al., Kluwer Academic Publishers, Dordrecht, 1996, pp. 389–408.
- *Arrangement of Central Points on the Faces of a Tetrahedron*, International Journal of Computer Discovered Mathematics 5 (2020), 13–41.
- *A Computer Algorithm for Proving Symmetric Homogeneous Triangle Inequalities*, International Journal of Computer Discovered Mathematics 7 (2022), 30–62.
- *The Shape of Central Quadrilaterals* (with Ercole Suppa), International Journal of Computer Discovered Mathematics 7 (2022), 131–180.

- *Relationships between a Central Quadrilateral and its Reference Quadrilateral* (with Ercole Suppa), International Journal of Computer Discovered Mathematics 7 (2022), 214–287.

Additional Remarks

Website: www.stanleyRabinowitz.com

Quang Hung Tran

Location
Born and working in Hanoi, Vietnam.

Year of Birth / Generation
Millennial (approx. 1981–1996).

Short Biography
Quang Hung Tran graduated in Mathematics from the University of Science, Vietnam National University, Hanoi. He is a mathematics teacher at the High School for Gifted Students, VNU University of Science, where he has devoted his career to educating and mentoring mathematically talented students. His primary interest lies in Euclidean geometry, especially in the context of mathematical olympiad training, while his broader research spans higher-dimensional and non-Euclidean geometry, the geometry of the Golden ratio and Fibonacci sequences, and the aesthetic, historical, and logical aspects of mathematics. Outside his academic work, he values family life and enjoys reading and spending time with his two sons.

Themes and Interests

- Euclidean geometry
- Mathematical olympiad problems and gifted student education
- Classical geometric inequalities and triangle geometry
- Notable points, circles, and projective methods (harmonic division, isogonal conjugation)
- Higher-dimensional Euclidean geometry
- Non-Euclidean geometry
- Golden ratio and Fibonacci-related geometric structures
- Aesthetic, historical, logical, and recreational mathematics

Selected Publications (Representative)

- *A Napoleon-like theorem for quadrilaterals*, American Mathematical Monthly, 2022.
- *Another Simple Proof of Pascal's Theorem*, Mathematics Magazine, 2023.
- *A generalization of the Pythagorean theorem via Ptolemy's theorem*, Mathematics Magazine, 2023.
- *A Generalization of de Gua's Theorem with a Vector Proof*, The Mathematical Intelligencer.
- *A family of weighted Erdös–Mordell inequality and applications*, Journal of Geometry, 2021.

- *Some strengthened versions of Klamkin's inequality and applications*, Geometriae Dedicata, 2021.
- *A synthetic proof of the Morley trisector theorem using congruent and similar triangles*, Elemente der Mathematik, 2025.
- *A generalisation of Sylvester's theorem with an application*, The Mathematical Gazette, 2025.
- Tran, Q. H. & Herrera, B., *n-Dimensional Generalizations of a Thébault Conjecture*, Mathematical Notes, 2024.
- *A Generalized Volume Formula for Tetrahedra with Congruent Facet Pairs*, The Mathematical Intelligencer, 2025.

Additional Remarks

He is deeply interested in the geometry of quadrilaterals—whether viewed as configurations of four lines, four points, or four angles—and in polygonal geometry more broadly. He notes that as one moves to higher-order polygons, the complexity of problems increases dramatically. Within this rich field, he is delighted and honored to have contributed to the development of the nL–n–Tf1: nL–Orthopole, documented at:
www.chrisvantienhoven.nl/epg/n-geometry/ngeom/nl-n-tf1/

Seiichi Kirikami (1949?–2023)

Location
Japan.

Year of Birth / Generation
Exact year unknown; passed away on 11 December 2023.

Short Biography
In daily life Seiichi worked as a mechanical engineer in the Thermal Power Division of Hitachi, Ltd. In his free time he enriched the geometry community with original ideas, elegant constructions, and generous participation in many collaborative discussions. From 2013 to 2018, Seiichi contributed intensively to the development of Quadri- and Poly-Geometry within the Quadri-Figures Group. His insights, constructions, and discussions were instrumental in the group's formative years, and his contributions helped define several of the key geometric notions that emerged in those years. Later on he contributed extensively to other groups such as Anopolis, Hyacinthos, Romantics of Geometry, ADGEOM, and the Encyclopedia of Triangle Centers (ETC), where many of his ideas became foundational. His work was characterized by simplicity, depth, and a unique ability to see geometric structures from unexpected angles. He was also known for his humility, kindness, and willingness to help others — qualities remembered fondly by colleagues and friends.

Themes and Interests

- Euclidean and projective geometry
- Triangle geometry and classical configurations
- Geometric problem creation and exploration
- OEIS contributions and combinatorial structures
- Applied mathematics, including epidemiological modelling
- Engineering and turbine-related innovations (patents)

Selected Contributions
Seiichi Kirikami's geometric ideas inspired many theorems, conjectures, and new terminology. Among the most notable:

- The *Kirikami six-circles configuration*, which led to the Hatzipolakis–Moses Theorem.
- His prompting led to the introduction of the term *Cyclologic*, now established in triangle-geometry terminology.
- He suggested naming the line through $X(5)$ perpendicular to the Euler line the *Hatzipolakis axis*.
- Numerous contributions to ETC, Hyacinthos, and other geometry forums.
- 29 OEIS entries associated with his name.

- Several patents in turbine-engine technology.
- Publications in epidemiology, including COVID-related infection-spread modelling.

Community Tributes
Colleagues remembered Seiichi with deep affection:

- “Geometry is the poorer of his death.” — A. P. Hatzipolakis
- “He was a great expert in projective and triangle geometry, and a very kind and helpful person.” — E. Suppa
- “He had a way of seeing things from a different angle and presenting them simply.” — C. van Tienhoven
- “He inspired many of my problems.” — A. Altıntaş
- “He wrote many interesting and innovative messages in ADGEOM.” — F. J. García Capitán
- “He helped me in my beginnings with wise and generous advice.” — C. E. Lozada
- “He contributed to OEIS and had important work in epidemiology.” — P. Moses

Additional Remarks
Seiichi Kirikami is remembered as a gentle, insightful, and generous geometer whose ideas continue to inspire new discoveries. His legacy lives on in the many theorems, concepts, and geometric structures that bear his influence.

3 Subjects

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- **subject: New construction of a general cubic**:
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- **subject: Some short pdf files on rational geometry**:
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4 Messages

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4.2 Messages

Message: #2052
Date: 2024-01-01
From: bernard.keizer@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

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```
Dear Chris, dear Eckart
Back from Norway, I find Chris messages 2045 and 2047, which
I've immediately printed and begun to study.
It's a fantastic progress in our knowledge of properties of
general cubics.
First reactions:
1) it seems a general cubic can be monopartite or bipartite with
1 real asymptote, tripartite with 1 or 3 real asymptotes or
quadripartite with 3 real asymptotes.
In the examples, the cases 1,2, 4 and 5 are drawn, but the 3rd
is lacking.
Is it correct that the cases 2, 4 and 5 are always pivotal
isocubics? (4 anallagmaty centers and 3 Moebius centers)
What are the cases 1 and 3? (only 2 anallagmaty centers and 1
Moebius center) the QA is reduced to a segment ...
2) which projective transformation transforms the general cubic
in a circular cubic?
3) I dream of a return to EQF with an application of these
properties to QA general pivotal isocubics (pivot not in QA-P4)
or to QL-Cu2 (cubic stelloïd not circular with 3 foci in the
same point QL-P1)
Best regards
Bernard
```

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Message: #2053
Date: 2024-01-02
From: bernard.keizer@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

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Dear Chris, dear Eckart
Again congratulations to Chris for these last memos with beautiful properties!
The QF-conic, the foci being the vertices of the asytriangle ...
Perhaps the property that the point Z, center of the diametral conic is on the QF-conic could be mentionned
(the asymptotes of this diametral conic are ZQ, asymptote of the cubic and ZF).
Is it also correct that that the point N, middle of the segment of 2 centers of anallagmaty lies on the QF-conic?
This property was mentionned by Eckart, but I don't see an explanation ...
Basic question for both of you: how are you drawing your different general cubics?
I know with Geogebra only one way, it is drawing a curve through 9 identified points.
Best regards
Bernard
Perhaps it would be nice of Chris and useful for us to merge it's 3 basic memos #2026, #2045 and #2047.

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Message: #2054
Date: 2024-01-03
From: eckart_schmidt@t-online.de
Subject: Flexpoints and -lines for cubics

```
Dear Bernard, dear Chris,

studying Schröter's book, I start once more (see already #2050)
... an excursion in flexpoints and -lines for cubics.

Relevant for the collinear real flexpoints W1,W2,W3 will be
... Schröter's "harmonic polar axes" (page 242),
... already described as L1,L2,L3 in #2050,
... with a common point P,
... intersecting the cubic in P1,P2,P3
... with tangentials in the flexpoints, WiPjPk collinear.,
... tangents in Pi and Pj at the cubic intersect on Lk.

For a x-partite cubic take P1,P2,P3
... on the part with one asymptote
... (attached a bipartite example).

A cubic has 9 flexpoints,
... three are real W1,W2,W3 and give a real flexline,
... one of 12 flexlines, which give four flextrilaterals,
... one real but not with line W1W2W3,
... which is part of another flextrilateral
... with opposite point P (page 245),
... so there are four real flexlines, one W1W2W3,
... bearing all 9 flexpoints.

Perhaps the real flextriangle is one of the followings:
Consider triangles A1A2A3
... with A1 on L1, A2 = L2^A1W3, A3 = L3^A2W1,
... for all these triangles the trilinear polar of P is W1W2W3,
... the triangle P1P2P3 is of this type,
... further the trilateral of the flextangents Tg1,Tg2,Tg3, ...

Let us enrich the constellation
... with 3X2 CU-intersections S of PWi,
... which are coconic on a S-conic.
The polar axis of Wi wrt the cubic or wrt the S-conic is Li.
The polar axis of P wrt the S-conic is W1W2W3.
P, Wi lie harmonic wrt the two S-points on PWi.
Flextangents Tgi, Tgj intersect in Tk on Lk.
The 4th harmonic of Wi wrt Tj,Tk is the intersection of Li and
TjTk.

There will be more observations, it is amazing,
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... what three flexpoints arrange on the cubic!

Best regards Eckart
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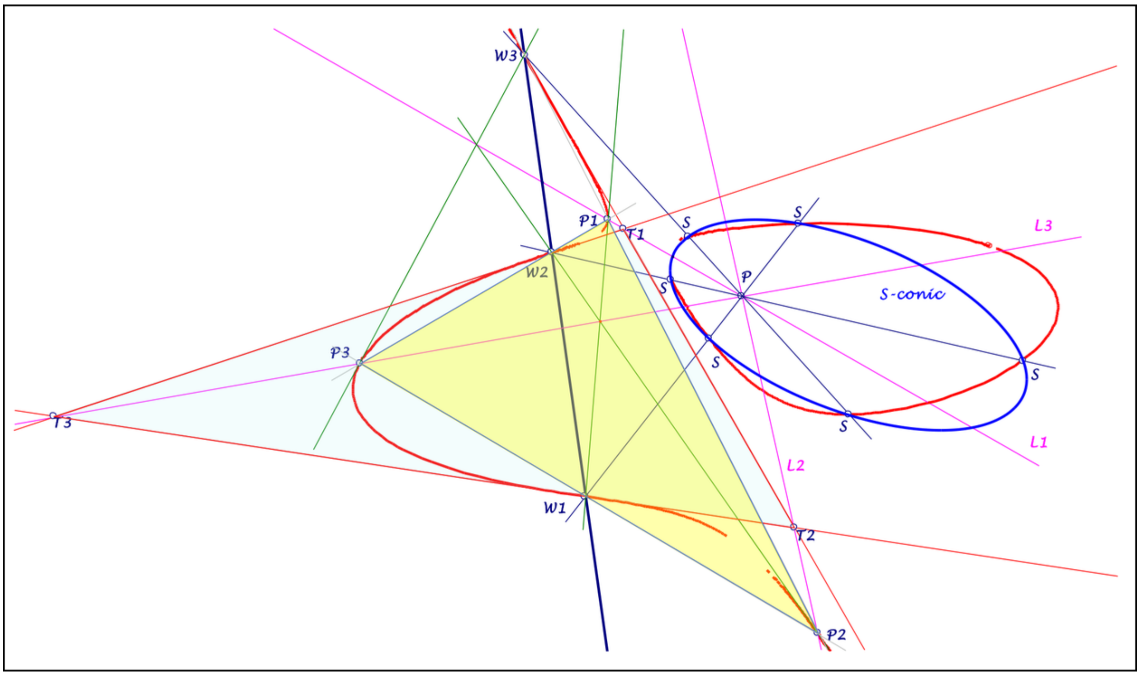

2024-01-03.pdf

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Message: #2055
Date: 2024-01-04
From: van10hoven@gmail.com
Subject: Re: Anallagmaty and Moebius

Dear Eckart,

Wrt your message #2041. You asked “Is there a relationship between these two transformations?” between the QMT and the Isoconjugation, where you also referred to #2035.

First of all, thank you very much for sharing the beautiful idea of Roland Stärk in QPG#2035 and your addition to it.
It made it clear for me that QA-Tf2 is an Isoconjugation wrt QA-Co1 circumscribing the QA-Diagonal Triangle.
I wrote at the end of my message QPG#2039 that *for any point P at CU* having 4 contact points of its tangents to CU, there is a pivotal QA-Tf2-transformation. Now I understand that this is an Isoconjugation.
Having said that, this implies that the transformation is not a general CU-transformation but a CU_P-transformation, dependent on P on CU.

Then your question if there is a relationship between a CU-QMT and an Isoconjugation.

Here are some remarks:
· There are 3x3=9 potential QMT-transformations on the general cubic and there are infinite many QA-Tf2 transformations on the general cubic.
· Nevertheless there is for every Infinity point IP1, IP2, IP3 also a QA-Tf2 and that one has some correlation with the QMT-transformations. Let’s call this transformations QA1-Tf2, QA2-Tf2, QA3-Tf2, referring to the 3 Quasi-Anallagmatic QA’s per infinity point: QA1, QA2, QA3. It appears that the QA-Tf2 transformations for this QA’s have this special property: QA1-Tf2(P)=P.IP1, QA2-Tf2(P)=P.IP2 and QA3-Tf2(P)=P.IP3 (Fred Lang’s notation).
· There are 3x3 QMT-transformations, some of them imaginary. Let’s call them QMT1a, QMT1b, QMT1c, QMT2a, QMT2b, QMT2c, QMT3a, QMT3b, QMTc, the numbers referring to IP1, IP2, IP3. We know every QMT1x-transformation is the sequence (P.M1x).IP1 or (P.IP1).M1x (Fred Lang’s notation). Therefore QMT1a(P)= (P.IP1).M1a = QA1-Tf2(P).M1a, etc.
· So the QAi-Tf2 transformations are part of all QMTix-transformations.
· I do not know about isoconjugations other than QA-Tf2, how they relate.

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*Then some special extra findings:*
1. QMT1a(QMT1b(QMT1c(P))) = P, etc.
2. QA1-Tf2(P) = Asy-1-Parallel-Point(P), etc.
3. QA-Co1 of QA1 = QF-conic relating to IP1, etc.

Maybe some of these items are already known.

Best regards,
Chris
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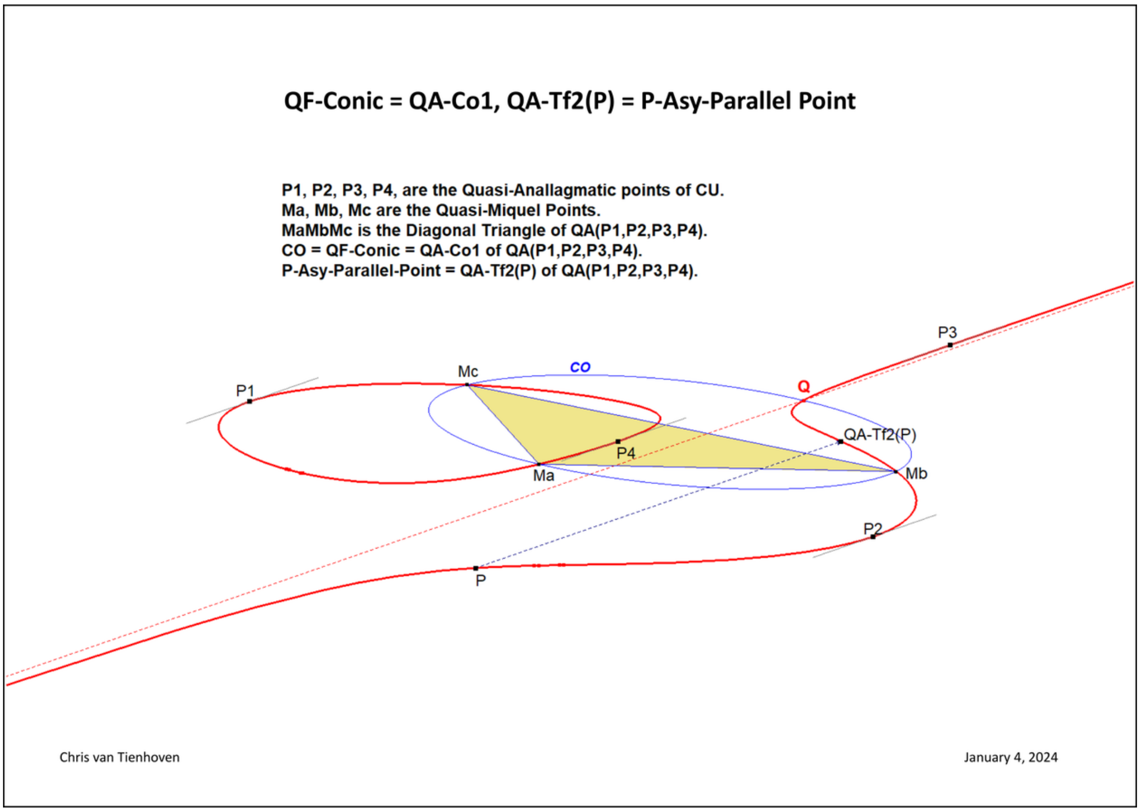

CU2-QF-Conic is QA-Co1 on Bipartite Cubic-01.pdf

Message: #2056
Date: 2024-01-04
From: bernard.keizer@gmail.com
Subject: Re: Flexpoints and -lines for cubics

```
Dear Eckart,
I'm very glad that you find new gems in Schröter's book!
It was a beautiful Christmas present.
Best regards
Bernard
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Message: #2057
Date: 2024-01-04
From: bernard.keizer@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

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```
Dear Chris, dear Eckart
Using all properties mentionned by Chris in his beautiful and
useful memo, I finally succeeded in drawing mono- and bipartite
circular or non circular cubics.
Let's give the points P1,P2, their middle N and a point P on
P1P2.
1) circular cubics
The perpendicular in M to P1P2 cuts the circle centered in N
through P1 and P2 in F1 and F2.
The 3 circles centered in P1, P2 and M through F1 and F2 are the
inversion circles C1 and C2 of 2 anallagmatic transformations
and the inversion circle C3 of the Moebius transformation
centered in M and swapping P1 and P2.
Let's now give a point Q as point where the cubic will cut it's
asymptote; the circle through M, N and Q is the circle C4,
QF-circle of Chris.
The focus F is the diametral point of Q on this circle.
For any point X, the inverses of X in the 2 circles are X1 and
X2 and the inverse of X1 in C2 or of X2 in C1 is Y, the Moebius
partner of X.
X, X1, X2 and Y are concyclic on a circle C5 centered on F1F2
and orthogonal to the 3 circles C1, C2 and C3.
This in in particular true for Q, it's 2 inverses Q1 and Q2 and
it's Moebius, which we name M'. C4 passes through the middles of
P1Q1 and P2Q2.
The asymptote of the searched cubic will be the parallel through
Q to MM', which cuts the circle C4 in Z.
The diametral conic, locus of the middles of the segments on
asyparallels is centered in Z and pass through P1, P2 and m, the
middle of MM'.
We are now able to find the points Q1' and Q2' such as the
middles q1 and q2 of Q1Q1' and Q2Q2' are on this diametral
conic.
The cubic through P1, P2, M, Q, M', Q1, Q2, Q1' and Q2' is the
searched circular cubic.
It is for any of it's points X a circular non focal cubic of the
QL P1P2XX1X2Y.
Variing Q gives mono- or bipartite circular cubics.
The polar conic of F wrt this cubic is a circle C6.
All this is well known or obvious, but the interest is to repeat
the construction with conics instead of circles.
2) non circular cubics
```

a) giving 2 points X and Y as QMT partners gives the points X1, intersection of P1X and P2Y and X2, intersection of P1Y and P2X.
Then we know how to draw 3 conics centered in P1, P2 and M, the 2 conics of QAT Co1 and Co2 and the conic of QMT Co3.
The 3 conics are similar with parallel axes and intersect in F1 and F2.
b) giving a point Q, the inverse of Q wrt Co1 is Q1, the inverse of Q wrt Co2 is Q2 and the inverse of Q1 wrt Co2 or of Q2 wrt Co1 is M'.
The conic Co4 or QF-conic of Chris passes through Q, M ,N and the middles of P1Q1 and P2Q2;the diametral point of Q is F.
The asymptote is the parallel to MM' through Q, which cuts the conic Co4 in Z.
The diametral conic is centered in Z with asymptotes ZQ and ZF and passes through P1, P2 and the middle m of MM'.
We have the same way Q1' and Q2'.
The cubic through the same 9 points is the searched non circular cubic.
It is in particular for any of it's points X a non circular circumcubic of the QL P1P2XX1X2Y.
Variing the point Q gives mono- or bipartite non circular cubics.
Adding the points X' and Y', inverses of X and Y wrt Co3, the conic Co5 passes through X, X1, X2, Y, X' and Y'.
It is centered on F1F2 and quasi orthogonal to the 3 conics Co1, Co2 and Co3 (the tangents from P1, P2 and M to Co5 are on Co1, Co2 and Co3).
The polar conic of F is the conic Co6.
The conics Co1 to Co6 are all similar with parallel axes.
Best regards
Bernard

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Message: #2058
Date: 2024-01-05
From: van10hoven@gmail.com
Subject: Re: Anallagmaty and Moebius

Dear Eckart, dear Bernard,

Did you find any Isoconjugate other than QA-Tf2, which we know from Triangle Geometry of Quadri Geometry, that is applicable for the general cubic.

This could bridge the gap with Bernard Gibert's CTC and Clark Kimberling’s ETC.

An interesting link in this regard is Singular Focus of Circular Cubics in Bernard Gibert's CTC (https://faculty.evansville.edu/ck6/encyclopedia/SingularFocusOfCubics.html).

Best regards,

Chris

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Message: #2059
Date: 2024-01-05
From: eckart_schmidt@t-online.de
Subject: Re: Anallagmaty and Moebius

```
Dear Chris,

wrt the question in #2058 perhaps of interest:
In #2054 you find a point P,
... intersection of the harmonic polar axes L1,L2,L3
    of the flexpoints,
... which intersect the cubic in points P1,P2,P3,
... (for x-partite cubics take the points
     not on the closed part).
The cubic is invariant wrt an isoconjugation
... with reference triangle P1P2P3 and fixed point P.

I just observed this property
... and hope, you can confirm it.

Best regards Eckart

PS: Every isoconjugation is QA-Tf2 of the QA of its fixed
points.
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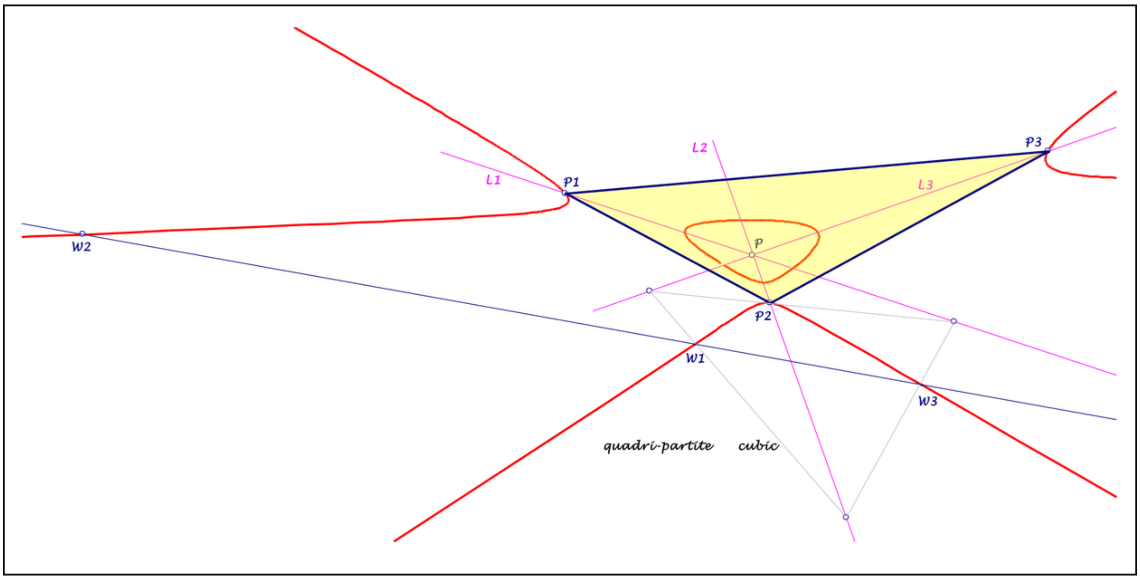

2024-01-05.pdf

Message: #2060
Date: 2024-01-05
From: bernard.keizer@gmail.com
Subject: General cubic

Dear Chris, dear Eckart
I begin to be lost with the over sixty messages of the item started with QAT and QMT ...
So I suggest respectuously to stop this item and start a new one.
For Eckart: what is a pivotal QMC-cubic? (messages 2042 and 2043)
For both of you; how do you draw your cubics? (question I asked in another item, but to which you didn't answer).
What I need now badly is a summary of the properties of the general cubic found so far ...
Best regards
Bernard

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Message: #2061
Date: 2024-01-05
From: bernard.keizer@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

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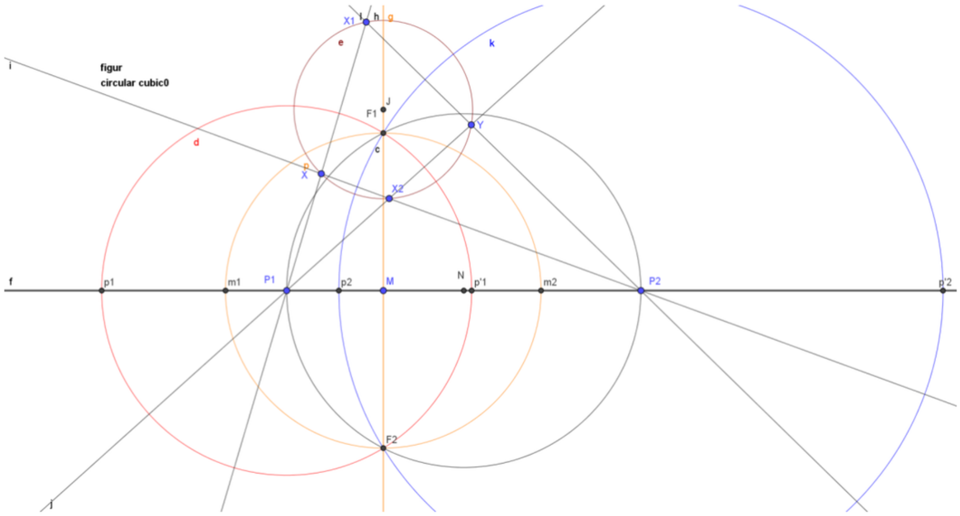
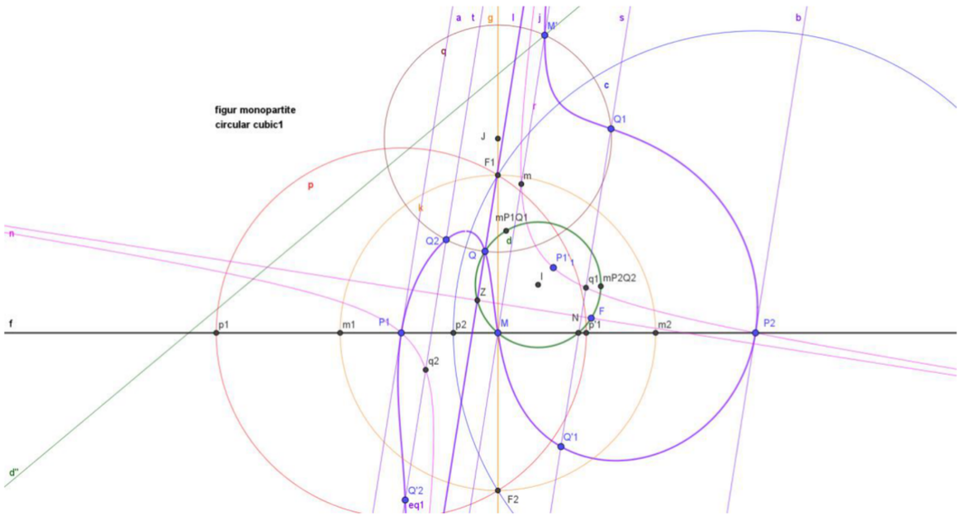
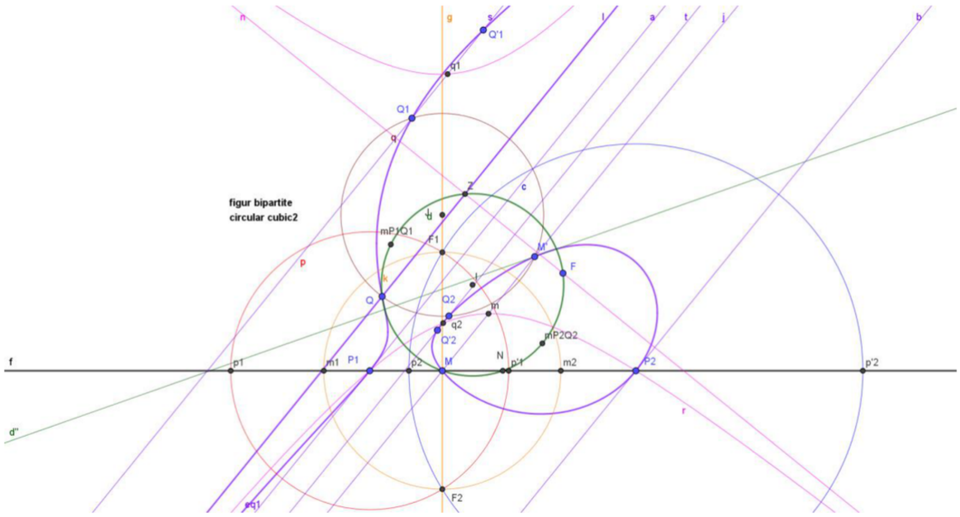
Dear Chris, dear Eckart
Before starting the same way the construction of circular and non circular cubics with 4 points P1, P2, P3 and P4, I realize that I forgot to put the attached file to my previous message.
With my apologies
Best regards
Bernard

Construction of circular cubics

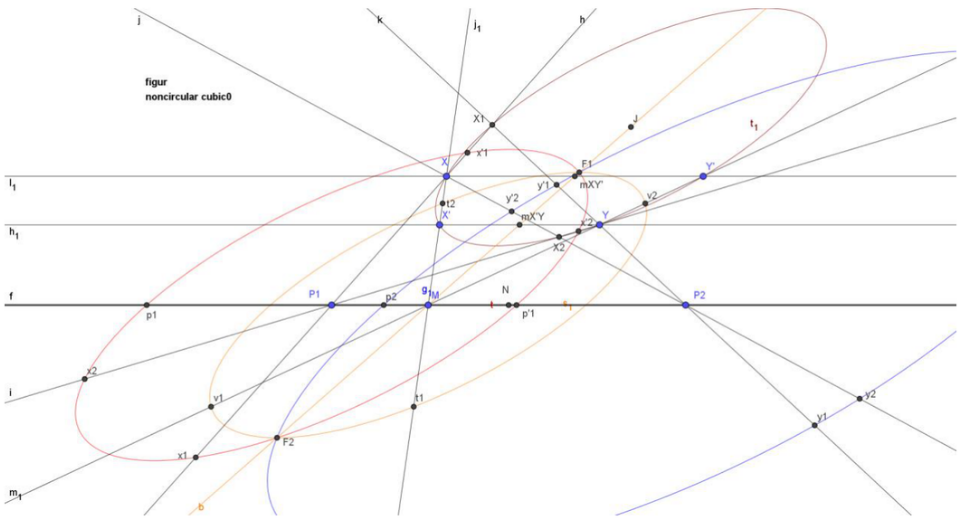




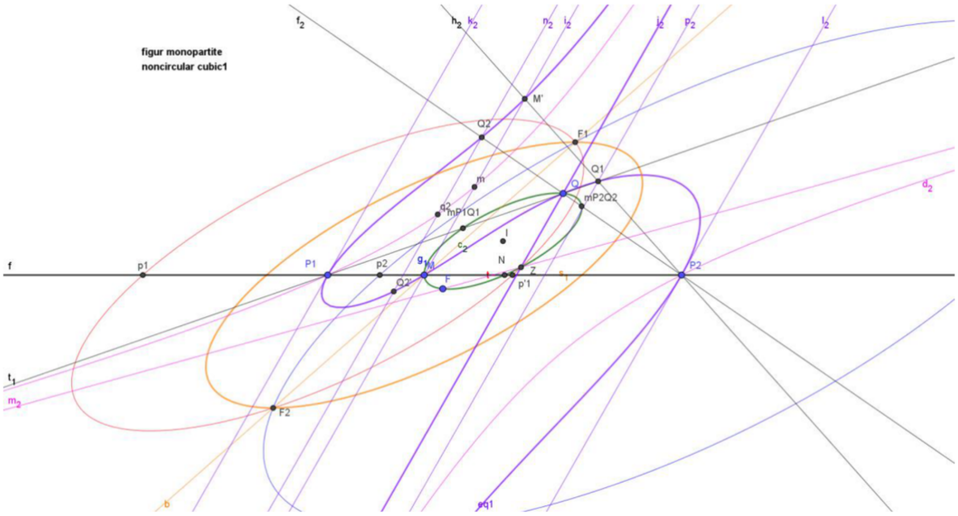
Circular and non circular cubics.pdf

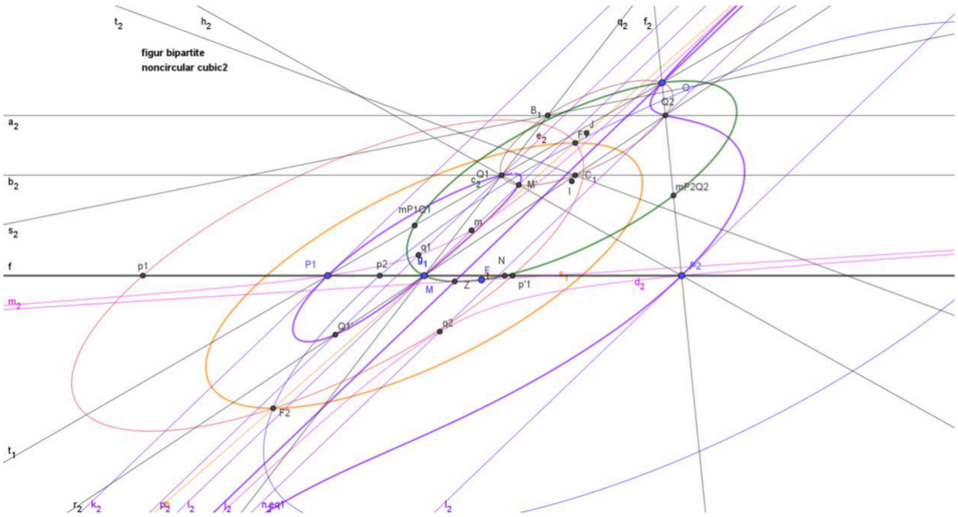


Construction of non circular cubics



Circular and non circular cubics.pdf





Circular and non circular cubics.pdf

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Message: #2062
Date: 2024-01-05
From: eckart_schmidt@t-online.de
Subject: Re: General cubic

```
Dear Bernard,

excuse if I don't answer ad hoc to your messages,
... but I need time for my own research activities,
... not able to handle all properties of cubics
... and get an informed overview.
Wrt your questions:
To draw a cubic I use a macro for given 9 points,
... time ago published by Chris in an old message.
QMC is a shortcut of "Quasi-Möbius-Conjugate".

Best regards Eckart
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Message: #2063
Date: 2024-01-05
From: van10hoven@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

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Dear Bernard,

I am glad that you're as excited as I am with the QF-conic and the diagram!
I will answer now your questions in #2052 and #2053.

#2052
1) Is it correct that the cases 2, 4 and 5 are always pivotal isocubics? (4 anallagmaty centers and 3 Moebius centers)
What are the cases 1 and 3? (only 2 anallagmaty centers and 1 Moebius center) the QA is reduced to a segment ...
You describe these types of cubics:
1. monopartite or
2. bipartite with 1 real asymptote,
3. tripartite with 1 asymptote
4. tripartite with 3 real asymptotes
5. quadripartite with 3 real asymptotes.

I think this are the basic shapes of cubics we encounter. Nevertheless, there is another aspect influencing the shape significantly, namely whether the asymptote intersects the cubic into infinity or not. This determines if the QF-conic degenerates or not and -I think- also the number of Quasi-Miquel points.

I don't know if it is correct that the cases 2, 4 and 5 are always pivotal isocubics. I suppose Eckart better can answer this question.
I think it is a good idea to make a table with these types of cubics and their properties:
· How many real asymptotes
· Per real asymptote whether there is a finite Q
· Per real asymptote how many real quasi-anallagmatic points (2 or 4)
· Per real asymptote how many real quasi-Miquel points (1 or 3)
· Is it a pivotal Isocubic?

This will show us how things relate per type of cubic.
I made a first start with several open ends. See attachment.
Remarks and additions are very welcome!
I will also attach an updated version of the diagram "Basic structure of the General Cubic CU".
Can you send me an example of a tripartite cubic with one asymptote?

2) “which projective transformation transforms the general cubic in a circular cubic?”
That very question was the basic question for me three months ago.
Ten years ago I studied projective transformations in order to be able to develop the theory of perspective fields.
There are several types of projective transformations:
1. A1B1C1D1 to A2B2C2D2, meaning that QA-1 is projected in QA-2 with different form. All other involved items are projected accordingly. There are 3 fixed points for this projective transformation in the plane that can be constructed I found out in a very interesting way.
2. ABCP1 to ABCP2, meaning that you choose the 3 fixed points of the transformation and another point changes position (P1 à P2). Again all other involved items can be projected accordingly.
I tried both projection types extensively, but to no avail. However it gave me the idea of the Quasi-Moebius Transformation. Then you came with the idea of the 2 x QAT (quasi anallagmaty) and Eckart found that the conics involved had parallel axes. Then I turned to the following type of projective transformation:

3. The “stretch”. Types 1 and 2 have the disadvantage that when you transform a conic with its center, that the transformed center no longer is the center of the transformed conic. The only way to bypass this problem, is by stretching the conic sideways in the direction of an axis. I attach a picture of the method.
This “stretch” can be applied for every circle in the CUc-environment.
What actually happens is that a circle is a conic (A,B,C,CI1,CI2), where CI1,CI2 are the Circular points at Infinity. After a “stretch” it becomes another conic (A’,B’,C’,CI1’,CI2’). Every circle undergoes this transformation CI1 à CI1’, CI2 à CI2’. Therefore there are so many similar and equally directed conics, because in the CUc-environment they are circles.
We might rename (CI1’,CI2’) with (SI1,SI2), where SI stands for Similarity points at Infinity.
It is not surprising that SI1 and SI2 are the imaginary infinity points of a monopartite or bipartite cubic. When the cubic is tripartite or quadripartite, then CI1’ and CI2’ become real infinity points (IP2 and IP3) and we have two extra asymptotes.
For the same reason a conic with the infinity points of a monopartite cubic on it will be an ellipse and a conic with the infinity points of a tripartite cubic or quadripartite cubic on it will be a hyperbola.

This also tells us that similar ellipses with equally directed axes share the same imaginary points at infinity. A bit strange to talk about infinity points on an ellipse, but they are imaginary after all.

3) “I dream of a return to EQF with an application of these properties to QA general pivotal isocubics”
That can be very interesting. I am looking forward to it. It also will be very interesting to apply our new items on the cubics of Bernard Gibert.

#2053
1. Perhaps the property that the point Z, center of the diametral conic is on the QF-conic could be mentioned
(the asymptotes of this diametral conic are ZQ, asymptote of the cubic and ZF).
True, beautiful property! It also lies on the related asymptote.
2. Is it also correct that that the point N, middle of the segment of 2 centers of anallagmaty lies on the QF-conic?
This property was mentioned by Eckart, but I don't see an explanation ...
Yes, right, that is also the cause that the QF-Conic is the 9-point-conic QA-Co1 of the 4 centers of anallagmaty!
3. “How are you drawing your different general cubics?”
What I basically do is pinpointing 9 reference points and so creating a 9P-cubic and then rearrange them until I find a cubic that suits me.
4. “Perhaps it would be nice of Chris and useful for us to merge it's 3 basic memos 2026, 2045 and 2047.”
What do you exactly expect in this?

In any case, I write a description of all the elements that we discover as a trio. It is a manuscript that is constantly edited and now has approximately 50 pages. In due course I may create a new encyclopedia of Cubic elements. But that is very labor intensive.

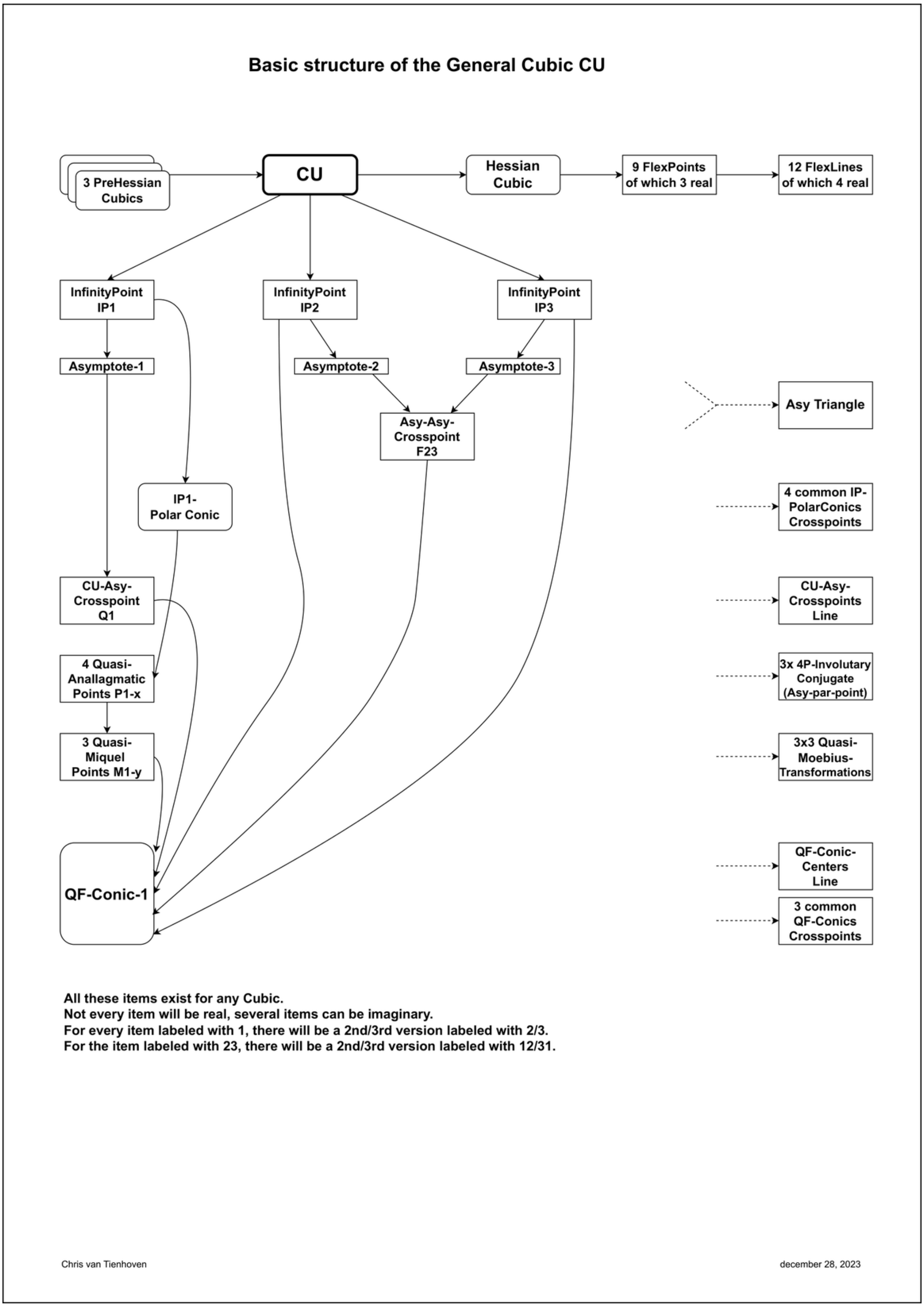
So far today.
Best regards,
Chris

TYPES OF CUBICS

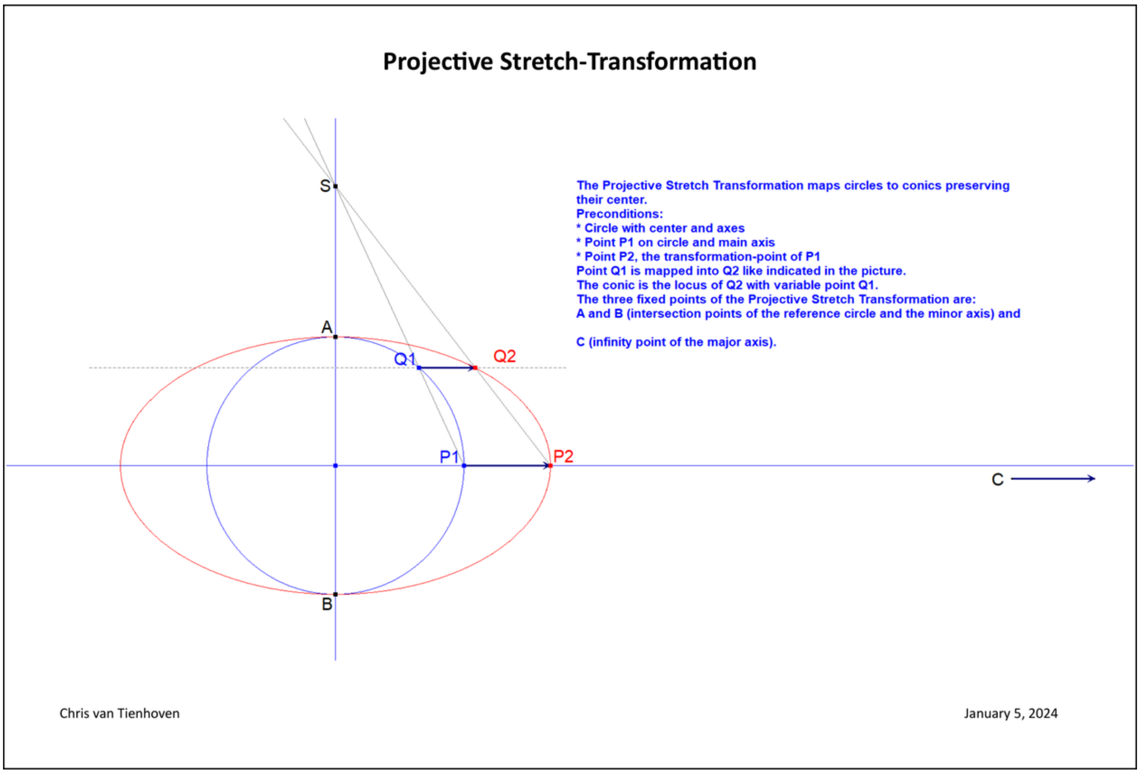
		Real Inf.Pts	Im.Inf.Pts	Real Asy's	CU-Asy-Crosspoints	Diametral Conic(s)	QF-Conic(s)	Anallagmatic Points	Miquel-points	Isocubic?	Comment
CUc1	**Monopartite Circular Cubic**	IP1	CI1,CI2	Asy1	Q1 finite	Hyperbola (P1,P2,IP1)	Circle (Q,F,M,Ce)	P1,P2	M	???	
CUc2	**Bipartite Circular Cubic**	IP1	CI1,CI2	Asy1	Q1 finite	Hyperbola	Circle	P1,P2,P3,P4	M1a,M1b,M1c	???	
CU1	**Monopartite cubic**	IP1	SI1,SI2	Asy1	Q1 finite	Hyperbola (P1,P2,IP1)	Ellipse (Q,F,M,Ce)	P1,P2	M	???	P1,P2,M collinear
CU2	**Bipartite cubic with 1 real asymptote,**	IP1	SI1,SI2	Asy1	**Q1 finite**	Hyperbola	Ellipse	P1,P2,P3,P4	M1a,M1b,M1c	???	
					Q1 infinite	Asy + 3P-Anall.Line	Inf.Line + 3P-Anall.Line	3 finite/1 infinite P's	2 finite/1 infinite M's	???	P1,P2,P3 collinear
CU3a	**Tripartite cubic with 1 asymptote**	???	???	???	???	??	???	???	???	???	
CU3b	**Tripartite cubic with 3 real asymptotes**	IP1,IP2,IP3	--	Asy1,Asy2,Asy3	**Q1,Q2,Q3 finite**	Hyperbola	Hyperbola (IPi,IPj)	how many?	how many?	???	
					Q1,Q2,Q3 infinite	Hyperbola	Inf.Line + MP-Anall.Line	how many?	how many?	???	
					1 or 2 Q's Infinite?	Hyperbola	How ?	how many?	how many?	???	
CU4	**Quadripartite cubic with 3 real asymptotes**	IP1,IP2,IP3	--	Asy1,Asy2,Asy3	**Q1,Q2,Q3 finite**	Hyperbola	Hyperbola (IPi,IPj)	how many?	how many?	???	
					Q1,Q2,Q3 infinite	Hyperbola	degenerated?	how many?	how many?	???	
					1 or 2 Q's Infinite?	Hyperbola	How ?	how many?	how many?	???	

Real Inf.Pts *Real Infinity Points of the Cubic*
IP1,IP2,IP3 *Infinity Points of the Cubic*
Im.Inf.Pts *Imaginary Infinity Points on the Cubic*
CI1,CI2 *Circular points at Infinity*
SI1,SI2 *Similarity points at Infinity*
Asy1,Asy2,Asy3 *Asymptote-names*
CU-Asy-Crosspoints *Intersection Points of the Asymptote with the Cubic*
Q1,Q2,Q3 *Names of the Intersection points of the cubic with one of its asymptotes*
Asy *Asymptote*
3P-Anall.Line **3P-Anall.Line** *Line through 3 collinear Anallagmatic Points*
MP-Anall.Line *Line through 1 Anallagmatic Point and 1 Miquel Point*
Inf.Line *Line at infinity*
Anallagmatic Points *Points where lines parallel to the asymptote touch the Cubic*
P1,P2,P3,P4 *Anallagmatic Points*
Miquel-points *Transformation Centers on the cubic*
Isocubic *Cubic with Isoconjugation property*

CU-Types of Cubics-Table-01.pdf



CU-BasicStructure-03.pdf



Projective Stretch-Transformation-01.pdf

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Message: #2064
Date: 2024-01-06
From: van10hoven@gmail.com
Subject: Re: Anallagmaty and Moebius

Dear Eckart,

Wrt your statement in #2059:
"The cubic is invariant wrt an isoconjugation with reference triangle P1P2P3 and fixed point P."

How can I check this? According to you P is a fixed point. Are there other fixed points?
What is the construction of the isoconjugation?
I also made a picture with W1,W2,W3, L1,L2,L3, P1,P2,P3 and P.
Why the choice "for x-partite cubics take the points not on the closed part"?
In another picture I noted that when drawing the Hessian too, that the Hessian has identical lines L1,L2,L3.

Best regards,
Chris

Ps I think too that we should observe the topics of our messages should be correct. It helps us and others to find related messages.

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Message: #2065
Date: 2024-01-07
From: eckart_schmidt@t-online.de
Subject: Re: Anallagmaty and Moebius

Dear Chris,

wrt your message 2064:
If you made a picture with W1,W2,W3, L1,L2,L3, P1,P2,P3 and P,
... you can test the isoconjugation as follows:
Consider the QA of P and its anticevians wrt P1P2P3
... and use QA-Tf2 (see the PS of my #2059).
Wrt the points Pi not on the closed part of the cubic,
... it is an observation, that I cannot prove up to now.

Best regards Eckart

PS: What about your PS? Is there something not correct in my message?
Is it not allowed, to send observations in drawings, but without prove?

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Message: #2066
Date: 2024-01-07
From: bernard.keizer@gmail.com
Subject: Re: Singular Focus, QF-conic, F-Polar Conic for a general cubic

---

Dear Chris,
Thanks for your long answer and many thanks for the table with the types of cubics.

1) Beginning with the drawing, I use personally the command Geogebra ImplicitCurve through 9 points, but I try to find 9 identified points.
Up to now I know 3 methods for pivotal isocubics:
a) a QA P1P2P3P4 with it's DT T1T2T3 and a pivot P give already 8 points, but the cevian triangle of P wrt DY gives 3 more points. Varying P gives different types of cubics.
b) using the same QA, but a pivot in an infinity point (parallels through T1, T2 and T3 give the cevian u1u2u3) gives cubics with the QA of contact points in P1P2P3P4 and the DT as Miquel triangle. Again, variing the direction of the asymptote gives different types of cubics. Q is the tangential of M1, M2 and M3 and F is the diametral point on the conic QA-Co1.
c) circles or conics of inversion for anallagmaty and Moebius give 7 circles or conics and the choose of the point Q gives a circular or noncircular cubic. Varying Q gives the different types.

2) We almost agree on the different types of cubics
a) circular cubics have only one asymptote and can be mono- and bipartite (2 or 4 anallagmatic centers).
b) non circular cubics can be monopartite or bipartite with 1 real asymptote, tripartite or quadripartite with 3 real asymptotes.
The case tripartite with 3 real asymptotes split in 2 cases CU3a 1 part with 1 asymptote and 2 parts with 2 asymptotes or CU3b 3 parts with 2 asymptotes. (there is no case of tripartite with 1 real asymptote). A trip in Bernard Gibert's catalogue shows these different types ... (K003 or K024 for tripartite and K011 for quadripartite).

3) A pivotal isocubic is pivotal isocubic with each of it's points as pivot and it's tangential QA as fixed points.
Therefore, the type CUc2, CU2, CU3a and b and CU4 are pivotal isocubics with 3*3 Miquel points and 3*4 anallagmatic points.
I still have a doubt for the characterization of CUc1 and CU1 (the QA is reduced to a segment and a point has only 1 tangential segment).

```
4) I hardly see the interest of the cases with Q infinite ...

Best regards
Bernard

PS As your table is almost perfect, you should define Ce
somewhere (center of the diametral conic) and name pivotal
isocubic instead of isocubic
```

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Message: #2067
Date: 2024-01-07
From: eckart_schmidt@t-online.de
Subject: Schroeter's types of cubics

```
Dear Bernard, dear Chris,

wrt Bernard's list of types of cubics in #2063:
In Schroeter's book (see PS) we find quite another list,
... at great length page 130-148.
Schroeter differentiates only between
... three "one-way" (einzügigen) types of cubics (page 138)
... and five "two way" (zweizügigen) types of cubics (page 142).

To understand this, we have to respect,
... that parts of cubics can be connected
    in a common infinity point,
... so our "tripartite with 3 real asymptotes" can be "one-way"
    (see below)!

Schroeter's differentiation is orientated
... at the two degenerated cubics "line and conic"
    and "three lines",
... solving the crosspoints in different ways,
... please have a look at Schroeder's figures wrt our types:
    1.  "monopartite" cubics are "one-way",
         fig 2, page 139,
    2.  "bipartite cubics with 1 real asymptote" are "two-way",
         fig 5, page 143,
    3.  "tripartite with 1 asymptote" not mentioned,
         not existing,
    4.  "tripartite with 3 real asymptotes",
         here we have to differentiate our list:
         a) one part with one asymptote: "two-way",
            fig 7, page 145,
```

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        b) all parts with two asymptotes: "one-way",
           fig 4a,4b, page 141,
    5.  "quadripartite with 3 real asymptotes" are "two-way",
        fig 9, page 147.

It is difficult to summarize 18 pages, I hope it is correct.
The main property of "one-way" cubics is (page 136),
... that every point is real tangential of two cubic points.
The main property of "two-way" cubics is (page 137),
... that every point of one part is no real tangential
... and every point of the other part is real tangential
    of 4 cubic points.

Conclusion: Every "two-way" cubic is a pivotal isocubic.

Best regards Eckart

PS: Heinrich Edward Schroeter: Die Theorie der ebenen Kurven
dritter Ordnung
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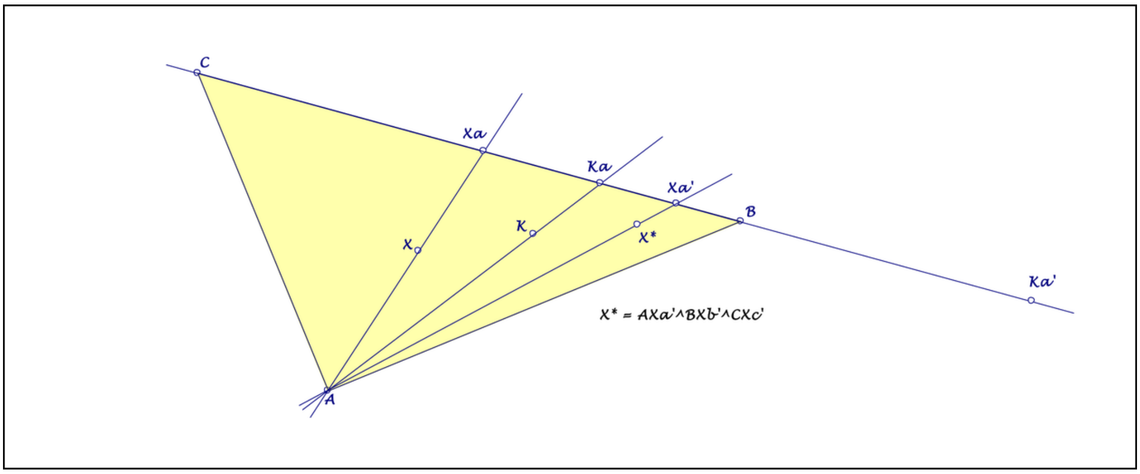
Message: #2068
Date: 2024-01-07
From: eckart_schmidt@t-online.de
Subject: Construction isoconjugation

Dear Chris,

in #2064 you ask for a construction of an isoconjugation.
An isoconjugation X-->X*can be defined
... for a triangle ABC and a fixed point K.
Let Ka = AK^BC, Xa = AX^BC,
... Ka' = 4th harmonic of Ka wrt BC,
... Xa' = 4th harmonic of Xa wrt KaKa',
... X* = AXa'^BXb'^CXc'.
I got this simple construction
... from Günther Pickert 20 years ago.

Best regards Eckart



2024-01-06.pdf

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Message: #2069
Date: 2024-01-07
From: bernard.keizer@gmail.com
Subject: Re: Schroeter's types of cubics

```
Dear Eckart,
Thanks for this clarification!
This time, it is not far from my last message 2066, in answer to
Chris message 2063.
I mentionned that your 3 doesn't exist, but I made a mistake for
your 4b (Chris 3b), which is not a pivotal isogonal cubic.
Best regards
Bernard
```

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Message: #2070
Date: 2024-01-08
From: van10hoven@gmail.com
Subject: Re: Anallagmaty and Moebius

Dear Eckart,

Wrt your question in # 2065
“What about your PS? Is there something not correct in my message.
Is it not allowed, to send observations in drawings, but without prove?”

There is nothing wrong with your messages.
It’s also perfectly fine to send observations without prove.
That’s the way we discovered many beautiful things.
It’s just about the way we deal with messages and topics.
Every message belongs to a topic.
The meaning of this is that when a person wants to study some topic he just has to choose that topic in question and he will find the messages about that topic.
When a question is replied the new message automatically gets the same topic.
Sometimes it happens that messages about some topic are replied with a message dealing about another topic and then the system fails.
Also it happens that a new subject has been started without naming a new topic at the Groups.io menu.
Then we get very long lists of messages not all belonging to the named topic.
I just want to say that it pays off when we pay attention to the rules of starting a new topic when we have a new subject and only reply on messages of the same topic you reply on, then we get neatly organized lists of messages per topic.
Being organized as we are, I don’t think it will to be a problem now we know it.

Best regards,

Chris

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Message: #2071
Date: 2024-01-08
From: van10hoven@gmail.com
Subject: Re: Anallagmaty and Moebius

Eckart,

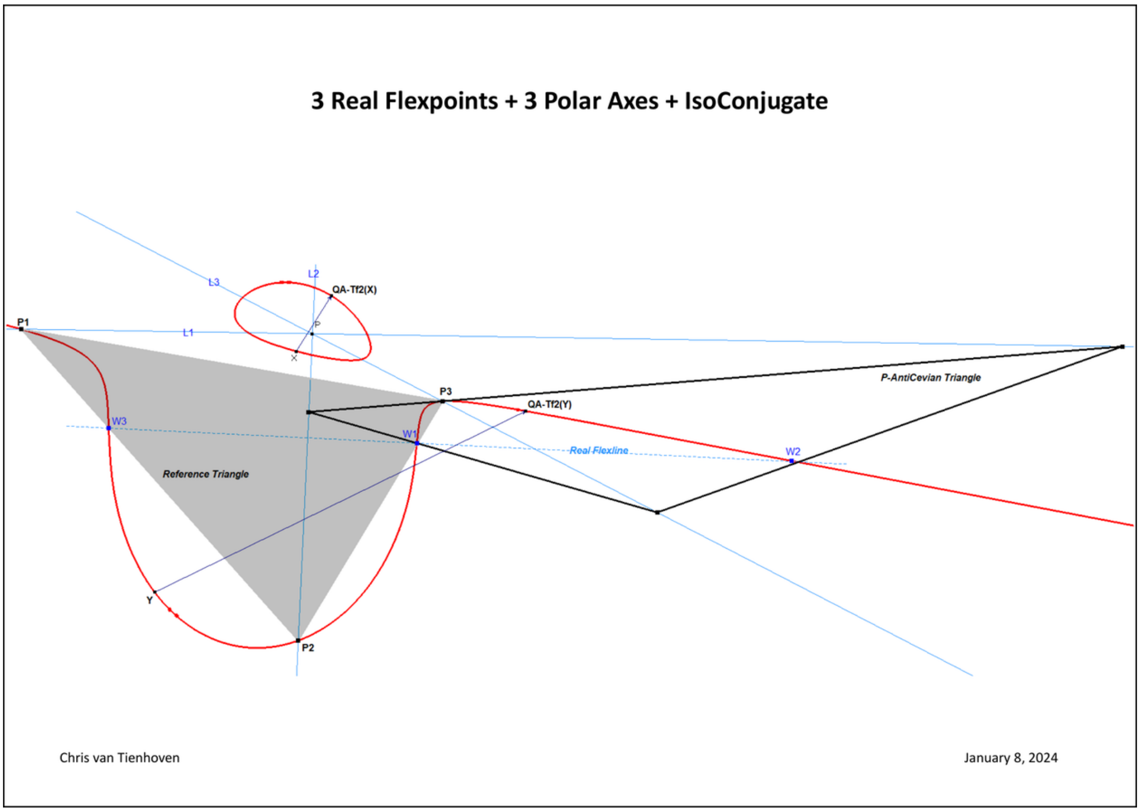
Regarding your messages #2059 and #2065. I checked in my drawing if indeed the cubic is invariant wrt an isoconjugation with reference triangle P1P2P3 and fixed point P. And I can confirm this is true for QA-Tf2 wrt QA(P,Pa,Pb,Pc), where PaPbPc is the Anticevian Triangle of P1P2P3. In my drawing I saw QA-Tf2(X) lying perfectly on the cubic for all X on the cubic.
See attached picture.
It is remarkable that the sides of the P-AntiCevian Triangle are lines through the Flexpoints W1, W2, W3 and are tangent to CU.

Regarding your message #2068.
Thanks for sharing this beautiful construction of an isoconjugation of Günther Pickert.
It is described as an harmonic conjugate of the 2nd degree.
Beautiful!
It also is the QA-Tf2 of QA(P, vertices AntiCevian Triangle).
However it is not the isoconjugate of Roland Stärk (your message #2035), neither is it the isoconjugate as mentioned in the paper of Bernard Gibert. According to my drawings they are different transformations.
How do you see this?
I thought it was Bernard Gibert who introduced the term Isocubic. Maybe I am wrong.

Bernard, by the way, why mentioning in the table “pivotal isocubic” instead of “isocubic”?

Best regards,

Chris



CU-9P1 CU-Flexpoints-13.pdf

Message: #2072
Date: 2024-01-09
From: eckart_schmidt@t-online.de
Subject: Re: Anallagmaty and Moebius

```
Dear Chris,

the constructions of isoconjugation
... by Pickert (#2068), Stärk (#2035) and Gibert
    (1.2.1 of his paper)
... give the same transformation:

Let us consider an isoconjugation
... defined by a fixed point K and a triangle ABC.
If we use the QA of K and its anticevians wrt ABC
... with the conic CO = QA-Co1,
... the constructions by Pickert and Stärk
... give the same isoconjugate.
Please serve a macro.

Gibert defines an isoconjugation
... with a point-image-pair wrt a triangle ABC.
Let (P, P*) be such pair, constructed with the macro above
... and follow Gibert's contruction M --> M*
    (1.2.1 in his paper):
... E0 = A.M ^ C.P*, E1 = M.P ^ B.C, Q = A.P ^ E0.E1,
    E2 = E0.Q ^ A.B,  M* = C.Q ^ P*.E2.
Pickert's and Stärk's construction give the same M* for M.

Best regards Eckart

PS: Thanks for the detailed remarks
... wrt isoconjugation and QMT,
... I think there will be no relevant connection.
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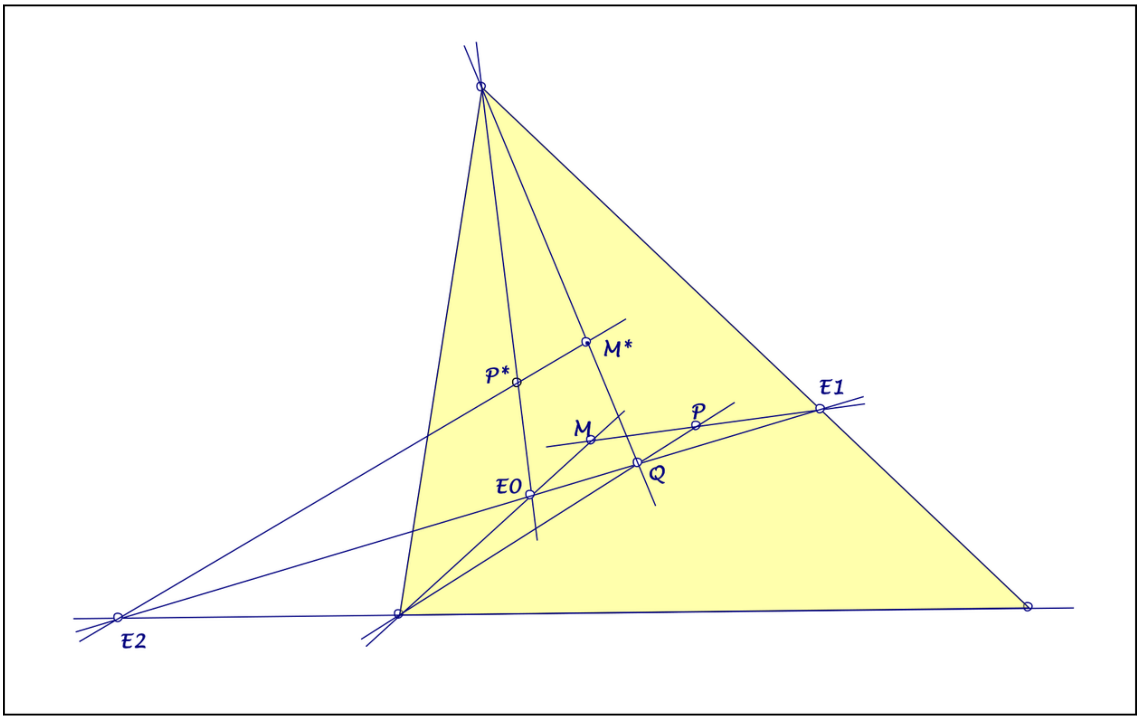

2024-01-09.pdf

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Message: #2073
Date: 2024-01-09
From: bernard.keizer@gmail.com
Subject: Re: General cubic

Dear Chris, dear Eckart
I'm convinced by Chris message 2070 of the necessity of organisation in our exchanges!
New messages about isoconjugation or flexpoints have nothing to do in the item anallagmaty and Moebius.
It makes only more difficult to find a message in an item of over 60 messages ...
Chris asked me what I was expecting from a merging of several memos; it is precisely a short summary of all properties found so far for a general cubic.
I'm lost in too many messages and often can no longer find something I remember, but hadn't printed ...
Looking again at some old figures, I have a question for Eckart about 2022 and 2023: could it be possible that the cubic drawn here is a quadripartite with hidden 4th part instead of a tripartite?
Best regards
Bernard

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Message: #2074
Date: 2024-01-09
From: bernard.keizer@gmail.com
Subject: Re: Construction isoconjugation

Dear Eckart,
You say in your message 2072 that your 3 constructions of the isoconjugation give the same result.
I found your construction with the dopple harmonic very interesting, I know the construction of Bernard Gibert, but I didn't find the 3rd one ...
If I'm not wrong, it works only with a real fixed point and the vertices of it's anticevian triangle.
It this case, having the pole of the isoconjugation (isoconjugate of the centroïd) with barycentric coordinates a^2,b^2,c^2 inside the triangle, the 4 real fixed points are +-a, +-b, +-c.
If the pole is outside the triangle, the fixed points are not real.
Best regards
Bernard

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Message: #2075
Date: 2024-01-09
From: bernard.keizer@gmail.com
Subject: Re: Flexpoints and -lines for cubics

Dear Chris, dear Eckart
Is it correct that we discussed already the flexpoints W1,2,3 and the polar lines intersecting in a point P as well as the vertices of it's anticevian triangle wrt the triangle of tangents in the item about hessian? Then some answers to Chris questions are already there. But you added the isoconjugation ...
Best regards
Bernard

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Message: #2076
Date: 2024-01-09
From: eckart_schmidt@t-online.de
Subject: Re: General cubic

Dear Bernard,

wrt your question in #2073: As I wrote
... the cubic in #2022 is tripartite, each part with two asymptotes,
... it cannot be quadripartite.

Best regards Eckart

PS: I try to use the same headline in my messages, when I answer.

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Message: #2077
Date: 2024-01-09
From: eckart_schmidt@t-online.de
Subject: Re: Construction isoconjugation

Dear Bernard,

references for the three isoconjugations I gave in #2072.
Stärk's construction
... includes also isoconjugations without real fixed points.

Best regards Eckart

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Message: #2078
Date: 2024-01-09
From: van10hoven@gmail.com
Subject: Re: Construction isoconjugation

Dear Eckart, dear Bernard,

Eckart, after your message #2072 it starts glooming in the dark.
Now I see that there are similarities AND differences between the different isoconjugates.
I will try to summarize how the information is coming to me now from different sources.
Let’s define:
· BG-IsoConjugate = IsoConjugate according to Bernard Gibert using P à P* as a transformation example
· GK-IsoConjugate = IsoConjugate according to Günther Pickert using a specific circumconic CO
· RS-IsoConjugate = IsoConjugate according to Roland Stärk using P as Center of the transformation

1. It is not difficult to find in constructions (assuming that we deal with the right GK-circumconic) that:
GK-IsoConjugate(X) = RS-IsoConjugate(X) wrt Reference Triangle ABC
2. Because the RS-IsoConjugate works with a simple variable point P instead of a yet elusive circumconic I think the RS-IsoConjugate is more manageable.
3. I myself many years ago found the same Conjugate and named it the P-Perspective Conjugate. This transformation worked with perspective reflections on the sides of the Reference Triangle. It is also very simple.
4. Noteworthy is that this Isoconjugate is also QA-Tf2(X) wrt QA(P+vertices P-AntiCevian Triangle) as Eckart mentioned before.
5. According to an old note of Bernard Gibert about QA-Tf2 it is also the P-Ceva Conjugate of X wrt the P-AntiCevian Triangle, which is the perspector of ABC and the X-CevianTriangle wrt the P-AntiCevian Triangle.
6. Interesting further is that when P=Incenter of ABC, then it is the IsogonalConjuagte and when P=Centroid of ABC, then it is the Isotomic Conjugate.

Then we come next to the BG-IsoConjugate.
This is actually a projective transformation, mapping X to X* and other points accordingly.
So when we substitute X* = Isogonal Conjugate(X) the BG-IsoConjugate will emulate the Isogonal Conjugate, when X*=Isotomic Conjugate(X), then BG-IsoConjugate will emulate the Isotomic Conjugate.

And also when we subsititute X* = RS-IsoConjugate(X), then
BG-Isoconjugate will emulate the RS-IsoConjugate. And all this because these Conjugates are preserved by a projective transformation.
But it also performs simple projective transformations when X*=some ordinary point Y, unrelated to other points.
So it looks like GK-IsoConjugate = RS-IsoConjugate = BG-IsoConjugate, but this is actually not quite true.
The BG-IsoConjugate is a transformation of a higher level that performs projective transformations or emulates other Conjugates that are preserved by projective transformations.
It is actually inconvenient that the word IsoConjugate is used for different concepts.
I think we should stick to the meaning of the BG-IsoConjugate, because the concept of IsoCubic is connected to it.

Now we come to the question what is an IsoCubic?
On page 7 of "Special Isocubics in the Triangle Plane" of Jean-Pierre Ehrmann and Bernard Gibert we find this definition:
By an isocubic we mean a circum-cubic which is invariant under an isoconjugation.
On page 4the construction of an IsoConjugation is given like Eckart described in former message #2072. And that is the construction that is actually the construction of a projective transformation.
Therefore can we say that an Isocubic is a cubic that is invariant under an isoconjugation,
· like the isogonal conjugate
· like the isotomic conjugate
· like the RS-conjugate
· or like any other conjugate that is preserved by a projective transformation?

Can we agree on this?
And when we agree on this:
What are examples of types of isoconjugates per type of general cubic?
And are these isoconjugates pivotal?
Maybe Eckart can answer this question right away.

Best regards,
Chris

Message: #2079
Date: 2024-01-10
From: eckart_schmidt@t-online.de
Subject: Re: Construction isoconjugation

```
Dear Chris,

thanks for the elaboration wrt " isoconjugations",
... but give me time, I am just working hard on another theme.
Having a first look, I have difficulties in understanding,
... beginning with the nomination:
Pickert uses a fixed point, Stärk uses a circumconic,
... but your definitions are in a different way.

Best regards Eckart
```

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Message: #2080
Date: 2024-01-10
From: eckart_schmidt@t-online.de
Subject: Hessian for a general cubic

Dear Bernard, dear Chris,

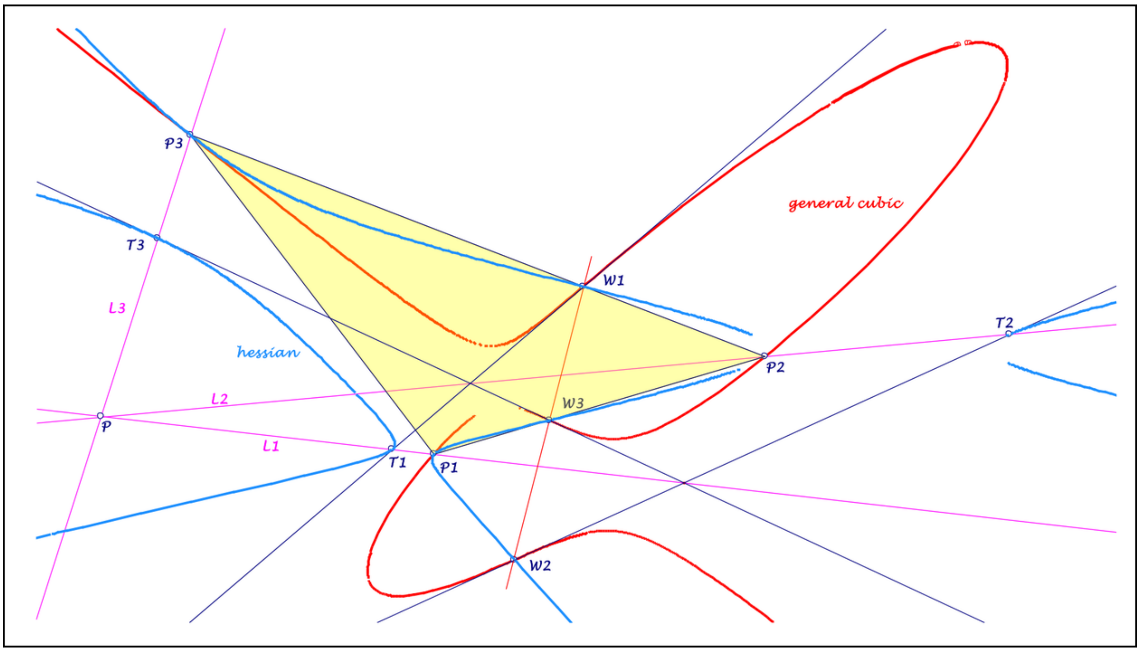
I can offer a "construction" for the hessian of a general cubic
... using the flexpoints with flextangents and harmonic polar axes,
... background is the book of Schroeter (pages in brackets)
... and the point-configuration in #2054.

The flexpoints of a cubic W1,W2,W3
... are also the flexpoints of the hessian (page197).
The flexpoints of a cubic are points with any three secants,
... intersecting the cubic coconic on a conic CO (page 242).
The polar axis of a flexpoint Wi wrt its conic CO
... is the harmonic polar axis Li for Wi (page 242).
The harmonic polar axes of W1,W2,W3
... have a common point P (page 245)
... and intersect the cubic in P1,P2,P3
... ... (not using points on the closed part).
Let T1,T2,T3 be the intersections
... of the flextangents and the harmonic polar axes L1,L2,L3.

The hessian is the cubic through the 9 points
... W1,W2,W3,P1,P2,P3,T1,T2,T3,
... contacting the cubic in P1,P2,P3,
... contacting the flextangents in T1,T2,T3.

The cubic and its hessian are invariant wrt
... an isoconjugation with triangle P1P2P3 and fixed point P.

Best regards Eckart



2024-01-10.pdf

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Message: #2081
Date: 2024-01-10
From: bernard.keizer@gmail.com
Subject: Re: Hessian for a general cubic

Dear Eckart,
If I'm not wrong, according to the theory, a cubic, it's
hessian, it's 3 prehessians and any cubic of the syzegetic
pencil intersect in 9 flexpoints, of which only 3 are real.
On your figure, the hessian passes through the 3 real flexpoints
W1, W2 and W3 and through the points T1, T2 and T3 where it
contacts the cayleyan and the flextangents, but cannot contact
the cubic in P1, P2 and P3.
Best regards
Bernard
PS See for example K003 and it's hessian K048 as well as the
prehessians in Bernard Gibert

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Message: #2082
Date: 2024-01-10
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic

Dear Bernard,

thanks for your observation,
... now I am quite doubtful,
... whether my cubic is the hessian,
... but my properties seem correct.
I have to study it once more.

Best regards Eckart

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Message: #2083
Date: 2024-01-11
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic

```
Dear Bernard,

you are right, my construction cannot give the hessian in #2080.
I apologize profusely my bad knowledge of hessians
... and my overhasty judgement.
But these derived cubics from the flexpoints seem interesting.

Best regards Eckart
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Message: #2084
Date: 2024-01-11
From: van10hoven@gmail.com
Subject: Re: General cubic

```
Dear Bernard,

You asked in #2073 for  a short summary of all properties found
so far for a general cubic.

Therefore I made a table of all items that we found in an
orderly way.

See attachment.

When you want me to add more references to earlier messages,
please let me know and I will add the message-numbers. Of course
other additions also are also very welcome.

Best regards,
Chris
```

Elements of the General Cubic

Ref.Code	---------------------------- NAME ----------------------------	Abbreviation	Validation	Remarks / References
	CU-GENERAL ITEMS			
CU	General Cubic	CU		
CU-Cu1	Hessian	He		
CU-3Cu1	3 PreHessians	pHe		
CU-9P1	Inflection points / Flexpoints	W1,W2,W3	3P=N	QPG#1829 and see [Fred Lang], page 137 4th item
CU-9L1	Flex Polar Lines	L1, L2, L3		
CU-27P1	Sextatic Points		2Q+P=N & 3Q=0 3P<>N	P is tangential of a flexpoint Q, but P is not a flexpoint [Fred Lang], page 137 11-14th item
CU-12L1	12 Flexlines of the Regular Cubic			[Fred Lang], page 137 5th item
CU-3L1	Asymptotes of the Regular Cubic	Asy1,Asy2,Asy3		QPG#1874+1877 & [Cuppens]
CU-3IP1	CU-Infinity Points	IP1, IP2, IP3		
CU-3P1	CU-Asy-Crosspoints	Q1, Q2, Q3		
CU-3Q1	Asy-Asy-Crosspoints	F1, F2, F3		
CU-3Co1	IP-Polar Conic / Diametral Conic			
CU-3Co2	QF-Conic			QA-Co1 of the 3 QA's of CU-3QA1
CU-3QA1	3 Sets of 4 Quasi-Anallagmatic Points	P1a,P1b,P1c,P1d P2a,P2b,P2c,P2d P3a,P3b,P3c,P3d		Points where Asy-parallels are tangent to CU
CU-3Tr1	3 Quasi-Miquel Triangles	M1a,M1b,M1c M2a,M2b,M2c M3a,M3b,M3c		Diagonal Triangles of the 3 QA's of CU-3QA1
CU-L1	CU-Asy-Crosspoints-Line	F1F2F3		Line through the 3 versions of CU-3P1
CU-L2	QF-Conic Centers Line			Line through the centers of 3 versions of CU-3Co2
CU-3Q1	Set of Common Intersection points of 3 QF-Conics			Common points of the versions of 3 CU-3Co1
CU-4Q1	Set of Common Intersection points of 3 IP-Polar Conics			QPG#1882-1883,#1897
CU-3Tf1	Set of 3 IP-Involutary Conjugates			for each QA in CU-3QA1 exists an Invol.Conjugate
CU-9Tf1	Set of 3x3 Quasi-Moebius Conjugates			for each vertex of CU-3Tr1 exists a Conjugate
	CU+P & CU+L derived items			
CU_P-P1	P-Tangential		N-2P	
CU_P-3P1	2nd intersection point of CU and Line parallel to CUc-Asy-i			QA-Tf2(P) wrt the 3 QA's of CU-3QA1
CU_L-L1	L-Tangential Line			(line P1.P2.P3 -> line tP1.tP2.tP3)
CU_P-Co1	P-Polar Conic of a Cubic			(analogon of 1st derivative)
CU_P-Tf1	P-Involutary Conjugate			QA-Tf2(X) of the QA of 4 P-points-of-tangency
CU_P-Tf2	1st CU_P Transformation			QPG#1805
	CU+2P derived items			
CU_2P-P1	3rd intersection point of CU and 2P-Line		N-PP2	
CU_2P-L1	Line through 2 given CU-points			
	CU+3P derived items			
CU_3P-P1	CU_3P PQR Addition Point		PP3	QPG#1781 picture
	CU+4P derived items			
CU_4P-P1	Cotterill's Point / Coresidual point		-N+PP4	QFG#2481 & [Cotterill] & [Cuppens] & [Fred Lang], page 137
CU_4P-P2	CU-CB4d Point		3N-2PP4	QPG#794-814 & #839 attachment 3.10
	CU+5P derived items			
CU_5P-P1	CU-6th Intersection point 5P-Conic with CU		2N-PP5	QPG#1806
CU_5P-Co1	CU-5P-Conic			
CU_5P-Co2	CU-5P-Tangential Conic (through the six Pi-Tangential Points)			[Fred Lang], page 137, 7th item
	CU+7P derived items			
CU_7P-P1	CU-7P-CB Pivot Point		-2N+PP7	
CU_7P-Tf1	CU-7P-Geiser Involution/Transformation		3N-P-PP7	
2CU_7P-L1	2CU-7P-Intersection Line			QFG#839 attachment 3.7
	CU+8P derived items			
CU_8P-P1	Cayley Bacharach Point for a Regular Cubic (CB Point)		3N-PP8	
	CU+9P derived items			
CUc_9P-P1	1st 9P-Sumpoint of the General Cubic		2N-PP9	
CUc_9P-P2	2nd 9P-Sumpoint of the General Cubic		-4N+PP9	

CU-Elements of the General Cubic-01.pdf

Message: #2085
Date: 2024-01-12
From: bernard.keizer@gmail.com
Subject: Re: General cubic

Dear Chris,
Thanks a lot, this is exactly what I expected!
It allowed me to revise my knowledge ...
I must confess that I probably missed or forgot some messages,
for example, what is the 7P Geiser involution and what are the
9P sumpoints?
Anyhow, I have to check all your items carefully. (But I don't
have access to Fred Lang' book, which you quote often).
My last wish (it's still Christmas) would be a catalogon of the
5 different shapes of general cubics we encountered
Best regards
Bernard

PS I suppose CU-L1 is Q1Q2Q3 and not F1F2F3 and the 9P items at
the end are for CU and not CUc

Message: #2086
Date: 2024-01-12
From: bernard.keizer@gmail.com
Subject: New construction of a general cubic

```
Dear Chris, dear Eckart
As I was in Paris these days, I made a last visit to the library
of the Institut Henri Poincaré in order to have a look again on
Salmon and Schröter.
Thanks to Eckart, I had this time the reference pages and I
found luminous the deformation of the 2 degenerated cubics line
+ conic and triangle ...
But I found also in the beginning a very interesting and simple
construction of a general cubic.
If we take 3 copples of points A,A',B,B' and C,C' not forming a
QL, it's possible to draw points step by step by completing the
QL's started with 2 copples.
AB and A'B' will intersect in D and AB' and A'B in D', the same
way E and E' from A,A',C and C' and F and F' from B,B',C and C'
and so on...
1) If the 3 copples form a QL, they define a Moebius
transformation MT.
By adding a point D and it's MT partner, we obtain a set of MT
invariant circular cubics (not necessary focal).
By adding 2 points not MT partners, we come back to the general
construction.
2) if the 3 copples don't form a QL, the construction gives a
unique general non circular cubic invariant in a QMT determined
from the 3 copples of points.
a) by giving 5 points A,A',B,B' and C and variing the point C'
for example, it's possible to find our 5 different shapes of
cubics.
b) the QMT associates to any point X the partner X' having the
same tangential T on the cubic; U being the 3rd intersection of
XX' with the cubic, T and U are QMT partners.
c) the construction used with the centers of quasi anallagmaty
P1 and P2 (where the tangents are asyparallels) works with any
copple of QMT partners X and X' and gives for any point Y
the same QMT partner Y'
d) it is remarkable that Fred Lang's calculation shows easily
that X' - X = A' - A = B' - B = C' - C = P2 - P1 = IP - M; then
the calculation will be the same as for MT on a circular cubic
Best regards
Bernard
```

Message: #2087
Date: 2024-01-12
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic

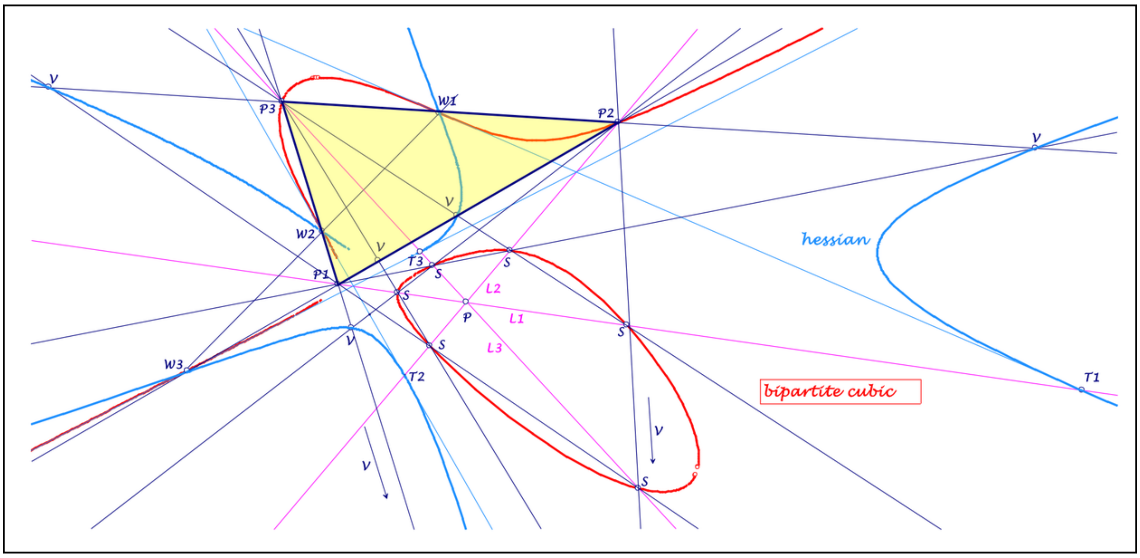
```
Dear Bernard, dear Chris,

I try it once more, up to now for a bipartite cubic,
... using the flexpoints with flextangents
    and harmonic polar axes,
... background is the book of Schroeter (pages in brackets)
... and the point-configuration in #2054.

The flexpoints of a cubic W1,W2,W3
... are also the flexpoints of the hessian (page197).
The flexpoints of a cubic are points with any three secants,
... intersecting the cubic coconic on a conic CO (page 242).
The polar axis of a flexpoint Wi wrt its conic CO
... is the harmonic polar axis Li for Wi (page 242).
The harmonic polar axes of W1,W2,W3
... have a common point P (page 245)
... and intersect the cubic in P1,P2,P3
    on the part with asymptote
... and in 6 further points S on the closed part.
Every Pi is two times collinear with two S-points not on Li,
... let the corresponding lines intersect PjPk in points V,
... which gives 6 V-points.
Finally let T1,T2,T3 be the intersections
... of the flextangents and the harmonic polar axes L1,L2,L3,
... T1,T2,T3 are contact points of the flextangents
    to the hessian (page 247).

The hessian is the cubic through the 12 points
... W1,W2,W3,T1,T2,T3 and the 6 V-points,
... contacting the flextangents in T1,T2,T3.

Best regards Eckart
```



2024-01-12.pdf

Message: #2088
Date: 2024-01-12
From: van10hoven@gmail.com
Subject: Re: General cubic

Dear Bernard,

Your didn't miss former messages.
The names of the 7P Geiser involution and the 9P-sumpoints were not mentioned before, but I intended to do so at due time.

1. The 7P Geiser involution is an old acquaintance of ours. It is actually 7P-s-Tf1, the transformation arising from the Cayley-Bacharach Point. I found it on this site.
Ueber zwei geometrische Probleme.
Journal für die reine und angewandte Mathematik /
Zeitschriftenband (1867) / Artikel / 78 - 89
http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002153149
And also:
https://www.researchgate.net/figure/The-Geiser-involution-interchanges-the-points-Q-and-Q_fig1_259010333

2. The 9P-SumPoints I will mention later. Right now I am revising the document about it. They are very interesting CU-9P points. For the construction the QF-Conic is needed.

3. Fred Lang's book is a digital paper and can be obtained here:
https://forumgeom.fau.edu/FG2002volume2/FG200217.pdf
I am sure you will like it.

4. Your PS at the end is quite true. Thanks for mentioning. I will amend it.

By the way, recently I also bought the books of Schröter and Salmon. But I didn't find time to study them yet, but I am sure that will come later.

Best regards,
Chris

Message: #2089
Date: 2024-01-13
From: bernard.keizer@gmail.com
Subject: Re: Hessian for a general cubic

Dear Eckart,
I admire your tenacity!
But despite all my efforts, I cannot reproduce your property ...
I'm not sure it holds. What makes you sure that this new cubic
is the hessian?
The only way to be sure is either that I make a new figure with
another cubic and it's hessian, giving the 3 flexpoints and
reconstituting the whole figur or that you make directly a
figure with the cubic and it's hessian.
On my figure, I have exactly the same points Wi, Ti and S and
properties of alignments, but your points V are not on my
hessian! (not far, but not exactly)
But thanks to you, I understand much better some properties.
For example, naming Si and S'i the points S on Li, there are 3
QMT's on the cubic swapping Wi and Pi and Si and S'i or Wi and
Si and Pi and S'i or Wi and S'i and Pi and Si:
the common tangential of Wi and Pi, as well as of Si and S'i is
Wi, which is it's own tangential and the 3rd intersection of
WiPi or SiS'i is Pi, the tangential of Pi being Wi ...
There is 1 QMT on the hessian, swapping Wi and Ti having the
same tangential Wi and the 3rd intersection of WiTi with the
hessian is Ti ...
The 6 points Wi and Ti form a QL (T1,T2 and W3 are aligned ...)
as well as the points Wi and Pi.
Best regards
Bernard

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Message: #2090
Date: 2024-01-13
From: bernard.keizer@gmail.com
Subject: Re: Hessian for a general cubic

```
Dear Eckart,
I made an exact construction with the cubic stelloïd K003 and
it's hessian K048.
I found easily the points P, Wi, Ti, Si and S'i and the
intersections V of the lines PiSjSk and PiS'jS'k with PjPk.
These points V are not on the hessian.
Best regards
Bernard
```

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Message: #2091
Date: 2024-01-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic:

```
Dear Bernard,

you are right, that I am wrong once more,
... I worked out this construction,
... studying a constructed hessian,
... but drawings with CABRI gave no clear verification,
... that the polar conics for the V-points degenerate
    in two lines.
Excuse my uncertainty, but thanks for your interest.

Best regards Eckart
```

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Message: #2092
Date: 2024-01-14
From: bernard.keizer@gmail.com
Subject: Re: General cubic

```
Dear Chris,
Thanks for the explanations and references
Best regards
Bernard
```

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Message: #2093
Date: 2024-01-15
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic

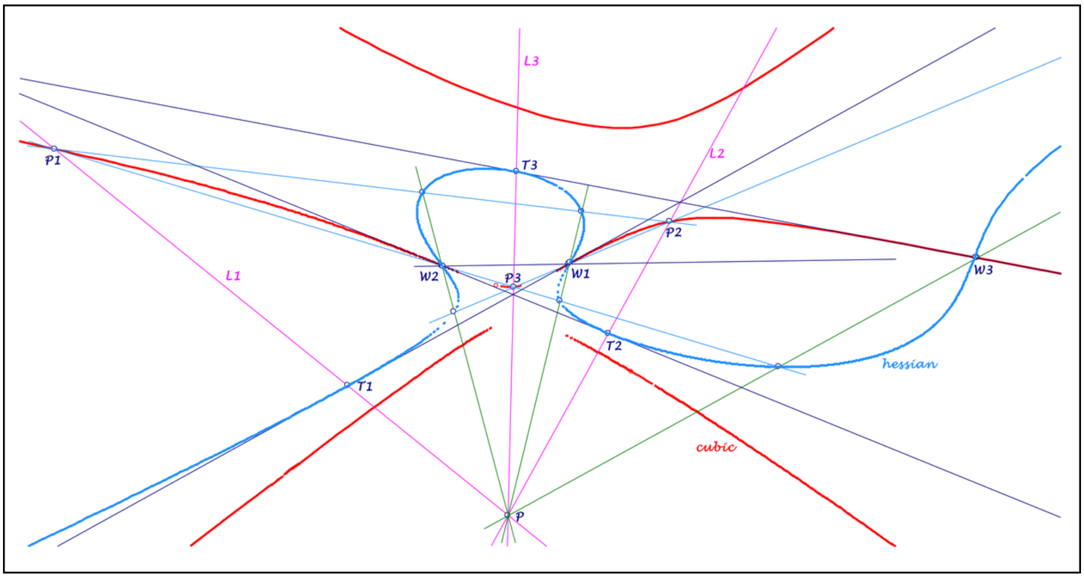
```
Dear Bernard,

I try it a third time for a bipartite cubic
... (not valid for a monopartite cubic),
... tested for your example K002/K048
... and several general bipartite cubics,
... as far as the precision of CABRI allows
With the nomination of #2087 and simpler as before:

The hessian for a bipartite cubic
... is the cubic through the 12 points
... W1,W2,W3,T1,T2,T3 and the 6 points
... PW1^P1P2, PW1^P1P3, PW2^P2P3, PW2^P2P1, PW3^P3P1,PW3^P3P2,
... contacting the flextangents in T1,T2,T3.
I hope, you can confirm this result.

Best regards Eckart

PS: Bipartite can here be Schröter's "zweizügig" (see #2067).
```



2024-01-15.pdf

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Message: #2094
Date: 2024-01-17
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

```
Dear Bernard, dear Chris,

I think, Chris' construction of the cubic hessian in #1944
... is already  explained in Schroeter's book,
... starting page 181,
    replace "konische Polare" by "polar conic",
... the "4 Grundpunkte" on page 186
    by the "4 real intersection points",
... every line PQ,QR,PR in Chris' construction gives such a QA
... and on page 191 is explained,
... that  the diagonals of such QA intersect the defining line
    on the hessian.
Chris construction is a very compact simple version of these
relations.

Background:
For two cubic points P,Q and their polar conics,
... intersecting in 4 points, which give a QA,
... the QA-diagonals intersect PQ on the hessian
... and the vertices of QA-Tr1 lie on the hessian.

Best regards Eckart
```

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Message: #2095
Date: 2024-01-19
From: bernard.keizer@gmail.com
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

Dear Eckart,
If I'm not wrong, Chris mentionned only the 9 DT vertices, but not the intersections between the DT sides and the defining lines.
(This property in Schröter is not in Cüppens).
Thanks to you (and Schröter), we have now 9 copples of corresponding points.
It's interesting to notice that one defining line is not enough, as the 3 copples form then a QL.
(See my message 2086 to which neither Chris nor you have answered).
Best regards
Bernard

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Message: #2096
Date: 2024-01-23
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic:

```
Dear Bernard, dear Chris,

apologizing once more,
... forget the hessians in my messages #2087, #2080, #2093,
... also the last one doesn't hold, I have to work more solid,
... but the point constellation,
    derived from the flexpoints of a cubic
... is very substantial and interesting:
With the nomination of #2087 for a bipartite/zweizügige cubic:
... starting with the flexpoints W1,W2,W3
    and flextangents tg1,tg2,tg3,
... harmonic polar axis L1,L2,L3 with common point P
... and intersections P1,P2,P3 with the infinity part
    of the cubic:

The cubic and L1,L2,L3 are invariant wrt an isoconjugation *
... with reference triangle TR = P1P2P3 and fixed point P
... with the flexpoints on the sidelines of TR.
The intersections Ti = Li^tgi are the contact points
... of the hessian and the flextangents,
... the points Ti* lie on Li, TiTi* bearing Pi.
Each Ti is one vertex of a final diagonal triangle
... in Chris' hessian construction for P1,P2,P3 in #1944,
Each Ti* is one vertex of a quadrangle in Chris' construction,
... TiWi and TiPi=Li are lines of these quadrangles ...
... so far, there will be more properties,

I hope, no fake observations are done again.

Best regards Eckart

PS: What are the polar conics of flexpoints?
As points of the cubic and its hessian
... they have to be double lines, bearing the flexpoint,
... one line is Li (see above), and the second line?
```

Message: #2097
Date: 2024-01-23
From: eckart_schmidt@t-online.de
Subject: Re: Construction isoconjugation

Dear Chris,

excuse, but I try in vain to understand the sense
... of your extensive discussion wrt "isoconjugate" in #2078,
... that is out of my geometric horizon.

Best regards Eckart

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Message: #2098
Date: 2024-01-23
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

Dear Bernard,

what do you mean in #2095 with "defining lines" for Chris'
construction?
Can you give the corresponding page in Schröter's book?
Thanks in advance.

Best regards Eckart

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Message: #2099
Date: 2024-01-23
From: bernard.keizer@gmail.com
Subject: Re: Hessian for a general cubic:

Dear Eckart,
I'm awfully sorry, but I didn't notice your message 2093! (Why are there now 2 items with the same name?)
Anyhow, I checked this last (3rd?) construction of the hessian and I can confirm that it doesn't hold either ...
But this time, it gave me a new idea, which could work in fact (so I hope).
I refer to your figur 2087.
Take as you do the 3 lines PW1, PW2 and PW3.
Then consider together the 9 contact points of the 3 tangents (other than the flextangents) from each Wi to the cubic.
They are on 9 lines, including the 3 lines Li (3 points on a line and 3 lines through a point).
In order to identify the points on your figure, I keep the points P1,2 and 3 and name Si the S point between P and Pi and S'i the other.
>From the 9 lines, you can eliminate the 3 lines Li; it remains P1S3S2 and P1S'2S'3, P2S3S1 and P2S'1S'3 and P3S1S'2 and P3S2S'1.
If I'm not wrong, your 6 searched points V could be the intersections of PW1 with P3S2S'1 and P2S'1S'3, PW2 with P1S'2S'3 and P3S1S'2 and PW3 with P1S3S2 and P2S3S1.
These new points V are in fact close to your old ones! (The real hessian is not very far from your old one).
If this is correct (which I hope), the key points would be S'1, S'2 and S3 ...
But I don't know how to identify them and how to interpret this!
May be you will have an idea?
Best regards
Bernard

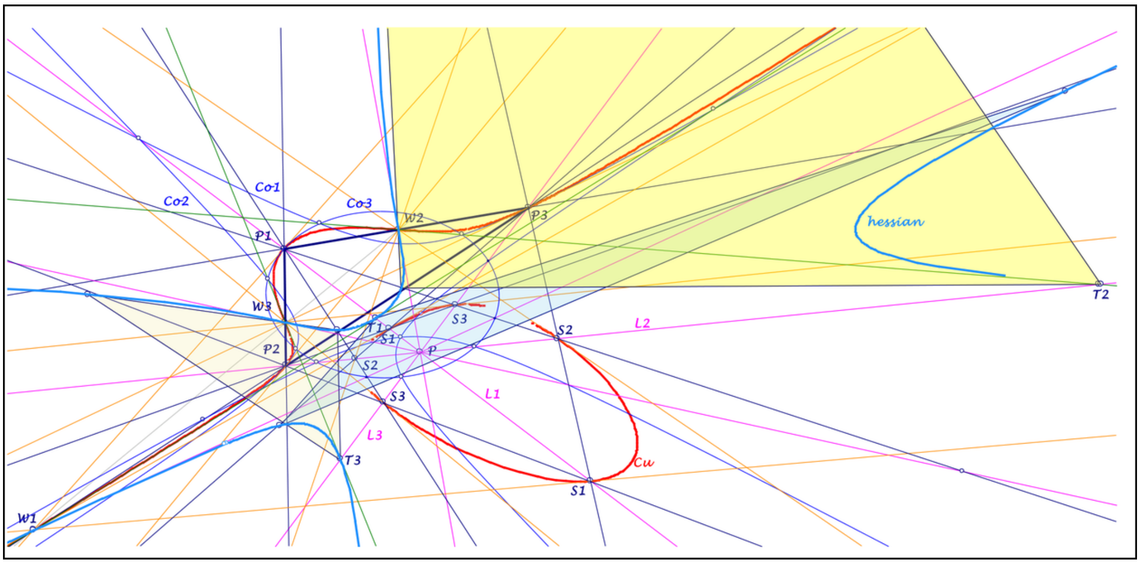
Message: #2100
Date: 2024-01-23
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic:

```
Dear Bernard,

thanks for interest, attached a drawing
... with Chris' construction and my point constellation,
... but I cannot apply your S-nomination
... for the collinearity necessary for the lines.

Is already mentioned, that Chris' final digital triangles
... are pairwise perspective wrt the flexpoints?

Best regards Eckart
```



2024-01-22.pdf

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Message: #2101
Date: 2024-01-23
From: bernard.keizer@gmail.com
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

```
Dear Eckart,
I hardly understand your question!
In your message 2094 above, you used yourself the word 'defining
line' and you mentionned the page 191 in Schröter ...
And you explained the background.
What is not clear?
Best regards
Bernard
PS The points P, Q and R are not necessary on the cubic
```

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Message: #2102
Date: 2024-01-24
From: van10hoven@gmail.com
Subject: Trio of Product Points Cubics

Dear Eckart, dear Bernard, dear friends,

Whilst I was looking for something else, I found this beautiful trio of cubics as a by-catch.
Given a hexagon with consecutive vertices P1, P2, P3, P4, P5, P6.
Consider the point pairs of opposite points (P1,P4), (P2,P5, (P3,P6).

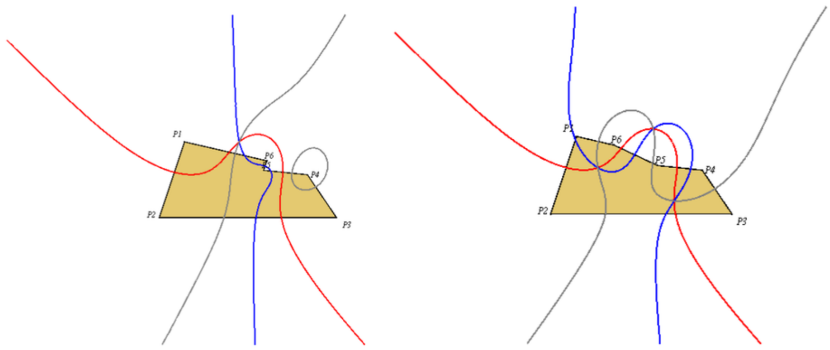
There are 3 cubics:
1. CU1=the locus of points S for which d(S,P1).d(S,P4) = d(S,P2).d(S,P5).
2. CU2=the locus of points S for which d(S,P1).d(S,P4) = d(S,P3).d(S,P6).
3. CU3=the locus of points S for which d(S,P2).d(S,P5) = d(S,P3).d(S,P6).

Common CU-points S have this property: d(S,P1).d(S,P4) = d(S,P2).d(S,P5) = d(S,P3).d(S,P6).
These cubics have 9 points in common (therefore CB-partners), most of them being imaginary points.
There are no general incidences of 3 points being collinear or 6 points being coconic (algebraically checked in a random configuration).

I do not know a way how to construct them, except by means of calculations in Mathematica and subsequent plotting. Maybe someone can help me.
See attached picture.

Best regards,
Chris

6G-s-3CU1 6G-Triple of Product Points Cubics
6G-s-9P1 6G-Product Points



Given a hexagon with consecutive vertices P1, P2, P3, P4, P5, P6.
Consider the point pairs of opposite points (P1,P4), (P2,P5), (P3,P6).
There are 3 cubics:

1. CU1=the locus of points S for which d(S,P1).d(S,P4) = d(S,P2).d(S,P5).
2. CU2=the locus of points S for which d(S,P1).d(S,P4) = d(S,P3).d(S,P6).
3. CU3=the locus of points S for which d(S,P2).d(S,P5) = d(S,P3).d(S,P6).

Common CU-points S have this property: d(S,P1).d(S,P4) = d(S,P2).d(S,P5) = d(S,P3).d(S,P6).
Algebraic calculations show that these cubics have 9 points in common (therefore CB-partners), most of them being imaginary points.
There are no general incidences of 3 points being collinear or 6 points being coconic (algebraically checked in a random configuration).

6G-s-3CU1 6G-Triple of Product Points Cubics.pdf

Message: #2103
Date: 2024-01-24
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

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```
Dear Bernard,

thanks for answer to #2098,
... but have a bit more understanding for my question,
... you started with Chris' construction,
... which is defined by three points P,Q,R,
... which have to be on the cubic.

Now I observed:
My "defining lines" PiPj intersect (see my drawing)
... the diagonal sides - not bearing a flexpoint - on the
hessian.

Best regards Eckart
```

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Message: #2104
Date: 2024-01-24
From: eckart_schmidt@t-online.de
Subject: Re: Trio of Product Points Cubics

```
Dear Chris,

I think, the following EQF-Note on my homepage
... will give some information:
...  2014-01-23.pdf (eckartschmidt.de)
<https://eckartschmidt.de/2014-01-23.pdf>

Best regards Eckart
```

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Message: #2105
Date: 2024-01-24
From: bernard.keizer@gmail.com
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

Dear Eckart,
Thanks for clarifying!
We perfectly agree that for 3 points P,Q and R, *not necessary on the cubic*:
1) we have 3 defining lines PQ, PR and QR
2) each line has 4 poles, intersections of all conics through these 4 points
3) these 4 poles form a QA
4) the DT vertices are on the hessian
5) the DT sides of the QA intersect the defining line on the hessian
6) each DT vertice and the intersection of the DT opposite side are conjugated points of the hessian
So we have all together 9 copples of conjugated points on the hessian
If the point 5) is your observation, congratulations!
I hope there is no longer any misunderstanding between us
Best regards
Bernard

PS I have neither Salmon nor Schröter at home, I've only consulted these books a copple of hours in the library of the Institut Poincaré in Paris

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Message: #2106
Date: 2024-01-25
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

```
Dear Bernard,

your property 5) in #2105 is not mine, see Schröter page 191.
My question in #2096 is also answered by Schröter page 196:
... the polar conic of a flexpoint Wi is the double-line
... of the flextangent tgi and the harmonic polar Li.

Wrt my point-constellation
(see my drawing in #2100, but the nomination of Coi is not
correct),
... starting with the flexpoints Wi
    and their harmonic polar-axes Li,
... intersecting the cubic in Pi on the non-closed part,
... we get three lines PiPj, each with 4 poles,
... which are the intersections
    of the polar conics of Pi and Pj,
... and define a QA,
    whose diagonal triangle has vertices on the hessian,
... one vertex Tk intersection of the flextangent tgk
    and the harmonic polar axis Lk.
The lines PiPj intersect the corresponding diagonal triangle
... in the flexpoint Wk and two other points of the hessian.
So we get 18 points of the hessian.

Best regards Eckart

PS: Wrt your formulation of property 2):
... the 4 poles of a line are the intersections
... of all polar conics for points on the line
    (Schröter, page 186).
```

Message: #2107
Date: 2024-01-26
From: bernard.keizer@gmail.com
Subject: Re: Hessian and generalised S-points for a pivotal isocubic

---

Dear Eckart,
I'm glad that we apparently understand each other again!
Thanks for correcting my formulation of property 2), which
wasn't correct.
I found in Cüppens the property that the polar conic of a
flexpoint Wi is formed of the tangent in the flexpoint (your
tgi) and the harmonic polar (your Li) intersecting in Ti;
what is now the polar conic of Tii (2 lines intersecting in Wi
and harmonic wrt the 2 tangents to the hessian in Wi and Ti (the
2nd is the line WiTi)?
I observed your figure 2100, but it is too complicate to be
reproduced.
I only notice that you start with the flexpoints, which is
rather a strange way!
I first draw the cubic and it's hessian in order to find
precisely the flexpoints ...
I suppose the property mentionned about the diagonal (and not
digital?) triangles is a consequence of the choice of the points
...
Best regards
Bernard

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Message: #2108
Date: 2024-01-26
From: bernard.keizer@gmail.com
Subject: Hessian and cayleyan

Dear Chris, dear Eckart,
For a cubic, the last step is the cayleyan, which is the holy grail!
The cayleyan is a curve of the 6th degree and of class 3, kind of dual of the hessian.
It is
1) the enveloppe of all lines through 2 corresponding points of the hessian
2) the enveloppe of all lines forming the degenerated polar conics of points of the hessian
(these 2 definitions are equivalent)
3) the locus of the contact point of the line with the curve, which is he harmonic conjugate of the complementary point wrt the 2 corresponding points
As the degenerated conics in 2 conjugated points are formed by 2 copples of lines, the 4 lines forms a QL.
By changing the order of the lines, it is possible to associate a line with one of the 3 others.
A cayleyan is therefore always the cayleyan of 3 different cubics (the 3 hessians are different, but the 4 lines and the cayleyan are the same).
It is remarkable that the 3 cubics and the 3 hessians belong to the same syzegetic pencil and have the same flexpoints!
Best regards
Bernard

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Message: #2109
Date: 2024-01-31
From: hoingason@gmail.com
Subject: Some short pdf files on rational geometry

Dear friends
Here are some short pdf files on rational geometry that I compiled
previously. Hope it is useful for you!
Best regards
-TXM-

Rational point whose distances are perfect squares

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
Here the author gives the solution for rational point whose distances are perfect squares

In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices.

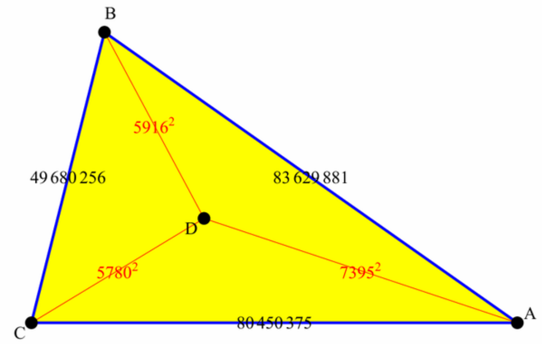
Suppose a, b, c are the lengths of the sides of a triangle with three rational medians (or rational centroid). Then we can set up a rational quadrilateral ABCD with the following lengths of sides:

BC = $a\sqrt{2b^2+2c^2-a^2}$, CA = $b\sqrt{2c^2+2a^2-b^2}$, AB = $c\sqrt{2a^2+2b^2-c^2}$, DA = a^2, DB = b^2, DC = c^2

Out[ ]=

x 2 y 3

|BC|	|DA|
49 680 256	7395^2
|CA|	|DB|
80 450 375	5916^2
|AB|	|DC|
83 629 881	5780^2



Rational point whose distances are perfect squares (TXM).pdf

General Rational Cyclic Polygons

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
Here the author gives a complete solution for general rational cyclic polygons

A general rational cyclic polygon is a cyclic polygon with a rational distance between any two vertices.

Let x, y, z be rational numbers with $x\,y\,z\,(x+y+z)>0$. Suppose three of the vertices have Cartesian coordinates

$M_1 = (z+x,\ 0)$,

$M_2 = \left(\frac{-x\,y+x\,z+y\,z+z^2}{z+x},\ \frac{2\sqrt{x\,y\,z(x+y+z)}}{z+x}\right)$,

$M_3 = (0,\ 0)$.

Suppose k_n is rational and set

$A_n = (x+z)\,(-1+k_n\,x\,z)\,(1+k_n\,y\,(x+y+z))\left(-1+k_n{}^2\,x\,y\,z\,(x+y+z)\right)$,

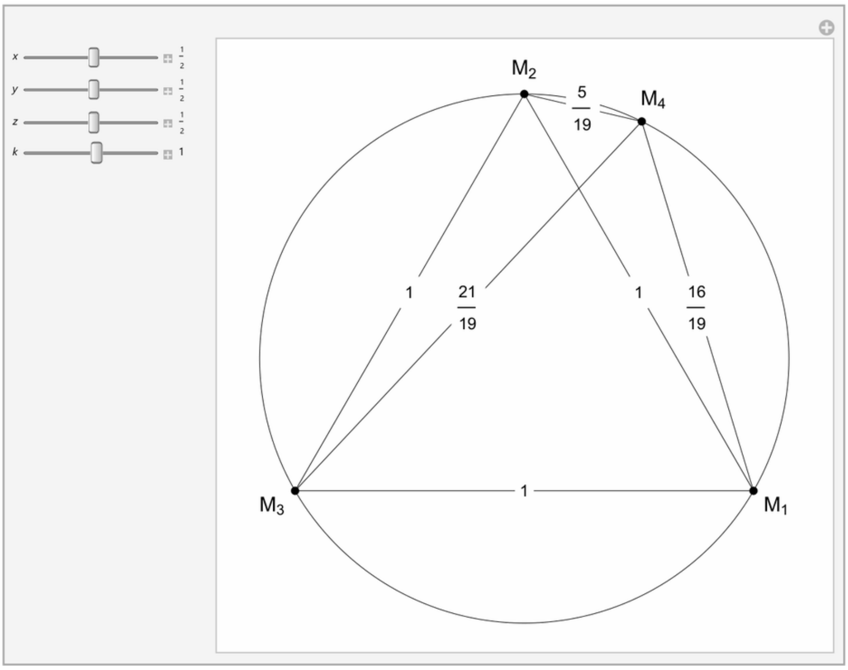
$B_n = -2\,k_n\,(x+z)\,\sqrt{x\,y\,z\,(x+y+z)}\,(-1+k_n\,x\,z)\,(1+k_n\,y\,(x+y+z))$,

$D_n = \left(1+k_n{}^2\,x\,y\,z\,(x+y+z)\right)^2$.

If $M_n = (A_n/D_n,\ B_n/D_n)$, $n = 4, 5, 6, \ldots$, then $M_1\,M_2\,M_3\,M_4 \ldots$ is a rational cyclic polygon.

The following illustration shows $M_1\,M_2\,M_3\,M_4$.

General Rational Cyclic Polygons (TXM).pdf



Related links

General Rational Cyclic Polygons

Rational Heptagon

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
Here the author gives a solution to the rational heptagon, which is also the answer to an interesting problem by Paul Erdős about seven points in the plane, no three on a line, no four on a circle with pairwise rational or integral distances.

In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices.

Suppose a, b, c are the lengths of the sides of a triangle with three rational medians (or rational centroid). Then we can set up a rational heptagon ABCDEFG with the coordinates of the vertices as follows:

$$A = \left(2a^4 - a^2 b^2 - 3a^2 c^2 + 3b^4 - 2b^2 c^2 + c^4,\ \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,\left(2a^2 - b^2 - c^2\right)\right)$$

$$B = \left(2b^2\left(c^2 + 2\left(a^2 + b^2 - c^2\right)\right),\ 0\right)$$

$$C = \left(\left(a^2 + b^2 - c^2\right)\left(c^2 + 2\left(a^2 + b^2 - c^2\right)\right),\ \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,\left(c^2 + 2\left(a^2 + b^2 - c^2\right)\right)\right)$$

$$D = \left(\frac{3}{2} b^2\left(3a^2 + b^2 - c^2\right),\ \frac{3}{2} b^2 \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\right)$$

$$E = \left(b^2\left(3a^2 + b^2 - c^2\right),\ b^2 \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\right)$$

$$F = \left(\frac{1}{2}\left(a^2 + b^2 - c^2\right)\left(2a^2 + 2b^2 - c^2\right),\ \frac{1}{2}\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,\left(c^2 + 2\left(a^2 + b^2 - c^2\right)\right)\right)$$

$$G = \left(b^2\left(c^2 + 2\left(a^2 + b^2 - c^2\right)\right),\ 0\right)$$

Rational Heptagon (TXM).pdf

Rational Malfatti Triangle

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract

Here the author gives a complete solution to the rational Malfatti triangle.

In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices.

Three circles packed inside a triangle such that each is tangent to the other two and to two sides of the triangle are known as Malfatti circles.

The triangle formed by the centers of three Malfatti circles is called the Malfatti triangle.

Given I, DEF is the incenter and Malfatti triangle of triangle ABC, respectively.

Then, DEF is a rational triangle if and only if IABC is a rational quadrilateral. Furthermore, IDEF is also a rational quadrilateral.

Suppose x, y, z are rational numbers such that

$$x\,y\,z\,(x+y)\,(x-z)\,(y-z)\,(x+z)\,(y+z)\left(x\,y-z^2\right)$$
$$(x\,y-x\,z-y\,z-z^2)\,(x\,y+x\,z+y\,z-z^2)\left(x^2+z^2\right)\left(y^2+z^2\right)\neq 0$$

Then we can set the Cartesian coordinates of the points A, B, C, D, E, F, I as follows:

$$A=\left(2\,y\,(y-z)\,(x+z)\,(y+z)^2\left(x\,y-z^2\right)\left(x^2+z^2\right)^2,\,0\right)$$

$$B=\left(\frac{(x+z)\left(x\,y-z^2\right)\left(\left(y^2+z^2\right)^4\left(x^3-x\,z^2\right)^2+\left(x^2+z^2\right)^4\left(y^3-y\,z^2\right)^2-(x+y)^2\left(-x\,y+z^2\right)^2\left(x^2\,y^2-\left(x^2+4\,x\,y+y^2\right)z^2+z^4\right)^2\right)}{y\,(y-z)\left(x^2+z^2\right)^2},\right.$$
$$\left.\frac{8\,x\,(x+y)\,(x-z)\,z\,(x+z)^2\,(y+z)\left(-x\,y+z^2\right)^2\,((y-z)\,z+x\,(y+z))\,(x\,(y-z)-z\,(y+z))}{\left(x^2+z^2\right)^2}\right)$$

$$C=(0,0)$$

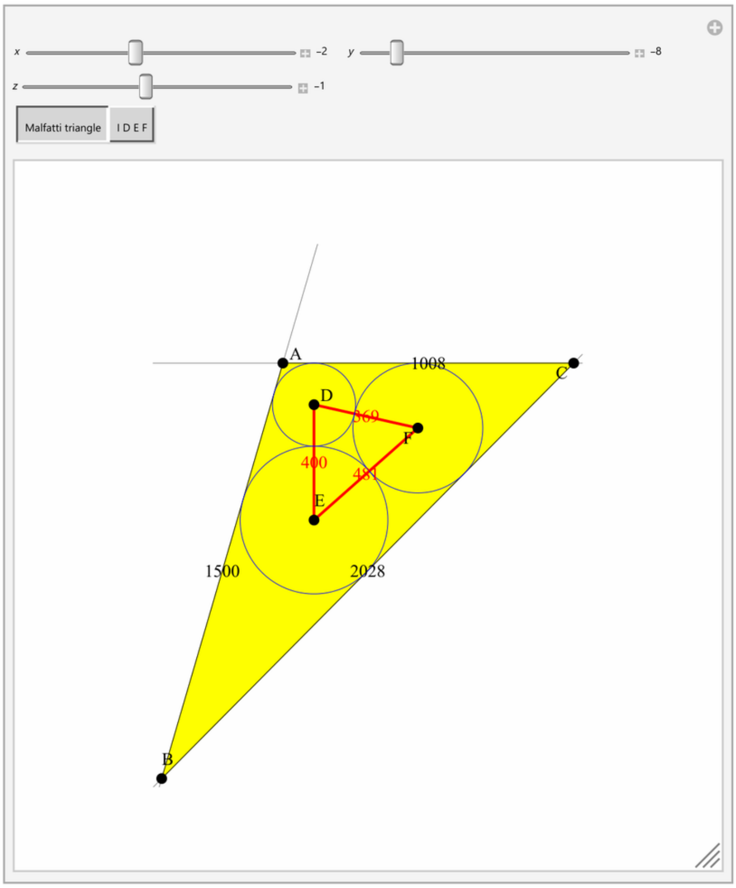
$$D=\left(2\,z\,(x+z)\,(y+z)^2\left(x\,y-z^2\right)\left(2\,x^2\,y-x\,(x+y)\,z+(-x+y)\,z^2\right)((y-z)\,z+x\,(y+z)),\right.$$
$$\left.-4\,x^2\,z\,(y+z)^2\left(-x\,y+z^2\right)^2\,(x\,(-y+z)+z\,(y+z))\right)$$

$$E=\left(-\frac{1}{\left(x^2+z^2\right)^2}\,2\,z\,(x+z)^2\left(-x\,y+z^2\right)(z\,(-y+z)-x\,(y+z))\right.$$
$$\left(-y\,z^5\,(y+z)^2-2\,x^2\,y\,z^3\left(y^2+2\,y\,z+5\,z^2\right)+x^4\,y\,z\left(-9\,y^2-2\,y\,z+15\,z^2\right)+\right.$$
$$\left.2\,x^3\,z^2\left(2\,y^3+9\,y^2\,z-3\,z^3\right)+x^5\left(2\,y^3-7\,y^2\,z+z^3\right)+x\,z^4\left(2\,y^3+y^2\,z+z^3\right)\right),$$
$$\left.-\frac{4\,z\,(x+z)^2\left(-x\,y+z^2\right)^2\left(x^4\,y^2+4\,x^3\,y\,(x+y)\,z+2\,x^4\,z^2-4\,x^2\,(x+y)\,z^3+\left(2\,x^2-y^2\right)z^4\right)(x\,(-y+z)+z\,(y+z))}{\left(x^2+z^2\right)^2}\right)$$

$$F=\left(2\,(x+y)\,z^2\,(x+z)^2\,(y+z)^2\left(x\,y-z^2\right)((y-z)\,z+x\,(y+z)),\right.$$
$$\left.-z\,(x+z)^2\,(y+z)^2\,((y-z)\,z+x\,(y+z))^2\,(x\,(-y+z)+z\,(y+z))\right)$$

$$I=\left(8\,x\,y\,(x+y)\,(x+z)\,(y+z)\left(-x\,y\,z+z^3\right)^2,\right.$$
$$\left.4\,x\,y\,z\,(x+z)\,(y+z)\left(x\,y-z^2\right)((y-z)\,z+x\,(y+z))\,(x\,(y-z)-z\,(y+z))\right)$$

Rational Malfatti Triangle (TXM).pdf



Rational Malfatti Triangle (TXM).pdf

Equation of the Form $px^2 + y^2 = n$

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract

Here the author gives a complete solution for quadratic equations of the form $n = p x^2 + y^2$

Let a, b, c, d, e, m, n be seven arbitrary integers such that

$$\begin{aligned}
&a(a^2-4b+4n)\Big(a^4c^4m^2+a^2\big(2bc^2(c^2+d^2-e^2)(e-2m)m+\\
&\quad\big(c^4(e-2m)^2+(d^2-e^2)^2(e-2m)^2+2c^2e^2(d+e-2m)(d-e+2m)\big)n\big)+\\
&\quad 8d^2(e-m)(-2b^2c^2m-b(c^2+d^2-e^2)(e-2m)n+2d^2(e-m)n^2)\Big)\neq 0.
\end{aligned}$$

Set

$$\begin{aligned}
x=\Big(&-a^6c^4m^2+64d^4(e-m)^2(b-n)^2n+\\
&a^4\big(8bc^4m^2+\big((d^2-e^2)^2(e-2m)^2+c^4(e^2+4em-8m^2)+2c^2e(-e^3+d^2(e-4m)+4em^2)\big)n\big)-\\
&4a^2\big(4b^2c^4m^2+4b(c^2+d^2-e^2)(e-2m)(d^2(e-m)+c^2m)n+\big((c^2-e^2)^2(e-2m)^2+\\
&\quad d^4(-3e^2+12em-8m^2)+2d^2e(e(e-2m)(3e-2m)+c^2(-3e+4m))\big)n^2\big)\Big)\Big/\\
\Big(&a(a^2-4b+4n)\big(a^4c^4m^2+a^2\big(2bc^2(c^2+d^2-e^2)(e-2m)m+\\
&\quad\big(c^4(e-2m)^2+(d^2-e^2)^2(e-2m)^2+2c^2e^2(d+e-2m)(d-e+2m)\big)n\big)+\\
&\quad 8d^2(e-m)(-2b^2c^2m-b(c^2+d^2-e^2)(e-2m)n+2d^2(e-m)n^2)\big)\Big)
\end{aligned}$$

and

$$\begin{aligned}
y=\Big(&-64bd^4(e-m)^2(b-n)^2n-\\
&a^6c^2m(bc^2m+2(c^2e+(d-e)(d+e)(e-2m))n)+a^4\big(8b^2c^4m^2-b\big((d^2-e^2)^2(e-2m)^2+\\
&\quad c^4(e^2-12em+8m^2)+2c^2(-e^2(e-6m)(e-2m)+d^2(e^2-12em+16m^2))\big)n+\\
&\quad 4(c^4e(e-2m)+e(-d^2+e^2)(e-2m)(d^2+e^2-2em)-2c^2(e-m)(e^3+4d^2m-2e^2m))n^2\big)-\\
&4a^2\big(4b^3c^4m^2+4b^2(c^2+d^2-e^2)(e-2m)(c^2m+d^2(-e+m))n+b\big((c^2-e^2)^2(e-2m)^2+\\
&\quad d^4(-3e^2-4em+8m^2)+2d^2(-e^2(5e-6m)(e-2m)+c^2(5e^2-20em+16m^2))\big)\\
&\quad n^2+8d^2(d^2e-(c-e)(c+e)(e-2m))(e-m)n^3\big)\Big)\Big/\\
\Big(&a(a^2-4b+4n)\big(a^4c^4m^2+a^2\big(2bc^2(c^2+d^2-e^2)(e-2m)m+\\
&\quad\big(c^4(e-2m)^2+(d^2-e^2)^2(e-2m)^2+2c^2e^2(d+e-2m)(d-e+2m)\big)n\big)+\\
&\quad 8d^2(e-m)(-2b^2c^2m-b(c^2+d^2-e^2)(e-2m)n+2d^2(e-m)n^2)\big)\Big)
\end{aligned}$$

Then the second – degree Diophantine equation of the following form follows :

$$n = (n a^2 - b^2) x^2 + y^2.$$

Related links

Equation of the Form $px^2 + y^2 = n$

Equation of the Form $\mathrm{px}^2 + y^2 = n(TXM).pdf$

Diophantine Equation—2nd Powers

Pell Equation

Diophantine Equation

Number Magic with Squares

A Three-Term Algebraic Identity with Squares or Quartics

A Two-Power Algebraic Identity

Seven Points with Integral Distances

Equation of the Form px$^2 + y^2 = n(TXM).pdf$

General Rational Quadrilateral

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract

Here the author gives the definition of the congruent factor and the formula that allows to construct a rational quadrilateral corresponding to any congruent factor.

In a rational polygon, the distance between any two vertices is a rational number.

The area of any rational polygon is of the form $a\sqrt{b}$; call $\sqrt{b}$ the congruent factor of that polygon.

Suppose x, y, z, u, v are rational numbers such that

$$x\,y\,z\,(x+y+z) > 0 \text{ and } (u+v\,x\,z)\,(u-v\,y\,(x+y+z))\left(-u\,v\,y\,z\,(y+z)+u^2\,(x+y+z)+v^2\,x^2\,y\,z\,(x+y+z)\right)\left(u^2\,x+u\,v\,y\,z\,(y+z)+v^2\,x\,y\,z\,(x+y+z)^2\right) \neq 0$$

Then we have rational quadrilateral ABCD with the lengths of sides as follows:

$\mathrm{BC} = y+z$

$\mathrm{CA} = x+z$

$\mathrm{AB} = x+y$

$$\begin{aligned}\mathrm{DA} = \Big(&(x+z)\left(u^2+v^2\,x\,y\,z\,(x+y+z)\right)\\
&\left(-u^2\,(x+y+z)+v^2\,x\,y\,z\,(x+y+z)^2+u\,v\,z\left(-2\,x^2-x\,(y+z)+y\,(y+z)\right)\right)\\
&\left(-u^2\,x+v^2\,x^2\,y\,z\,(x+y+z)+u\,v\,y\left(2\,x^2+3\,x\,(y+z)+y\,(y+z)\right)\right)\Big)\Big/\\
\Big(&(u+v\,x\,z)\,(u-v\,y\,(x+y+z))\left(-u\,v\,y\,z\,(y+z)+u^2\,(x+y+z)+v^2\,x^2\,y\,z\,(x+y+z)\right)\\
&\left(u^2\,x+u\,v\,y\,z\,(y+z)+v^2\,x\,y\,z\,(x+y+z)^2\right)\Big)\end{aligned}$$

$$\begin{aligned}\mathrm{DB} = \Big(&(y+z)\,(u-v\,y\,z)\,(u+v\,x\,(x+y+z))\\
&\big(u^4\,x\,(x+y+z)-4\,u^3\,v\,x\,y\,z\,(x+y+z)+4\,u\,v^3\,x^2\,y^2\,z^2\,(x+y+z)^2+v^4\,x^3\,y^2\,z^2\,(x+y+z)^3+\\
&\quad u^2\,v^2\,y\,z\left(2\,x^4+8\,x^3\,y+x^2\,(11\,y-z)\,(y+z)+6\,x\,y\,(y+z)^2+y^2\,(y+z)^2\right)\big)\Big)\Big/\\
\Big(&(u+v\,x\,z)\,(u-v\,y\,(x+y+z))\left(-u\,v\,y\,z\,(y+z)+u^2\,(x+y+z)+v^2\,x^2\,y\,z\,(x+y+z)\right)\\
&\left(u^2\,x+u\,v\,y\,z\,(y+z)+v^2\,x\,y\,z\,(x+y+z)^2\right)\Big)\end{aligned}$$

$$\begin{aligned}\mathrm{DC} = \Big(&u\,v\,(x+z)\,(y+z)\\
&\big(u^4\left((x+y)^3+y\,(3\,x+2\,y)\,z+y\,z^2\right)+v^4\,x^2\,y^2\,z^2\,(x+y+z)^2\left((x+y)^3+y\,(3\,x+2\,y)\,z+y\,z^2\right)-\\
&\quad 2\,u^3\,v\,y\,z\left(-2\,x^3+x\,(y-z)\,(y+z)+y\,(y+z)^2-2\,x^2\,(y+2\,z)\right)-\\
&\quad 2\,u\,v^3\,x\,y^2\,z^2\,(x+y+z)\left(2\,x^3-y\,(y+z)^2+2\,x^2\,(y+2\,z)+x\left(-y^2+z^2\right)\right)+\\
&\quad u^2\,v^2\,y\,z\big(2\,x^5+y^2\,z\,(y+z)^2-x\,y\,(y+z)^2\,(y+4\,z)+x^4\,(4\,y+6\,z)-\\
&\qquad x^2\,(y+z)\left(2\,y^2+13\,y\,z-z^2\right)+x^3\left(y^2-2\,y\,z+5\,z^2\right)\big)\big)\Big)\Big/\\
\Big(&(u+v\,x\,z)\,(u-v\,y\,(x+y+z))\left(-u\,v\,y\,z\,(y+z)+u^2\,(x+y+z)+v^2\,x^2\,y\,z\,(x+y+z)\right)\\
&\left(u^2\,x+u\,v\,y\,z\,(y+z)+v^2\,x\,y\,z\,(x+y+z)^2\right)\Big)\end{aligned}$$

General Rational Quadrilateral (TXM).pdf

Related links

General Rational Quadrilateral

Message: #2110
Date: 2024-01-31
From: van10hoven@gmail.com
Subject: Re: Some short pdf files on rational geometry

Dear Trinh Xuan Minh,

Thanks for posting your documents.
I like your efforts to find geometric figures with rational distances.
Especially your first attachment with “Rational point whose distances are perfect squares” I find very appealing. It has all properties to become a classic problem. I wonder if there are more of these examples.

Best regards,
Chris van Tienhoven

← Previous Next → ↩ Message Index ↑ Subjects

Message: #2111
Date: 2024-01-31
From: van10hoven@gmail.com
Subject: CU-inscribed complete n-Gons

Dear Eckart and Bernard,

Here is another bycatch.
It is about Complete Polygons that can be spanned into a cubic.
See attachment.
Also included a very simple construction of the P-tangent and P-tangential at a general cubic.
I already found two applications of the Complete Hexagon spanned into a Cubic, where I was working on in the first place, but I have to work it out better first.

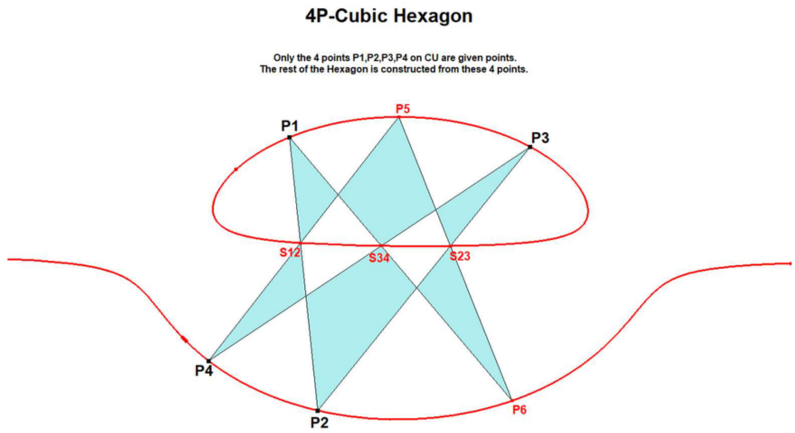
Best regards,
Chris

CU-inscribed complete n-Gons

Definition 1: A *hexagon*, or 6-gon, is a geometrical figure composed of six consecutive vertices, with no three of them being collinear.

Definition 2: A *complete hexagon* is a geometric figure formed by six consecutive vertices, with no three of them being collinear. Additionally, it includes the points of intersection of the 3 pairs of opposite sides.
In other words, a complete hexagon consists of six consecutive vertices and the points of intersection of the 3 pairs of opposite sides, creating a set of nine points.

Theorem: It is possible to inscribe a complete hexagon into a cubic using just four starting points.



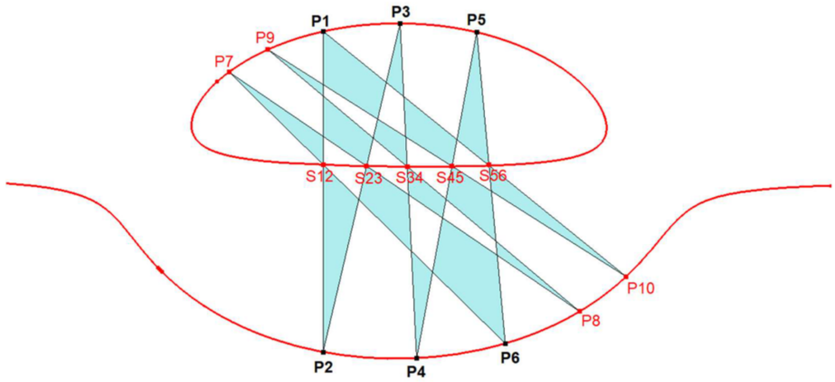
The procedure to construct the CU-inscribed Hexagon (6-gon) using 4 random points on CU is:
1. Start with 4 random points P1,P2,P3,P4 on a reference cubic CU.
2. Draw line P1P2 intersecting CU again in S12.
3. Draw line P2P3 intersecting CU again in S23,
4. Draw line P3P4 intersecting CU again in S34,
5. Draw line P4S12 intersecting CU again in P5,
6. Draw line P5S23 intersecting CU again in P6.
7. Draw line P6S34 intersecting CU again in P1 !!!

In a similar way it is possible to inscribe a complete decagon into a cubic using just six starting points.
1. Start with 6 random points P1,P2,P3,P4,P5,P6 on a reference cubic CU.
2. Draw line P1P2 intersecting CU again in S12.
3. Draw line P2P3 intersecting CU again in S23,
4. Draw line P3P4 intersecting CU again in S34,
5. Draw line P4P5 intersecting CU again in S45,
6. Draw line P5P6 intersecting CU again in S56,
7. Draw line P7S12 intersecting CU again in P8,
8. Draw line P8S23 intersecting CU again in P9.
9. Draw line P9S34 intersecting CU again in P10,
10. Draw line P10S45 intersecting CU again in P11.
11. Draw line P11S56 intersecting CU again in P1 !!!

CU-Inscribed Complete n-Gons-01.pdf

6P-Cubic Decagon

Only the 6 points P1,P2,P3,P4,P5,P6 on CU are given points.
The rest of the Decagon is constructed from these 6 points.



Extrapolating, it appears that there is a CU-inscribed 2P-2-gon, 4P-6-gon (Hexagon), 6P-10-gon (Decagon), 8P-14-gon, etc.

Construction of 2-Gon/6-Gon/10-Gon/etc.
Construction of a CU-inscribed 4P-6-gon in a more general way:
1. Start with 4 random points P(1),P(2),P(3),P(4) on the reference cubic CU.
2. Draw lines P(i)P(i+1) intersecting CU in S(i,i+1) for i=1,2,3
3. Draw lines P(i+3)S(i,i+1) intersecting CU in P(i+4) for i=1,2,3
Finally P(7) coincides with P(1).

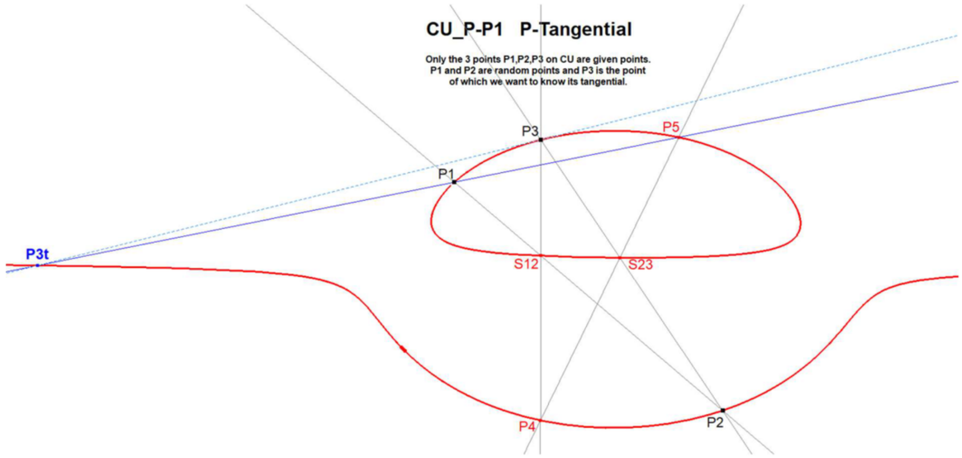
Similar construction of a CU-inscribed (n+1)P-2n-gon, valid for n=3,5,7, etc.
1. Start with n+1 random points P(1),…,P(n+1) on a cubic CU.
2. Draw lines P(i)P(i+1) intersecting CU in S(i,i+1) for i=1,..,n
3. Draw lines P(i+n)S(i,i+1) intersecting CU in P(i+n+1) for i=1,…,n
Finally P(2n+1) coincides with P(1).

4-Gons/8-Gons/12-Gons/etc.
Note that using the same procedure for a CU-inscribed 3P-4-gon, 5P-8-gon, 7P-12-gon, etc. it doesn't deliver the original starting point:
In the case of a 3P-4-gon, P5 doesn't coincide with P1, like it does with the 4P-hexagon and 6P-decagon, etc.
However P1P5 is the Tangential of P3.
In the case of a 5P-8-gon, P9 doesn't coincide with P1, like it does with the 4P-hexagon and 6P-decagon, etc.
However P1P9 is the Tangential of P5.

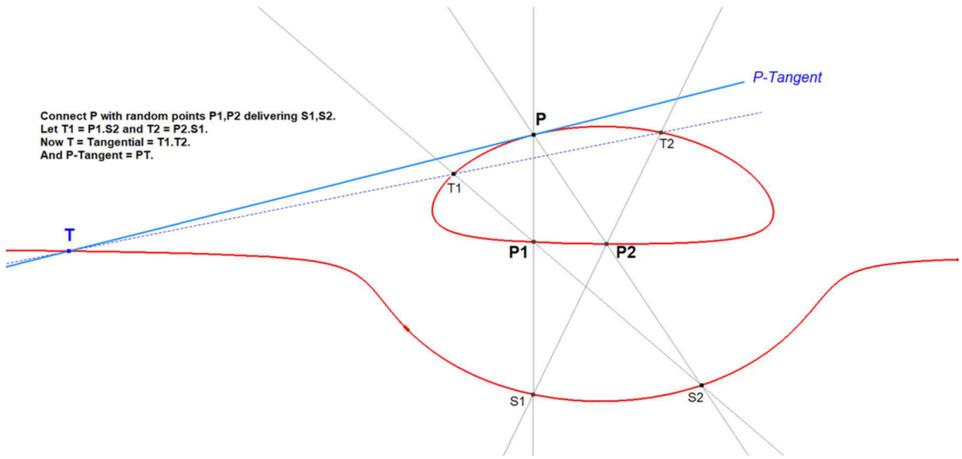
Connection with Eckart's Hexagon Theorem
Eckart's Hexagon Theorem (QPG#1799) states that given 3 collinear points P,Q,R and a starting point X1, then a closed Hexagon can be constructed. The 6 vertices will be coconic.
The way of construction of the Hexagon is similar to the construction of the CU_4P-Hexagon, only the starting point is different, there are 3 collinear starting points.

Relationship with P-Tangent/P-Tangential
In general, when using n=2,4,6, … the construction also can be done, however P(2n+1) will not coincide with P(1).
P(2n+1) will be determined by this relationship: P(1)P(2n+1) = the Tangential of P(n+1).
So there is a CU-inscribed 2n-gon created for n=3,5,7, …
But there is not a CU-inscribed 2n-gon created for n=2,4,6, …

By the way, the property that P(1)P(2n+1) = the Tangential of P(n+1) for n=2,4,6,… can be used perfectly to construct the P-tangent or the P-tangential.
Let’s take n=2, that means that P(1)P(5)=the Tangential of P(3).



Rearranging the point names gives this simple construction:



CU-Inscribed Complete n-Gons-01.pdf

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Message: #2112
Date: 2024-01-31
From: bernard.keizer@gmail.com
Subject: Re: CU-inscribed complete n-Gons

Dear Chris,
Apparently, you didn't read my message 2086! Neither did Eckart, although I told, I found this construction in Schröter ...
All this is rather obvious ...
For the hexagon, P1 and P4, P2 and P5 and P3 and P6 are 3 copples of corresponding points defining a cubic and it's QMT.
P1P2 and P4P5 intersect in S12, but also P1P5 and P2P4 intersect in T12 on the cubic.
P1 and P4, P2 and P5 and P3 and P6 have the same tangentials on the cubic, corresponding points of the 3rd intersections of P1P4, P2P5 and P3P6.
S12 and T12 are themselves corresponding points, having the same tangential ...
Best regards
Bernard
I'm waiting impatiently now on your complete memo about general cubics ...

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Message: #2113
Date: 2024-02-01
From: hoingason@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] Some short pdf files on rational

---

Dear Mr. Chris van Tienhoven
Thank you, the file you mentioned is also the only solution for "Rational
point whose distances are perfect squares”
Best Regards
-TXM-

Vào 0:48, Th 5, 1 thg 2, 2024 Chris <van10hoven@gmail.com> đã viết:

> Dear Trinh Xuan Minh,
> Thanks for posting your documents.
> I like your efforts to find geometric figures with rational distances.
> Especially your first attachment with “Rational point whose distances are
> perfect squares” I find very appealing. It has all properties to become a
> classic problem. I wonder if there are more of these examples.
> Best regards,
> Chris van Tienhoven

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Message: #2114
Date: 2024-01-31
From: bernard.keizer@gmail.com
Subject: Re: CU-inscribed complete n-Gons

```
Dear Chris,
Please forget my previous message!
In fact, P1 and P4 have no reason to be corresponding points!
Your property of closed hexagon is simply  a result of Fred
Lang's calculation ...
Best regards
Bernard
```

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Message: #2115
Date: 2024-02-01
From: eckart_schmidt@t-online.de
Subject: Real cubic elements

```
Dear Bernard, dear Chris,

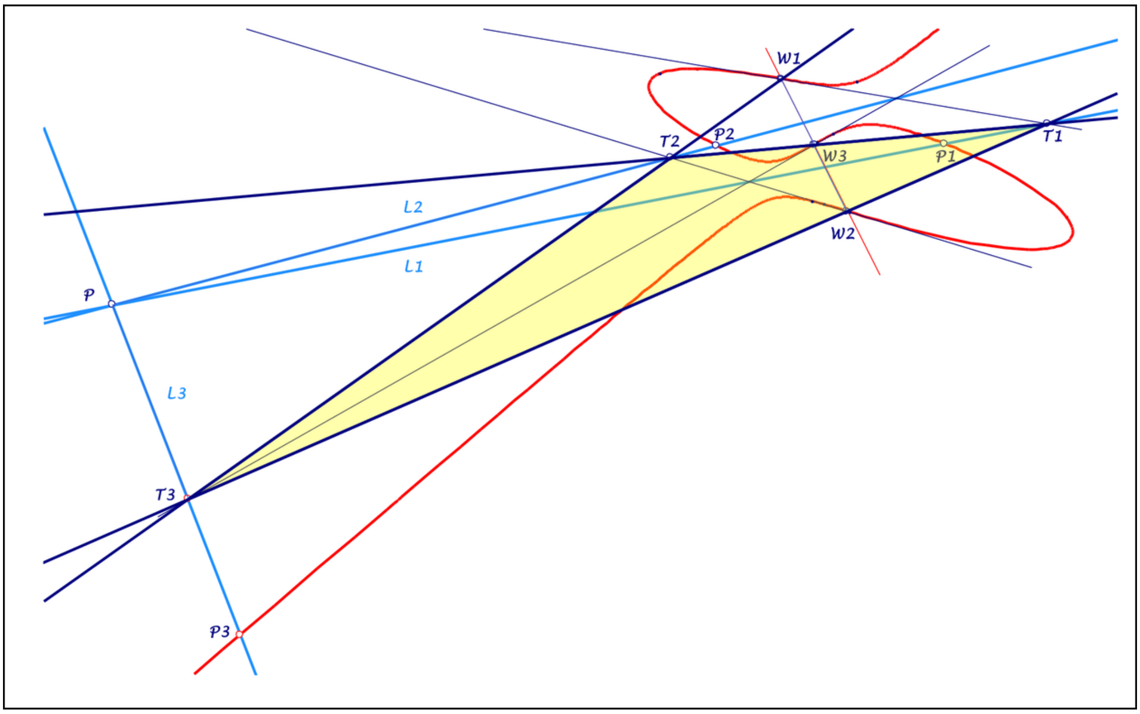
please remember my point-constellation for a general cubic:
W1,W2,W3  real flexpoints,
L1,L2,L3  harmonic polars of the real flexpoints,
P1,P2,P3  intersections of L1,L2,L3
          and the not closed part(s) of the cubic,
P         common point of L1,L2,L3,
T1,T2,T3  intersections of L1,L2,L3 and the flextangents.
P,Pi,Ti   collinear, Pi,Pj,Wk collinear, Ti,Tj,Wk collinear.

The 3 real flexpoints W1,W2,W3 are collinear on a line L,
dual: the 3 harmonic polars L1,L2,L3 have a common point P.
Wi and Li are dual elements wrt EQF, QA-8
... for the QA = PT1T2T3 and for the QL = TiTjWiWj.

Finally:
There are 3 of the 9 flexpoints real and collinear on a
flexline,
... only 4 of the12 flexlines are real,
    one bears the 3 real flexpoints,
... the other three give the only real flex-trilateral
... with a real flexpoint on each side ...
What about this unique real flex-trilateral, may it be T1T2T3?

Best regards Eckart

PS. References: Schröter, page 239.
```



2024-02-01.pdf

Message: #2116
Date: 2024-02-01
From: eckart_schmidt@t-online.de
Subject: Re: CU-inscribed complete n-Gons

```
Dear Chris,

thanks for your paper,
... I used your method already 2005,
... but with three collinear points S12, S23, S34
    and a starting point,
... so that the six points are coconic.
The main result of your paper
... was the fantastic simple construction of the cubic tangent,
... that was new for me, thanks!

Best regards Eckart
```

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Message: #2117
Date: 2024-02-01
From: eckart_schmidt@t-online.de
Subject: Re: CU-inscribed complete n-Gons

```
Dear Chris,

I just noticed, if we take for S12, S23, S34 the flexpoints
... we get hexagons with circumconics,
... whose polar axes for the flexpoints
    are the harmonic polars L1, L2, L3.

Best regards Eckart
```

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Message: #2118
Date: 2024-02-03
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,
We begin to know this figure perfectly, as you put it already half a dozent time on the Forum!
But you add this time the notion of duality, which I found very promising ...
It took me time to realise that the duality cannot be at the same time wrt the QA PT1T2T3 and wrt the QL W1W2W3T1T2T3.
Let's consider the points Xi, Yi and Zi on Li: the Xi are the vertices of the anticevian triangle of P wrt the Ti triangle,
the Yi are the vertices of the cevian triangle of P wrt the Ti triangle.
The Zi are the intersections of Li and YjYk (through Wi).
(Remember, P and the Xi are the 4 poles of the flexline wrt the cubic)
There are plenty of alignments and harmonic divisions.
I found 3 possible dualities QA/QL (but there are perhaps others ...)
1) QA is PT1T2T3 (as you mentionned)
DT is the cevian triangle of P wrt T1T2T3, id est Y1Y2Y3
QL is Z1Z2Z3 with the flexline W1W2W3as cevian
2) QL is T1T2T3 with the flexline as cevian (as you mentionned)
DT is the triangle X1X2X3
QA is P and it's anticevian triangle wrt X1X2X3
in this case, the dual of WiTi is Xi
3)DT is T1T2T3
QA is the anticevian triangle of P wrt T1T2T3, id est X1X2X3
QL is Y1Y2Y3 with the flexline as cevian
in this case, the dual of WiTi is Yi
I said at the beginning promising because the dual of the cayleyan (sextic of class 3 tangent in the Ti to the lines WiTi) will be a cubic through the Xi in the 2nd case and to the Yi in the 3rd case!
But are you interested in the cayleyan?
Best regards
Bernard

Message: #2119
Date: 2024-02-03
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Bernard,

thanks for interest, I have reproduced your remarks,
... they are correct.
The reason for our misunderstanding will be,
... that I mean with QL = TiTjWiWj
    a QL with lines TiTj, TjWi, WiWj, WjTi.
Excuse if I am not familiar with cayleyan.

Best regards Eckart
```

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Message: #2120
Date: 2024-02-05
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,
Always on your figure 2115: Pi is the unique contact point of the tangent from Wi to the cubic; it is the point Ti for the unique real prehessian of your cubic!
Drawing the 4 tangents (other than TiWi) from Ti to the cubic, it gives 4 points aligned in pairs with Wi; the polar conic of Wi is formed by the 2 lines Li and tgi, which are harmonic wrt each pair of tangents and the 2 lines through Wi form the polar conic of Ti wrt the cubic, which are harmonic wrt WiPi and?
Concerning the cayleyan, we dicussed it in old times about the cubic stelloïd QL-Cu2 and it's hessian QL-Cu1.
It is rather complicate to find 26 points in order to draw this curve.
Suppose now thay you have a duality wrt any QA/QL
It's then easier to find 9 lines tangent to the curve and to draw the cubic through the 9 dual points of the lines; the cayleyan will be the dual of this cubic, id est the locus of the dual points of the tangents to the cubic (the dual of a curve is either the envelop of the dual lines of it's points or the locus of the dual points of it's tangents).
Best regards
Bernard

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Message: #2121
Date: 2024-02-09
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris, dear Bernard,

wrt hessian and cayleyan of a cubic I found the following property:

498. Lässt man eine Gerade G alle möglichen Lagen in der Ebene annehmen und bestimmt jedesmal ihre vier Pole a b c d bezüglich einer Curve 3. O., so beschreiben die Diagonalpunkte des vollständigen Vierecks a b c d die Hesse'-sche Curve und die Seiten dieses Vierecks hüllen die Cayley'sche Curve ein.
Reference: H. Durege (1871): Die ebenen Kurven dritter Ordnung, Eine Zusammenstellung ihrer bekannteren Eigenschaften. Page 264.

Best regards Eckart

PS. I hope you will understand the German text.
In a reprinting of hansebooks the author is written: H. Duerge.

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Message: #2122
Date: 2024-02-09
From: van10hoven@gmail.com
Subject: Re: New construction of a general cubic

Dear Bernard,

I started to study your message #2086.
However I don't understand it.
Did you find it at Salmon or Schröter? At what pages?
You tell it is a simple construction method of a general cubic.
Then you start with 6 points 'et voila' there is the general
cubic with the little help of an MT or QMT. I think I miss some
points.
After all a unique general cubic can't be constructed with less
than 9 points.
Could you explain a bit more and give me the reference to which
book, which pages.
Best regards,

Chris

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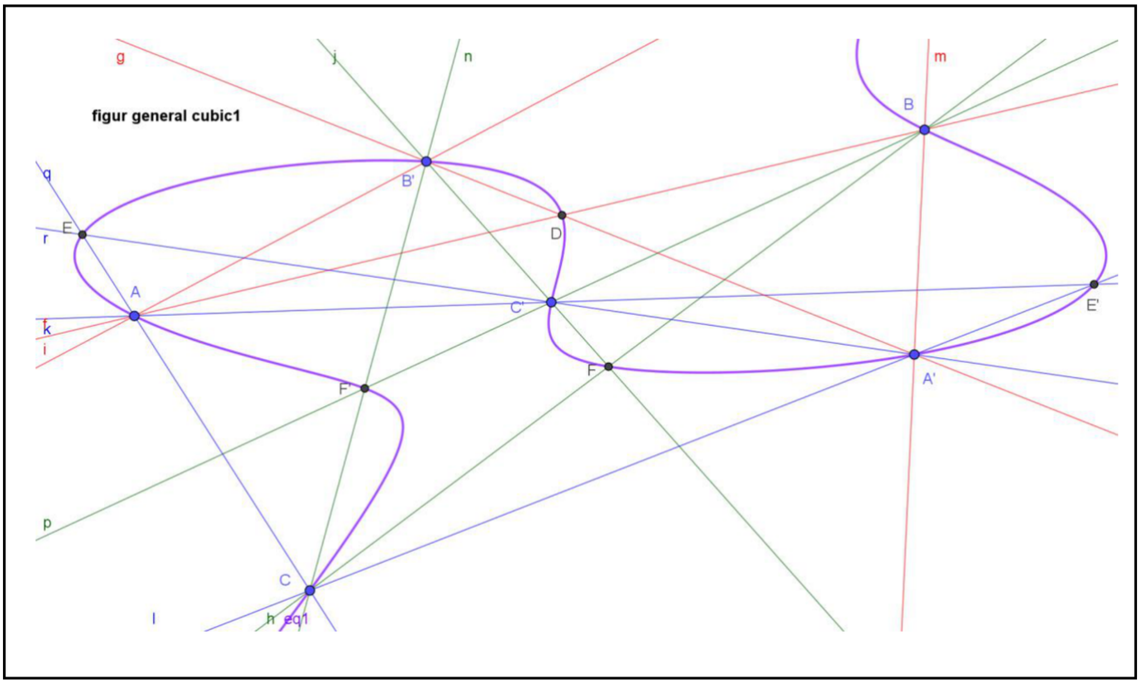
Message: #2123
Date: 2024-02-11
From: bernard.keizer@gmail.com
Subject: Re: New construction of a general cubic

```
Dear Chris,
Thanks for this interest!
I found this construction in Schröter in the first pages.
I don't start with 6 points, but with 3 copples of points, not
forming a QL!
I send you a picture, I suppose it is more convincing than a
long message ...
Best regards
Bernard
```



figur general cubic.pdf

Message: #2124
Date: 2024-02-11
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
This is indeed a good way of presenting the properties of the hessian and the cayleyan!
But thanks to you and Schröter, we know already more than that ...
1) the DT vertices are on the hessian
2) the intersections of the DT sides with the initial line are also on the hessian
3) the DT vertices and the 3 points on the line form 3 copples of conjugated points of the hessian
4) the 3 lines through these 3 copples of points are also tangent to the cayleyan
If you consider now the dual QL of the QA of the 4 poles, the caleyan is the dual of a cubic through the QL vertices and through the dual points of the 3 lines in point 4).
But I suppose these 9 points form a CB system ...
Best regards
Bernard

PS If you take as the initial line a line through 2 conjugated points of the hessian, the 4 poles are 2 copples of points of the hessians of 2 other cubics having the same cayleyan.
The DT of the QA of the poles is formed by the 2 conjugated points and their tangential on the 1st hessian. I'll try to prepare a short memo ...

Message: #2125
Date: 2024-02-12
From: van10hoven@gmail.com
Subject: Re: New construction of a general cubic

Dear Bernard,

Thanks, now I understand.

One can construct a cubic from 3 pairs of points (A,A'), (B,B'), (C,C').
By finding extra QL-diagonal-points and constructing then the 9P-cubic.
The MT is the QMT-transformation as usual.
And M is the 3rd intersection point of P1P2, where P1 and P2 are the Anallagmatic points.
The QMT-associate of X is the point with same tangential.

Best regards,
Chris

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Message: #2126
Date: 2024-02-12
From: van10hoven@gmail.com
Subject: Re: Hessian for a general cubic:

```
Dear Eckart,

I am catching up with older messages.
So I found your message 2100, where you say:
--------------------------------------------
“Is already mentioned, that Chris' final digital triangles
... are pairwise perspective wrt the flexpoints?”
I suppose you mean "diagonal triangles".
This is not what I see in the drawing of my message #1944.
Do you have a drawing with this feature?

Best regards,

Chris
```

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Message: #2127
Date: 2024-02-12
From: eckart_schmidt@t-online.de
Subject: Re: Hessian for a general cubic:

```
Dear Chris,

excuse, I forgot in #2100, that the observation was only made
... for a construction,
    starting with the points P1,2,3 in my point constellation,
... P1,2,3 are the intersections of the harmonic polars
    of the flexpoints
... and the non closed part of the cubic.
This can be controlled in the attached drawing,
... the triangles are colored.

Best regards Eckart
```

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Message: #2128
Date: 2024-02-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

what do you mean with your introduction of #2124,
... no doubt, we know more ...
... your property 1)
    is already part of Chris' construction of the hessian,
... your properties 2) and 3) were new for me, thanks,
... but I needed some days to reproduce
    your concept for the cayleyan.

I have drawn the dual cubic wrt the special case
... for a line PiPj in my point constellation
... and tested whether known tangents of the cayleyan
... have QA-dual points on this cubic,
... but I struggle with the precision ...
... up to now I cannot confirm your construction.

Are the following tangents to the cayleyan correct?
1) tangents and harmonic polars of the flexpoints
... as part of their degenerated polar conics,
2) common tangents of cubic and its hessian,
3) tangents to the hessian in the flexpoints.

Can you give references for your cayleyan concept?

Best regards Eckart
```

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Message: #2129
Date: 2024-02-14
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

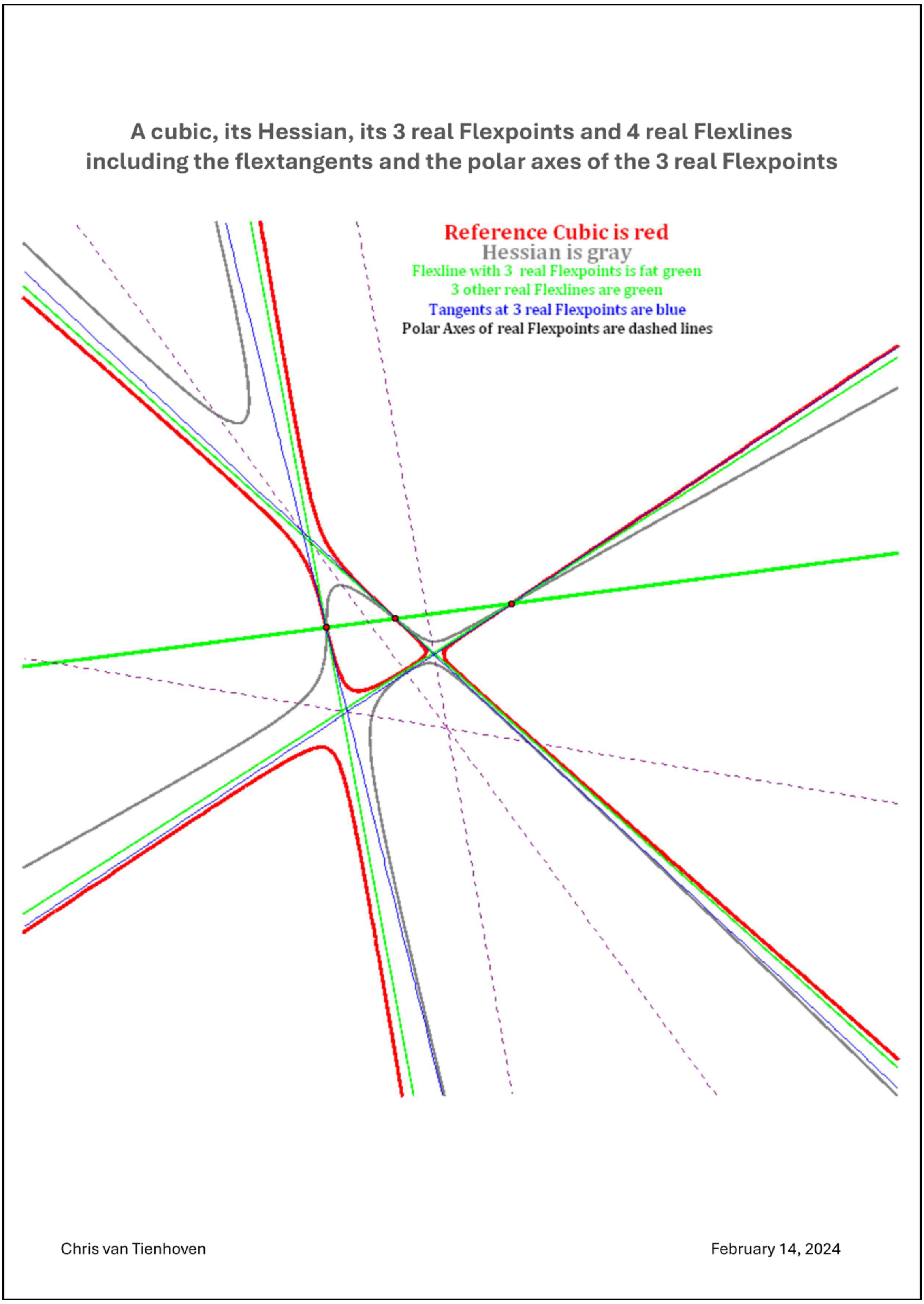
Regarding your message #2115, where you ask:

“What about this unique real flex - trilateral, may it be T1T2T3?”

I investigated this in an algebraic setting (with Mathematica), depicting some cubic and its Hessian, then determining the real flex points, the flextangents and the polar axes for these points and last but not least calculated the 3 additional real flexlines.
The big advantage of calculations with Mathematica is that you can deal with imaginary points and imaginary lines just as easily as with real points and real lines.

>From attached picture you can see that T1T2T3 is not the flex-trilateral.
Nevertheless the flexlines-trilateral and the flextangents-trilateral are perspective, where the polar axes are the conductive lines meeting at P.
I do not see right away a manner to construct the flexlines in a not-algebraic way.

Best regards,
Chris



CU-12L1 real Flexlines and real Flexpoints-01.pdf

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Message: #2130
Date: 2024-02-15
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
It's hard to progress this way!
How can you say the properties 2) and 3) are new for you?
They are in Schröter and already mentionned by you in #2094 and
by me in #2105 and #2106 ...
Best regards
Bernard

For your questions 1) and 3) certainly 2) I don't know.
I have no special references, cayleyan is a well-known concept
and I gave the definitions in the starting message #2108 of this
item.
```

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Message: #2131
Date: 2024-02-15
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,
thanks for reproaches, excuse my interest in your message.
Eckart
```

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Message: #2132
Date: 2024-02-16
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

many thanks for clearing my question
... and the drawing gathering the results.
The perspectivity is a relevant property
... and the point P becomes more and more significant.
Your drawing shows two triangles
... with vertices on the harmonic polars L1,2,3
    of the real flexpoints W1,2,3,
... bearing on each side a flexpoint:
(1) triangle of the flextangents Tgi of the real flexpoints,
(2) triangle of the real flexlines unequal L = W1W2W3.

There are more triangles X1X2X3 with this property
... X1 on L1, X2 = L2 ^ X1W3 on L2 and X3 = L3 ^ X1W2 on L3:
(3) triangle with vertices in the intersections
    of Li and flextangent Tgi,
(4) the anticevian triangle of P wrt any triangle of this type
    is also of this type.

Finally we can consider the variable
... QA = P plus X1, X2, X3 and QL = L plus X1X2,X2X3,X3X1
... with dual elements P and L as well as Wi and Li.

It is very interesting to study this point constellation,
... starting with the flexpoints,
... a last example: locus for CSC(P) is a circle ...

Best regards Eckart
```

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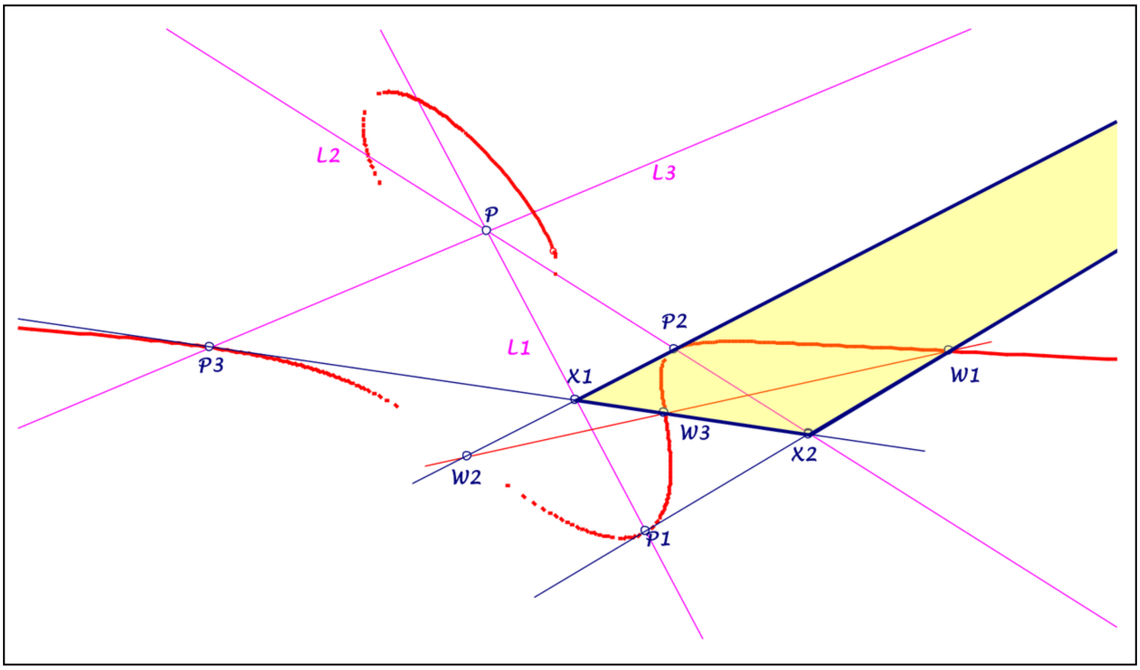
Message: #2133
Date: 2024-02-17
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

wrt the three other real flexlines:
Can you proof with your calculations for the drawing in #2129,
... whether the cubic tangents WiPi in Pi
    are the other real flexlines?
Pi = intersection of the harmonic polar Li of Wi and unclosed
parts of the cubic.
This cannot clear be controlled in your drawing.
Thanks in advance.

Best regards Eckart

PS. Correction for the last observation in #2128:
... locus for CSC(P) are two circles.
```



2024-02-17.pdf

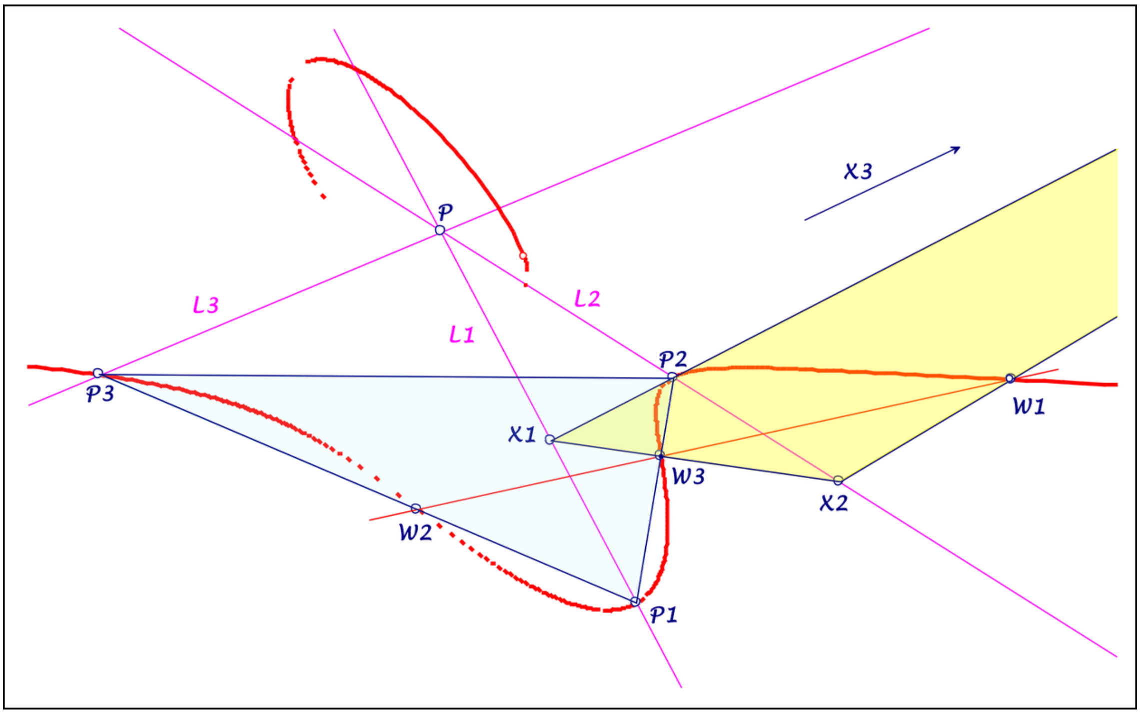
Message: #2134
Date: 2024-02-18
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

further aspect of the triangle in #2133,
... whose sidelines are the cubic tangents in P1,2,3,
... with vertices X1,2,3 in the anticevians of P wrt P1P2P3

Perhaps already mentioned,
... the cubic is invariant wrt the isoconjugation QA-Tf2
    for PX1X2X3,
... swapping pairwise points with asyparallel
    tangents on different parts of the cubic
... and maps the harmonic polars Li of the flexpoints to itself.

Best regards Eckart
```



2024-02-18.pdf

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Message: #2135
Date: 2024-02-19
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart,

Regarding your question in #2133:
“Can you proof with your calculations for the drawing in #2129,
... whether the cubic tangents WiPi in Pi
    are the other real flexlines?”

When I look to my picture in QPG#2129, then I only see one
intersection point of The Polar Axes Li with the cubic.
This point clearly can be seen does not deliver a green flexline
combined with one of the flexpoints Wi.

Best regards,
Chris
```

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Message: #2136
Date: 2024-02-19
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

if I look in your picture, there are two intersections P1,P2
... of the cubic (red)  and the polar axes (dashed lines),
... a third one will be down outward the paper.

In  P1 and P2 the real flexlines (green) seem to contact the
cubic,
... so I think my assumption perhaps holds.

Best regards Eckart

PS: In Schröder's classification your cubic is monopartite.
```

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Message: #2137
Date: 2024-02-20
From: bernard.keizer@gmail.com
Subject: A magical QL construction

Dear Chris, dear Eckart
I found accidentally this construction, which I find interesting.
I have no special references, it all pure personal invention!
Naturally, it is possible to draw the cayleyan as dual of it's dual by drawing the locus of dual points of the tangents to the dual cubic ...
Best regards
Bernard

A magical QL construction

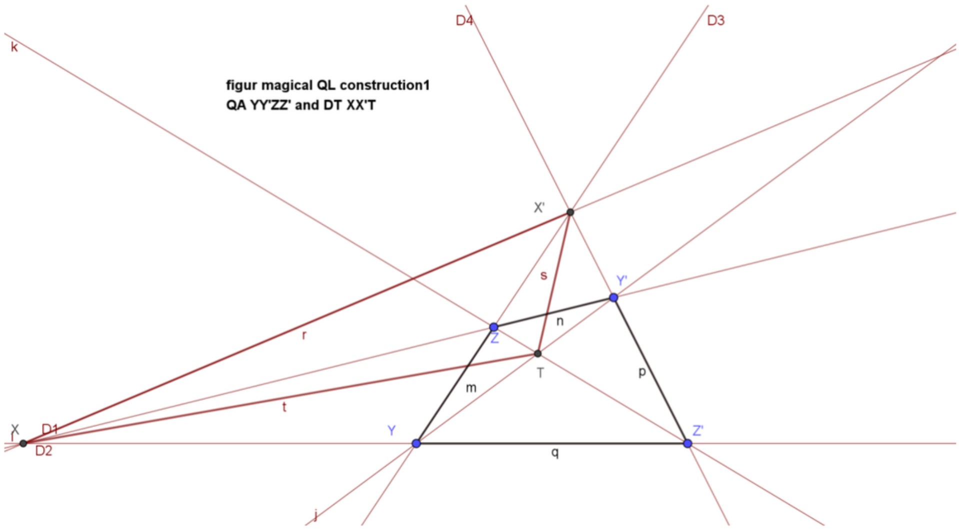
1. Figure 1 QA and DT

Let's start with a copple of conjugated points X and X' on the hessian H1 of a cubic K1.

According to the definition of the hessian, the polar conics of X and X' wrt K1 are 2 copples of lines through X and X', the poles of XX' wrt K1 are Y, Y' Z and Z' and the DT of the QA of the poles is XX'T. The 4 lines are tangent to the cayleyan.

These lines pass also through Y and Y' and through Z and Z', but in a different order (D1 and D2 through X and D3 and D4 through X', D2 and D3 through Y and D1 and D4 through Y' or D1 and D3 through Z and D2 and D4 through Z').

Let's assume that Y and Y' are conjugated points on a 2nd hessian H2 of a cubic K2 and Z and Z' are conjugated points of a 3rd hessian H3 of a cubic K3 ; K1, K2 and K3 will have the same cayleyan.



A magical QL construction.pdf

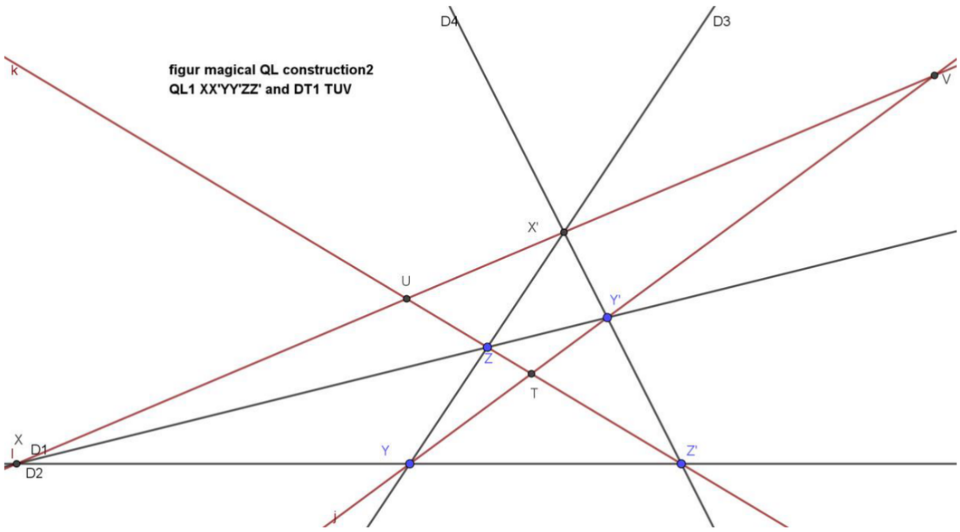
2. Figure 2 QL1 and DT1

Let's continue with the QL XX'YY'ZZ' and it's DT TUV.

TU and TV are harmonic wrt the lines TX and TX' and the same way UT and UV are harmonic wrt UY and UY' and VT and VU are harmonic wrt VZ and VZ'.

Let's assume that T is the tangential of X and X' on H1 and TU and TV form the polar conic wrt K1 of a point T' of XX' ; the same way, U is the tangential of Y and Y' on H2 and UT and UV form the polar conic wrt K2 of a point U' on YY' and V is the tangential of Z and Z' on H3 and VZ and VZ' form the polar conic wrt K3 of a point V' on ZZ'.

The 4 lines of the QL and the 3 sides of DT will be tangent to the cayleyan.



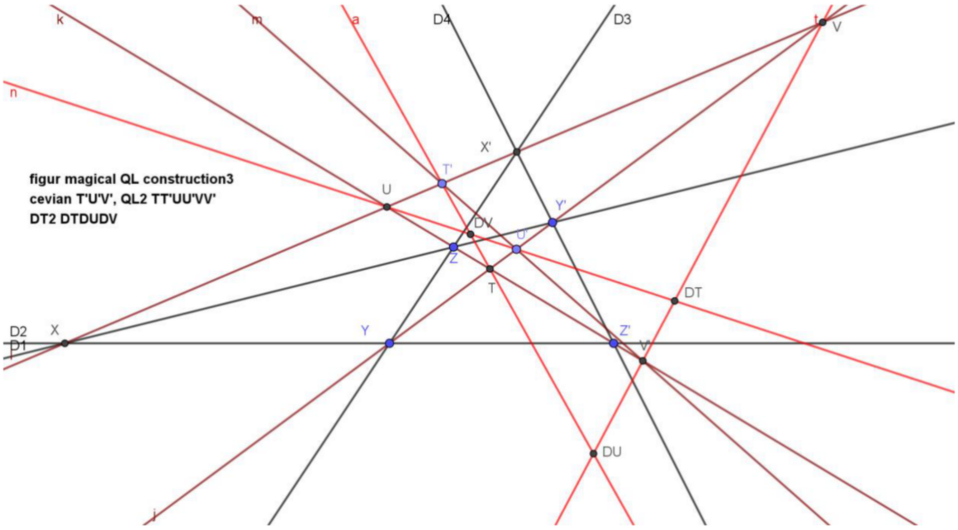
A magical QL construction.pdf

3. Figure 3 QL2 and DT2

Let's add now a cevian T'U'V' to the DT1 triangle TUV, in order to form a 2nd QL TT'UU'VV'.

The properties of QL1 hold for QL2 and TT', UU' and VV' are tangent to the cayleyan and intersect in DT, DU and DV, forming the DT2 of QL2.

Now, we have 11 lines tangent to the cayleyan, the sides of the 2 QL's QL1 and QL2 and the sides of DT2 ; the cayleyan being a curve of class 3 is entirely determined.



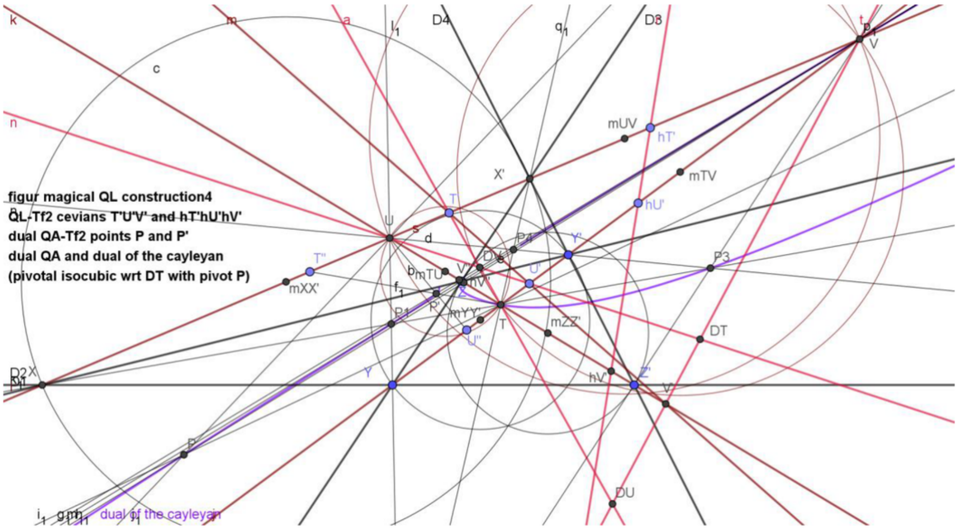
A magical QL construction.pdf

4. Figure 4 QL and dual of cayleyan

The dual of the line T'U'V' wrt QL1 is a point P.

The dual of the cayleyan wrt QL1 passes through the 4 vertices P1, P2, P3 and P4 of the dual QA of QL1, through the vertices T, U and V of DT1, through the dual point P of T'U'V' and through the dual points of TT', UU' and VV'.

It is a pivotal isocubic wrt DT1, the fixed points of the isoconjugation being P1, P2, P3 and P4 and the pivot and the isopivot being the point P and it's QA-Tf2 P' wrt the QA.



5. Figure 5 QL and 3 hessians

We may proceed the same way once more, by adding a 4th line to DT2 : it must be QL-Tf2 of T'U'V' wrt QL1 and we have now 3 QL's XX'YY'ZZ', TT'UU'VV' and DTDT'DUDU'DVDV'.

This 3rd QL has in turn as DT DDTDDUDDV (intersections of DUDU' and DVDV', DTDT' and DVDV' and DTDT' and DUDU').

We have to proceed a last time the same way by adding a 4th line to this DT3 : it must be QL-Tf2 of DT'DU'DV' wrt QL2. We have finally 4 QL's and a last DT …

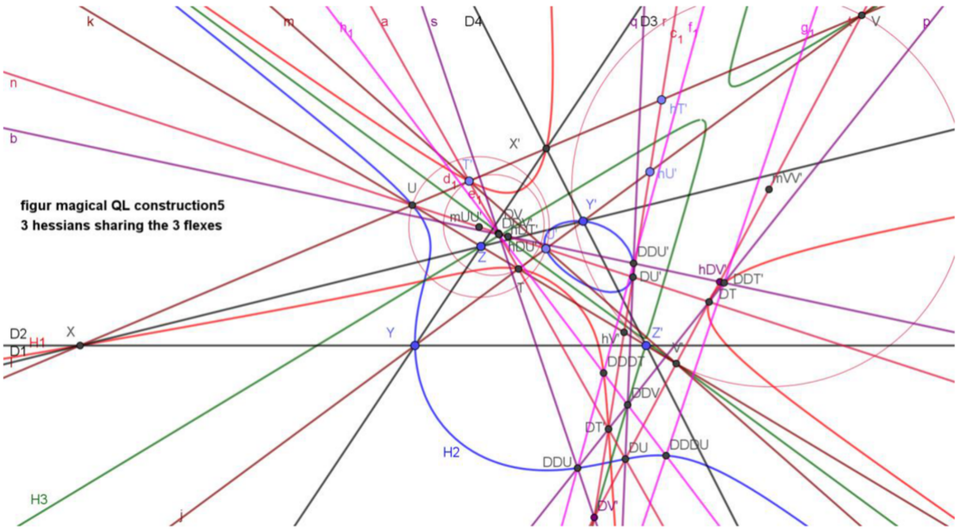
The hessian 1 is defined by X, X', T, T', DT, DT', DDT, DDT' and DDDT.

It's easy to check that T is the tangential of X and X', DT is the tangential of T and T', DDT is the tangential of DT and DT' and DDDT is the tangential of DDT and DDT'.

(If we start with the hessian H1, all these points are perfectly defined and the construction determines H2 and H3, but it's more interesting to start with a QL and a random line T'U'V').

The hessians 2 and 3 are defined the same way.

Naturally, the 3 hessians share the 3 same flexes.



A magical QL construction.pdf

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Message: #2138
Date: 2024-02-20
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

Regarding your question in QPG#2133:
"Can you proof with your calculations for the drawing in #2129,
... whether the cubic tangents WiPi in Pi are the other real flexlines?
Pi = intersection of the harmonic polar Li of Wi and unclosed parts of the cubic."

Answer according to my calculations:
The real points P1, P2, P3 do not lie on the 3 flexlines in question.
Neither do the tangents at P1,P2,P3 coincide with the 3 flexlines in question.
I found a bit awkward extra little gem:
L1,L2,L3 are the polars of W1,W2,W3 wrt conics through extra intersection points of 3 lines with CU through W1,W2,W3.

Instead of these conics we might take degenerate conics comprised by the other two L-lines.
It appears:
* L1 is the W1-polar of Conic(L2,L3).
* L2 is the W2-polar of Conic(L3,L1).
* L3 is the W3-polar of Conic(L1,L2).
In a way, the system is self-polar.
Explanation: Conic(L2,L3) passes through points that are crosswise intersection points of 2 out of 4 flexlines through W1. Etc.

Best regards,
Chris

← Previous Next → ↔ Message Index ↑ Subjects

Message: #2139
Date: 2024-02-21
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

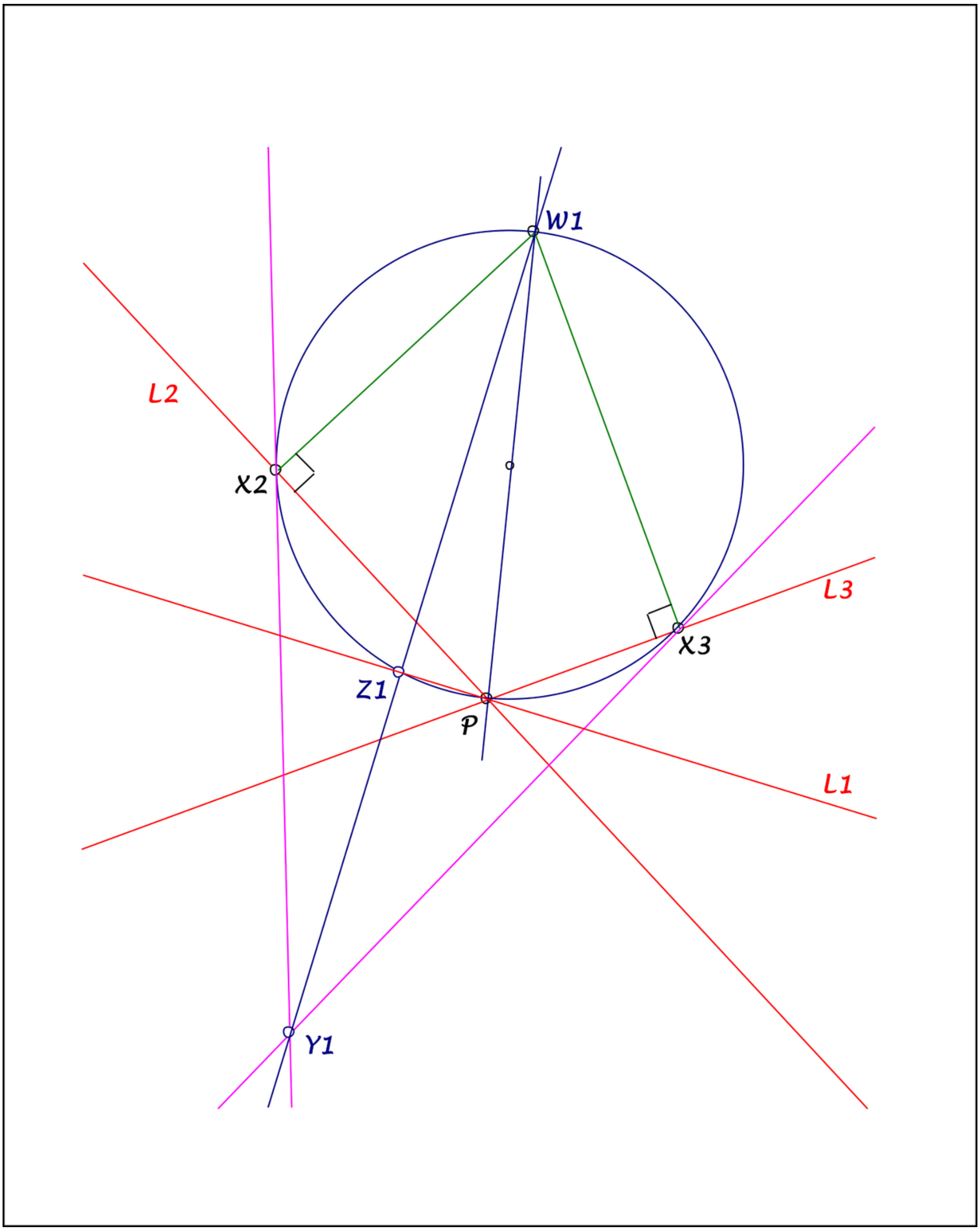
```
Dear Chris,

thanks once more for clearance of my question,
... I shall give up this problem.

Question: What are
"... crosswise intersection points
     of 2 out of 4 flexlines through W1..."?
Do you mean such a following construction of
"... the W1-polar of Conic(L2,L3)"?

X2, X3 = pedal points of W1 on L2,L3,
... CI = circumcircle (W1,X2,X3), P = CI-diametral of W1,
... CI-tangents in X2,X3, intersecting in Y1,
... Z1 = 2nd intersection of CI and W1Y1,
... PZ1 = L1 (attached).

Best regards Eckart
```



2024-02-21.pdf

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Message: #2140
Date: 2024-02-21
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

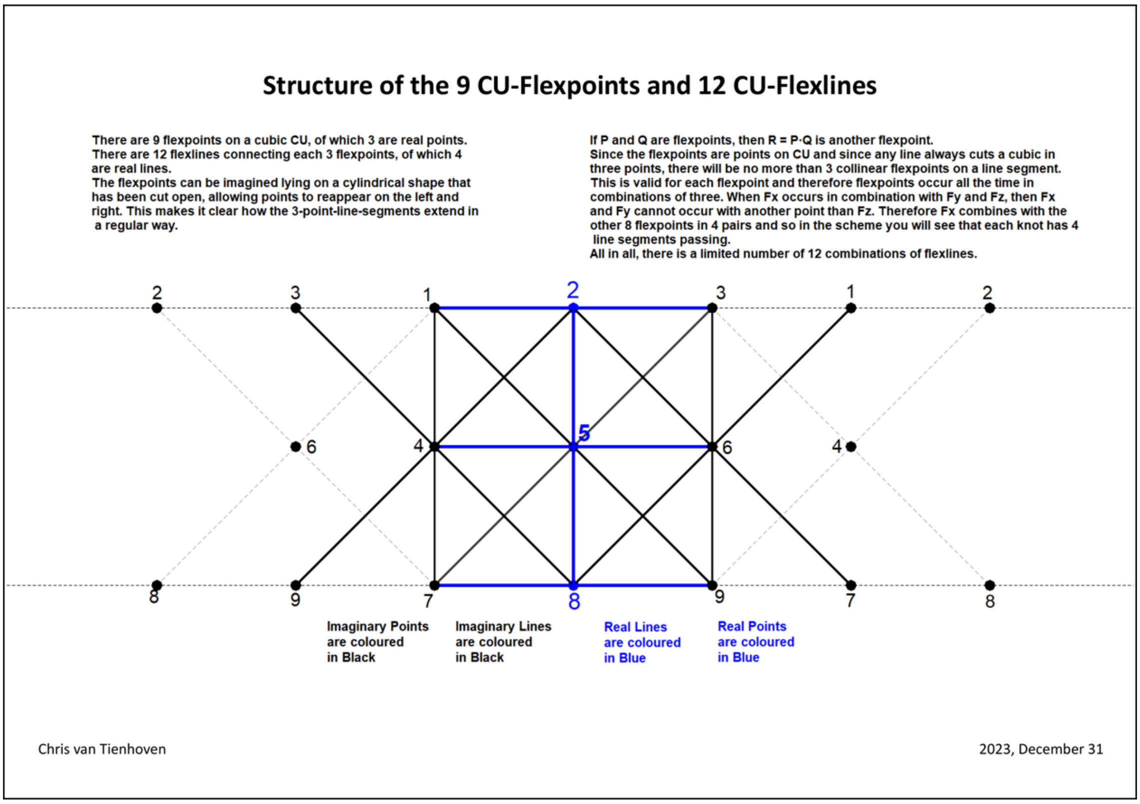
Dear Eckart,

QPG#2139
Question: What are "... crosswise intersection points of 2 out of 4 flexlines through W1..."?

Well maybe my statement was a bit cryptic. Perhaps because I have been confined to my home for days due to an extremely severe flu that I am suffering from. It was a miracle I could think anyhow.

What I meant is this:
See the picture I already sent in QPG#2049.
Look at the Flexpoints 1,2,3,4,5,6,7,8,9 and the flexlines passing through them in a regular system.
Now we concentrate on these three flexlines through point 5: (1,5,9), (3,5,7), (2,5,8).
With these lines through Flexpoint 5 we should have a conic through (1,9,3,7,2,8) (property of a flexpoint like you told me Schröter taught us). However look again at the picture: (1,2,3) and (7,8,9) are both collinear. Therefore the conic is a degenerate one.
So in the picture one of the conics (there are more versions) that you can use to determine the 5-Polar is the degenerate conic consisting of the other two real flexlines not passing through 5.
Isn’t it pretty . . .
What is valid for 5 (a real Flexpoint) is also valid for the other two real Flexpoints.
Then an extra remark still wrt your question in QPG#2133.
It is not possible for any flexline (real or imaginary) to be tangent to CU, because it already passes through three CU-Flexpoints. It cannot touch or cross CU in any point, apart -of course- from the known flexpoints.

Best regards,
Chris



CU-12L1 Flexlines-scheme 03.pdf

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Message: #2141
Date: 2024-02-24
From: hoingason@gmail.com
Subject: Objects related to rational triangles with three rational medians

---

```
As the title says, here is my short note on objects related to
rational
triangles with three rational medians. Hope it's useful to
someone.
Best Regards
--TXM--
```

Rational triangle with three rational medians

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract. In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices. In this short note I give some theorems and problems related to triangles with three rational medians.

Theorem 1. Rational Isogonal Conjugates

Given a rational triangle ABC, draw lines from the vertices to an arbitrary point X. Reflect those three lines in the angle bisectors . The three new lines intersect in a point X' called the isogonal conjugate of X. Then X is a rational point of ABC if and only if X' is also a rational point of ABC.

Theorem 2. Rational point whose distances are perfect squares

Suppose a, b, c are the lengths of the sides of a triangle with three rational medians (or rational centroid). Then we can set up a rational quadrilateral ABCD with the following lengths of sides:
$BC = a\sqrt{2b^2 + 2c^2 - a^2}$, $CA = b\sqrt{2c^2 + 2a^2 - b^2}$, $AB = c\sqrt{2a^2 + 2b^2 - c^2}$, $DA = a^2$, $DB = b^2$, $DC = c^2$

Theorem 3. Rational Heptagon

Suppose a, b, c are the lengths of the sides of a triangle with three rational medians (or rational centroid). Then we can set up a rational heptagon ABCDEFG with the coordinates of the vertices as follows:

$$A = \left(2a^4 - a^2b^2 - 3a^2c^2 + 3b^4 - 2b^2c^2 + c^4,\ \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,(2a^2 - b^2 - c^2)\right)$$

$$B = \left(2b^2(c^2 + 2(a^2 + b^2 - c^2)),\ 0\right)$$

$$C = \left((a^2 + b^2 - c^2)(c^2 + 2(a^2 + b^2 - c^2)),\ \sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,(c^2 + 2(a^2 + b^2 - c^2))\right)$$

$$D = \left(\tfrac{3}{2}b^2(3a^2 + b^2 - c^2),\ \tfrac{3}{2}b^2\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\right)$$

$$E = \left(b^2(3a^2 + b^2 - c^2),\ b^2\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\right)$$

$$F = \left(\tfrac{1}{2}(a^2 + b^2 - c^2)(2a^2 + 2b^2 - c^2),\ \tfrac{1}{2}\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}\,(c^2 + 2(a^2 + b^2 - c^2))\right)$$

$$G = \left(b^2(c^2 + 2(a^2 + b^2 - c^2)),\ 0\right)$$

Theorem 4. Rational Gergonne point

Rational triangle with three rational medians (TXM).pdf

Suppose a, b, c are the lengths of the sides of a triangle with three rational medians (or rational centroid). Then the triangle DEF with the rational Gergonne point has the following side lengths:
$EF = a^2 (b^2 + c^2 - a^2)$, $FD = b^2 (c^2 + a^2 - b^2)$, $DE = c^2 (a^2 + b^2 - c^2)$

*Note that according to theorem 1, the isogonal conjugate point of Gergonne point is also a rational point of triangle DEF.

Theorem 5. Rational triangle with three rational medians

If x and y are rational and $x\,y\,(2y - x)(x^2 - y^2)(2x - y)(2x^2 + 5xy - 6y^2)(3x^2 + 5xy - 4y^2)$.
$(2x^2 - y^2)(3x^2 - 5xy + y^2)(4x^2 - 5xy + 3y^2)(x^2 - 10xy + 7y^2) \geq 0$

Then we can construct a rational triangle ABC with three rational medians as follows:

$BC = | -6x^5 - x^4 y - 69x^3 y^2 + 126x^2 y^3 - 64xy^4 + 6y^5 |$

$CA = | 18x^5 - 8x^4 y - 78x^3 y^2 + 93x^2 y^3 - 53xy^4 + 18y^5 |$

$AB = | 12x^5 - 29x^4 y + 63x^3 y^2 - 21x^2 y^3 - 47xy^4 + 24y^5 |$

*Note that according to theorem 1, the isogonal conjugate point of centroid (Symmedian point) is also a rational point of triangle ABC.

Related links

Rational Isogonal Conjugates

Message: #2142
Date: 2024-02-24
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

thanks for explications
... and the amazing argumentation with flexpoints, compliment!

Best regards Eckart
```

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Message: #2143
Date: 2024-02-24
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Chris, dear Eckart
I hardly follow your discussion, as I don't practice
Mathematica.
But Chris demo that the flexlines cannot be tangent to CU is
luminous!
It applies immediately to the hessian too, as the 9 flexes
belong also to the hessian, as well as to any cubic of the same
syzegetic pencil.
By the way, figures are always useful and suggesting.
Looking at the figure in message 2129, it seems obvious that the
polar axes of the real flexpoints wrt the hessian are the same
lines L1, L2 and L3.
The point P is also the same and each Li cut the flexline in the
harmonic of Fi wrt the 2 other flexes.
The degenerated polar conics wrt the initial cubic in the 3
flexpoints Fi are formed by the line Li and the tangent in Fi.
The degenerated polar conics wrt the hessian in the 3 flexpoints
are formed by the line Li and the tangent to the hessian in Fi,
which is in turn tangent to the hessian of the hessian ...
I don't remember this property has been mentionned before ...
Best regards
Bernard
```

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Message: #2144
Date: 2024-02-25
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Please read the degenerated polar conics *of* (and not in) the 3 flexpoints and in the points Ti (vertices of the triangle of the flextangents)

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Message: #2145
Date: 2024-02-25
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart,
Following remark is perhaps also interesting.
Drawing the 3 tangents from each Flexpoint Fi to the blue hessian, it appears that one is the tangent in Fi to the red reference cubic.
The 2 others are tangent in Fi to the 2 other prehessians of this hessian!
As each Li cuts apparently the red cubic in only one point, it seems that this cubic has only one real prehessian ...
Best regards
Bernard

PS Sorry again, the vertices of the triangle of the flextangents are the Xi (P and the Xi are the poles of the flexline wrt the reference cubic) and the Ti are the vertices of the DT of the QA of the 4 poles (in other words the vertices of the cevian triangle of P wrt the triangle of the Xi). I hope this last definition is correct this time.

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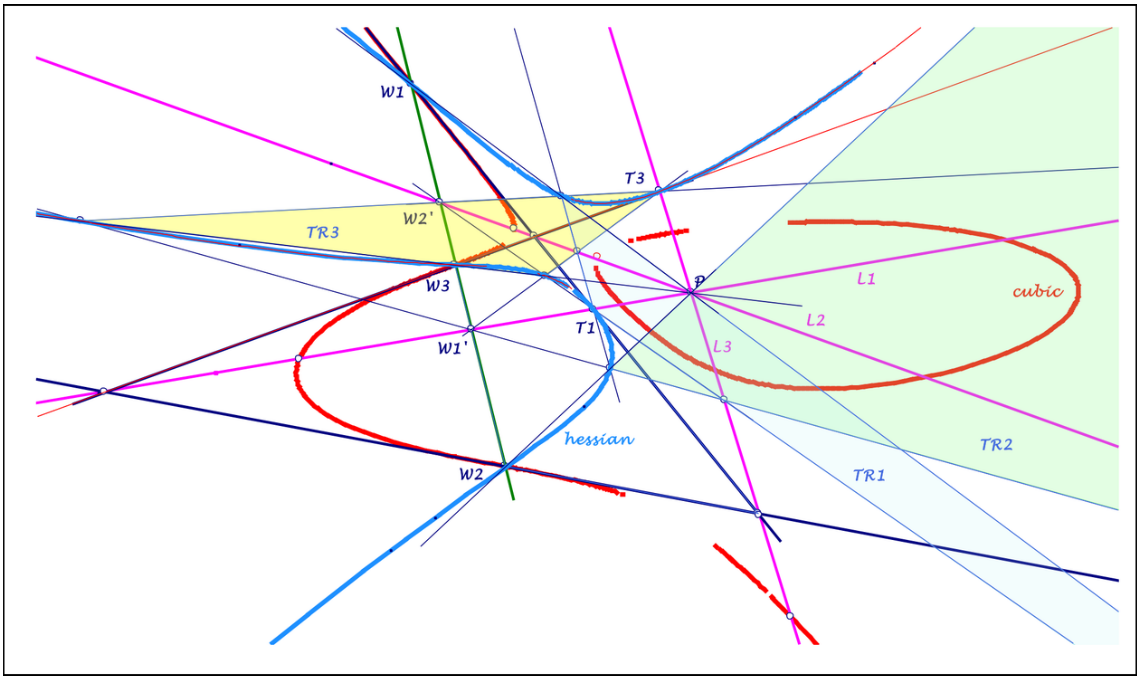
Message: #2146
Date: 2024-02-25
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris, dear Bernard,

in addition to Bernard's messages
... some remarks wrt the three final diagonal triangles TRi
... in Chris' construction of the hessian:

The three triangles are pairwise perspektiv wrt the flexpoints
W1,W2,W3.
Let Wi' be the 4th harmonic of Wi wrt Wj,Wk
... and Ti the intersection of the flextangent in Wi
    and the harmonic polar Li.
Ti is one vertex, TiWj' and TiWk' are two sidelines
... and PWi the third sideline of TRi.
Perhaps helpful for drawing the hessian (attached).

Best regards Eckart
```



2024-02-25.pdf

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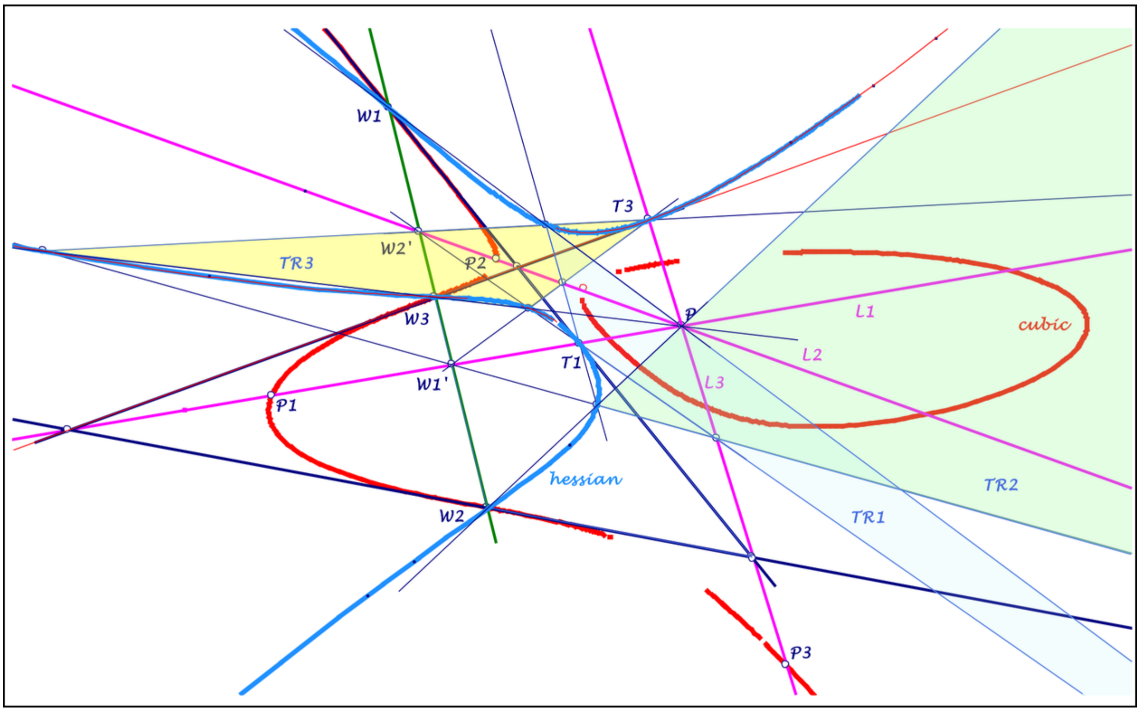
Message: #2147
Date: 2024-02-25
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris, dear Bernard,

in my last message the three final diagonal triangles
... in Chris' construction of the hessian
... are for the starting points P1,P2,P3
... intersections of the harmonic polars L1,L2,L3
... with the nonclosed parts of the cubic.

Excuse, that I forgot this information.

Best regards Eckart
```



2024-02-25a.pdf

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Message: #2148
Date: 2024-02-25
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Chris, dear Eckart
Eckart, I regret that you don't confirm that the lines Li and
the point P are the same for the hessian as well as for any
cubic of the syzegetic pencil.
Chris, the 9 flexes (3 real + 6 imaginary) form obviously a CB
system.
In order to draw a cubic of the syzegetic pencil, I suppose you
just need a 10th point with Mathematica.
For example, the flex tangents of the hessian intersect the 3
lines Li in 3 points ti on the hessian of the hessian ...
Best regards
Bernard
```

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Message: #2149
Date: 2024-02-26
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris, dear Bernard,

I can confirm Bernard's observation,
... that the lines Li and the point P
    are the same for the hessian.
In addition to my last message wrt the three triangles TRi:
P and Wi lie harmonic on one sideline of each triangle,
... PWi ^ TjWk' and PWi ^ TkWj'
    are the corresponding triangle points
... and therefore points on the hessian.

Best regards Eckart
```

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Message: #2150
Date: 2024-02-26
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart,
Thanks for confirmation
Beautiful construction indeed!
Provided you have the 3 real flexpoints on your red cubic, the 3
triangles give you 9 other points (including the 3 Ti).
I have a last question:
why do you say this construction is related to the points Pi?
The 9 points of contact with the tangents from the Wi play
symmetric roles (3 Pi and 6 Si on your figure 2100).
There are plenty of alignments like PiPjWk, SiSjWk or SiSjPk (it
would be better to distinguish the 2 Si in Qi and Ri for
example) ...
Thanks in advance
Best regards
Bernard
```

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Message: #2151
Date: 2024-02-26
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Bernard,

wrt your question "...why do you say this construction is
related to the points Pi?"

If you take P1,P2,P3 for Chris' hessian construction,
... you get finally the three triangles
    (see my last message 2147).

Best regards Eckart
```

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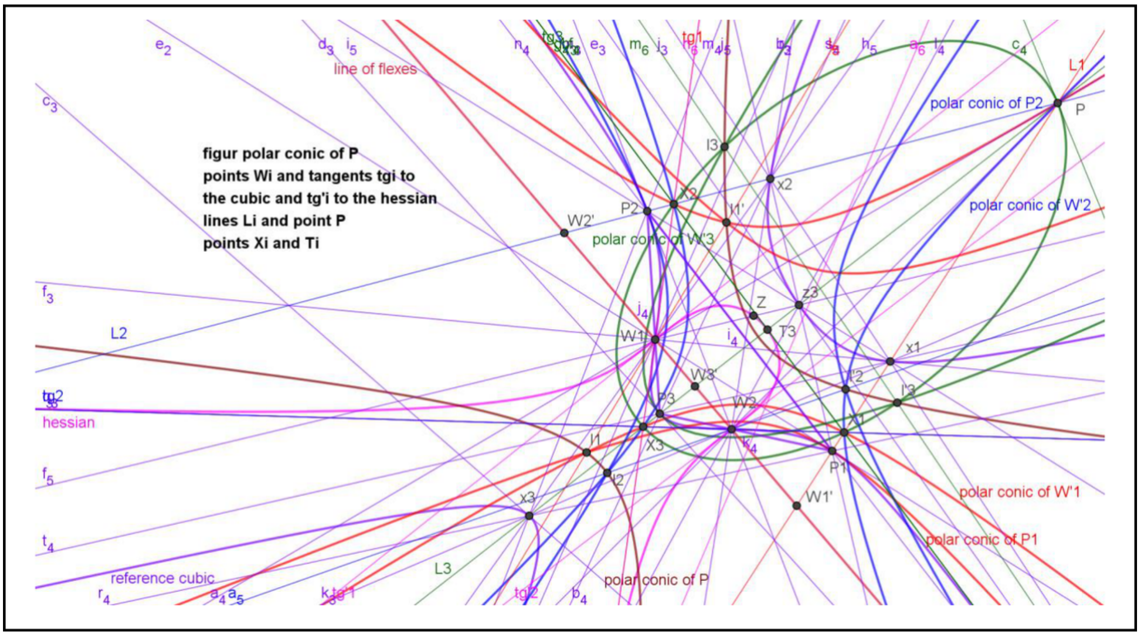
Message: #2152
Date: 2024-02-29
From: bernard.keizer@gmail.com
Subject: polar conic of P

Dear Chris, dear Eckart
Long time ago, I asked Eckart what the polar conics of Wi and Ti were ...
According to the the theory, the polar conic in a point of the hessian is formed of 2 lines harmonic the 2 tangents to the hessian in this point.
The polar conic of Wi is formed by Li and tgi, intersecting in Ti and tgi is already one of the tangents in Ti to the hessian ...
The polar conic of Ti is formed by 2 lines intersecting in Wi and harmonic wrt the 2 tangents to the hessian in Wi, one is tgi, let's name the other tg'i.
I tried to enlighten this question by drawing the polar conic of P wrt the cubic; this conic passes through the poles of the lines Li.
Using Chris method in 1944, I get the hessian of the cubic, the flexes Wi, the lines Li, the tangents in Wi to the cubic tgi and to the hessian tg'i.
The triangle of the flextangents tgi is X1X2X3 and the cevian triangle of P wrt this triangle is T1T2T3.
I use finally Eckart's points P1, P2 and P3.
Drawing the polar conics of P1, P2 and P3 (I suppose they are the Co1,2 and 3 in Eckart's message 2100), it appears that these 3 conics intersect 2 by 2 in 4 points being the poles of the lines PiPj.
As PiPj passes through Wk, these 4 points are by pairs on the lines Lk and tgk intersecting in Tk.
Drawing then the polar conics of W'i, it appears that the 3 conics pass through X1, X2, X3 and P (poles of the flexline) and that the polar conic of W'i is tangent in P to PWi.
Last, the intersection of the polar conics in Pi and W'i gives 4 points being the poles of Li.
As Li passes through Ti, these 4 points are on 2 lines through Wi forming the polar conic of Ti.
The polar conic of P passes through these 12 points.
Unfortunately, these points are not necessary all real and I have on my drawing only 6 points li and l'i.
Drawing such figures takes some time, but gives a certain pleasure ...
Best regards
Bernard



polar conic of P.pdf

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Message: #2153
Date: 2024-02-29
From: van10hoven@gmail.com
Subject: Re: Objects related to rational triangles with three rational medians

---

```
Dear Trinh Xuan Minh,

Thank you for your beautiful paper on 'Rational triangle with
three rational medians'.
I think there is a whole world of rational points to be
discovered.

Best regards,
Chris van Tienhoven
```

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Message: #2154
Date: 2024-03-01
From: hoingason@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] Objects related to rational triangles

---

```
Thank you for your praise! Yes, I think so too. Have a good day.
Best Regards
--TXM--

Vào 13:57, Th 6, 1 thg 3, 2024 Chris <van10hoven@gmail.com> đã
viết:
> Dear Trinh Xuan Minh,
> Thank you for your beautiful paper on 'Rational triangle with
three
> rational medians'.
> I think there is a whole world of rational points to be
discovered.
> Best regards,
> Chris van Tienhoven
```

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Message: #2155
Date: 2024-03-01
From: bernard.keizer@gmail.com
Subject: Re: polar conic of P

```
Naturally, it's easy to check that the polar line of P wrt the
polar conic of P wrt the cubic is the polar line of P wrt the
cubic, id est the flexline.
```

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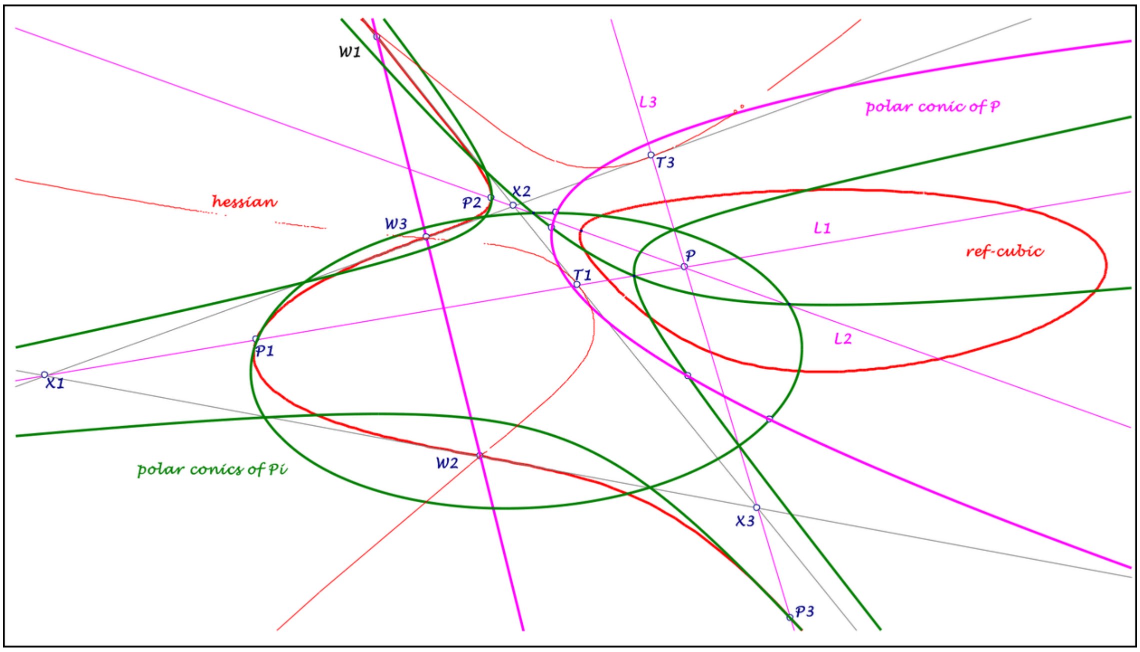
Message: #2156
Date: 2024-03-03
From: eckart_schmidt@t-online.de
Subject: Re: Polar conic of P

```
Dear Bernard,

thanks for interest and gathered properties for my point
configuration,
... but the polar conics of Wi are already mentioned in #2106.
In addition:
The conjugated points Ti of the flexpoints Wi,
... intersections of flextangent and Li,
... are P1,2,3 for the hessian
... and contact points of the hessian and the cayleyan
... with common tangent in the flextangent
    (Duerge 514, page 270).
As sign, that I studied your message,
... I made a drawing with relevant elements.
Can it be, that the polar of P wrt its polar conic is the
flexline?

Best regards Eckart
```



2024-03-03.pdf

Message: #2157
Date: 2024-03-03
From: eckart_schmidt@t-online.de
Subject: Re: Polar conic of P

```
Dear Bernard,

excuse my last question,
... I just recognized your last message.

Best regards Eckart
```

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Message: #2158
Date: 2024-03-04
From: hoingason@gmail.com
Subject: Rational Bicentric Quadrilateral

```
In this short note, I give a formula for a rational bicentric
quadrilateral
that has rational sides and diagonals, and has a rational area.
Hope it's
useful to someone.
Best regards
--TXM--
```

Rational Bicentric Quadrilateral

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract. In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices. In this short note we will talk about bicentric quadrilaterals that have rational sides and diagonals, and have a rational area.

A bicentric quadrilateral is a convex quadrilateral that has both an incircle and a circumcircle. Suppose $x,\ y$ are rational numbers such that

$$x\,y\,(x^2-y^2)\,(x^8-8x^6y^2+30x^4y^4-8x^2y^6+y^8)^2\,(x^4-4x^2y^2-y^4)$$
$$(x^4+4x^2y^2-y^4)\,(x^4-2x^3y-2x^2y^2-2xy^3+y^4)\,(x^4+2x^3y+2x^2y^2-2xy^3+y^4)$$
$$(x^4+2x^3y-2x^2y^2+2xy^3+y^4)\,(x^4-2x^3y+2x^2y^2+2xy^3+y^4)\neq 0$$

Then we can set the coordinates of the 4 vertices of a bicentric quadrilateral ABCD such that it has a rational area and a rational distance between any two vertices as follows :

$$A=\left(-(x^8-8x^6y^2+30x^4y^4-8x^2y^6+y^8)^2\,(x^8+8x^6y^2-2x^4y^4+8x^2y^6+y^8),\ 0\right)$$

$$B=\Bigg(-\frac{1}{x^8+8x^6y^2-2x^4y^4+8x^2y^6+y^8}\,(x^4-2x^3y-2x^2y^2-2xy^3+y^4)$$
$$(x^4+2x^3y+2x^2y^2-2xy^3+y^4)\,(x^4+2x^3y-2x^2y^2+2xy^3+y^4)$$
$$(x^4-2x^3y+2x^2y^2+2xy^3+y^4)\,(x^8+16x^5y^3-18x^4y^4-16x^3y^5+y^8)$$
$$(x^8-16x^5y^3-18x^4y^4+16x^3y^5+y^8),\ (32x^3(x-y)y^3(x+y)$$
$$(x^{24}-8x^{22}y^2-6x^{20}y^4+24x^{18}y^6-241x^{16}y^8+2032x^{14}y^{10}+492x^{12}y^{12}+2032x^{10}y^{14}-$$
$$241x^8y^{16}+24x^6y^{18}-6x^4y^{20}-8x^2y^{22}+y^{24}))\,/\,(x^8+8x^6y^2-2x^4y^4+8x^2y^6+y^8)\Bigg)$$

$$C=(0,\ 0)$$

$$D=\Bigg(-\frac{1}{x^8+8x^6y^2-2x^4y^4+8x^2y^6+y^8}\,2\,(x^2-y^2)^2$$
$$(x^{28}-14x^{26}y^2+11x^{24}y^4+244x^{22}y^6+729x^{20}y^8-3410x^{18}y^{10}+11\,547x^{16}y^{12}-1832x^{14}y^{14}+$$
$$11\,547x^{12}y^{16}-3410x^{10}y^{18}+729x^8y^{20}+244x^6y^{22}+11x^4y^{24}-14x^2y^{26}+y^{28}),$$
$$-\Big(\Big(8\,x\,y\,(x^2-y^2)^3\,(x^{24}-8x^{22}y^2-6x^{20}y^4+24x^{18}y^6-241x^{16}y^8+2032x^{14}y^{10}+$$
$$492x^{12}y^{12}+2032x^{10}y^{14}-241x^8y^{16}+24x^6y^{18}-6x^4y^{20}-8x^2y^{22}+y^{24})\Big)\Big/$$
$$(x^8+8x^6y^2-2x^4y^4+8x^2y^6+y^8)\Big)\Bigg)$$

Suppose (O, R) and (I, r) are the circumcircle and incircle of ABCD, respectively. Then we have :

$$O = \left(-\frac{1}{2}\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2\left(x^8 + 8x^6y^2 - 2x^4y^4 + 8x^2y^6 + y^8\right),\right.$$
$$\left. -\frac{\left(x^4 - 6x^2y^2 + y^4\right)\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2\left(x^8 + 8x^6y^2 - 2x^4y^4 + 8x^2y^6 + y^8\right)}{8x(x-y)y(x+y)}\right)$$

$$I = \Big(-\Big(\Big((x-y)^2(x+y)^2\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2$$
$$\left(x^{32} - 16x^{30}y^2 - 40x^{28}y^4 + 592x^{26}y^6 + 4828x^{24}y^8 - 22\,160x^{22}y^{10} + \right.$$
$$5608x^{20}y^{12} + 21\,584x^{18}y^{14} + 44\,742x^{16}y^{16} + 21\,584x^{14}y^{18} + 5608x^{12}y^{20} -$$
$$\left. 22\,160x^{10}y^{22} + 4828x^8y^{24} + 592x^6y^{26} - 40x^4y^{28} - 16x^2y^{30} + y^{32}\right)\Big)\Big/$$
$$\Big(\left(x^2+y^2\right)^2\left(x^8 - 16x^6y^2 + 46x^4y^4 - 16x^2y^6 + y^8\right)^2\left(x^8 + 8x^6y^2 - 2x^4y^4 + 8x^2y^6 + y^8\right)\Big)\Big),$$
$$\Big(32x^3(x-y)^3y^3(x+y)^3\left(x^4 - 6x^2y^2 + y^4\right)\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2$$
$$\left(x^{16} - 8x^{14}y^2 - 20x^{12}y^4 + 136x^{10}y^6 + 38x^8y^8 + 136x^6y^{10} - 20x^4y^{12} - 8x^2y^{14} + y^{16}\right)\Big)\Big/$$
$$\Big(\left(x^8 - 16x^6y^2 + 46x^4y^4 - 16x^2y^6 + y^8\right)^2\left(x^8 + 8x^6y^2 - 2x^4y^4 + 8x^2y^6 + y^8\right)\Big)\Big)$$

$$R = \frac{\left(x^2+y^2\right)^2\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2\left(x^8 + 8x^6y^2 - 2x^4y^4 + 8x^2y^6 + y^8\right)}{8xy\left(x^2 - y^2\right)}$$

$$r = \frac{1}{x^4 - 6x^2y^2 + y^4}$$
$$2xy\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)\left(x^{18} - 9x^{16}y^2 - 12x^{14}y^4 + 156x^{12}y^6 - 98x^{10}y^8 + \right.$$
$$\left. 98x^8y^{10} - 156x^6y^{12} + 12x^4y^{14} + 9x^2y^{16} - y^{18}\right)$$

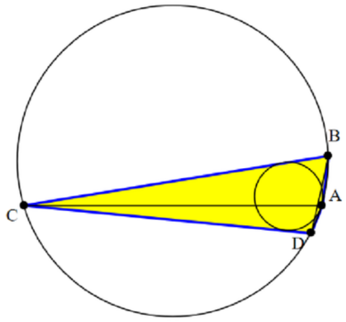
The area of quadrilateral ABCD is

$$4xy\left(x^2+y^2\right)^2\left(x^8 - 8x^6y^2 + 30x^4y^4 - 8x^2y^6 + y^8\right)^2$$
$$\left(x^{26} - 9x^{24}y^2 + 2x^{22}y^4 + 30x^{20}y^6 - 265x^{18}y^8 + 2273x^{16}y^{10} - 1540x^{14}y^{12} + \right.$$
$$\left. 1540x^{12}y^{14} - 2273x^{10}y^{16} + 265x^8y^{18} - 30x^6y^{20} - 2x^4y^{22} + 9x^2y^{24} - y^{26}\right)$$

Rational Bicentric Quadrilateral (TXM).pdf

|AB|	|CD|	|AC|
4 786 400	27 705 150	28 644 481
|BC|	|DA|	|BD|
29 613 727	2 877 823	7 604 641

area
106 307 357 184 600



Related links

A Rational Bicentric Quadrilateral with a Rational Area

Message: #2159
Date: 2024-03-05
From: bernard.keizer@gmail.com
Subject: Re: Polar conic of P

Dear Eckart,
Thanks for the answer with the figure!
My question was double: what are the polar conics of Wi and Ti.
As PiPj are aligned with Wk, the intersections of the polar conics of Pi and Pj are on the polar conic of Wk, id est est the 2 lines Lk and tgk intersecting in Tk.
The polar conic of Ti is made of 2 lines intersecting in Wi and harmonic wrt tgi and tg'i (tangents in Wi to the hessian).
As Ti, Pi and P are aligned on Li, the polar conic of P and the polar conic of Pi intersect in 4 points (poles of Li) on the 2 lines forming the polar conic of Ti.
It happens on your figure the same than on mine that only 2 poles of Li seem to be real, giving only one of the 2 searched lines ...
Best regards
Bernard

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Message: #2160
Date: 2024-03-05
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Bernard and Eckart,

Last weeks I was busy understanding the Cayleyan “the holy grail”.
I tried to make a picture in Mathematica, but to no avail.
It did not become easier with so many lines enveloping the curve. Thanks for mentioning this all!
Moreover you both mentioned that “The degenerated polar conics wrt the initial cubic in the 3 flexpoints Fi are formed by the line Li (Harmonic polars) and the tangent in Fi.”
I wondered if this is true for the 3 real flexpoints or for all 9 real or imaginary points.
In a numerical example in Mathematica I found out that this indeed is true for all 9 real or imaginary points Fi. As a consequence The Cayleyan is enveloped by the 9 CU-Fi-tangents, whilst the reference cubic also is enveloped by the same 9 CU-Fi-tangents. Hmmm . . . .

I wondered how the picture would look like.
Several attempts constructing the Cayleyan failed, which stopped me every time for a full day in despair. I wondered what I could believe being true in my understanding.
After thinking and rethinking this morning I had another attempt in Cabri. And this time I saw appearing a curve of 6 th degree.
Great was the joy, when I saw that the CU-Fi-Tangents touch the Hessian and the Cayleyan in exactly the same points where also the Hessian and the Cayleyan touch mutually.
Attached you will find a full picture with reference cubic, Hessian, Caylean, real Fi-Tangents and real Fi-harmonic polars.
It all falls beautifully together.

The construction I used looks like my earlier mentioned construction of the *Hessian*.
In short the construction of the Hessian:
1. Given Reference Cubic CU.
2. Choose 3 random points P, Q, R (not necessary on CU) such that their Polar Conics mutually intersect in 4 real points.
3. Draw the Diagonal Triangles of the 3 QA's formed by the mutual intersection points of the 3 Polar Conics.
4. The cubic constructed from the 3x3 vertices of the Diagonal Triangles will be the Hessian of CU.

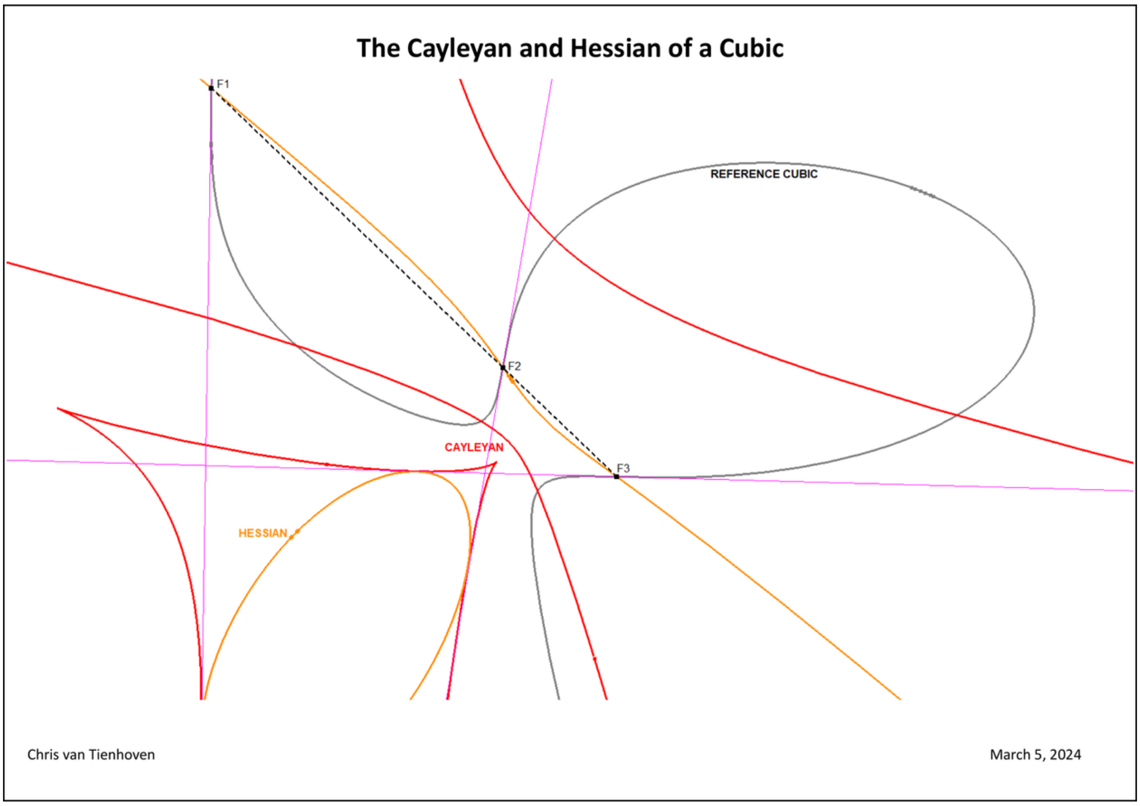
And now the construction I used for the *Cayleyan*:
1. Given Reference Cubic CU.
2. Choose 3 random points P, Q, R (not necessary on CU) such that their Polar Conics mutually intersect in 4 real points.
3. The three polar conics will intersect mutually in 3 sets of 4 points. So we have 3 QA’s.

For the construction we just need 2 of these QA’s.
4. Map per QA with QA-Tf10 the 6 sidelines in 6 points Ti. Two QA’s give 2x6 points Ti. As reference-QA just any 4 reference points can be chosen.
5. Draw the cubic CUx through 9 of these 12 points Ti and it will appear that the other 3 points Ti also will lie on this cubic CUx.
6. Determine a variable point X on the just constructed cubic Cux and draw the tangent Tx at X to Cux.
7. Map Tx with QA-Tf10 into point Px.
8. The locus of Px will be the Cayleyan.

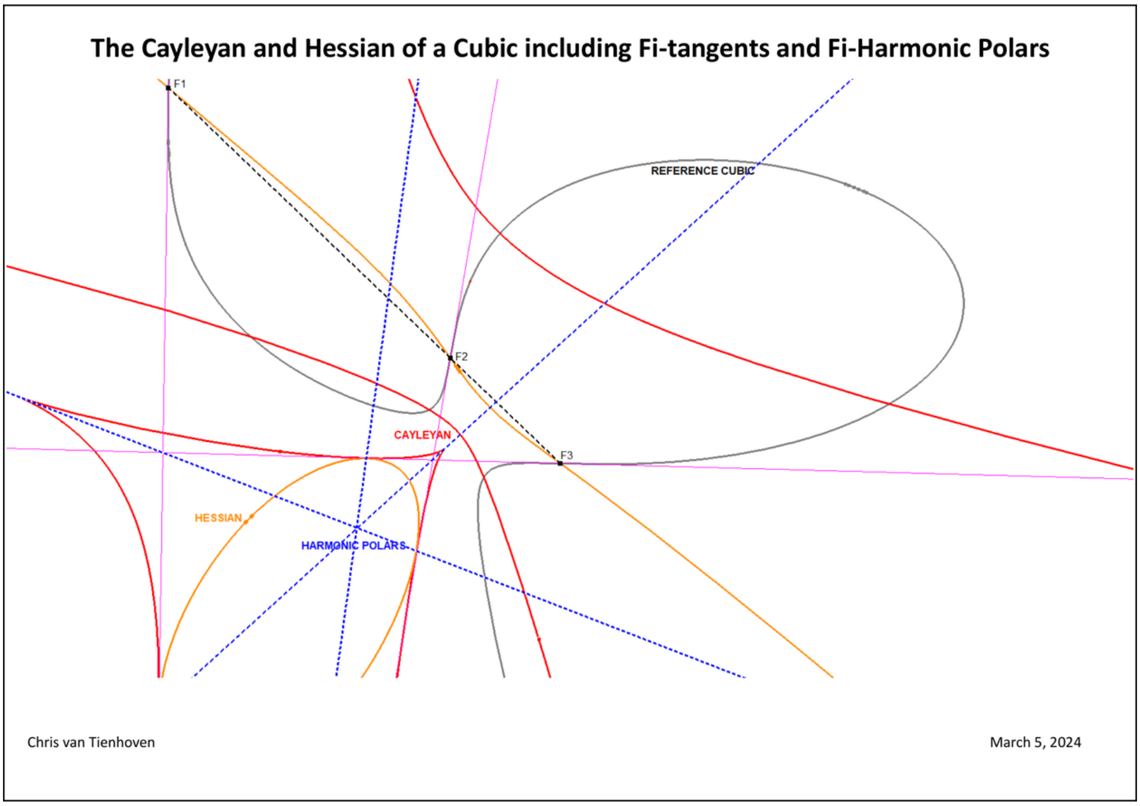
For Cabri it was the limit of its possibilities, but it managed. For me one basic question remains: what exactly is the function of the Cayleyan? For the Hessian it is clear. The hessian is the counterpart of the 2nd derivative, intersecting the reference cubic in the flexpoints. But what is the function of the Cayleyan?

Best regards,

Chris



CU-Cayleyan-plus-Hessian-01.pdf



CU-Cayleyan-plus-Hessian-01.pdf

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Message: #2161
Date: 2024-03-05
From: van10hoven@gmail.com
Subject: Re: Rational Bicentric Quadrilateral

Dear TXM,

Thanks again for your contribution!
Best regards,

Chris van Tienhoven

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Message: #2162
Date: 2024-03-06
From: hoingason@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] Rational Bicentric Quadrilateral

---

Thank you Mr. Chris van Tienhoven
--TXM-

Vào Th 4, 6 thg 3, 2024 vào lúc 03:19 Chris
<van10hoven@gmail.com> đã
viết:
> Dear TXM,
> Thanks again for your consribution!
> Best regards,
> Chris van Tienhoven

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Message: #2163
Date: 2024-03-06
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Chris,
Wunderbar!
So you finally succeeded in drawing the cayleyan, the holy grail of the cubic.
Congratulations!
Now, we have a beautiful figure, which allows to stufy seriously this curve.
For example, in any point of the hessian, there are 3 tangents to the cayleyan, one with the point as conjugated point (through the conjugate) and 2 with the point as complementary point (the 2 conjugated points being the 2 other intersections of these 2 lines with the hessian).
Doing this for the Fi gives 2 lines other than FiTi forming the polar conic of Ti.
The cayleyan has 3 turning points, let say Yi. Could P be by any chance the orthocenter of the triangle of the Yi?
For the function, I don't know; any curve with degree 3 and more has a hessian as locus of points in which the polar conic degenerates in 2 lines and a cayleyan as envelop of these lines ...
Best regards
Bernard

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Message: #2164
Date: 2024-03-06
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

wonderful result, my admiration,
... I tried in vain to construct the cayleyan,
... my result beside much frustration only a sextic
... tangent to the flextangents and the harmonic polars ...

Your drawing shows good,
... that the cayleyan touch the hessian
... in the intersections of the flextangents and harmonic polars
... and intersect the reference cubic in the contact points
... of common tangents - beside the flextangents -
... of the hessian and the reference cubic.

Best regards Eckart
```

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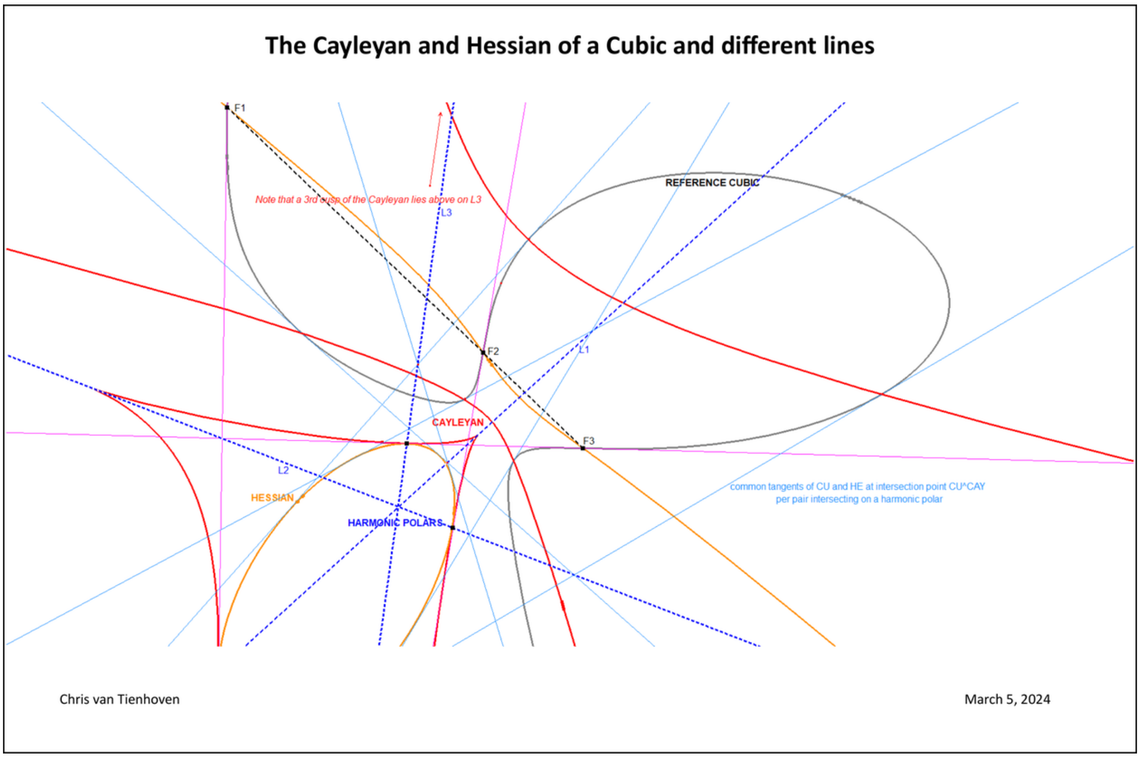
Message: #2165
Date: 2024-03-07
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Bernard and Eckart,

Thank you both for your inspiring response!
I made a new picture with some new lines and comments, inspired by your comments. See attachment.
Bernard you asked: “The cayleyan has 3 turning points, let say Yi. Could P be by any chance the orthocenter of the triangle of the Yi?”.
I tried it in Cabri but it certainly is not the orthocenter. I also tried ETC-points X(1)-X(13), neither of them is the point P of triangle Y1Y2Y3.
Moreover there are 3 turning points (cusps). Invisible in the picture is the 3rd cusp, lying above outside of the picture on Harmonic Polar L3.
Best regards,

Chris



CU-Cayleyan-plus-Hessian-02.pdf

Message: #2166
Date: 2024-03-07
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

looking  on your wonderful drawings,
... I studied the intersections
... of the reference cubic and the cayleyan,
... which are contact points of common tangents
... at the reference cubic and the hessian.
Connecting these six intersections with the flexpoints
... we get nine lines, by three through the flexpoints,
... each line bearing two intersections.
These six intersections seem to lie on a conic.
What about this conic?

Best regards Eckart
```

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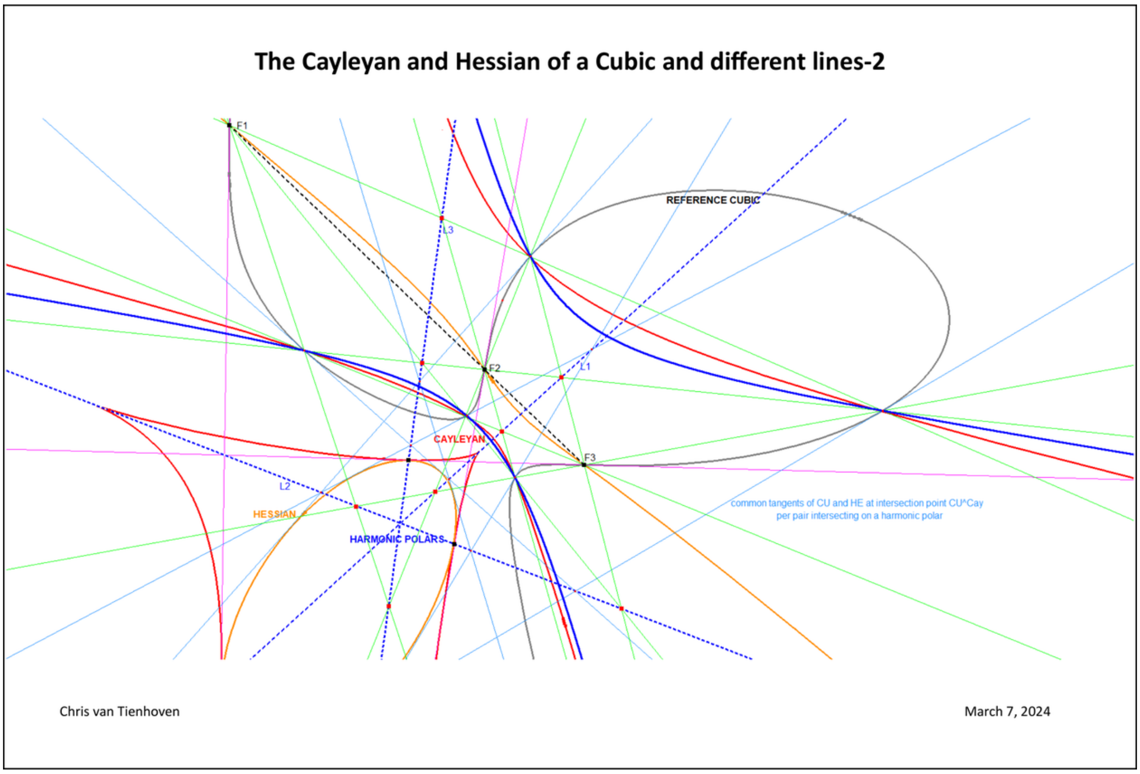
Message: #2167
Date: 2024-03-07
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
About your question:
   "Connecting these six intersections with the flexpoints
    ... we get nine lines, by three through the flexpoints,
    ... each line bearing two intersections.
   These six intersections seem to lie on a conic.
   What about this conic?"
I drew the nine lines and was surprised to see that 2 of these
six intersection points of CU and Cayleyan are always collinear
with one of the real flexpoints. Therefore there are “just” nine
lines. They also connected on the Harmonic Polars.
I drew also the conic (see attached picture). However the 6 th
intersection points of CU and Cayleyan lies out of reach of the
picture. But in my picture it does not look like the conic
passes through this point. But it can be a precision deviation.
I don’t see any special items with this conic (yet).
Best regards, Chris



CU-Cayleyan-plus-Hessian-03.pdf

Message: #2168
Date: 2024-03-08
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

can you control, whether the 4th harmonic of a flexpoint
... wrt two intersections of cubic and cayleyan
... is a point on a harmonic polar?

Best regards Eckart
```

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Message: #2169
Date: 2024-03-08
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

excuse the question in my last message, it's evident:
The harmonic polar Li is part of the polar conic of the
flexpoint Wi.

Best regards Eckart
```

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Message: #2170
Date: 2024-03-09
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart and Bernard,

Some more observations. I don't know if you mentioned them before.

1. In my drawing somehow the Harmonic Polars intersect the reference cubic in only one point, which is the point where the only tangent from Fi is tangent to CU. (Fi = Flexpoint i; F1,F2,F3 are the real Flexpoints)

2. There are 12 flexlines of which 4 are real. The only flex line through 3 real flexpoints is F1F2F3, another real flexline passes through F1, another real flexline through F2, another real flexline passes through F3. In the same way there are 12 mutual intersection points Pi of the 9 Fi-Harmonic Polars Li (of which L1,L2,L3 are the real Fi-Harmonic Polars). One point is P0=L1^L2^L3, another point P1 lies on L1, another point P2 lies on L2, another point P3 lies on L3. It is uncertain yet how to determine these last 3 points.

3. Already mentioned by both of you that the real Flexline through (F1,F2,F3) is the polar of P0 wrt the CU-P0-Polar Conic. In the same way the Flexline through F1 will be the polar of point P1 wrt the CU-P1-Polar Conic, the Flexline through F2 will be the polar of point P2 wrt the CU-P2-Polar Conic, the Flexline through F3 will be the polar of point P3 wrt the CU-P3-Polar Conic. So when we know points P1, P2, P3, then we know the elusive 2nd, 3rd, 4th flexline. As already said P1,P2,P3 are intersection points of Fi-Harmonic Polars.

Best regards,

Chris

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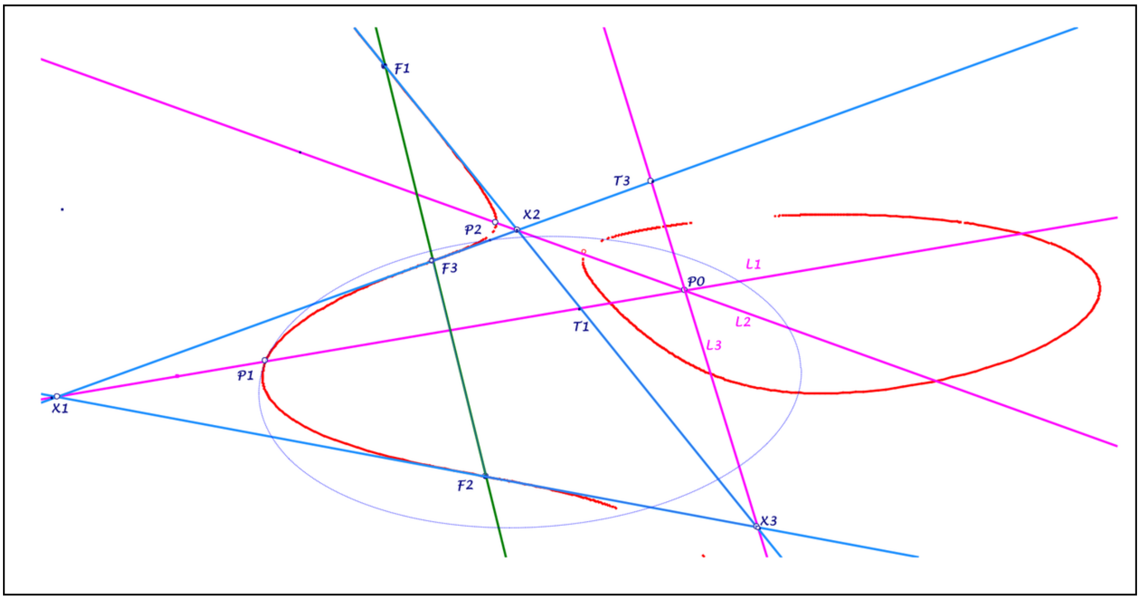
Message: #2171
Date: 2024-03-10
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

I have difficulties with your last message (see attached
drawing).
wrt 1. The harmonic polars can intersect the cubic
       in three points,
   ... you have to take the intersection with
       the non closed part of the cubic
   ... for the points P1,P2,P3.

wrt 3. The 2nd real flexline through F1 cannot be
   ... "the polar of point P1 wrt the CU-P1-Polar Conic",
   ... see attached drawing,
   ... I searched in vain for these 2nd real flexlines
       for F1,F2,F3.
Where are my misunderstandings?

Best regards Eckart
```



2024-03-10.pdf

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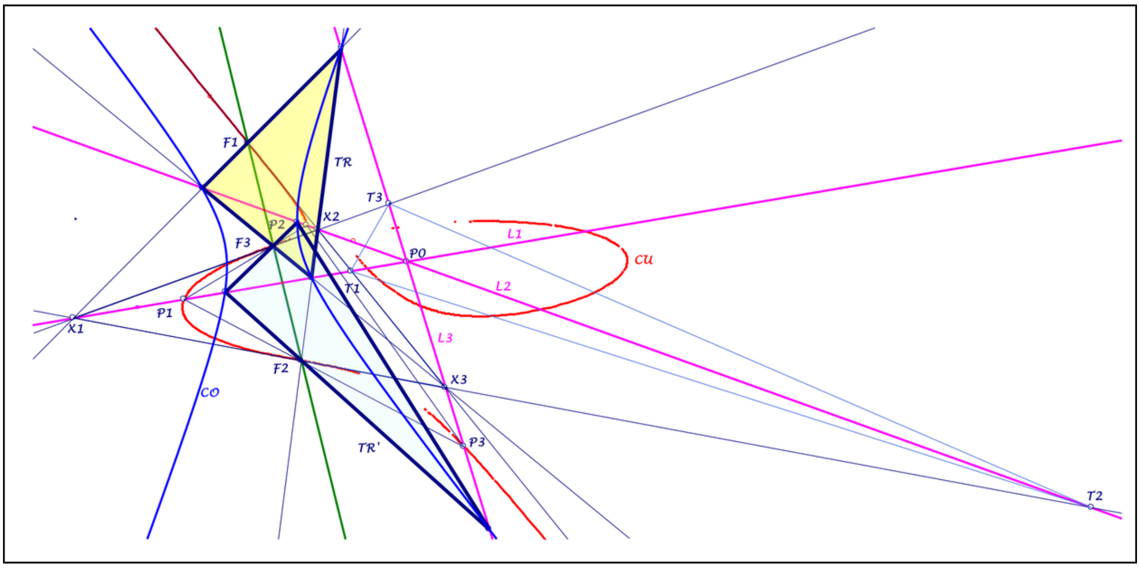
Message: #2172
Date: 2024-03-10
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

following observations are only made with your drawings,
... perhaps some observations don't hold:
(1) The polars of the flexpoints Fi wrt the conic CO
... of intersections of reference cubic and its cayleyan
... are their harmonic polars Li wrt the cubic.
(2) For the intersections Xi of flextangents
... the lines FiXi intersect Lj and Lk on the cayleyan
... with cayleyan tangentials Fi.
(3) The triangles P1P2P3, X1X2X3,T1T2T3
... are pairwise perspective wrt P0,
... have their vertices on L1,L2,L3,
... bearing on their sides the flexpoints.
(4) The 6 intersections of the lines Li and the conic CO
... give two such triangles,
... one TR with sidelines FiXi
... and vertices in the anticevians of P wrt X1X2X3
... with CO-tangents through the flexpoints.
(5) The 2nd triangle TR' has vertices
... in the 4th harmonic of P wrt Xi and the intersection
... of Li and the flexline L.
In this way the conic CO and its intersections
... with the cubic and its cayleyan
... can be found without a construction of the cayleyan.

I hope these observations hold, excuse if not.

Best regards Eckart
```



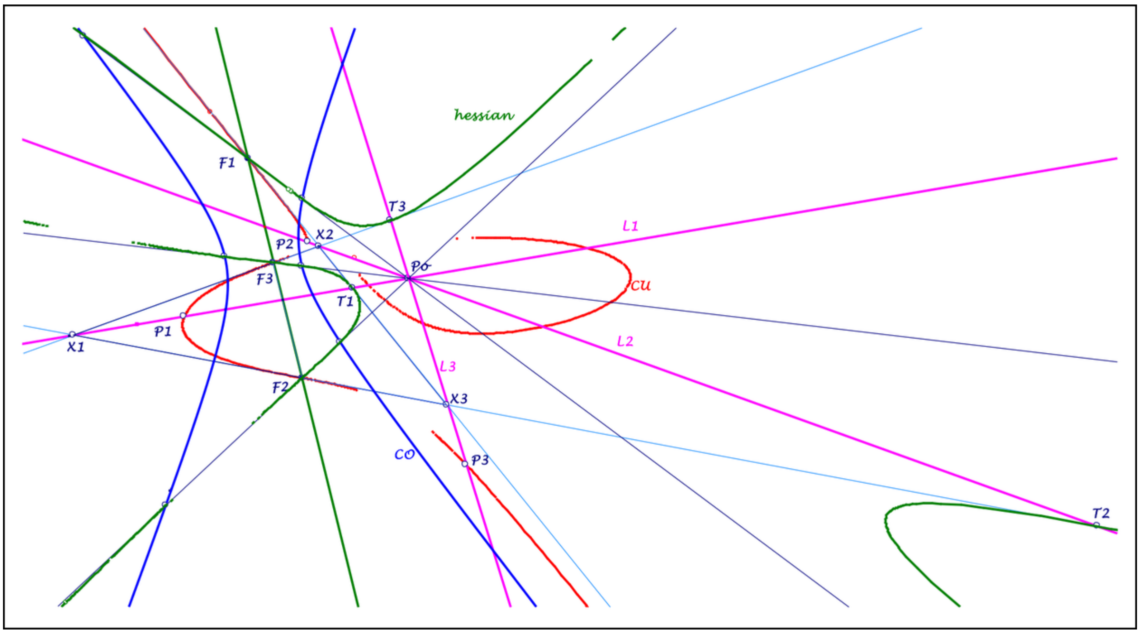
2024-03-10a.pdf

Message: #2173
Date: 2024-03-10
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris,

with the conic CO, described in my last message,
... you can "construct" the hessian of the cubic:
The 3 lines Fi.Po intersect the conic CO in 6 points,
... take further the intersections Ti of Li
and the flextangents tgi,
... these nine points define the hessian.

Best regards Eckart



2024-03-10b.pdf

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Message: #2174
Date: 2024-03-11
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

the discussed conic CO in the last messages
... is the image of the flexline F1F2F3
... wrt an isoconjugation
... with reference triangle in the anticevians of Po wrt X1X2X3
... and fixpoint Po.

Best regards Eckart
```

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Message: #2175
Date: 2024-03-11
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

the hessian is a nonpivotal isocubic
... wrt the reference triangle T1T2T3
... an isoconjugation with fixed point Po and root Po
... and a hessian point Q, for example 4th harmonic of Po
... ... wrt Xi and intersection of Li and the flexline F1F2F3.

Best regards Eckart
```

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Message: #2176
Date: 2024-03-11
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
Fascinating properties of your conic C0 and this description of
the hessian!
I try in vain to follow, but it goes too fast for me ...
I still have a question:
do you need or not the hessian in order to have the flexes? (see
Chris method)
as you need the lines Li and the points P and Xi and Ti for the
drawing of the hessian with your conic C0, I don't understand
the reasoning
Thanks in advance for your answer
Best regards
Bernard
```

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Message: #2177
Date: 2024-03-11
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

the flexpoints can very precise approximated
... by drawing three lines through a cubic point
... and searching that point
... with coconic 6 further intersections with the cubic,
... next step the harmonic polar ...

Best regards Eckart
```

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Message: #2178
Date: 2024-03-11
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

Regarding your message #2171.

Wrt 1 You are right. The harmonic polars can intersect the cubic in three points. In my drawing I only saw one intersection point and that one was the point where the only tangent from Fi is tangent to CU. I suppose you are right that with >1 intersection points, then take the intersection with the non closed part of the cubic.

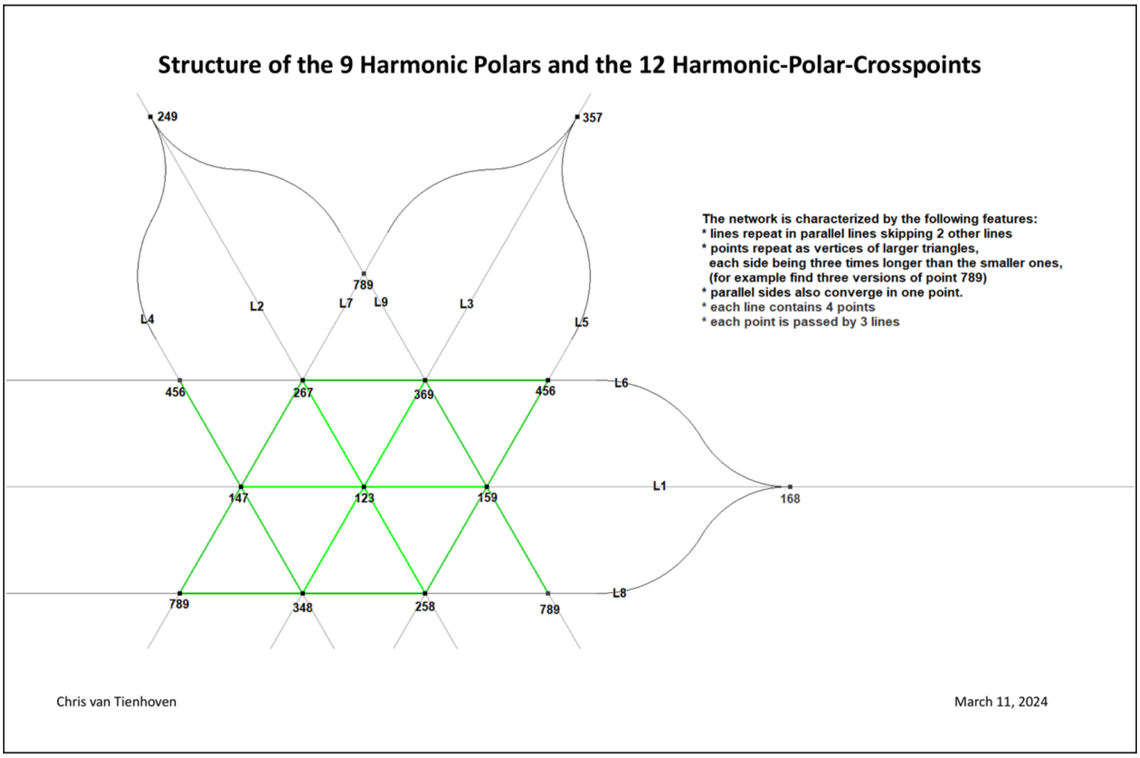
Wrt 3 I suppose you are right that in your drawing the 2nd real flexline through F1 cannot be the polar of point P1 wrt the CU-P1-Polar Conic, because your definition of P1 is the intersection of the F1-Harmonic Polar with the non closed part of the cubic.

In my message #2170 I defined P0, P1, …, P11 being the mutual intersection points of the Harmonic-Polars.
They are a kind of duals with the 12 flexlines passing through the 9 flexpoints.
The 9 Flexpoints lie on 12 Flexlines.
The 9 Harmonic Polars intersect in 12 Harmonic-Polar-Crosspoints.
The 12 Harmonic-Polar-Crosspoints and the 12 Flexlines have a Pole/Polar-relationship wrt the CU-Pi-Polar Conic. (Pi in my definition)

In message #2049 as well as #2140 I sent a drawing of the structure of the flexpoint/flexline-network.
I attach now another picture of the structure of the Harmonic Polar/Harmonic Polar Crosspoints-network. There are real as well as imaginary points/lines involved. I checked it in numeric examples in Mathematica.
Again it is a special line-point-network.

Best regards,

Chris



CU-12P1 Harmonic Polar Crosspoints-scheme-01.pdf

Message: #2179
Date: 2024-03-12
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard, dear Chris,

I just noticed, that #2173 doesn't hold
... for monopartite cubics, see Chris drawings,
... I have to prove my results once more for monopartite cubics.
Excuse my uncertainty!

Best regards Eckart
```

Message: #2180
Date: 2024-03-12
From: van10hoven@gmail.com
Subject: XY-Polar and PoloConica

Dear Eckart and Bernard,

I found an interesting paper from a Dutch Professor who lived one century ago:
J. de Vries, On polar figures with respect to a plane cubic curve.
Available at:
https://www.dwc.knaw.nl/DL/publications/PU00013494.pdf

There are two interesting items:
1. Given a general Cubic CU and two random points X and Y on the cubic, then the X-Polar if the Y-Polar Conic = the Y-Polar if the X-Polar Conic.
Let's call this polar: XY-Polar coded Lxy.
2. Theorem:
Given 2 points X,Y the locus of points Z for wich Lxy, Lyz, Lxz will concur is a conic he named Poloconica.
So far I have no clue how to construct "Poloconica".
I found 5 points and constructed so a preliminary conic but did not find special properties so far.
I hope you have better ideas or have extra background regarding these items.

Best regards,
Chris

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Message: #2181
Date: 2024-03-12
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
Thanks a lot for your answer in 2177!
I understand now better the logic of your construction ...
And naturally, all fit's together.
Best regards
Bernard

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Message: #2182
Date: 2024-03-12
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
I you begin with the notions of polarity and apolarity, there is a wide field!
For point 1), X and Y are not necessary on the cubic.
The polar of X wrt the polar conic of Y and the polar of Y wrt the polar conic of X are the same line Lxy often named the mixed polar of X and Y.
(By the way, the polar of a point P wrt the polar conic of P wrt the cubic is the polar of P wrt the cubic).
For point 3), I knew only that the poloconic of the polar conic of a point P wrt a cubic is the polar conic of P wrt the hessian.
If the 2 conics coïncide, it is an autopoloconic.
I found the article of de Vries on polar figures with respect to a plane cubic 1910.
I'll try to read it, but all these notions, which seemed elementary to geometers one century ago, are at the limits of my comprehension ...
Best regards
Bernard

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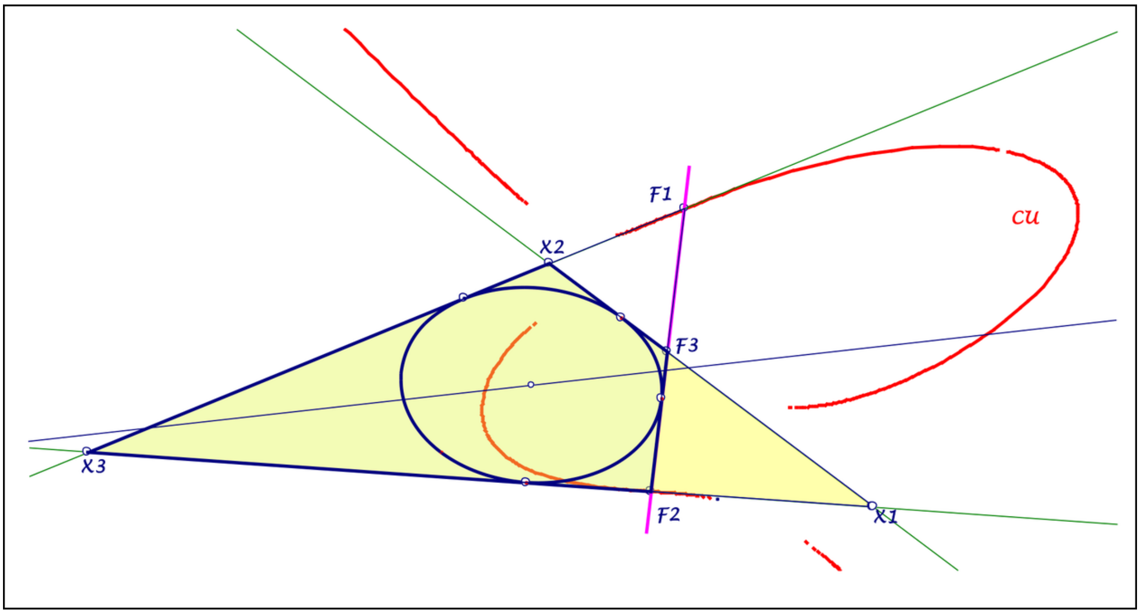
Message: #2183
Date: 2024-03-13
From: eckart_schmidt@t-online.de
Subject: Re: XY-Polar and PoloConica

```
Dear Chris,

references for "poloconic"
... in Schröter page 178, in Duerge page 198.
I had only a look on property 318 Duerge, page183,
... used for the flexline F1F2F3:
This poloconic is an inscribed conic of X1X2X3,
... tangent to the flexline,
... centered on QL-L1 of the QG = FiFjXiXj
... and contact point for a convex QG
... ... in the 4th harmonic of Fk wrt FiFj.

Best regards Eckart
```



2024-03-13.pdf

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Message: #2184
Date: 2024-03-13
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
In fact, the poloconic mentionned here is the poloconic of the degenerated conic formed by the 2 lines XY and the mixed polar Lxy of X and Y.
This leads to the notions of polar quadrangle and polar triangle wrt the cubic, which were new for me.
Very interesting, indeed!
Best regards
Bernard
PS a cubic intersects the hessian in 6 points; the poloconic of the conic passes through the conjugates of these 6 points

---

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Message: #2185
Date: 2024-03-13
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris, dear Eckart
Here some perhaps useful references found on the Net (3 articles and 2 books)
1) Paul Appell Courbes autopolaires
2) P.C. Delens Sur la théorie invariante des cubiques planes
3) Henry S. White Conics and cubics connected with a plane cubic by certain covariant relations
4) Surendramohan Ganguli Lectures on the theory of plane curves (see in particular part II cubic and quartic forms)
5) Plana algebraic curves Jason Cantarella
Best regards
Bernard

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Message: #2186
Date: 2024-03-13
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Bernard, dear Chris,

this is a new version of my messages #2172 - #2175:

Nominations:
F1,F2,F3 flexpoints of the cubic,
L1,L2,L3 their harmonic polars,
Po common point of L1,L2,L3,
P1,P2,P3 intersections of L1,L2,L3
... with the nonclosed part of the cubic,
X1,X2,X3 intersections of the flextangents,
T1,T2,T3 intersections of Li and the flextangents.
CO conic, image of the flexline F1F2F3 wrt an isoconjugation
... with fixpoint Po and reference triangle
... with vertices in the anticevians of Po wrt X1X2X3.

Properties:
(a) The polars of the flexpoints Fi wrt the conic CO
... are their harmonic polars Li wrt the cubic.
(b) The triangles P1P2P3, X1X2X3,T1T2T3
... are pairwise perspective wrt Po,
... have their vertices on L1,L2,L3,
... bearing on their sides the flexpoints.
(c) The six intersections of the lines Li and the conic CO
... give two such triangles.
(d) One triangle with sidelines FiXi
... and vertices in the anticevians of Po wrt X1X2X3
... with CO-tangents through the flexpoints.
(e) The 2nd triangle has vertices
... in the 4th harmonic of Po
... wrt Xi and the intersection of Li and the flexline L.
(f) Nine points of the hessian
... F1,F2,F3 and T1,T2,T3 and PiXj ^ PjXi on Lk.
(g) The hessian is a nonpivotal isocubic
... wrt the reference triangle T1T2T3,
... an isoconjugation with fixed point Po and root Po
... and any hessian point Q, see(f).

Please forget the old messages #2172 - #2175!

Best regards Eckart

PS: Can Chris once more control
... with the new definition of CO

```
... whether CO bears the intersections
... of the cubic and its cayleyan?
     Thanks in advance!
```

Message: #2187
Date: 2024-03-13
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard and Eckart,

Thank you very much for the explanations and references for the poloconic.
It helped me a lot to understand the notion of a poloconic.
Especially the paper of Durege (pages 161 and 178) helped me to find the definition of a Poloconic.
The poloconic of line Lp wrt reference cubic CU is the envelope of line-polars (= 2nd Polar of P wrt CU) mapped from points P on line Lp wrt CU. It is called the poloconic of line Lp wrt CU.
And Bernard your remark “(By the way, the polar of a point P wrt the polar conic of P wrt the cubic is the polar of P wrt the cubic)” was helpful to me.

Then I worked on the paper of Prof. J. de Vries, "On polar figures with respect to a plane cubic curve".
On his first page he states:
So the poloconica of two lines is the locus of the point Z which with relation to the points of intersection X,Y of this conic with one of the given lines are in such a position that the polar lines pxz and pyz concur on the other one of the given lines, which is then at the same time polar line of X and Y.

This text is hard to understand.
However next simple construction is derived from it (after many trials and errors in interpretating these words):
1. Given a line L1 with 2 fixed points (X,Y) on it.
2. Determine some other line L2 with variable point V on it.
3. The Poloconica is the locus of pxv^pyv with variable V on L2.
(pxv and pyv just need one auxiliary V-Polar Conic and then construct the polars from X and Y wrt this conic giving pxv and pyv).

I’m left with these 2 questions:
1. how can this “Poloconica” be seen as a “Poloconic” given above definition? Which lines envelope this conic?
2. What extra properties has this Poloconica?
I will read more of your messages later.

Best regards,
Chris

Message: #2188
Date: 2024-03-14
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
The 2 lines form a degenerated conic!
The poloconic is the poloconic of this degenerated conic (and not of a line, Duerge's definition is not relevant here) ...
Best regards
Bernard
PS in my message 2174, please read in the PS conic instead of cubic

---

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Message: #2189
Date: 2024-03-14
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
I checked your construction step by step! Congratulations
Fantastic and I think it opens a wide range of other constructions, but that's for another time ...
I have just 2 questions at this stage:
1) I know that you like the points Pi on the non closed part of the cubic.
Bu you could have 2 more triangles Qi and Ri with the 6 other contact points, provided Qi and Ri on Li and QiRj passing through Fk
2) why did you change the point (f)?
In 2172, you had 6 intersections of the lines PoFi with the conic lying on the hessian (naturally with the points Fi and Ti).
Is it wrong?
Now you have the intersections of PiXj and PjXi on Lk, which gives only 3 points ...
Best regards
Bernard

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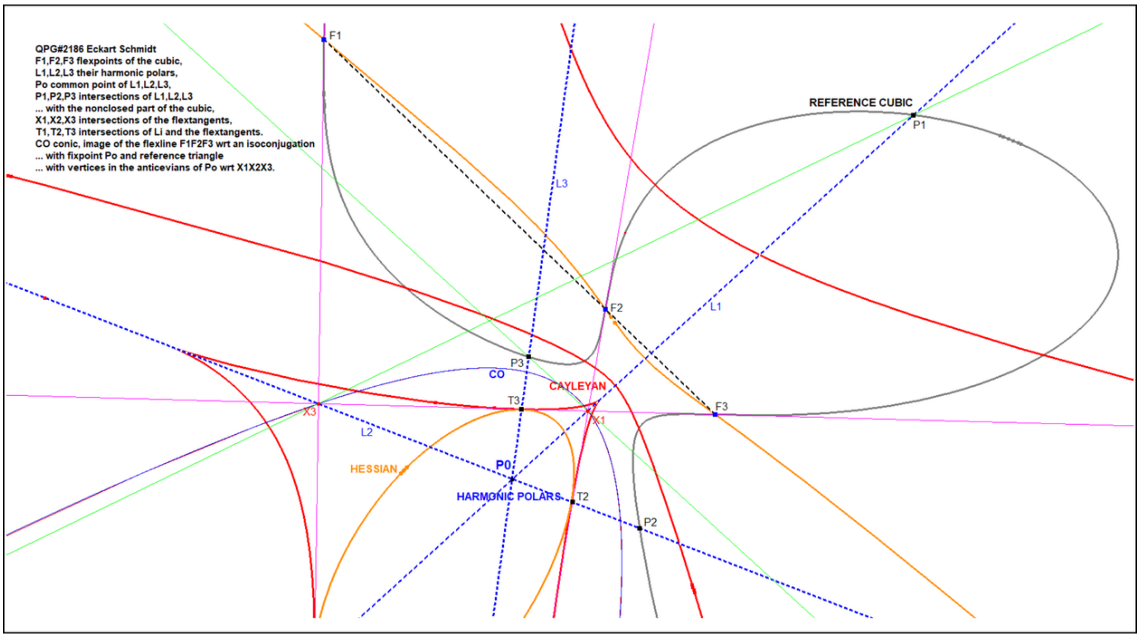
Message: #2190
Date: 2024-03-14
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,

According to my drawing CO does not bear the intersections of
the cubic and its cayleyan.
See attachment.

Best regards,
Chris
```



CU-Cayleyan-41-TryoutES.pdf

Message: #2191
Date: 2024-03-14
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
it's not your conic, is it?
Best regards
Bernard
```

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Message: #2192
Date: 2024-03-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

thanks for your drawing,
... but please prove whether your conic is CO conic,
    image of the flexline F1F2F3 wrt an isoconjugation
... with fixpoint Po and reference triangle
... with vertices in the anticevians of Po wrt X1X2X3.

Best regards Eckart
```

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Message: #2193
Date: 2024-03-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

thanks for interest!
Wrt 1) For a monopartite cubic,
... there are only three intersections of the Li and the cubic.
Wrt 3) The property in #2173 doesn't hold, see already #2179.

Best regards Eckart
```

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Message: #2194
Date: 2024-03-14
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,

I used the isoconjugation of Günther Pickert that you mentioned
in QPG#2068.

Attached the Cabri-macro I made for a triangle ABC with fixed
point K mapping X into X’.
(editorial note: Cabri-macro omitted because it is no
print-file)

Best regards,
Chris
```

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Message: #2195
Date: 2024-03-14
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

To be precise (like you asked), I used Günther Pickert's Isoconjugation for points on the flexline F1F2F3 with fixpoint Po, like described in your message QPG#2068.

I don't know what you mean specifically with "with vertices in the anticevians of Po wrt X1X2X3" since in your message QPG#2068 you don't use these items in the construction.

Best regards,

Chris

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Message: #2196
Date: 2024-03-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris,

```
your conic in #2190 cannot be my conic CO in #2186,
... the isoconjugate of a line is a circumconic
    of the reference triangle,
... your conic is circumscribed X1X2X3,
... but the reference triangle of CO
... shall have vertices in the anticevians of Po wrt X1X2X3.
```

Best regards Eckart

```
PS: Excuse, I couldn't open your CABRI file,
... my macro for isoconjugation has the Pickert construction.
```

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Message: #2197
Date: 2024-03-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

perhaps of interest, further points of the hessian:
... Yk = PiXj ^ Lk = PjXi ^ Lk,
... Zk = TiYj ^ Lk = TjYi ^ Lk,
... the triangles Y1Y2Y3 and Z1Z2Z3 are also of type (b)
    in #2186
... perhaps the only real flexline trilateral
    will be also of this type?

Best regards Eckart
```

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Message: #2198
Date: 2024-03-14
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
Thanks for your help, but I'm totally lost!
There are now too many contradictory drawings ...
1) on your drawing 2173, the 3 lines PoFi cut the hessian in 6 points, which are coconic! what's this conic?
2) on your new message2186, unfortunately without figure, you mention apparently the intersection Yk on Lk of PiXj and PjXi as being on the hessian, but I can't reproduce this property, as Lk cuts the hessian in only one point, which is Tk.
What is my misunderstanding?
What makes you sure you have this time the real hessian and not a proxy?
I have now difficulties with Geogebra ...
Would you be kind enough to make a figure with the cubic, the (real) hessian and the conic (and naturally, P, Li, tgi, Fi, Ti, Xi and your last Yi and Zi).
Many thanks in advance.
Best regards
Bernard
PS For the cayleyan, it's more difficult, but any duality suits, it's not necessary a QA/QL duality (for example a pole/polar wrt a conic)

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Message: #2199
Date: 2024-03-15
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
Always working on your last conic, I looked again at Chris 2nd figure of message 2160.
It's easy to put the points Pi (unique point), Xi and Ti as well as the turning points Yi of the cayleyan on the lines Li.
X3 and Y3 are out of the figure, but P1X2 and P2X1 intersect clearly on L3, but not on the hessian.
Can you help me?
Best regards
Bernard
PS Here Li intersects the reference cubic in 1 point, but the hessian in 3 points, the turning point of the cayleyan being the harmonic of Ti wrt the 2 others.

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Message: #2200
Date: 2024-03-15
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

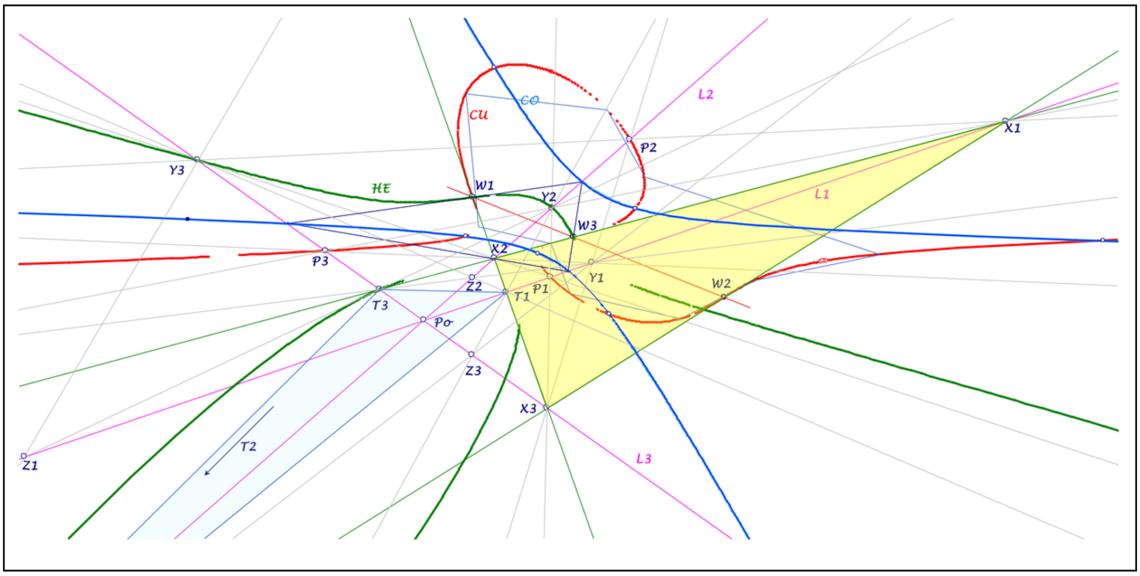
```
Dear Bernard,

I try, to clear your questions,
... but it's difficult for me to keep a clear view.

Wrt 1) I don't know, whether the property in #2173 holds for
bipartite cubics,
... for monopartite cubics Chris' drawing shows,
... that such a cubic cannot be my CO,
... therefore I begged to forget the message 2173.

Wrt 2) Thanks for your question, I have to correct:
... the property will only be valid for monopartite cubics,
... I tried a drawing for this case in your sense.

Best regards Eckart
```



2024-03-15.pdf

Message: #2201
Date: 2024-03-15
From: hoingason@gmail.com
Subject: Rational Orthodiagonal Quadrilateral

```
In this short note, I give a method for generating all rational
orthodiagonal quadrilaterals. Hope it's useful to someone.
Best regards
--TXM--
```

Rational Orthodiagonal Quadrilateral

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices. An orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals or with a perpendicular pair of opposite sides. In this short note the author gives a method to create all such rational orthodiagonal quadrilaterals.

Theorem.

Suppose x, y, z, t, k are rational numbers such that

$$t^4 - 32\, t^2 x^2 y^2 z^2 \left(x^4 y^2 + y^2 z^4 - 2 x^2 \left(y^4 - y^2 z^2 + z^4\right)\right) + 256\, x^4 y^8 z^4 \left(x^2 - z^2\right)^4 = k^2$$

Then we can set the coordinates of the 4 vertices of every rational orthodiagonal quadrilateral ABCD as follows :

$A = \left(8\, t\, x\, y^2 z \left(x^2 - z^2\right), 0\right)$

$B = \left(0, 16\, t\, x^2 y^2 z^2\right)$

$C = \left(8\, t\, x^2 y\, z \left(z^2 - y^2\right), 0\right)$

$D = \left(0, 16\, x^2 y^4 z^2 \left(x^2 - z^2\right)^2 - t^2\right)$

Example:

For $t = -\dfrac{8\, x\, y^3 z^2 \left(x^2 + z^2\right)}{y^2 + z^2}$, after reduction we get the following solution :

$A = \left(4\, y\, z \left(y^2 + z^2\right)\left(-x^4 + z^4\right), 0\right)$

$B = \left(0, -8\, x\, y\, z^2 \left(x^2 + z^2\right)\left(y^2 + z^2\right)\right)$

$C = \left(4\, x\, z \left(x^2 + z^2\right)\left(y^4 - z^4\right), 0\right)$

$D = \left(0, -4\, y^2 z^2 \left(x^2 + z^2\right)^2 + \left(x^2 - z^2\right)^2 \left(y^2 + z^2\right)^2\right)$

Related links

Rational Cyclic Orthodiagonal Quadrilateral

Rational Orthodiagonal Quadrilateral (TXM).pdf

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Message: #2202
Date: 2024-03-15
From: van10hoven@gmail.com
Subject: Re: Rational Orthodiagonal Quadrilateral

```
Dear Trinh Xuan Minh,
Thanks again for your contribution!
Best regards,
Chris van Tienhoven
```

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Message: #2203
Date: 2024-03-15
From: hoingason@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] Rational Orthodiagonal Quadrilateral

---

```
Thank you Mr. Chris van Tienhoven
Wishing you a day filled with joy.
Best Regards
--TXM--

Vào 21:33, Th 6, 15 thg 3, 2024 Chris <van10hoven@gmail.com> đã
viết:
> Dear Trinh Xuan Minh,
> Thanks again for your contribution!
> Best regards,
> Chris van Tienhoven
```

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Message: #2204
Date: 2024-03-15
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
Thanks a lot for the figure, it helps me greatly to understand!
I simply observe that your cubic is monopartite and the hessian bipartite, like the ones of Chris in 2160.
According to your message 2197, the points Yi and Zi should be on the hessian.
1) The property doesn't hold on Chris figure (we have P1, P2 and P3 and X1 and X2, hence Y3, which is not on the hessian)
2) Your drawn hessian pass through W1, W3, Y1, Y2, Y3, T1, T2 (I suppose), T3 and Z2, but not through W2, Z1 and Z3.
What makes you think this cubic is the hessian?
I'm desperately wanting to understand ...
Best regards
Bernard

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Message: #2205
Date: 2024-03-15
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Bernard,

please accept my apologies, at the moment I am very confused,
... I can neither help you nor me,
... for it seems, that I use a false hessian.
Your observation is correct in #2199
... and my drawing shows not a correct tangency
... of the hessian and the flextangents in Ti.
The observation f) in #2186 doesn't hold.

Best regards Eckart

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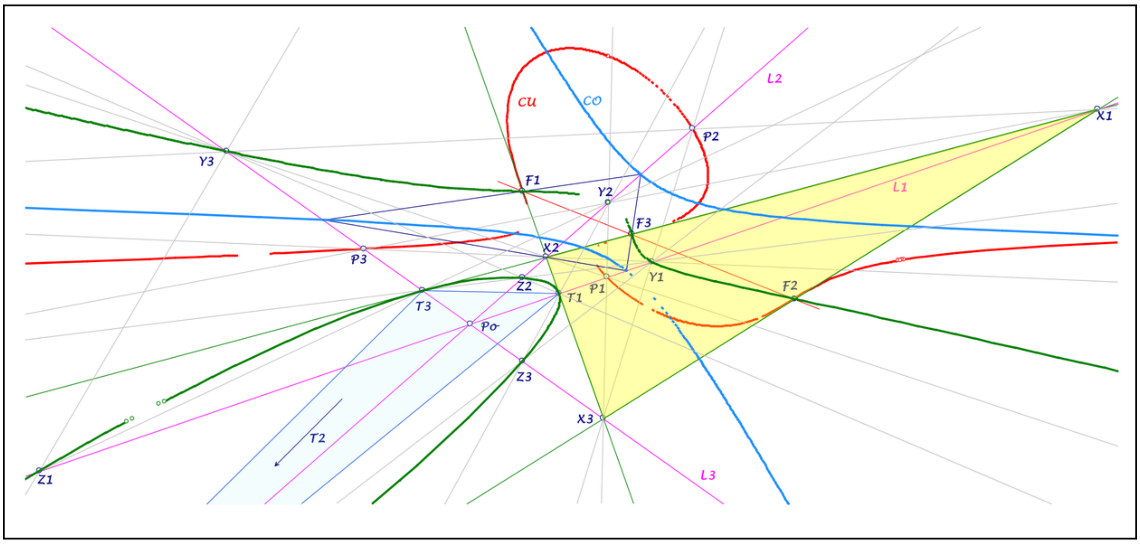
Message: #2206
Date: 2024-03-15
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

excuse once more, my actual work is miserable!
I haven't seen the mistake in the last figure,
... a last stretching will be the reason.
Drawing the 9P-cubic once more,
... the tangency in Ti to the flextangents is better,
... it seems that the points Y1,Y2,Y3 and Z1,Z2,Z3
... are points of the hessian.

Best regards Eckart
```



2024-03-15.pdf

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Message: #2207
Date: 2024-03-16
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
This time, all seems to fit together!
Congratulations and many thanks for the figure.
But I still wonder why this property doesn't work on Chris 2nd
figure in message 2160!
Let's say it clearly, one of your 2 hessians is wrong ...
Best regards
Bernard
PS Georges Polya said long time ago that geometry is the art of
correct reasoning on incorrect figures
```

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Message: #2208
Date: 2024-03-16
From: bernard.keizer@gmail.com
Subject: Hessian and prehessians

```
Dear Chris, dear Eckart
The 9 flexes form a CB system and the cubics passing through
these 9 flexes belong to the syzegetic pencil.
To identify one of this cubic, we just need a point or a
tangent.
All these cubics share the lines Li, the point P, the 3 real Fi
and the 3 lines PFi, each PFi being the harmonic of Li wrt Lj
and Lk.
Each Li cuts the flexine L in a point F'i harmonic of Fi wrt Fj
and Fk.
P is the dual of L, Fi the dual of Li and F'i the dual of PFi.
1) For any point M (not on L or Li), the line FiM cuts Li in Mi
and the harmonic of M wrt Fi and Mi give 3 points M'i.
Doing this again from these 3 new points gives 2 other points so
that we have 6 points 2 by 2 aligned with the Fi.
The 6 points are coconic (we have an example on Chris figure
2160 with the intersections of the cubic and it's cayleyan).
2) The lines Li cut the reference cubic in 3 points Pi or 9
points Pi, Qi and Ri, contact points of the tangents from Fi to
the cubic choosen such as PiPj or QiQj or RiRj are  aligned with
Fk.
```

They cut the same way the hessian in 3 points Ti or 9 points Ti, Ui and Vi, contact points of the tangents from the Fi to the hessian with the same rule.
Now it's easy to understand 2 main consequences:
1) If the Ui and Vi are real, the lines FiUi and FiVi are the tangents in Fi to the 2 cubics having the same hessian as the reference cubic.
If they aren't real, the reference cubic is the only real cubic having this hessian
Note that the 2 cubics having the same hessian have not the same cayleyan
2) If the Pi, Qi and Ri are real, the lines FiPi, FiQi and Fi Ri are the tangents in Fi to the 3 prehessians of the reference cubic.
If the Qi and Ri are not real, the reference cubic has only one real prehessian.
Note that the 3 cubics having the same hessian have 3 different cayleyans
Naturally, in all the cases, the Xi triangles of the tangents in the flexes are determined as the anticevian triangles of the corresponding Ui, Vi, Pi, Qi and Ri triangles.
Best regards
Bernard
PS for Eckart If I'm not wrong, my points Ui and Vi are your points Yi and Zi

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Message: #2209
Date: 2024-03-16
From: hoingason@gmail.com
Subject: Rational Rectangle-Point

In this short note, I give a method for generating all rational rectangle-point. Hope it's useful to someone.
Best regards
--TXM--

Rational Rectangle-Point

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
If a rectangle ABCD has rational side lengths and the distances from a point M to the vertices are also rational, the rectangle-point combination is sometimes called a rational rectangle. In this short note the author gives a method to create all such rational rectangle.

Theorem.

Suppose x, y, z, t, k are rational numbers such that

$$t^4 - 32\,t^2 x^2 y^2 z^2 \left(x^4 y^2 + y^2 z^4 - 2\,x^2 \left(y^4 - y^2 z^2 + z^4\right)\right) + 256\,x^4 y^8 z^4 \left(x^2 - z^2\right)^4 = k^2$$

Then we can set the coordinates of five vertices of all rational rectangle-point combinations ABCD-M as follows:

$$A = \left(8\,t\,x\,y^2 z\left(x^2 - z^2\right),\ 16\,t\,x^2 y^2 z^2\right)$$
$$B = \left(-8\,t\,x^2 y\,z\left(y^2 - z^2\right),\ 16\,t\,x^2 y^2 z^2\right)$$
$$C = \left(-8\,t\,x^2 y\,z\left(y^2 - z^2\right),\ 16\,x^2 y^4 z^2\left(x^2 - z^2\right)^2 - t^2\right)$$
$$D = \left(8\,t\,x\,y^2 z\left(x^2 - z^2\right),\ 16\,x^2 y^4 z^2\left(x^2 - z^2\right)^2 - t^2\right)$$
$$M = (0,\ 0)$$

Example:

For $t = \dfrac{8\,x\,y^3 z^2\left(x^2 + z^2\right)}{y^2 + z^2}$, after reduction we get the following solution :

$$A = \left(-4\,y\,z\left(y^2 + z^2\right)\left(-x^4 + z^4\right),\ 8\,x\,y\,z^2\left(x^2 + z^2\right)\left(y^2 + z^2\right)\right)$$
$$B = \left(4\,x\,z\left(x^2 + z^2\right)\left(-y^4 + z^4\right),\ 8\,x\,y\,z^2\left(x^2 + z^2\right)\left(y^2 + z^2\right)\right)$$
$$C = \left(4\,x\,z\left(x^2 + z^2\right)\left(-y^4 + z^4\right),\ \left(y^2 - z^2\right)^2\left(x^4 + z^4\right) - 2\,x^2 z^2\left(y^4 + 6\,y^2 z^2 + z^4\right)\right)$$
$$D = \left(-4\,y\,z\left(y^2 + z^2\right)\left(-x^4 + z^4\right),\ \left(y^2 - z^2\right)^2\left(x^4 + z^4\right) - 2\,x^2 z^2\left(y^4 + 6\,y^2 z^2 + z^4\right)\right)$$
$$M = (0,\ 0)$$

Related links

Constructing a Rational Rectangle

Rational Rectangle-Point (TXM).pdf

Message: #2210
Date: 2024-03-16
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,
I've made my own figur with a general cubic and it's hessian.
These 2 curves are exact, but each of the curves is made with a cubic of 9 points with the command of Geogebra Implicit curve.
Unfortunately, Geogebra doesn't give the exact intersections of the 2 curves and I have to approximate them.
Nevertheless, the figur shows clearly that your points Yi as intersection of PjXk and PkWj are certainly on Li, but not on the hessian!
Again, what makes you think that your curve is the hessian *?*
Did you draw it exactly or is it pure speculation?
I spent many hours in order to check your construction and I would not like to die stupid ...
Thanks in advance for your answer
Best regards
Bernard

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Message: #2211
Date: 2024-03-16
From: bernard.keizer@gmail.com
Subject: Cocayleyan cubics

```
Dear Chris, dear Eckart
In my previous message, I found a way to draw the 2 cohessian
cubics of a reference cubic and it's 3 prehessians.
I think the same is also possible in order to find the 2
cocayleyan cubics of the reference cubic (which have not the
same hessian).
I refer again to Chris beautiful figure in his message 2160.
There are 3 tangents from Fi to the cayleyan, one is FiTi, let's
name the 2 others FiWi and FiZi, Wi and Zi being on Li.
Then FiWi and FiZi are the tangents in Fi to the 2 cocayleyan
cubics and in Wi and Zi to the corresponding hessians.
Best regards
Bernard
PS I would be very happy if Chris could confirm one day these 2
cohessian cubics and cocayleyan cubics with Mathematica
(Again, the 2 cohessian cubics are not necessary real, but the 2
cocayleyan are always real)
```

Message: #2212
Date: 2024-03-17
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard,

Regarding your message QPG#2184, where you say:
“In fact, the poloconic mentioned here is the poloconic of the degenerated conic formed by the 2 lines XY and the mixed polar Lxy of X and Y.
This leads to the notions of polar quadrangle and polar triangle wrt the cubic, which were new for me.”
Which polar quadrangle and polar triangle (construction) do you exactly mean?

Regarding your message QPG#2185, thanks for the references!
Especially 5) Plana algebraic curves Jason Cantarella, (written by Hilton), is very accessible.

Regarding your message QPG#2188, where you say:
“The 2 lines form a degenerated conic!
The poloconic is the poloconic of this degenerated conic (and not of a line, Duerge's definition is not relevant here) ...”
So you say that the Poloconic is derived from conic. Can you tell me what the official definition for this is?

By the way the reference of Eckart in QPG#2121 Reference: H. Durege (1871): Die ebenen Kurven dritter Ordnung, where he tells that in a reprinting of Hansebooks the author is written: H. Duerge. I think that “Duerge” is a misspelling, because his book is sold under the name H. Durege: https://www.amazon.nl/Die-Eben en-Curven-Dritter-Ordung/dp/116130746X.

Best regards,
Chris

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Message: #2213
Date: 2024-03-17
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

you ask:
"...what makes you think that your curve is the hessian?"

The green cubic in #2206 is constructed as 9P-cubic
... of F1,F2,F3,T1,T2,T3,Y1,Y2,Y3 (Yi = PjXk^PkXj),
... it is tangent to the flextangents in Ti,
... it has the same flexpoints
    and harmonic polars with common Po.
I think this are properties for the conclusion, you ask for.

Best regards Eckart
```

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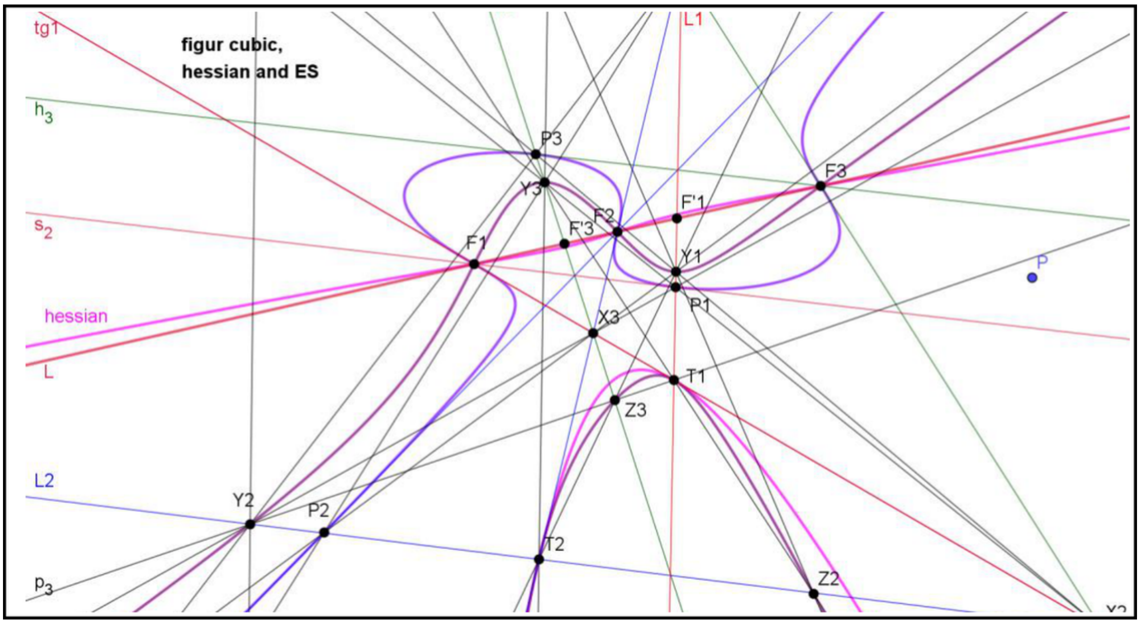
Message: #2214
Date: 2024-03-17
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
As you had the courtesy of sending me a complete figure, I do
the same.
I succeeded finally in drawing the curve and it's hessian and I
added without difficulty your cubic (ES) with the points Yi and
Zi.
All your triangles Pi, Xi, Ti, Yi, Zi and the like share the 2
properties:
1) vertices on the Li
2) sides through the Fi
The combination of 2 such triangles gives a 3rd one of the same
kind ...
In fact, your cubic ES has the same flexes and the same polar
lines concouring in the same point P and is tangent in Ti to the
flextangents and naturally to the hessian and to the cayleyan.
But it is not the hessian! (the flextangents are different)
It seems you have drawn another different curve and you will
certainly find other properties ...
Best regards
Bernard
```



Cubic, hessian and ES.pdf

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Message: #2215
Date: 2024-03-17
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

thanks ,thanks, thanks, I give up!
This gives a lot of clearence, my cubic ES is not the hessian!
Proving single points, I found also disagreements.
But what is the deeper geometry of the cubic ES,
... there are a lot of properties.

Best regards Eckart
```

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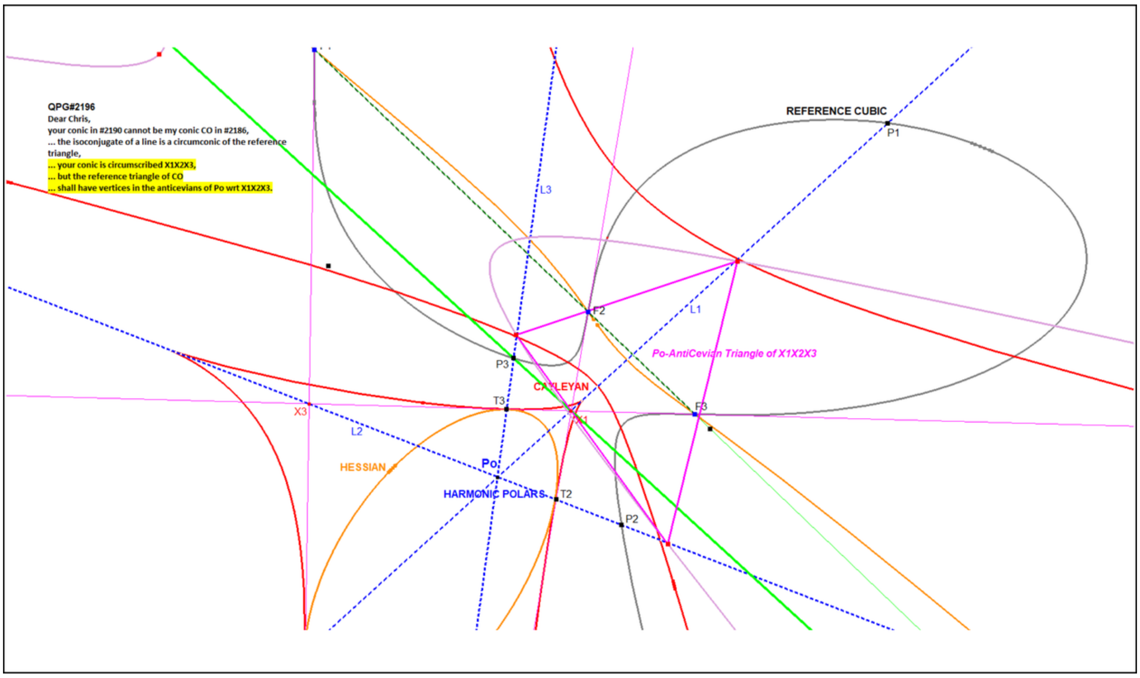
Message: #2216
Date: 2024-03-17
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

Regarding your message #2196.
In #2186 you wrote
“CO conic, image of the flexline F1F2F3 wrt an isoconjugation
... with fixpoint Po and reference triangle
... with vertices in the anticevians of Po wrt X1X2X3.”

I made a new picture with Pickert’s construction with the Po-Anticevian Triangle wrt X1X2X2 as reference triangle. See attachment.
The conic is the one through the vertices of Po-Anticevian Triangle wrt X1X2X2.
This time the polar of Fi wrt this conic is NOT the Fi-Harmonic Polar.
In my former picture the polar of Fi wrt that conic was indeed the Fi-Harmonic Polar.

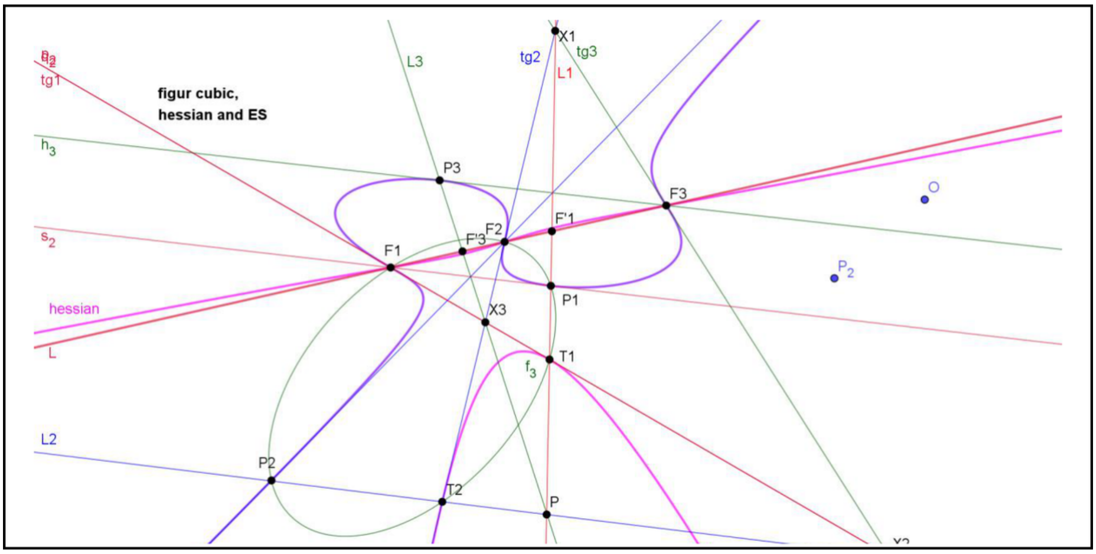
Best regards,
Chris



CU-Cayleyan-42-TryoutES.pdf

Message: #2217
Date: 2024-03-18
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

```
Dear Chris,
I was very busy these days/weeks
1) with Eckart's last elucubrations about a new wrong
construction of the hessian
2) with my own last elucubrations about the 2 cohessians, the 3
prehessians and the 2 cocayleyans of a reference cubic
So I had almost forgotten the aticle of de Vries
I thought you wanted to study it!
I don't mean exactly any quadrangle or any triangle and I don't
know a construction.
I just read in de Vries in the 2nd paragraph down on 1st page
and above on 2nd page the definitions of these 2 items.
I found an example on your figure 2160 or on my reproduced figur
(see attached file).
F1F2T1T2 form a polar quadrangle and it's DT F3X3T3 form a polar
triangle.
The poloconic of the conic formed by the flexline L and the line
T1T2 (through F3) is certainly through F1, F2, T1 and T2
(perhaps through p1 and P2?).
The polar of F3 wrt this conic is L3.
Best regards
Bernard
PS There are probably definitions of the poloconic of a line or
of a conic, but I don't know them, I've told you everything I
knew ...
```



Cubic and hessian.pdf

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Message: #2218
Date: 2024-03-18
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
My apologise, I lost my most elementory reflexes!
The polar conic of the conic formed by the flexline L and the line T1T2 (through F3) cuts the hessian in points which are conjugates on the hessian of the intersections of the initial conic with the hessian, id est the points T1, T2, T3 (conjugates of F1, F2 and F3) and F1 and F2 (conjugates of T1 and T2).
It is therefore the degenerated conic formed by the 2 lines T3F1 (through T2) and T3F2 (through T1).
I hope you will agree with that.
The same goes naturally for the 2 other conics formed the same way by the flexline L and T1T3 or T2T3.
Best regards
Bernard

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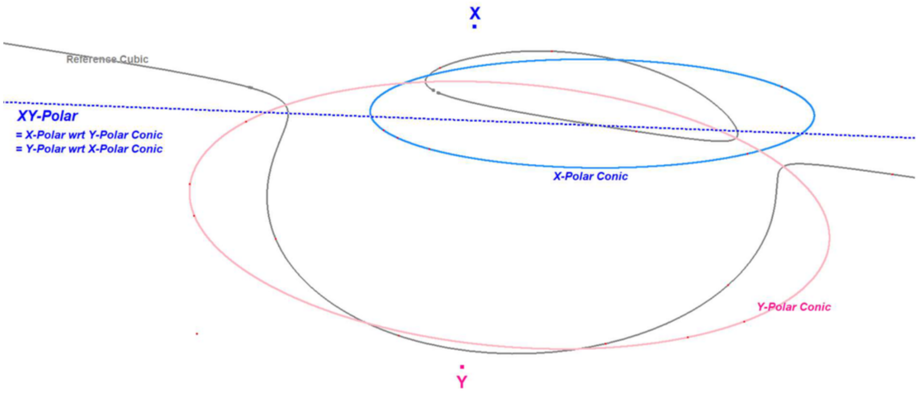
Message: #2219
Date: 2024-03-18
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard,
Thanks for your remarks about the Polar Quadrangle and its Diagonal Triangle.
It made me realize again that any conic through the vertices maps as a polar any vertex of the QA-DT to its opposite DT-side.
I can confirm your example, that the conic through F1, F2, T1, T2 and P1, P2 produces as polar of F3 the line L3.
Today I researched further these items:
CU_2P-L2 Mixed XY-Polar
CU_2L-Co1 Poloconica
Attached a description.
If you have any remarks, please let me know.
Best regards,
Chris

CU_2P-L2 Mixed XY-Polar

Given a general Cubic CU and two random points X and Y on the cubic, then:
the X-Polar of the Y-Polar Conic = the Y-Polar of the X-Polar Conic.
This Polar is called the Mixed Polar or XY-Polar or Mixed XY-Polar.
A description of the P-Polar Conic can be found at CU_P-Co1. It is the conic through P (not necessary on CU) and the 4 points of tangency from the tangents at P to CU.
The P-Polar is just the polar of some point P wrt some conic.



Properties

1. It isn't maybe easy to construct the Polar Line of an infinity point IP wrt an X-Polar Conic. This now easily can be constructed as the X-Polar Line wrt the IP-Polar Conic. And the IP-Polar Conic we know, it is the Diametral Conic CU-IP-Co1.
2. Given 2 points (X,Y) on a line L1 and another point Z on a 2^{nd} line L2. The locus of the intersection point of the Mixed YZ-Polar and the Mixed XZ-Polar with variable Z on L2 is a conic called the Poloconica CU_2L-Co1. Every point Z on this conic has the property that given any 2 points (X,Y) on L1 or L2 and another random point Z on the not used line of (L1,L2) the XY-Polar, YZ-Polar and the XZ-Polar will concur in one point.

CU_2P-L2 XY-Polar and CU_2L-Co1 PoloConica-01.pdf

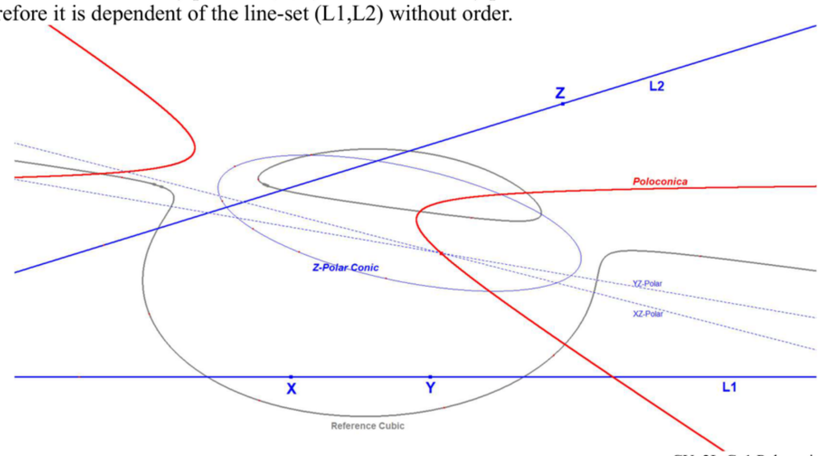
CU_2L-Co1 Poloconica

[J. de Vries, On polar figures with respect to a plane cubic curve] describes these items:
1. Given a general Cubic CU and two random points X and Y not necessary on the cubic, then the X-Polar of the Y-Polar Conic = the Y-Polar of the X-Polar Conic.
This the XY-Polar Lxy. See CU_2P-L2.
2. Theorem:
Given 2 points X,Y, then the locus of points Z for which Lxy, Lyz, Lxz concur is a conic he named Poloconica.
There are more properties. Here is the construction.

Construction

- Given 2 points (X,Y) on some line L1 and another point Z on some line L2.
- The locus of the intersection point of the YZ-Polar and the XZ-Polar with variable Z on L2 is the conic called the Poloconica CU_2L-Co1.

This conic is the same for any pointset (X,Y) on L1 and also the same for any point Z on L2.
It is also the same for any pointset (X,Y) on L2 and for any point Z on L1.
Therefore it is dependent of the line-set (L1,L2) without order.



CU_2L-Co1 Poloconica-20.fig

Actually the lines L1,L2 and the Poloconica CO form a triple (L1,L2,CO).
L1, L2, CO are interchangeable for these properties:

1. The X- and Y-Polar (X,Y on one item of (L1,L2,CO)) wrt Z-Polar Conic (Z on another item of (L1,L2,CO)) concur in a point on last item of (L1,L2,CO).
2. The X-Polar (X on one item of (L1,L2,CO)) wrt Y-Polar Conic (Y on another item of (L1,L2,CO)) passes through a fixed point Z of last item of (L1,L2,CO) with variable X or Y, the one that is on a line.

Construction of L2 when (L1,CO) are known
Given L1 and CO.

1. Let X and Y be 2 random points on L1.
2. Let V and W be 2 random points on CO.
3. Let Lxv be the polar of X wrt the V-Polar Conic.
   Let Lyv be the polar of Y wrt the V-Polar Conic.
   Let Lxw be the polar of X wrt the W-Polar Conic.
   Let Lyw be the polar of Y wrt the W-Polar Conic.
4. Now L2 = Lxv^Lyv connected with Lxw^Lyw.

CU_2P-L2 XY-Polar and CU_2L-Co1 PoloConica-01.pdf

Message: #2220
Date: 2024-03-18
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

```
Dear Chris,

Congratulation for your pedagogical effort!
But why don't you go straight to the point .
page 2 at the top
1) is ok
2) The locus of Z for which Lxy, Lxz and Lyz concur is the
poloconic of the lines XY and Lxy

Then, you may present successively
1) the mixed polar conic Lxy
2) the poloconic of 2 lines L1 and L2
   (I trust you with the construction
    of the poloconic of 2 lines)
   1) and 2) are already done
3) the poloconic of XY and Lxy
4) the intersections W and Z of Lxy with the conic
5) the polar QA XYWZ
6) it's DT, which is a polar triangle
   3), 4), 5 and 6) are to be done

Best regards
Bernard
```

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Message: #2221
Date: 2024-03-19
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard,

I was glad with your remarks. Writing a paper it is easy to get in your own writing bubble and then it is good to get feedback. I changed your mentioned point 2).
About the general term “Poloconic”, I am still cautious using this term, because it is also used by many others and we still don’t know the exact definition. Therefore I use the special term “Poloconica” that J. de Vries made up.
About the Polar Quadrangle I am not sure if it is functional to mention it. I wonder about the added value in this case and about the special properties in this environment. Do you know about any special properties of the Polar-QA and -DT in this case?

J. de Vries mentions:
* X,Y,Z,W form a closed group, so that each side of the quadrangle determined by them is the polar line of the vertices not lying on it, therefore (it is) a polar quadrangle (Reye).
Out of considerations ensues that a polar quadrangle is determined by two of its vertices, but also by two of its opposite sides. In the last case the vertices are determined by the poloconica of the given lines; in the former case we can use the poloconica belonging to the polar line of the given points and their connecting line.”

As a matter of fact I do not understand much of his description (like many other parts of his article). Can you brew something from his words?

Here are my problems:
1. “each side of the quadrangle determined by them is the polar line of the vertices not lying on it” I cannot confirm this, can you?
2. I suppose he means when he says “is determined by two of its vertices, but also by two of its opposite sides” that these cases occur when the Poloconica intersects the two defining lines. But there is also the case that the Poloconica doesn’t intersect the two defining lines in any real points.
3. And I still wonder what is the added value of this.
After finishing this part I will start with studying your message #2211.

Best regards,
Chris

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Message: #2222
Date: 2024-03-19
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
I don't know how to help you !
After all, it is your article and you do what you want ...
But it seems to me, your article is not finished!
You present the mixed polar Lxy and the poloconic of a conic
formed by 2 lines (points 1 and 2).
Fair enough ... (but the construction of L2 knowing L1 and CO
has no interest)
But the 3rd part is not here, *the poloconic of the lines XY and
Lxy (L1 is XY, L2 is Lxy)*
You will notice that this conic passes through X and Y
The rest follows
4) W and Z are the intersections of Lxy with the poloconic
5) X, Y , W and Z form a polar QA.
This means that for one vertice W for example Lxy, Lxz and Lyz
concur in W (4 possibilities) and for one side XY for example
Lxy is WZ (3 possibilities)
By the way, the poloconic always intersect XY and Lxy
6) It's DT is a polar triangle wrt the cubic
For me, the polar QA and the polar triangle wrt a cubic are as
interesting as the polar triangle wrt a conic!
And as you decided to study the properties of the cubic, I
supposed you would be interested ...
Best regards
Bernard

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Message: #2223
Date: 2024-03-19
From: hoingason@gmail.com
Subject: General Rational Quadrilateral and Elliptic Curves (TXM)

```
In this short note, the author gives a method for generating all
rational
quadrilaterals. Hope it's useful to someone.
Best regards
--TXM--
```

General Rational Quadrilateral and Elliptic Curves

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices. If at least 1 vertex of a rational polygon is removed, the resulting polygon is called a sub-polygon of the original polygon. The area of any rational polygon is of the form $a\sqrt{b}$; call $\sqrt{b}$ the congruent factor of that polygon. In this short note, the author gives a method for generating all rational quadrilaterals.

Theorem 1.
A polygon is rational if and only if it shares the same congruent factor with all its sub-polygons.

Theorem 2.

Suppose x, y, z, u, v, k are rational numbers such that

$$\begin{aligned}
&\big(u^4 + 8u^3(x+z)^2 - 128uxyz(x+z)^2(x+y+z) + 256x^2y^2z^2(x+y+z)^2 + \\
&\qquad 16u^2\big(x^4 + 4x^3z + z^4 + 2x^2z(y+3z) + 2xz\big(y^2 + yz + 2z^2\big)\big)\big)v^4 + \\
&\quad 4\big(128uxyz(x+z)^2(x+y+z)(x(y-z) - z(y+z)) - 512x^2y^2z^2(x+y+z)^2(x(y-z) - z(y+z)) + \\
&\qquad 2u^4(x(-y+z) + z(y+z)) + 8u^3(x+z)^2(x(-y+z) + z(y+z)) - \\
&\qquad 64u^2xyz(x+y+z)\big(2x^2 + z(-y+z) + x(y+3z)\big)\big)v^3 + \\
&\quad 16\big(-16u^3xyz(x+z)^2(x+y+z) + 256ux^2y^2z^2(x+z)^2(x+y+z)^2 + 32u^2xyz(x+y+z) \\
&\qquad\quad \big(x^2\big(x^2 + 4xy + y^2\big) - 4xy^2z + \big(-5x^2 - 6xy + y^2\big)z^2 - 2(3x+y)z^3 - 2z^4\big) + \\
&\qquad u^4\big(x^2y^2 - 4xy(x+y)z + (x-y)^2z^2 + 2(x+y)z^3 + z^4\big) + \\
&\qquad 256x^2y^2z^2(x+y+z)^2\big(x^2y^2 - 4xy(x+y)z + (x-y)^2z^2 + 2(x+y)z^3 + z^4\big)\big)v^2 - \\
&\quad 64xyz(x+y+z)\big(128uxyz(x+z)^2(x+y+z)(x(y-z) - z(y+z)) - \\
&\qquad 512x^2y^2z^2(x+y+z)^2(x(y-z) - z(y+z)) + 2u^4(x(-y+z) + z(y+z)) + \\
&\qquad 8u^3(x+z)^2(x(-y+z) + z(y+z)) - 64u^2xyz(x+y+z)\big(2x^2 + z(-y+z) + x(y+3z)\big)\big)v + \\
&\quad 256x^2y^2z^2(x+y+z)^2\big(u^4 + 8u^3(x+z)^2 - 128uxyz(x+z)^2(x+y+z) + \\
&\qquad 256x^2y^2z^2(x+y+z)^2 + 16u^2\big(x^4 + 4x^3z + z^4 + 2x^2z(y+3z) + 2xz\big(y^2 + yz + 2z^2\big)\big)\big) = k^2
\end{aligned}$$

Then we can set the coordinates of four vertices of all rational quadrilateral ABCD as follows:

$A = (x + z,\ 0)$

$$B = \left(y + z - \frac{2xy}{x+z},\ \frac{2\sqrt{xyz(x+y+z)}}{x+z}\right)$$

$C = (0,\ 0)$

$$D = \left(\left(u^2 (v - 4xy)(v + 4z(x+y+z)) - 16xy(v-4xy)z(x+y+z)(v+4z(x+y+z)) + 4u(x+z)^2 \left(v^2 - 16xyz(x+y+z)\right) \right) / (4(u+v)(x+z)(uv - 16xyz(x+y+z))), - \frac{2u\sqrt{xyz(x+y+z)}\,(v+4yz)(-v+4x(x+y+z))}{(u+v)(x+z)(uv-16xyz(x+y+z))} \right)$$

Example:

For $v = -\dfrac{4y(x+y+z)\left(u^2 x - 16x^2 yz(x+y+z) + 4uz\left(2x^2 + x(y+z) + y(y+z)\right)\right)}{-u^2(x+y+z) + 16xyz(x+y+z)^2 + 4uy\left(2x^2 + 3x(y+z) + (y+z)(y+2z)\right)}$,

after reduction we get the following solution :

$$A = \left((x+z)^2 (u + 4xz)(u - 4y(x+y+z)) \left(-4uyz(y+z) + u^2(x+y+z) + 16x^2 yz(x+y+z)\right)\left(u^2 x + 4uyz(y+z) + 16xyz(x+y+z)^2\right), 0\right)$$

$$B = \left((u+4xz)(x(-y+z) + z(y+z))(u - 4y(x+y+z)) \left(-4uyz(y+z) + u^2(x+y+z) + 16x^2 yz(x+y+z)\right)\left(u^2 x + 4uyz(y+z) + 16xyz(x+y+z)^2\right), 2\sqrt{xyz(x+y+z)}\,(u+4xz)(u-4y(x+y+z)) \left(-4uyz(y+z) + u^2(x+y+z) + 16x^2 yz(x+y+z)\right)\left(u^2 x + 4uyz(y+z) + 16xyz(x+y+z)^2\right)\right)$$

$$C = (0, 0)$$

$$D = \left(-4u(x+z)^2 \left(u^4 \left(x^3(-y+z) - 3x^2 y(y+z) - 3xy(y+z)^2 - y(y+z)^3\right) + 128uxy^2 z^2 (x+y+z)\left(4x^4 + 8x^3(y+z) + 4x^2(y+z)^2 - x(y-z)(y+z)^2 - y(y+z)^3\right) - 256x^2 y^2 z^2 (x+y+z)^2 \left(x^3(y-z) + 3x^2 y(y+z) + 3xy(y+z)^2 + y(y+z)^3\right) + 8u^3 yz\left(-4x^4 - 8x^3(y+z) - 4x^2(y+z)^2 + x(y-z)(y+z)^2 + y(y+z)^3\right) + 16u^2 yz\left(6x^5(y-z) + 10x^4(2y-z)(y+z) + 5x^3(5y-z)(y+z)^2 + x^2(14y-z)(y+z)^3 - y^2 z(y+z)^3 + xy(y+z)^2\left(3y^2 + 5yz + 4z^2\right)\right)\right), -8u(x+z)^2 \sqrt{xyz(x+y+z)}\left(-u^2(x+y+z) + 16xyz(x+y+z)^2 + 4uz\left(-2x^2 - x(y+z) + y(y+z)\right)\right) \left(-u^2 x + 16x^2 yz(x+y+z) + 4uy\left(2x^2 + 3x(y+z) + y(y+z)\right)\right)\right)$$

Theorem 3.

If (x_1, y_1) is a rational point of curve (E) : $y^2 = ax^4 + bx^3 + cx^2 + dx + e$,
then we have a rational point (x_2, y_2) on (E) as follows :

$$x_2 = \left(\left(b^4 - 8ab^2 c + 16a^2 c^2 - 64a^3 e\right)x_1^9 - 24a\left(b^2 d - 4acd + 8abe\right)x_1^8 - 12\left(b^3 d - 4abcd - 12a^2 d^2 + 20ab^2 e\right)x_1^7 - 8\left(b^2 cd - 4ac^2 d - 18abd^2 + 12b^3 e + 32abce - 48a^2 de\right)x_1^6 + 6\left(5b^2 d^2 + 20acd^2 - 36b^2 ce - 16ac^2 e + 48abde + 64a^2 e^2\right)x_1^5 + 24\left(2bcd^2 + 3ad^3 - 8bc^2 e - b^2 de + 8acde + 16abe^2\right)x_1^4 + 4\left(4c^2 d^2 + 9bd^3 - 16c^3 e - 28bcde + 60ad^2 e + 12b^2 e^2 + 64ace^2\right)x_1^3 + 24d\left(cd^2 - 4c^2 e + 2bde + 16ae^2\right)x_1^2 + 3\left(3d^4 - 8cd^2 e - 16c^2 e^2 + 32bde^2 + 64ae^3\right)x_1 + 8e\left(d^3 - 4cde + 8be^2\right)\right) / \left(8a\left(b^3 - 4abc + 8a^2 d\right)x_1^9 - 3\left(-3b^4 + 8ab^2 c + 16a^2 c^2 - 32a^2 bd - 64a^3 e\right)x_1^8 - \right.$$

$$24\,b\left(-b^2\,c+4\,a\,c^2-2\,a\,b\,d-16\,a^2\,e\right)x_1^7-$$
$$4\left(-4\,b^2\,c^2+16\,a\,c^3-9\,b^3\,d+28\,a\,b\,c\,d-12\,a^2\,d^2-60\,a\,b^2\,e-64\,a^2\,c\,e\right)x_1^6-$$
$$24\left(-2\,b^2\,c\,d+8\,a\,c^2\,d+a\,b\,d^2-3\,b^3\,e-8\,a\,b\,c\,e-16\,a^2\,d\,e\right)x_1^5-$$
$$6\left(-5\,b^2\,d^2+36\,a\,c\,d^2-20\,b^2\,c\,e+16\,a\,c^2\,e-48\,a\,b\,d\,e-64\,a^2\,e^2\right)x_1^4-$$
$$8\left(b\,c\,d^2+12\,a\,d^3-4\,b\,c^2\,e-18\,b^2\,d\,e+32\,a\,c\,d\,e-48\,a\,b\,e^2\right)x_1^3-$$
$$12\left(b\,d^3-4\,b\,c\,d\,e+20\,a\,d^2\,e-12\,b^2\,e^2\right)x_1^2-$$
$$24\,e\left(b\,d^2-4\,b\,c\,e+8\,a\,d\,e\right)x_1+d^4-8\,c\,d^2\,e+16\,c^2\,e^2-64\,a\,e^3\Big)$$

$$y_2=$$
$$\Big(y_1\Big(-3\,d^8+32\,c\,d^6\,e-96\,c^2\,d^4\,e^2-96\,b\,d^5\,e^2+768\,b\,c\,d^3\,e^3-384\,a\,d^4\,e^3+256\,c^4\,e^4-1536\,b\,c^2\,d$$
$$e^4-512\,b^2\,d^2\,e^4+2048\,a\,c\,d^2\,e^4+2048\,b^2\,c\,e^5-2048\,a\,c^2\,e^5-2048\,a\,b\,d\,e^5+$$
$$4096\,a^2\,e^6-16\big(c\,d^7-12\,c^2\,d^5\,e+6\,b\,d^6\,e+48\,c^3\,d^3\,e^2-48\,b\,c\,d^4\,e^2+$$
$$96\,a\,d^5\,e^2-64\,c^4\,d\,e^3+96\,b\,c^2\,d^2\,e^3-16\,b^2\,d^3\,e^3-512\,a\,c\,d^3\,e^3+$$
$$64\,b^2\,c\,d\,e^4+512\,a\,c^2\,d\,e^4+512\,a\,b\,d^2\,e^4-384\,b^3\,e^5-1024\,a^2\,d\,e^5\big)x_1-$$
$$8\big(2\,c^2\,d^6+9\,b\,d^7-24\,c^3\,d^4\,e-96\,b\,c\,d^5\,e+216\,a\,d^6\,e+96\,c^4\,d^2\,e^2+$$
$$336\,b\,c^2\,d^3\,e^2-192\,b^2\,d^4\,e^2-768\,a\,c\,d^4\,e^2-128\,c^5\,e^3-384\,b\,c^3\,d\,e^3+$$
$$736\,b^2\,c\,d^2\,e^3-640\,a\,c^2\,d^2\,e^3+384\,a\,b\,d^3\,e^3+128\,b^2\,c^2\,e^4+1024\,a\,c^3\,e^4-$$
$$1728\,b^3\,d\,e^4+5632\,a\,b\,c\,d\,e^4-3072\,a^2\,d^2\,e^4-4608\,a\,b^2\,e^5-2048\,a^2\,c\,e^5\big)x_1^2-$$
$$112\big(b\,c\,d^6+6\,a\,d^7-12\,b\,c^2\,d^4\,e-16\,b^2\,d^5\,e+16\,a\,c\,d^5\,e+48\,b\,c^3\,d^2\,e^2+64\,b^2\,c\,d^3\,e^2-$$
$$160\,a\,c^2\,d^3\,e^2-64\,a\,b\,d^4\,e^2-64\,b\,c^4\,e^3-112\,b^3\,d^2\,e^3+640\,a\,b\,c\,d^2\,e^3-$$
$$256\,a^2\,d^3\,e^3-64\,b^3\,c\,e^4+512\,a\,b\,c^2\,e^4-640\,a\,b^2\,d\,e^4-1024\,a^2\,b\,e^5\big)x_1^3+$$
$$\big(364\left(b^2-8\,a\,c\right)d^6+5824\,a\,b\,d^5\,e+1344\,b\left(3\,b^2-20\,a\,c\right)d^3\,e^2+19\,712\,b^3\,c\,d\,e^3-$$
$$129\,024\,a\,b\,c^2\,d\,e^3+286\,720\,a^2\,b\,d\,e^4+224\,d^4\,e\left(13\,b^2\,c+26\,a\,c^2+128\,a^2\,e\right)-$$
$$448\,d^2\,e^2\left(51\,b^2\,c^2-48\,a\,c^3-124\,a\,b^2\,e+64\,a^2\,c\,e\right)+$$
$$1792\,e^3\left(12\,b^2\,c^3+4\,a\,c^4+3\,b^4\,e-16\,a\,b^2\,c\,e-32\,a^2\,c^2\,e+64\,a^3\,e^2\right)\big)x_1^4+$$
$$112\big(13\,b^2\,c\,d^5-52\,a\,c^2\,d^5-40\,b^2\,c^2\,d^3\,e+160\,a\,c^3\,d^3\,e+30\,b^3\,d^4\,e+$$
$$112\,a\,b\,c\,d^4\,e+144\,a^2\,d^5\,e-48\,b^2\,c^3\,d\,e^2+192\,a\,c^4\,d\,e^2-96\,b^3\,c\,d^2\,e^2-$$
$$960\,a\,b\,c^2\,d^2\,e^2+240\,a\,b^2\,d^3\,e^2+64\,a^2\,c\,d^3\,e^2+416\,b^3\,c^2\,e^3+144\,b^4\,d\,e^3-$$
$$768\,a\,b^2\,c\,d\,e^3-1536\,a^2\,c^2\,d\,e^3+2688\,a^2\,b\,d^2\,e^3+3072\,a^3\,d\,e^4\big)x_1^5+$$
$$\big(56\,b\left(23\,b^2-32\,a\,c\right)d^5+2688\,a^2\,d^6+8064\,b^3\,c^2\,d\,e^2-35\,840\,a\,b\,c^3\,d\,e^2-$$
$$86\,016\,a^2\,b\,c\,d\,e^3+1792\,b\,d^3\,e\left(-2\,b^2\,c+a\,c^2+106\,a^2\,e\right)-$$
$$112\,d^4\left(-15\,b^2\,c^2+60\,a\,c^3-196\,a\,b^2\,e-336\,a^2\,c\,e\right)+$$
$$1792\,c\,e^2\left(-b^2\,c^3+4\,a\,c^4+29\,b^4\,e+36\,a\,b^2\,c\,e-32\,a^2\,c^2\,e+64\,a^3\,e^2\right)+$$
$$896\,d^2\,e\left(-7\,b^2\,c^3+28\,a\,c^4+3\,b^4\,e-162\,a\,b^2\,c\,e-168\,a^2\,c^2\,e+480\,a^3\,e^2\right)\big)x_1^6+$$
$$\Big(32\,a\left(143\,b^2+280\,a\,c\right)d^5+30\,720\,b^4\,c\,d\,e^2+16\,384\,a\,c\,d\,e\left(-c^2+4\,a\,e\right)^2+$$
$$16\,b\,d^4\left(173\,b^2\,c-564\,a\,c^2+5968\,a^2\,e\right)-$$
$$128\,b\,d^2\,e\left(69\,b^2\,c^2-212\,a\,c^3+438\,a\,b^2\,e+1320\,a^2\,c\,e\right)-$$
$$512\,b^2\,d\,e\left(8\,c^4+171\,a\,c^2\,e+72\,a^2\,e^2\right)-$$
$$256\,d^3\left(-4\,b^2\,c^3+16\,a\,c^4+3\,b^4\,e+82\,a\,b^2\,c\,e-4\,a^2\,c^2\,e-1144\,a^3\,e^2\right)+$$
$$256\,b\,e^2\left(-3\,b^2\,c^3+12\,a\,c^4+87\,b^4\,e+680\,a\,b^2\,c\,e-96\,a^2\,c^2\,e+192\,a^3\,e^2\right)\Big)x_1^7+$$

$$\big(19\,872\,a^2\,b\,d^5+40\,960\,a\,b\,c^4\,d\,e+19\,872\,b^5\,d\,e^2-328\,704\,a^2\,b\,c^2\,d\,e^2+$$
$$86\,016\,a^3\,b\,d\,e^3+768\,b^3\,c\,d\,e\left(-14\,c^2+43\,a\,e\right)+$$
$$6\,d^4\left(157\,b^4+80\,a\,b^2\,c+1136\,a^2\,c^2+20\,160\,a^3\,e\right)+$$
$$128\,b\,d^3\left(22\,b^2\,c^2-84\,a\,c^3-183\,a\,b^2\,e+258\,a^2\,c\,e\right)-32\,d^2\big(-8\,b^2\,c^4+32\,a\,c^5-$$
$$15\,b^4\,c\,e+636\,a\,b^2\,c^2\,e+576\,a^2\,c^3\,e+5040\,a^2\,b^2\,e^2-7296\,a^3\,c\,e^2\big)+$$
$$32\,e\big(-32\,b^2\,c^5+128\,a\,c^6+213\,b^4\,c^2\,e-576\,a\,b^2\,c^3\,e-976\,a^2\,c^4\,e+$$
$$3780\,a\,b^4\,e^2+7296\,a^2\,b^2\,c\,e^2+1664\,a^3\,c^2\,e^2+768\,a^4\,e^3\big)\big)\,x_1^8+$$
$$\big(-768\,a\,b\left(b^2-40\,a\,c\right)d^4+22\,272\,a^3\,d^5+3072\,a^2\,d\,e\left(-c^2+4\,a\,e\right)^2-$$
$$512\,a\,b^2\,c\,d\,e\left(-53\,c^2+330\,a\,e\right)+64\,b^4\,d\,e\left(-141\,c^2+1492\,a\,e\right)-$$
$$16\,d^3\left(-173\,b^4\,c+552\,a\,b^2\,c^2+48\,a^2\,c^3+3504\,a^2\,b^2\,e-10\,880\,a^3\,c\,e\right)+$$
$$32\,b\,d^2\left(32\,b^2\,c^3-128\,a\,c^4+143\,b^4\,e-656\,a\,b^2\,c\,e-2736\,a^2\,c^2\,e-1152\,a^3\,e^2\right)+$$
$$256\,b\,e\big(-16\,b^2\,c^4+64\,a\,c^5+35\,b^4\,c\,e+4\,a\,b^2\,c^2\,e-$$
$$512\,a^2\,c^3\,e+1144\,a^2\,b^2\,e^2+1024\,a^3\,c\,e^2\big)\big)\,x_1^9+$$
$$\big(56\,b\left(23\,b^4-64\,a\,b^2\,c+144\,a^2\,c^2\right)d^3+896\,a^2\left(3\,b^2+58\,a\,c\right)d^4-1792\,b^5\,c\,d\,e-$$
$$35\,840\,a^2\,b\,c^3\,d\,e-86\,016\,a^3\,b\,c\,d\,e^2+1792\,a\,b^3\,d\,e\left(c^2+106\,a\,e\right)-$$
$$112\,d^2\left(-15\,b^4\,c^2+56\,a\,b^2\,c^3+16\,a^2\,c^4-196\,a\,b^4\,e+1296\,a^2\,b^2\,c\,e-576\,a^3\,c^2\,e\right)+$$
$$448\,e\big(-15\,b^4\,c^3+56\,a\,b^2\,c^4+16\,a^2\,c^5+6\,b^6\,e+84\,a\,b^4\,c\,e-$$
$$336\,a^2\,b^2\,c^2\,e-128\,a^3\,c^3\,e+960\,a^3\,b^2\,e^2+256\,a^4\,c\,e^2\big)\big)\,x_1^{10}+$$
$$112\,\big(13\,b^5\,c\,d^2-40\,a\,b^3\,c^2\,d^2-48\,a^2\,b\,c^3\,d^2+30\,a\,b^4\,d^3-96\,a^2\,b^2\,c\,d^3+$$
$$416\,a^3\,c^2\,d^3+144\,a^3\,b\,d^4-52\,b^5\,c^2\,e+160\,a\,b^3\,c^3\,e+192\,a^2\,b\,c^4\,e+$$
$$112\,a\,b^4\,c\,d\,e-960\,a^2\,b^2\,c^2\,d\,e+240\,a^2\,b^3\,d^2\,e-768\,a^3\,b\,c\,d^2\,e+144\,a\,b^5\,e^2+$$
$$64\,a^2\,b^3\,c\,e^2-1536\,a^3\,b\,c^2\,e^2+2688\,a^3\,b^2\,d\,e^2+3072\,a^4\,b\,e^3\big)\,x_1^{11}+$$
$$\big(448\,a^2\,b\left(9\,b^2+44\,a\,c\right)d^3+5376\,a^4\,d^4+5824\,a\,b^5\,d\,e-26\,880\,a^2\,b^3\,c\,d\,e-$$
$$129\,024\,a^3\,b\,c^2\,d\,e+286\,720\,a^4\,b\,d\,e^2+$$
$$28\,d^2\left(13\,b^6+104\,a\,b^4\,c-816\,a^2\,b^2\,c^2+768\,a^3\,c^3+1984\,a^3\,b^2\,e-1024\,a^4\,c\,e\right)+$$
$$224\,e\big(-13\,b^6\,c+26\,a\,b^4\,c^2+96\,a^2\,b^2\,c^3+32\,a^3\,c^4+$$
$$128\,a^2\,b^4\,e-128\,a^3\,b^2\,c\,e-256\,a^4\,c^2\,e+512\,a^5\,e^2\big)\big)\,x_1^{12}+$$
$$112\,\big(-b^6\,c\,d+12\,a\,b^4\,c^2\,d-48\,a^2\,b^2\,c^3\,d+64\,a^3\,c^4\,d+16\,a\,b^5\,d^2-64\,a^2\,b^3\,c\,d^2+$$
$$112\,a^3\,b^2\,d^3+64\,a^4\,c\,d^3-6\,b^7\,e-16\,a\,b^5\,c\,e+160\,a^2\,b^3\,c^2\,e+64\,a^2\,b^4\,d\,e-$$
$$640\,a^3\,b^2\,c\,d\,e-512\,a^4\,c^2\,d\,e+640\,a^4\,b\,d^2\,e+256\,a^3\,b^3\,e^2+1024\,a^5\,d\,e^2\big)\,x_1^{13}+$$
$$\big(-72\,b^7\,d+768\,a\,b^5\,c\,d+3072\,a^3\,b\,c^3\,d+13\,824\,a^4\,b\,d^3-45\,056\,a^4\,b\,c\,d\,e-$$
$$384\,a^2\,b^3\,d\left(7\,c^2+8\,a\,e\right)+256\,a^2\,d^2\left(6\,b^4-23\,a\,b^2\,c-4\,a^2\,c^2+144\,a^3\,e\right)+$$
$$16\,\big(-b^6\,c^2+12\,a\,b^4\,c^3-48\,a^2\,b^2\,c^4+64\,a^3\,c^5-108\,a\,b^6\,e+384\,a^2\,b^4\,c\,e+$$
$$320\,a^3\,b^2\,c^2\,e-512\,a^4\,c^3\,e+1536\,a^4\,b^2\,e^2+1024\,a^5\,c\,e^2\big)\big)\,x_1^{14}+$$
$$16\,\big(-b^7\,c+12\,a\,b^5\,c^2-48\,a^2\,b^3\,c^3+64\,a^3\,b\,c^4-6\,a\,b^6\,d+48\,a^2\,b^4\,c\,d-$$
$$96\,a^3\,b^2\,c^2\,d+16\,a^3\,b^3\,d^2-64\,a^4\,b\,c\,d^2+384\,a^5\,d^3-96\,a^2\,b^5\,e+$$
$$512\,a^3\,b^3\,c\,e-512\,a^4\,b\,c^2\,e-512\,a^4\,b^2\,d\,e+1024\,a^5\,b\,e^2\big)\,x_1^{15}-$$
$$\big(3\,b^8-32\,a\,b^6\,c+96\,a^2\,b^4\,c^2-256\,a^4\,c^4+96\,a^2\,b^5\,d-768\,a^3\,b^3\,c\,d+$$
$$1536\,a^4\,b\,c^2\,d+512\,a^4\,b^2\,d^2-2048\,a^5\,c\,d^2+384\,a^3\,b^4\,e-$$

$$2048\,a^4\,b^2\,c\,e + 2048\,a^5\,c^2\,e + 2048\,a^5\,b\,d\,e - 4096\,a^6\,e^2\big)\,x_1^{16}\big)\big)\Big/$$

$$\big(-d^4 + 8\,c\,d^2\,e - 16\,c^2\,e^2 + 64\,a\,e^3 + 24\,e\,(b\,d^2 - 4\,b\,c\,e + 8\,a\,d\,e)\,x_1 +$$
$$12\,(b\,d^3 - 4\,b\,c\,d\,e + 20\,a\,d^2\,e - 12\,b^2\,e^2)\,x_1^2 +$$
$$8\,(b\,c\,d^2 + 12\,a\,d^3 - 4\,b\,c^2\,e - 18\,b^2\,d\,e + 32\,a\,c\,d\,e - 48\,a\,b\,e^2)\,x_1^3 +$$
$$6\,(-5\,b^2\,d^2 + 36\,a\,c\,d^2 - 20\,b^2\,c\,e + 16\,a\,c^2\,e - 48\,a\,b\,d\,e - 64\,a^2\,e^2)\,x_1^4 +$$
$$24\,(-2\,b^2\,c\,d + 8\,a\,c^2\,d + a\,b\,d^2 - 3\,b^3\,e - 8\,a\,b\,c\,e - 16\,a^2\,d\,e)\,x_1^5 +$$
$$4\,(-4\,b^2\,c^2 + 16\,a\,c^3 - 9\,b^3\,d + 28\,a\,b\,c\,d - 12\,a^2\,d^2 - 60\,a\,b^2\,e - 64\,a^2\,c\,e)\,x_1^6 -$$
$$24\,b\,(b^2\,c - 4\,a\,c^2 + 2\,a\,b\,d + 16\,a^2\,e)\,x_1^7 +$$
$$3\,(-3\,b^4 + 8\,a\,b^2\,c + 16\,a^2\,c^2 - 32\,a^2\,b\,d - 64\,a^3\,e)\,x_1^8 -$$
$$8\,a\,(b^3 - 4\,a\,b\,c + 8\,a^2\,d)\,x_1^9\big)^2$$

Related links

General Rational Quadrilateral

Message: #2224
Date: 2024-03-19
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris, dear Bernard,

thanks to Chris for reacting on my #2196,
... but I have a question to his drawing in #2216:
Shall the thin lined conic round
... the Po-anticevian triangle of X1X2X3 be my CO
    described in #2186?

I couldn't open your CABRI file in #2190,
... so I "copied" your drawing with transparent paper,
... to get nearly the same cubic in a CABRI file for me
... and constructed the conic CO, described in #2186
... which is absolute not yours (1st attached).

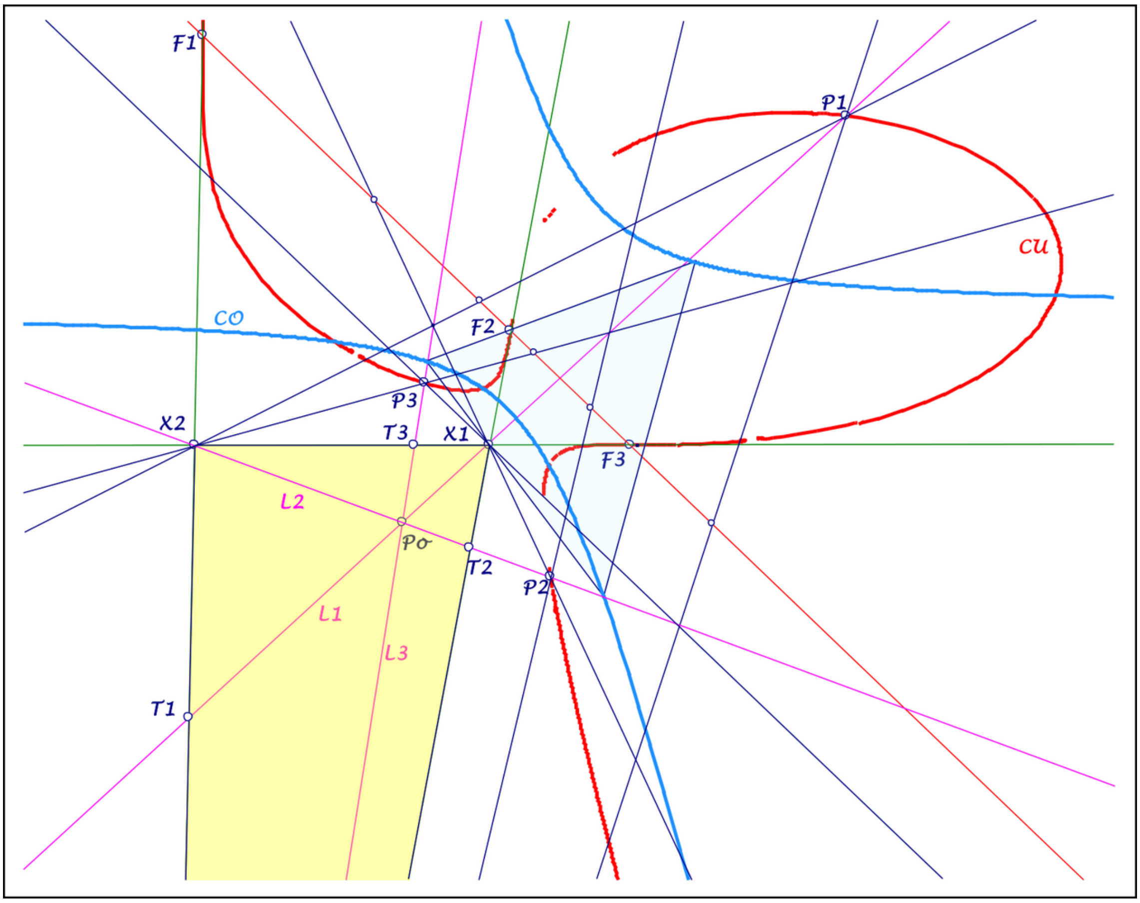
How can it be, that we handle more than ten years
isoconjugations
... and cannot draw the same image?

Finally attached our discussed curves (without the cayleyan)
... and ES my fake hessian and a new cubic XX,
    further the conic CO:

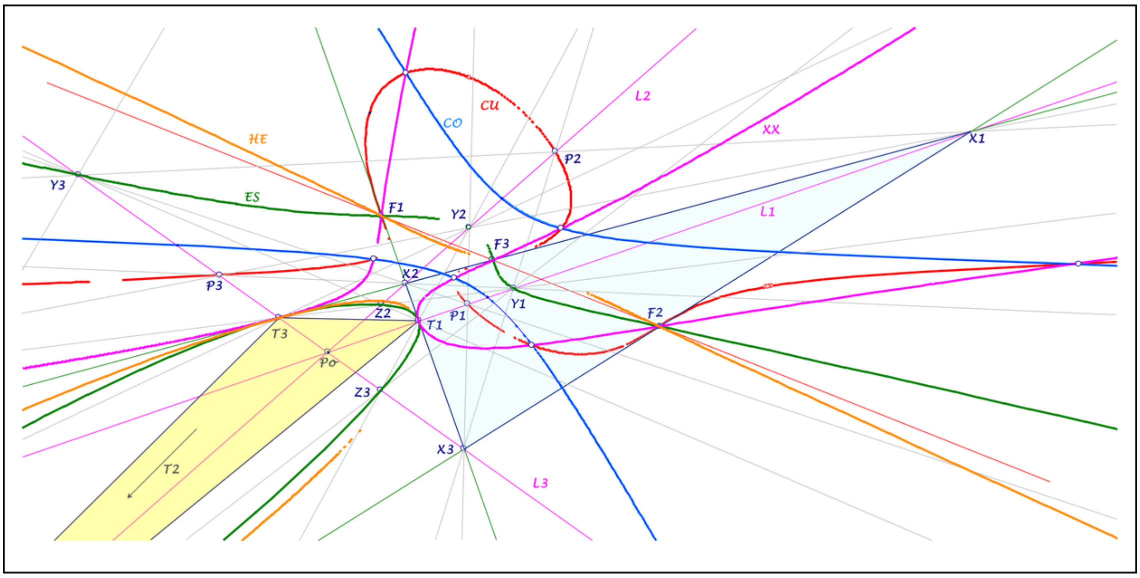
The cubics HE, ES, XX are nonpivotal isocubics,
... wrt the reference triangle T1T2T3,
... an isoconjugation with fixed point Po and root Po,
... constructed for any given point of the cubic,
... which is for the new cubic XX an intersection of CO and CU.

Best regards Eckart

PS: In Chris' drawing in message #2190,  X3 will be X2.
Excuse, I cannot get or send CABRI files, I don't know why.
```



2024-03-18.pdf



2024-03-19.pdf

← Previous Next → ↔ Message Index ↑ Subjects

Message: #2225
Date: 2024-03-20
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
I was waiting impatiently for a new completed version of your paper?
By the way, I found another little point to correct: at the beginning, the description of the P-polar conic is true only if P is on the cubic; otherwise the conic is not through P.
Meanwhile, I read old notes taken years ago by reading plenty of articles of the web.
I think I can give you some useful explanations and definitions:
1) an ordinary cubic has 2 nets of conics, the net of the polar conics and the net of the apolar conics
the conic is a curve of 2nd order (number of intersections with a line) and of the 2nd class (number of tangents from a point)
any polar conic cuts any apolar conic harmonically
For example for the cubic stelloïd associated to a QL (QL-Cu2), the polar conics are rectangular hyperbolas ad the apolar conics are the conics inscribed in the QL.
2) the mixed polar conic Lxy of 2 points is the locus of the points such as their polar conics cut harmonically the conic reduced to the 2 points X and Y
in other words, the line XY intersects the polar conic of the points of Lxy in 2 points harmonic wrt X and Y
3) the mixed polar conic of 2 lines of more generally the poloconic of a conic is the locus of the points such as their polar conics are apolar to the initial conic (id est cut it harmonically)
I suppose you may be able to find a confirmation in Durège and to check the properties on your figures.
Naturally, these properties may be generalised and you will find polocubic of a cubic wrt a quartic, but let's concentrate on cubics ...
Best regards
Bernard
PS I didn't find the poloconic of a line wrt a cubic, which is in Durège, but if you find something, please don't hesitate to tell me

---

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Message: #2226
Date: 2024-03-20
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard, dear Chris,

is this anywhere published?

Any given non degenerated cubic without knot
... is a nonpivotal isocubic
... with reference triangle P1P2P3,
... root Po, isoconjugation with fixed point Po.

By the way: This isoconjugation maps the flexline F1F2F3
... to a P1P2P3-circumscribed conic
    tangent to the cubic CU in P1,P2,P3,
... Po-polar wrt this conic is the flexline F1F2F3.

Best regards Eckart
```

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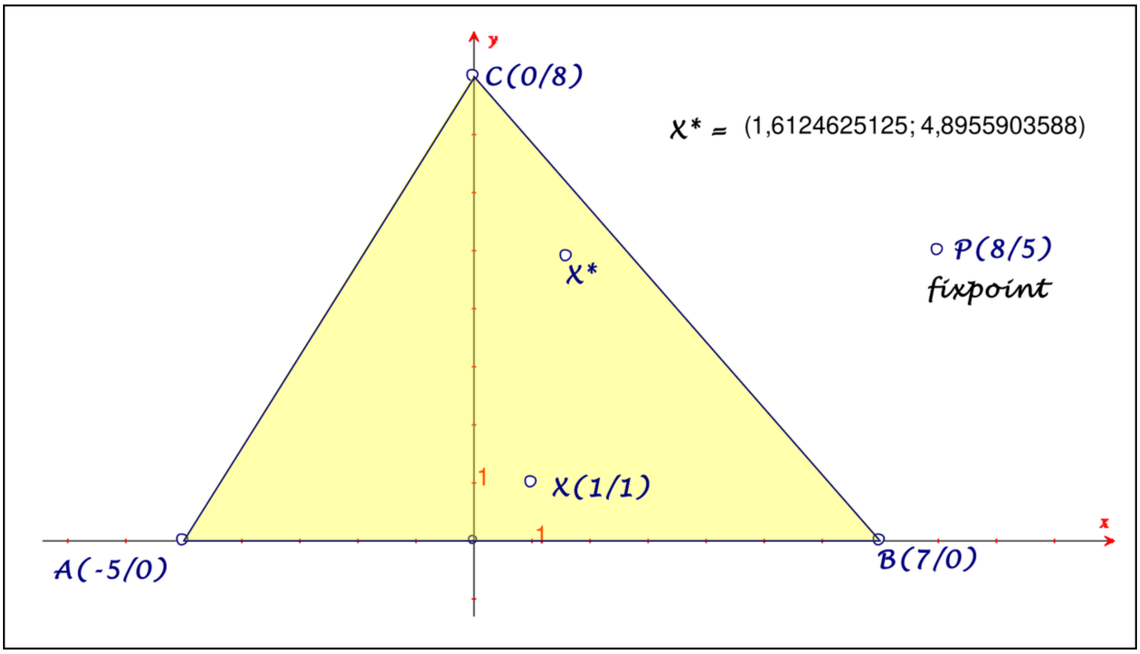
Message: #2227
Date: 2024-03-22
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

do you agree with the attached drawing
... for an isoconjugation with fixed point?

Best regards Eckart
```



2024-03-22.pdf

Message: #2228
Date: 2024-03-22
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard, dear Chris,

perhaps of interest, an application of #2226,
... given QL-Cu1 as nonpivotal isocubic:
... reference triangle P1P2P3 with Pi = CSC(Fi),
... isoconjugation with fixed point Po = L1^L2^L3
... with Li = QL-Tf2(FiPi), root Po.

Background: isoconjugate of cubic points is CSC.
Gibert's circle (see 1.5.3) degenerates to QL-L1.

Best regards Eckart
```

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Message: #2229
Date: 2024-03-22
From: bernard.keizer@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
For a general non degenerated cubic without knot, the QMT
(quasi-Moebius transformation) is an isoconjugation!
All these considerations have a direct link with my messages
2208 and 2211 about cohessians, prehessians and cocayleyans ...
Best regards
Bernard
```

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Message: #2230
Date: 2024-03-22
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard,

I did not react earlier because I was overwhelmed by uncertainties around the poloconica.
When I say Poloconica, then I mean the conic as described by J. de Vries in his paper.
The word Poloconic I reserve for other definitions given by various authors.
I came to the conclusion that there were discrepancies in my drawings. So I had to go back to the “drawing board”.

After looking at everything I came to these conclusions:

1. My construction was wrong. It created an interesting conic but not the poloconica being the locus of points Z for which Lxy, Lxz and Lyz concur.

2. The Poloconica described as the poloconic of the lines XY and Lxy does not describe a 1-1 dependency of Poloconica of these lines, just that the Poloconica is related to these lines.

3. The poloconica is dependent of just two points (X,Y) and of course the reference cubic CU.

4. Basic points X, Y do not have to lie on CU. When they lie on CU, then the intersection of Lxy with XY will be the 3rd intersection point of XY with CU.

I did not find a new proper construction of the Poloconica yet and I withdraw everything I wrote about the Poloconica. I am sorry that I have not been carefully enough.

Then about your message QPG#2217 where you write:
“I found an example on your figure 2160 or on my reproduced figure (see attached file)
F1F2T1T2 form a polar quadrangle and it's DT F3X3T3 form a polar triangle
The poloconic of the conic formed by the flexline L and the line T1T2 (through F3) is certainly through F1, F2, T1 and T2 (perhaps through p1 and P2?)
The polar of F3 wrt this conic is L3”

I find this example amazing. Especially that the polar of F3 wrt this conic is L3.
My compliments.

However I cannot understand that conic (F1,F2,T1,T2,P1,P2) is the poloconic of flexline L and the line T1T2. I checked if points Z on this conic have the property that Lzf1 and Lzf2 and Lf1f2 concur in 1 point and they do not. Neither do Lzt1 and Lzt2 and Lt1t2 concur in one point. So I think it can’t be the poloconica in this configuration. What made you reason that it is the poloconic?

Further I looked to the Polar Quadrangle in general. What is special to it?
Can it be said that lines become points and point become lines in the pole-polar relation?
So a quadrangle becomes an quadrilateral and a quadrilateral becomes a quadrangle wrt some conic? Or are there more special properties (evt. meant by J. de Vries)?

Bernard, you wrote much more, but now I am re-evaluating everything, so I can't comment on it right now. Next time more on the subject.

Best regards,
Chris

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Message: #2231
Date: 2024-03-23
From: hoingason@gmail.com
Subject: Rational Trapezoid (TXM)

In this short note, the author gives a method for generating all
rational
trapezoids. Hope it's useful to someone.
Best regards
--TXM--

Rational Trapezoid

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices. If at least 1 vertex of a rational polygon is removed, the resulting polygon is called a sub-polygon of the original polygon. The area of any rational polygon is of the form $a\sqrt{b}$; call $\sqrt{b}$ the congruent factor of that polygon. In this short note, the author gives a method for generating all rational trapezoid.

Theorem 1.

If (x_1, y_1) is a rational point of curve $(E): y^2 = a x^4 + b x^3 + c x^2 + d x + e$,
then we have a rational point (x_2, y_2) on (E) as follows :

$$\begin{aligned}
x_2 = \big(&(b^4 - 8ab^2c + 16a^2c^2 - 64a^3e)x_1^9 - \\
&24a(b^2d - 4acd + 8abe)x_1^8 - 12(b^3d - 4abcd - 12a^2d^2 + 20ab^2e)x_1^7 - \\
&8(b^2cd - 4ac^2d - 18abd^2 + 12b^3e + 32abce - 48a^2de)x_1^6 + \\
&6(5b^2d^2 + 20acd^2 - 36b^2ce - 16ac^2e + 48abde + 64a^2e^2)x_1^5 + \\
&24(2bcd^2 + 3ad^3 - 8bc^2e - b^2de + 8acde + 16abe^2)x_1^4 + \\
&4(4c^2d^2 + 9bd^3 - 16c^3e - 28bcde + 60ad^2e + 12b^2e^2 + 64ace^2)x_1^3 + \\
&24d(cd^2 - 4c^2e + 2bde + 16ae^2)x_1^2 + \\
&3(3d^4 - 8cd^2e - 16c^2e^2 + 32bde^2 + 64ae^3)x_1 + 8e(d^3 - 4cde + 8be^2)\big) / \\
\big(&8a(b^3 - 4abc + 8a^2d)x_1^9 - 3(-3b^4 + 8ab^2c + 16a^2c^2 - 32a^2bd - 64a^3e)x_1^8 - \\
&24b(-b^2c + 4ac^2 - 2abd - 16a^2e)x_1^7 - \\
&4(-4b^2c^2 + 16ac^3 - 9b^3d + 28abcd - 12a^2d^2 - 60ab^2e - 64a^2ce)x_1^6 - \\
&24(-2b^2cd + 8ac^2d + abd^2 - 3b^3e - 8abce - 16a^2de)x_1^5 - \\
&6(-5b^2d^2 + 36acd^2 - 20b^2ce + 16ac^2e - 48abde - 64a^2e^2)x_1^4 - \\
&8(bcd^2 + 12ad^3 - 4bc^2e - 18b^2de + 32acde - 48abe^2)x_1^3 - \\
&12(bd^3 - 4bcde + 20ad^2e - 12b^2e^2)x_1^2 - \\
&24e(bd^2 - 4bce + 8ade)x_1 + d^4 - 8cd^2e + 16c^2e^2 - 64ae^3\big)
\end{aligned}$$

$$y_2 = \sqrt{a x_2^4 + b x_2^3 + c x_2^2 + d x_2 + e}$$

Theorem 2.

Suppose x, y, z, t, k are rational numbers such that $t\,(x+z)\,(y+z) \neq 0$ and $x\,y\,z\,(x+y+z) \geq 0$ and

$$\begin{aligned}
&256t^4x^2y^2z^2(x+y+z)^2 + 128t^3xyz(y+z)^2(x+y+z) + \\
&\quad 16t^2(y^4 + 2x^2yz + 2xy^2z + 4y^3z + 2xyz^2 + 6y^2z^2 + 4yz^3 + z^4) - 8t(y+z)^2 + 1 = k^2
\end{aligned}$$

Then we can set the coordinates of four vertices of all rational trapezoid ABCD (AB//CD) as follows:

$A = (0, 0)$

Rational Trapezoid (TXM).pdf

$$B = \left(-\frac{(1+4\,t\,x\,y)\,(x\,(y-z)-z\,(y+z))\,(-1+4\,t\,z\,(x+y+z))}{4\,t\,(x+z)\,(y+z)^2},\right.$$
$$\left.\frac{(1+4\,t\,x\,y)\,\sqrt{x\,y\,z\,(x+y+z)}\,(-1+4\,t\,z\,(x+y+z))}{2\,t\,(x+z)\,(y+z)^2}\right)$$

$$C = (z+x,\ 0)$$

$$D = \left(x-y+\frac{2\,x\,y}{x+z},\ -\frac{2\,\sqrt{x\,y\,z\,(x+y+z)}}{x+z}\right)$$

Example:

For $t = -\frac{(x+y)\,(y-z)}{4\,y\,z\,(x+z)\,(2\,x+y+z)}$, after reduction we get the following solution :

$$A = (0,\ 0)$$

$$B = \left(\left(x\,y-x\,z-y\,z-z^2\right)\left(3\,x^2\,y+3\,x\,y^2+y^3-x^2\,z+2\,x\,y\,z+y^2\,z-x\,z^2\right)\right.$$
$$\left(x^2\,y+x\,y^2-3\,x^2\,z-2\,x\,y\,z-3\,x\,z^2-y\,z^2-z^3\right),\ -2\,\sqrt{x\,y\,z\,(x+y+z)}$$
$$\left.\left(3\,x^2\,y+3\,x\,y^2+y^3-x^2\,z+2\,x\,y\,z+y^2\,z-x\,z^2\right)\left(x^2\,y+x\,y^2-3\,x^2\,z-2\,x\,y\,z-3\,x\,z^2-y\,z^2-z^3\right)\right)$$

$$C = \left((x+y)\,(y-z)\,(x+z)^3\,(y+z)^2\,(2\,x+y+z),\ 0\right)$$

$$D = \left((x+y)\,(y-z)\,(x+z)\,(y+z)^2\,(2\,x+y+z)\left(x^2+x\,y+x\,z-y\,z\right),\right.$$
$$\left.-2\,(x+y)\,(y-z)\,(x+z)\,(y+z)^2\,\sqrt{x\,y\,z\,(x+y+z)}\,(2\,x+y+z)\right)$$

Related links

General Rational Quadrilateral

Message: #2232
Date: 2024-03-23
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart ,

First regarding your message #2224.
Yes, the thin-lined conic round the Po-anticevian triangle X1X2X3 was the conic I thought you asked for using the Pickert-construction.
I thought you specifically asked for this conic, because the conic I produced in #2190 passed through vertices X1,X2,X3. You said they had to pass though the vertices of the Po-Anticevian Triangle.
Actually I was surprised like you, but ok I can produce such a conic.

Now back to the drawing-board.
According to me the Pickert-construction is actually another version of QA-Tf2. In our case QA=(Po and the vertices of the Po-Anticevian triangle wrt X1X2X3). When it is QA-Tf2 as mentioned, then the mapping of a line is a conic, not specifically passing through any earlier mentioned vertices.
I checked my earlier drawing from #2190 again, this time using the QA-Tf2-construction instead of the Pickert-construction and the same conic came out. So I still think my construction in #2190 was plausible. Moreover the Polars of F1,F2,F3 wrt this conic are the Fi-Harmonic Polars L1,L2,L3.
Then I checked your picture in #2227. My construction using Pickert's method gave exactly the same point (1,61/4,90) like your point. Special isn't it?

About Cabri. I use two ways to import Cabri-macro's:
1. Double-click on a macro-name in a Windows-map.
2. In Cabri open > file > type of file=*.mac

Maybe it gives you an idea. Or you can update the Cabri-app.

Best regards,
Chris

Message: #2233
Date: 2024-03-23
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Chris,
1) If I understand correctly, the only sure construction of your paper is Lxy for 2 points X and Y!
(Did you check the property that the polar conic of any point on Lxy was cut by the line XY in 2 points harmonic wrt X and Y?)
2) You don't know how to construct the poloconic of 2 lines
3) Therefore, you don't how to construct the poloconic of XY and Lxy ...
I'll wait until next time
Last, the poloconic of the 2 lines L and T1T2 is not a conic through P1 and P2, as I conjectured wrongly in 2217 but the degenerated conic through T3, as corrected in 2218.
Best regards
Bernard

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Message: #2234
Date: 2024-03-23
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

can it be, that you use my CO construction in a false manner?

My description:
CO conic, image of the flexline F1F2F3 wrt an isoconjugation
... with fixpoint Po and reference triangle
... with vertices in the anticevians of Po wrt X1X2X3.

In your QA-Tf2 interpretation:
The reference triangle for the isoconjugation
... is the Po-anticevian triangle wrt X1X2X3,
... for your QA-Tf2-application as isoconjugation
    you have to take
... QA = Po plus the Po-anticevian triangle
    wrt the Po-anticevian triangle wrt X1X2X3.

I hope, this will end our misunderstanding.
Thanks for CABRI-advices, I shall try it.

Best regards Eckart
```

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Message: #2235
Date: 2024-03-23
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard,

can you refer a message with an example
... of a QMT, which is an isoconjugation,
... it seems I lost a clear view!
I think this property is independent
... of my message 2226.
Excuse that I don't follow your discussion
... for cohessians, prehessians and cocayleyans,
... I have to respect my limits.

Best regards Eckart
```

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Message: #2236
Date: 2024-03-23
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,

I suppose that:
QA = Po plus the Po-anticevian triangle wrt the Po-anticevian
triangle wrt X1X2X3.
means
QA = Po plus the Po-anticevian triangle wrt X1X2X3.
If so, yes that’s what I did.
About the macro’s of Cabri it is also important that a
macro-file has the extension .mac.
Sometimes the extension gets lost and then the file can’t be
interpreted by Cabri.

Best regards,
Chris
```

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Message: #2237
Date: 2024-03-24
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

QA = Po plus the Po-anticevian triangle wrt the Po-anticevian
triangle wrt X1X2X3.
   means NOT
QA = Po plus the Po-anticevian triangle wrt X1X2X3.

I think too, you did the last, but the first means:
Name the Po-anticevian triangle wrt X1X2X3 than Y1Y2Y3,
... name the Po-anticevian triangle wrt Y1Y2Y3 than Z1Z2Z3
... and take QA = Po plus Z1Z2Z3
    for your QA-Tf2-version of the isoconjugation.

Excuse, that I could not accept the incorrect interpretation.

Best regards Eckart
```

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Message: #2238
Date: 2024-03-24
From: van10hoven@gmail.com
Subject: Re: XY-Polar and PoloConica

Dear Bernard,

Regarding your messages #2218 and #2233:

QPG#2218 BK "The polar conic of the conic formed by the flexline L and the line T1T2 (through F3) is the degenerated conic formed by the 2 lines T3F1 (through T2) and T3F2 (through T1).
I hope you will agree with that."

QPG#2233 BK "Last, the poloconic of the 2 lines L and T1T2 is not a conic through P1 and P2, as I conjectured wrongly in 2217 but the degenerated conic through T3, as corrected in 2218."

I summarize what I read in own words (correct me if I am wrong):
The poloconica of points (T1,T2) with mixed T1T2-Polar the line F1F2F3 is the degenerate conic formed by the 2 lines T3F1 (through T2) and T3F2 (through T1).
First of all I checked if F1F2F3 was the mixed T1T2-Polar, which was confirmed in my drawing.
Then I checked if the poloconica of T1T2 (which is the locus of points Z with the property that the mixed polars Lt1t1, Lzt1, Lzt2 concur in one point) is indeed the locus of . . .
I placed a point Z on degenerate conic (T3,T1,T2,F1,F2). Then the ZT1-Polar and ZT2-Polar did not concur on the T1T2-Polar, which is F1F2F3.

I don't know if I checked things in your meaning, but this is what I found.

Best regards,

Chris

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Message: #2239
Date: 2024-03-24
From: bernard.keizer@gmail.com
Subject: Re: XY-Polar and PoloConica

```
Dear Chris,
For me, the case is over now!
For 2 points X and Y, we deal with the poloconic (without a) of
the 2 lines XY and Lxy.
In fact, the flexline is Lt1t2 as well as Lt1t3 and Lt2t3.
But I can't understand why you couldn't check the property that
for Z on the conic fomed by the 2 ines T3F1 and T3F2 the lines
Lt1z and Lt2z intersect on Lt1t2 ...
It works for Z in T3 (the 3 lines are the same), for F1
(intersection in F2) and F2 (intersection in F1)
Best regards
Bernard
```

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Message: #2240
Date: 2024-03-24
From: bernard.keizer@gmail.com
Subject: Transformations on a cubic

```
Dear Chris, dear Eckart
Any non degenerated cubic without node has 3 transformations
(one if the cubic is monopartite, 3 if it is bipartite).
1) The cubic may be drawn with one of the transformations, given
3 copples of conjugated points (see my message 2086)
2) The transformation swaps 2 points having the same tangential
(the 3rd intersection is the conjugate of this tangential)
3) This transformation was studied as QMT with 2 conjugate
points in which the tangent is parallel to the asymptote
4) The transformation is also an isoconjugation wrt a certain
triangle with one fixed point in P0 (configuration studied by
Eckart)
For example, if the triangle is T1T2T3, the 3 other fixed points
are the vertices X1, X2 and X3 of the anticevian triangle of P0
xrt T1T2T3.
It is remarkable that the triangle X1X2X3 is the triangle formed
by the flextangents of a reference cubic, whereas the
transformation belongs to the hessian ...
Hence the considerations about the 2 cohessians and the 3
prehessians
Best regards
Bernard
```

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Message: #2241
Date: 2024-03-24
From: bernard.keizer@gmail.com
Subject: Fred Lang's dual calculation

```
Dear Chris, dear Eckart
The cayleyan being a curve of class 3, it is possible to make
with it's tangents the same kind of calculations as with the
points of a cubic.
L1 + L2 + L3 = N means that the 3 lines concur in a point as F1
+ F2 + F3 = N means that the 3 points are on a line.
The same way for 6 tangents tangent to a conic or 9 tangents
tangent to a cubic ...
By the way, the cayleyan has 9 turning points, 3 real and 6
imaginary, they are one the 9 harmonic lines of the 9 flexes.
These 9 harmonic lines (3real and 6 imaginary) form a CB system.
I hope this will give you plenty of new ideas ...
Best regards
Bernard
PS I'm now leaving home for a month
```

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Message: #2242
Date: 2024-03-24
From: van10hoven@gmail.com
Subject: Re: Fred Lang's dual calculation

```
Dear Bernard,

Beautiful!
Have a nice time.
Best regards,

Chris
```

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Message: #2243
Date: 2024-03-25
From: bernard.keizer@gmail.com
Subject: Re: Cocayleyan cubics

Dear Chris, dear Eckart
It is easy to complete this drawing with the drawing of the 2 hessians of the 2 cocayleyans, as the fixed points for the isoconjugations of these 2 hessians are precisely again P0 and the vertices of the anticevian triangles of P0 wrt the Wi and Zi triangles (exactly like the Xi are the vertices of the anticevian triangle of the Ti triangle).
Best regards
Bernard

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Message: #2244
Date: 2024-03-28
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris, dear Bernard,

the hessian HE and the reference cubic CU
... have the same flexpoints F1,F2,F3
... the same harmonic polars L1,L2,L3
... and the same common point Po,
... but different intersections of L1,L2,L3
... and the unclosed part of the hessian HE,
... what about these points P1',P2',P3'?

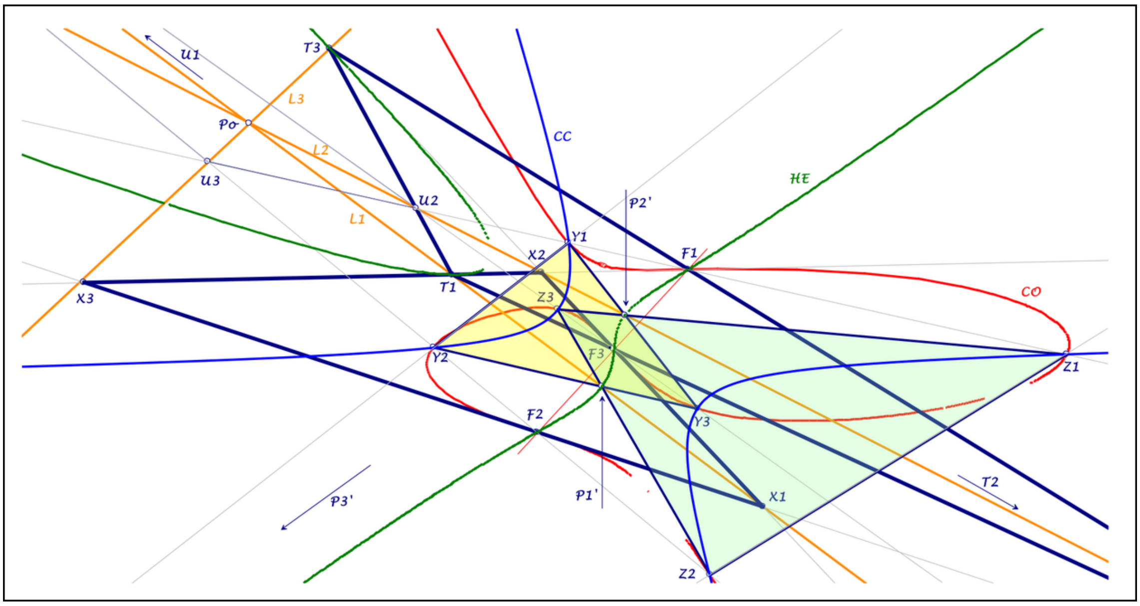
Here a construction, not always real, but without drawing the
hessian,
... I hope you can confirm my observations:

Let us start with the triangle T1T2T3 with vertices
... in the intersections of L1,L2,L3
    and corresponding flextangents,
... and consider the cevian triangle U1U2U3 of Po.
The lines UiUjFk intersect the cubic CU
... in two further points Yk and Zk,
... so we get six points on the cubic CU,
... which are coconic on a conic CC.
These six intersections are two 3-times perspective triangles
... wrt the flexpoints F1,F2,F3,
... please use the notation so that
... the triangles are Y1Y2Y3 and Z1Z2Z3,
    with Yi, Zi Fi collinear,
... then YiYj ^ ZiZj on Lk are Pk'.

The conic CC is the isoconjugate of the flexline F1F2F3
... wrt an isoconjugation with fixpoint Po
... and reference triangle Y1Y2Y3 or Z1Z2Z3.
Perhaps the intersections of CC  and CU
... are also the intersections of CU and the cayleyan?

Now the hessian can be described and drawn
... by 9 points F1,F2,F3,T1,T2,T3,P1',P2',P3',
... or as nonpivotal isocubic with reference triangle T1T2T3
... ... and isoconjugation with fixed point Po and root Po
... ... and one of the points P1',P2',P3' or another HE-point,
... or by Chris' first construction with three polar conics,
... ... structural better, but also not always real.

Best regards Eckart
```



2024-03-28.pdf

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Message: #2245
Date: 2024-03-29
From: hoingason@gmail.com
Subject: Congruent factor and rational polygons

```
After 15 centuries (since the time of Brahmagupta), I have
discovered the
symmetric solution formula for all Heron triangles. Hope you
will like it.
Best regards
--TXM--
```

Congruent factor and rational polygons

Trinh Xuan Minh
Email: hoingason@gmail.com

Abstract
As the title states, in this short note I give some theorems related to the concept of *congruent factor*. This leads to a completely symmetric parametric solution for all Heron triangles.

In a rational polygon, the distance between any two vertices is a rational number and each vertex is called the rational point of the polygon formed by the other vertices . If at least 1 vertex of a rational polygon is removed, the resulting polygon is called a sub-polygon of the original polygon . The area of any rational polygon is of the form $a\sqrt{b}$; call $\sqrt{b}$ the congruent factor of that polygon.

Theorem 1.
A polygon is rational if and only if it shares the same congruent factor with all its sub-polygons. In other words, every rational polygon is packed by rational triangles with the same congruent factor.

Theorem 2.
Suppose m, n, p, t are rational numbers and $t > 0$. Then the sides a, b, c of all rational triangles with *congruent factor* $\sqrt{t}$ can be expressed as follows:

$a = |(n-p)(n\,p\,t+1)(m^2 t+1)|$

$b = |(p-m)(p\,m\,t+1)(n^2 t+1)|$

$c = |(m-n)(m\,n\,t+1)(p^2 t+1)|$

The area of the above triangle is

$|(m-n)(n-p)(m-p)(m\,n\,t+1)(n\,p\,t+1)(m\,p\,t+1)|\sqrt{t}$

Corollary 1:

For $t = q^2$, we get parametric solution of symmetrical form for all Heron triangles

$a = |(n-p)(n\,p\,q^2+1)(m^2 q^2+1)|$

$b = |(p-m)(p\,m\,q^2+1)(n^2 q^2+1)|$

$c = |(m-n)(m\,n\,q^2+1)(p^2 q^2+1)|$

The area of the above triangle is

$|q(m-n)(n-p)(m-p)(m\,n\,q^2+1)(n\,p\,q^2+1)(m\,p\,q^2+1)|$

Corollary 2 :

For $\{m, n, p, t\} = \left\{\frac{1}{u\,w^2(u+v)(u\,v-w^2)}, 0, -\frac{1}{u\,v(u\,v-w^2)^2}, u^2 v^2 w^2 (u+v)^2 (u\,v-w^2)^2\right\}$,

after reduction we get Brahmagupta' s solution for the Heron triangles

Congruent factor and rational polygons (TXM).pdf

$a = (u + v)(u v - w^2)$

$b = u(v^2 + w^2)$

$c = v(u^2 + w^2)$

Related links

Heronian triangle

Message: #2246
Date: 2024-04-04
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Bernard, dear Chris,

in #2244 I described two triangles Y1Y2Y3 and Z1Z2Z3 for a
cubic,
... these two triangles lead to an interesting transformation
... CB = 7P-s-Tf1 wrt Y1,Y2,Y3,Z1,Z2,Z3 and one flexpoint Fi.
(a) X and CB(X) are collinear with Fi,
(b) CB swaps the other two flexpoints Fj, Fk,
(c) the cubic is invariant wrt CB,
(d) the hessian is also invariant wrt CB,
(e) CB swaps Lj and Lk, Li bears fixpoints,
(f) CB swaps Tj and Tk as well as Xj and Xk.,
(g) CB swaps flextangents of Fj and Fk,
             flextangent  of F1 invariant.

I just found a general version:
Take on a cubic CU two points U and V
... and consider the triangles U1U2U3 and V1V2V3
... with Ui = 3rd intersection of CU and UFi,
         Vi = 3rd intersection of CU and VFi,
... further W = the 3rd intersection of CU and UV,
... then 7P-s-Tf1 wrt U1U2U3V1V2V3W lets the cubic invariant
... with pivot in the tangential of W.

Or more general:
Take three lines through a cubic point U,
... which give six further CU- intersections,
... the CB-transformation 7P-s-Tf1 wrt these seven points
... maps the cubic to itself with pivot in the tangential of U.

I hope you can confirm these observations.

Best regards Eckart

PS: The properties are only for the first version.
```

Message: #2247
Date: 2024-04-04
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,

Regarding your message #2237:
I was surprised about your statement:
QA = Po plus the Po-anticevian triangle wrt the Po-anticevian
triangle wrt X1X2X3,
meaning
Name the Po-anticevian triangle wrt X1X2X3 than Y1Y2Y3,
... name the Po-anticevian triangle wrt Y1Y2Y3 than Z1Z2Z3
... and take QA = Po plus Z1Z2Z3
    for your QA-Tf2-version of the isoconjugation.

I don't understand the background for the double AntiCevian
Triangle in this case.
Anyhow I used this double-AntiCevian-construction, but again I
got the same conic that I drew in #2216.

My surprise was fed by the fact that:
The isoconjugate of X wrt triangle Tr0 and fixed point K
= the "Pickert-conjugate" of X wrt Tr0 with fixed point K
  (according to you)
= QA-Tf2(X) wrt QA(K, vertices K-AnticevianTriangle)
  (checked by me)
So I don't understand the use of the double application of the
Po-anticevian triangle.
Could you please explain.

Thank you for your persistence in clearing up ambiguities.
I do appreciate that.

Best regards,
Chris
```

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Message: #2248
Date: 2024-04-05
From: eckart_schmidt@t-online.de
Subject: Re: Transformations on a cubic

```
Dear Bernard,

in #2240 you cited a cubic-isoconjugation
... with ref-triangle T1T2T3 and fixpoint Po,
... but not the cubic, but its hessian remains invariant.
The cubic remains invariant for an isoconjugation
... with ref-triangle P1P2P3 and fixpoint Po.
To draw the cubic in this way you need the root Po
... and a further cubic point Q.

Best regards Eckart
```

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Message: #2249
Date: 2024-04-05
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

I regret our isoconjugation-discussion.
In #2234 I described a conic:
CO conic, image of the flexline F1F2F3 wrt an isoconjugation
... with fixpoint Po and reference triangle
... with vertices in the anticevians of Po wrt X1X2X3.
Your interpretation of "Pickert-conjugate" is correct,
... TrO is the Po-anticevian triangle wrt X1X2X3 and K = Po,
... for QA-Tf2(X)
    wrt QA(K, vertices of K-anticevian triangle wrt TrO)
... you have to use the double application
    of the Po-anticevian triangle.

Best regards Eckart
```

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Message: #2250
Date: 2024-04-05
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

```
Dear Eckart,
Thanks for explaining.
Thanks also for your patience!
Best regards,
Chris
```

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Message: #2251
Date: 2024-04-07
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

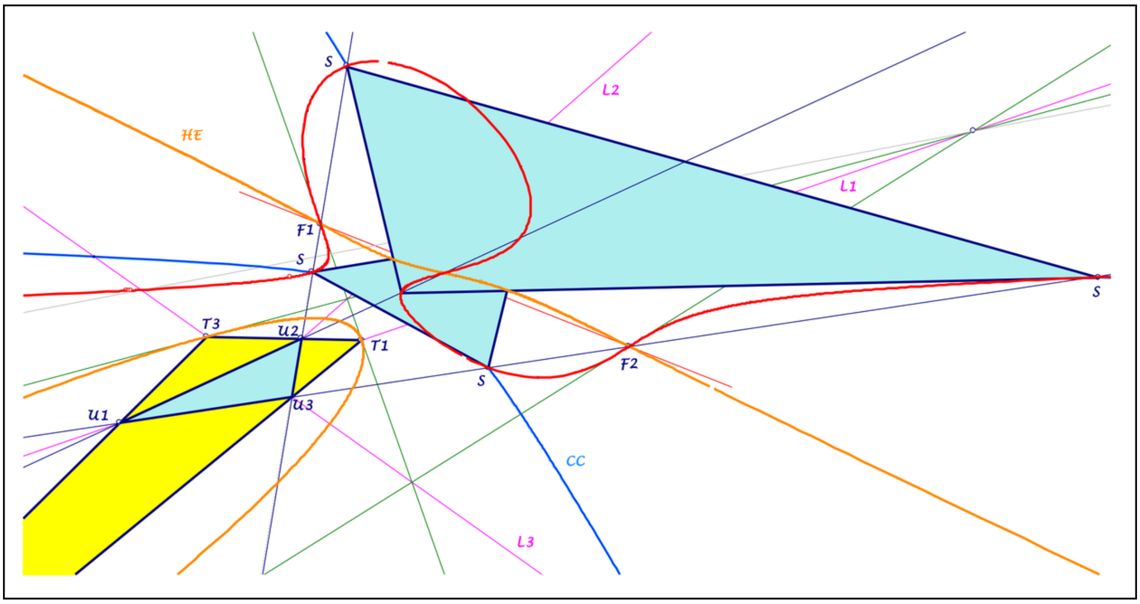
```
Dear Chris,

can you please control the following "construction"
... of the six intersections of a cubic and its cayleyan:
Let us start with a cubic and its flexpoints F1,F2,F3,
... the common point Po of their harmonic polars L1,L2,L3,
... further the triangle T1T2T3
    with vertices in the intersections
... of flextangents and harmonic polars L1,L2,L3,
... finally the cevian triangle U1U2U3 of Po wrt T1T2T3.
The sidelines UiUjFk intersect the cubic further in two points S
...  and the resulting 6 coconic intersections S will be
... the intersections of the cubic and its cayleyan.
I tested this observation with the cubic contacts
... of common tangents of cubic and hessian,
... which are points of the cayleyan.
Finally perhaps please have a look in #2244 ...

Best regards Eckart

PS: The conic for the 6 intersections
... is not the earlier discussed conic C0.

Excuse the docx-format for the drawing,
... pdf showed breaks.
```



2024-04-07.docx

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Message: #2252
Date: 2024-04-07
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Bernard, dear Chris,

in # 2244 I described the points P1',P2',P3' for the hessian,
... so we get an isoconjugation wrt P1'P2'P3'
    with fixed point Po,
... which maps the hessian to itself.
The hessian bears the points T1,T2,T3,
... intersections of flextangents and harmonic polars L1,L2,L3,
... the isoconjugates of Ti will be new points Ti*
    of the hessian.
The Cayley-Bacharach transformation CB in #2246 wrt Fi
... swaps Tj* and Tk* an lets Ti* fixed.

Best regards Eckart

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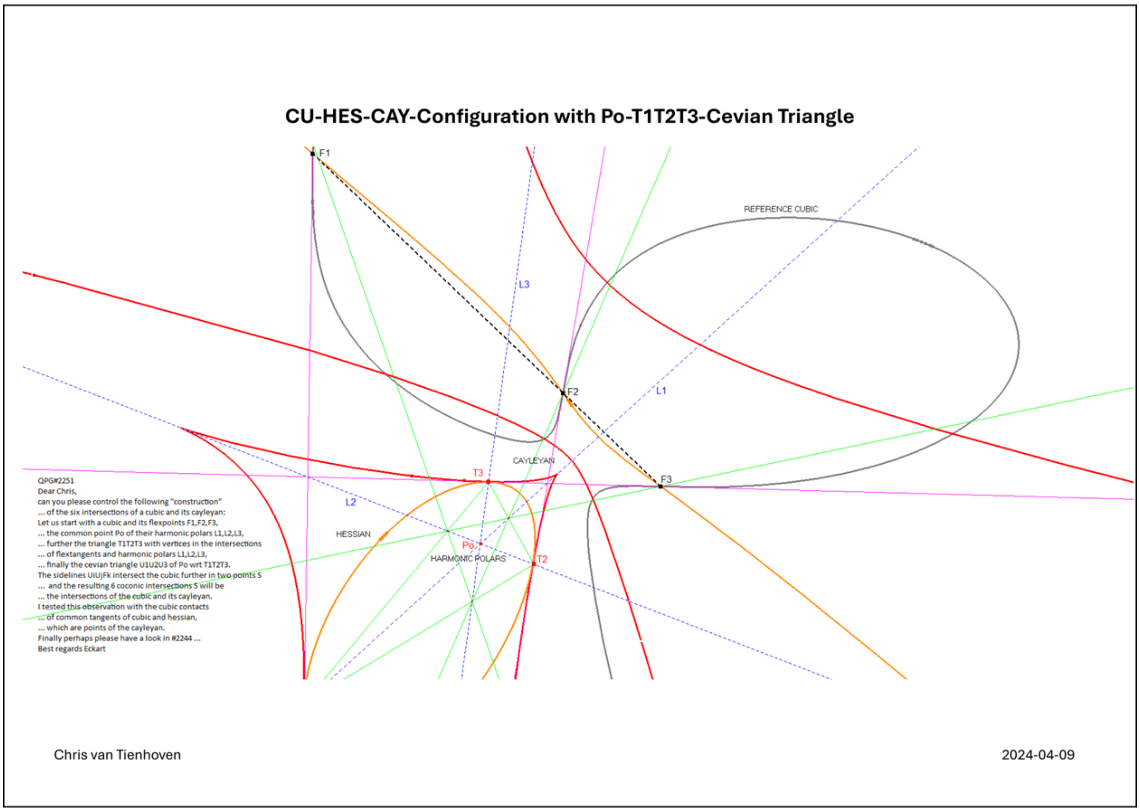
Message: #2253
Date: 2024-04-09
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

Regarding your message QPG#2251.
It's remarkable how many special features you've found in this configuration so far.
I attach a picture with the Cayleyan and the cevian triangle U1U2U3. As you can see, in my picture the sidelines of U1U2U3 don't exactly pass the intersections of CU and its Cayleyan.
But that does not mean they don’t, because due to the many complicated constructions, the picture is at the end of its accuracy. I don't rule out that your conjecture is correct.
By the way, can you make a synopsis of all properties you found thus far in this CU-HES-CAY-configuration?

Best regards,
Chris



CU-Cayleyan-45-TryoutES.pdf

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Message: #2254
Date: 2024-04-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Bernard, dear Chris,

in "Encyklopädie der mathematischen Wissenschaften"
... I found a reference for constructing cubics
... in a very elementary way:
    G. Salmon, Lond. Trans.148 (1858), p.535:

"Die Berührungspunkte der Tangenten,
... welche sich von einem Punkte O
... an die Kegelschnitte eines Büschels legen lassen,
... erfüllen eine Kurve dritter Ordnung.
... Der Punkt O ist auf der zu erzeugenden Kurve
    beliebig wählbar,
... als Basispunkt des Kegelschnittbüschels
    sind die Berührungspunkte
... der vier von ihm ausgehenden Kurventangenten zu wählen."

These cubics will be QA- and QA-Tr1-circumscribed
... QA-Tf2-invariant with pivot O
... and therefore pivotal and bipartite.,
... but not necessary 7P-cubics.
For O = QA-P4 we get QA-Cu1.

Best regards Eckart
```

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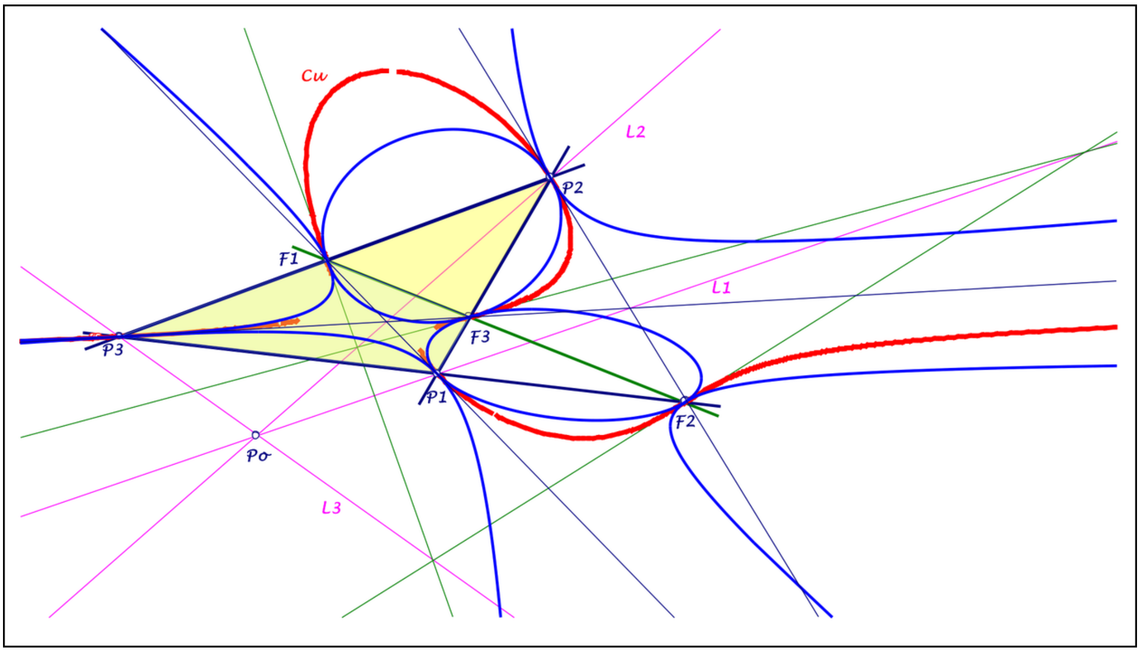
Message: #2255
Date: 2024-04-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

Dear Bernard, dear Chris,

the triangle P1P2P3 with vertices in the intersections
... of the harmonic polars L1,L2,L3 of the flexpoints
... and the nonclosed part of the cubic is very special,
... not only for its isoconjugation with fixpoint Po,
... which lets the cubic invariant,
... but the following properties:
... sides PiPj bear the flexpoint Fk, which is tangential of Pk,
... therefore the trilateral P1P2P3 and the flexline F1F2F3
... give a QL on the cubic:
If we consider lines through Po,
... their QL-Tf2 image lines envelope a conic,
... tangent to the cubic in P1,P2,P3,
... further: every trilateral of the QL has a circumconic
... tangent to the cubic in its vertices
... (see also eckartschmidt.de/Zirkul..pdf, 9f).

Best regards Eckart

PS: Excuse the bad drawing in pdf, I don't know the reason.



2024-04-13.pdf

Message: #2256
Date: 2024-04-14
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

thanks for drawing conic and cayleyan in #2253,
... accuracy is also my problem,
... but I think the discussed property holds.
Now I look for the three real flexlines beside F1F2F3,
... but I found no reference for construction,
... vertices must be real but no flexpoints.
There is further a real unknown point,
... vertex of a flextriangle with opposite sideline F1F2F3.

Can you give any tip?

Best regards Eckart
```

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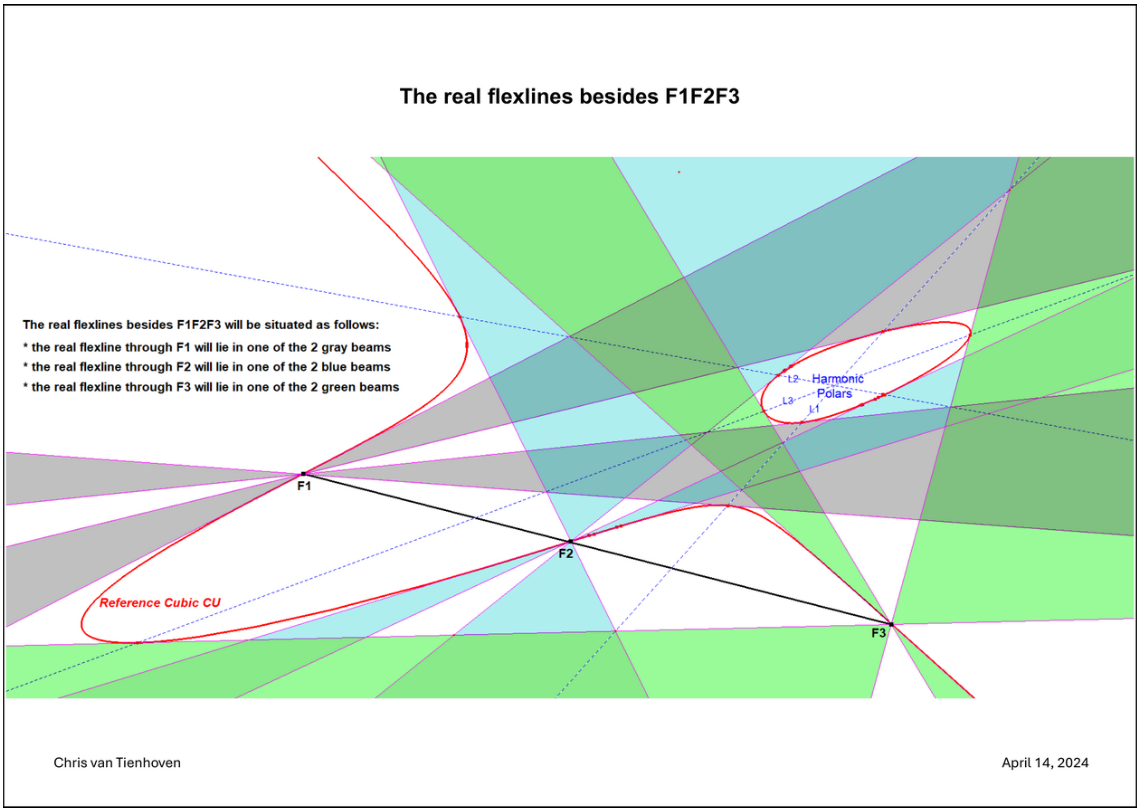
Message: #2257
Date: 2024-04-14
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

Because the real flexlines through F1, F2, F3 already pass through other imaginary flexpoints, they can't intersect the reference cubic in any other points, therefore they lie between the tangents from F1, F2, F3 to the cubic.
I made a picture of it. See attachment.
That's all I know so far.
What do you mean with:
"There is further a real unknown point,
 ... vertex of a flextriangle with opposite sideline F1F2F3."?

Best regards,
Chris



CU-12L1 Real Flexlines-10.pdf

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Message: #2258
Date: 2024-04-14
From: van10hoven@gmail.com
Subject: Re: Hessian and prehessians

Dear Bernard,

Thanks for your excellent analyses of prehessians, cohessians and cocayleyans in #2208 and #2211.

The logic behind Prehessians and the Cohessians
Permit me to rephrase it partly in my own words and add some new thoughts, as yet only for the Prehessians and the Cohessians.
1. The CU-tangent at Fi is also tangent to HE at Ti.
2. The CU-Fi-tangent also touches HE. HE is also a cubic.
3. CU is the Hessian of its 3 Prehessians pHE1, pHE2, pHE3
   and they all share the same Flexpoints Fi (i=1-9).
4. Therefore the pHE1-Fi-tangent must touch CU.
5. The question is at which point.
6. The Fi-points of tangency are determined by the Fi-Polar
   conics, which are degenerated in Li and the CU-Fi-tangent.
7. Therefore Pij, Qij, Rij being the intersection points of Li
   with CU are these points where FiPij are pHEj-Pij-tangents
   (i=1-9, j=1-3).
8. So Prehessian pHE1 is defined by Fi (F1, ...,F9),
   being 9 points forming a CB-system and tangents
   F1P11, F2P21, F3P31, F4P41, ..., F9P91
   (last 6 tangents are imaginary).
9. Analogous properties for prehessians pHE2 and pHE3.

Further remarks and questions.

In #2208 you say:
2) If the Pi, Qi and Ri are real, the lines FiPi, FiQi and Fi Ri are the tangents in Fi to the 3 prehessians of the reference cubic.

Perfect! That sounds reasonable. However, how do you determine cutting the cubic with a harmonic polar, giving 3 intersection points, which point is Pi, which point is Qi and which point is Ri? That is important to know, because in the next step you will use corresponding FiPi for Prehessian-1, corresponding FiQi for Prehessian-2, corresponding FiRi for Prehessian-3.

Then, you also asked in #2211 if I could confirm one day these 2 cohessian cubics and cocayleyan cubics with Mathematica.

I’d love to do that in Mathematica, but you can help me with some calculation method for the Cayleyan. For the Hessian there is this well-known determinant of a matrix with 2nd derivative components. Do you happen to know something similar for the Cayleyan? You always also have thorough theoretical knowledge, so I hope you can help me.
Accidentally in Cabri there is a Cuppens-described macro construction of a cubic through F1,F2,F3 when the real tangents t1,t2,t3 at these flexpoints are known, knowing as well as an extra point. For me there is one uncertainty, I don’t know if the cubic defined this way also will pass through the 6 imaginary flexpoints and will be tangent there analogously.
However constructing Prehessian-1 in this way, using F1P1/F2P2/F3P3 as tangent at F1/F2/F3 with extra known points P1,P2,P3, then I find a cubic degenerated into 3 lines F1P1, F2P2, F3P3.
Analogues properties for Prehessian-2 and Prehessian-3.

Constructing Cohessian-1 in this way, using F1U1/F2U2/F3U3 is tangent at F1/F2/F3, then I find a cubic degenerated into 3 lines F1U1, F2U2, F3U3.
Analogues properties for Cohessian-2. That is not an expected result.
This makes me think.
Like you commented, in general, the 3 Prehessians and 2 Cohessians are identified by the 9 flexpoints (forming a CB-system) and one extra item (a point or a tangent).
The Cuppens method identifies a cubic through the 3 real flexpoints and corresponding tangents and one extra point on the wished cubic.
Do both methods define the same cubic?
So much for my thoughts.

Best regards,
Chris

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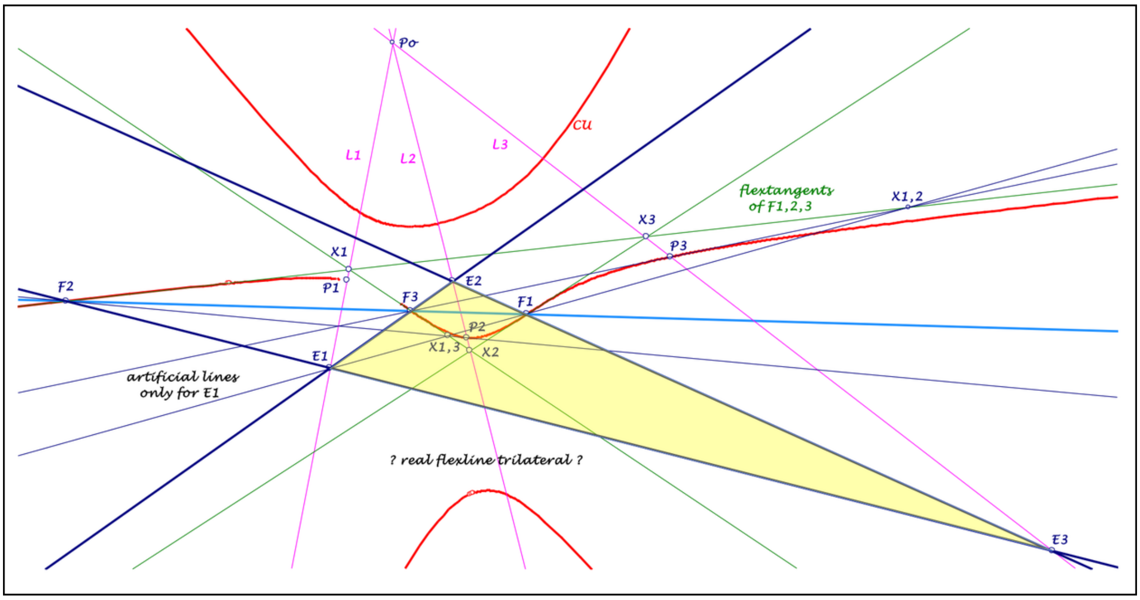
Message: #2259
Date: 2024-04-16
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris,

many thanks for your observation
... for the real flexlines in #2257.
I made a new drawing and marked the regions,
... where the vertices of the real flexlateral have to be,
... using my conjecture, that they lie on L1,L2,L3,
... collinear pairwise with a real flexpoint,
... I found the following possibility
... to put up for discussion:
Let us start with a general cubic CU (attached)
... with P1,P2,P3 and X1,X2,X3 (as in previous messages)
... and XiFj ^ FkPk = Xi,j
    and XiFk ^ FjPj = Xi,k
    and Xi,jXi,k ^ Li = Ei,
... then E1,E2,E3 may be the vertices of the real flexlateral.
I made several drawings and your property holds.
Can you prove my assumption?

Best regards Eckart

PS: Wrt your final question in #2257 see Schröter, p.239.



2024-04-15.pdf

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Message: #2260
Date: 2024-04-16
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Chris, dear Eckart,
Beautiful construction of Salmon!
It's easy to check that the contact points of the tangents from
a point to the circumscribed conics of the 4 vertices of a QA
describe a pivotal isocubic.
That was new for me.
Question: this is obvious for 4 real points defined as
intersection of 2 conics, the cubic being bipartite.
What if we start with 2 conics with 2 points as intersection or
with no intersection at all?
As Chris already said, imaginary points can be manipulated as
real points with Mathematica ...
Could this explain the case of monopartite cubics?
Best regards
Bernard
```

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Message: #2261
Date: 2024-04-16
From: bernard.keizer@gmail.com
Subject: Re: Hessian and prehessians

Dear Chris, dear Eckart,
Alleluiah, at last! I no longer expected any answer, as Eckart said it was beyong he's limits ...
Seriously, it's always the same logic: giving the 3 flexes Fi and the 3 harmonic lines Li intersecting in P,
any triangle T1T2T3 with vertices Ti on Li gives a reference cubic CU and it's hessian HE belonging to the syzegetic pencil with FiTi tangent in Fi to CU and in Ti to HE.
Wrt Chris message,
1) all the remarks are correct
2) it's easy to make a distinction between the 9 points as we must have PiQjRk aligned (their tangentials are Fi, Fj and Fk, which are aligned)
if the cubic CU is monopartite, we have only P1, Q1 and R1
if it is bipartite, take P1, Q1 and R1 of the unclosed part, then you have only 2 possibles choises for P2, which gives the rest
3) I have no idea about the cayleyan's calculation (see in Salmon or Schröter something with jacobian ...)
4) there is a mystake in your last construction, which explains the bad result
the points P1, P2 and P3 are not on the prehessian (the same way, the points T1, T2 and T3 are on the hessian, but not on the reference cubic)
You may check Cuppens construction by testing the cubic having the 3 flexes Fi, the 3 lines FiTi as flextangents and the point *P1* as extra point.
I'm sure you will get the reference cubic CU.
The problem for the construction of the 2 cohessians, the 3 prehessians and the 2 cocayleyans is that we have the 3 flexes and the 3 flextangents, but not an extrapoint!
Best regards
Bernard

Message: #2262
Date: 2024-04-17
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
Taking 2 real points as basis points of a set of circles
(centered on the perpendicular bisector of the segment joining
the 2 points), the 2 imaginary points are the circular points.
Then the locus of contact points of tangents of a point M to
these circles is a monopartite QL-Cu1 (M being the focus).
Taking the same points as Poncelet points of a set of circles
(centered on the line through the 2 points), the 2 points of
intersection of the circles are also imaginary.
Then the locus of contact points of tangents of a point M to
these circles is a bipartite QL-Cu1 (M being the focus).
Best regards
Bernard

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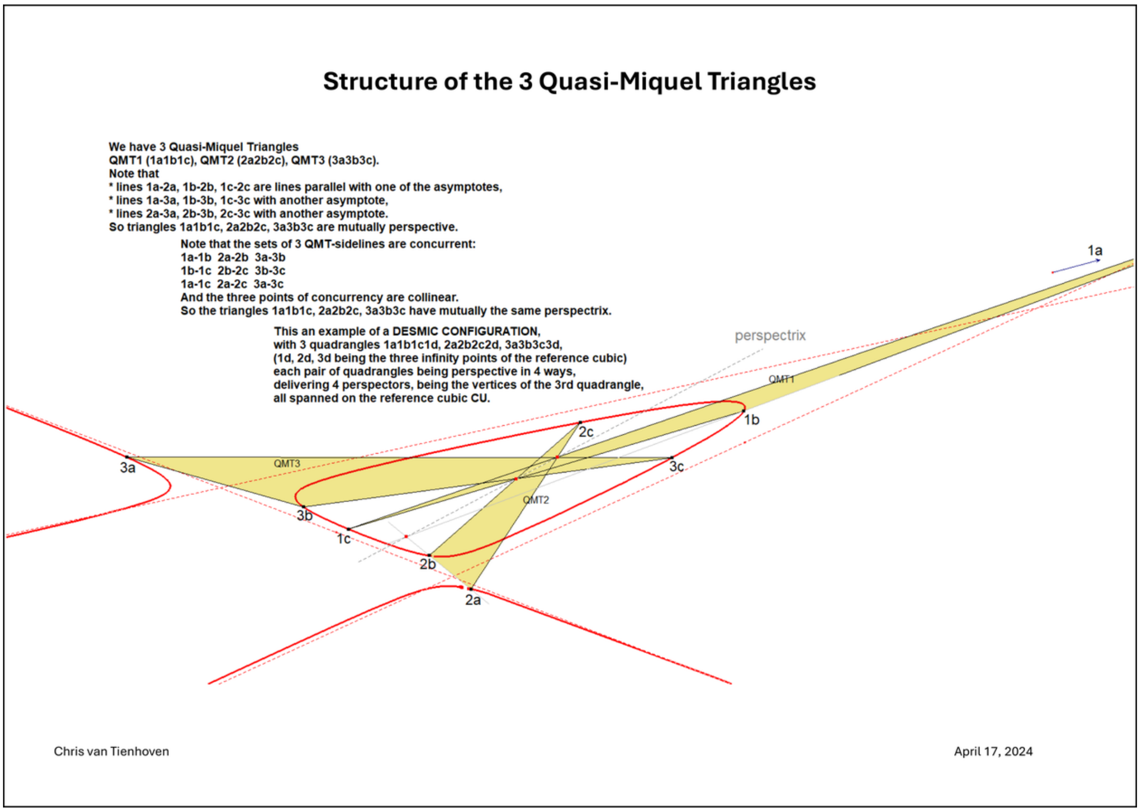
Message: #2263
Date: 2024-04-17
From: van10hoven@gmail.com
Subject: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

Dear Eckart and Bernard,

Just a message in between about the Quasi-Miquel Triangles.
I found that they form a Desmic Configuration
(https://www.chrisvantienhoven.nl/qa-items/qa-triangles/qa-tr-1)
combined with the infinity points of the reference cubic.
See attached picture.

Best regards,
Chris



CU4-Quadripartite Cubic-with Diametral Conics-08-plus QMTs.pdf

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Message: #2264
Date: 2024-04-18
From: eckart_schmidt@t-online.de
Subject: Real flexline trilateral

Dear Bernard, dear Chris,

my assumption wrt the real flexline trilateral in # 2259 doesn't hold:
... I found a counter-example, drawing a cubic and its hessian,
... which have the same flexpoints and -lines
... and therefore the same real flexline trilateral,
... what I could not confirm, excuse.
But there are new aspects ...

Best regards Eckart

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Message: #2265
Date: 2024-04-18
From: bernard.keizer@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

Dear Chris,
If I'm not wrong, the vertices of the Miquel triangles are with the infinity points of the asymptotes the tangentials of the 3 points Q where the cubic cuts these asymptotes.
The 3 infinity points are aligned on the infinity line, the 3 points Q are therefore also aligned and the 3 Miquel triangles form with the infinity points a Desmic system, also named Reye configuration.
Best regards
Bernard

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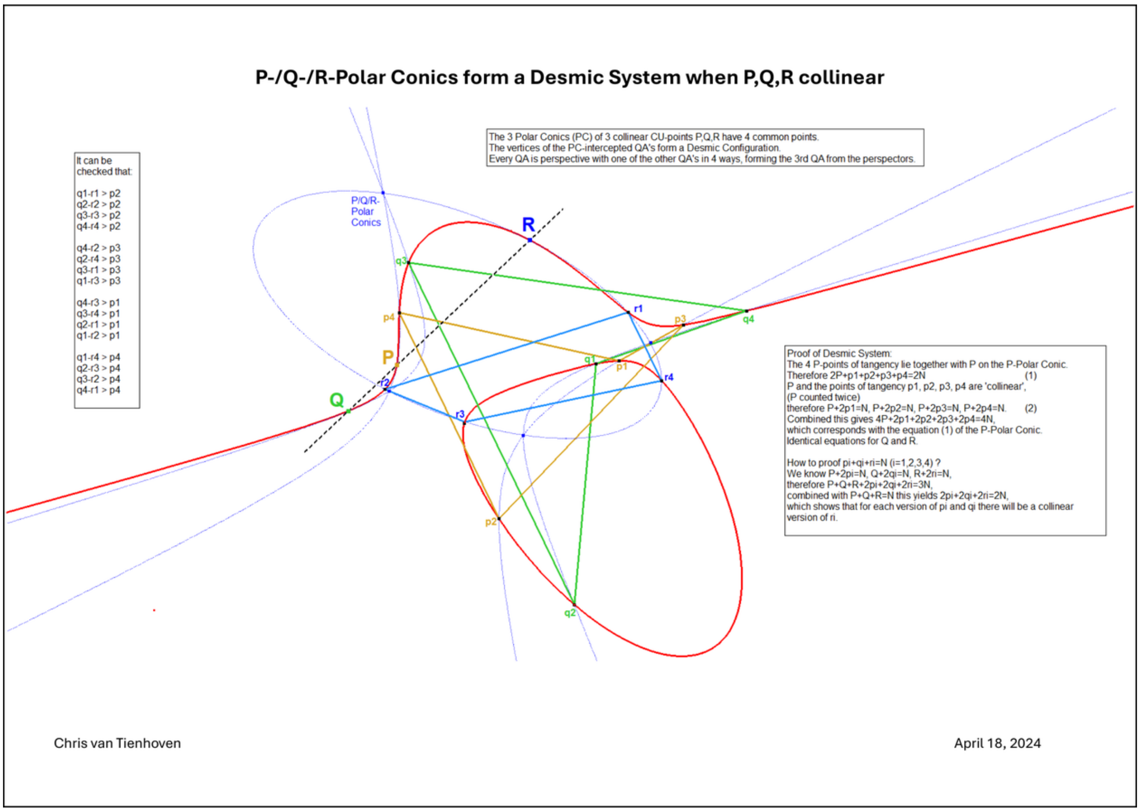
Message: #2266
Date: 2024-04-18
From: van10hoven@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

```
Dear Bernard and Eckart,

Here another example of a Desmic system.
When we have the Polar Conics of points P, Q, R on CU, then the
vertices of the 3 Polar-Conic-intercepted QA's form a Desmic
Configuration. Every QA is perspective with one of the other
QA's in 4 ways, forming the 3rd QA with its perspectors.
See attached picture for description and proof.
In this general case it also appeared that the 3 mentioned Polar
Conics have 4 points in common. We already discussed this in the
case of the Polar Conics of the CU-Infinity Points.
But now it appears to be also true for 3 collinear points P,Q,R.

Best regards,
Chris
```



CU_L-12P1 PQR-PolarConics-DesmicSystem-02.pdf

Message: #2267
Date: 2024-04-18
From: bernard.keizer@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

```
Dear Chris,
It's exactly the same property!
p1, p2 p3 and p4 have the same tangential, which is P.
If the 3 tangentials are aligned, then
1) the 3 polar conics pass through the 4 same points, which are
the poles of the line PQR
2) the 12 points pi, qi and ri (i = 1 to 4) form a Reye
configuration or a Desmic system and are on the same pivotal
cubic
Best regards
Bernard
```

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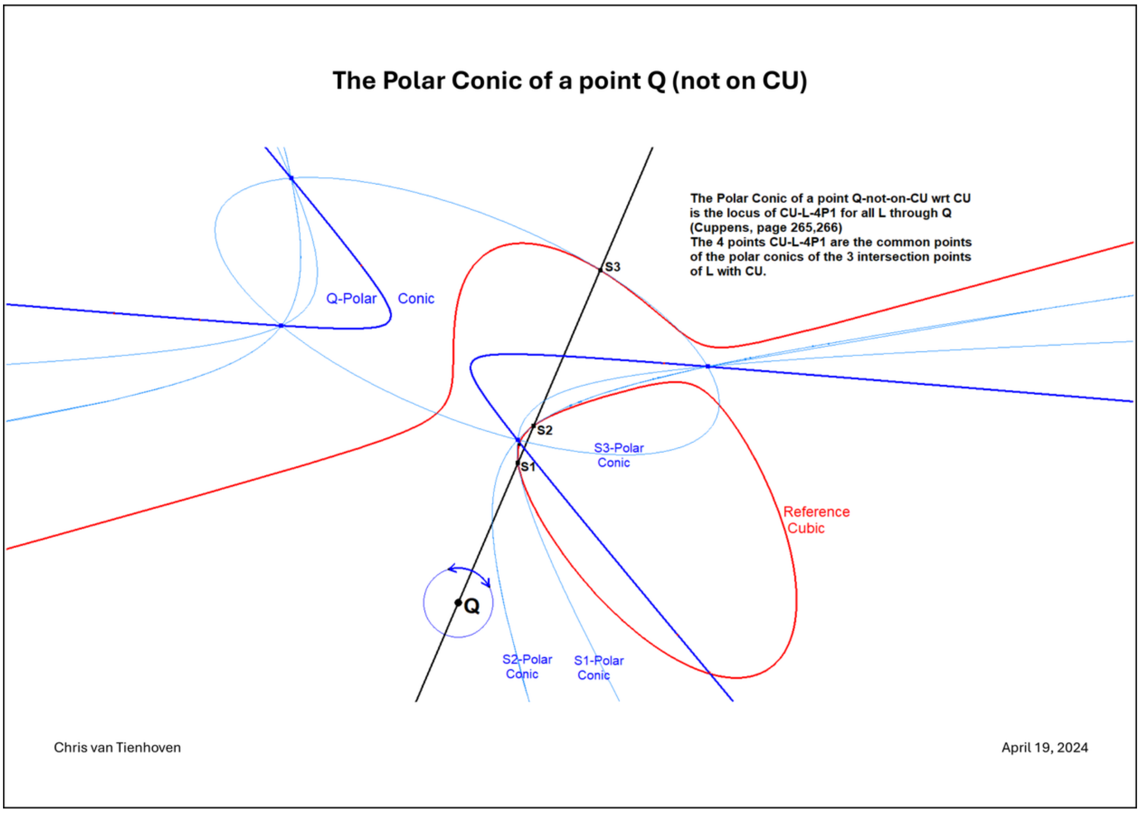
Message: #2268
Date: 2024-04-18
From: van10hoven@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

Dear Bernard,

You are right, 1) the 3 polar conics pass through the 4 same points, which are the poles of the line PQR.
I looked it up at [Cuppens].
There I also found this beautiful feature:
The Polar Conic of a point Q-not-on-CU wrt CU
is the locus of the 4 poles of L for all L through Q
[Cuppens, page 265,266]

Construction of Polar Conics
This makes it possible to construct the Q-Polar Conic when Q is not on CU.
See attached picture.
When Q is on CU we have construction method of [Cundy and Parry], Journal of geometry Vol. 53 (1995), 2.15 page 45.
Best regards,
Chris



CU_Q-Co1 Q-Polar Conic of a Cubic-02.pdf

Message: #2269
Date: 2024-04-19
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and prehessians

```
Dear Bernard, dear Chris,

if we use mono- and bipartite cubic in the sense of Schröter,
... every not degenerated cubic is mono- or bipartite.
Is it correct, that the hessian of a monopartite cubic is
bipartite
... and the hessian of a bipartite cubic in monopartite?
Then the points P1',P2',P3' of the hessian of a bipartite
ref-cubic
... are the points T1,T2,T3 of the ref-cubic.

Nevertheless:
In general the hessian is invariant wrt an isoconjugation
... for the triangle T1T2T3 with fixed point Po.

Best regards Eckart
```

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Message: #2270
Date: 2024-04-19
From: bernard.keizer@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

```
Dear Chris,
I used this method in order to draw the polar conic of P, by
searching the poles of L1, L2 and L3 ...
Best regards
Bernard
```

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Message: #2271
Date: 2024-04-19
From: van10hoven@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

Dear Bernard,

Which method did you use to find the poles of L1,L2,L3?

Best regards,
Chris

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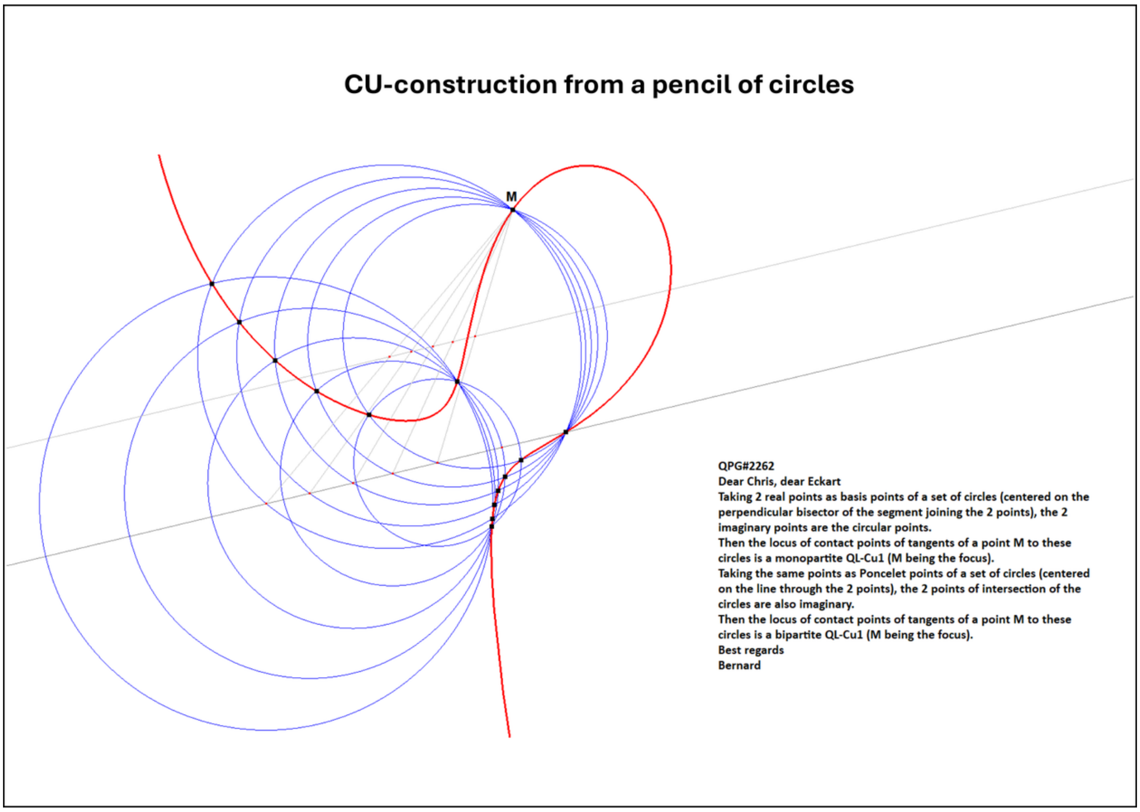
Message: #2272
Date: 2024-04-19
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Bernard,

Thanks for the nice construction of a QL-Cu1 type of cubic.
I like all these simple and special construction of cubics.
However the cubic will not (always) be bipartite. See enclose picture.
Is there a reference to this construction?

Note: let T1 and T2 be the basis points of the pencil of circles.
CU is constructed such that the lines MT1 and MT2 will be tangents at T1 and T2.

Best regards,
Chris



CU-construction-from-pencil-of-circles-01.pdf

Message: #2273
Date: 2024-04-20
From: eckart_schmidt@t-online.de
Subject: Common tangents of cubic and hessian

```
Dear Bernard, dear Chris,

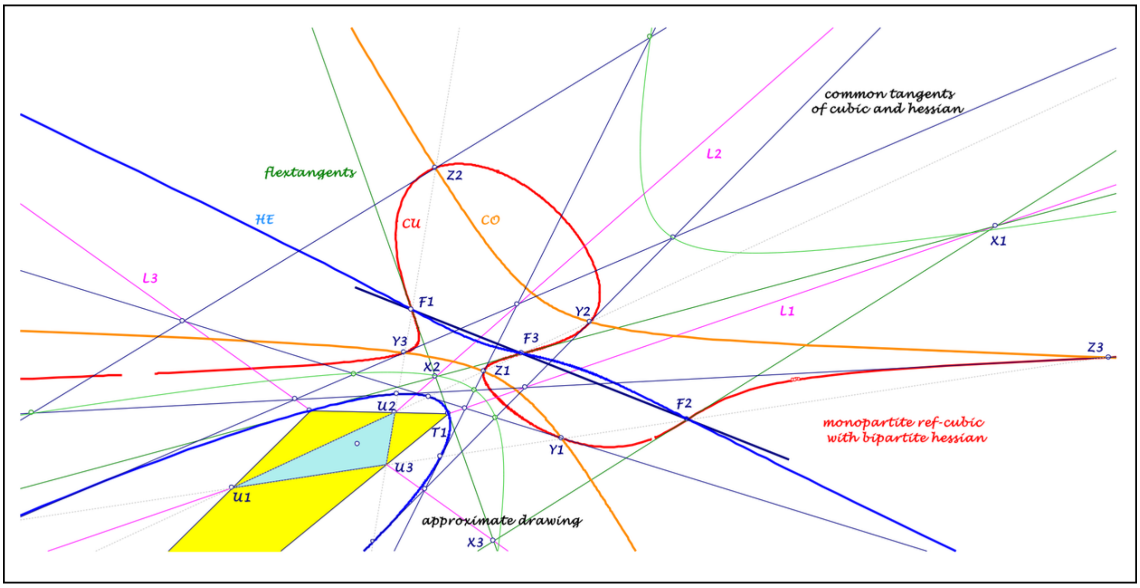
already mentioned in #2244 and #2251 a conic CC for a cubic CU,
... bearing 6 intersections with the cubic,
... on the sidelines of the cevian triangle U1U2I3
    of Po wrt  triangle T1T2T3.
These 6 points are also the intersections of the cubic and the
cayleyan,
... for they are the contact points on the ref-cubic
... for common tangents at the cubic and its hessian
... unequal the flextangents.

These 6 contact points are pairwise collinear with the
flexpoints
... and give two 3-times perspective triangles
    wrt the flexpoints,
... please use the notation Y1Y2Y3 and Z1Z2Z3 for the triangles,
... so that  Yi, Zi, Fi collinear and Yi, Zj, Fk on UiUj
    and YiYj ^ ZiZj on Lk,
... please have a look in the attached drawing.

For a monopartite ref-cubic the lines UiUj
... intersect also the hessian in their contact points
    (unequal Fk).
The 6 common tangents intersect in 9 = 3X3 points on L1,L2,L3:
... the tangents in Yi and Zi intersect on Li,
... the tangents in Yi and Zj intersect on Lk.
For a bipartite ref-cubic there are further 6 intersections
coconic on the hessian:
... intersections of the tangents in Yi and Yj
    as well as in Zi and Zj.
For a monopartite ref-cubic these further 6 intersections
... are coconic on a circumconic of X1X2X3
    with tangents in Xi through Fi.
So far, there will be more properties,
... especially wrt 4th harmonic points.

Best regards Eckart

PS: Attached only an approximate drawing for a monopartite
cubic,
... I hope someone can confirm these observations.
Excuse the attached word file, but it has a better quality.
```



2024-04-20.docx

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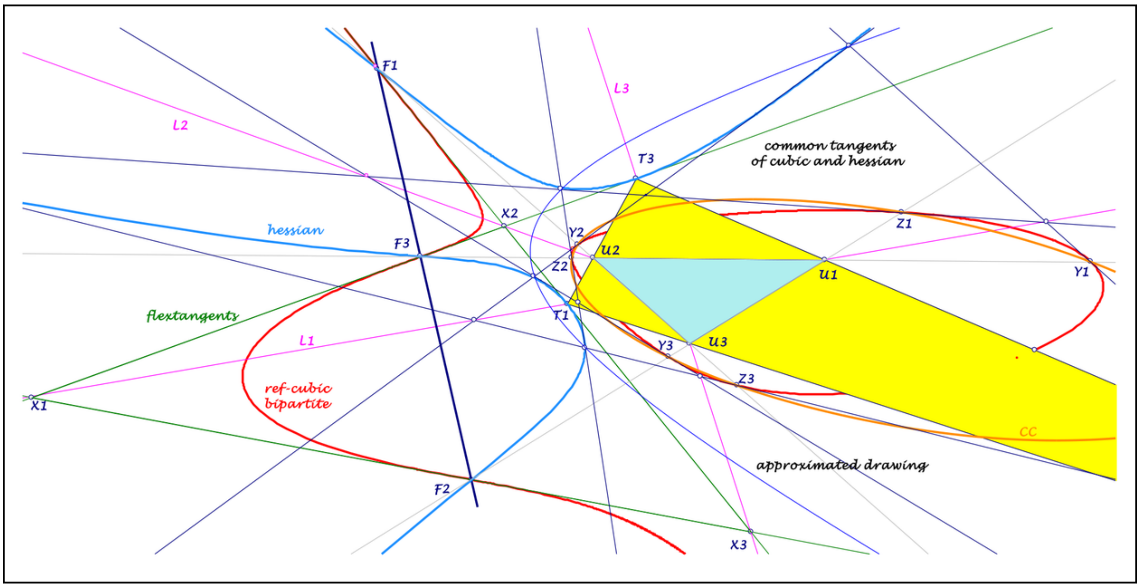
Message: #2274
Date: 2024-04-20
From: eckart_schmidt@t-online.de
Subject: Re: Common tangents of cubic and hessian

```
Dear Bernard, dear Chris,

attached an approximate drawing for a bipartite ref-cubic,
... not so good, for there is a further part of the cubic
... right down outside the page.

Best regards Eckart
```



2024-04-21.docx

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Message: #2275
Date: 2024-04-20
From: bernard.keizer@gmail.com
Subject: Re: Common tangents of cubic and hessian

Dear Eckart,
It seems very interesting, but I have difficulties with the Word file!
Could you please send a pdf file for both cases, it's easier to print them.
I looked in Bernard Gibert where there are many examples of cubics, hessians and prehessians and I think you are right that a monopartite cubic has a bipartite hessian and vice-versa (monopartite and bipartite in the sense of Schröter). Naturally, the prehessian(s) are of the same type as the hessian ...
Best regards
Bernard

PS I regret you use Ui, Yi and Zi, with I already used in my messages about cohessians, prehessians and cocayleyans #2208 and #2211.
Why don't you name titjtk the vertices of the cevian triangle of P wrt TiTjTk and Mi,j,k and Ni,j,k the 6 coconic points where CU intersect the cayleyan.

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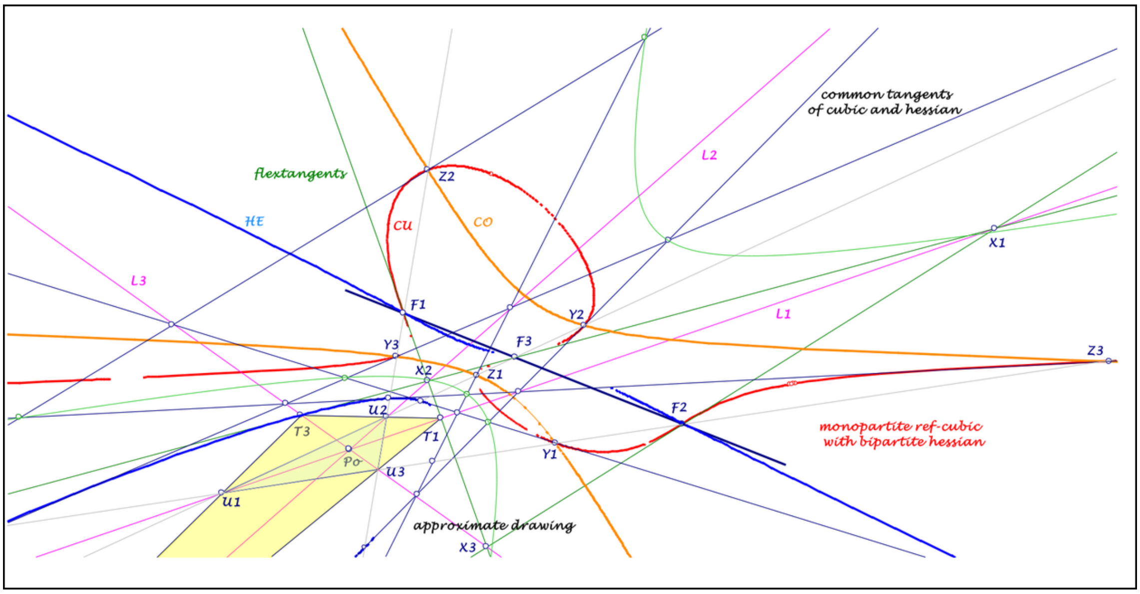
Message: #2276
Date: 2024-04-20
From: eckart_schmidt@t-online.de
Subject: Re: Common tangents of cubic and hessian

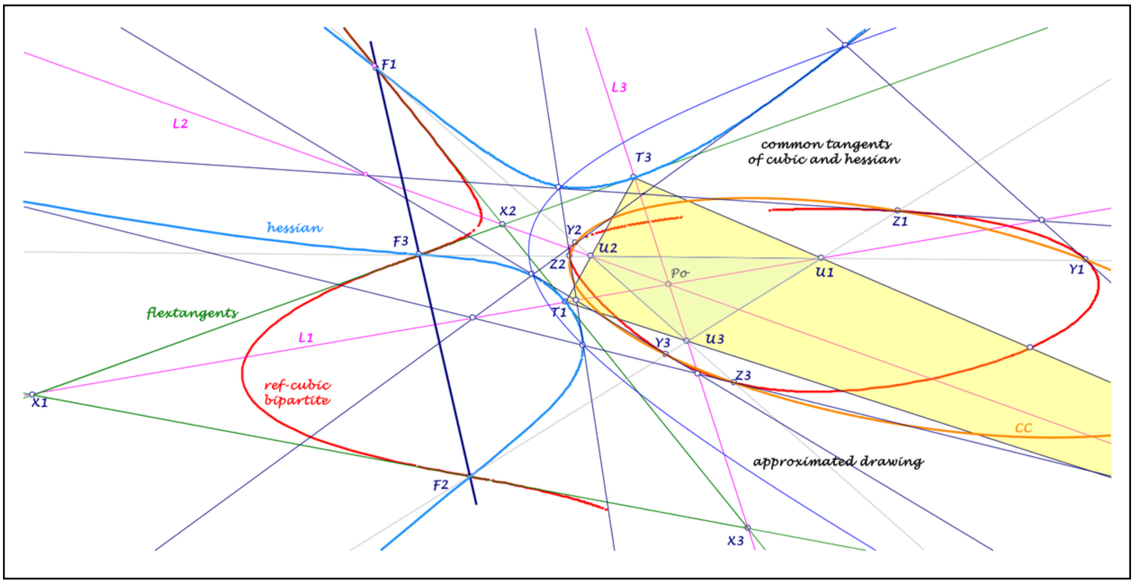
```
Dear Bernard,

attached the two pdf-files, I don't know,
... why the cubics don't appear complete.

Best regards Eckart
```



2024-04-20.pdf



2024-04-21.pdf

Message: #2277
Date: 2024-04-20
From: bernard.keizer@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

```
Dear Chris,
I explained the method in my message 2152 with a figure!
Best regards
Bernard
```

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Message: #2278
Date: 2024-04-20
From: bernard.keizer@gmail.com
Subject: Cayleyan and dual cubic

```
Dear Chris,
Geogebra offers apparently less faculties than Cabri!
Would you please be kind enough to draw on the same figur a
cubic CU, it's hessian HE and it's cayleyan Ca (like you already
did in your message 2160) *with the cubic used as QA-Tf10 of the
cayleyan* (naturally with the flexes Fi, the poles P, X1, X2 and
X3 of the flexline and the harmonic lines Li).
I suppose the mentionned cubic belongs also to the syzegetic
pencil and has the same flexes, but I would like to be sure of
that before making a more precise assumption.
Is it correct that the QA-Tf10 of all QAs formed as poles of
lines wrt the reference cubic give the same transformation?
Considering the QA of the poles of flexes, it's sides are the
flextangents and the harmonic lines.
What are the QA-Tf10 of these lines? (the flexes and the points
Xi?)
Many thanks in advance for your help, it's difficult for me to
progress without a precise figure ...
Best regards
Bernard
```

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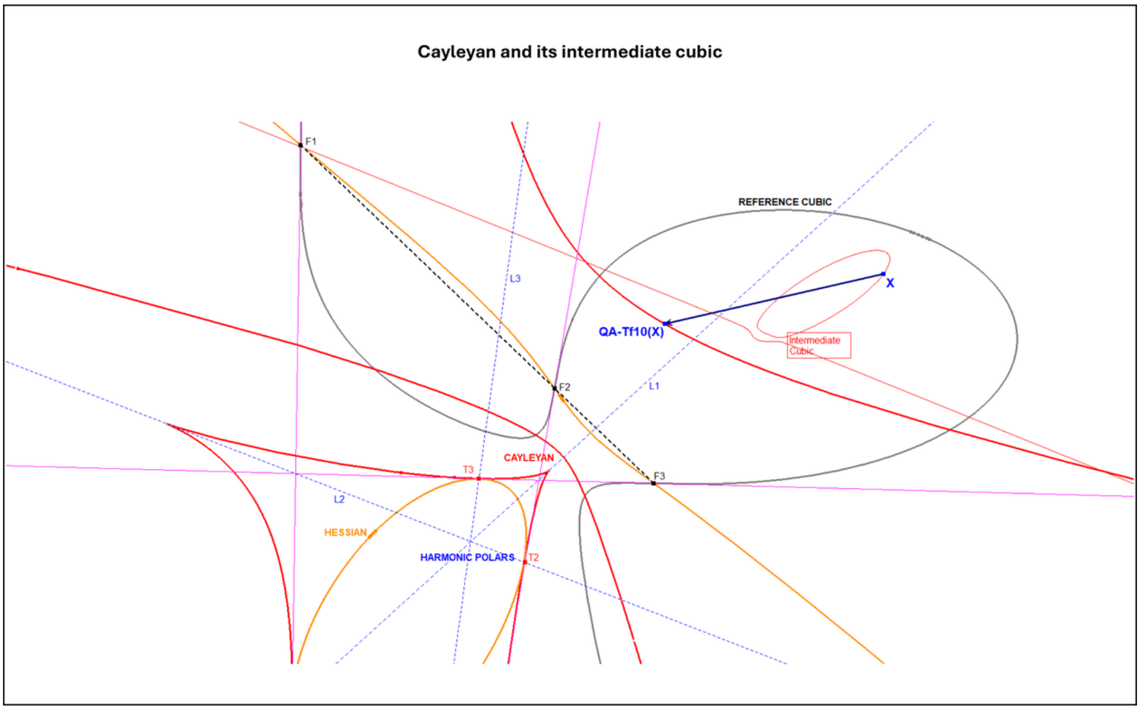
Message: #2279
Date: 2024-04-22
From: van10hoven@gmail.com
Subject: Re: Cayleyan and dual cubic

```
Dear Bernard,

Attached the asked picture.
As you can see the cubic does not pass through the Flexpoints.
In the construction of the QA-Tf10 points I used a random QA.
Varying this QA also the cubic varies.
I hope it helps you in your further considerations.

Best regards,
Chris
```



CU-Cayleyan-16a-plus-Hessian-F-Tangents-Polars.pdf

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Message: #2280
Date: 2024-04-23
From: van10hoven@gmail.com
Subject: Re: The set of 3 Quasi-Miquel Triangles form a Desmic System

---

```
Dear Bernard,

I tried to understand your message #2152.
I cannot find a general method for constructing the 4 poles of a
line in it.

By the way your attached drawing maybe of use for yourself.
For me it is a blur of lines and curves. It is even hard to
distinguish the reference cubic.
I think it would help when you hide certain lines and/or curves
or make two pictures where you highlight different topics.
Of course with all due respect for your beautiful topics.

Best regards,
Chris
```

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Message: #2281
Date: 2024-04-23
From: eckart_schmidt@t-online.de
Subject: Re: Common tangents of cubic and hessian

```
Dear Bernard, dear Chris,

the following property in #2273 wrt intersections
... of common tangents for cubic and hessian:
"For a bipartite ref-cubic there are further 6 intersections
coconic on the hessian:
... intersections of the tangents in Yi and Yj
    as well as in Zi and Zj."
... doesn't hold in general, excuse:

But there are three conics for
(1) the contact points on the ref-cubic,
(2) the contact points on the hessian,
(3) the intersections of the common tangents in Yi, Zj
    (i unequal j)
... all three with the property: polar of Fi is Li.

Best regards Eckart
```

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Message: #2282
Date: 2024-04-25
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

Dear Chris, dear Eckart
First many thanks to both of you for your beautiful figures and my apologies for my own ugly figures!
Here I'm trying to gather most of my observations/conjectures, hoping it will give you new ideas.
1) Naturally, Chris method in order to draw the cayleyan CA is correct: any QA, it's QA-Tf10 swapping CA and a cubic defined by 9 points QA-Tf10 of 9 tangents to CA ...
2) Any line has wrt the reference cubic CU 4 poles forming a QA and
a) the DT vertices of this QA as well as the intersections of the DT sides and the reference line are on the hessian HE
b) the 6 sides of the QA are tangent to the cayleyan CA
3) I use since the beginning the following notations
Li intersect CU in Pi,Qi and Ri, HE in Ti,Ui and Vi and CA in Ti, Wi and Zi (tangents from Fi) and in Yi (turning points tangents from P)
Li intersects tgi (flextangent in Fi) in Ti (contact point of HE and CA).
The flextangents in Fj and Fk intersect on Li in Xi
The points Ti are the vertices of the cevian triangle of P wrt the Xi triangle
I added the points ti, vertices of the cevian triangle of P wrt the Ti triangle and AXi, vertices of the anticevian triangle of P wrt the Xi triangle
As mentionned already, tgi is the harmonic line of Li wrt Lj and Lk
Therefore, ti is the harmonic of Fi wrt Tj and Tk, Ti is the harmonic of Fi wrt Xj and Xk and Xi is the harmonic of AXj and AXk
4) Since the QAs of the poles envelop CA, it seems appropriate to start with one of them as reference QA for the QA-Tf10
Let's start with the QA PXiXjXk,formed by the 4 poles of the flexline L
The DT is T1T2T3, which is self dual and the dual QL is formed by the flexes Fi and the points ti
Fi is the dual of Li and ti the dual of XjXk (or FiTi)
I suppose the dual cubic of CA belongs to the syzegetic pencil and has flextangents T2T3, T1T3 and T1T2
5) We could have the same idea with the QA formed by P and the points AXi
This time, the DT is X1X2X3, which is self dual and the dual QL is formed by the the flexes Fi and the points Ti

The dual QL will be the hessian HE, but it's dual curve is not CA
6) Finally, my conjecture is that CU and HE belongs to same syzegetic pencil of cubics having the he same flexes and that CA belongs the same way to a pencil of curves of 3rd class having the same tangents Li in their turning points. There is a curve of 3rd class in this pencil which has HE as it's CA and CA as it's HE, but it is another story ...
7) To complete the construction, it is convenient to search the poles of well known lines:
a) I name F'i the harmonic of Fi wrt Fj and Fk; F'i is the intersection of Li and L
drawing the polar conic of P shows no difficulty on Chris reference figur, as we have 6 tangents
drawing the polar conic of F'i is even easier, as it passes already through P, X1, X2 and X3 and we just need to draw a tangent
The intersections of these 2 conics are the 4 poles of Li
We may notice that the DT of the QA of the 4 poles is FiUiVi (the 2 lines through Fi form the polar conic of Ti and are tangent to CA; the lines are FiWi and FiZi)
b) The polar conic of Fi is formed by the 2 lines tgi and Li, which intersect the polar conic of P in the 4 poles of PFi
We may notice that one of the DT vertices is Fi
c) Searching the poles of tgi, it appears that we know already the polar conics of Fi (see point b) and Ti (see point a)
The intersection of the 4 lines gives the 4 poles
That's all for today!
Best regards
Bernard

Message: #2283
Date: 2024-04-26
From: van10hoven@gmail.com
Subject: IP-SumPoints of the set (P1,...,P9)

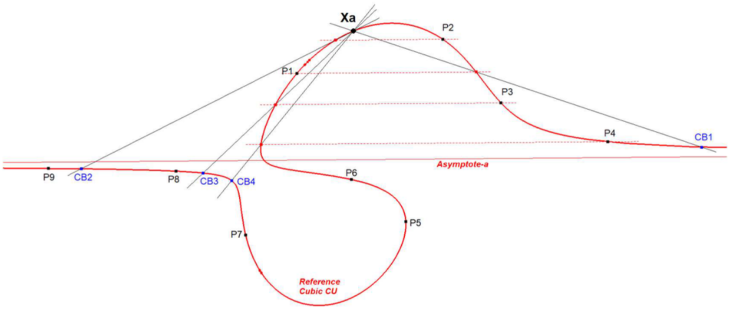
```
Dear Eckart and Bernard,

Attached you will find a surprising new CU-point,
which calculated is a version of (P1+...+P9).

Best regards,
Chris
```

CU_9P-3P1 IP-SumPoints of the set (P1,…,P9)

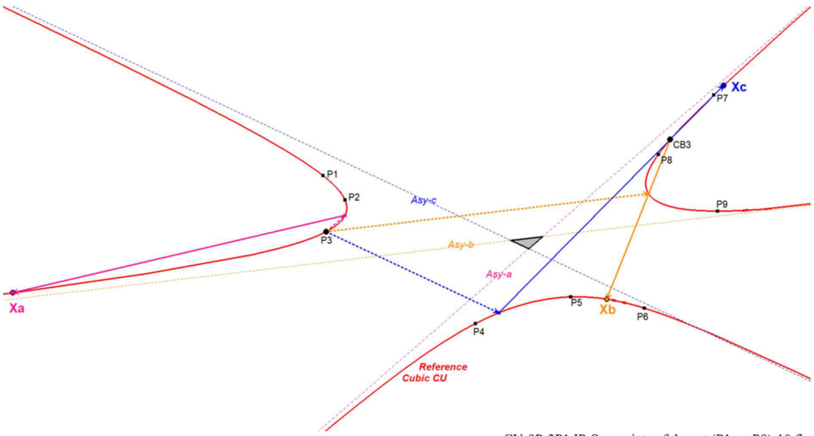
Given reference cubic CU with 9 random points P1, …, P9 on CU.
Let aPi be the 2nd intersection point of CU with the Asy-a-Parallel through Pi (i=1,…,9).
Xa = aPi.CBi, which is a fixed point for i=1, ... , 9.



CU-9P-3P1 IP-Sumpoints of the set (P1,…,P9)-01.fig

Validation:
Calculation for IPa and CB9:
P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + CB9 = 3N (1)
P9n + P9 + IPa = N (2)
CB9 + P9n + Xa = N (3)
From (1) and (2) and (3) it follows that:
Xa = IPa + PP9 -3N, where PP9 = P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + P9

In the same way there are fixed points Xb and Xc, when there are 3 real CU-asymptotes.
Xa, Xb, Xc are the IPa/b/c-Sumpoints of the set (P1,...,P9).



CU-9P-3P1 IP-Sumpoints of the set (P1,…,P9)-10.fig

CU_9P-3P1 IP-SumPoints of the set _P1,...,P9_-01.pdf

Message: #2284
Date: 2024-04-26
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

```
Dear Bernard,

many thanks for gathering your observations,
... that will be a great help for me,
... but next week start 3 weeks holiday with my family,
... there after I shall try to follow your discussion
    with Chris.

Best regards Eckart
```

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Message: #2285
Date: 2024-04-26
From: eckart_schmidt@t-online.de
Subject: Re: IP-SumPoints of the set (P1,...,P9)

```
Dear Chris,

there is an analogon for 7P wrt cb.

Best regards Eckart
```

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Message: #2286
Date: 2024-04-27
From: eckart_schmidt@t-online.de
Subject: Re: IP-SumPoints of the set (P1,...,P9)

```
Dear Chris,

your described point Xa in #2283
... constructed analog for a 7P wrt cb
... gives the 3rd intersection of 7P-s-Cu1 and  Q.7P-s-Tf1(Q) ,
... Q = intersection of the cubic and its asymptote.

Best regards Eckart
```

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Message: #2287
Date: 2024-04-27
From: bernard.keizer@gmail.com
Subject: Re: IP-SumPoints of the set (P1,...,P9)

```
Dear Chris,
The property works by replacing IPa by any fixed point of the
cubic (independant from the 9 points), for example IPb and IPc,
as you did, or Q, the point(s) where the curve cuts it's
asymptote or M, the quasi-Miquel point(s) or the centers of
quasi-anallagmaty (where the tangent is parallel to the
asymptote ...
Best regards
Bernard
```

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Message: #2288
Date: 2024-05-02
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard, dear Chris,

before holiday on the way to the real flextrilateral?

Let us consider the following set of triangles A1A2A3
... with vertices on the harmonic polars of the flexpoints,
... bearing on each side a flexpoint:
... A1 on L1, A2 = L2 ^ A1F3, A3 = L3 ^ A1F2.

These set bears several well known triangles:
... P1P2P3, T1T2T3, X1X2X3 and their cevian triangles wrt Po,
... the flexline F1F2F3 as degenerated trilateral,
... trilateral F1P1.F2P2.F3P3, tangential to the cubic,
... F1T1.F2T2.F3T3 = X1X2X3 tangential to the cubic
    and the hessian ...
Trilaterals of this set will be the degenerated cubics
... with the same flexpoints and its harmonic polars.
... I suppose, that the real flextrilateral will be also of this
type.

The last example shows a connection to common tangents of cubic
and hessian,
... in #2244 and #2273 a conic CC is described,
... intersecting the cubic in six points, contacts of
... common tangents with the hessian
    (unequal the flexlines of F1,F2,F3).
These 6 common tangents have 15 intersections (# in the
drawing),
... 9 lie on L1,L2,L3 and give three triangles of the set above,
... 6 further intersections Tg lie on a conic CT,
... an Tg-intersection connected with a flexpoint
... ... bears further another Tg-intersection.
The CT-polar of Fi is Li and the polar of Po is F1F2F3.
For a monopartite cubic the conic CT
... is a circumconic of X1X2X3, image of F1F2F3
... for an isoconjugation wrt X1X2X3 with fixed point Po.

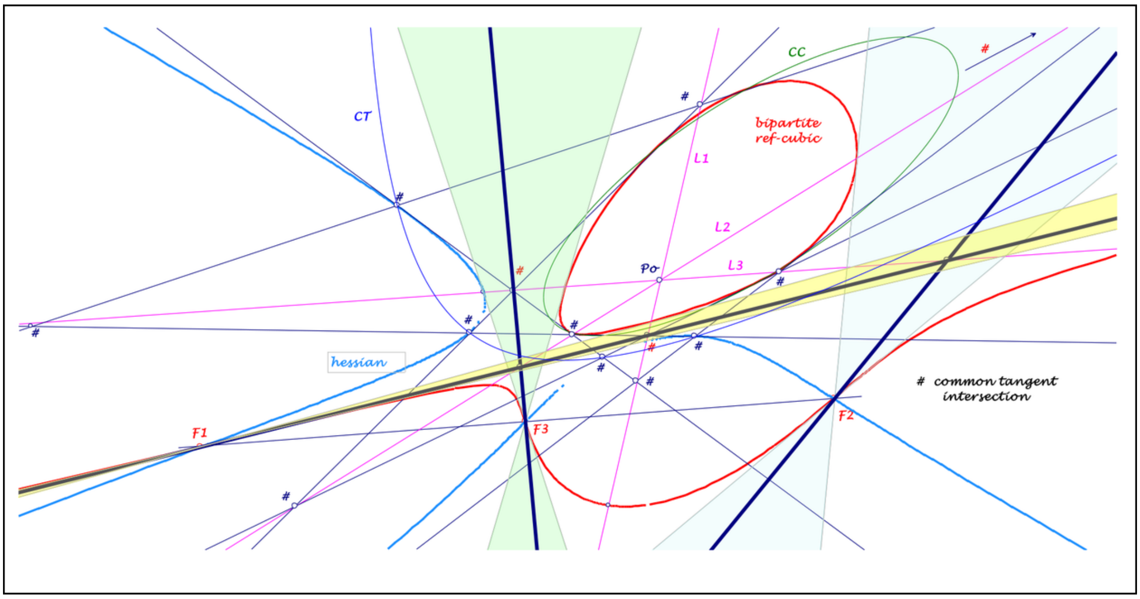
But back to the problem of the real fextrilateral:
... Cubic and its hessian have the same flexpoints,
... 3 are real and collinear, the other 6 flexpoints
... lie on 3 real lines through the real flexpoints.
These 3 real flexlines have no further intersections
... with the cubic and its hessian,
```

... so there exist regions RGi for these flexlines,
    centered in Fi,
... see attached drawing and Chris' remarks in #2257.
Each region RGi contains on Li one intersection (# in the drawing),
... these intersections connected with Fi
    may be the real flexlines,
... which intersect on Li in the overlapping regions.
Are there any criterions to decide whether this is true?

Finally: What about the 6 coconic Tg-intersections on CT,
... which seem to be points on the hessian
    for a bipartite cubic,
... already mentioned in #2273 and canceled in #2281 by me,
... but here once more to see?
They seem to be tangent and intersecting the cubic.

Best regards Eckart

PS: It is difficult, to make a complete and precise drawing,
... so you have sometimes to extrapolate.



2024-05-02.pdf

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Message: #2289
Date: 2024-05-03
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Chris, dear Eckart
Thanks to Eckart, it's a beautiful and interesting figur!
Is it possible, by any chance, that the triangle found by Eckart
is the triangle I named Y1Y2Y3 (cusps of the cayleyan)?
The points # on Li would be the vertices of the cevian triangle
of P wrt this triangle and the harmonic of Fi wrt Yj and Yk ...
Best regards
Bernard

Message: #2290
Date: 2024-05-07
From: van10hoven@gmail.com
Subject: Set of 9 Tritangent Conical Points

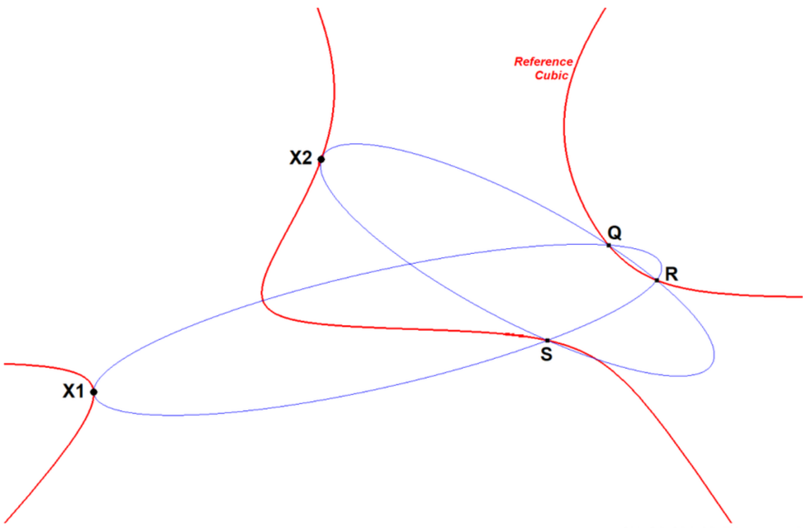
Dear Eckart and Bernard,

Just in between:
For each set of 3 points P,Q,R on Reference Cubic Cu there
exists a set of 9 points X1, …, X9 that
produce a conic through P,Q,R that will be tritangent to CU
[Fred Lange, page 137, 10.].
See attachment.

Best regards,
Chris

CU_3P-9P1 Set of nine 3P-Tritangent Conical Points

For each set of 3 points P,Q,R on Reference Cubic Cu there exists a set of 9 points X1, …, X9 that produce a conic through P,Q,R that will be tritangent to CU.
See [Fred Lange, page 137, 10.].



CU_3P-9P1 3P-Tritangent Points-01.fig

Validation
The equation 3X+Q+R+S=2N has nine solutions.

CU_3P-9P1 3P-Tritangent Points-01.pdf

Message: #2291
Date: 2024-05-07
From: van10hoven@gmail.com
Subject: Re: Set of 9 Tritangent Conical Points

Dear Eckart and Bernard,

The mentioned property is valid, but not my picture with X1 and X2.
The shown points X1,X2 are not tritangent, because the conics have an extra interesection point with CU, apart from Q,R,S and Xi.
Maybe you can help me with a petter picture.
Best regards,

Chris

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Message: #2292
Date: 2024-05-07
From: van10hoven@gmail.com
Subject: Re: CU-inscribed complete n-Gons

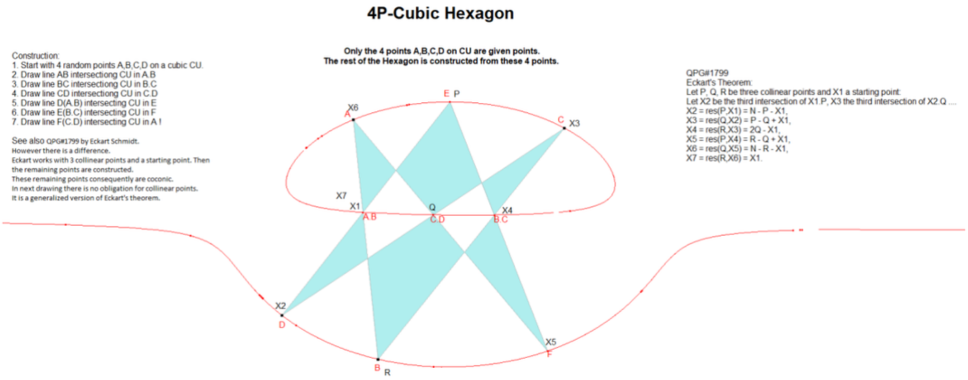
Dear Bernard and Eckart,

I found several examples of the 4P-Hexagon.
Special about it is that they connect Cayley-Bacharach Points (CBi), Pivot Points (Sij, Tij, Uij) and derived 9P-Sumpoints with the term PP9=P1+P2+P3+P4+P5+P6+P7+P8+P9.
The derived 9P-sumpoints are new.
Some were known with the Circular Cubic. They are new for a General Cubic. They can be constructed with the QF-Conic (general cubic) instead of the QF-circle (circular cubic).
When points occur in a 4P-Hexagon, only 4 points are enough to construct the rest of all 9 points.
See attachment.

Best regards,
Chris

Examples 4P-Hexagon

General Construction of the 4P-Hexagon

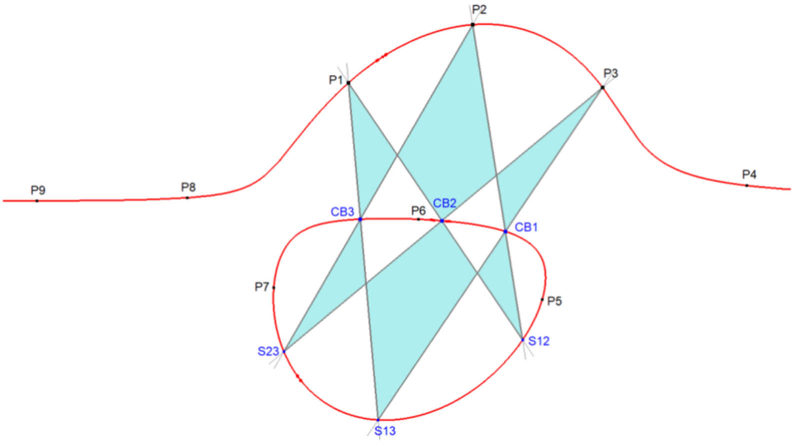


CU_4P Cubic Hexagon-01.fig

CU_4P Cubic Hexagon-20-Examples-STU-points.pdf

Example 1

Given 9 random points P1, ..., P9 on reference cubic CU.
Let CB1, CB2, CB3 the Cayley-Bacharach points of P1, P2, P3 wrt the set (P1, ...,P9).
Let S12=P1CB2^P2CB1, S13=P1CB3^P3CB1, S23=P2CB3^P3CB2.
S12, S13, S23 are points on CU.
The points P1,P2,P3, CB1,CB2,CB3, S12, S13, S23 form a perfect CU-4P-Hexagon.



CU_7P-CB Pivot Point-10.fig

CU_4P Cubic Hexagon-20-Examples-STU-points.pdf

Example 2
Given 9 random points P1, ..., P9 on reference cubic CU.
Let CB1, CB2, CB3 the Cayley-Bacharach points of P1, P2, P3 wrt the set (P1, ...,P9).

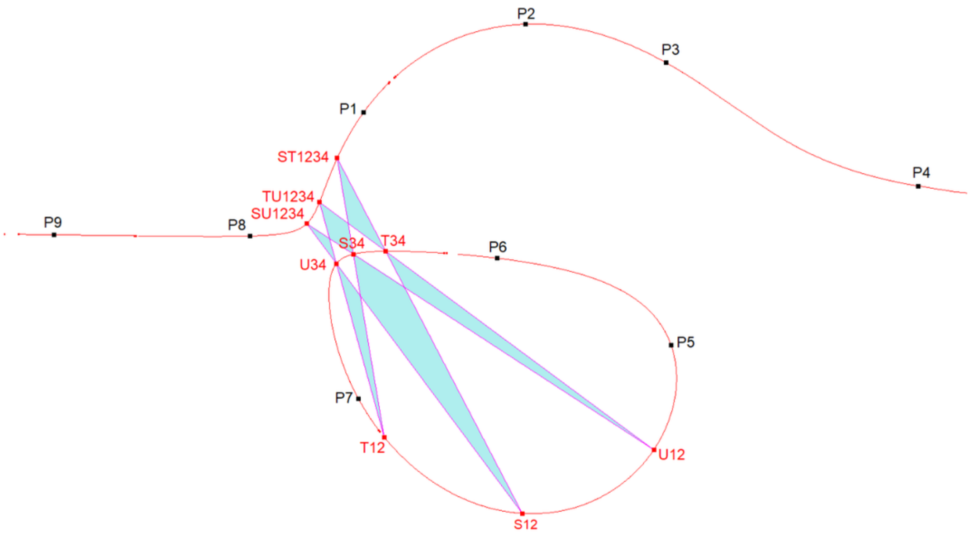
	Validation
Sij = Pi.CBj ^ Pj.CBi = CB-Pivot Point	-2N-Pi-Pj+PP9 = -2N+PP7
Tij = common point of conics (Pi,Pj,Pv,Pw,Svw),	4N-Pi-Pj -PP9
Uij = common point of conics (CBi,CBj,CBv,CBw,Svw),	-8N-Pi-Pj+3PP9

where (i,j) are fixed and (v,w) are variable different numbers from the range (1,...,9).

Skl = Pk.CBl ^ Pl.CBk,	-2N-Pk-Pl+PP9 = -2N+PP7
Tkl = common point of conics (Pk,Pl,Pv,Pw,Svw),	4N-Pk-Pl -PP9
Ukl = common point of conics (CBk,CBl,CBv,CBw,Svw),	-8N-Pk-Pl+3PP9

where (k,l) are fixed and (v,w) are variable different numbers from the range (1,...,9).

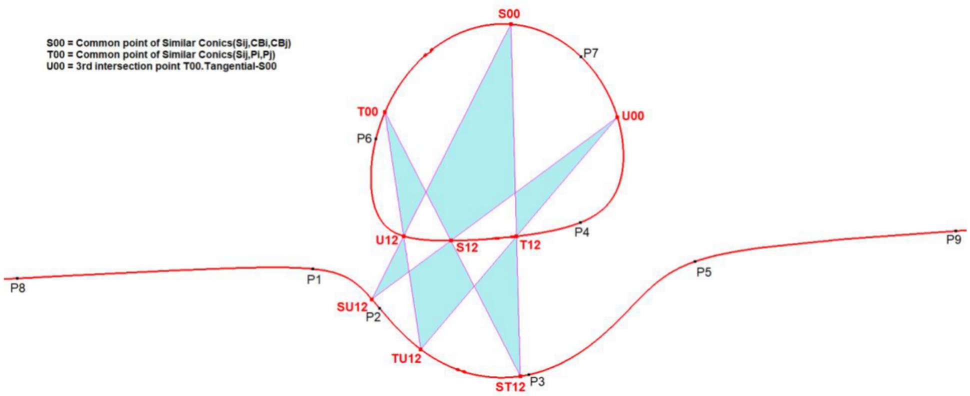
STijkl = Sij.Tkl ^ Skl.Tij = Cotterill's Point	-N+Pi+Pj+Pk+Pl = -N+PP4
TUijkl = Tij.Ukl ^ Tkl.Uij	5N+Pi+Pj+Pk+Pl-2PP9
SUijkl = Sij.Ukl ^ Skl.Uij	11N+Pi+Pj+Pk+Pl-4PP9



CU_4P Cubic Hexagon-11-Example-STUij-STUkl.fig

Example 3

Given 9 random points P1, ..., P9 on reference cubic CU.
Let CB1, CB2, CB3 the Cayley-Bacharach points of P1, P2, P3 wrt the set (P1, ...,P9).



CU_9P-P1 and -P2 SumPoints-50.fig

	Validation
Sij = Pi.CBj ^ Pj.CBi = CB-Pivot Point	-2N-Pi-Pj+PP9 = -2N+PP7
Tij = common point of conics (Pi,Pj,Pv,Pw,Svw),	4N-Pi-Pj -PP9
Uij = common point of conics (CBi,CBj,CBv,CBw,Svw),	-8N-Pi-Pj+3PP9

where (i,j) are fixed and (v,w) are variable different numbers from the range (1,...,9).

S00 = Sij for which (Pi,Pj) represent two points at infinity of CU	-2N-IP2-IP3+PP9 = -3N + IP1 + PP9
T00 = Tij for which (Pi,Pj) represent two points at infinity of CU	4N-IP2-IP3-PP9 = +3N + IP1 - PP9
U00 = Uij for which (Pi,Pj) represent two points at infinity of CU	-8N-IP2-IP3+3PP9 = -9N + IP1 - PP9

S00, T00, U00 can be constructed as follows:
S00 = Common point of Similar Conics (Sij,CBi,CBj) *)
T00 = Common point of Similar Conics (Sij,Pi,Pj) *)
U00 = 3rd intersection point T00.Tangential-S00

STij = Sij.T00 ^ S00.Tij	-IP1 + 12 N + Pi + Pj - 4 PP9
TUij = Tij.U00 ^ T00.Uij	-IP1 + 6 N + PI + Pj - 2 PP9
SUij = Sij.U00 ^ S00.Uij	-IP1 + Pi + Pj

*) Similar Conics (P1,P2,P3) are defined as the conics (P1,P2,P3,IP2,IP3), where IP2 and IP3 are the not current (imaginary) infinity points. They can be constructed as conics through (P1,P2,P3) similar and equally directed to the IP1-QF-Conic of CU.

Especially next points are interesting:

Sij = CB-Pivot Point	-2N+PP7
SUij = Sij.U00 ^ S00.Uij (can be constructed simply as (Pi.Pj).IP1)	-IP1 + Pi + Pj

But also the other points are interesting because they have the term
PP9=(P1+P2+P3+P4+P5+P6+P7+P8+P9).

CU_4P Cubic Hexagon-20-Examples-STU-points.pdf

Message: #2293
Date: 2024-05-07
From: anopolis72@gmail.com
Subject: KOG

```
The Croatian Society for Geometry and Graphics
<http://master.grad.hr/hdgg/index-en.html>
publishes the journal KOG

The issues 1 - 27 can be found here
https://master.grad.hr/hdgg/kog_stranica/

In the last issue 27
https://master.grad.hr/hdgg/kog_stranica/kog27/KoG27.pdf
there is an interesting article

BORIS ODEHNAL, A Miquel-Steiner Transformation

ABSTRACT Each complete quadrilateral has three Miquel-Steiner
points. Any triangle together with an arbitrarily chosen point
not on a triangle side also defines a complete quadrilateral,
and thus, this pivot point defines three MiquelSteiner points.
These three Miquel points form a triangle which is perspective
with the base triangle. The mapping that assigns to the pivot
point the uniquely defined perspector is a quadratic and not
involutive Cremona transformation and shall be called
Miquel-Steiner transformation. We shall study the action of the
Miquel-Steiner transformation and its inverse.

APH
```

Message: #2294
Date: 2024-05-07
From: van10hoven@gmail.com
Subject: Re: KOG

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Dear Antreas,

Thank you for posting.

Best regards,
Chris van Tienhoven
```

← Previous Next → ↔ Message Index ↑ Subjects

Message: #2295
Date: 2024-05-08
From: van10hoven@gmail.com
Subject: Constructions of a cubic given a conic and 5 points

```
Dear Eckart and Bernard,

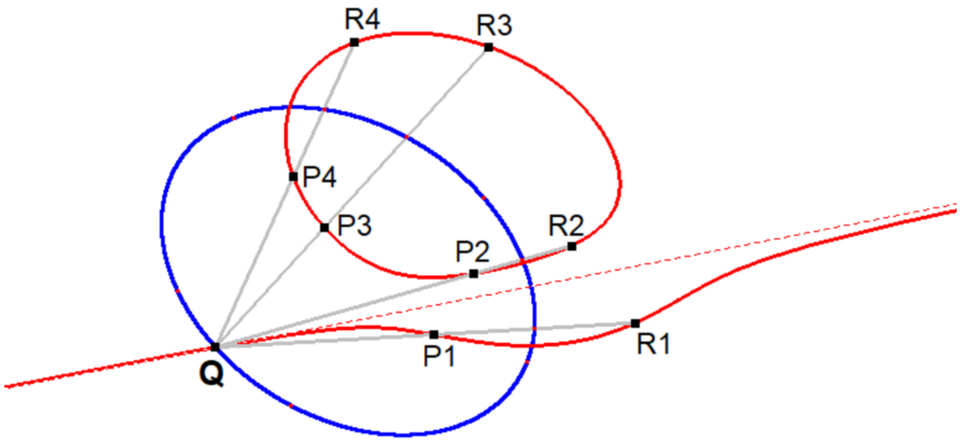
Attached a small paper about constructions of a cubic given a
conic and 5 points.

Best regards,
Chris
```

Several constructions of a cubic with a conic and 5 points

Example with an ellipse

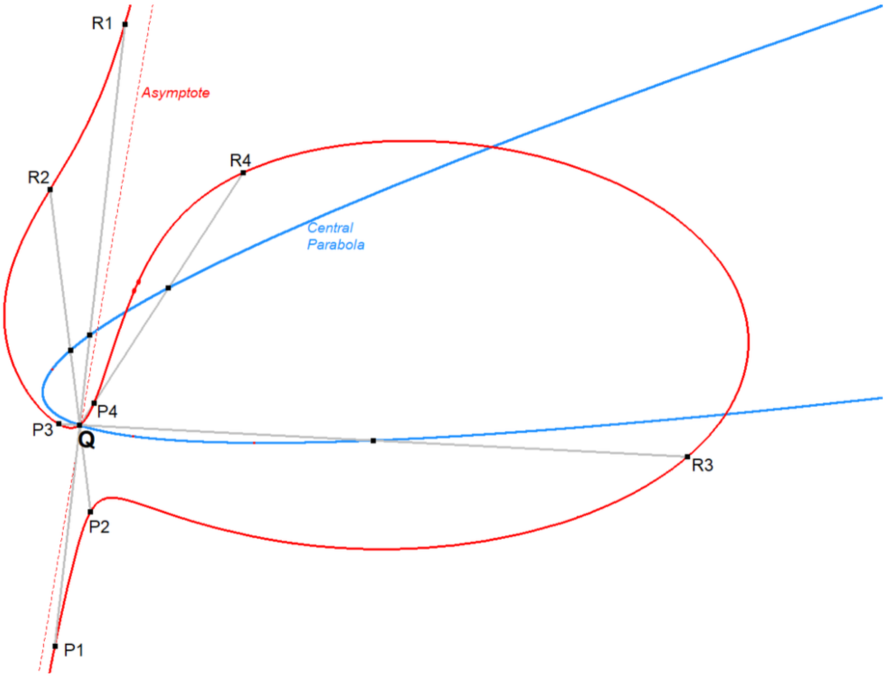
Given a conic and a point Q on the conic.
Given 4 other points P1, P2, P3, P4 not on the conic.
Construct 4 points R1, R2, R3, R4 as the "reflections" about the conic:
Ri is the point reflection of Pi about the intersection point of half-line QPi with the reference conic.
Then draw the unique cubic through Q, P1,P2,P3,P4, R1,R2,R3,R4.
In this case Q is be the intersection point of one of the asymptotes with the newly constructed cubic CU and therefore the reference conic will be the QF-conic of CU.



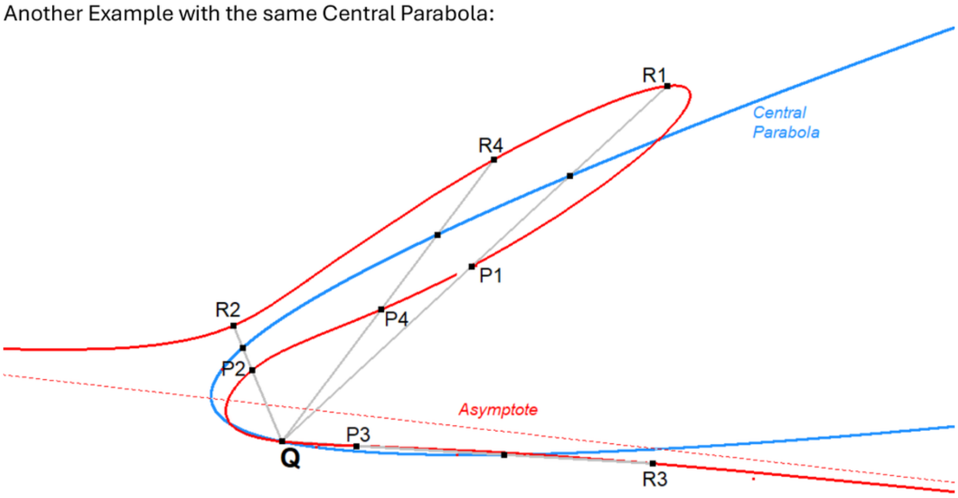
Construction of a cubic with a conic and 5 points-01.pdf

Examples with a parabola

When we start with a parabola for the construction we find a cubic with a parabola as QF-Conic. The asymptote of the cubic passes through Q.



Another Example with the same Central Parabola:

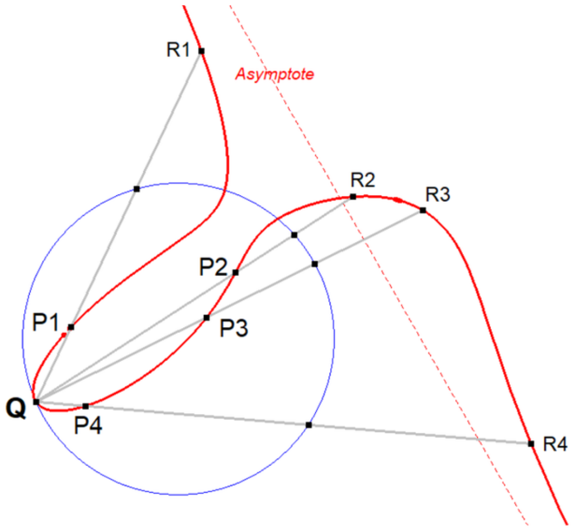


This time the asymptote does not pass through Q.

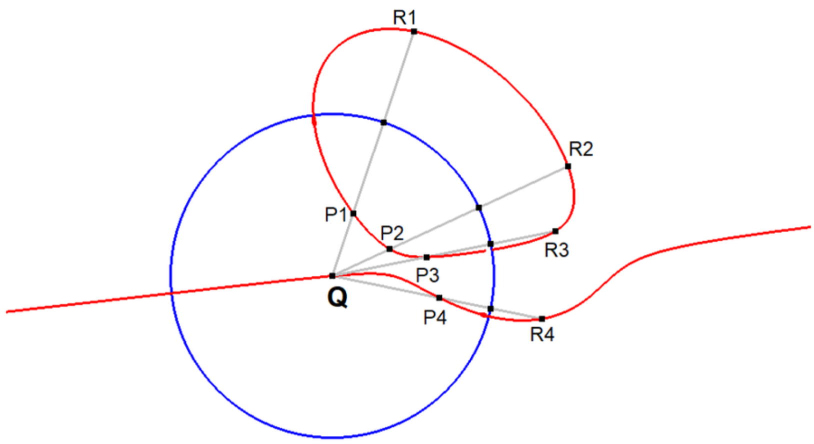
Construction of a cubic with a conic and 5 points-01.pdf

Examples with a circle

Given a circle and a point Q in its center.
Given 4 other points P1, P2, P3, P4 not on the circle.
We find a cubic with one asymptote not passing through Q.
So it seems that it is not common in this construction that the asymptote passes through Q.



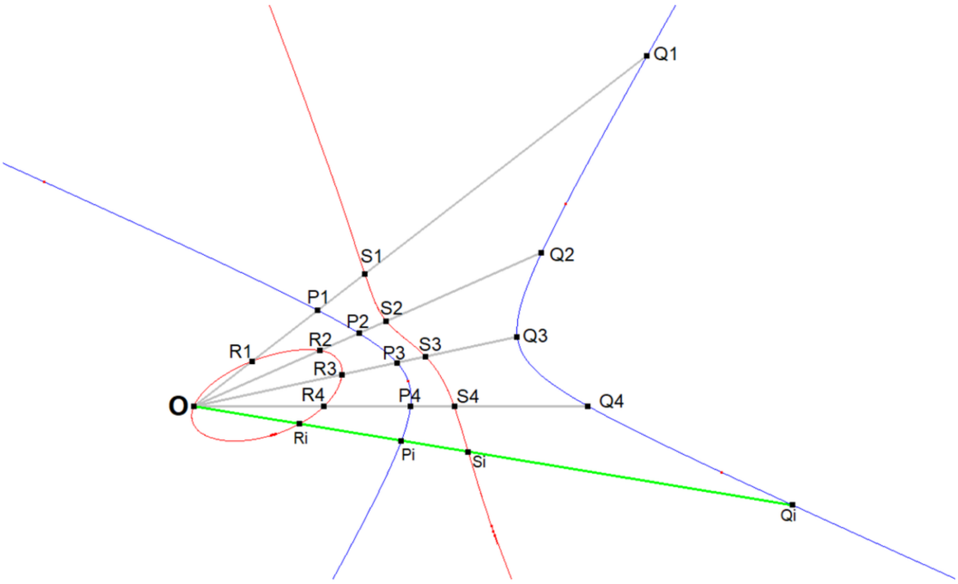
Given a circle and a point Q in its center.
Given 4 points P1, P2, P3, P4.
Construct 4 points R1, R2, R3, R4 as the "reflections" about the circle.
Ri is the point reflection of Pi about the intersection point of half-line QPi with the reference circle.
Then draw the unique cubic through Q, P1,P2,P3,P4, R1,R2,R3,R4.



Construction of a cubic with a conic and 5 points-01.pdf

Example with a hyperbola

Given a hyperbola and a point O not on the hyperbola.
Given 4 other points P1, P2, P3, P4 on the hyperbola.
Let Q1, Q2, Q3, Q4 be the 2nd intersection points of OP1, OP2, OP3, OP4 with the hyperbola.
Let R1, R2, R3, R4 be random points on OP1, OP2, OP3, OP4.
Construct point S1 = the harmonic conjugate of R1 wrt (P1,Q1).
Construct points S2, S3, S4 analogously.
Then draw the unique cubic through O, R1,R2,R3,R4, S1,S2,S3,S4.
Now every point Pi on the hyperbola will produce a 2nd intersection point Qi with line OPi.
And intersection points (Ri,Si) of OPi with the newly constructed cubic will be harmonic conjugated with (Pi,Qi).



So a hyperbola and a point O and 4 given points R1, R2, R3, R4 (all points not on the hyperbola) produce a unique cubic.

Construction of a cubic with a conic and 5 points-01.pdf

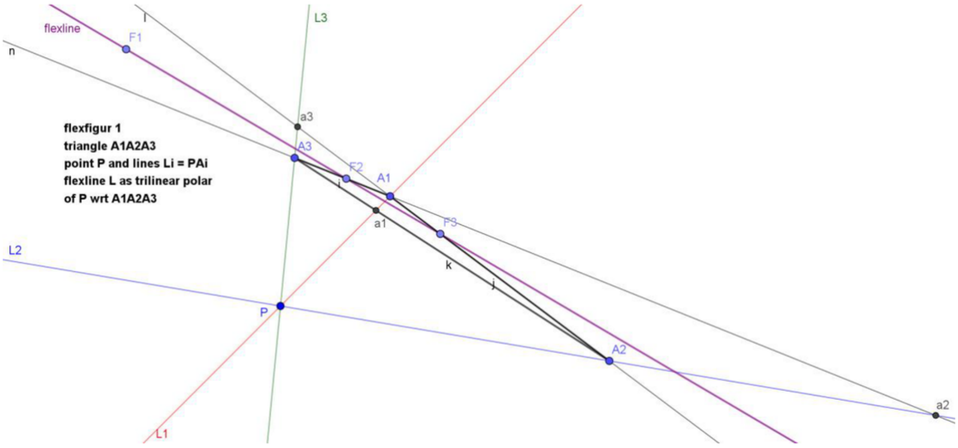
Message: #2296
Date: 2024-05-10
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

Dear Chris, dear Eckart
It took me time (as Geogebra is not Cabri), but I finally succeeded in drawing the cayleyan of a reference cubic!
This time I explained all my constructions on rather ugly figures, but summarized them in a last simple figure, which looks like Chris figure in message 2160.
Many thanks in advance for your comments, remarks or critics!
Best regards
Bernard

PS I don't remember we had already noticed that for any triangle like A1A2A3 (cf Eckart), Li is the trilinear polar of Fi and the flexline L the trilinear polar of P

Cubic, hessian and cayleyan

1) Wrt a triangle A1A2A3, the flexline L is the trilinear polar of a point P and the lines Li are the trilinear polars of the flexes Fi.



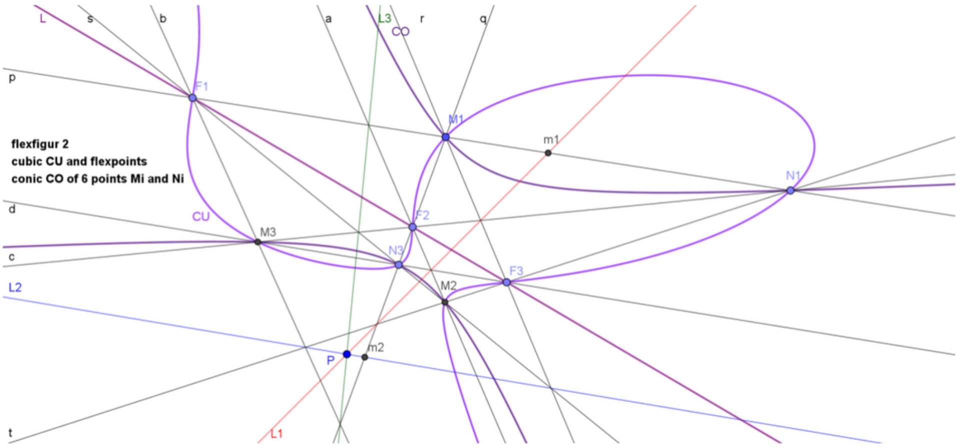
2) Starting with a point M1, F1M1 cuts L1 in m1 and F2M2 cuts L2 in m2.

Let N1 be the harmonic of M1 wrt F1 and m1 and N3 be the harmonic of M1 wrt F2 and m2.

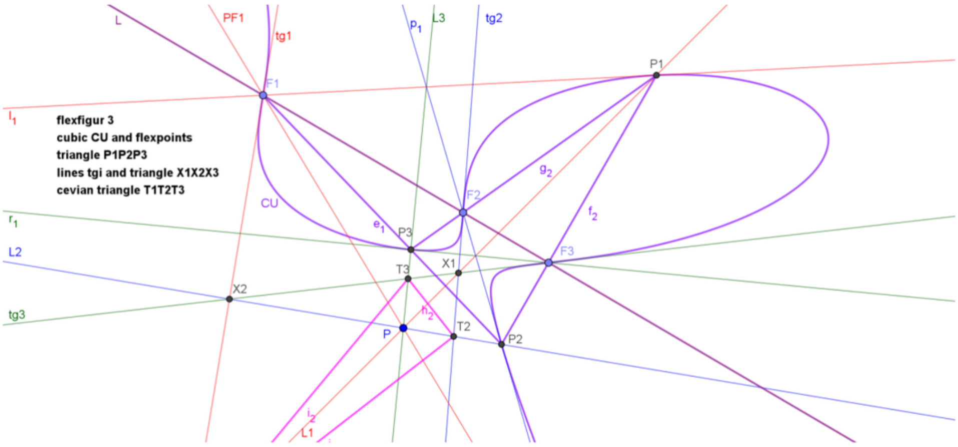
F1N3 and F3N1 intersect in M2 and F2N1 and F3N3 intersect in M3.

Last, F1M3, F2M2 and F3M1 intersect in N2.

The 6 points Mi and Ni are on a conic CO and determine with the 3 flexes Fi the reference cubic CU.



3) The tangents in Fi to the cubic give the triangle of the Pi, the flextangents tgi in Fi give the triangle of the Xi, the cevian of P wrt this triangle is the triangle of the Ti.

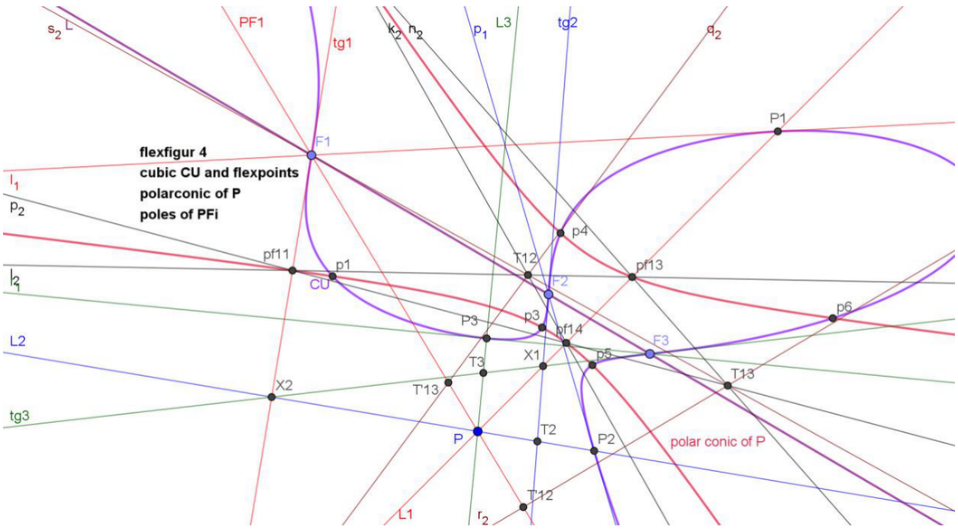


4) We have then the interesting lines PFi, Li and tgi, for which we will search the pôles.

The tangents from P to the cubic CU give 6 contact points pi and the polar conic of P wrt the cubic.

The polar conic of Fi is made of Li and tgi through Ti, which intersect the polar conic of P in 4 points.

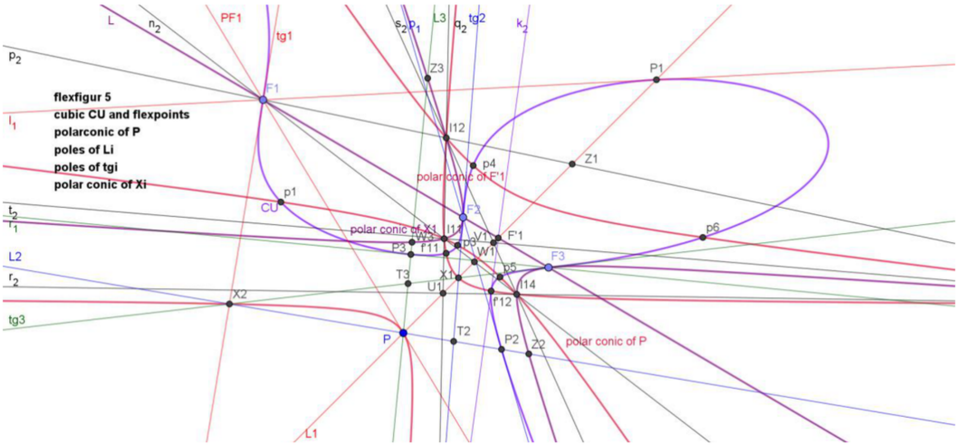
Hence the poles of PF1 are pf11, pf12, pf13 and pf14 with sides tangent to the cayleyan and DT T1T12T13 on the hessian, as well as the intersections of PF1 with the diagonals T1, T’12 and T’13.



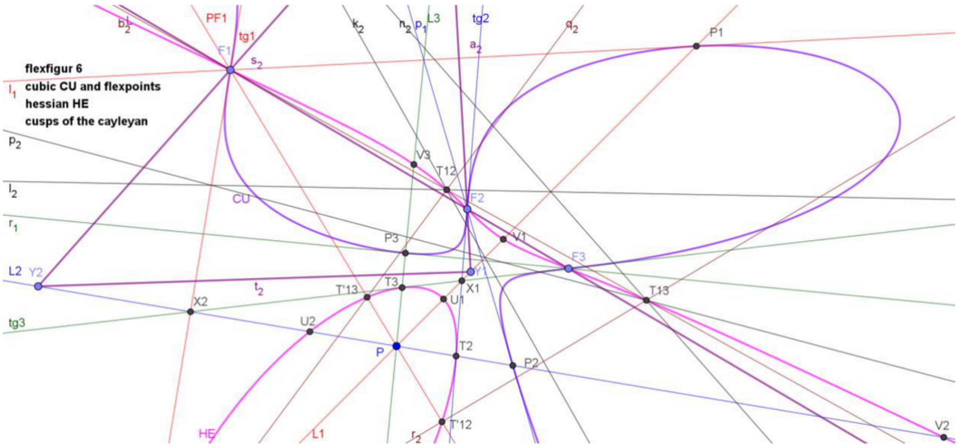
5) The same way, we find the poles of the Li as intersection between the polar conic of P and the polar conic of F'i, the harmonic of Fi wrt Fj and Fk, which passes already through P and the Xi.

Here the points l11, l12, l13 and l14, pôles of L1 ; 2 sides of this QA are F1W1 and F1Z1, tangents from F1 to the cayleyan and the DT of this QA is F1U1V1 (U1, V1, W1 and Z1 on L1 with W1 and Z1 harmonic wrt U1 and V1).

The 2 lines F1W1 and F1Z1 form the polar conic of T1 wrt the cubic CU. The pôles of tg1 are therefore the intersection with the polar conic of F1 and are F1 as dobble point and W1 and Z1. We may notice that the polar conic of X1 passes through l11, l12, l13 and l14 as well as F2 and F3, W2 and W3 and Z2 and Z3.



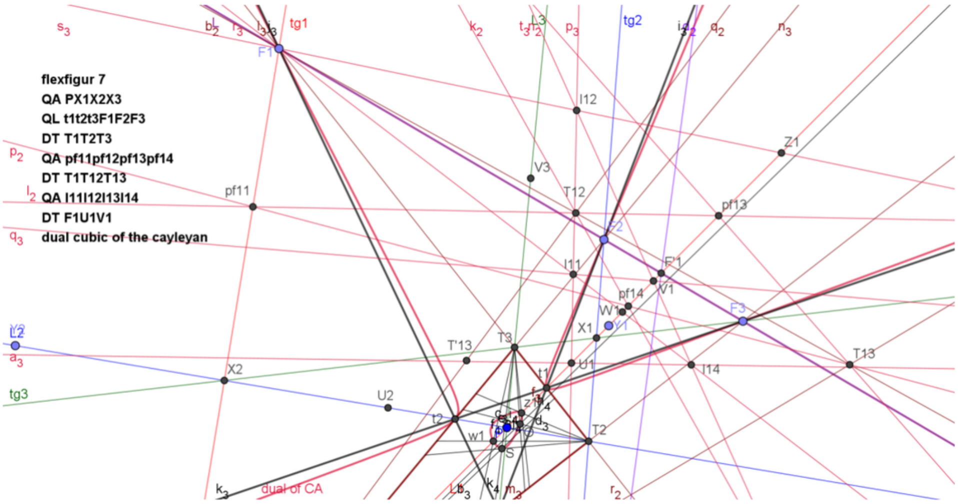
6) We had after figure 4 already enough elements in order to draw the hessian HE through the Fi, the Ti, T12 and T13, T'12 and T'13.



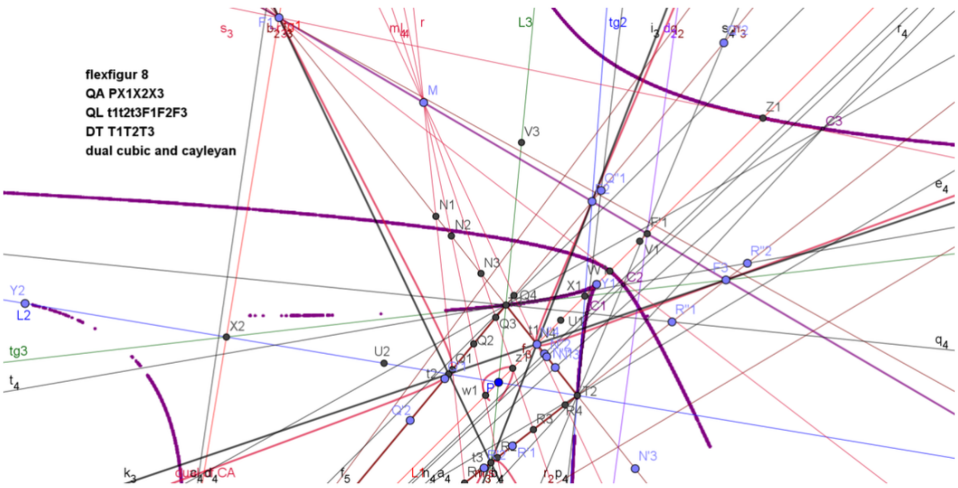
What now about the cayleyan ? We have already 12 points, the Ti, the Wi, the Yi and the Zi. We have also 12 tangents the lines Li, FiTi, FiWi and FiZi.

As the cayleyan is a curve of 3rd class, we have to use a duality which will transform this curve in a cubic : either a QA/QL duality or a simple pole/polar duality wrt a conic.

7) Using the duality wrt the QA PX1X2X3/QL F1F2F3t1t2t3 with DT T1T2T3, where the ti are the vertices of the cevian triangle of P wrt the triangle of the Ti, we find this 1rst cubic circumscribed to the QL, as the cayleyan is inscribed in the QA.



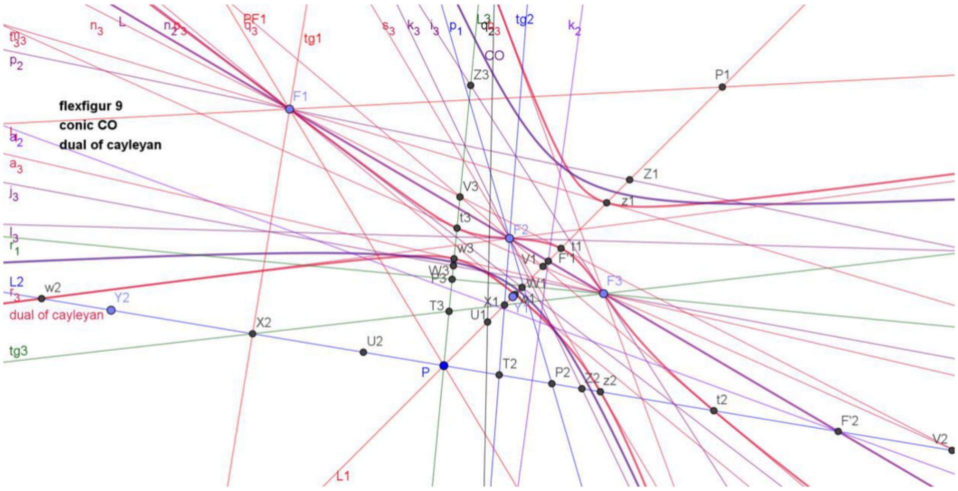
We find the cayleyan as the dual of this cubic as locus of the dual points of the tangents to the cubic.(Here, the tangents are drawn from a variable point M of the flexline)



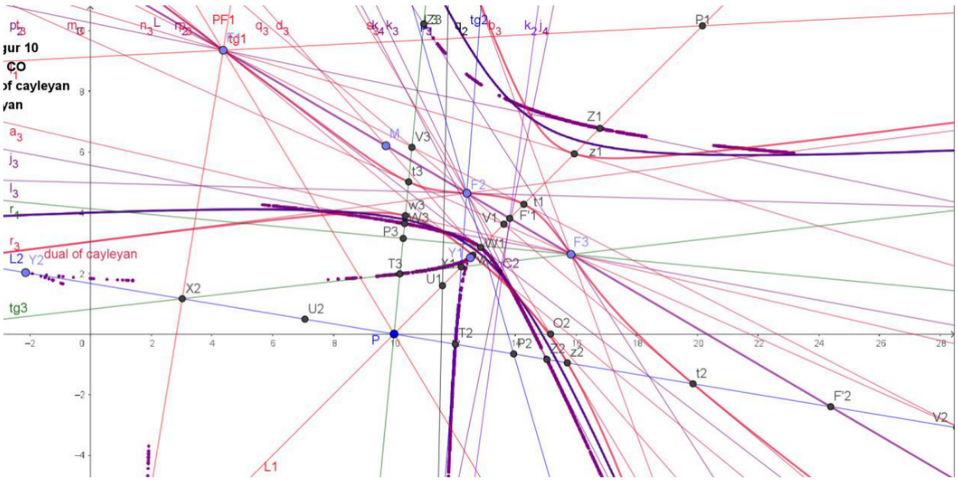
Cubic, hessian and cayleyan.pdf

9) We may use the duality pole/polar conic wrt any conic.

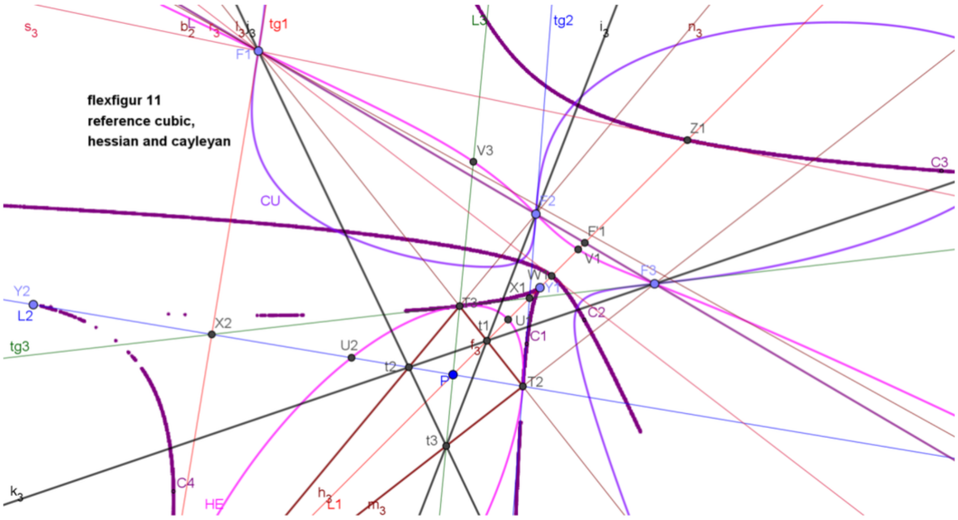
But it seems convenient to use a conic like CO in the 2nd figure, as the flexline is the polar of P and the flexes Fi the pôles of Li ; here, the ti, wi and zi are the pôles of the lines FiTi, FiWi and FiZi.



Again, the cayleyan is the dual of the cubic in the same duality. (Here, the same way, the tangents are drawn from a variable point M on the flexline).



Now we may have on the same figure the reference cubic CU, the hessian HE and the cayleyan CA.



Looking at the figure on my screen, I can confirm your statement that Eckart’s conjecture about the intersections of the reference cubic and it’s cayleyan lying on the sides of the triangle t1t2t3 (cevian triangle of P wrt T1T2T3) doesn’t hold.

Message: #2297
Date: 2024-05-13
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

```
Dear Bernard,

... back from holiday I had a first look in your paper,
... but I was surprised of the last sentence.

I think your drawing can also be interpreted
... as a confirmation of my conjecture (see #2253 of Chris),
... it shows no significant deviation.

Best regards Eckart
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Message: #2298
Date: 2024-05-13
From: bernard.keizer@gmail.com
Subject: Cohessians, prehessians and cocayleyans

```
Dear Chris, dear Eckart,
After my memo about cubic, hessian and cayleyan, I've tried to
gather my ideas about cohessians, prehessians and cocayleyans.
All these cubics belong to the well-known syzegetic pencil of
cubics having the same 9 flexes (3 real and 6 imaginary).
I largely used the QA-Tf2 transformation mentionned by Eckart;
this transformation is the same as the QMT we already studied.
Best regards
Bernard
```

Cohessians, prehessians and cocayleyans

1) Let's start with a cubic CU, it's hessian HE and it's cayleyan CA.
   If CU is monopartite, HE will be bipartite and vice-versa (Eckart).
   The flexpoints are Fi, the harmonic lines Li, intersecting in P, the flexline L, the flextangents tgi, which are tangent in Fi to CU and in Ti to HE and CA.
2) For any triangle A1A2A3 such as FiAjAk are aligned, L is the trilinear polar of F and Fi the trilinear polar of Li.
3) Li intersects CU bipartite in Pi, Qi and Ri (monopartite only in Pi), contact points of the tangents from Fi, HE bipartite in Ti, Ui and Vi (monopartite only in Ti) and CA in Ti, Wi and Zi (contact points of the tangents from Fi) and Yi, cusp of CA and harmonic of Ti wrt Ui and Vi. Wi and Zi are harmonic wrt Ui and Vi.
4) Each cubic is invariant in 1 (monopartite) or 3 (bipartite) transformations swapping to a point of the curve the point(s) having the same tangential :
   It is the QMT (quasi-Moebius transformation) already studied.
   It is also a QA-Tf2 transformation wrt the triangle of contact points of the tangents from Fi to the cubic with fixed points P and the vertices of the anticevian triangle of P wrt this triangle.
5) As immediate application, CU is invariant in 1 (monopartite) or 3 (bipartite) transformations wrt the triangles of the Pi, the Qi or the Ri.
   HE is invariant in 1 (monopartite) or 3 (bipartite) transformations wrt the triangles of the Ti, the Ui or the Vi.
6) The 2 cohessians (hessian bipartite) have cayleyans tangent to HE in Ui or Vi and cusps in the harmonics of Ui wrt Ti and Vi or Vi wrt Ti and Ui.
   If the hessian is monopartite, there are no cohessians, CU (bipartite) is the only cubic having HE as hessian.
7) The 2 cocayleyans have hessians tangent to CA in Wi and Zi, the flextangents are FiWi and FiZi. It's interesting to notice that the for X and X' conjugates on the 1rst hessian HE, the poles of XX' wrt CU are Y and Y' describing the 2nd hessian and Z and Z' describing the 3rd hessian. XX'YY'ZZ' form a QL and the 4 lines tangent to the cayleyan are the same for the 3 hessians, but in a different order.
8) The 3 prehessians having CU (bipartite) as hessian have cayleyans tangent to CU in Pi, Qi or Ri. If CU is monopartite, it has only but one prehessian.

Cohessians, prehessians and cocayleyans.pdf

Message: #2299
Date: 2024-05-15
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart and Bernard,

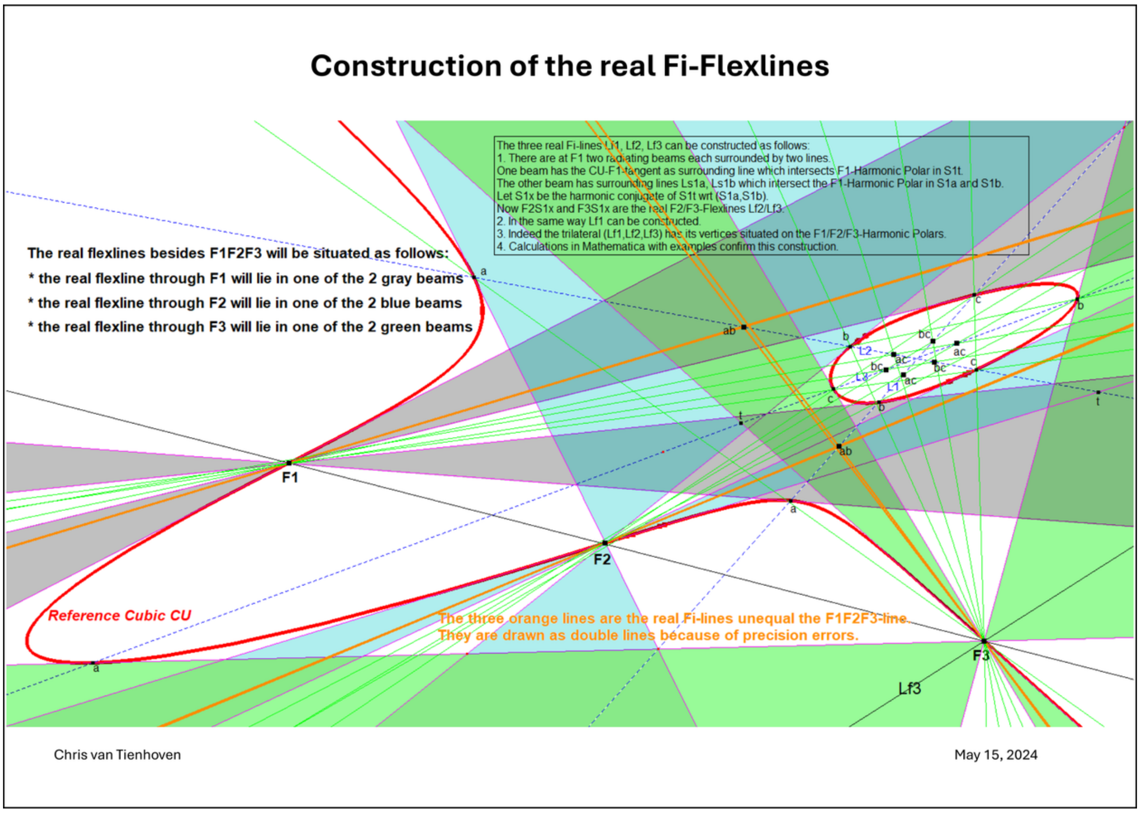
I found a construction of the real Flexlines through F1, F2, F3 other than F1F2F3.
I checked it algebraically in Mathematica. Without Mathematica I wouldn't have been able to find the result.
The key aspect is that the Fi-Flexlines will be positioned on one of two beams originating from Fi (where i=1,2,3). Therefore the bounding lines of these beams are of importance and used in the construction.

Construction:
1. Let CU be the Reference Cubic.
Let F1, F2, F3 be the three real Flexpoints.
Let L1, L2, L3 be the Harmonic Polars of F1, F2, F3 wrt CU.
2. Let S1t be the intersection point of L1 with the CU-F1-tangent.
3. Let S1a, S1b, S1c be the three intersection points of L1 with CU.
4. The lines F1S1a, H1S1b, F1S1c and F1S1t form the bounding lines of the two beams in which the real F1-Flexline can occur.
One beam has bounding line F1S1t and let us arrange points in the set (S1a, S1b, S1c) such that F1S1c will be the other bounding line of this beam. So (S1a,S1b) can be used as bounding lines F1S1a, F1S1b of the 2nd F1-beam.
5. Let H1c be the Harmonic Conjugate of S1t wrt (S1a,S1b).
Let H1b be the Harmonic Conjugate of S1t wrt (S1c,S1a).
Let H1a be the Harmonic Conjugate of S1t wrt (S1b,S1c).
Now we have 3 harmonic points H1a, H1b, H1c.
Look for which point H1x (x=a, b or c) the lines F2H1x and F3H1x are lines within the designated F2- and F3-beams. In my pictures it was H1c all the time. So H1x=H1c, but it may be possibly H1x=H1a or H1b. Whatever the outcome is denote U1=H1x.
6. Now F2U1 & F3U1 are the real flexlines from F2 and F3 other than F1F2F3.
7. In an identical way F1U2 & F3U2 as well as F1U3 & F2U3 can be obtained.

It will appear that:
F1U2 = F1U3 = real flexline from F1 other than F1F2F3.
F2U1 = F2U3 = real flexline from F2 other than F1F2F3.
F3U1 = F3U2 = real flexline from F3 other than F1F2F3.

Note:
The above construction only works when the 3 points of the set (S1a, S1b, S1c) in step 3 are real points. I have not yet devised a construction for cases where two of these points are imaginary.

Best Regards,
Chris



CU-12L1 Real Flexlines-12-REAL Fi-Lines.pdf

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Message: #2300
Date: 2024-05-16
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

```
Dear Chris,

have you read my message #2288
(next to last passage and drawing),
... where I described a "construction" of the real flexlines,
... which seems nearly to be the same as yours?

Best regards Eckart
```

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Message: #2301
Date: 2024-05-16
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Chris, dear Eckart,
I prefer to answer your 2 last messages in this new item
1) because the other is completely full!
2) because part of the answer is in this syzegetic pencil of
cubics, mentionned in this item
All the cubics of the pencil have the same 9 flexes (real and
imaginary), in particular the curve and it's hessian.
So, if the cubic is monopartite, the hessian is bipartite and we
could apply Chris construction ...
Dear Chris, just have a look on your figure 2160 or on my
flexfigure 6 in my memo.
It appears that your 3 real flexlines must be in the triangles
P2F2T2 and P3F3T3 and outside the triangle P1F1T1.
I don't think it works with your construction applied to the
hessian HE.
But the triangle Y1Y2Y3 of the cusps of the cayleyan (Yi
harmonic of Ti wrt Ui and Vi) works perfectly (as already
suggested in my message 2289 in answer to Eckart's message 2288)
...
I have no explanation and no proof and it remains only a
suggestion! But, who knows?
Best regards
Bernard
```

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Message: #2302
Date: 2024-05-16
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

Dear Chris, dear Eckart,
Thanks to this reflexion, I realise I made a mystake in my message 2298: the cubic, it's hessian as well as the cohessians and prehessians and hessian of the hessian etc belong to the syzegetic pencil of cubics having the same 9 flexes (real and imaginary), but not the cocayleyans and their hessians!
Best regards
Bernard

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Message: #2303
Date: 2024-05-16
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

About your question in #2300.
I read your message #2288 globally, but at the time I did not find a clear solution in it for the real flexlines. You proposed three lines, but as far as I can see they are not the right lines.

So I went on with the idea of "beams" that I already mentioned to you in #2257 and some extra ideas that were lingering on for a long time in my mind. I don't think I used special ideas from you. If so I would have mentioned it and given you the credits for sure (like I always do).

With your special talent I think you would have found some solution when you would have had the opportunity to check solutions in Mathematica like I did. That makes me humble.

Best regards,
Chris

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Message: #2304
Date: 2024-05-16
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris,

can you give me your drawing in #2299 with the hessian of your ref-cubic?
... and is my starting condition correct?
..."These 3 real flexlines have no further intersections
... with the cubic and its hessian,..."

Thanks in advance.
Best regards Eckart

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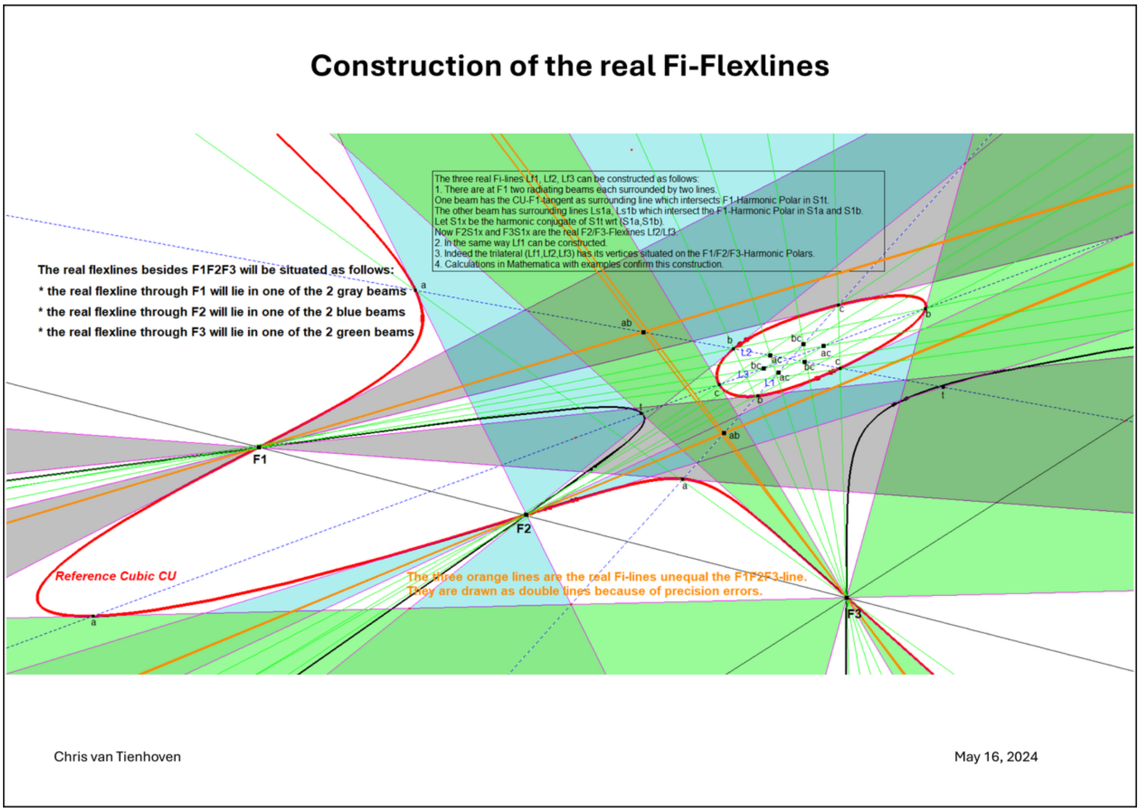
Message: #2305
Date: 2024-05-16
From: van10hoven@gmail.com
Subject: Re: Hessian and cayleyan

Dear Eckart,

Attached you will find the drawing of #2299 including the hessian of the ref-cubic.

Since the Hessian has the same Flexpoints as the Reference Cubic, for the three real flexlines there can't be any further intersection points with ref-cubic or Hessian than the imaginary flexpoints and consequently in any real picture no other real intersection point can't be seen.

Best regards,
Chris



CU-12L1 Real Flexlines-13-REAL Fi-Lines.pdf

Message: #2306
Date: 2024-05-16
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

Dear Chris,
Using again my flexfigur 6 in my memo (which looks like your 2160, I drew the tangents to HE in Fi, which intersect the lines Li in 3 points li (with Filjlk aligned).
Then I searched the harmonics of li wrt Ti and Vi and I found the points Hi (with FiHjHk aligned).
The sides of H1H2H3 are in the authorized beams (they have no further intersection with the cubic CU and it's hessian HE) and must be your searched flexlines.
I believe you if you say, it is confirmed by Mathematica.
My congratulations, the construction is amazing!
Best regards
Bernard

PS It is correct to use your construction directly if the cubic is bipartite and to use it with the bipartite hessian if the cubic is monopartite

---

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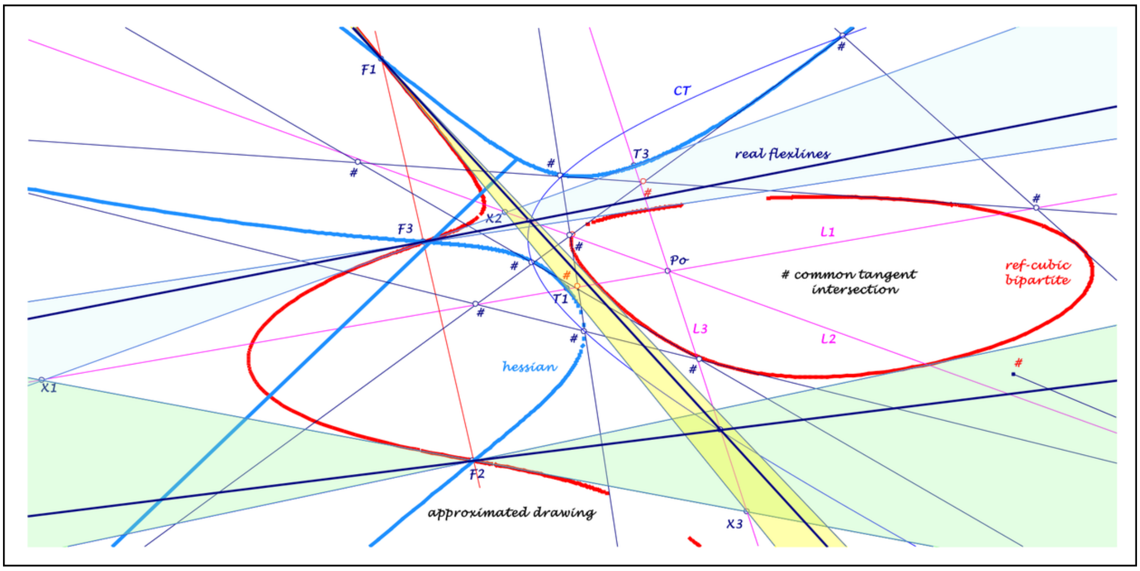
Message: #2307
Date: 2024-05-17
From: eckart_schmidt@t-online.de
Subject: Re: Hessian and cayleyan

Dear Chris,

you are right, my construction for the real flexlines
... is not yours and therefore wrong,
... perhaps it describes the cusps of the caylean
... as Bernard assumes,
...excuse my overhasty statement.
Some remarks wrt your construction:
(1) The bounding lines of the beams
... will be easier described, using also the hessian,
... see my drawing.
(2) In two drawings I got the assumption,
... that the intersections of the conic CT (see #2288)
... with the lines Li in the allowed regions
... are the vertices of the real flexline triangle.

Best regards Eckart

PS: Thanks for the added hessian in the drawing.



2024-05-17.pdf

Message: #2308
Date: 2024-05-17
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Chris,
I hope I understood and applied correctly your construction.
I found this morning a complementory property and I send it to
you with a figur.
Do you agree with all this?
Best regards
Bernard
```

Flexfigur 6bis with 3 real flexlines

1) **Construction 1**
   (Chris instructions applied to HE yesterday evening)

Flextangents to HE in Fi intersect Li in Ii

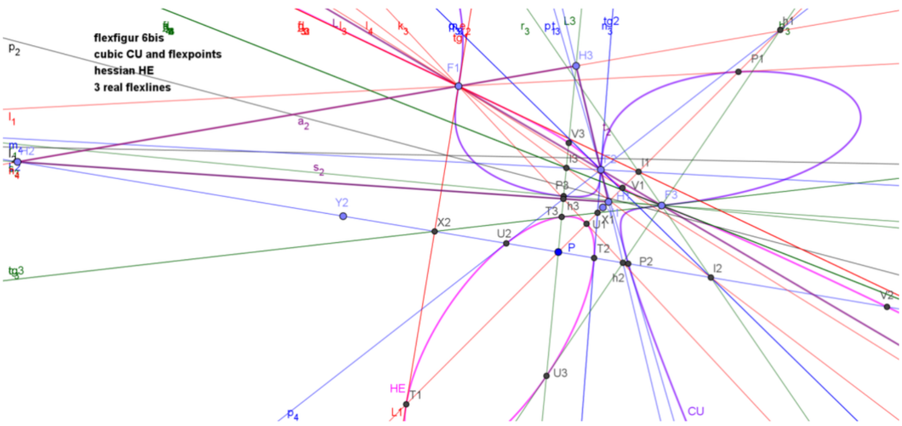
Harmonics of Ii wrt Ti and Vi give Hi

Lines HjHk are the 3 real flexlines (other than L)

2) **Construction 2**
   (found this morning)

Lines FiUi intersect in 3 points hi on Li

The triangle of the Hi (real flexlines) is the anticevian triangle of P wrt the triangle of the hi



3 real flexlines.pdf

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Message: #2309
Date: 2024-05-17
From: van10hoven@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

Dear Bernard,

I tried again and again to understand your notifications in last messages.
Your pictures are very elaborate and hard to have access to.
I wonder if Geogebra does not have the possibility to hide less important intermediate results.
Then you just drop some sentences without clarifying what means what.
Of course you have explained them in earlier messages, but which ones.
Can't you just give an overview how things fit together, how you constructed in which order, with which notifications?
I don't have the overview.
I give up for now.

Best regards,
Chris

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Message: #2310
Date: 2024-05-18
From: eckart_schmidt@t-online.de
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Bernard,
I can only repeat Chris' message,
... the lines FiUi in your drawing intersect Li in Ui,
... which are not your points hi ...
Best regards Eckart
```

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Message: #2311
Date: 2024-05-18
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Eckart,
Let's start with the beginning: I didn't say: Lines Li intersect
Li, but intersect on Li.
Perhaps not very clear, but it means that FjUj and FkUk
intersect in hi on Li.
For example F2U2 and F3U3 intersect in h1 on L1.
In other words, h1h2h3 is the anticevian of P wrt U1U2U3 and
H1H2H3 is the anticevian of the anticevian of U1U2U3.
I respectuously asked and I'm still asking if this property was
correct!
(Surprisingly, this doesn't work apparently for the other
harmonics of li wrt Ti and Ui or Ui and Vi).
Thanks for your interest and attention
Best regards
Bernard
```

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Message: #2312
Date: 2024-05-18
From: eckart_schmidt@t-online.de
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Bernard,

thanks for your clearence,
... but what are your points Ui and Vi,
... what is their difference?
Are they contact points of tangents from Fi to HE?

Best regards Eckart
```

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Message: #2313
Date: 2024-05-18
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

Dear Chris,
I found your message perfectly unjustified! You ask at the same time for more explanations and more simple figures ...
I've precisely tried, in order to answer your questions, in my last memo (message 2296) to explain step by step all my constructions and conclusions and to give at the end a figur without these intermediate constructions.
I got no comments, no remarks and no critics.
But as you said once, you're not my teacher and I'm not your pupil ...
For your problem, frankly, it's not so complicate!
For a bipartite CU, let's name, since I do since the beginning, Pi, Qi and Ri the intersections of Li with the curve and Ti the contact point of HE and CA, intersection of Li and the flextangent in Fi.
1) Then, if I understand correctly, you consider the 3 possible harmonics of Ti wrt 2 of the points P,Q,R i, which gives 3 triangles and choose the one which suits in the authorized beams.
Is my interpretation of your construction correct?
2) It seems that the choosen triangle H1H2H3 is the anticevian of the anticevian of the triangle of the remaining point P, Q or R i
Is this property correct?
3) If the cubic is monopartite, the hessian is bipartite and the notations become li instead of Ti and Ti, Ui and Vi instead of Pi, Qi and Ri.
The construction is the same ...
Best regards
Bernard
PS I hope our cooperation will go on, but efforts have to come from both sides ...

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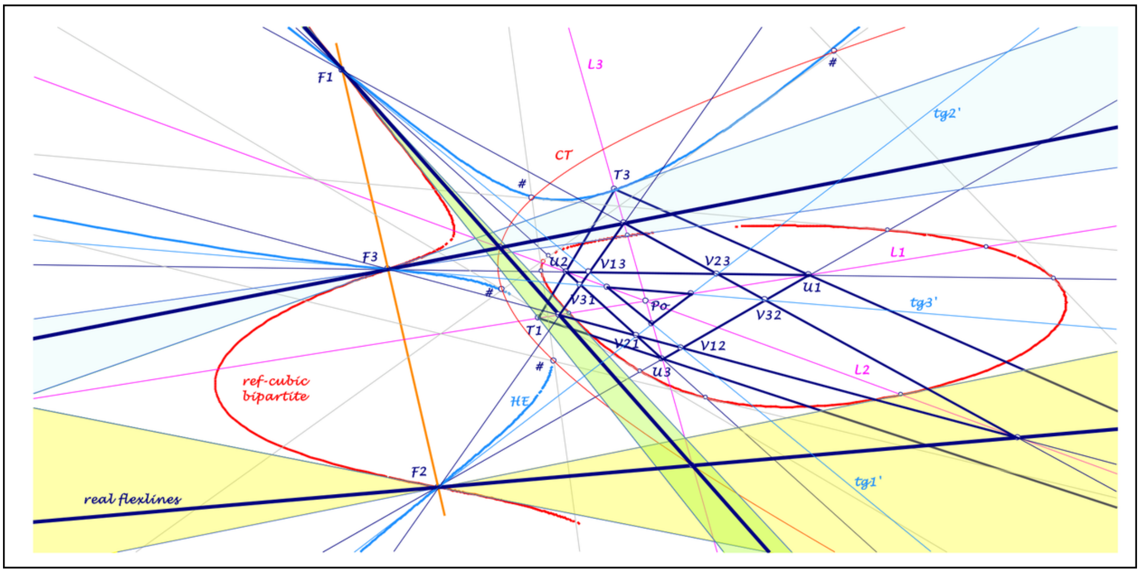
Message: #2314
Date: 2024-05-18
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard, dear Chris,

perhaps construction 3 and 4
...for the real flexlines of a bipartite cubic:
Consider the common tangents of cubic and hessian,
... special their 6 intersections # not on Li (see attached),
... which seem coconic HE-points on a conic CT (see #2288),
... CT intersects in the allowed regions the lines Li
... in the intersections Hi of the real flexlines.
Or:
Let U1U2U3 be the cevian triangle of Po wrt T1T2T3,
... let the real flextangents of the hessian tgi'
    intersect UiUj in Vik,
... Vij, Vji, Fk are collinear,
... VijVji ^ VikVki on Li is a point of the real flexline
    through Fi.

Perhaps someone can confirm my observations,
... only considered for one example, see attached.

Best regards Eckart
```



2024-05-18.pdf

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Message: #2315
Date: 2024-05-18
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

Dear Eckart,
Ti, Ui and Vi are indeed the contact points of the tangents from Fi to HE and the intersections of Li with HE.
I used Ti as contact point of the flextangent in Fi to CU, Ui as the 2nd point on the closed part of HE and Vi as the point on the infinity part.
Best regards
Bernard

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Message: #2316
Date: 2024-05-18
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

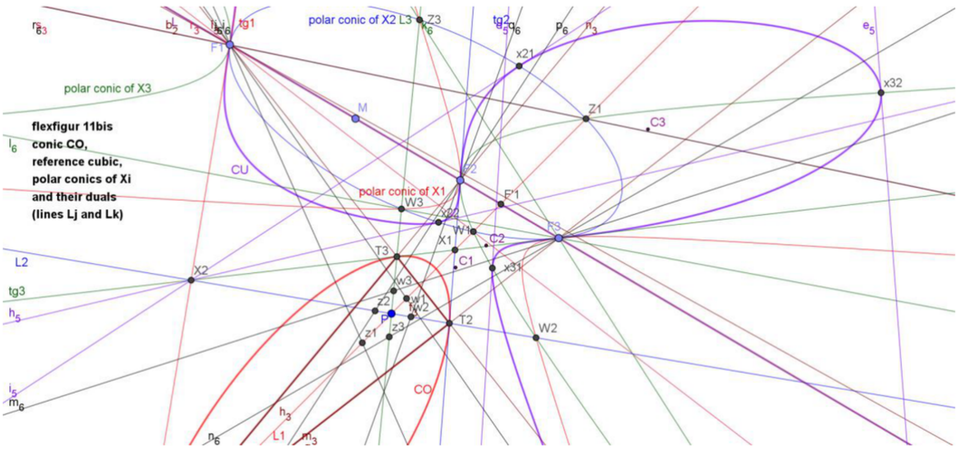
Dear Eckart,
I realise I didn't react to your last comment. I hoped it wouldn't be the only comment to this memo ...
I would desperately have been happy to confirm your conjecture, but as I told, when looking at the figure on my screen, the deviation is real.
Perhaps my whole construction is wrong, but I doubt.
Meanwhile, I hoped you would draw yourself a cayleyan and get a certitude.
I made a last construction with a simple duality pole/polar wrt a new conic.
I choosed this time the conic inscribed in X1X2X3 and tangent to the sides in T1, T2 and T3.
We need in fact a limited number of tangents to the cayleyan ...
Here is the result of this new attempt, I hope it will give you ideas ...
Best regards
Bernard

A new dual of the cayleyan

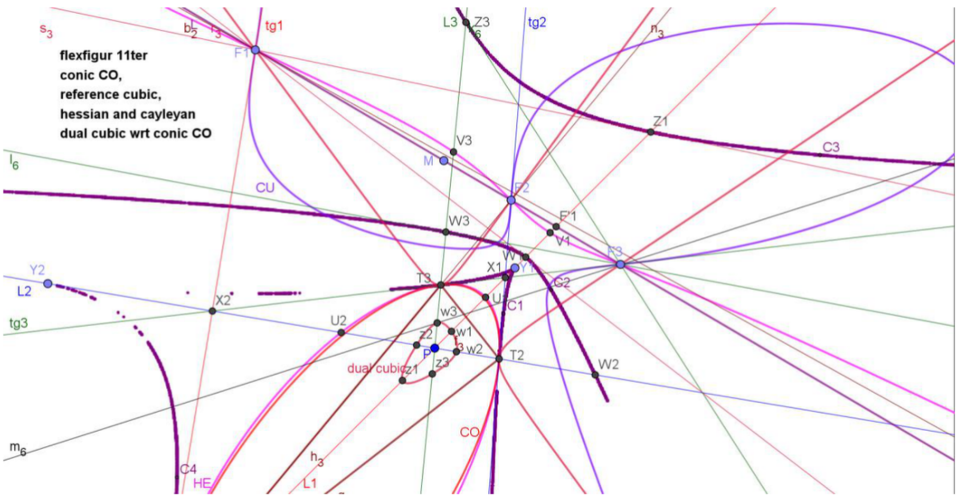
Let's consider the conic CO inscribed in X1X2X3 and tangent to the sides in T1, T2 and T3 and the duality pole/polar wrt this conic.

Let's draw the polar conics of X1, X2 and X3 wrt CU. They intersect pairwise in the points Fi and the points Wi and Zi and are tangent in the Fi to the flextangents tgi to the cubic in Fi.

The dual conics of these 3 conics are degenerated conics formed by 2 lines Li.



Knowing the 3 lines Li and the 9 tangents FiTi, FiWi and FiZi to the cayleyan, it's not difficult to draw the cubic through the Fi (pôles of li) and the points Ti, wi and zi (pôles of FiTi, FiWi and FiZi). The cayleyan can be draw as the locus of the poles of the tangents to the cubic wrt CO.



A new dual cubic and the cayleyan.pdf

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Message: #2317
Date: 2024-05-19
From: eckart_schmidt@t-online.de
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Bernard,

thanks for clearance wrt Ui in #2315:
... Ui = intersection beside Ti of Li
         and the closed part of the hessian,
... now I can reproduce your construction 2,
... which give the real flexlines for a monopartite cubic,
... for a bipartite cubic with monopartite hessian
    has no closed part.

Best regards Eckart
```

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Message: #2318
Date: 2024-05-19
From: bernard.keizer@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Eckart,
A million thanks for this confirmation of the construction 2!
Naturally, as a curve and it's hessian have the same 9 flexes,
the construction goes for both monopartite (applied to the
hessian) and bipartite (applied to the curve) ...
Best regards
Bernard
```

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Message: #2319
Date: 2024-05-19
From: van10hoven@gmail.com
Subject: Re: Cohessians, prehessians and cocayleyans

```
Dear Bernard,

Thanks for your further explanation in #2313!
Wrt your point 1) I can agree. My points S1t, S2t, S3t are T1,
T2, T3 indeed.
Wrt your point 2) I cannot confirm this in my drawing.

Best regards,
Chris
```

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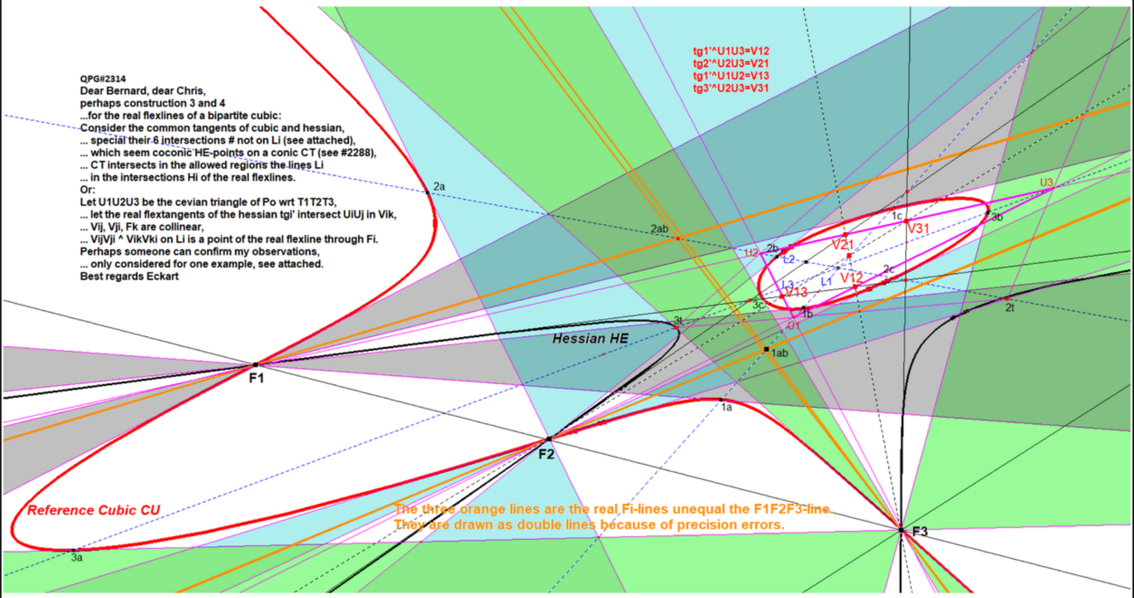
Message: #2320
Date: 2024-05-19
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,

I checked your second construction.
I drew
   tg1'^U1U3=V12
   tg2'^U2U3=V21
   tg1'^U1U2=V13
   tg3'^U2U3=V31
Lines V12V21 and V13V31 intersect in L1 indeed.
But not in a point of the (orange) real flexlines.
So I cannot confirm your statement.
See attached figure.
I hope I followed your description well.

Best regards,
Chris
```



Lines V12V21 and V13V31 intersect in L1 indeed. But not in a point of the (orange) real flexlines.

Chris van Tienhoven May 19, 2024

CU-12L1 Real Flexlines-17-REAL Fi-Lines.pdf

Message: #2321
Date: 2024-05-19
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

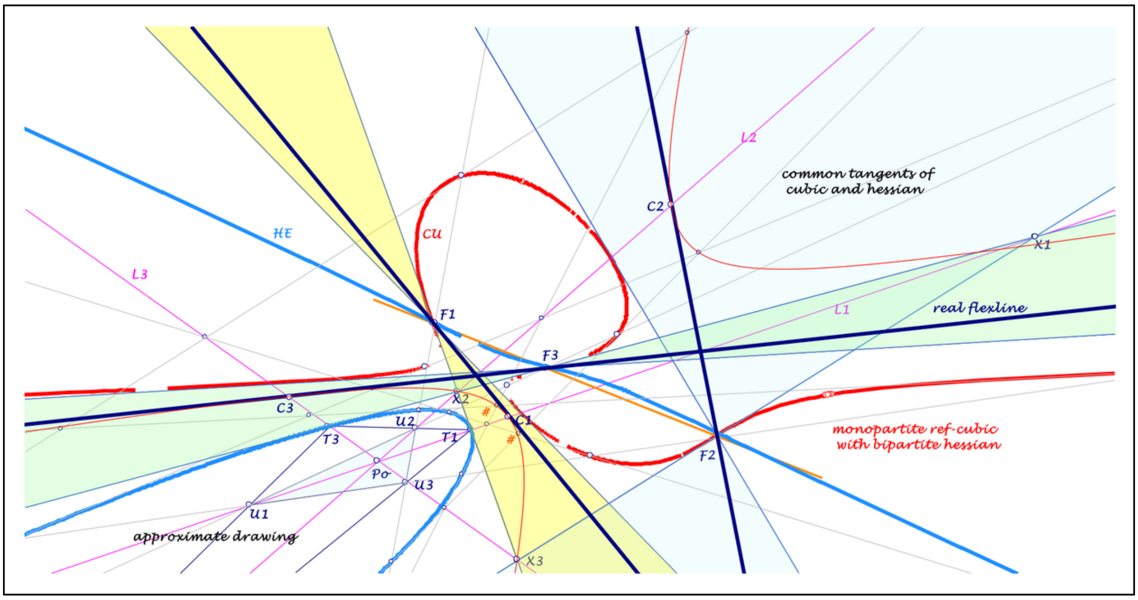
Dear Chris,

thanks for #2320 with a prove of construction 4,
... which seems not to hold,
    so let us forget this curious construction.
I hope, that construction 3 holds for bipartite cubics,
... here in addition this construction for monopartite cubics:

Consider the common tangents of cubic and hessian,
... special their 6 intersections # not on Li (see attached),
... which seem coconic points on a conic CT,
... which is a circumconic of X1X2X3
... (Xi intersection of CU-flextangents in Fj and Fk),
... image of the flexline F1F2F3
... ... for an isoconjugation wrt X1X2X3 with fixed point Po.
CT intersects in the allowed regions the line Li
... in a point Ci of the real flexline through Fi.

The attached drawing is very inexact,
... so the result very insecure, excuse,
... drawings often need hours.
Thanks in advance for prove.

Best regards Eckart



2024-05-19.pdf

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Message: #2322
Date: 2024-05-20
From: van10hoven@gmail.com
Subject: The unique Polar Circle of a cubic

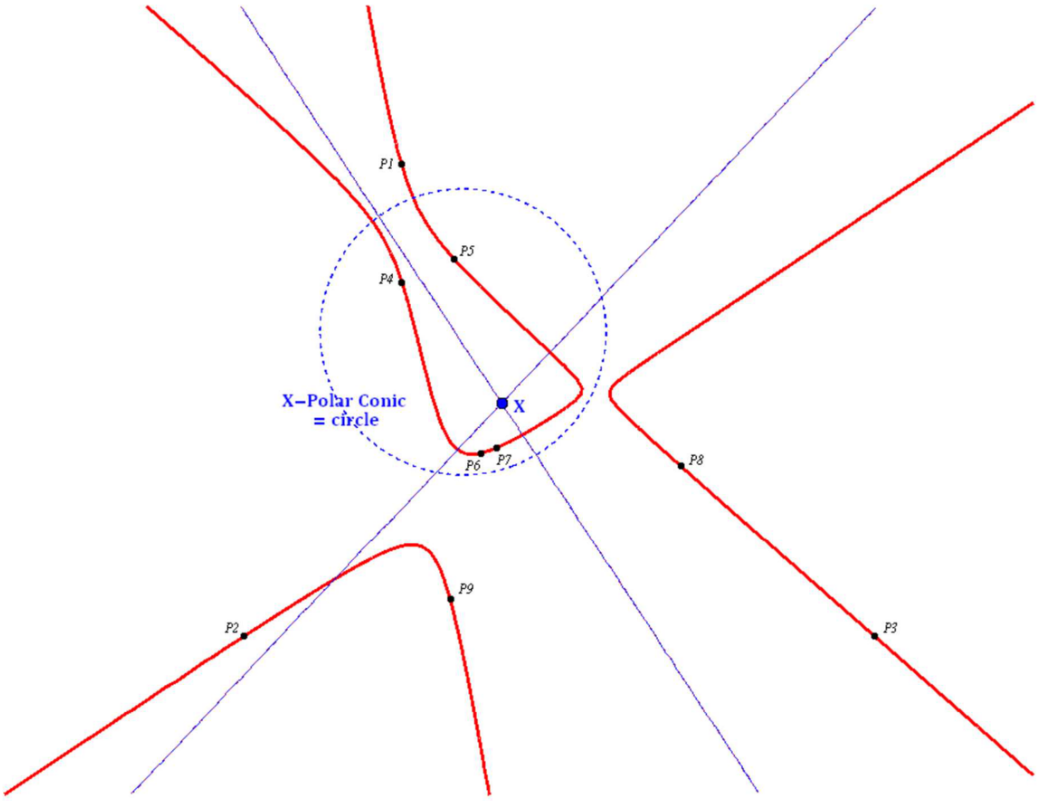
Dear Eckart and Bernard,

I wondered if there is a point X for which the X-Polar Conic is a Circle.
Maybe there are even more of these points.
I worked it out algebraically and came to a simple conclusion:
Since The Polar Conic at X(p,q,r) algebraically is derived from:
    {D[Cubic, x], D[Cubic, y], D[Cubic, z]} . (p,q,r)
we have a linear equation in (p,q,r) when (x,y,z) is substituted by the coordinates of the circular points at infinity CI1 or CI2.

Consequently the 2 substitutions with CI1 and Ci2 give 2 linear equations in (p,q,r).
Both equations must satisfy for X, so only the intersection point of both lines coming forth from these equations will give a circular X-Polar Conic.
So there is always 1 point X for which the X-Polar Conic is a Circle.
I made a picture to check it all.
See attachment.

Best regards,
Chris

There exists one and only one point X for which the X-Polar Conic wrt a cubic is a circle.
Proof:
Since a Polar Conic at X(p,q,r) algebraically is derived from:
(D[Cubic, x], D[Cubic, y], D[Cubic, z]) . (p,q,r)
we have a linear equation in (p,q,r) when (x,y,z) is substituted by the coordinates of the circular points at infinity CI1 or CI2.
Consequently the 2 substitutions with CI1 and Ci2 will give 2 linear equations in (p,q,r).
Both equations must satisfy for a point X having a circular X-Polar Conic, so only the intersection point of both lines coming forth from these equations will give a circular X-Polar Conic.
So there is always one and only one point X for which the X-Polar Conic wrt a cubic is a circle.



Properties:

- When the Reference Cubic is a Circular Cubic (CUc), then X will be the real Infinity Point of CUc.

CU-Ci1 The unique CU-Polar Circle-01.pdf

Message: #2323
Date: 2024-05-20
From: eckart_schmidt@t-online.de
Subject: Re: The unique Polar Circle of a cubic

```
Dear Chris,

only a reference for this circle:
Bernard Gibert: Special isocubics ... 2.4.4 or 2.2.4 circles
with figure 2.3
I cannot reach the actually version, the old has several times
changed.

Best regards Eckart
```

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Message: #2324
Date: 2024-05-20
From: van10hoven@gmail.com
Subject: Re: The unique Polar Circle of a cubic

```
Dear Eckart and Bernard,

I was overhasty in my comment that:
* When the Reference Cubic is a Circular Cubic (CUc),
  then X will be the real Infinity Point of CUc.

It should be:
* When the Reference Cubic is a Circular Cubic (CUc),
  then X will be the Focus of CUc.

Best regards,
Chris
```

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Message: #2325
Date: 2024-05-20
From: van10hoven@gmail.com
Subject: Re: The unique Polar Circle of a cubic

Dear Eckart,

Thanks for the reference.
The extra properties:

· If the cubic has three real asymptotes, then X is the Lemoine point (aka Symmedian Point) of the triangle formed by the asymptotes and, obviously, when they concur, it is the point of concurrence. See [Gibert, Isocubics, page 21].

· The parallels to these asymptotes passing through X meet the cubic again at six points lying on a same circle. This circle is analogous to the (first) Lemoine circle obtained when the cubic is the union of the sidelines of triangle ABC. See [Gibert, Isocubics, page 21].

are also very noteworthy.

Best regards,
Chris

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Message: #2326
Date: 2024-05-20
From: van10hoven@gmail.com
Subject: Re: The unique Polar Circle of a cubic

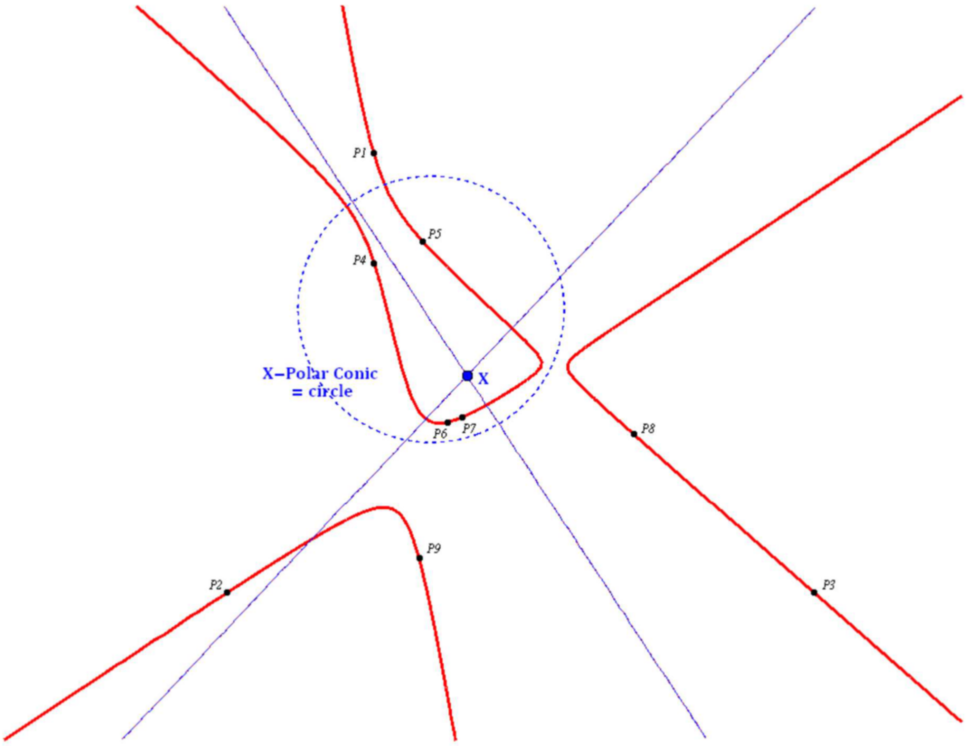
Dear Eckart and Bernard,

Attached an updated description of the unique CU-X-Polar Circle.
I also checked algebraically that point X is X(6) of the Asy-Triangle.

Best regards,
Chris

CU-Ci1 The unique CU-Polar Circle

There exists one and only one point X for which the X-Polar Conic wrt a cubic is a circle.
Proof:
Since a Polar Conic at X(p,q,r) algebraically is derived from:
(D[Cubic, x], D[Cubic, y], D[Cubic, z]) . (p,q,r)
we have a linear equation in (p,q,r) when (x,y,z) is substituted by the coordinates of the circular points at infinity CI1 or CI2.
Consequently the 2 substitutions with CI1 and Ci2 will give 2 linear equations in (p,q,r).
Both equations must satisfy for a point X having a circular X-Polar Conic, so only the intersection point of both lines coming forth from these equations will give a circular X-Polar Conic.
So there is always one and only one point X for which the X-Polar Conic wrt a cubic is a circle.

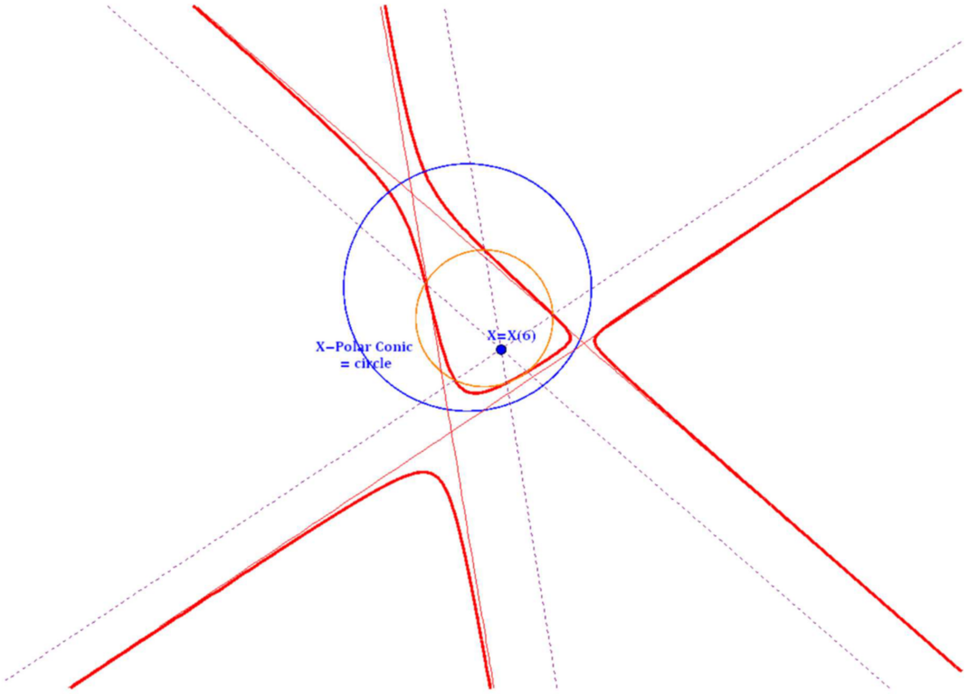


CU-Ci1 Unique Polar Circle of a Cubic-01.jpg

References
See QPG#2322-2324 and [Gibert, Isocubics, page 21].

Properties
- When the Reference Cubic is a Circular Cubic (CUc), then X will be the Focus of CUc.
- If the cubic has three real asymptotes, then X is the Lemoine point aka Symmedian Point of the triangle formed by the asymptotes and, obviously, when they concur, it is the point of concurrence. See [Gibert, Isocubics, page 21].
- The parallels to these asymptotes passing through X meet the cubic again at six points lying on a related circle. This circle is the first Lemoine circle of the Asy-triangle. See [Gibert, Isocubics, page 21].

CU-Ci1 The unique CU-Polar Circle-01.pdf



CU-Ci1 Unique Polar Circle of a Cubic-02.jpg

CU-Ci1 The unique CU-Polar Circle-01.pdf

Message: #2327
Date: 2024-05-23
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

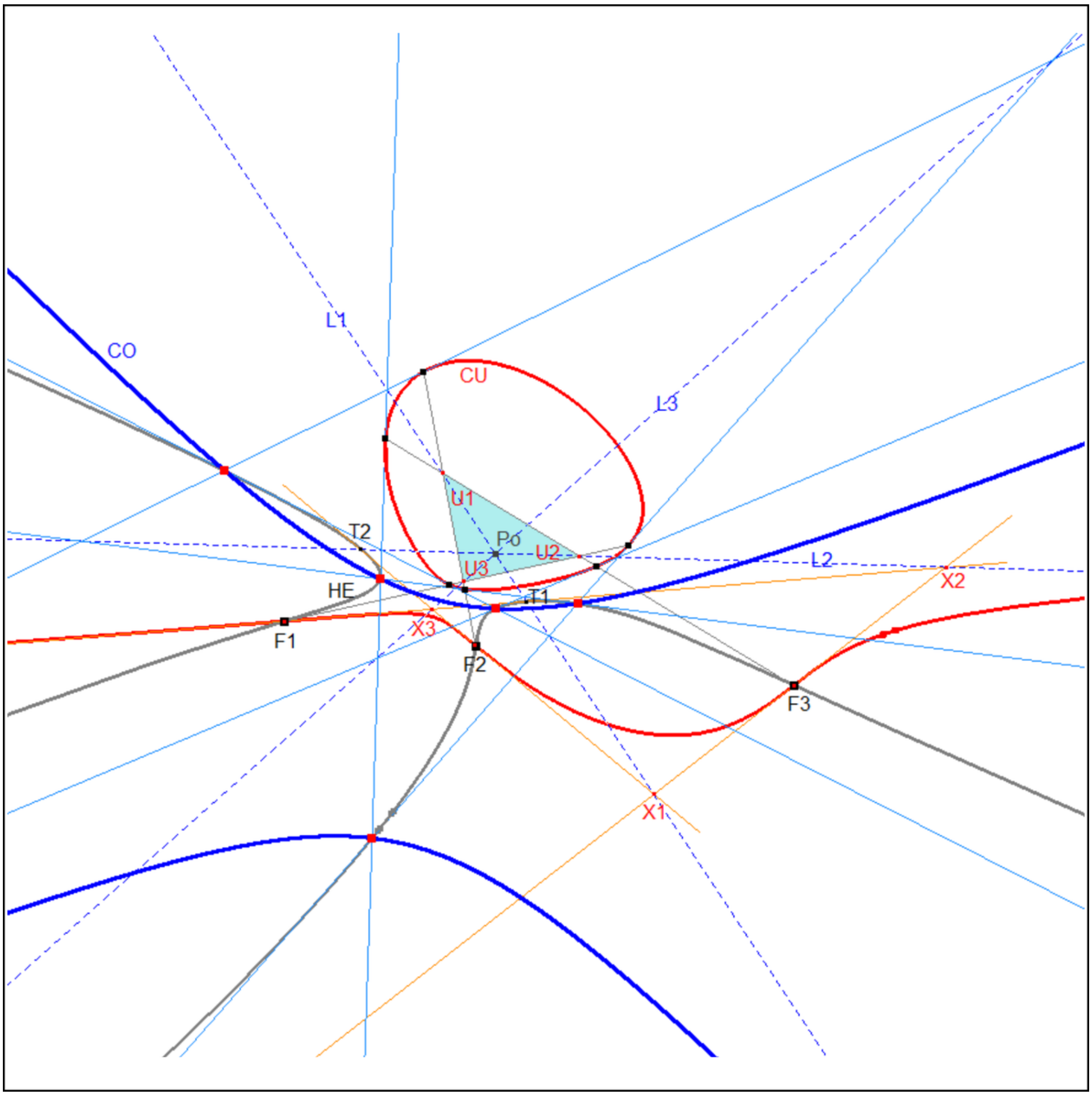
Dear Eckart,

I did my best to draw your configuration like you described in #2321.
I started with a new figure and you are right, it takes hours to make such a figure.
But it was a good exercise.

Here are some remarks;
1. I noticed that the curve of your reference cubic is incomplete in your drawing. Probably a consequence of the Cabri-macro you use. Also you say it is a monopartite cubic. Seeing the incomplete curve I have my doubts. Especially because the most left part of the curve tends to curl downwards to the right part of the curve. Therefore I think that your incomplete cubic is actually a bipartite cubic with an oval part. It will look like the drawing I made in attached picture. It is not that interesting for the rest of what we are studying, but just an observation.
2. It was interesting for me to draw the common tangents of cubic and hessian myself. It is a beautiful discovery that these common tangents touch CU at the intersection points of CU and the sidelines of triangle U1U2U3, which is the Po-cevian triangle of T1T2T3 (Po=common point of the 3 Fi-Harmonic Polars, T1,T2,T3 being the intersection points of CU-Fi-tangent and Fi-Harmonic Polar (i=1,2,3)).
3. In my drawing I made following observations wrt the common tangents not through F1, F2, F3. Each of these common tangents intersects with 3 other common tangents at the 3 Fi-Harmonic Polars and twice with another common tangent at HE and not at a Fi-Harmonic Polar. The last intersections should be your points #. Only I observe these points lie on HE, which I didn’t find in your notes. See my attached picture.
4. Drawing a conic through the points #, in my drawing, the conic does not pass though the points X1,X2,X3, being the vertices of the CU-Fi-tangents.
5. However I did find that these intersection points lie on the conic: U1U2^X3X1, U1U2^X3X2, U2U3^X1X3, 2U3^X1X2, U3U1^X2X3, U3U1^X2X1.
6. From here I stopped, because of the difference in our findings.

```
I am not sure if we did the same construction. Therefore I
described my construction extensively and I'm curious about your
thoughts on this.

Best regards,
Chris
```



CU-Cu1 Hessian-104.pdf

Message: #2328
Date: 2024-05-23
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for your interest and remarks:
wrt 1. There is no doubt, that my cubic is monopartite,
... if I print my attached drawing, there is no incompleteness,
... but we have clearly to differ between mono- and bipartite.

You discuss a bipartite cubic,
which I describe in #2314 (as mentioned),
... #2321 was given for a monopartite cubic.

wrt 3. Your observation is mentioned in #2314.
wrt 4. The property is not mentioned by me for bipartite
       cubics,
       ... it holds only for monopartite cubics.
wrt 5. This description of CT for bipartite cubics is new,
       thanks.

So far for clearance of my construction 3 for the real flexline
trilateral.

Best regards Eckart
```

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Message: #2329
Date: 2024-05-25
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,

I created a precise picture in Mathematica to better evaluate your conjecture that Harmonic Polars and Real Flexlines intersect on the conic CT.

The picture shows that the intersection points are indeed situated very close to the conic CT, but unfortunately, they do not lie exactly on the conic CT.

In the attached document, I have detailed my method and results.

Best regards,

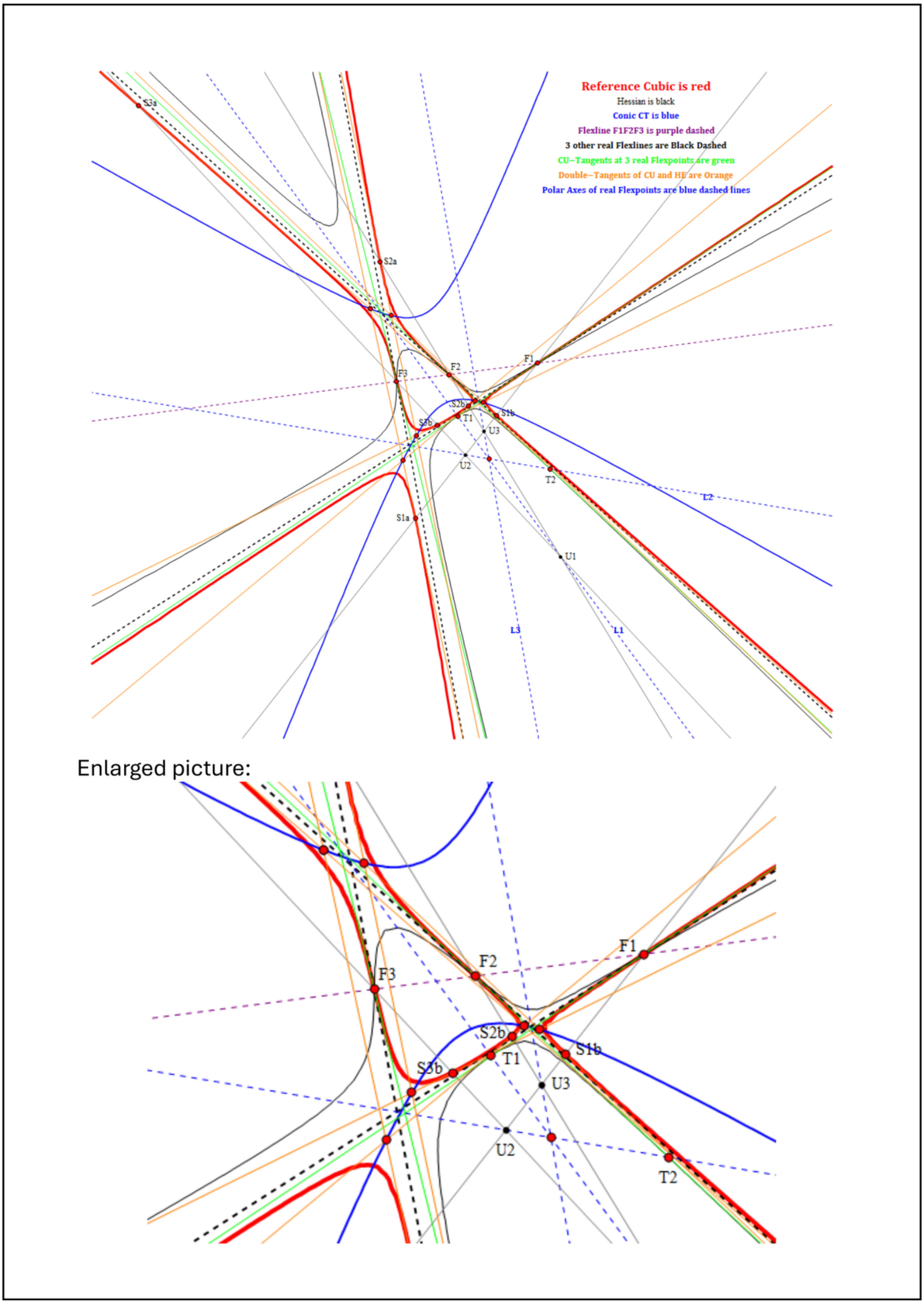
Chris

Working method to determine wether intersection points of Fi-Harmonic Polars and real Fi-Flexlines lie on conic CT.

1. Construct the CU-Fi-tangents forming a triangle with vertices X1, X2, X3.
2. Construct the 3 concurrent Fi-Harmonic Polar Li (i = 1, 2, 3)) with Po = common point.
3. Let T1, T2, T3 be the intersection points of CU-Fi-tangent and Fi-Harmonic Polar (i = 1, 2, 3)).
4. Let U1U2U3 be the Po-Cevian Triangle of T1T2T3.
5. Construct the intersection points of CU and the sidelines of triangle U1U2U3, which are the touchpoints of the common tangents of CU and HE.
6. Each of these common tangents (other than the CU-Fi-tangents) intersects:
* Three times with another common tangent at one of the 3 Fi-Harmonic Polars delivering 6 D-points.
* Twice with another common tangent at HE delivering 6 E-points.
7. Drawing a conic CT through the E - points, the conic according to ES will pass though the points X1, X2, X3, being the vertices of the CU-Fi-tangents when CU is bipartite.
8. CT∩L1 will give intersection points (C1a, C1b),
   CT∩L2 will give intersection points (C2a, C2b),
   CT∩L3 will give intersection points (C3a, C3b).
9. C1x C2x C3x will be the triangle of the real Flextangents or not. It appears not.

See pictures next page.

CU-12L1 Construction of the 3 real Fi-Flextangents-01.pdf



CU-12L1 Construction of the 3 real Fi-Flextangents-01.pdf

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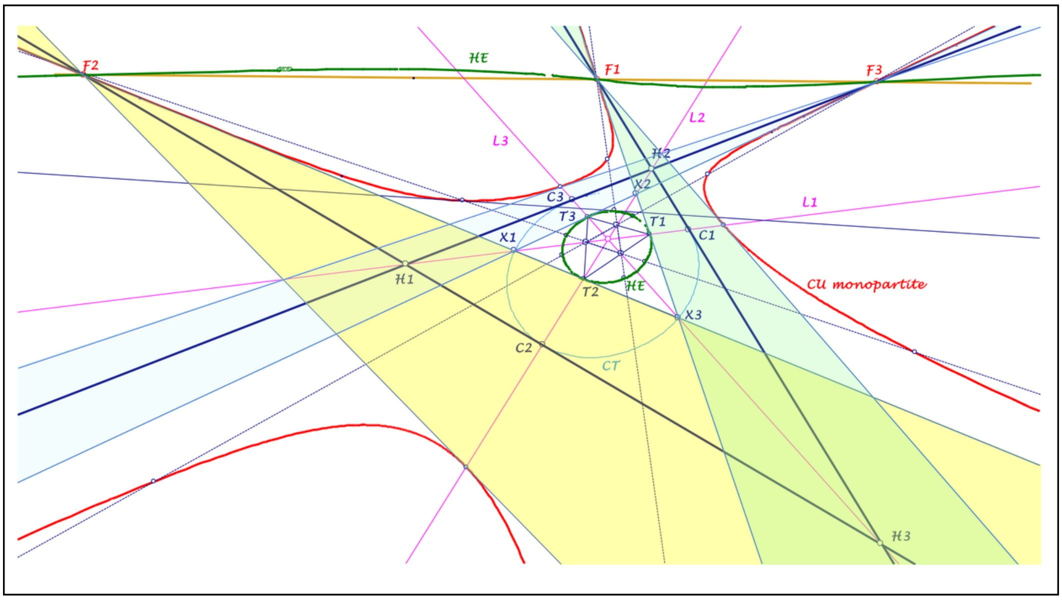
Message: #2330
Date: 2024-05-27
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for your precise drawings in #2290 with description,
... your remarks concern monopartite cubics,
... by me described in #2321,
... as construction 3 of the real flexline trilateral.
Wrt 7. I think
       you have to replace "bipartite" by "monopartite".
Wrt 9. The intersections Ci of CT and Li in the allowed regions
... have to be connected with Fi, to give a real flexline
    (see #2321),
... so your last sentence is evident, but I think you mean
... the intersections of Li and the real flexlines
    don't lie on the conic CT.
Excuse, if I am not convinced, even the enlarged picture
... shows the intersections without significant distance
    in all three cases.
Attached my sight of this theme, described in #2321,
... but my real flexlines remain doubtful,
... how do you draw these lines for monopartite cubics,
... your construction in #2299 cannot be applied,
... for there are no three intersections of Li and CU?

Best regards Eckart
```



2024-05-26.pdf

Message: #2331
Date: 2024-05-27
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

my conjecture for the real flexlines unequal F1F2F3:
1.  Cubic and its hessian define three beams for the flexlines
through Fi.
2.  The common tangents of cubic and hessian
... give 6 special intersections not on Li,
    which define a conic CT.
3.  The conic CT intersects Li
... for monopartite cubics in the Fi-beam in a point Ci
    on the real flexline through Fi,
... for bipartite cubics in the common part of Fj- and Fk-beam
... ... in a point Hi, intersection of the real flexlines
        through Fj, Fk.

Best regards Eckart

PS. Constructions for CT in #2321 and #2327.
```

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Message: #2332
Date: 2024-05-29
From: bernard.keizer@gmail.com
Subject: Syzygetic pencil

Dear Chris, dear Eckart
I'm no longer trying to compete with you on drawings of cubic, hessian, cayleyan and real flexlines.
But I would be interested in pursuing the reflexion about these inflexion points.
1) Any cubic has 9 inflexion points, 3 real and 6 imaginary forming a CB system
2) All the cubics sharing these 9 inflexion points form the syzygetic pencil
3) The 9 points can be determined as intersection of 2 cubics of the pencil, for example the cubic and it's hessian
4) The equation of any cubic of the pencil can be written as a linear combination of 2 cubics of the pencil
Taking CU and HE will give tCU + (1-t)HE with t a real constant
4) The triangle of real flexlines is a degenerated cubic belonging to the pencil, as it contains the 9 flexes
5) Let's name RFi the flexlines and their equations
The equation of the cubic formed by the real 3 flexlines can be written as CRF = RF1*RF2*RF3 = 0
6) Therefore, it exists a real constant t0 for which t0CU + (1-t0)HE = CRF
7) Then t0(CU - CRF) + (1-t0)(HE - CRF) = 0
8) Last, it follows that the equation of any cubic CU can be written under the form CRF + K, where K is a constant
Is all this correct? (may be it depends on the system of coordinates ...)
Chris, does it look like the calculations you made on Mathematica by checking your construction?
Thanks in advance for your attention
Best regards
Bernard

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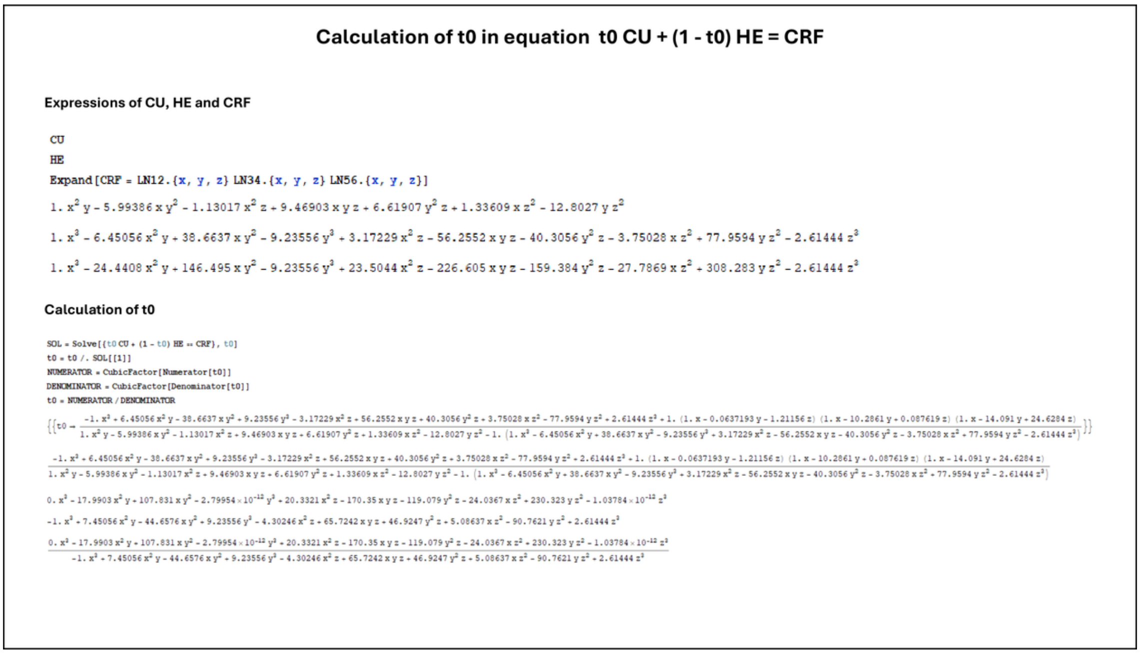
Message: #2333
Date: 2024-05-29
From: van10hoven@gmail.com
Subject: Re: Syzegetic pencil

```
Dear Bernard,
I made a calculation in Mathematica (wrt the Syzegetic Pencil)
of t0 in equation  t0 CU + (1 - t0) HE = CRF.
See attachment.
It is not a simple value of t0 that came out in Mathematica.
Best regards,
Chris
```



Calculation of t0 in equation t0 CU + (1 - t0) HE = CRF

Expressions of CU, HE and CRF

```
CU
HE
Expand[CRF = LN12.{x, y, z} LN34.{x, y, z} LN56.{x, y, z}]
```

$$1.\,x^2 y - 5.99386\,x y^2 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2$$

$$1.\,x^3 - 6.45056\,x^2 y + 38.6637\,x y^2 - 9.23556\,y^3 + 3.17229\,x^2 z - 56.2552\,x y z - 40.3056\,y^2 z - 3.75028\,x z^2 + 77.9594\,y z^2 - 2.61444\,z^3$$

$$1.\,x^3 - 24.4408\,x^2 y + 146.495\,x y^2 - 9.23556\,y^3 + 23.5044\,x^2 z - 226.605\,x y z - 159.384\,y^2 z - 27.7869\,x z^2 + 308.283\,y z^2 - 2.61444\,z^3$$

Calculation of t0

```
SOL = Solve[{t0 CU + (1 - t0) HE == CRF}, t0]
t0 = t0 /. SOL[[1]]
NUMERATOR = CubicFactor[Numerator[t0]]
DENOMINATOR = CubicFactor[Denominator[t0]]
t0 = NUMERATOR / DENOMINATOR
```

$$\left\{\left\{t0 \to \frac{-1.\,x^3 + 6.45056\,x^2 y - 38.6637\,x y^2 + 9.23556\,y^3 - 3.17229\,x^2 z + 56.2552\,x y z + 40.3056\,y^2 z + 3.75028\,x z^2 - 77.9594\,y z^2 + 2.61444\,z^3 + 1.\,(1.\,x - 0.0637193\,y - 1.21156\,z)\,(1.\,x - 10.2861\,y + 0.087619\,z)\,(1.\,x - 14.091\,y + 24.6284\,z)}{1.\,x^2 y - 5.99386\,x y^2 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2 - 1.\,(1.\,x^3 - 6.45056\,x^2 y + 38.6637\,x y^2 - 9.23556\,y^3 + 3.17229\,x^2 z - 56.2552\,x y z - 40.3056\,y^2 z - 3.75028\,x z^2 + 77.9594\,y z^2 - 2.61444\,z^3)}\right\}\right\}$$

$$\frac{-1.\,x^3 + 6.45056\,x^2 y - 38.6637\,x y^2 + 9.23556\,y^3 - 3.17229\,x^2 z + 56.2552\,x y z + 40.3056\,y^2 z + 3.75028\,x z^2 - 77.9594\,y z^2 + 2.61444\,z^3 + 1.\,(1.\,x - 0.0637193\,y - 1.21156\,z)\,(1.\,x - 10.2861\,y + 0.087619\,z)\,(1.\,x - 14.091\,y + 24.6284\,z)}{1.\,x^2 y - 5.99386\,x y^2 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2 - 1.\,(1.\,x^3 - 6.45056\,x^2 y + 38.6637\,x y^2 - 9.23556\,y^3 + 3.17229\,x^2 z - 56.2552\,x y z - 40.3056\,y^2 z - 3.75028\,x z^2 + 77.9594\,y z^2 - 2.61444\,z^3)}$$

$$0.\,x^3 - 17.9903\,x^2 y + 107.831\,x y^2 - 2.79954\times10^{-12}\,y^3 + 20.3321\,x^2 z - 170.35\,x y z - 119.079\,y^2 z - 24.0367\,x z^2 + 230.323\,y z^2 - 1.03784\times10^{-12}\,z^3$$

$$-1.\,x^3 + 7.45056\,x^2 y - 44.6576\,x y^2 + 9.23556\,y^3 - 4.30246\,x^2 z + 65.7242\,x y z + 46.9247\,y^2 z + 5.08637\,x z^2 - 90.7621\,y z^2 + 2.61444\,z^3$$

$$\frac{0.\,x^3 - 17.9903\,x^2 y + 107.831\,x y^2 - 2.79954\times10^{-12}\,y^3 + 20.3321\,x^2 z - 170.35\,x y z - 119.079\,y^2 z - 24.0367\,x z^2 + 230.323\,y z^2 - 1.03784\times10^{-12}\,z^3}{-1.\,x^3 + 7.45056\,x^2 y - 44.6576\,x y^2 + 9.23556\,y^3 - 4.30246\,x^2 z + 65.7242\,x y z + 46.9247\,y^2 z + 5.08637\,x z^2 - 90.7621\,y z^2 + 2.61444\,z^3}$$

CU-Cu1 Calculation wrt Syzegetic Pencil.pdf

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Message: #2334
Date: 2024-05-30
From: bernard.keizer@gmail.com
Subject: Re: Syzegetic pencil

Dear Chris,
Thanks a lot for interest and quick answer.
The naming syzegetic pencil is not correct (my fault, I used it this way ...)
The correct naming is syzygetic pencil or Hesse pencil.
Meanwhile, I found a definition in Wikipedia and a reference ...
There is certainly something to find, as CU, HE and CRF belong to the pencil, which is defined by 2 cubics of the pencil!
You started with these 3 real flexlines carrying the real as well as the imaginary flexes.
I hope I have opened a new topic of a great interest.
I will try to study it a little more ...
Best regards
Bernard
PS What is your reference triangle for barycentrics? Could you use the vertices H1, H2 and H3 of CRF? The equation 3 would be XYZ = 0

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Message: #2335
Date: 2024-05-30
From: bernard.keizer@gmail.com
Subject: Re: Syzegetic pencil

Dear Chris,
My idea is correct!
Each equation is obviously defined up to a multiplying factor ...
With your equations, it's not difficult to find that HE - CRF is something like 18*CU. It fits term to term.
So there is in fact a linear combination between the 3 curves.
Alleluiah
Best regards
Bernard

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Message: #2336
Date: 2024-05-30
From: van10hoven@gmail.com
Subject: Re: Syzegetic pencil

Dear Bernard,

I just found the same result.
See attachment.

Best regards,
Chris

Relationship between CU, HE and CRF

CU = Reference Cubic
HE = Hessian of CU
CRF = Cubic degenerated in 3 real flexlines, each containing 3 flexpoints, totaling all 9 flexpoints

Expressions of CU, HE and CRF after making first coefficient = 1

```
CU
HE
Expand[CRF = LN12.{x, y, z} LN34.{x, y, z} LN56.{x, y, z}]
```

$$1.\,x^2 y - 5.99386\,x y^2 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2$$

$$1.\,x^3 - 6.45056\,x^2 y + 38.6637\,x y^2 - 9.23556\,y^3 + 3.17229\,x^2 z - 56.2552\,x y z - 40.3056\,y^2 z - 3.75028\,x z^2 + 77.9594\,y z^2 - 2.61444\,z^3$$

$$1.\,x^3 - 24.4408\,x^2 y + 146.495\,x y^2 - 9.23556\,y^3 + 23.5044\,x^2 z - 226.605\,x y z - 159.384\,y^2 z - 27.7869\,x z^2 + 308.283\,y z^2 - 2.61444\,z^3$$

Relationship between CU, HE and CRF

```
Factor[HE - CRF]
CU
```

$$17.9903\,(1.\,x^2 y - 5.99386\,x y^2 + 1.55614\times10^{-13}\,y^3 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2 + 5.76888\times10^{-14}\,z^3)$$

$$1.\,x^2 y - 5.99386\,x y^2 - 1.13017\,x^2 z + 9.46903\,x y z + 6.61907\,y^2 z + 1.33609\,x z^2 - 12.8027\,y z^2$$

Therefore in this case HE - CRF = 17.9903 CU

CU-Cu1 Calculation wrt Syzegetic Pencil-02.pdf

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Message: #2337
Date: 2024-05-30
From: bernard.keizer@gmail.com
Subject: Re: Syzegetic pencil

Dear Chris,
Naturally, this gives a complete validation of your construction!
It is remarkable that any cubic defines alone the triangle of real flexlines and that any cubic of the syzygetic pencil leads to the same triangle.
It is more than a triangle, as in fact any cubic defines a QL with the 4 real flexlines.
The elements of this QL (Miquel point, Miquel circle, Newton Line, Steiner Line ...) are the same for all cubics of the pencil ...
Beautiful geometry!
Best regards
Bernard

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Message: #2338
Date: 2024-05-30
From: van10hoven@gmail.com
Subject: Re: Syzygetic pencil

Dear Bernard,

Beautiful geometry indeed.
This also gives the possibility in Mathematica to draw any cubic of the Syzygetic Pencil when only one point other than the flexpoints is known.

Best regards,
Chris

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Message: #2339
Date: 2024-05-31
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,

Regarding your messages #2328, # 2330 and #2331.
First of all, I like your beautiful drawing in #2330.
Nevertheless, I do not know how to make progress.
If I understand you correctly, your conjecture is that when the reference cubic is monopartite, then:
· X1,X2,X3 are on conic CT
· The intersection points of Li and the real flexlines lie on conic CT.

For me, this is not logical because:
1. I don't see why certain points will concur on a CU-indicated conic only when it is monopartite. Can you provide any reason why this should or could be the case?
2. The last picture I did send to you, #2329, was very accurate. According to your definition the cubic depicted can be categorized as monopartite because there is no 2nd part of the curve (oval). Still, both concurrences cannot be seen. Additionally, Mathematica works by default with 16 decimals. My experience with Mathematica (unlike Cabri) is that incidences are clearly translated into very precise drawings.

In #2330 you reasonably ask:
... how do you draw these lines for monopartite cubics,
... your construction in #2299 cannot be applied,
... for there are no three intersections of Li and CU?
I mentioned in #2299 (last remark) that I have not yet devised a construction for cases where two of these points are imaginary. This is still the case. I hope that as we continue to work together, we will find such a construction.

Best regards,
Chris

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Message: #2340
Date: 2024-06-01
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,

About your question in #2330:
... how do you draw these lines (flexlines)
    for monopartite cubics,
... your construction in #2299 cannot be applied,
... for there are no three intersections of Li and CU?

In your drawing #2330, Li intersects CU at only one real point.
However, Li intersects HE at three real points.
Therefore, the triplet of three real Flexlines of HE can be
constructed according to #2299.
Since CU and HE have identical Flexpoints, they also have
identical Flexlines.
Therefore, in this case, the triplet of three real Flexlines can
still be constructed.

Best regards,
Chris
```

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Message: #2341
Date: 2024-06-01
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for your remarks in #2339, I think, you are right with
your doubts wrt
... "The intersection points of Li and the real flexlines
     lie on conic CT."
I found monopartite cubics,
... with no intersections of Li and CT in the allowed regions,
... so my conjecture 3. in #2331 for monopartite cubics
    doesn't hold, excuse.

Wrt "monopartite": I use the classification of Steiner,
... who holds parts of cubics as connected,
... if they have a common point at infinity,
... so we have only mono- and bi-partite cubics.

Best regards Eckart

PS: I just saw your last message #2340,
    so I can test my conjectures.
```

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Message: #2342
Date: 2024-06-01
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,

About the mentioned classification of Steiner, do you have a
reference for this classification?

Best regards,
Chris
```

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Message: #2343
Date: 2024-06-02
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

excuse my confusion,
replace Steiner by Schröter (page 136 and &17).

Best regards Eckart
```

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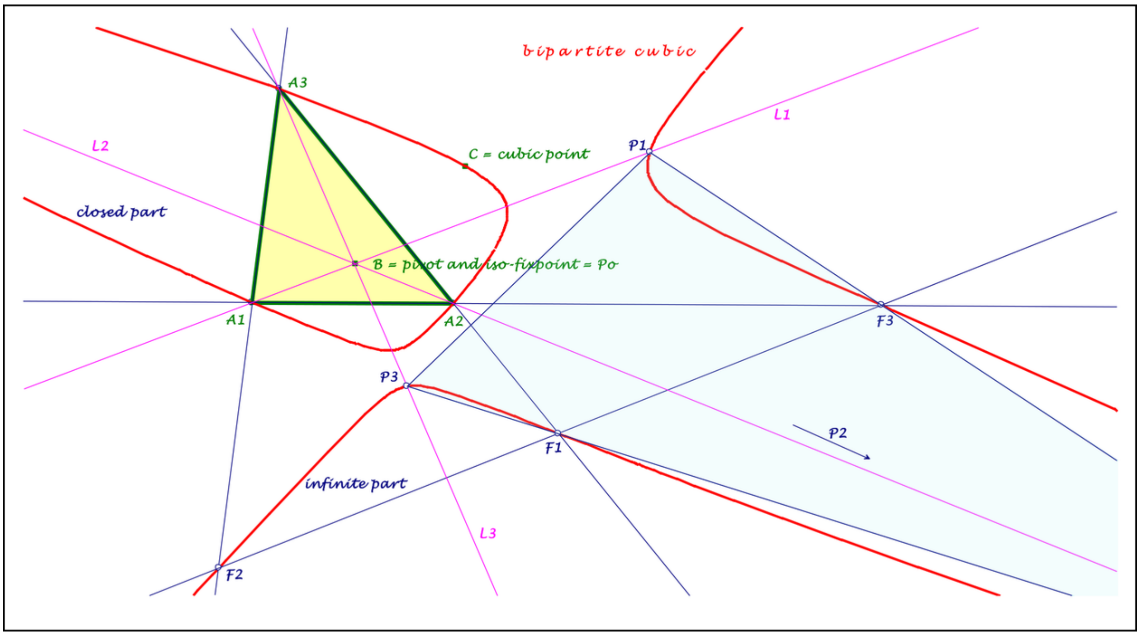
Message: #2344
Date: 2024-06-12
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

are  the following properties correct
... and perhaps already well known?
 1) A non degenerated cubic
... is a nonpivotal isocubic with ref-triangle P1P2P3
... with isoconjugation-fixpoint Po and root Po
    and a cubic-point Q.
 2) The hessian of a non-degenerated cubic
... is a non-pivotal isocubic wrt ref-triangle T1T2T3
... with isoconjugation-fixpoint Po and root Po
    and a hessian point Q.
This will be helpful, to draw a cubic as attached:
... starting with a ref-triangle A1A2A3, a point B
    and a point C,
... construct the non-pivotal isocubic wrt ref-triangle A1A2A3,
... isoconjugation-fixpoint B and root B and cubic-point C,
... for a monopartite cubic A1,2,3 will be P1,2,3 with B = Po,
... for a bipartite cubic Pi will be the intersection
... ... of AiB and the infinite part of the cubic with B = Po.

Best regards Eckart
```



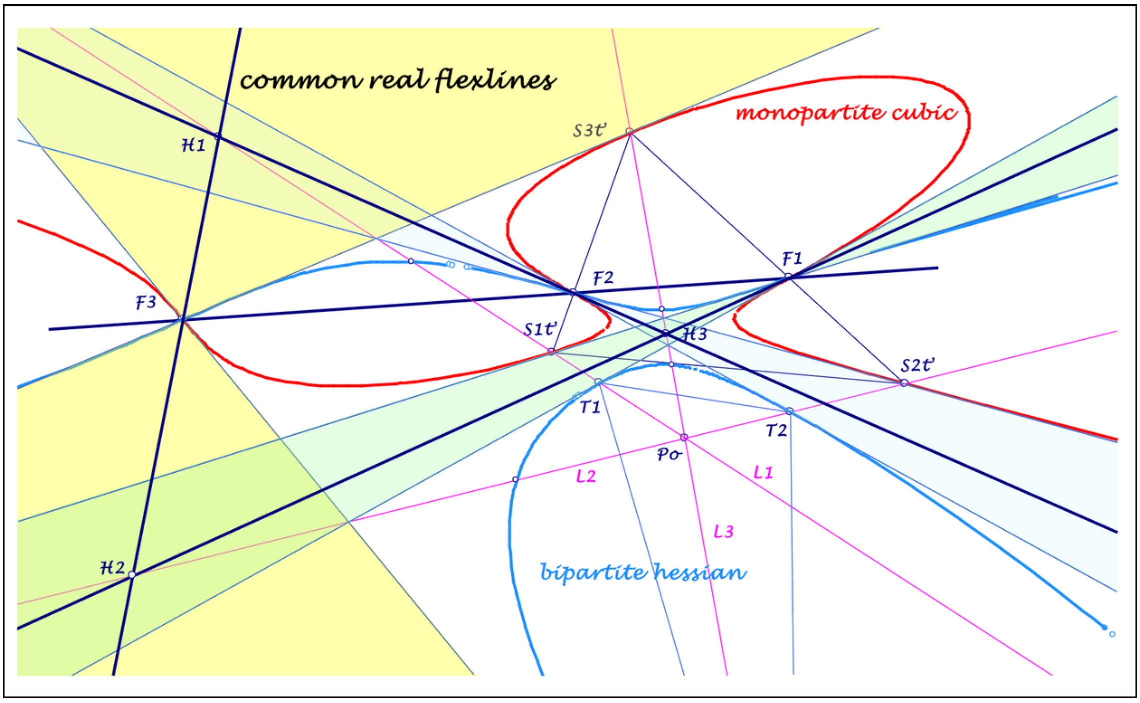
2024-06-12.pdf

Message: #2345
Date: 2024-06-15
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

attached a drawing of the real flexlines
... for a monopartite cubic,
... using the bipartite hessian,
... thanks for the instruction.

Best regards Eckart
```



2024-06-15.pdf

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Message: #2346
Date: 2024-06-17
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

Regarding your message #2344,
could you briefly describe how you constructed the non-pivotal isocubic with respect to the reference triangle A1A2A3?

Best regards,
Chris

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Message: #2347
Date: 2024-06-17
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,

Congratulations on the drawing of the real flexlines for a monopartite cubic!
I know these constructions are quite a job.

Best regards,
Chris

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Message: #2348
Date: 2024-06-18
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

Dear Chris,

wrt the construction of a nonpivotal isocubic in #2344
... starting with a triangle A1A2A3 a point B and a point C.
More than 10 years ago I made a macro for nonpivotal isocubics,
... but I don't remember details, background was Bernard Gibert's 1.5.4.
You can take his construction, using
... A1A2A3 = ABC, B = root P, C = Q,
... than Q* will be QA-Tf2(Q) for QA = root P plus its anticevians.

Best regards Eckart

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Message: #2349
Date: 2024-06-18
From: van10hoven@gmail.com
Subject: Mean Intersect Line

Dear Bernard and Eckart,

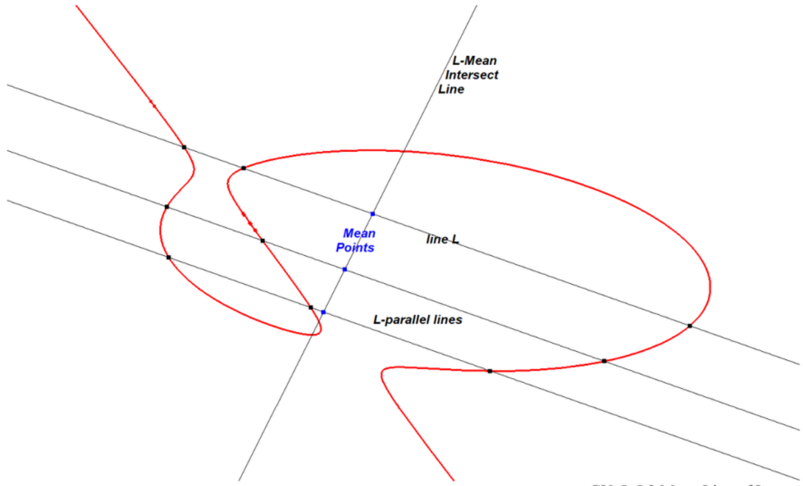
There is a very special line transformation mapping a random line in a “mean line”.
Given reference cubic CU and a random line L. The mean point of the intersection points of CU and all L-parallel lines are collinear. See E. de Jonquiere - Mélanges de géométrie, page 197.
Available at https://books.google.nl/books?id=UCMNXC7gFlEC&pg=PA197&hl=nl&source=gbs_toc_r&cad=2#v=onepage&q&f=false
According to him this theorem originates from Newton. It is not only valid for conics and cubics, but also for all curves of n th degree!
That will give us an opportunity when we might investigate general quartics and quintics later on.
I called this line the L-Mean Intersect Line and found that all these lines are tangent to the Steiner InEllipse of the CU-Asy-Triangle. In other words, they envelop the Steiner InEllipse of the CU-Asy-Triangle.
The special thing is that this line gives us the opportunity to determine the Steiner InEllipse of the CU-Asy-Triangle, especially when there is no triangle because there is only one asymptote.
After all it is the envelope of L-Mean Intersect Lines. It appears that when there is only one CU-asymptote, the analogon of the Steiner InEllipse of the CU-Asy-Triangle is a Hyperbola touching the one and only CU-asymptote.
Furthermore I found some more special properties in relationship with the Diametral Conic and the QF-Conic.
I described it all in attached small paper.
I hope you will enjoy, just like I did and find some more properties.

Best regards,
Chris

CU_L-L2 L-Mean Intersect Line

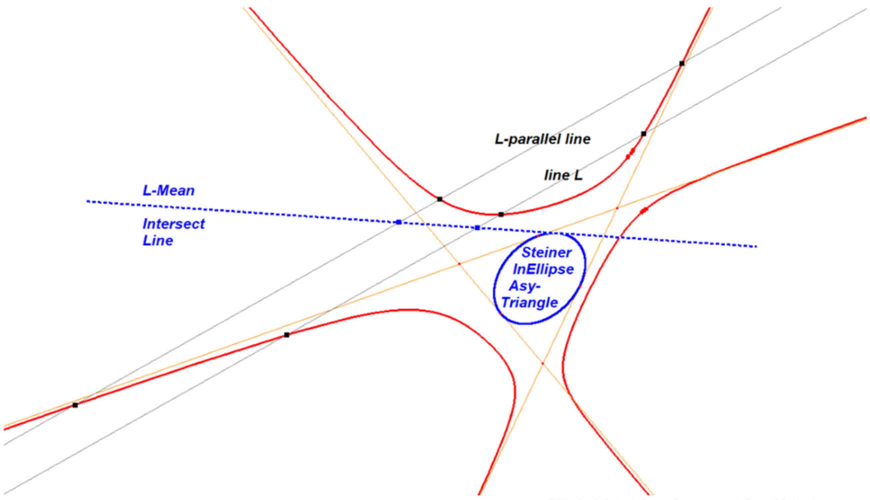
Given reference cubic CU and a random line L.
The mean point of the intersection points of CU and all L-parallel lines are collinear. See E. de Jonquiere - Mélanges de géométrie, page 197. According to him this theorem originates from Newton. It is not only valid for conics and cubics, but also for all curves of n^{th} degree.

The locus of the mean point of intersection points of CU and L-parallel lines is called the L-Mean Intersect Line CU_L-L2.



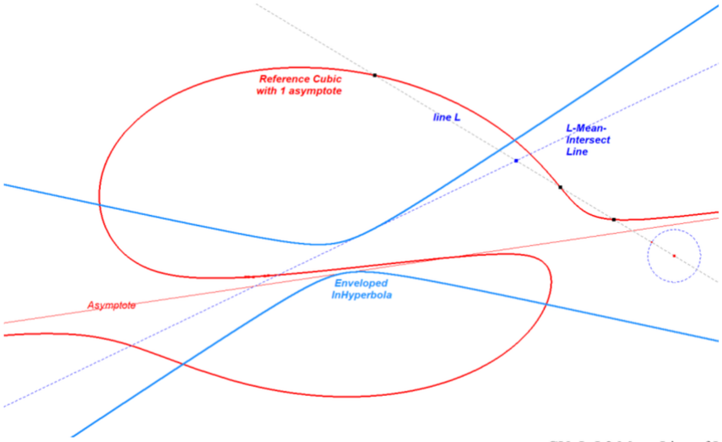
CU_L-L2 Mean Line of Intersection-02.fig

A special property of all L-Mean Intersect Lines is that they are tangent to the Steiner InEllipse of the CU-Asy-Triangle CU-Tr1. In other words, they envelop the Steiner InEllipse of the CU-Asy-Triangle. This is only visible when CU has three real asymptotes.



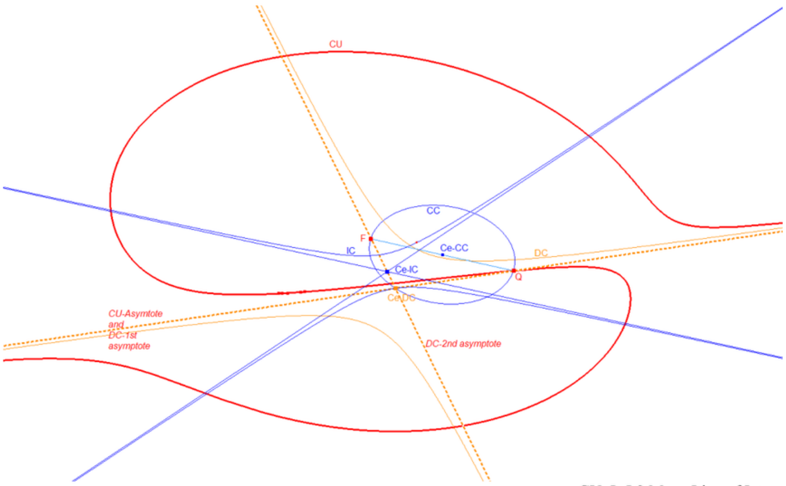
CU_L-L2 L-Mean Intersect Line-10-3-Asymptotes.fig

Now that we know this property, when CU has only one real asymptote, the Steiner InEllipse can be constructed as the envelope of L-Mean Intersect Lines. In this case, instead of an InEllipse, it becomes an InHyperbola touching the one and only real asymptote. As a matter of algebraic consistency, the InHyperbola will also touch the imaginary asymptotes, but in the imaginary realm.



CU_L-L2 Mean Line of Intersection-30.fig

This inconic that can be determined now for all appearances of a cubic has several incidences with CU and 2 other conics, the Diametral Conic CU-IP-Co1 and the Central Conic/QF Conic CU-IP-Co2.



CU_L-L2 Mean Line of Intersection-31.fig

The center Ce-DC of the Diametral Conic DC lies on IC.
The 2 asymptotes of the Diametral Conic are Asy1 and another line passing through the center of IC.
This 2nd DC-asymptote contains:
* obviously the Center of IC
* the center of DC
* the Point F diametral to Q on the Central Conic CC
The Center of DC also lies on the Central Conic CC.
Note: F is also the intersection point of the imaginary 2nd and 3rd asymptotes of reference cubic CU.

CU_L-L2 Mean Line.pdf

Message: #2350
Date: 2024-06-19
From: bernard.keizer@gmail.com
Subject: Re: Mean Intersect Line

Dear Chris, the property is mentionned in Cüppens P.227, where your L-Mean Intersect Line is called diameter associated to a direction d.
(In fact, it is the polar line wrt the cubic of the infinity point of the initial line L).
But Cüppens doesn't mention the Steiner Inellipse or the InHyperbola as envelop of these diameters ...
Best regards
Bernard

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Message: #2351
Date: 2024-06-19
From: van10hoven@gmail.com
Subject: Re: Mean Intersect Line

Dear Bernard,

Thanks for the reference of Cuppens.

I found his notion of 'diameter associated to a direction d' at pages 255, 256 and not at page 227.
There he also mentions (page 256) that the mean point of the CU-L-intersect-points coincides with the mean point of the intersection points of L with the three CU-asymptotes.
I think that is quite remarkable.

Best regards,
Chris

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Message: #2352
Date: 2024-06-19
From: bernard.keizer@gmail.com
Subject: Re: Mean Intersect Line

Dear Chris,
Thanks to you, I've revised Cüppens!
In fact, the diameter associated to a direction is introduced on page 227, as I told, but other properties are developped on the pages 255, 256 and 257, as you mention.
In particular, the polar conic of the infinity line is the envelop of the diameters of a general cubic on page 256 and the Steiner inellipse for a cubic with 3 real asymptotes on page 257.
Remarkable, indeed!
Best regards
Bernard

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Message: #2353
Date: 2024-06-22
From: van10hoven@gmail.com
Subject: Re: Cayleyan and dual cubic

Dear Bernard,

I was reviewing the many messages we exchanged over the past months and came across your remarkable construction of the Cayleyan in GeoGebra.
Congratulations on this impressive work!

Best regards,
Chris

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Message: #2354
Date: 2024-06-23
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

Regarding your message #2344.
I still don't get your construction.
Thanks for explaining the construction of the Isoconjugate wrt
the construction of a nonpivotal isocubic in #2348. That helps.
I understand from your words:
1. You start with a non-degenerated reference cubic CU. Because
you mention an isoconjugation-fixpoint Po and root Po and a
cubic-point Q, I suppose you construct it with the help of the
isoconjugate-construction in #2348. I assume that you start with
a random triangle P1P2P3, a random point Po. Having these items
and the knowledge of the isoconjugation-construction, which
construction method you use to construct the non-degenerated
cubic?
2. Then you construct the hessian HE with as reference cubic CU.
Again you intruduce a ref-triangle A1A2A3, a point B and a point
C. Are these items randomly chosen? And then you construct the
nonpivotal cubic with these items. Again I wonder which
construction method you use?

I hope you understand my confusion.

Best regards,
Chris

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Message: #2355
Date: 2024-06-23
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

I try to describe my points 1) and 2) in #2344 once more:

1)
a) Consider a non-degenerated cubic,
... its harmonic polars L1,2,3 with common point Po
... and their intersections P1,2,3
    with the infinite part of the cubic.
b) Consider now the ref-triangle P1P2P3
... and the isoconjugation * with fixed point Po,
... further a non special point Q of the cubic,
... and construct the nonpivotal isocubic wrt triangle P1P2P3
... the isoconjugation *, root also Po and the point Q
    (see Gibert 1.5.4)
... and you will get the cubic back.

2)
a) see above.
b) see above,
... but replace the ref-triangle by T1T2T3,
... (cevian triangle of the flextangent triangle),
... take now a non-special point Q of the hessian
... and the corresponding nK will the hessian.

Last passage of #2344 (independent of above):
This construction can be used,
... to get a cubic without the nine-point-construction.
I hope these observations are correct.

Best regards Eckart

PS. You ask once more for a construction method for a nK:
It is Gibert's 1.5.4 for a ref-triangle P1P2P3 for 1) or T1T2T3
for 2),
... isoconjugation * with fixed point Po (or its cevians),
... that means X* = QATf2(X) for QA = Po plus its anticevians,
... finally a given cubic point Q and its Q*.
```

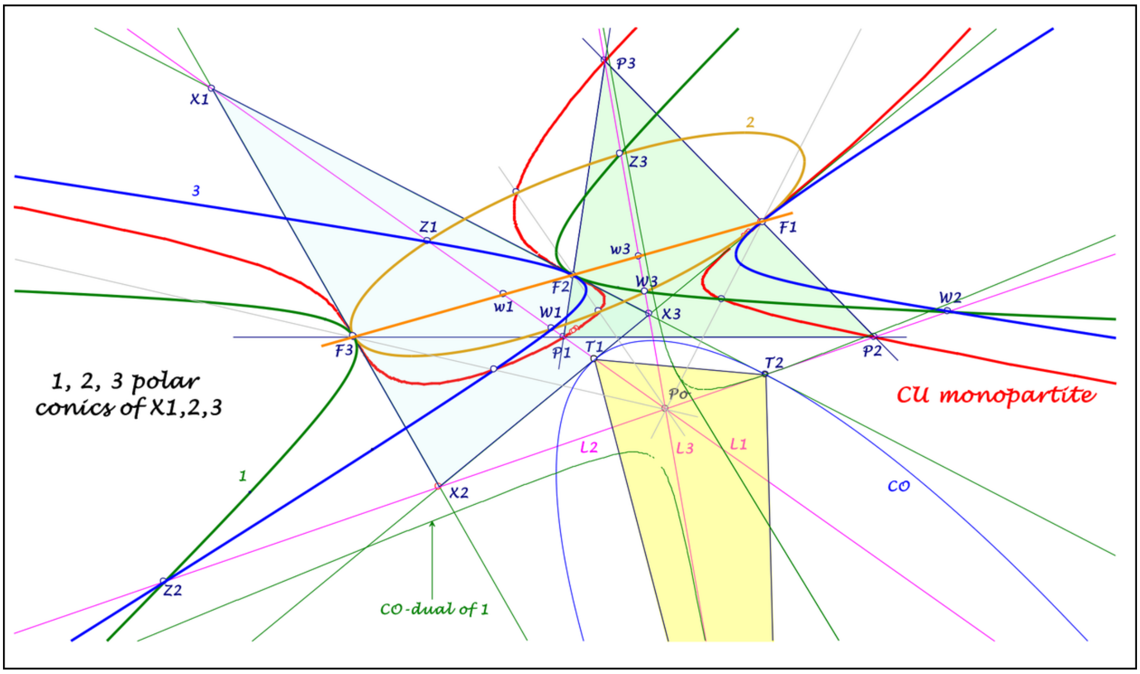
Message: #2356
Date: 2024-06-25
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

Dear Bernard,

I study desperately the first drawing in your paper
..."A new dual of the cayleyan" in #2316,
... one question: Is the polar conic of Xi
... bearing the flexpoints Fj and Fk,
... tangent to the flexlines tgj and tgk,
... bearing the contact points of tangents from Xi to the cubic,
... which are the further intersections of Po.Fi and the cubic?
If this is correct, as your drawing seems to show,
... I can not confirm the third sentence in your paper:
... "The dual conics of these 3 conics are degenerated conics
formed by 2 lines Li."

Best regards Eckart

PS. CABRI shows in the attached drawing
... a false additional line for the CO-dual of 1, excuse.



2024-06-25.pdf

Message: #2357
Date: 2024-06-25
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

It looks like this topic is as complicated as an IKEA manual,
but let's try to decipher it together!
You asked me in #2344 if the properties mentioned by you were
correct. I think I understand what you are writing. However,
there is just one item I do not know what to do about.
In #2344 you say:
draw a cubic as attached:
... starting with a ref-triangle A1A2A3,
    a point B and a point C,
... construct the non-pivotal isocubic wrt ref-triangle A1A2A3,
... isoconjugation-fixpoint B and root B and cubic-point C.

I asked you how to construct the cubic, assuming that this is
what you describe, just knowing a reference triangle, a root B,
and a cubic-point C. Then you explain how the isoconjugate
works.
However, that is just one part of constructing an isocubic. You
do not describe the second part. Maybe it's obvious to you, but
apparently not to me.
In QA-Cu-1 I described once the construction of 3 types of
Isocubics.
In all cases you need a line Lv through a pivot as follows:
Let Lv be some line through V.
Let IC(Lv) be the Involutary Conjugate QA-Tf2 of Line Lv.
IC(Lv) is a conic since QA-Tf2 is a transformation of the 2nd
degree.
The locus of the intersection IC(Lv) ^ Lv is a QA-Cubic Type 1.
Because of its function V is called the Pivot Point of the
QA-Cubic Type 1.
Is this the way you construct the cubic or is there another way?
I tried this construction method using the shape of your
reference triangle and similar position of B and C, but did not
find a similar isocubic.

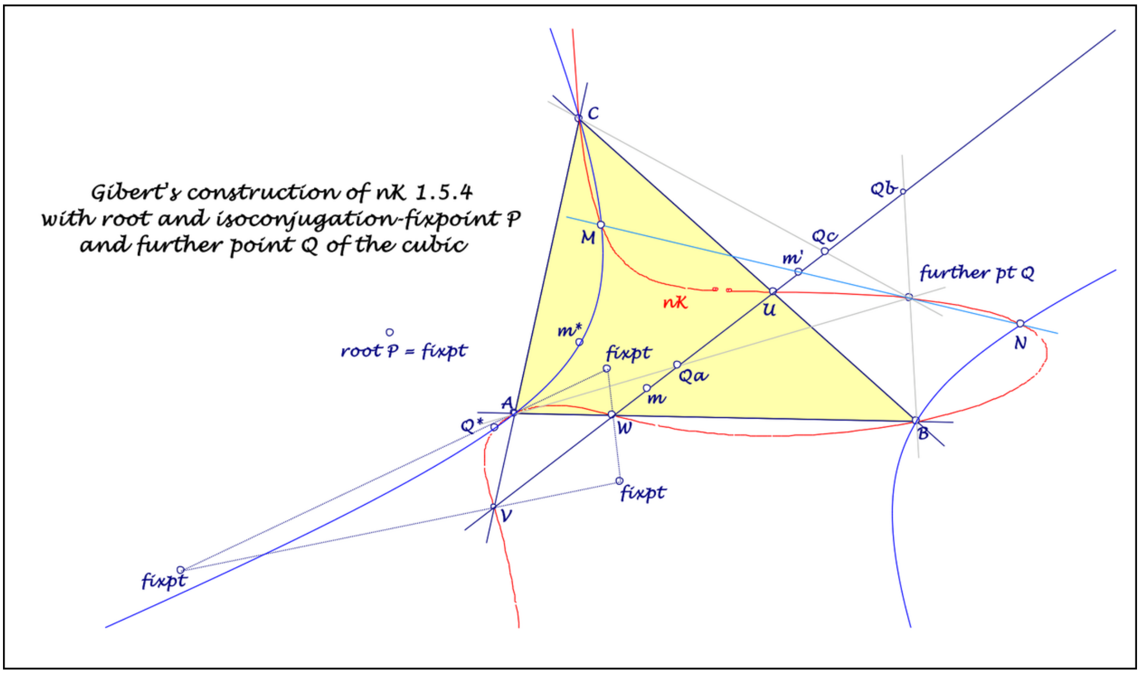
Best regards,
Chris

Message: #2358
Date: 2024-06-26
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

I don't find three versions of nK in Gibert's paper,
... your QA-CU-1 was new for me,
... but you describe cubics for quadrangles,
    Gibert uses ref-triangles
... and isoconjugations, which can be defined by a fixpoint,
... the anticevians of the fixpoint are also fixpoints.
For a nK we need further the root and a non-special point of the
cubic.
If you start "... just knowing a reference triangle, a root B,
and a cubic-point C..."
... you forget, that the root should here also be a fixed point
    for the isoconjugation,
... then the construction of Gibert 1.5.4 can be done
    (attached),
... what is the "second part" of the construction?

Best regards Eckart
```



2024-06-26.pdf

Message: #2359
Date: 2024-06-26
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

```
Dear Eckart,
Thanks for your interest and your attention after so many time!
You are perfectly right, the dual conics of the polar conics of
Xi are conics through Tj and Tk.
But I hope you didn't stop at this point.
The dual cubic of the cayleyan is correct with the 9 points Ti,
wi and zi dual of FiTi, FiWi and FiZi (a beautiful quadripartite
cubic) as well as the cayleyan, dual of this cubic.
The cayleyan is either the envelop of the dual of the points of
the cubic or the locus of the dual of it's tangents.(Dual is
here pole/polar wrt the conic CO).
So I suppose you are able now to draw your own cayleyan ...
I'm impatiently waiting on a beautiful figur of yours with
Cabri.
Best regards
Bernard
```

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Message: #2360
Date: 2024-06-27
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

```
Dear Bernard,

I think there is a typo in your last message:
The polar conics of Xi don't pass through Tj and Tk,
... but through further intersections of Po.Fi and the cubic,
... nevertheless I succeeded in drawing the cayleyan,
... struggling with the limits of my CABRI.
Thanks for your paper.

Best regards Eckart
```

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Message: #2361
Date: 2024-06-27
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

```
Dear Eckart,
Not the polar conic of Xi, but it's dual passes through Tj and
Tk!
Your own figure shows it, the dual conic of the polar conic of
X1 passes through T2 and T3!!!
Can you show your cayleyan? then the intersection with the
reference cubic ...
Best regards
Bernard
```

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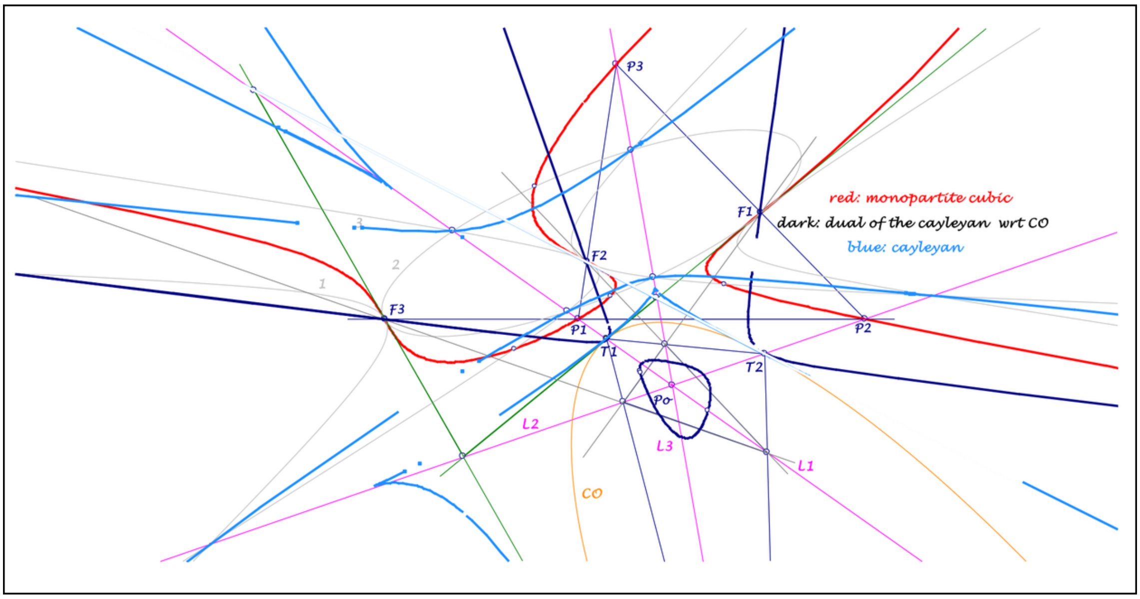
Message: #2362
Date: 2024-06-29
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

```
Dear Bernard,

attached my drawing of the cayleyan,
... it needs hours of CABRI limits,
... and the result is not a complete drawing,
... perhaps good for some observations,
... the intersections of cayleyan and cubic
... remain further conjectures.

Best regards Eckart

PS: Excuse, that I overlook the property "dual" in #2359.
But I think it's new, that the polar conics of Xi
... bear the further intersections of Po.Fi and the cubic.
```



2024-06-29.pdf

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Message: #2363
Date: 2024-06-29
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

```
Dear Eckart,
1) Sorry, I forgot to confirm your statement in message 2356
The 2 further intersections (other than Fi) of PFi are contact
points of tangents from Xi to the cubic (as well as Fj and Fk).
2) Congratulations for your drawing of the cayleyan
I thought Cabri was easier to handle as Geogebra, but apparently
it isn't true!
Thanks anyway for your tenacity
Best regards
Bernard
```

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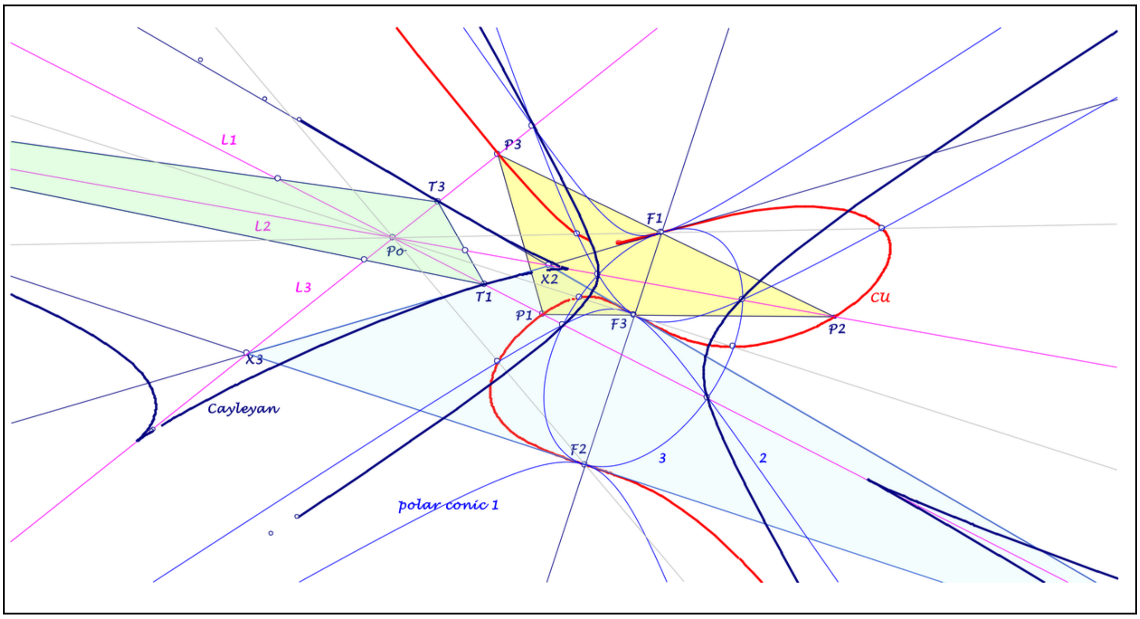
Message: #2364
Date: 2024-07-02
From: eckart_schmidt@t-online.de
Subject: Re: Cayleyan and dual cubic

Dear Bernard,

I have modified your Caylean construction in #2316,
... with a macro for a not quite exact drawing:
Let us consider your cubic, dual of the cayleyan wrt the conic CO,
... as nonpivotal isocubic wrt the triangle T1T2T3,
... root Po, fixpoint of the isoconjugation Po and a point Q.
Then we can construct points of the cubic
... and their polars wrt your conic CO,
... inscribed X1X2X3, tangent to the sides in T1,T2,T3,
... which envelope the cayleyan.
This gives a macro , depending of the point Q.
The polar conics of Xi are tangent to the flextangents in Fj, Fk,
... bearing the intersections of PoFi and the cubic,
... they give 6 points of the cayleyan.
It is easy to arrange the point Q,
... so that the macro above gives the caylean,
... attached an example.

Best regards Eckart



2024-07-02.pdf

Message: #2365
Date: 2024-07-03
From: bernard.keizer@gmail.com
Subject: Re: Cayleyan and dual cubic

Dear Eckart,
I don't want to start a new argument with you (there are other
more disturbing events in France by the time), but frankly
1) I don't see why you need an approximate constrction in order
to draw a cubic for which you already know 12 points
2) I don't understand either why it is easier to draw the
cayleyan as envelop of lines raher than as locus of points
Best regards
Bernard

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Message: #2366
Date: 2024-07-05
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

Your construction of a nonpivotal isocubic in #2348 looks very
interesting.
Nevertheless I can't follow your brief instructions.

A complete description of a construction would be like:
1. given triangle A1A2A3, B=root P and an extra point C
2. given isoconjugate QA-Tf2(Q) with QA=root P plus its
anticevians.
3. now nK is the locus of QA-Tf2(X) with variable point X on
line L???/curve Cv??? (WHICH LINE OR CURVE?)
I miss the description of step 3.
When you read your own message #2348 you should agree that you
don't describe this step.

Best regards,
Chris

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Message: #2367
Date: 2024-07-06
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

excuse my lack of understanding for your point 3), I can only
repeat:
If we accept, that an isoconjugation * wrt a triangle ABC
... is defined by a fixed point P,
... we can draw a nonpivotal isocubic with root P and a point Q
... with Gibert's 1.5.4 without any line or curve L.

Best regards Eckart

PS:
Wrt 2): I think, there is a typo, replace Q by X.
Wrt 3): nK is invariant wrt QA-Tf2.
```

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Message: #2368
Date: 2024-07-07
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

wrt your starting in #2366
..." Your construction of a nonpivotal isocubic in #2348 ...":
It is not my construction, but Gibert's 1.5.4.
Have you read my message #2358 meanwhile,
... where I gave a drawing for Gibert's construction?

Best regards Eckart
```

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Message: #2369
Date: 2024-07-09
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart,

Thanks to your perseverance I now understand where I
misunderstood.
I considered your message #2348 to be a construction of the nK
of your own.
When I reread you message for the umpteenth time I saw my
misinterpretation.
I am sorry for the misunderstanding.

In the meantime I made a construction of the nK according to
Bernard Gibert’s description.
He describes several points of the nK to be constructed. I did
do so and applied a macro for constructing a cubic with 9
points.
I checked that the cubic was invariant wrt QA-Tf2 indeed.

Then I made a macro. Unfortunately it appeared that the macro
doesn’t work for every constellation of reference triangle, root
and extra point.
Therefore I was not able to check if CU was a non-pivotal
isocubic wrt ref-triangle T1T2T3 with isoconjugation-fixpoint Po
and root Po and a hessian point Q.
I'm going to see if I can achieve something else in a different
way.

Best regards,
Chris
```

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Message: #2370
Date: 2024-07-12
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart and Bernard.

Eckart, thanks for recent support, ideas and comments!
I wonder if you effectively constructed the nK from T1T2T3 with root Po and any point of CU, resulting in CU being delivered as nK?
Then, I promised to see if I could achieve something else in a different way regarding your message #2344. More came out than I expected.

Setup in Mathematica
I made a setup of a random cubic CU with 9 random points. Then I calculated
HE and 9 intersection points Fi (i=1,…9) of which 3 points are real. and T1,T2, T3 being the intersection points of Harmonic Polars Li.
Let (P1a,P1b,P1c) be the 3 points of intersection of CU with the Harmonic Polars L1 (2 points can be imaginary).
Let (P2a,P2b,P2c) be the 3 points of intersection of CU with the Harmonic Polars L2 (2 points can be imaginary).
Let (P3a,P3b,P3c) be the 3 points of intersection of CU with the Harmonic Polars L3 (2 points can be imaginary).
Here are my conclusions based on EMPIRICAL PROOF:

Conclusion-1
Empirical proof shows that:
The XYZ-Po-isoconjugation (XYZ=Reference Triangle, Po=Root) maps a point of CU to another point of CU and maps a point of HE to another point HE,
where XYZ = T1T2T3 or P1xP2yP3y (x,y,z can be replaced by any letter of the set (a,b,c)). There is no pivot point.
P1xP2yP3z-Isoconjugation (for all possible P1xP2yP3z-reference-triangles) wrt CU-points all deliver the same CU-isoconjugate point.
Also the T1T2T3-Isoconjugation wrt CU-points delivers the same CU-isoconjugate point.
P1xP2yP3z-Isoconjugation (for all possible P1xP2yP3z-reference-triangles) wrt HE-points all deliver the same HE-isoconjugate point.
Also the T1T2T3-Isoconjugation wrt HE-points delivers the same HE-isoconjugate point.
The Isoconjugation-property wrt CU or HE holds also when points of the reference triangle are imaginary.

So CU and HE are both non-pivotal cubics of wrt reference triangles T1T2T3 as well as P1xP2yP3z with root Po, where:
* P1x is any of the points (P1a, P1b, P1c)
* P2y is any of the points (P2a, P2b, P2c)
* P2z is any of the points (P3a, P3b, P3c)
For example P1aP2aP3a or P1aP2bP3c or P1bP2aP3c are valid Reference Triangles.
There are 3^3 of these possible P1xP2yP3z-Reference Triangles.

Conclusion-2
Since the isoconjugation wrt T1T2T3 or P1xP2yP3z and root Po is valid for CU and HE, it will be that:
* Any of the 9 crosspoints CU^HE (which are the 9 flexpoints Fi (i = 1, ..., 9)) will be mapped into another point that is a common point.
* Since there are general principles for general points it cannot be that Fi is mapped into Fj with i <> j, because there are no general rules that effectuate this in a general way.
For example when F1 is mapped into F2, there will be a correlation between these points, whilst they are mutually independent points.
So it only can be that F1 is mapped into F1, F2 into M2, etc.
Calculations confirm this.
So All 9 Flexpoints are fixed points of the T1T2T3 - Po - Isoconjugation or P1xP2yP3z - Po - Isoconjugation.

Conclusion-3
Since the T1T2T3-Po-Isoconjugation relates to (T1,T2,T3) being a subset of only the real points of the full set (T1,...,T9), there also will be more TiTjTk-Po-Isoconjugates where Po will be the common point of Harmonic Polars (Li,Lj,Lk).
Let's call Po in this case Pijk. The 9 Harmonic Polars form a network with 12 crosspoints of 3 Harmonic Polars (Li,Lj,Lk) intersecting in Pijk. For each root Pijk there are corresponding vertices of a triangle TiTjTk. So there are 12 TiTjTk-Pijk-Isoconjugations (of which at least one is based upon real points).
Subsequently there are also 12 isoconjugations wrt Reference triangles PixPjyPkz.

Final Conclusion
The degrees of freedom of the Isoconjugation regarding the chosen reference triangle and the chosen root even become greater:
· There are 12 Roots Pijk (of which at least 1 is real, which is Po)
· There are 12 TiTjTk-Triangles (often with imaginary vertices)
· There are 12 Pijk-Cevian Triangles UiUjUk of TiTjTk

· There are 12*27 PixPjyPkz-Triangles (Pia=1st intersection point of CU^Li, Pib=2nd intersection point of CU^Li, i=1-9, (x,y,z) can be replaced by (a,b,c), etc.)
Chose one of the many Triangles and chose one of the 12 Roots.
Apply the isoconjugation ICx wrt the chosen triangle and the chosen root and the outcome will be the same for any other choice.
When a point P is chosen on Reference Cubic CU, then ICx(P) will lie on CU.
When a point P is chosen on the Hessian HE, then ICx(P) will lie on HE.
The Isoconjugation ICx is non-pivotal.

Conjecture
Since the T1T2T3-Po-Isoconjugation has these 9 fixpoints, it will be that all Cubics through the 9 Flexpoints, that means all cubics of the Syzygetic Pencil, will all be Isocubics wrt the described set of Isoconjugations!
I hope this will give plenty of new ideas.

Best regards,
Chris

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Message: #2371
Date: 2024-07-12
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart and Bernard,

I want to generalize the properties of my last message #2370.
I made a new set up in Mathematica for empirical proof of next general properties:

1. CU is a nonpivotal Isocubic wrt any triangle made up from vertices spanned on any 3 (out of 9) Harmonic Polars and with any point in the plane functioning as Root.

2. Apply the isoconjugation IC wrt the chosen triangle and some random root and the outcome will be the same for any other choice of Harmonic Polars and Root.

3. When a point P is chosen on Reference Cubic CU, then IC(P) will lie on CU.

4. When a point P is chosen on the Hessian HE, then IC(P) will lie on HE.

5. The Isoconjugation IC is non-pivotal.

Conjecture : All these properties are valid for all cubics of the Syzygetic Pencil.

Best regards,
Chris

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Message: #2372
Date: 2024-07-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

thanks for your extensive messages,
... here only a remark wrt the first sentence of #2370:
"I wonder if you effectively constructed the nK from T1T2T3
... with root Po and any point of CU, resulting in CU
    being delivered as nK?"
Is there a misunderstanding, CU has to be replaced by HE (see
#2344).
To get the hessian of a cubic CU,
... you have to take as reference triangle T1T2T3 of CU,
... iso-fixed points beside Po are X1,X2,X3 of CU,
... Po of the hessian is Po of CU,
... finally you need a point Q of the hessian.
Now I shall study your conclusions.

Best regards Eckart
```

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Message: #2373
Date: 2024-07-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

what are your points "...T1,T2, T3 being the intersection points
of Harmonic Polars Li."
... in #2370, Setup in Mathematica?
I used always Ti as intersection of Li and the flextangent in
Fi,
... Pi as intersection of Li and the nonclosed part of CU,
... Xi as the intersection of the flextangents in Fj,Fk
... for real flexpoints F1,F2,F3.

Best regards Eckart
```

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Message: #2374
Date: 2024-07-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

excuse my questions wrt conclusion-1, is your
... "XYZ-Po-isoconjugation (XYZ=Reference Triangle, Po=Root)"
... an isoconjugation with fixed point Po?
Then the following doesn't hold
... "P1xP2yP3z-Isoconjugation (...) wrt CU-points
     all deliver the same CU-isoconjugate point".
If the reference triangles P1xP2yP3z for a bipartite cubic
... lie on different parts of the cubic, it seems,
... that the cubic is invariant, but wrt different image points.
I am total confused by your conclusions.
Cubic and its hessian need different ref-triangles and
isoconjugations
... to be described as nonpivotal isocubics.

Best regards Eckart
```

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Message: #2375
Date: 2024-07-13
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

I try in vain to understand your last message #2371,
starting with 1.
"CU is a nonpivotal Isocubic wrt any triangle made up from
vertices
... spanned on any 3 (out of 9) Harmonic Polars
... and with any point in the plane functioning as Root."
The root defines a tripolar, intersecting the sidelines of the
triangle in U,V,W ...
... but what about your isoconjugation IC?
What is the definition of your isoconjugation IC?
Can you give me two isoconjugated points or fixpoints of the
isoconjugation,
... to study your properties?
Best regards Eckart
```

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Message: #2376
Date: 2024-07-13
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Eckart,

Regarding your message #2373: I Meant with T1,T2,T3 the same as you: the intersection of Li and the flextangent in Fi. I am sorry about my incomplete description.
In your descriptions Pi = intersection of Li and the nonclosed part of CU. In my description P1x, P2y, P3z are all 3 intersection points of Li and CU. I don't distinguish in my description a closed and non-closed part of CU, but just all 3 intersection points. Because it is difficult to distinguish closed and non-closed algebraically.

Regarding your message #2374 about my use of the word "Root". I thought I used your definition, like you use it in #2364 "root Po, fixpoint of the isoconjugation Po".
For me an XYZ-Po-isoconjugation (XYZ=Reference Triangle, Po=Root) is QA-Tf2, like you explained before, where QA=(Po + vertices of Po-Anticevian-triangle of XYZ).

Further my conclusions were based upon calculations in Mathematica. They all pointed in the same direction. I felt very confident about the results.
However I did not check them in Cabri.
Maybe there are discrepancies. Maybe they represent different worlds. I will check it next days.
I hope it fits together.

Best regards,
Chris

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Message: #2377
Date: 2024-07-14
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

Dear Chris,

thanks for clearence, so we consider the same isoconjugation
... wrt P1xP2yP3z with fixpoint Po, but the following doesn't
hold:
"P1xP2yP3z-Isoconjugation (...) wrt CU-points all deliver the
same CU-isoconjugate point."

Consider a bipartite cubic with 27 possible ref-triangles
... which give different isoconjugates for the same cubic-point.
I only considered the ref-triangle on the infinite part P1P2P3.

If we consider ref-triangles on the closed part,
... there is one triangle with the flexpoints on its sidelines,
... then the corresponding isoconjugate of a cubic-point
... is again a cubic-point, but different.

Best regards Eckart

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Message: #2378
Date: 2024-07-14
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
I don't know where to start in your discussion, it's perhaps already too late!
1) the root doesn't play an important role, it is simply the trilinear pole of the line through the 3rd intersections of the sides of the reference triangle.
For all our triangles, this line is the flexline and the root is P0. Here the point which matters is that P0 is also one of the fixed points of the isoconjugation.
2) the QA-Tf2, their the reference triangles and their fixed points other than P0 are not the same for CU and HE and the QA-Tf2 of the flexes have no reason to be the same point.
Anyhow, an isoconjugation can have only 4 points (P0 and the vertices of the anticevian triangle of P0 wrt the reference triangle) and they can certainly not be the flexes!
(For example, on CU, QA-Tf2(Fi) is Pi with Eckart's definition and on HE, QA-Tf2(Fi) is Ti ...)
3) An isoconjugation is neither pivotal nor non pivotal, the cubic invariant in this isoconjugation is a pivotal pK or a non pivotal nK ...
By the way, a bipartite cubic can be a nK for certain triangles and a pK for other.
Maybe the reading of Bernard Gibert's chapter 1 (not only the construction 1.5.4) could be useful
Best regards
Bernard

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Message: #2379
Date: 2024-07-14
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart and Bernard,

Like I already told I felt very confident about the results of
my messages #2370 and #2371. However I did not check them in
Cabri.
I did some quick checks now In Cabri an they do not match with
my calculations in Mathematica.
So I have to go back to the drawing board and look after the
mismatches.
I am terribly sorry that I did not do that in advance before
sending you my messages.
So please forget my messages #2370 and #2371. I'll let you know
what remains after more extensive checks.

Best regards,
Chris
```

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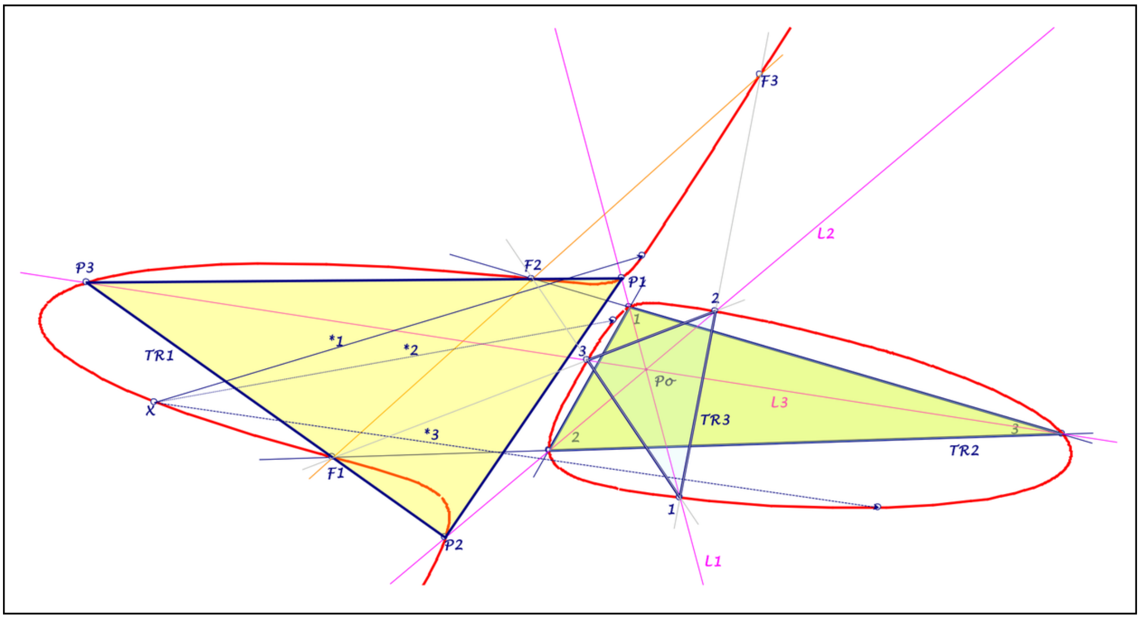
Message: #2380
Date: 2024-07-15
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Chris,

I have to correct my #2377:
Let us consider a bipartite cubic CU (attached)
... with 9 CU-intersections of L1,L2,L3,
... which give three triangles A1A2A3, AiAj bearing Fk,
... one triangle is P1P2P3 on the infinite part,
... the other two lie on the closed part.
Each triangle gives an isoconjugation with fixed point Po,
... which lets the cubic invariant.

Best regards Eckart
```



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Message: #2381
Date: 2024-07-15
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Chris, dear Eckart
This seems more reasonable!
In fact, a bipartite cubic is caracterised by 3 QA-Tf2.
The 3 images of any point of the cubic in these 3
transformations give 3 points having the same tangential as the
initial point.
The 4 points are the fixed points of a 4th isoconjugation and
the cubic is a pivotal isocubic with pivot the common
tangential.
The bipartite cubic is a triple nK in these 3 transformations
and a pK for an infinity of isoconjugations ...
Best regards
Bernard
```

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Message: #2382
Date: 2024-07-15
From: bernard.keizer@gmail.com
Subject: Approximate construction of the hessian

Dear Eckart,
You have asked many times for an easy approximate construction of the hessian of a cubic.
Here is a simple one.
Once having the real flexes Fi and the harmonic lines Li as well as the points Xi and Ti, take a random point M1 and find the 5 other points M2, M3, N1, N2 and N3 such as Fi, Mi and Ni are aligned as well as Fi, Mj and Nk (I explained this construction already in an old memo, Ni is the 4th harmonic of Mi wrt Fi and the intersection of FiMi with Li ...).
The 6 points Mi and Ni are coconic and define with the 3points Fi a cubic having the Fi as real flexes, but not the 6 other flexes (this cubic doesn't belong to the syzygetic pencil).
But it's not too difficult to choose the point M1 such as the cubic passes through the points Ti ...
This time it is the hessian belonging to the syzygetic pencil!
And it's also easy to check that this hessian is invariant in the isocojugation wrt the triangle of the Ti with fixed points P0 and the points Xi.
Best regards
Bernard

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Message: #2383
Date: 2024-07-16
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
The isoconjugations in which CU or HE are invariant (1 for a monopartite cubic, 3 for a bipartite cubic) are the same transformations as the QMT we have already studied!
They associate to a point the point(s) having the same tangential ...,
Best regards
Bernard

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Message: #2384
Date: 2024-07-17
From: eckart_schmidt@t-online.de
Subject: Re: Approximate construction of the hessian

Dear Bernard,

I thought, you don't want to start a new argument with me, see #2265 ...
Nevertheless: Are Mi, Ni the points in your paper "Cubic, hessian and cayleyan"?
Then these 6 points and the 3 Fi define a conic and a line as degenerated cubic,
... and changing M1 gives another conic and the same line,
... but not the hessian, where is my wrong conclusion?

Best regards Eckart

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Message: #2385
Date: 2024-07-17
From: bernard.keizer@gmail.com
Subject: Re: Approximate construction of the hessian

Dear Eckart,
Yes, the points are the Mi and Ni of my paper "cubic, hessian and cayleyan".
Naturally, you are right, the 6 coconic points form a CB system with the Fi and a CB system with the Ti.
Normally, Geogebra should return an undetermined cubic.
Surprisingly,with the Ti, it returns for any point Mi a cubic through the Ti and the Fi, tangent in Ti to FiTi!
My construction doesn't work in fact.
Thanks for your comment
Best regards
Bernard

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Message: #2386
Date: 2024-07-21
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart and Bernard,

I re-reviewed, adjusted, and re-ran all my calculations in
Mathematica.
It turned out that there was a very essential Mathematica macro
that was not functioning properly. After recalculation I arrived
at a confirmation of what Eckart stated in #2344:
1) A non-degenerated cubic
... is a nonpivotal isocubic with ref-triangle P1P2P3
... with isoconjugation-fixpoint Po and root Po
    and a cubic-point Q.
2) The hessian of a non-degenerated cubic
... is a non-pivotal isocubic wrt ref-triangle T1T2T3
... with isoconjugation-fixpoint Po and root Po
    and a hessian point Q.

Just like that, no more.
Once again, apologies for jumping too fast to conclusions and
the fuzz it caused.

I checked also your statement in #2380:
about a bipartite cubic CU
... with 9 CU-intersections of L1,L2,L3,
... which give three triangles A1A2A3, AiAj bearing Fk,
... one triangle is P1P2P3 on the infinite part,
... the other two lie on the closed part.
Each triangle gives an isoconjugation with fixed point Po,
... which lets the cubic invariant.

Calculations in Mathematica confirm this.
Calculations also confirm that three triangles can be formed for
a cubic of every shape (not only bipartite) with A1A2A3, with
AiAj bearing Fk. Only then we have to deal with imaginary
intersection points.
Calculations also confirm that this is not only true for the 3
real Flexpoints, but also for combinations with imaginary
Flexpoints, leading to imaginary Harmonic Polars, etc.
```

Actually for any set of 3 Collinear Flexpoints out of the set of 9 Flexpoints (F1, …, F9), leading to 3 concurrent Harmonic Polars (Li,Lj,Lk) out of the set of 9 Harmonic Polars (L1, …, L9), real or imaginary, leading to 3 sets of 3 intersection points (Aix,Ajy,Akz), with AixAjy bearing Fk, AjyAkz bearing Fi, AkzAix bearing Fj, it will be that CU is a non-pivotal Isocubic with reference triangle AixAjyAkz with isoconjugation-fixpoint Li^Lj^Lk.
There are 12 possible sets of 3 collinear Flexpoints out of the set of 9 Flexpoints (F1, …, F9), so there will be 12 different isoconjugations, each mapping any CU-point into another CU-point.

Best regards,
Chris

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Message: #2387
Date: 2024-07-22
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris,
Amazing and magical, these properties with imaginary points and lines!
Congratulations for your calculations with Mathematica
1) and 2) from Eckart's message 2344 nothing new, but it's good that you confirm
3) Eckart's statement in message 2380 the same
For the real flexline, as I put 10 times on the Forum, without reaction from you or Eckart
1) these 3 transformations (real or imaginary) are the same as the QMT we already studied
2) they associate to a point of the cubic the point(s) having the same tangential
3) they refer to the 3 prehessians (one at last being real) of the cubic, as a point and it's transformed are corresponding points in the sense of Salmon or Schröter, meaning that each one is the center of the degenerated polar conic of the other wrt the prehessian (see Cüppens pages 239/240).
For example, for the hessian HE of a reference cubic CU, the points Ti (or Ui or Vi, intersections of Li with HE) are the corresponding points of the Fi, have the same tangential (Fi being it's own tangential) and are the centers of the degenerated polar conics of Fi wrt CU (formed by Li and FiTi), Fi being conversely the center of the degenerated polar conic of Ti wrt CU (formed by FiWi and FiZi).
Now, for the 9 flexes on 12 lines (3 flexes on a line and 4 lines through each flex), you have 9 harmonic lines(naturally all tangents to the cayleyan) through 12 points P0 (each point on 3 lines and each line through 4 points) and 12 isoconjugations.
But I have some doubts that any point of any cubic could be the tangential of more than 4 points and that a cubic could have more than 3 prehessians.
Is it by any chance possible that the 12 isoconjugations give in fact 4 times the same isoconjugate? (each point on the curve and in particular each flex having in fact only 3 corresponding points)?
Best regards
Bernard

Message: #2388
Date: 2024-07-23
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Bernard,

I was just about to check if the 12 isoconjugations (described in my last message) possibly could have identical outcomes.
Then I saw your message coming in with the very same question.
I need some time to check it properly.

About your statement that these 3 transformations (real or imaginary) are the same as the QMT, I have some doubts.
The QMT as we defined it earlier is strictly a transformation centered in one of the vertices of one of the 3 Quasi-Miquel Triangles, which isn't the case here.
Can you shed some light on this?

You also say "They associate to a point the point(s) having the same tangential ...". That part I can agree with, but that doesn't make it a QMT.

Best regards,
Chris

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Message: #2389
Date: 2024-07-23
From: eckart_schmidt@t-online.de
Subject: Re: Real cubic elements

```
Dear Bernard, dear Chris,

wrt the three isoconjugations in #2380, perhaps evident:
Their product is the identity for cubic-points.

Best regards Eckart
```

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Message: #2390
Date: 2024-07-23
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Eckart,
Yes, it is evident!
As each transformation swaps a point with another point having
the same tangential and as there are 4 points having the same
tangential,
1 gives 2, 2 gives 3, 3 gives 4 and 4 gives 1!
Best regards
Bernard
```

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Message: #2391
Date: 2024-07-24
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
In the message 2387, I gathered the 3 properties of these 3 fundamental transformations for any cubic.
As QMT and the isoconjugations have the same property 2 with points having the same pretangential, it is naturally and undoubtely the same transformation!
You don't have necessay to start the QMT with the QM points (3rd intersection of the line through 2 points having the infinity point of the asymptote as pretangential).
A simple Fred Lang's calculation shows that it works in fact for any copple of points X and X' having the same pretangential T and the 3rd intersection S, which is the transform of T. X'-X = T-S = Ω-M ... We discussed all this stuff already!
Perhaps a better formulation for Eckart's property:
If 1, 2, 3 and 4 are the 4 points having the same pretangential and Tf1, Tf2 and Tf3 are the 3 transformations mentionned in Salmon, Schröter, Cüppens or Bernard Gibert,
Tf1 swaps 1 and 2 (and 3 and 4)
Tf2 swaps 1 and 3 (and 2 and 4)
Tf3 swaps 1 and 4 (and 2 and 3)
and obviously Tf3 = Tf1*Tf2 and the like and Tf1*Tf2*Tf3 = I(dentity)
Best regards
Bernard

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Message: #2392
Date: 2024-07-24
From: eckart_schmidt@t-online.de
Subject: Gibert's circle 1.5.3 for nK isocubics

```
Dear Bernard, dear Chris,

if we accept, that a non-special cubic is a nonpivotal isocubic,
... we can study Gibert's circle in the theorem 1.5.3:
"Any nK can be considered as the locus of point M
... such that the points M and M* are conjugated
... wrt to a fixed circle."
Therefore a cubic can be defined
... by a circle CI and an isoconjugation *,
... (defined by a triangle ABC and a fixpoint K).
An approximated drawing of the cubic is possible,
... looking for points X, which lie on the circle-polar of X*
... and using a 9-point construction with A,B,C.

Let us consider such a cubic:
(1) The cubic is invariant wrt the isoconjugation *.
(2) AB intersects the circle-polar of C on the cubic in W,
    analog for U, V.
(3) U,V,W are collinear, root R is the tripol of the line UVW.
(4) The isoconjugate conic CO of UVW
    contacts the cubic in A,B,C.
(5) Gibert's circle is orthogonal to all circles
    with diameter XX*.

On the other hand, starting with a 9-point cubic,
... there are different isoconjugations,
    which let the cubic invariant
... and give different Gibert-circles, not always real.

Best regards Eckart
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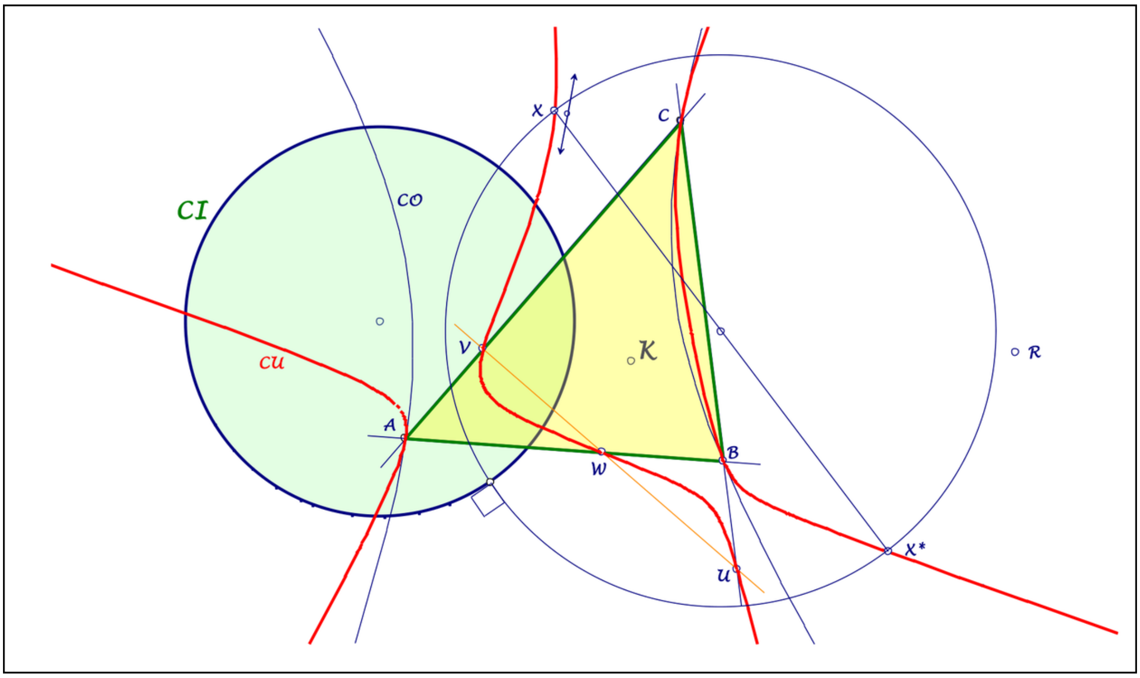

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Message: #2393
Date: 2024-07-25
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

Dear Bernard,

Now I understand what you mean.
QMT and the isoconjugations are the same type of tran
sformation.
Let IC be the name for the Isoconjugations of Eckart with
fixpoint Po.
Let QMT be the name of the Quasi-Moebius Transformation,
centered in one of the vertices of the Quasi-Miquel Triangles.
They are the same type of transformation but not the same
transformation, because:
1. QMT is based upon a Quasi-Miquel Triangle-vertex; there are 3
vertices per triangle giving 3 QMT's,
2. IC is based upon a triangle and a fixpoint; there are 3
triangles with an identical fixpoint giving 3 IC's.
Therefore, they are different transformations.

However, they have in common:
1. There are 3 transformations that, when combined, produce the
identity.
2. Any cubic point P and its 3 mapped points P1, P2, P3 share
the same tangential.

Therefore, they are the same type of transformation, producing
an identical outcome.

Best regards,
Chris

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Message: #2394
Date: 2024-07-26
From: bernard.keizer@gmail.com
Subject: Re: Gibert's circle 1.5.3 for nK isocubics

```
Dear Eckart
Beautiful property!
I applied it without difficulty to a CU with triangle P1P2P3 and
it's HE with triangle T1T2T3.
But I found no particular definition for the center of the
Gibert's circles ...
Best regards
bernard
```

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Message: #2395
Date: 2024-07-26
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

```
Dear Chris,
You are playing with words!
The transformations associating to a point always the same point
on a cubic are not the same type of transformation, but the same
transformation!
Any cubic is a nK wrt any inscribed triangle, with different
roots and fixed points (it can even be a pK if the fixed points
are on it).
But it remains that all these nK's swapping 2 points having the
same tangential are the same transformation ...
The same goes for the QMT's, which can be defined wrt all the QM
points of the tangential QA's of the cubic.
All these QMT's are the same transformation ...
Each cubic is defined by it's 3 fundamental transformations
associating the vertices of any tangential QA.
Please reread Salmon, Schröter, Cüppens or Bernard Gibert in
order to be convinced, I personnally give up!
Best regards
Bernard
```

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Message: #2396
Date: 2024-07-26
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Bernard,

Indeed it's a question of words.
Maybe even a difference in the use of words in French and Dutch
language.
You call it the same transformation and I call it the same type
of transformation.
Because of my interpretation I did not understand you, but now I
do. I hope you understand.
It was in no way meant personal or in any irritation.

By the way, there is another difference between the same
transformation with IC-construction and the QMT-construction:
The QMT-construction is also applicable for points not on a
cubic.
Do you have any references in pages for Salmon, Schröter,
Cüppens and Bernard Gibert?
I did not know these authors describing this transformation.

Best regards,
Chris
```

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Message: #2397
Date: 2024-07-26
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris,
Thanks for your kind message!
The main thing is that we agree and understand eachother ...
I think, you're right, different IC or QMT are different transformations.
But different transformations can have the same result (swapping the same points).
By the way, IC construction also works for points not on the cubic ...
There are only 3 types of transformations which leave a cubic globally invariant.
I'm stil waiting impatiently for your checking with Mathematica that the 12 isoconjugations mentionned in your message 2386 give in fact 4times the same conjugate (as asked in my message 2387 and promised in your message 2388).
So long
Best regards
Bernard

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Message: #2398
Date: 2024-07-26
From: van10hoven@gmail.com
Subject: lost papers Forum Geometricorum

Dear friends,

I recently heard that all papers from Forum Geometricorum are no longer available at their usual site.

In case you are interested, Francisco Javier García Capitán has posted a link to the files here:
https://garciacapitan.blogspot.com/2020/12/links-related-to-triangle-geometry.html

Fred Lang's paper, which we used so much, is one of these papers. And if I am not mistaken, there is also a paper from Eckart among them.
Obviously, this happens at a time with the most valuable papers.

Bernard and Eckart, is it possible for me to get the papers you published on your own site so that at least at one other place your valuable information will be stored?
If it is a big file, you can send them (eventually compressed as a zip-file) to my email address or otherwise to this forum.

Best regards,
Chris

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Message: #2399
Date: 2024-07-26
From: garciacapitan@gmail.com
Subject: Re: lost papers Forum Geometricorum

```
Hello everybody,

I am contacting FAU Departament of Mathematics, perhaps the
website could be restored.
I wish it with all my heart.

Francisco Javier.
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Message: #2400
Date: 2024-07-27
From: anopolis72@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] lost papers Forum Geometricorum

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Hmmm.... what will happen one day in the future to ETC or CTP?

The only one who took care of the survival of his database is
Neil Sloane with OEIS Foundation

APH

On Fri, Jul 26, 2024 at 10:54 PM Francisco Javier García Capitán
via
groups.io <garciacapitan=gmail.com@groups.io> wrote:

> Hello everybody,
> I am contacting FAU Department of Mathematics,
> perhaps the website could be restored.
> I wish it with all my heart
> Francisco Javier.
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Message: #2401
Date: 2024-07-26
From: bernard.keizer@gmail.com
Subject: Re: Gibert's circle 1.5.3 for nK isocubics

Dear Chris, dear Eckart
In message 2086, I gave the construction found in Schröter of a cubic, starting with 3 copples of points.
The cubic is a nK with the transformation swapping A and A', B and B' and C and C'.
Then Gibert's circle is centered in the radical center of the 3 circles with diameters AA', BB' and CC' and orthogonal to these 3 circles.
By completing the QL's with AA'BB', AA'CC' and BB'CC', the center of the Gibert's circle is the intersection of the 3 Steiner Lines of the 3 QL's.
Best regards
Bernard

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Message: #2402
Date: 2024-07-26
From: van10hoven@gmail.com
Subject: Re: lost papers Forum Geometricorum

Dear Francisco, Antreas and friends,
We are once again confronted with the fact that painstakingly collected information can suddenly disappear into the maelstrom of time.
I believe the best possibility for preserving this information arises when a reputable mathematical organization sets the goal of preserving it.
I have tried to organize this before, but the timing wasn't right.
It would be helpful if someone with good contacts at such an organization could make them aware of this need.
There are also benefits for the organization, as the number of hits on their website would greatly increase. It doesn't have to be difficult because most website operators often have their information stored compactly elsewhere. The most important thing is that they provide storage capacity.
For the time being, we are still dependent on individuals who store this type of information privately like you and me.
Best regards, Chris

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Message: #2403
Date: 2024-07-27
From: eckart_schmidt@t-online.de
Subject: Re: Gibert's circle 1.5.3 for nK isocubics

```
Dear Bernard,

thanks for interest,
... wrt your first property see already 5) in my message,
... in addition to your second property:
... Gibert's circle is orthogonal to all QL-Ci5 of your QLs.

Best regards Eckart
```

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Message: #2404
Date: 2024-07-30
From: van10hoven@gmail.com
Subject: Re: Real cubic elements

```
Dear Bernard,

#2397
“I'm stil waiting impatiently for your checking with Mathematica
that the 12 isoconjugations mentionned in your message 2386 give
in fact 4times the same conjugate (as asked in my message 2387
and promised in your message 2388).”

I will gladly do this in the foreseeable future after working
out another very interesting subject.

Best regards,

Chris
```

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Message: #2405
Date: 2024-07-30
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
I checked, as a confirmation of properties already mentionned, following property:
considering the triangle H1H2H3 of the flexpoints and the isoconjugations wrt P1P2P3 for CU and T1T2T3 for HE with fixed points P0 and the vertices of the anticevian triangle of P0 wrt P1P2P3 or T1T2T3, the isoconjugates of the sides HjHk are conics through P1,P2 and P3 and T1,T2 and T3 tangent to CU and HE in Pi or Ti with no other intersection with CU and HE.
It means that the isoconjugates of the 2 imaginary flexes on HjHk are also imaginary on these conics.
Best regards
Bernard

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Message: #2406
Date: 2024-07-31
From: bernard.keizer@gmail.com
Subject: Re: Real cubic elements

Dear Chris, dear Eckart
Naturally, the IC of the flexline wrt P1P2P3 and T1T2T3 are conics inscribed in the anticevian triangles and tritangent to CU and HE in P1,P2and P3 and T1,T2 and T3.
Best regards
Bernard

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Message: #2407
Date: 2024-07-31
From: van10hoven@gmail.com
Subject: Re: lost papers Forum Geometricorum

Dear Bernard,

Thanks for sending me your 9 papers on my email-address.
I stored them on my computer.
Now your papers are stored at least at one other place.

Best regards,
Chris

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Message: #2408
Date: 2024-08-01
From: bernard.keizer@gmail.com
Subject: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris, dear Eckart
1) Let's consider a cubic CU and it's hessian HE
Li cuts CU in Pi, Qi and Ri and the hessian in Ti, FiTi on XjXk
being tangent in Ti to the hessian and the cayleyan
The flextangents intersect in Xi, Xj and Xk, the triangle T1T2T3
being the cevian triangle of P wrt X1X2X3
Then Pi, Qi and Ri are the points Ti for the 3 prehessians of CU
All this is wellknown
2) Let's consider now the triangle H1H2H3 of the 3 real
flexlines (other than the flexline F1F2F3)
Hi is the harmonic of Ti wrt Qi and Ri
h1h2h3 is the cevian triangle of P wrt H1H2H3
This triangle H1H2H3 is a degenerated cubic RF belonging to the
syzygetic pencil, as it contains the 9 flexes (3 real F1, F2 and
F3 and 6 imaginary)
For this cubic, hi is Ti as well as Pi and Hi is Xi as well as
Qi and Ri (naturally, Hi is the harmonic of hi wrt Hi and Hi)
Then the hessian of this cubic contains the 9 flexes and hi, it
is therefore the cubic RF itself, which is selfhessian
This cubic RF has 3 prehessians, one has Fihi as flextangents,
it is naturally the cubic RF again (it's hessian being RF), the
2 others are the same cubic with FiHi as flextangents.
3) There are 12 lines through the 9 flexes, 4 being real and the
8 other imaginary
There are 3 other triangles like H1H2H3, with the flexline
F1F2F3 and the 8 imaginary lines as sides
These 3 triangles have the same property as H1H2H3, as they pass
also through the 9 flexes
They form also 3 degenerated cubics of the pencil, with the same
property that they are their own hessian and prehessian (each
having another prehessian).
Best regards
Bernard

Message: #2409
Date: 2024-08-03
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Bernard,

Thanks for your precise description in #2408. Very interesting!

I cannot follow all of your reasoning about (pre)hessians, but what stands out to me is your remark 3), especially that there are three other triangles H1H2H3. I happened to study this from a different angle. These four triangles H1H2H3 are the key to combining the network you described from 9 Flexpoints and 12 Flexlines (3 points per line, 4 lines per point) with the network from 9 Harmonic Polars (HP) and 12 HP-Crosspoints (4 points per line, 3 lines per point). By having one network, it is possible to determine the other using these four triangles H1H2H3.
Very special also is that H1, H2, and H3 are HP-Crosspoints. The vertices of the four triangles together happen to be the 12 HP-Crosspoints. Right now, I am properly describing this in a small paper and hope to post it soon.

It is nice to see that the triples of the sidelines of the four triangles also function as degenerate cubics belonging to the syzygetic pencil."

Best regards,
Chris

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Message: #2410
Date: 2024-08-04
From: van10hoven@gmail.com
Subject: Flex-network and Harmonic Polar-network

Dear Bernard and Eckart,

There are several special features about the Flexpoints and Harmonic Polars of a cubic.
We are familiar with the networks they form, but I have discovered how these networks can be integrated and even derived from each other using 4 Triangles.

Here is a summary:
There are two networks regarding Flexpoints and Harmonic Polars (HP) wrt a cubic CU:
1. The Flex-network, containing 9 Flexpoints and 12 Flexlines (3 points per line, 4 lines per point).
2. The HP-network, containing 9 Harmonic Polars (HP) and 12 HP-Crosspoints (4 points per line, 3 lines per point).
Both networks are dual with respect to each other, as one has 3 points per line and 4 lines per point, while the other has 4 points per line and 3 lines per point.
It is shown that one network can be determined from the other and vice versa. Moreover both networks can be integrated.

Key is a configuration of 4 triangles:
- Whose 4x3 vertices are the 12 HP-Crosspoints.
- Whose 4x3 sidelines are the 12 Flexlines.
As a consequence:
- Flexlines not only contain 3 Flexpoints (on CU) but also 2 HP-Crosspoints (not on CU).
- HP-Crosspoints are passed not only by 3 Harmonic Polars but also by 2 Flexlines.

Please see the attached paper for further details.
I hope you like it as much as I do.

Best regards,
Chris

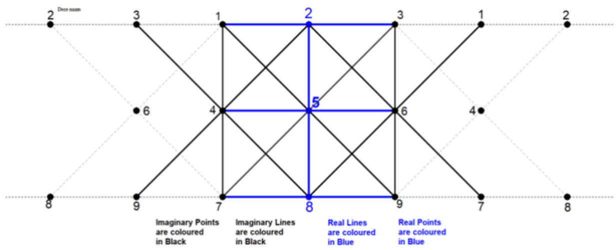
Flexpoints on a cubic, Flexlines, Harmonic Polars and HP-Crosspoints

Summary
There are two networks regarding Flexpoints and Harmonic Polars (HP) wrt a cubic CU:
1. The Flex-network, containing 9 Flexpoints and 12 Flexlines (3 points per line, 4 lines per point).
2. The HP-network, containing 9 Harmonic Polars (HP) and 12 HP-Crosspoints (4 points per line, 3 lines per point).
Both networks are dual with respect to each other, as one has 3 points per line and 4 lines per point, while the other has 4 points per line and 3 lines per point.
It is shown that one network can be determined from the other and vice versa. Moreover both networks can be integrated.
Key is a configuration of 4 triangles:
- Whose 4x3 vertices are the 12 HP-Crosspoints.
- Whose 4x3 sidelines are the 12 Flexlines.
As a consequence:
- Flexlines not only contain 3 Flexpoints (on CU) but also 2 HP-Crosspoints (not on CU).
- HP-Crosspoints are passed not only by 3 Harmonic Polars but also by 2 Flexlines.

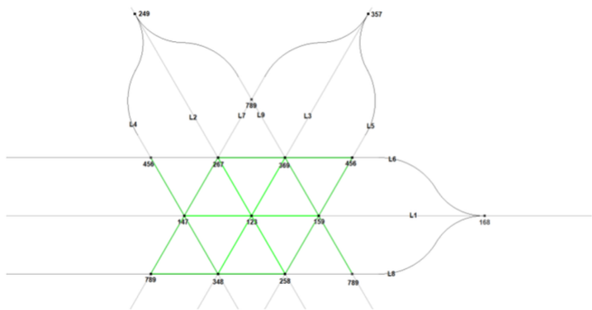
Real and Imaginary items
Note that not all these items can always be seen in pictures, as several items may be imaginary.
Familiar Flexpoints and the only real ones are F1, F2, F3.
Familiar Flexlines and the only real ones are the line F1F2F3 and the 3 other real Flexlines passing through H1, H2, H3.
Familiar Harmonic Polars and the only real ones are L1, L2, L3.
Familiar HP-Crosspoints and the only real ones are Po and H1, H2, H3.

CU-9P Flex-Network HP-Network-integration-01.pdf

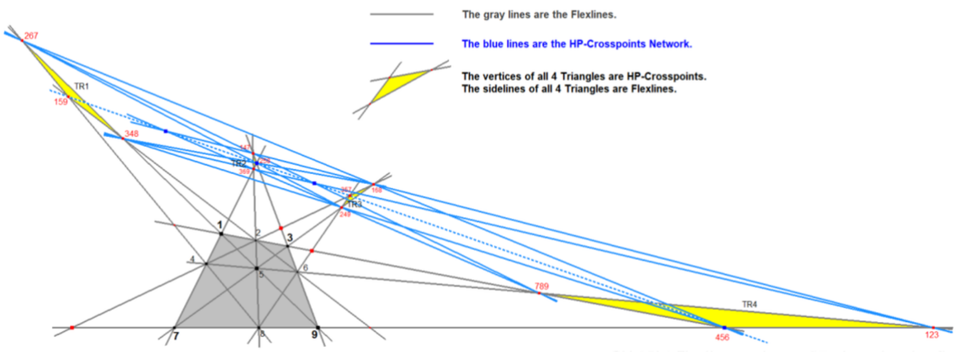
Normalized picture of the Flex-network
(3 points per line, 4 lines per point)
Each point is denoted by 1 number, each line by 3 numbers (of the 3 points on the line).



Normalized picture of the HP-network
(4 points per line, 3 lines per point).
Each line is denoted by 1 number, each point by 3 numbers (of the 3 passing lines).



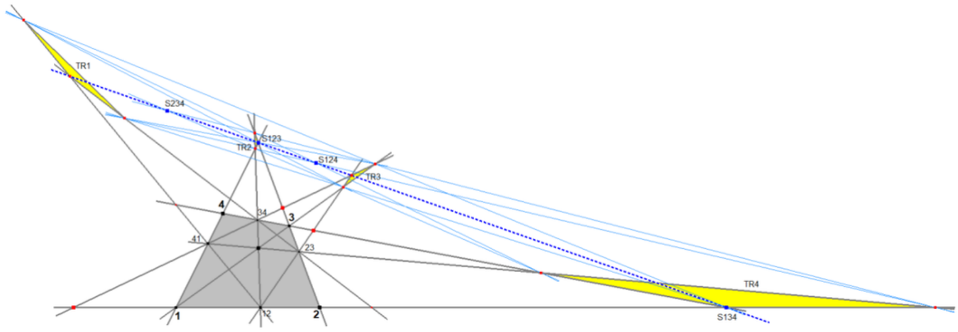
Not normalized picture of both networks integrated, showing the 4 connecting triangles



CU-12L1 Flexlines-scheme-27-plus-triangles.fig

CU-9P Flex-Network HP-Network-integration-01.pdf

Integration of both Networks



CU-12L1 Flexlines-scheme-22-plus-triangles.fig

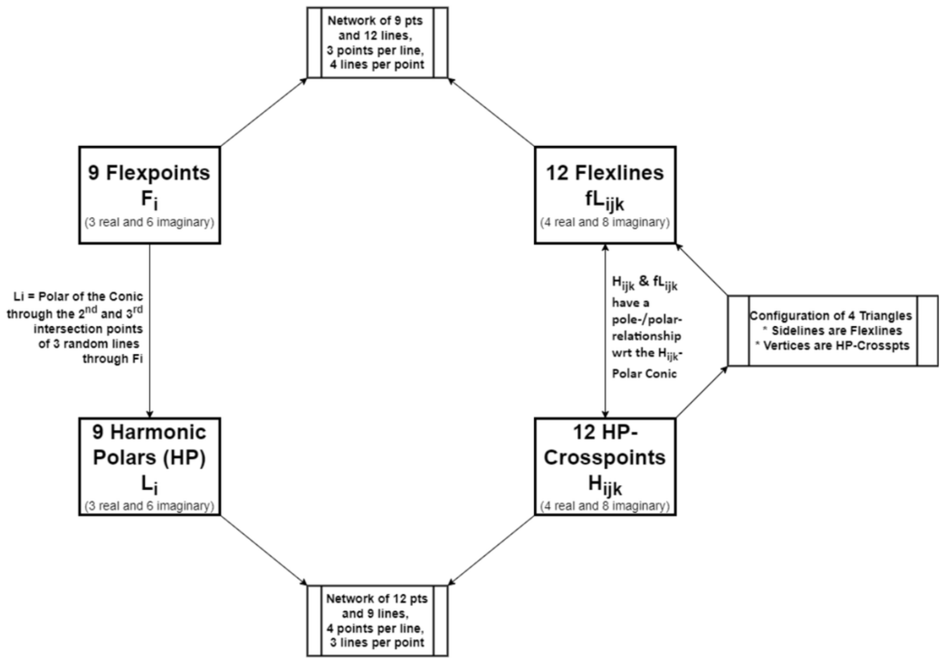
Given a quadrigon 1.2.3.4 with quasi-midpoints (12, 23, 34, 41) on its 4 sidelines, where quadrigons 1.2.3.4 and 12.23.34.41 are situated in such a way that they share the same diagonal crosspoint.
Now, 4 triplets of lines (1.3, 12.23, 34.41), (2.4, 23.34, 41.12), (1.2, 3.4, 32.41), and (4.1, 2.3, 12.34) form 4 triangles.
When quadrigon 1.2.3.4 is a square and (12, 23, 34, 41) are the real midpoints, the 4 triplets of lines will be 4 sets of 3 parallel lines meeting at a point at infinity. Therefore, we will call these triplets "sets of quasi-parallel lines."
Since 1.2.3.4 is not a square, each triplet of quasi-parallel lines will form a triangle.
We call the 4 triplets of lines (2.4, 23.34, 41.12), (4.1, 2.3, 12.34), (1.3, 12.23, 34.41), (1.2, 3.4, 32.41), resp. TR1, TR2, TR3, TR4.
Triangles TR1, TR2, TR3 and TR4 are per 3 triangles perspective:
  * TR1, TR2, TR3 have as common perspector S123.
  * TR1, TR2, TR4 have as common perspector S124.
  * TR1, TR3, TR4 have as common perspector S134.
  * TR2, TR3, TR4 have as common perspector S234.
S123, S124, S134, S234 are all collinear on the 3rd diagonal of the reference quadrigon 1234.
Moreover
  * S134 is crosspoint 1.2^3.4 and one of the vertices of TR4, and
  * S123 is crosspoint 1.4^2.3 and one of the vertices of TR2.
S124 and S234 are points situated on the 3rd diagonal S123.S134 and dependent on the location of quasi-midpoints (12, 23, 34, 41).

From above picture it shows that:
having the Flex-Network it is possible to construct the HP-Network, using the Quasi-parallel-triangles.
The method of doing so is by
1. numbering the Flexpoints, which give Flexlines denoted by 3 numbers
2. combine 4 times 3 quasi-parallel Flexlines, which give 12 vertices denoted by the 3 missing numbers from the set (1,...,9) after skipping the numbers of the Flexlines involved. These vertices are the 12 HP-Crosspoints.
3. lines can be drawn per 3 triangles with 1 identical number, for example points 267, 168, 456 from resp. TR1, TR3 and TR4 have the number '6' in common, which makes the line L6.

Conversely it is possible having the HP-Network to construct the Flex-Network (again using the Quasi-parallel-triangles) using the same rules only switching points and lines.

CU-9P Flex-Network HP-Network-integration-01.pdf

Diagram of Flexpoints, Harmonic Polars and their derived Networks



<u>The configuration of 4 Triangles</u>

Each Flexline fL_{ijk} is defined by the three Flexpoints F_i,F_j,F_k it contains.
Through F_i there are 4 Flexlines each containing 2 other Flexpoints.
In the Flex-network for each Flexline fL_{ijk} there are just 2 other Flexlines fL_{lmn} and fL_{pqr} that have mutually no common Flexpoints.
fL_{ijk}, fL_{lmn} and fL_{pqr} are here called a set of 3 quasi-parallel Flexlines.
There are 4 sets of 3 quasi-parallel Flexlines in the Flex-network forming 4 Flexline-Triangles.
Per Flexline-Triangle each sideline is a Flexline and therefore denoted by (i,j,k).
Therefore 2 Flexlines are denoted by (i,j,k) and (l,m,n)
and therefore a vertex is denoted by (i,j,k) and (l,m,n).
These are 6 of 9 different indices with missing indices (p,q,r).
So a vertex also can be denoted by (p,q,r).
It appears that vertex (p,q,r) = HP-Crosspoint H_{pqr}.
So all 12 HP-crosspoints are also a vertex of the 4 Flexline-Triangles.
Conversely each HP-Crosspoint is also the crosspoint of 2 Flexlines.
Note that because of this special feature any Flexline contains not only 3 Flexpoints, but also 2 HP-crosspoints.

<u>Real Elements</u>

One of the 4 Flexline-Triangles is real. It is the triangle formed by the 3 real Flexlines. The other real Flexline is the line through the 3 real Flexpoints.
The vertices of the real Flexline-Triangle are 3 of the 4 real HP-Crosspoints. The other real HP-Crosspoints is the crosspoint of the 3 real Harmonic Polars.

CU-9P1 Diagram-Flexpoints-HarmonicPolars-02a.png

CU-9P Flex-Network HP-Network-integration-01.pdf

Message: #2411
Date: 2024-08-05
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

On Sat, Aug 3, 2024 at 11:56 PM, Chris wrote:

> Dear Bernard,
> Thanks for your precise description in #2408. Very interesting!
> I cannot follow all of your reasoning about (pre)hessians, but what stands
> out to me is your remark 3), especially that there are three other
> triangles H1H2H3. I happened to study this from a different angle. These
> four triangles H1H2H3 are the key to combining the network you described
> from 9 Flexpoints and 12 Flexlines (3 points per line, 4 lines per point)
> with the network from 9 Harmonic Polars (HP) and 12 HP-Crosspoints (4
> points per line, 3 lines per point). By having one network, it is possible
> to determine the other using these four triangles H1H2H3.
> Very special also is that H1, H2, and H3 are HP-Crosspoints. The vertices
> of the four triangles together happen to be the 12 HP-Crosspoints. Right
> now, I am properly describing this in a small paper and hope to post it
> soon.
> It is nice to see that the triples of the sidelines of the four triangles
> also function as degenerate cubics belonging to the syzygetic pencil."
> Best regards,
> Chris

Dear Chris,
I hardly understand what you cannot follow wrt hessian and prehessian.
Before answering your complete message about HP and HP crosspoints, let's try an example.
Let's consider the triangle H1H2H3.
Simple calculation shows that the cubic $X^3 + Y^3 + Z^3 = 0$ wrt this triangle has as hessian $XYZ = 0$ and that the cubic $XYZ = 0$ has also the cubic $XYZ = 0$ as hessian.

Hence the 1st cubic and the 3rd cubic are prehessians of the 2nd (the 3rd being the same as the 2nd).
This defines a syzygetic pencil: any cubic on the form $X^3 + Y^3 + Z^3 + k XYZ = 0$ has as hessian $X^3 + Y^3 + Z^3 + k' XYZ = 0$, with $k' = - (108 + k^3)/ 3 k^2$
Here, all these cubics have the same asymptotes, which are the sides of H1H2H3 and the same real flexes, which are the infinity points of the asymptotes.
P0 is the centroïd of H1H2H3 and the lines Li are the medians of this triangle.
For a different P0 with a finite trilinear polar as flexline, I suppose we must have a cubic through the 3 real flexes having also XYZ = 0 as hessian and the same calculation.
Unfortunately, I wasn't able to calculate an example. Maybe you could help me ...
Best regards
Bernard

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Message: #2412
Date: 2024-08-06
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
I can finally understand that you are not immediately convinced!
I was myself surprised with my results.
In fact, the triangle H1H2H3 is not an ordinary cubic. It has an infinity of flexes on each side !!!
This particulary cubic is therefore the hessian of an infinity of cubics and belongs to an infinity of syzygetic pencils.
Each cubic and each pencil are determined by a point P0, giving the line of real flexes as trilinear polar of P0 wrt the triangle.
A geometric approach is given by the real flexes Fi and the flextangents FiHi, which determine a unique cubic.
An analytic approach consists in finding the conditions under which the triangle is the hessian of the cubic through the Fi.
Unfortunately, the calculations are a little boring.
I hope these few explanations will be of some help ...
Best regards
Bernard

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Message: #2413
Date: 2024-08-07
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

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Dear Bernard,

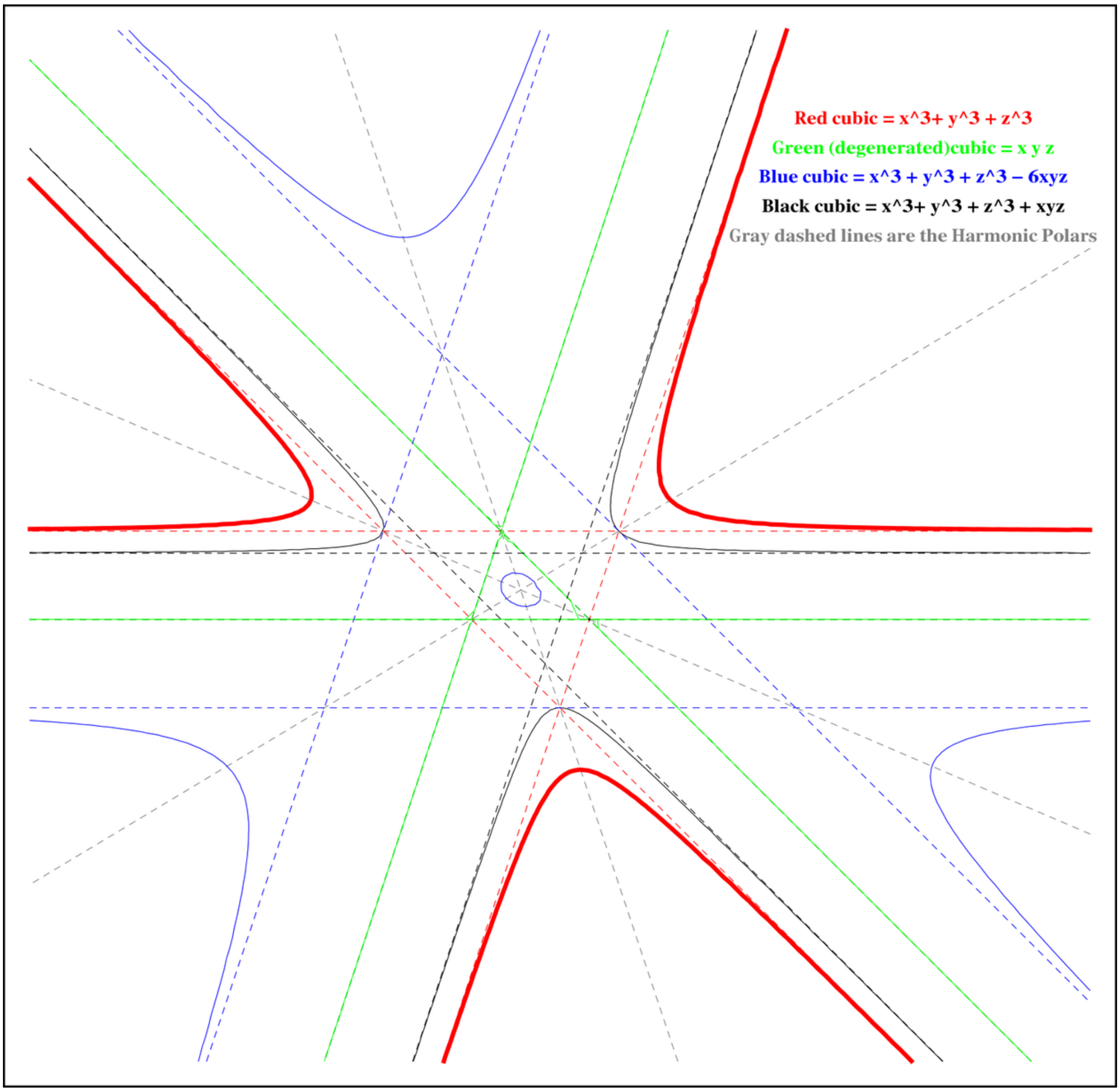
I made a picture of your calculated cubics
  $X^3 + Y^3 + Z^3 = 0$, $XYZ = 0$ and
  $X^3 + Y^3 + Z^3 + k XYZ = 0$,
including their asymptotes.
See attachment.
It can be seen that the asymptotes of these cubics are parallel.

Further:
· The cubic XYZ = 0 is degenerated into 3 lines.
· The real Flexline-triple bounding H1H2H3 coincide
  with these 3 lines.
· The real Flexline F1F2F3 is the line at infinity.
· The 3 real Flexpoints F1, F2, F3 are the infinity points
  of the asymptotes.
· The Harmonic Polars Li (intersecting in common point P0)
  are the medians of this triangle.
· P0 is the centroid of H1H2H3 and the lines Li
  are the medians of this triangle.

I think this confirms almost everything you wrote in #2411.

Best regards,

Chris



CU-Cu1 Hessian-plus Flexlines-01-simple-example.pdf

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Message: #2414
Date: 2024-08-07
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

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```
Dear Chris,
Thanks a lot for your interest and attention
1) you corrected my formulation about the asymptotes (parallel
to the sides of H1H2H3)
2) you add an example with k = -6 for CU and k' = 1 for HE
So you corrected or confirmed completely the properties
mentionned in my message 2411
It's a real pleasure to progress this way ...
What about the message 2412?
Best regards
Bernard
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Message: #2415
Date: 2024-08-07
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

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```
Dear Bernard,

About your message #2412.
I do not understand your remark “In fact, the triangle H1H2H3 is
not an ordinary cubic. It has an infinity of flexes on each side
!!!”.

According to my calculations the flexes for all shown cubics are
the same set of 9 flexpoints with barycentric coordinates:
   {-1, 1, 0}
   {(-1)^(1/3), 1, 0}
   {-(-1)^(2/3), 1, 0}
   {-1, 0, 1}
   {(-1)^(1/3), 0, 1}
   {-(-1)^(2/3), 0, 1}
   {0, -1, 1}
   {0, (-1)^(1/3), 1}
   {0, -(-1)^(2/3), 1}

Best regards,
Chris
```

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Message: #2416
Date: 2024-08-08
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Apparently, you don't understand what I mean.
Naturally, your calculation of the flexes is correct for all cubics of this syzygetic pencil!
But there are an infinity of different syzygetic pencils having the same triangle H1H2H3 as hessian of a main cubic.
Each is defined by a point P0 and it's trilinear polar, which is the line of the *real finite flexes F1, F2 and F3* on the sides of H1H2H3.
The main cubic is defined by $X^3 + Y^3 + Z^3 + a1\ X^2Y + a2\ XY^2 + a3Y^2Z + a4YZ^2 + a5X^2Z + a6XZ^2 = 0$ (no term in XYZ).
The cubic passes through the 3 real finite points F1, F2 and F3.
The hessian of this cubic must be XYZ = 0.
Having P0, F1, F2 and F3, it gives a certain number of conditions which might identify the ai (i = 1 to 6) and the cubic.
As already mentionned, the flextangents to this main cubic should be FiHi.
I hope this will be clear now ...
Best regards
Bernard

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Message: #2417
Date: 2024-08-09
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Bernard,
Thanks for further explaining.
I checked the hessian of main cubic

$$X^3 + Y^3 + Z^3 + a1\ X^2Y + a2\ XY^2 + a3\ Y^2Z + a4\ YZ^2 + a5\ X^2Z + a6\ XZ^2 = 0.$$

According to my calculations it is:
(-a2 a5^2 - a1^2 a6 + 3 a2 a6) x^3 + (-a1^2 a4 + 3 a2 a4 + 2 a1 a3 a5 - 3 a5^2 + 9 a6 - a1 a2 a6) x^2 y + (-3 a3^2 + 9 a4 - a1 a2 a4 + 2 a2 a3 a5 + 3 a1 a6 - a2^2 a6) x y^2 + (-a1 a3^2 + 3 a1 a4 - a2^2 a4) y^3 + (-3 a1^2 + 9 a2 + 2 a1 a4 a5 - a3 a5^2 + 3 a3 a6 - a2 a5 a6) x^2 z - 3 (-9 + a1 a2 + a3 a4 - a2 a4 a5 - a1 a3 a6 + a5 a6) x y z + (9 a1 - 3 a2^2 - a1 a3 a4 - a3^2 a5 + 3 a4 a5 + 2 a2 a3 a6) y^2 z + (9 a3 - 3 a4^2 + 3 a2 a5 + 2 a1 a4 a6 - a3 a5 a6 - a2 a6^2) x z^2 + (3 a1 a3 - a1 a4^2 + 9 a5 - a3 a4 a5 + 2 a2 a4 a6 - 3 a6^2) y z^2 + (3 a3 a5 - a4^2 a5 - a3 a6^2) z^3 = 0.

When a1=1, a2=2, a3=3, a4=4, a5=5, a6=6, then the hessian is:
 -20 x^3 + 17 x^2 y + 55 x y^2
 - 13 y^3 - 26 x^2 z + 72 y^2 z - 105 x z^2 - 34 y z^2 - 143 z^3
 + 69 x y z = 0.

Anyhow it is not x y z = 0 like you predicted.
Then you state “the flextangents to this main cubic should be FiHi”.
I calculate for FiHi:
 {1,1,0}
 {1,0,1}
 {0,1,1}
I calculate for the real flextangents:
 {3 - 2 a1 + a2, 3 + a1 - 2 a2, a3 + a5}
 {3 - 2 a5 + a6, a1 + a4, 3 + a5 - 2 a6}
 {a2 + a6, 3 - 2 a3 + a4, 3 + a3 - 2 a4}
But when you mean the real flexlines instead of the flextangents, they are indeed:
 {1,1,0}
 {1,0,1}
 {0,1,1}

Best regards, Chris

Message: #2418
Date: 2024-08-09
From: unidentifiedlethargicorganism@gmail.com
Subject: Quadrigon, diagonals, angle bisectors, conic

Dear Geometers,

I already posted this result on Romantics of Geometry on April 12, 2023.
See RoG #12268 https://www.facebook.com/share/p/8oQCAGvaSCmwW82w/
See also a solution by Francisco Javier García Capitán https://www.facebook.com/share/p/RxpknAuevjsfHWkk/

Let P1 P2P3P4 be a quadrigon.
Let M1 be the angle bisector of ∠P4P1P2, and define M2, M3, M4 cyclically.
Let N1 be the reflection in M1 of P1P3.
Let N2 be the reflection in M2 of P2P4.
Let N3 be the reflection in M3 of P1P3.
Let N4 be the reflection in M4 of P2P4.

Then, in general there exists a conic that is tangent to the six lines P1P3, P2P4, N1, N2, N3, N4.

I don't think there are any interesting properties regarding this conic....

Best Regards,
Keita Miyamoto

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Message: #2419
Date: 2024-08-09
From: van10hoven@gmail.com
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

Dear Keita Miyamoto and Francisco,

Keita Miyamoto, welcome to our forum!
I checked your beautiful inscribed conic.
I noticed that its center lies on the line QA-P1.QG-P2.

Best regards,
Chris van Tienhoven

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Message: #2420
Date: 2024-08-09
From: unidentifiedlethargicorganism@gmail.com
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

Dear Chris van Tienhoven,

Thank you very much for your interest and attention.
There may also exist a conic that is tangent to the six lines P1P3, P2P4, M1, M2, M3, M4. (I have not checked that whether this still holds or not when interior angle bisectors are replaced by exterior angle bisectors)
This conjecture is based on my GeoGebra sketch, and not confirmed algebraically. Sorry if I am wrong.

Best Regards,
Keita Miyamoto

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Message: #2421
Date: 2024-08-09
From: van10hoven@gmail.com
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

```
Dear Keita Miyamoto,

I checked your conjecture in a drawing and I think it holds.

Best regards,
Chris van Tienhoven
```

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Message: #2422
Date: 2024-08-09
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

```
Dear Chris,
Thanks for your calculation of the hessian.
I'm now looking for a system of ai such as H1H2H3 (XYZ = 0) is
the hessian!
Your calculation gives 10 coefficients for the hessian, 9 have
to be equal to 0, in order to have only the term in XYZ !!!
Is there a solution?
The calculation with ai = i has no sense, I suppose it is a
wrong reading of my sentence ai (i = 1 to 6); I didn't mean ai =
1 to 6 ...
The calculation of FiHi with the Fi the infinite points of the
sides of H1H2H3 has no sense either, as I'm precisely looking
for other cubics with finite real flexes on the trilinear polar
of a point P0 (not in the centroïd).
I suppose it would be easier to be together in front of a board
with a peace of chalk, but it's not possible ...
I refer to your own construction (points 1 and 2 of my message
#2408) and I'm looking for the prehessian of the triangle H1H2H3
taken as cubic.
Best regards
Bernard
```

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Message: #2423
Date: 2024-08-10
From: eckart_schmidt@t-online.de
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

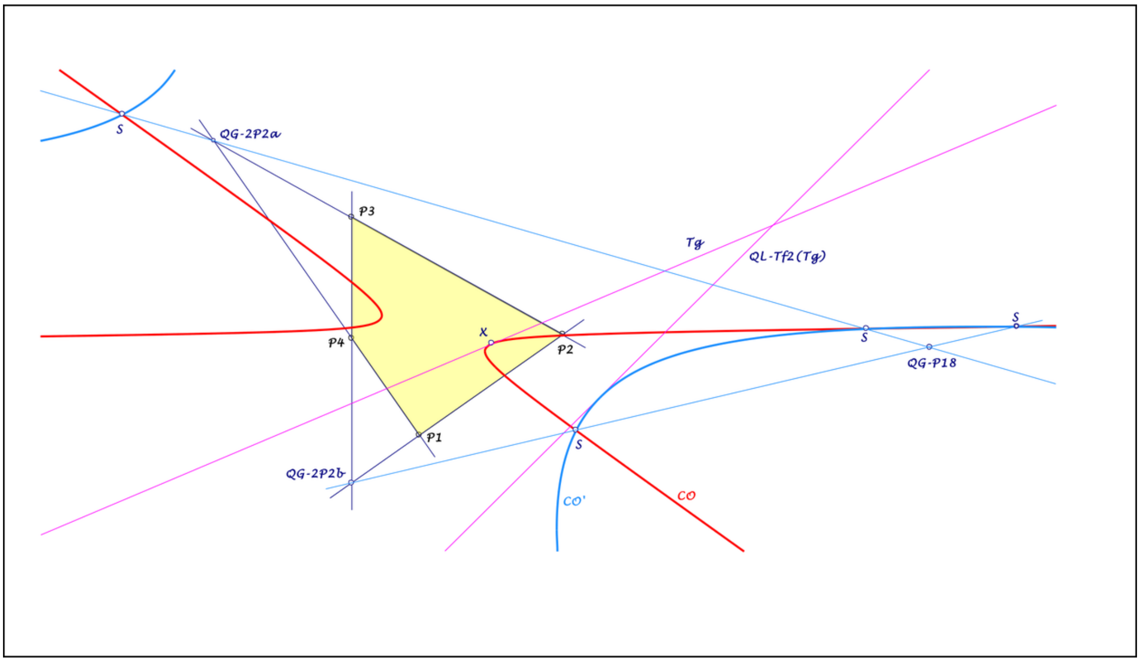
```
Dear Keita Miyamoto,

I studied the conic in #2418, perhaps of interest:

Your conic is a quadrigon element,
... if we use QL-Tf2 for the tangents Tg of your conic CO,
... they envelope a new conic CO',
... intersecting CO in points S on QG-P18.QG-2P2a,b.

If tangents Tg of CO and its images are orthogonal,
... they intersect in the contact point X of Tg.

Best regards Eckart
```



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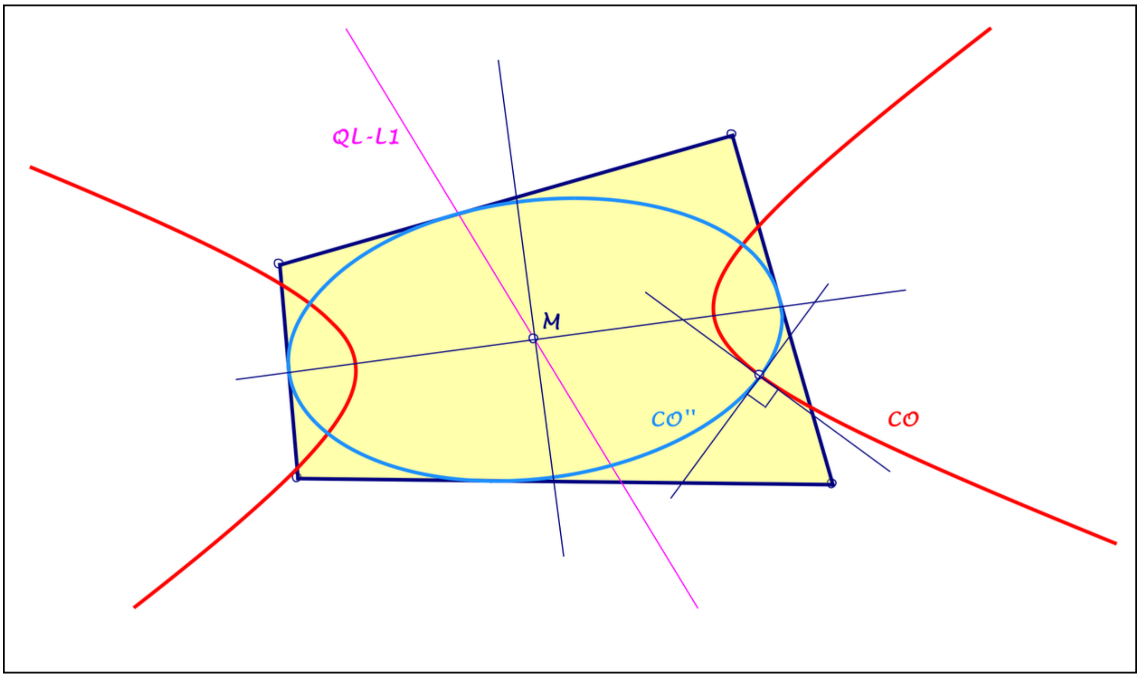
Message: #2424
Date: 2024-08-10
From: eckart_schmidt@t-online.de
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

```
Dear Keita Miyamoto,

a further observation for your conic CO in #2418:
The conic is centered in M on QL-L1
... and is center of a QG-inscribed conic CO'',
... intersecting CO orthogonal.

Best regards Eckart
```



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Message: #2425
Date: 2024-08-10
From: van10hoven@gmail.com
Subject: Re: Quadrigon, diagonals, angle bisectors, conic

Dear Eckart and Keita Miyamoto,

Beautiful property and picture in #2424!

Best regards,
Chris

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Message: #2426
Date: 2024-08-11
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Thanks to you, I've revised my knowledge in calculations of the hessian of a cubic and of tangent in a point to a cubic.
Reading again all our exchanges, I realise that I wasn't perhaps clear enough and that my example of the cubic $X^3 + Y^3 + Z^3 = 0$ was rather misleading.
Starting with a reference triangle H1H2H3 and a point P (1/u,1/v,1/w), I suppose we agree that the line of real flexes has as equation uX + vY + wZ =0.
Then the Fi are (0,-w,v), (-w,0,u) and (-v,u,0) and the Li are vY = wZ, uX = wZ and uX = vY.
I'm looking for the cubic having the Fi as flexes and having the triangle H1H2H3 taken as a cubic as hessian.
I know already that the cubic passes through the Fi, I'm sure it has FiHi as flextangents and I know how to calculate the hessian of a general cubic.
All these elements should give enough conditions to identify this cubic.
But as I'm doing all these calculations by hand I'm rather discouraged ...
Can you help me? First of all, is my reasoning correct?
Thanks in advance
Best regards
Bernard
PS there is perhaps another possibility: starting with a cubic CU and calculating it's hessian HE, you were able to identify the triangle H1H2H3 as the cubic RF.
Then the searched cubic is a linear combination of CU and HE (all belonging to the syzygetic pencil) and has RF as hessian
...

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Message: #2427
Date: 2024-08-11
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Bernard,

About your remark in #2426:
“Starting with a reference triangle H1H2H3 and a point P (1/u,1/v,1/w), I suppose we agree that the line of real flexes has as equation uX + vY + wZ =0”.

First let vertices of reference triangle H1H2H3 be {1,0,0}, {0,1,0}, (0,0,1}.
When you mean with P the point P0, then I think it should be {1,1,1} and not {1/u,1/v,1/w}, unless you perform a projective transformation, but then also the reference cubic will be transformed.
I am still not sure what you are aiming to do. I think you want to find a neat way to determine the pre-Hessians. Is that correct?
If so, what steps do you plan to take? When you formulate it in a way that I can calculate it in Mathematica, I can do so. Also for more complicated cubics. Also with combining CU and HE.

Best regards,
Chris

Message: #2428
Date: 2024-08-12
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Many thanks for your offer of calculations!
What I'm aiming to do is simply to find the cubic which is the prehessian of RF ...
Let's start with the beginning.
You were able to find the equations of a CU, it's HE and the trianglecubic RF (H1H2H3) wrt a reference triangle ABC (which triangle did you choose?).
According to the theory, we should have a real cubic FE (for Fermat) such as CU = FE + kRF and HE = FE + k'RF with k' = -(108 + k^3)/3k^2
Then we have CU - HE = (k-k')RF and k'CU - kHE = (k'-k)FE
I suppose you have CU, HE, RF, P0 and the real Fi
1) Can you check that RF is it's own hessian?
2) Can you find FE as linear combination of CU and HE having RF as hessian?
3) Can you check that the flextangents of FE are FiHi?
Last, but not least, can you make a changing in the reference triangle of your barycentric coordinates?
Taking the vertices of RF as reference (H1 = 1,0,0, H2 = 0,1,0 and H3 = 0,0,1), RF becomes XYZ = 0, what about P0 and the Fi, CU, HE and FE?
All this is impossible by hand, but I suppose possible with Mathematica ...
Thanks in advance for your precious help, but don't hesitate to tell me it is too complicate or simply too timeconsuming.
Best regards
Bernard

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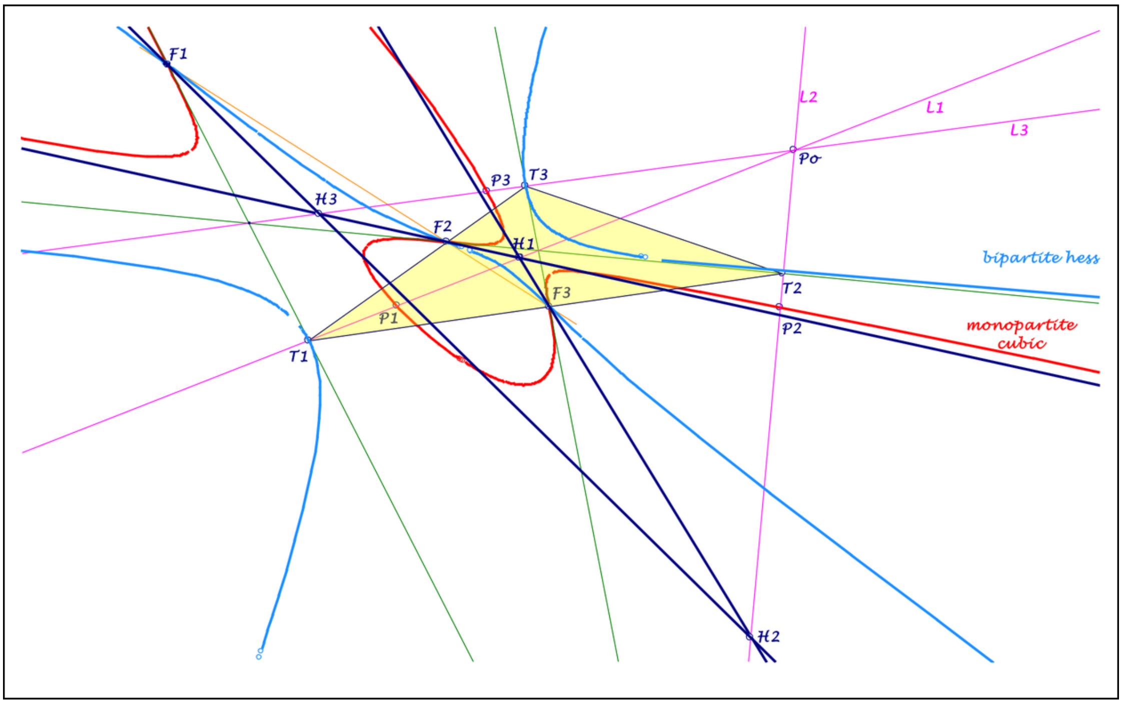
Message: #2429
Date: 2024-08-13
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

I need your help, for I am total confused:
We found for the real flexlines beside F1F2F3 in a real picture,
... that they intersect the cubic only in a real flexpoint
    (#2305)
... and that cubic and its hessian have the same real flexlines.
In #2299 you describe a construction of real flexlines for
bipartite cubics,
... for a monopartite cubic we get the real flexlines
... as real flexlines of the bipartite hessian.
So I started with a monopartite cubic (attached)
... and constructed the bipartite hessian,
... but the common real flexlines with your construction
... intersect the starting cubic not only in the real
    flexpoints.
I got the same observation with a second drawing.
Can you please prove this constellation in a similar case,
... for I cannot send CABRI-files, excuse.
I searched in vain for a mistake in my constructions.

Best regards Eckart
```



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Message: #2430
Date: 2024-08-13
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,
If the flexlines to the hessian intersect the lines Li in ti,
the points Hi are the harmonics of ti wrt the 2 points of
intersection of Li with the hessian other than Ti.
I don't see on your drawing the flextangents to the hessian and
the points ti ...
Best regards
Bernard
```

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Message: #2431
Date: 2024-08-14
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

you describe only Chris' construction,
... which I have used in my drawing,
... but I haven't shown the artificial lines.

Have you constructed H1,2,3 for a similar monopartite ref-cubic?

Best regards Eckart
```

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Message: #2432
Date: 2024-08-14
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

starting your #2430, I think you mean "flextangents"
... instead of "flexlines", to describe Chris' construction.

Best regards Eckart
```

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Message: #2433
Date: 2024-08-14
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,
My apologise!
Naturally, I meant flextangents, tangents in the 3 real flexes
to the hessian.
Looking again at your figure, it seems it works.
The sides of the correct triangle H1H2H3 (the real flexlines)
lie between FiTi and Fiti.
In this case, there is no other intersection than the Fi between
these real flexlines and the initial cubic (the same goes for
it's hessian).
Best regards
Bernard
```

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Message: #2434
Date: 2024-08-14
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart and Bernard,

When CU is monopartite the construction of the
Real-Flexline-Triple can be performed by doing exactly the same
construction steps with the Hessian HE instead of the Reference
Cubic CU.
Fortunately they share the same Harmonic Polars L1, L2, L3.
As a consequence:
1. (P,Q,R) will not be the 3 intersection points of L1 with CU,
but will be the 3 intersection points of L1 with HE, and
2. T1, T2, T3 will not be the intersection points of Li with the
CU-Fi-Tangents but he intersection points of Li with the
HE-Fi-tangents.

In summary:
Construction of the Real-Flexline-Triple
When CU is not monopartite:
1. Let (P,Q,R) be the 3 intersection points of L1 with CU.
2. Let T1 be the intersection point of L1 with the
CU-F1-tangent.
3. There are these Harmonic Conjugates H1a=T1 wrt (P,Q), H1b=T1
wrt (Q,R), H1c=T1 wrt (R,P).
4. H1 will be the one for which line F1H1x has besides F1 no
further real intersections with CU.
5. In the same way H2 and H3 will be determined.
6. (H1H2, H2H3, H3H1) will be the Real-Flexline-Triple.

When CU is monopartite:
1. Let (P,Q,R) be the 3 intersection points of L1 with HE.
2. Let T1 be the intersection point of L1 with the
HE-F1-tangent.
3. There are these Harmonic Conjugates H1a=T1 wrt (P,Q), H1b=T1
wrt (Q,R), H1c=T1 wrt (R,P).
4. H1 will be the one for which line F1H1x has besides F1 no
further real intersections with HE or CU.
5. In the same way H2 and H3 will be determined.
6. (H1H2, H2H3, H3H1) will be the Real-Flexline-Triple.

I hope this solves your problem.

Best regards,
Chris

Message: #2435
Date: 2024-08-14
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

it is not, that I haven't understood your construction,
... but have you done it for a similar monopartite cubic,
... whether my observed contrarieties are possible?

Best regards Eckart
```

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Message: #2436
Date: 2024-08-15
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

```
Dear Bernard,

Regarding your message #2428.

I think the essence of your question is that you want to split a
general cubic CU into two parts FE and RF, in order to define a
convenient Syzygetic pencil FE + kRF (which will help finding
prehessians and cohessians).

One part is the general part FE, the other part the degenerate
cubic RF made up from the lines FiHi.

If this is not your aim please let me know.

Anyhow I started with a general cubic
CU = a1 x^3 + a2 y^3 + a3 z^3 + a4 x^2 y + a5 x^2 z + a6 x y^2 +
a7 y^2 z + a8 x z^2 + a9 y z^2 + a0 x y z.

Now the calculated Hessian
```

HE = -2 (3 a0^2 a1 - 4 a0 a4 a5 + 4 a5^2 a6 + 4 a4^2 a8 - 12 a1 a6 a8) x^3 + 2 (a0^2 a4 - 12 a2 a5^2 - 12 a0 a1 a7 + 8 a4 a5 a7 + 36 a1 a2 a8 - 4 a4 a6 a8 - 4 a4^2 a9 + 12 a1 a6 a9) x^2 y - 2 (12 a0 a2 a5 - a0^2 a6 - 8 a5 a6 a7 + 12 a1 a7^2 - 12 a2 a4 a8 + 4 a6^2 a8 - 36 a1 a2 a9 + 4 a4 a6 a9) x y^2 - 2 (3 a0^2 a2 - 4 a0 a6 a7 + 4 a4 a7^2 - 12 a2 a4 a9 + 4 a6^2 a9) y^3 - 2 (12 a3 a4^2 - a0^2 a5 - 36 a1 a3 a6 + 4 a5^2 a7 + 4 a5 a6 a8 - 12 a1 a7 a8 + 12 a0 a1 a9 - 8 a4 a5 a9) x^2 z + 2 (a0^3 + 108 a1 a2 a3 - 12 a3 a4 a6 - 4 a0 a5 a7 - 12 a2 a5 a8 - 4 a0 a6 a8 + 12 a4 a7 a8 - 4 a0 a4 a9 + 12 a5 a6 a9 - 12 a1 a7 a9) x y z + 2 (36 a2 a3 a4 - 12 a3 a6^2 + a0^2 a7 - 4 a5 a7^2 - 12 a0 a2 a8 + 8 a6 a7 a8 + 12 a2 a5 a9 - 4 a4 a7 a9) y^2 z - 2 (12 a0 a3 a4 - 12 a3 a5 a6 - 36 a1 a3 a7 - a0^2 a8 + 4 a5 a7 a8 + 4 a6 a8^2 - 8 a4 a8 a9 + 12 a1 a9^2) x z^2 + 2 (36 a2 a3 a5 - 12 a0 a3 a6 + 12 a3 a4 a7 - 12 a2 a8^2 + a0^2 a9 - 4 a5 a7 a9 + 8 a6 a8 a9 - 4 a4 a9^2) y z^2 - 2 (3 a0^2 a3 - 12 a3 a5 a7 + 4 a7 a8^2 - 4 a0 a8 a9 + 4 a5 a9^2) z^3

In order to find RF we first need to know the flexpoints.
This can be done by solving the equations CU=0 & HE=0.
However this calculation is too consuming for Mathematica.
So I started a numerical example:
CU = x^3 + 2 y^3 + 3 z^3 + 4 x^2 y + 5 x^2 z + 6 x y^2 + 7 y^2 z + 8 x z^2 + 9 y z^2 + 10 x y z

Now the Hessian HE = 67683 x^3 + 87718 x^2 y + 216045 x y^2 + 247990 y^3 + 120206 x^2 z - 371981 x y z - 194924 y^2 z + 259904 x z^2 - 298884 y z^2 + 315673 z^3.

The flexpoints can be calculated:
F1 = {1, -5.67702, 4.12841}
F2 = {1, -0.188152, -0.545668}
F3 = {1, -0.527378, -0.256798}
F4 = {1, -0.0943149 + 0.31483 i, -0.107809 - 0.238896 i}
F5 = {1, -0.0943149 - 0.31483 i, -0.107809 + 0.238896 i}
F6 = {1, -0.503738 + 0.0446798 i, -0.512235 - 0.48278 i}
F7 = {1, -0.503738 - 0.0446798 i, -0.512235 + 0.48278 i}
F8 = {1, -0.539595 + 0.518318 i, -0.528218 - 0.0257362 i}
F9 = {1, -0.539595 - 0.518318 i, -0.528218 + 0.0257362 i}

The line through (F1,F2,F3) is:
L123 = x + 1.20636 y + 1.41665 z.

The other 3 real flexlines are:
L145 = x + 4.23028 y + 5.57488 z
L289 = x + 0.0894636 y + 1.80177 z
L367 = x + 1.81441 y + 0.167918 z

Finally RF = L145 L289 L367 = x^3 + 6.13415 x^2 y + 8.21624 x y^2 + 0.686674 y^3 + 7.54457 x^2 z + 22.2303 x y z + 14.7979 y^2 z + 11.2833 x z^2 + 19.5887 y z^2 + 1.68667 z^3.

Then your questions :

1) Can you check that RF is its own hessian?

The Hessian of RF = 1. x^3 + 6.13415 x^2 y + 8.21624 x y^2 + 0.686674 y^3 + 7.54457 x^2 z + 22.2303 x y z + 14.7979 y^2 z + 11.2833 x z^2 + 19.5887 y z^2 + 1.68667 z^3,
which has the same equation as RF (like you predicted).

2) Can you find FE as linear combination of CU and HE having RF as hessian?
There are 9 combinations (t1,t2) for which RF = the Hessian of t1 CU + t2 HE:
{-0.123083 - 0.213186 i, -0.000726313 - 0.00125801 i},
{-0.123083 + 0.213186 i, -0.000726313 + 0.00125801 i},
{-0.0794269 + 0.137571 i, 0.0010738 - 0.00185988 i},
{-0.0794269 + 0.137571 i, 0.0010738 - 0.00185988 i},
{-0.0794269 - 0.137571 i, 0.0010738 + 0.00185988 i},
{-0.0794269 - 0.137571 i, 0.0010738 + 0.00185988 i},
{0.158854, -0.00214761},
{0.158854, -0.00214761},
{0.246166, 0.00145263}
Combinations 8 and 9 are identical.

3) Can you check that the flextangents of FE are FiHi?
The tangents at Flexes F1, F2, F3 are: TnF1 = x - 0.0445114 y - 0.303432 z, TnF2 = x - 1.80484 y + 2.45494 z, TnF3 = x + 3.12305 y - 2.5196 z.

The lines FiHi are: F1H1 = L145 = x + 4.23028 y + 5.57488 z, F2H2 = L289 = x + 0.0894636 y + 1.80177 z, F3H3 = L367 = x + 1.81441 y + 0.167918 z.
They are not identical to the Flextangents.

So far my elaborations.
1. Is this what you expected? How to proceed?
2. What are your conclusions?

Best regards,
Chris

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Message: #2437
Date: 2024-08-15
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

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```
Dear Bernard,

Here an addition to my last message.
I forgot your question 4).

1) Last, but not least, can you make a changing in the reference
triangle of your barycentric coordinates? Taking the vertices of
RF as reference (H1 = 1,0,0, H2 = 0,1,0 and H3 = 0,0,1), RF
becomes XYZ = 0, what about P0 and the Fi, CU, HE and FE?

First of all I calculated FE = t1 CU + t2 HE.
I told in last message that there are 9 solutions for (t1,t2).
After substituting these solutions for (t1,t2) it appeared that
there were duplicates and actually there are only 2 different
versions of FE:

FE1 = 1. x^3 + 2.57503 x^2 y + 4.52022 x y^2 + 2.87691 y^3 +
3.30099 x^2 z + 1.83381 x y z + 1.79337 y^2 z + 5.80774 x z^2 +
1.92995 y z^2 + 3.87691 z^3

FE2 = 1. x^3 + 6.13415 x^2 y + 8.21624 x y^2 + 0.686674 y^3 +
7.54457 x^2 z + 22.2303 x y z + 4.7979 y^2 z + 11.2833 x z^2 +
19.5887 y z^2 + 1.68667 z^3

Their hessians are RF.
Note FE2 is RF!
So actually there is here only one FE-cubic.

Taking the vertices of RF as Reference Triangle (H1 = 1,0,0, H2
= 0,1,0 and H3 = 0,0,1), these are the converted items:

CU-conv =
   x^3 + 0.000696947 y^3 + 0.000466517 z^3 + 0.0484133 x y z
HE-conv =
   x^3 + 0.000696947 y^3 + 0.000466517 z^3 - 0.0211317 x y z
RF-conv = x y z

FE1-conv = x^3 + 0.000696947 y^3 + 0.000466517 z^3
FE2-conv = x y z

The Flexpoints converted are :
F1-conv = {0, 1., -1.14317}
F2-conv = {1, 0, -12.8937}
```

```
F3-conv = {-0.0886611, 1., 0}
F4-conv = {0, 1., 0.571585 + 0.990014 i}
F5-conv = {0, 1., 0.571585 - 0.990014 i}
F6-conv = {1, 5.63945 + 9.76781 i, 0}
F7-conv = {1, 5.63945 - 9.76781 i, 0}
F8-conv = {1, 0, 6.44685 - 11.1663 i}
F9-conv = {1, 0, 6.44685 + 11.1663 i}

P0 converted:
P0-conv = {1, 11.2789, 12.8937}
```

I am surprised about the results.
It is very beautiful.
I wonder how we can make a construction of the FE-cubic of a general cubic.

Best regards,
Chris

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Message: #2438
Date: 2024-08-15
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Wunderbar!
Amazing!
I'm totally bluffed by your capacity of calculations.
I think it was worth it!
I will continue to study it ...
I think there was a mystake in your 1st message just at the beginning and repeated in your answer to the point 3.
The degenerated cubic RF is not made of the lines FiHi.
Fi is on the side HjHk, but FiHi is not HjHk.
I'm still convinced that the flextangents to FE are the FiHi
The flextangents to RF are the sides of H1H2H3
The 2nd part of my quest was to find a link between the coordinates of P0 wrt H1H2H3 and the coefficients of the terms of FE and RF.
(In other words, the syzygetic pencil is determined by the point P0)
But it is another story ...
I won't be home for 2 or 3 weeks, so I wanted to answer immediately your beautiful message!
Many, many thanks for all this huge work
Best regards
Bernard
PS I was wrong when thinking the coefficients of x^3, y^3 and z^3 were equal ...

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Message: #2439
Date: 2024-08-15
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

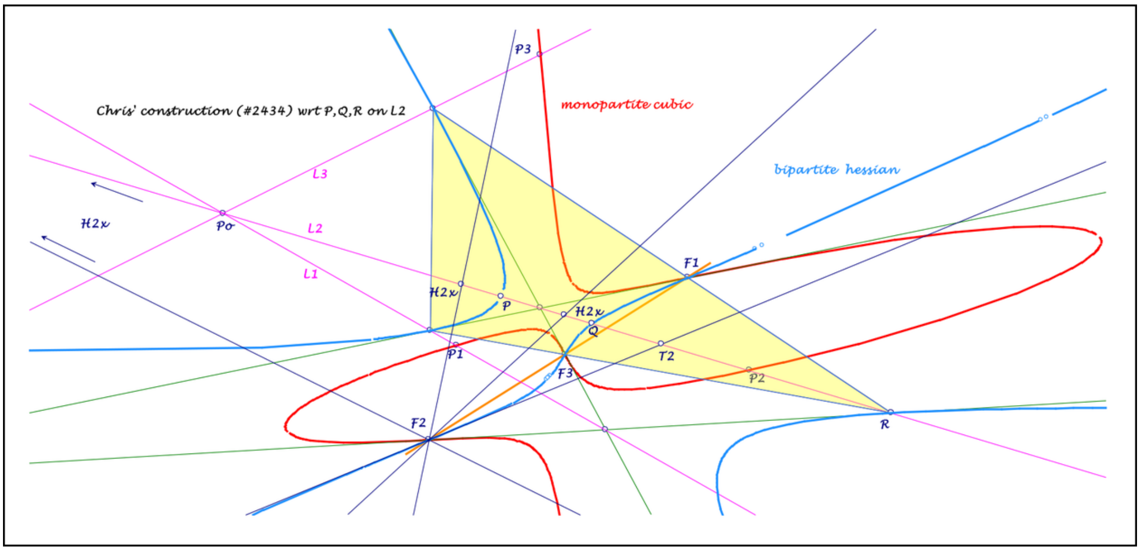
Dear Chris,
I was so excited with your results that I couldn't wait to answer your message.
Now I've made a few calculations myself.
I tell you where I am and I hope you will do something with that.
Starting with a triangle H1H2H3, any cubic of the form a1X^3 + a2Y^3 + a3Z^3 = 0 defines a FE and a syzygetic pencil.
RF is XYZ = 0
Any cubic like FE + kRF belongs to the pencil and it's hessian is given by FE + k'RF with k' = -(108a1a2a3 + k^3)/3k^2.
F1 is given by X = 0 and Z^3/Y^3 = - a1/a3 and the like.
So the point P0 defines the pencil, the line of real flexes being it's trilinear polar wrt H1H2H3.
Starting with P0 gives the line of real flexes and the coefficients of FE.
Naturally, if P0 is the centroïd of H1H2H3, a1 = a2 = a3 and we find my 1st example with the real flexes as infinity points.
My mystake was to think these 3 coefficients were always equal and to search other coefficients for X^2Y, XY^2 ...
In fact, your calculations show that there are only these 3 coefficients, which are not equal.
I thank you very much for all these calculations and for this new understanding, which is a real discovery for me.
I hope you will continue all these reflexions and calculations, but I think we've found the key of these mysterious syzygetic pencils.
Best regards
Bernard

Message: #2440
Date: 2024-08-16
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

attached your construction in #2434
... for a monopartite cubic with P,Q,R on L2,
... there is no F2H2x only intersecting CU and HE in F2.

Best regards Eckart
```



2024-08-15.pdf

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Message: #2441
Date: 2024-08-16
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

```
Dear Chris,
Geogebra gives the possibility of drawing triangle curves given
barycentric equation.
I drew without difficulty the cubic
    FE 8X^3 + 125Y^3 + 27 Z^3 = 0.
The line of real flexes is 2X + 5Y -3Z = 0.
Hence P0 and the Li.
The flextangents to FE are indeed FiHi.
Note that FE is monopartite and RF bipartite.
Perhaps forgotten, but obvious:
    in the pencil, FE is k = 0 and RF is k = ∞ !
Best regards
Bernard
```

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Message: #2442
Date: 2024-08-16
From: van10hoven@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

```
Dear Bernard,

I am just as enthusiastic as you are about the results.
I checked three other things:
#2438 “I'm still convinced that the flextangents to FE are the
FiHi”.
I can confirm this from numerical calculations.
Then I checked if the real Flexline F1F2F3 is the Trilinear
Polar of P0 wrt H1H2H3.
There was a perfect match in numerical calculations.
I also checked QPG#2439 “Any cubic like FE + kRF belongs to the
pencil and it's hessian is given by FE + k'RF with k' =
-(108a1a2a3 + k^3)/3k^2.”
I calculated it and again it is confirmed from numerical
calculations.
Have a nice holiday!

Best regards,
Chris
```

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Message: #2443
Date: 2024-08-17
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Just before leaving, a last curiosity!
For k = 3a1a2a3, we have k' = k.
The corresponding cubic, which is also selfhessian is made of 3times the line of real flexes $(a1\ X + a2\ Y + a3\ Z)^3 = a1^3\ X^3 + a2^3\ Y^3 + a3^3\ Z^3 + 3\ a1\ a2\ a3\ X\ Y\ Z = 0$.
I hope you will make one of your complete surveys of all these beautiful properties of these syzygetic pencils.
Best regards
Bernard

PS How do you calculate the equation of the cayleyan?

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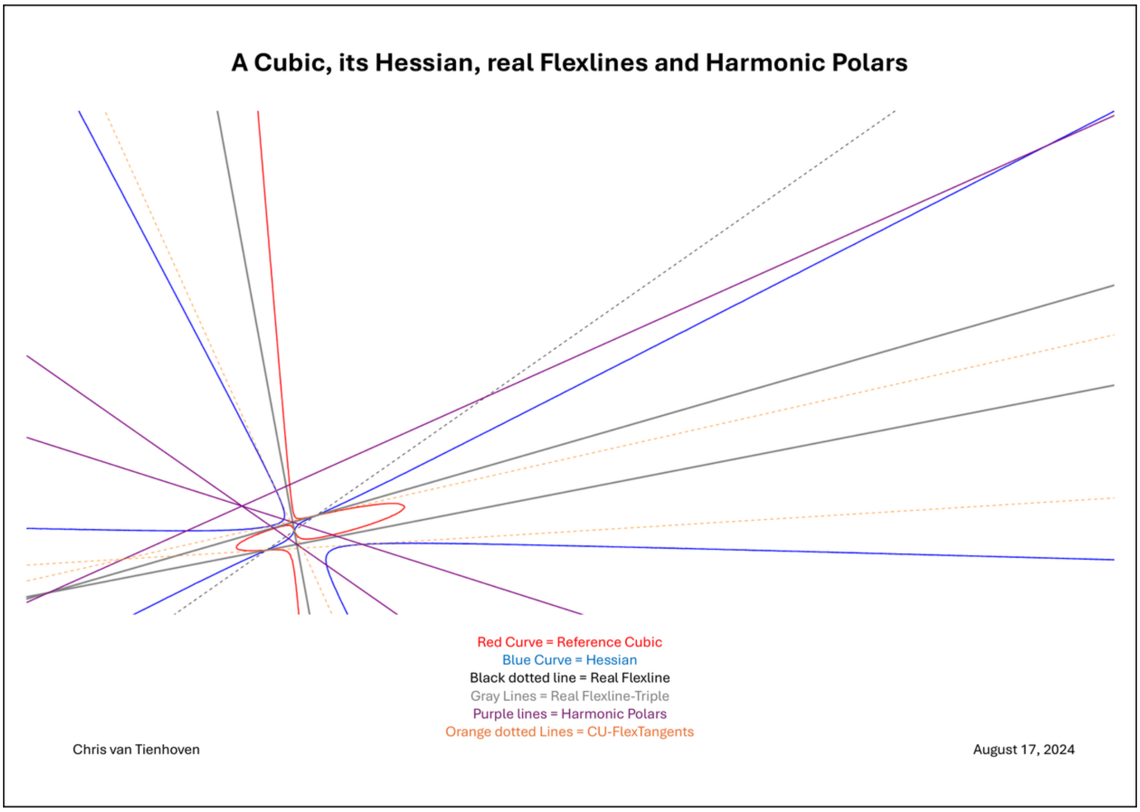
Message: #2444
Date: 2024-08-17
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,

Regarding your message QPG#2435:
I did not construct a similar monopartite cubic, but I traced
your drawing and created a picture of it in Mathematica. Please
see the attachment.
As you can see, my drawing closely matches yours. However, there
are some differences. I believe that the Harmonic Polar L3, in
particular, is different. It deviates, creating other
intersection points with the Hessian.
I hope this may help explain the discrepancy. If not, I can take
it a step further and perform the construction steps
explicitly."

Best regards,
Chris
```



CU-CU1-Cubic-Hessian-real Flexlines-Harmonic Polars-01.pdf

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Message: #2445
Date: 2024-08-19
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for the drawing,
... is it calculated or constructed?
My doubts are regarding your construction.
If you accept in my drawing #2440 the hessian, Po and L1,2,3,
... and it seems they are correct, then you see at once,
... that your construction in #2434 cannot give the real
    flexline-triple.
Sorry, I cannot control my doubts in your minimized drawing,
... where I miss the flextangents of the hessian.
Excuse my difficulties in understanding!
Best regards Eckart

PS: After reading once more your construction in #2434, passage
4, I think, you have to replace F1H1x by F2H1x and F3H1x,
... for they shall be the real flexlines with the used property.
But my drawing does not contain a H2x with this property,
... and my figure seems very precise.
```

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Message: #2446
Date: 2024-08-19
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,

My drawing was entirely calculated in Mathematica. The images
generated in Mathematica are more precise than those produced in
Cabri.
You are correct that in my message #2434, F1H1x should be
replaced by F2H1x and F3H1x. I apologize for the inaccuracy.
I will look into it as soon as possible to determine the source
of the discrepancy. I am confident we will find it.

Best regards,
Chris
```

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Message: #2447
Date: 2024-08-21
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

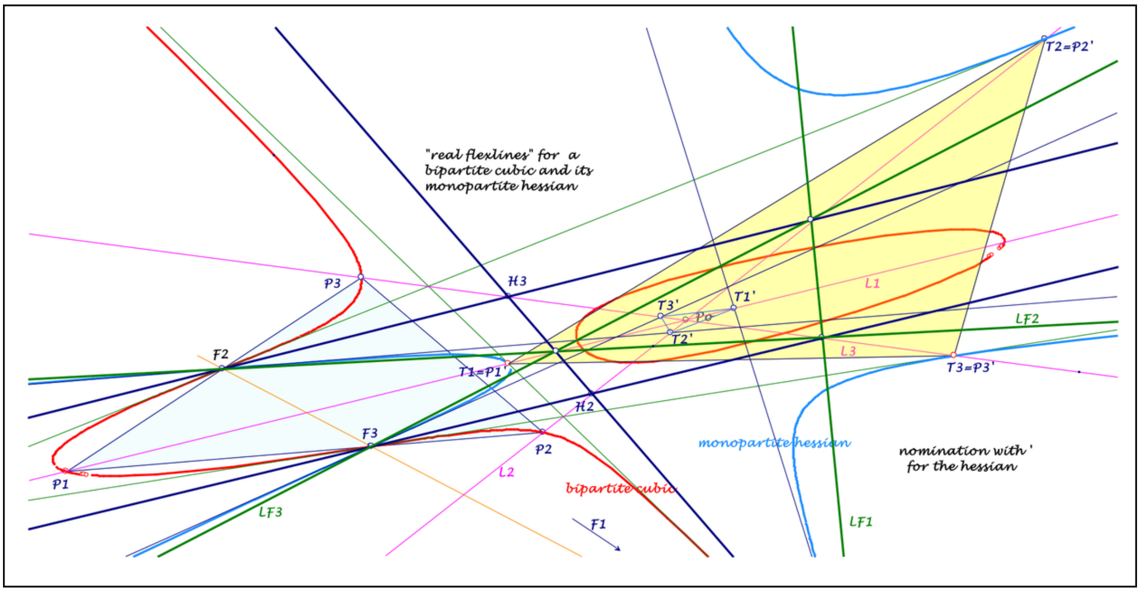
can you confirm my conjecture for the real flexlines unequal
F1F2F3?
Let LFi be the line
... Fi connected with the 4th harmonic of Po wrt Pi, Ti.
 a) For monopartite cubics: LF1,2,3 are the "real flexlines".
 b) For bipartite cubics: LFi intersects Lj and Lk
... on the "real flexlines" through Fj and Fk.

These "real flexlines" are the same for a cubic and its hessian
... and intersect the cubic and its hessian
    only in the real flexpoints.

I tested it for a bipartite cubic and its hessian (attached),
... the drawing shows more properties.

Best regards Eckart

PS: Ti intersections of harmonic polar Li and flextangent in Fi.
```



2024-08-21.pdf

Message: #2448
Date: 2024-08-23
From: analgeomatica@gmail.com
Subject: A square is generated from any quadrilateral

Dear Mr Chris and friends,

It's been a while since I posted to the group, but I've been following the group's news since it moved to group.io. The geometry of quadrilaterals and polygons has always been a big interest of mine. Thank you very much for the useful group work.

When I come back I would like to introduce to you two articles of mine about constructing a square from any quadrilateral. This is considered as we construct an equilateral triangle from any triangle, that is the famous Napoleon theorem.

A Napoleon-like Theorem for Quadrilaterals
<https://www.tandfonline.com/doi/pdf/10.1080/00029890.2022.2116247>:

Main theorem: Let ABCD be a convex quadrilateral. Let squares ADA*D* and
BCB*C* be erected externally on the sides AD and BC, and let parallelograms
ABA'B' and CDC'D' be erected externally on the sides AB and CD in such a
way that AB' is equal and perpendicular to CD and CD' is equal and
perpendicular to AB. Then the centers X, Y, Z, and W of the quadrilaterals
ADA*D*, ABA'B', BCB*C*, and CDC'D' form a square.

The asymmetric propeller with squares, and some extensions
<https://www.cambridge.org/core/journals/mathematical-gazette/article/abs/asymmetric-propeller-with-squares-and-some-extensions/44A92EE2B45BEBD2A94061ABC94FF051>:

Main theorem: Let ABCD be an arbitrary quadrilateral.
Erect four squares ABEF, BCGH, CDKL, and DAPQ outside ABCD.
Let X, Y, Z, and W be the centroid of quadrilaterals PFEH, EHGL, GLKQ, and KQPF,
respectively. Then,
 i) quadrilateral XYZW is a square,
ii) the center of square XYZW coincides with the centroid of quadrilateral ABCD.

Here the term "quadrilateral" that I commonly use is "quadrigon" in EQF <https://chrisvantienhoven.nl/quadrigon-objects>.

I look forward to your comments.

Sincerely yours,
Tran Quang Hung

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Message: #2449
Date: 2024-08-25
From: van10hoven@gmail.com
Subject: Re: A square is generated from any quadrilateral

Dear Tran Quang Hung,

It's great to hear from you again!
I’ve reviewed your configurations, and I noticed something special about the center of the square in your first configuration.
It lies on the line QA-P1.QG-P7.QG-P9, which runs parallel to QG-P1.QG-P6 and QL-P2.QL-P7.
As we can observe in a Quadrigon, QL-Points (and QA-Points) also play a significant role.

Best regards,
Chris

P.S. I once registered several of your nG-Points. I'm not sure if you’re aware of this.
Please see the Menu nG - Objects (https://www.chrisvantienhoven.nl/ng-items/ng-geninf/ng-0).

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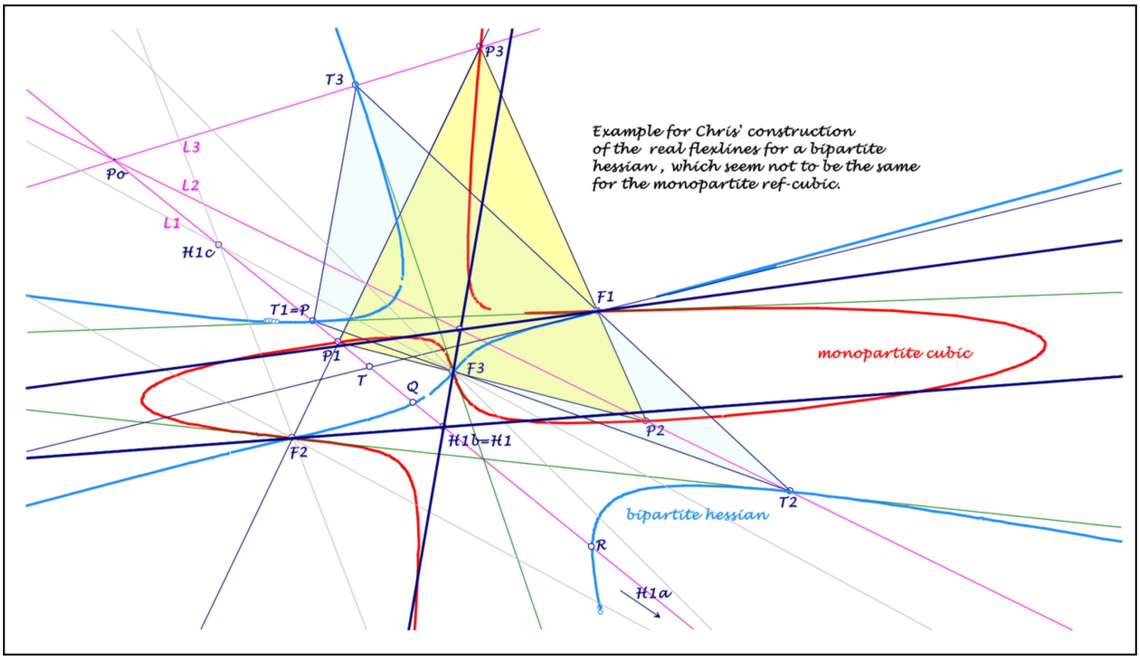
Message: #2450
Date: 2024-08-25
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

attached once more a drawing, showing my doubts
... for your construction of the real flexlines
    (#2434 corrected)
... for a bipartite hessian, which should give also
... the real flexlines of the monopartite ref-cubic
    (Schröter, p.197).

Best regards Eckart
```



2024-08-25.pdf

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Message: #2451
Date: 2024-08-26
From: analgeomatica@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] A square is generated from any

---

Dear Mr Chris and friends,

Thank you very much for your interest and review. I look forward to continuing to contribute new constructions to EQF and EPF. I am very happy to see my constructions for 5-gon and 6-gon appearing on your site. I still follow the discussions on the group closely and I always would like to contribute more.

I hope you will be interested in my second construction for quadrigon.

All the best,
Tran Quang Hung

Message: #2452
Date: 2024-08-28
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,

This time I am totally confused too.
Finally I motivated myself to draw the elaborate construction in Cabri of:
1. cubic CU (resembling yours in #2440),
2. hessian HE,
3. the 3 real flexpoints (F1,F2,F3), being CU^HE
4. the 3 harmonic polars (L1,L2,L3),
5. CU-tangents at the 3 real Flexpoints (Tn1,Tn2,Tn3),
6. HE-tangents at the 3 real Flexpoints (Tn1’,Tn2’,Tn3’),
7. the intersection points Li^Tni (T1,T2,T3),
8. the intersection points Li^Tni’ (T1’,T2’,T3’)
9. the intersection points HE^Li unequal Ti called (P’,Q’).
10. and finally the Harmonic Conjugates Hi’ of Ti’ wrt (P’,Q’).
Indeed, just like your drawing, I saw the intended real Flexlines just crossing CU.
The place they should be deviates just some degrees (viewed from Fi) when I compare it with my Mathematica drawing in message #2444.
This could be due to precision errors of Cabri.
I then revisited my initial calculations in Mathematica. There I calculated points H1, H2, H3 and they were almost spot on for all three points, with deviations of less than 1%, which I attributed to precision errors in Mathematica.
As I mentioned earlier, I traced your drawing onto the Mathematica drawing from message #2444. As a follow-up to this I then recalculated points H1, H2, and H3, but this time there were larger deviations, which left me confused.
I was surprised that the results were often so close.
I took me some time to process this. I couldn't find any misinterpretations in my calculations. I could hardly imagine that something was wrong. However, just like you, I can't ignore these deviations. This puts the entire construction in jeopardy.
I have come to the PRELIMINARY CONCLUSION THAT THE CONSTRUCTION AS MENTIONED IN MESSAGE #2299 DOES NOT DELIVER THE REAL FLEXLINE TRILATERAL. The question of finding a proper construction of the real flextrilateral is open again.
Thank you for your continued efforts!
Best regards,
Chris

Message: #2453
Date: 2024-08-31
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Chris, dear Eckart
There is a french expression Don't throw the baby with the water
of the bath!
I regret your conclusion of abandoning your construction, as it
works most of the time and is proved by your calculations.
Nevertheless, another construction is perhaps possible; who
knows?
Best regards
Bernard
```

Message: #2454
Date: 2024-08-31
From: bernard.keizer@gmail.com
Subject: Re: Triangle of real flexlines as self hessian cubic of the syzygetic

---

Dear Chris,
Please forget my previous message 2443, as the reasoning and the calculation are wrong!
There are 4 triangles RF like H1H2H3, one is H1H2H3, the 2nd has a real side, the line of real flexes and 2 imaginary sides intersecting in a real vertice, P0 and the 3rd and 4th have 3 imaginary sides and vertices.
For k = 0, we have FE and and for k = ∞ it's hessian RF
For k = 6a1a2a3, we have FE1 and for k = - 3a1a2a3 it's hessian RF1 (the 2nd mentionned RF)
The 2 last RF are imaginary
The 4 RF have the same property that they are their own hessian and that they have a 2nd prehessian, which is double.
For k = -6a1a2a3, k' = a1a2a3 and for k = 3a1a2a3, k' = -5a1a2a3, giving 2 ordinary members of the pencil and their hessians.
Best regards
Bernard

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Message: #2455
Date: 2024-08-31
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
Beautiful summary!
1) The flex points and the harmonic polars form 2 dual configurations
2) The 4 triangles like H1H2H3 = RF have 12 vertices and 12 sides which are also dual
3) The flexpoints form a CB system and all the cubics of the syzygetic pencil (or Hesse pencil) have the same 9 flexes (3 real and 6 imaginary).
The curves of the 3rd order tangent to the 9 harmonic polar belong to the dual syzygetic pencil.
The hessian of a cubic of the pencil is a cubic of the same pecil and it's cayleyan is a curve of the 3rd order belonging to the dual pencil.
The converse is true: the hessian of a curve of the 3rd of the dual pencil is a curve of the 3rd order of the same pencil and it's cayleyan is a cubic of the 1st pencil.
There is a correspondance between cubics of the 1st pencil and curves of the 3rd order of the 2nd pencil, the 2 curves swap their hessians and cayleyans.
The cayleyan of a cubic of the 1st pencil is the hessian of a curve of the 3rd order of the 2nd pencil and the cayleyan of this curve is the hessian of the initial cubic.
Best regards
Bernard
PS Are you able to calculate with Mathematica the equation of the cayleyan of a cubic in barycentric coordinates wrt H1H2H3?

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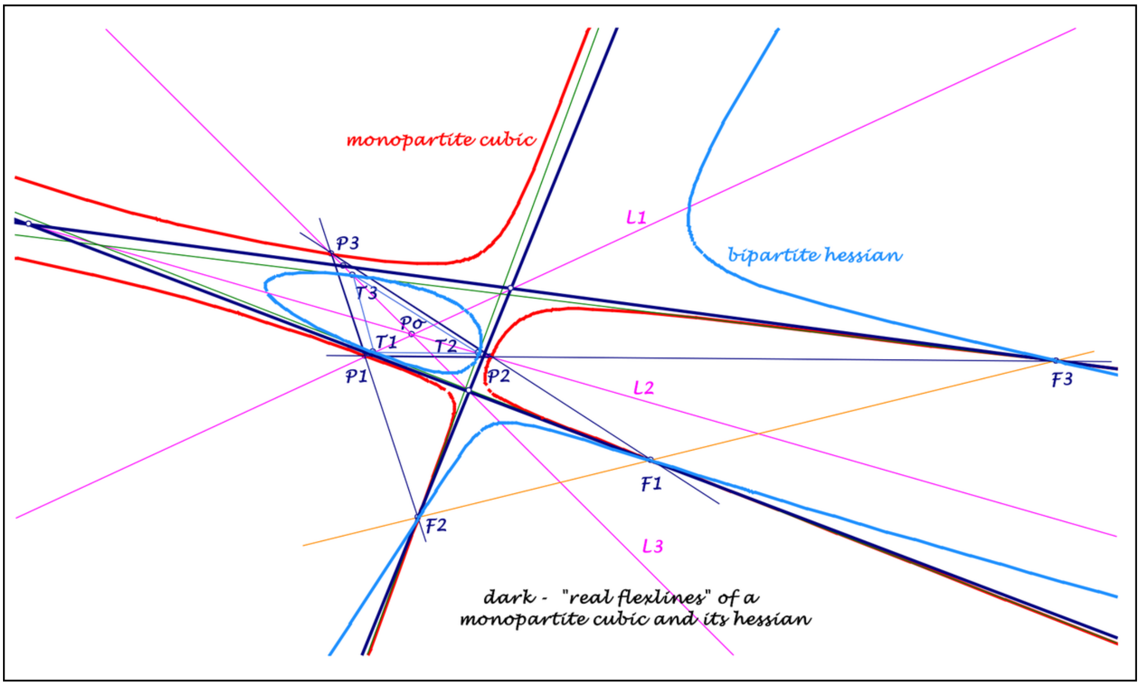
Message: #2456
Date: 2024-08-31
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for respecting my doubts,
... but my convention in #2447 will be not correct,
... perhaps the first part holds,
... using further the lines LFi:
... connection of Fi  and the 4th harmonic of Po wrt Pi, Ti
... (Ti intersection of Li and flextangent in Fi).
a) For monopartite cubics Lf1,2,3 are the "real flexlines".
Attached a drawing for a monopartite cubic.

Best regards Eckart
```



2024-08-31.pdf

Message: #2457
Date: 2024-09-05
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
I'm waiting impatiently for your reactions!
The barycentric equation of a line is uX + vY + wZ = 0
The barycentric equation of a cubic of the pencil wrt H1H2H3 is FE + kRF = 0,
where FE is $a1X^3 + a2Y^3 + a3Z^3 = 0$ and RF is XYZ = 0.
The hessian belongs to the same pencil and it's equation is FE + khRF = 0.
The cayleyan of a cubic belongs to the dual pencil and it's tangential equation wrt H1H2H3 is DFE + kcDRF, where DFE is $a1u^3 + a2v^3 + a3w^3 = 0$ and DRF is uvw = 0.
Note that uvw = 0 is the same triangle H1H2H3 defined by it's sides!
I suppose that the barycentric equation of the cayleyan has the same way the same coefficients as the tangential equation of the cubic.
Hence my question: what's the tangential equation of the cubic FE?
Many thanks in advance if you can answer this question ...
Best regards
Bernard

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Message: #2458
Date: 2024-09-06
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Chris,
Please cancel the end of my previous message 2457!
The beginning is correct:
the tangential equation of curves of the 3rd order of the dual
pencil has the same coefficients as the barycentric equation of
the cubics of the initial pencil.
But the converse is not true, as the curves of the 2 dual
pencils are not dual themselves!
Best regards
Bernard
PS Therefore my question in the PS of my message 2455 still
holds (barycentric equation of the cayleyan wrt H1H2H3)
```

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Message: #2459
Date: 2024-09-06
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Bernard,

You mentioned in #2443: “I hope you will make one of your comprehensive surveys of all these beautiful properties of these syzygetic pencils.”

Attached, you will find a brief survey outlining the beautiful properties we discovered, as well as some results about the Cayleyan. Calculating the Cayleyan, in particular, was quite a challenge.
I have also included additional calculation results for Hessians and preHessians.
This is just a preliminary version.
I plan to add more illustrations and further theoretical background.
Additionally, I hope we will eventually determine the values of (k, l, m) in the normalized expression of the Cayleyan (see the last page).

Then some questions:

1. Could you share with me where you first encountered the idea of splitting CU into FE + RF and the normalization idea? Preferably with references.

2. What role did Fermat play in this?

3. Then another question: do you think that every general cubic can be normalized to this expression: a1 x^3 + a2 y^3 + a3 z^3 + a0 x y z = 0? Does this mean anything new about solving third degree equations?

Best regards,
Chris

A different frame of reference for a Cubic, Flexpoints, Hessian and Caylean

Conjecture
Any Cubic CU with the general equation

$$\mathbf{c_1\, x^3 + c_2\, y^3 + c_3\, z^3 + c_4\, x^2\, y + c_5\, x^2\, z + c_6\, x\, y^2 + c_7\, y^2\, z + c_8\, x\, z^2 + c_9\, y\, z^2 + c_0\, x\, y\, z = 0}$$

with coordinates referring to a reference triangle ABC with vertices (1,0,0), (0,1,0) and (0,0,1) ,
can be rewritten with respect to a reference triangle corresponding to the real flex triangle $H_1H_2H_3$,.
This is the triangle bounded by the 3 real flexlines, each passing through one of the 3 real flexpoints,
with this equation: $\mathbf{a_1\, x^3 + a_2\, y^3 + a_3\, z^3 + a_0\, x\, y\, z = 0}$.

So when we work with the real flex triangle $H_1H_2H_3$ as reference triangle then there are coefficients (a_1, a_2, a_3, a_0) such that:

$$\mathbf{CU = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + a_0^3\, x\, y\, z = 0}$$

Next coordinates/equations follow from calculations.
The *9 flexpoints* are

$F_1 = \{0, -a_3, a_2\}$	real point
$F_2 = \{-a_3, 0, a_1\}$	real point
$F_3 = \{-a_2, a_1, 0\}$	real point
$F_4 = \{0, -i_1\, a_3, a_2\}$	imaginary point
$F_5 = \{0, -i_2\, a_3, a_2\}$	imaginary point
$F_6 = \{-i_1\, a_3, 0, a_1\}$	imaginary point
$F_7 = \{-i_2\, a_3, 0, a_1\}$	imaginary point
$F_8 = \{-i_1\, a_2, a_1, 0\}$	imaginary point
$F_9 = \{-i_2\, a_2, a_1, 0\}$	imaginary point

where $i_1 = -(-1)^{1/3}$ and $i_2 = (-1)^{2/3}$. Note that $i_1\, i_2=1$, $i_1^2=i_2$ and $i_1^3=1$.
The coordinates of all flexpoints show that they all lie on one of the sidelines of the reference triangle.

The *12 Flexlines* are:

$L_{123} = \{a_1, a_2, a_3\}$	real Flexline $F_1F_2F_3$: $a_1\, x + a_2\, y + a_3\, z = 0$
$L_{145} = \{1, 0, 0\}$	real Flexline $F_1F_4F_5$: $x = 0$
$L_{267} = \{0, 1, 0\}$	real Flexline $F_2F_6F_8$: $y = 0$
$L_{389} = \{0, 0, 1\}$	real Flexline $F_3F_7F_9$: $z = 0$
$L_{179} = \{i_1\, a_1, a_2, a_3\}$	imaginary Flexline $F_1F_7F_9$
$L_{168} = \{i_2\, a_1, a_2, a_3\}$	imaginary Flexline $F_1F_6F_8$
$L_{258} = \{a_1, i_1\, a_2, a_3\}$	imaginary Flexline $F_2F_5F_8$
$L_{249} = \{a_1, i_2\, a_2, a_3\}$	imaginary Flexline $F_2F_4F_9$
$L_{346} = \{a_1, a_2, i_1\, a_3\}$	imaginary Flexline $F_3F_4F_6$
$L_{357} = \{a_1, a_2, i_2\, a_3\}$	imaginary Flexline $F_3F_5F_7$
$L_{478} = \{a_1, i_1\, a_2, i_2\, a_3\}$	imaginary Flexline $F_4F_7F_8$
$L_{569} = \{a_1, i_2\, a_2, i_1\, a_3\}$	imaginary Flexline $F_5F_6F_9$

where $i_1 = -(-1)^{1/3}$ and $i_2 = (-1)^{2/3}$. Note that $i_1\, i_2=1$, $i_1^2=i_2$ and $i_1^3=1$.

The *9 Harmonic Polars* are:

$L_1 = \{0, a_2, -a_3\}$	real line
$L_2 = \{a_1, 0, -a_3\}$	real line
$L_3 = \{a_1, -a_2, 0\}$	real line
$L_4 = \{0, a_2, i_3\, a_3\}$	imaginary line
$L_5 = \{0, a_2, i_4\, a_3\}$	imaginary line
$L_6 = \{a_1, 0, i_3\, a_3\}$	imaginary line
$L_7 = \{a_1, 0, i_4\, a_3\}$	imaginary line
$L_8 = \{a_1, i_3\, a_2, 0\}$	imaginary line
$L_9 = \{a_1, i_4\, a_2, 0\}$	imaginary line

where $i_3 = {}^1/_2\,(1 + 3^{1/2}\, i)$ and $i_4 = {}^1/_2\,(1 - 3^{1/2}\, i)$. Note that $i_3\, i_4=1$, $i_3^2= -i_4$, $i_4^2= -i_3$, $i_3^3= -1$ and $i_4^3= -1$.

CU-Flex-NormalizedConfiguration-01.pdf

Harmonic Polar-Crosspoint P0
The 3 real Harmonic Polars are concurrent in **P0 $\{1/a_1, 1/a_2, 1/a_3\}$.**
P0 is the trilinear pole wrt the reference triangle $H_1H_2H_3$. of the real Flexline $F_1F_2F_3$: L_{123}.
Conversely L_{123} is the trilinear polar of P0 wrt the reference triangle $H_1H_2H_3$.

The Hessian of CU
We know:

$$\mathbf{CU = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + a_0^3\ x\ y\ z}$$

The Hessian HE of CU is:

$$\mathbf{HE = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + (a_0^6 / 3 + (36\ a_1^3\ a_2^3\ a_3^3) / a_0^3)\ x\ y\ z}$$

The Syzygetic Pencil
All Cubics passing through the 9 Flexpoints of CU are called cubics of the Syzygetic Pencil.
CU and its Hessian mutually intersect in the 9 Flexpoints of CU. Therefore they are called members of the Syzygetic Pencil.
The 3 sidelines of the real flex triangle $H_1H_2H_3$ (being the 3 real flexlines resp. through F_1, F_2, F_3) form together also a degenerate cubic, which will be called here RF. On these 3 lines all 9 Flexpoints occur. So it is also a member of the Syzygetic Pencil.
Since it is a pencil every Cubic in it can be described as t CU_1 + (1-t) CU_2, where CU_1 and CU_2 are also members of the pencil.
We now ask the question of finding a cubic FE as linear combination of CU and HE such that RF is its hessian.
Specifically we need to find CUx = Hessian of t CU + (1-t) HE.
It appears that there are 9 values of t that provide a solution.
However, upon substituting these values of t, it turns out that there are only 2 distinct solutions.

$$\mathbf{FE = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3}$$

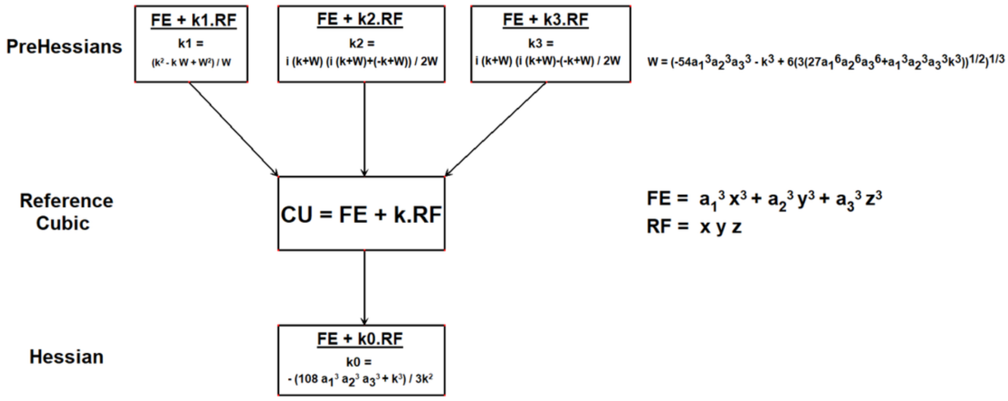
and

$$\mathbf{RF = x\ y\ z.}$$

FE is the main part and will be called here the *Core Curve*, RF is the degenerate part and will be called here the *Residual Curve*.
RF is actual the degenerate cubic consisting of the three sidelines (x=0, y=0, z=0) of the reference triangle, which are the three real Flexlines, which indeed contain the 9 Flexpoints.
Moreover FE and RF are derived from the pencil t CU + (1-t) HE and therefore both pass through the 9 Flexpoints and consequently are members of the Syzygetic Pencil.
It also appears that FE and RF share the same Hessian, namely RF.

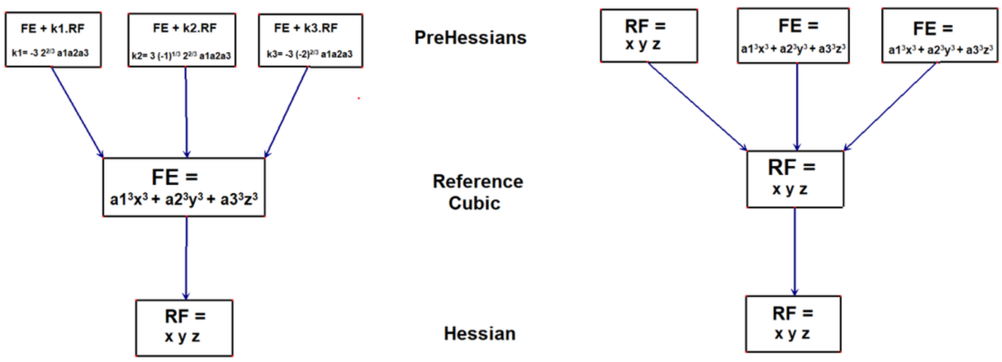
Hessians and PreHessians

Since a Hessian is also a cubic, it is also possible to calculate the Hessian of the Hessian, and further downwards.
Moreover it is possible, knowing some cubic CU, to find out for which upper cubic CU is the Hessian.
There are 3 of these cubics called the PreHessians pHE_1, pHE_2, pHE_3.
So per definition the Hessian of all three cubics pHE_1, pHE_2, pHE_3 will be CU.
So "upwards" there are 3 PreHessians of a cubic and "downwards" there is 1 Hessian per cubic.



Examples:

The 3 PreHessians of FE are also of the form FE + k.RF. Its Hessian is RF.
The 3 PreHessians of RF are RF itself and FE (counting twice).



The Cayleyan

Let CU = $\mathbf{a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + a_0^3\ x\ y\ z.}$

Numerical analyses show that the Cayleyan of CU has this expression:
CAY =

$$\mathbf{a_1^6\ x^6 + a_2^6\ y^6 + a_3^6\ z^6}$$
$$\mathbf{+\ k\ (a_1^3\ a_2^3\ x^3\ y^3 + a_1^3\ a_3^3\ x^3\ z^3 + a_2^3\ a_3^3\ y^3\ z^3)}$$
$$\mathbf{+\ l\ (a_1^4\ a_2\ a_3\ x^4\ y\ z + a_1\ a_2^4\ a_3\ x\ y^4\ z + a_1\ a_2\ a_3^4\ x\ y\ z^4)}$$
$$\mathbf{+\ m\ (a_1^2\ a_2^2\ a_3^2\ x^2\ y^2\ z^2)}$$

k, l, m are fixed expressions with a_1, a_2, a_3, a_0 in it.
It still isn't clear which expressions they are.

CU-Flex-NormalizedConfiguration-01.pdf

Message: #2460
Date: 2024-09-06
From: eckart_schmidt@t-online.de
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Chris,

reading your interesting paper:
Is it already mentioned, that P0
... is the 4th real intersection of flexlines,
... as Schröter describes on page 239?
With your nomination  P0 = L478 ^ L569 = (a2a3,a3a1,a1a2).

Best regards Eckart
```

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Message: #2461
Date: 2024-09-08
From: eckart_schmidt@t-online.de
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Chris,

I try to calculate with your results,
... but are there different hessians
... in the terms of HE on page 2 and 3 for k = a0^3?

Best regards Eckart
```

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Message: #2462
Date: 2024-09-08
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Eckart,
The expression on page 2 is wrong!
The expression on page 3 is correct.
I'm preparing a complete answer to Chris ...
Best regards
Bernard
```

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Message: #2463
Date: 2024-09-08
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Eckart,

You are right there are some differences.

At page 3 it says
Hessian is FE + k0.RF,
where k0 = - (108 a1^3 a2^3 a3^3 + k^3) / (3k^2)
Substituting k = a0^3 makes
k0 = -(108 a1^3 a2^3 a3^3 + a0^9) / 3a0^6
   = -(36 a1^3 a2^3 a3^3) / a0^6) + a0^3 / 3
This indeed is the right expression.

Therefore at page 2 instead of
HE = a1^3 x3 + a2^3 y3 + a3^3 z3
   + (a0^6 / 3 + (36 a1^3 a2^3 a3^3) / a0^3) x y z
it should be
HE = a1^3 x3 + a2^3 y3 + a3^3 z3
   - (a0^3 / 3 + (36 a1^3 a2^3 a3^3) / a0^6) x y z

I am sorry for the typos.

Best regards,
Chris
```

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Message: #2464
Date: 2024-09-08
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Eckart,

About your question in #2460:

Is it already mentioned, that P0
... is the 4th real intersection of flexlines,
... as Schröter describes on page 239?
With your nomination P0 = L478 ^ L569 = (a2a3,a3a1,a1a2).

It is mentioned in my message #2410:
“Flexlines not only contain 3 Flexpoints (on CU) but also 2
HP-Crosspoints (not on CU).”
(HP = Harmonic Polar)
That was the reason for my small paper at that time.

Best regards,
Chris

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Message: #2465
Date: 2024-09-08
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Bernard,

"Regarding your message #2453 about not throwing the baby with the bathwater, we have the same expression in Dutch, so I understand you very well.

I don't want to abandon the construction because it yields some interesting results. However, my conclusion is that it has not been proven rigorously enough through calculations. Like you, I hope that another construction will emerge.

Nonetheless, the triangle H1H2H3 still exists, though not constructed as we initially thought. Instead, it represents the true flextrilateral, bounded by the three distinct real flexlines, F1F2F3, which we still do not know how to construct.

Best regards

Chris

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Message: #2466
Date: 2024-09-08
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
I answer your message as quickly as possible!
Thanks a lot for your first preliminary version, which is naturally not complete, but the 1st step is there!
1) Long time ago, I put on the Forum the following reference:
Henry S. White Conics and cubics connected with a plane cubic by certain covariant relations.
Hard stuff (autopoloconics and autopolar conics), but I had no reaction from you, nor from Eckart and I gave up ...
Recently, all came together: you showed the triangle H1H2H3, which is a cubic through the 9 flexes, I found on Internet the definition of the Hesse (or syzygetic) pencil with a reference of the article by Michela Artebani The Hesse pencil of plane cubic curves (which contains calculations for the hessian and the cayleyan).
First I drew with Geogebra the cubic $X^3 + Y^3 + Z^3 = 0$ and P0 as the centroïd of the triangle RF (you remeber certainly our exchanges)
Last, I had the idea of adapting the expressions to a general cubic with P0 = (1/a1,1/a2,1/a3) and here we are (I'm convinced it fits to any non singular cubic, but I don't see the link with the resolution of equations of the 3rd degree).
Pierre de Fermat worked on such curves with n instead of 3 ...
2)page 1
I think you don't need i1, i2, i3 and i4
All your elements can be described by using the cubic roots of the unit, which are 1, j and j^2 with $j = ½ (1+ i\sqrt{3})$ and $j^2 = ½ (1-i\sqrt{3})$
We are in 2 dual systems:
a) flexes and harmonic polars
b) flexlines and HP cross-points (here I agree with Eckart, it was in your previous memo, but it's worth repeating it) respectively sides and vertices of 4 triangles ...
c) cubics through the flexes and curves of the 3rd order tangent to the harmonic polars, which have the same barycentric and tangential equations
3) page 2
I think I've found a better expression $CU = a1^3X^3 + a2^3Y^3 + a3^3Z^3$ ka1a2a3XYZ, which avoids the a0 or $a0^3$ ...
Then HE belongs to the pencil with $k' = -(108 + k^3)/3k^2$
There are 4 cubics like FE and 4 cubics like RF, sharing the property that FE and RF are the prehessians of RF (k = 0, 6, 6j or 6j) and k' = ∞, -3, -3j or $-3j^2$)

4) page 3
I liked your prehessians of FE (one is real and the 2 others imaginary)
5) page 4
I hope the simplification I propose will help to identify the missing coefficients
I only notice that all the terms you calculated could come from the development of CU^2
I think in the dual system we will have DCU = DFE + $Ka1^2a2^2a3^2$DRF
You will find in Artebani the cayleyan of CU(k), curve of the dual pencil with rank k'' = $(54 - k^3)/9k$
Then k'' = K' and this cayleyan is the hessian in the dual system of DCU(K) with K = -18/k
Conversely, the cayleyan of DCU is the hessian HE of CU and k' = K''
The 2 curves swap their hessian and cayleyan
The 3 cubics having the same hessian correspond to 3 curves of 3rd order having the same cayleyan
The 3 cubics having the same cayleyan correspond to 3 curves of 3rd order having the same hessian
I'm glad we could join our efforts in order to understand all these calculations ...
Naturally, Eckart you will be welcome to the club if you are interested!
Best regards
Bernard
PS Ces choses-là sont rudes, il faut pour les comprendre avoir fait des études ...

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Message: #2467
Date: 2024-09-15
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Bernard and Eckart,

Bernard, thank you for your detailed feedback in #2466. I especially appreciated the references, which were very helpful. I also found a new reference that I believe will be quite useful:

Araceli Bonifant and John Milnor - On Real and Complex Cubic Curves
Available at: https://arxiv.org/pdf/1603.09018

I have incorporated as many of your comments as possible. Attached, you will find a revised version of the survey on cubics in normalized form.

There are still a few challenges I'm facing:

- Finding (a1,a2,a3,k) in terms of (c1,c2,c3,c4,c5,c6,c7,c8,c9,c0): See page 1 of the attached paper. The method of calculation is clear, but my computer consistently encounters memory issues after hours of processing.

- Finding (u, v, w) for the Cayleyan of a reference cubic in Hesse's form: See page 5 of the attached paper. Despite numerous attempts, I haven't been able to obtain a satisfactory solution. It's more complex than anticipated.

- Creating visualizations of the Hessians and pre-Hessians.

Best regards,

Chris

A different frame of reference for a Cubic, Flexpoints, Hessian and Caylean

General form of a cubic equation

A cubic equation in barycentric coordinates can be expressed in general form as:

$$\mathbf{c_1 x^3 + c_2 y^3 + c_3 z^3 + c_4 x^2 y + c_5 x^2 z + c_6 x y^2 + c_7 y^2 z + c_8 x z^2 + c_9 y z^2 + c_0 x y z = 0}$$

where the coordinates refer to a reference triangle ABC with vertices (1,0,0), (0,1,0) and (0,0,1).

First normal form of a cubic equation

When the reference triangle is changed, the equation will change accordingly, depending on the new reference triangle. This transformation can be achieved using a projective transformation of points. For a projective transformation, four basic points must be mapped to four other points. In our case, these four points are the vertices of the reference triangle and its barycenter.

Projective Transformation

The projective transformation pTf applied for any distinct point X(x,y,z), identified in reference triangle with vertices (1,0,0), (0,1,0) and (0,0,1), to reference triangle P1(P1x,P1y,P1z), P2(P2x,P2y,P2z), P3(P3x,P3y,P3z) is:

pTf (P1, P2, P3, X) =
(Det[(X, P2, P3)] (p1x+p1y+p1z),
Det[(P1, X, P3)] (p2x+p2y+p2z),
Det[(P1, P2, X)] (p2x+p2y+p2z)),

where Det[(P,Q,R)] is defined as the determinant of the 3x3-matrix formed by the 3 sets of coordinates of the 3 points P,Q,R.

The projective transformation applied for all points (x,y,z) on CU from reference triangle (1,0,0), (0,1,0) and (0,0,1) to reference triangle P1,P2,P3 will be Tf^{-1}.

Let H1H2H3 be the real flex triangle $H_1H_2H_3$ of the reference cubic. This is the triangle bounded by the 3 real flexlines, each passing through one of the 3 real flexpoints.

When the reference triangle is changed to this flex triangle, the equation of the cubic simplifies to a form, here called the first normal form:

$$\mathbf{b_1 x^3 + b_2 y^3 + b_3 z^3 + b_0 x y z = 0}.$$

In order to connect to later developments the first normal form is rearranged to:

$$\mathbf{a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k a_1 a_2 a_3 x y z = 0}$$

So when we work with the real flex triangle $H_1H_2H_3$ as reference triangle then there are coefficients (a_1, a_2, a_3, k) such that:

$$\mathbf{CU = a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k a_1 a_2 a_3 x y z = 0}$$

Second normal form of a cubic equation

Another projective transformation can be applied mapping $(\mathbf{a_1 x}) \rightarrow x$, $(a_2 \mathbf{y}) \rightarrow y$, $(\mathbf{a_3 z}) \rightarrow z$, resulting in the second normal form:

$$\mathbf{x^3 + y^3 + z^3 + k x y z = 0}$$

This form is known as **Hesse's Form**.

Thus, after performing two projective transformations, the equation of a cubic in its general form is transformed into Hesse's form. A sequence of two projective transformations can be considered as one projective transformation.

For a proof of the existence of a projective mapping from any smooth cubic in general form to Hesse's form see [1], page 2 and [2], page 4.

Flexpoints, Flexlines and Harmonic Polars in first normal form

After the projective transformation next coordinates/equations follow from consequent calculations.
The *9 flexpoints* are

$F_1 = (0, -a_3, a_2)$	real point
$F_2 = (-a_3, 0, a_1)$	real point
$F_3 = (-a_2, a_1, 0)$	real point
$F_4 = (0, -i_2\, a_3, a_2)$	imaginary point
$F_5 = (0, -i_1\, a_3, a_2)$	imaginary point
$F_6 = (-i_2\, a_3, 0, a_1)$	imaginary point
$F_7 = (-i_1\, a_3, 0, a_1)$	imaginary point
$F_8 = (-i_2\, a_2, a_1, 0)$	imaginary point
$F_9 = (-i_1\, a_2, a_1, 0)$	imaginary point

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
The coordinates of all flexpoints show that they all lie on one of the sidelines of the reference triangle.

The *12 Flexlines* are:

$L_{123} = (a_1, a_2, a_3)$	real Flexline $F_1F_2F_3$: $a_1\, x + a_2\, y + a_3\, z = 0$
$L_{145} = (1, 0, 0)$	real Flexline $F_1F_4F_5$: $x = 0$
$L_{267} = (0, 1, 0)$	real Flexline $F_2F_6F_8$: $y = 0$
$L_{389} = (0, 0, 1)$	real Flexline $F_3F_7F_9$: $z = 0$
$L_{179} = (i_2\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_7F_9$
$L_{168} = (i_1\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_6F_8$
$L_{258} = (a_1, i_2\, a_2, a_3)$	imaginary Flexline $F_2F_5F_8$
$L_{249} = (a_1, i_1\, a_2, a_3)$	imaginary Flexline $F_2F_4F_9$
$L_{346} = (a_1, a_2, i_2\, a_3)$	imaginary Flexline $F_3F_4F_6$
$L_{357} = (a_1, a_2, i_1\, a_3)$	imaginary Flexline $F_3F_5F_7$
$L_{478} = (a_1, i_1\, a_2, i_1\, a_3)$	imaginary Flexline $F_4F_7F_8$
$L_{569} = (a_1, i_2\, a_2, i_2\, a_3)$	imaginary Flexline $F_5F_6F_9$

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
Note that after the projective transformation indeed the triangle bounded by L_{145}, L_{267}, L_{389} is the new reference triangle with sidelines (1,0,0), (0,1,0), (0,0,1), and consequently with vertices H1(1,0,0), H2(0,1,0), H3(0,0,1) of the new reference triangle.

The *9 Harmonic Polars* are:

$L_1 = (0, a_2, -a_3)$	real line
$L_2 = (a_1, 0, -a_3)$	real line
$L_3 = (a_1, -a_2, 0)$	real line
$L_4 = (0, a_2, -i_2\, a_3)$	imaginary line
$L_5 = (0, a_2, -i_1\, a_3)$	imaginary line
$L_6 = (a_1, 0, -i_2\, a_3)$	imaginary line
$L_7 = (a_1, 0, -i_1\, a_3)$	imaginary line
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where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.

Harmonic Polar-Crosspoint P0
The 3 real Harmonic Polars are concurrent in **P0 ($1/a_1$, $1/a_2$, $1/a_3$).**
P0 is the trilinear pole wrt the reference triangle $H_1H_2H_3$. of the real Flexline $F_1F_2F_3$: L_{123}.
Conversely L_{123} is the trilinear polar of P0 wrt the reference triangle $H_1H_2H_3$.

CU-Flex-NormalizedConfiguration-02.pdf

The Hessian of CU

We know:

$$\mathbf{CU = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

After calculation the Hessian HE of CU appears as:

$$\mathbf{HE = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k'\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

where $k' = (108 + k^3) / 3$

The Syzygetic Pencil

All Cubics passing through the 9 Flexpoints of CU are called cubics of the Syzygetic Pencil.
CU and its Hessian mutually intersect in the 9 Flexpoints of CU. Therefore they are called members of the Syzygetic Pencil.
The 3 sidelines of the real flex triangle $H_1H_2H_3$ (being the 3 real flexlines resp. through F_1, F_2, F_3) form together also a degenerate cubic, which will be called here RF. On these 3 lines all 9 Flexpoints occur. So it is also a member of the Syzygetic Pencil.
Since it is a pencil every Cubic in it can be described as $t\ CU_1 + (1\text{-}t)\ CU_2$, where CU_1 and CU_2 are also members of the pencil.
We now ask the question of finding a cubic FE as linear combination of CU and HE such that RF is its hessian.
Specifically we need to find CUx = Hessian of t CU + (1-t) HE.
It appears that there are 9 values of t that provide a solution.
However, upon substituting these values of t, it turns out that there are only 2 distinct solutions.

$$\mathbf{FE = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3}$$

and

$$\mathbf{RF = x\, y\, z.}$$

FE is the main part and will be called here the *Core Curve*, RF is the degenerate part and will be called here the *Residual Curve*.
RF is actual the degenerate cubic consisting of the three sidelines (x=0, y=0, z=0) of the reference triangle, which are the three real Flexlines, which indeed contain the 9 Flexpoints.
Moreover FE and RF are derived from the pencil t CU + (1-t) HE and therefore both pass through the 9 Flexpoints and consequently are members of the Syzygetic Pencil.
It also appears that FE and RF share the same Hessian, namely RF.

Hessians and PreHessians

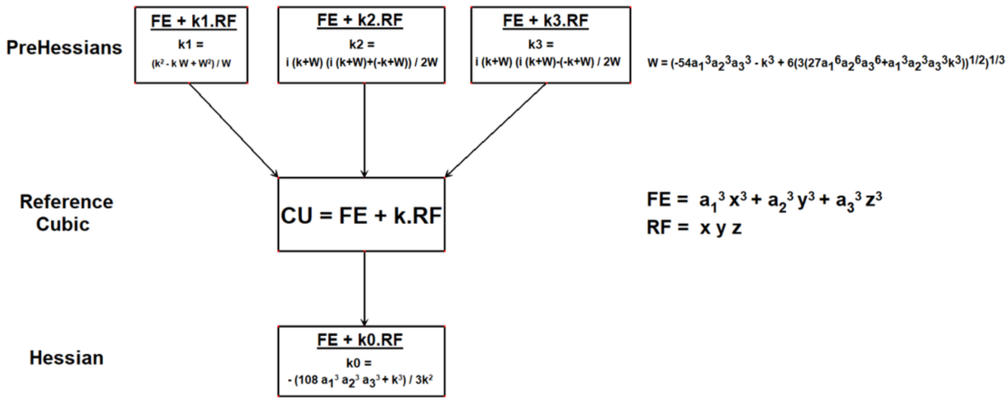
Since a Hessian is also a cubic, it is also possible to calculate the Hessian of the Hessian, and further downwards.

Moreover it is possible, knowing some cubic CU, to find out for which upper cubic CU is the Hessian.

There are 3 of these cubics called the PreHessians pHE_1, pHE_2, pHE_3.

So per definition the Hessian of all three cubics pHE_1, pHE_2, pHE_3 will be CU.

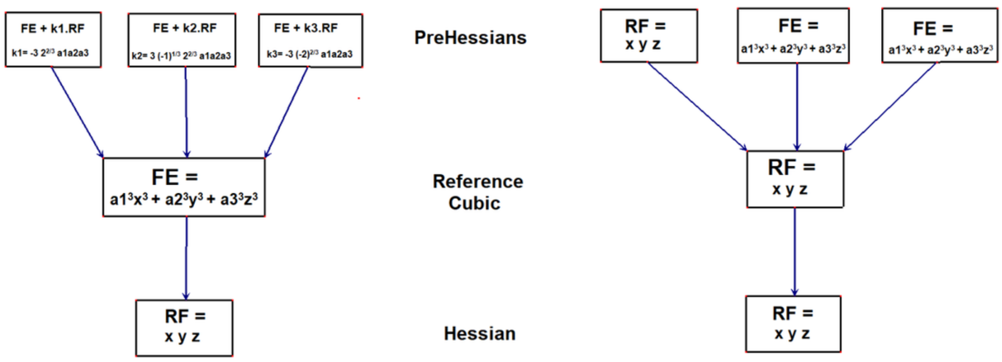
So "upwards" there are 3 PreHessians of a cubic and "downwards" there is 1 Hessian per cubic.



Examples:

The 3 PreHessians of FE are also of the form FE + k.RF. Its Hessian is RF.

The 3 PreHessians of RF are RF itself and FE (counting twice).



The Cayleyan

Let CU = $\mathbf{a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$

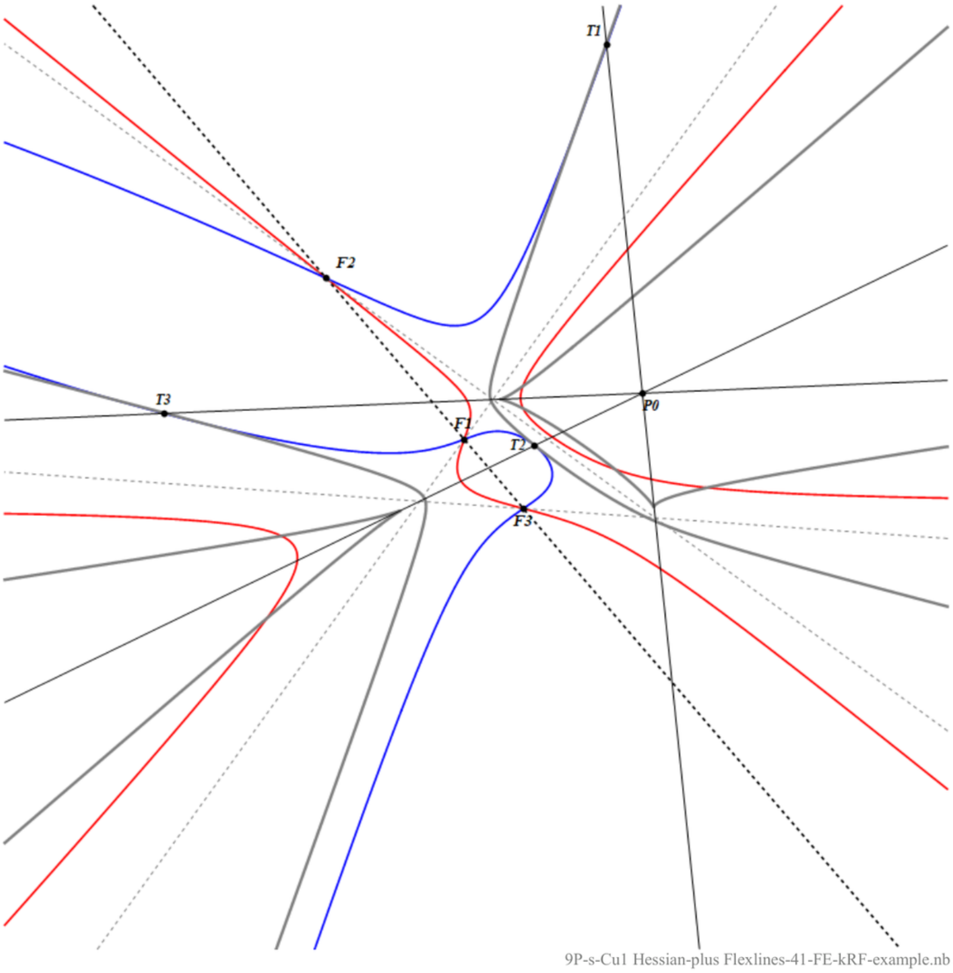
Numerical analyses show that the Cayleyan of CU has this expression:
CAY =

$$\mathbf{a_1^6\, x^6 + a_2^6\, y^6 + a_3^6\, z^6}$$
$$\mathbf{+\ u\, (a_1^3\, a_2^3\, x^3\, y^3 + a_1^3\, a_3^3\, x^3\, z^3 + a_2^3\, a_3^3\, y^3\, z^3)}$$
$$\mathbf{+\ v\, (a_1^4\, a_2\, a_3\, x^4\, y\, z + a_1\, a_2^4\, a_3\, x\, y^4\, z + a_1\, a_2\, a_3^4\, x\, y\, z^4)}$$
$$\mathbf{+\ w\, (a_1^2\, a_2^2\, a_3^2\, x^2\, y^2\, z^2)}$$

u, v, w are fixed expressions with probably a_1, a_2, a_3, a_0 in it.
It still isn't clear which expressions they are.

CU-Flex-NormalizedConfiguration-02.pdf

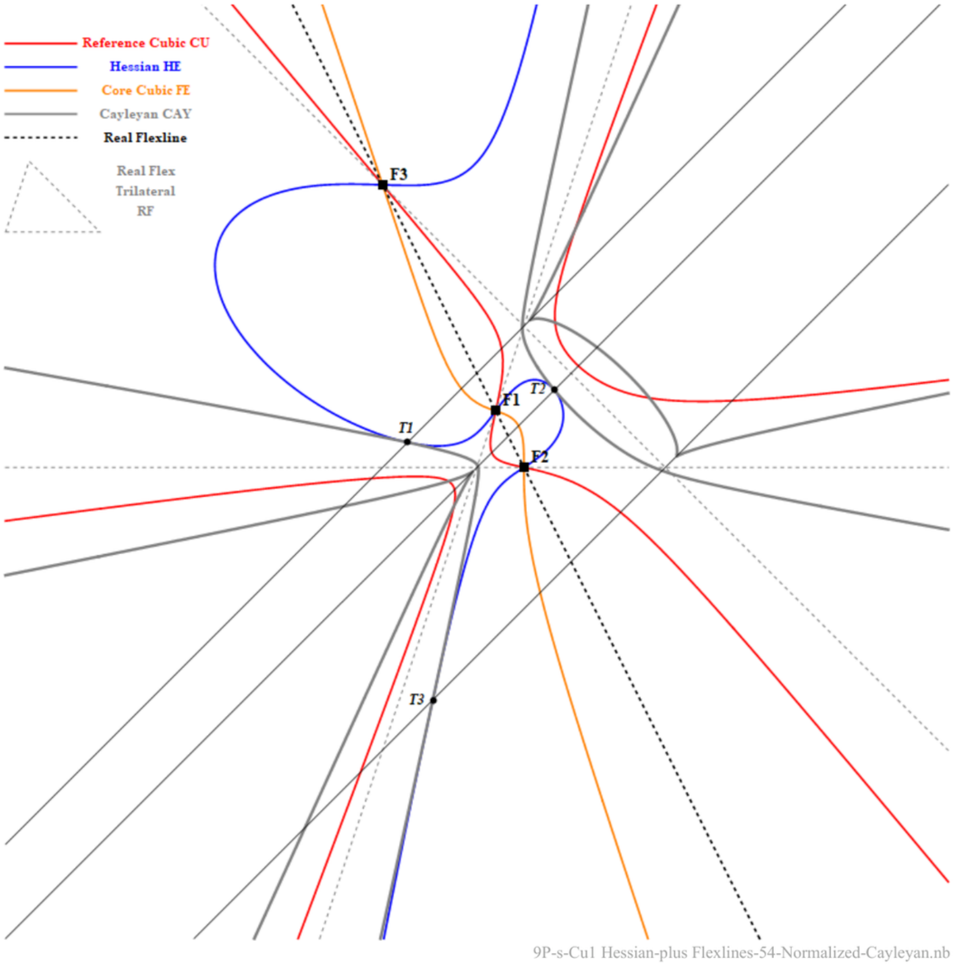
Picture of CU / HE / CAY



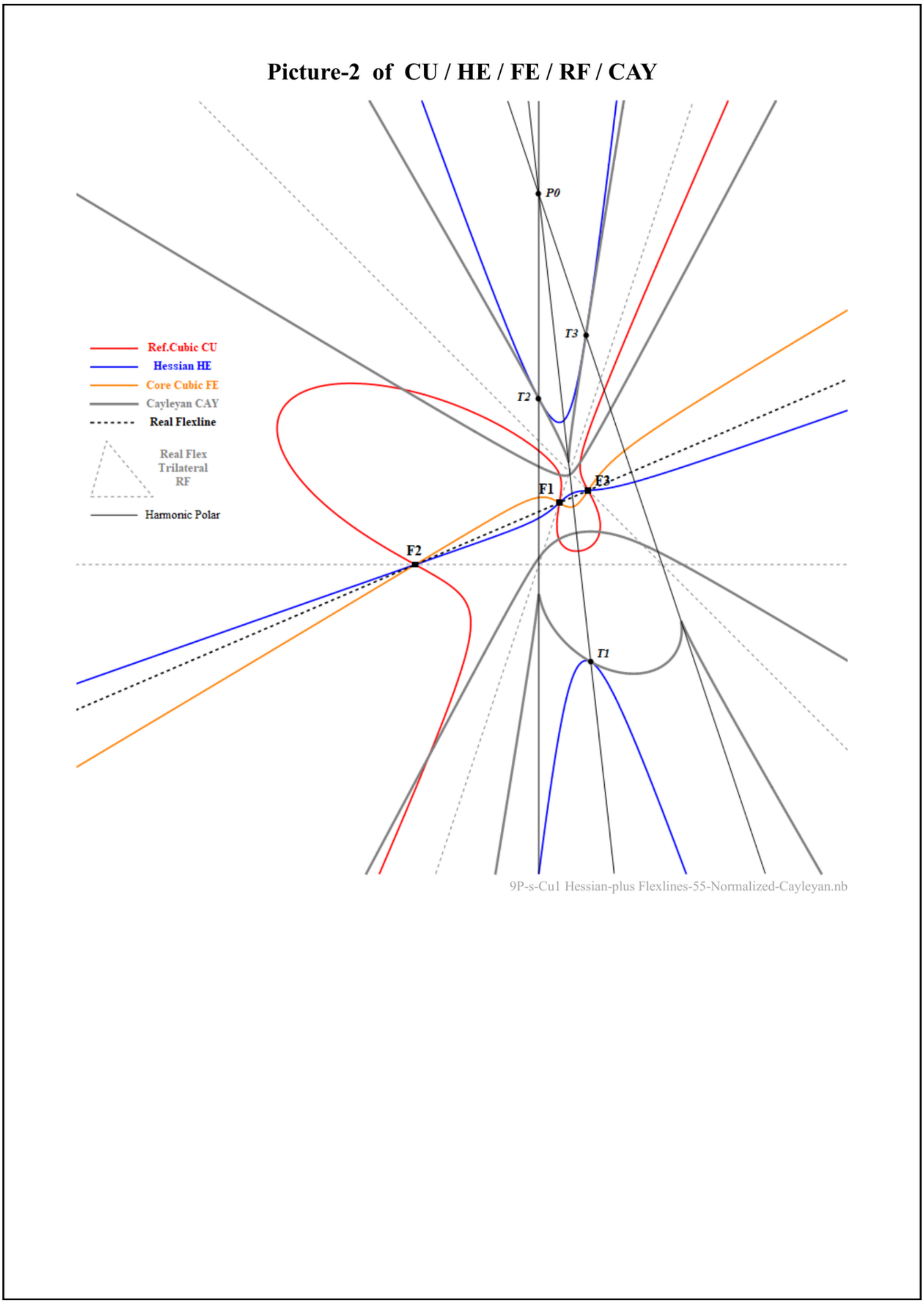
- Reference Cubic CU is red
- Hessian HE is blue
- Cayleyan CAY is gray
- F1, F2, F3 are the real Flexpoints
- The 1 gray dotted line is the real Flexline F1F2F3
- The 3 light gray dotted lines are the real Flexlines
- The 3 black lines are the 3 Fi-Harmonic Polars
- T1, T2, T3 are intersection points of corresponding Fi-Tangent and Fi-Harmonic Polar
- P0 is the common intersection point of the 3 real Fi-Harmonic Polars

CU-Flex-NormalizedConfiguration-02.pdf

Picture-1 of CU / HE / FE / RF / CAY



CU-Flex-NormalizedConfiguration-02.pdf



CU-Flex-NormalizedConfiguration-02.pdf

Note 1.
A “primitive cube root of unity” or “primitive third root of 1” is a complex number that, when raised to the power of 3, equals 1, and it is not equal to 1 itself.
There are exactly three cube roots of unity:
1. 1
2. i1 = -1/2 + i sqrt[3]/2
3. i2 = $i1^2$ = -1/2 - i sqrt[3]/2

Among these, i1 and i2=i1^2 are called primitive cube roots of unity because they generate all cube roots of unity when raised to successive powers, and they themselves are not equal to 1.
The primitive cube roots of unity have the following properties:
- $i1^3$ = 1
- i1 <> 1 and i2 <>1
- 1 + i1 + i2 = 0

Geometrically, they are the points on the complex plane that form the vertices of an equilateral triangle inscribed in the unit circle, with one vertex at 1.

References:
[1] Araceli Bonifant and John Milnor - On Real and Complex Cubic Curves
Available at: https://arxiv.org/pdf/1603.09018

[2] Michela Artebani and Igor Dolgachev - The Hesse Pencil of Plane Cubic Curves
Available at: https://arxiv.org/abs/math/0611590

Message: #2468
Date: 2024-09-16
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,

Thanks for your new version!
On page 3, the k' is not correct, it fails the sign -
I reget that you didn't put the calculation of k'' for the cayleyan.
If CU is a cubic (curve of the 3rd degree or 3rd order) of a Hesse pencil (sharing the 9 flexes),
CA is a curve of 3rd class of the dual Hesse pencil (sharing the 9 harmonic polars as tangents).
For CU under the form

$(a1X)^3 + (a2Y)^3 + (a3Z)^3 + k(a1X)(a2Y)(a3Z) = 0$,

HE is the same with k' instead of k

the tangential equation of CA is

$(u/a1)^3 + (v/a2)^3 + (w/a3)^3 + k''(u/a1)(v/a2)(w/a3) = 0$.

It's remarkable that $k' = -k/3 - (6/k)^2$ and $k'' = 6/k - (k/3)^2$
and for k and -18/k, k' and k'' swap mutually
There are still so many properties to mention.
For example, a cubic is tangent to the 3 cayleyans of it's 3 prehessians.
Conversely, a curve of the 3rd class is tangent to the 3 hessians of the 3 cubics having this curve as cayleyan
The beat goes on!

Best regards
Bernard

Message: #2469
Date: 2024-09-19
From: unidentifiedlethargicorganism@gmail.com
Subject: Hexalateral, Kantor-Hervey points, collinear

Dear Geometers,

THIS IS NOT MY OWN FINDING but I would like to share it in this group.
The source (information like who discovered it) is unknown, but seemingly it is one of the problems from a Chinese plane geometry exercise called □□□□.
What is the following line called in EPG? Is it already known in this group?

Let L_1L_2L_3L_4L_5L_6 be a general hexalateral in $\mathbb{R}^2$ (a 6-Line figure. For simplicity, we assume that any 2 lines are non-parallel and any 3 lines are non-concurrent).
Let K_(i, j)≡K_(j, i):= QL-P3 Kantor-Hervey point of a quadrilateral L_kL_lL_mL_n (i, j, k, l, m, n are distinct elements of {1, 2, 3, 4, 5, 6})
Then, ∀p∈{1, 2, 3, 4, 5, 6}, ∀q∈{1, 2, 3, 4, 5, 6}□{p}, ∃Q_p∈$\mathbb{R}^2$; K_(p, q)Q_p⊥L_q.
Moreover, ∃L={A+λ*v | A∈$\mathbb{R}^2$, λ∈ℝ, v∈$\mathbb{R}^2$□0*}; {Q_p}∈L.

In other words,
M_(p , q):= perpendicular line to L_q through K_(p , q) (p , q are 2 distinct elements from {1, 2, 3, 4, 5, 6})

Q_ p := point of concurrency of 5 lines M_(p , q), M_(p. r), M_(p , s), M_(p, t), M_(p , u)
(These 5 lines are concurrent. q, r, s, t, u are distinct elements of {1, 2, 3, 4, 5, 6} any of which is different from p.)

Then, the 6 points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 lie on the same line L.

Which is this line in EPG?

Sincerely,
Keita Miyamoto

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Message: #2470
Date: 2024-09-20
From: van10hoven@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

Dear Keita,

Thank you very much for your contribution. It looks very interesting. However, I am not quite sure if I understand you correctly. You state: ∀ p ∈ {1, 2, 3, 4, 5, 6}, ∀ q ∈ {1, 2, 3, 4, 5, 6} \ {p}, ∃ Q_p ∈ℝ ^{2}; K_(p, q)Q_p ⊥ L_q.

I paraphrase this as follows: There is a point Q_p with the property that K_(p,q)Q_p is perpendicular to L_q. Stated this way, every point on the line perpendicular to L_q through K_(p,q) could be Q_p. I assume you don't mean this, and perhaps you mean that the point Q_p also lies on L_p. In that case, Q_p is defined as the intersection point of L_p and the line perpendicular to L_q through K_(p,q).

In that case, accordingly, there will be a corresponding point L_q, and the line L_p.L_q will have the properties you defined for all occurrences of (p,q) from (1,2,3,4,5,6). However, in my first drawing, I cannot confirm that L_p.L_q is the same for all occurrences.

Finally, to clarify, do you mean that QL-P3 refers to the definition of EQF:

The circumcenters of the 4 component triangles of the Reference Quadrilateral are concyclic (on the Miquel Circle). These circumcenters Oi (i=1,2,3,4) form a Quadrangle with 4 component triangles whose Orthocenters HOi (i=1,2,3,4) are also concyclic (on the Hervey Circle) with circumcenter QL-P3, the Kantor-Hervey Point.

I look forward to your response.

Best regards,

Chris

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Message: #2471
Date: 2024-09-20
From: unidentifiedlethargicorganism@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

Dear Chris,

I am so sorry for the confusion.

QL-P3 Kantor-Hervey point is the point of concurrency of 4 lines each of which is the perpendicular bisector of the segment joining the circumcenter and orthocenter of a component triangle of a reference quadrilateral.

Let me clarify the statement as follows:

Let L_1L_2L_3L_4L_5L_6 be a 6-Line figure.

Let K_(1,2) be the Kantor-Hervey point
of the quadrilateral L_3L_4L_5L_6.
Let K_(1,3) be the Kantor-Hervey point
of the quadrilateral L_2L_4L_5L_6.
Let K_(1,4) be the Kantor-Hervey point
of the quadrilateral L_2L_3L_5L_6.
Let K_(1,5) be the Kantor-Hervey point
of the quadrilateral L_2L_3L_4L_6.
Let K_(1,6) be the Kantor-Hervey point
of the quadrilateral L_2L_3L_4L_5.
Let K_(2,3) be the Kantor-Hervey point
of the quadrilateral L_1L_4L_5L_6.
Let K_(2,4) be the Kantor-Hervey point
of the quadrilateral L_1L_3L_5L_6.
Let K_(2,5) be the Kantor-Hervey point
of the quadrilateral L_1L_3L_4L_6.
Let K_(2,6) be the Kantor-Hervey point
of the quadrilateral L_1L_3L_4L_5.
Let K_(3,4) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_5L_6.
Let K_(3,5) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_4L_6.
Let K_(3,6) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_4L_5.
Let K_(4,5) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_3L_6.
Let K_(4,6) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_3L_5.
Let K_(5,6) be the Kantor-Hervey point
of the quadrilateral L_1L_2L_3L_4.

Let M_(1,2) be the perpendicular line to L_2 through K_(1,2).
Let M_(1,3) be the perpendicular line to L_3 through K_(1,3).
Let M_(1,4) be the perpendicular line to L_4 through K_(1,4).
Let M_(1,5) be the perpendicular line to L_5 through K_(1,5).
Let M_(1,6) be the perpendicular line to L_6 through K_(1,6).
Then,
M_(1,2), M_(1,3), M_(1,4), M_(1,5), M_(1,6) are concurrent.
Let Q_1 be the point of concurrency.

Let M_(2,1) be the perpendicular line to L_1 through K_(1,2).
Let M_(2,3) be the perpendicular line to L_3 through K_(2,3).
Let M_(2,4) be the perpendicular line to L_4 through K_(2,4).
Let M_(2,5) be the perpendicular line to L_5 through K_(2,5).
Let M_(2,6) be the perpendicular line to L_6 through K_(2,6).
Then,
M_(2,1), M_(2,3), M_(2,4), M_(2,5), M_(2,6) are concurrent.
Let Q_2 be the point of concurrency.

Let M_(3,1) be the perpendicular line to L_1 through K_(1,3).
Let M_(3,2) be the perpendicular line to L_2 through K_(2,3).
Let M_(3,4) be the perpendicular line to L_4 through K_(3,4).
Let M_(3,5) be the perpendicular line to L_5 through K_(3,5).
Let M_(3,6) be the perpendicular line to L_6 through K_(3,6).
Then,
M_(3,1), M_(3,2), M_(3,4), M_(3,5), M_(3,6) are concurrent.
Let Q_3 be the point of concurrency.

Let M_(4,1) be the perpendicular line to L_1 through K_(1,4).
Let M_(4,2) be the perpendicular line to L_2 through K_(2,4).
Let M_(4,3) be the perpendicular line to L_3 through K_(3,4).
Let M_(4,5) be the perpendicular line to L_5 through K_(4,5).
Let M_(4,6) be the perpendicular line to L_6 through K_(4,6).
Then,
M_(4,1), M_(4,2), M_(4,3), M_(4,5), M_(4,6) are concurrent.
Let Q_4 be the point of concurrency.

Let M_(5,1) be the perpendicular line to L_1 through K_(1,5).
Let M_(5,2) be the perpendicular line to L_2 through K_(2,5).
Let M_(5,3) be the perpendicular line to L_3 through K_(3,5).
Let M_(5,4) be the perpendicular line to L_4 through K_(4,5).
Let M_(5,6) be the perpendicular line to L_6 through K_(5,6).
Then,
M_(5,1), M_(5,2), M_(5,3), M_(5,4), M_(5,6) are concurrent.
Let Q_5 be the point of concurrency.

Let M_(6,1) be the perpendicular line to L_1 through K_(1,6).
Let M_(6,2) be the perpendicular line to L_2 through K_(2,6).
Let M_(6,3) be the perpendicular line to L_3 through K_(3,6).

```
Let M_(6,4) be the perpendicular line to L_4 through K_(4,6).
Let M_(6,5) be the perpendicular line to L_5 through K_(5,6).
Then,
M_(6,1), M_(6,2), M_(6,3), M_(6,4), M_(6,5) are concurrent.
Let Q_6 be the point of concurrency.

Then, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 are collinear.

Sincerely,
Keita Miyamoto
```

Message: #2472
Date: 2024-09-20
From: Stan.Rabinowitz@comcast.net
Subject: four normals to an ellipse

```
Is this a known result?

Suppose P is a point outside an ellipse such that there are 4
normals from P to the ellipse.
Let the feet of these normals be W, X, Y, and Z.
Then
(a) The Steiner-Gergonne point (QA-P3) of quadrilateral WXYZ is
O, the center of the ellipse.
(b) The conic through P, W, X, Y, Z passes through O.
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Message: #2473
Date: 2024-09-20
From: eckart_schmidt@t-online.de
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

```
Dear Keita,

with great interest I constructed the described line for a 6L,
... it was new for me, thanks,
... but I found no properties wrt well known 6L-elements,
... up to now ...

Best regards Eckart
```

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Message: #2474
Date: 2024-09-21
From: van10hoven@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

Dear Keita,

Thanks for your further explanation.
Not in the least to my own surprise, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 and the line through these points are familiar acquaintances in QPG.
Actually the 6 Q-i are the 6 versions of 5L-o-P1 in a 6-Line, and indeed, they are collinear on a line called 6L-e-L1.
Let me explain.
5L-o-P1 is the 5-version of nL-o-P1: nL-Morley's 1st Orthocenter. The infix ‘o’ stands for ‘odd’. There is a sequence of these points 3L-o-P1, 5L-o-P1, 7L-o-P1, etc., for n-Laterals with n=odd.
5L-o-P1 is the version of nL-o-P1 for the 5-Lateral or 5L.
When you look to the first picture in QPG at nL-o-P1: nL-Morley's 1st Orthocenter, you will see that every odd version of nL-o-P1 arises from an even version (n-1)-e-P1 (‘e’ stands for even).
In our case 5L-o-P1 arises from 4L-e-P1, which happens to be QL-P3!
When you analyze the construction, you will see that:
· 4L-e-P1 = QL-P3
· 5L-n-P1 = Q_i
· 6L-e-L1 = the line through (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6)

To summarize, the construction is known. It was described by Morley in a general way. Similar constructions exist for an 8-Line, 10-Line, etc.

Best regards,
Chris

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Message: #2475
Date: 2024-09-21
From: unidentifiedlethargicorganism@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

Dear Chris,

Thank you very much for your response.
I had a hard time understanding some of the definitions of n-Line Lines / Points, and I couldn't identify this line (6L-e-L1) by myself. The polygon geometry is too difficult for me....
Thank you for your help.

Sincerely,
Keita Miyamoto

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Message: #2476
Date: 2024-09-21
From: van10hoven@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

Dear Keita,

I must admit, this is not easy material. It's remarkable how Morley discovered all these properties simply by analyzing the structure of formulas involving points and lines in n-Lines (reference figures made from n lines) and drawing conclusions from there. Bernard Keizer, Eckart Schmidt, and I took an entire year to fully understand his descriptions—partly because the subject contains many layers of complexity. Furthermore, Morley used a style of description that may have been common in his time but was unfamiliar to us.
I fear my own explanations may also be somewhat complicated. So, I will attempt to explain this topic in a hopefully simpler, less formal manner.

Let's begin with a figure made from 3 lines. The three perpendiculars drawn from the three 3L-vertices to the three reference lines concur at 3L-o-P1, which is also 3L-n-p1.

Note 1:
- The infix "-o-" stands for an odd point, meaning that this point exists only for 3L, 5L, 7L, etc.

- The infix “-n-” stands for a general point, which exists in 3L, 4L, 5L, 6L, etc.

Note 2:
- The suffix “-P1” in 3L-o-P1 is written with an uppercase “P,” indicating a normal point.
- The suffix “-p1” in 3L-n-p1 uses a lowercase “p,” indicating a point named in the style Morley used. See *nL-n-pi (http://www.chrisvantienhoven.nl/nl-items/nl-obj/nl-pts/nl-n-pi)*: nL-Morley's intermediate recursive pi points. This distinction adds complexity, as these points are generated in layers. It took us a long time to fully grasp this. However, once you understand this, you should be able to tackle the rest.

Now, consider the following procedure:
- In a 3L: The perpendicular bisector of 3L-n-p0 (Circumcenter) and 3L-n-p1 (Orthocenter) will be 3L-o-L1.
- In a 4L, there are four 3-lines. The four versions of 3L-o-L1 concur at 4L-e-P1 (which, by the way, is also QL-P3).
- In a 5L, there are five 4-lines. The perpendiculars from the five 4L-e-P1 to the unused sideline of the 4L will concur at 5L-o-P1. The perpendicular bisector of 5L-o-P1 and 5L-n-p1 will be 5L-o-L1.
- In a 6L, there are six 5-lines. The six versions of 5L-o-L1 concur at 6L-e-P1.
- In a 7L, there are seven 6-lines. The perpendiculars from the seven 6L-e-P1 to the unused sideline of the 6L will concur at 7L-o-P1. The perpendicular bisector of 7L-o-P1 and 7L-n-p2 will be 7L-o-L1.
- In an 8L, there are eight 7-lines. The eight versions of 7L-o-L1 concur at 8L-e-P1.
- In a 9L, there are nine 8-lines. The perpendiculars from the nine 8L-e-P1 to the unused sideline of the 8L will concur at 9L-o-P1. The perpendicular bisector of 9L-o-P1 and 9L-n-p3 will be 9L-o-L1.
- And so on...

As you can see, at each level, a point or line is generated that is then used at the next level. The only independently generated points are 3L-n-p0, 5L-n-p1, 7L-n-p2, 9L-n-p3, etc. These are the *nL-n-pi (http://www.chrisvantienhoven.nl/nl-items/nl-obj/nl-pts/nl-n-pi)*: nL-Morley's intermediate recursive pi points, which can be studied at the nL-n-pi page of QPG.
If you have questions, please let me know.
Best regards,
Chris

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Message: #2477
Date: 2024-09-22
From: van10hoven@gmail.com
Subject: Re: Hexalateral, Kantor-Hervey points, collinear

```
Dear Keita and Eckart,

Regarding Keita's message #2471 I noticed another special
feature.

It appears that the conics (K12,K13,K14,K15,K16),
(K21,K23,K24,K25,K26), (K31,K32,K34,K35,K36),
(K41,K42,K43,K45,K46), (K51,K52,K53,K54,K56),
(K61,K62,K63,K64,K65) have three common points, forming a
6L-Triangle.

I couldn't find any other relationship with other 6L-items.

Best regards,
Chris
```

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Message: #2478
Date: 2024-09-22
From: van10hoven@gmail.com
Subject: Re: four normals to an ellipse

```
Dear Stanley,

Regarding your message #2472.
This property is new to me.
It is a beautiful result.
I noticed both properties are also valid for the feet of 4
normals from P to a hyperbola.

Best regards,
Chris
```

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Message: #2479
Date: 2024-09-23
From: Stan.Rabinowitz@comcast.net
Subject: Re: four normals to an ellipse

```
I have located a reference to result (b): An Introduction to
Projective Geometry, 2nd edition, by Louis Napoleon George
Filon, 1908.
Page 204 states "The feet of the normals from any point O to a
conic s lie on a rectangular hyperbola through O and the centre
of s, whose asymptotes are parallel to the axes of s."
```

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Message: #2480
Date: 2024-09-23
From: Stan.Rabinowitz@comcast.net
Subject: Re: four normals to an ellipse

```
The hyperbola is known as the hyperbola of Apollonius.
```

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Message: #2481
Date: 2024-09-23
From: van10hoven@gmail.com
Subject: Re: four normals to an ellipse

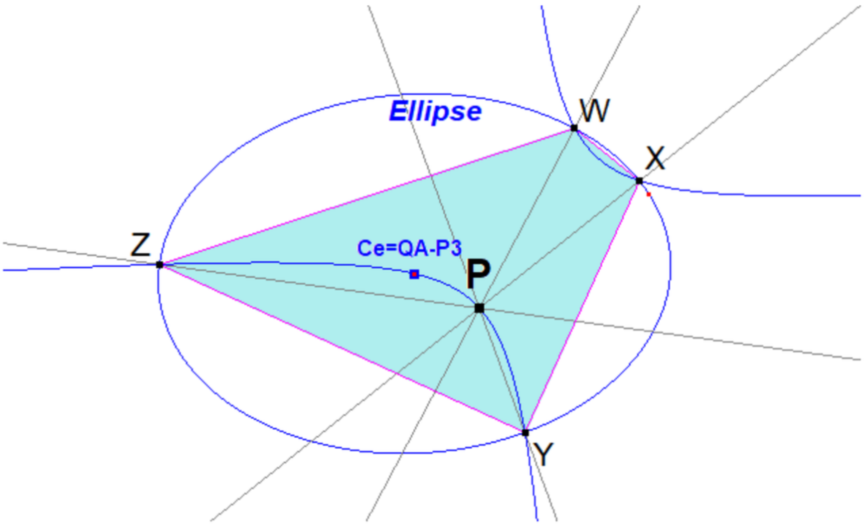
```
Dear Stanley,

Thanks for your special results and the reference.
I had some more ideas.
You start with an ellipse. I wondered what would happen when we
start with the four feet of the normals and consider them as a
quadrangle.
See attachment.
It was nice to delve into these subjects again.

Best regards,
Chris
```

A special property about 4 normals from P to an ellipse

QPG-message #2472, 20-9-2024, Stanley Rabinowitz
Is this a known result?
Suppose P is a point outside an ellipse such that there are 4 normals from P to the ellipse.
Let the feet of these normals be W, X, Y, and Z.
Then
(a) The Steiner-Gergonne point (QA-P3) of quadrilateral WXYZ is O, the center of the ellipse.
(b) The conic through P, W, X, Y, Z passes through O.

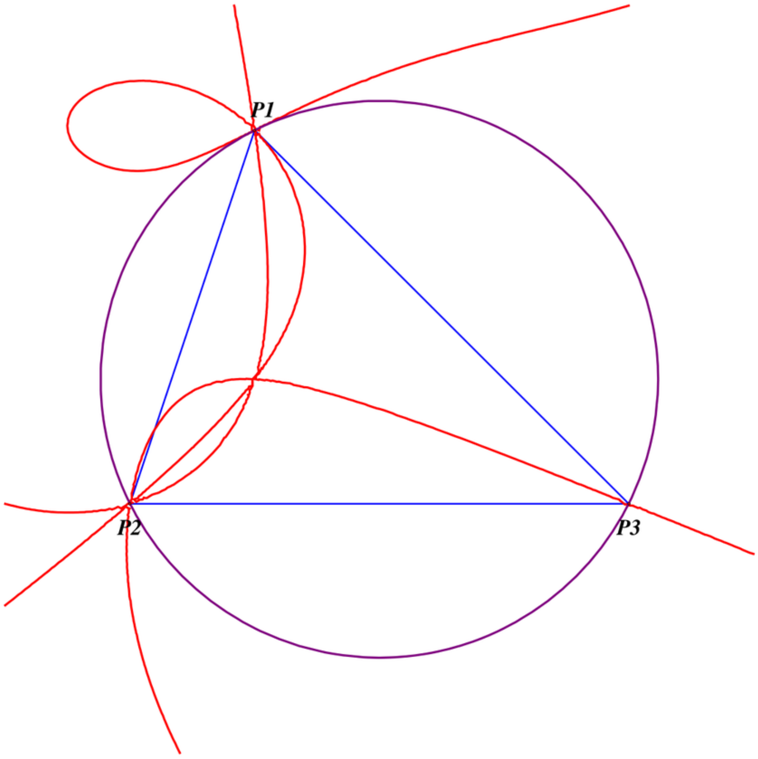


Another approach
How does it look like when we start with 4 points (P1,P2,P3,P4) forming the vertices of a quadrangle QA, intended to be the points of the conic/ellipse, from where the normals at these points will coincide.
We already noticed that the center of this circumscribed conic will be QA-P3, also known as the Gergonne-Steiner Point.
Therefore, the reflections of (P1,P2,P3,P4) in QA-P3 will also be points of the conic.
Calculation shows that this conic is QA-Co3, the Gergonne-Steiner Conic, being the conic with the least eccentricity circumscribing QA(P1,P2,P3,P4).

Concurring Normal Lines
Now, let's look at the normal lines at (P1,P2,P3,P4) to QA-Co3. When do they concur?
Calculations suggest that the following conditions make that the normals of 4 points coincide:
* a=b+c, b=c+a, c=a+b (when P1,P2,P3 are collinear)
* $a^2=b^2+c^2$, $b^2=c^2+a^2$, $c^2=a^2+b^2$ (when the triangle P1P2P3 is right-angled)
* P4 lies on the circumcircle of (P1,P2,P3), meaning P1P2P3P4 forms a Cyclic Quadrangle
Reference from "An Introduction to Projective Geometry, 2nd edition, by Louis Napoleon George Filon, 1908", Page 204 confirms this condition.
* Lastly, when P4 lies on next septic the normals at (P1,P2,P3,P4) also coincide.

The equation of the septic is:

a^4 (a^2 - 3 b^2 + 3 c^2) y^4 z^3 -
a^4 (a^2 + 3 b^2 - 3 c^2) y^3 z^4 +
x^4 (-c^4 (3 a^2 - 3 b^2 - c^2) y^3 + c^2 (4 a^4 - 2 a^2 b^2 - 2 b^4 - 5 a^2 c^2 + 5 b^2 c^2 + c^4) y^2 z - b^2 (4 a^4 - 5 a^2 b^2 + b^4 - 2 a^2 c^2 + 5 b^2 c^2 - 2 c^4) y z^2 + b^4 (3 a^2 - b^2 - 3 c^2) z^3) +
x^3 (-c^4 (3 a^2 - 3 b^2 + c^2) y^4 + 2 (a - b) (a + b) c^2 (3 a^2 + 3 b^2 - 5 c^2) y^3 z - (b - c) (b + c) (10 a^4 - 11 a^2 b^2 + b^4 - 11 a^2 c^2 + 14 b^2 c^2 + c^4) y^2 z^2 - 2 b^2 (a - c) (a + c) (3 a^2 - 5 b^2 + 3 c^2) y z^3 + b^4 (3 a^2 + b^2 - 3 c^2) z^4) +
x^2 (c^2 (2 a^4 + 2 a^2 b^2 - 4 b^4 - 5 a^2 c^2 + 5 b^2 c^2 - c^4) y^4 z + (a - c) (a + c) (a^4 - 11 a^2 b^2 + 10 b^4 + 14 a^2 c^2 - 11 b^2 c^2 + c^4) y^3 z^2 - (a - b) (a + b) (a^4 + 14 a^2 b^2 + b^4 - 11 a^2 c^2 - 11 b^2 c^2 + 10 c^4) y^2 z^3 - b^2 (2 a^4 - 5 a^2 b^2 - b^4 + 2 a^2 c^2 + 5 b^2 c^2 - 4 c^4) y z^4) +
x (a^2 (a^4 - 5 a^2 b^2 + 4 b^4 + 5 a^2 c^2 - 2 b^2 c^2 - 2 c^4) y^4 z^2 - 2 a^2 (b - c) (b + c) (5 a^2 - 3 b^2 - 3 c^2) y^3 z^3 - a^2 (a^4 + 5 a^2 b^2 - 2 b^4 - 5 a^2 c^2 - 2 b^2 c^2 + 4 c^4) y^2 z^4)



When the 4th point is taken on the purple circumcircle or the red septic, the four normals at (P1,P2,P3,P4) to the Gergonne-Steiner Conic QA-Co3 will be concurrent.
It is interesting to find out if any or which ETC-points lie on this septic.

It appears that in the range X1-X5372 that the following points lie on the septic:
(X(4), X(74), X(671), X(1156), X(1320),
Note: Series of ETC-points will be listed from here as follows: {4,74,671,895,1156,1320}.

Here follows a list of the QA's made up with vertices: P1,P2,P3, P4=Xi.
We actually work with a reference triangle P1,P2,P3 with an extra ETC-point Xi added to make it a Quadrangle.
For these ETC-points Xi there will be a Gergonne-Steiner conic COi (QA-Co3 in EQF) and its center Oi (QA-P3 in EQF), a point Ni where the normals at P1,P2,P3,P4 concur and an extra rectangular conic CO2i (QA-Co2 in EQF) through (P1,P2,P3,P4,Oi,Xi)

Xi = X4 Orthocenter
CO4 = undefined
O4 = undefined
N4 = undefined
CO2-4 = undefined

Xi = X74
CO74 = circumcircle through
O74 = X3
N74 = X3
CO2-74 = conic through {3,576,671}

Xi = X671
CO671 = Steiners circumconic
O671 = X2
N671 = X4
CO2-671 = conic through
{2,4,10,13,14,17,18,76,83,94,96,98,226,262,275,321,485,486,598,671,801,1029,1131,1132,1139,1140,1327,1328,1446,1676,1677,1751,1916,2009,2010,2051,2052,2394,2592,2593,2671,2672,2986,2996,3316,3317,3366,3367,3370,3373,3374,3381,3382,3387,3388,3391,3392,3397,3399,3406,3407,3413,3414,3424,3429,3590,3591,3597,4049,4052,4080,4444}

Xi = X895
CO895 = conic through
{110,287,648,651,677,895,1331,1332,1797,1813,1814,1815,2986,2987,2988,2989,2990,2991,4558,4563}
O895 = X6
N895 = X64
CO2-895 = conic through
{3,4,6,54,64,65,66,67,68,69,70,71,72,73,74,248,265,290,695,879,895,1173,1175,1176,1177,1242,1243,1244,1245,1246,1439,1798,1903,1942,1987,2213,2435,2574,2575,2992,2993,3426,3431,3519,3521,3527,3531,3532,3657,4846}

Xi = X1165
CO1165 = conic through
{88,100,162,190,651,653,655,658,660,662,673,771,799,823,897,1156,1492,1821,2349,2580,2581,3257,4598,4599,4604,4606,4607}
O1165 = X9
N1165 = X1
CO2-1165 = conic through
{1,4,7,8,9,21,79,80,84,90,104,177,256,294,314,885,941,943,981,983,987,989,1000,1039,1041,

1061,1063,1156,1172,1251,1320,1389,1392,1476,1896,1937,2298,2320,2335,2344,2346,2481,
2648,2997,3062,3065,3254,3255,3296,3307,3308,3427,3467,3495,3551,3577,3680,4180,4866,
4876,4900}

Xi = X1320

CO1320 = conic through
{100,643,644,664,1120,1280,1320,1897,3699,3903,4614}
O1320 = X1
N1320 = X84
CO2-1320 = conic through
{1,4,7,8,9,21,79,80,84,90,104,177,256,294,314,885,941,943,981,983,987,989,1000,1039,1041,
1061,1063,1156,1172,1251,1320,1389,1392,1476,1896,1937,2298,2320,2335,2344,2346,2481,
2648,2997,3062,3065,3254,3255,3296,3307,3308,3427,3467,3495,3551,3577,3680,4180,4866,
4876,4900}

Reference

(An Introduction to Projective Geometry, 2nd edition, by Louis Napoleon George Filon, 1908. Page 204)

214. The hyperbola of Apollonius. Let the quadrangle $ABCD$, which defines a pencil of conics, be inscribable in a circle. Ω, Ω' are conjugate points in the involution determined by the pencil on the line at infinity. The double points of this involution are therefore determined by two rectangular directions. These give the axes of the two parabolas through the four points

and these are parallel to the asymptotes of the centre locus. The centre locus is then a rectangular hyperbola.

The same rectangular hyperbola is the locus of points conjugate to points at infinity with regard to the pencil of conics.

If then I^∞ be a point at infinity in the direction of one of the axes of a conic s of the pencil, the conjugate point I' is the intersection of the diameters of s and of the circle c about $ABCD$ which are conjugate to the direction of I^∞. But both of these are perpendicular to the direction of I^∞. Hence they meet at I'^∞ at infinity in the perpendicular direction. The conics of the pencil have therefore axes parallel to the asymptotes of the centre locus.

Let now I^∞ be any point at infinity, I' its conjugate point with regard to the pencil; take for s the conic of the pencil through I', and let C, O be the centres of s, c respectively.

From the property of the point I', the diameters conjugate to the direction defined by I^∞ with regard to s and c pass through I'. They are therefore CI' and OI'.

But OI' is perpendicular to its conjugate direction by a property of the circle. Thus OI' is perpendicular to the tangent at I' to the conic s, for this tangent is parallel to the diameter conjugate to CI' and therefore passes through I^∞. Hence the normal at I' to the conic through I' passes through O, or the rectangular hyperbola which is the locus of centres is also the locus of the feet of perpendiculars from O on the conics of the system.

Now let s be any conic, O any point. A circle centre O meets s in four points A, B, C, D, and considering the above results for the pencil of conics through A, B, C, D we obtain the theorem:

The feet of the normals from any point O to a conic s lie on a rectangular hyperbola through O and the centre of s, whose asymptotes are parallel to the axes of s. Since this rectangular hyperbola meets s in four points, four normals can in general be drawn from a point to a conic.

This hyperbola is known as the hyperbola of Apollonius for the point O and the conic s.

Transcription last phrase:
Now let s be any conic, O any point. A circle centre O meets s in four points A, B, C, D, and considering above results for the pencil of conics through A, B, C, D we obtain the theorem:
The feet of the normal from any point O to a conic s lie on a rectangular hyperbola through O and the center of s, whose asymptotes are parallel to the axes of s, whose asymptotes are parallel to the axes of s. Since this rectangular hyperbola meet s in four points, four normal can in general be drawn from a point to a conic.
This hyperbola is known as the hyperbola of Apollonius for the point O and the conic s.

Chris van Tienhoven, September 23, 2024

CO-Ce-is-QA-P3 of P-Normal-Feet - 01.pdf

Message: #2482
Date: 2024-09-23
From: Stan.Rabinowitz@comcast.net
Subject: Re: four normals to an ellipse

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I'm glad my "discovery" spurred on further research.

Another relevant reference is Baker's Principles of Geometry
(volume II), 1930, around pages 92-93.
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Message: #2483
Date: 2024-09-23
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

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Dear Chris,
I'm still hoping that you will succeed in identifying the
barycentric equation of the cayleyan wrt the reference triangle
H1H2H3.
As I don't know exactly what you are able to draw with Cabri or
to compute with Mathematica, I hardly can imagine how you will
proceed ...
But I'm bluffed by your figures in the 2nd version of your memo.
I suppose the whole figure can be parametrised wrt a triangle
H1H2H3 with a1, a2, a3 and k.
I tried my self following figures with H1 = (20, 0), H2
=(-20,0), H3 = (10,30) in an orthonormal coordinate system,
then in the barycentric coordinates H1 = (1,0,0), H2 = (0,1,0)
and H3 = (0,0,1) a1 = 2, a2 = 5 and a3 = -3, k = -6, k' = 1 and
k'' = -5 or k = 3, k' = -5 and k'' = 1.
For example, P0 = ( 1/2,1/5,-1/3) ...
Are you able to reproduce the same cubics CU1 and HE1 with the
cayleyan CA1 and CU2 and HE2 with CA2 and to compute the
equation of CA1 and CA2.
I remember I found the beginning of my comprehension of this
Hesse pencil with splitting FE and RF after you have provided a
numerical example ...
Just tell me where you are in your quest.
Thanks in advance
Best regards
Bernard
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Message: #2484
Date: 2024-09-23
From: van10hoven@gmail.com
Subject: Re: four normals to an ellipse

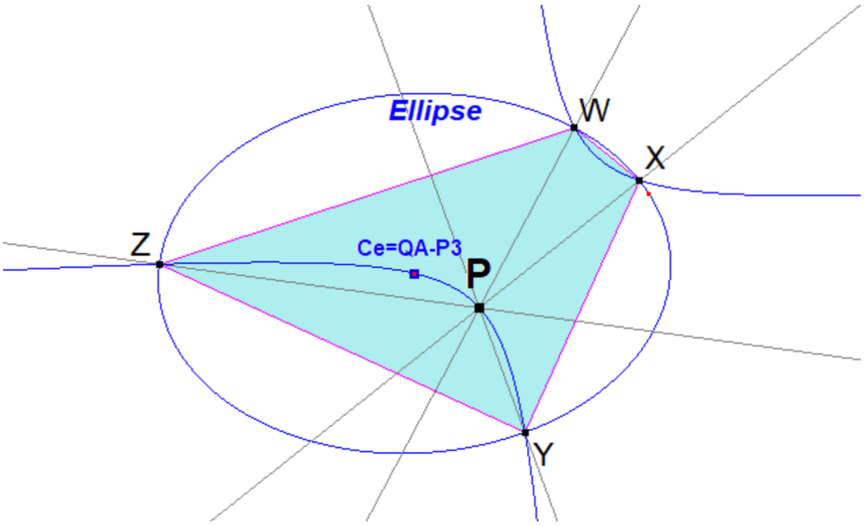
Dear Stanley,

You are right, there was a mistake with X(74) about the Jerabek Hyperbola.
I made an improved version of the document. See attachment.

Best regards,
Chris

A special property about 4 normals from P to an ellipse

QPG-message #2472, 20-9-2024, Stanley Rabinowitz
Is this a known result?
Suppose P is a point outside an ellipse such that there are 4 normals from P to the ellipse.
Let the feet of these normals be W, X, Y, and Z.
Then
(a) The Steiner-Gergonne point (QA-P3) of quadrilateral WXYZ is O, the center of the ellipse.
(b) The conic through P, W, X, Y, Z passes through O.



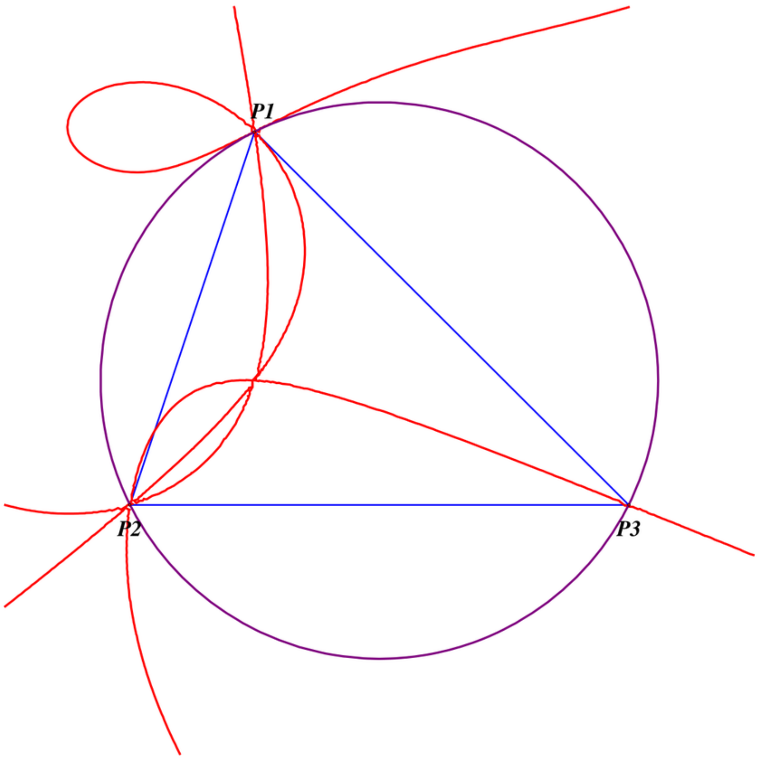
Another approach
How does it look like when we start with 4 points (P1,P2,P3,P4) forming the vertices of a quadrangle QA, intended to be the points of the conic/ellipse, from where the normals at these points will coincide.
We already noticed that the center of this circumscribed conic will be QA-P3, also known as the Gergonne-Steiner Point.
Therefore, the reflections of (P1,P2,P3,P4) in QA-P3 will also be points of the conic.
Calculation shows that this conic is QA-Co3, the Gergonne-Steiner Conic, being the conic with the least eccentricity circumscribing QA(P1,P2,P3,P4).

Concurring Normal Lines
Now, let's look at the normal lines at (P1,P2,P3,P4) to QA-Co3. When do they concur?
Calculations suggest that the following conditions make that the normals of 4 points coincide:
* a=b+c, b=c+a, c=a+b (when P1,P2,P3 are collinear)
* $a^2=b^2+c^2$, $b^2=c^2+a^2$, $c^2=a^2+b^2$ (when the triangle P1P2P3 is right-angled)
* P4 lies on the circumcircle of (P1,P2,P3), meaning P1P2P3P4 forms a Cyclic Quadrangle
Reference from “An Introduction to Projective Geometry, 2nd edition, by Louis Napoleon George Filon, 1908”, Page 204 confirms this condition.
* Lastly, when P4 lies on next septic the normals at (P1,P2,P3,P4) also coincide.

CO-Ce-is-QA-P3 of P-Normal-Feet - 02.pdf

The equation of the septic is:

a^4 (a^2 - 3 b^2 + 3 c^2) y^4 z^3 -
a^4 (a^2 + 3 b^2 - 3 c^2) y^3 z^4 +
x^4 (-c^4 (3 a^2 - 3 b^2 - c^2) y^3 + c^2 (4 a^4 - 2 a^2 b^2 - 2 b^4 - 5 a^2 c^2 + 5 b^2 c^2 + c^4) y^2 z - b^2 (4 a^4 - 5 a^2 b^2 + b^4 - 2 a^2 c^2 + 5 b^2 c^2 - 2 c^4) y z^2 + b^4 (3 a^2 - b^2 - 3 c^2) z^3) +
x^3 (-c^4 (3 a^2 - 3 b^2 + c^2) y^4 + 2 (a - b) (a + b) c^2 (3 a^2 + 3 b^2 - 5 c^2) y^3 z - (b - c) (b + c) (10 a^4 - 11 a^2 b^2 + b^4 - 11 a^2 c^2 + 14 b^2 c^2 + c^4) y^2 z^2 - 2 b^2 (a - c) (a + c) (3 a^2 - 5 b^2 + 3 c^2) y z^3 + b^4 (3 a^2 + b^2 - 3 c^2) z^4) +
x^2 (c^2 (2 a^4 + 2 a^2 b^2 - 4 b^4 - 5 a^2 c^2 + 5 b^2 c^2 - c^4) y^4 z + (a - c) (a + c) (a^4 - 11 a^2 b^2 + 10 b^4 + 14 a^2 c^2 - 11 b^2 c^2 + c^4) y^3 z^2 - (a - b) (a + b) (a^4 + 14 a^2 b^2 + b^4 - 11 a^2 c^2 - 11 b^2 c^2 + 10 c^4) y^2 z^3 - b^2 (2 a^4 - 5 a^2 b^2 - b^4 + 2 a^2 c^2 + 5 b^2 c^2 - 4 c^4) y z^4) +
x (a^2 (a^4 - 5 a^2 b^2 + 4 b^4 + 5 a^2 c^2 - 2 b^2 c^2 - 2 c^4) y^4 z^2 - 2 a^2 (b - c) (b + c) (5 a^2 - 3 b^2 - 3 c^2) y^3 z^3 - a^2 (a^4 + 5 a^2 b^2 - 2 b^4 - 5 a^2 c^2 - 2 b^2 c^2 + 4 c^4) y^2 z^4)



When the 4th point is taken on the purple circumcircle or the red septic, the four normals at (P1,P2,P3,P4) to the Gergonne-Steiner Conic QA-Co3 will be concurrent.
It is interesting to find out if any or which ETC-points lie on this septic.

It appears that in the range X1-X5372 that the following points lie on the septic:

X(4), X(74), X(671), X(1156), X(1320).

Note: Series of ETC-points will be listed from here as follows: {4,74,671,895,1156,1320}.

Here follows a list of the QA's made up with vertices: P1,P2,P3, P4=Xi.
We actually work with a reference triangle P1,P2,P3 with an extra ETC-point Xi added to make it a Quadrangle.
For these ETC-points Xi there will be a Gergonne-Steiner conic COi (QA-Co3 in EQF) and its center Oi (QA-P3 in EQF), a point Ni where the normals at P1,P2,P3,P4 concur and an extra rectangular conic CO2i (QA-Co2 in EQF) through (P1,P2,P3,P4,Oi,Xi)

Xi = X4 Orthocenter
CO4 = undefined
O4 = undefined
N4 = undefined
CO2-4 = undefined

Xi = X74
CO74 = circumcircle
O74 = X3
N74 = X3
CO2-74 = conic through
{3,4,6,54,64,65,66,67,68,69,70,71,72,73,74,248,265,290,695,879,895,1173,1175,1176,1177, 1242,1243,1244,1245,1246,1439,1798,1903,1942,1987,2213,2435,2574,2575,2992,2993,3426, 3431,3519,3521,3527,3531,3532,3657,4846}

Xi = X671
CO671 = Steiners circumconic
O671 = X2
N671 = X4
CO2-671 = conic through
{2,4,10,13,14,17,18,76,83,94,96,98,226,262,275,321,485,486,598,671,801,1029,1131,1132, 1139,1140,1327,1328,1446,1676,1677,1751,1916,2009,2010,2051,2052,2394,2592,2593,2671, 2672,2986,2996,3316,3317,3366,3367,3370,3373,3374,3381,3382,3387,3388,3391,3392,3397, 3399,3406,3407,3413,3414,3424,3429,3590,3591,3597,4049,4052,4080,4444}

Xi = X895
CO895 = conic through
{110,287,648,651,677,895,1331,1332,1797,1813,1814,1815,2986,2987,2988,2989,2990,2991, 4558,4563}
O895 = X6
N895 = X64
CO2-895 = conic through
{3,4,6,54,64,65,66,67,68,69,70,71,72,73,74,248,265,290,695,879,895,1173,1175,1176,1177, 1242,1243,1244,1245,1246,1439,1798,1903,1942,1987,2213,2435,2574,2575,2992,2993,3426, 3431,3519,3521,3527,3531,3532,3657,4846}

Xi = X1165
CO1165 = conic through
{88,100,162,190,651,653,655,658,660,662,673,771,799,823,897,1156,1492,1821,2349,2580, 2581,3257,4598,4599,4604,4606,4607}
O1165 = X9

N1165 = X1
CO2-1165 = conic through
{1,4,7,8,9,21,79,80,84,90,104,177,256,294,314,885,941,943,981,983,987,989,1000,1039,1041,1061,1063,1156,1172,1251,1320,1389,1392,1476,1896,1937,2298,2320,2335,2344,2346,2481,2648,2997,3062,3065,3254,3255,3296,3307,3308,3427,3467,3495,3551,3577,3680,4180,4866,4876,4900}

Xi = X1320
CO1320 = conic through
{100,643,644,664,1120,1280,1320,1897,3699,3903,4614}
O1320 = X1
N1320 = X84
CO2-1320 = conic through
{1,4,7,8,9,21,79,80,84,90,104,177,256,294,314,885,941,943,981,983,987,989,1000,1039,1041,1061,1063,1156,1172,1251,1320,1389,1392,1476,1896,1937,2298,2320,2335,2344,2346,2481,2648,2997,3062,3065,3254,3255,3296,3307,3308,3427,3467,3495,3551,3577,3680,4180,4866,4876,4900}

Reference

(An Introduction to Projective Geometry, 2nd edition, by Louis Napoleon George Filon, 1908. Page 204)

214. The hyperbola of Apollonius. Let the quadrangle $ABCD$, which defines a pencil of conics, be inscribable in a circle. Ω, Ω' are conjugate points in the involution determined by the pencil on the line at infinity. The double points of this involution are therefore determined by two rectangular directions. These give the axes of the two parabolas through the four points

and these are parallel to the asymptotes of the centre locus. The centre locus is then a rectangular hyperbola.

The same rectangular hyperbola is the locus of points conjugate to points at infinity with regard to the pencil of conics.

If then I^∞ be a point at infinity in the direction of one of the axes of a conic s of the pencil, the conjugate point I' is the intersection of the diameters of s and of the circle c about $ABCD$ which are conjugate to the direction of I^∞. But both of these are perpendicular to the direction of I^∞. Hence they meet at I'^∞ at infinity in the perpendicular direction. The conics of the pencil have therefore axes parallel to the asymptotes of the centre locus.

Let now I^∞ be any point at infinity, I' its conjugate point with regard to the pencil; take for s the conic of the pencil through I', and let C, O be the centres of s, c respectively.

From the property of the point I', the diameters conjugate to the direction defined by I^∞ with regard to s and c pass through I'. They are therefore CI' and OI'.

But OI' is perpendicular to its conjugate direction by a property of the circle. Thus OI' is perpendicular to the tangent at I' to the conic s, for this tangent is parallel to the diameter conjugate to CI' and therefore passes through I^∞. Hence the normal at I' to the conic through I' passes through O, or the rectangular hyperbola which is the locus of centres is also the locus of the feet of perpendiculars from O on the conics of the system.

Now let s be any conic, O any point. A circle centre O meets s in four points A, B, C, D, and considering the above results for the pencil of conics through A, B, C, D we obtain the theorem:

The feet of the normals from any point O to a conic s lie on a rectangular hyperbola through O and the centre of s, whose asymptotes are parallel to the axes of s. Since this rectangular hyperbola meets s in four points, four normals can in general be drawn from a point to a conic.

This hyperbola is known as the hyperbola of Apollonius for the point O and the conic s.

Transcription last phrase:
Now let s be any conic, O any point. A circle centre O meets s in four points A, B, C, D, and considering above results for the pencil of conics through A, B, C, D we obtain the theorem:
The feet of the normal from any point O to a conic s lie on a rectangular hyperbola through O and the center of s, whose asymptotes are parallel to the axes of s, whose asymptotes are parallel to the axes of s. Since this rectangular hyperbola meet s in four points, four normal can in general be drawn from a point to a conic.
This hyperbola is known as the hyperbola of Apollonius for the point O and the conic s.

Chris van Tienhoven, September 23, 2024

Message: #2485
Date: 2024-09-26
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Bernard,

Thank you for your message and contribution.
You asked where I am in my quest so far.
For several days, I have been focused on creating appealing illustrations of pre-Hessians.
I was particularly interested in making images that show the three real pre-Hessians of a reference cubic.

This led me to question the conditions under which three real pre-Hessians exist versus when there is only one real pre-Hessian and two imaginary ones.
I believe I have found the answer, but I also need to organize all my research materials.
Creating these images require a precise and well-organized setup.

I will be on holiday until the 5th of October, after which I plan to continue my work.
That's where things stand for now.

Best regards,
Chris

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Message: #2486
Date: 2024-09-26
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
A bipartite cubic has 3 real prehessians, which are monopartite
A monopartite cubic has only one real prehessian, which is bipartite ...
Good holidays
Best regards
Bernard

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Message: #2487
Date: 2024-09-27
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
In our magical Hesse pencil, a cubic CU = FE + k'RF has 3 real prehessians if the 3 roots k1, k2 and k3 of the equation $k^3 + 3k^2k' + 108 = 0$ are real.
For example, FE has only one real prehessian and RF has 3 real prehessians ...
This means that FE is monopartite, whereas RF is bipartite?
Best regards
Bernard

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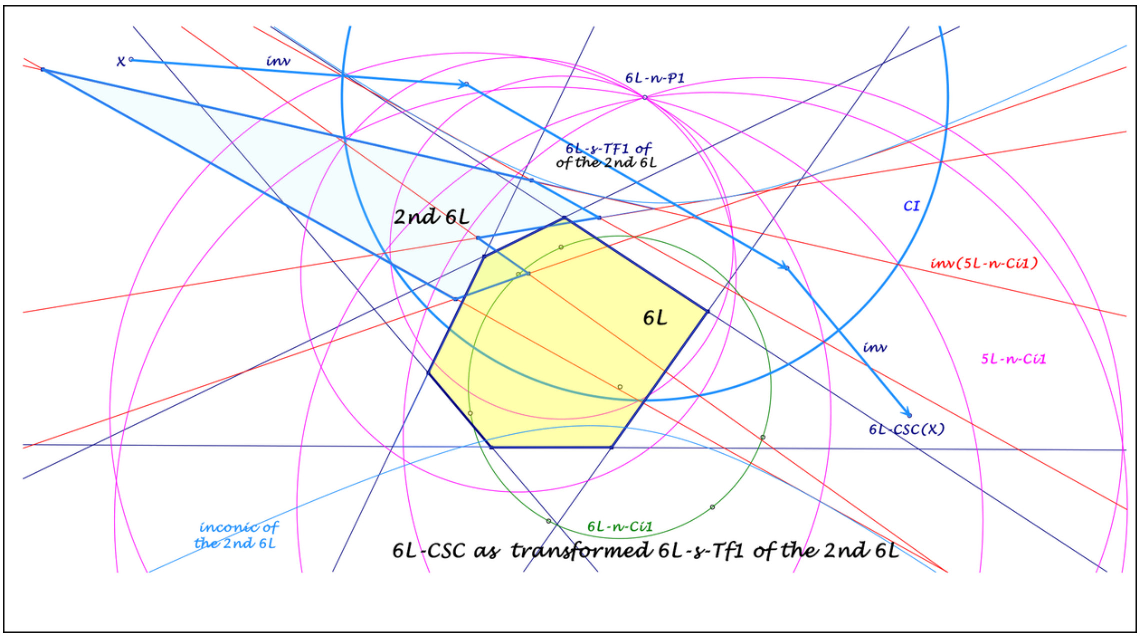
Message: #2488
Date: 2024-09-30
From: eckart_schmidt@t-online.de
Subject: CSC for 2n-line

Dear Chris,

what about 2n-CSC, already 2015 described in 2015-01-08.pdf
(eckartschmidt.de)
<https://www.eckartschmidt.de/2015-01-08.pdf> but not in EPG?
The only, but relevant CSC there is QL-Tf1 = 4L-s-Tf1,
... the transformation 6L-s-Tf1 is only
... a Möbius transformation for 6L with inconic,
... it can be generalized for 2n-lines,
... but here described in EPG-nomination for 6L-lines,
... see attached construction:
(1) Inversion wrt any circle CI round 6L-n-P1,
(2) 6L-s-Tf1 for a 2nd 6L with inverse lines of 5L-n-Ci1,
(3) again Inversion wrt the circle CI.
In the general 2n-case replace (2) by
... Morley's involution wrt the 2nd 2n-line (Ref: [48], p.111).

Best regards Eckart



2024-09-29.pdf

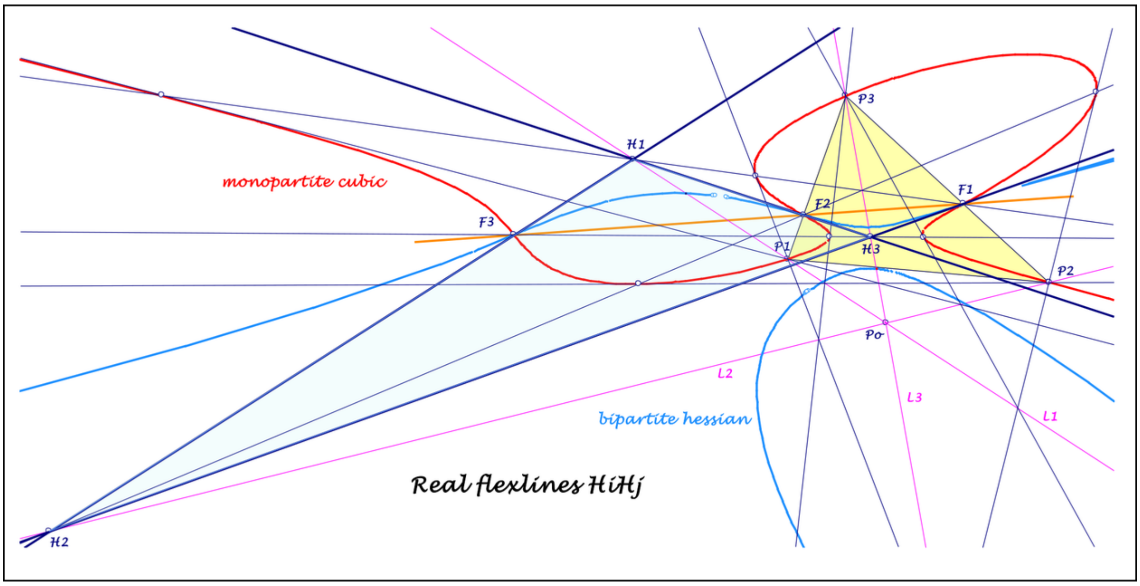
Message: #2489
Date: 2024-10-05
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

Dear Chris,

this will be a construction for the real flexlines unequal
F1F2F3
... for a monopartite cubic:
Connect the two contact points of tangents from Pi at the cubic,
... the connecting line, bearing Fi, will cut the line Li in Hi.
The attached figure is very precise, drawn and calculated.

Best regards Eckart

PS. I have proved the drawing,
... measured the coordinates of Po wrt H1H2H3,
... to get the coefficients a1,a2,a3 in your formula,
... and measured the coordinates for two cubic points,
... controlling whether they give the same k.



2024-10-05.pdf

Message: #2490
Date: 2024-10-06
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,
Wunderbar! Congratulations
This method, if it works, is amazingly simple!
Chris method for a bipartite cubic was also amazingly simple ...
Did you check that your method for a monopartite cubic gave the
same result as Chris method for the bipartite hessian?
(or naturally the converse, Chris method for a bipartite cubic
and your method for the monopartite hessian?)
Could you please, if it is not too complicate, give your
calculations?
Do you start with a cubic wrt a certain triangle? with which
equation?
Many thanks in advance for your answer
Best regards
Bernard
```

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Message: #2491
Date: 2024-10-07
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

thanks for interest,
... but what about Chris' method,
... please have a look in #2452, next to last sentence.
Wrt my "calculations": they are described in the PS of #2489.
I hope someone can confirm my conclusion for monopartite cubics.

Best regards Eckart
```

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Message: #2493
Date: 2024-10-09
From: contiwa.goma3@gmail.com
Subject: Quartic Eq. and QL

Greetings!

I'm M@IMF. (Let me use a handle.)
I'm interested in geometrical meaning of algebraic equations.

According to Lill's method,
when 5 points F(m, f - p/2), F~(q, p/2 - r), X'(x', -p/2),
X"(x", p/2), Y'(0, y') satisfy
FX'_|_X'Y'_|_Y'X"_|_X"F~, the slope of the line X'Y' is the
solution of the quartic equation
ft^4 + mt^3 + pt^2 + qt + r = 0.

If the quartic equation has 4 real solutions, then X'Y's and
Y'X"s give quadrilaterals:
QL1 = the quadrilateral formed by the 4 lines X'Y's.
QL1~ = the quadrilateral formed by the 4 lines Y'X"s.
Their QL-P1s and QL-L3s are as follows:

QL-P1	QL-L3	
QL1	F	X'-axis (y = -p/2)
QL1~	F~	X"-axis (y = p/2)

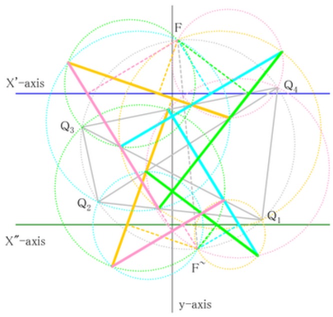
Let Q1, Q2, Q3, Q4 be the antipodes of F wrt the circumcircles
of the component triangles of QL1.
(They are also the antipodes of F~ wrt the circumcircles of the
component triangles of QL1~.)
Then y-axis (x=0) is the orthopolar line of FF~ wrt the
quadrangle Q1Q2Q3Q4 (QA0).
Some correspondences between QL and QA objects are in the
attached file.

To reconstruct the QL from some points and lines described using
f, m, p, q, r by ORIGAMI (paper-folding),
we need to construct a triangle, which means solving a cubic
equation.
I'm wondering what triangle is good.

Best regards
M@IMF

Quartic Equation and Quadrilateral

M@IMF



$Q_i(-ft_i - r/t_i\ ,\ ft_i^2 + mt_i + p/2)$

Let f, m, p, q, r be real numbers.
Given 2 points F(m, f − p/2) and F~(q, p/2 − r) in the xy-plane.
Let points X'_i, X''_i, Y'_i (i=1,2,3,4) lie on X'-axis (y = −p/2), X''-axis (y = p/2), y-axis, respectively.
If $FX'_i \perp X'_iY'_i \perp Y'_iX''_i \perp X''_iF^{\sim}$,
then the slope of the line $X'_iY'_i$ is the solution of the quartic equation

$$ft^4 + mt^3 + pt^2 + qt + r = 0.$$

(Lill's method)

Define

QL1 = the quadrilateral formed by the 4 lines X'Y's,
QL1~ = the quadrilateral formed by the 4 lines Y'X''s.

Let Q_1, Q_2, Q_3, Q_4 be the antipodes of F wrt the circumcircles of the component triangles of QL1 and define QA0 as the quadrangle $Q_1Q_2Q_3Q_4$.
Some QL objects correspond to QA objects as follows:

Points, Lines	QL1 / QL1~	QA0
((m+q)/2, (f−r)/2)	QL-P4''	Circumcenter
(0, (r−f)/2)	QL-P5''	Anticenter
((m+q)/4, 0)	QL-P6''	Centroid
Y-axis	QL-L1''	Orthopolar line of FF~
X'-axis	QL-L3 / ---	*Simson line* of F
X''-axis	--- / QL-L3	*Simson line* of F~

QL-Zn'' is homothetic with QL-Zn (ratio 2, center QL-P1), where Z = P, L, Ci, etc.

From above equation, we get

$$r(-1/t)^4 + (-q)(-1/t)^3 + p(-1/t)^2 + (-m)(-1/t) + f = 0.$$

You will see the meaning of this when you rotate the figure 180 degrees around the origin.

p.s.
When 6 points F(m − q/2, f − p/2), F~(q/2 − s, p/2 − r), X'(x', −p/2), X''(x'', p/2), Y'(−q/2, y'), Y''(q/2, y'') satisfy
$FX' \perp X'Y' \perp Y'X'' \perp X''Y'' \perp Y''F^{\sim}$, the slope of the line X'Y' is the solution of the quintic equation $ft^5 + mt^4 + pt^3 + qt^2 + rt + s = 0$.

Quartic Eq. and QL_v2_.pdf

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Message: #2494
Date: 2024-10-10
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,
You don't give much information about your construction!
What is your starting cubic? Is it drawn or calculated or both?
The only possible ''proof'' of your construction would be the drawing of your cubic with the barycentric equation wrt H1H2H3 calculated with the coefficients a1, a2, a3 and k ...
Naturally, you can calculate a1, a2 and a3 for any triangle with vertices on the Li and sides through the Fi, but that gives different Hesse pencils!
And I still do not know if Cabri allows such constructions.
Doing the converse construction with Geogebra of cubics with the mentionned equation, I've drawn a dozent of cubics of the pencil and have made plenty of boring barycentric calculations ...
If I understand you correctly, FiHi intersects the cubic in 2 points other than Fi and the 2 points have the point Pi on Li as common tangential (they are corresponding points in the sense of Salmon or Schröter).
I think this property doesn't hold! (The 2 tangents intersect on Li, but not on CU).
If you take simply for example the cubic named FE (with k = 0), it is easy to draw and to calculate that the flextangent in Fi is precisely FiHi, which has no other intersection with the cubic!
But FE is monopartite, the point Pi exists, as well as the 2 contact points of the tangents other than PiFi from Pi to the cubic.
Best regards
Bernard
PS Chris, what do you think of this construction?

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Message: #2495
Date: 2024-10-10
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

I have drawn monopartite cubics as nonpivotal isocubics
... for a reference triangle P1P2P3,
    a point Po and a cubic point Q
... and an isoconjugation with fixed points
    in the anticevians of Po wrt P1P2P3.

The lines PoPi will be the the harmonic polars Li
... and the intersections of PiPj and the cubic
    are the flexpoints F1,F2,F3.

The orientated areas of PoPiPj will be the barycentric
coordinates1/aj of Po,
... so I can calculate k for Chris' CU-formula
    a1^3*x^3+a2^3*y^3+a3^3*z^3+k*x*y*z = 0,
... k has to be the same for all points of the cubic,

This can very precise be arranged by varying triangles H1H2H3
... with H1 on L1, H2 = H1F3 ^ L2, H3 = H2F1 ^ L3,
... so that two cubic points give the same k.

Best regards Eckart
```

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Message: #2496
Date: 2024-10-10
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

excuse, there is a mistake in my description, here the right
version:
I have drawn monopartite cubics as nonpivotal isocubics
... for a reference triangle P1P2P3, a point Po and a cubic
point Q
... and an isoconjugation with fixed points in the anticevians
of Po
    wrt P1P2P3.
The lines PoPi will be the the harmonic polars Li
... and the intersections of PiPj and the cubic
    are the flexpoints F1,F2,F3.
Let us consider variable triangles H1H2H3
... with H1 on L1, H2 = H1F3 ^ L2, H3 = H1F2 ^ L3.
The orientated areas of PoHiHj
give the barycentric coordinates1/aj of Po wrt H1H2H3,
... so I can calculate k for Chris' CU-formula
    a1^3*x^3+a2^3*y^3+a3^3*z^3+k*x*y*z = 0,
... k has to be the same for all points of the cubic,
    if H1H2H3 is the real flexline trilateral,
... this can very precise be arranged by varying H1H2H3
... so that two cubic points give the same k.

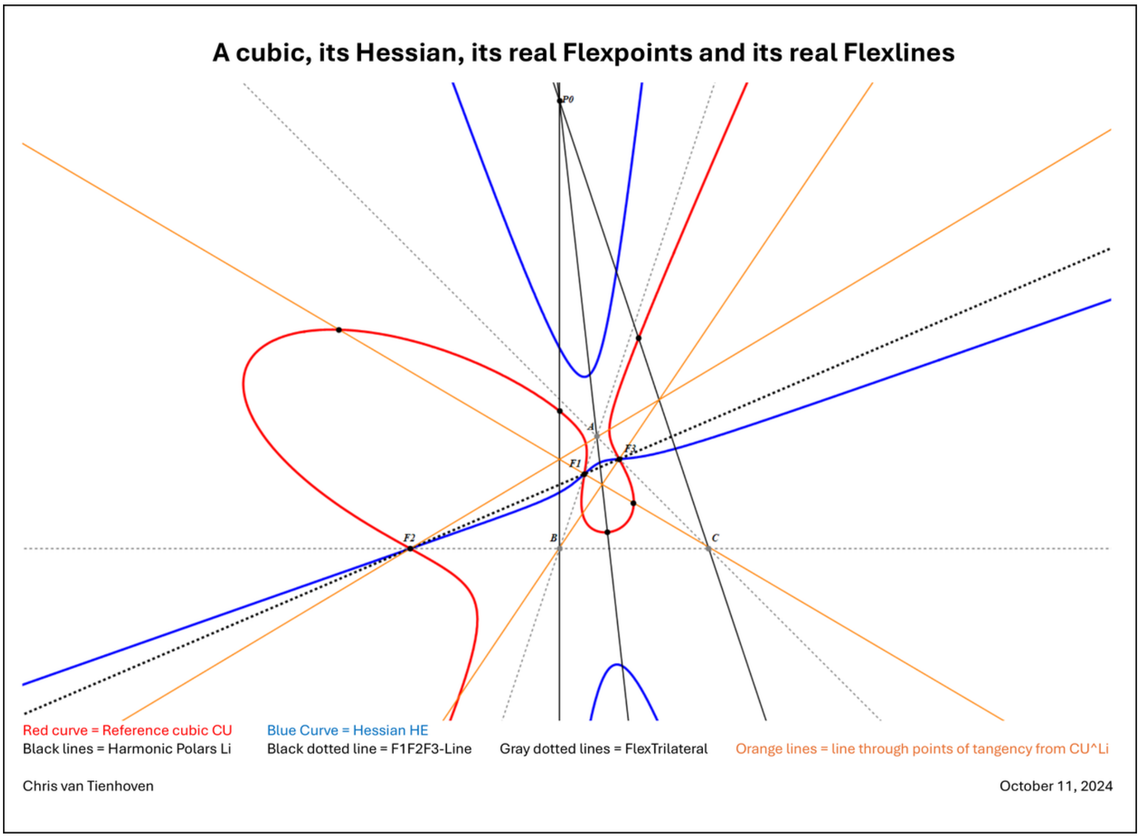
Best regards Eckart
```

Message: #2497
Date: 2024-10-11
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart and Bernard,

I was pleasantly surprised by the new construction of the real FlexTrilateral by Eckart.
I have extensively researched the subject in Mathematica to avoid ending up with another 'almost but not quite' construction. The first plot I drew seemed to confirm Eckart’s conjecture. However, I could not find any algebraic confirmation that the line through the tangent points of the tangents from CU^Li would pass through Hi. I checked in many ways to ensure I didn’t make any computational or interpretational errors, but it seems to me that the lines in question are once again almost-lines. As shown in the attached plot, it indeed appears that the orange lines pass through (A=H2, B=H3, C=H1).
Therefore, I enlarged the plot so you can see that they actually do not pass exactly through the mentioned points.
Another argument making this unlikely is the following reasoning:
A reference cubic and its Hessian have 9 inflection points, of which only 3 are real. Depending on the shape of the reference cubic, there are either 2 or 4 tangent points of the tangents from CU^Li. Algebraically, it doesn’t matter whether the points are real or imaginary. So, if something holds for 2 tangent points, it should also hold for 4 tangent points, and suddenly there are 6 connecting lines between the tangent points. Do all of them pass through Hi? Or only a selection of these lines? On what basis? I investigated this for a cubic with 4 real tangent points, but the 6 connecting lines were sometimes almost-lines and sometimes not. There was no clear pattern.
These are my findings so far.
When you think my results are valid, don’t get disappointed, we’ll find a way or not. At least we have fun.

Best regards,
Chris



CU-HE-12L1abc-02-Numerical.pdf

Message: #2498
Date: 2024-10-11
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Chris, dear Eckart
First thanks to Eckart for his precise description of his construction/calculation.
Second, thanks to Chris for confirming my idea that this construction was another 'almost but not quite' construction (I like this expression ...).
Last, could you confirm that Cabri doesn't allow the construction of a curve of a given barycentric equation wrt a certain triangle (Geogebra offers this possibility).
That's precisely by drawing and calculating the cubic FE that I was able to say that the flextangent in Fi is FiHi, which has no other intersections with the cubic!
Chris, you had already valided this property ... And this conter-example is enough to prove that Eckart's property doesn't hold!
Best regards
Bernard

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Message: #2499
Date: 2024-10-11
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Bernard,

Indeed, Cabri does not allow the construction of a curve with a
given barycentric equation with respect to a specific triangle.
You are fortunate that GeoGebra offers this option:).

Best regards,
Chris
```

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Message: #2500
Date: 2024-10-11
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for the drawing,
... but I look for a significant counterexample.
Wrt your final reasoning,
... have you overlooked,
    that I only consider monopartite cubics,
... which have always exact two real tangents
    from a cubic point to the cubic,
... see Schröter, page 251.

Best regards Eckart
```

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Message: #2501
Date: 2024-10-12
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

in #2494 you wrote:
... "FiHi intersects the cubic in 2 points other than Fi and
     the 2 points have the point Pi on Li as common tangential.
     I think this property doesn't hold!
    (The 2 tangents intersect on Li, but not on CU)."
But Pi is the intersection of Li and the cubic (see attachment
of #2494).
Further: Can you give me a drawing of FE (for k = 0),
... which shows your described properties?

Best regards Eckart
```

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Message: #2502
Date: 2024-10-13
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

I don't need further a drawing of FE,
... I got myself an approximate curve,
... but what about monopartite?

Best regards Eckart
```

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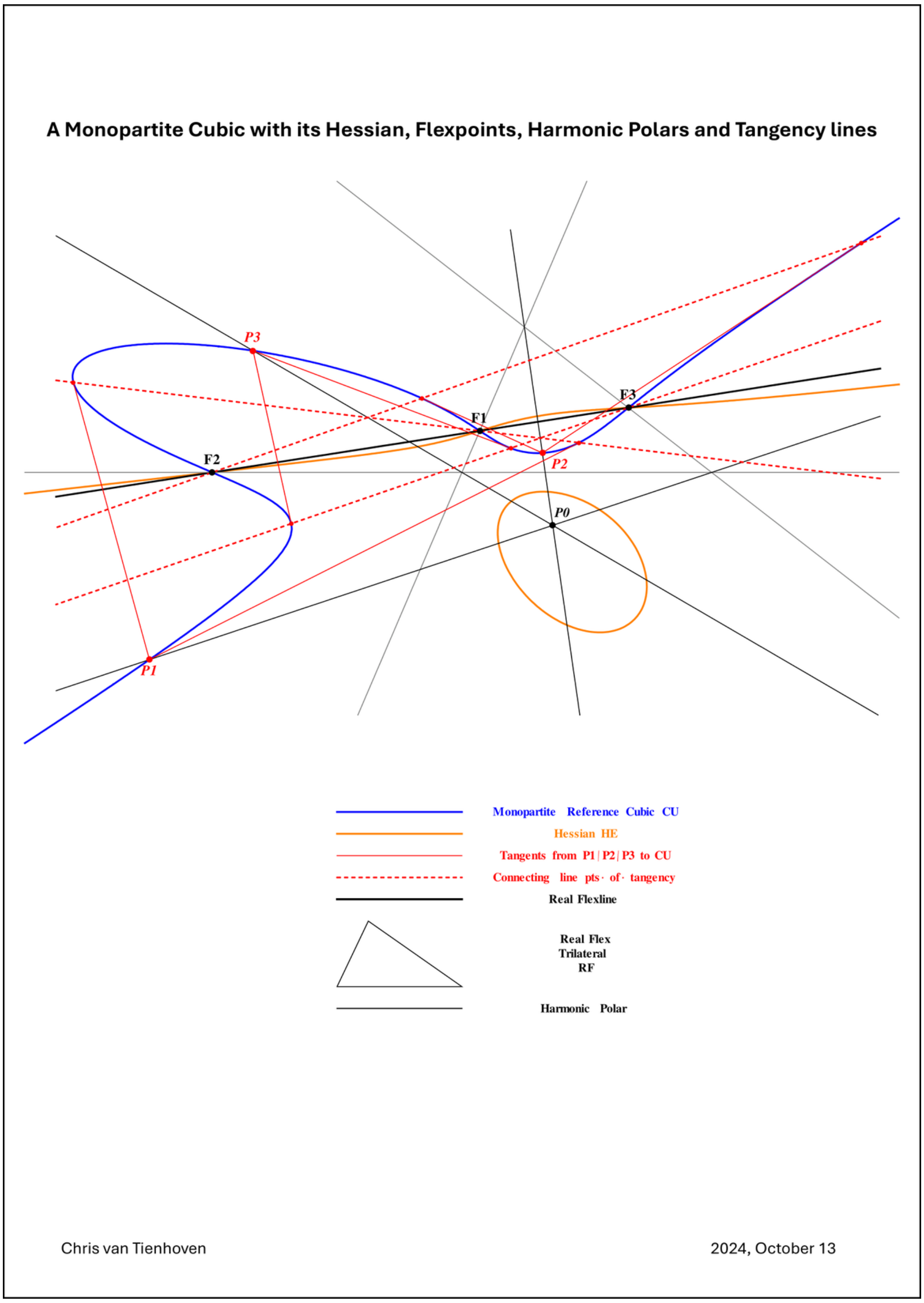
Message: #2503
Date: 2024-10-13
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,
Attached a significant counterexample.
Best regards,
Chris
```



CU-HE-12L1abc-04-Numerical.pdf

Message: #2504
Date: 2024-10-13
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

many thanks for the really "significant" counterexample,
... I have drawn a similar cubic and proved my construction
    as described in #2496,
... and the aberration was unexpected higher
    as in earlier examples.

Best regards Eckart
```

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Message: #2505
Date: 2024-10-13
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Chris, dear Eckart
I've decided in answer to Eckart's request to send you a little
memo on the Hesse pencil with significant figures.
I hope I didn't make any mystake and hope also that it will give
you ideas in order to find a new indiscutable construction of
H1H2H3.
Best regards
Bernard
```

Hesse pencil

Reference triangle H1H2H3

P0 = (1/a1,1/a2,1/a3)

Line of real flexes a1X + a2 Y + a3Z = 0

Cubic FE $a1^3X^3 + a2^3Y^3 + a3^3Z^3 = 0$

Hessian RF XYZ = 0

Cubic CU FE + ka1a2a3RF = 0 (equation F)

Hessian HE FE + k'a1a2a3RF = 0 with $k' = -36/k^2 - k/3$

Everything is then calculable

Real flexes are F1 (0,-a3,a2) ...

Harmonic lines are L1 Y/a2 – Z/a3 =0 ...

Tangent in (x,y,z) of a point of CU is XF'x + Y F'y + ZF'z = 0

Flextangent in F1 is $X(-ka1a2a3*a2a3) + Y*3a2^3a3^2 + Z*3a3^3a2^2 = 0$

The flextangent in F1 intersects the line L1 in the point T1 (6/ka1,1/a2,1/a3)

This point T1 is on HE, as $216/k^3 +1 +1 + k'*6/k = 0$

(It's a new proof of $k' = -(108 + k^3)/3k^2$

For k = 0, T1 is (1,0,0), id est the point H1

For k = 6, T1 is the point P0

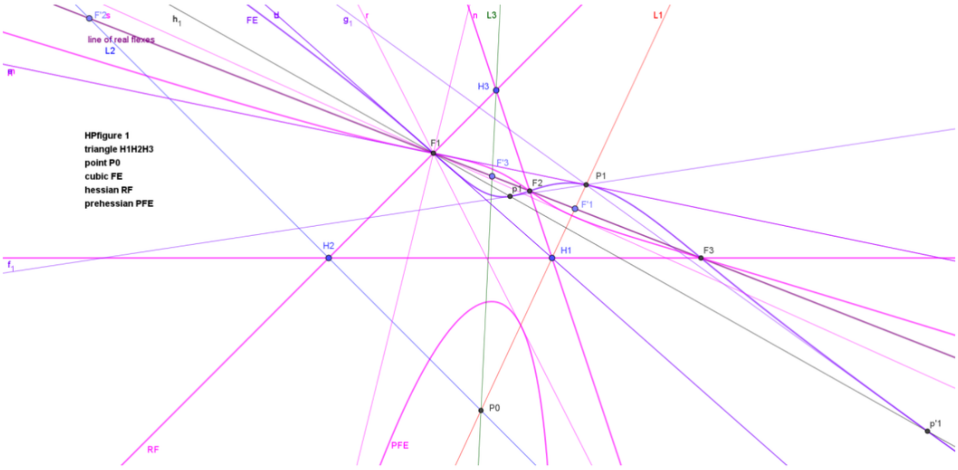
(In this case, CU is the cubic FE1, which has for hessian the triangle RF1, having a real side, which is the line of real flexes, and 2 imaginary sides, which intersect in P0)

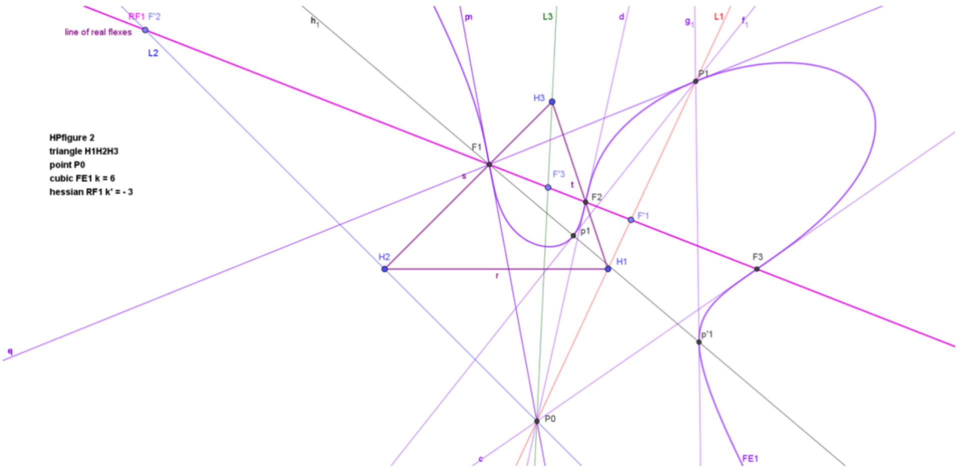
It follows 5 different figures

1) FE, RF and PFE k = 0, k' = ∞ and k= 3 cbr(4)
2) FE1 and RF1 k = 6 and k' = - 3
3) CU and HE for k = -6 and k' = 1
4) CU and HE for k = 3 and k' = - 5
5) CU and HE for k = - 10 and k' = - 3.693

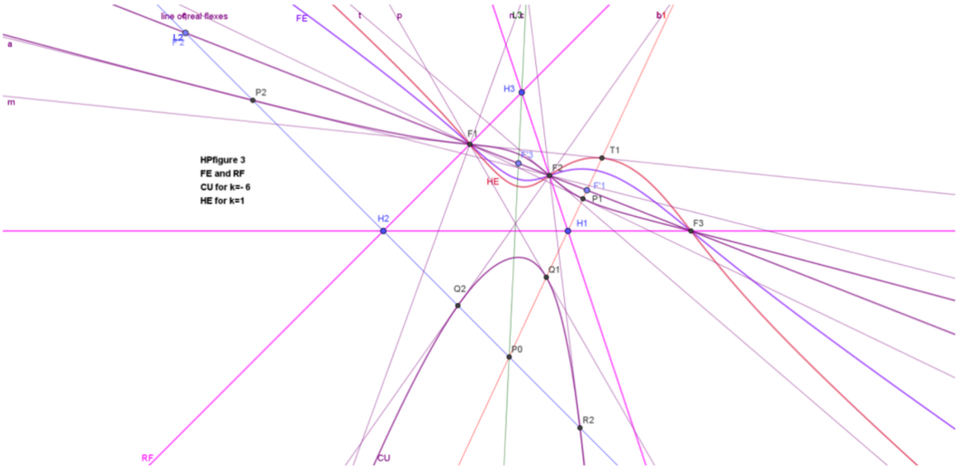
The figures are on a triangle H1 = (20,0), H2 = (-20,0) and H3 = (10,30) on rectangular axes and with a1 = 2, a2 = 5 and a3 = -3

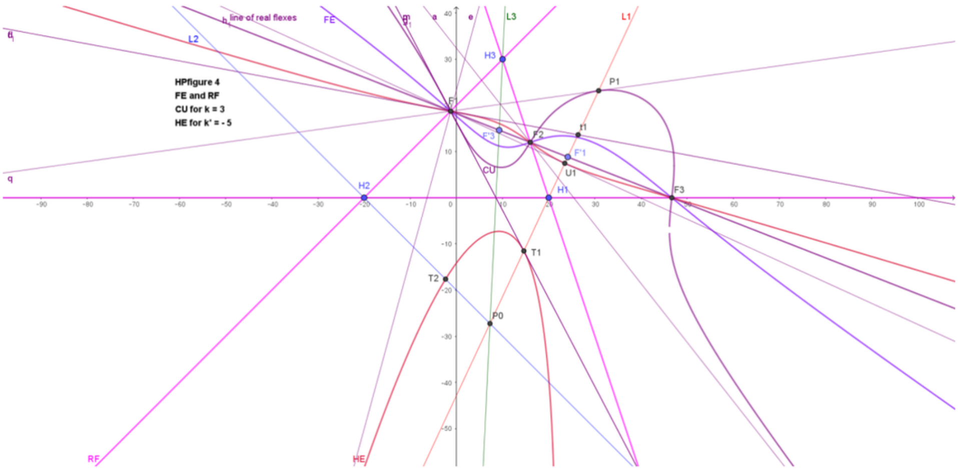
Hesse pencil.pdf



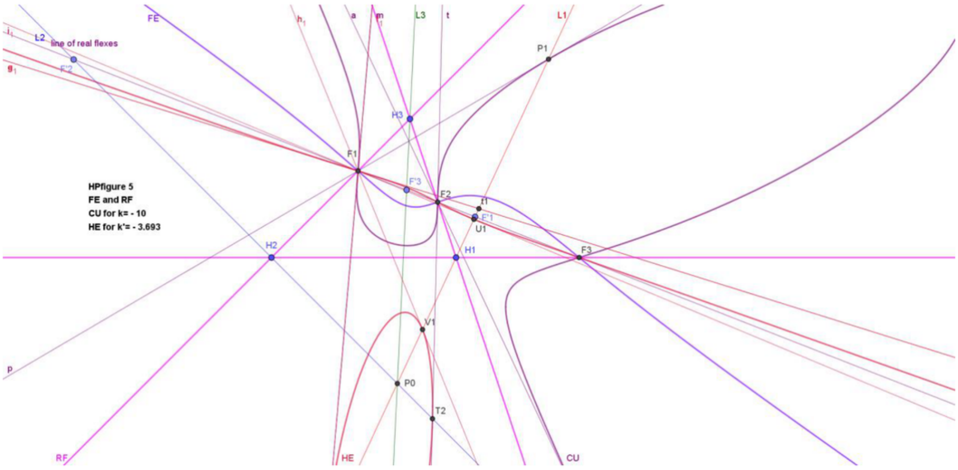


Hesse pencil.pdf





Hesse pencil.pdf



Hesse pencil.pdf

Message: #2506
Date: 2024-10-13
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

Dear M@IMF,

Thank you for your interesting configuration.
We have already discussed this outside the forum, but I am repeating my earlier message here for completeness and to allow others to join the conversation.
Let’s define the following:
- QL1 = the quadrilateral formed by the 4 lines X'Y's.
- QA1 = Q1Q2Q3Q4 (a cyclic quadrangle).

Note that a quadrangle QA is a reference figure consisting of 4 points, while a quadrilateral QL consists of 4 lines.
Now some aspects:
- F is the Miquel point QL-P1 with respect to QL1.
- The circumcenter of QA1: Since F and F~ are antipodes on the QA1-circle, its center could indeed be (F+F~)/2 = ((m+q)/2, (f-r)/2). However, it is not QL-P4 with respect to QL1. It is the reflection of QL-P1 (=F) about QL-P4 with respect to QL1.
- The anticenter of QA1: This could be (0, (r-f)/2) (although I haven’t verified this), but it is not QL-P5 with respect to QL1. It is the reflection of QL-P1 (=F) about QL-P5 with respect to QL1.
- The centroid of QA1: This could be ((m+q)/4, 0) (again, unverified), but it is not QL-P6 with respect to QL1. It is the reflection of QL-P1 (=F) about QL-P6 with respect to QL1.
- The Newton line (QL-L1) of QL1 seems to align with the Y-axis at half the distance of QL-P1.
- The orthopolar line QA-Tf8 of FF~ with respect to QA1 appears to be the Y-axis.
- The orthopolar line QL-Tf5 of FF~ with respect to QL1 appears to be the X'-axis.
- The QL-pedal line QL-L3 of FF~ with respect to QL1 also appears to be the X'-axis.

Additional observation:
- If x is the x-coordinate of X'i, then the slope of X'iY'i is (f-p) / (m-x) (calculated in Mathematica).

When starting the configuration with quadrilateral QL1, we can construct F=QL-P1.
Next, we can choose some point F~, then construct the Y-axis as the orthopolar line of FF~, and the X’-axis as the QL-pedal line of QL1.

I’m still unsure about the construction of the X’’-axis, but you mentioned it as QL-L3 (presumably of the quadrilateral formed by the lines Y’iX’’i).

Best regards,
Chris van Tienhoven

p.s.
You describe the plane as a Cartesian coordinate system with the x-axis as QL-L3 and the y-axis as QL-L1.
In EQF, this possibility within a QL was previously described in QL-L-1
(http://www.chrisvantienhoven.nl/ql-items/ql-lines/ql-l-1): The Railway Watcher.

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Message: #2507
Date: 2024-10-14
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

Dear Bernard,

many thanks for the drawings
... but especially for the summary on the first page,
... a great help for me.

Best regards Eckart

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Message: #2508
Date: 2024-10-14
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear Chris,

Thank you for your support.
I post the reply to your mail (a bit modified), too.
I'm going to post an additional explanation later.

<QL-Zn">
QL-Zn" is homothetic with QL-Zn (ratio 2, center QL-P1).
Zn expresses Pn, Ln, Cin, etc.
QL-Pn" is the reflection of QL-P1 in QL-Pn, so QL-P4" is the antipode of QL-P1 wrt QL-Ci3.
As for QL-Ln",

QL1	QL1~ (the quadrilateral formed by the 4 lines Y'X"s)	
QL-P1	F(m, f - p/2)	F~(q, p/2 - r)
QL-L1	x = m/2	x = q/2
QL-L1"	x = 0	x = 0
QL-L3	y = -p/2	y = p/2 (X"-axis)
QL-L3"	y = -p/2 - f	y = p/2 + r

<Construction of X"-axis>
I studied quartic equation and Lill's method, then related it to QL(QA).
I haven't considered construction from QL1.
Anyway, F~ must be the reflection of F in QL-P4".
After constructing QA1, construct QL1~ as Simson lines of F~ wrt the component triangles of QA1.
X"-axis is QL-L3 of QL1~. (And Simson line of F~ wrt QA1.)
cf. http://eckartschmidt.de/2013-01-05.pdf

<Derivation of Quartic Eq.>

Line	Slope	
FX'	-f/(x' - m)	x' = m + ft
X'Y'	-(y' + p/2)/x'	y' = -x't - p/2
Y'X"	-(y' - p/2)/x"	y' = x"/t + p/2
X"F~	r/(x" - q)	x" = q + r/t

>From above, we can obtain -(ft + m)t - p/2 = (q + r/t)/t + p/2, and the desired equation.
(Similarly, the quintic eq.)

Best regards
M@IMF

p.s.

```
I should have use other name for the quadrangle Q1Q2Q3Q4.
I myself call it CQA".
```

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Message: #2509
Date: 2024-10-14
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear all,

Let me explain more.
Let's consider the quadrilateral QL1 only and change the coordinate.
(I want to call the axis of QL-Co1 QL-L0.)

<QL1>
Let QL1 be formed by the tangents of the parabola x^2 = 4fy with focus F(0, f)
Li: y = tix - fti^2 (i = 1,2,3,4)
Define
m = -f(t1 + t2 + t3 + t4)
p = f[(t1 + t2)(t3 + t4) + t1t2 + t3t4]
q = -f[(t1 + t2)t3t4 + (t3 + t4)t1t2]
r = ft1t2t3t4.
Then the slope of Li is the solutions of the quartic equation
ft^4 + mt^3 + pt^2 + qt + r = 0.

<Tij, Mij, Kij>
Define
Tij = the intersection point of Li and Lj = (f(ti + tj), ftitj)
Mij = the midpoint of Tij and Tkl = (-m/2, (Kij + p)/2)
Kij = -f(ti + tj)(tk + tl).
where (i,j,k,l) = (1,2,3,4) or their permutation.
Note that
Tij = Tji, Mij = Mji = Mkl = Mlk, Kij = Kji = Kkl = Klk.
Kij is the solution of the cubic equation
K^3 + 2pK^2 + (p^2 - 4rf + mq)K - (fq^2 - mpq + rm^2) = 0.

<QL-Pn, Ln>
Some QL points and lines are
QL-P1(0, f) , QL-L2: y = -f , QL-L3: y = 0 ,
QL-L1: x = -m/2 , QL-P12(-m/2, p/6) , QL-P4 ((q - m)/4, (3f - r + p)/4) ,
QL-L4: x = -(q + 3m)/4 , QL-P2 (-(q + 3m)/4, -f) , QL-P6 ((q - 3m)/8, (2f + p)/4) ,
QL-P5(-m/2, (f + r + p)/4) , QL-P3 (-(q + 3m)/4, (3r - f + p)/4).

QL-P4 is obtained as follows.
For point P(x, y), let z[P] = x + iy, where i is the imaginary unit. (Sorry for the confusion.)

According to CSC (QL-Tf1),
(z[QL-P4] - z[F])(z[E] - z[F]) = (z[Tij] - z[F])(z[Tkl] - z[F])
where point E(0, -3f) is the reflection of F in QL-L2.
Since
LHS = -4if(z[QL-P4] - if)
RHS = (f^2)[(ti + tj) + i(titj - 1)][(tk + tl) + i(tktl - 1)] =
f[(p - r - f) + i(m - q)],
we get
z[QL-P4] = [(q - m) + i(3f - r + p)]/4.

QL-2P3a is the point W such that triangles F.E.W and F.W.QL-P4
are similar.
QL-2P3b is the reflection of QL-2P3a in F.

<Reconstruction of QL1>
QL1 is reconstructed from F(QL-P1), QL-P4, QL-L3, and Mij.
(1) Construct QL-2P3a,b
(2) Tij is the point T such that triangles Mij.QL-2P3a.T and
Mij.T.QL-2P3b are similar.
Tkl is the reflection of Tij in Mij.
(3) Let the intersection points of the circle F.Tij and QL-L3 be
X'i and X'j.
Li(Lj) is the line Tij.X'i(Tij.X'j). Similarly Lk and Ll.

Best regards
M@IMF

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Message: #2510
Date: 2024-10-15
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Chris, dear Eckart,
I continued to calculate and it's easy to prove that the tangent to HE in T1 passes through F1 ...
The equation of the flextangent in F1 is (see my memo) -ka1X + 3a2Y + 3 a3Z = 0
The coordinates of X1, intersection of the flextangents in F2 and F3, are (k-3/3a1,1/a2,1/a3).
Now, I suppose it should be possible to have the coordinates of H1, H2 and H3 wrt the triangle X1X2X3 knowing the coordinates of the 4 points P0, X1, X2 and X3 wrt the triangle H1H2H3 and identify this way a1, a2, a3 and k.
Remember these 4 points are the poles of the line of real flexes wrt the cubic CU!
Best regards
Bernard

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Message: #2511
Date: 2024-10-16
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Chris and Eckart,
I also like these easy calculations:
the polar conic of F1 wrt CU is degenerated in the 2 lines L1 and flextangent in Fi, intersecting in T1
the polar conic of F'1 (harmonic of F1 wrt F2 and F3, on L1) contains the 4 points P0, X1,X2 and X3
the polar line of P0 wrt any CU of the pencil is XF"x(1/a1) + YF"y(1/a2) + ZF"z(1/a3) = 0, giving the line of real flexes (k vanishes).
Best regards
Bernard

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Message: #2512
Date: 2024-10-17
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Chris and Eckart
My last calculation wasn't complete, but it is relatively easy
to show that the polar line of the 4 points P0, X1, X2 and X3 is
the line of real flexes, as expected.
(The coordinates of X1 are in my message 2510, X2 and X3 are the
like).
Best regards
Bernard
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Message: #2513
Date: 2024-10-18
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard, dear Chris,

what about the equation a1^3*x^3+a2^3*y^3+a3^3*z^3 + k*x*y*z = 0
... wrt H1H2H3 for a monopartite cubic?
I have drawn some monopartite cubics (one attached)
... with a variable triangle H1'H2'H3' on L1,L2,L3
    with Fi on Hj'Hk'.
Then I have determined  the barycentric coordinates wrt
H1'H2'H3'
... for Po(1/a1,1/a2,1/a3)  and some points of the cubic,
... to get the constant k in the equation by calculation,
... but I found no H1'H2'H3', so that the k-values coincide
... for all cubic points with acceptable accuracy.

Is there a reference for the cubic equation above?

Best regards Eckart

PS: Curious observation:
... Two cubic points collinear with Fi give the same k,
... starting with a cubic-point,
    you get 6 coconic points with the same k.
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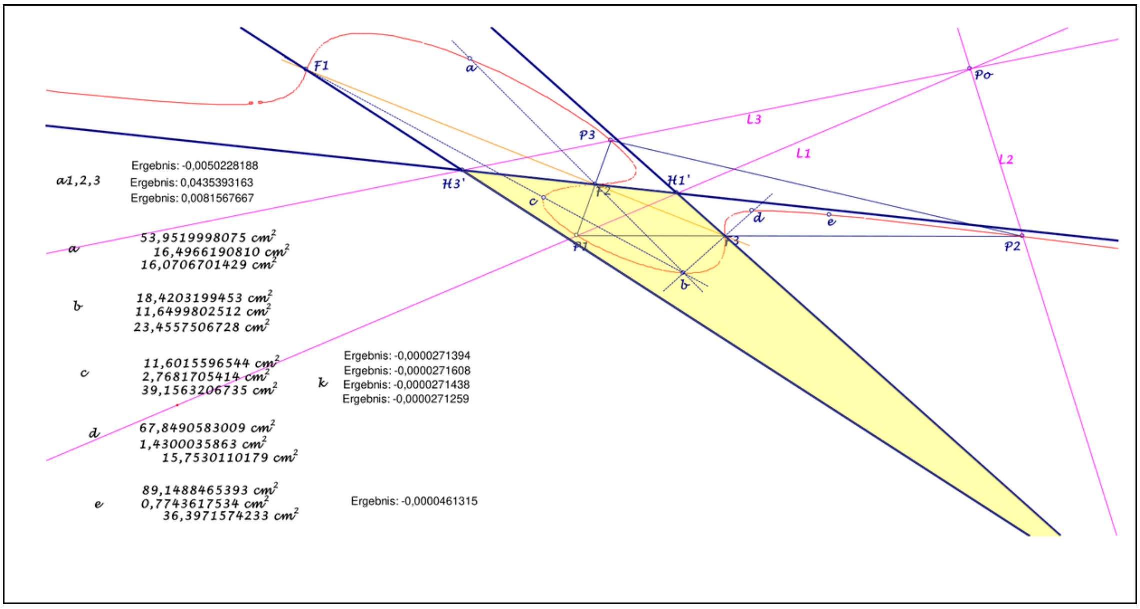

2024-10-18.pdf

Message: #2514
Date: 2024-10-18
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Eckart,
Once again, your equation is wrong!
See please my memo in 2505.
The last term is not kxyz, but ka1a2a3xyz.
Best regards
Bernard
```

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Message: #2515
Date: 2024-10-19
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard,

if I am not totally confused,
... ka1a2a3 has to be as well constant as k for all points
    of the cubic.

Best regards Eckart
```

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Message: #2516
Date: 2024-10-19
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,
Naturally, you are right, k or ka1a2a3 must be constant for all points of the cubic!
The idea of calculating the barycentric coordinates of points of the cubic wrt variable H1H2H3 is great.
But I suppose in this case you are able to calculate the barycentric coordinates of any point in the plane, not only P0.
Why don't you try my idea in the message 2510?
Having the coordinates of P0, X1, X2 and X3 should allow to calculate a1, a2, a3 and k ...
Normally, having calculated the coordinates of X1, X2 and X3 wrt the reference triangle H1H2H3 should allow to calculate the coordinates of H1, H2 and H3 wrt the triangle X1X2X3. (Or you could also take T1T2T3).
I suppose it is a calculation possible in Mathematica by inversion of matrices and Chris could be helpful. Unfortunately I'm not able to do it by hand ...
Best regards
Bernard

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Message: #2517
Date: 2024-10-21
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear all,

Let me continue to explain.
To obtain Kij, we need to solve the cubic equation.

<Lill's method>
When 4 points P(a, e), P~(c, b), X(x', 0), Y(0, y') satisfy PX_|_XY_|_YP~,
the slope of the line XY is the solution of the cubic equation et^3 + at^2 + bt + c = 0.
If the cubic equation has 3 real solutions ti (i = 1,2,3), then XYs give a triangle (TR0),
and its verices are Ti(-eti, -c/ti).
The segment PP~ is the diameter of the circumcircle of TR0,
and Simson line of P(P~) wrt TR0 is x(y)-axis.
The orthocenter and centroid of TR0 are H(-c, -e) and G(a/3, b/3), respectively.
P is the focus of the inscribed parabola (x - a)^2 = 4ey (directrix: y = -e),
and P~ is that of (y - b)^2 = 4cx (directrix: x = -c).

<ORIGAMI (paper-folding)>
Given the focus F and directrix d of a parabola on a piece of paper.
Fold the paper so that F lies on d, then the crease is a tangent of the parabola.
Let F1 and d1 be the focus and directrix of an inscribed parabola of a triangle,
and let F2 and d2 be those of another inscribed parabola of the triangle.
Make a fold that places F1 onto d1 and F2 onto d2 simultaneously, then the crease
coincides a side of the triangle (, which is one of the common tangents to the two parabolas.)

<QL-Tr2>
QL-Co1 is inscribed in QL-Tr2, so QL-Tr2 is constructed by ORIGAMI from
F1(0, f) = QL-P1 , d1: y = -f (QL-L2)
F2((q - 3m)/4, p/2) = QL-P6" , d2: x = -(q + 3m)/4 (QL-L4),
where QL-P6" is the reflection of QL-P1 in QL-P6 and its Simson line is x = -3m/4.
The slopes of the sides of QL-Tr2 are the solutions of the cubic equation

ft^3 + (3m/4)t^2 + (p/2)t + q/4 = 0.   (i.e. 4ft^3 +
3mt^2 + 2pt + q = 0!)
I wonder whether we can reconstruct QL1 from its QL-Tr2 and some
points and lines described using f, m, p, q, r (without
ORIGAMI).

<QL-Tr1>
The line TijTkl is the side of QL-Tr1, and its equation is
y = (q'/2Kij' - m/4f)x + (Kij + p' + mq'/2Kij')/2,
where Kij' = Kij + (m^2)/4f, p' = p - (m^2)/4f, q' = q -
(m/2f)p',
and (i,j,k,l) = (1,2,3,4) or their permutations.
Mij is the intersection point of this line and QL-L1.
For simplicity, let m = 0, then the cubic equation of K becomes
K^3 + 2pK^2 + (p^2 - 4rf)K - fq^2 = 0,
and we obtain
(-2f)(q/2K)^3 + g(q/2K)^2 + p(q/2K) + q/4 = 0,
where g = (p^2 - 4rf)/q.
QL-Tr1 is costructed from
F1(g, -2f - p/2) , d1: y = 2f - p/2
F2(q/4, p/2) = QL-P16 , d2: x = -q/4,
since Simson line of F1 is y = -p/2.
When m is non-zero, the expressions are complicated.

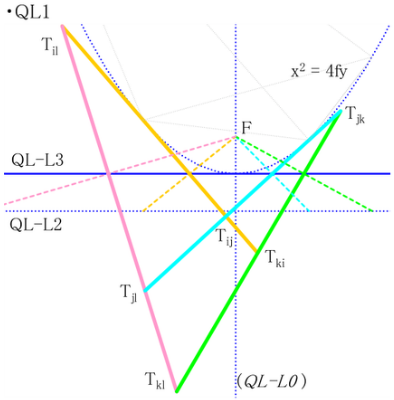
<DT'>
Kij' is the solution of the cubic equation
K'^3 + [2p' - (m^2)/4f]K'^2 + (p'^2 - 4rf + mq')K' - fq'^2 = 0,
and we obtain
(-2f)(q'/2K')^3 + (g' + m)(q'/2K')^2 + (p' - Am/2)(q'/2K') +
q'/4 = 0
where g' = (p'^2 - 4rf)/q', A = m/4f.
A triangle (DT') is constructed from
F1(g' + m/2 , -2f - p'/2 + Am/2) , d1: y = 2f - p'/2 + Am/2
F2(q'/4 - m/2 , p'/2) , d2: x = -q'/4 - m/2.
The two Simson lines intersect at (-m/2, -p'/2 + Am/2), which is
the centroid of the quadrangle formed by the points of tangency
of QL-Co1 and QL1.
M14, M24, M34 are the feet of the perpendiculars from F2 onto
the edges of DT'.
I'm not sure DT' is worth thinking about.

<Subnormal series>
1   <- C2 <- V4 <- S4
{ti} -> {Tij} -> {Kij} -> {f,m,p,q,r}

Best regards
M@IMF
p.s. I'm very sorry for my typo in Chris's name in message
#2508.

Quadrilateral and ORIGAMI (Paper-Folding) M@IMF

•QL1



<QL1>

Let QL1 be formed by the tangents of the parabola $x^2 = 4fy$

L_i: $y = t_i x - ft_i^2$ (i = 1,2,3,4),

and let t_is be the solutions of the quartic equation

$ft^4 + mt^3 + pt^2 + qt + r = 0$.

Define

T_{ij} = the intersection point of L_i and L_j = $(f(t_i + t_j), ft_it_j)$

M_{ij} = the midpoint of T_{ij} and T_{kl} = $(-m/2, (K_{ij} + p)/2)$

$K_{ij} = -f(t_i + t_j)(t_k + t_l)$,

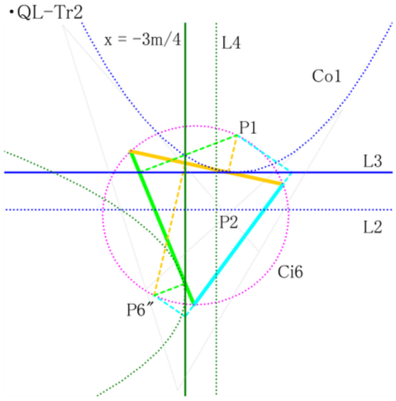
where (i,j,k,l) = (1,2,3,4) or their permutations.

K_{ij} is the solution of the cubic equation

$K^3 + 2pK^2 + (p^2 - 4rf + mq)K - (fq^2 - mpq + rm^2) = 0$.

L_i and L_j are constructed by making creases that place F(focus) onto QL-L2 and pass through T_{ij}.

•QL-Tr2



<Lill's method and ORIGAMI>

When 4 points P(a, e), $P^\sim$(c, b), X(x', v), Y(u, y') satisfy PX⊥XY⊥Y$P^\sim$, the slope of the line XY is the solution of the cubic equation

$(e - v)\tau^3 + (a - u)\tau^2 + (b - v)\tau + (c - u) = 0$.

If it has 3 real solutions τ_i (i = 1,2,3), then X_iY_is give a triangle whose vertices are $T_i(u - e\tau_i, v - c/\tau_i)$,

where $X_i(a + (e - v)\tau_i, v)$ and $Y_i(u, b + (c - u)/\tau_i)$ lie on Simson line of P and $P^\sim$ wrt the triangle, respectively.

For QL-Tr2,

P = P1(0, f) , Simson line: y = 0 (L3)

$P^\sim$ = P6″((q − 3m)/4, p/2) , Simson line: x = −3m/4,

where P6″ is the reflection of P1 in P6. ("QL−" is omitted.)

The slopes of the sides of QL-Tr2 are the solutions of

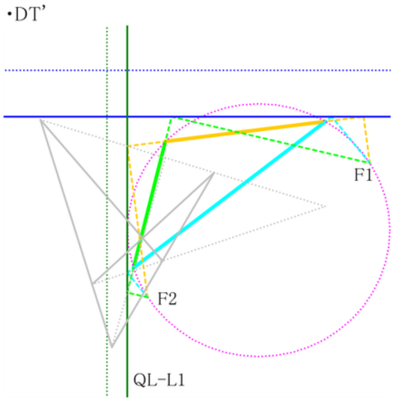
$f\tau^3 + (3m/4)\tau^2 + (p/2)\tau + q/4 = 0$.

I can't express τ_i in terms of t_1, t_2, t_3, t_4, so I don't know how to number the verties of QL-Tr2.

QL-Tr2 is constructed by making creases that place P1 onto L2 and P6″ onto L4 simultaneously.

(We can use P17&L6 instead of P6″&L4.)

•DT'



<QL-L1 and DT'>

The line $T_{ij}T_{kl}$ is the side of QL-Tr1, and its equation is

$y = (q'/2K_{ij}' - m/4f)x + (K_{ij} + p' + mq'/2K_{ij}')/2$,

where $K_{ij}' = K_{ij} + m^2/4f$, $p' = p - m^2/4f$, $q' = q - (m/2f)p'$.

M_{ij} is the intersection point of this line and QL-L1.

K'_{ij} satisfies

$(-2f)(q'/2K')^3 + (g' + m)(q'/2K')^2 + (p' - Am/2)(q'/2K') + q'/4 = 0$, where $g' = (p'^2 - 4rf)/q'$, $A = m/4f$.

A triangle (DT') is constructed from

F1(g' + m/2 , −2f − p'/2 + Am/2) , y = 2f − p'/2 + Am/2

F2(q'/4 − m/2 , p'/2) , x = −q'/4 − m/2.

M_{ij} corresponds to Y.

DT' coincides QL-Tr1 when m = 0.

(The vertex of the inscribed parabola with focus F2 is QL-P23(−m/2, p'/2).)

QL and ORIGAMI.pdf

Message: #2518
Date: 2024-10-22
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

```
Dear Chris and Eckart,
I've found a mystake in my memo wrt the last figure HP5
The figure is correct, but CU is for k = 10 (and not - 10 as
mentionned) and HE is for k' = - 3.693 (as mentionned).
For k = -10, k' = 2.973 and this is another figure!
Best regards
Bernard
PS What about my idea of calculating the barycentric coordinates
of H1, H2 and H3 wrt the reference triangle X1X2X3 (function of
a1, a2, a3 and K?
```

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Message: #2519
Date: 2024-10-26
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

Dear M@IMF,

I tried to make an overview of your very special properties regarding Lill's Method.
Forgive when I use some new notation.
I made for readers not familiar with *Lill's method* a small description of the method. See attachment.
What now follows describes the setting for a solution of an equation of the 4th degree.
We deal with a set of 5 consecutive perpendicular line segments (5L1, 5L2, 5L3, 5L4, 5L5), starting at point A0 and ending at point A5. Let's call it a 5-chain.
Then we make a 4-chain, this time a set of 4 consecutive perpendicular line segments,
with its line segments bouncing perpendicularly at the 5Li-lines.
According to Lill's theory there are 4 possible 4-chains that start at A0 and end in A5.
So let's say that:
· 4L1a, 4L2a, 4L3a, 4L4a
  describes the consecutive lines of 4-chain-a,
· 4L1b, 4L2b, 4L3b, 4L4b
  describes the consecutive lines of 4-chain-b,
· 4L1c, 4L2c, 4L3c, 4L4c
  describes the consecutive lines of 4-chain-c,
· 4L1d, 4L2d, 4L3d, 4L4d
  describes the consecutive lines of 4-chain-d.
Thanks to you I know these special properties:
Let QL2 = QL(4L2a,4L2b,4L2c,4L2d) and QL3 = QL(4L3a,4L3b,4L3c,4L3d).

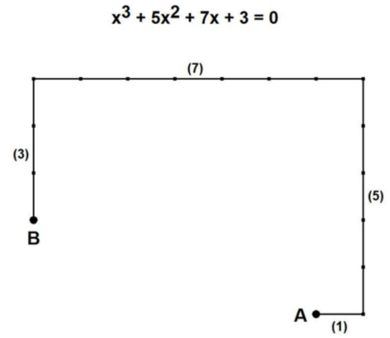
Now
· QL2-P1 = A0 and QL3-P1 = A5 (Miquel Points of QL2 and QL3).
· QL2-L3 = 5L2 and QL3-L3 = 5L4 (QL-Pedal Lines of QL2 and QL3).
· QL2-L1 = perpendicular bisector of line segment 5L2
  and QL3-L1 = perpendicular bisector of line segment 5L4
  (QL-Newton Lines of QL2 and QL3).
· There are many QL-points that can be described in a cartesian coordinate system for QL2 using QL2-L1 and QL2-L3 as axes and for QL3 using QL3-L1 and QL3-L3 as axes. (see attached picture).
The question arises if it is possible to construct 4-chain-a, 4-chain-b, 4-chain-c, 4-chain-d knowing the reference 5-chain.

Best regards, Chris

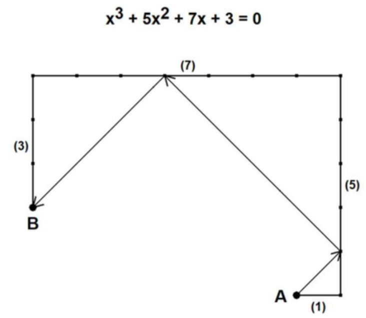
Lill's Method

This method comes from a paper from E. Lill:
Lill, E.. "Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but." *Nouvelles annales de mathématiques : journal des candidats aux écoles polytechnique et normale* 6 (1867): 359-362. <http://eudml.org/doc/98167>.
For further visualization and understanding, refer to this video:
[Lill's Method on YouTube] (https://www.youtube.com/watch?v=IUC-8P0zXe8).

Here is an equation: x^3 + 5x^2 + 7x + 3 = 0.
Let's have a starting point A.
The leading coefficient of our polynomial is 1.
This tells us to advance 1 unit to the right.
Make a quarter turn counterclockwise direction.
The second coefficient is 5 which tells us to advance 5 units.
The next coefficient is 7. Another turn counterclockwise and advance 7 units.
The last coefficient is 3, so again advance 3 units counterclockwise and we end in point B.
This leaves us with this figure.



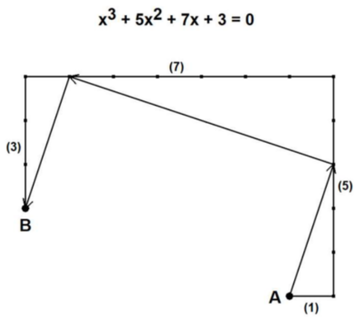
Now we aim to bounce a ball from A to B, bouncing perpendicularly at the constructed contour sides. (a bit like billiard players are doing, with the exception that the bouncing is perpendicular instead of following reflecting path lines).
Now we get this picture.



What it brings us here is that the negative tangent of the first bouncing line gives a solution of the equation where the figure is based upon. The negative tangent in this case is -1, therefore x=-1 will be a solution of the equation. By substituting x=-1 in above equation it becomes clear that the outcome is 0.

Lill_s Method-description-01.pdf

There is another solution with bouncing at right angles:



What it brings us here is that the negative tangent of the first bouncing line gives a solution of the equation where the figure is based upon. The negative tangent in this case is -3, therefore x=-3 will be a solution of the equation. By substituting x=-3 in above equation it becomes clear that the outcome is 0 indeed.

Above example is for an equation of degree 3 , implicating an equation with 4 coefficients and just as much line segments are used.
As a consequence for an equation of degree 4, 5 line segments are used, for an equation of degree 5, 6 line segments are used, etc..

Bouncing rules
Own interpretation of the bouncing rules.
Given polynomial: a0 + a1 x + a2 x2 + + a3 x3
0. Have starting point A0
1. Go north a0 units, ending up in A1
2. Go east a1 units, ending up in A2
3. Go south a2 units, ending up in A3
4. Go west a3 units, ending up in A4
5. Go north a4 units, ending up in A5
. . . and so on

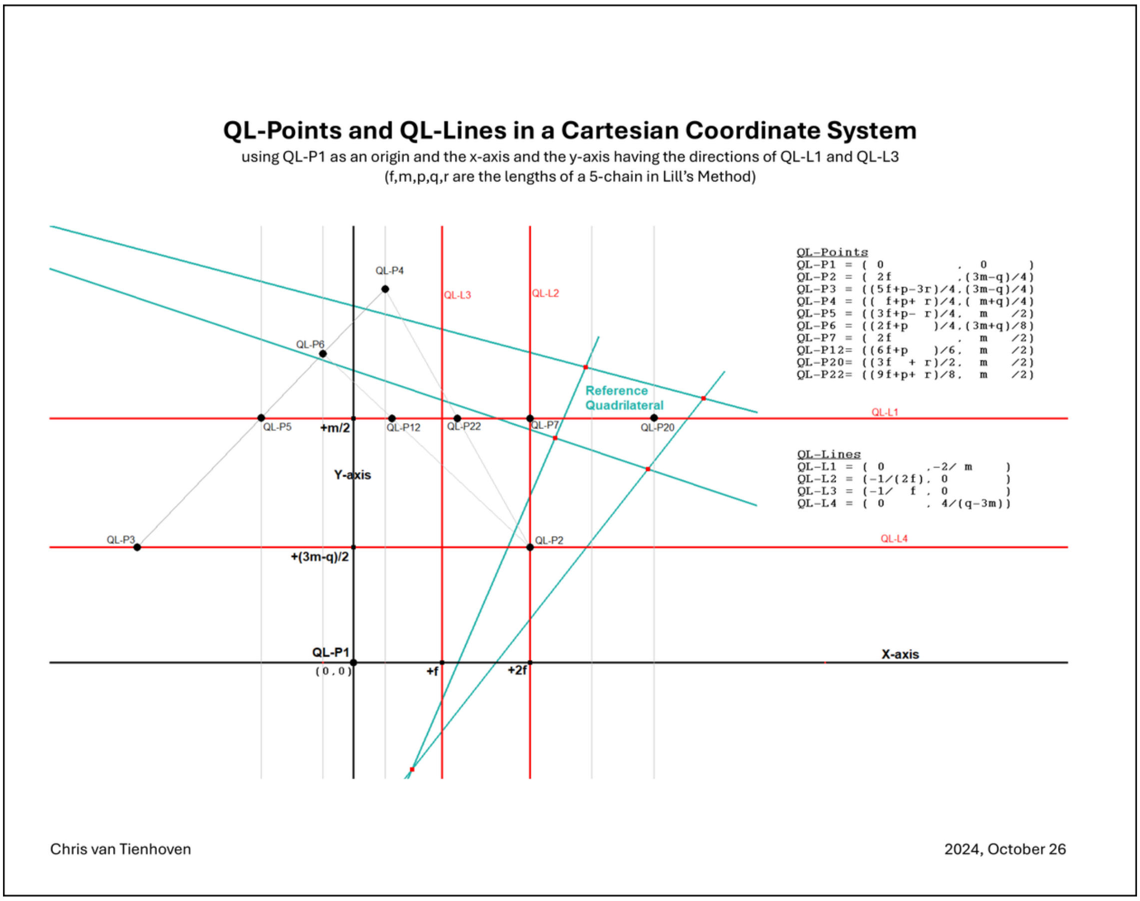
Of course you can also start with east or south or west.
You can also go the other way around: west > south > east > north > west > etc.
It is important to be consistent in the method.
When the number to be measured on a line segment is negative, just measure it the other way, but after that proceed in the North-East-South-West direction that is chosen.
The bouncing is at right angles.
Bouncing is not specifically at a line segment but at the full extended line.
For a zero coefficient the direction makes a double direction change (e.g. after north follows south) for measuring the next following coefficient.

Lill_s Method-description-01.pdf

Example of the 4 solutions for an equation of 4th degree, where the lengths of A0A1, A1A2, A2A3, A3A4, A4A5 are the coefficients of this equation.

The negative tangents of the 4 first line segments from A0 will be the 4 solutions of the equation of 4th degree.

Lill_s Method-description-01.pdf



QL-Points and QL-Lines in a Cartesian Coordinate System-01.pdf

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Message: #2520
Date: 2024-10-26
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

```
Dear Chris,

I admire your excellent ability to understand and express
accurately.
For your question, let
    A5 = (m + q, f + p + r)
    A4 = (m + q, f + p)
    A3 = (m, f + p)
    A2 = (m, f)
    A1 = (0, f)
    A0 = (0, 0),
then the slopes of QL2 are solutions of
    ft^4 + mt^3 + (-p)t^2 + (-q)t + r = 0,
and the slopes of QL3 are solutions of
    rs^4 + (-(-q))s^3 + (-p)s^2 + (-m)s + f = 0.
Of course, both equations are equivalent (s = -1/t).

Constructing 4-chains mean solving these quartic equation
geometrically.
In general, a quartic equation not always has a real solution,
while an equation of the n(odd number)th degree has at least one
real solution.
We assume the equation has 4 real solutions.

According to msg#2509, we can construct QL
when Mij (the midpoint of a diagonal) is given.
Mij is obtained by solving the cubic equation of K', which means
constructing QL-Tr1 or DT' (or others) by ORIGAMI (msg#2517).

cf. T.C.Hull, "Solving Cubics With Creases: The Work of Beloch
and Lill"
http://origametry.net/papers/amer.math.monthly.118.04.307-hull.p
df
(The link is broken in
https://en.wikipedia.org/wiki/Lill%27s_method)

Best regards
M@IMF
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Message: #2521
Date: 2024-10-28
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

```
Dear Chris,
About a month ago, I asked you where you were in your quest so
far.
Since then, I've drawn a lot of curves and made a lot of
calculations.
Every point, line or curve can be calculated wrt H1H2H3 as
function of a1, a2, a3 and k.
(for example, the points Fi, Ti, Xi, the flextangents, the lines
Li, the polar conics of Fi and Ti, the 2 other tangents in Fi to
the cayleyan ...).
But unfortunately, I'm still not able to find a barycentric
equation of the cayleyan known only by it's tangential equation
(see my message 2468).
I was hoping you would find it, or at last provide me examples
of your calculations on page 5 of your memo.
In your message 2467, you enonced the 3 challenges you were
facing.
As you can imagine, I was particularly interested in the 2 first
1) for a general cubic, how identify H1, H2 and H3 and a1, a2,
a3 and k? (you said, the method is clear, but could you please
describe it shortly)
2) how find the barycentric equation of the cayleyan? (as
function of a1, a2, a3 and k or k")
Did you find new ideas or developments or references?
I just hope you didn't give up!
I'm impatiently waiting for your answer
Best regards
Bernard
```

Message: #2522
Date: 2024-10-28
From: van10hoven@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Bernard,

Thank you for your message.

Unfortunately, I've been very busy with the holidays and various other projects. I can understand your impatience. I haven't forgotten it and will certainly shed some light on several of the open questions soon.

In the meantime, I'm curious about your opinion on Lill's method, which is the subject of our new member M@IMF. It ties in with many topics we've discussed before, including quadrilaterals, n-gons, and curves of the nth degree."

Best regards,

Chris

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Message: #2523
Date: 2024-10-28
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear all,

Here's an additional information.

As Chris explained, Lill's method can be applied to an algebraic equation of n-th degree.
(https://en.wikipedia.org/wiki/Lill%27s_method)
Maybe its configuration will relate to "Simson line" wrt cyclic n-angle, which is like Clifford's chain.
I read the theorem was discovered by de Longchamps (1877), Langley (1890).
M.Kobayashi, "A geometrical treatment of complex numbers."

Any n-th degree equation can be solved by multi-fold origami.
R.Alperin, R.Lang, "One-, Two-, and Multi-Fold Origami Axioms."
J.Koenig, D.Nedrenco, "Septic Equations are Solvable by 2-fold Origami."

Best regards
M@IMF

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Message: #2524
Date: 2024-10-31
From: bernard.keizer@gmail.com
Subject: Re: Flex-network and Harmonic Polar-network

Dear Chris,
I'm rather disappointed and this is an understatement!
Here some reflexions, before stopping for a while
1) Having the general barycentric equations of a cubic CU and it's hessian HE allows you to calculate with Mathematica the coordinates of the 9 flexes (3 real and 6 imaginary). These 9 points are on 12 lines, 4 real and 8 imaginary. It's then easy to identify the 4 real lines, one is the line of the 3rel flexes, the 3 other are the sides of the triangle H1H2H3. If I'm not wrong, this was your method.
The point P0 gives the coefficients a1, a2 and a3 and the points X1, X2 and X3 or T1, T2 or T3 give the coefficient k.
2) Another possibility
Forming the equation of CU + tHE gives only 4 values of t giving 4 triangles, one with 3 real sides is RF, a 2nd with one real side (the line of real flexes) and 2 imaginary and the 2 others with 3 imaginary sides. The 12 sides of the 4 triangles intersect in the 9 flexes (3 real and 6 imaginary).
3) We know now the barycentric equations of CU = FE + ka1a2a3RF and HE = FE + k'a1a2a3RF and the tangential equation of CA DFE + k''/a1a2a3 * DRF.
The equation of DFE, G(u,v,w) is the same as FE F(X,Y,Z) replacing a1X by u/a1, a2Y by v/a2 and a3Z by w/a3 and the equation of DRF is uvw = 0.
4) For any curve having the barycentric equation F(X,Y,Z) = 0 (1) and tangential equation G(u,v,w) = 0 (1) , there is a naturally a link between both equations.
The tangent is uX + vY + wZ = 0 (2)
Writing a system of 4 equations F(X,Y,Z) as (1), the tangent as (2) and F'x/u = F'y/v = F'z/w as (3) and (4), it is possible to find G(u,v,w) by eliminating X, Y and Z.
Conversely, having G(u,v,w) as (1), the same tangent as (2) and G'u/X = G'v/Y = G'w/Z as (3) and (4), it is possible to find F(X,Y,Z) by eliminating u, v and w.
It is possible to find the tangential equation of the cubic CU or the barycentric equation of the cayleyan CA.
The terms on page 5 of your memo are perfectly correct, but the fixed coefficients are only function of k" and not of a1, a2 and a3. Unfortunately, I've not been able so far to calculate them (they are rather complicated ...)
Best regards Bernard

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Message: #2525
Date: 2024-11-03
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Bernard,

I am looking into your emails again and try to respond to your remarks and questions.
I first start with your remark in QPG#2486:
A bipartite cubic has 3 real prehessians, which are monopartite
A monopartite cubic has only one real prehessian, which is bipartite ...

I calculated some bipartite cubic, but did not find 3 real preHessians.
Where is your information from?
Do you have an example of numbers (a1,a2,a3,k) for which there are 3 preHessians?

Then I proceed with QPG#2510.
I continued to calculate and it's easy to prove that the tangent to HE in T1 passes through F1 ...
The equation of the flextangent in F1 is (see my memo) -ka1X + 3a2Y + 3 a3Z = 0
The coordinates of X1, intersection of the flextangents in F2 and F3, are (k-3/3a1,1/a2,1/a3).
Now, I suppose it should be possible to have the coordinates of H1, H2 and H3 wrt the triangle X1X2X3 knowing the coordinates of the 4 points P0, X1, X2 and X3 wrt the triangle H1H2H3 and identify this way a1, a2, a3 and k.

I calculated the coordinates of H1, H2 and H3 wrt the triangle X1X2X3:
H1c = {3 (a1 a2 + a1 a3 - a2 a3) k + a2 a3 k^2, -9 (a1 a2 - a1 a3 + a2 a3) - 3 a1 a3 k, 9 (a1 a2 - a1 a3 - a2 a3) - 3 a1 a2 k}
H2c = {-9 (a1 a2 + a1 a3 - a2 a3) - 3 a2 a3 k, 3 (a1 a2 - a1 a3 + a2 a3) k + a1 a3 k^2, 9 (a1 a2 - a1 a3 - a2 a3) - 3 a1 a2 k}
H3c = {-9 (a1 a2 + a1 a3 - a2 a3) - 3 a2 a3 k, -9 (a1 a2 - a1 a3 + a2 a3) - 3 a1 a3 k, -3 (a1 a2 - a1 a3 - a2 a3) k + a1 a2 k^2}

While I was working on that, I converted right away some other related items:

F1c = {0, -3 (a1 a2 - a1 a3 + a2 a3) - a1 a3 k, -3 (a1 a2 - a1 a3 - a2 a3) + a1 a2 k}
F2c = {-3 (a1 a2 + a1 a3 - a2 a3) - a2 a3 k, 0, -3 (a1 a2 - a1 a3 - a2 a3) + a1 a2 k}

```
F3c = {-3 (a1 a2 + a1 a3 - a2 a3) - a2 a3 k, 3 (a1 a2 - a1 a3 +
a2 a3) + a1 a3 k, 0}
Poc = {3 (a1 a2 + a1 a3 - a2 a3) + a2 a3 k, 3 (a1 a2 - a1 a3 +
a2 a3) + a1 a3 k, -3 (a1 a2 - a1 a3 - a2 a3) + a1 a2 k}
CUc = Aa^2 Ab^2 Ac^2 (-162 + 90 k - 12 k^2 + k^3) x y z
+ (9 - 3 k + k^2) (Ab^3 Ac^3 x^3 + Aa^3 Ac^3 y^3 + Aa^3 Ab^3 z^3
+ 3 Aa Ab^2 Ac^3 x^2 y + 3 Aa^2 Ab Ac^3 x y^2 + 3 Aa Ab^3 Ac^2
x^2 z + 3 Aa^3 Ab Ac^2 y^2 z + 3 Aa^2 Ab^3 Ac x z^2 + 3 Aa^3
Ab^2 Ac y z^2),
HEc = Aa^2 Ab^2 Ac^2 (-486 + 162 k - 54 k' + 54 k k' - 9 k^2 k'
+ k^3 k') x y z
+ 3 Aa Ab Ac (9 - 3 k + k^2) (3 + k')
(Ab Ac^2 x^2 y + Aa Ac^2 x y^2 + Ab^2 Ac x^2 z + Aa^2 Ac y^2 z +
Aa Ab^2 x z^2 + Aa^2 Ab y z^2)
+ (27 + 27 k - 9 k^2 + k^3 - 27 k' + 9 k k')
(Ab^3 Ac^3 x^3 + Aa^3 Ac^3 y^3 + Aa^3 Ab^3 z^3)

where
Aa = 3 a1 a2 + 3 a1 a3 - 3 a2 a3 + a2 a3 k
Ab = 3 a1 a2 - 3 a1 a3 + 3 a2 a3 + a1 a3 k
Ac = -3 a1 a2 + 3 a1 a3 + 3 a2 a3 + a1 a2 k
```

I hope I understood your message well.
In the coming days I will delve further into your other
remarks/questions.
Best regards,

Chris

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Message: #2526
Date: 2024-11-03
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Chris,
I see with pleasure that you didn't abandon the item!
Congratulations for the calculations, which should interest Eckart.
Let's now answer your initial question
A CU has 3 real prehessians
a) if it is bipartite
b) if there are 4 tangents from a point of CU to CU
c) if it has 3 real QMT's swapping 2 by 2 the 4 contact points of the these 4 tangents
d) if the equation $k^3 + 3k^2k' + 108 = 0$ has 3 real roots in k
It has 1 real prehessian
a) if it is monopartite
b) if there are 2 tangents from a point of CU to CU
c) if it has only one QMT swapping the 2 contact points
   of these 2 tangents
d) if the equation has only one real root
For the point d),
   it appears that there are 3 real roots if $k' < -3$
For $k' = -3$, the solutions in k are $k = -3$ and $k = 6$, which is a dobble solution
For $k' = \infty$, let's divide by k', then the solutions in k are $k = -3$ and $k = 0$, which is a dobble solution (we have then FE and RF)
There is only one root if $k' > -3$
All the curves in my memo are for a1 = 2, a3 = 5 and a3 = - 3, there are several examples for different k ...
Best regards
Bernard
PS I will be in Venice for the 3 coming weeks ...

Message: #2527
Date: 2024-11-04
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

```
Dear all,

Relationship between Simson line wrt cyclic n-angle and
configuration for n-th degree eq. is examined (n=3,4,5).

<Simson line>
Given concyclic points F, Q1, Q2, Q3, ...,
Simson line of F wrt cyclic n-angle is inductively defined.
Let
Sim{ij} = line QiQj
Fij = the foot of the perpendicular from F onto Sim{ij}.

n=3
Sim{ijk} = line through Fij, Fik, Fjk
Fijk = the foot of the perpendicular from F onto Sim{ijk}.
Sim{ijk} is Simson line of F wrt triangle QiQjQk.

n=4
Sim{ijkl} = line through Fijk, Fijl, Fikl, Fjkl
Fijkl = the foot of the perpendicular from F onto Sim{ijkl}.
Sim{ijkl} is Simson line of F wrt cyclic quadrangle QiQjQkQl.

n=5
Sim{ijklh} = line through Fijkl, Fijkh, Fijlh, Fiklh, Fjklh
Fijklh = the foot of the perpendicular from F onto Sim{ijklh}.
Sim{ijklh} is Simson line of F wrt cyclic pentangle QiQjQkQlQh.
....

<Cubic eq.>
Relationship to the cubic eq. (#2517) is as follows:
   (F = P, F~ = P~, Qi = Ti)
   Xk = Fij
   Yk = F~ij
   X-axis = Sim{123} of F
   Y-axis = Sim{123} of F~,
where (i,j,k) = (1,2,3) or their permutations.
The intersection point of the two Simson lines is the orthopole
of FF~ wrt triangle Q1Q2Q3.

<Quartic eq.>
Relationship to the quartic eq. (#2493) is as follows:
   Tij (Li^Lj) = Fkl
   T~ij (L~i^L~j) = F~kl
   Ll (line X'lY'l) = Sim{ijk} of F         [line 4L2d]
```

L~l (line Y'lX"l) = Sim{ijk} of F~ [line 4L3d]
X'l = Fijk [4L1d^4L2d]
X"l = F~ijk [4L3d^4L4d]
X'-axis = Sim{1234} of F [line 5L2]
X"-axis = Sim{1234} of F~ [line 5L4]
X'0 = F1234 [A1]
X"0 = F~1234 [A4],

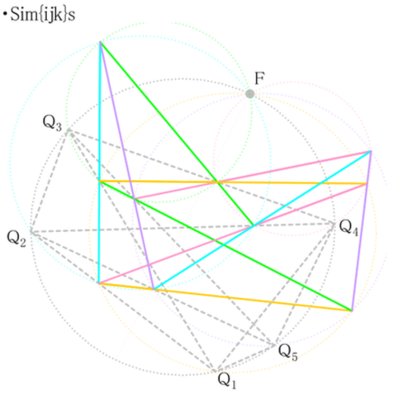
where (i,j,k,l) = (1,2,3,4) or their permutations. (L~l, T~ij, X'0, X"0 are defined here.)
Chris's notations (#2519) are also shown in [ ].
Y'l (Ll^L~l) lies on Y'-axis, which is the orthopolar line of FF~ wrt quadrangle Q1Q2Q3Q4 and perpendicular to X' and X"-axis.
Note that three quadrigon F.X'i.X'j.Tij, Tij.Y'i.Y'j.T~ij, T~ij.X"i.X"j.F~ are similar.

<Quintic eq.>
Let f, m, p, q, r, s be real numbers.
Given 2 points
F (m - q/2, f - p/2) and
F~(q/2 - s, p/2 - r) in the xy-plane.
Let points X'i, Y'i, X"i, Y"i (i=1,2,3,4,5)
lie on X'-axis (y = -p/2),
Y'-axis (x = -q/2),
X"-axis (y = p/2),
Y"-axis (x = q/2), respectively.
If FX'i_|_X'iY'i_|_Y'iX"i_|_X"iY"i_|_Y"iF~,
then the slope of the line X'iY'i is the solution of the quintic equation
ft^5 + mt^4 + pt^3 + qt^2 + rt + s = 0.
(Allow me to use "ordinary" Cartesian-coordinate.)

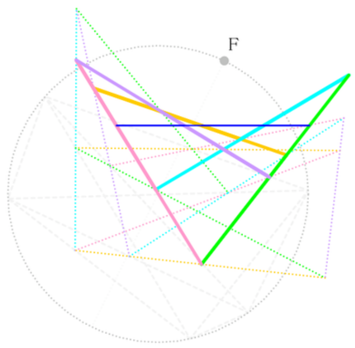
There are five points Q1, ..., Q5 on the circle FF~ such that line X'hY'h(X"hY"h) is Simson line of F(F~) wrt quadrangle QiQjQkQl,
where (i,j,k,l,h) = (1,2,3,4,5) or their permutations.
Then
X'h = Fijkl
Y"h = F~ijkl
X'-axis = Sim{12345} of F
Y"-axis = Sim{12345} of F~.
Y'hX"h is the orthopolar line of FF~ wrt quadrangle QiQjQkQl.
I'm not sure geometrical meaning of X"-axis and Y'-axis.

By the way, if we discuss beyond QL and quartic eq.,
it might be better to change the topic, like "nL and Lill's method" or something else.
Best regards,
M@IMF

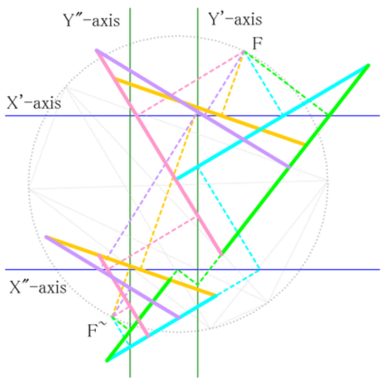
•Sim{ijk}s



•Sim{ijkl}s and Sim{12345}



•Lill's method for quintic eq.



<Simson line>

Given concyclic points F, Q_1, Q_2, Q_3, Q_4, Q_5,
Simson line of F wrt cyclic pentangle is inductively defined.
In the following, (i,j,k,l,h) = (1,2,3,4,5) or their permutations.

Let
  Sim{ij} = line Q_iQ_j
  F_{ij} = the foot of the perpendicular from F onto Sim{ij}.
There are 10 F_{ij}s.

n=3
  Sim{ijk} = line through F_{ij}, F_{ik}, F_{jk}
  F_{ijk} = the foot of the perpendicular from F onto Sim{ijk}.
Sim{ijk} is Simson line of F wrt triangle $Q_iQ_jQ_k$.
There are 10 Sim{ijk}s.
F_{ij} lies on Sim{ijk}, Sim{ijl}, Sim{ijh}.

n=4
  Sim{ijkl} = line through F_{ijk}, F_{ijl}, F_{ikl}, F_{jkl}
  F_{ijkl} = the foot of the perpendicular from F onto Sim{ijkl}.
Sim{ijkl} is Simson line of F wrt cyclic quadrangle $Q_iQ_jQ_kQ_l$.
There are 5 Sim{ijkl}s.
F_{ijk} lies on Sim{ijkl} and Sim{ijkh}.

n=5
  Sim{12345} = line through F_{1234}, F_{1235}, F_{1245}, F_{1345}, F_{2345}
  F_{12345} = the foot of the perpendicular from F onto Sim{12345}.
Sim{12345} is Simson line of F wrt cyclic pentangle $Q_1Q_2Q_3Q_4Q_5$.

<Quintic eq.>

Let f, m, p, q, r, s be real numbers. Given 2 points
F(m − q/2, f − p/2) and $F^{\sim}$(q/2 − s, p/2 − r) in the xy-plane.
Let points X'_i, Y'_i, X''_i, Y''_i (i=1,2,3,4,5) lie on X'-axis (y = −p/2),
Y'-axis (x = −q/2), X″-axis (y = p/2), Y″-axis (x = q/2), respectively.
If $FX'_i \perp X'_iY'_i \perp Y'_iX''_i \perp X''_iY''_i \perp Y''_iF^{\sim}$,
then the slope of the line $X'_iY'_i$ is the solution of the quintic equation

$$ft^5 + mt^4 + pt^3 + qt^2 + rt + s = 0.$$

Relationship to Simson line is as follows:
  Line $X'_hY'_h$ = Sim{ijkl} of F
  Line $X''_hY''_h$ = Sim{ijkl} of $F^{\sim}$
  $X'_h = F_{ijkl}$
  $Y''_h = F^{\sim}_{ijkl}$
  X'-axis = Sim{12345} of F
  Y″-axis = Sim{12345} of $F^{\sim}$.
$Y'_hX''_h$ is the orthopolar line of $FF^{\sim}$ wrt quadrangle $Q_iQ_jQ_kQ_l$.
I'm not sure geometrical meaning of X″-axis and Y'-axis.

Simson Line and Quintic Eq.pdf

Message: #2528
Date: 2024-11-06
From: contiwa.goma3@gmail.com
Subject: 5L and Lill’s method

Dear all,

This is a continuation of msg#2527(Re: Quartic Eq. and QL).

Let
   5L2 = 5L formed by X'Y's
   5L3 = 5L formed by Y'X"s
   5L4 = 5L formed by X"Y"s
   Oq = the midpoint of F and F~
Circle Oq = the circumcircle of pentangle Q1Q2Q3Q4Q5,
then
   5L2-n-P3 = the midpoint of F and Oq
   5L3-n-P3 = Oq
   5L4-n-P3 = the midpoint of F~ and Oq,
and the radii of 5L2-n-Ci1, 5L3-n-Ci1, 5L4-n-Ci1 are half the radius of Circle Oq.

<Quintic eq. (from #2527)>
Let f, m, p, q, r, s be real numbers.
Given 2 points
   F(m - q/2, f - p/2) and
   F~(q/2 - s, p/2 - r) in the xy-plane.
Let points X'i, Y'i, X"i, Y"i (i=1,2,3,4,5)
   lie on X'-axis (y = -p/2),
   Y'-axis (x = -q/2),
   X"-axis (y = p/2),
   Y"-axis (x = q/2), respectively.
If FX'i_|_X'iY'i_|_Y'iX"i_|_X"iY"i_|_Y"iF~,
then the slope of the line X'iY'i is the solution
of the quintic equation
    ft^5 + mt^4 + pt^3 + qt^2 + rt + s = 0.
(Lill's method)

Let QLh-P4" be the reflection of F in QL-P4 of QL{X'iY'i, X'jY'j, X'kY'k, X'lY'l},
where (i,j,k,l,h) = (1,2,3,4,5) or their permutations.
Then Qh is the reflection of F in QLh-P4".
See also #2493.

<Relationship to Chris's notation>
Rotate above configuration 180 degrees around the origin,
and displace so that F comes at the origin,
then we get
   A6 = (m - q + s, f - p + r)   [F~]

```
   A5 = (m - q, f - p + r)          [Y"0]
   A4 = (m - q, f - p)              [Y"-axis^X"-axis]
   A3 = (m, f - p)                  [X"-axis^Y'-axis]
   A2 = (m, f)                      [Y'-axis^X'-axis]
   A1 = (0, f)                      [X'0]
   A0 = (0, 0)                      [F].
(cf. #2519, #2520)

Best regards,
M@IMF
```

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Message: #2529
Date: 2024-11-15
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear all,

Some properties of the quadrangle Q1Q2Q3Q4 (#2493) are shown.
Recall that
QL1 = QL{L1,L2,L3,L4}
F = QL1-P1
X'i = Li^QL1-L3
Tij = Li^Lj
Mij = the midpoint of diagonal TijTkl
Ql = the antipode of F wrt the circumcircle of the component
triangle formed by {Li,Lj,Lk}.
QA0 = the cyclic quadrangle Q1Q2Q3Q4
F~ = the antipode of F wrt QL-Ci3" (the circumcircle of QA0),
where (i,j,k,l) = (1,2,3,4) or their permutations.

<Similarity>
Let Vl be the foot of the perpendicular from Ql onto FF~, and
let Vl' be the reflection of Vl in F.Ql.
Then the pentangles {F,Tjk,Tki,Tij,Vl'} and {F,Qi,Qj,Qk,Ql} are
similar.

Let Wlk be the foot of the perpendicular from Ql onto line QiQj.
Then the 3 pentangles {Mjk",Wil,Wjk,Wkj,Wli},
{Mki",Wik,Wjl,Wki,Wlj}, {Mij",Wij,Wji,Wkl,Wlk}
are homothetic with homothetic center QA0-P2 (QL-P5"),
where Mij" is the reflection of F in Mij.
Also they are inversely similar to the pentangle
{F,Qi,Qj,Qk,Ql}.

<CSC (QL1-Tf1)>
CSC(Qi) = X'i
CSC(F~) = X'0 (the foot of the perpendicular from F onto QL1-L3)
CSC(vertice of QL1-Tr1) = vertice of TR0 (the pedal triangle of
F wrt QA0-Tr1)
CSC(vertice of QA0-Tr1) = vertice of TR1 (the pedal triangle of
F wrt QL1-Tr1)
Note that TR0 and TR1 are inversely similar.

<Deltoid>
If the cubic eq. (f-r)t^3 + mt^2 + (p-2r)t + q = 0 has 3 real
solutions,
QL-Ci3" has an inscribed triangle whose Simson lines envelop
QL1-Qu2.
The triangle is constructed by ORIGAMI from

F(m, f - p/2), y = 2r - f - p/2
F~(q, p/2 - r), x = -q.
Li is the Simson line of Qi~ (the reflection of Qi in QA0-P3).
The quadrilateral formed by Simson lines of Q1,Q2,Q3,Q4 is the reflection of QL1~ in QA0-P2.

<ORIGAMI+C>
Let QL' be the quadrilateral formed by tangents of QL-Ci3" at the vertices of QA0.
Given QA0-P11 somehow, QA0-P13 is the midpoint of QA0-P11 and QA0-P3.
QL'-P1 is the second intersection point of QA0-Ci2 and line QA0-P1.P3 (QA0-L1).
Let l1 be the reflection of QA0-L1 in Steiner axis of QL1 (angle bisector between lines F.X'0 and FF~),
then QL'-L2 is the line through QA0-P11 perpendicular to l1.
(Note that the line through QA0-P3 parallel to l1 is QL'-L1, which is a tangent of QA0-Co1.)

Draw a Circle with center QA0-P3 and radius twice that of QL-Ci3".
Make creases that place QL'-P1 onto QL'-L2 and QA0-P3 onto the Circle simultaneously.
The points of tangency of the creases and the Circle are the vertices of QA0.
The line through Qj perpendicular to QL1-L1 and the line F.X'j intersect on QL-Ci3".

Best regards,
M@IMF

← Previous Next → ↔ Message Index ↑ Subjects

Message: #2530
Date: 2024-11-15
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

Dear M@IMF,

Thanks for all your beautiful new insights.
I find the subject very interesting.
At the moment, I’m quite busy, but I will respond in due course.

Chris

← Previous Next → ↔ Message Index ↑ Subjects

Message: #2531
Date: 2024-11-24
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Bernard and Eckart,

Attached a new version of my document on Hesse's Form.
This time I added a picture with 3 real preHessians and related content.
In my message #2467 I mentioned 3 challenges:
- Finding (a1,a2,a3,k) in terms of (c1,c2,c3,c4,c5,c6,c7,c8,c9,c0): See page 1 of the attached paper. The method of calculation is clear, but my computer consistently encounters memory issues after hours of processing.
- Finding (u, v, w) for the Cayleyan of a reference cubic in Hesse's form: See page 5 of the attached paper. Despite numerous attempts, I haven't been able to obtain a satisfactory solution. It's more complex than anticipated.
- Creating visualizations of the Hessians and pre-Hessians.
At least the 3rd challenge is fulfilled now.

Bernard you asked in message #2521
As you can imagine, I was particularly interested in the 2 first
1) for a general cubic, how identify H1, H2 and H3 and a1, a2, a3 and k? (you said, the method is clear, but could you please describe it shortly)
2) how find the barycentric equation of the cayleyan? (as function of a1, a2, a3 and k or k")
Did you find new ideas or developments or references?
I am afraid I have to be a bit technical.

1)
1. We have a general cubic
CU = c1 x^3 + c2 y^3 + c3 z^3 + c4 x^2 y + c5 x^2 z + c6 x y^2 + c7 y^2 z + c8 x z^2 + c9 y z^2 + c0 x y z
First we have to calculate the Hessian HE by calculating the determinant of the Hessian Matrix.
We find a curve of 3rd degree. Then I would like to find the 9 intersection points of CU and HE, which are the 9 Flexpoints.
I stopped Mathematica after a day computing.

2)
I start with a normalized cubic in Hesse's Form.
Then I do the usual steps for calculating the Cayleyan:
1. Determine 3 points (P1,P2,P3) on the reference cubic CU
2. Determine the Polar Conics (PCn1,PCn2,PCn3) of these 3 points wrt to CU

3. Determine of Polar Conics PCn1 and PCn2 their 4 intersection points (S12a,S12b,S12c,S12d).
4. Determine of the QA(S12a,S12b,S12c,S12d) its 6 sidelines (ab12,ac12,ad12,bc12,bd12,cd12).
5. Analogously determine of the QA(S13a,S13b,S13c,S13d) its 6 sidelines (ab13,ac13,ad13,bc13,bd13,cd13).
6. Now we have 2*6=12 sidelines enveloping the Cayleyan. These sidelines are tangent to the Caylean.
7. We know in this special case that the dual of the Caylean is a cubic.
Therefore QA-Tf10 of all tangents to the Caylean will be points on a subsidiary cubic CUX.
Therefore we determine of 9 of these QA-sidelines their QA-Tf10 point mapping. The mapped points will lie on CUX.
We check that QA-Tf10 of other corresponding sidelines also lie on CUX.
8. Then we know that every tangent to CUX will be transformed by QA-Tf10 into a point on the Cayleyan CAY.
Let QA-Tf10(CUXtg)=Pt={Ptx,Pty,Ptz}.
9. Now we solve from these equations (CUX(x,y,z)==0, X=Ptx(x,y,z), Y=Pty(x,y,z), Z=Ptz(x,y,z)) the points (X,Y,Z) satisfying these conditions.
The locus of these points (X,Y,Z) will be the Cayleyan.
Here it stops at step 9 (almost ready), because “No more memory available.”

I fear no progress will be made here until new ideas emerge.
Nonetheless, I hope you find the attached document informative.

Best regards,
Chris

A different frame of reference for a Cubic, Flexpoints, Hessian and Caylean

General form of a cubic equation

A cubic equation in barycentric coordinates can be expressed in general form as:

$$\mathbf{c_1 x^3 + c_2 y^3 + c_3 z^3 + c_4 x^2 y + c_5 x^2 z + c_6 x y^2 + c_7 y^2 z + c_8 x z^2 + c_9 y z^2 + c_0 x y z = 0}$$

where the coordinates refer to a reference triangle ABC with vertices (1,0,0), (0,1,0) and (0,0,1).

First normal form of a cubic equation

When a reference triangle is chosen, the corresponding reference points play a pivotal role. Each target point is defined by coordinates relative to these reference points. To express the same target point in relation to a second reference triangle, a point reference transformation is required. This transformation pTf can be formulated as follows.

Let target point X(x,y,z) be identified with respect to a reference triangle with vertices (1,0,0), (0,1,0) and (0,0,1). Let triangle P1(p1x,p1y,p1z), P2(p2x,p2y,p2z), P3(p3x,p3y,p3z) be the second reference triangle. Then the new coordinates of X with respect to triangle P1P2P3 will be defined by:

pTf (P1, P2, P3, X) =
(Det[(X, P2, P3)] (p1x+p1y+p1z),
Det[(P1, X, P3)] (p2x+p2y+p2z),
Det[(P1, P2, X)] (p2x+p2y+p2z)),

where Det is the determinant of the 3x3-matrix formed by the 3 sets of point coordinates indicated.
The point reference transformation applied for all points (x,y,z) on CU from reference triangle (1,0,0), (0,1,0) and (0,0,1) to reference triangle P1,P2,P3 will use pTf^{-1} for alle points on CU.

Let H1H2H3 be the real flex triangle $H_1H_2H_3$ of the reference cubic. This is the triangle bounded by the 3 real flexlines, each passing through one of the 3 real flexpoints.

When the reference triangle is changed to this flex triangle, the equation of the cubic simplifies to a form, here called the first normal form:

$$\mathbf{b_1 x^3 + b_2 y^3 + b_3 z^3 + b_0 x y z = 0}.$$

In order to connect to later developments the first normal form is rearranged to:

$$\mathbf{a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k a_1 a_2 a_3 x y z = 0}$$

So when we work with the real flex triangle $H_1H_2H_3$ as reference triangle then there are coefficients (a_1, a_2, a_3, k) such that:

$$\mathbf{CU = a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k a_1 a_2 a_3 x y z = 0}$$

Second normal form of a cubic equation

Another transformation can be applied mapping $(\mathbf{a_1 x}) \rightarrow x$, $(a_2 \mathbf{y}) \rightarrow y$, $(\mathbf{a_3 z}) \rightarrow z$, resulting in the second normal form:

$$\mathbf{x^3 + y^3 + z^3 + k x y z = 0}$$

This form is known as **Hesse's Form**.

Thus, after performing two transformations, the equation of a cubic in its general form is transformed (projectively equivalent) into Hesse's form.

For a proof of the existence of a projectively equivalent mapping from any smooth cubic in general form to Hesse's form see [1], page 2 and [2], page 4.

CU-Flex-NormalizedConfiguration-03.pdf

Flexpoints, Flexlines and Harmonic Polars in first normal form

After the projective transformation next coordinates/equations follow from consequent calculations.
The *9 flexpoints* are

$F_1 = (0, -a_3, a_2)$	real point
$F_2 = (-a_3, 0, a_1)$	real point
$F_3 = (-a_2, a_1, 0)$	real point
$F_4 = (0, -i_2\, a_3, a_2)$	imaginary point
$F_5 = (0, -i_1\, a_3, a_2)$	imaginary point
$F_6 = (-i_2\, a_3, 0, a_1)$	imaginary point
$F_7 = (-i_1\, a_3, 0, a_1)$	imaginary point
$F_8 = (-i_2\, a_2, a_1, 0)$	imaginary point
$F_9 = (-i_1\, a_2, a_1, 0)$	imaginary point

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
The coordinates of all flexpoints show that they all lie on one of the sidelines of the reference triangle.

The *12 Flexlines* are:

$L_{123} = (a_1, a_2, a_3)$	real Flexline $F_1F_2F_3$: $a_1\, x + a_2\, y + a_3\, z = 0$
$L_{145} = (1, 0, 0)$	real Flexline $F_1F_4F_5$: $x = 0$
$L_{267} = (0, 1, 0)$	real Flexline $F_2F_6F_8$: $y = 0$
$L_{389} = (0, 0, 1)$	real Flexline $F_3F_7F_9$: $z = 0$
$L_{179} = (i_2\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_7F_9$
$L_{168} = (i_1\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_6F_8$
$L_{258} = (a_1, i_2\, a_2, a_3)$	imaginary Flexline $F_2F_5F_8$
$L_{249} = (a_1, i_1\, a_2, a_3)$	imaginary Flexline $F_2F_4F_9$
$L_{346} = (a_1, a_2, i_2\, a_3)$	imaginary Flexline $F_3F_4F_6$
$L_{357} = (a_1, a_2, i_1\, a_3)$	imaginary Flexline $F_3F_5F_7$
$L_{478} = (a_1, i_1\, a_2, i_1\, a_3)$	imaginary Flexline $F_4F_7F_8$
$L_{569} = (a_1, i_2\, a_2, i_2\, a_3)$	imaginary Flexline $F_5F_6F_9$

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
Note that after the projective transformation indeed the triangle bounded by L_{145}, L_{267}, L_{389} is the new reference triangle with sidelines (1,0,0), (0,1,0), (0,0,1), and consequently with vertices H1(1,0,0), H2(0,1,0), H3(0,0,1) of the new reference triangle.

The *9 Harmonic Polars* are:

$L_1 = (0, a_2, -a_3)$	real line
$L_2 = (a_1, 0, -a_3)$	real line
$L_3 = (a_1, -a_2, 0)$	real line
$L_4 = (0, a_2, -i_2\, a_3)$	imaginary line
$L_5 = (0, a_2, -i_1\, a_3)$	imaginary line
$L_6 = (a_1, 0, -i_2\, a_3)$	imaginary line
$L_7 = (a_1, 0, -i_1\, a_3)$	imaginary line
$L_8 = (a_1, -i_2\, a_2, 0)$	imaginary line
$L_9 = (a_1, -i_1\, a_2, 0)$	imaginary line

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.

Harmonic Polar-Crosspoint P0
The 3 real Harmonic Polars are concurrent in **P0 ($1/a_1$, $1/a_2$, $1/a_3$).**
P0 is the trilinear pole wrt the reference triangle $H_1H_2H_3$. of the real Flexline $F_1F_2F_3$: L_{123}.
Conversely L_{123} is the trilinear polar of P0 wrt the reference triangle $H_1H_2H_3$.

The Hessian of CU
We know:

$$\mathbf{CU = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + k\ a_1\ a_2\ a_3\ x\ y\ z = 0}$$

After calculation the Hessian HE of CU appears as:

$$\mathbf{HE = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + k'\ a_1\ a_2\ a_3\ x\ y\ z = 0}$$

where $k' = -(108 + k^3) / 3$

The Syzygetic Pencil
All Cubics passing through the 9 Flexpoints of CU are called cubics of the Syzygetic Pencil.
CU and its Hessian mutually intersect in the 9 Flexpoints of CU. Therefore they are called members of the Syzygetic Pencil.
The 3 sidelines of the real flex triangle $H_1H_2H_3$ (being the 3 real flexlines resp. through F_1, F_2, F_3) form together also a degenerate cubic, which will be called here RF. On these 3 lines all 9 Flexpoints occur. So it is also a member of the Syzygetic Pencil.
Since it is a pencil every Cubic in it can be described as $t\ CU_1 + (1\text{-}t)\ CU_2$, where CU_1 and CU_2 are also members of the pencil.
We now ask the question of finding a cubic FE as linear combination of CU and HE such that RF is its hessian.
Specifically we need to find CUx = Hessian of t CU + (1-t) HE.
It appears that there are 9 values of t that provide a solution.
However, upon substituting these values of t, it turns out that there are only 2 distinct solutions.

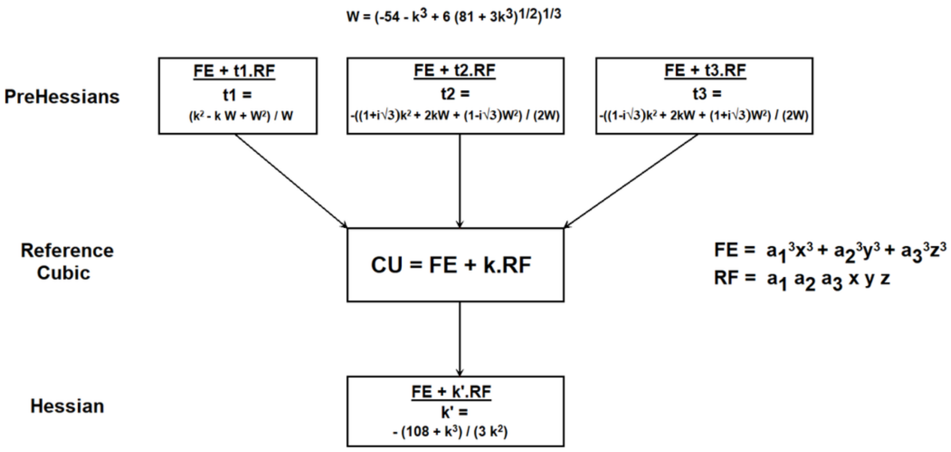
$$\mathbf{FE = a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3}$$

and

$$\mathbf{RF = a_1\ a_2\ a_3\ x\ y\ z.}$$

FE is the main part and will be called here the *Core Curve*, RF is the degenerate part and will be called here the *Residual Curve*.
RF is actual the degenerate cubic consisting of the three sidelines (x=0, y=0, z=0) of the reference triangle, which are the three real Flexlines, which indeed contain the 9 Flexpoints.
Moreover FE and RF are derived from the pencil t CU + (1-t) HE and therefore both pass through the 9 Flexpoints and consequently are members of the Syzygetic Pencil.
It also appears that FE and RF share the same Hessian, namely RF.

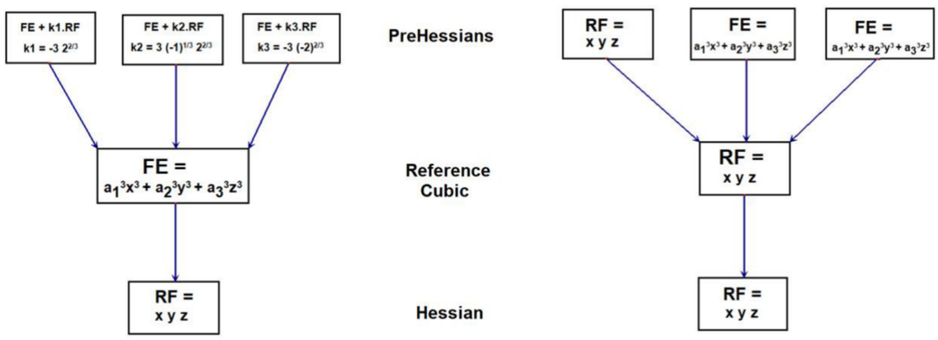
Hessians and PreHessians
Since a Hessian is also a cubic, it is also possible to calculate the Hessian of the Hessian, and further downwards.
Moreover it is possible, knowing some cubic CU, to find out for which upper cubic CU is the Hessian.
There are 3 of these cubics called the PreHessians pHE_1, pHE_2, pHE_3.
So per definition the Hessian of all three cubics pHE_1, pHE_2, pHE_3 will be CU.
So “upwards” there are 3 PreHessians of a cubic and “downwards” there is 1 Hessian per cubic.



CU-3Cu1 CU-HE-pHE1-pHE2-pHE3 picture-03.fig

From the equations of t1, t2, t3 it follows that there will be three real preHessians when k <= -3. When k > -3, there will be one real preHessian and two imaginary preHessians.

Examples:
The 3 PreHessians of FE are also of the form FE + k.RF. Its Hessian is RF.
The 3 PreHessians of RF are RF itself and FE (counting twice). Its Hessian is again RF.



CU-3Cu1 CU-HE-pHE1-pHE2-pHE3 picture-02.fig

The Cayleyan

Let CU = $\mathbf{a_1^3\ x^3 + a_2^3\ y^3 + a_3^3\ z^3 + k\ a_1\ a_2\ a_3\ x\ y\ z = 0}$

Numerical analyses show that the Cayleyan of CU has this expression:
CAY =

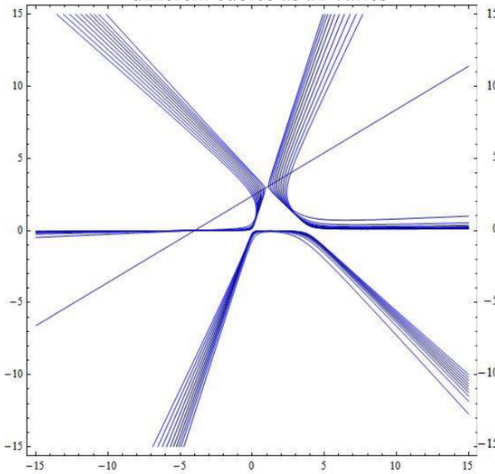
$$\begin{aligned}
&\mathbf{a_1^6\ x^6 + a_2^6\ y^6 + a_3^6\ z^6}\\
&\mathbf{+\ u\ (a_1^3\ a_2^3\ x^3\ y^3 + a_1^3\ a_3^3\ x^3\ z^3 + a_2^3\ a_3^3\ y^3\ z^3)}\\
&\mathbf{+\ v\ (a_1^4\ a_2\ a_3\ x^4\ y\ z + a_1\ a_2^4\ a_3\ x\ y^4\ z + a_1\ a_2\ a_3^4\ x\ y\ z^4)}\\
&\mathbf{+\ w\ (a_1^2\ a_2^2\ a_3^2\ x^2\ y^2\ z^2)}
\end{aligned}$$

u, v, w are fixed expressions probably have variables a_1, a_2, a_3, and k in it.
It still isn't clear which expressions they have.

Pictures of CU = $a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k\, a_1 a_2 a_3\, xyz$

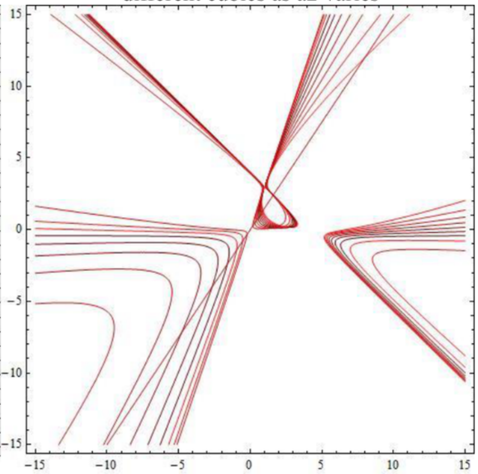
for different values of a1, a2, a3, k

different cubics as a1 varies



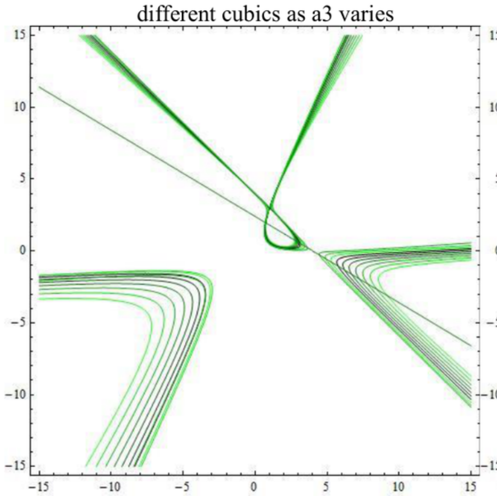
a1 = -10 -> 0, a2 = -2, a3 = -4, k = 100
when a1=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a2 varies



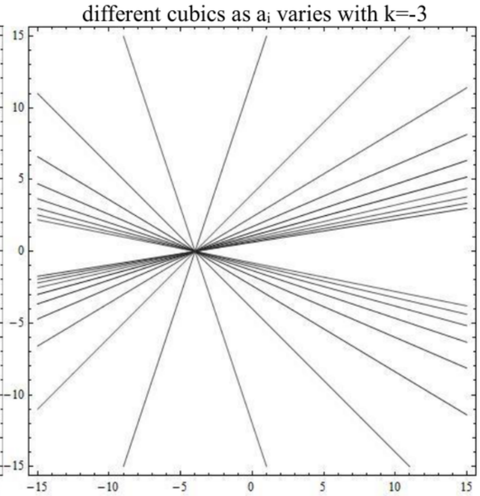
a1 = 1, a2 = -10 -> 0, a3 = -4, k = 100
when a2=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a3 varies



a1 = 1, a2 = -2, a3 = -10 -> 0, k = 100
when a1=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a_i varies with k=-3



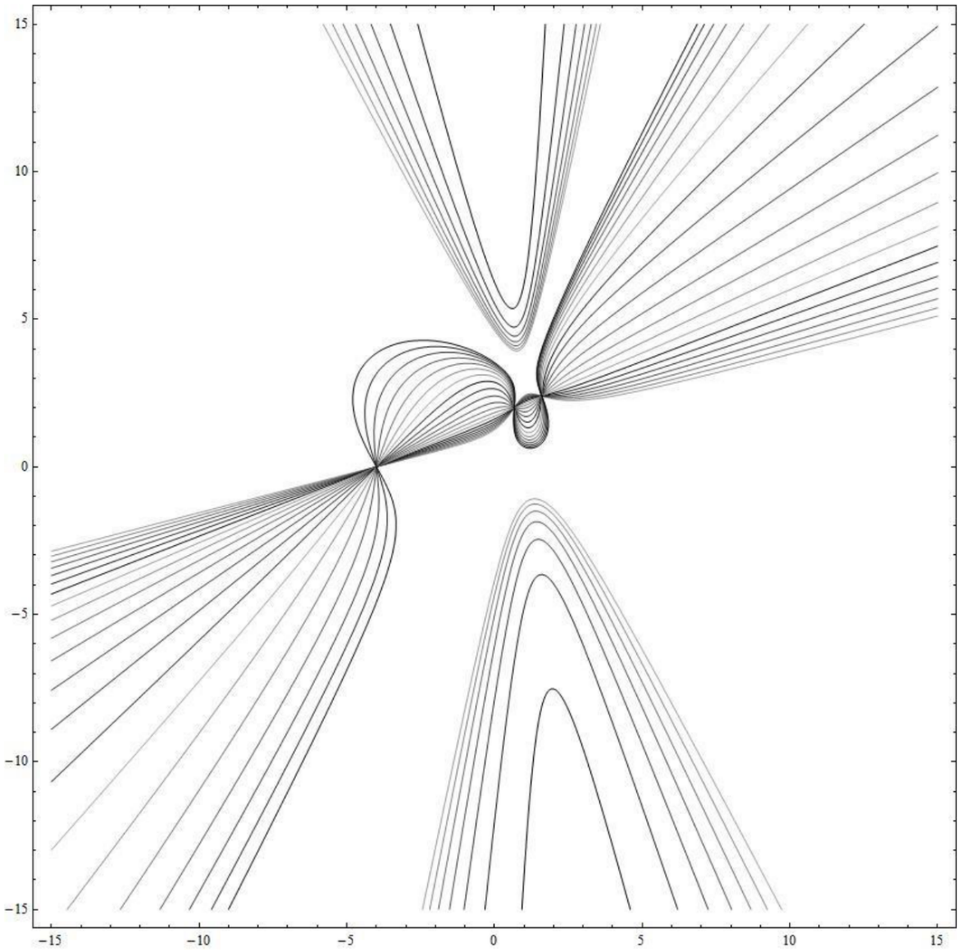
a1 = -10 -> 7, a2 = -2, a3 =-4, k = -3
all cubics degenerate in lines when k = -3
for all values of (a1,a2,a3)

CU-a1is-10-to-0 a2is-2 a3is-4 kis100.jpg
CU-a1is1 a2is-2 a3is-10-to-0 kis100

CU-a1is1 a2is-10-to-0 a3is-4 kis100.jpg
CU-a1is-10-to-0 a2is-2 a3is-4 kis-3.jpg

CU-Flex-NormalizedConfiguration-03.pdf

different cubics as variable k varies

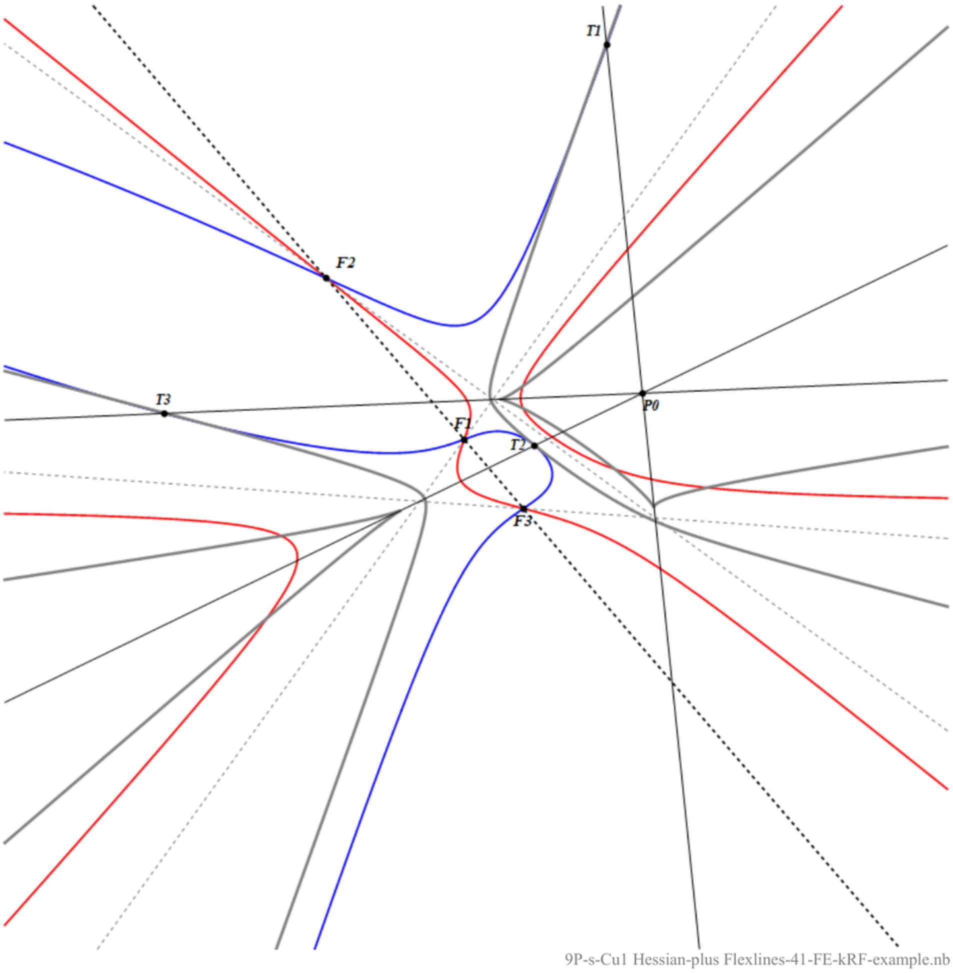


CU-a1is1 a2is-2 a3is-4 kis-10to10.jpg

a1 = 1, a2 = -2, a3 = -10 -> 0, k = -10 -> 10
all cubics pass through the 3 real Flexpoints
when k = -3 the cubic degenerates in a real line and an (imaginary) conic

CU-Flex-NormalizedConfiguration-03.pdf

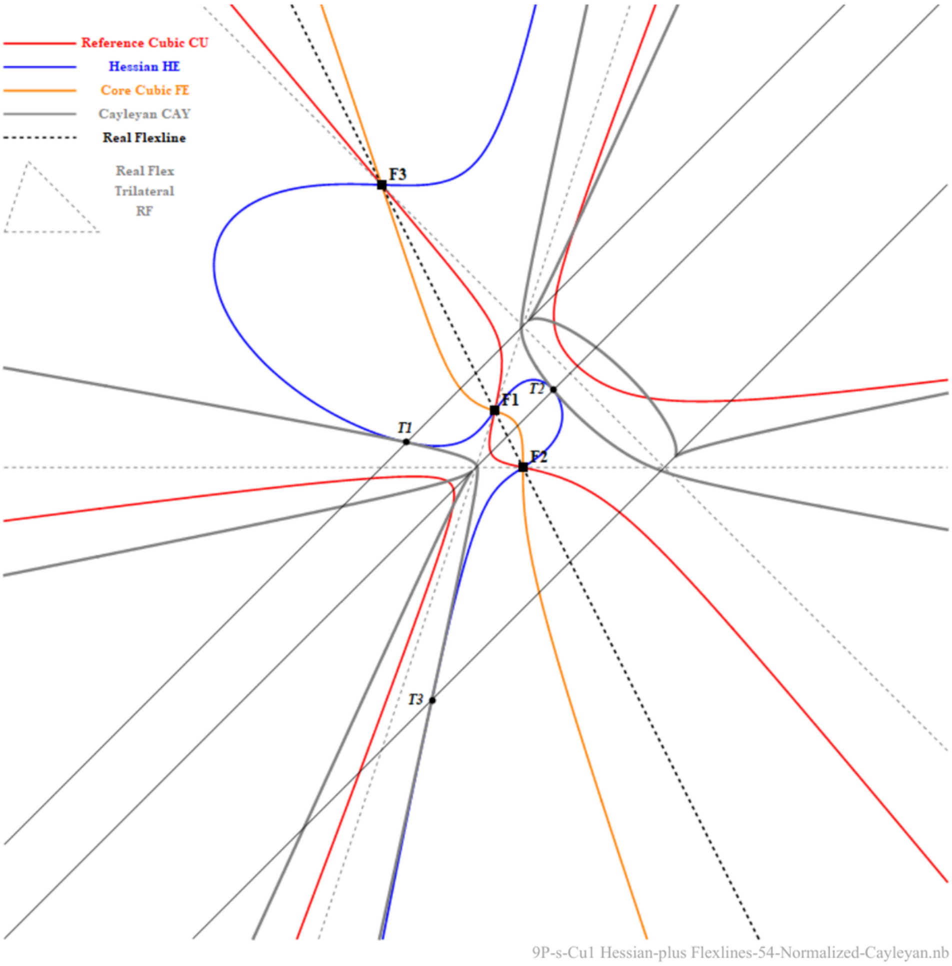
Picture of CU / HE / CAY



- Reference Cubic CU is red
- Hessian HE is blue
- Cayleyan CAY is gray
- F1, F2, F3 are the real Flexpoints
- The 1 gray dotted line is the real Flexline F1F2F3
- The 3 light gray dotted lines are the real Flexlines
- The 3 black lines are the 3 Fi-Harmonic Polars
- T1, T2, T3 are intersection points of corresponding Fi-Tangent and Fi-Harmonic Polar
- P0 is the common intersection point of the 3 real Fi-Harmonic Polars

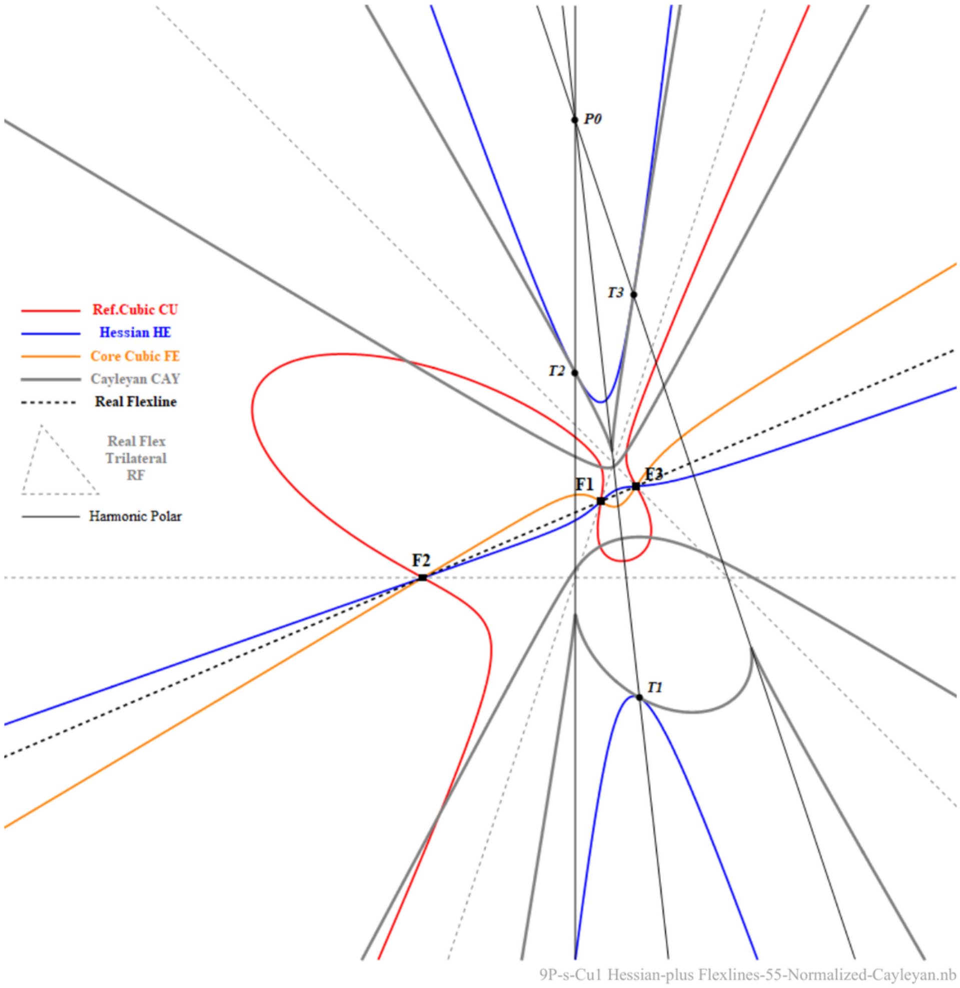
CU-Flex-NormalizedConfiguration-03.pdf

Picture-1 of CU / HE / FE / RF / CAY



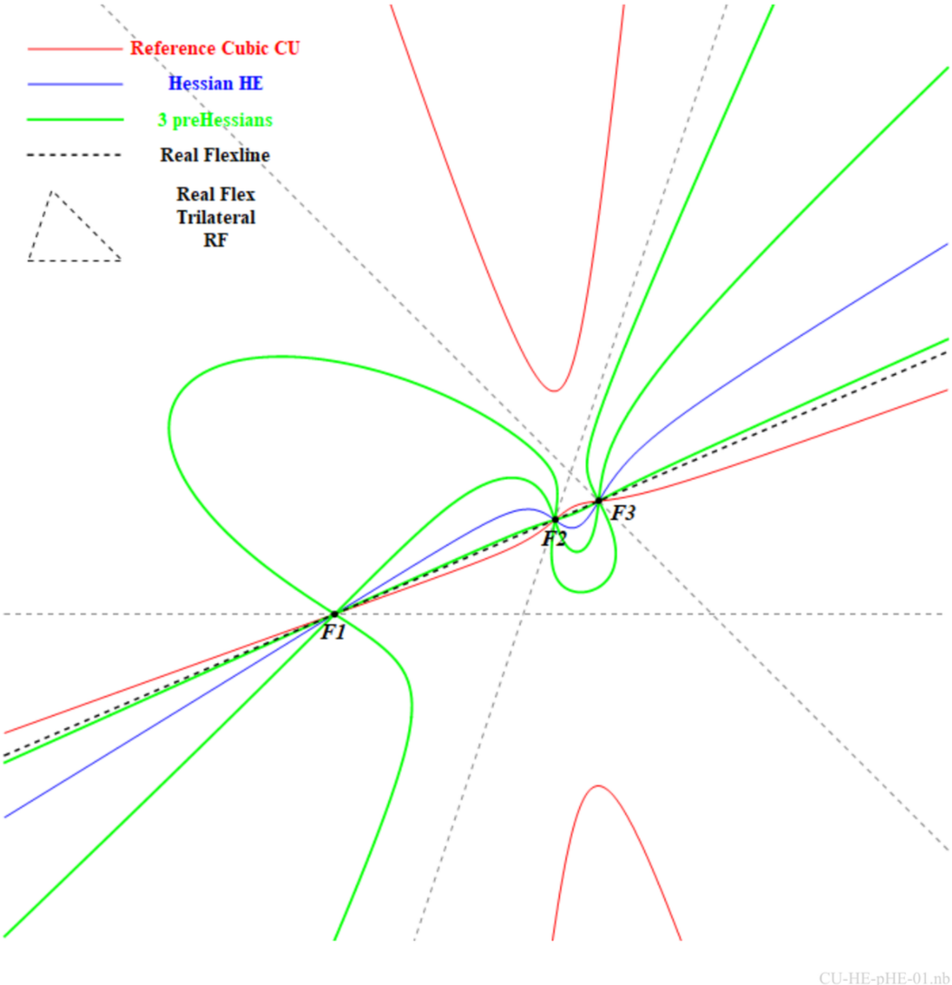
CU-Flex-NormalizedConfiguration-03.pdf

Picture-2 of CU / HE / FE / RF / CAY



CU-Flex-NormalizedConfiguration-03.pdf

Picture of CU / HE / pHE1, pHE2, pHE3 (preHessians)



CU-Flex-NormalizedConfiguration-03.pdf

Note 1.
A "primitive cube root of unity" or "primitive third root of 1" is a complex number that, when raised to the power of 3, equals 1, and it is not equal to 1 itself.
There are exactly three cube roots of unity:
1. 1
2. i1 = -1/2 + i sqrt[3]/2
3. i2 = $i1^2$ = -1/2 - i sqrt[3]/2

Among these, i1 and i2=i1^2 are called primitive cube roots of unity because they generate all cube roots of unity when raised to successive powers, and they themselves are not equal to 1.
The primitive cube roots of unity have the following properties:
- $i1^3$ = 1
- i1 <> 1 and i2 <>1
- 1 + i1 + i2 = 0

Geometrically, they are the points on the complex plane that form the vertices of an equilateral triangle inscribed in the unit circle, with one vertex at 1.

References:
[1] Araceli Bonifant and John Milnor - On Real and Complex Cubic Curves
Available at: https://arxiv.org/pdf/1603.09018

[2] Michela Artebani and Igor Dolgachev - The Hesse Pencil of Plane Cubic Curves
Available at: https://arxiv.org/abs/math/0611590

CU-Flex-NormalizedConfiguration-03.pdf

Message: #2532
Date: 2024-11-29
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear all,

Let's solve the quartic equation in other ways.
(This is a continuation of msg#2517.)

<QA and Quartic eq.>
Let x = 2ft, then solving the quartic equation
   ft^4 + mt^3 + pt^2 + qt + r = 0 is equivalent to
solving the two equations:
   x^2 = 4fy   (1)
   y^2 + (m/2f)xy + py + (q/2)x + rf = (K/4f)(x^2 - 4fy) (2)
where K is a real number.

Let ti (i=1,2,3,4) be the solutions of the quartic equation,
then intersection points of (1) and (2) are Zi(fti, fti^2).
((1) and (2) are circumconics of quadrangle Z1Z2Z3Z4.)
QL1 is formed by the tangents of (1) at Zis.
Using
    y' = y + (m/4f)x, K' = K + (m^2)/4f,
    p' = p - (m^2)/4f, q' = q - (m/2f)p',
(2) becomes
    y'^2 + p'y' + (q'/2)x + rf = (K'/4f)(x^2 - 4fy).   (3)
When K' = 0,
    this is QL1-Co3,
    otherwise it is an ellipse (K'<0) or a hyperbola (K'>0):
    [y' + (K' + p')/2]^2 - (K'/4f)[x - (fq'/K' - m/2)]^2
    = Φ(K')/4K',   (4)
    where Φ(K') =
    K'^3 + [2p' - (m^2)/4f]K'^2 + (p'^2 - 4rf + mq')K' - fq'^2.
Denote uij = (ti + tj - tk - tl)/4,
   then Kij' = 4f(uij)^2 and Φ(Kij') = 0,
where (i,j,k,l) = (1,2,3,4) or their permutations.
When K' is K'ij, (4) becomes two lines:
    y' + (Kij' + p')/2 +/- uij[x - (fq'/Kij' - m/2)] = 0,
which are ZiZj and ZkZl. Their intersection point is
    Sij(fq'/Kij' - m/2, -(Kij + p' + mq'/2Kij')/2).
Zi and Zj are obtained as intersection points of ZiZj
and (1) by solving a quadratic equation.
Similary Zk and Zl.

Reciprocal relationship wrt x^2 = 4fy (QL1-Co1):
   pole   Zi   Tij = Li^Lj   Sij = ZiZj^ZkZl
   polar   Li   ZiZj   SjkSki = TijTkl

<QL-inconic and ORIGAMI+C>
The polar curve of (4) wrt QL1-Co1 is
the envelope of polars of points of (4) wrt QL1-Co1:
(1/fK')(x + m/2)^2 + (4K'/Φ(K'))[y - (K'+p')/2
- (q'/2K' - m/4f)(x + m/2)]^2 = 1. (5)
This is an inconic of QL1, so we can construct QL1
as the common tangents of this curve and QL1-Co1 by "ORIGAMI+C."

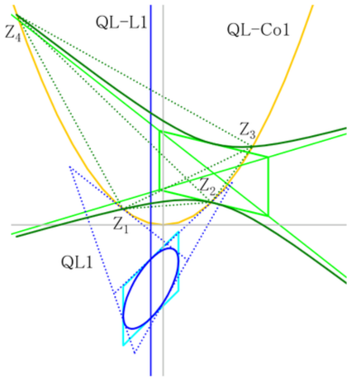
1. Since the foci of an inconic of QL are mutual CSCs, they can be constructed from the inconic center and QL-2P3 (msg#2509).
2. Let F1 and F2 be the foci, and F1' be the reflection of F1 in a tangent of the inconic.
3. Draw a Circle with center F2 passing through F1' (with a Compass).
4. Make creases that place QL1-P1 onto QL1-L2 and F1 onto the Circle simultaneously.

If the inconic has a vertical tangent,
it is x = -m/2 +/- (fK')^(1/2),
otherwise we can use a tangent y =
(q'/2K' - m/4f)x + (K + p' + mq'/2K')/2 +/- [Φ(K')/4K']^(1/2).

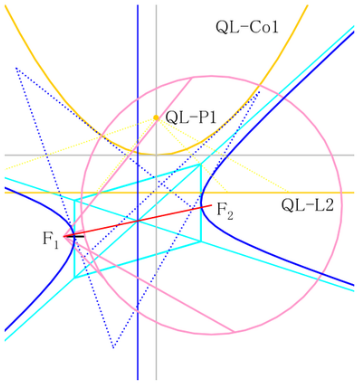
References:
A.Kasem, F.Ghourabi, T.Ida, "Origami axioms and circle extension."
Y.Nishimura, "Axioms of geometric constructions by origami with compass (in Japanese)."
I learned ORIGAMI by Japanese translation of R.Geretschlaeger's book, "Geometric Constructions in Origami."

Best regards,
M@IMF

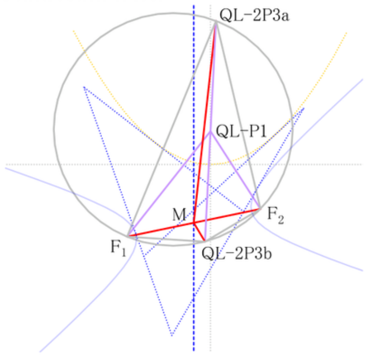
・QA-circumconic and QL-inconic



・ORIGAMI+C



・Construction of foci



<QA-circumconic>

Let Z_1,Z_2,Z_3,Z_4 be the intersection points of two conics:

$x^2 = 4fy$ (1)

$y^2 + (m/2f)xy + py + (q/2)x + rf = 0,$ (2)

then a circumconic of the quadrangle $Z_1Z_2Z_3Z_4$ (QA1) is

$y^2 + (m/2f)xy + py + (q/2)x + rf = (K/4f)(x^2 - 4fy),$ (3)

where K is a real number.

Using

$K' = K + m^2/4f,\ p' = p - m^2/4f,\ q' = q - (m/2f)p',$

$y' = y + (m/4f)x,$

$\Phi(K') = K'^3 + (2p' - m^2/4f)K'^2 + (p'^2 - 4rf + mq')K' - fq'^2,$

(3) becomes

$[y' + (K' + p')/2]^2 - (K'/4f)[x - (fq'/K' - m/2)]^2$
$= \Phi(K')/4K'.$ (4)

<QL-inconic>

Let Co1, Co2 be conics. The polar curve of Co2 wrt Co1 is the envelope of polars of points of Co2 wrt Co1.
This is another conic Co3, and Co2 is the polar curve of Co3 wrt Co1. The polar curve of (4) wrt (1) is

$(4K'/\Phi(K'))[y - (K'+p')/2 - (q'/2K' - m/4f)(x + m/2)]^2$
$+ (1/fK')(x + m/2)^2 = 1.$ (5)

Let QL1 be a quadrilateral formed by the tangents of (1) at the vertices of QA1, then (1) is QL1-Co1 and (5) is an inconic of QL1. When K'=0, (4) becomes QL1-Co3 and (5) becomes QL1-Co2.

<ORIGAMI+C>

We can construct QL1 as the common tangents of (5) and (1) by "ORIGAMI+C."

1. Construct the foci of (5) from the inconic center M and QL-2P3. (See below.)
2. Let F_1 and F_2 be the foci, and F_1' be the reflection of F_1 in a tangent of (5).
3. Draw a Circle with center F_2 passing through F_1'.
4. Make creases that place QL1-P1 onto QL1-L2 and F_1 onto the Circle simultaneously.

If (5) has a vertical tangent, it is $x = -m/2 \pm \sqrt{(fK')}$,
otherwise we can use a tangent

$y = (q'/2K' - m/4f)x + (K + p' + mq'/2K')/2$
$\pm \sqrt{(\Phi(K')/4K')}.$

<Construction of foci>

Since the foci of an inconic of QL are mutual CSCs,

$\triangle$QL1-P1.F_1.QL-2P3a/b ∽ $\triangle$QL1-P1.QL-2P3a/b.F_2,

and the quadrangle {F_1,QL-2P3a,F_2,QL-2P3b} is a harmonic quadrangle. So F_1 is the point T such that

$\triangle$M.QL-2P3a.T ∽ $\triangle$M.T.QL-2P3b.

F_2 is the reflection of T in M.

QA-circumconic:	QL-Co1	ellipse	QL-Co3	s-hyperbola	t-hyperbola
QL-inconic:	QL-Co1	t-hyperbola	QL-Co2	s-hyperbola	ellipse

("S-hyperbola" has two vertical tangents, while "t-hyperbola" has none.)

QL-Inconic and ORIGAMIplusC.pdf

Message: #2533
Date: 2024-12-07
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

perhaps can you confirm my conjecture
... for the real flex trilateral H1H2H3 of a bipartite cubic
... with flexpoints F1,F2,F3, their harmonic polars L1,L2,L3
... with common point Po and intersections P1,P2,P3
    with the infinite part of the cubic:
Let Fia and Fib be the intersections of FiPo and the closed part
of the cubic,
... let Pia and Pib be the intersections of PiPo
    and the closed part of the cubic.

These 12 points can be named and ordered on the closed part of
the cubic
... as follows:P1a,F2a,P3b,F1b,P2a,F3a,P1b,F2b,P3a,F1a,P2b,F3b,
... this can be done in two versions, I cannot give a decision,
... but one gives 4-times intersections as follows:
... H1 = P2a.F2a ^ P3a.F3b ^ P2b.F1a ^ P3b.F1b on L1,
... H2 = P3a.F3a ^ P1a.F1b ^ P1b.F2b ^ P3b.F2a on L2,
... H3 = P1a.F1a ^ P2a.F2b ^ P1b.F3a ^ P2b.F3b on L3,
... which will be the vertices of the real flex trilateral,
... for I made several drawings and "proved" with the control
    in #2489
... without significant difference.

Best regards Eckart

PS.  There is short time for me for geometry, I have to care for
my wife.
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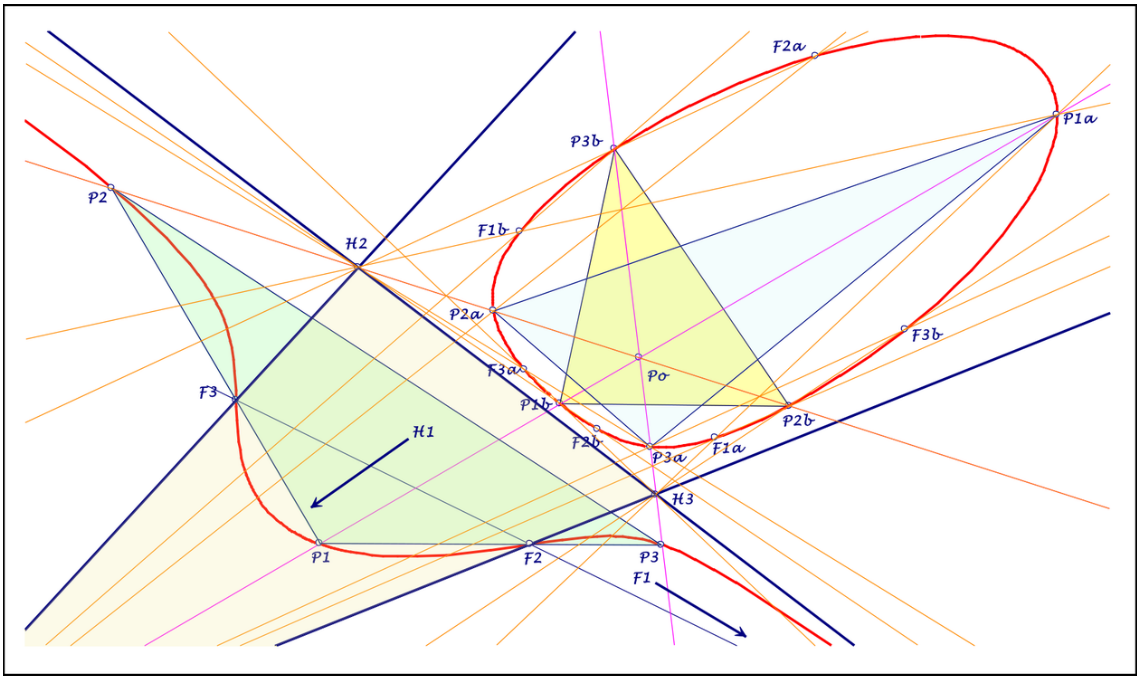

2024-12-07.pdf

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Message: #2534
Date: 2024-12-09
From: van10hoven@gmail.com
Subject: SEIICHI KIRIKAMI

Dear friends,

Message from Antreas Hatzipolakis
Antreas shared this in Euclid:

I learned with great sadness that my Japanese friend, Seiichi Kirikami, passed away on 11/12/2023.
Seiichi contributed valuable geometric problems and theorems to various groups (Hyacinthos, Quadri-Figures, Romantics of Geometry) and to ETC.
Geometry is the poorer of his death.
May his memory be eternal.

My Thoughts
Seiichi was a wonderful friend.
In 2012, we first got in touch about the Encyclopedia of Quadri-Figures , which had just been introduced. Between 2012 and 2018, he made many meaningful contributions to it.

Seiichi had a way of seeing things from a slightly different angle and presenting them in the simplest terms. That was his gift. He was always humble, kind, and eager to contribute.

I introduced him to Antreas and the forum Anopolis (which later became Euclid).
I still remember what he told me when he retired in November 2013. His words were so simple:
"Yesterday was the last day in the office. Near its end, I gave some of my data on the PC to people for whom I thought they would be useful. After notifying my retirement by email, I erased all of my data and emails. So today is the start of my new life. But I feel nothing has changed now."

Over time, his interests grew broader. As far as I know, he contributed to other forums too.

I will always remember him as a very pleasant person with wonderful ideas and a great sense of simplicity. May he rest in peace.

Chris van Tienhoven

Message: #2535
Date: 2024-12-10
From: unidentifiedlethargicorganism@gmail.com
Subject: QA-L9, QA-Cu1, coaxal circles

Dear Geometers,

Are the following results known in EQF / QFG / QPG?

Let P1P2P3P4 be a quadrangle, and S1S2S3 be its diagonal triangle, where S1=P1P2∩P3P4, S2=P1P3∩P2P4, S3=P1P4∩P2P3.
Let L be a line parallel to QA-L9 (= perpendicular bisector of Euler-Poncelet point and isogonal center).
Let Q1=P1P2∩L, Q1'=P3P4∩L, Q2=P1P3∩L, Q2'=P2P4∩L, Q3=P1P4∩L, Q3'=P2P3∩L.
Let (O1), (O2), (O3) be the circumcircles of S1Q1Q1', S2Q2Q2', S3Q3Q3', resp.

Then,
1) (O1), (O2), (O3) are coaxal. Let Lr be the radical axis, and Lc be the line through the centers of (O1), (O2), (O3). Let Q=Lr∩Lc (this point is often called "radical trace" of (coaxal) circles).
2) (O1), (O2), (O3) are tangent to QA-Cu1 at their respective vertices of the diagonal triangle, S1, S2, S3.
3) The real intersection points of (O1), (O2), (O3) lie on QA-Cu1. Conversely, QA-Cu1 is the locus of the real intersection points of (O1), (O2), (O3) when L moves.
4) Lr passes through a fixed point T, which is the real intersection point of QA-L9 and QA-Cu1. T is also QA-Tf1(QA-L9).
5) Lc is tangent to a fixed parabola. In other words, the envelope of Lc when L moves is a parabola. Here, we denote this parabola QA-CoX.
6) QA-CoX is tangent to QA-L2.
7) The locus of Q when L moves is a rational circular cubic (a circular cubic with a unique real singularity). Here, we denote this cubic QA-CuX.
One of the well-known geometric interpretations of a rational circular cubic is that it is the pedal curve of a parabola. In this case, Lr passes through a fixed point T, and Lc is tangent to a fixed parabola QA-CoX, and Lr and Lc are orthogonal, and Q is their intersection point. Therefore, the locus of Q is a rational circular cubic (= a pedal of QA-CoX with pole T).
For more information, see https://mathcurve.com/courbes2d.gb/cubiccirculairerationnelle/cubiccirculairerationnelle.shtml.

8) QA-CuX passes through QA-P6 (= midpoint of Euler-Poncelet point and isogonal center), T (= QA-Tf1(QA-L9)), circular points at infinity. Note that QA-CuX can be acnodal (T can be an isolated point).
9) QA-CuX is tangent to QA-CoX, although a tangency point is not necessarily unique.
10) QA-CuX degenerates to a circle (or a line) when P1P2P3P4 is concyclic.
11) The real asymptote of QA-CuX is parallel to the real asymptote of QA-Cu1 (which is also parallel to QA-L4 = QA-P1-P6 line).

These results are based on my experiments and observations on GeoGebra, and not confirmed by computation. I am sorry if there are any mistakes.
(Generalized) Cayley-Bacharach theorem might be useful to prove these results.

Original problem on RG
https://www.facebook.com/groups/parmenides52/permalink/8882497791863848/
on AoPS
https://artofproblemsolving.com/community/c6h3458745_quadrangle_coaxal_circles_locus

Sincerely,
Keita Miyamoto

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Message: #2536
Date: 2024-12-10
From: bernard.keizer@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,
I find always your imagination very amazing!
Unfortunately, I cannot confirm your construction.
As you know, I'm able to draw with Geogebra cubics wrt a certain triangle H1H2H3 and choosen coefficients a1, a2, a3 and k.
Your construction doesn't give the vertices H1, H2 and H3.
I fear it is again an ''almost but not quite'' construction, as Chris put it.
Personnally, I don't think there is such a simple construction and the elements H1, H2, H3 as well as a1, a2, a3 and k have to be calculated.
Naturally, I would be glad to be wrong ...
Best regards
Bernard
PS If you give me the coordinates of H1, H2, H3, F1, F2 and F3 (and naturally P0, which follows), I could try to reproduce your cubic and it's hessian by variing k ...

Message: #2537
Date: 2024-12-11
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Bernard, dear Chris,

I think, Bernard is right, that #2536
... is not  a construction of H1,H2,H3.

After starting with a bipartite cubic,
... constructing my points H1,H2,H3,
... measuring the coordinates of one cubic point,
... calculating k and looking for points with the same k,
... there appeared significant aberrations.

But what about my constructed 4-times intersections H1,H2,H3?

Wrt my last message #2533, perhaps of interest:
The 12 intersections Pia,Pib,Fia,Fib (i=1,2,3)
... of Li and FiPo with the closed part of a bipartite cubic,
... nominated in the order of #2533, are part of a lot of conics
... through two flexpoints and corresponding harmonic poles,
    example attached:
... F1,F2,P1,P2,F1a,F2b and F1,F2,P1,P2,F1b,F2a and
F1,F2,P1,P2,F3a,F3b,
... F1,F2,P1,P2,P1a,P2a and F1,F2,P1,P2,P1b,P2b,
... F1,F2,P1,P2 with CU-contact point P3a or P3b
... and finally F1,F2,P1,P2,H1,H2.
All points Y1,Y2 are coconic with F1,F2,P1,P2,
... if Y1 on L1, Y2 on L2 and F3 on Y1Y2, as H1 and H2.
The centers of these conics lie on QA-Co1 of F1F2P1P2,
... we get three such conics QA-Co1
    with three common intersections,
... what about these triangle (attached),
... also to be found in the monopartite case.
The  vertices of this triangle lie coconic with P1,P2,P3,Po.

Best regards Eckart
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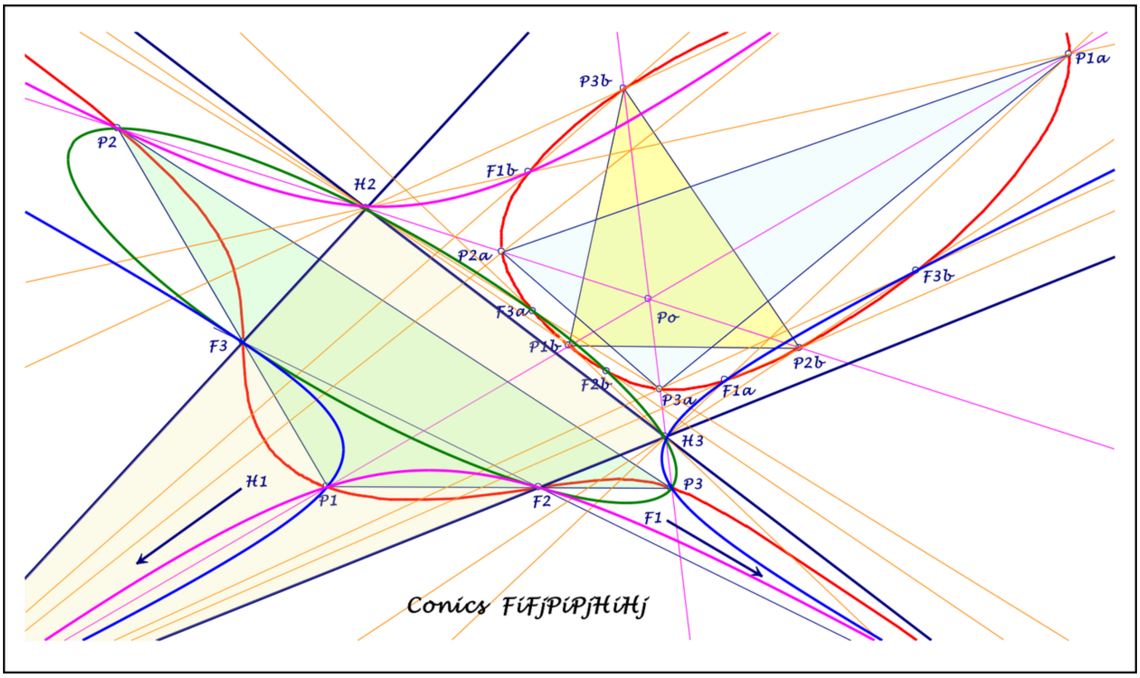

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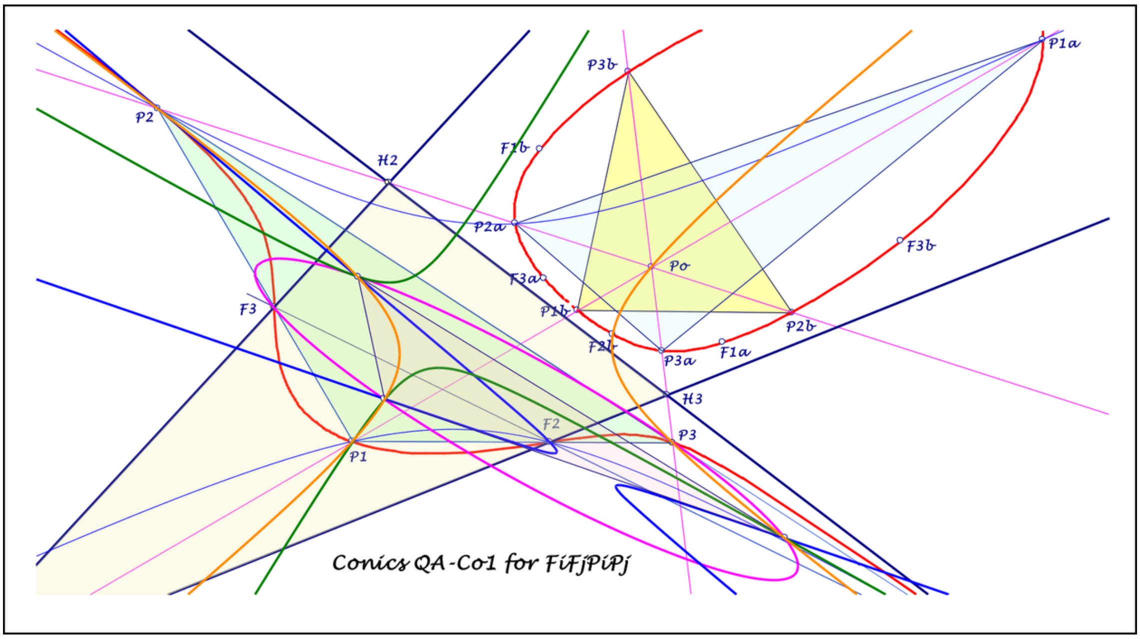


2024-12-11b.pdf

Message: #2538
Date: 2024-12-11
From: van10hoven@gmail.com
Subject: Re: Real flexline trilateral

Dear Eckart,

I admire your perseverance.

Once again, you’ve come up with a beautiful construction that made me think: this could be it.

I set up a scenario with a numerical example and a bipartite cubic in Mathematica. Just preparing this took quite some time. Following your instructions, finding and seeing the set of four converging lines to Hi was very impressive. Please see the attached picture.

However, when testing this numerical example, I encountered a drawback: all four lines were "almost, but not quite" converging. Initially, I thought this might still be due to precision errors—after all, we are working with numerous calculations involving cubic equations.

But the general approach should have confirmed the result. I tested every possible combination of lines (Fjx, Pky) in the general approach to see if they would pass through H1, H2, or H3. Unfortunately, I couldn’t find any collinearity among Hi, Fjx, and Pky.

To ensure you’re comfortable accepting this result, I searched for an example where the four lines didn’t converge to Hi—and I found one. Please see the second attached picture.

This is all I was able to deduce.

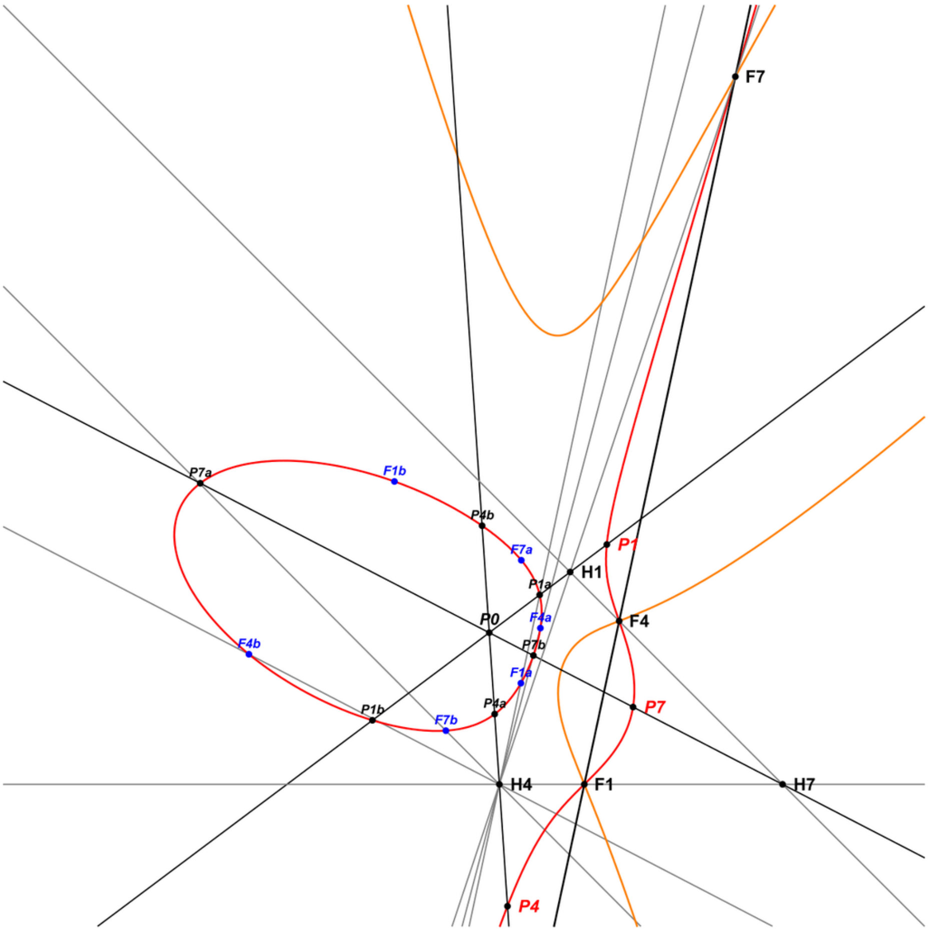
It makes me wonder: how is it that we encounter so many "almost, but not quite" solutions?

Are all the roads leading to Rome, or are they merely leading near to Rome?

Best regards,
Chris

Tryout finding points H1, H2, H3 with a bipartite cubic

picture 1



Note: The real Flexpoints are F1, F4 and F7

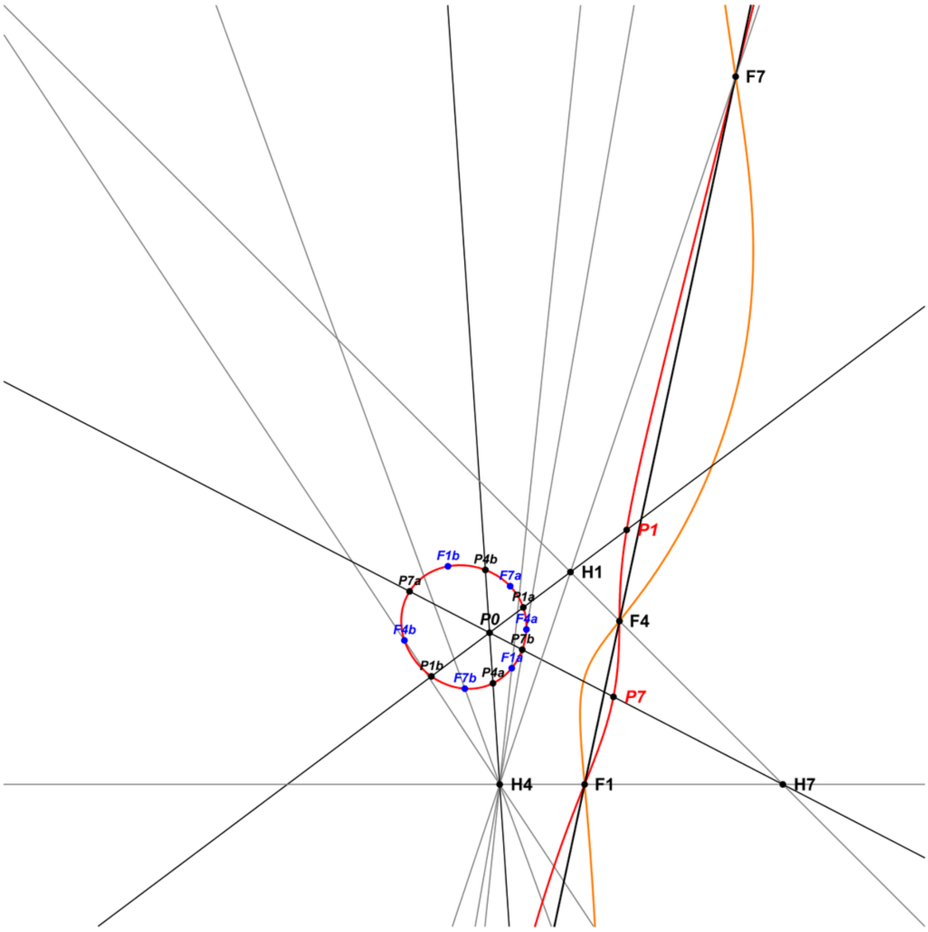
Reference cubic is red, Hessian is orange.

Chris van Tienhoven 2024, december 11

CU-HE Tryout finding points H1-H2-H3 - 01.pdf

Tryout finding points H1, H2, H3 with a bipartite cubic

picture 2



Note: The real Flexpoints are F1, F4 and F7

Reference cubic is red, Hessian is orange.

Chris van Tienhoven

2024, december 11

CU-HE Tryout finding points H1-H2-H3 - 01.pdf

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Message: #2539
Date: 2024-12-12
From: bernard.keizer@gmail.com
Subject: 2 dual Hesse pencils

Dear Chris, dear Eckart
Chris asked recently for new ideas.
Here are some!
In particular, I found the barycentric equation of the cayleyan and of a curve Γ mentionned in Artebani ...
(I hope Chris will be able to confirm these expressions)
This allows me to draw these curves with Geogebra.
All this activity takes definitively (too) much time, but gives great pleasure!
I hope this Christmas gift will please you and give you new ideas
Merry Christmas and Happy New Year!
Best regards
Bernard

1. 2 dual Hesse pencils wrt a triangle H1H2H3

The 1rst pencil contains the cubics having the same 9 flexes (3 real and 6 imaginary). The real flexes are F1, F2 and F3

The 2nd pencil contains the curves of the 3rd order tangent to the 9 harmonic polars (3 real) The harmonic polars are L1, L2 and L3 ; they intersect in P0

2. 4 curves

2 curves belong to the 1rst pencil : the cubic CU and it's hessian HE

2 curves belong to the 2nd pencil the cayleyan CA and a curve Γ associated to the cubic CU

CA is the hessian of Γ and HE is it's cayleyan (in other words, CU and Γ swap their hessian and cayleyan)

3. Calculations

The barycentric coordinates of P0 are 1/a1, 1/a2 and 1/a3 and the barycentric equation of the line of the 3 real flexes is a1x + a2y + a3z = 0.

All the curves of both pencils have an equation of the form $x^3 + y^3 + z^3 + kxyz = 0$, considering it as a barycentric equation for the 1rst pencil and replacing x, y and z by a1x, a2y and a3z and as a tangential equation for the 2nd pencil and replacing x, y and z by u/a1, v/a2 and w/a3.

In particular, if the cubic CU has the constant k, then HE has the constant $k' = -(6/k)^2 - k/3$, CA has the constant $k'' = 6/k - (k/3)^2$ and Γ has the constant $k^* = -18/k$.

It's obvious that $k'(k^*) = k''(k)$ and $k''(k^*) = k'(k)$, which explains the swapping of HE and CA for the curves CU and Γ.

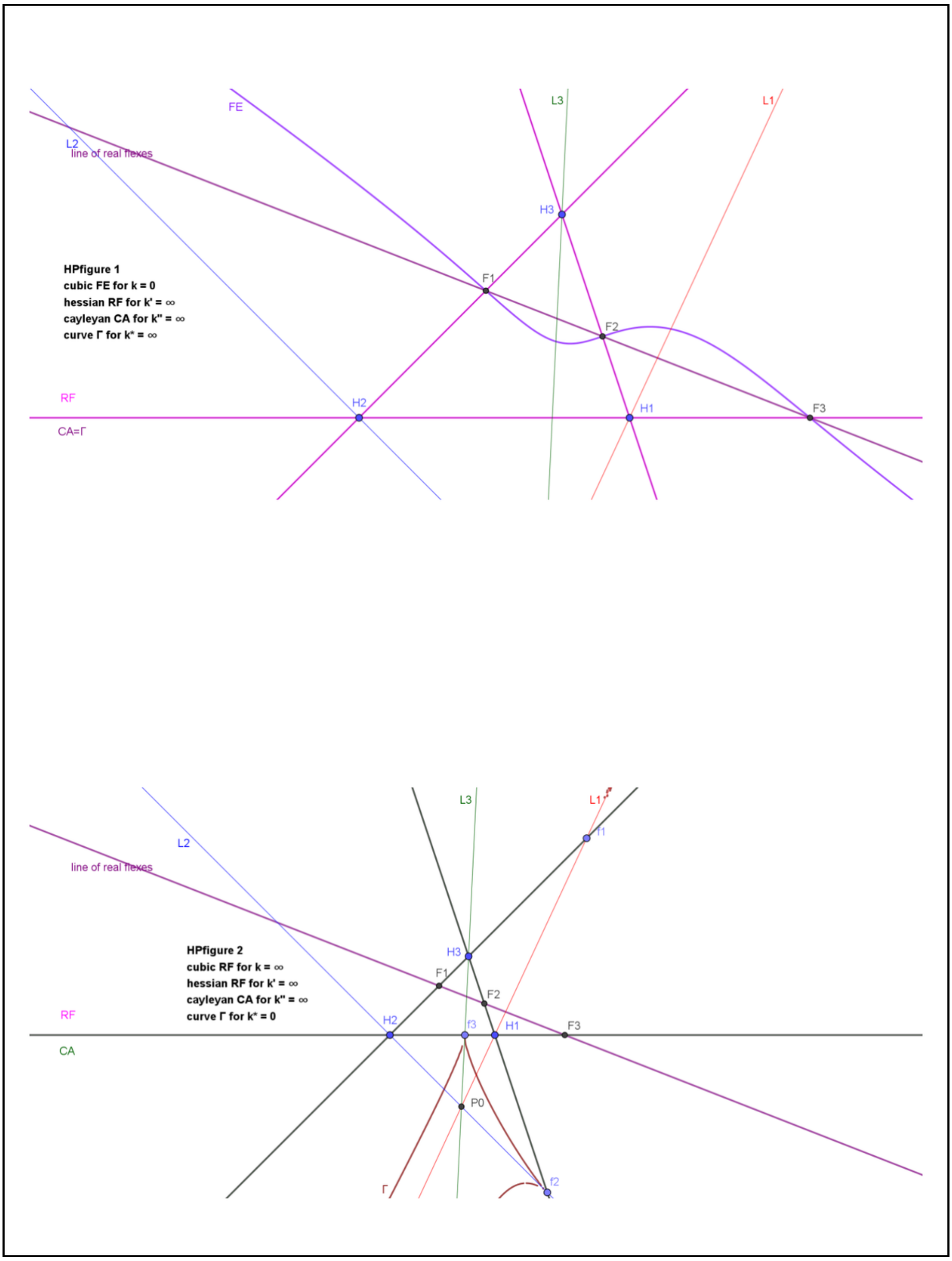
It's possible to find the tangential equation of a curve defined by a barycentric equation and vice-versa, the calculation being exactly the same. We find equations of the 6th degree[1].

In particular, the barycentric equation of CA is $27 (x^3 + y^3 + z^3)^2 - 4 (27 + k''^3) (x^3y^3 + x^3z^3 + x^3y^3) - 18k''^2 xyz (x^3 + y^3 + z^3) - k''(108 + k''^3) x^2y^2z^2 = 0$ and the same for Γ replacing k'' by k*.

4. Properties

Drawing this way a certain number of curves, it appears that the cusps of the curves Γ are the

2 dual Hesse pencils.pdf



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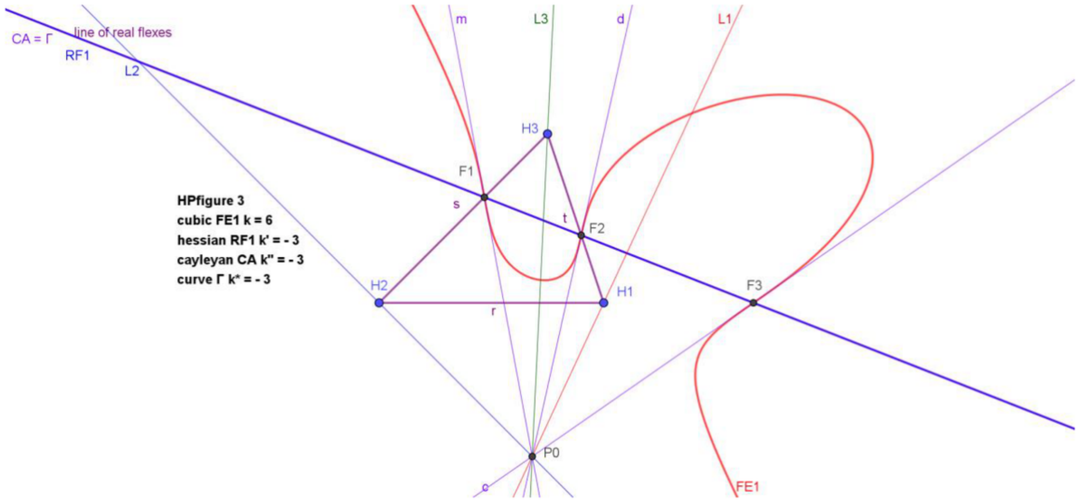

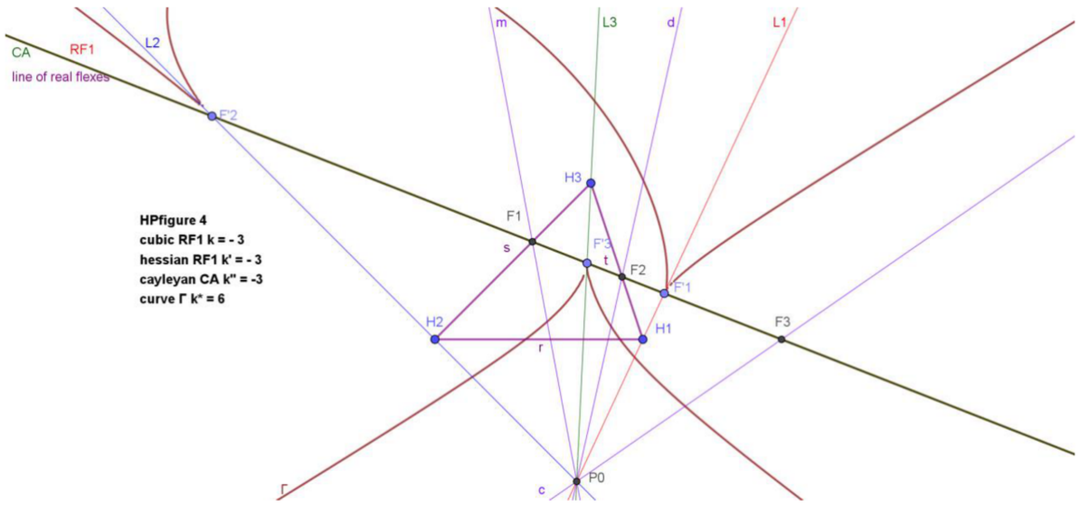

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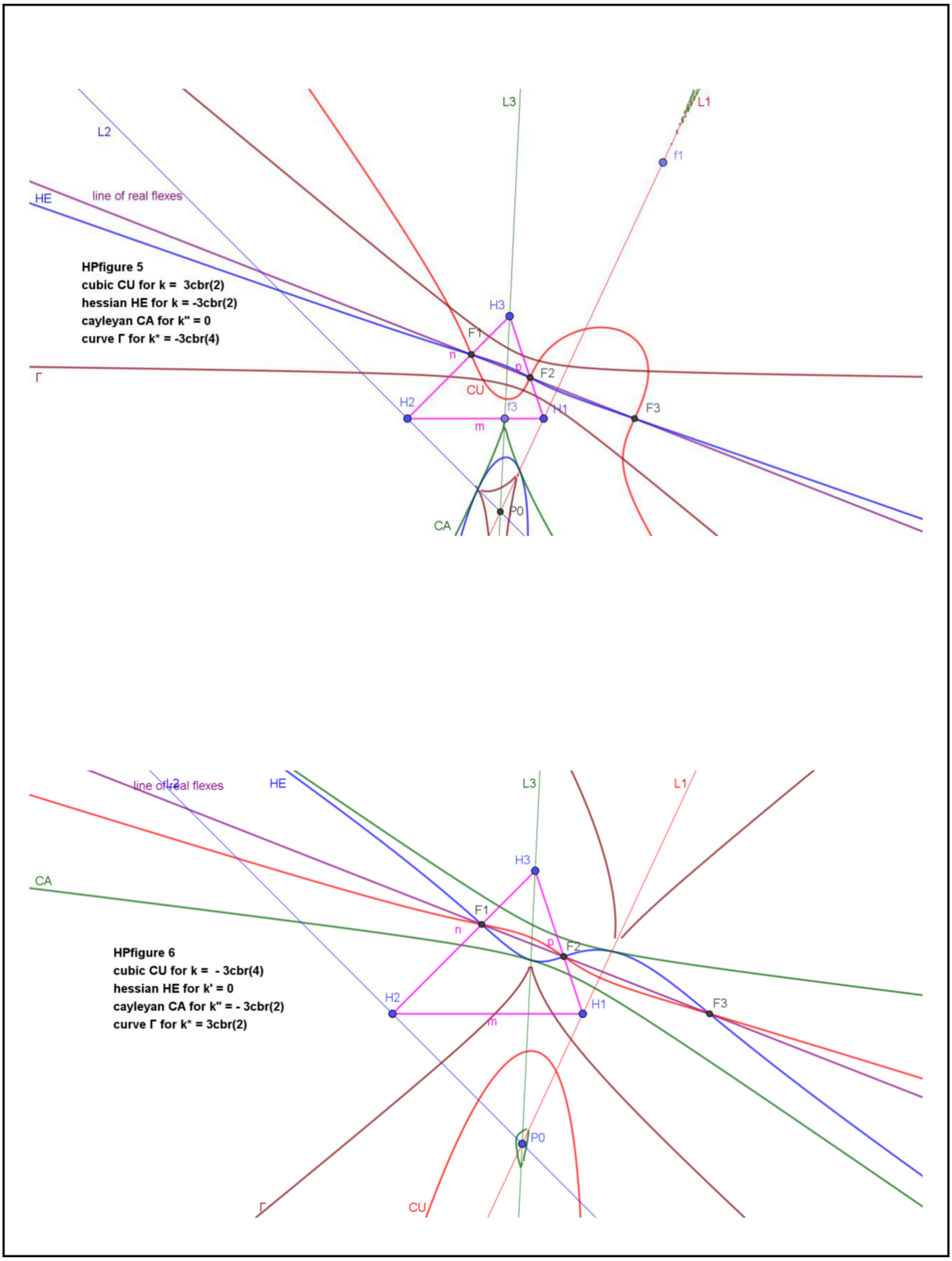

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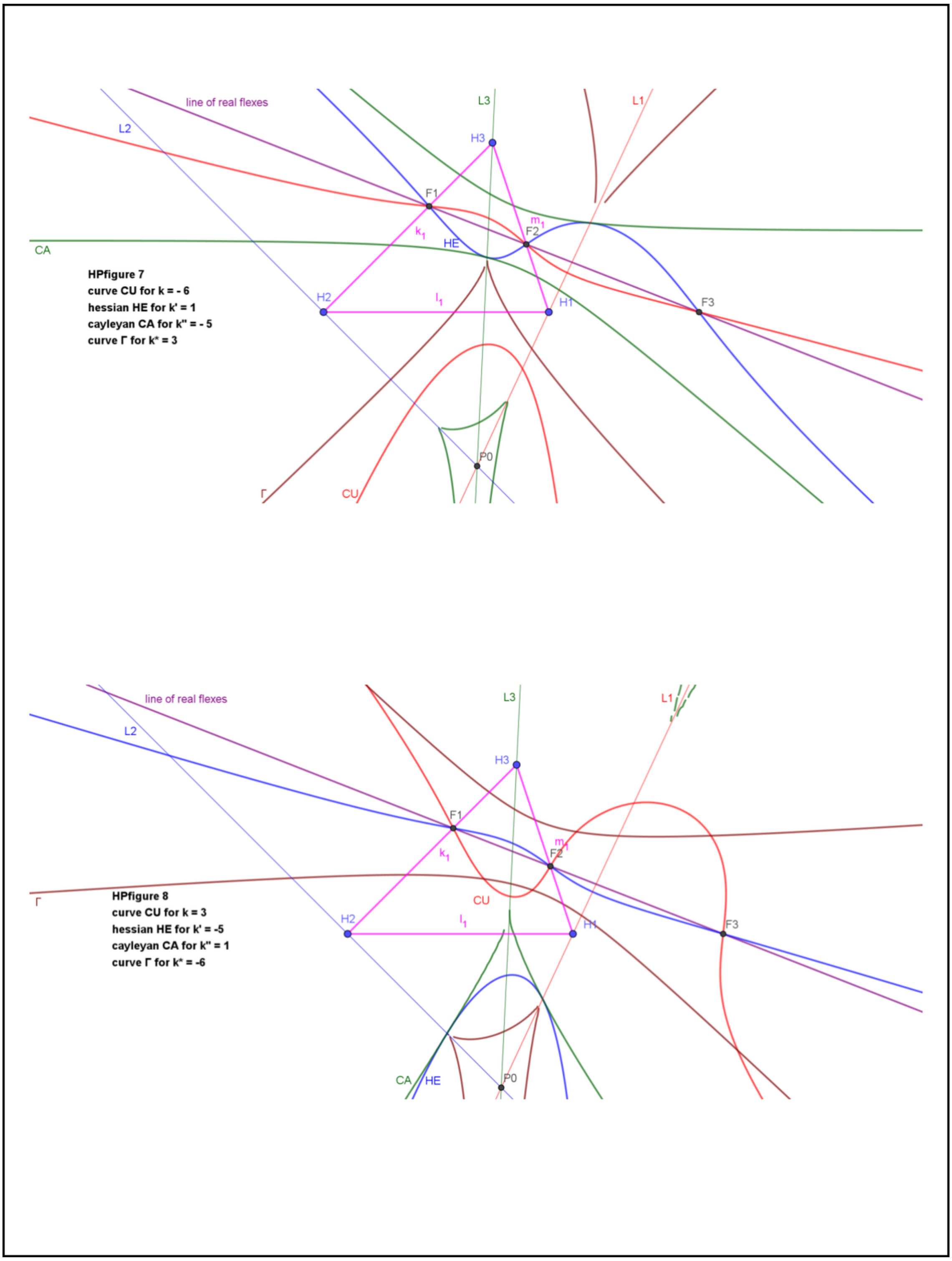

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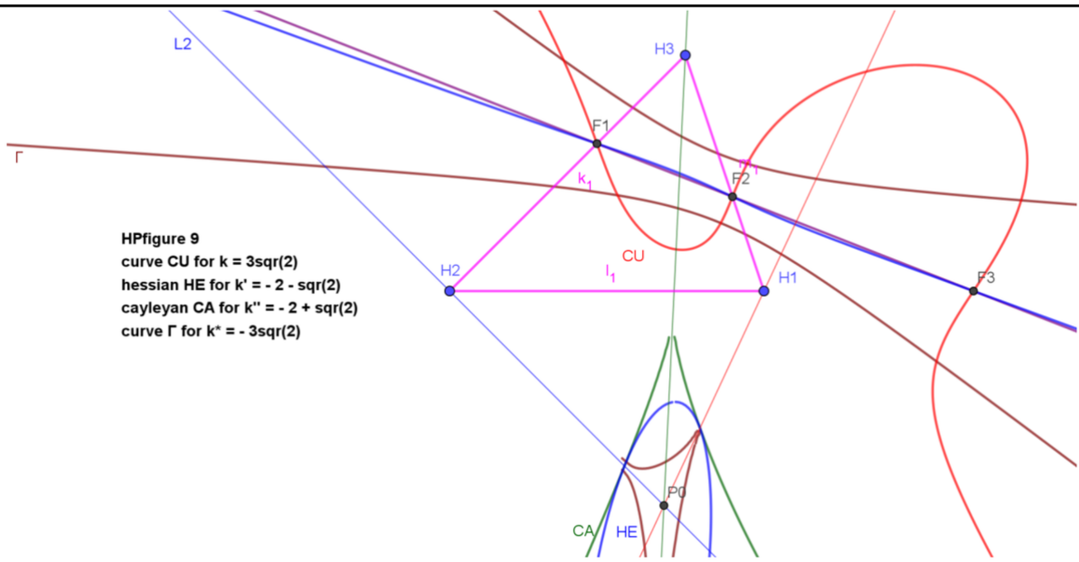

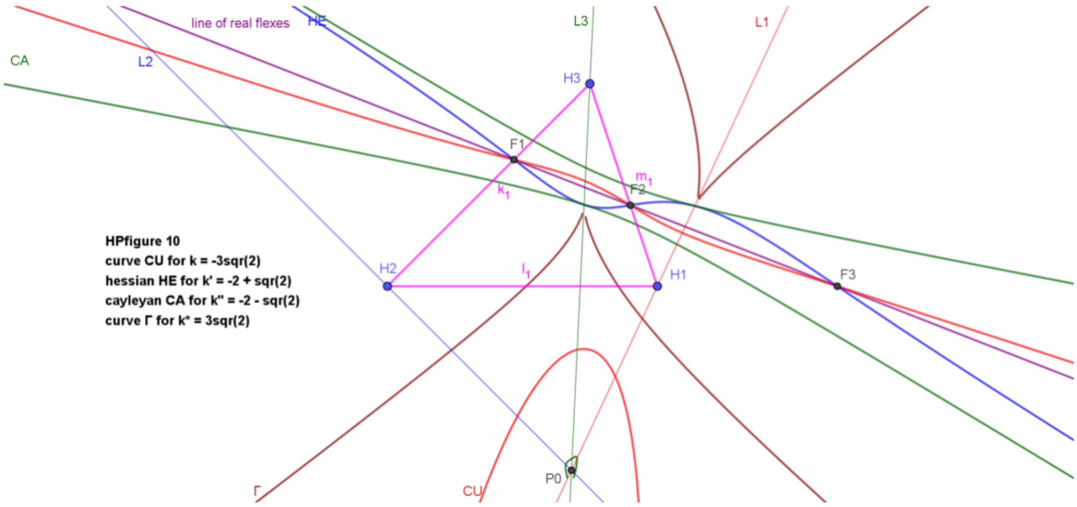

1 The calculations are long and boring !

Considering the tangential equation G(u,v,w) of the curve and the tangent ux + vy + wz = 0, we know that G'u/x = G'v/y = G'w/z and must eliminate u, v and w in order to find the barycentric equation F(x,y,z) of the curve.

2 dual Hesse pencils.pdf

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Message: #2540
Date: 2024-12-12
From: van10hoven@gmail.com
Subject: Re: QA-L9, QA-Cu1, coaxal circles

Dear Keita,

The circular cubic QA-Cux you describe, which passes through QA-P6 and the intersection point QA-Cu1 ∩ QA-L9, is entirely new to me.
Moreover, you mentioned that its asymptote is parallel to both QA-L9 and the asymptote of QA-Cu1.
It is a beautiful cubic to look at.
Additionally, the property that a rational circular cubic is the pedal curve of a parabola is also new to me.
Thank you for your contribution.

Best regards,
Chris

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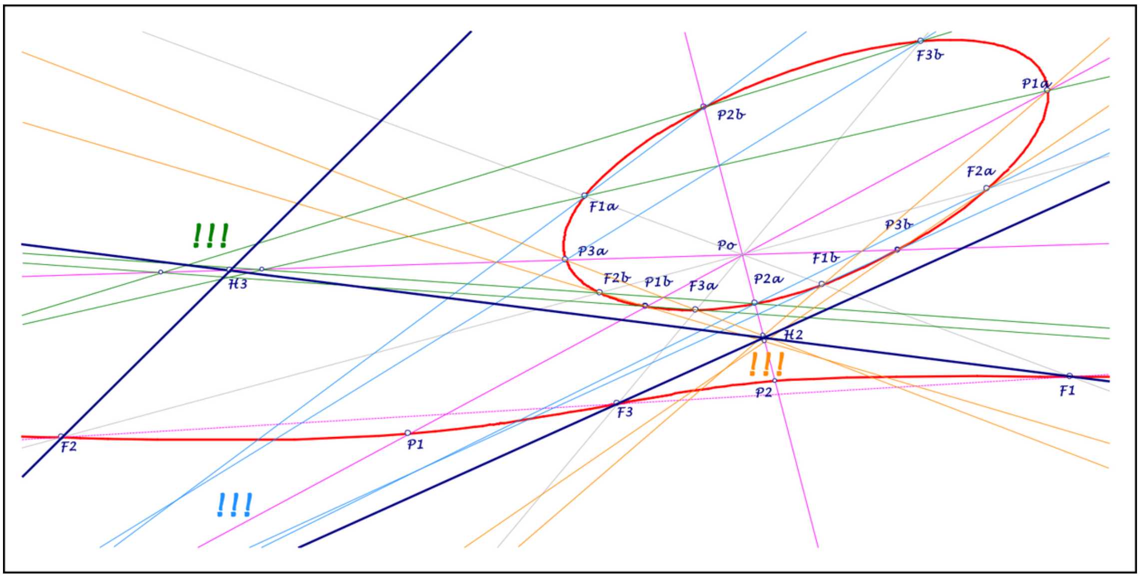
Message: #2541
Date: 2024-12-12
From: eckart_schmidt@t-online.de
Subject: Re: Real flexline trilateral

```
Dear Chris,

thanks for interest and drawing,
... but I don't understand your nomination.
So I made a similar figure,
... which shows significant,
... that my "4-times intersections" are already not correct,
... they are two intersections on the harmonic polar,
... more explicit in H1 not in the drawing.
Excuse again my false conjecture!

Best regards Eckart

PS. Many thanks for your private sympathy, a new stage in my
life.
```



2024-12-12.pdf

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Message: #2542
Date: 2024-12-13
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

```
Dear M@IMF ,

Thank you for all your messages.
I've noticed that I'm having some difficulty keeping track of
everything.
Could you kindly summarize your recent messages in 3-6
sentences?
Looking forward to hearing from you.

Best regards,
Chris
```

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Message: #2543
Date: 2024-12-13
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

```
Dear Chris,

My recent posts:

Lill's method
#2520  Reply to your question (#2519)
#2523  Additional information (Simson line, multi-fold origami)
#2528  5L and Lill's method

Simson line
#2527  Relationship between Simson line
       and n-th degree eq. (n=3,4,5)

Cyclic QA
#2529  Some properties of QA0

QA,QL-conic
#2532  Quartic eq. and QA-circumconic
QL-inconic and ORIGAMI+C

Best regards,
M@IMF

p.s.
I'm very sad about Mr.Kirikami's death.
I saw his name many times when I googled something about
geometry.
```

Message: #2544
Date: 2024-12-14
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear Chris,

Did I misunderstand your request?
I hope this will be helpful (not confuse you).

#2520 (Reply to your question)
Constructing 4-chains means constructing QL2(QL3).
To obtain QL{L1,L2,L3,L4}, construct QL-Tr1 by ORIGAMI (#2517)
from some points and lines given by reference 5-chain.
This gives Mij (the intersection point of QL-Tr1 and QL-L1).
Then, Tij (= Li^Lj) is constructed from Mij and QL-2P3.
Finally, Li is constructed from Tij and QL-P1&L3 (#2509).

#2532 (QA,QL-conic)
The quartic eq. is solved by finding Zi (the intersection points of two QA1-circumconics).
Zi is the pole of Li, ZiZj is the polar of Tij wrt QL1-Co1, and QA1-Tr1 is QL1-Tr1.
The quartic eq. is also solved by constructing the common tangents of QL1-Co1
and QL1-inconic, which is the polar curve of QA1-circumconic wrt QL1-Co1.
(Caution: Green hyperbola in the figure is NOT conic (2) of the attached file.)
ORIGAMI+C means paper-folding with Circle (or Compus).

#2529 (Some properties of QA0)
Some pentangles are similar to pentangle {F,Q1,Q2,Q3,Q4}.
CSC of Qi, F~, vertice of QL1-Tr1 and QA0-Tr1 are given.
QL-Ci3" has an inscribed triangle whose Simson lines envelop QL1-Qu2, if any.
Tangents of QL-Ci3" at Qi is constructed by ORIGAMI+C.

#2523 (Additional information)
Lill's method can be applied to n-th degree eq.
and its configuration will relate to "Simson line" wrt cyclic n-angle.
Any n-th degree eq. can be solved by multi-fold origami.

#2527 (Simson line)
Simson line wrt cyclic n-angle is defined inductively.
Relationship to the configuration for n-th degree eq. are
X/Y-axis = sim{123} of F/F~ (cubic eq.)

```
X'/X"-axis = sim{1234} of F/F~  (quartic eq.)
X'/Y"-axis = sim{12345} of F/F~  (quintic eq.).

#2528  (5L and Lill's method)
5L2-n-P3 = the midpoint of F and Oq
5L3-n-P3 = Oq
5L4-n-P3 = the midpoint of F~ and Oq,
where Oq is the midpoint of F and F~.

Best regards,
M@IMF
```

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Message: #2545
Date: 2024-12-16
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

Dear M,

Thank you for the summaries in messages #2543 and #2544. They were really helpful to me.

I have another question to start with:

In message #2520, you wrote, “According to msg#2509, we can construct QL when Mij (the midpoint of a diagonal) is given.”

I don’t quite understand your description of how to construct this QL. While I understand the principle of folding, I’m unclear on the process of starting with a reference 5-chain and obtaining four 4-chains using the point Mij. Could you please explain this to me in simple steps?

Additionally, you mention QL-Tr1 and QL-Tr2. Are you referring to the QL-Diagonal Triangle and QL-NPC Triangle (in EQF terms)?

Best regards,

Chris

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Message: #2546
Date: 2024-12-16
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

```
Dear Chris,

First, let me explain Mij => Li (QL) in two steps.

<Mij => Tij>
Since Tij (= Li^Lj) and Tkl (= Lk^Ll) are mutual CSCs,
triangles QL-P1.Tij.QL-2P3a/b and QL-P1.QL-2P3a/b.Tkl are
similar,
and the quadrangle {Tij, QL-2P3a, Tkl, QL-2P3b} is a harmonic
quadrangle,
where (i,j,k,l) = (1,2,3,4) or their permutations.
So, triangles Mij.QL-2P3a.Tij and Mij.Tij.QL-2P3b are similar.
(Similarly Tkl.)
You can construct Tij from Mij and QL-2P3 without ORIGAMI.
Please see the 3rd figure of "QL-Inconic and ORIGAMIplusC.pdf"
attached #2532.

<Tij => Li>
Let the intersection points of QL-L3 and the circle with
diameter QL-P1.Tij
be X'i and X'j. Then Li is the line Tij.X'i and Lj is the line
Tij.X'j. (Similarly Lk and Ll.)
You can also construct Li from Tij and QL-P1&L2 by ORIGAMI.
Please see the 1st figure of "QL and ORIGAMI.pdf" attached
#2517.

Next, I'll explain 5-chain => Mij (Kij').
(I'm not good at English, so it takes some time to write English
messages.)
As for QL-Tr1 and QL-Tr2, you're right.

Best regards,
M@IMF

p.s.
Please use "MIMF" instead of "M@IMF" if @ is annoying.
I use it simply because someone might think, "Oh, 007's boss was
headhunted by Impossible Mission Force?"
```

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Message: #2547
Date: 2024-12-16
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear Chris,

The process of 5-chain => 4-chain will be
5-chain => {F1&d1, F2&d2} => TrX => Mij => Tij => Li => 4-chain.

<TrX = QL-Tr1>
Let
F1 = the anticomplement of QL-P1 wrt QL-Tr1
d1 = the line through QL-P10 perpendicular to QL-L1
F2 = QL-P16 (the antipode of F1 wrt QL-Ci1)
d2 = the line through QL-P10 parallel to QL-L1,
then QL-Tr1 is constructed by making creases that place F1 onto d1
and F2 onto d2 simultaneously.
The intersection points of QL-Tr1 and QL-L1 are M14, M24, M34.

<Reference 5-chain>
Let
A0 = (0, f)   [F]
A1 = (0, 0)   [X'0]
A2 = (-m, 0)   [X'-axis^Y-axis]
A3 = (-m, p)   [X"-axis^Y-axis]
A4 = (q - m, p)   [X"0]
A5 = (q - m, p - r)   [F~],
then
F1 = (g' - m/2 , -2f - p'/2 - Ag')
d1: y = 2f - p'/2 - Ag'
F2 = (q'/4 + Ap' + g'A^2 , p'/2 + Ag')
d2: x = -q'/4 - Ap' - g'A^2,
where p' = p - (m^2)/4f, q' = q - (m/2f)p', g' = (p'^2 - 4rf)/q', A = m/4f.
Note that I assume any points and lines described using f, m, p, q, r can be constructed.

<4-chains and QL>
4-chains are constructed when we know the QL{4L2a,4L2b,4L2c,4L2d},
which can be obtained from Mij (the midpoint of the diagonal).
Let ti (i=1,2,3,4) be the slopes of the QL, then they are solutions of
ft^4 + mt^3 + pt^2 + qt + r = 0,
and
Mij = (-m/2, (Kij + p)/2)

```
Kij = -f(ti + tj)(tk + tl),
where (i,j,k,l) = (1,2,3,4) or their permutations.

<TrXs>
The side of QL-Tr1 is
y = (q'/2Kij' - m/4f)x + (Kij + p' + mq'/2Kij')/2,
where Kij' = Kij + (m^2)/4f.
Kij' is the solution of the cubic equation
K'^3 + [2p' - (m^2)/4f]K'^2 + (p'^2 - 4rf + mq')K' - fq'^2 = 0.
Mij can be also obtained from DT' (,whose slope of side is
q'/2Kij') of #2517.
QL-Tr2 is constructed from QL-P1&L2 and QL-P6"&L4,
where QL-P6" is the reflection of QL-P1 in QL-P6.
But I don't know how to construct Mij from QL-Tr2.

Best regards,
M@IMF
```

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Message: #2548
Date: 2024-12-19
From: van10hoven@gmail.com
Subject: well wishes and reflections about the past year

Dear Friends,

As the end of the year approaches, it is time for well wishes and a moment of reflection.

Over the past year, *Bernard Keizer, Eckart Schmidt,* and I have mainly been engaged with Cubics. Along the way, other interesting topics also crossed our path, including remarkable contributions from:

* Keita Miyamoto (#2418, conic in Quadri-Geometry, #2469 Hexalateral, Kantor-Hervey points, #2535 QA-L9, QA-Cu1, coaxal circles)
* Tran Quang Hung (#2448 A Napoleon-like Theorem for Quadrilaterals, The asymmetric propeller with squares, and some extensions)
* Stanley Rabinowitz (#2471, four normals to an ellipse)
* Antreas Hatzipolakis (#2293, A Miquel-Steiner Transformation)
* Trinh Xuan Minh (#2109, …,#2245, several items on Rational Triangles, Polygons)
* M@IMF (#2491, Lill’s Method on solving equations of nth degree)

We were also saddened by the passing of *Seiichi Kirikami* , our dear friend who contributed so much to Quadri Geometry during the early days of the EQF.

Another notable event was the disappearance of Forum Geometricorum from the internet. Thankfully, Francisco Javier Garcia Capitan had a complete archive of 20 years’ worth of articles available on his website.

Personally, I have noticed a renewed interest in the topic of *Perspective Fields*. Clark Kimberling now mentions the methodology in the introduction of the ETC. I plan to write a paper on Perspective Fields.

I also received the news that my website for EQF, EPG, and Perspective Fields is now too outdated under Joomla. It needs a complete overhaul. Therefore, I have decided to rebuild it entirely with my son, no longer using Joomla as engine but moving to WordPress. At the same time, I intend to include our results on Cubics. Since these deviate from Quadri-Figures and Polygon Geometry, I want to compile all our findings into a new encyclopedia titled *"Encyclopedia of Poly Geometry."* The word "Poly" is used here to signify "many" in the sense of many (n) reference frameworks for n-Points, n-Lines, n-Gons, as well as Curves of the nth degree.

This is a massive project, but it is already underway.

In short, I won't be bored in the new year. As a result, I may not always respond to messages as quickly as I would like, but my undivided interest remains.

The topic raised by M@IMF on "Lill's Method for solving equations of nth degree" is highly relevant to Poly Geometry. It deals with functions f(x) of the n th degree, how they can be solved synthetically in a surprisingly simple way, and—interestingly—makes extensive use of results from Quadri-Figures and Polygon Geometry. This fits perfectly with our group's focus. I highly recommend delving into this topic, starting with messages #2519, #2493. We are only at the beginning.

I wish everyone a fine Christmas and a happy New Year, and may your wishes come true.

Warm regards,
Chris van Tienhoven

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Message: #2549
Date: 2024-12-19
From: anopolis72@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] well wishes and reflections about the

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On Thu, Dec 19, 2024 at 10:26 AM Chris via groups.io
<van10hoven=
gmail.com@groups.io> wrote:

> Personally, I have noticed a renewed interest in the topic of
> *Perspective Fields*.
> Clark Kimberling now mentions the methodology in the
> introduction of the *ETC*.
> I plan to write a paper on Perspective Fields.

Dear Chris

Let me mention your valuable contributions to ETC

Part 1: 24
Part 2: 10
Part 3: 6
Part 4: 46
Part 5: 3
Part 6: 3
Part 18: 3
Part 30: 2
Part 34: 8

It would be much more if you had the time to do more

Wishing you a healthy and creative new year

Greetings from Greece
APH

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Message: #2550
Date: 2024-12-19
From: van10hoven@gmail.com
Subject: Re: Quartic Eq. and QL

Dear Mimf,

Thank you for all your messages and the further explanations.
I read your note about the name M@IMF. Initially, I didn't understand the hidden meaning behind it, but now I see the playful construction. You must have been fascinated by the old 007 and Mission Impossible series. However, it's still unclear to me whether Mimf is your first name or another part of your full name.
It's great to see how QL-geometry is now being applied in another discipline.
I'm also impressed by how well-versed you've become in QL-geometry.
I must admit that I'm still getting used to the principle of folding (making creases).
Nonetheless, I've started revisiting Lill's method.
I've tried to delve into your messages, though I haven't fully worked through everything yet.

Regarding your messages, the sequence of your construction from a 5-Chain to a set of four 4-Chains is as follows:
"5-chain => {F1&d1, F2&d2} => TrX => Mij => Tij => Li => 4-chain."

I understand the first part, "5-chain => {F1&d1, F2&d2} => TrX," but it becomes a bit complex beyond that.
It's also unclear to me whether you've found a complete construction from the 5-Chain to a set of 4-Chains, particularly given your final remark: "I don't know how to construct Mij from QL-Tr2."

Additionally, I've done some research on the occurrence of the points nL-n-P1 and nL-n-P3 from EPG in the n-Chains of Lill.
I noticed that you were already exploring that direction yourself.
I've attached a working document with my progress so far. Not verything is worked oud very well yet.

Looking forward to hearing your thoughts!

Best regards,
Chris

Properties 3-Chains, 4-Chains, etc.

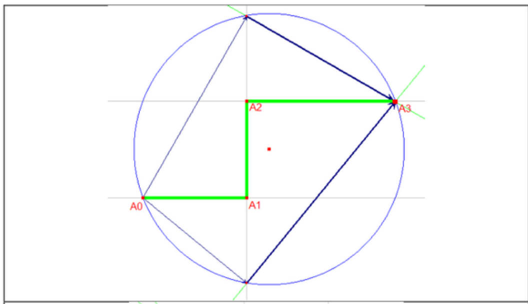
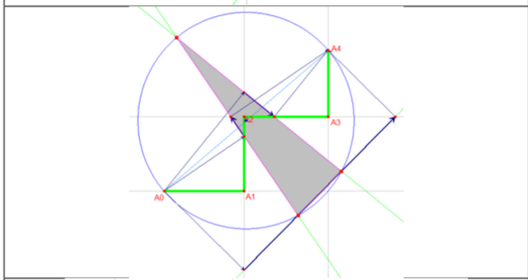
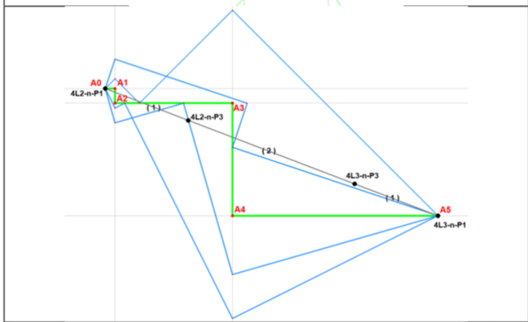
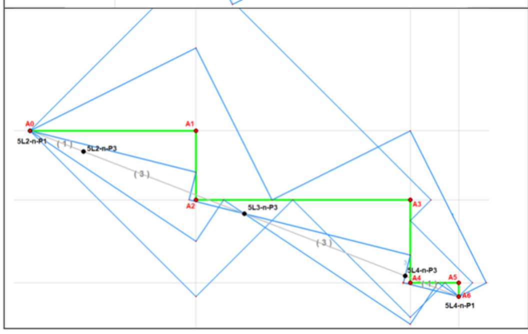
NOTATIONS
Let (6L1, 6L2, 6L3, 6L4, 6L5, 6L6) represent the Reference 6-Chain, and (5L1a, 5L2a, 5L3a, 5L4a, 5L5a, 5L6a) the *a*-version of the inscribed 5-Chains. The other 5-Chains are the *b*-, *c*-, *d*-, and *e*-versions and are denoted accordingly.
The vertices of the 6-Chain are A0, A1, A2, A3, A4, A5, A6.
The vertices of the *a*-version of the inscribed 5-Chain are A0, B1a, B2a, B3a, B4a, A6.
The vertices of the *b*-, *c*-, *d*-, and *e*-versions are denoted accordingly.
The 5-Line (5L) formed by (5L2a, 5L2b, 5L2c, 5L2d, 5L2e) is denoted as 5L2.
The 5-Line formed by (5L3a, 5L3b, 5L3c, 5L3d, 5L3e) is denoted as 5L3.
The 5-Line formed by (5L4a, 5L4b, 5L4c, 5L4d, 5L4e) is denoted as 5L4.
A point in EPG (Encyclopedia of Poly Geometry) such as nL-n-Px used in 5L2 is denoted as 5L2-n-Px.
Now:

- A0 = 5L2-n-P1
- A6 = 5L4-n-P1
- Midpoint (A0, A6) = 5L3-n-P1 = 5L3-n-P3 ???
- The points (A0, 5L2-n-P3, 5L3-n-P3, 5L4-n-P3, A6) are collinear with mutual distance ratio (1 : 3 : 3 : 1).

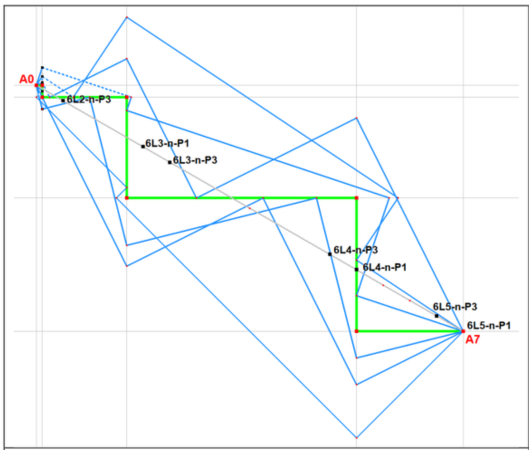
There are corresponding properties for 3-Chains, 4-Chains, 5-Chains, 6-Chains, etc.

See next table.

Properties 3-Chains 4-Chains-etc-01.pdf

	**3-Chain** * There are 2 inscribed 2-Chains. * Vertices lie on a circumcircle.
	**4-Chain** * There are 3 inscribed 3-Chains. * A0 = ?; A4 = ? * 3L2-n-P3 = Midpoint A0.A4
	**5-Chain** * There are 4 inscribed 4-Chains. * A0=4L2-n-P1; A5=4L4-n-P1 * The points (A0, 4L2-n-P3, 4L3-n-P3, A5) are collinear with mutual distance ratio (1 : 2 : 1).
	**6-Chain** * There are 5 inscribed 5-Chains (blue). * A0=5L2-n-P1; A6=5L4-n-P1 * Midpoint (A0, A6) = 5L3-n-P1 = 5L3-n-P3 ??? The points (A0, 5L2-n-P3, 5L3-n-P3, 5L4-n-P3, A6) are collinear with mutual distance ratio (1 : 3 : 3 : 1).

Properties 3-Chains 4-Chains-etc-01.pdf

	**7-Chain** * There are 6 inscribed 6-Chains (blue). * A0=6L2-n-P1; A7=6L5-n-P1 * The points (A0=6L2-n-P1, 6L2-n-P3, 6L3-n-P1, 6L3-n-P3, 6L4-n-P3, 6L4-n-P1, 6L5-n-P3, 6L5-n-P1=A7) are all collinear with mutual distance ratio (1 : 3 : 1 : 6 : 1 : 3 : 1).

Properties 3-Chains 4-Chains-etc-01.pdf

Message: #2551
Date: 2024-12-19
From: contiwa.goma3@gmail.com
Subject: Re: Quartic Eq. and QL

Dear Chris,

I think constructing QL-Tr1 or DT' gives a complete construction from the 5-Chain
to a set of 4-Chains although the expressions of {F1,d1,F2,d2} are unsatisfactory.
I hope QL can be constructed from QL-Tr2.
The reason I can't so far is simply because I'm not good at synthetic geometry.
(That's why my messages are full of fomulae.)
Being defined as the intersection points of conics is another difficulty for me.
But I'll continue to study.

By the way, I found a mistake in "QL and ORIGAMI.pdf" attached #2517.
<QL-L1 and DT'> should be <QL-Tr1 and DT'>.

As for 6-Chain, I'll post to "5L and Lill's method."

Best regards,
M@IMF

p.s.
MIMF is an abbreviation of a phrase that is secret so far.

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Message: #2552
Date: 2024-12-20
From: contiwa.goma3@gmail.com
Subject: Re: 5L and Lill's method

```
Dear Chris,

This is reply to #2550.
First let me correct my mistake in #2528.

Let
5L2 = 5L formed by X'Y's (= {5L2a, 5L2b, 5L2c, 5L2d, 5L2e})
5L3 = 5L formed by Y'X"s (= {5L3a, 5L3b, 5L3c, 5L3d, 5L3e})
5L4 = 5L formed by X"Y"s (= {5L4a, 5L4b, 5L4c, 5L4d, 5L4e})
Oq = the midpoint of F(A0) and F~(A6)
Circle Oq = the circumcircle of pentangle Q1Q2Q3Q4Q5,
then
5L2-n-P3 = the point dividing the segment F.Oq
           in the ratio 1:3
5L3-n-P3 = Oq
5L4-n-P3 = the point dividing the segment F~.Oq
           in the ratio 1:3.
The radii of 5L2-n-Ci1, 5L3-n-Ci1, 5L4-n-Ci1
           are R/4, R/2, R/4, respectively,
           where R is the radius of Circle Oq.

Some comments:
* 5L3-n-P1 = 5L3-n-P3 means the radius of QL-Ci3 of each
component QL of 5L3 is R/2.
* nL2 and nLn-1 are understandable because they are Simson lines
of cyclic nP.
Also they are tangents of one parabola, respectively. On the
other hand,
nLi (i=3,...,n-2) is unsure because there is no correspondent in
5-Chain.
* If we express the Reference 6-Chain as (6L1f, 6L2f, 6L3f,
6L4f, 6L5f, 6L6f),
the relationship to 7-Chain will be clear and we don't mix-up
them with 6-Line in 7-Chain.
* 7-Chain ... that's over my head. I only vaguely imagined, but
now I can see it in my real eyes!
* I'm curious about the regularity:
  (A0, 3L2-n-P3, A4)    (1 : 1)
  (A0, 4L2-n-P3, 4L3-n-P3, A5)    (1 : 2 : 1)
  (A0, 5L2-n-P3, 5L3-n-P3, 5L4-n-P3, A6)    (1 : 3 : 3 : 1)
  (A0, 6L2-n-P3, 6L3-n-P3, 6L4-n-P3, 6L5-n-P3, A7)    (1 : 4 : 6
  : 4 : 1)
  ....
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```
  and
  (A0, 5L3-n-P1, A6)    (1 : 1)
  (A0, 6L3-n-P1, 6L4-n-P1, A7)    (1 : 2 : 1)
  ....

Best regards,
M@IMF
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Message: #2553
Date: 2024-12-22
From: unidentifiedlethargicorganism@gmail.com
Subject: Quadrilateral, 4 orthotransversals, inconic

```
Dear Geometers,

This is a cross-post from RG.
https://www.facebook.com/groups/parmenides52/permalink/895923116
7523843/

Let L1L2L3L4 be a general quadrilateral, and P be a point in the
plane.
Let P{1,2} = L1 ∩ L2,
    P{1,3} = L1 ∩ L3,
    P{1,4} = L1 ∩ L4,
    P{2,3} = L2 ∩ L3,
    P{2,4} = L2 ∩ L4,
    P{3,4} = L3 ∩ L4.

Let L1' be the orthotransversal of P wrt the triangle formed by
L2, L3, L4.
Define L2', L3', L4' cyclically (for the other component
triangles of the quadrilateral).
Let P{1,2}' = L1' ∩ L2',
    P{1,3}' = L1' ∩ L3',
    P{1,4}' = L1' ∩ L4',
    P{2,3}' = L2' ∩ L3',
    P{2,4}' = L2' ∩ L4',
    P{3,4}' = L3' ∩ L4'.

Then, there exists a conic simultaneously tangent to the 6 lines
    P{1,2}P{1,2}',
    P{1,3}P{1,3}',
    P{1,4}P{1,4}',
    P{2,3}P{2,3}',
    P{2,4}P{2,4}',
    P{3,4}P{3,4}'.

Sincerely,
Keita Miyamoto
```

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Message: #2554
Date: 2024-12-22
From: unidentifiedlethargicorganism@gmail.com
Subject: quadrilateral, concurrent orthocentric transversals,

---

```
Dear Geometers,

This is a cross-post from RG.
https://www.facebook.com/groups/parmenides52/permalink/895928532
0851761/

Let L1L2L3L4 be a general quadrilateral, and P be a point in the
plane.
Let P{1,2} = L1 ∩ L2,
    P{1,3} = L1 ∩ L3,
    P{1,4} = L1 ∩ L4,
    P{2,3} = L2 ∩ L3,
    P{2,4} = L2 ∩ L4,
    P{3,4} = L3 ∩ L4.

Let H1 be the orthocenter of the triangle formed by L2, L3, L4.
Let q{1,2} be the line through H1 perpendicular to PP{3,4}.
Let Q{1,2} = L2 ∩ q{1,2}.
Let q{1,3} be the line through H1 perpendicular to PP{2,4}.
Let Q{1,3} = L3 ∩ q{1,3}.
Let q{1,4} be the line through H1 perpendicular to PP{2,3}.
Let Q_{1,4} = L4 ∩ q{1,4}.

Then, Q{1,2}, Q{1,3}, Q{1,4} lie on the same line q1.
We call q1 "the orthocentric transversal of P wrt the triangle
formed by L2, L3, L4."
Define q2, q3, q4 cyclically (for the other component triangles
of the quadrilateral).

Then, the 4 lines q1, q2, q3, q4 concur in a point Q, and the
mapping f: P → Q is an involution, i.e., f(f(P)) = P. Hence, P
and Q are considered as a pair of conjugate points wrt the
reference quadrilateral.

Sincerely,
Keita Miyamoto
```

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Message: #2555
Date: 2024-12-24
From: eckart_schmidt@t-online.de
Subject: Re: quadrilateral, concurrent orthocentric transversals,

---

```
Dear Keita Miyamoto,

only a short look in your message #2554:
Your transformation maps lines
... to conics through the Plücker points QL-2P1.
Points on QL-Ci5 are mapped diametral.
Merry Christmas

Eckart
```

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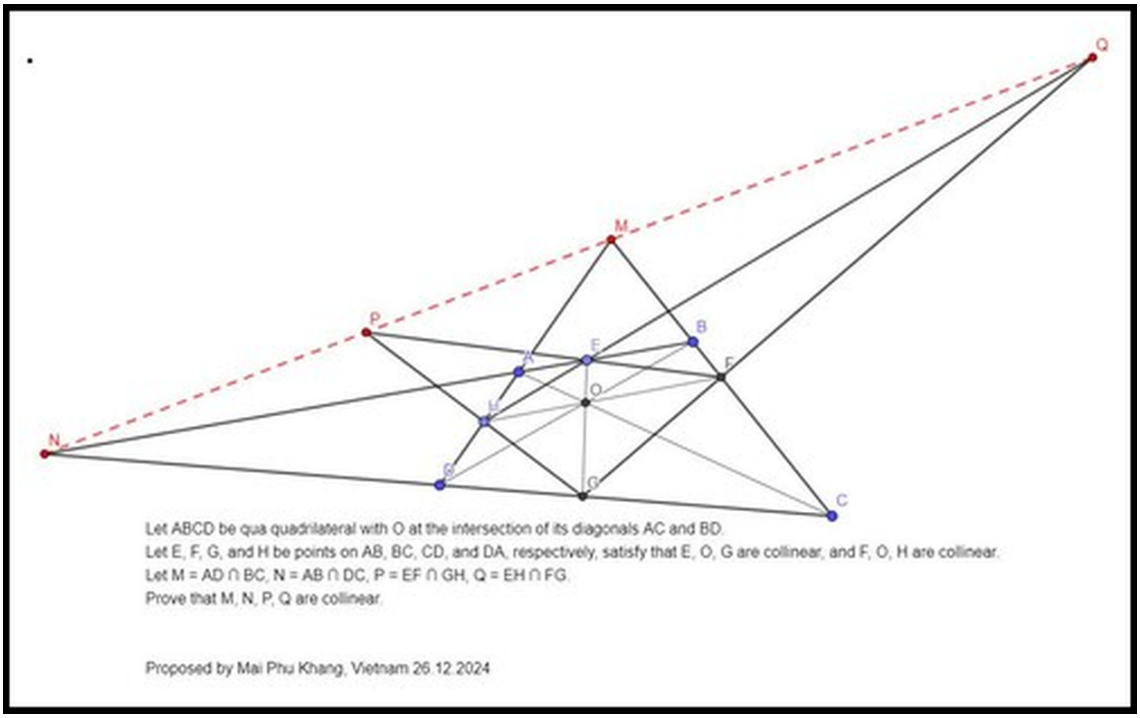
Message: #2556
Date: 2024-12-27
From: anopolis72@gmail.com
Subject: A line in a quadrangle

```
Problem by Phú Khang
<https://www.facebook.com/photo/?fbid=1131634365289552&set=gm.89
86676398112653&idorvanity=1019808738132832>
```



Phu_Khang.jpg

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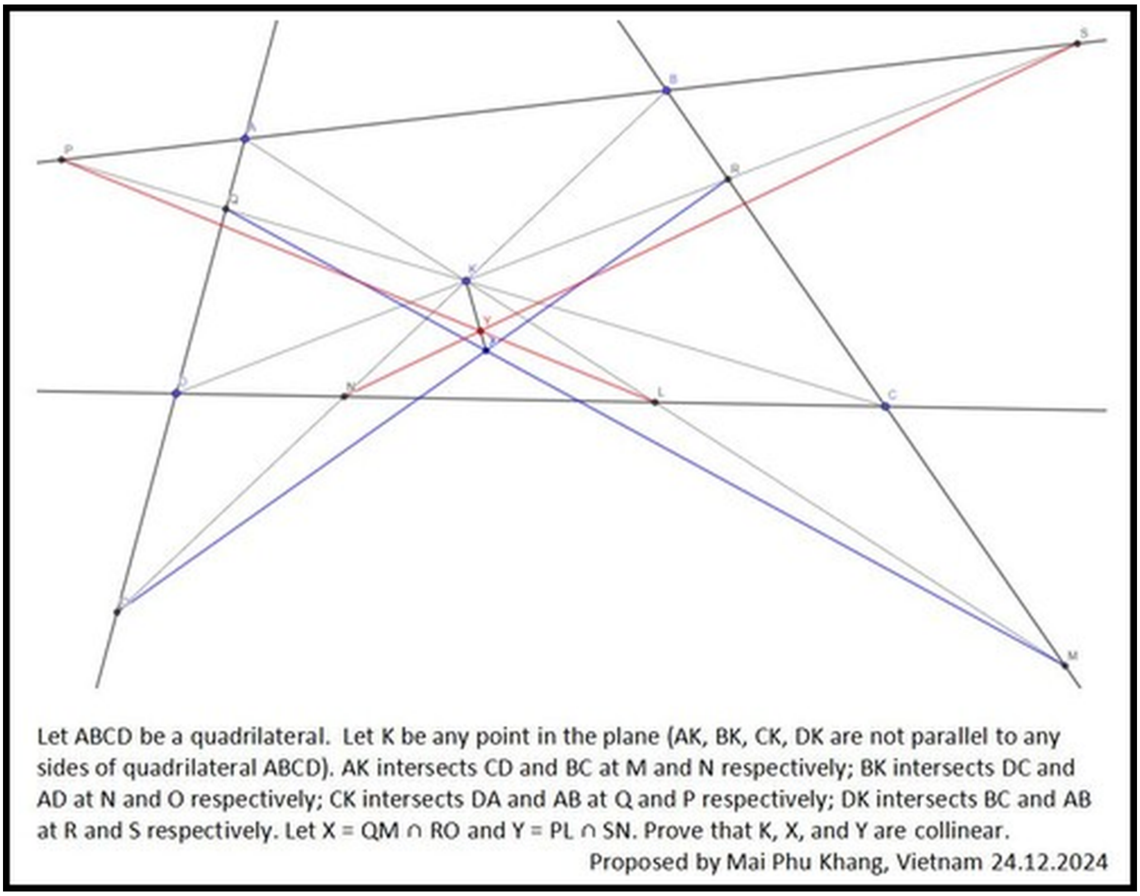
Message: #2557
Date: 2024-12-27
From: anopolis72@gmail.com
Subject: Re: [Quadri-and-Poly-Geometry] A line in a quadrangle

---

```
Another problem by Phú Khang
<https://www.facebook.com/photo?fbid=1130111228775199&set=gm.8974526142661012&idorvanity=1019808738132832>

On Fri, Dec 27, 2024 at 2:34 PM αντρέας χατζηπολάκης via
groups.io
<anopolis72=gmail.com@groups.io> wrote:

> Problem by Phú Khang
```



Phu_Khang2.jpg

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Message: #2558
Date: 2024-12-27
From: unidentifiedlethargicorganism@gmail.com
Subject: Re: A line in a quadrangle

```
Dear Antreas Hatzipolakis, and Mai Phu Khang,

The first one is just QG-L1 = the polar of any inscribed
QG-conic.

Sincerely,
Keita Miyamoto
```

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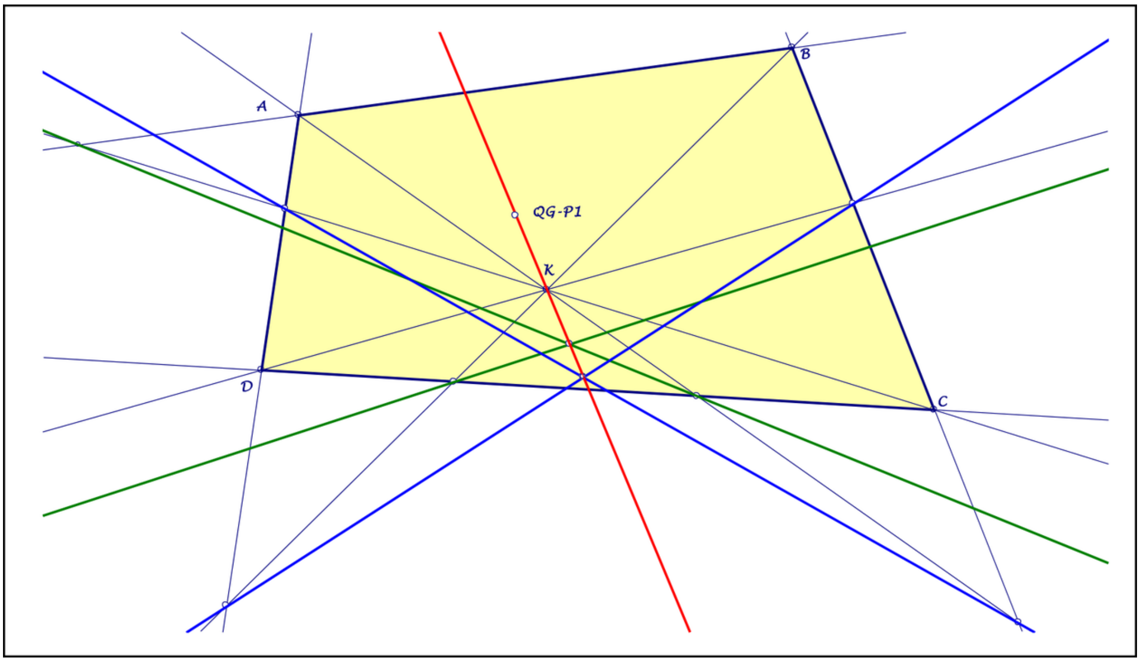
Message: #2559
Date: 2024-12-27
From: eckart_schmidt@t-online.de
Subject: Re: A line in a quadrangle

Dear Mai Phu Khang,

I think that you consider in #2557 a quadrigon ABCD,
... it seems that your description is not correct,
... I hope my attached drawing is right,
... then the resulting line is the connection of K
... and the diagonal intersection QG-P1.

Best regards Eckart



2024-12-27.pdf

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5 Keyword Index

The list below shows several keywords along with numbers of related messages. Click on a number to go to the corresponding page.

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6 Colophon

Sources & Contact

Web address (QPG Forum): https://groups.io/g/Quadri-and-Poly-Geometry
EPG Encyclopedia (content reference): https://www.chrisvantienhoven.nl
Editorial correspondence: van10hoven@gmail.com

Journal of the Quadri- and Poly-Geometry Group

ISSN: (to be assigned)
Published by: Uitgeverij Varenboom
Editorial Board: Chris van Tienhoven

Published Volumes:

- Volume 7 (2025), messages #2560–#2897
- Volume 6 (2024), messages #2052–#2559
- Volume 5 (2023), messages #1545–#2051
- Volume 4 (2022), messages #1295–#1544
- Volume 3 (2021), messages #631–#1294
- Volume 2 (2020), messages #61–#630
- Volume 1 (Nov. 2019–Dec. 2019), messages #1–#60

Predecessor Journal:
Journal of the Quadri-Figures Group

ISSN: (to be assigned)
Published by: Uitgeverij Varenboom
Editorial Board: Chris van Tienhoven

Volumes of the predecessor journal:

- Volume 7 (Jan. 2019–Oct. 2019), messages #3280–#3906
- Volume 6 (2018), messages #2780–#3299
- Volume 5 (2017), messages #2170–#2799
- Volume 4 (2016), messages #1403–#2169
- Volume 3 (2015), messages #917–#1402
- Volume 2 (2014), messages #394–#916
- Volume 1 (2013), messages #1–#393