Encyclopedia of Cubic Elements

Exploring curves of 3^{rd} degree.

Content

CU-P-Co1 P-Polar Conic of a Cubic (P on CU)
CU-Q-Co1 Q-Polar Conic of a Cubic (Q not on CU)
CU-P-Tf1 P-Involutary Conjugate of X
CU-P-Tf2 1st CU_P Transformation
CU-L-4P1 4 Poles of a Line
CU-L-12P1 L-Desmic System
CU-L-L1 L-Tangential Line (with 3 Pi-Tangential Points)
CU-L-L2 L-Mean Line of Intersection

CU+2P derived items
CU-2P-P1 3rd intersection point of CU and 2P-Line
CU-2P-L1 Line through 2 given CU-points
CU-2P-L2 Mixed XY-Polar

CU+3P derived items
CU-3P-P1 CU_3P PQR Addition Point
CU-3P-9P1 9 3P-Osculating Conic Points

CU+4P derived items
CU-4P-P1 Cotterill's Point / Coresidual point
CU-4P-P2 CU-CB4d Point
CU-4P-cHe1 CU-inscribed Complete Hexagon
2CU-4P-Co1 CU-4P-Common Points Conic

CU+5P derived items
CU-5P-P1 CU-6th Intersection point 5P-Conic with CU
CU-5P-Co1 CU-5P-Conic
CU-5P-Co2 CU-5P-Tangential Conic (with 6 Pi-Tangntl Pts)

CU+6P derived items
CU-6P-cDe1 CU-inscribed Complete Decagon

CU+7P derived items
CU-7P-P1 CU-7P-CB Pivot Point
CU-7P-Tf1 CU-7P-Geiser Involution/Transformation
2CU-7P-L1 7P-Cubics Intersection Line

CU+8P derived items
CU-8P-P1 Cayley Bacharach Point for a Regular Cubic

CU+9P derived items
CU-9P-P4 Tangential of CU_9P-P1
CU-9P-3P1 IP-Sumpoints of the set (P1,…,P9)

Note: Not all items have been incorporated into this paper thus far.

CU General Cubic

CU-1: Notes on the General Cubic

General Approach to Cubics
Many papers have been written about cubic curves, often enriched with examples involving triangles, quadrilaterals, special points, and other geometric configurations.
But what are the intrinsic geometric properties of a cubic when viewed purely as a third-degree polynomial curve?
In 2023, Eckart Schmidt, Bernard Keizer, and Chris van Tienhoven (the author of this Encyclopedia) began a renewed exploration into the fundamental nature of cubics. Having collaborated for many years on the geometry of triangles, quadri-figures, and n-figures, we encountered numerous remarkable properties of cubic curves. Eventually, we decided it was time to focus exclusively on the cubic itself—stripped of external references—to uncover its core characteristics as a third-degree polynomial curve.
The central question became:
What are the general properties of the general cubic?

Historical Context
Many well-known books and websites have been written about cubics.
Here are a few examples:

- George Salmon, Arthur Cayley. A Treatise on the Higher Plane Curves: Intended as a sequel to A Treatise on Conic Sections.
- George Samon. Traité de géomètrie analytique (courbes planes). ISBN: 1-4212-0859-8
- Julian Lowell Coolidge, A treatise on Algebraic Curves. ISBN: 0-486-49576-0
- Heinrich Edward Schroeter, Die Theorie de ebenen Kurven dritter Ordnung. ISBN: 9783743307506.
- Bernard Gibert made a tremendous contribution by cataloging many types of cubics, most often using a triangle as reference.
- Roger Cuppens, Faire de la Géomètrie supérieure en jouant avec Cabri-Géomètre II. Tome II. APMEP Brochure no 125, ISBN: 2-912846-38-2.

Undoubtedly, I've omitted many names of researchers who have made significant contributions in the ongoing quest to understand the cubic, each from their own unique perspective.

A Different Approach
In the *Encyclopedia of Polygon Geometry*, we considered a cubic as a curve defined by nine points—or seven, in the case of a circular cubic.
Let us now define a cubic simply as a third-degree polynomial curve, abbreviated as **CU**.
Let **9P** represent a set of nine distinct points in the complex projective plane.
When comparing the cardinality of the set of all 9P's to the set of all CU's, we observe that:
|9P| > |CU|,
where **|X|** denotes the cardinality (i.e., the number of elements) in set X.
This observation reveals two key facts:

- Given a 9P, there exists a unique cubic CU passing through it.
- Given a CU, there are many possible 9P's lying on it.

This implies that a cubic cannot be uniquely described by a single 9P.
Therefore, neither the BG-style nor the QPG-style fully captures the essence of a general cubic.
In our search for general properties, we must remain aware of the cardinality relationships between cubics and sets like |3P|, |4P|, |7P|, |9P|, etc.

Cardinality in Transformations
We encounter similar issues with the **Scimemi Transformation CO–Tf3**.
Originally introduced by Benedetto Scimemi as a 5P-transformation, it was later recognized as a conical transformation. It is now described as CO–Tf3, with properties valid for any set of five points on a reference conic.
Another example of cardinality's importance is the relationship between **Quadrangles (QA)**, **Quadrilaterals (QL)**, and **Quadrigons (QG)**:
|QA| < |QG| and **|QL| < |QG|**.
One cannot properly describe quadrilaterals without accounting for these cardinality rules.

Complex Projective Plane
In EPG, we work within the **Complex Projective Plane**, where we distinguish between finite and infinite points, as well as between real and imaginary points.

Main Results
Some key findings about general cubics include:

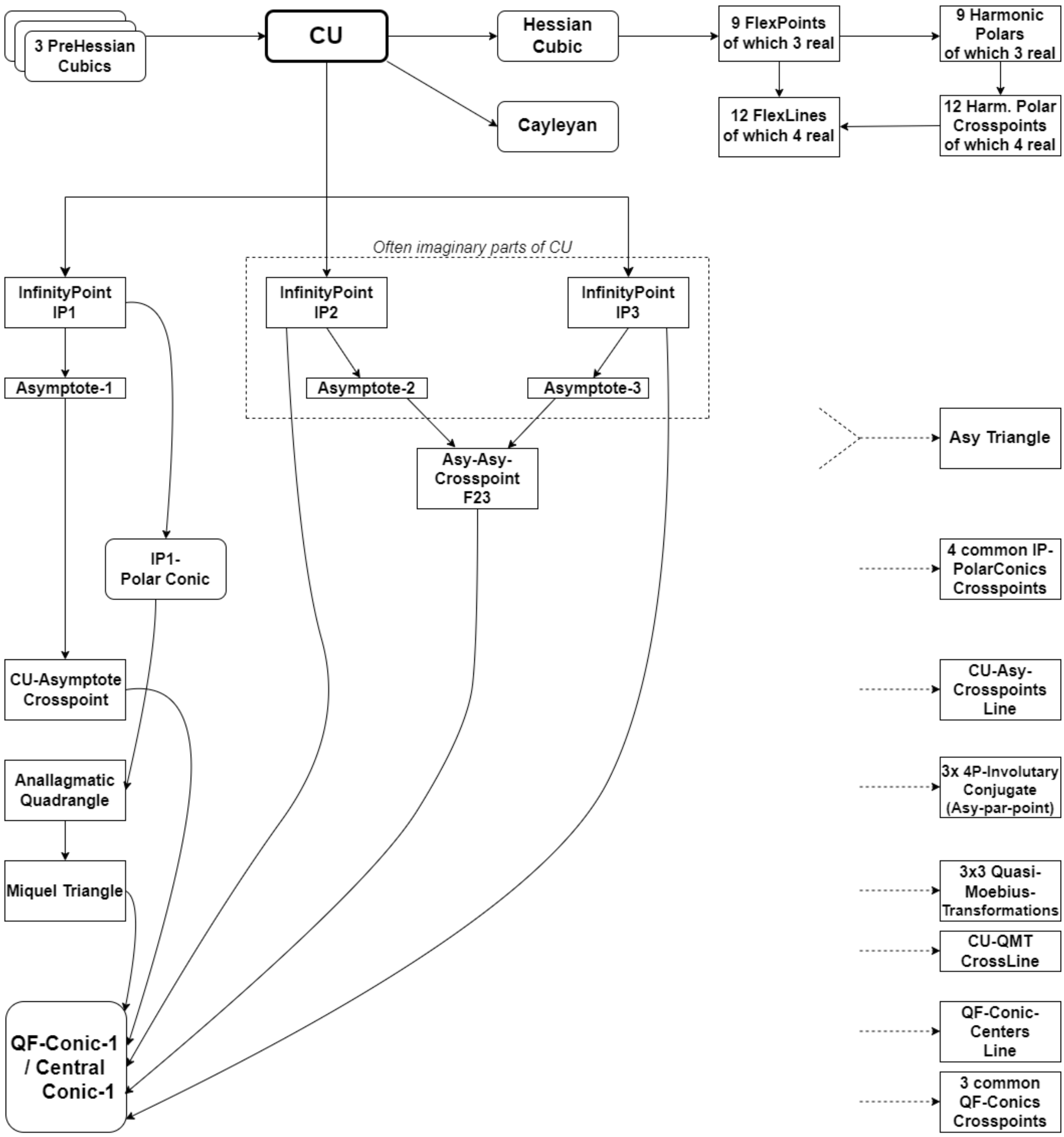
- A cubic has at least **1 real Asymptote** (out of 3).
- A cubic has at least **2 real Anallagmatic Points** (out of 12).
- A cubic has at least **1 real Miquel point** (out of 9).
- A cubic has at least **1 real Möbius transformation** (out of 9).
- A cubic has at least **1 real Central Conic (QF-conic)** (out of 3).
- A cubic has at least **1 real Diametral Conic** (out of 3).
- A cubic has **3 real Flexpoints** (out of 9) and **3 real Harmonic Polars** (out of 9).
- A cubic has **4 real Flexlines** (out of 12) and **4 real Harmonic Polar Crosspoints** (out of 12).

All of these elements can be constructed for any cubic.
This list is not exhaustive.

Note:
The concepts of A*nallagmatic points*, *Miquel point*, and *Möbius transformation* are treated here as **quasi-versions** of their classical definitions, meaning that there is a new 'extended' definition, implied by the cubic environment. For clarity, we refer to them as such when associated with general cubics. The specific meaning of each quasi-version is explained in detail within the Encyclopedia.

CU-2:

Basic structure of the General Cubic CU



All these items exist for any Cubic.
For every item mentioned under IP1, there will be a corresponding items for IP2 & IP3.
When IP2 & IP3 are imaginary, then these corresponding items will also be imaginary.
When IP2 & IP3 are the circular points at infinity, then CU will be a circular cubic.
There will also be Asy-Asy-Crosspoints Asy1-Asy2 & Asy1-Asy3.
Asy-Asy-Crosspoints are real when the referred asymptotes are both real or both imaginary.

CU-3 Table with Types of Cubics

TYPES OF CUBICS

		1-/2-parts	Real Inf.Pts	Im.Inf.Pts	Real Asy's	CU-Asy-Crosspoints	Diametral Conic(s)	Central Conic(s)	Anallagmatic Points	Miquel-points	Isocubic	Comment
CUc1	Monopartite Circular Cubic	1-part	IP1	CI1,CI2	Asy1	Q1 finite	Hyperbola (P1,P2,IP1)	Circle (Q,F,M,Ce)	P1,P2	M	no	
CUc2	Bipartite Circular Cubic	2-part	IP1	CI1,CI2	Asy1	Q1 finite	Hyperbola	Circle	P1,P2,P3,P4	M1a,M1b,M1c	yes	
CU1	Monopartite cubic	1-part	IP1	SI1,SI2	Asy1	Q1 finite	Hyperbola (P1,P2,IP1)	Ellipse (Q,F,M,Ce)	P1,P2	M	no	P1,P2,M collinear
CU2	Bipartite cubic with 1 real asy	2-part	IP1	SI1,SI2	Asy1	Q1 finite	Hyperbola	Ellipse	P1,P2,P3,P4	M1a,M1b,M1c	yes	
						Q1 infinite	Asy + 3P-Anall.Line	Inf.Line + 3P-Anall.Line	3 finite/1 infinite P's	2 finite/1 infinite M's		P1,P2,P3 collinear
CU3a	Tripartite cubic with 3 real asy's	1-part	IP1,IP2,IP3	--	Asy1,Asy2,Asy3	Q1,Q2,Q3 finite	Hyperbola	Hyperbola (IPi,IPj)	how many?	how many?	???	
						Q1,Q2,Q3 infinite	Hyperbola	Inf.Line + MP-Anall.Line	how many?	how many?	???	
						1 or 2 Q's Infinite?	Hyperbola	How ?	how many?	how many?	???	
CU3b	Tripartite cubic with 1 part-one asy	2-part									yes	
CU4	Quadripartite cubic with 3 real asy's	2-part	IP1,IP2,IP3	--	Asy1,Asy2,Asy3	Q1,Q2,Q3 finite	Hyperbola	Hyperbola (IPi,IPj)	how many?	how many?	???	
						Q1,Q2,Q3 infinite	Hyperbola	degenerated?	how many?	how many?	???	
						1 or 2 Q's Infinite?	Hyperbola	How ?	how many?	how many?	???	

1-part *Cubic-parts can be connected in their infinty points, making that for example a tripartite cubic is made up from one part. 1-way cubics have always 2 P-tangential points.*
2-part *But a cubic with an oval is made up from 2 parts. See QPG#2067, property Schröter. 2-way cubic have 0 or 4 P-tangential points.*
Real Inf.Pts *Real Infinity Points of the Cubic*
IP1,IP2,IP3 *Infinity Points of the Cubic*
Im.Inf.Pts *Imaginary Infinity Points on the Cubic*
CI1,CI2 *Circular points at Infinity*
SI1,SI2 *Similarity points at Infinity*
Asy1,Asy2,Asy3 *Asymptote-names*
CU-Asy-Crosspoints *Intersection Points of the Asymptote with the Cubic*
Q1,Q2,Q3 *Names of the Intersection points of the cubic with one of its asymptotes*
Asy *Asymptote*
3P-Anall.Line
3P-Anall.Line *Line through 3 collinear Anallagmatic Points*
MP-Anall.Line *Line through 1 Anallagmatic Point and 1 Miquel Point*
Inf.Line *Line at infinity*
Anallagmatic Points *Points where lines parallel to the asymptote touch the Cubic*
P1,P2,P3,P4 *Anallagmatic Points*
Miquel-points *Transformation Centers on the cubic*
Isocubic *Cubic with Isoconjugation property*

CU-Types of Cubics-Table-02

CU-4 CU Point Addition Method

Summary
This method describes a mathematical operation on points that lie on a specific cubic curve (CU). Through a geometric construction, a group structure is defined in which two points on the curve are combined to form a third. The CU Point Addition Method forms the core of this structure and provides a consistent way to perform point additions within the domain of the curve, giving rise to numerous applications in which conjectures can be tested and validated.

Historical Context
The method of point addition on elliptic curves originates from Niels Henrik Abel (1802–1829). Abel studied elliptic integrals and discovered that they cannot always be expressed in terms of elementary functions.
His work led to the concept of elliptic functions, which were later directly connected to elliptic curves. He introduced the idea of a group structure on curves via analytic functions.
Other mathematicians such as Carl Gustav Jacobi (1804–1851), Karl Weierstrass (1815–1897), and Henri Poincaré (1854–1912) expanded the theory.
Fred Lang wrote the article *Geometry and Group Structures of Some Cubics*, published in *Forum Geometricorum*, Vol. 2 (2002). Lang generalizes the method for broader classes of cubic curves in the projective plane. Among other things, he introduces a neutral element O.
See [77].

What does point addition involve?
On an elliptic curve, two points P and Q can be added as follows:

- Draw a line through P and Q.
- Determine the third intersection point of that line with the curve.
- Reflect that point across the x-axis (in the real plane).
- The result is the point P + Q.

This was the known method.
Lang generalizes this approach for broader classes of cubic curves in the projective plane. He introduces a neutral element O and defines the group operation as:
$P + Q = (P . Q) . O$
where $P . Q$ is the third intersection point of the line through P and Q, and $. O$ is an additional operation involving the neutral point.

Some notable insights from Lang's work:

- The operation is commutative, but not always associative without additional conditions.
- He uses this structure to analyze properties of classical cubic curves such as those of Thomson, Darboux, and Lucas.
- Connections are made between collinearity of points and their sum in the group:
  - Three CU-points are collinear ⇔ their sum equals a fixed point N.
  - Six CU-points lie on a conic ⇔ their sum is 2N.
  - Nine CU-points lie on another cubic curve ⇔ their sum is 3N.

It is remarkable how many problems can be solved using these simple rules.

ELABORATION

Definitions and conventions

- Let CU be a nonsingular reference cubic curve without cusp or node in the complex projective plane.
- Let O be some point on CU functioning as Origin.
- Points Pi (P1, P2, P3, etc.) are supposed to be reference points on CU.
- Two operations are used, the . operation and the + operation.
  - *P1·P2 is defined as the third intersection of the line P1P2 with CU.*
  - *P1+P2 is defined as (P1.P2).O*
- The + operation defines a group structure on the points on CU, satisfying closure, associativity ((a + b) + c = a + (b + c)), identity and inverse. Here the identity element is 0, and the inverse is the negative. The group is abelian, meaning the operation is also commutative (a + b = b + a).
- The tangential point of a point P is the intersection of CU with the tangent at P.
- Denote the sum P1+P2+ … +Pi by PPi.

Rules for adding points

1. Any point O on a reference cubic CU can be choden as the origin. Let N be the tangential point of O.
2. You can assign values to any point on the cubic. O has value 0 (zero). N has value N, and other points on the cubic are valued bu their name.
3. -P1 = O.P1
4. N-P1-P2 = P1.P2
5. P1+P2 = O.(P1.P2)
6. When P1,P2,P3 on CU are collinear then P1+P2+P3 = N
7. When P1 through P6 on CU are coconic then P1+P2+P3+P4+P5+P6 = 2N
8. When P1 through P9 on CU lie on another cubic, then P1+P2+ … +P9 = 3N
9. For intersection points of CU with curves of degree n (n>3), then P1+ … +Pm=nN, where m=3n.
10. The equation kX = Q has k^2 (eventually complex) solutions for X. Such a point is called a **torsion point**.

These rules serve as general principles for point addition on a cubic, derived from group theoretic properties.

APPLICATIONS

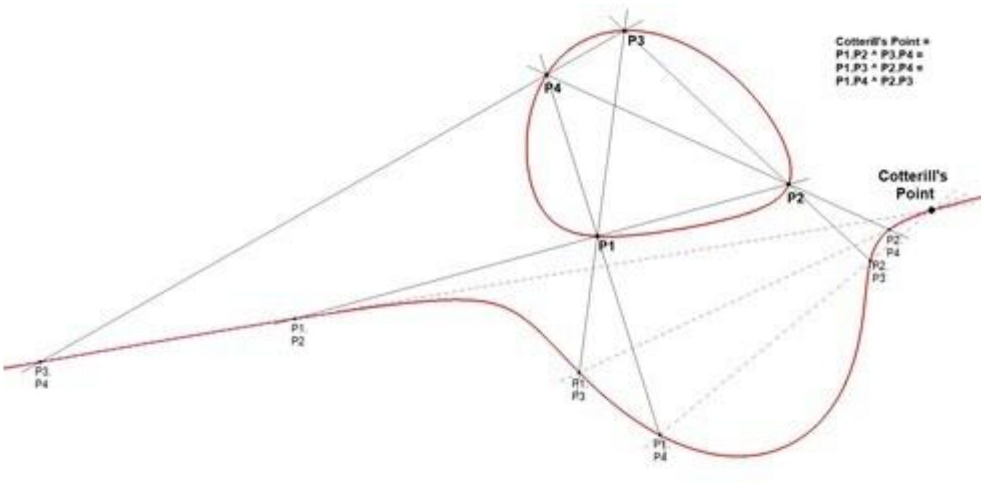
1. Cotterill's Point of P1 through P4 is P1+P2+P3+P4 – N, abbreviated as (PP4–N).
2. The Cayley-Bacharach point of P1 through P8 is (3N-PP8).
3. The 7P-pivot.

1. Cotterill's Point

Cotterill's Point of P1, P2, P3, P4 is (P1 . P2) . (P3 . P4).
It is also equal to (P1 . P3) . (P2 . P4), and to (P1 . P4) . (P2 . P3), making it a special point.
Using the point-value rules (especially rule 6), we find:

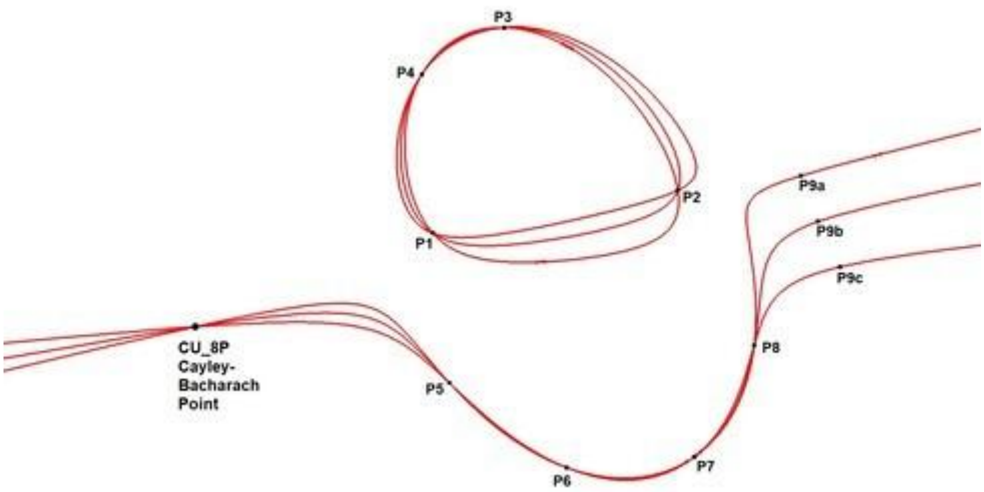
- (P1 . P2) . (P3.P4) = N – (N - P1 - P2) – (N - P3 - P4) = P1 + P2 + P3 + P4 – N
- (P1 . P3) . (P2.P4) = N – (N - P1 - P3) – (N - P2 - P4) = P1 + P2 + P3+ P4 – N
- (P1 . P4) . (P2.P3) = N – (N - P1 - P4) – (N - P2 - P3) = P1 + P2 + P3+ P4 – N

This conforms that all three constructions yield the same point-values, this proves the existence of a unique point - **Cotterill's Point**.



2. Cayley-Bacharach point

Cayley-Bacharachs Theorem states that if P1 through P8 lie on a cubic, then every other cubic passing through these eight points also passes through a common ninth point.
The 9th point is the Cayley-Bacharach Point of the eight.
Define CB9 =be the Cayley-Bacharach Point of P1 through P8.
From rule 8: P1 + P2 + … + P8 + CB9 = 3N.
So: CB9 = 3N - PP8.
This leads to a construction method:
CB9 = (P1 . P2) . (P3 . P4) . (P5 . P6) . (P7 . P8)
Here, Pi . Pj means the third intersection point of CU and line PiPj".
Using rule 6, this construction confirms CB9 = 3N - PP8.
Any permutation of the eight points yields the same CB9, proving its uniqueness.



3. The 7P-Pivot

Let P1 through P7 be fixed reference points on CU. L
et Px and Py be two additional random points on CU.
Each set of eight points define a Cayley-Bacharach Point:

- CBx for (P1, …, P7, Px)
- CBy for (P1, …, P7, Py)

Their intersection point lies on CU and turns out to be a fixed point—the 7P-Pivot.
This point was contributed by Eckart Schmidt.

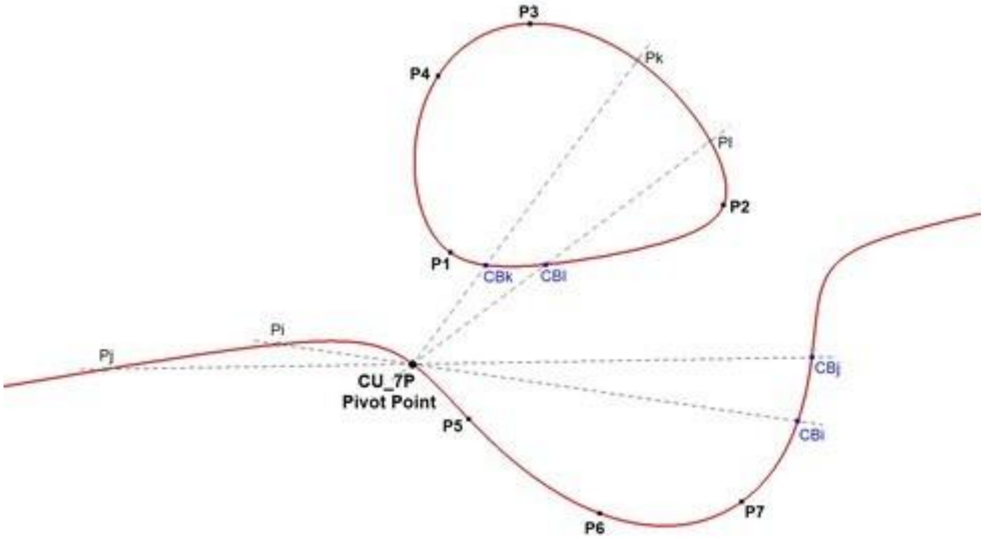
Proof
Recall:

- Pi.Pj = (Pi + Pj) - N
- CBi = 3N - (PP9 - Pi)

Let:

- Tx = Px . CBx, then Tx = (Px + (3N - (PP7 + Px)) - N) = 2N - PP7
- Ty = Py . CBy, then Ty = (Py + (3N - (PP7 + Py)) - N) = 2N - PP7

Both Tx and Ty are constructed both as points on CU and have the same value: 2N – PP7.
Since Px and Py do not appear in the final expression, the point is independent of them.
Thus, the 7P-Pivot is a unique point on CU determined solely by the seven reference points.



CU-5 Normal Form of a Cubic Equation

General form of a cubic equation

A cubic equation in barycentric coordinates can be expressed in general form as:

$$\mathbf{c_1\, x^3 + c_2\, y^3 + c_3\, z^3 + c_4\, x^2\, y + c_5\, x^2\, z + c_6\, x\, y^2 + c_7\, y^2\, z + c_8\, x\, z^2 + c_9\, y\, z^2 + c_0\, x\, y\, z = 0}$$

where the coordinates refer to a reference triangle ABC with vertices (1,0,0), (0,1,0) and (0,0,1).

First normal form of a cubic equation

When a reference triangle is chosen, the corresponding reference points play a pivotal role. Each target point is defined by coordinates relative to these reference points. To express the same target point in relation to a second reference triangle, a point reference transformation is required. This transformation pTf can be formulated as follows.

Let target point X(x,y,z) be identified with respect to a reference triangle with vertices (1,0,0), (0,1,0) and (0,0,1). Let triangle P1(p1x,p1y,p1z), P2(p2x,p2y,p2z), P3(p3x,p3y,p3z) be the second reference triangle. Then the new coordinates of X with respect to triangle P1P2P3 will be defined by:

pTf (P1, P2, P3, X) =
(Det[(X, P2, P3)] (p1x+p1y+p1z),
Det[(P1, X, P3)] (p2x+p2y+p2z),
Det[(P1, P2, X)] (p2x+p2y+p2z)),

where Det is the determinant of the 3x3-matrix formed by the 3 sets of point coordinates indicated. The point reference transformation applied for all points (x,y,z) on CU from reference triangle (1,0,0), (0,1,0) and (0,0,1) to reference triangle P1,P2,P3 will use pTf^{-1} for alle points on CU.

Let H1H2H3 be the real flex triangle $H_1H_2H_3$ of the reference cubic. This is the triangle bounded by the 3 real flexlines, each passing through one of the 3 real flexpoints.

When the reference triangle is changed to this flex triangle, the equation of the cubic simplifies to a form, here called the first normal form:

$$\mathbf{b_1\, x^3 + b_2\, y^3 + b_3\, z^3 + b_0\, x\, y\, z = 0}.$$

In order to connect to later developments the first normal form is rearranged to:

$$\mathbf{a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

So when we work with the real flex triangle $H_1H_2H_3$ as reference triangle then there are coefficients (a_1, a_2, a_3, k) such that:

$$\mathbf{CU = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

Second normal form of a cubic equation

Another transformation can be applied mapping **(a_1 x)** → x, (a_2 **y)** → y, **(a_3 z)** → z, resulting in the second normal form:

$$\mathbf{x^3 + y^3 + z^3 + k\, x\, y\, z = 0}$$

This form is known as **Hesse's Form**.

Thus, after performing two transformations, the equation of a cubic in its general form is transformed (projectively equivalent) into Hesse's form.

For a proof of the existence of a projectively equivalent mapping from any smooth cubic in general form to Hesse's form see [1], page 2 and [2], page 4.

Flexpoints, Flexlines and Harmonic Polars in first normal form

After the projective transformation next coordinates/equations follow from consequent calculations.
The *9 flexpoints* are

$F_1 = (0, -a_3, a_2)$	real point
$F_2 = (-a_3, 0, a_1)$	real point
$F_3 = (-a_2, a_1, 0)$	real point
$F_4 = (0, -i_2\, a_3, a_2)$	imaginary point
$F_5 = (0, -i_1\, a_3, a_2)$	imaginary point
$F_6 = (-i_2\, a_3, 0, a_1)$	imaginary point
$F_7 = (-i_1\, a_3, 0, a_1)$	imaginary point
$F_8 = (-i_2\, a_2, a_1, 0)$	imaginary point
$F_9 = (-i_1\, a_2, a_1, 0)$	imaginary point

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
The coordinates of all flexpoints show that they all lie on one of the sidelines of the reference triangle.

The *12 Flexlines* are:

$L_{123} = (a_1, a_2, a_3)$	real Flexline $F_1F_2F_3$: $a_1\, x + a_2\, y + a_3\, z = 0$
$L_{145} = (1, 0, 0)$	real Flexline $F_1F_4F_5$: $x = 0$
$L_{267} = (0, 1, 0)$	real Flexline $F_2F_6F_8$: $y = 0$
$L_{389} = (0, 0, 1)$	real Flexline $F_3F_7F_9$: $z = 0$
$L_{179} = (i_2\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_7F_9$
$L_{168} = (i_1\, a_1, a_2, a_3)$	imaginary Flexline $F_1F_6F_8$
$L_{258} = (a_1, i_2\, a_2, a_3)$	imaginary Flexline $F_2F_5F_8$
$L_{249} = (a_1, i_1\, a_2, a_3)$	imaginary Flexline $F_2F_4F_9$
$L_{346} = (a_1, a_2, i_2\, a_3)$	imaginary Flexline $F_3F_4F_6$
$L_{357} = (a_1, a_2, i_1\, a_3)$	imaginary Flexline $F_3F_5F_7$
$L_{478} = (a_1, i_1\, a_2, i_1\, a_3)$	imaginary Flexline $F_4F_7F_8$
$L_{569} = (a_1, i_2\, a_2, i_2\, a_3)$	imaginary Flexline $F_5F_6F_9$

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.
Note that after the projective transformation indeed the triangle bounded by L_{145}, L_{267}, L_{389} is the new reference triangle with sidelines (1,0,0), (0,1,0), (0,0,1), and consequently with vertices H1(1,0,0), H2(0,1,0), H3(0,0,1) of the new reference triangle.

The *9 Harmonic Polars* are:

$L_1 = (0, a_2, -a_3)$	real line
$L_2 = (a_1, 0, -a_3)$	real line
$L_3 = (a_1, -a_2, 0)$	real line
$L_4 = (0, a_2, -i_2\, a_3)$	imaginary line
$L_5 = (0, a_2, -i_1\, a_3)$	imaginary line
$L_6 = (a_1, 0, -i_2\, a_3)$	imaginary line
$L_7 = (a_1, 0, -i_1\, a_3)$	imaginary line
$L_8 = (a_1, -i_2\, a_2, 0)$	imaginary line
$L_9 = (a_1, -i_1\, a_2, 0)$	imaginary line

where i1 and i2 are the primitive cube roots of unity: $i_1 = (-1)^{2/3}$ and $i_2 = -(-1)^{1/3}$. See Note-1.

Harmonic Polar-Crosspoint P0
The 3 real Harmonic Polars are concurrent in **P0 ($1/a_1$, $1/a_2$, $1/a_3$).**
P0 is the trilinear pole wrt the reference triangle $H_1H_2H_3$. of the real Flexline $F_1F_2F_3$: L_{123}.
Conversely L_{123} is the trilinear polar of P0 wrt the reference triangle $H_1H_2H_3$.

The Hessian of CU

We know:

$$\mathbf{CU = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

After calculation the Hessian HE of CU appears as:

$$\mathbf{HE = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k'\, a_1\, a_2\, a_3\, x\, y\, z = 0}$$

where $k' = -(108 + k^3) / 3$

The Syzygetic Pencil

All Cubics passing through the 9 Flexpoints of CU are called cubics of the Syzygetic Pencil.
CU and its Hessian mutually intersect in the 9 Flexpoints of CU. Therefore they are called members of the Syzygetic Pencil.
The 3 sidelines of the real flex triangle $H_1H_2H_3$ (being the 3 real flexlines resp. through F_1, F_2, F_3) form together also a degenerate cubic, which will be called here RF. On these 3 lines all 9 Flexpoints occur. So it is also a member of the Syzygetic Pencil.
Since it is a pencil every Cubic in it can be described as t CU_1 + (1-t) CU_2, where CU_1 and CU_2 are also members of the pencil.
We now ask the question of finding a cubic FE as linear combination of CU and HE such that RF is its hessian.
Specifically we need to find CUx = Hessian of t CU + (1-t) HE.
It appears that there are 9 values of t that provide a solution.
However, upon substituting these values of t, it turns out that there are only 2 distinct solutions.

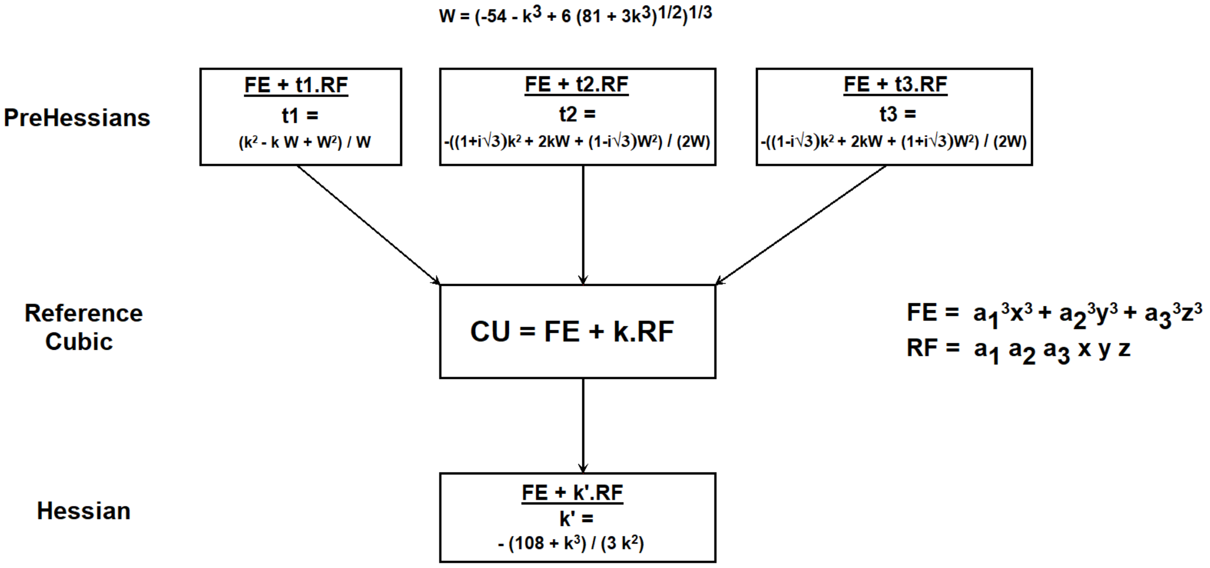
$$\mathbf{FE = a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3}$$

and

$$\mathbf{RF = a_1\, a_2\, a_3\, x\, y\, z.}$$

FE is the main part and will be called here the *Core Curve*, RF is the degenerate part and will be called here the *Residual Curve*.
RF is actual the degenerate cubic consisting of the three sidelines (x=0, y=0, z=0) of the reference triangle, which are the three real Flexlines, which indeed contain the 9 Flexpoints.
Moreover FE and RF are derived from the pencil t CU + (1-t) HE and therefore both pass through the 9 Flexpoints and consequently are members of the Syzygetic Pencil.
It also appears that FE and RF share the same Hessian, namely RF.

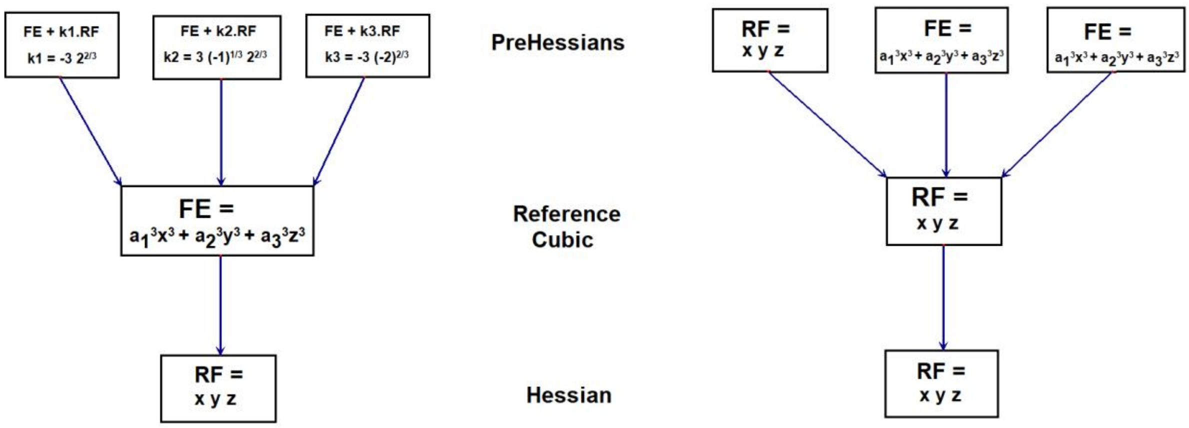
Hessians and PreHessians
Since a Hessian is also a cubic, it is also possible to calculate the Hessian of the Hessian, and further downwards.
Moreover it is possible, knowing some cubic CU, to find out for which upper cubic CU is the Hessian.
There are 3 of these cubics called the PreHessians pHE_1, pHE_2, pHE_3.
So per definition the Hessian of all three cubics pHE_1, pHE_2, pHE_3 will be CU.
So "upwards" there are 3 PreHessians of a cubic and "downwards" there is 1 Hessian per cubic.



CU-3Cu1 CU-HE-pHE1-pHE2-pHE3 picture-03.fig

From the equations of t1, t2, t3 it follows that there will be three real preHessians when $k \le -3$. When $k > -3$, there will be one real preHessian and two imaginary preHessians.

Examples:
The 3 PreHessians of FE are also of the form FE + k.RF. Its Hessian is RF.
The 3 PreHessians of RF are RF itself and FE (counting twice). Its Hessian is again RF.



CU-3Cu1 CU-HE-pHE1-pHE2-pHE3 picture-02.fig

The Cayleyan

Let CU = $\mathbf{a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3 + k\, a_1\, a_2\, a_3\, x\, y\, z = 0}$

The Cayleyan of CU has this expression:
CAY =

$$\begin{aligned}&27\,(a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3)^2\\ &- 4\,(27 + k_2^3)\,(a_1^3\, a_2^3\, x^3\, y^3 + a_1^3\, a_3^3\, x^3\, z^3 + a_2^3\, a_3^3\, y^3\, z^3)\\ &- 18\, k_2^2\, a_1\, a_2\, a_3\, x\, y\, z\,(a_1^3\, x^3 + a_2^3\, y^3 + a_3^3\, z^3)\\ &- k_2\,(108 + k_2^3)\, a_1^2\, a_2^2\, a_3^2\, x^2\, y^2\, z^2 = 0,\end{aligned}$$

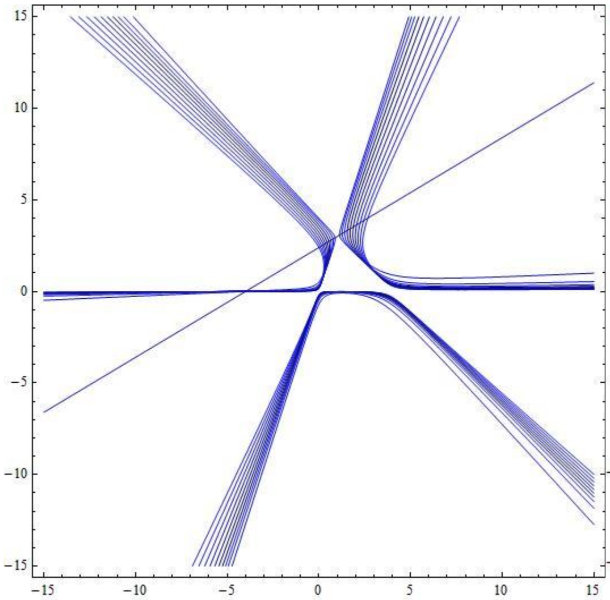
where $k_2 = 6/k - (k/3)^2$

See [66], QPG#2819, #2826, #2827.

Pictures of $CU = a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + k\, a_1 a_2 a_3\, xyz$

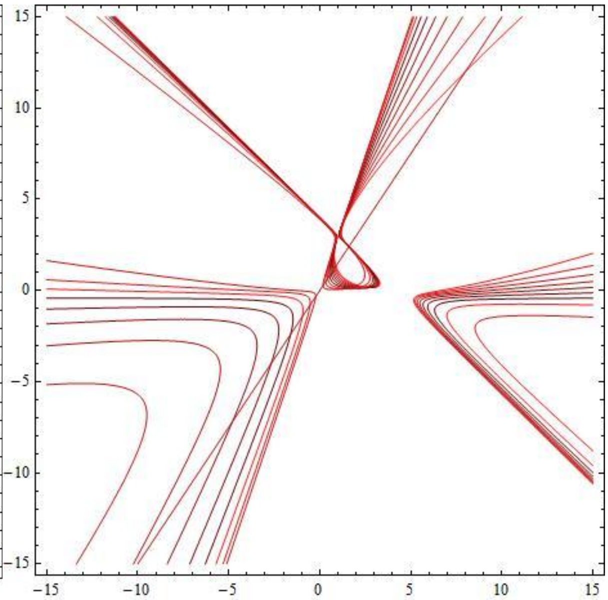
for different values of a1, a2, a3, k

different cubics as a1 varies



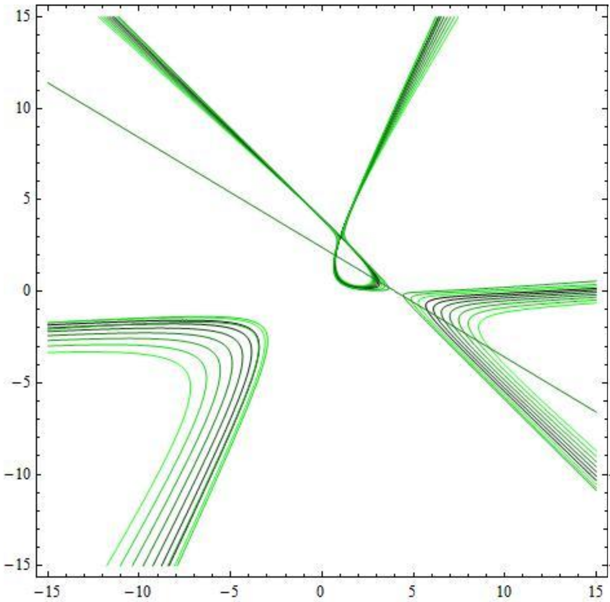
a1 = -10 -> 0, a2 = -2, a3 = -4, k = 100
when a1=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a2 varies



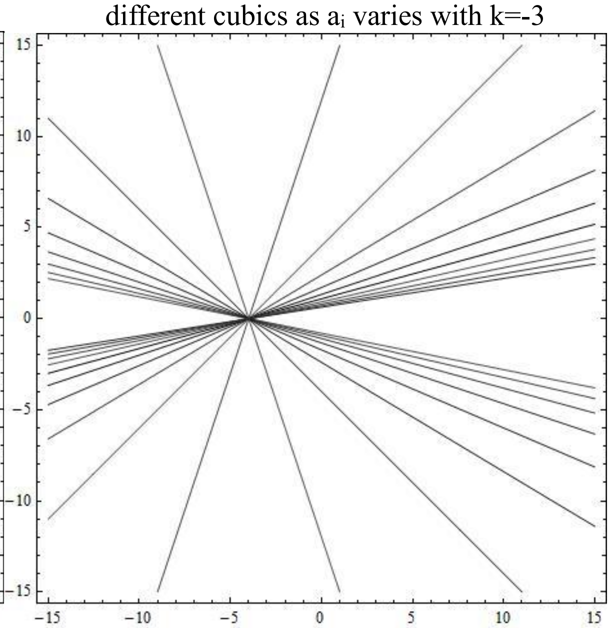
a1 = 1, a2 = -10 -> 0, a3 = -4, k = 100
when a2=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a3 varies



a1 = 1, a2 = -2, a3 = -10 -> 0, k = 100
when a1=0 the cubic degenerates in 3 lines,
one of which is a real flexline

different cubics as a_i varies with k=-3

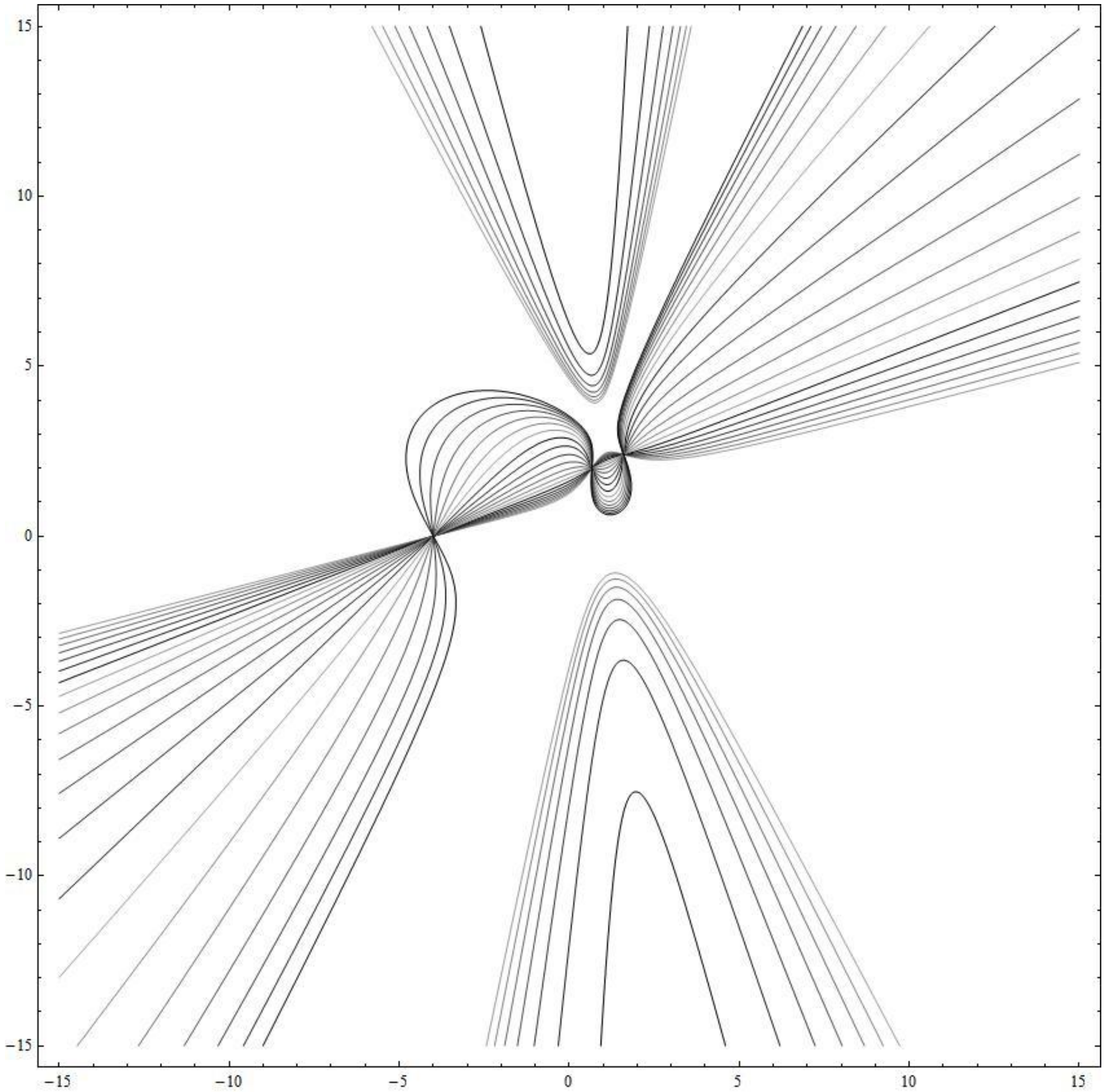


a1 = -10 -> 7, a2 = -2, a3 =-4, k = -3
all cubics degenerate in lines when k = -3
for all values of (a1,a2,a3)

CU-a1is-10-to-0 a2is-2 a3is-4 kis100.jpg
CU-a1is1 a2is-2 a3is-10-to-0 kis100

CU-a1is1 a2is-10-to-0 a3is-4 kis100.jpg
CU-a1is-10-to-0 a2is-2 a3is-4 kis-3.jpg

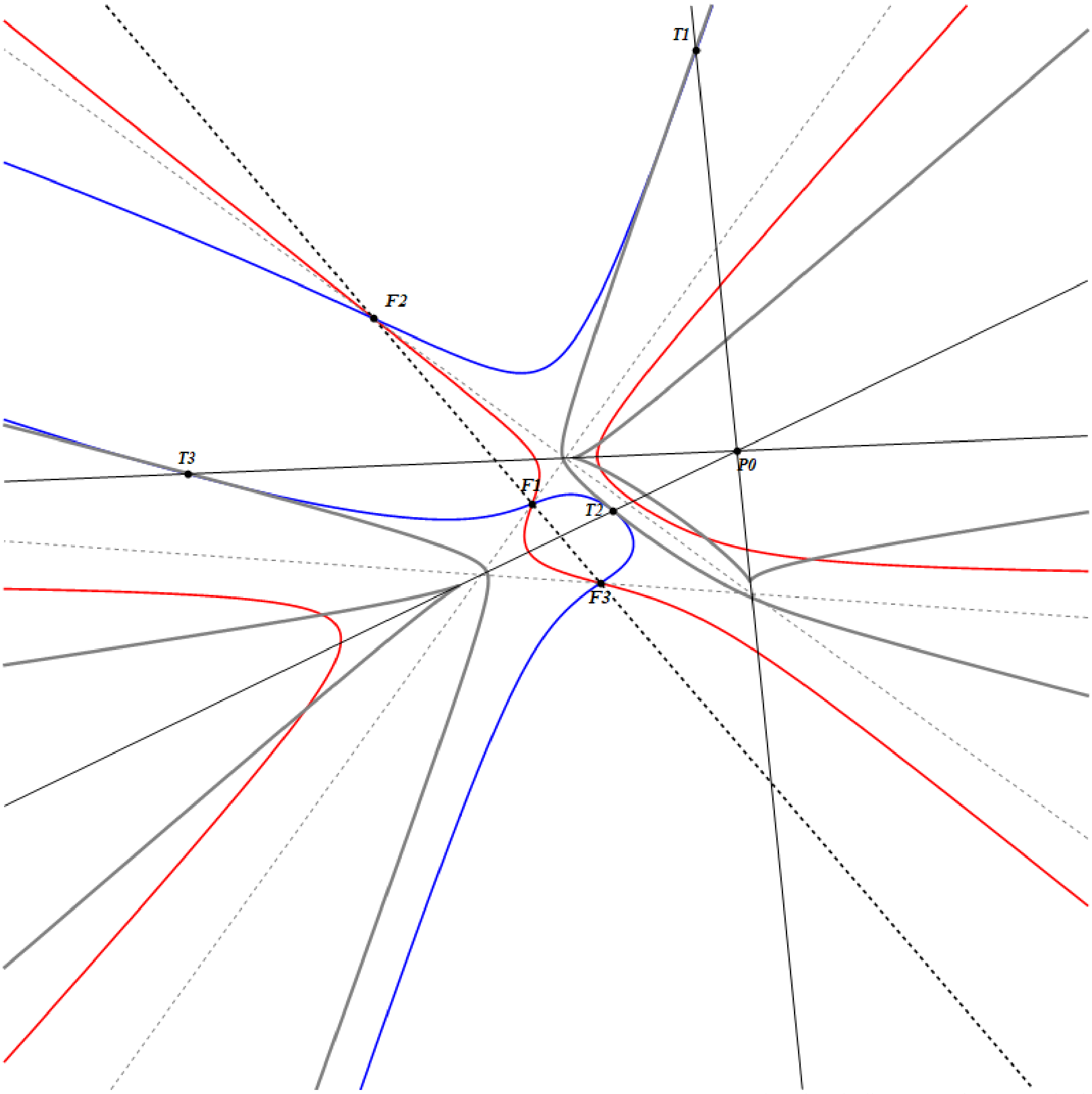
different cubics as variable k varies



CU-a1is1 a2is-2 a3is-4 kis-10to10.jpg

a1 = 1, a2 = -2, a3 = -10 -> 0, k = -10 -> 10
all cubics pass through the 3 real Flexpoints
when k = -3 the cubic degenerates in a real line and an (imaginary) conic

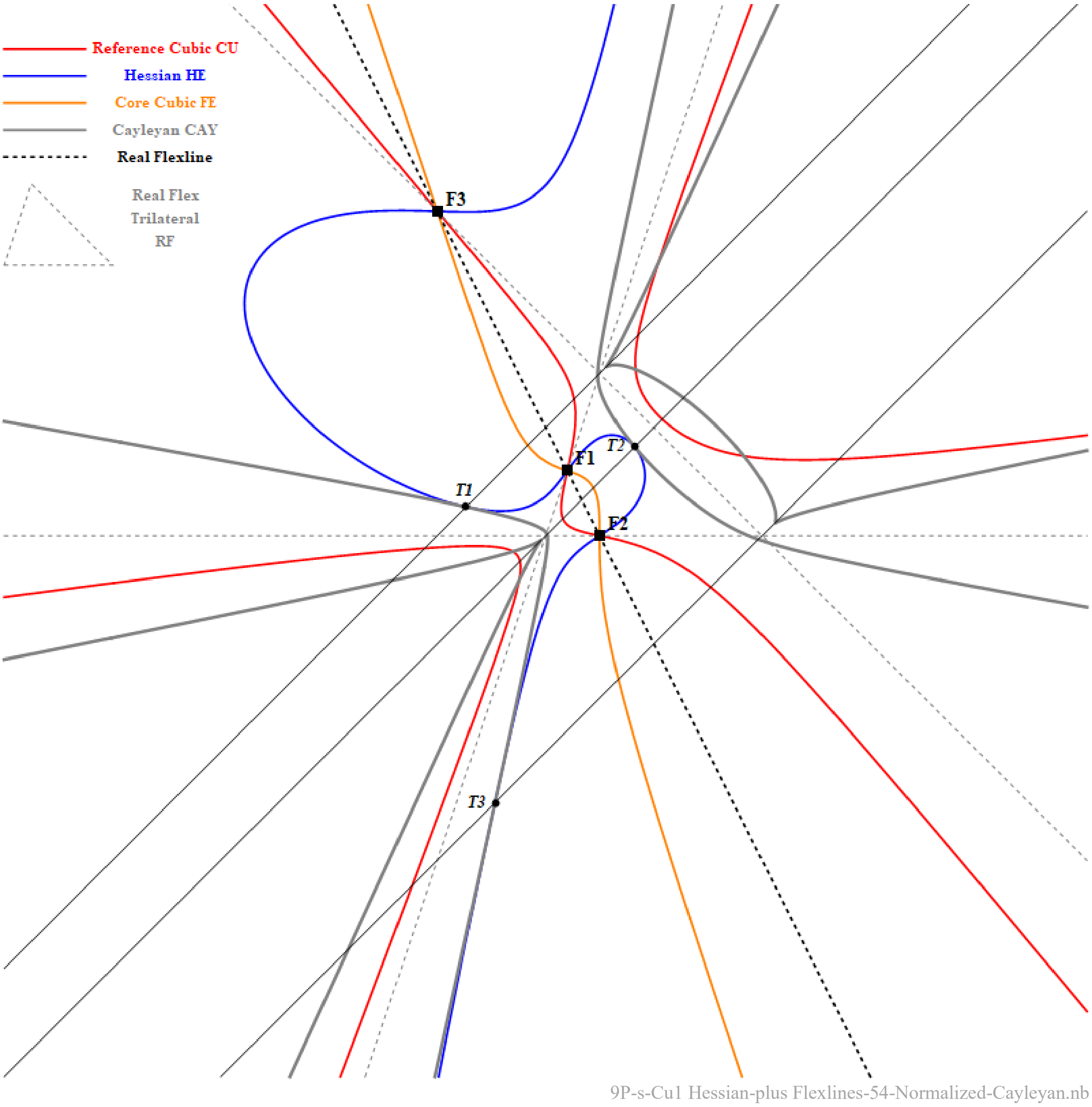
Picture of CU / HE / CAY



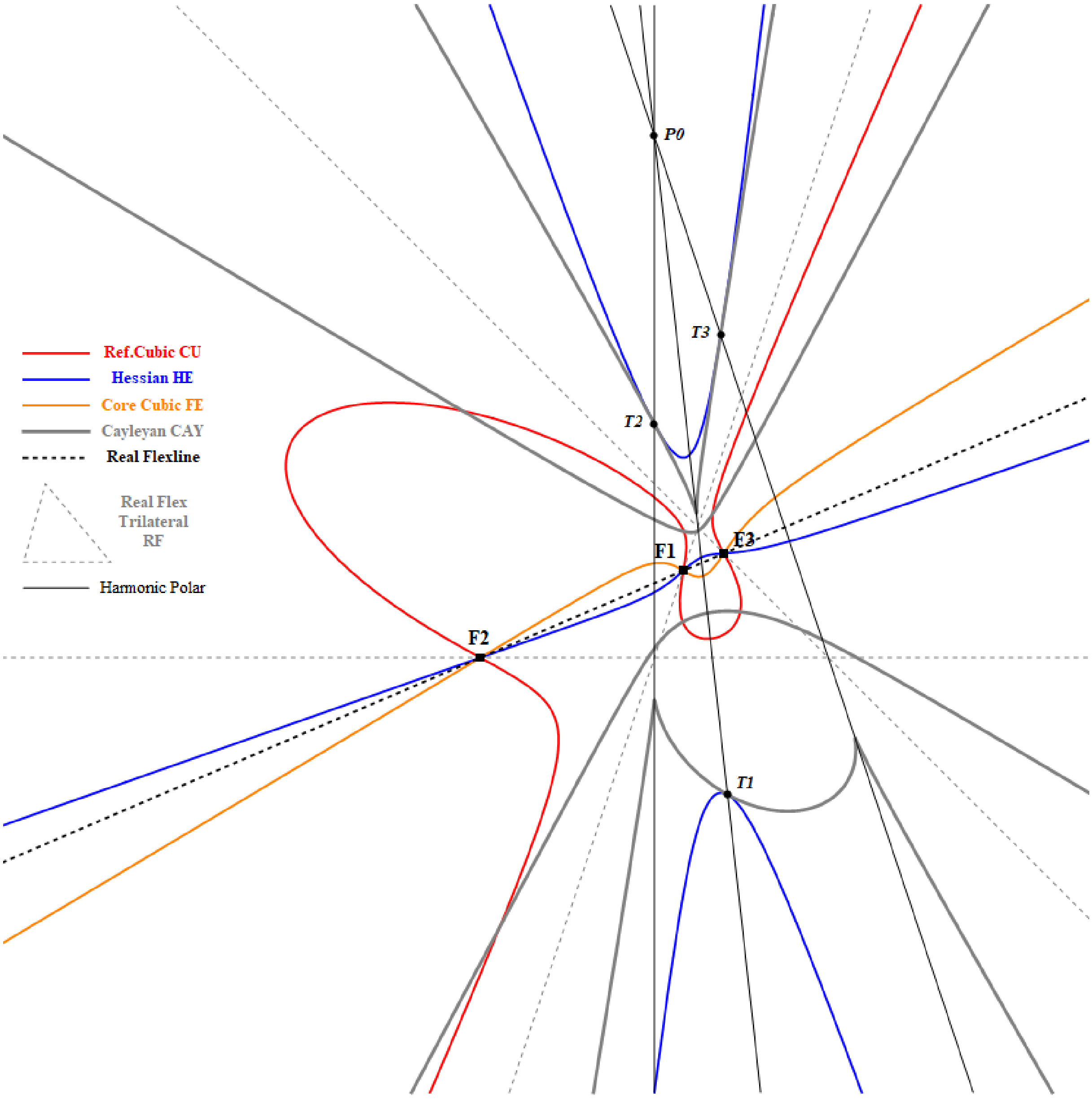
9P-s-Cu1 Hessian-plus Flexlines-41-FE-kRF-example.nb

- Reference Cubic CU is red
- Hessian HE is blue
- Cayleyan CAY is gray
- F1, F2, F3 are the real Flexpoints
- The 1 gray dotted line is the real Flexline F1F2F3
- The 3 light gray dotted lines are the real Flexlines
- The 3 black lines are the 3 Fi-Harmonic Polars
- T1, T2, T3 are intersection points of corresponding Fi-Tangent and Fi-Harmonic Polar
- P0 is the common intersection point of the 3 real Fi-Harmonic Polars

Picture-1 of CU / HE / FE / RF / CAY

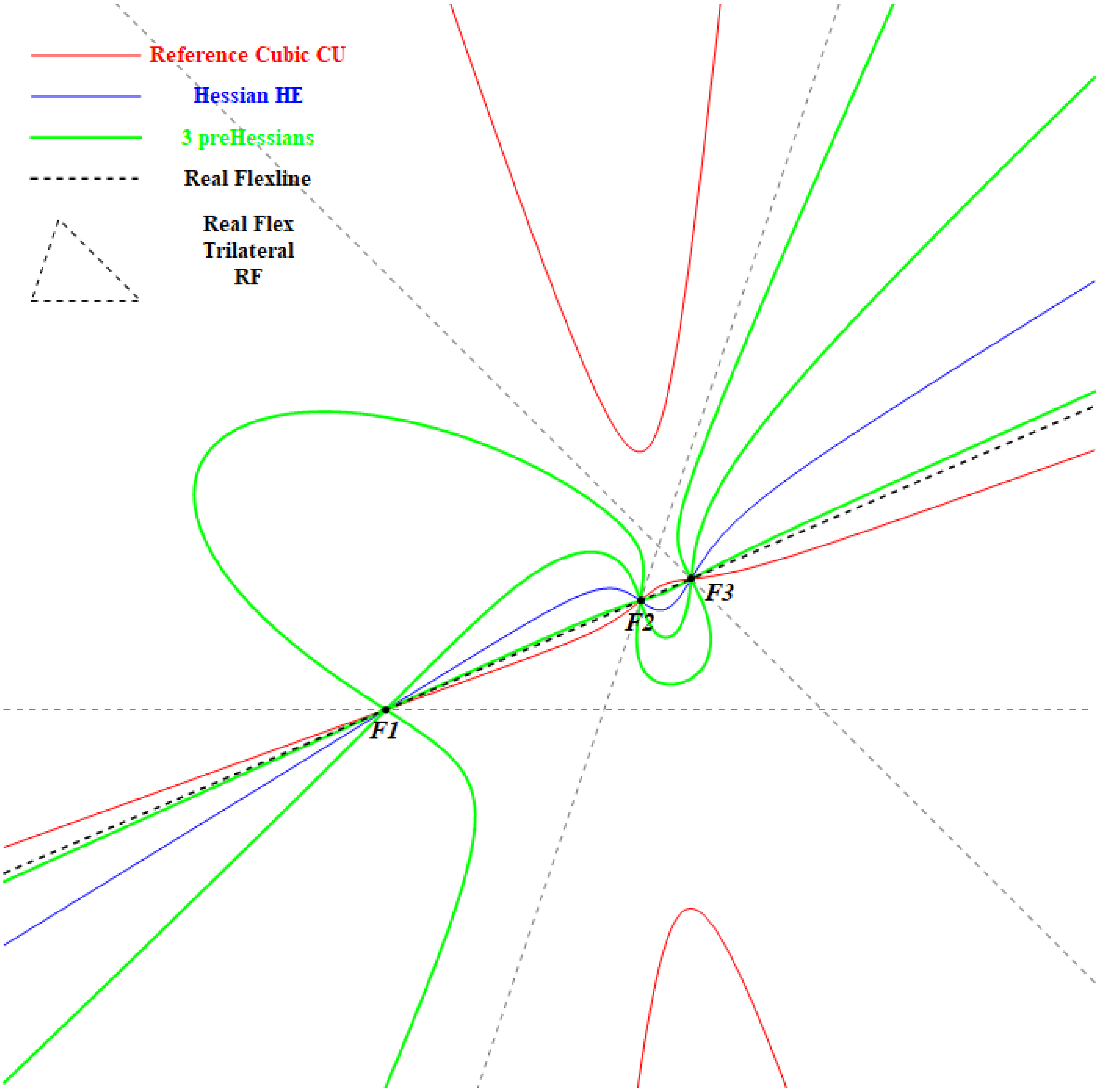


Picture-2 of CU / HE / FE / RF / CAY



9P-s-Cu1 Hessian-plus Flexlines-55-Normalized-Cayleyan.nb

Picture of CU / HE / pHE1, pHE2, pHE3 (preHessians)



CU-HE-pHE-01.nb

Note 1.
A “primitive cube root of unity” or “primitive third root of 1” is a complex number that, when raised to the power of 3, equals 1, and it is not equal to 1 itself.
There are exactly three cube roots of unity:
1. 1
2. i1 = -1/2 + i sqrt[3]/2
3. i2 = $i1^2$ = -1/2 - i sqrt[3]/2

Among these, i1 and i2=i1^2 are called primitive cube roots of unity because they generate all cube roots of unity when raised to successive powers, and they themselves are not equal to 1.
The primitive cube roots of unity have the following properties:
- $i1^3 = 1$
- i1 <> 1 and i2 <>1
- 1 + i1 + i2 = 0

Geometrically, they are the points on the complex plane that form the vertices of an equilateral triangle inscribed in the unit circle, with one vertex at 1.

References:
[1] Araceli Bonifant and John Milnor - On Real and Complex Cubic Curves
Available at: https://arxiv.org/pdf/1603.09018

[2] Michela Artebani and Igor Dolgachev - The Hesse Pencil of Plane Cubic Curves
Available at: https://arxiv.org/abs/math/0611590

CU-Cu1 The CU-Hessian

Summary
The Hessian of a cubic is the locus of points P for which the P-Polar Conic CU-P-Co1 is degenerated in two lines.
It is also a cubic and it intersects the Reference Cubic CU in 9 points, which are the 9 inflection points (flexpoints CU-9P1) of the cubic.

Further description

1. The locus of points P for which the CU-Polar Conic (CU-P-Co1) degenerates into 2 lines (meeting at Q) is called the Hessian HE (which is also a cubic).
2. Also the CU-Q-Polar Conic will degenerate into 2 lines meeting at P.
3. P and Q are referred to as *corresponding points*.
4. The tangents to HE at P and Q share the same tangential R.
5. Let S be the *corresponding point* of R.
6. S is called the *complementary point* of P and Q.
7. The CU-P-Polar Line is the same as the HE-Q-tangent.

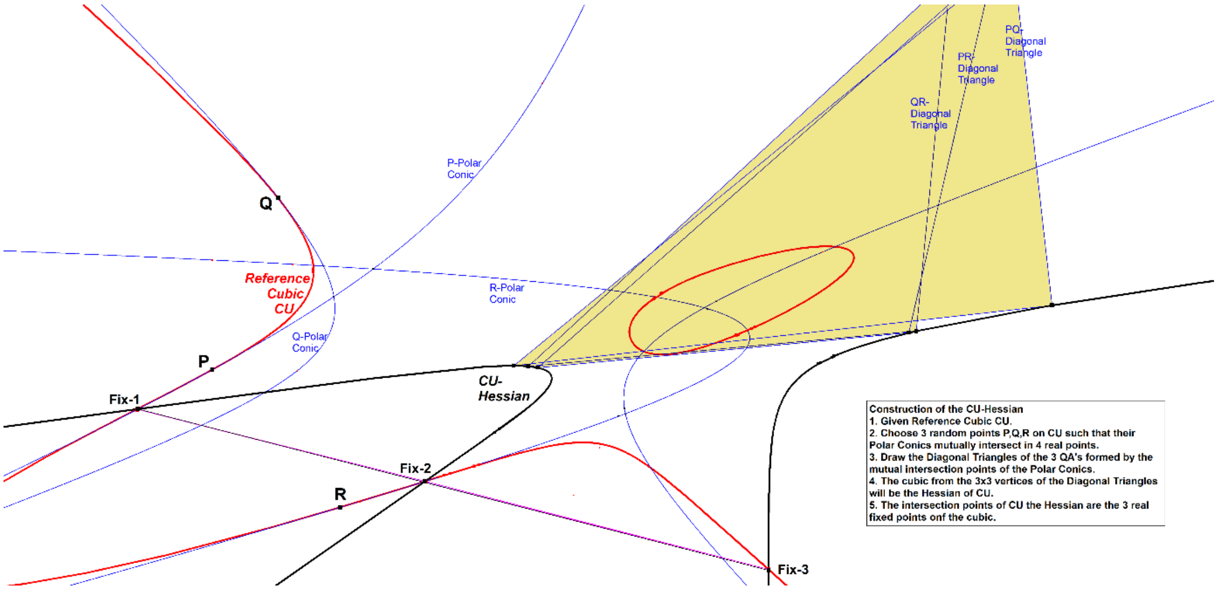
See [Jean-Pierre Ehrmann and Bernard Gibert, Isocubics 2.2.6, page 24].

There is also the Hessian matrix. It is a square matrix of second-order partial derivatives of a scalar-valued function, or scalar field. It describes the local curvature of a function of many variables. The Hessian matrix was developed in the 19th century by the German mathematician Ludwig Otto Hesse (1811-1874) and later named after him. The Hessian Matrix is used to find the equation of the Hessian Curve by calculating the determinant of the Hessian Matrix.

$$\text{CU-Hessian} = \text{DET} \begin{vmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx\,dy} & \frac{d^2 f}{dx\,dz} \\ \frac{d^2 f}{dy\,dx} & \frac{d^2 f}{dy^2} & \frac{d^2 f}{dy\,dz} \\ \frac{d^2 f}{dz\,dx} & \frac{d^2 f}{dz\,dy} & \frac{d^2 f}{dz^2} \end{vmatrix} = 0$$

Where **f** represents the equation of CU in (x,y,z).

Construction
Besides calculating this curve it is also possible to construct the Hessian as follows.
1. Given Reference Cubic CU.
2. Choose 3 random points P, Q, R (not necessary on CU) such that their Polar Conics mutually intersect in 4 real points. For the construction of a P-Polar Conic, see CU-P-Co1.
3. Draw the Diagonal Triangles of the 3 QA's formed by the mutual intersection points of the 3 Polar Conics.
4. The cubic constructed from the 3x3 vertices of the Diagonal Triangles will be the Hessian of CU.



CU-Cu1 Hessian-50-BipartiteCubic

See QPG-messages #1923 and #1944 and [Jean-Pierre Ehrmann and Bernard Gibert, Isocubics].
See for more points on the Hessian als QPG-messages #2105-#2107.

Properties

- CU (Reference Cubic) and HE (Hessian) intersect in the 9 inflection points of the reference cubic. Only three of these inflection points are real collinear points, the other 6 points are always imaginary points.
- The Hessian itself is also a cubic and has therefore its own inflection points but they are the same 9 points.
- Any cubic is the Hessian of 3 other cubics named Prehessians (not necessary real). See QPG#1945. See [BG, Isocubics].
- Any cubic shares its Hessian with 2 other cubics. See QPG#1945. See [BG, Isocubics].
- The initial cubic, it's 3 Prehessians as well as it's Hessian and the 2 other cubics sharing this Hessian form the *Syzygetic pencil of cubics* through the 9 flexes, 3 of them being real and aligned and the 6 others imaginary. See QPG#1945. See [BG, Isocubics]. See also MICHELA ARTEBANI AND IGOR DOLGACHEV - THE HESSE PENCIL OF PLANE CUBIC CURVES, https://dept.math.lsa.umich.edu/~idolga/hesserev.pdf.

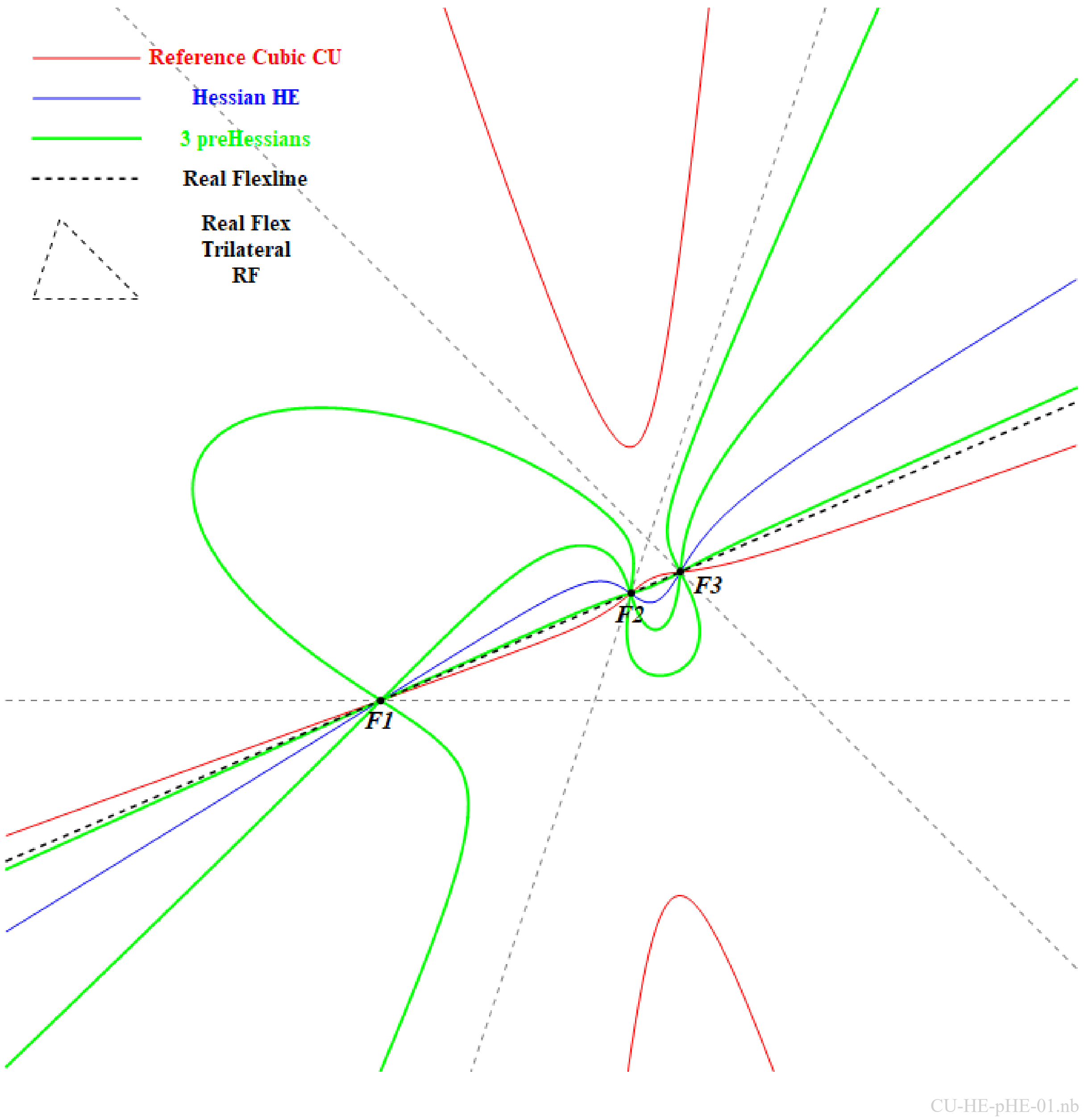
CU-3Cu1 The three PreHessians

Since a Hessian is also a cubic, it is also possible to calculate the Hessian of the Hessian, and further downwards.
Moreover it is possible, knowing some cubic CU, to find out for which upper cubic CU is the Hessian.
There are 3 of these cubics called the PreHessians pHE_1, pHE_2, pHE_3.
So per definition the Hessian of all three cubics pHE_1, pHE_2, pHE_3 will be CU.
So "upwards" there are 3 PreHessians of a cubic and "downwards" there is 1 Hessian per cubic.
For a formal description of the construction of the three PreHessians see [16b].

Actual calculation of the prehessians

- Let the Reference Cubic be of the form (Hesse's Form) CU = FE + k a_1 a_2 a_3 RF, where FE = $a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3$ and RF = x y z .
- Then a preHessian should be of the form pHEt = FE + t a_1 a_2 a_3 RF
- The calculated Hessian of preHessian pHEt will be $3 a_1^3 t^2 x^3 + 3 a_2^3$ t^2 $y^3 + 3 a_3^3 t^2 z^3 - a_1 a_2 a_3 (108 + t^3) x y z$
  and should be CU = $a_1^3 x^3 + a_2^3 y^3 + a_3^3 z^3 + a_1 a_2 a_3 k x y z$
- They are identical when $3 (a1^3) (t^2) / a1^3 = - a_1 a_2 a_3 (108 + t^3) / (a_1 a_2 a_3 k)$
  That is the case for t = t1, t2, t3, with
  - $t1 = -k + k^2/W + W$
  - $t2 = -k - ((1 + i \sqrt{3}) k^2)/(2 W) + 1/2 i (i + \sqrt{3}) W$
  - $t3 = -k + (i (i + \sqrt{3}) k^2)/(2 W) - 1/2 (1 + i \sqrt{3}) W$

  where $W = (-54 - k^3 + 6\sqrt{3}(\sqrt{(27 + k^3)}))^{1/3}$
- As can be seen the roots are independent of (a_1, a_2, a_3) and only depend on k.
- It appears that:
  - t1 is real when k < -3
  - t2 is real for all k
  - t3 is real when k < -3


Reference Cubic CU
Hessian HE
3 preHessians
Real Flexline
Real Flex
Trilateral
RF
F1
F2
F3
CU-HE-pHE-01.nb

CU-Sx1 The CU-Cayleyan

The Cayleyan and Its Geometric Context

The Hessian is the locus of points P for which the P-Polar Conic degenerates into 2 straight lines. The Cayleyan (sometimes spelled Cayleyian) is the envelope of these degenerate lines – a curve formed by their continuous variation.

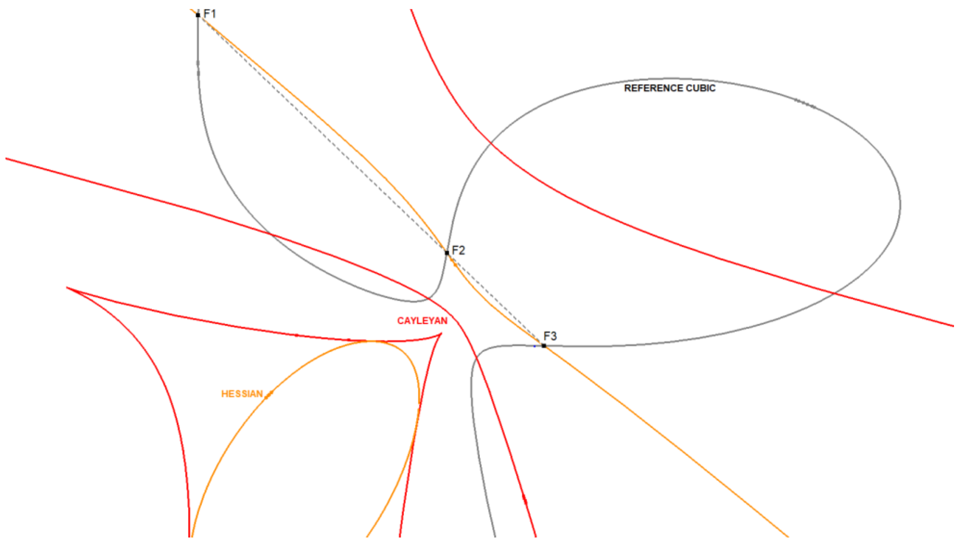
Concepts used

The following notions will be referenced throughout this discussion:

- The Reference Cubic CU
- The Cayleyan CAY of CU: CU-Sx1
- The Hessian HE of CU: CU-Cu1
- The Harmonic Polars of CU: CU-9L1
- The Polar Conic of a point P with respect to CU: CU-P-Co1
- The 4 Poles of a line L with respect to CU: CU-L-4P1
- The Quadri-Pole of a line L: QA-Tf10(L)

Properties of the Cayleyan

- The **Cayleyan is a sextic curve**—a curve of **degree 6**.
  - This means that **any line** in the plane will intersect the Cayleyan in **exactly six points**, which may be real or imaginary.
- The **Cayleyan is also of class 3**.
  - This implies that **from any point** in the plane, there exist **exactly three tangents** to the Cayleyan, again possibly imaginary.



CU-Cayleyan-14-plus-Hessian-F-Tangents-Polars.fig

Additional properties

Furthermore the Cayleyan is:

1) The envelope of all lines through 2 corresponding points of the Hessian. These corresponing points (H1,H2) of the Hessian HE are some point H1 on HE and the intersection point H2 of the straight lines of the degenerated H1-Polar Conic wrt Reference Cubic CU.

2) The envelope all lines forming the degenerated Polar Conics of points of the Hessian (these 2 definitions are equivalent)

3) The locus of the contact point of the line with the curve, which is the harmonic conjugate of the complementary point wrt the 2 corresponding points.
For further clarification of the terms *corresponding points* and *complementary point*, see:

- **Gibert – Isocubics** [17b], pages 20–21
- **Cuppens** [63], pages 267–270

Classical Description by Durège

The most concise description of the Cayleyan comes from Durège. See [84], page 264.
He writes:

> *498. Lässt man eine Gerade G alle möglichen Lagen in der Ebene annehmen*
> *und bestimmt jedesmal ihre vier Pole a b c d bezüglich einer Curve 3. O.,*
> *so beschreiben die Diagonalpunkte des vollständigen Vierecks a b c d*
> *die Hesse'sche Curve und die Seiten dieses Vierecks hüllen die Cayley'sche Curve ein.*
> *– Denn diese Seiten bilden die sämmtlichen conischen Polaren, welche aus Geradenpaaren bestehen.*

Translation:

> *498. Consider a line G at all possible positions in the plane*
> *and determine its four poles a b c d with respect to a curve of the 3rd degree,*
> *then the diagonal points of the complete quadrangle a b c d will describe*
> *the Hessian Curve and the sides of this quadrangle will envelope the Cayleyan Curve.*
> *Because of these sides forming all the polar conics, which consist of pairs of straight lines.*
> The four poles of a line like described by Durège are CU-L-4P1.

Summary

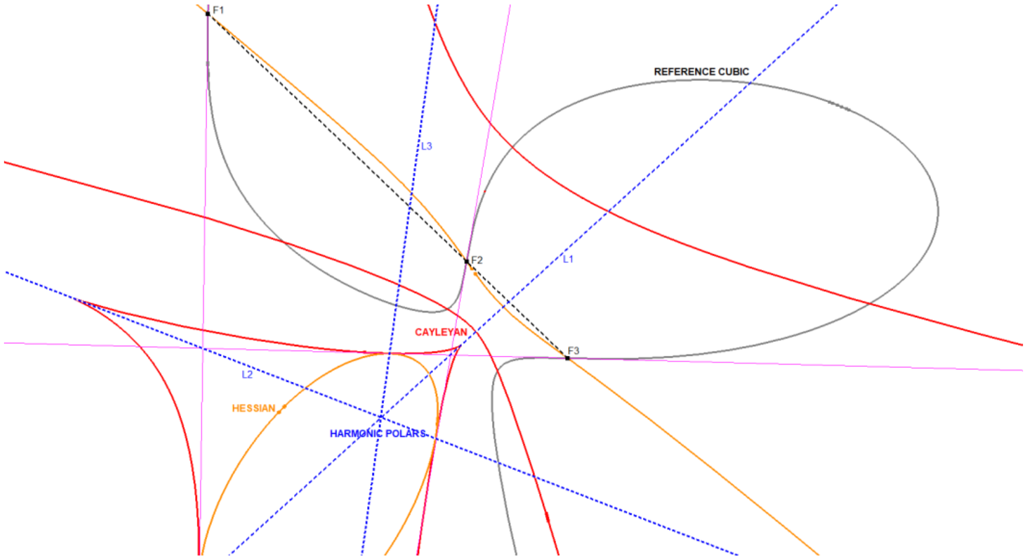
- All P-Polar Conics wrt CU will intersect in 4 common fixed points, when P is a point on some fixed line L in the plane.
- These 4 common fixed points are called the 4 poles of L wrt CU.
- The Hessian is the locus of points P where the P-Polar Conics wrt CU degenerates into 2 lines.
- The Cayleyan is the curve enveloped by these straight lines.

Construction

1. Given Reference Cubic CU.
2. Choose 3 random points P, Q, R (not necessary lying on CU) such that their Polar Conics mutually intersect in 4 real points.
3. These three polar conics will intersect mutually in 3 distinct sets of 4 points each. Thus, we obtain 3 quadrangles QA's. For the construction, only two of these QA's are needed.
4. For each QA (quadrangle), apply the QA-Tf10 transformation *) to map the 6 QA-sides to 6 points Ti. Two QA's yield a total of 2×6 points Ti.
5. As the reference QA, just any 4 reference points mat be selected.
6. Construct a cubic CUx that passes through 9 of the 12 points Ti. It will then be observed that the remaining 3 points Ti also lie on this cubic CUx.
7. Choose a variable point X on the constructed cubic Cux and draw the tangent Tx to Cux at point X.
8. Map Tx the QA-Tf10 transformation *) into point Px.
9. The locus of Px traces out the Cayleyan.

See reference: QPG#2160.

Note
*) Since any Reference Quadrangle can be chosen fot the QA-Tf10 transformation, a convenient choice for barycentric coordinatecal culations is the quadrangle with vertices (1:0:0), (0:1:0), (0:0:1), and (p:q:r).
In this case QA-Tf10 [(p:q:r)] = (2p+q+r : p+2q+r : p+q+2r).

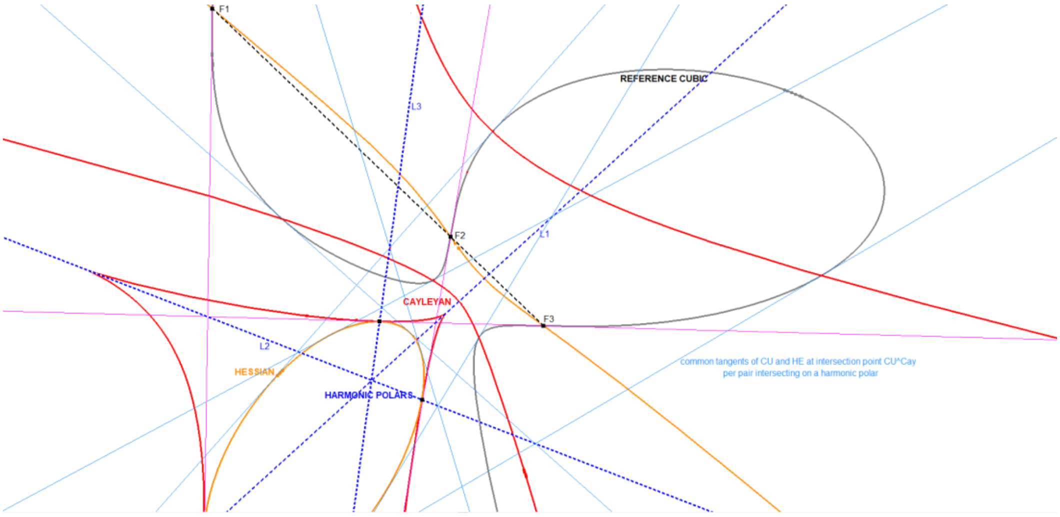


CU-Cayleyan-16-plus-Hessian-F-Tangents-Polars.fig

The Reference Cubic CU, the Hessian HE and the Cayleyan CAY are intricately connected in several geometric ways.

Intersection of CU and HE
As shown in the image above, CU and HE intersect in 3 real Flexpoints F1, F2, F3, which lie on a straight line (see CU-9P1).
The tangents to CU at these points are also tangent to CAY and HE at the same locations. These points are where the Harmonic Polars Li and the Fi-Tangents converge.

Intersection of CU and CAY
In the next image, CU and CAY intersect in 6 real points.
The tangents to CU at these points are also tangent to HE, per pair of tangents intersects on a harmonic polar (see CU-9L2).



CU-Cayleyan-17-plus-Hessian-F-Tangents-Polars.fig

Further reading

More information about the Cayleyan can be found in:

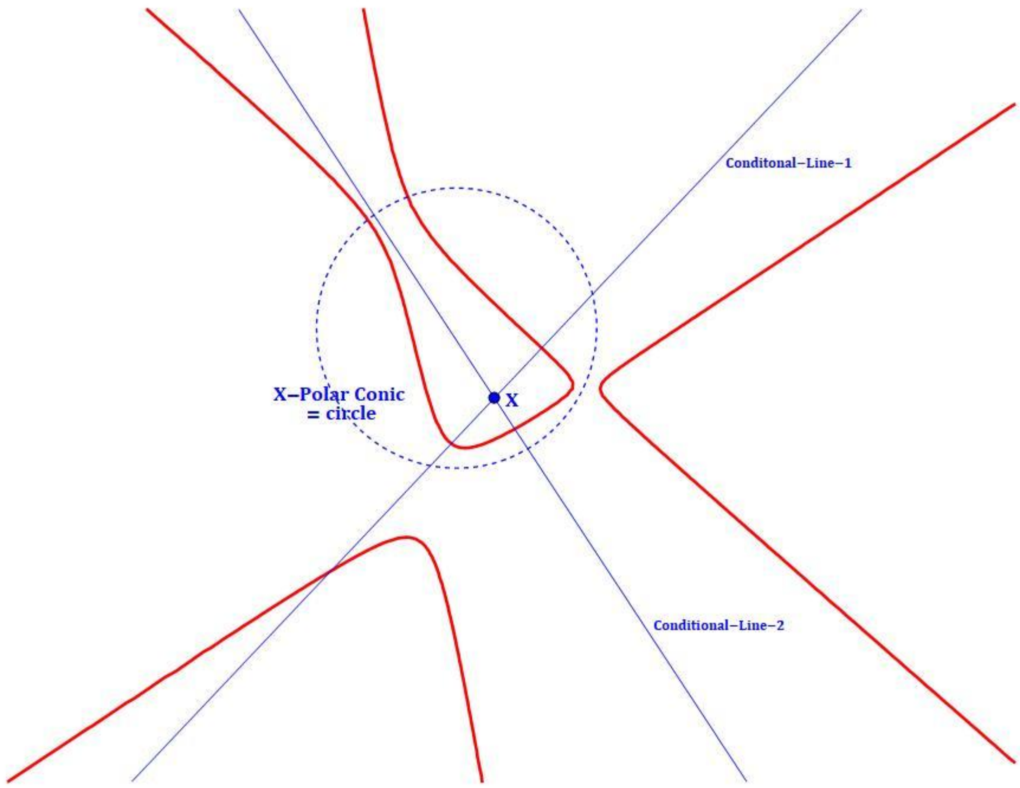
- QPG references #2108, #2118-2121, #2124, #2128, #2137, #2156, #2160, #2163-2169
- Literature:
  - Cuppens [63] – pages 271,272,
  - Hilton [85] – page 107.

CU-Ci1 The unique CU-Polar Circle

There exists one and only one point X for which the X-Polar Conic (CU-P-Co1) is a circle.
Proof:
Since a Polar Conic wrt a Cubic at X(p,q,r) algebraically is derived from:

(D[Cubic, x], D[Cubic, y], D[Cubic, z]) . (p,q,r)

we have a linear equation in (p,q,r) when (x,y,z) is substituted by the coordinates of the circular points at infinity CI1 or CI2.
Consequently the 2 substitutions with CI1 and Ci2 will give 2 linear equations in (p,q,r).
Both equations must satisfy for a point X having a circular X-Polar Conic, so only the intersection point of both conditional lines coming forth from these 2 equations will give a circular X-Polar Conic.
So there is always one and only one point X for which the X-Polar Conic wrt a cubic is a circle.

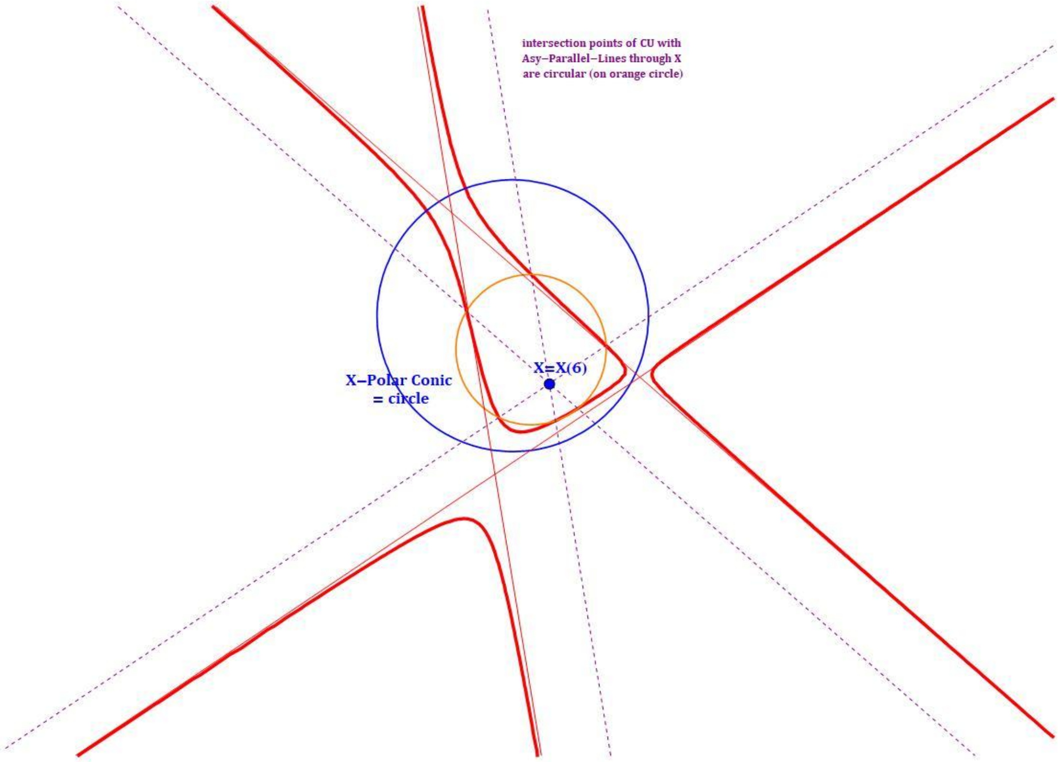


CU-Ci1 Unique Polar Circle of a Cubic-01.jpg

References
See QPG#2322-2324 and Gibert, Isocubics, [17b], page 21.

Properties

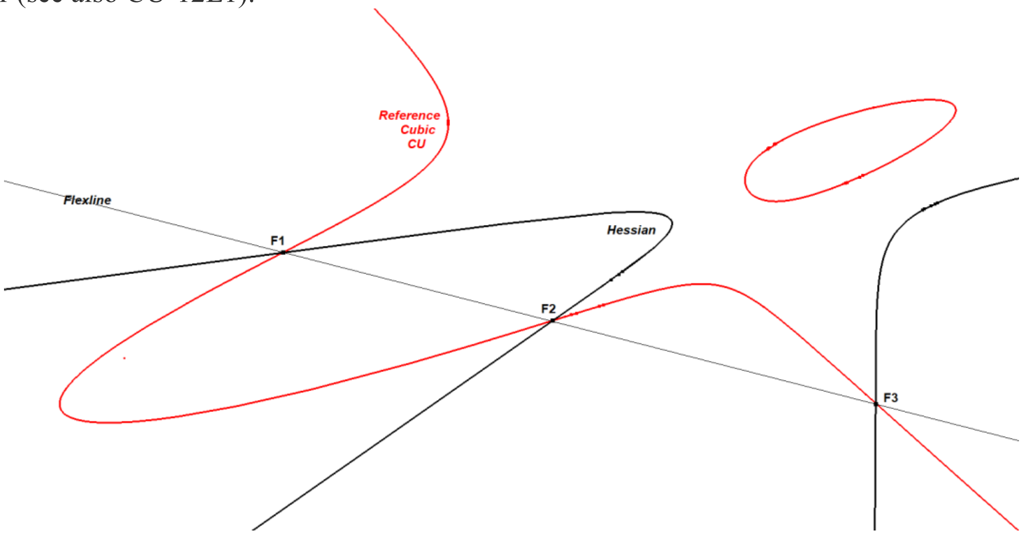
- When the Reference Cubic is a Circular Cubic (CUc), then X will be the Singular Focus of CUc.
- If the cubic has three real asymptotes, then X is the Lemoine point aka Symmedian Point (X(6) in the Encyclopedia of Triangle Centers) of the triangle formed by the asymptotes and, obviously, when they concur, it is the point of concurrence. See [Gibert, Isocubics, page 21].
- The parallels to these asymptotes passing through X meet the cubic again at six points lying on a related circle. This circle is the first Lemoine circle of the Asy-triangle CU-3L1. See [Gibert, Isocubics, page 21].



CU-Ci1
Unique Polar Circle of a Cubic-02.jpg

CU-9P1 Set of 9 Flexpoints of the Cubic

Each Cubic has 9 Flexpoints, also known as Flexes.
They are the inflection points of the cubic. An inflection point is a point on the curve where the curvature changes - either from increasing to decreasing or vice versa.
Flex points are the intersection points of the cubic CU with its Hessian curve, which is also of degree three (see CU-Cu1). Therefore there are nine such intersections, corresponding to the nine inflection points.
Of these nine inflection points, three are real and six are imaginary. The three real points are always collinear (see also CU-12L1).



CU-12L1a Real Flexline-01

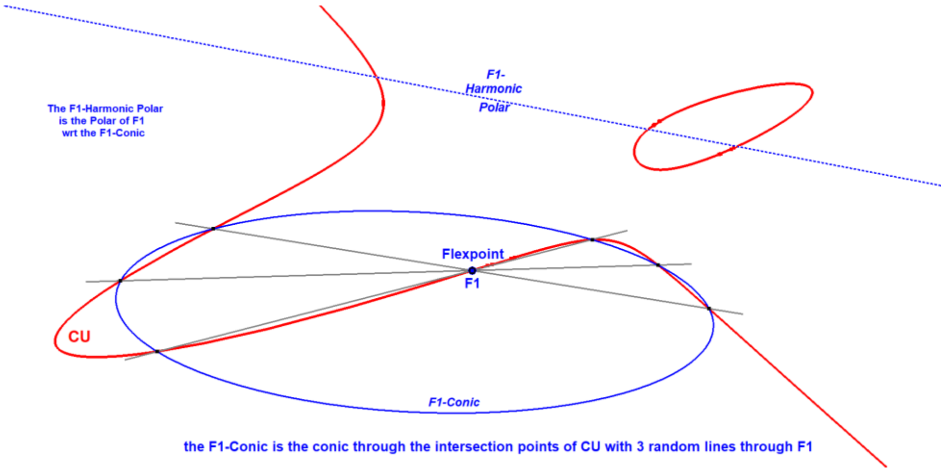
Quick construction to approximate the 3 real flexpoints *(QPG#2046)*
This is based on Schröter's book [82], page 242.
Start with an arbitrary point F on the cubic with three secants,
... which give 3x2 cubic intersections,
... if they are coconic on a conic CO, then F is an inflection point.

Construction of the 3 real flexpoints
The standard method for determining the 3 real flex points is by drawing the Hessian of CU. Refer to CU-Cu1 for details.
The Hessian is also a cubic. Consequently, the Reference Cubic CU and its Hessian intersect at nine points. These points constitute the nine flex points, of which only three will be real and collinear.

Network of the 9 flexpoints
There is an entangled network of 9 flexpoints and 12 connecting flexlines with 3 points per line and 4 lines per points. See CU-12L1.

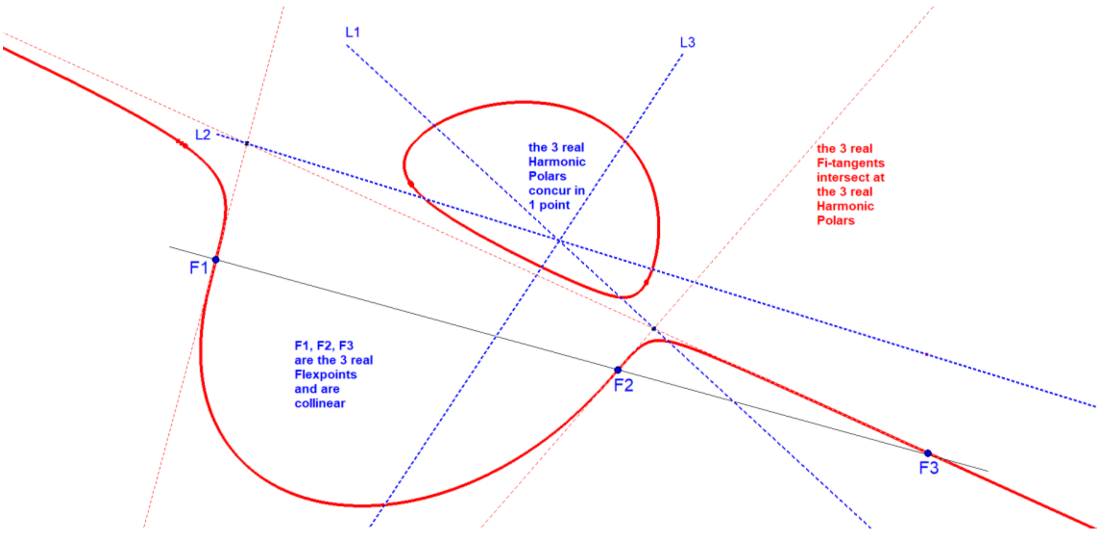
CU-9L1 Harmonic Polars

Let P be a point on the cubic curve CU.
Draw three lines (Lp1,Lp2,Lp3) through P, each intersecting the reference cubic in two additional points. This yields six points: (P1a,P1b), (P2a,P2b), (P3a,P3b).
According to Schröter (see [82], page 242) these six points are coconic when P is a flexpoint. The form of the conic will be different for each set of (Lp1,Lp2,Lp3). However the Harmonic Polar of P wrt all these different conics will be a fixed line.
These polars are called the Harmonic Polar of Fi (Fi being one of the 9 Flexpoints CU-9P1).
Other names are “Polar Axis of a flexpoint” or “Flexpolar”.
See also QPG#2046, #2050, #2054.



CU-9L1 Fi-Harmonic Polars-10.fig

The 3 Harmonic Polars of the 3 real Flexpoints F1, F2 and F3 concur in a fixed point.

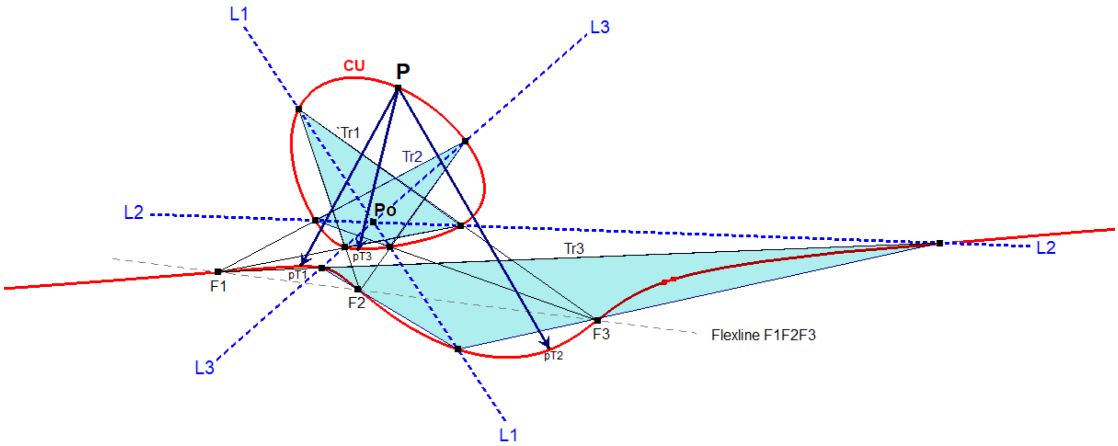


CU-9L1 Fi-Harmonic Polars-21.fig

Since there are 9 Flexpoints (3 of them real and 6 imaginary) there also will be 9 Harmonic Polars (3 of them real and 6 imaginary).

Properties

- The polar conic of a flexpoint Fi wrt CU is a conic degenerated in two lines: the Fi-Harmonic Polar and the CU-Fi-Tangent.
- The three CU-Fi-Tangents form a triangle with its vertices on the three real Fi-Harmonic Polars.
- The three real CU-Fi-Flexlines (see CU-12L1) form a triangle with its vertices on the three real Fi-Harmonic Polars.
- The Harmonic Polars L1, L2, L3 intersect CU each in 3 points forming 3 triangles, per triangle with their sides through resp. F1, F2, F3. These triangles are mutually perspective, all with the common intersection point of (L1,L2,L3) as perspector and the line through (F1,F2,F3) as perspectrix.

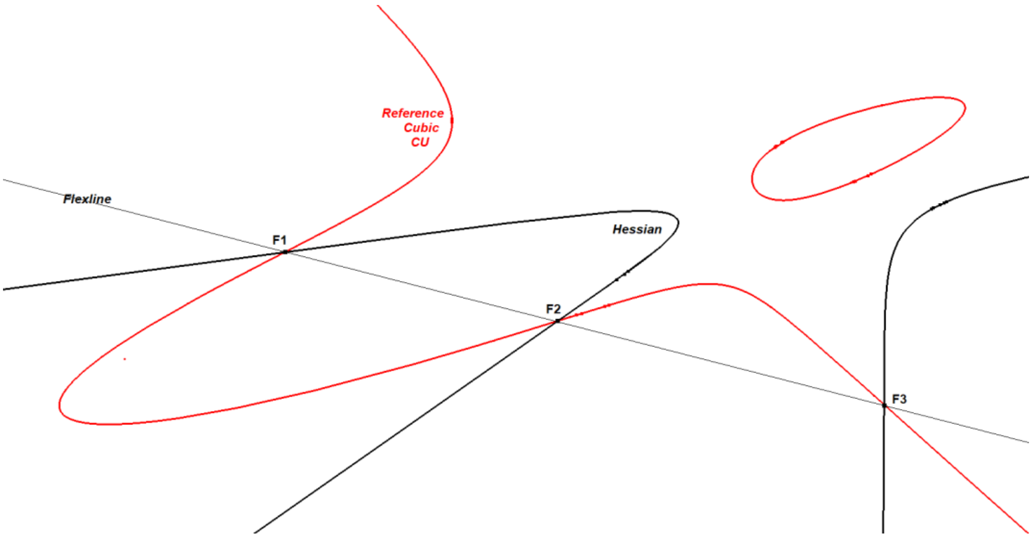


CU-Cu1 Hessian-112a-Pi-Isoconjugates.fig

CU is a non-pivotal Isocubic wrt each of these triangles and with fixpoint the common intersection point Po of (L1,L2,L3). The product of the isoconjugates of some cubic-point P wrt the 3 triangles is the identity, where the isoconjugate is QA-Tf2 with QA=(Po+vertices of anticevian triangle of reference triangle). See QPG#2380,#2381,#2386-#2388,#2390.

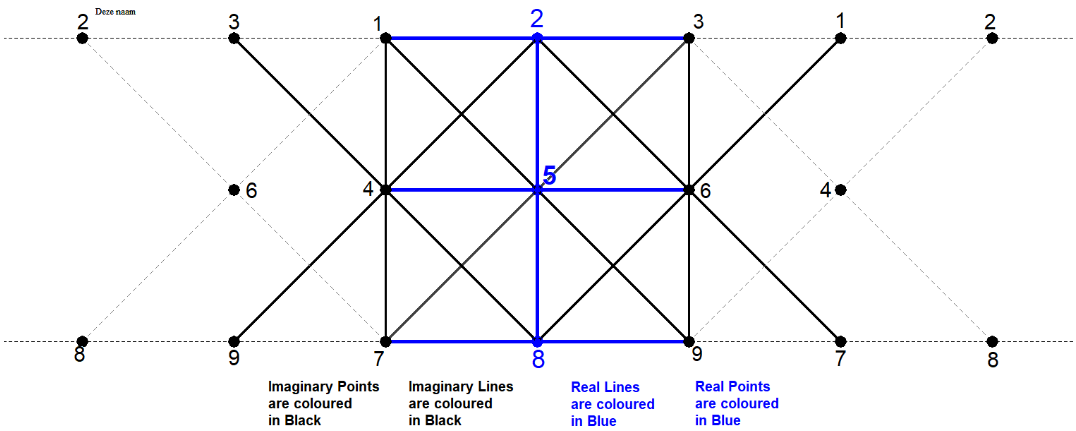
CU-12L1 Set of 12 Flexlines of the Cubic

Like described at CU-9P1, there is a set of 9 Flexpoints on the cubic.
However on any cubic only 3 of these Flexpoints will be real points and they also will be collinear.
The other 6 Flexpoints will be imaginary points and there will be 12 Flexlines of which 4 real lines.
The other flexlines are imaginary lines.



CU-12L1a Real Flexline-01

Structure of the Flexlines
The 9 Flexpoints (3 real and 6 imaginary) lie mutually in a strict order on 12 Flexlines (4 real and 8 imaginary).
If P and Q are flexpoints, then R = P·Q is another flexpoint. See [FL], page 137.
Since the flexpoints are points on CU and since any line always cuts a cubic in three points, there will be no more than 3 collinear flexpoints on a line segment.
This is valid for each flexpoint and therefore flexpoints occur all the time in combinations of three.
When P occurs in combination with Q and R, then P and Q cannot occur with another point than R.
Therefore P combines with the other 8 flexpoints in 4 pairs and so in the scheme below you will see that each knot has 4 line segments passing.
All in all, there is a limited number of 12 combinations of flexlines.



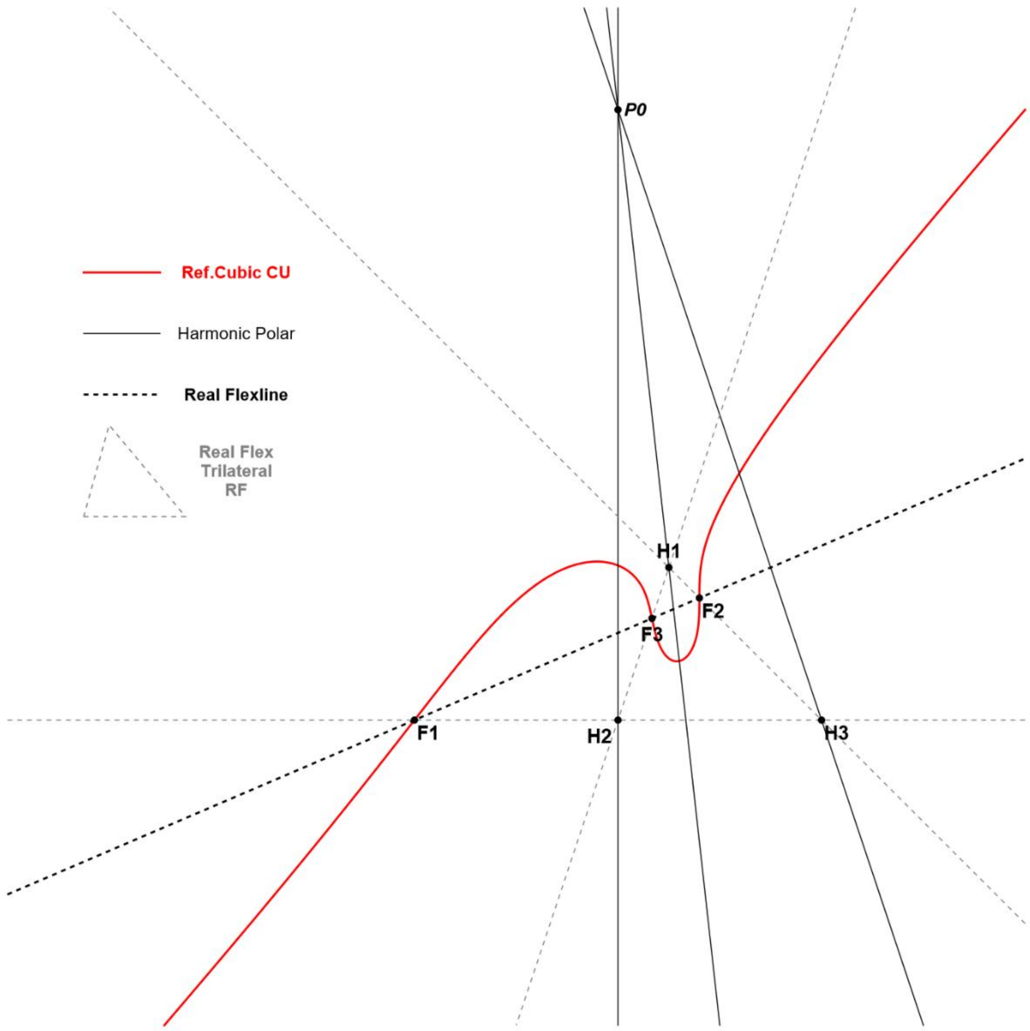
CU-12L1 Flexlines-scheme 03.fig

The flexpoints can be imagined lying on a cylindrical shape that has been cut open, allowing points to reappear on the left and right. This makes it clear how the 3-point-line-segments extend in a regular way.
In the picture above the 12 combinations are shown schematically.
For more information see [75] and [79], page 17.

There is a relationship with the Harmonic Polars Network. See CU-12P1 and QPG#2410.

Real and Imaginary Flexlines
Of the 12 Flexlines, there are always 4 real and 8 imaginary ones for all cubics.
The 4 real Flexlines are:

- *The main real Flexline*. The standard method for determining the flexpoints involves drawing the Hessian of CU. Refer to CU-Cu1 for details. The Hessian is also a cubic, meaning that the Reference Cubic CU and its Hessian intersect at nine points. These points constitute the nine flex points, of which only three are real and collinear, thereby constituting the main real flexline.
- The triplet of real Flexlines (each passing through one real Flexpoint), also described as the real *Flexline Trilateral*. Eckart Schmidt discovered their construction, as outlined in QPG#2635 for the bipartite cubic and in QPG#2747 for the general cubic.



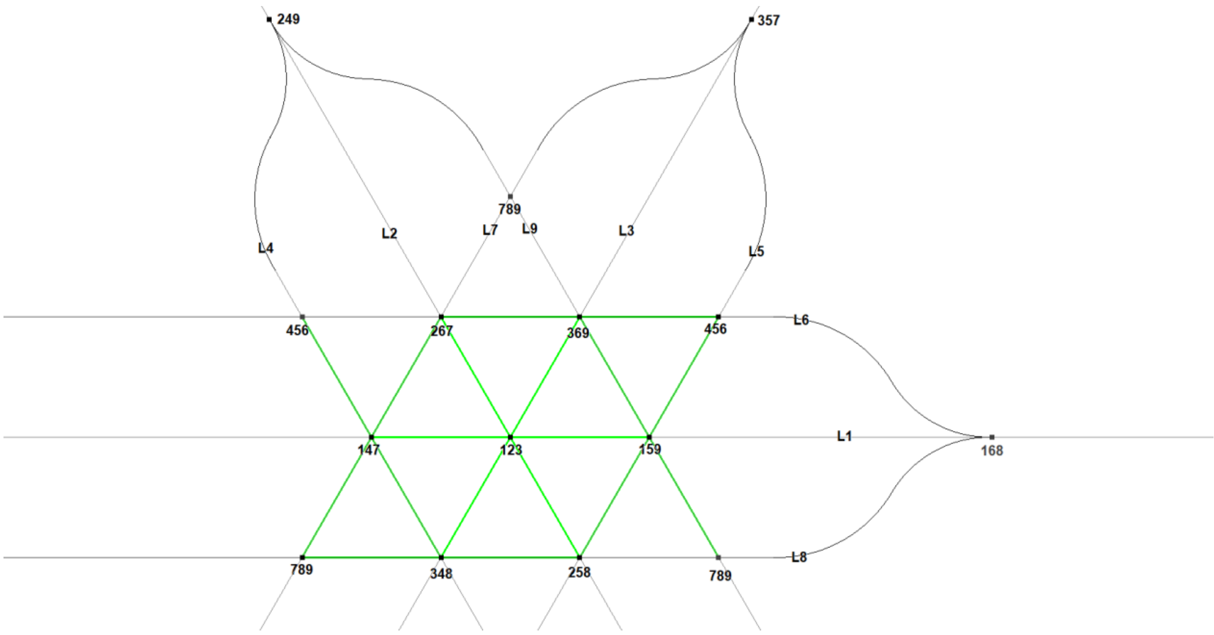
CU-HE-12L1abc-32-bipartite cubic.png

CU-12P1 Harmonic Polar Crosspoints

Let P1, …, P12 be the 12 mutual intersection points of the Harmonic-Polars L1, …, L9 (see CU-9L1). They are a kind of duals with the 12 flexlines (CU-12L1) passing through the 9 flexpoints (CU-9P1). The 9 Flexpoints (3 of them real points) lie on 12 Flexlines (4 of them real lines). The 9 Harmonic Polars (3 of them real lines) intersect in 12 Harmonic-Polar-Crosspoints (4 of them real points).

The 12 Harmonic-Polar-Crosspoints and the 12 Flexlines have a Pole/Polar-relationship wrt the CU-Pi-Polar Conic.
Note that there are always 3 real Harmonic Polars and 6 imaginary Harmonic Polars.
Note also that there is 1 real Harmonic Polar Crosspoint, being the common intersection point of the 3 real Harmonic Polars. Beyond that there are three other real Harmonic Polar Crosspoints, each lying on just 1 real Harmonic Polar and 2 imaginary Harmonic Polars. The remaining Harmonic Polar Crosspoints are all imaginary points, each lying on three imaginary Polars.

The set of 9 Harmonic Polars (real or imaginary) are structured in a point-network as follows:



CU-9L1 Harmonic Polars-scheme of intersection points-02.fig

It is a spacious triangular net characterized by the following features:
* there are effectively 9 lines, each line contains 4 points
* there are effectively 12 points, each point is passed by 3 lines
In order to make intersections appear like they (according to calculations) do:
* lines repeat in parallel lines skipping 2 other lines
* points repeat as vertices of larger triangles,
  each side being three times longer than the smaller ones,
  (for example find three versions of point 789)
* parallel lines also converge in one point.
That's all because this is a system of points and lines in the complex projective plane being projected to a noncomplex affine plane.

There is a relationship with the Flexpoint/Flexline-Network. See CU-12L1 and QPG#2410.

CU-27P1 The 27 Sextatic Points of a Cubic Curve

The 27 Sextactic Points on CU
In classical algebraic geometry, a **sextactic point** on a plane curve is a point where the curve has unusually high contact with a conic.
The study of sextactic points dates back to 19th-century projective geometry, notably in the work of Cayley, Salmon, and Clebsch.
See also [77], page 137 (option 13.) and [80].

For a smooth irreducible cubic curve, the sextactic points are defined as follows:

Definition
A point P on CU is called a sextactic point if there exists a conic Ci such that:

- Q osculates CU at P to order six, i.e., the intersection multiplicity ≥6.
- This contact is exceeds the generic case, since a general conic intersects a cubic curve in 6 points (by Bézout's theorem), but not all at one point.

Properties

- A smooth cubic curve has **exactly 27 distinct sextactic points** over an algebraically closed field of characteristic zero.
- These points are intrinsically defined by the geometry of the curve and do not depend on any embedding or parametrization.
- The sextactic points are **invariant under projective transformations** that preserve the curve.

Geometric Interpretation

- Sextactic points are the analogues of **inflection points**, but for conics instead of lines.
- While an inflection point is where the tangent line meets the curve with multiplicity ≥ 3, a sextactic point is where a conic meets the curve with multiplicity ≥ 6 at a single point.

CU Point Validation

- 6P = 2N has 36 solutions,
- Nine of these are flex points, the others 27 are the sextatic points.

CU-IP-P1 Infinity points of CU

According to Bézout's theorem, a line (a curve of degree 1) and a cubic curve (degree 3) intersect in exactly three points, counted with multiplicity.
In the context of the complex projective plane, there exists a line at infinity, which contains all points "at infinity" — both real and imaginary. Since this is a line, it must intersect any cubic curve in three points. These are referred to as the points at infinity of the cubic curve CU.
These points at infinity may be real or imaginary. Algebraic analysis shows that imaginary points at infinity always occur in conjugate pairs. As a result, a cubic curve must have either zero or two imaginary points at infinity, and consequently either one or three real points at infinity.

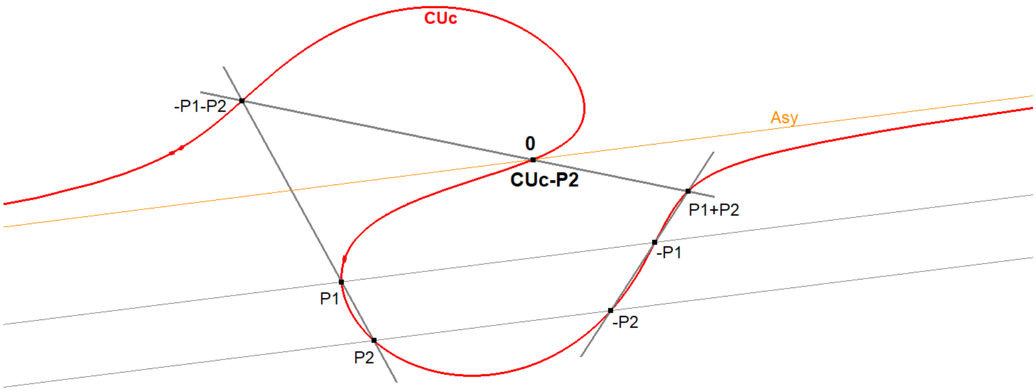
Case of a Circular Cubic
When the cubic is a circular cubic, it contains the two circular points at infinity — which are complex conjugate and imaginary. Therefore, a circular cubic always has exactly one real point at infinity, and thus one real asymptote.

CU-IP-P2 Intersection point of CU with the CU-asymptote (CU-Asy-Crosspoint)

Each cubic CU has at least one real asymptote that intersects the CU itself at a fixed point, referred to here as the CU-Asy-Crosspoint.
The other two asymptotes may be real or imaginary, each with its own CU-Asy-Crosspoint.
CU-Asy-Crosspoints can be either finite or infinite.

General Construction

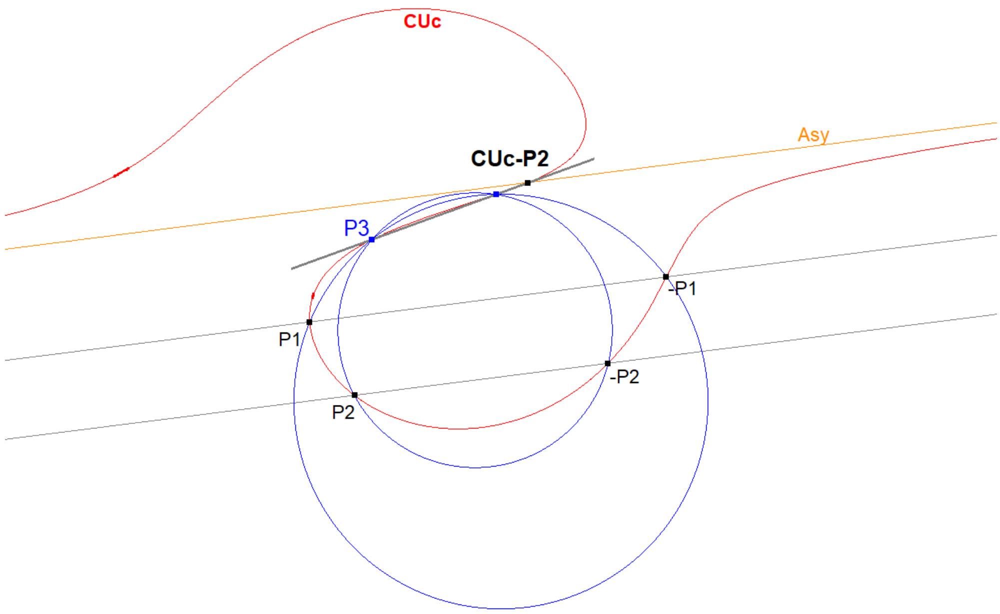
- In this construction, we have two reference points, P1 and P2, on CU, and we know the direction of the asymptote.
- Draw the points –P1 and –P2 as the intersection points of the asymptote-parallels passing through P1 and P2.
- Draw the point –P1–P2 as the third intersection point of the line P1·P2 with CUc.
- Draw the point +P1+P2 as the third intersection point of the line –P1·–P2 with CUc.
- The third intersection point of the line connecting –P1–P2 and +P1+P2 is CU-IP-P2.



CUc-P2 CUc-Asy-Crosspoint-10-simple construction.fig

Validation

Construction-1 on a Circular Cubic
In this construction we have 3 reference points P1,P2,P3 on CU.

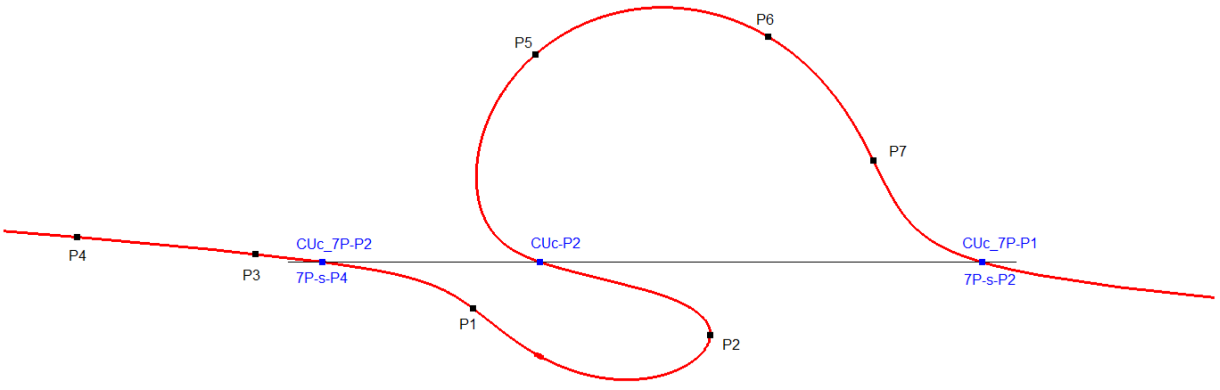


CUc-P2 CUc-Asy-Crosspoint-20-simple construction.fig

Validation

Construction-2 on a Circular Cubic

In this construction we have 7 reference points on CUc.
CUc-P2 = 3rd intersection point CUc_7P-P1.CUc_7P-P2 (in EPG 7P-s-P2.7P-s-P4)
In this method the direction of the asymptote is not used.
See QPG#787



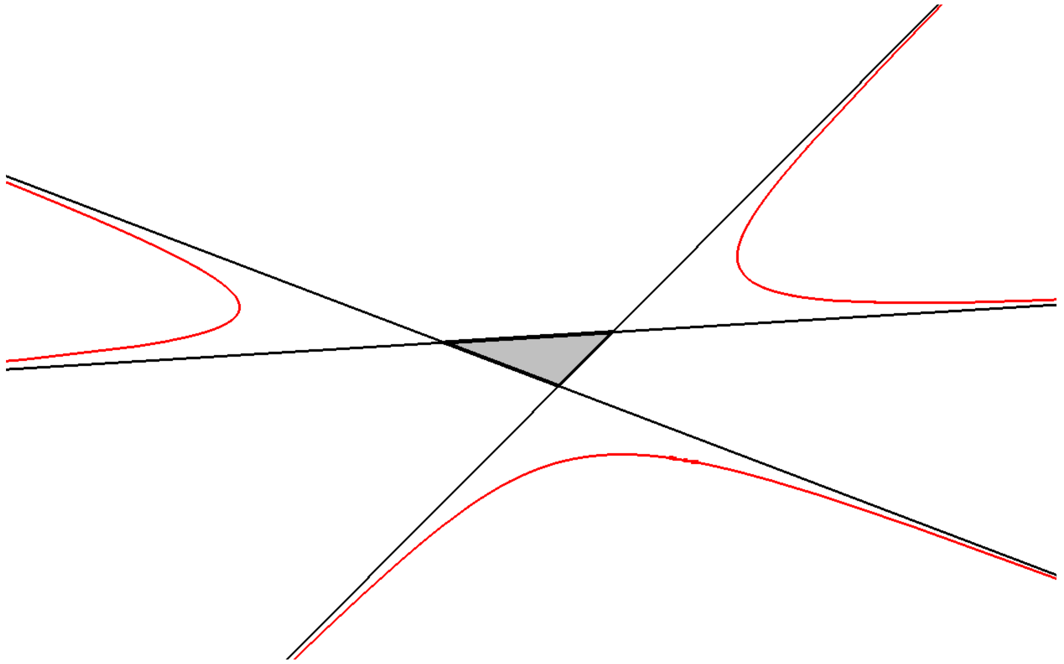
7P-s-P5 Intersection point 7P-Cubic and its Asymptote-00.fig
CUc-P2 CUc-Asy-Crosspoint-30-construction.fig

Validation

Properties

CU-IP-Q1 Vertices CU-Asymptotes Triangle or Asy-Asy-Crosspoints

The 3 asymptotes of a general cubic form a triangle with vertices CU-IP-Q1a, CU-IP-Q1a, CU-IP-Q1a. When there is only one real version then the only version is called F. Each version is a point diametral to Q (CU-IP-P2) on the corresponding version of a QF-conic (CU-IP-Co2), which also contains the vertices of a Quasi-Miquel Triangle (CU-IP-3P1).
The Polar Conic of a Vertex of the Asy-Triangle contains the same infinity points as the two asymptotes through this vertex and the same infinity points as the corresponding Central Conic/QF-Conic.
There are more properties. See QPG#2045.



CU-IP-Q1 CU-Asy-Triangle-01.fig

Occurrences for different types of CU

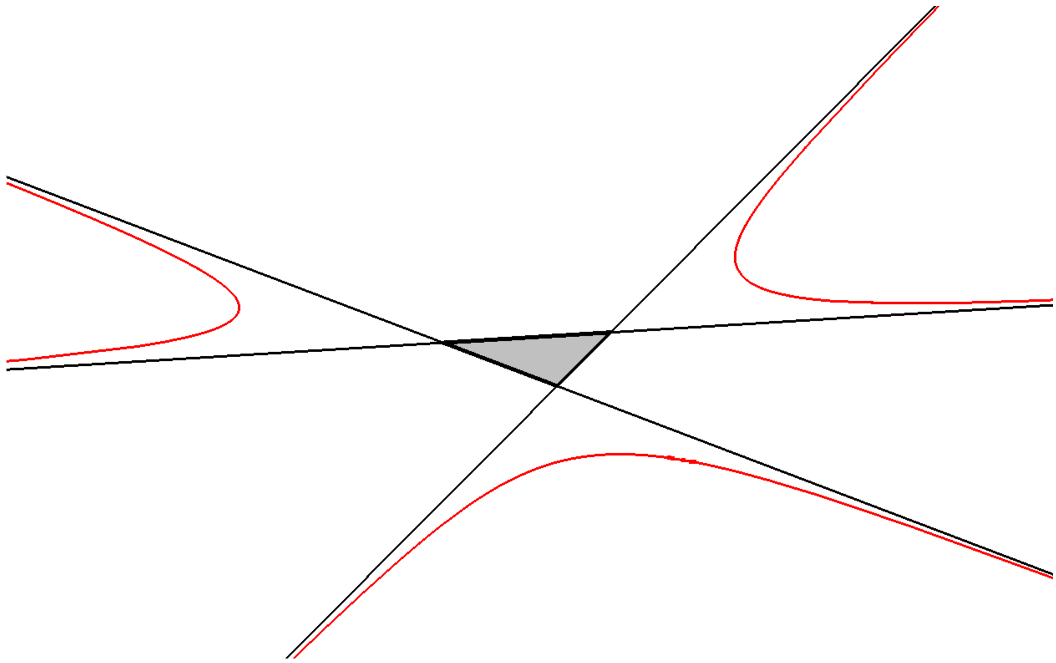
- A real general cubic has three asymptotes. Either one or all three of them are real.
- These asymptotes intersect pairwise at three points (CU-IP-Q1), which may also be either one or three real points.
- One real intersection point can be considered the counterpart of the singular focus found in a circular cubic.
- In some cases, two of the asymptotes may be imaginary lines that intersect at a real vertex, denoted CU-IP-Q1a.

Relationship with CUc

On a circular cubic there is only one real asymptote and only one real vertex F. This single vertex is the real intersection point of the two imaginary asymptotes of the circular cubic and called (the singular) focus of the circular cubic. It is also the common intersection point of the Perpendicular Bisectors of the remaining intersection points of CU and all lines though Q (CU-IP-P2).

CU-IP-L1 Asymptotes of the Cubic

The 3 asymptotes of a general cubic are CU-IP-L1a, CU-IP-L1b, CU-IP-L1c.
The points at infinity (IP1,IP2,IP3) of the reference cubic are also the points at infinity of the asymptotes.
The asymptote triangle and its vertices of the are described at CU-IP-Q1.
For a real Cubic there is at least one real asymptote. Two of them can be imaginary.



CU-IP-L1 CU-Asy-Triangle-01.fig

Construction
A construction of all 3 asymptotes given 9 points of the cubic can be found at [Cuppens, pages 212,242,243].

Construction of the 2nd and 3rd CU-asymptote
When one asymptote is already known, then the 2nd and 3rd asymptote can be constructed as follows.
Let IP1 be the infinity point of the known asymptote.
Note that the IP1-Central Conic (CU-IP-Co2) has infinity points IP2 and IP3 on it.
Therefore when IP1, IP2, IP3 are real points and the IP1-asymptote is known, then the IP2- and IP3-asymptotes con be constructed as follows:

1. Construct the IP1-Central Conic Co1 as described at CU-IP-Co2.
2. Construct its CO1-asymptotes. They will be parallel to the CU-asymptotes with infinity points IP2 and IP3.
3. Construct the CU-asymptotes Asy-2 and Asy-3 using the general construction method described at CU-IP-P2.

Properties:

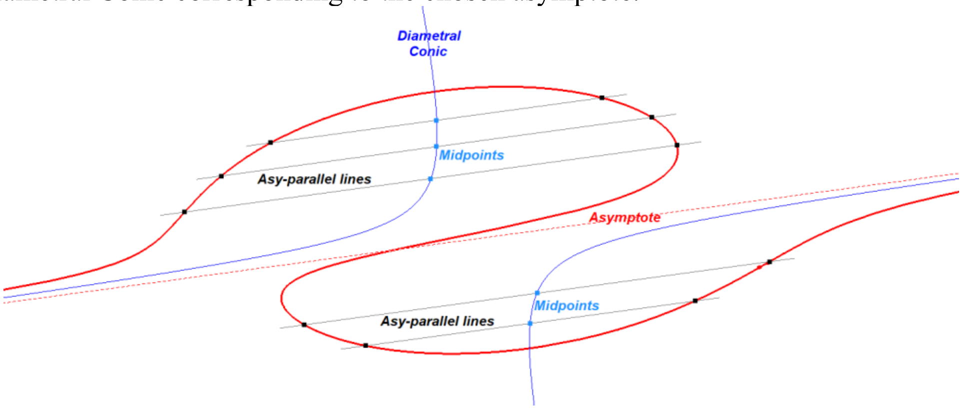
- If the cubic has three real asymptotes, then the Lemoine point aka Symmedian Point (X(6) in ETC) of the triangle formed by the asymptotes has a special function. The polar conic of this point wrt the cubic is a circle. See CU-Ci1 and Gibert, Isocubics, [17b], page 21.

CU-IP-Co1 IP-Diametral Conic / IP-Polar Conic

CU-IP-Co1 is the Polar Conic of one of an infinity point of a reference cubic CU.
It contains the infinity point of the reference cubic it refers to, which makes it a hyperbola with one asymptote in common with CU.
There are 3 versions of the IP-Diametral Conic, for each infinity point there will be one version.
Two of these versions can be imaginary.
When CU has just 1 real asymptote, then there will be just 1 real IP-Diametral Conic.
When CU has 3 real asymptotes, then there will be 3 real IP-Diametral Conics.
It is mentioned by Roger Cuppens in a general context for general cubics and called by him "Conique Diamétrale". See [63], page 262.

Construction

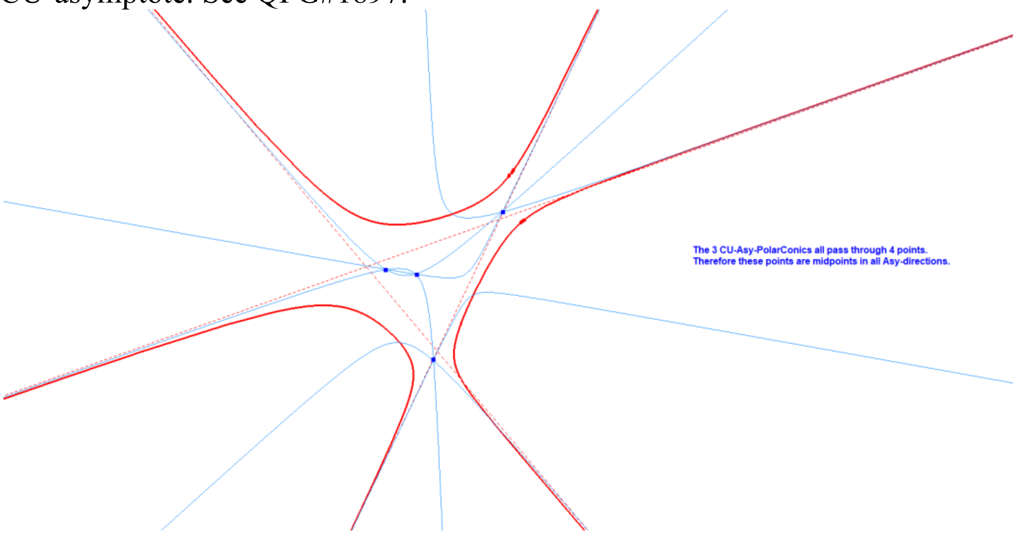
1. Draw 5 lines parallel to one chosen asymptote each having a set of 2 finite intersection points with CU.
2. The conic through the 5 midpoints of the sets of the 2 finite intersection points will be the IP-Diametral Conic corresponding to the chosen asymptote.



CU-IP-Co1 DiametralConic-Construction-01.fig

Properties

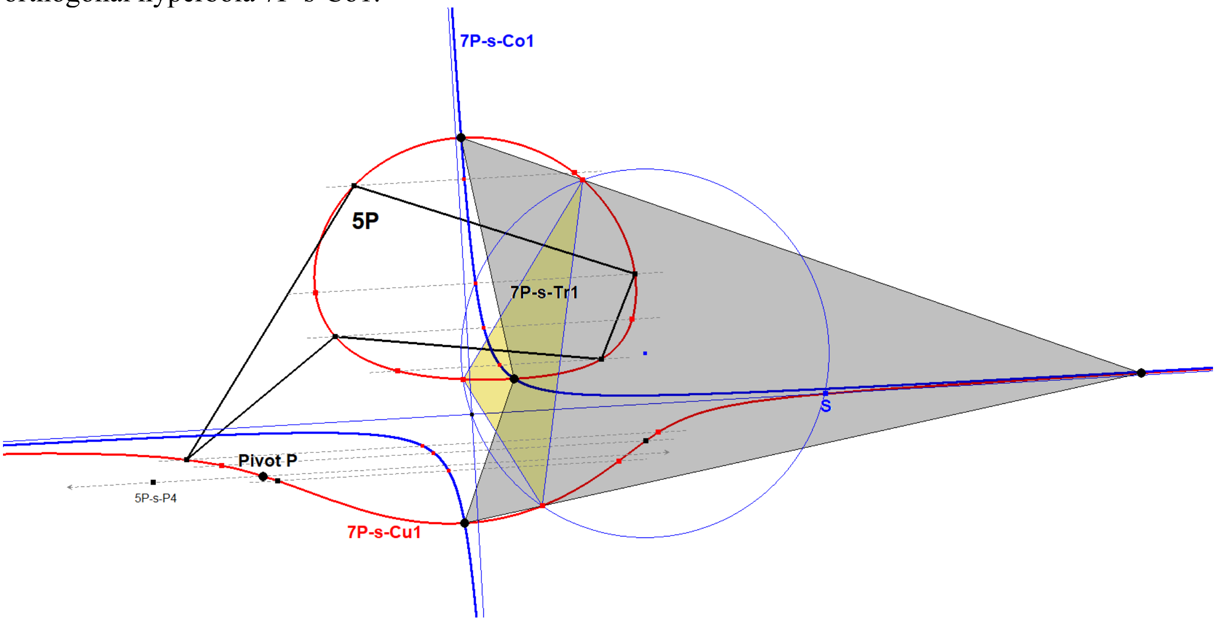
1. CU-IP-Co1 is the locus of midpoints of the 2 intersection points of Asy-parallel lines with CU. The Asy-parallel line that coincides with the asymptote will deliver as intersection points twice the CU-Infinity point. Their midpoint will be the CU-Infinity point. Therefore each version of CU-IP-Co1 will share the CU-asymptote it refers to.
2. The 3 versions of CU-IP-Co1 share 4 common points. These points have the special property that they are midpoint of 2 finite intersection points of any line parallel to a CU-asymptote. See QPG#1897.



CU-IP-Co1 Set of 3 CU-DiametralConics-13.png

3. Each version of CU-IP-Co1 cuts CU in the 4 anallagmatic points CU-IP-4P1 where the parallels of an asymptote touch the cubic. Two of these points can be imaginary.

Case Circular Cubic
On a circular cubic CU-IP-Co1 is an orthogonal hyperbola.
In the context of a circular cubic 7P-s-Cu1 it was mentioned by Eckart Schmidt in QPG-message #837.
Given a bipartite circular cubic CUc with 5 points as 5P, the remaining two points (Pi,Pj) give a fixed cubic pivot P = Pi.5P-s-Tf6(Pi) ^ Pj.5P-s-Tf6(Pj).
The line P.5P-s-P4 is parallel to the asymptote of 7P-s-Cu1.
Therefore parallels to P.5P-s-P4 intersect the cubic in two finite points, whose midpoints give the orthogonal hyperbola 7P-s-Co1.

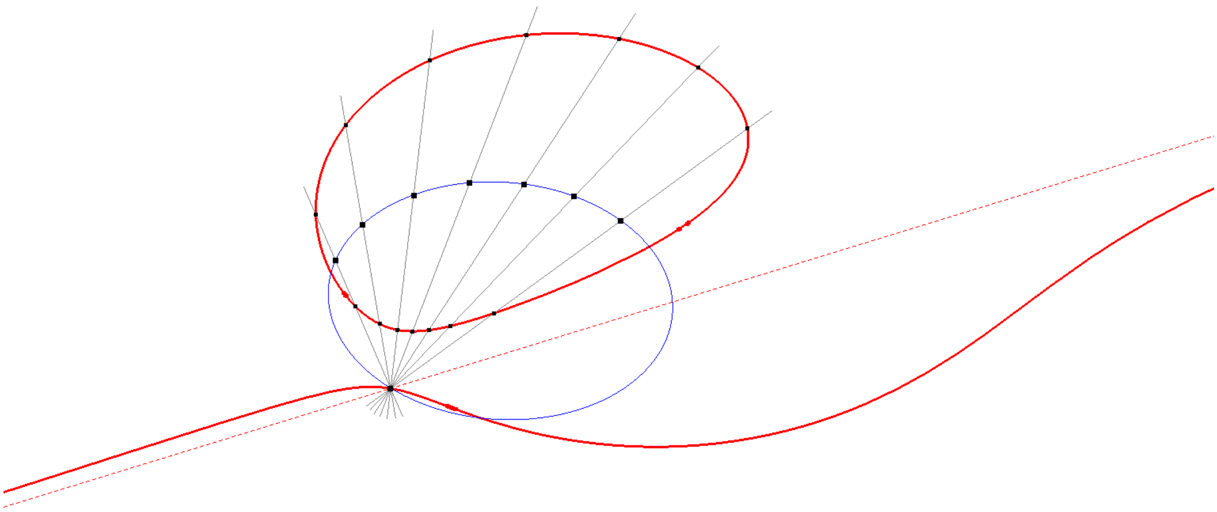


Properties

- one asymptote coincides with the asymptote of the cubic
- the hyperbola intersects the cubic in four points CUc2-4P, which are the in- and excenters of the CUc2-Tr1 Miquel Triangle. See QPG#1862.
- the hyperbola intersects the cubic in four points, the vertices of an orthocentric quadrangle (each vertex is orthocenter of the other three), whose diagonal triangle ABC is 7P-s-Tr1, and its circumcircle bears the center of 7P-s-Co1 and the intersection Q (7P-s-P5) of the cubic and its asymptote (7P-s-L1). See QPG#837.

CU-IP-Co2 Central Conic or QF-Conic

There are three Central Conics, each associated with one of the asymptotes (infinity points) of the cubic. When two asymptotes of a given cubic are not real, their corresponding Central Conics are also not real.
The construction of a Central Conic is straightforward. Given a reference cubic CU, let Asy1 be one of its asymptotes, corresponding to Infinity Point IP1, and intersecting CU at Q1. Draw a pencil of lines through Q1, each intersecting CU at two additional points, Si and Ti. Let Mi be the midpoint of Si and Ti. The locus of Mi forms the Central Conic of CU associated with IP1.
For the first description of the QF-Conic/Central Conic, see QPG#2045.
To indicate that the conic belongs to a specific cubic CU and an infinity point IP1, it can be referred to as the CU-IP1-Central Conic or simply the IP1-Central Conic.



CU-IP-Co2 IP-Central Conic-01.fig

The IP-Central Conic serves as a mold for the shape of the cubic and as a similarity model for other auxiliary conics related to the cubic (see CU-IP-3Tf1 and CU-9P-P1/P2/P3).
Its shape can be either an ellipse or a hyperbola. It forms an ellipse when the other two infinity points are imaginary, and a hyperbola when they are real. This occurs because these two infinity points lie on the conic itself, meaning that IP2 and IP3 belong to the IP1-Central Conic. Note that (IP1, IP2, IP3) can be cyclically interchanged.

Construction of the Second and Third CU-Asymptotes
The IP1-Central Conic contains the infinity points IP2 and IP3.
Therefore, when IP1, IP2, and IP3 are real points and the IP1-asymptote is known, the IP2- and IP3-asymptotes can be constructed as follows:

1. Construct the IP1-Central Conic Co1 as described above.
2. Determine its Co1-asymptotes, which will be parallel to the CU-asymptotes associated with the infinity points IP2 and IP3.
3. Construct the CU-asymptotes Asy-2 and Asy-3 using the general construction method outlined in CU-IP-P2.

Special Property
The Polar Conic of a vertex of the Asy-Triangle contains the same infinity points as the two asymptotes passing through this vertex. Consequently, it also shares these infinity points with the corresponding Central Conic/QF-Conic. This follows from the definition that a P-polar conic passes through the P-points of tangency, which, in the case of a point on a CU-asymptote, are the CU-infinity points.

QF-Circle/Central Circle at a Circular Cubic

The IP-Central Conic is a circle in the case of a Circular Cubic CUc.
A QF-Conic of a General Cubic is derived from the QF-circle in a circular cubic CUc.
There is only one real QF-Conic/-Circle because a Circular Cubic has only one real asymptote.
This circle contains several familiar CU-points viz

- Point Q, the CU-Asy Crosspoint CU-IP-P2a
- Point F, the Singular Focus of a Circular Cubic, which is the intersection point of the tangents at the circular points at infinity, which actually are the two imaginary asymptotes of CUc, which makes it the intersection point of CU-IP-L1b and CU-IP-L1c.
- Points M1a,M1b,M1c, being the 3 Quasi-Miquel Points (vertices of CU-IP-3P1a), of which 1 or 3 are real points.

Apart from these properties the QF-Circle is the locus of midpoints (S1,S2), being the 2nd and 3rd intersection points of a pencil of lines through Q.

QF-Conic/Central Conic at a General Cubic CU

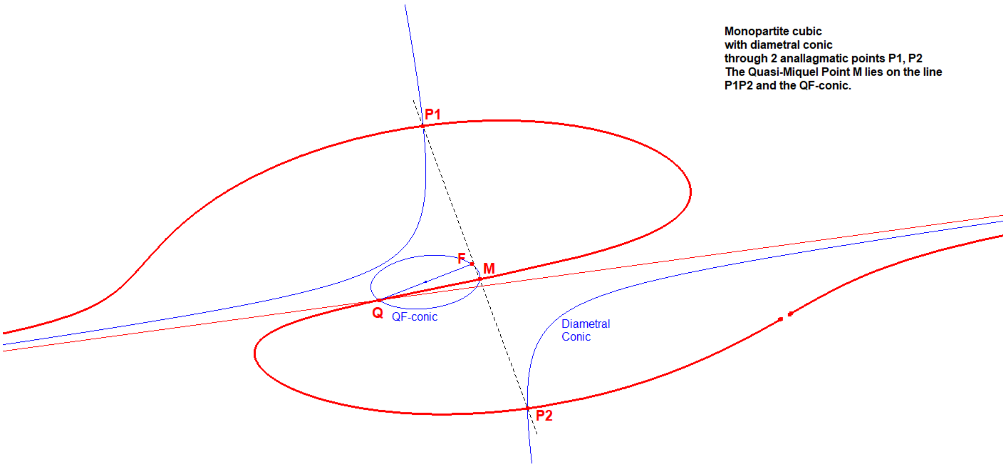
This QF-Circle also does exist for the General Cubic, but in this case it is a conic with these properties:

- It is a conic instead of a circle.
- There are 3 of them, for each CU-asymptote there will be one QF-Conic, but sometimes 2 of them (CU-IP-L1b and CU-IP-L1c) will be imaginary.
- The CU-Asy Crosspoint CU-IP-P2a lies on QF-conic CU-IP-Co2a (the a-version of the conic).
- The Asy-Asy-Crosspoint CU-IP-Q1a (being CU-IP-L1b^CU-IP-L1c) lies on the a-version of the QF-conic. This point will be the point opposite of CU-IP-P2a on the QF-Conic.
- The infinity points of the not involved asymptotes CU-IP-L1b an CU-IP-L1c also lie on the a-version of the QF-conic. When they are real points the QF-Conic will be a hyperbola, when they are imaginary then the QF-Conic will be an ellipse.
- Points M1a,M1b,M1c, being the 3 Quasi-Miquel Points (vertices of CU-IP-3P1a) lie on the a-version of the QF-conic, where 1 or 3 of them will be real points.
- The QF-Conic is the locus of midpoints (S1,S2), where (S1,S2) are the 2nd and 3rd intersection points of a pencil of lines through CU-IP-P2a.

Just like there is an a-version of the QF-conic, there also will be a b-version and a c-version of the QF-conic, cyclologically shifting items.
When the reference cubic has 3 asymptotes without finite CU-Asy Crosspoints Qi, then the QF-Conic will degenerate into two lines, the line-at-infinity as well as the line through CU-IP-3P1a and CU-IP-L1b^CU-IP-L1c.
See introduction and further explanation of this QF-Conic at QPG#2025.

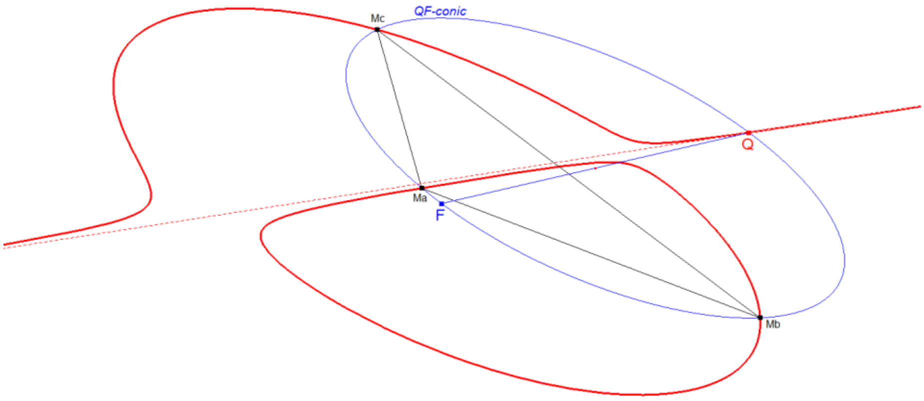
PICTURES

QF-Conic on a Monopartite Cubic:



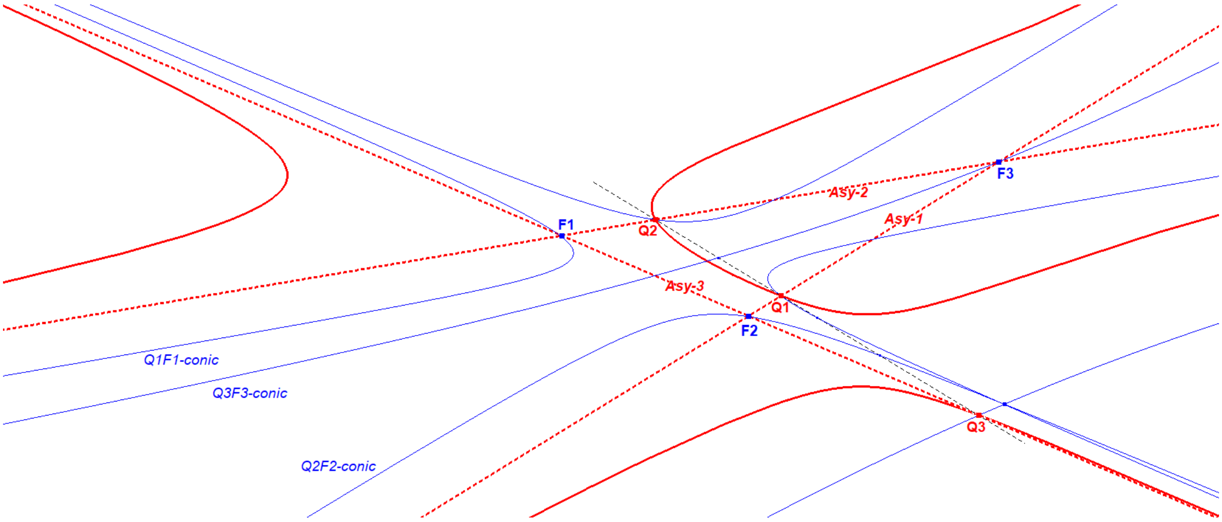
CU1-Monopartite Cubic-02.fig

QF-Conic on a Bipartite Cubic:



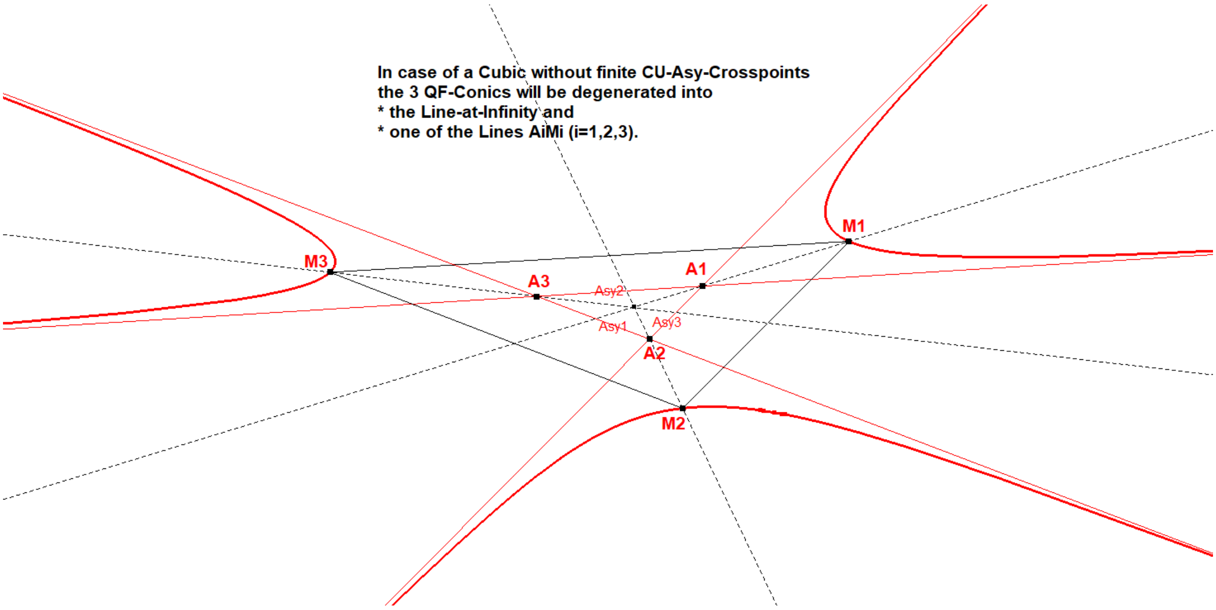
CU2-Bipartite Cubic-10-QF-conic.fig

QF-Conic on a Tripartite Cubic:



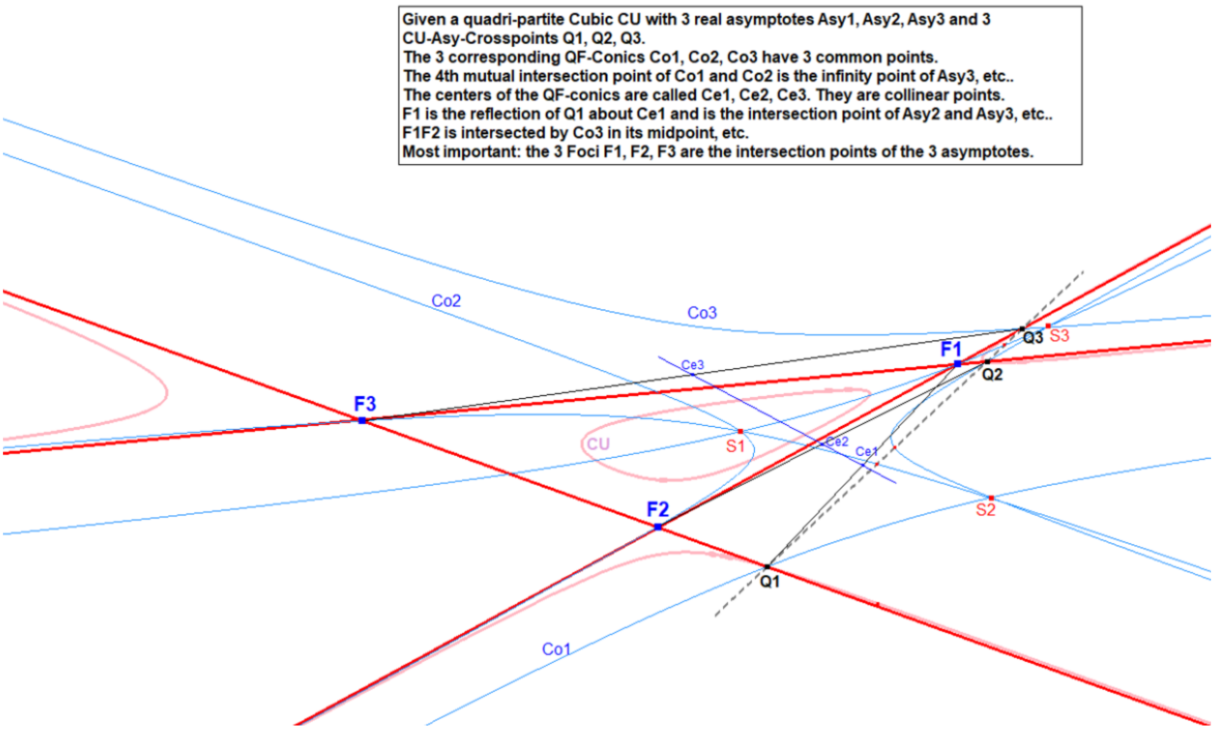
CU3-Tripartite Cubic-10-QF-conic.fig

QF-Conic on a Tripartite Cubic without finite CU-Asy-Crosspoints:



CU3-Tripartite Cubic-2A-2A-2A-02-degenerate QF-Conic.fig

QF-Conic on a Quadripartite Cubic:

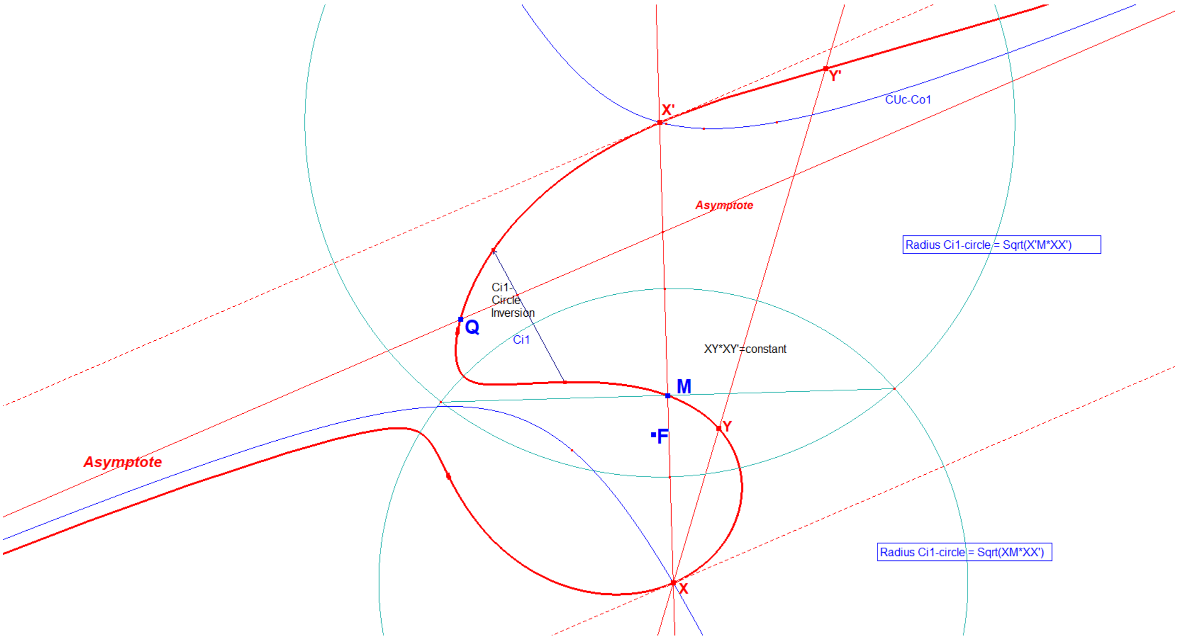


CU4-Quadripartite Cubic-2A-2A-2A-3A-11-Focus Points.fig

CU-IP-4P1 Anallagmatic Points

Anallagmatic means: without change (from the Greek allagma "change").
A curve is said to be anallagmatic if it is globally invariant by an inversion, meaning that the inversion of a point on the curve will be another point on the curve.
This notion was studied by Moutard in 1864.
Normally an inversion is related to an inversion-circle. However here we also consider the inverse of a point X wrt an inversion-conic. One might call this a ‘quasi-inversion’ and the conic center a ‘quasi-anallagmatic point’.
The (quasi-)inverse of P wrt inversion-conic CO with center O is constructed as follows: Draw a half-line OP and let S be the intersection point of OP with CO. Now construct X such that OS^2=OP*OX.
It is said that all Circular Cubics are Anallagmatic Curves, meaning that there are circle center(s) on the curve for which its circle functions as an inversion-circle. Every point on the curve inverted wrt this inversion-circle will result in another point on the curve.

Picture of 2 Anallagmatic Points on a Circular Cubic working with inversion-circles



CUc1-Tf1-MonopartiteCubic-CSC Moebius Trf-28-Anallagmatic property.fig

General rules for Anallagmatic points

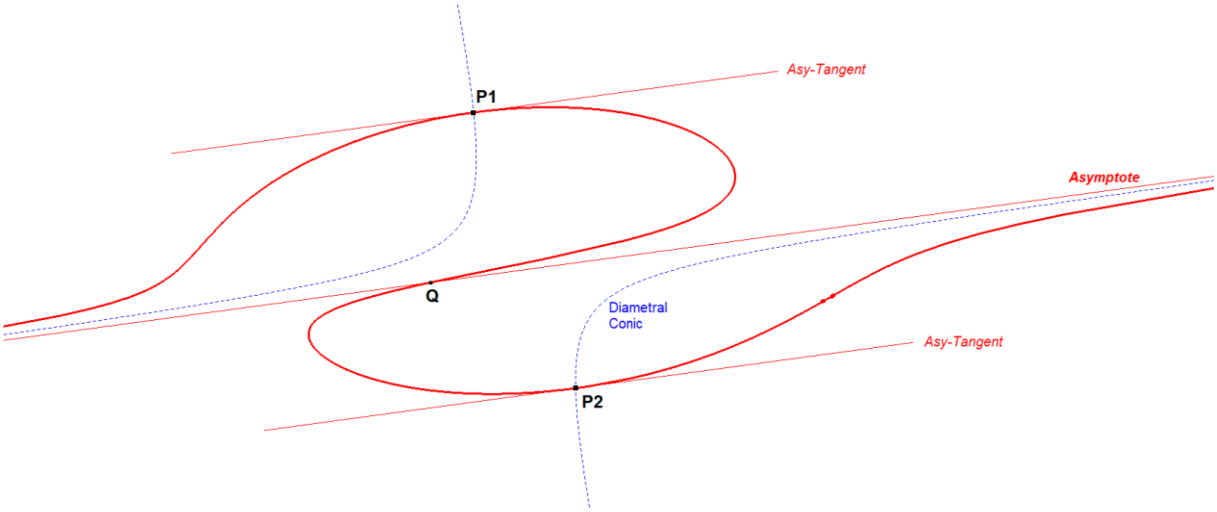
1. Strictly speaking, anallagmatic points on a cubic only occur on circular cubics.
2. However, there is a quasi-variant occurring on regular non-circular cubics. Instead of inversion-circles, there will be inversion-conics.
3. In general, a cubic has 3 asymptotes, two of which can be imaginary.
4. Per asymptote there are 4 anallagmatic points, of which 2 can be imaginary in the case of a real asymptote. All of them will be imaginary in the case of imaginary asymptotes. Ultimately, it is therefore possible that a cubic ends up with just 2 real anallagmatic points (case of a monopartite cubic). But it is also possible there will be 3x4 real anallagmatic points (case of a tripartite or quadripartite cubic).
5. Since there are always per asymptote 4 anallagmatic points, we talk here about an Anallagmatic Quadrangle, bearing in mind that 2 or even 4 of them can be imaginary. In the extreme case we even are dealing with an imaginary quadrangle.

General construction of Anallagmatic points

1. Given a reference cubic CU.
2. This cubic has 3 asymptotes, at least one of which is real.
3. Choose a real CU-Asymptote.
4. There are 4 points P1, P2, P3, P4 on the cubic where the tangent is parallel to this asymptote. These 4 points are called the anallagmatic points. Two of them can be imaginary points, for example on a monopartite cubic, leaving only 2 real anallagmatic points.
5. They are the intersection points of CU with the Diametral Conic (CU-IP-Co1) corresponding with the chosen CU-asymptote. This is because the Diametral Conic is a P-Polar Conic and a P-Polar Conic intersects the cubic at the points of tangency of all possible tangents to P. Clearly, P here is the CU- infinity point of the chosen asymptote.
6. The Diametral Conic is a Hyperbola sharing an asymptote with the chosen CU-Asymptote.
7. Cubic and Hyperbola are mutually tangent at their common infinity point, making this point a double intersection point of both curves. According to Bezout's Theorem there are 6 intersection points of a conic and a cubic and therefore 4 finite intersection points remain and these are the Anallagmatic points Pi (i=1,2,3,4).
8. When there are 3 real asymptotes, then there will be 3 different sets of 4 Anallagmatic Points (Anallagmatic Quadrangles), in sets of 2 possibly being imaginary points.

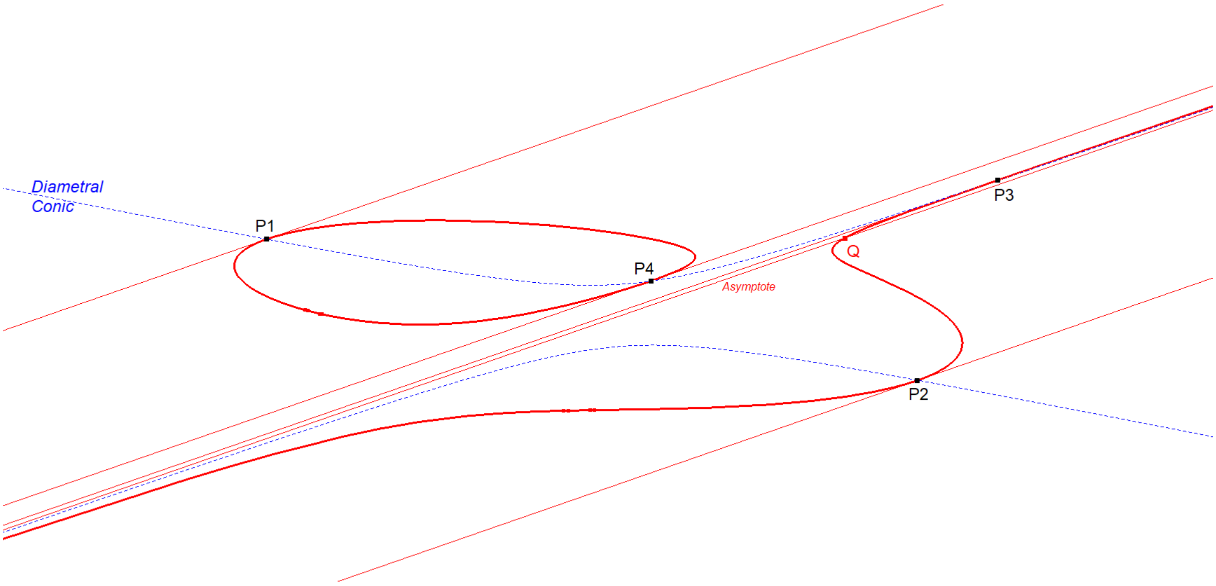
Picture Anallagmatic Points on a Monopartite Cubic

There is 1 real asymptote and there are just 2 real Anallagmatic Points.



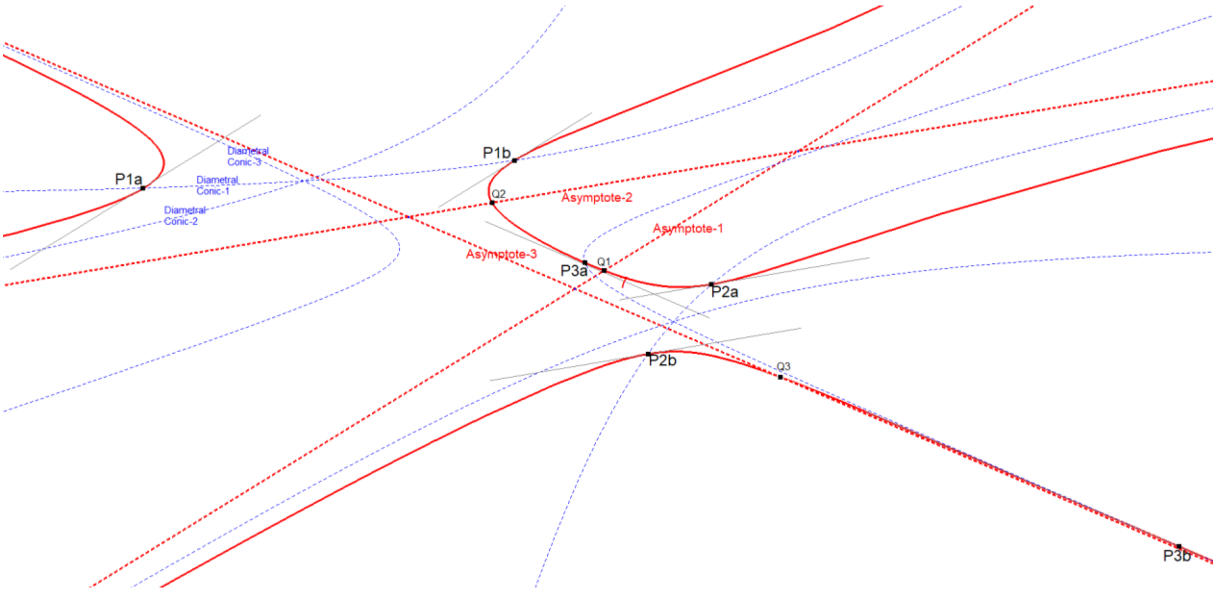
CU1-Monopartite Cubic-04-Anallagmatic Points.fig

Picture Anallagmatic Points on a Bipartite Cubic
There is an example with 1 real asymptote and there are 4 real Anallagmatic Points.



CU2-Bipartite Cubic-20-Anallagmatic Points.fig

Picture Anallagmatic Points on a Tripartite Cubic
Here there are 3 real asymptotes and 3x2 real Anallagmatic Points.
Note: There are also Tripartite Cubics generating sets of 4 real Anallagmatic Points.



CU3-Tripartite Cubic-41-DiametralConics-Anall-Points.fig

Properties

- Each set of 4 Anallagmatic Points forms a Quadrangle whose vertices of the Diagonal Triangle also lie on the cube. This triangle is called the Quasi-Miquel triangle (CU-IP-3P1).
- Each vertex of this Quasi-Miquel Triangle (CU-IP-3P1) is the center of a Quasi-Moebius Transformation (CU-IP-3Tf1).
- Tangents at the Anallagmatic Points enclose strips/segments within which the cubic ‘moves’. This can easily be seen in the images above.

CU-IP-3P1 Set of 3 Quasi-Miquel Triangles

Each cubic has 3 special inscribed triangles. They are called the Quasi-Miquel Triangles, because they are related to the Miquel Triangle (QA-Tr2) in a Quadrangle.
Their locations are fixed on the cubic. Every cubic has its own fixed set of 3 Quasi-Miquel Triangles.
However not every vertex of these triangles is always real. There can be imaginary vertices depending on the shape of the cubic.
In principle, every asymptote of the cubic has such a related triangle.
When the reference cubic has only one real asymptote, then there will be one such triangle of which at least one vertex will be real.
When the reference cubic has three real asymptote, then there will be three such triangles of which per asymptote at least one vertex will be real.
So it can be said that each cubic has three of these triangles with a total of 3x3 vertices, but in the minimum case only one vertex will be real and in the maximum case there will be 3x3 vertices real.
It's all depending on the shape of the cubic.
From here the different options will be discussed.

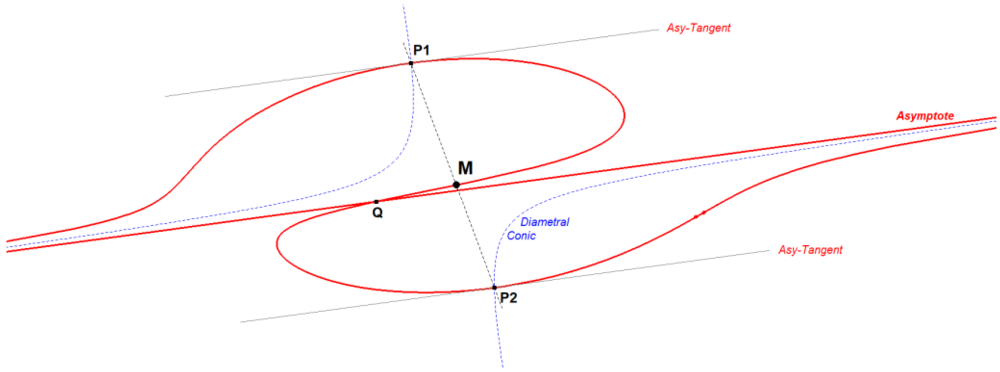
The Quasi-Miquel Triangles are actually the Diagonal Triangles of the three Quadrangles formed by the sets of 4 Anallagmatic Points (CU-IP-4P1) that can be constructed per CU-asymptote.
Also within the sets of 4 Anallagmatic Points (CU-IP-4P1) not always all points will be real.
There are these situations:

1. When the reference cubic has only one real asymptote, then there will be 2 or 4 real Anallagmatic points.
   - When there are 2 real Anallagmatic points P1 and P2, then there will be one real vertex of the corresponding Quasi-Miquel Triangle. It is the 3rd intersection point of P1P2 with CU.
   - When there are 4 real Anallagmatic points P1, P2, P3 and P4, then there will be 3 real vertices of the corresponding Quasi-Miquel Triangle. They are the vertices of the Diagonal Triangle (QA-Tr1) of the Quadrangles formed by (P1,P2,P3,P4).
2. When three asymptotes are real, then there will be 2 or 4 real anallagmatic points per asymptote. Again the same rules 1a and 1b apply for each asymptote.

Pictures of QMT’s on different types of Cubics

Picture Anallagmatic Points and Quasi-Miquel Point on a Monopartite Cubic

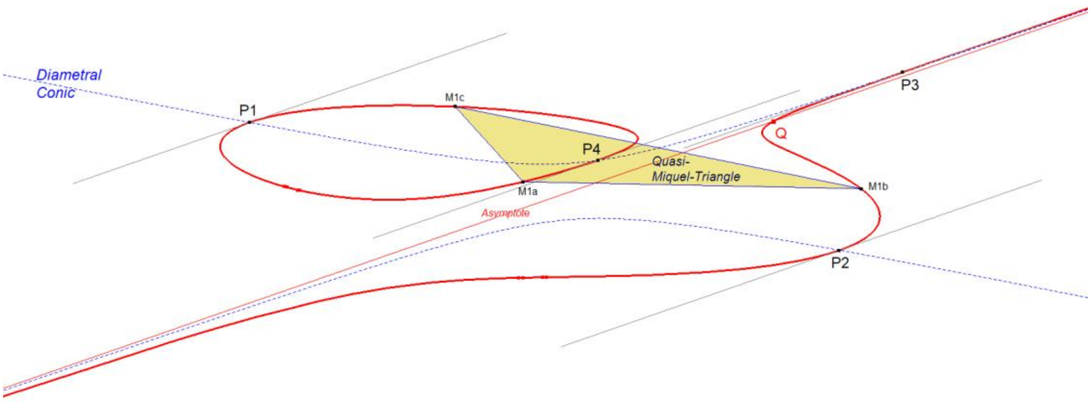
There is 1 real asymptote and there are 2 real Anallagmatic Points P1 and P2 and 1 real Quasi-Miquel-Point M (the only real vertex of a Quasi-Miquel Triangle).



CU1-Monopartite Cubic-05-Anal-Miq-Point.fig

Picture Anallagmatic Points and Quasi-Miquel Triangle on a Bipartite Cubic

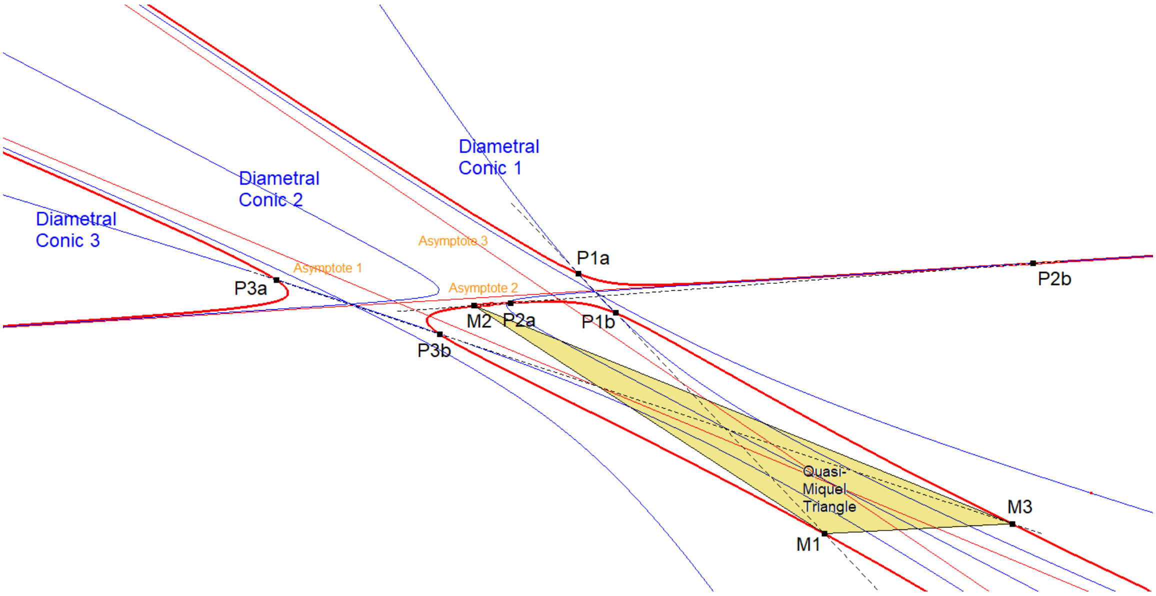
There is 1 real asymptote and there are 4 real Anallagmatic Points and a Quasi-Miquel-Triangle with 3 real vertices.



CU2-Bipartite Cubic-21-Anall-Miquel-Points.fig

Picture Anallagmatic Points and Quasi-Miquel Triangle on a Tripartite Cubic

There are 3 real asymptotes witch each 2 real Anallagmatic Points (P1a,P1b, P2a,P2b, P3a,P3b) and each 1 Quasi-Miquel-Triangle with each 1 real vertex (M1, M2, M3).



CU3-Tripartite Cubic-50-DiametralConics-Anall-Miq-Points.png

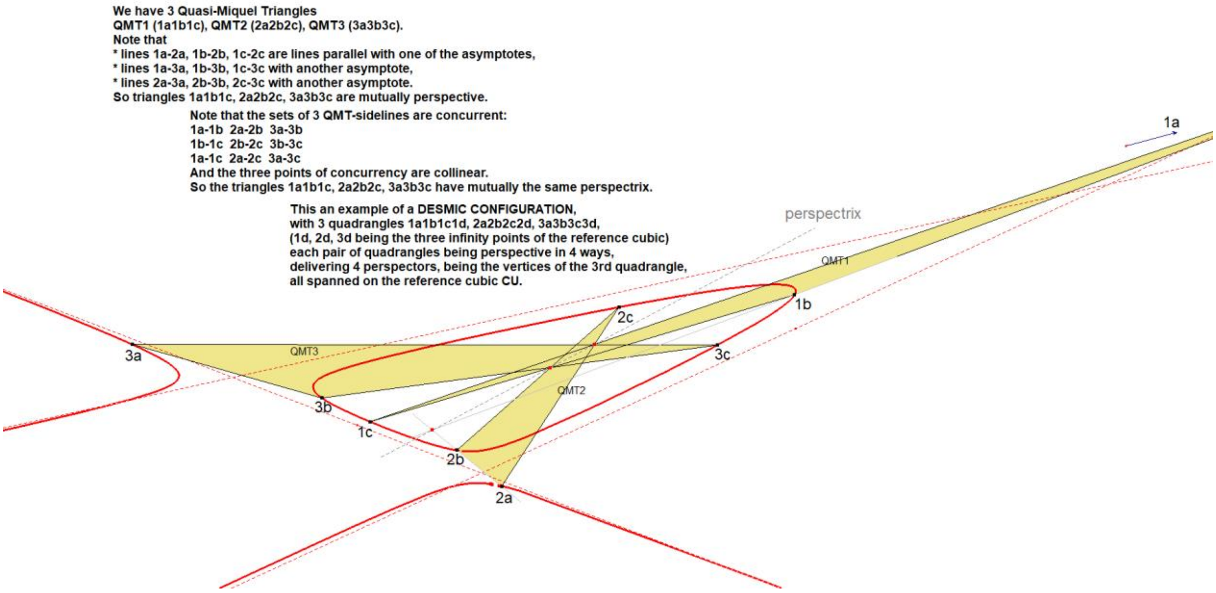
Picture Anallagmatic Points and Quasi-Miquel Triangle on a Bipartite Cubic

Now there are 3 Quasi-Miquel Triangles.

Special property:

The set of 3 Quasi-Miquel Triangles in combination with the CU-infinity points form a Desmic Configuration.

See QPG#2263. See also EQF/QA-Tr-1.



CU4-Quadripartite Cubic-with Diametral Conics-08-plus QMTs.fig

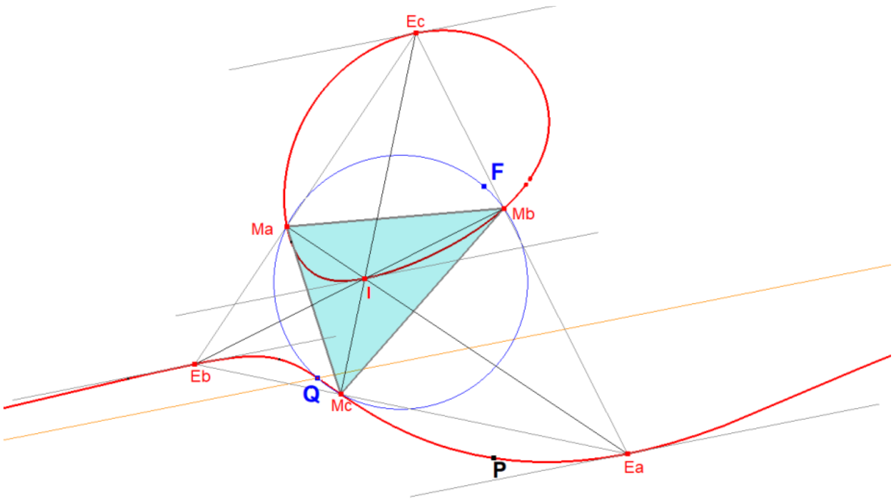
Note:

The 4[th] branch of the Quadripartite Cubic is not visible at this picture. It passes at the left through the utmost vertex of QMT1.

QUASI-MIQUEL TRIANGLE ON A CIRCULAR CUBIC

On the Circular Cubic there are additional special features with respect to the Quasi-Miquel Triangle.

1. Given a Bipartite Circular Cubic CUc2.
2. CUc2 has a Focus F (CU-IP-Q1a) and an asymptote intersecting CUc2 in Q.
3. The circle with Diameter QF intersects CUc2 in 4 points: Q, M1, M2, M3. Q is the intersection point of CUc2 with its asymptote and M1, M2, M3 are the vertices of the Quasi-Miquel Triangle for a Circular Cubic.
4. The internal and external bisectors of the Miquel Triangle intersect in the incenter I and excenters E1, E2, E3 of the Miquel Triangle.
5. The tangents at I, E1, E2, E3 are lines parallel to the CUc2-asymptote.
6. Therefore points I, E1, E2, E3 are the Anallagmatic Points (CU-IP-4P1) of CUc2.
7. I, E1, E2, E3 lie all on the IP-Polar Conic of CUc2 (IP=Infinity Point of CUc2), which is the Orthogonal Hyperbola, described by Eckart Schmidt in QPG#837/#1538. This allows for an easy construction of this Orthogonal Hyperbola. It is also called the Diametral Conic (CU-IP-Co1) for the Circular Cubic.

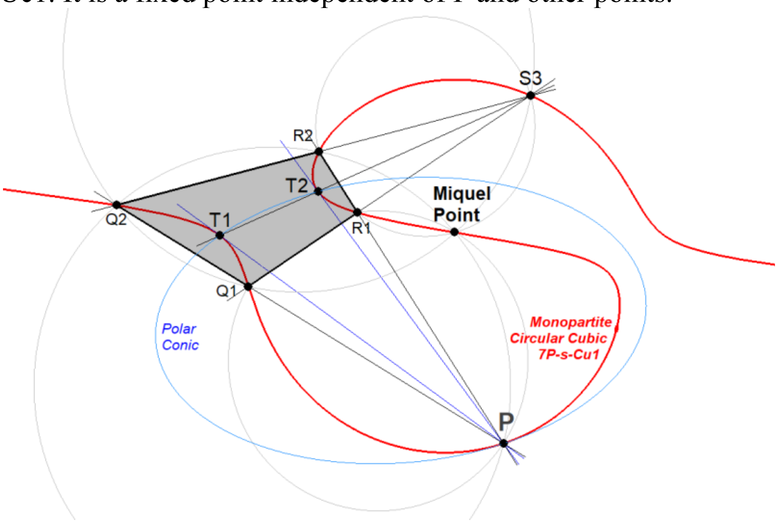


CUc2-4P In- and Excenters Miquel Triangle-01a.fig

Another construction of the Quasi-Miquel Triangle on a Circular Cubic

The monopartite case

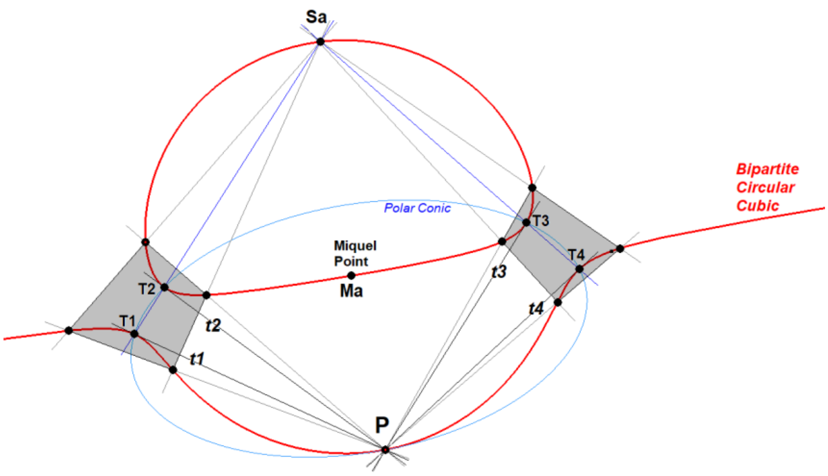
1. Given a random point P on Monopartite Cubic CUc1.
2. Let T1 and T2 be the contact points of the tangents from P to the monopartite CUc1.
3. Let S3 be the 3rd intersection point of T1T2 intersecting CUc1.
4. Then draw a line through P intersecting CUc1 in Q1 and Q2.
5. Let R1 be the 3rd intersection point of Q1S3 with CUc1.
6. Let R2 be the 3rd intersection point of Q2S3 with CUc1.
7. R1R2 will be a line through P.
8. The Miquel point of the Quadrilateral with defining lines Q1Q2, R1R2, Q1R1, Q2R2 will be a point on CUc1. It is a fixed point independent of P and other points.



Note: The Miquel Point of a Quadrilateral (QL-P1 in EQF) is the common point of the circumcircles of each triangle formed by 3 out of 4 lines of the Quadrilateral.

The bipartite case

1. Given a random point P on Bipartite Cubic CUc2.
2. When CUc2 is bipartite then 4 tangents can be drawn from a random point P to the cubic and there will be 4 tangents t1, t2, t3, t4 with resp. contact points T1, T2, T3 and T4.
3. In a similar construction each pair of tangents (t1,t2) and (t3,t4) deliver a Miquel Point, which appear to be identical, which is point Ma in the picture.



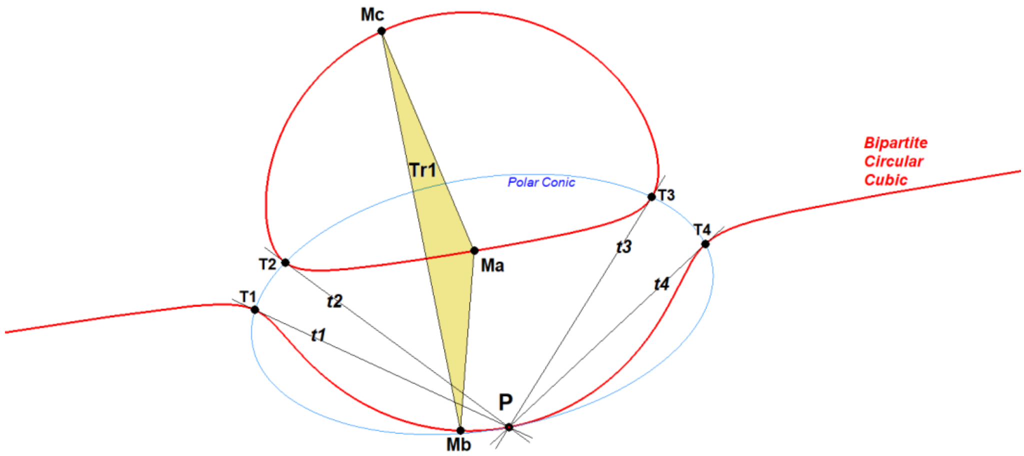
CUc2-Tr1 Miquel Triangle-02.jpg

4. Likewise the pairs of tangents (t1,t3) and (t2,t4) deliver a 2nd Miquel Point Mb and the pairs of tangents (t1,t4), (t2,t3) deliver a 3rd Miquel point Mc. Together they form the Cuc2-Miquel

Triangle. It is a similar triangle as occurring in a Quadrangle, where it is called the Miquel Triangle QA-Tr1.

5. In addition it appears that MaMbMc is the Miquel Triangle of the Quadrangle T1T2T3T4. Point P will be point QA-P4 of Quadrangle T1T2T3T4.

The easiest way of construction of MaMbMc in the bipartite case is by constructing the P-Polar Conic and its 4 cubic intersecting points T1, T2, T3, T4 and construct from Quadrangle T1T2T3T4 the Miquel Triangle (QA-Tr2 in EPG).



CUc2-Tr1 Miquel Triangle-01.fig

Tr1 is a triangle, unique for a bipartite circular cubic, not dependent of the choice of any point on the cubic, but only of the bipartite circular cubic itself.

Conversely Tr1 can be regarded as a reference triangle for CUc2.
In terms of Bernard Gibert's Catalogue for Cubics in the Triangle Plane [17], the cubic is an isogonal circular pK (X(6),P) wrt Tr1, P being the pivot which lies on the infinity line and is the infinity point of the asymptote of the cubic. The Isogonal Conjugate with respect to triangle Tr1 of any cubical point P on the cubic will also lie on the cubic.
Further remarks about this configuration can be found at [66], QPG-messages #825-#852.

CU-IP-3Tf1 Set of 3 Quasi-Möbius Transformations (QMT)

There exist at every Cubic 3 Quasi-Miquel Triangles with each 3 vertices (CU-IP-3P1). Some of them are real, others are imaginary points depending on the shape of the cubic.
Nevertheless each vertex is the center of a Quasi-Möbius Transformation (QMT). Of course the real vertices are appealing because they are visible in the real plane and so are the real QMT's.
Each QMT maps a point on the cubic to another point on the cubic. It is often described that these points are 'swapped.' Additionally, it is stated that the cubic is invariant with respect to the QMT, meaning that the QMT maps any point on the cubic to another point on the cubic.
There are two main ways of construction for these QMT's.

1. The limited and straightforward construction, which only works for points on the cubic.
2. The extended and intricate construction, which not only works for points on the cubic but also *involves points outside of the cubic*. It swaps points on the cubic in the same way as the limited construction does.

The limited construction

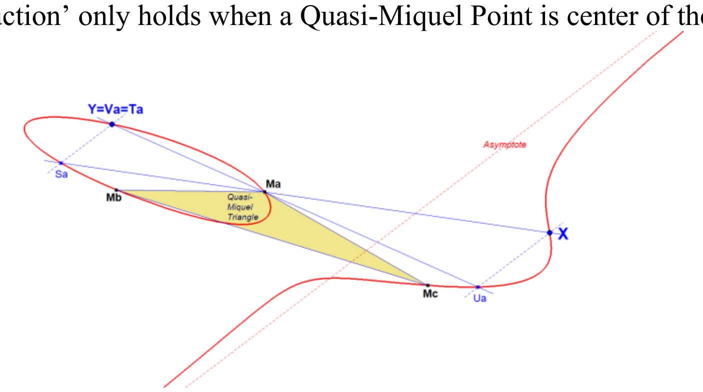
We will choose one of the vertices of the three Quasi-Miquel Triangles to illustrate our construction. Each Quasi-Miquel Triangle is associated with one of the asymptotes. Let's designate our vertex as being related to asymptote-1, abbreviated as Asy1, and identify it as the 'a-vertex.' Therefore, we will refer to our vertex as Ma. The construction will be carried out with this vertex as the center. Let X be the point to be swapped into Y.
Now the construction is very simple:

1. Draw XMa and let Sa be the 3rd intersection point of XMa with the cubic.
2. Draw a line through Sa parallel to Asy1 and let Ta be the 2nd finite intersection point of this parallel line with the cubic (the 3rd intersection point will be the infinity point of the asymptote).
3. Then Y=Ta is the point of transformation.

There is even a resembling alternative for this construction:

1. Draw a line through X parallel to Asy1 and let Ua be the 2nd finite intersection point of this parallel line with the cubic (the 3rd intersection point will be the infinity point of the asymptote).
2. Draw MaUa and let Va be the 3rd intersection point of MaUa with the cubic.
3. Then Y=Va=Ta is the point of transformation.

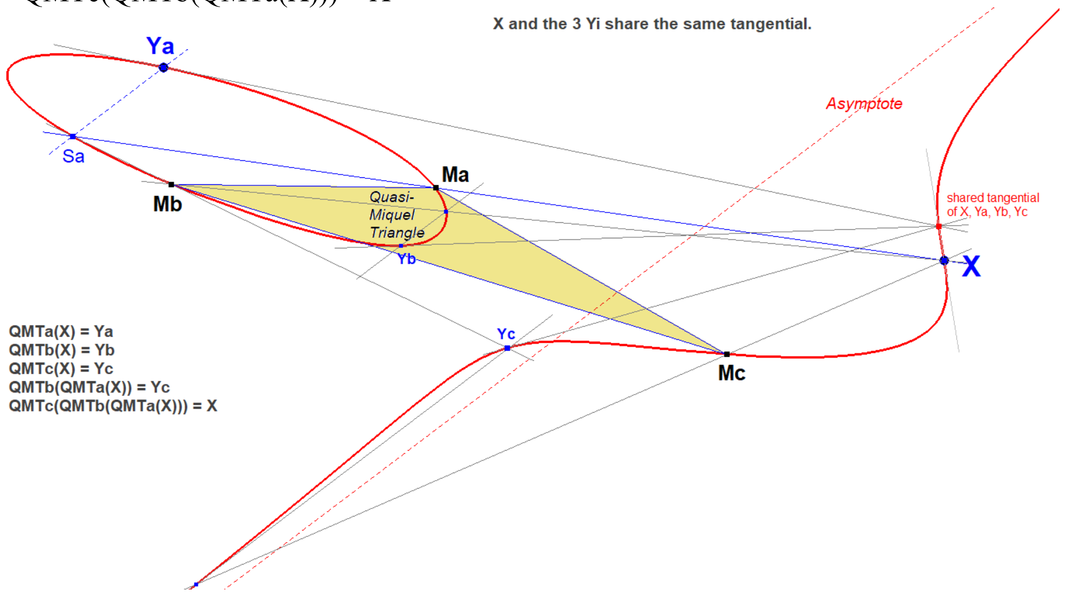
This 'trapezium-construction' only holds when a Quasi-Miquel Point is center of the trapezium.



CU-IP-Tf1-Quasi-Moebius Transformation-01.fig

Note that vertex has its own point of transformation mapped by QMTa, QMTb, QMTb. Let's call the mapped points Ya, Yb, Yc.

1. X, Ya, Yb, Yc share the same tangential.
2. QMTc(QMTb(QMTa(X))) = X



CU-IP-Tf1-Quasi-Moebius Transformation-02.fig

The extended construction

There exist(s) at every Circular Cubic 1 or 3 Quasi-Miquel Point(s). See CU-IP-3P1, special case 'Circular Cubics'.
Each of their vertices is the center of a Möbius Transformation.
This Möbius Transformation maps a point by inversion about a circle with center M (Quasi-Miquel Point) completed with a reflection about an axis through M.
There is an equivalent for this Möbius Transformation for the general cubic called the Quasi-Möbius-Transformation (abbreviated by 'QMT').
The difference is that a Quasi-Möbius-Transformation maps a point by inversion about a ***conic*** with center M completed with a reflection about an axis through M.
It is found and developed at QPG.

The inverse wrt a conic
For this we must define the notion of the inverse X of a point P wrt a conic.
For a circle we draw a half-line from O (the center of the circle) to P intersecting the circle in S, and measuring X on half-line OP such OS^2 = OP * OX.
For a conic we do exactly the same: we draw a half-line from O (the center of the conic) to P intersecting the conic in S, and measuring X on half-line OP such OS^2 = OP * OX.

Construction of the Quasi-Möbius-Transformation CU-IP-3Tf1(P)
Given a **bipartite** reference cubic CU.
This cubic has 3 asymptotes, from which at least one is real.
Choose a real asymptote.

Anallagmatic Points
There are 4 points P1, P2, P3, P4 on the cubic where the tangent is parallel to this asymptote.
These points are called the Anallagmatic points (see CU-IP-4P1). Some of these points can be imaginary.
They are anallagmatic because these points are the center of a ***circle*** inverting each point on CU in another point on CU. This actually is only the case when the cubic is circular (having the circular points at infinity on it).
However here we extent the definition of Anallagmatic to being also the center of a ***conic*** inverting each point on CU in another point.

Inversion conic and anti-inversion conic
3 of the 4 anallagmatic points are centers of an "inversion conic", meaning that the 2 other CU-intersection points of each line through this center lie inverse wrt the conic.
1 of the 4 anallagmatic points is center of an "anti-inversion conic", meaning that the 2 other intersection points of each line through this center lies inverse wrt the conic and will then be reflected about the center.
Therefore this single point has a special status.
Let's denote the anallagmatic point with the special status P1 and the other 3 anallagmatic points P2, P3, P4.
Note: On a circular cubic this point is the incenter of the Diagonal Triangle of QA(P1,P2,P3,P4) and the other 3 anallagmatic points are the excenters of the Diagonal Triangle of QA(P1,P2,P3,P4).

Inversion conics
Each anallagmatic point is center of an "(anti-)inversion conic".
Denote these conics with center P1,P2,P3,P4 as Co1, Co2, Co3, Co4.
These conics are similar (same eccentricity) and have parallel axes.
Their positioning is ***quasi-orthogonal***, meaning that per set of two crossing conics (let's say Co1 and Co2) the 2 Co1-tangents at their intersection points intersect in the center of Co2 and vice versa.
Co1, Co2, Co3, Co4 have a quasi-orthogonal positioning, which appears to be a condition to mutually combine pairwise these conics.

Pairwise intersecting of the inversion conics
Like mentioned they pairwise intersect in two real intersection points, defining a Radical Axis.
Denote the intersection points of Co1 and Co2 as F12a&F12b, etc.
The 4 conics can mutually cross 6 times.
All 6 radical axes pass through P1.
Furthermore the 3 mutual radical axes of Co2, Co3, Co4 pass through some extra points.
Summarizing:
F12a-F12b only passes through P1
F13a-F13b only passes through P1
F14a-F14b only passes through P1
F23a-F23b passes through P1, P4, M14
F24a-F24b passes through P1, P3, M13
F34a-F34b passes through P1, P2, M12
Where M12, M13, M14 are the vertices of the Diagonal Triangle of QA(P1,P2,P3,P4).

Pairwise combining the inversion conics
Inversion conics can be pairwise combined into another conic, which is called the mutual product of these two inversion conics.
Let Co1 and Co2 be two inversion-conics and INV1 and INV2 the inversion-transformations wrt Co1 and Co2. Let Co12 be the product-inversion-conic Co1xCo2 and INV12 the inversion-transformations wrt Co12.
Now CO12 is the conic for which INV12(X) = INV1(INV2(X)) = INV2(INV1(X)).
It appears that wrt inversion-conics Co1, Co2, Co3, Co4 that Co12=Co34, Co13=Co24, Co14=Co23.
Therefore we will call these inversion-conics resp. Co12-34, Co13-24, Co14-23 with resp. centers M12-34, M13-24, M14-23:
Specifications of these product conics are:
Co12-34 = Co1 x Co2 as well as Co3 x Co4

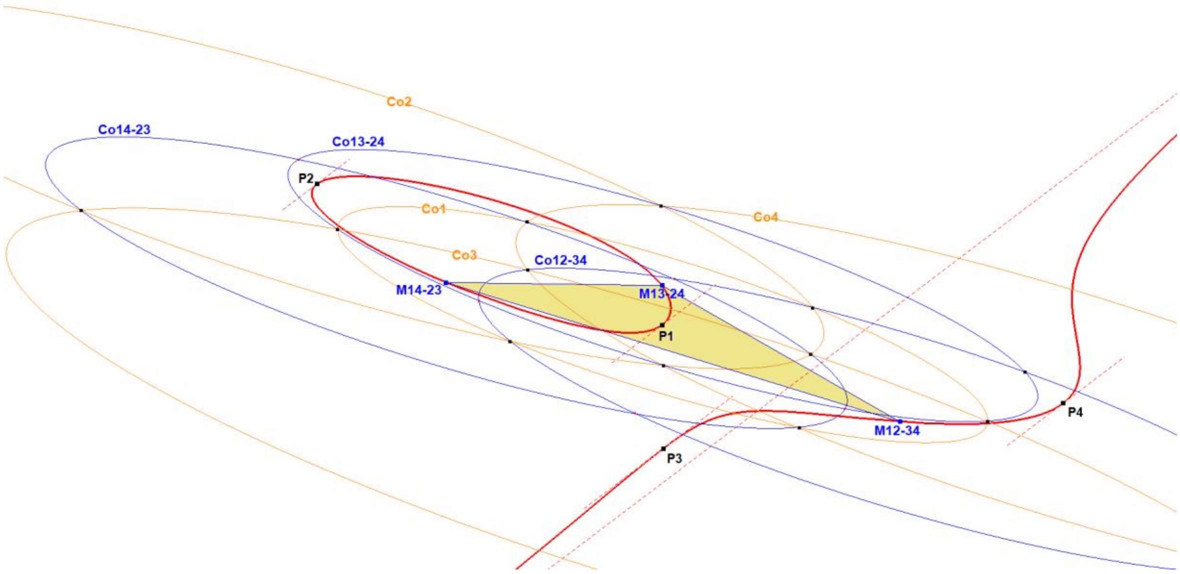
- with center M12-34=P1P2xP3P4,
- passing through the intersection points of Co3xCo4 (with tangents parallel to P3P4)
- passing through the intersection points of Co1xCo2 (with tangents through P2)
- with QMT-axis=P1P2 and QMT-target-point=infinity point P3P4.

Co13-24 = Co1 x Co3 as well as Co2 x Co4

- with center M13-24=P1P3xP2P4,
- passing through the intersection points of Co2xCo4 (with tangents parallel to P2P4)
- passing through the intersection points of Co1xCo3 (with tangents through P3)
- with QMT-axis=P1P3 and QMT-target-point=infinity point P2P4.

Co14-23 = Co1 x Co4 as well as Co2 x Co3

- with center M14-23=P1P4xP2P3,
- passing through the intersection points of Co2xCo3 (with tangents parallel to P2P3)
- passing through the intersection points of Co1xCo4 (with tangents through P4)
- with QMT-axis=P1P4 and QMT-target-point=infinity point P2P3.



CU-IP-3P1a-Quasi-Miquel Triangle-25-Trf-Conics

Construction of the product conics
Since we know the center of these 3 product conics, as well as their axes and in all cases 4 intersection points on it, the product conics can easily be constructed.

Definition Quasi-Möbius Transformation
Since we know the center of these 3 product conics, as well as their axes we also can define the Quasi-Möbius Transformations as being the (anti-)inverse in a given product conic completed by the reflection about the given axis.

Quasi-Möbius Transformations (QMT's) wrt product-conics
Denote the Quasi-Möbius-Transformation wrt product-conic Co12-34 by QMT12-34.
QMT12-34(P) = INV2(AINV1(P)) = AINV1(INV2(P)) = INV3(INV4(P)) = AINV4(INV3(P)), where

- AINV1(P) is the Anti-Inverse of P wrt Conic1 (with center P1),
- INV2(P) is the Inverse of P wrt Conic2 (with center P2),
- INV3(P) is the Inverse of P wrt Conic3 (with center P3),
- INV4(P) is the Inverse of P wrt Conic4 (with center P4).

Similar rules are valid for QMT13-24(P) wrt Co13-24 and QMT14-23(P) wrt Co14-23.

Properties of the QMT's
So finally we have three QMT's:

- QMT12-34 swaps (P1,P2), (P3,P4), (M13-24,M14-23)
  wrt conic Co12-34 with center M12-34,
  with QMT-axis P1P2 and QMT-target-point=infinity point P3P4.
- QMT13-24 swaps (P1,P3), (P2,P4), (M12-34,M14-23)
  wrt conic Co13-24 with center M13-24,
  with QMT-axis P1P3 and QMT-target-point=infinity point P3P4.
- QMT14-23 swaps (P1,P4), (P2,P3), (M12-34,M13-24)
  wrt conic Co14-23 with center M14-23,
  with QMT-axis P1P4 and QMT-target-point=infinity point P2P3.

Apart from the mentioned swapping-pairs also all points on the cubic can be swapped with other points on the cubic, as well as points not on the conic can be swapped with other points not on the cubic.

See these QPG-messages #1968, #1971-#1974, #1983-#2020, #2026, #2028-#2031 by Bernard Keizer, Eckart Schmidt and Chris van Tienhoven where you can find the discussion that caused above theory.

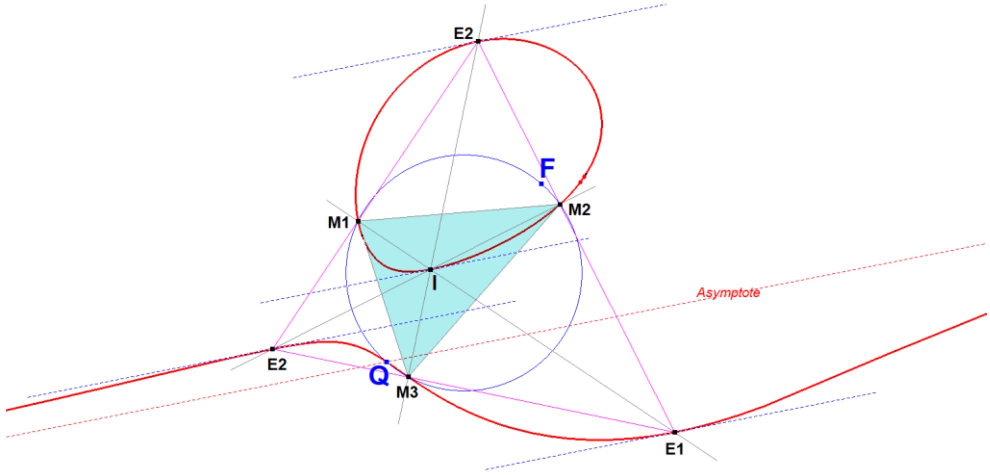
Properties

- QMT's on a general cubic map lines and conics into conics
- QMT's on a circular cubic map lines into circles and map circles into lines.
- QPG#1795: X and X' have the same tangential.
- QPG#1791: for a bipartite circular cubic there are three of this transformations giving for a cubic point P three images P1, P2, P3. The points P, P1, P2, P3 have the same tangential.

QMT's on Circular Cubics

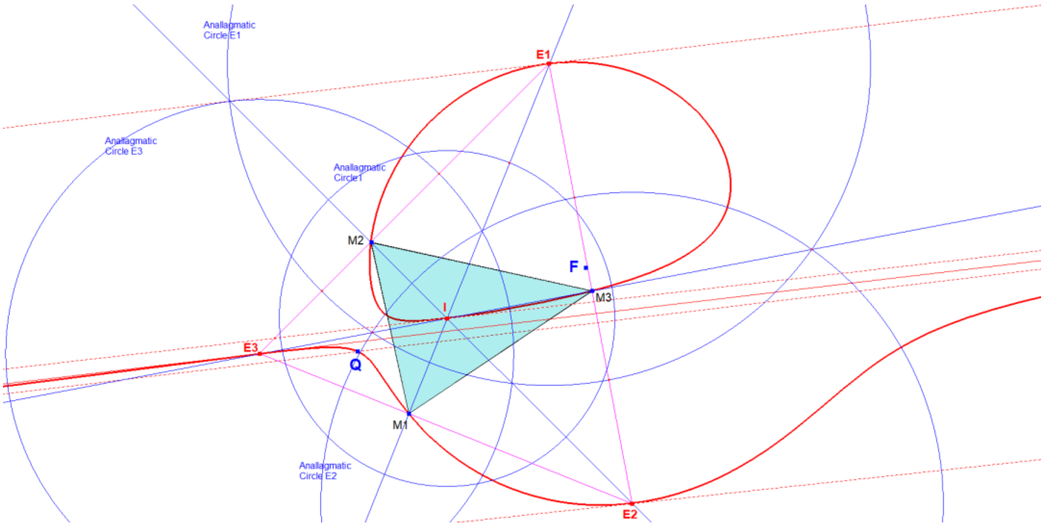
In QPG#1862 there is a resumé about a QMT on the circular cubic with two branches.
It is full of special properties.
Here are the results.

1. Given a Bipartite Circular Cubic CUc2.
2. CUc2 has a Focus F and an asymptote intersecting CUc2 in Q.
3. CUc2 has a unique Miquel Triangle M1M2M3 lying on CUc2 and the circle with diameter FQ.



CUc2-4P In- and Excenters Miquel Triangle-03.fig

4. Every vertex M1, M2 and M3 is the center of a Moebius Transformation Tf1, Tf2, Tf3, described by Bernard in QPG#1795.
5. Let m1, m2, m3 be the lengths of the sides of triangle M1M2M3. Let r1, r2, r3 be the radius of Moebius-config-circles around M1, M2, M3.
6. Tf1 swaps (M2,M3). Tf2 swaps (M3,M1). Tf3 swaps (M1,M2).
7. Therefore r1*r1 = m2*m3, r2*r2 = m3*m1, r3*r3 = m1*m2.
8. Also m1*r1*r1 = m2*r2*r2 = m3*r3*r3 = m1*m2*m3.
9. Because of 6. the axes of the transformations are the internal angle bisectors of the Miquel Triangle. Consequently the perpendiculars of the axes in M1, M2, M3 are the external angle bisectors.
10. The internal and external bisectors of the Miquel Triangle intersect in the incenter I and excenters E1, E2, E3 of the Miquel Triangle. These are all points of CUc2.
11. The tangents at I, E1, E2, E3 are lines parallel to the CUc2-asymptote.
12. All points on CUc2 lie within two strokes bounded by 2 times 2 of these tangents. The oval part in one stroke, the unlimited part in the other stroke.
13. I, E1, E2, E3 lie all on the IP-Polar Conic of CUc2 (IP=Infinity Point of CUc2), which is the Orthogonal Hyperbola, described by Eckart in QPG#837/#1538. This allows for an easy construction of this Orthogonal Hyperbola.

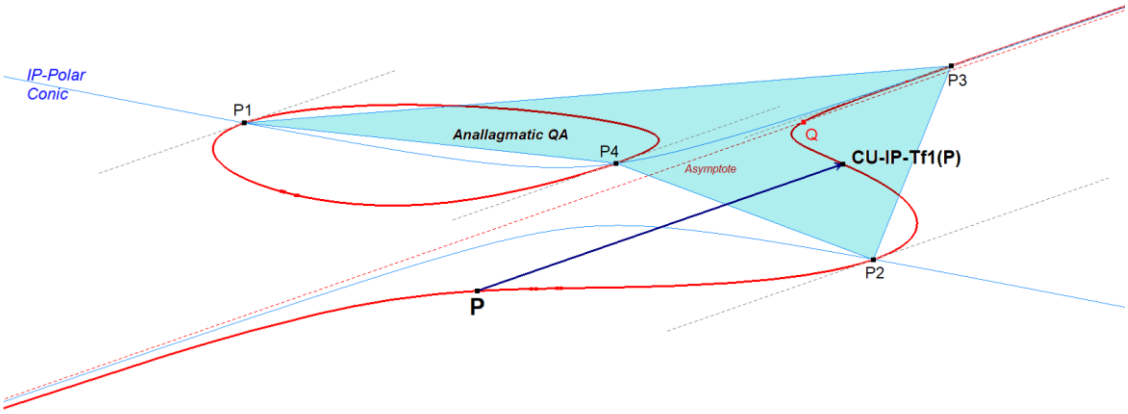


CUc2-4P-In-Ex-Centers-Anallagmatic-Points-03-Anallagmatic Circles

CU-IP-Tf1 Set of 3 IP-Involutary Conjugates

On a general cubic CU we have 3 asymptotes, of which at least one will be real.
We can draw lines parallel to these asymptotes denoted as Asy-parallel lines.
In the projective plane an Asy-parallel line will always pass through the infinity point (IP) of the related asymptote, which also is an infinity point of the general cubic. Therefore any asy-parallel line will have no more than 2 real intersection points with CU. When it passes through a given real CU-point, there will be just one other real intersection point of the Asy-parallel line with CU. This point is called here the P-Asy-parallel Point.

On a general cubic we have 4 points where Asy-parallel lines touch the cubic, the so-called Anallagmatic Points (CU-IP-4P1). They are the intersection points of CU and the IP-Polar Conic (CU-IP-Co1) related to the infinity point IP of the related asymptote. These 4 points form a Quadrangle QA. For each such QA exists an Involutary Conjugate (see QA-Tf2 in EQF) mapping any point on CU to another point on CU.
The special thing about this Involutary Conjugate is that it maps a point P on CU to its P-Asy-parallel Point. See also QPG#2055.



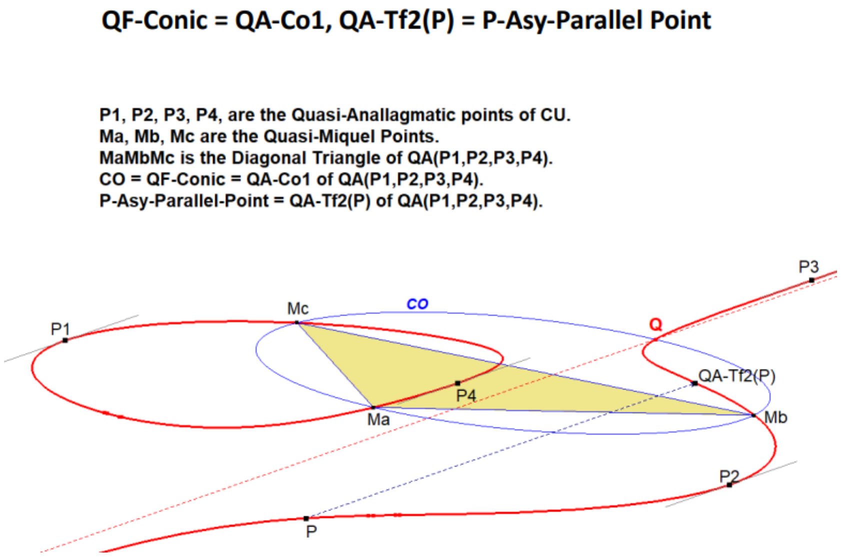
CU2-Bipartite Cubic-30-AnallagmPoints-plu-Invol-Conj.fig

CU-IP-P-P1 P-Asy-Parallel Crosspoint

CU-IP-P-P1 is the P-Asy-Parallel Crosspoint or stated in another way the 2nd intersection point of CU and the line through P parallel to the IP-asymptote.

QPG#1869, also QA-Tf2(P) wrt the 3 QA's of CU-IP-4P1

QPG#2055

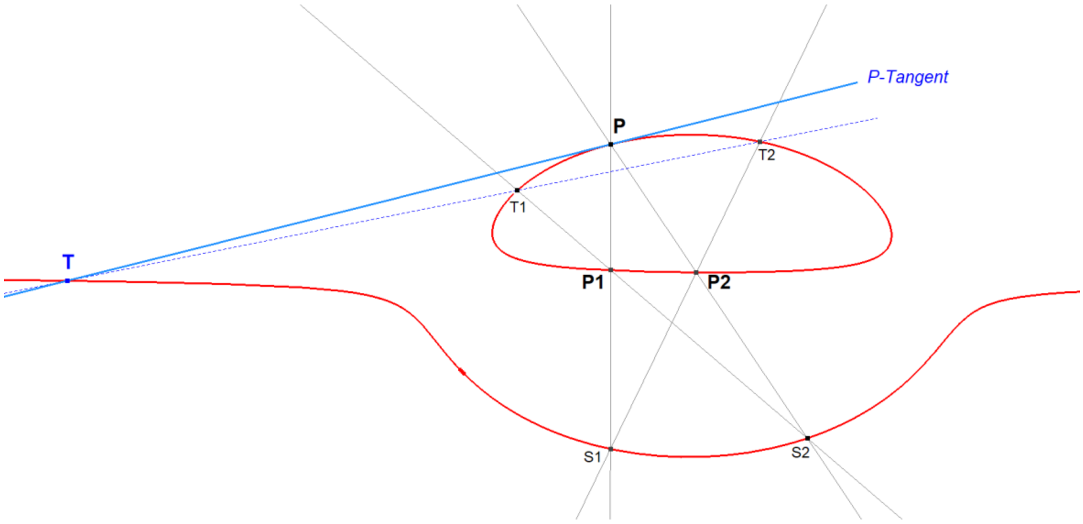


CU-P-P1 Tangential of P

The tangential of a point P on CU is the intersection of the tangent at P with CU. Since any line always intersects a cubic at three points (real or imaginary), and a point of tangency counts twice, each point on CU will have exactly one tangential.

Construction

There is a simple construction:
1. Start with P and 2 random points P1,P2 on a cubic CU.
2. Draw line PP1 intersecting CU in S1.
3. Draw line PP2 intersecting CU in S2.
4. Draw line S1P2 intersecting CU in T2.
5. Draw line S2P1 intersecting CU in T1.
Let T be the 3rd intersection point of T1T2 with CU.
T is the Tangential of P (CU-P-P1) and PT will be the Tangent at P to CU.
See also QPG#2111.



CU-P-P1 P-Tangential-00.fig

CU-P-L1 Tangent at P

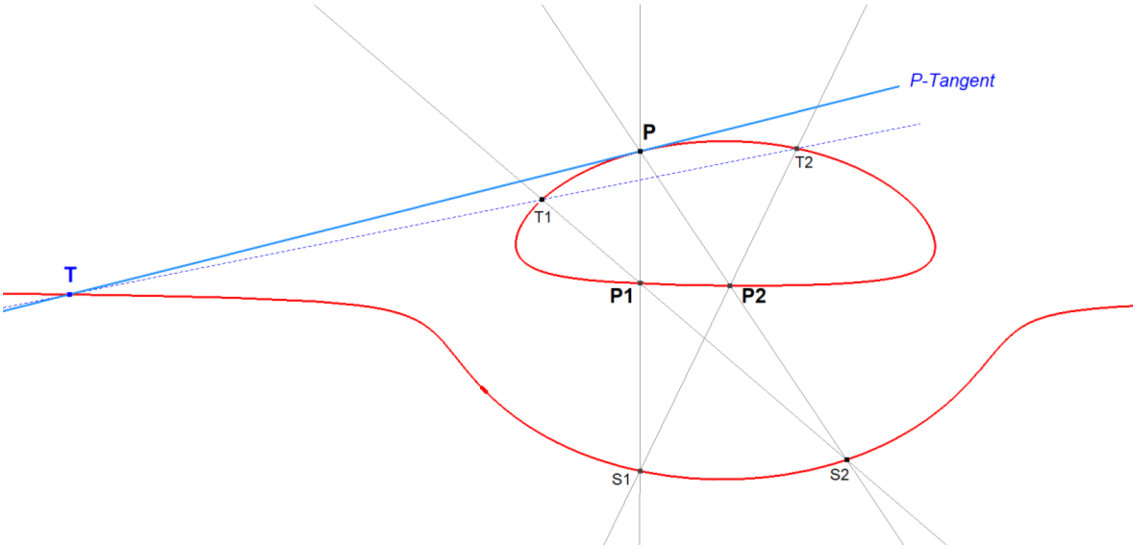
The Tangent at a point P on CU or the P-Tangent is the tangent at P to CU.

Construction

There is a simple construction:

1. Start with P and 2 random points P1,P2 on a cubic CU.
2. Draw line PP1 intersecting CU in S1.
3. Draw line PP2 intersecting CU in S2.
4. Draw line S1P2 intersecting CU in T2.
5. Draw line S2P1 intersecting CU in T1.

Let T be the 3rd intersection point of T1T2 with CU.
T is the Tangential of P (CU-P-P1).
PT will be the Tangent at P to CU.
See also QPG#2111.



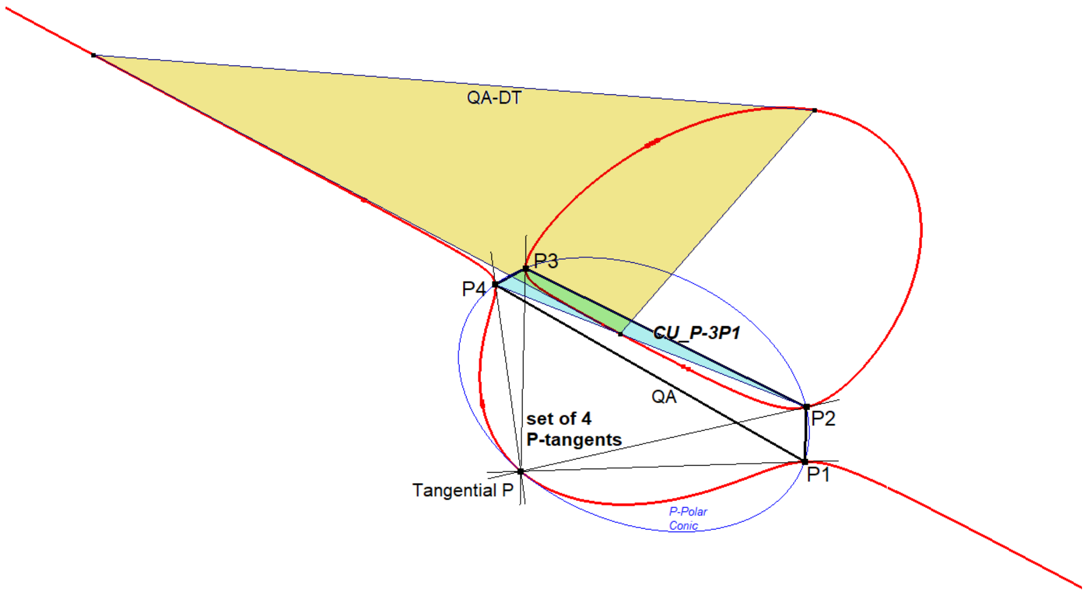
CU-P-L1 P-Tangent-00.fig

CU-P-3P1 The co-tangentials of P

On a general cubic CU we have 4 points of tangency (CU-P-4P1) where tangents from a point P touch the cubic. They are the intersections of CU with the P-Polar Conic (CU-P-Co1).
On the other hand, when one of these points of tangency is known, then the remaining three *co-tangentials* can be constructed as follows:

Construction 1

1. Let P1 be a known point on CU and let P2,P3,P4 be the other 3 CU-points that share the same tangential point P.
2. Since only P1 is known on the Cubic, we first construct its tangential point P (see CU-P-P1).
3. Next, construct the P-Polar Conic (CU-P-Co1), which touches the cubic at each of the four points of tangency with respect to P. If the construction is done correctly, P1 will be among the four points intersection with CU.
4. The remaining 3 intersection points are the members of CU-P-3P1.
5. To avoid ambiguity caused by possible reordering of intersection points in dynamic geometry software, one may prefer to construct CU-P-3P as the AntiCevian Triangle of P1 wrt the Diagonal Triangle QA-DT of the Quadrangle (P1, P2, P3, P4).



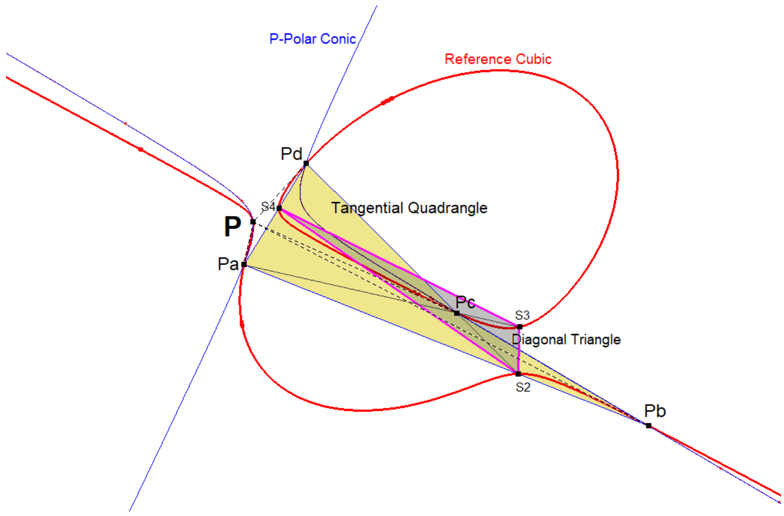
CU-P-3P1 CU-points with same tangential - 01.fig

Construction 2

The next construction is based on the following property (see QPG#2722):

- Let P be an arbitrary point on CU, and let (P1,P2,P3,P4) be the four points of tangency of the tangents from P to CU (also called *tangentials*).
- Let (P1a,P1b,P1c,P1d) be the four tangentials of P1.
- Now it turns out that the set (P2,P3,P4) coincides with the vertices (Q1a,Q1b,Q1c) of the Diagonal Triangle of the Quadrangle QA(P1a,P1b,P1c,P1d).

Therefore, the set CU-P-3P1 consists of the vertices of Diagonal Triangle of the Quadrangle formed by the four tangentials of P.



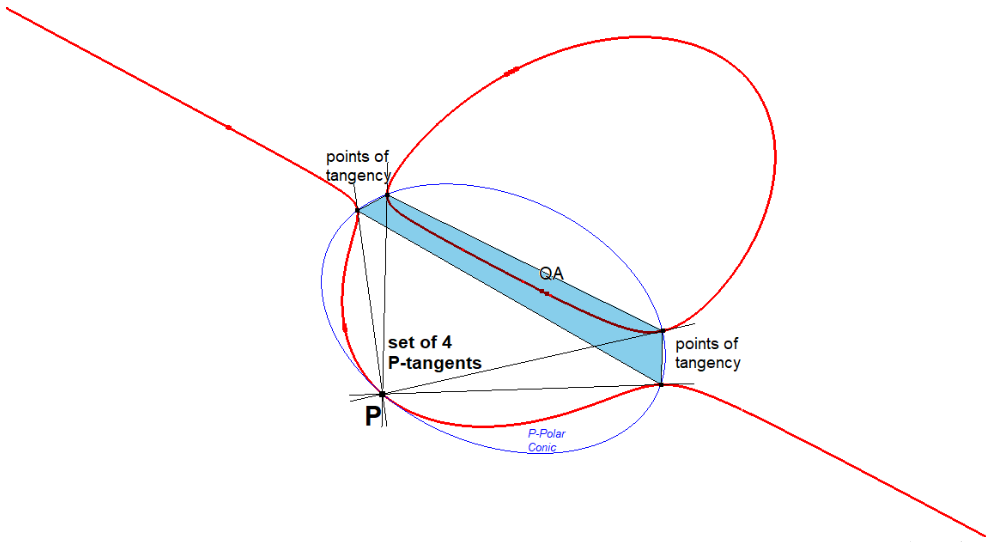
CU-P-3P1 P-points with same tangential-01.fig

Note

Note that not for every cubic there will be 4 real intersection points of the P-Polar Conic CU-P-Co1. For example when CU is a monopartite cubic, then there will be 2 real intersection points and 2 imaginary intersection points. As a consequence, when P1 is a real point on a monopartite cubic CU, then CU-P-3P1 will consist of 1 real point and 2 imaginary points.

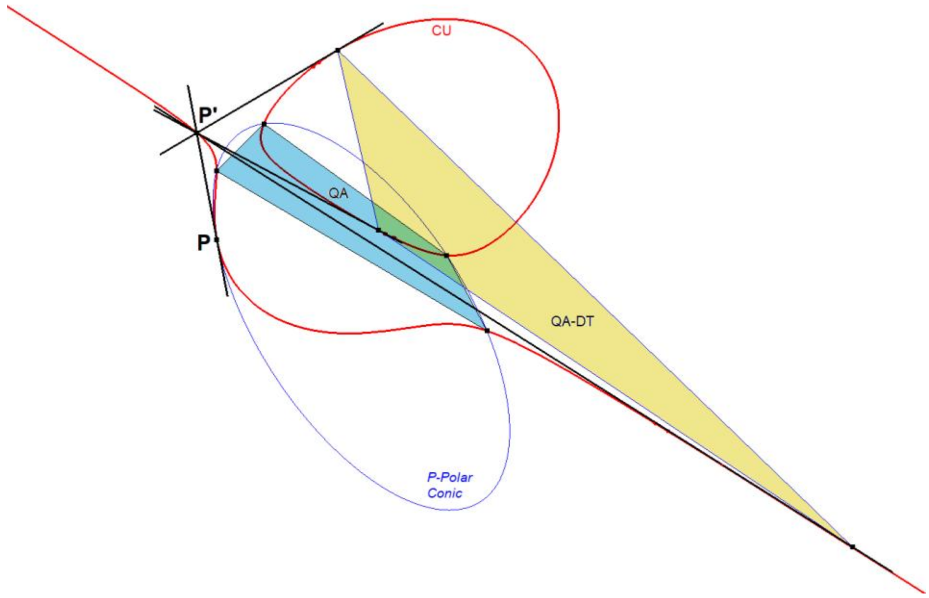
CU-P-4P1 Set of 4 tangency points from P to CU

On a general cubic CU we have 4 points of tangency of tangents from some point P to the cubic. They are the intersection points of CU and the P-Polar Conic (CU-P-Co1).



CU-P-4P1 P-points of tangency-01.fig

There is an extra property for the Diagonal Triangle of the Quadrangle QA formed by the touchpoints of the tangents from some point P on CU. The vertices of its Diagonal Triangle also lie on CU and share the same tangential P' with P, which is QA-Tf2(P).



CU-P-4P1 P-points of tangency-02-plus DT.fig

Validation

2X + P = N has 4 solutions

Properties

- When P = IP (Infinity Point of CU), then CU-P-4P1 = CU-IP-4P1.
- The 4 points CU-P-4P1 form a Quadrangle QA. The Diagonal Triangle QA-DT of QA also has its vertices on CU.
- For each such QA exists an Involutary Conjugate (CU-P-Tf1) mapping any point on CU to another point on CU.

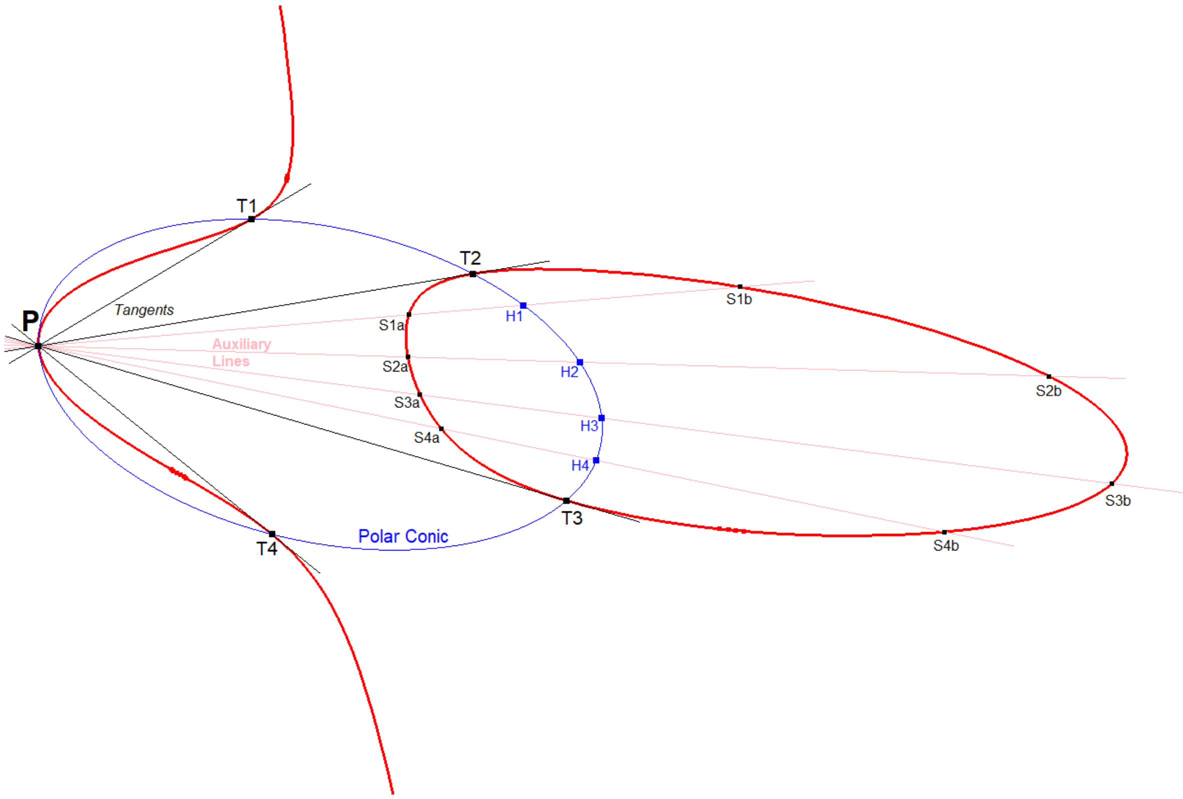
CU-P-Co1 P-Polar Conic of a Cubic

The Polar Conic of some point P wrt to a Cubic is the equivalent of the Polar Line of some point P wrt to a Conic.
In algebraic geometry, the first polar, or simply polar of an algebraic plane curve C of degree n with respect to a point P is an algebraic curve of degree n-1 which contains every point of C whose tangent line passes through P.
When the plane curve is a cubic it is the 1st derivative of the function of the Reference Cubic CU being a function of 2nd degree or a conic. This conic will contain every point of CU whose tangent line passes through P.

Construction

1. Given Reference Cubic CU and a random point P on CU.
2. Draw 4 lines through P intersecting CU in a 2nd and 3rd point resp. (S1a, S1b), (S2a, S2b), (S3a, S3b), (S4a, S4b).
3. Then construct the points Hi=Harmonic Conjugate of P wrt (Sia, Sib) for i=1,2,3,4.
4. The conic (P, H1, H2, H3, H4) will be the P-Polar Conic of P wrt CU.

This specific construction method for the Polar Conic wrt a cubic can be found at [83], 2.15 page 45.



CU-P-Co1 CU P-Polar Conic-01.fig

As can be seen in the picture

- the P-Polar Conic is tangent to CU at P
- the P-Polar Conic intersects CU in the points of tangency of the tangents from P to CU.

The total number of intersection points of cubic and conic should be 6 according to Bezout's Theorem. This is exactly what happens. There are these 6 points: P (counting twice because of tangency), T1, T2, T3, T4.

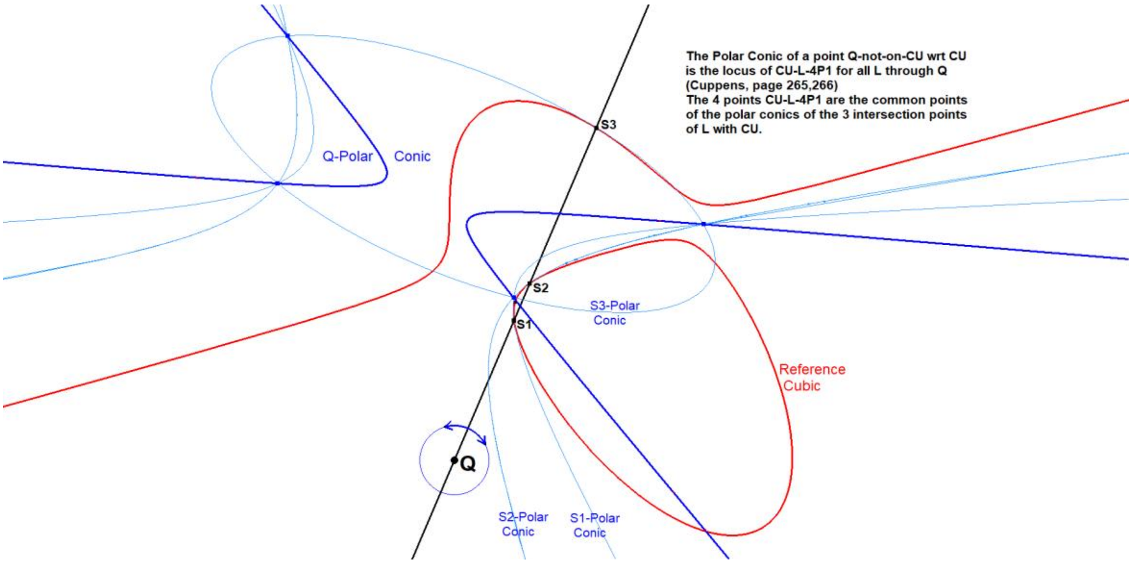
Applications

- A P-Polar Conic yields 4 P-Points of tangency P1,P2,P3,P4 forming a quadrangle QA(P1,P2,P3,P4). The vertices of the Diagonal Triangle QA-Tr1 of this QA also lie on the cubic. The involutary Conjugate (QA-Tf2) of this QA maps a point P on CU to another point on CU. See CU-P-Tf1 and CU-IP-Tf1.
- The IPi-Polar Conic (IPi is the Infinity Point of the i^{th} Asymptote (i=1,2,3) of the cubic) CU-IP-Co1 is called “Conique Diamétrale” by Roger Cuppens. See [63], page 262. It cuts CU in IPi and the 4 Anallagmatic Points CU-IP-4P1.

CU-Q-Co1 Q-Polar Conic of a Cubic

The Polar Conic of some point Q (not on CU) wrt to CU is a conic through the 4 contact points of the tangents from Q to the reference cubic CU.
Two or four of these contact points can be imaginary. In that case these points won't be visible in the picture, but there will be a conic anyhow and it will only pass through the visible contact points.

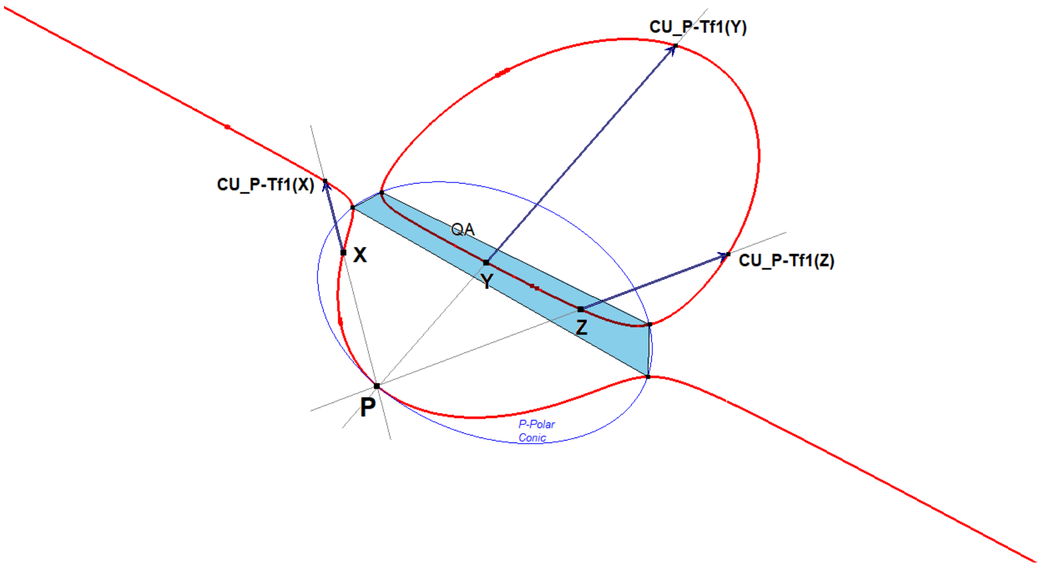
Construction
There is a construction method for the P-Polar Conic when P is on CU. See CU-P-Co1.
This method can be used to construct the Q-Polar Conic with Q not on CU.



It uses the property that the Polar Conic of a point Q-not-on-CU wrt CU = the locus of CU-L-4P1 for all L through Q (see Cuppens [63], page 265,266). The 4 points CU-L-4P1 are the common points of the polar conics of the 3 intersection points of L with CU.

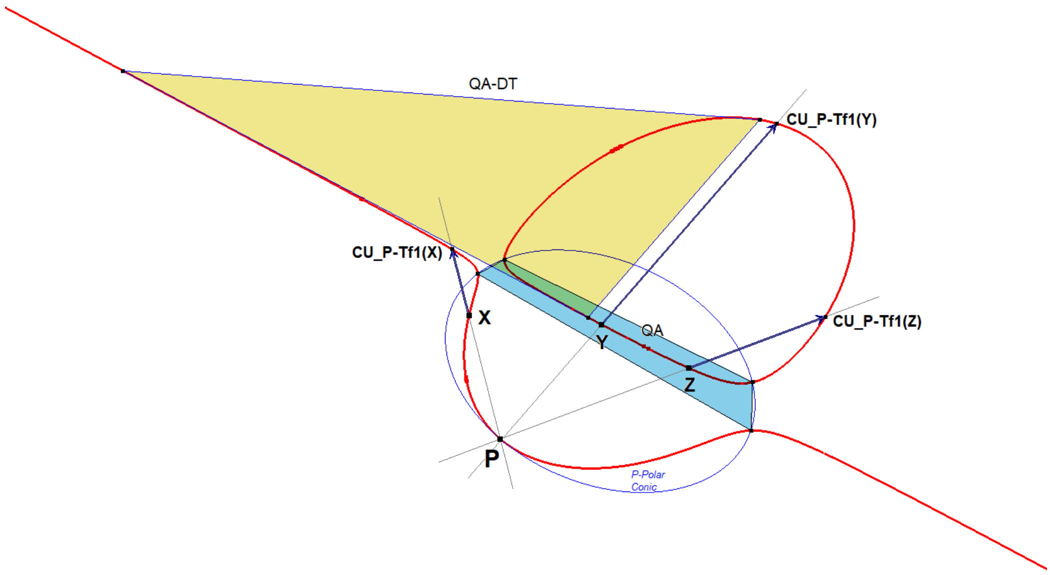
CU-P-Tf1 P-Involutary Conjugate

On a general cubic we have 4 points of tangency (CU-P-4P1) of tangents from some point P to the cubic. They are the intersection points of CU and the P-Polar Conic (CU-P-Co1).
These 4 points form a Quadrangle QA. For each such QA exists an Involutary Conjugate (see QA-Tf2 in EQF) mapping any point on CU to another point on CU.
It is a P-pivotal transformation because all points to be mapped and mapped points are lined up with (pivot-)point P.



CU-P-Tf1 P-pivotal QA-Tf2-00.fig

An isocubic is defined as a circumcubic wrt some reference triangle which is invariant under an isoconjugation. The Diagonal Triangle QA-DT of the defined QA above has its points on CU and therefore CU is a circumcubic of QA-DT and can function as reference triangle. The vertices of QA are the 4 fixed points of QA-Tf2, meaning that these points are transformed to itself by QA-Tf2.



CU-P-Tf1 P-pivotal QA-Tf2-01.fig

It is said that QA-Tf2 is an isoconjugation and therefore any general cubic with some point P having 4 real points of tangency (CU-P-4P1) on the cubic will be an isocubic.

Two constructions

Eckart Schmidt makes note of two other triangular constructions wrt the QA-DT at QPG#2035 and QPG#2068 that bring about the same transformation as QA-Tf2.

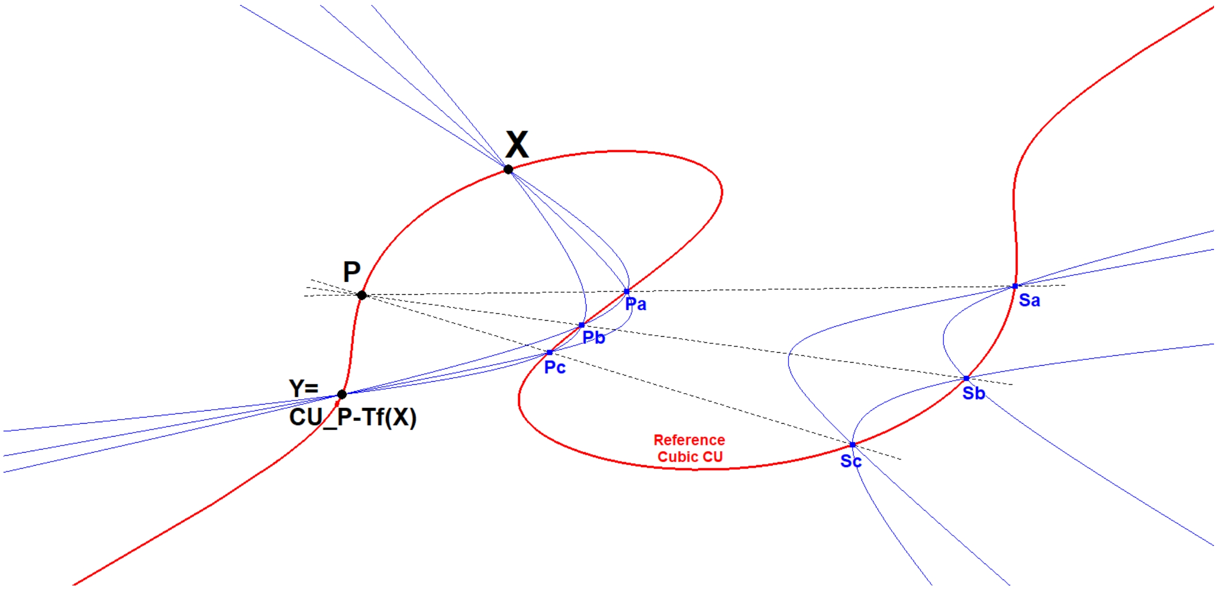
Here is a summary of these two constructions:

QPG#2035, December 14, 2023
There is this an isoconjugation (reference is a private correspondence of Roland Stärk, 22.11.2002):
An isoconjugation X --> X* wrt a triangle ABC
... can be defined by a circumscribed conic CO,
... starting with a point X, take the intersection A' = XA ^ CO
... and consider the intersection Xa of CO
... and a parallel to BC through A',
... analog we get Xb and Xc with the image X* = AXa ^ BXb ^CXc.
Examples of isoconjugations for a triangle:
... isogonal for CO = circumcircle,
... isotomic for CO = Steiner circumellipse,
... QA-Tf2 for a QA: CO = QA-Co1,
... in general for an isoconjugation: CO = image of the line at infinity.

QPG#2068, January 7, 2024
An isoconjugation X-->X* can be defined
... for a triangle ABC and a fixed point K.
Let Ka = AK^BC, Xa = AX^BC,
... Ka' = 4th harmonic of Ka wrt BC,
... Xa' = 4th harmonic of Xa wrt KaKa',
... X* = AXa'^BXb'^CXc'.
I got this simple construction
... from Günther Pickert 20 years ago.

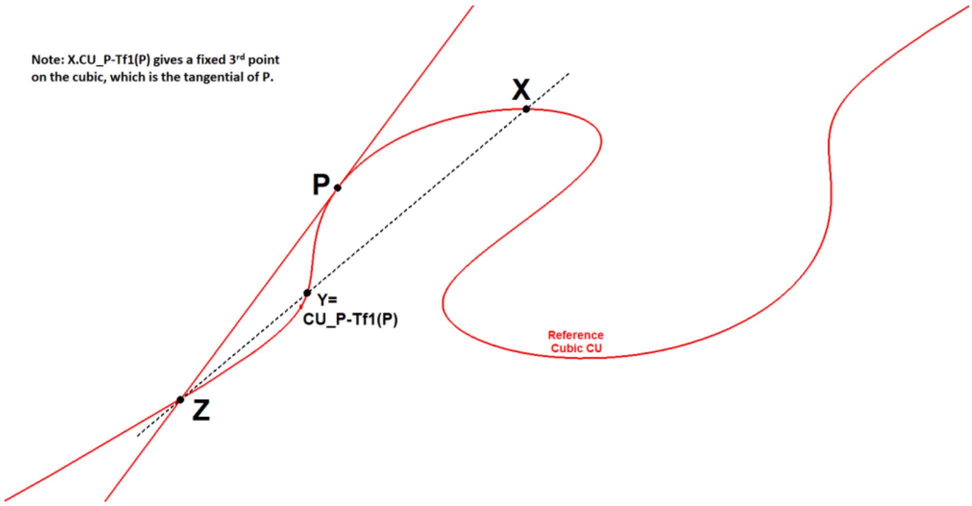
CU-P-Tf2 1st CU-P Transformation

Given a Reference Cubic CU and a random reference point P on the cubic.
Let X, Pa, Pb, Pc be four random points on CU.
Let Sa, Sb, Sc be the 3rd intersection points of CU and resp. PPa, PPb, PPc.
Construct conics (X, Pa, Sa, Pb, Sb), (X, Pa, Sa, Pc, Sc), (X, Pb, Sb, Pc, Sc). These conics will be shown to concur in a common cubic point Y=CU-P-Tf2(X), which is independent of Pa, Pb, Pc.



Validation
Lines: P+Pa+Sa=N, P+Pb+Sb=N, P+Pc+Sc=N
Conic: X+Pa+Sa+Pb+Sb+Y=2N → X+Pa+(N-P-Pa)+Pb+(N-P-Pb)+Y=X+2N-2P+Y=2N → Y=2P-X
This validation shows that transformation point Y is a point on CU independent of Pa,Pb,Pc,Sa,Sb,Sc.

Properties
Let Z be 3rd intersection point of the CU-cubic with line XY: Z+X+Y=N.
Subsititute Y=2P-X → Z+X+2P-X=N → Z+2P=N, which shows that the 3 points Z,P,P are collinear and therefore Z is the tangential of P.



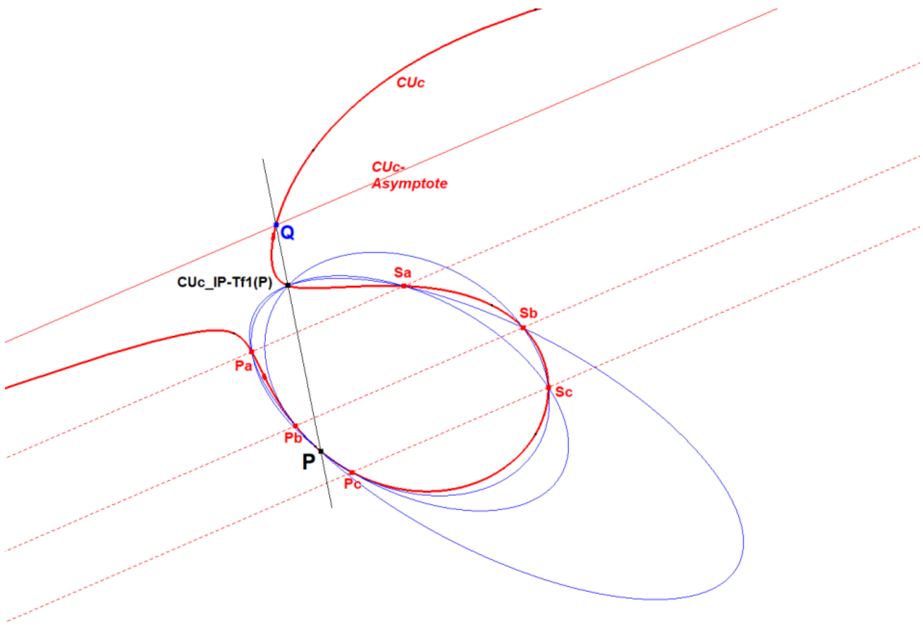
Construction of P-tangent and P-tangential
This gives a construction method for the tangent at some point P1 to the cubic as well as the tangential of that point P0 on the cubic.

1. Let P be some point on the cubic for which we want to construct the tangent.
2. Let X, Pa, Pb, Pc be four random points on the cubic.
3. Let Sa,Sb,Sc be the 3rd intersection points of PPa,PPb,PPc with the cubic.
4. Let Y be the 4th intersection point of the conics (X,Pa,Sa,Pb,Sb) and (X,Pa,Sa,Pc,Sc).
5. Let Z be the 3rd intersection point of XY with the cubic.

Z will be the tangential of P0 wrt the cubic and PZ will be the tangent at P to the cubic.

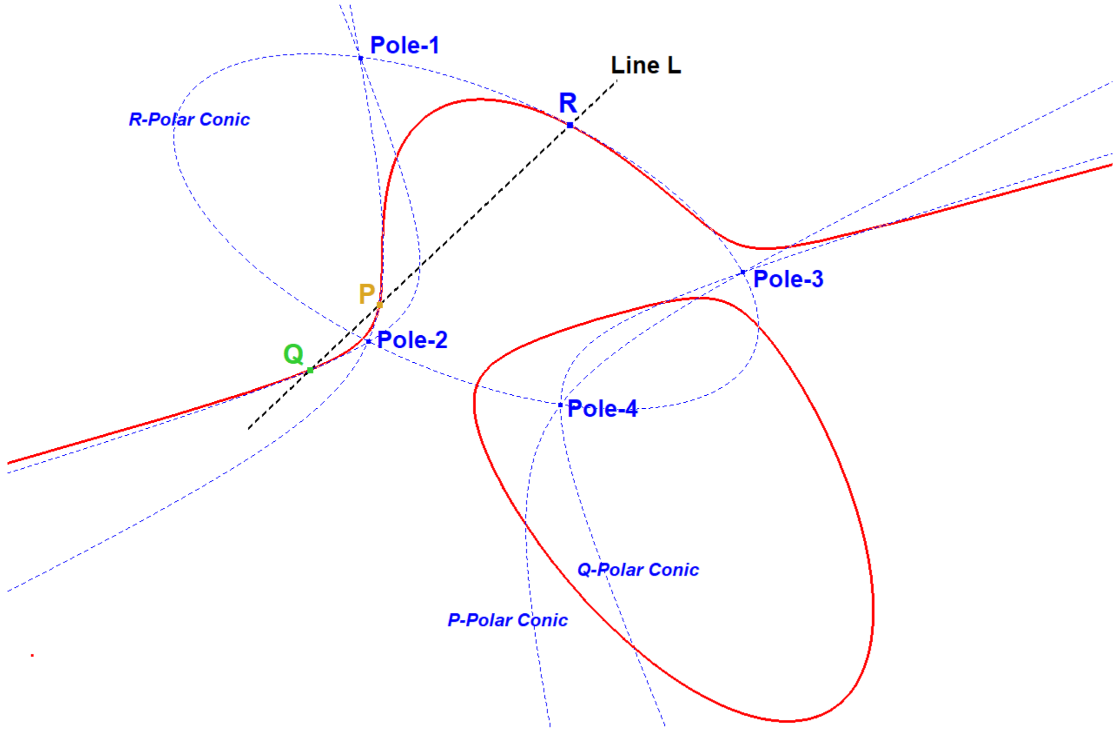
Another Property

When the cubic is cicular and the fixed point is IP (infinity Point), then CU-P-Tf1(P)=Q=intersection point of asymptote with CUc. This offers a method to contruct Q, the intersection point of the CUc-asymptote with CUc.



CU-L-4P1 Poles of a line

Given a line L intersecting reference cubic CU in points P,Q,R,
The Polar Conics (CU-P-Co1) of P, Q, R will have 4 common points called the 4 poles of line L. See [Cuppens, page 265].



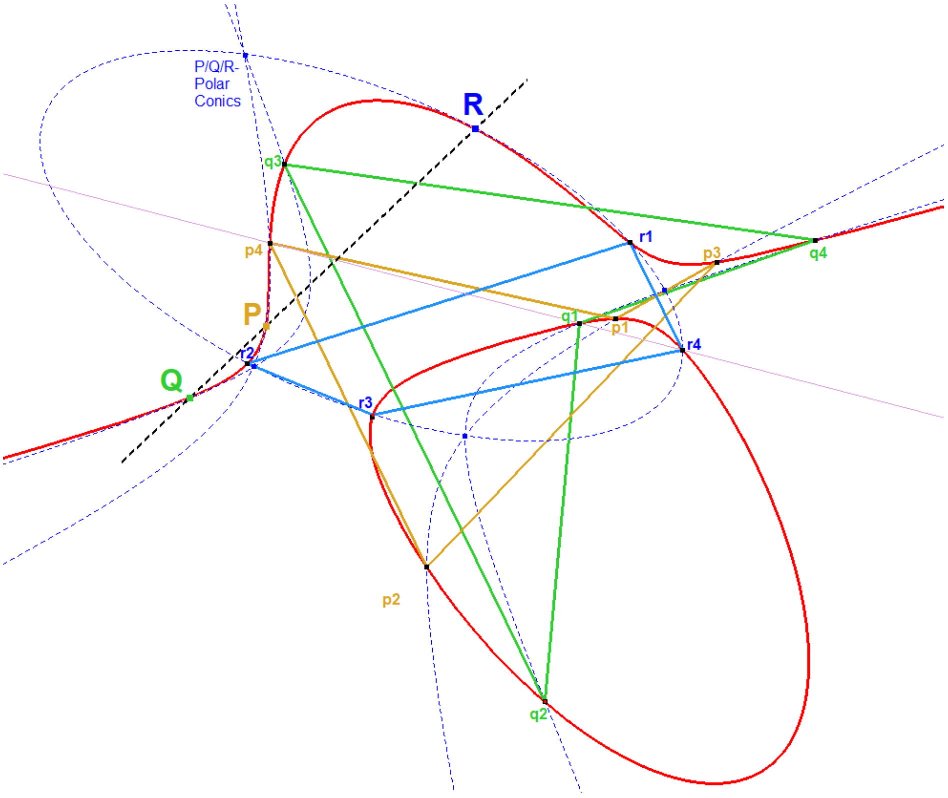
CU-L-4P1 four poles of a line-01.fig

Properties

- Not only the Polar Conics of P,Q,R share these 4 poles, but also the Polar Conics of all points on line L share these 4 poles.

CU-L-12P1 PQR-Polar Conics Desmic System

Given a line L intersecting reference cubic CU in points P,Q,R.
Draw the Polar Conics (CU-P-Co1) of P, Q and R wrt CU.
The vertices of the 3 PC-intercepted QA's (PC=Polar Conic, QA=Quadrangle) form a Desmic Configuration. See QA-Tr-1.
Every QA is perspective with one of the other QA's in 4 ways, forming the 3rd QA from its perspectors, which makes it a Desmic System.
See QPG#2266.



CU-L-12P1 PQR-PolarConics-DesmicSystem-02.fig

Desmic System made visible
Let p1, p2, p3, p4 be the intersection points of the P-Polar Conic with CU, apart from P.
Let q1, q2, q3, q4 be the intersection points of the Q-Polar Conic with CU, apart from Q.
Let r1, r2, r3, r4 be the intersection points of the R-Polar Conic with CU, apart from R.
In the drawing next collinearities can be checked:

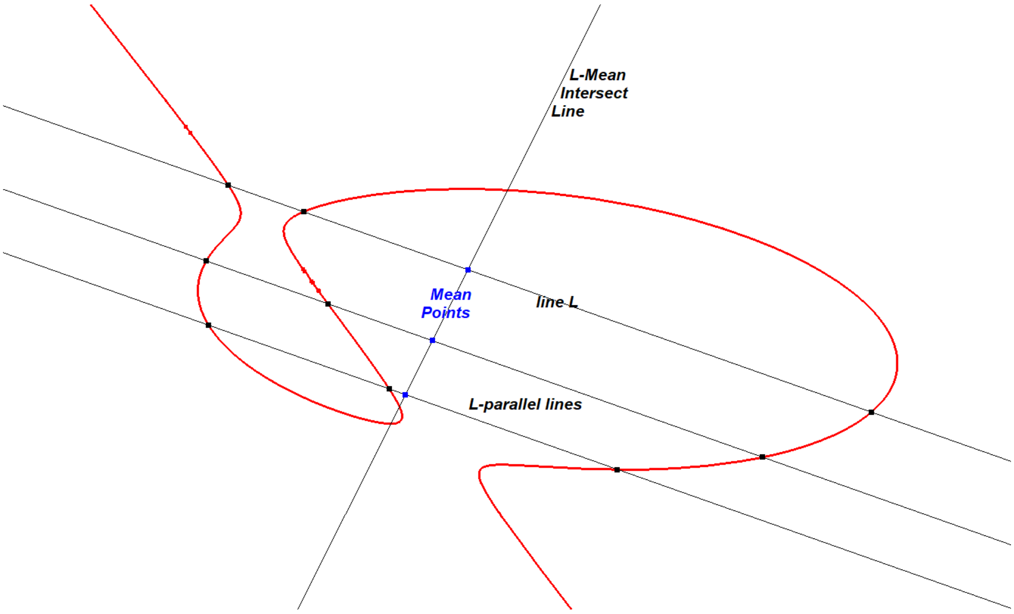
p1-q4-r3, p1-q3-r4, p1-q2-r1, p1-q1-r2
p2-q1-r1, p2-q2-r2, p2-q3-r3, p2-q4-r4
p3-q4-r2, p3-q2-r4, p3-q3-r1, p3-q1-r3
p4-q1-r4, p4-q2-r3, p4-q3-r2, p4-q4-r1.

Which makes it a Desmic System.

Proof of Desmic System using Point-calculation.
Because P and the points of tangency p1, p2, p3, p4 are 'collinear' (points of tangency counted twice), we know P+2pi=N (i=1,2,3,4) and also Q+2qi=N and R+2ri=N, therefore P+Q+R+2pi+2qi+2ri=3N. (1)
We know P+Q+R=N because P, Q and R are collinear. (2)
Combining (1) and (2) this yields 2pi+2qi+2ri=2N.
This shows that for each version of pi and qi there will be a collinear version of ri (i=1,2,3,4).

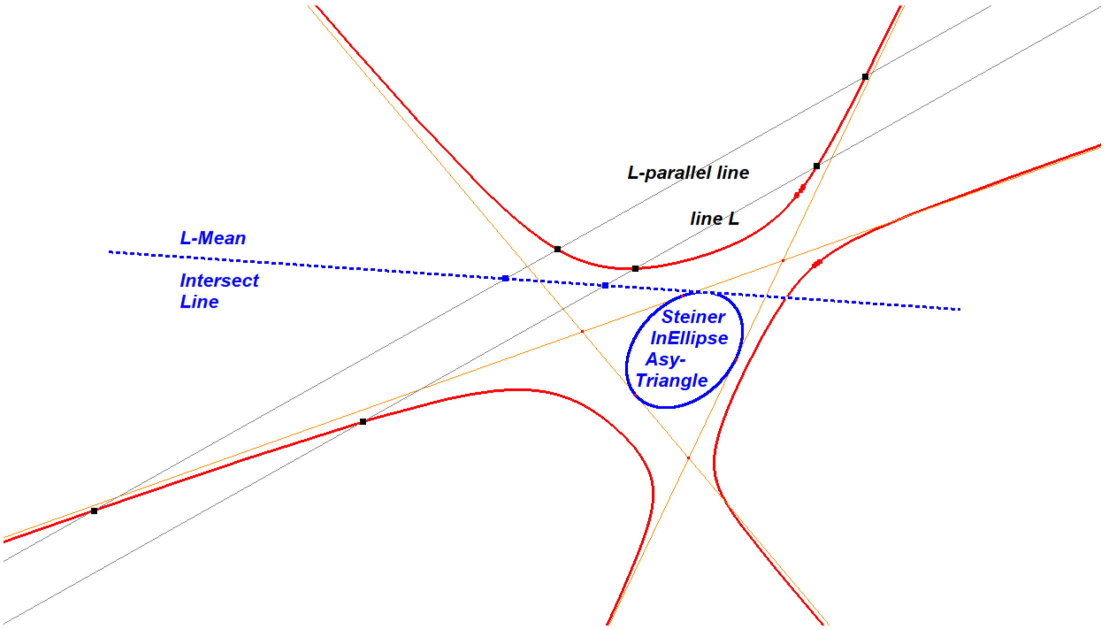
CU-L-L2 L-Mean Intersect Line

Given reference cubic CU and a random line L.
The mean point of the intersection points of CU and all L-parallel lines are collinear. See E. de Jonquiere - Mélanges de géométrie, page 197. According to him this theorem originates from Newton. It is not only valid for conics and cubics, but also for all curves of n^{th} degree.
It is also mentioned by Cuppens at [Cuppens, page 255]. Here it is called the diameter associated to a direction d. Cuppens also mentions at page 256 that the mean point of the intersection points of L with CU coincides with the mean point of the intersection points of L with the three CU-asymptotes.
The locus of the mean point of intersection points of CU and L-parallel lines is called here the L-Mean Intersect Line CU-L-L2.



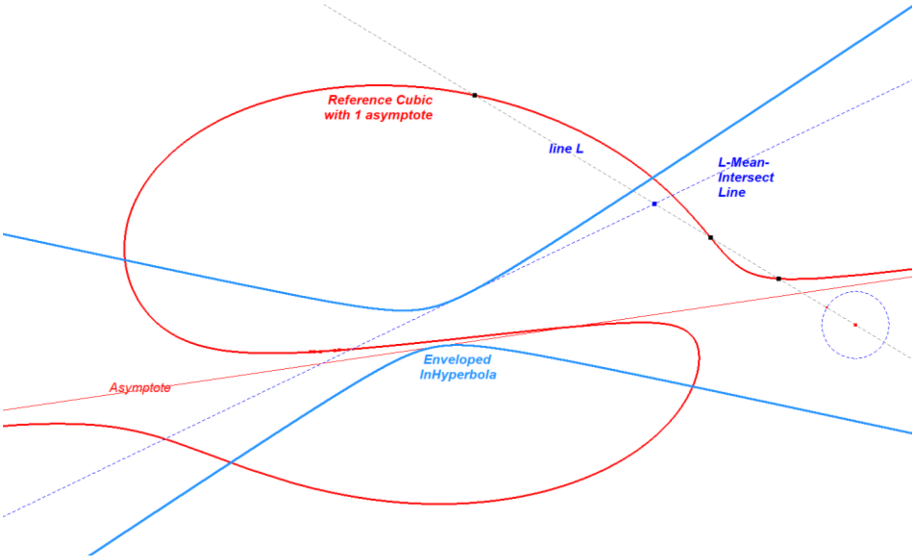
CU-L-L2 Mean Line of Intersection-02.fig

A special property of all L-Mean Intersect Lines is that they are tangent to the Steiner InEllipse of the CU-Asy-Triangle CU-Tr1. In other words, they envelop the Steiner InEllipse of the CU-Asy-Triangle. This is only visible when CU has three real asymptotes.



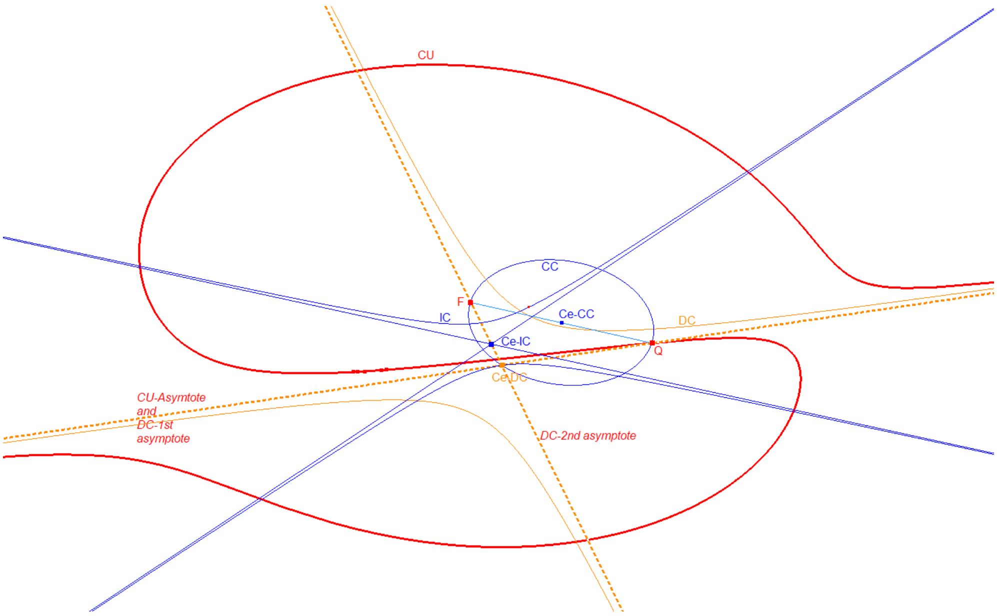
CU-L-L2 L-Mean Intersect Line-10-3-Asymptotes.fig

Now that we know this property, when CU has only one real asymptote, the Steiner InEllipse can be constructed as the envelope of L-Mean Intersect Lines. In this case, instead of an InEllipse, it becomes an InHyperbola touching the one and only real asymptote. As a matter of algebraic consistency, the InHyperbola will also touch the imaginary asymptotes, but in the imaginary realm.



CU-L-L2 Mean Line of Intersection-30.fig

This inconic that can be determined now for all appearances of a cubic has several incidences with CU and 2 other conics, the Diametral Conic CU-IP-Co1 and the Central Conic/QF Conic CU-IP-Co2.



CU-L-L2 Mean Line of Intersection-31.fig

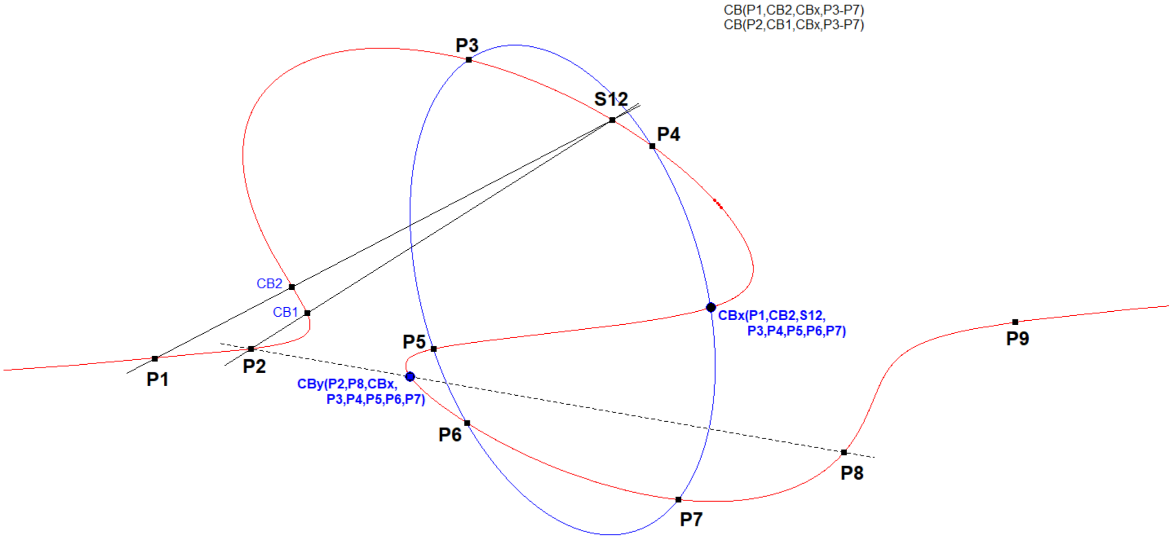
The center Ce-DC of the Diametral Conic DC lies on IC.
The 2 asymptotes of the Diametral Conic are Asy1 and another line passing through the center of IC.
This 2nd DC-asymptote contains:

- obviously the Center of IC
- the center of DC
- the Point F diametral to Q on the Central Conic CC

The Center of DC also lies on the Central Conic CC.
Note: F is also the intersection point of the imaginary 2nd and 3rd asymptotes of reference cubic CU.

CU-2P-P1 3rd intersection point of P1P2-line

Given two random points P1 and P2 on reference cubic CU.
According to Bezout's theorem any line (curve of 1st degree) has three intersection points with a cubic (curve of 3rd degree) in the projective complex plane. These intersection points of a line can be real or imaginary as well as finite or infinite.
Consequently, when we have 2 points P1 and P2, there will be a 3rd intersection point of the line through P1 and P2 with CU.
The existence of this point is obvious, but its construction (needed for exact positioning) is not that simple.



CU-2P-P1 3rd Intersection Point with 2P-Line-01.fig

Construction
We use for the construction these cubic properties:

1. When 2 cubics intersect in 8 points, then they will intersect in a fixed 9th point, that will be the 9th intersection point for all cubics through these 8 points. This is called the 8P-Cayley Bacharach Point CU-8P-P1. Denote CB=Cayley-Bacharach Point. See CU-8P-P1 for its construction.
2. Given 7 points on a reference cubic CU. Then for every point Pi there will be a Cayley-Bacharach Point CBi(P1,…,P7,Pi) with special property that 3rd intersection point P.CBi with CU will be fixed for every general location of Pi. This point is called CU-7P-P1, the 7P-Pivot Point. See CU-7P-P1 for its construction.
3. When 2 cubics intersect in 9 points and 3 are collinear then the other 6 will be coconic.
4. When 2 cubics intersect in 9 points and 6 are coconic then the other 3 will be collinear.

Now we can proceed to the actual construction:

1. Determine 9 points P1, …, P9 on the cubic.
2. Construct CB1=CB(P2,P3, …, P9) and CB2=CB(P1,P3, …, P9). Lines P1.CB2 and P2.CB1 will meet in 7P-Pivot Point S12 on CU.
3. Construct CBx=CB(P1,CB2,S12,P3,P4,P5,P6,P7). The first three points are collinear, therefore CBx will be the 6th intersection point of conic(P3,P4,P5,P6,P7).
4. When we want to know the 3rd intersection point of P2.P8 then this will be CBy = CB(P2,P8,P3,P4,P5,P6,P7), because the last 6 points are coconic. The points P2 and P8 can be replaced by random points Px, Py.

5. So we have a method for finding the 3rd intersection point of PxPy with CU given 9 help points on CU.

There is another construction described at [Cuppens, pages 243,244].

CU-2P-L1 Line through 2 given CU-points

This line CU-2P-L1 is obvious, but needs to be mentioned for reasons of completeness.
Two points on a cubic define a line through these points which is CU-2P-L1.

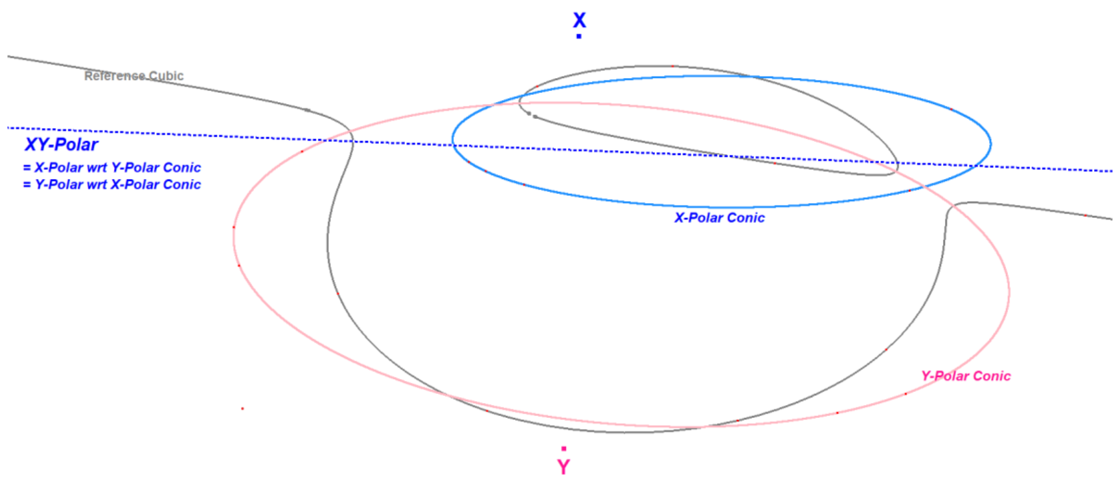
Because of Bezout's Theorem any line crossing a cubic has 3 intersection points. Therefore there is a 3^{rd} intersection point of this line with the reference cubic for which it is not obvious to construct this point. Nevertheless it is possible to construct this point by the ruler only knowing 7 other points of the reference cubic which is described at CU-2P-P1. See [63].

CU-2P-L2 Mixed XY-Polar

Given a general Cubic CU and two random points X and Y on the cubic, then:

- the X-Polar of the Y-Polar Conic = the Y-Polar of the X-Polar Conic.

This Polar is called the Mixed Polar or XY-Polar or Mixed XY-Polar.
A description of the P-Polar Conic can be found at CU-P-Co1. It is the conic through P (not necessary on CU) and the 4 points of tangency from the tangents at P to CU.
The P-Polar is just the polar of some point P wrt some conic.



Properties

1. It isn't maybe easy to construct the Polar Line of an infinity point IP wrt an X-Polar Conic. This now easily can be constructed as the X-Polar Line wrt the IP-Polar Conic. And the IP-Polar Conic we know, it is the Diametral Conic CU-IP-Co1.
2. Given 2 points (X,Y) on a line L1 and another point Z on a 2^{nd} line L2. The locus of the intersection point of the Mixed YZ-Polar and the Mixed XZ-Polar with variable Z on L2 is a conic called the Poloconica CU-2L-Co1. Every point Z on this conic has the property that given any 2 points (X,Y) on L1 or L2 and another random point Z on the not used line of (L1,L2) the XY-Polar, YZ-Polar and the XZ-Polar will concur in one point.
3. J. de Vries (On polar figures with respect to a plane cubic curve) (see [81]) describes these items:
   1. Given a general Cubic CU and two random points X and Y not necessary on the cubic, then the X-Polar of the Y-Polar Conic = the Y-Polar of the X-Polar Conic. This the XY-Polar Lxy.
   2. Theorem:
   The locus of points Z for which Lxy, Lyz, Lxz concur is a conic he called Poloconica.

CU-3P-9P1 Nine 3P-Oscilating Conic Points

In projective geometry, given three distinct points Q,R,S one can define a special configuration of nine points that arise from the osculation of conics constrained by this triple.
See [77], page 137 (option 10.) and [80], pages 4-8.

Definition
Let Q,R,S be three fixed, non-collinear points. The **Nine Osculating Conic Points** associated with the triple (Q,R,S) are defined as the set of points P1,…,P9 such that:

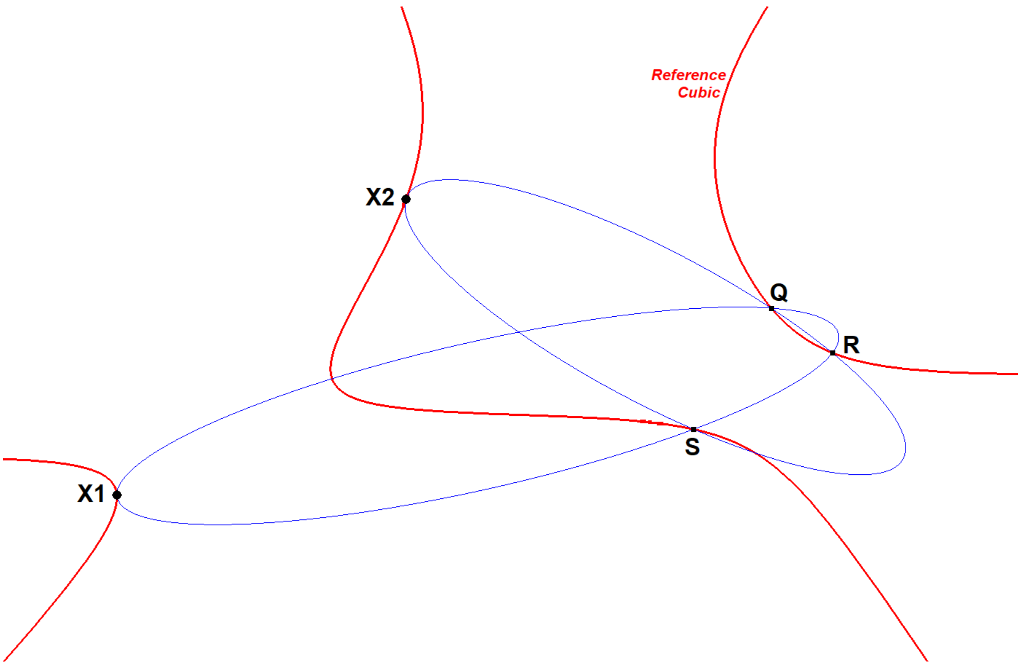
- Each point Pi lies on a unique conic Ci that **osculates** a given cubic CU at Pi to order ≥ 5 (to the extent that it exists). Each point Pi lies on a unique conic Ci that has contact of order at least 5 with a given cubic CU at Pi; that is, the conic matches the cubic's tangent direction, curvature, and higher derivatives up to fifth order at that point.
- Each conic Ci passes through the fixed triple Q,R,S, i.e., Q,R,S ∈ Ci.
- The conic Ci has maximal contact with CU at Pi under the constraint that it passes through Q,R,S.

Geometric Interpretation

- These nine points represent the **locus of maximal constrained osculation**: they are the points where conics through Q,R,S "kiss" the curve CU most tightly.
- The configuration depends on both the geometry of CU and the choice of the triple (Q,R,S), but the number of such points is always nine under generic conditions.

Properties

- The nine points are **distinct** and **algebraically determined** by the intersection of the osculating condition and the conic constraint.
- They form a **projectively invariant configuration**: any projective transformation preserving Q,R,S will map the nine points accordingly.



CU-3P-9P1 3P-Tritangent Points-01.fig

CU Point Validation
The equation 3X+Q+R+S=2N has nine solutions.

CU-4P-P1 Cotterill’s Point

Definition
Let Px and Py be points on the cubic curve CU, and denote (Px.Py) as the third point of intersection of the line PxPy and the curve CU.
The Cotterill’s Point associated with four points P1, P2, P3, P4 on CU is defined as

(P1.P2).(P3.P4).

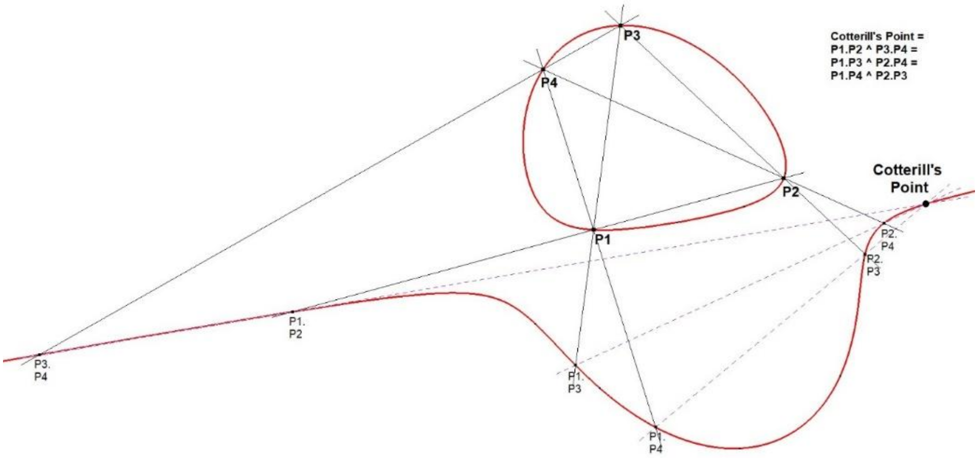
However it is also equal to

(P1.P3).(P2.P4)

and

(P1.P4).(P2.P3).

This invariance under pairings makes it a ditinguished point.



Hostorical Content
Cotterill’s Point is known from these papers:

1. Thomas Cotterill himself published in 1851: A Geometrical Property of Curves of the third Order, The Cambridge and Dublin Mathematical Journal, Vol VII, p.14, 1851. See [78].
2. H. Durège used the terminology “gegenuberliegende Punkt” and published in 1871. Die Ebenen Curven Dritter Ordung, Leipzig, 1871.
3. R. Deaux used the word “coresidual” and published in 1953. Cubiques anallagmatiques, Mathesis, 62 (1953) 193–204.

It looks like Cotterill was the first one to publish about this point.
See also Cuppens [63], page 208 below.

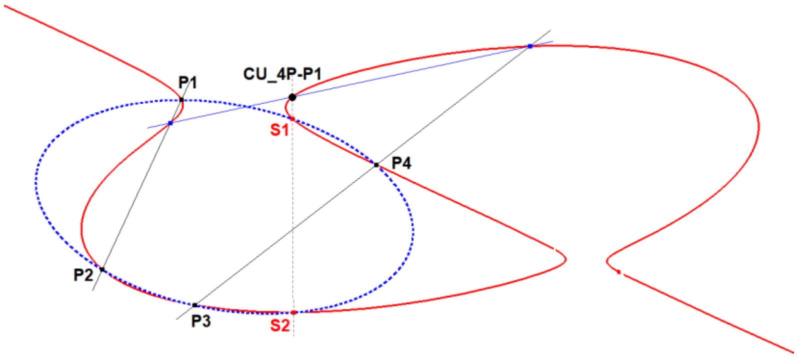
CU Point Validation

(P1.P2).(P3.P4) = N – (N-P1-P2) – (N-P3-P4) = P1+P2+P3+P4 – N,
(P1.P3).(P2.P4) = N – (N-P1-P3) – (N-P2-P4) = P1+P2+P3+P4 – N,
(P1.P4).(P2.P3) = N – (N-P1-P4) – (N-P2-P3) = P1+P2+P3+P4 – N,

which proves that these three construction methods (P1.P2).(P3.P4), (P1.P3).(P2.P4), (P1.P4).(P2.P3) yield a point of identical value.
So CU-4P-P1 = PP4 – N.

Properties

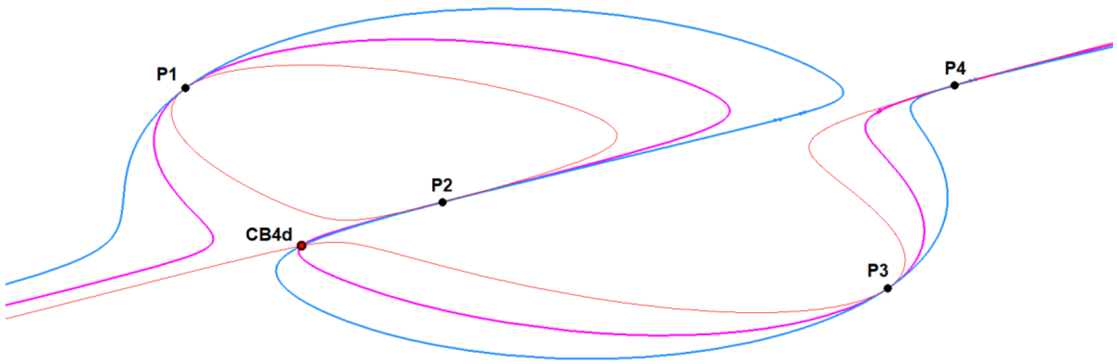
- Every Conic passing through 4 given points on a cubic will cut it again in 2 points (according to Bezout’s Theorem), such that the straight line joining them will pass through a fixed point being Cotterill’s point. See [78].



P1
CU_4P-P1
S1
P4
P2
P3
S2

CU-4P-P2 CB4d-Point

Given a general reference cubic CU and 4 random points (P1,P2,P3,P4) on it.
A situation is conceivable that two or more cubics are mutually tangent at these 4 points.
All P1-P2-P3-P4-circumscribed cubics sharing the same tangent at P1, P2, P3 and P4 have a common point CB4d. It is a special case of the Cayley-Bacharach Theorem (see CU-8P-P1) for general cubics circumscribed about a quadrangle.
See QPG#839 (attachment §3.10) and QPG#794-814 for more information.



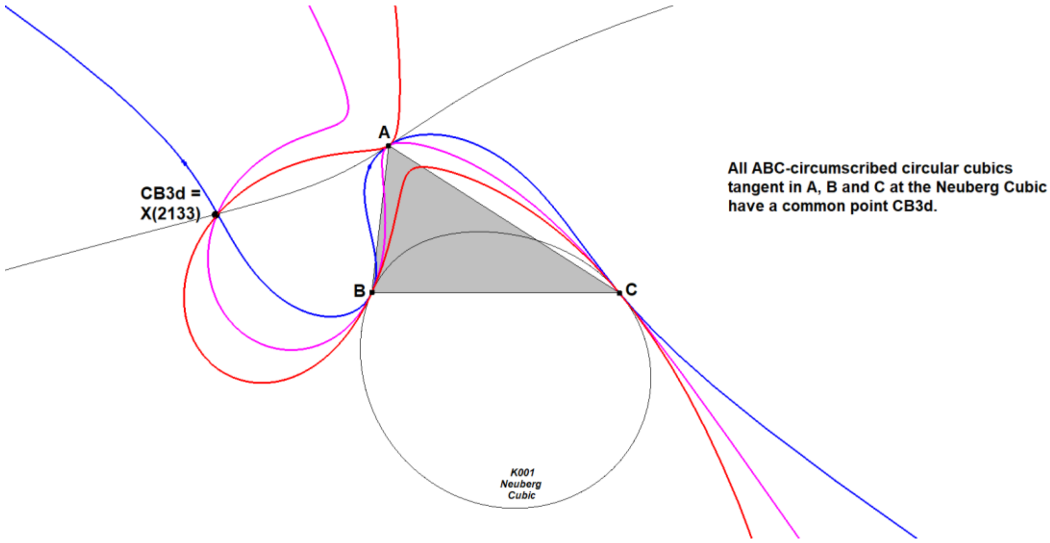
CU-4P-P2 CB4d-Point-01.fig

Circular Cubics

When the reference cubic is a circular cubic then 3 random points with mutually tangent cubics at these points produce a CB3d-point. This is a special case of the Cayley-Bacharach Theorem (see CU-8P-P1) for circular cubics circumscribed about a triangle.

Relationship with CTC and ETC

The point CB3d on the Neuberg cubic K001 is X(2133). See QPG#797.



CU-4P-P2 CB4d-Point-Case Neuberg Cubic-10.fig

Relationship with EQF

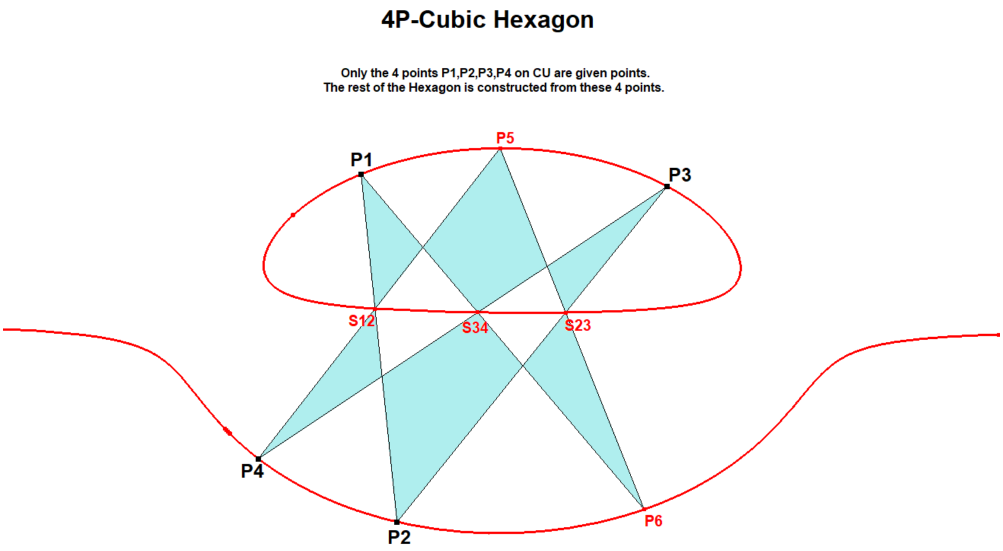
The point CB4d on the cubic QA-Cu1 is the tangential of QA-P41. See QPG#800.

CU-4P-cHe1 CU-inscribed Complete Hexagon

Definition 1: A *hexagon*, or 6-gon, is a geometrical figure composed of 6 consecutive vertices, with no three of them being collinear.

Definition 2: A *complete hexagon* is a geometric figure formed by 6 consecutive vertices, with no three of them being collinear. In addition, it includes the points of intersection of the 3 pairs of opposite sides.
In other words, a complete hexagon comprises 6 consecutive vertices and the points of intersection of the 3 pairs of opposite sides, creating a set of 9 points.

Theorem: A complete hexagon can be inscribed in a cubic curve using only four initial points.



The procedure to construct the CU-inscribed Hexagon (6-gon) using 4 random points on CU is:
1. Start with 4 random points P1,P2,P3,P4 on a reference cubic CU.
2. Draw line P1P2 intersecting CU again in S12.
3. Draw line P2P3 intersecting CU again in S23,
4. Draw line P3P4 intersecting CU again in S34,
5. Draw line P4S12 intersecting CU again in P5,
6. Draw line P5S23 intersecting CU again in P6.
7. Draw line P6S34 intersecting CU again in P1 !
Last step easily can be proven using the CU Point Addition Method CU-4.

CU Point Validation

1. Use the property that for 3 collinear points P, Q, and R, the sum P + Q + R = N.
2. S12 = N – P1 – P2
3. S23 = N – P2 – P3
4. S34 = N – P3 – P4
5. P5 = N – P4 – S12 = N – P4 – (N – P1 – P2) = P1 + P2 – P4
6. P6 = N – P5 – S23 = N – P5 – (N – P2 – P3) = P2 + P3 – P5
7. Px = N – P6 – S34 = N – (P2 + P3 – P5) – (N – P3 – P4) = -P2 + P4 + P5 = -P2 + P4 + (P1 +P2 – P4) = P1

Connection with Eckart's Hexagon Theorem

Eckart's Hexagon Theorem (QPG#1799) states that given 3 collinear points P,Q,R and a starting point X1, then a closed Hexagon can be constructed. The 6 vertices will be coconic.

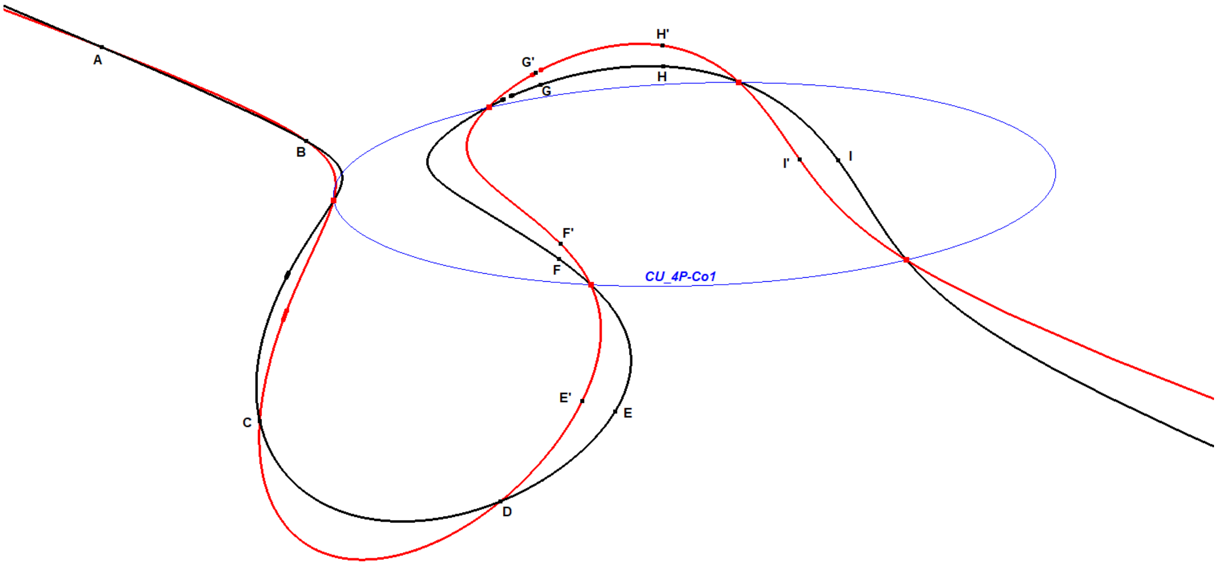
The way of construction of the Hexagon is similar to the construction of the CU-4P-Hexagon, only the starting points are different, there are 3 collinear points and 1 other point.

Applications

1. The 3 Quasi-Miquel Triangles (CU-IP-3P1) form together a CU-inscribed Complete Hexagon.
2. The 9P-SumPoints and surrounding points form a CU-inscribed Complete Hexagon. See CU-9P-P1.

2CU-4P-Co1 CU-4P-Common Points Conic

Given 2 cubics CU1 & CU2 having 4 points A,B,C,D in common.
Other defining points E,F,G,H,I on CU1 differ from defining points E',F',G',H',I' on CU2.
Since 2 cubics always have 9 points in common and since A,B,C,D are known common points, there have to be 5 other common points. These points lie on a conic CU-4P-Co1 defined by these 5 other common points.
This conic and its construction is described by Roger Cuppens. See [63], p.236



2CU-4P-Co1 CU-4P-Common Points Conic-02

Construction
Given 14 points A, B, C, D, E, F, G, H, I, E',F',G', H', and I'.
Consider the pencil F of conics passing through the 4 points A, B, C and D determine:

- the tangents e, e', f, f', g, g', h, h', i and i' at point A to the conics from the pencil F passing respectively through the points E, E', F, F', G, G', H, H', I and I';
- the center of homography S (resp. S') from the set of {e,f,g,h,i} and {E,F,G,H,I} (resp. {e',f',g',h',I'} and {E',F',G',H',I'});
- the transformed e'' (resp. f'', resp. g'') of the line e' (resp. f', resp. g') according to the homography which maps e into (SE), f into (SF), and g into (SG);
- the intersection point U (resp. V, resp. W) of the lines e'' and (S'E') (resp. f'' and (S'F'), resp. g'' and (S'G')).

The conic g passing through the 5 points S, S', U, V and W is the conic we are looking for.

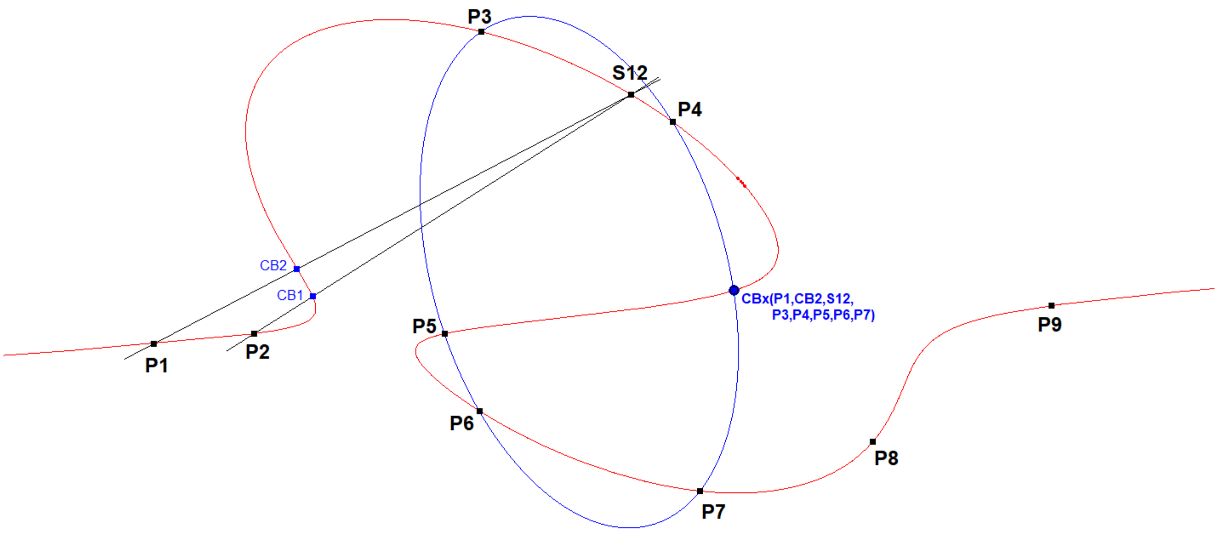
Properties

- 2CU-4P-Co1 has another point in common with CU1 as well as CU2. This will be Cotterill's Point CU-4P-P1(A,B,C,D) wrt CU1 and CU2.

Validation
A+B+C+D+S1+S2+S3+S4+S5=3N
S1+S2+S3+S4+S5+S6=2N
Therefore A+B+C+D+(2N-S6)=3N → S6 = -N+A+B+C+D,
which is Cotterill's Point CU-4P-P1(A,B,C,D)

CU-5P-P1 6th intersection point of a 5P-conic

Given five random points P1, …, P5 on reference cubic CU.
According to Bezout's theorem any conic (curve of 2nd degree) has six intersection points with a cubic (curve of 3rd degree) in the projective complex plane. These intersection points of a conic can be real or imaginary as well as finite or infinite.
Consequently, when we have 5 points P1, …, P5, there will be a 6th intersection point of the conic through P1, …, P5 with CU.
The existence of this point is obvious, but its construction is not that simple.



CU-5P-P1 6th Intersection Point with 5P-Conic-01.fig

Construction

1. We use for the construction these cubic properties:
2. When 2 cubics intersect in 8 points, then they will intersect in a fixed 9th point, that will be the 9th intersection point for all cubics through these 8 points. This is called the 8P-Cayley Bacharach Point CU-8P-P1. Denote CB=Cayley-Bacharach Point. See CU-8P-P1 for its construction.
3. Given 7 points on a reference cubic CU. Then for every point Pi there will be a Cayley-Bacharach Point CBi(P1,…,P7,Pi) with special property that 3rd intersection point P.CBi with CU will be fixed for every general location of Pi. This point is called CU-7P-P1, the 7P-Pivot Point. See CU-7P-P1 for its construction.
4. When 2 cubics intersect in 9 points and 3 are collinear then the other 6 will be coconic.
5. When 2 cubics intersect in 9 points and 6 are coconic then the other 3 will be collinear.

Now we can proceed to the actual construction of the 6th intersection point of conic (P3,P4,P5,P6,P7) with CU.

6. Determine 9 points P1, …, P9 on the cubic.
7. Construct CB1=CB(P2,P3, …, P9) and CB2=CB(P1,P3, …, P9).
   Lines P1.CB2 and P2.CB1 will meet in 7P-Pivot Point S12 on CU.
8. Construct CBx=CB(P1,CB2,S12,P3,P4,P5,P6,P7). The first three points are collinear, therefore CBx will be the 6th intersection point of conic(P3,P4,P5,P6,P7).

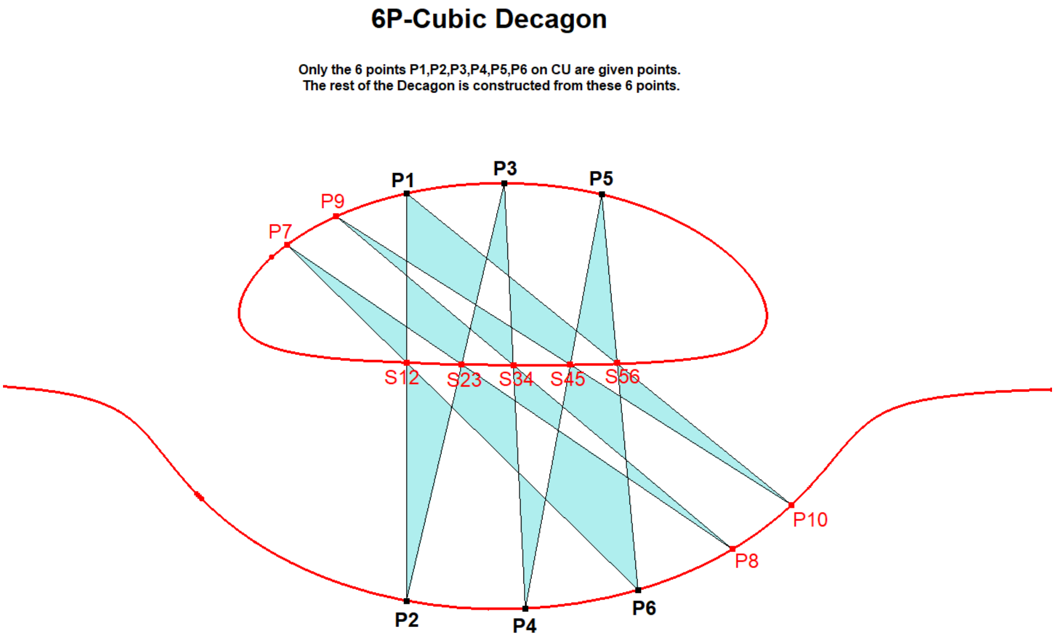
CU-6P-cDe1 CU-inscribed Complete Decagon

In a similar way as the construction of the CU-inscribed Complete Hexagon it is possible to inscribe a Complete Decagon into a cubic using just six starting points.

Definition 1: A *decagon*, or 10-gon, is a geometrical figure composed of 10 consecutive vertices, no three of them being collinear.

Definition 2: A *complete decagon* is a geometric figure formed by 10 consecutive vertices, no three of them being collinear and additionally including the intersection points of the 5 pairs of opposite sides. In other words, a complete hexagon consists of 10 consecutive vertices and the points of intersection of the 5 pairs of opposite sides, creating a set of 15 points of the Reference Cubic.

Theorem: A complete decagon can be inscribed in a cubic curve using only six initial points.



CU-6P Cubic Decagon-02.fig

Construction

The procedure to construct the CU-inscribed Decagon (10-gon) using 6 random points on CU is:
1. Start with 6 random points P1,P2,P3,P4,P5,P6 on a reference cubic CU.
2. Draw line P1P2 intersecting CU again in S12.
3. Draw line P2P3 intersecting CU again in S23,
4. Draw line P3P4 intersecting CU again in S34,
5. Draw line P4P5 intersecting CU again in S45,
6. Draw line P5P6 intersecting CU again in S56,
7. Draw line P6S12 intersecting CU again in P7,
8. Draw line P7S23 intersecting CU again in P8.
9. Draw line P8S34 intersecting CU again in P9,
10. Draw line P9S45 intersecting CU again in P10.
11. Draw line P10S56 intersecting CU again in P1 !

Last step easily can be proven using the CU Point Addition Method.

CU Point Validation

1. Use the property that for 3 collinear points P, Q, and R, the sum P + Q + R = N.
2. S12 = N – P1 – P2
3. S23 = N – P2 – P3

4. S34 = N – P3 – P4
5. S45 = N – P4 – P5
6. S56 = N – P5 – P6
7. P7 = N – P6 – S12 = N – P6 – (N – P1 – P2) = P1 + P2 – P6
8. P8 = N – P7 – S23 = N – P7 – (N – P2 – P3) = P2 + P3 – P7
9. P9 = N – P8 – S34 = N – P8 – (N – P3 – P4) = P3 + P4 – P8
10. P10 = N – P9 – S45 = N – P9 – (N – P4 – P5) = P4 + P5 – P9
11. Px = N – P10 – S56 = N – P10 – (N – P5 – P6) = P5 + P6 – P10 = P5 + P6 – (P4 + P5 – P9) = P6 + P9 – P4 = P6 + (P3 + P4 – P8) – P4 = P3 + P6 – P8 = P3 + P6 – (P2 + P3 – P7) = P6 + P7 – P2 = P6 + (P1 + P2 – P6) – P2 = P1

Construction of 2-Gon/6-Gon/10-Gon/etc.
Extrapolating, it appears that there is a CU-inscribed 2P-2-gon, 4P-6-gon (Hexagon), 6P-10-gon (Decagon), 8P-14-gon, etc.
Construction of a CU-inscribed 4P-6-gon in a more general way:
1. Start with 4 random points P(1),P(2),P(3),P(4) on the reference cubic CU.
2. Draw lines P(i)P(i+1) intersecting CU in S(i,i+1) for i=1,2,3
3. Draw lines P(i+3)S(i,i+1) intersecting CU in P(i+4) for i=1,2,3
Finally P(7) coincides with P(1).

Similar construction of a CU-inscribed (n+1)P-2n-gon, valid for n=3,5,7, etc.
1. Start with n+1 random points P(1),…,P(n+1) on a cubic CU.
2. Draw lines P(i)P(i+1) intersecting CU in S(i,i+1) for i=1,..,n
3. Draw lines P(i+n)S(i,i+1) intersecting CU in P(i+n+1) for i=1,…,n
Finally P(2n+1) coincides with P(1).
This also can be proven using the CU Point Addition Method CU-4.

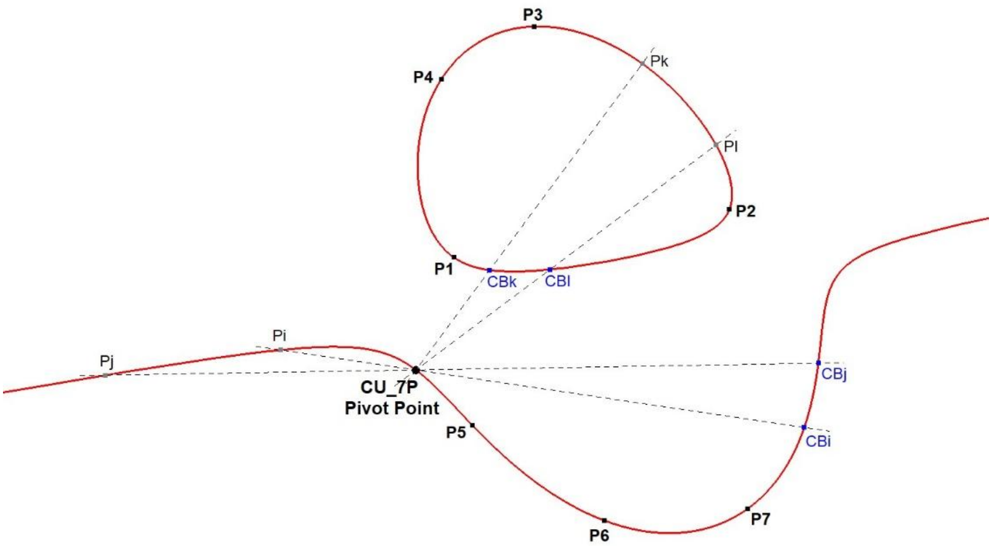
4-Gons/8-Gons/12-Gons/etc.
Note that using the same procedure for a CU-inscribed 3P-4-gon, 5P-8-gon, 7P-12-gon, etc. it doesn’t deliver the original starting point:
In the case of a 3P-4-gon, P5 doesn't coincide with P1, like it does with the 4P-hexagon and 6P-decagon, etc.
However P1P5 is the Tangential of P3.
In the case of a 5P-8-gon, P9 doesn't coincide with P1, like it does with the 4P-hexagon and 6P-decagon, etc.
However P1P9 is the Tangential of P5. This also can be proven using the CU Point Addition Method CU-4.

CU-7P-P1 CB-Pivot Point

Definition

- Given a fixed set of 7 random reference points Pi (i=1, …, 7) on CU.
- Given two extra random points Px and Py on CU.
- Every set of 8 points define a Cayley-Bacharach Point (CU-8P-P1) of these 8 points.
- We can compose two sets of 8 points by adding Px and Py to the fixed set of 7 points.
- Let CBx be the CB-point of (P1, …, P7,Px) and CBy be the CB-point of (P1, …, P7,Py).
- The intersection point of lines PxCBx and PyCBy will appear to be a fixed point on CU.
- This point is the CB-Pivot Point.

This point was contributed by Eckart Schmidt.
It is a unique point on CU determined by just 7 reference points on CU.



CU Point Validation

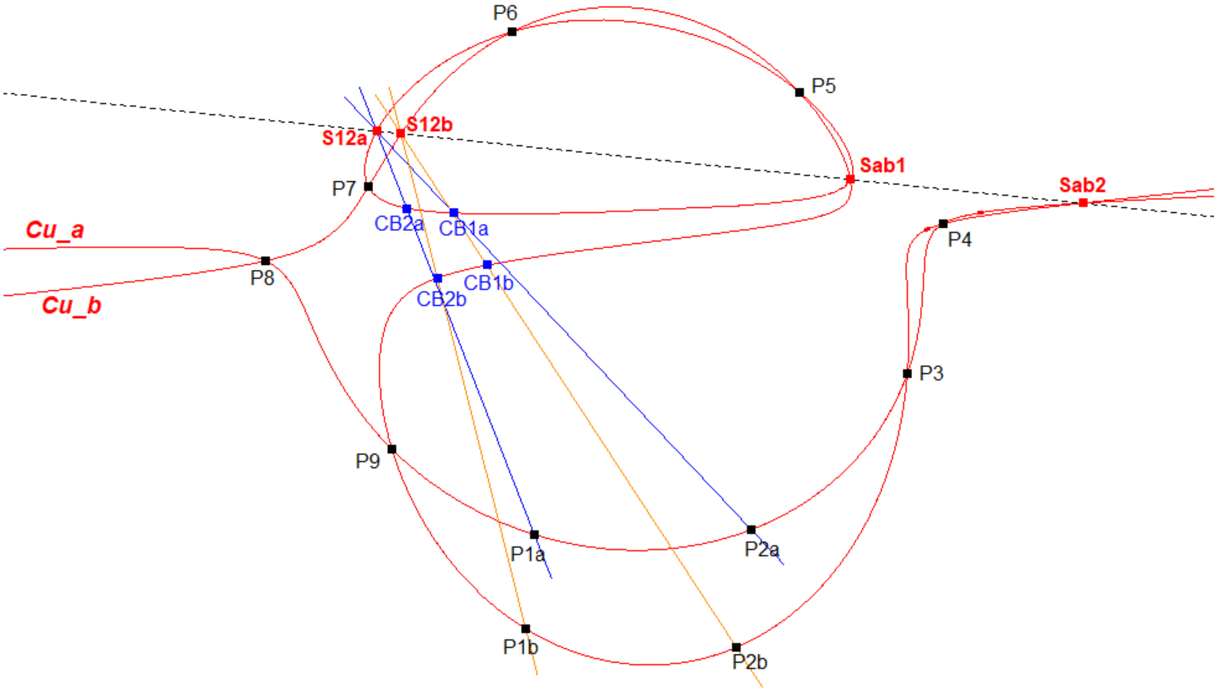
- Pi.Pj=(Pi+Pj)-N and CBi=3N-(PP9-Pi);
- Let Tx=Px.CBx, then Tx=(Px+(3N-(PP7+Px))-N)=2N-PP7
- Let Ty=Py.CBy, then Ty=(Py+(3N-(PP7+Py))-N)=2N-PP7
- Tx and Ty are constructed both as a point on CU and are identical points since they have the same validation. So CU-7P-P1 = 2N – PP7.
- From the formula it is clear that this point is independent of Px and Py, since they do not appear in the formula.

2CU-7P-L1 7P-Cubics Intersection Line

Given 2 cubics having 7 points in common. Since 2 cubics always share 9 intersection points, both cubics will have 2 other intersection points. The line through these 2 points is the same line as the line through the two 7P-Pivots (CU-7P-P1) determined per cubic.

Circular Cubics
The same property holds for circular cubics having 5 points in common. Then the line through the remaining 2 intersection points is the same line as the line through the two 5P-Pivots (analogon of CU-7P-P1 for circular cubics) determined per circular cubic. See QPG#839 (attachment §3.5).

Clarification:
Let CU-a and CU-b be two regular 9P-cubics with 7 points P3, … , P9 in common and individual points P1a, P2a on CU-a and P1b, P2c on CU-b.
Let CBi = Cayley-Bacharach Points of Pi wrt the other 8 points.
Let Sij = Pi.CBj ^ Pj.CBi. It is the pivot point 7P-s-P2(P3,P4,P5,P6,P7,P8,P9). (7P-s-P2=CB_7P-P1)
Specifically let S12a = P1aCB2a ^ P2a.CB1a and let S12b = P1bCB2b ^ P2b.CB1b.
CU-a and CU-b have 7 points in common, therefore there must be 2 extra intersection points. Denote them with Sab1 and Sab2.
Theorem:
S12a ,S12b, Sab1, and Sab2 are collinear.
Proof:
In general CB(P1,...,P7,S1) will be a point S2 on **CU-a** when S1 is on **CU-a**.
In general CB(P1,...,P7,S1) will be a point S2 on **CU-b** when S1 is on **CU-b**.
Therefore when S1 lies on CU-a as well as CU-b, S2 also will lie on CU-a as well as CU-b. This is only the case for intersection points, therefore Sab1 and Sab2 will be CB-partners.
And since CB-partners on CU-a pass through the pivot point S12a and since CB-partners on CU-b pass through the pivot point S12b, it must be that S12a, S12b, Sab1, and Sab2 are collinear.
See QPG#839 (attachment §3.7).

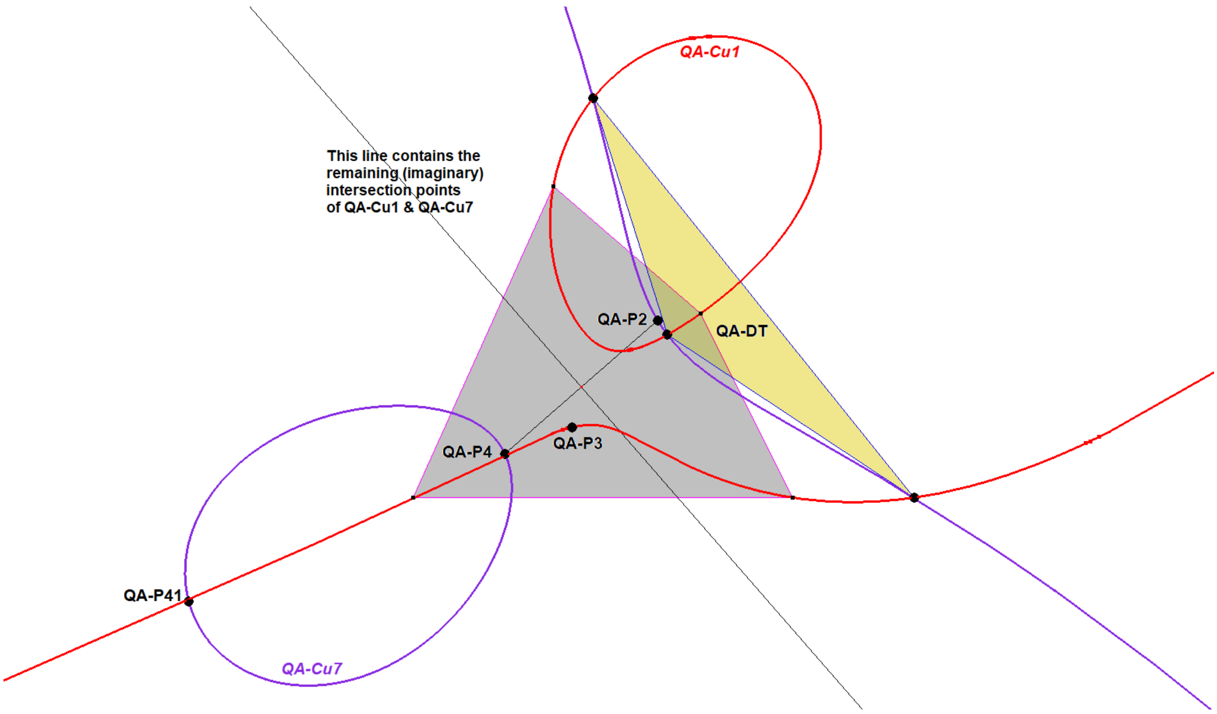


9P-s-Cu1 Two 9P-Cubics-7 points in common-intersectionpoints-Line-02.fig

Extra property
We looked to the intersection points of two cubics with 7 common points.
When we have three cubics with 7 points in common, then:
If three curves of the third order pass through seven and no more points, the lines connecting their remaining pairs of intersections meet in pairs on the three curves.
(which happen to be the 7P-CB-pivot points (7P-s-P2=CB_7P-P1) of the 7 points on the three cubics.
See [Julian Lowell Coolidge - A Treatise on Algebraic Curves, page 33].

Application in EQF
Cubics QA-Cu1 and QA-Cu7 intersect in 5 real points, the vertices of the Diagonal Triangle DT, QA-P4 and QA-P41 and the 2 circular points at infinity.
The question is: What about the 2 remaining intersection points? See QPG#861-868.

Here the 7P-Cubics Intersection Line comes in. This line connects two pivot-points 7P_7P-P1, per cubic one pivot point.
In our situation the pivot point of QA-Cu7 is the infinity point of its asymptote, the pivot point of QA-Cu1 is a finite point X on QA-Cu1.
Therefore, the 6th and 7th intersection points of QA-Cu1 and QA-Cu7 will lie on the line through point X on QA-Cu1, parallel to the asymptote of QA-Cu7. It turns out that this line is the perpendicular bisector of QA-P2 and QA-P4. See QPG#866.
However, looking at the picture below, no real intersection points of QA-Cu1 and QA-Cu7 (other than the DT vertices, QA-P4 and QA-P41, and the two circular points) can be seen. Obviously, these are two imaginary points, nevertheless lying on a real line. Calculations in Mathematica for a numerical case confirm this.
Finally, the answer to the question is that the two remaining intersection points of QA-Cu1 and QA-Cu7 are two imaginary points lying on the perpendicular bisector of QA-P2 and QA-P4, thanks to the theorem of the 7P-Cubics Intersection Line.

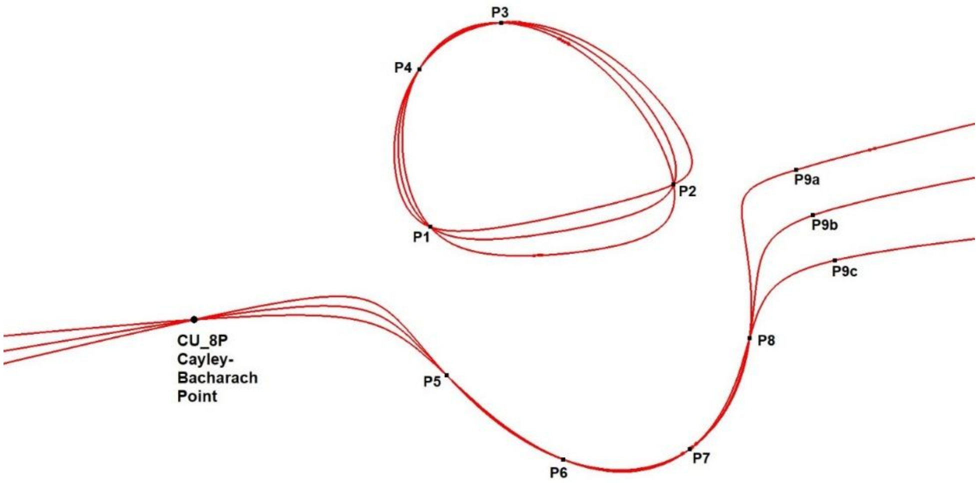


QA-Cu1 intersected with QA-Cu7-10.fig

Note: That 2 imaginary points possibly will lie on a real line can easily be seen from next example.
Given P1 = (1, +i, -i) and P2 = (1, -i, +i). Their connecting line is the cross-product of P1 and P2, being (0, 1, 1), which is a real line indeed.

CU-8P-P1 Cayley-Bacharach point

Cayley-Bacharach's theorem states that when P1,P2,P3,P4,P5,P6,P7,P8 are points on a cubic, that all other cubics passing through these 8 points will have a common 9th point.
The 9th point is then the Cayley Bacharach Point of the other 8 points.
Define CB9 = Cayley-Bacharach Point of P1, …, P8.
We know from the CU Point Addition Method (CU-4) that P1+P2+P3+P4+P5+P6+P7+P8+CB9=3N.
Therefore CB9=3N-PP8.
Knowing this formula of the Cayley-Bacharach Point we can deduct this construction method CB9=(P1.P2).(P3.P4).(P5.P6).(P7.P8), where the . operation Pi.Pj means "construct 3rd intersection point of Cu and line PiPj".
It is easy to calculate that CB9=3N-PP8 following this construction and using rule 6. It is also clear that any point of the set of 8 points can be taken as P1, etc. So the construction can be done in multiple ways delivering the same point CB9.
Moreover it proves in a simple way the existence of the Cayley-Bacharach point.

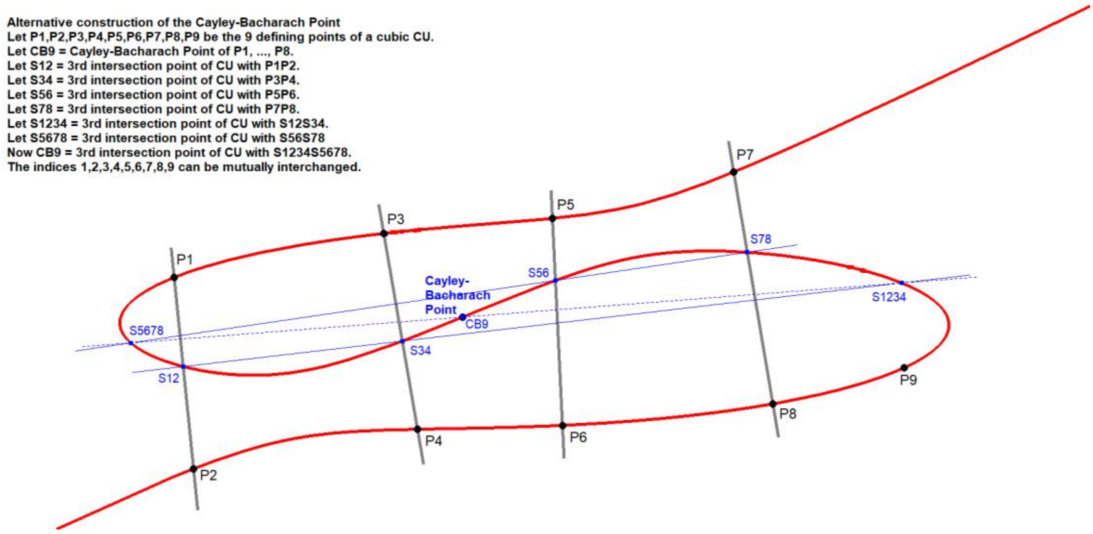


CU-8P-P1 CU-8P Cayley-Bacharach Point-01.fig

CU Point Validation

- P1+P2+P3+P4+P5+P6+P7+P8 + CU-8P-P1 = 3N
- Therefore CU-8P-P1 = 3N – PP8.

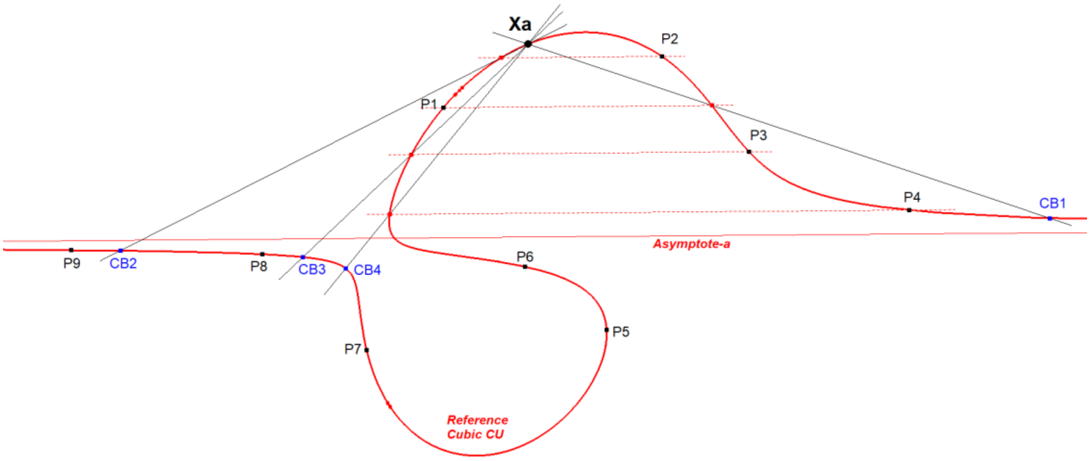
From the CU Point Validation, a construction for the Cayley-Bacharach Point emerges by successively intersecting connection lines among the eight reference points as follows: for each pair of points, a third CU-point is determined; these CU-points are then paired and connected again to yield new CU-points. This hierarchical process continues until the final intersection of the last two CU-points produces a point that coincides with the Cayley-Bacharach Point.



8P-s-P1-Cayley-Bacharach Point-11-Alternative-Construction.png

CU-9P-3P1 IP-SumPoints of the set (P1,…,P9)

Given reference cubic CU with 9 random points P1, …, P9 on CU.
Let aPi be the 2nd intersection point of CU with the Asy-a-Parallel through Pi (i=1,…,9).
Xa = aPi.CBi, which is a fixed point for i=1, ... , 9.



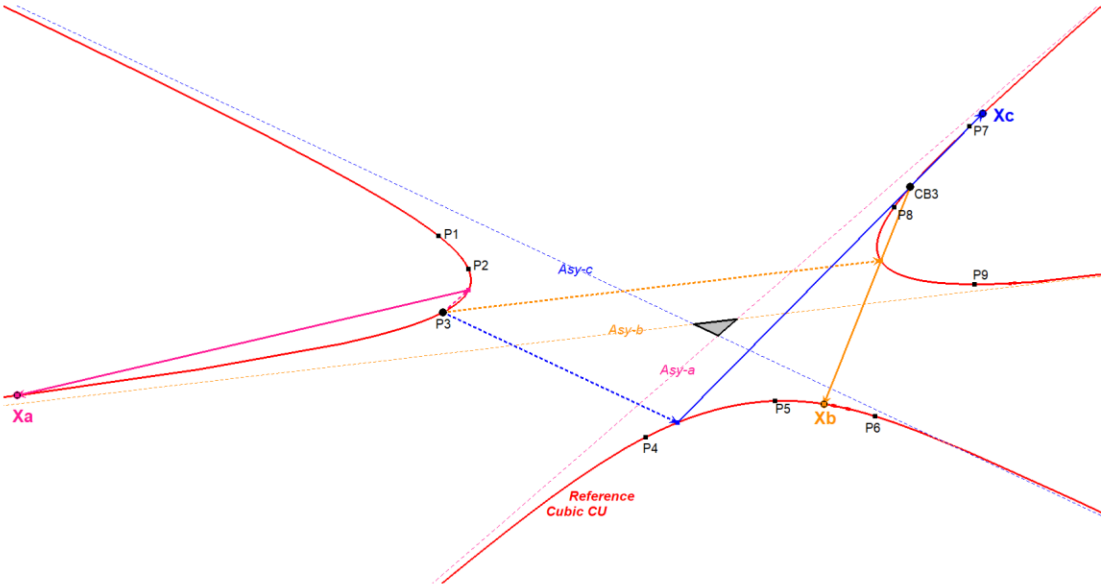
CU-9P-3P1 IP-Sumpoints of the set (P1,…,P9)-01.fig

Validation:
Calculation for IPa and CB9:

P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + CB9 = 3N (1)
P9n + P9 + IPa = N (2)
CB9 + P9n + Xa = N (3)

From (1) and (2) and (3) it follows that:
Xa = IPa + PP9 -3N, where PP9 = P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + P9

In the same way there are fixed points Xb and Xc, when there are 3 real CU-asymptotes.
Xa, Xb, Xc are the IPa/b/c-Sumpoints of the set (P1,...,P9).



CU-9P-3P1 IP-Sumpoints of the set (P1,…,P9)-10.fig