Conical Information

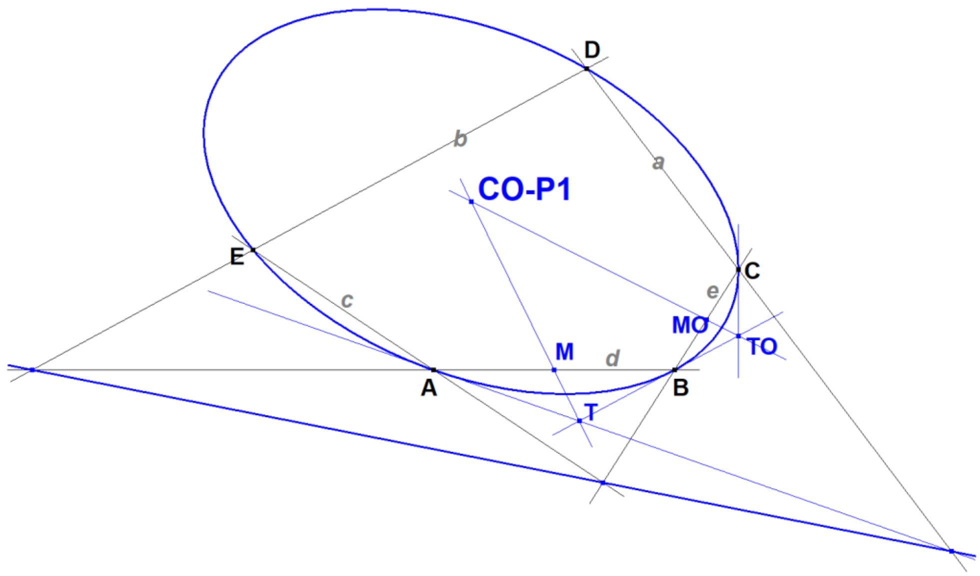
CO-P1 Conic Center

CO-P1 is the Center of the Reference Conic.

Construction ***(****See Ref-19):*
1. Let A, B, C, D, E be 5 points on the Reference Conic.
2. Let the tangents at A, B meet at T, and those at B, C meet at TO.
3. Let M, MO be the midpoints of AB and BC, then the center O is MT.MOTO.

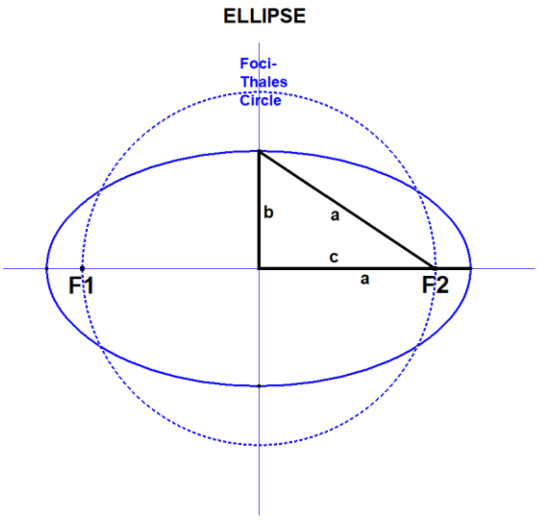
Construction of Conic Tangents:

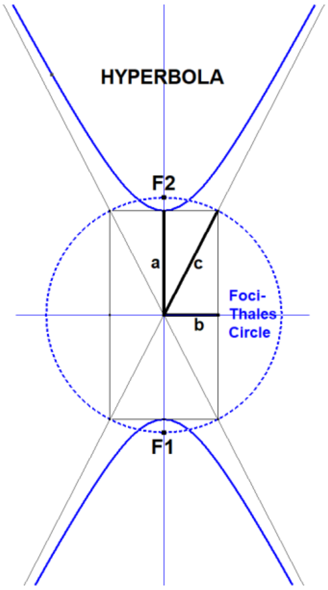
4. Let d = AB, e = BC, a = CD, b = DE, c = EA, then bd.ce cuts a in a point lying on the tangent at A.



CO-Ci1 Foci-Thales Circle

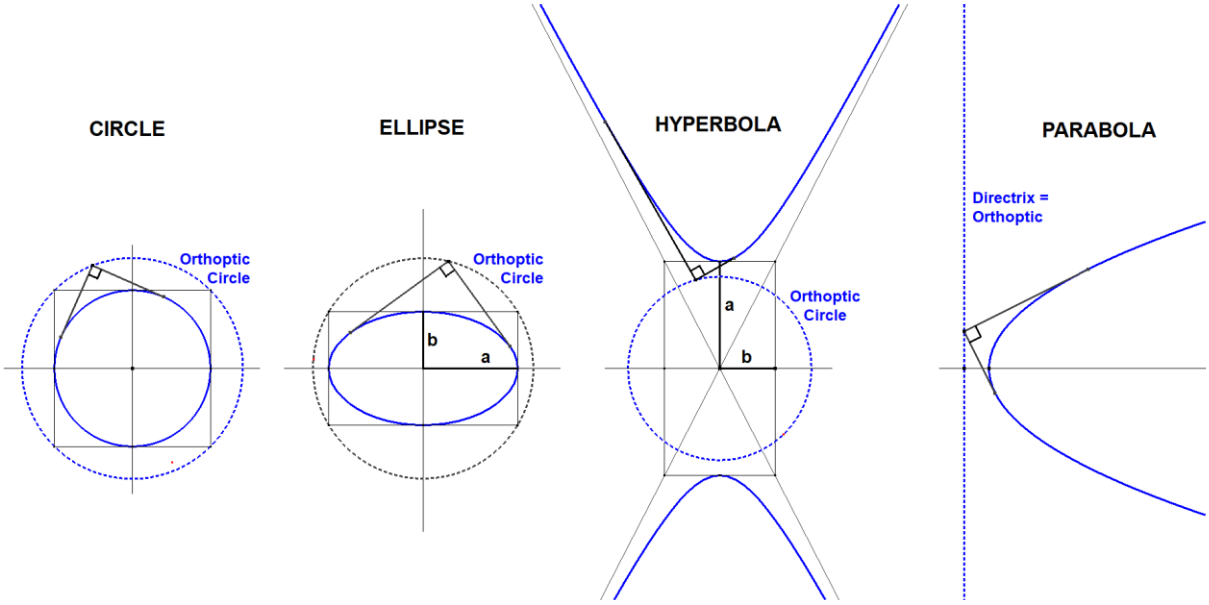
The Thales Circle is a circle with a given diameter.
CO-Ci1 is the Thales Circle with the foci of the Reference Conic as diameter.
Since only the ellipse and hyperbola have two foci, this circle only exists for these types of conics.





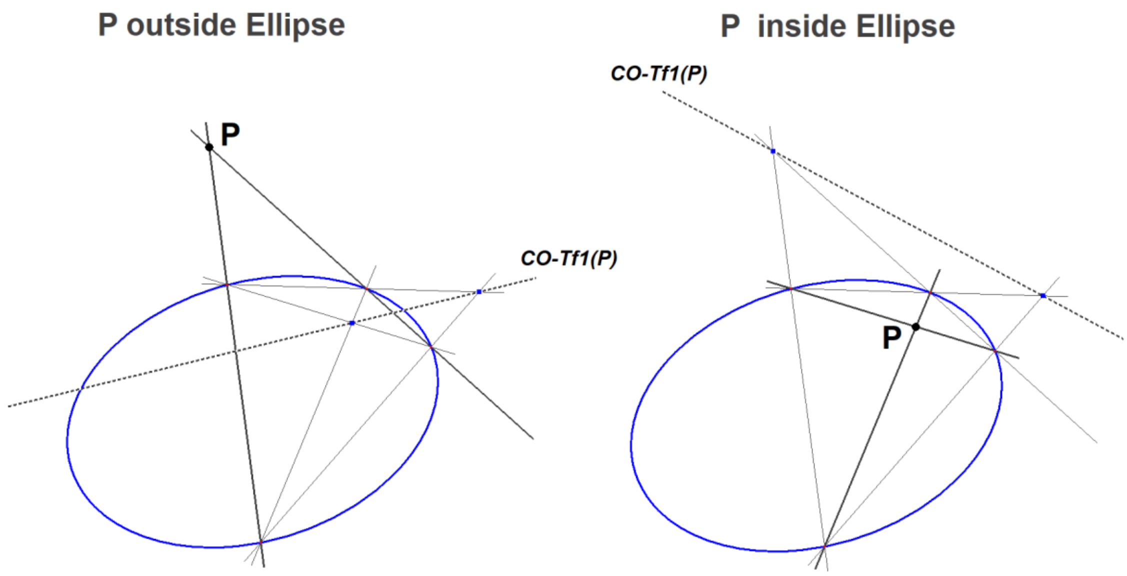
CO-Ci2 CO-Orthoptic Circle

An orthoptic is the locus of points for which two tangents of a given curve meet at a right angle.
CO-Ci2 is the orthoptic of the reference conic.
The orthoptic of an ellipse $x^2/a^2 + y^2/b^2 = 1$, is the director circle $x^2 + y^2 = a^2 + b^2$.
The orthoptic of a hyperbola $x^2/a^2 - y^2/b^2 = 1$, $a > b$, is the circle $x^2 + y^2 = a^2 - b^2$.
The orthoptic of a parabola is its directrix.



CO-Tf1 Conical Polar

The polar of some point P wrt some conic CO is the line L connecting the points of tangency of the tangents through P at CO.
In this construction P is called the pole and L is called the polar.
This construction is very intuitive. However there is a flaw in the definition because the polar cannot be constructed under all circumstances. For example when CO is some ellipse and P is drawn within the ellipse, then the construction fails.
Therefore another construction is made that includes the result of the first construction:
The polar of some point P wrt some conic is the 3rd diagonal (see QG-L1 in EQF) of any quadrigon formed by the intersection points of any 2 lines through P with the conic.

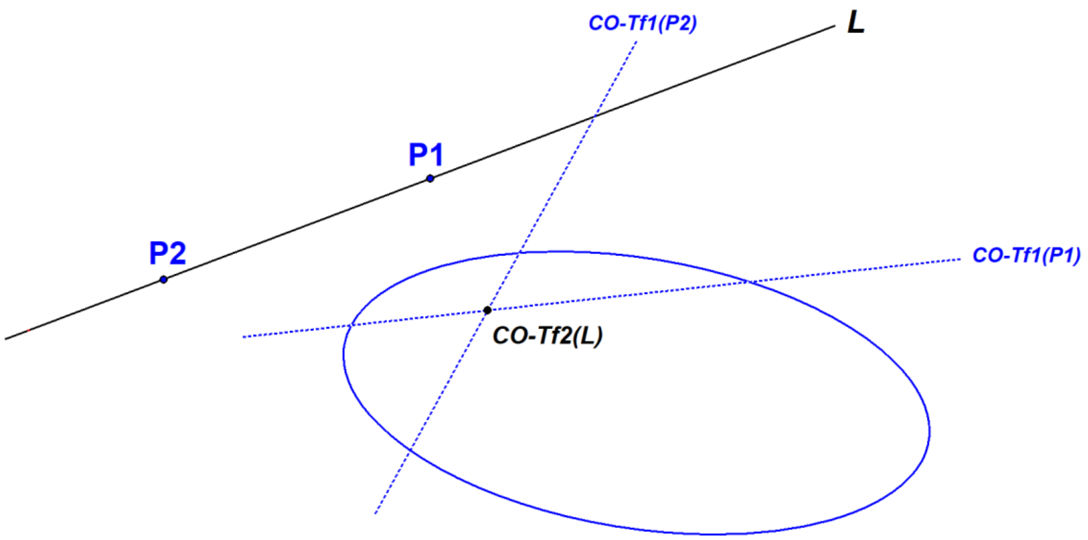


Properties:

- The polar line of the focus of a parabola is the directrix

CO-Tf2 Conical Pole

The Conical Pole P of a line L wrt some conic CO is the intersection point of the tangents to CO at the intersection points of CO with L. In this construction P is called the pole and L is called the polar.
This construction is very intuitive. However there is a flaw in the definition because the pole cannot be constructed under all circumstances. For example when CO is some ellipse and L is not intersecting the ellipse, then the construction fails.
Therefore another construction is made that includes the result of the first construction:
The pole of some line L wrt some conic (see CO-Tf1) is the intersection point of the polars of two random points from line L.



Another construction from Eckart Schmidt. See Ref-34, QFG#2811.
1. Let L be the line to be transformed.
3. Let Pl be the pole (Co-Tf2) of L.
4. Let Lp be the line through Pl perpendicular to L.
5. Let Lc be the reflection of P.Co-P1 in the main axis of Co.
6. Then Co-Tf4(L) is the polar of the intersection point of Lp and Lc.

CO-Tf3 Scimemi Transformation

The Scimemi-transformation CO-Tf3 is a point-to-point mapping defined by a conic CO.
This transformation and its action on pentagons was presented by Benedetto Scimemi at Feb. 2005 in Bloomington as a homage to Douglas Hofstadter for his 60th birthday.
This page has been written with the contribution of Benedetto Scimemi.

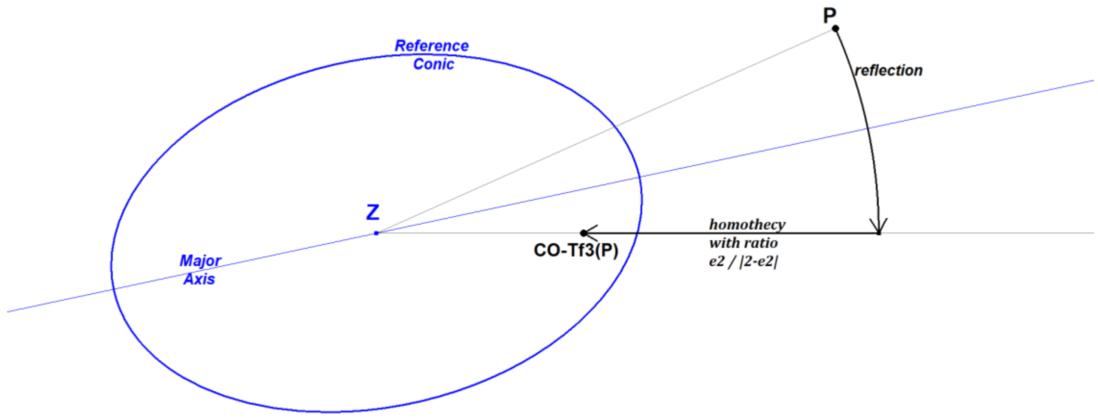
CO-Tf3 is defined wrt any conic CO, except circles and rectangular hyperbolas. Extensions to these cases, as limits, will be discussed below.
For a central conic CO, the mapping P → CO-Tf3(P) can be described as:

1. a reflection of P about an axis of the conic, followed (or preceded) by
2. a homothety with fixed point in the conic center Z and scale factor r > 0.
   The choice of the reflection axis and the value r only depend on the *shape* of the conic (i.e. they are the same for similar conics; see below).

When CO is a parabola the homothety is replaced by a translation.
The mapping CO-Tf3^{-1} , the inverse CO-Tf3, has the same definition, provided the factor r is replaced by its inverse r^{-1}.
It appears from the definition that

CO-Tf3 and CO-Tf3^{-1} are negative (odd) similarities.

Their fixed point is Z, their fixed lines are the CO-axes.
Notice that one gets the same mappings for Z-homothetic conics CO.



Definition (Ref-34, B. Scimemi, QFG#935 and #1252):
When CO is an *Ellipse*:

then CO-Tf3 is the product of the reflection in the major axis and the homothety, centered in Z, with scale factor $(a^2 - b^2)/(a^2 + b^2)$. Here a > b are the axes lengths.
Equivalently $r = \cos[\alpha]$, where α is the angle under which a vertex views the minor axis.

When CO is a *(non-orthogonal) Hyperbola:*

call "principal" the axis which lies inside the smaller angle α formed by the asymptotes (N.B. not always the principal axis is the focal axis),
then CO-Tf3 is the product of the reflection in the principal axis and the homothety, centered in Z, with scale factor $r = (a^2 + b^2) / (a^2 - b^2)$, where a > b are the axes lengths. In this case $r = \cos^{-1} [\alpha]$.

When CO is a *Parabola*:

then CO-Tf3 will be an isometry, the product of the reflection in the parabola axis and a parallel translation which amounts to twice the (oriented) distance directrix -> focus.

When CO is degenerated into two (non-orthogonal) lines crossing in Z:

then CO-Tf3 is the product of the reflection in the angle bisector of the smaller angle α formed by the lines; and the homothety, centered in Z, with scale factor cos [α].

When CO is degenerated into two parallel lines:

then CO-Tf3 is the reflection in the mid-line (parallel, equidistant).

Limit-cases:

One can extend CO-Tf3 and by CO-Tf3^{-1}, as limits, to orthogonal hyperbola and circles, but the resulting mappings are not invertible.

For an orthogonal hyperbola, CO-Tf3(P) tends the infinity point of the line Z.P, reflected in the principal axis; CO-Tf3^{-1}(P) tends to the conic center for all P.

As for circles, CO-Tf3 maps each point P into the circle center; CO-Tf3^{-1} can't be constructed.

Eccentricity

By introducing the eccentricity e, the homothety factor $((a^2 - b^2)/(a^2 + b^2))^{\pm 1}$ can be unified into the single formula $r = e^2/|2-e^2|$ holding for all types of conics.

See Ref-34, QFG#935 and #1252.

CO-Tf3 Chord Lemma

- This Chord Lemma is a basic property of Co-Tf3 (Benedetto Scimemi, 2018, February 8, personal mail).
- For any chord P1P2 of any conic CO, let M be its midpoint. Then then CO-Tf3(M) lies on the perpendicular bisector of P1P2 .

In particular, for any point P of CO, the normal to CO in P is the line P. CO-Tf3(P).

Conversely: For any point X, if the line through X normal to X.CO-Tf3(X) cuts CO in two (real) points P1, P2, then X is the midpoint of P1.P2.

Here are some consequences:

1. Let Tr be any CO-inscribed triangle , O its circumcenter, N its ninepoint circle. Then O lies on the circle CO-Tf3(N).
2. Let Tr be any triangle, O its circumcenter. For any conic CO circumscribed to Tr, the point CO-Tf3^{-1}(O) lies on the nine-point circle of Tr.
3. Let CO be a parabola, X1, X2 any points, Mi the midpoint of Xi . CO-Tf3(Xi). Then the line M1M2 is the axis of the parabola.
4. For any quadrangle Q inscribed in any conic CO, the CO-Tf3 image of QA-P2 is QA-P4.
5. For any pentangle PA = P1P2P3P4P5 let QAi the Component Quadrangle, obtained by ignoring the point Pi. Let Hi = QAi-P2, Oi = QAi-P4. Then the pentangles H1H2H3H4H5, O1O2O3O4O5 are negatively similar, the similarity being CO-Tf3 : Hi -> Oi, where CO is the conic circumscribing PA.

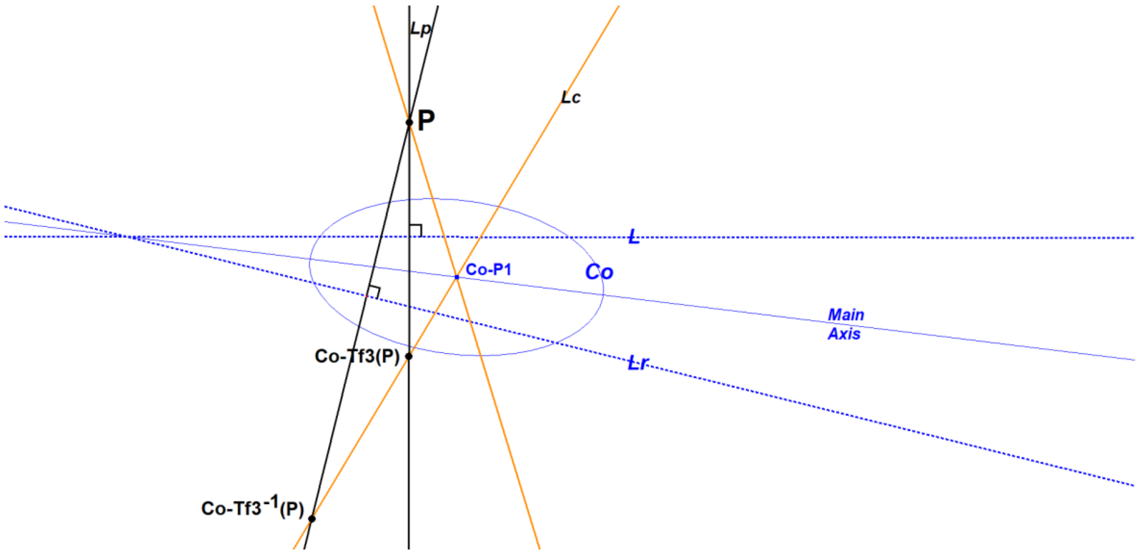
It was the proof of the last statement which gave origin to the whole subject of CO-Tf3.

Proofs of the above properties can be derived by synthetic arguments from the Chord Lemma. A proof of the Lemma itself is very easy if one represents CO by P = [a cos ∂, b sin ∂] or similar (see below, in the Section Coordinates).

Constructions

Schmidt-Construction for CO-Tf3(P):
This construction is based upon Eckart Schmidt's construction at Ref-34, QFG#2811.

1. Let L be the polar CO-Tf1 of P.
2. Let Lp be the line through P perpendicular to L.
3. Let Lc be the reflection of P.CO-P1 in the principal axis of CO.
4. Finally CO-Tf3(P) = the intersection point of Lp and Lc.



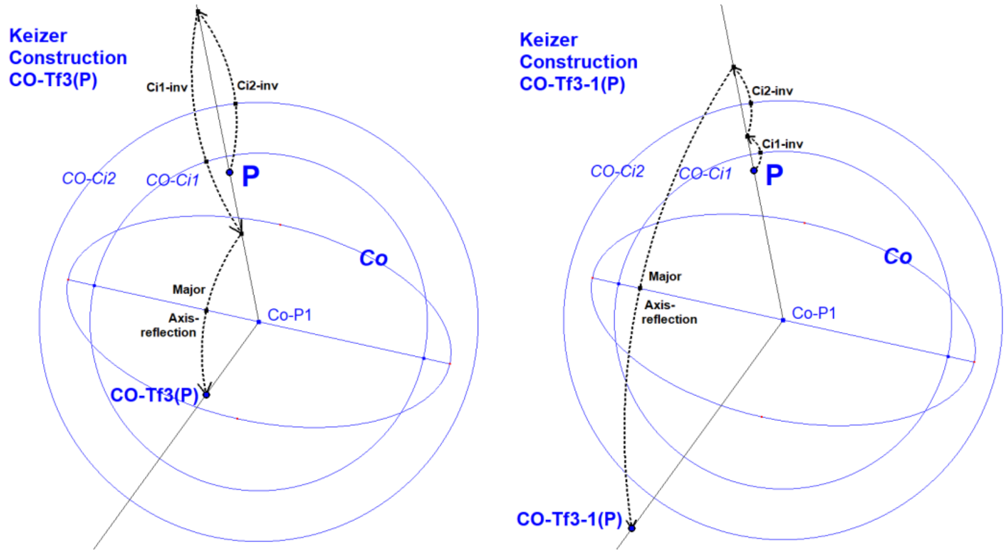
Schmidt-Construction for CO-Tf3^{-1}(P):
This construction was found by Eckart Schmidt. See Ref-34, QFG#2811.

1. Let L be the polar CO-Tf1 of P.
2. Let Lr be the reflection of L in the main axis of CO.
3. Let Lp be the line through P perpendicular to Lr.
4. Let Lc be the reflection of P.CO-P1 in the principal axis of CO.
5. Finally CO-Tf3^{-1}(P) = the intersection point of Lp and Lc.

Keizer-Construction for CO-Tf3(P):
This construction is based upon Bernard Keizer's construction at Ref-34, QFG#2826.

1. Let P1 be the inverse of P in the Orthoptic Circle CO-Ci2
2. Let P2 be the inverse of P1 in the Circle CO-Ci1 with the foci as diameter
3. CO-Tf3(P) = the reflection of P2 in the principal axis of CO.

Note: When the principal axis does not connect the foci, the orthoptic circle is not real. Replace it by the circle with radius $(a^2 - b^2)^{1/2}$.



Keizer-Construction for CO-Tf3^{-1}(P):

This construction is based upon Bernard Keizer's construction at Ref-34, QFG#2826.

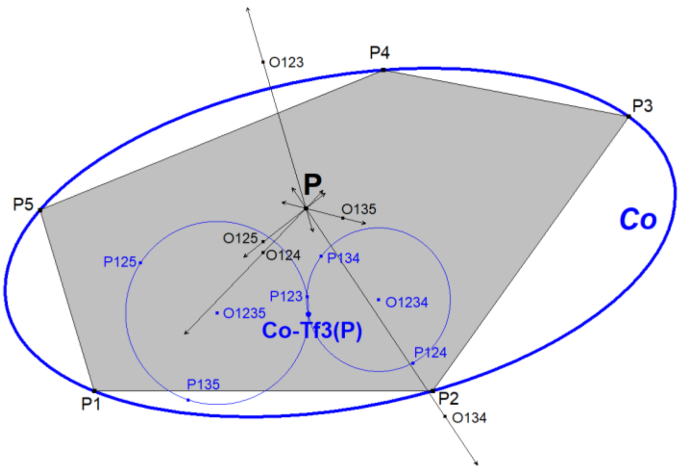
1. Let P1 be the inverse of P in the Circle CO-Ci1 with the foci as diameter
2. Let P2 be the inverse of P1 in the Orthoptic Circle CO-Ci2
3. CO-Tf3^{-1}(P) = the reflection of P2 in the principal axis of CO.

Note: When the principal axis does not connect the foci, the orthoptic circle is not real. Replace it by the circle with radius $(a^2 - b^2)^{1/2}$.

Another Construction for CO-Tf3^{-1}(P):

Next construction for CO-Tf3^{-1}(P) is based upon the construction of 5P-s-Tf3, which is a construction of Telv Cohl. See [33], Anopolis #1986.

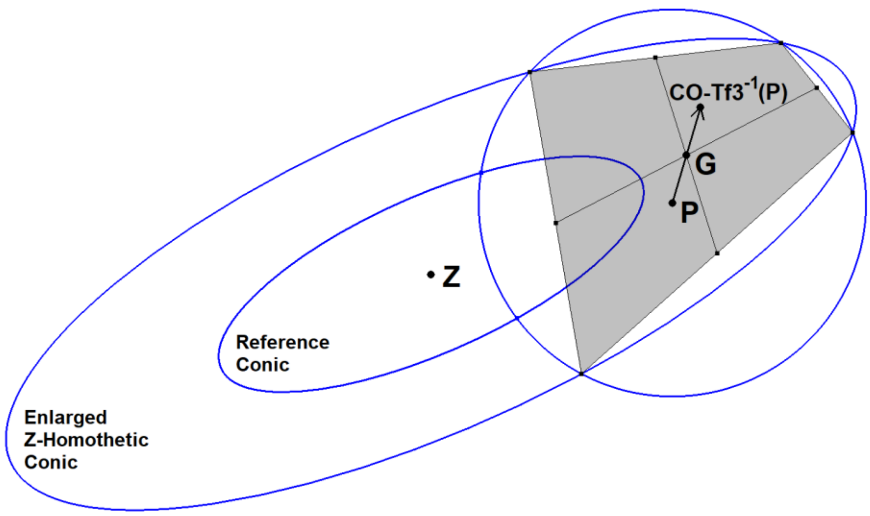
1. Let P be the point to be transformed wrt some reference conic CO.
2. Span some Pentangle P1.P2.P3.P4.P5 into Conic CO.
3. Let P_{ijk} be the Orthopole of line P.O_{ijk} wrt triangle Pi.Pj.Pk (O_{ijk} = center circumcircle Pi.Pj.Pk), where (i,j,k) are different numbers from (1,2,3,4,5).
4. Let Ci_{1234} be the circle through P_{123}, P_{124}, P_{134}, having center O_{1234}.
   Let Ci_{1235} be the circle through P_{123}, P_{125}, P_{135}, having center O_{1235}.
   These circles have point P_{123} in common.
5. CO-Tf3^{-1}(P) will be the 2nd intersection point of Ci_{1234} and Ci_{1235}.



Construction for CO-Tf3^{-1} from Benedetto Scimemi:

(a quick recipe for CABRI and Geogebra users)

1. Choose a circle Ci centered in P, such that CO and Ci have 4 intersections A1, A2, A3, A4. This is always possible when CO is a hyperbola; in other cases, it may be necessary to replace CO by a (larger or smaller) conic CO* obtained from CO by a convenient Z-homothety.
2. Construct the centroid QA-P1 of the quadrangle QA = A1A2A3A4.
3. Then P ~ QA-P4 and CO-Tf3^{-1}(P) ~ QA-P2 will be the reflection of P on QA-P1 (QA-Centroid).

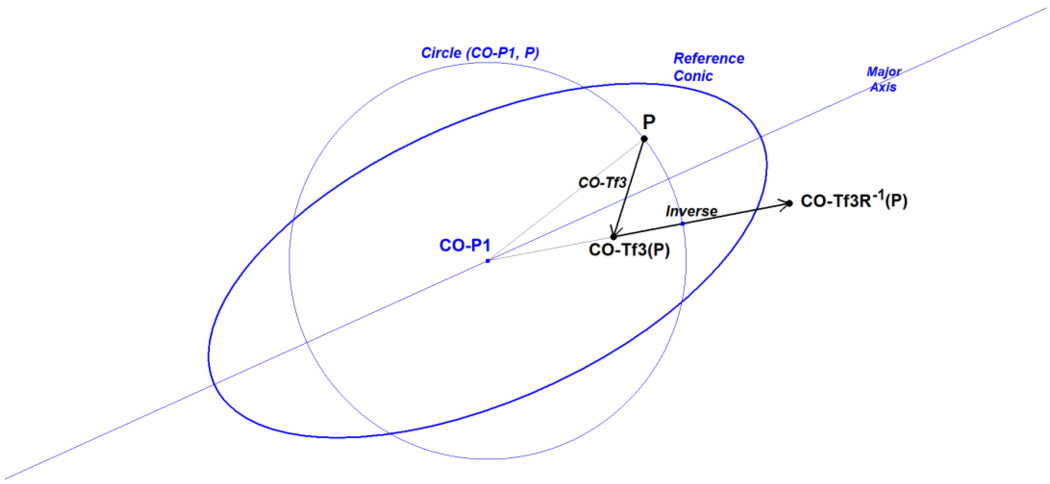


This construction is based on what can be considered the main property of CO-Tf3 / CO-Tf3^{-1}: they swap QA-P2 and QA-P4 for all quadrangles QA inscribed in CO (see the final Table).

Construction, when knowing CO-Tf3(P) or CO-Tf3^{-1}(P), of their inverse:

The construction of CO-Tf3^{-1}(P) is, knowing the construction of CO-Tf3(P), very simple.

1. Let Cix be the circle through P with center CO-P1.
2. CO-Tf3R^{-1} (P) is the inverse of CO-Tf3(P) wrt Cix.



The construction of CO-Tf3(P), when knowing CO-Tf3^{-1}(P), is identical.

Relationship with Frégier's Point

For each point P on any Conic CO-Tf3(P) coincides with Frégier's Point.
Frégier was a French mathematician who published several articles in the "Annales de Gergonne" around 1810. He discovered a remarkable point transformation with respect to a conic.
Let P be some point on conic CO. Draw a number of right angles having this point as their vertices. Then the intersected chords have in common a point, being called Frégier's Point. See Ref-13, keyword "Frégier's Theorem".
Frégier's Point is only defined for points on a conic. CO-Tf3 is defined for all points in the plane. This makes CO-Tf3 a generalization of Frégier's Point.

Relationship with Orthotransversal Line and Pedal Circle wrt a CO-inscribed Triangle

For any CO-inscribed triangle Tr and ***any P on CO***:

- CO-Tf3(P) lies on the orthotransversal line of P w.r.t. Tr
- $CO\text{-}Tf3^{-1}(P)$ lies on the pedal circle of P w.r.t. Tr.

Therefore, given any two triangles inscribed in the same conic CO, the two orthotransversal lines of P meet in CO-Tf3(P), the two pedal circles of P meet in $CO\text{-}Tf3^{-1}(P)$. In particular, this holds when we deal with Component Triangles of a Quadrangle and it proves (by choosing CO circumscribed to the Quadrangle and passing through P) that
(1) the 4 orthotransversal lines have a common point. See Ref-59, AoPS, VU Thanh Tung, 2015, June 26 and Ref-34, QFG#1233).
(2) the 4 pedal circles have a common point. See Ref-59, AoPS, Luis Gonzales, 2011, August 7 and Ref-34, QFG#1218).
For a description of the connection of both properties see Ref-34, QFG#1252, Benedetto Scimemi, August 2015).

Barycentric Coordinates:

Let Reference Conic CO be defined by 5 points (1:0:0), (0:1:0), (0:0:1), (p:q:r), (u:v:w), then CO has equation **X y z + Y z x + Z x y**,
where X = p u (q w - r v), Y = q v (r u - p w), Z = r w (p v - q u).

and CO-Tf3[(x:y:z)] will have these barycentric coordinates:

$$\begin{aligned}(X (+a^2 (x + y + z) - b^2 (x + y - z) - c^2 (x - y + z)) - 2 a^2 (Y z + y Z) :\\ Y (-a^2 (x + y - z) + b^2 (x + y + z) + c^2 (x - y - z)) - 2 b^2 (X z + x Z) :\\ Z (-a^2 (x - y + z) + b^2 (x - y - z) + c^2 (x + y + z)) - 2 c^2 (X y + x Y))\end{aligned}$$

The barycentric coordinates of $CO\text{-}Tf3^{-1}$[(x:y:z)] are much longer.

Cartesian Coordinates:

Cartesian Coordinates for both of CO-Tf3 and $CO\text{-}Tf3^{-1}$ are very easily written if one represents CO by P = [a cos ∂, b sin ∂] for ellipses or [a / cos ∂, b tan ∂] for hyperbolas.
Then CO-Tf3([x,y]) = [x, - y]$((a2 -b2)/(a2 + b2))^{\pm 1}$.
If CO is the parabola y = x^2 then CO-Tf3([x,y]) = [-x, y+1].

Examples of CO-Tf3 and CO-Tf3^{-1} Transformations:

The action of CO-Tf3 can be read from left to right.

The action of CO-Tf3^{-1} can be read from right to left.

	Reference Conic			
Triangle	Steiner Inellipse / Steiner Circumellipse	X(5988)	X(1)	
		X(2)	X(2)	X(2)
		X(114)	X(3)	X(1352)
		X(98)	X(4)	X(6776)
		X(6036)	X(5)	X(182)
		X(115)	X(6)	X(2549)
		X(1281)	X(8)	
			X(10)	X(3923)
			X(11)	X(5091)
		X(6109)	X(13)	X(10654)
		X(6108)	X(14)	X(10653)
		X(6115)	X(15)	
		X(6114)	X(16)	
		X(1513)	X(98)	
		X(230)	X(115)	
Quadrangle	Any Conic circumscribing Reference Quadrangle	QA-P2	QA-P4	
	QA-Co1	QA-P6	QA-P2	QA-P23
		QA-P36	QA-P11	
		QA-L1	QA-L4	QA-L1
		QA-Lx *)	QA-L2	
	QA-Co3	QA-P23	QA-P6	
		QA-P30	QA-P12	
		QA-P3.QA-P4	QA-L1	
		QA-P2.QA-P23	QA-L2	
		QA-Ly *)	QA-L4	
		QA-L8	QA-L8	
Pentangle	Circumscribed Conic of Pentangle	5P-s-P1	5P-s-P1	
		5P-s-Px **)	5P-s-Py **)	

*) QA-Lx = line through QA-P6 parallel to QA-P2.QA-P23
QA-Ly = line through QA-P23 parallel to QA-L1

**) 5P-s-Px = Conic Center of the five 5P-versions of QA-P2
5P-s-Py = Conic Center of the five 5P-versions of QA-P4
Warning: In a Pentangle the Scimemi Transformation CO-Tf3 wrt the circumscribed conic is identical to the inverse of the 5P-s-Tf3 Transformation.

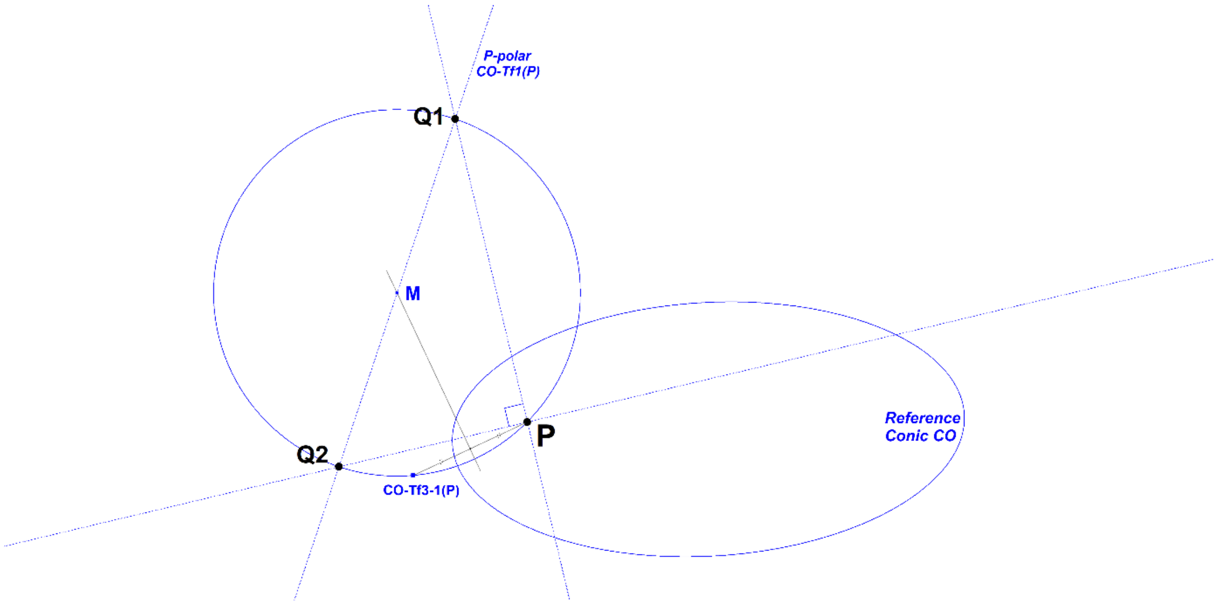
These results were found by Eckart Schmidt, Benedetto Scimemi and the author of EQF/QPG. There will be probably many more appealing examples.

CO-Tf4 Rectangular Selfpolar Triangle

Let CO be the Reference Conic and P be a random point.
There are two points Q1 and Q2 on the polar of P wrt CO, such that triangle P.Q1.Q2 is a rectangular selfpolar triangle wrt CO,
meaning that:

- Q1.Q2 is the polar of P wrt CO,
- P.Q1 is the polar of Q2 wrt CO,
- P.Q2 is the polar of Q1 wrt CO,
- P.Q1 ⊥ P.Q2.

The rectangular selfpolar triangle P.Q1.Q2 can be constructed as follows:
1. Construct CO-Tf3^{-1}(P), the inverse of the Scimemi Transformation CO-Tf3 of P wrt CO.
This point lies on the circumcircle of P.Q1.Q2.
2. Let M be the intersection point of the P-polar CO-Tf(P) and the perpendicular bisector of P.CO-Tf3^{-1}(P)).
3. Q1 and Q2 are the intersection points of the P-polar and the circle centered in M through P.



Properties:

- Let 5P be a 5-Point and P a random point. Construct in each of the 5 Component Quadrangles of the 5P the 4P-Cubic(P). The 4P-Cubic(P) is constructed conform QA-Cu7, only with a variable pivot P instead of the fixed pivot QA-P4.
  Apart from P, there will be 2 points Q1 and Q2 through which all Cubics pass. P, Q1, Q2 form the Rectangular Selfpolar Triangle of P wrt the 5P-circumscribed conic. See [34], QFG#3445, #3640, #3641, # 3642.

Constructions of Conical items using the Scimemi Transformation

Construction of Center and Axes of a Conic using the Scimemi Transformation

When the Scimemi Transformation of some point P is known the center and the minor and major axes of a conic can be easily constructed.
Let P be some point on the conic CO.
Let P1 = Scimemi Transformation of P wrt CO.
Let P2 = 2^{nd} generation Scimemi Transformation of P wrt CO.
Then line P.P2 is a line through the Conic Center
By applying twice this construction we get 2 lines crossing in the Conic Center and consequently CO-P1 is determined.
By drawing a line through CO-P1 parallel to the angle bisector of angle P.P1.P2 we get the minor axis. By drawing the line through CO-P1 perpendicular to the minor axis we get the major axis.

Construction of the Normal and Tangent at P to a Conic using the Scimemi Transformation

Let P be some point on reference conic CO. Let P1 = 1^{st} Scimemi Transformation of P wrt CO.
Then:

1. P.P1 will be the Normal at P to the conic.
2. the line through P perpendicular at P.P1 will be the tangent at P to the conic.

Construction of the Parabola Axis

Let P1,P2 be some random points wrt some parabola CO.
Then Parabola Axis = Midpoint(P1,CO-Tf3(P1)).Midpoint(P2,CO-Tf3(P2)).

CO-Tf4 Conical Orthopolar

This transformation maps a line into another line.
CO-Tf4 is the equivalent of 5P-s-Tf4.

Construction of CO-Tf4(L):

1. Let L be a random line.
2. CO-Tf4(L) is the locus of CO-Tf3(P) with P varying on L.

Another construction of CO-Tf4(L):

This construction was found by Eckart Schmidt. See Ref-34, QFG#2811.

1. Let L be the line to be transformed.
2. Let Pl be the pole (CO-Tf2) of L.
3. Let Lp be the line through Pl perpendicular to L.
4. Let Lc be the reflection of P.CO-P1 in the main axis of CO.
5. Then CO-Tf4(L) is the polar of the intersection point of Lp and Lc.

CO-2P1 Foci of a Conic

CO-2P2 Vertices of a Conic

CO-2L1 Axes of a Conic

CO-2L2 Directrices of a Conic

CO-2L3 Asymptotes of a Conic