Encyclopedia of Poly Geometry

Everything is documented with references when known.
When no references are mentioned the information is found and described by the author of EPG. Even then it is possible that items and properties were known before. When you know this kind of information or when you have other remarks or questions please let me know by mail.
Several items are proven. When known there will be a reference to the corresponding article. However especially with Poly-figures it is often very hard to give a full proof for the validity of involved item.
Therefore when things are very likely and "proven" with drawing software or algebraic software they still will be mentioned in EPG, often with reference to discussions and waiting for a person who delivers the full proof.

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Table of contents

Introduction

n-Lines

nL-n-Luc5f nL-Par1-Per1-Construction
nL-n-Luc5g nL-Par1-Per2-Construction
nL-n-Luc5h nL-Par2-Per1-Construction
nL-n-Luc5i nL-Par2-Per2-Construction
nL-n-Luc5j nL-Per1-Per2-Construction

nL-e: Even recursive Objects in an n-Line
nL-e-P1 nL-Morley's EnnaDeltoid Center
nL-e-P2 nL-Clifford's Point
nL-e-L1 nL-Morley's Ortho Directrix
nL-e-L2 nL-Morley's Alternate Line of Orthocenters
nL-e-Cv1 nL-Morley's Inscribed EnnaDeltoid

nL-o: Odd recursive Objects in an n-Line
nL-o-P1 nL-Morley's 1st Orthocenter
nL-o-P2 nL-Clifford's Circle Center
nL-o-L1 nL-Line of Inscribed EnnaDeltoid Centers
nL-o-L2 nL-Morley's Line of Orthocenters
nL-o-Ci1 nL-Clifford's Circle
nL-o-2L1 nL-Orthogonal Ortho Axes

nL-s: Specific Objects in an n-Line

5L-s: Specific Objects in 5-Lines or Pentalaterals
5L-s-P1 5L-Inscribed Conic Center
5L-s-P2 5L-X(186)-Hofstadter Point
5L-s-P3 5L-X(265)-Hofstadter Point
5L-s-P4 5L-Center of the Anticenter Circle
5L-s-P5 5L-OH Division Point
5L-s-P6 5L-QL-P3 Par1/Par2-Homothetic Center
5L-s-P7 5L-QL-P20 Par1/Par2-Homothetic Center
5L-s-P8 5L-QL-P27 Ref/Par1/Par2-Homothetic Center
5L-s-P9 5L-QL-P28 Par1/Per2-Homothetic Center
5L-s-P10 5L-QL-P29 Par1/Per2-Homothetic Center

5L-s-Co1 5L-Inscribed Conic
5L-s-L1 5L-Newton Line
5L-s-L2 5L-Isoconjugate Line
5L-s-L3 5L-PAP Miquel Points Line

5L-s-Tf1 5L-Schmidt Transformation

6-Lines
6L-s-P1 6L-Conical Center
6L-s-P2 6L-Conical Ref-Par2 Homothetic Center
6L-s-P3 6L-Conical Ref-Per2 Homothetic Center
6L-s-P4 6L-Conical Par2-Per2 Homothetic Center
6L-s-Co1 6L-Conical Center Conic
6L-s-Tf1 6L-Schmidt Transformation

7-Lines
7L-s-3P1 7L-Conical Triplet Points
7L-s-3Ci1 7L-Conical Triplet Circle

8-Lines
8L-s-P1: 8L-Conical Center

n-Points

nP-n: General recursive Objects in an n-Point
nP-n-P1 nP-MVP-Centroid
nP-n-P2 nP-MVP-Circumcenter
nP-n-P3 nP-MVP-Orthocenter
nP-n-P4 nP-MVP-Nine-point Center

nP-n-L1 nP-MVP-Eulerline
nP-n-L2 nP-LSD Line

nP-n-Luc Level-up constructions
nP-n-Luc1 nP-Mean Vector Point

nP-s: Specific Objects in an n-Point

5P-s: Specific Objects in 5-Points or Pentangles
5P-s-P1 5P-Circumscribed Conic Center
5P-s-P2 5P-Involutary Center

5P-s-Co1 5P-Circumscribed Conic
5P-s-Tf1 5P-Line of Involution Centers
5P-s-Tf2 5P-Line of Involutary Conjugates
5P-s-Tf3 5P-Orthopole
5P-s-Tf4 5P-Orthopolar Line

8P-s: Specific Objects in 8-Points or Octangles
8P-s-P1 Cayley Bacharach Point

n-Gons

5-Gons

5G-s Specific Objects in a 5-Gon
5G-s-P1 5G-Common Newton Lines Point
5G-s-P2 5G-Bicircles Crosspoint
5G-s-P3 5G-Inner Miquel Points Center

INTRODUCTION

This encyclopedia is giving illustrations and a description of properties related to Poly-Figures.
Here a Poly-Figure is defined as a plane geometrical figure of n points and/or n lines, where n is a natural number > 1.
In this Encyclopedia we will use these terms:

- **n-Gon**, meaning a figure consisting of n points and n lines connected in a fixed order,
- **n-Point**, meaning a figure consisting of n random points without order,
- **n-Line**, meaning a figure consisting of n random lines without order,

where n is a natural number >1.

For the different values of n we have these Poly-Figures:

n = 3 Triangle
Well-known figure consisting of 3 points and 3 lines.
* Triangle = 3-Point = 3-Line = 3-Gon.
Described in the Encyclopedia of Triangle Centers ETC. See Ref-12.

n = 4 Quadri-Figure
* Quadrangle (4-Point),
* Quadrilateral (4-Line),
* Quadrigon (4-Gon).
Described in the Encyclopedia of Quadri Figures EQF.

n = 5 Penta-Figure
* Pentangle (5-Point)
* Pentalateral (5-Line)
* Pentagon (5-Gon)

Familiar names for Polygons or n-Gons when n>5 are:

n = 6 Hexagon
n = 7 Heptagon
n = 8 Octagon
n = 9 Nonagon or Enneagon
n = 10 Decagon
n = 11 Hendecagon
n = 12 Dodecagon
n =13 Tridecagon
n = 14 Tetradecagon
n = 15 Pentadecagon
n = 16 Hexadecagon
n = 17 Heptadecagon
n = 18 Octadecagon
n = 19 Enneadecagon
n = 20 Icosagon
n = 30 Triacontagon
n = 100 Hectogon
etc.

n-Lines, n-Points and n-Gons
In EPG three basic platforms occur:

- **n-Line**, meaning a figure consisting of n random lines without order. All objects of an n-Line will be prefixed with "nL-".
- **n-Point**, meaning a figure consisting of n random points without order. All objects of an n-Point will be prefixed with "nP-".
- **n-Gon**, meaning a figure consisting of n points and n lines cyclically connected in a fixed order. All objects of an n-Gon will be prefixed with "nG-".

Recursive processes
Like described before a Polygon has n variable points/lines whether or not in a fixed order.
There are similar points in Polygons for all n > 2.
For example when n = 3 we have a 3-Line or a triangle and in a triangle we have a circumcircle and a circumcircle has a circumcenter X(3).
When n = 4 we have a 4-Line or a quadrilateral containing 4 component 3-Lines/Triangles, each having a circumcenter X(3). These 4 circumcenters lie on a circle called "4L-Centercircle". This "4L-Centercircle" has a circumcenter QL-P4.
When n = 5 we have a 5-Line or a Pentalateral containing 5 component 4-Lines/Quadrilaterals, each having a 4-Line circumcenter. These 5 circumcenters lie on a circle called "5L-Centercircle". This "5L-Centercircle" has a 5L-circumcenter.
Etc.
This takes a lot of words just describing a relatively simple recursive construction method.
By using simple notation techniques we can simplify the wording of these kind of statements.
For example above sentences can be simplified by stating:
"An n-Line contains n (n-1)-Lines. So in an n-Line n (n-1)L-circumcenters can be constructed. These circumcenters are always concyclic and therefore define an nL-center-circle having an nL-circumcenter. This recursive process starts with the circumcircle of a 3-Line."

m-Lines and p-Lines
By introducing another notation we can even simplify these wordings.
In the recursive process we often deal with (n-1)-Lines or (n+1)-Lines.
To indicate that we talk about an n-Line of a lower or upper level we can talk about *m-Lines* or *p-Lines*, where "m" and "p" resp. are meaning "minus 1" and "plus 1".
So now we can say:
"An n-Line contains n *m-Lines*. So in an n-Line n *mL*-circumcenters can be constructed."
instead of:
"An n-Line contains n *(n-1)-Lines*. So in an n-Line n *(n-1)L*-circumcenters can be constructed.
This notation will be used whenever convenient.

The neos-system: n-Points, e-Points, o-Points and s-Points
In the Encyclopedia of Polygon Figures different types of points will be distinguished.
an n-Point is a recursive point that occurs in all n-Lines for n=natural number >2.
an e-Point is a recursive point that occurs in all n-Lines for n=even number >2.
an o-Point is a recursive point that occurs in all n-Lines for n=odd number >2.
an s-Point is a non-recursive but specific point only occurring in an n-Line for n = specific natural number > 2.
For example the notation for a point will be:

- nL-n-P1, meaning **g**eneral-recursive Point 1 in an n-Line, where n can be 3,4,5,6,….
- nL-e-P1, meaning **e**ven-recursive Point 1 in an n-Line, where n can be even numbers 4,6,8,10,…

- nL-o-P1, meaning **o**dd-recursive Point 1 in an n-Line, where n can be odd numbers 3,5,7,9,….
- nL-s-P1, meaning **s**pecific non-recursive Point 1 in an n-Line, where n is specifically 3,4,5,6,….
- Etc.

This implies that we will have different sets of points.
n-Points, e-Points, o-Points will be described in general.
s-Points will be described for the fixed number of n it occurs with.
The same infixes -n-, -e-, -o-, -s- will also be used for Lines, Circles, Cubics, Quartics, Transformations, etc.

Here are some examples for n-Lines:

5L-n-P1, meaning general-recursive Point 1 in a 5-Line
6L-e-P1, meaning even-recursive Point 1 in a 6-Line
7L-o-P1, meaning odd-recursive Point 1 in a 7-Line
8L-s-P1, meaning specific non-recursive Point 1 in an 8-Line

Central Points/Centers

Described points in the Encyclopedia of Polygon Figures actually will be “central points” or “centers”. The meaning of a central point/center best can be given with an example.
For example in 3-Line/triangle we have 3 lines L1,L2,L3.
We have the intersection point S_{12} of L1 and L2. This not a central point.
We have the circumcenter O of the circumcircle. This is a central point.
S12 is not a central point in a 3-Line because it is not equally dependent on the 3 basic elements of a 3-Line, namely L1,L2,L3.
However O is a central point in a 3-Line because it is equally dependent on the 3 basic elements of a 3-Line, namely L1,L2,L3.
The same can be done in a 4-Line figure, etc.
In literature little is written about central points in a Polygon.
Clark Kimberling defines in ETC (see Ref-12) a triangle center like this:

> Suppose a point P has a trilinear representation
> f(A,B,C) : g(A,B,C) : h(A,B,C) such that
> (i) g(A,B,C) = f(B,C,A)
> and h(A,B,C) = f(C,A,B);
> (ii) f(A,C,B) = f(A,B,C);
> (iii) if P is written as
> u(a,b,c) : u(b,c,a) : u(c,a,b), where a,b,c are the side lengths of triangle ABC, then u is homogeneous in the variables a,b,c. (By the law of sines and (i), such u must exist.)
> Then P is a *triangle center*, or simply a *center*.

Benedetto Scimemi proposes in his document “Central Points of the Complete Quadrangle” (see Ref-36):

> Let $\mathbf{E}$ be the Euclidean plane; a ($n$gonal)central point P is a symmetric mapping: $\mathbf{E}_n\_ \to \mathbf{E}$ which commutes with all similarities φ (in the sense that $P(\varphi(A_i))=\varphi(P(A_i))$). Likewise one defines central lines, central scalars, central conics etc. If this definition is studied analytically, some interesting algebraic questions naturally arise.

Important for the Encyclopedia of Polygon Figures is that we only describe Central Points/Centers and related Central Lines, Central Conics, etc.

How many Poly-Points/Centers do potentially exist?

When using the notion of Point here we actually mean a Central Point or Center. See paragraph just before.

In *Triangle Geometry* very many Points are described in Ref-12, the Encyclopedia of Triangle Centers (ETC). And that's only just the beginning.
Points can be combined giving rise to other points. So it looks like there is no end in the number of Triangle points.

In *Quadri Geometry* less points are described.
However there is the **DT-method** (Diagonal Triangle-method) for making 4L-points/4P-points from ETC-points (being 3L-points/3P-points):

> For Quadrilaterals (4-Line figure) as well as Quadrangles (4-Point figure) we have a Diagonal Triangle (resp. QL-Tr1 and QA-Tr1). These are Triangles and every ETC-point in these triangles become a Quadrilateral-/Quadrangle-Point in the system of the Reference Quadrilateral-/Quadrangle because their construction is strictly symmetric. Maybe they are not very interesting points because generally there are hardly relations with existing Quadrilateral-/Quadrangle-Points. However the principle counts.

Then there is also the **Ref/Per2-method** of making 4L-points from ETC-points (3L-points):

> Let Ref be a Reference Quadrilateral of lines L-1,L-2,L-3,L-4.
> Let P-i = ETC-point Px of triangle (Lj.Lk.Ll), where (i,j,k,l) are different numbers from (1,2,3,4).
> Let Lp-i be the perpendiculars from P-i on L-i (i=1,2,3,4).
> Now we have a 1st generation perpendicular quadrilateral Per1.
> By doing the same construction on Per1 (instead of on Ref) we get a 2nd generation perpendicular quadrilateral Per2.
> For several ETC-points it has been checked that all the time Ref is homothetic with Per2 (except for cases with extremities). Because the construction is strictly symmetric there will be for all these ETC-points Px a QL-homothetic center (HC) QL-Px.
> See QFG#1937,#1938.

This is not only true for Quadri Geometry but also for *Poly Geometry*.
There is also at least the **MVP-method** for making nL-points and nP-points from ETC-points (3L-points):

> Every Triangle Center can be transferred to a corresponding point in an n-Line by a simple recursive construction. The resulting point which will be called an nL-MVP Center, where MVP is the abbreviation for Mean Vector Point.
> *Definition:* A Mean Vector Point (MVP) is the mean of a bunch of vectors with identical origin. It is constructed by adding these vectors and then dividing the resultant vector by n. The Mean Vector Point is the endpoint of the divided resultant vector. In all n-Lines we can use any random point as origin. The endpoint of the resultant vector will be invariant for all different origins.
> When X(i) is a triangle Center we define the nL-MVP X(i)-Center as the Mean Vector Point of the n (n-1)L-MVP X(i)-Centers.
> When the (n-1)L-MVP X(i)-Centers aren't known they can be constructed from the MVP X(i)-Centers another level lower, according to the same definition. By applying this definition to an increasingly lower level finally the level is reached of the 3L-MVP X(i)-Center, which simply is the X(i) Triangle Center. Then it can be "rolled up" to the starting level.
> See QFG#869,#873,#878,#881.

Still it is too early to say that *there are more Poly-Figure-points then Triangle-points* because possibly there are general mechanisms creating ETC-points from nL- or nP-points.

n-Lines

nL-1: General recursive Objects in an n-Line

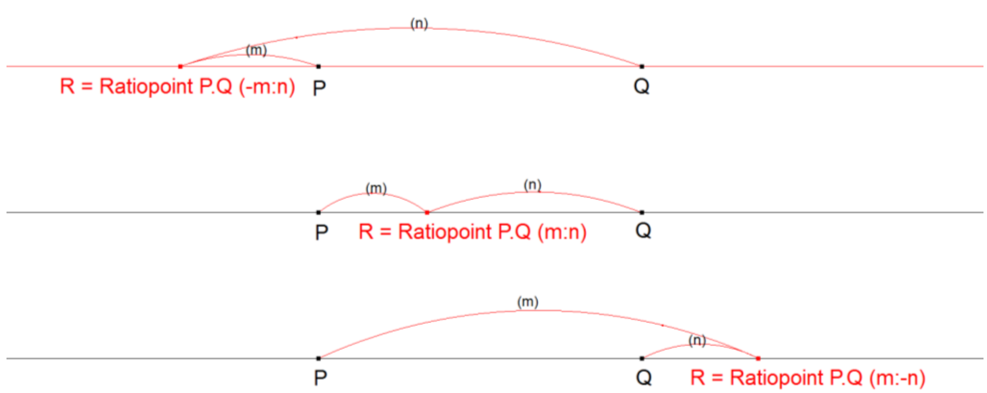
Many general objects in an n-Line are described by Prof. Frank Morley in the period 1886-1930. Morley's discoveries were all made purely by algebraic approaches. In his time it even wasn't possible to check his discoveries in drawings because of the complicated recursive character.
His documents are often hard to understand in detail. However involved documents were piece by piece "decoded" at the end of 2014 by Bernard Keizer, Eckart Schmidt and the author of this encyclopedia in a discussion at the Yahoo Quadri-Figures Group (Ref-34). See especially messages #826-#917. Accordingly they are mentioned in EPG and completed with drawings using Cabri or Mathematica software.
For quick insight pictures of n-Lines often are represented by figures bounded by n line-segments.

How many (n-1)-lines can be made up from an n-Line?

Many of the recursive constructions are based upon the property that from an n-Line exactly n different (n-1)-Lines can be made up. This can easily be deduced by omitting one line from the n-Line. This will leave behind an (n-1)-Line. Since exactly n Lines can be omitted there will be n different (n-1)-Lines contained in an n-Line. The (n-1)-Lines in an n-Line will be called **the Component (n-1)-Lines.** The remaining line after choosing an (n-1)-Line in an n-Line will be called **the omitted line**.
In descriptions we say "an n-Line contains n (n-1)-Lines" or "an n-Line has n Component (n-1)-Lines". When we want to indicate different objects occurring in (n-1)-Lines we say that there are n versions of these (n-1)L-objects.
The n versions of an object often will be noted with a suffix consisting of un underscore and a number 1, …, n, indicating the number of the omitted line. For example a 5-Line contains 5 4-Lines and therefore has 5 4L-MVP-Centroids (4L-n-P8). They will be noted as 4L-n-P8_1, 4L-n-P8_2, 4L-n-P8_3, 4L-n-P8_4 and 4L-n-P8_5. The suffix number at the end is the number of the omitted line.

Ratiopoint

A Ratiopoint R is a point collinear to two other given points X,Y and with the distances to these two other points in a given ratio.
This method is for example used for nL-n-P5, nL-n-P7, nL-n-pi.



Recursive Eulerline situated points in an n-Line

There are different ways of construction of Eulerline points to a higher n-Line level.
All methods are based upon the central property that from any n-Line n versions of (n-1)-Lines can be constructed.
Triangle points are X(2)=Centroid, X3=Circumcenter, X(4)=Orthocenter, X(5)=Nine-point Center.

Morley's points:

3L-point	*4L-point*	*5L-point*	*6L-point*	
X(2)	QL-P22	5L-n-P2	6L-n-P2	Etc.
X(3)	QL-P4	5L-n-P3	6L-n-P3	Etc.
X(4)	QL-P2	5L-n-P4	6L-n-P4	Etc.
X(5)	QL-P30	5L-n-P5	6L-n-P5	Etc.

Distance ratios for Morley's Eulerline points are not preserved at the different n-levels.

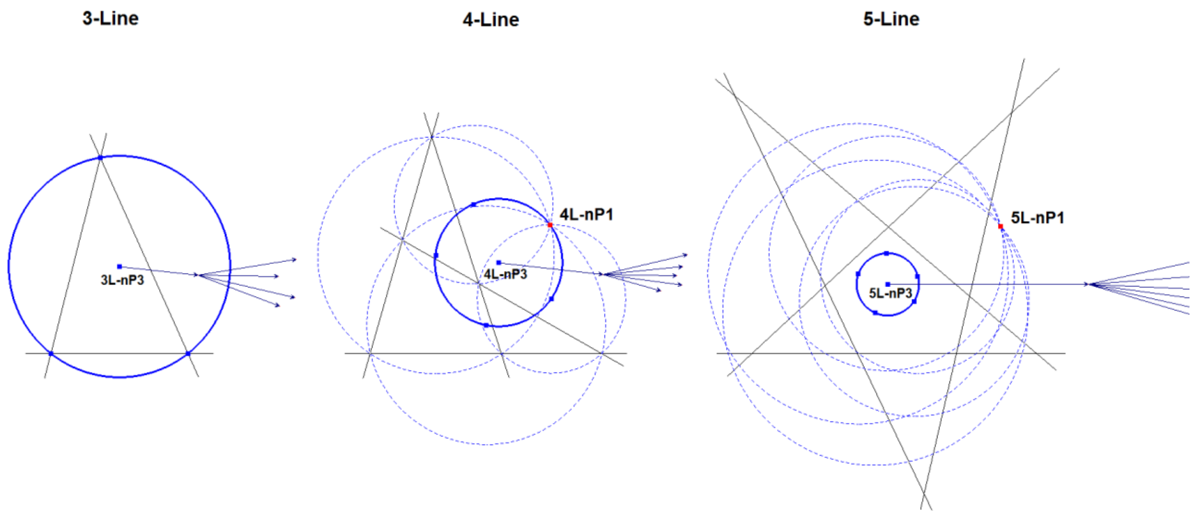
MVP Points: Multi Vector Points:

3L-point	*4L-point*	*5L-point*	*6L-point*	
X(2)	4L-n-P8 = QL-P12	5L-n-P8	6L-n-P8	Etc.
X(3)	4L-n-P9 = QL-P6	5L-n-P9	6L-n-P9	Etc.
X(4)	4L-n-P10 = QL-P2	5L-n-P10	6L-n-P10	Etc.
X(5)	4L-n-P11 = Midpoint (QL-P2,QL-P6)	5L-n-P11	6L-n-P11	Etc.

Distance ratios for MVP Eulerline points are preserved at the different n-levels.

nL-n-P1: nL-Centric Focus

A *Triangle (3-Line)* has a circumcircle. Morley in Ref-49 calls this circle a Centercircle.
In a *Quadrilateral (4-Line)* there are 4 Component 3-Lines whose 3L-Centercircle centers are concyclic on the 4L-Centercircle. Moreover the 4 3L-Centercircles have a common point, the 4L-Centric Focus.
In a *Pentalateral (5-Line)* there are 5 Component 4-Lines whose 4L-Centercircle centers are concyclic on the 5L-Centercircle. Again the 5 4L-Centercircles have a common point, the 5L-Centric Focus.
Etc.
Goormaghtigh (Ref-55) named this point the Centric Focus because in a 4-Line this point is the Focus of the inscribed parabola (QL-Co1). In a 4-Line it also is the node of Morley's Mono Cardioid (QL-Qu1).
Moreover nL-n-P1 is the node of the generalized Mono Cardioid also called the nL-Mono EnnaCardioid described by Morley in Ref-47. This nL-Mono EnnaCardioid is a curve circumscribing all (n-1)L-Centercircles. See nL-n-Cv1.



The 3 vertices of the 3-Line define the 3L-Centercircle.

The 4 centers of the 4 3L-Centercircles define the 4L-Centercircle.
The 4 3L-Centercircles are concurrent in the 4L-Centric Focus.

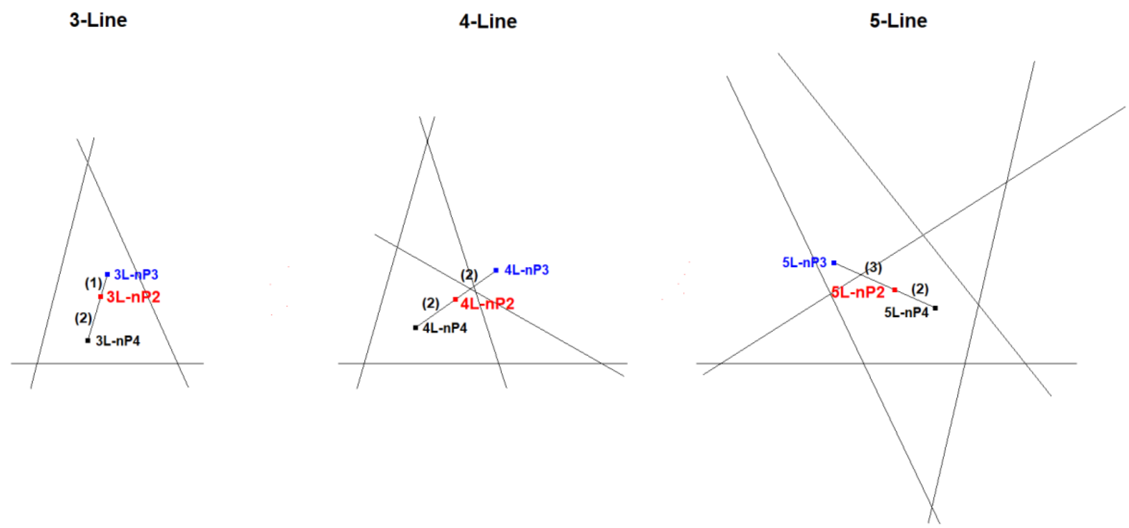
The 5 centers of the 5 4L-Centercircles define the 5L-Centercircle.
The 5 4L-Centercircles are concurrent in the 5L-Centric Focus.

Correspondence with ETC/EQF:
When n=4, then nL-n-P1 = QL-P1.

Properties:
- 5L-n-P1 is collinear with 5L-o-P2 and 5L-n-P3.
- 5L-n-P1 is a point on 5L-o-Ci1 and inversion of 5L-o-P2 wrt 5L-n-Ci1 (QFG#722, October 6, 2014, Eckart Schmidt).
- 5L-n-P1 is a node of the Mono EnnaCardioid nL-n-Cv1 circumscribing n (n-1)L-EnnaCardioids (n-1)L-n-iCv1. See Ref-37.

nL-n-P2: nL-Morley's Centroid

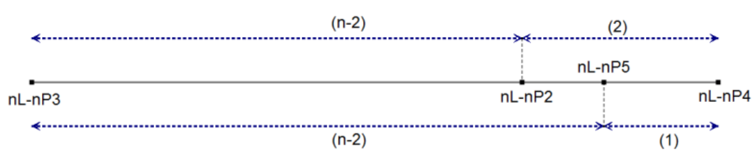
In accordance to the Encyclopedia of Triangle Centers (Ref-12) the 2nd center is a centroid.
Morley describes in Ref-49 an nL-Circumcenter (nL-n-P3), an nL-Orthocenter (nL-n-P4) and an nL-n-Nine-point Center (nL-n-P5) but he doesn't describe a Centroid of the n-Line.
Eckart Schmidt describes in Ref-34, QFG#880 an nL-Centroid related to Morley's nL-Circumcenter (nL-n-P3) and nL-Orthocenter (nL-n-P4): **nL-n-P2 = Ratiopoint nL-n-P3.nL-n-P4 (n-2 : 2)**. For explanation of Ratiopoint see nL-1.
This centroid is also the Homothetic Center of the Reference n-Line and the n-Line formed by the lines through nL-n-P5 parallel to Li. See Level-up Construction nL-n-Luc5a.
Because it is derived from Morley's Centers it is called Morley's Centroid.



3L-n-P2 = 3L-n-P3.3L-n-P4 (1 : 2) **4L-n-P2 = 4L-n-P3.4L-n-P4 (2 : 2)** **5L-n-P2 = 5L-n-P3.5L-n-P4 (2 : 3)**

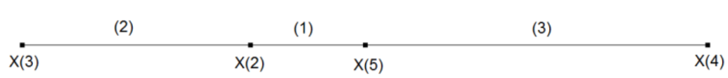
ALLOCATION OF POINTS ON THE EULERLINE

GENERAL n-Line



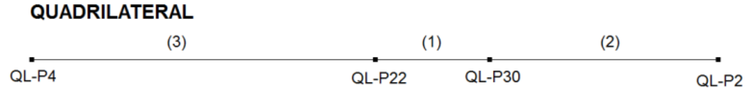
Situation in an n-Line:
nL-nP2 = nL-Morley Centroid
nL-nP3 = nL-Morley's Circumcenter
nL-nP4 = nL-Morley's 2nd Orthocenter
nL-nP5 = nL-Morley's 2nd Circlecenter

TRIANGLE



Situation in a TRIANGLE (n=3):
X(2) = Triangle Centroid
X(3) = Triangle Circumcenter
X(4) = Triangle Orthocenter
X(5) = Triangle Nine-point Center

QUADRILATERAL



Situation in a QUADRILATERAL (n=4):
QL-P2 = QL-Morley Point
QL-P4 = QL-Miquel Circumcenter
QL-P22 = QL-NPC Homothetic Center
QL-P30 = QL-Morley's Second Circle Center

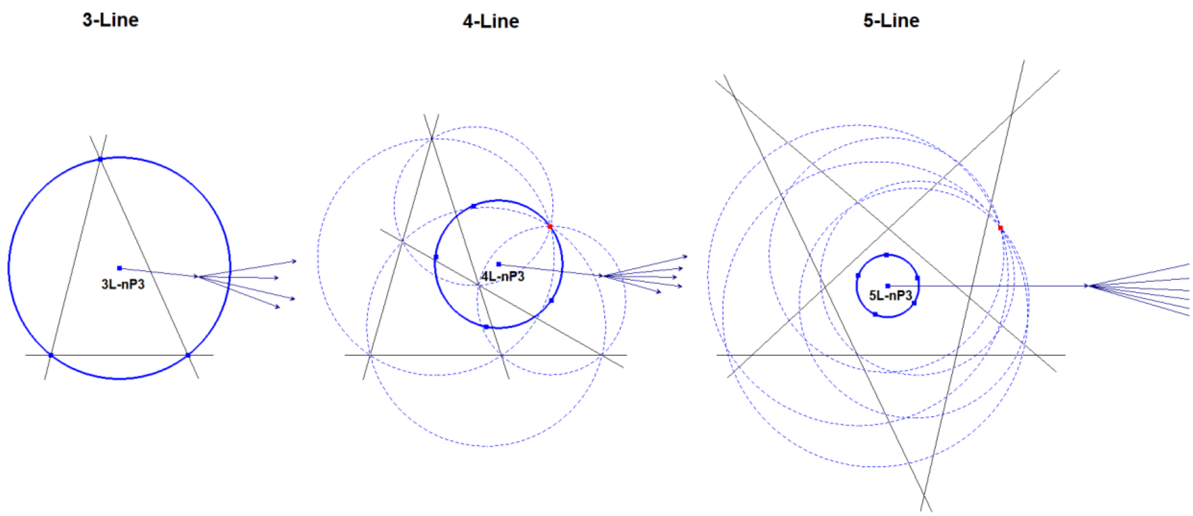
Correspondence with ETC/EQF:
When n=3, then nL-n-P2 = X(2).
When n=4, then nL-n-P2 = QL-P22.

Properties:

- nL-n-P2 is also the Homothetic Center of the Reference n-Line and the n-Line formed by the lines through the n (n-1)L-versions of nL-n-P2 parallel to the omitted Line.

nL-n-P3: nL-Morley's Circumcenter / Centric Center

A *Triangle (3-Line)* has a circumcircle. Morley in Ref-49 calls this circle a Centercircle.
In a *Quadrilateral (4-Line)* there are 4 component 3-Lines whose 3L-Centercircle Centers are concyclic on the 4L-Centercircle.
In a *Pentalateral (5-Line)* there are 5 component 4-Lines whose 4L-Centercircle Centers are concyclic on the 5L-Centercircle. Etc.
Morley proved in Ref-49 that there exists a Centercircle in an n-Line for all n, built from the centers of the Centercircles from the Component m-Lines.
The Center of this Centercircle is shortly named the Centric Center by Goormaghtigh in Ref-55.
This nL-Centric Center is the basis for several other Morley points.
Morley uses the letter "a_1" or "p_0" for this point in Ref-49.
Morley describes in Ref-49 some recursive points "p_i" for i=1, .. , n/2 (see nL-n-pi) that are useful for constructing some other points. In this notation nL-n-P3 = p_0 (or in EPG notation: nL-n-p0).



The 3 vertices of the 3-Line define the 3L-Centercircle.

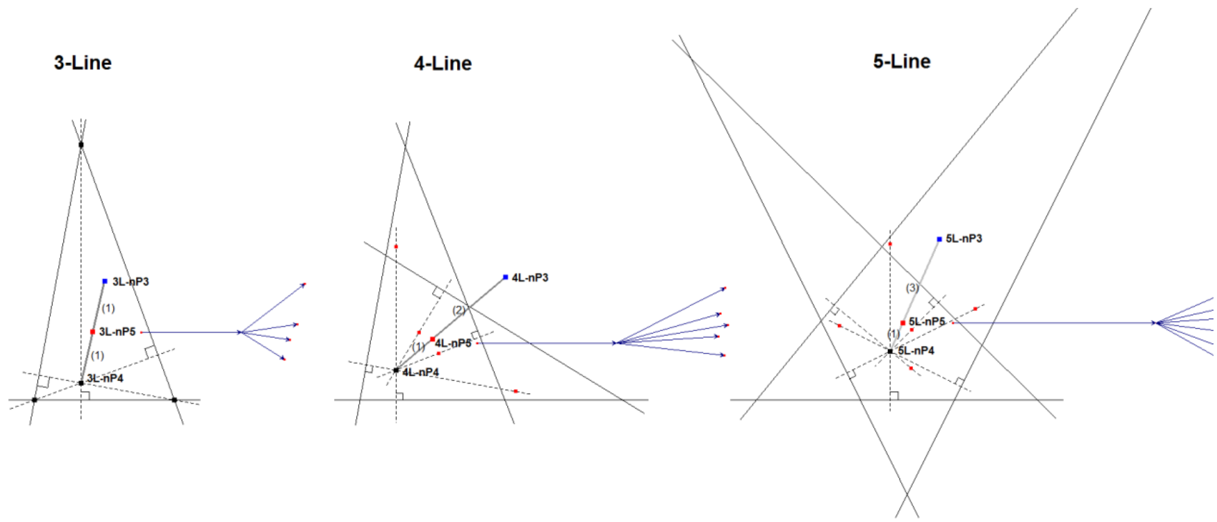
The 4 centers of the 4 3L-Centercircles define the 4L-Centercircle.

The 5 centers of the 5 4L-Centercircles define the 5L-Centercircle.

Correspondence with ETC/EQF:
When n=3, then nL-n-P3 = X(3).
When n=4, then nL-n-P3 = QL-P4.

nL-n-P4: nL-Morley's 2nd Orthocenter

Morley's 2nd Orthocenter is described by Morley as the common point of the perpendiculars (Level-up Construction nL-n-Luc1) from the n points nL-n-P5 of the Component m-Lines to the omitted line of the Reference n-Line.
Morley uses the letter "h" for this point in Ref-49.



The perpendiculars from the 3 vertices of the 3-Line coincide in 3L-nP4.

Define 3L-nP5 = 3L-nP4.3L-nP3 (1:1).

The perpendiculars from the 4 versions of 3L-nP5 to the omitted line coincide in 4L-nP4.

Define 4L-nP5 = 4L-nP4.4L-nP3 (1:2).

The perpendiculars from the 5 versions of 4L-nP5 to the omitted line coincide in 5L-nP4.

Define 5L-nP5 = 5L-nP4.5L-nP3 (1:3).

Correspondence with ETC/EQF:
When n=3, then nL-n-P4 = X(4).
When n=4, then nL-n-P4 = QL-P2.

Properties:

- nL-n-P4 is also the External Homothetic Center of nL-n-Ci1 & nL-n-Ci2. See Ref-49.

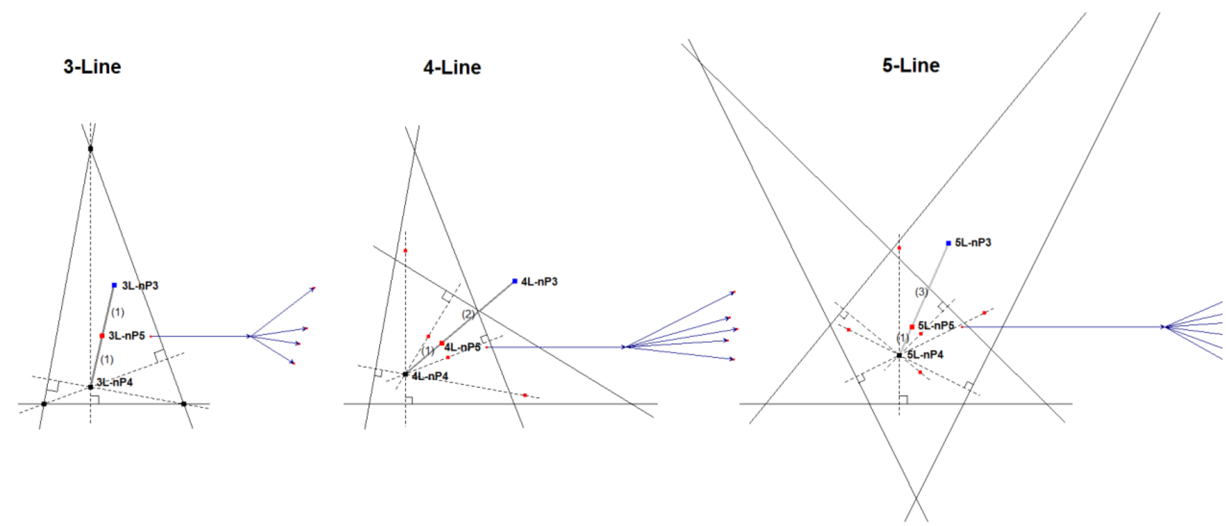
nL-n-P5: nL-Morley's 2nd Circle Center

Morley defines a 2nd circle with radius 1/(n-1) radius of the centric circle (1/2 for the triangle and 1/3 for the quadrilateral, etc.). The location of the center of this second circle is defined as the Ratiopoint nL-n-P4.nL-n-P3 (1 : n-1).

nL-n-P5 can be constructed in a recursive way:

- Construct the perpendiculars of the n versions of (n-1)L-n-P5 of the Component (n-1)-Lines to the omitted line. They will concur in nL-n-P4.
- Construct nL-n-P5 = nL-n-P4.nL-n-P3 (1 : n-1).

For the triangle, the 2nd circle is the Euler Circle (or Nine-point Circle or Feuerbach Circle) with center the Nine-point Center X(5) being 3L-n-P4.3L-n-P3 (1:1).
The 4 perpendiculars drawn from the 4 points of the Component 3-Lines to the 4th line concur in the 2nd Orthocenter of the 4-Line being 4L-n-P4. The center of the 2nd circle in the 4-Line will be at 1/3 on the segment 4L-n-P4.4L-n-P3 of the 4-Line.
The 5 perpendiculars drawn from the 5 points of the Component 4-Lines to the 5th line concur in the 2nd Orthocenter of the 5-Line being 5L-n-P4. The center of the 2nd circle in the 5-Line will be at 1/4 on the segment 5L-n-P4.5L-n-P3 of the 4-Line, etc.



The perpendiculars from the 3 vertices of the 3-Line coincide in 3L-nP4.

Define 3L-nP5 = 3L-nP4.3L-nP3 (1:1).

The perpendiculars from the 4 versions of 3L-nP5 to the omitted line coincide in 4L-nP4.

Define 4L-nP5 = 4L-nP4.4L-nP3 (1:2).

The perpendiculars from the 5 versions of 4L-nP5 to the omitted line coincide in 5L-nP4.

Define 5L-nP5 = 5L-nP4.5L-nP3 (1:3).

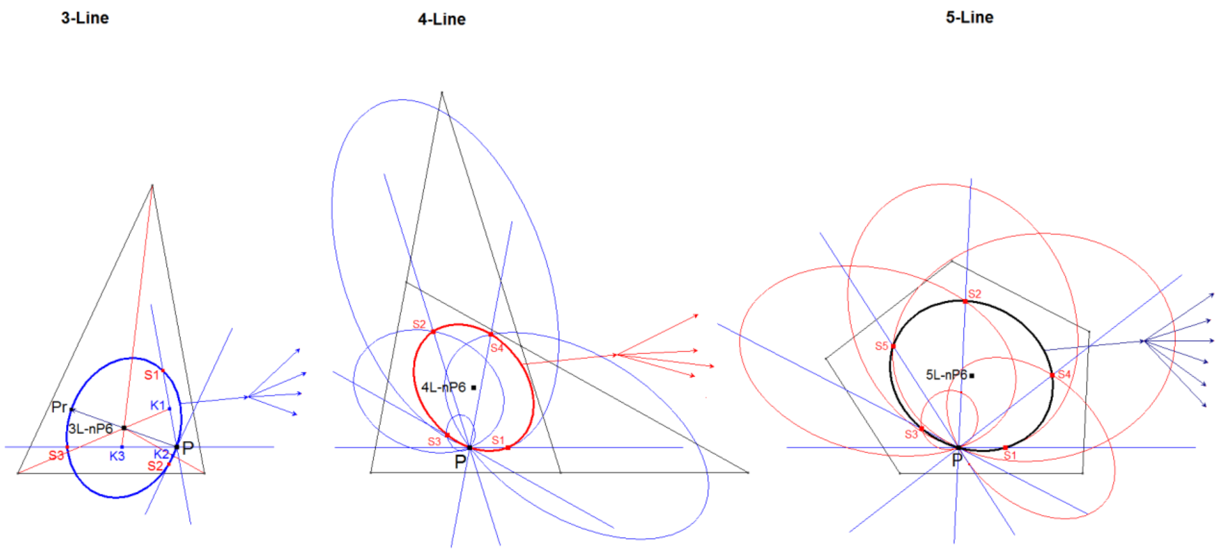
Correspondence with ETC/EQF:
When n=3, then nL-n-P5 = X(5).
When n=4, then nL-n-P5 = QL-P30.

Properties:

- nL-n-P5 is the center of Morley’s Second Circle QL-n-Ci2.

nL-n-P6: nL-Least Squared Distances Point

nL-n-P6 is the unique point in an n-Line with the Least Sum of Squared Distances to its n Lines.
nL-n-P6 can be constructed in a recursive way:



The nL-n-P6 point can be constructed because in an n-Line the points with an equal sum of squared distances lie on an ellipse. See Ref-34, QFG #1617, #1622.
This ellipse with a Fixed Sum of Squared Distances is called here an FSD-ellipse and when passing through P it is called the P-FSD-ellipse. The point with Least Sum of Squared Distances is the center of any FSD-ellipse.
A P-FSD-ellipse can be constructed by drawing lines in an n-Line through P parallel to the n Lines. Now on each of these parallel lines there will be a second point next to P with the same fixed sum of squared distances also lying on the P-FSD-ellipse. When we find 5 of these second points we have defined a conic and this conic should be the P-FSD-ellipse. Actually 4 points will be enough for construction because P is per definition also on the conic.
Per level a P-FSD-ellipse is constructed that will be transferred to the next level. The center of the P-FSD-ellipse is nL-n-P6 for that level.

Construction in a 3-Line (triangle):
This is the lowest level which differs from the general case.
1. Let P be some arbitrary point and K be the Symmedian Point X(6) in a triangle. X(6) is the nL-Least Squares Point of a triangle.
2. Draw lines Lp1,Lp2,Lp3 through P parallel to the sidelines L1,L2,L3,
3. Let K1,K2,K3 be the intersection points of Lp1,Lp2,Lp3 and the resp. symmedians through the triangle vertices L2^L3, L3^L1, L1^L2.
4. Let S1,S2,S3 be the reflections of P in K1,K2,K3.
5. Let Pr be the reflection of P in K.
6. The conic through P, Pr, S1, S2, S3 will be the P-FSD-ellipse.

In a similar way we can construct a P-FSD-ellipse in a 4-Line (quadrilateral).
1. Let L1,L2,L3,L4 be the 4 defining lines of the 4-Line.
2. Draw lines Lp1,Lp2,Lp3,Lp4 through arbitrary point P parallel to the sidelines L1,L2,L3,L4.

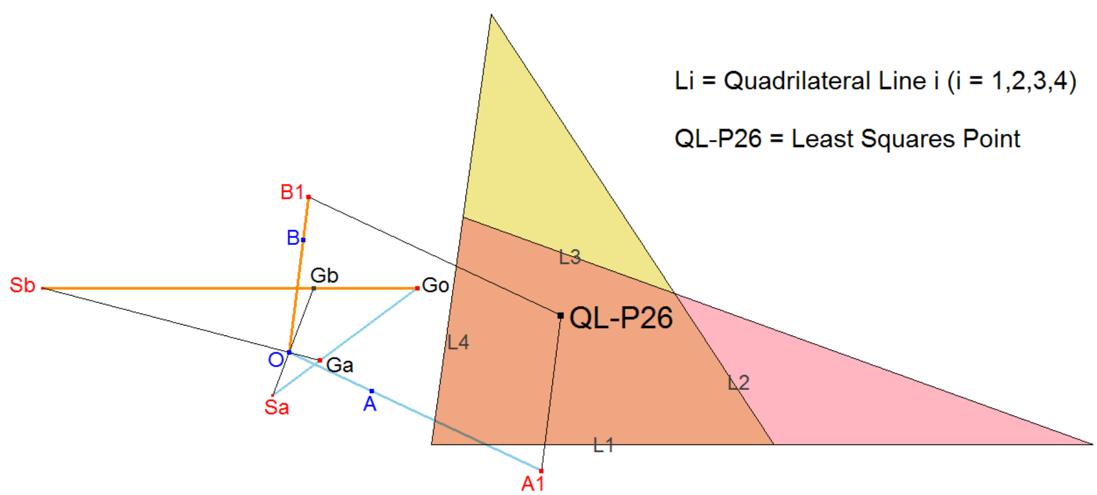
3. We are searching for the second point on Lp1 with same sum of squared distances to L1,L2,L3,L4 as P has. When we vary P on Lp1 at least the distance to L1 is fixed. So we have to find the point with fixed sum of squared distances to L2,L3,L4. This is the FSD-triangle problem like described here before. So construct the P-FSD-ellipse wrt triangle L2.L3.L4. Let S1 be the 2nd intersection point of this P-FSD-ellipse with Lp1. S1 has the same fixed sum of squared distances to L1,L2,L3,L4 as P.
4. Accordingly we can construct S2, S3, S4.
5. The conic through P, S1, S2, S3, S4 will be the P-FSD-ellipse in a 4-Line (quadrilateral).
6. The center of this ellipse is QL-P26 indeed. See EQF.

In a similar way we can construct a P-FSD-ellipse in a 5-Line (pentalateral).
1. Let L1,L2,L3,L4,L5 be the 5 defining lines of the 5-Line (pentalateral).
2. Draw lines Lp1,Lp2,Lp3,Lp4,Lp5 through arbitrary point P parallel to the sidelines L1,L2,L3,L4,L5.
3. We are searching for the second point on Lp1 with same sum of squared distances to L1,L2,L3,L4,L5 as P has. When we vary P on Lp1 at least the distance to L1 is fixed. So we have to find the point with fixed sum of squared distances to L2,L3,L4,L5. This is the FSD-triangle-problem for a 4-Line like described here before. So construct the P-FSD-ellipse wrt quadrilateral L2.L3.L4.L5. Let S1 be the 2nd intersection point of this P-FSD-ellipse with Lp1. S1 has the same fixed sum of squared distances to L1,L2,L3,L4,L5 as P.
4. Accordingly we can construct S2, S3, S4, S5.
5. The conic through P, S1, S2, S3, S4 will be the P-FSD-ellipse in a 4-Line (quadrilateral). It will appear that S5 is also on the conic.
6. The center of this ellipse will be the LSD-point of a 5-Line.

In a recursive way P-FSD-ellipses can be constructed in every n-Line (n>2) and the center of this P-FSD-ellipse will be the LSD-points of the n-Line.

Another Construction:

Coolidge describes in Ref-25 a general method for constructing this point in an n-Line.
In this picture an example is given in a 4-Line, where nL-n-P6 = QL-P26.



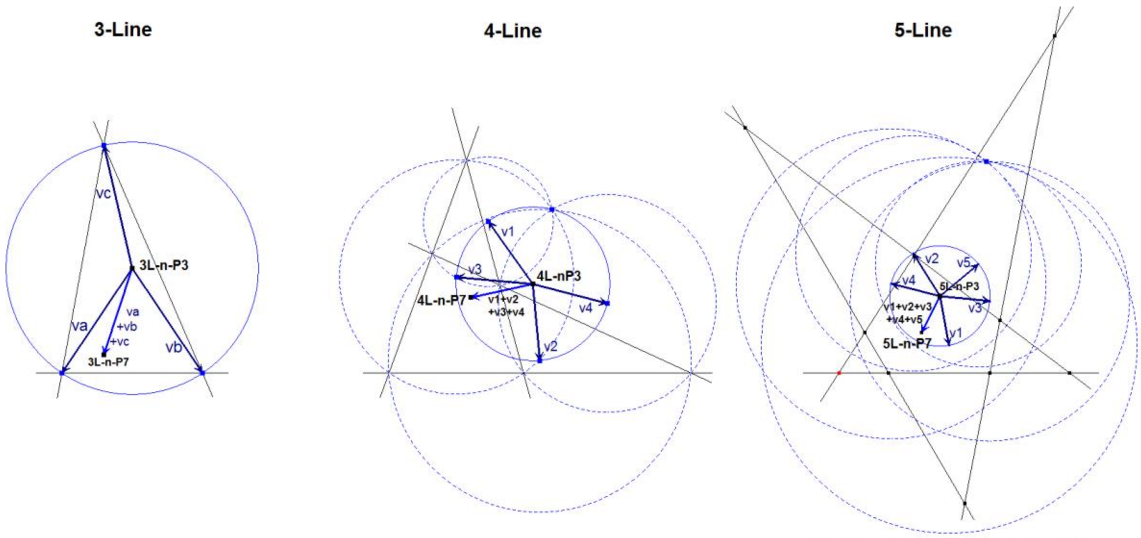
This construction is a modified version of the construction of Coolidge.

Let O (origin), A and B be *random* non-collinear points.
Go = Quadrangle Centroid of the projection points of O on the n basic lines of the Reference n-Line.
Ga = Quadrangle Centroid of the projection points of O on the n lines through point A parallel to the n basic lines of the Reference Quadrilateral.
Gb = Quadrangle Centroid of the projection points of O on the n lines through point B parallel to the n basic lines of the Reference Quadrilateral.
Let Sa = Ga.Go ^ O.Gb and Sb = Gb.Go ^ O.Ga.
Construct A1 on line O.A such that Sa.Ga : Ga.Go = O.A : A.A1.
Construct B1 on line O.B such that Sb.Gb : Gb.Go = O.B : B.B1.
Construct P such that O.A1.P.B1 is a parallelogram and where O and P are opposite vertices. P is the Least Squares Point nL-n-P6.

Correspondence with ETC/EQF:
When n=3, then nL-n-P6 = X(6).
When n=4, then nL-n-P6 = QL-P26.

nL-n-P7: nL-Hervey Point

The Hervey Point is defined by Morley as the endpoint of the sum of n vectors (Level-up Construction nL-n-Luc3) with common origin nL-n-P3 and endpoints the n (concyclic) lower level points (n-1)L-n-P3.
Goormaghtigh describes this n-Line point in Ref-55 referring to Morley's document Ref-49 and calls it the Hervey Point, because in a Quadrilateral this point coincides with a point earlier described by Hervey (QL-P3).
Morley describes in Ref-49 some recursive intermediate points "p_i" for i=0, .. , n/2 (see nL-n-pi) that are useful for constructing other points. In this notation nL-n-P7 = p_1 (or in EPG notation: nL-n-p1).



The 3 vertices of the 3-Line define the 3L-Centercircle.
3L-n-P7 is the endpoint of the sum of the 3 corresponding vectors in that circle.

The 4 centers of the 4 3L-Centercircles define the 4L-Centercircle.
4L-n-P7 is the endpoint of the sum of the 4 corresponding vectors in that circle.

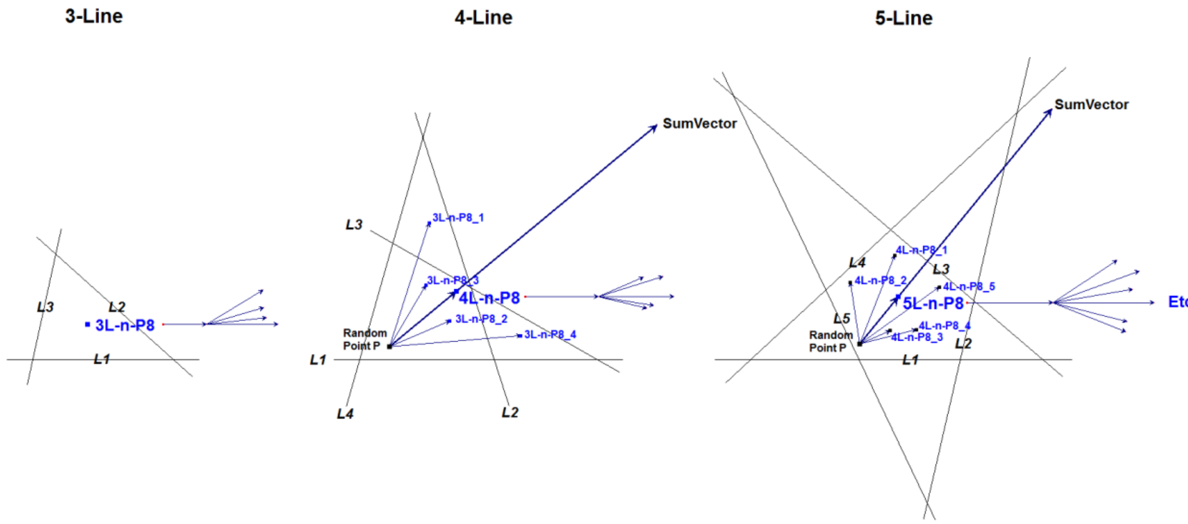
The 5 centers of the 5 4L-Centercircles define the 5L-Centercircle.
5L-n-P7 is the endpoint of the sum of the 5 corresponding vectors in that circle.

Correspondence with ETC/EQF:
When n=3, then nL-n-P6 = X(4).
When n=4, then nL-n-P6 = QL-P3.

Properties:
- The ratio d(nL-n-P7, (n-1)L-n-P3) : d(nL-n-P7, (n-1)L-n-P7) is fixed for all n Component m-Lines, where m=(n-1). According to Goormaghtigh (see Ref-55) this ratio = Radius-of-Centercircle : d(nL-n-P1,nL-n-P3).
- In a 5-Line construct the 5 versions of QL-L4 (Morley Line) in each Component 4-Line. They form the 1st generation Morley 5-Line. Do the same process in the 1^{st} generation Morley 5-Line. This leads to the 2^{nd} generation Morley 5-Line consisting of 5 lines concurring in 5L-n-P7. This point is also the common point QL-P3 of all Component 4-Lines of the 1^{st} generation Morley 5-Line. See Ref-34, QFG # 826.
- In a 6-Line the six 5L-versions of 5L-n-P7 are coconic.

nL-n-P8: nL-MVP Centroid

nL-n-P8 is the nL-Mean Vector Point (see nL-n-Luc4) of X(2), the Triangle Centroid.

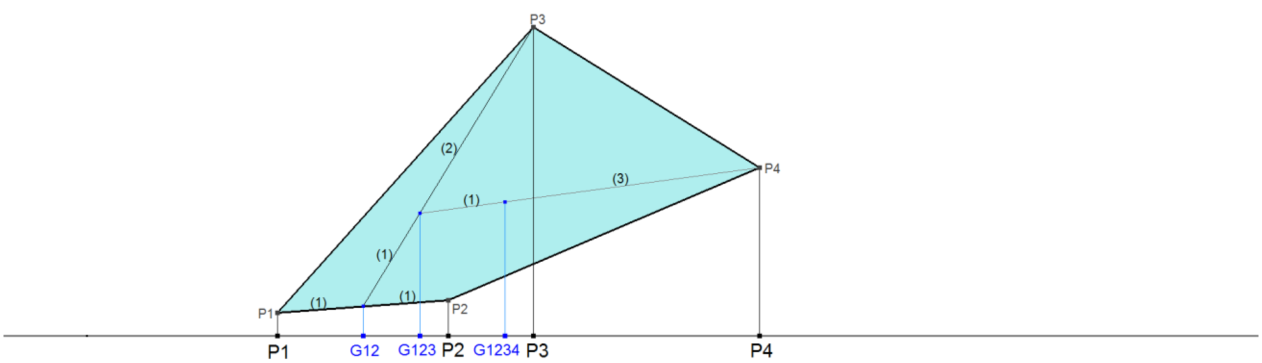


3L-n-P8 is the 3L-Centroid.

P is a random point and the origin of the vectors with endpoints the 4 3L-Centroids.
4L-n-P8 is the endpoint of the SumVector / 4.
4L-n-P8 is independent of the location of P.

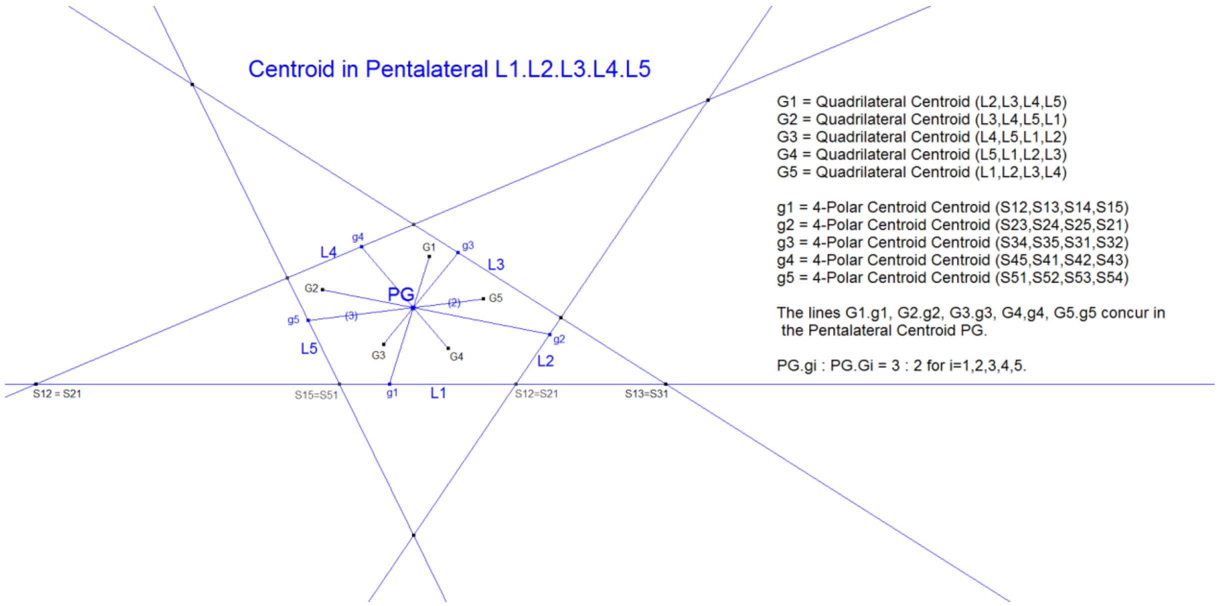
P is a random point and the origin of the vectors with endpoints the 5 4L-MVP-Centroids.
5L-n-P8 is the endpoint of the SumVector / 5.
5L-n-P8 is independent of the location of P.

Another construction of nL-MVP Centroid:
The 4L-MVP-Centroid QL-P12 is constructed from 3L-MVP-Centroid X(2) using 4-polar Centroids.
The 5L-MVP-Centroid can be generated from 4L-MVP-Centroid QL-P12 in a similar way using 5-polar centroids.
Each line of the 5 lines in a 5-Line has 4 intersection points with the 4 other lines.
These 4 collinear points can be seen as a "flat" quadrangle and have a corresponding Quadrangle Centroid also called here a 4-polar Centroid.



4-polar centroid of 4 collinear points P1,P2,P3,P4
determine point G12 on P1,P2 (1:1) = midpoint (P1,P2)
determine point G123 on G12,P3 (1:2) = tripolar centroid (P1,P2,P3)
determine point G1234 on G123,P4 (1:3) = 4-polar centroid (P1,P2,P3,P4)

In a 5-Line there are 5 Component Quadrilaterals. The lines connecting the QL-Centroids of these Component Quadrilaterals with their corresponding 4-polar centroids concur in one point being 5L-n-P8.



In the same way the 6L-MVP-Centroid also can be generated from the 5L-MVP-Centroid, etc.

Correspondence with ETC/EQF:

In a 3-Line:

- 3L-n-P8 = 3L-MVP Centroid = X(2)
- 3L-n-P9 = 3L-MVP Circumcenter = X(3)
- 3L-n-P10 = 3L-MVP Orthocenter = X(4)
- 3L-n-P11 = 3L-MVP Nine-point center = X(5)

In a 4-Line we find:

- 4L-n-P8 = 4L-MVP Centroid = QL-P12 (4L-Centroid)
- 4L-n-P9 = 4L-MVP Circumcenter = QL-P6 (Dimidium Point)
- 4L-n-P10 = 4L-MVP Orthocenter = QL-P2 (Morley Point)
- 4L-n-P11 = 4L-MVP Nine-point center = Midpoint (QL-P2,QL-P6)

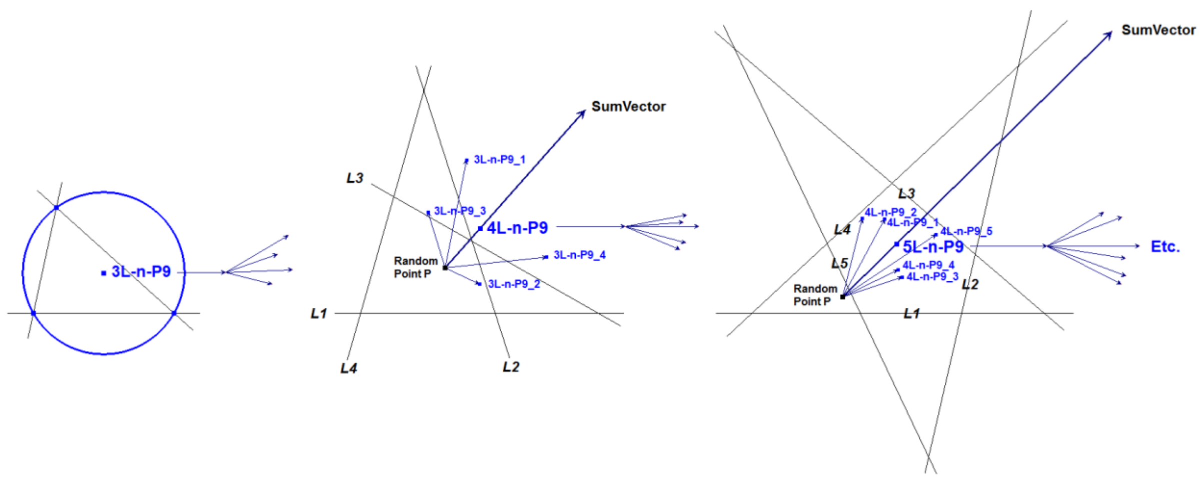
Properties:

- nL-n-P8, nL-n-P9, nL-n-P10 and nL-n-P11 are collinear. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).
- nL-n-P8 is also the Homothetic Center of the Reference n-Line and the n-Line formed by the lines through the n (n-1)L-versions of nL-n-P8 parallel to the omitted Line. So starting with X(2) in a triangle it can be gradually constructed up to all higher n-levels in this way.

nL-n-P9: nL-MVP Circumcenter

nL-n-P9 is the nL-Mean Vector Point (see nL-n-Luc4) of X(3), the Triangle Circumcenter.

3-Line | 4-Line | 5-Line



3L-n-P9 is the 3L-Circumcenter.

P is a random point and the origin of the vectors with endpoints the 4 3L-Circumcenters.
4L-n-P9 is the endpoint of the SumVector / 4.
4L-n-P9 is independent of the location of P.

P is a random point and the origin of the vectors with endpoints the 5 4L-MVP-Circumcenters.
5L-n-P9 is the endpoint of the SumVector / 5.
5L-n-P9 is independent of the location of P.

Correspondence with ETC/EQF:

In a 3-Line:

3L-n-P8 = 3L-MVP Centroid = X(2)
3L-n-P9 = 3L-MVP Circumcenter = X(3)
3L-n-P10 = 3L-MVP Orthocenter = X(4)
3L-n-P11 = 3L-MVP Nine-point center = X(5)

In a 4-Line we find:

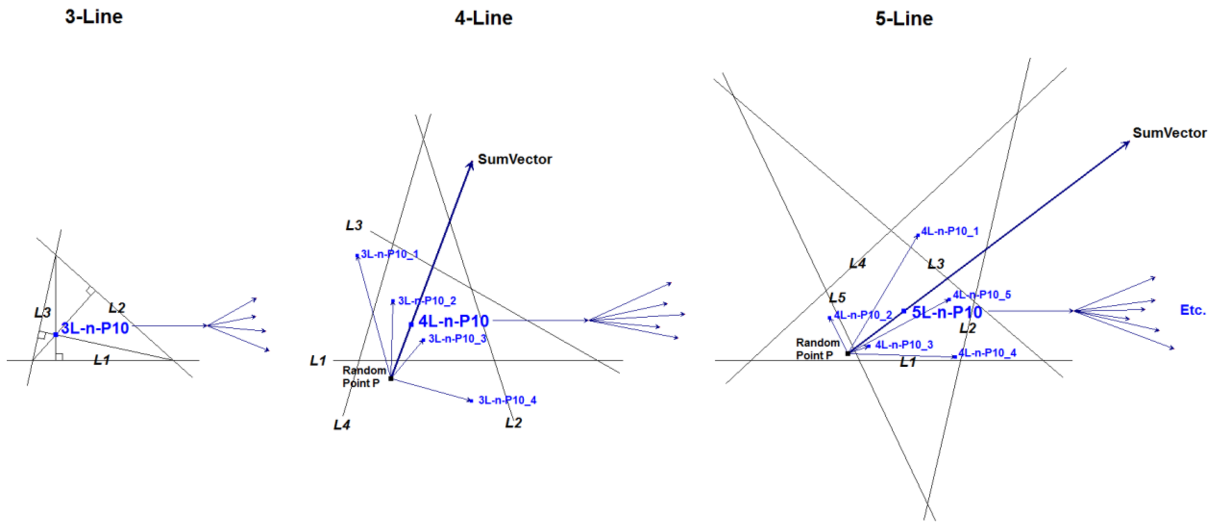
4L-n-P8 = 4L-MVP Centroid = QL-P12 (4L-Centroid)
4L-n-P9 = 4L-MVP Circumcenter = QL-P6 (Dimidium Point)
4L-n-P10 = 4L-MVP Orthocenter = QL-P2 (Morley Point)
4L-n-P11 = 4L-MVP Nine-point center = Midpoint (QL-P2,QL-P6)

Properties:

- nL-n-P8, nL-n-P9, nL-n-P10 and nL-n-P11 are collinear. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nL-n-P10: nL-MVP Orthocenter

nL-n-P10 is the nL-Mean Vector Point (see nL-n-Luc4) of X(4), the Triangle Orthocenter.



3L-n-P10 is the 3L-Orthocenter.

P is a random point and the origin of the vectors with endpoints the 4 3L-Orthocenters.
4L-n-P10 is the endpoint of the SumVector / 4.
4L-n-P10 is independent of the location of P.

P is a random point and the origin of the vectors with endpoints the 5 4L-MVP-Othocenters.
5L-n-P10 is the endpoint of the SumVector / 5.
5L-n-P10 is independent of the location of P.

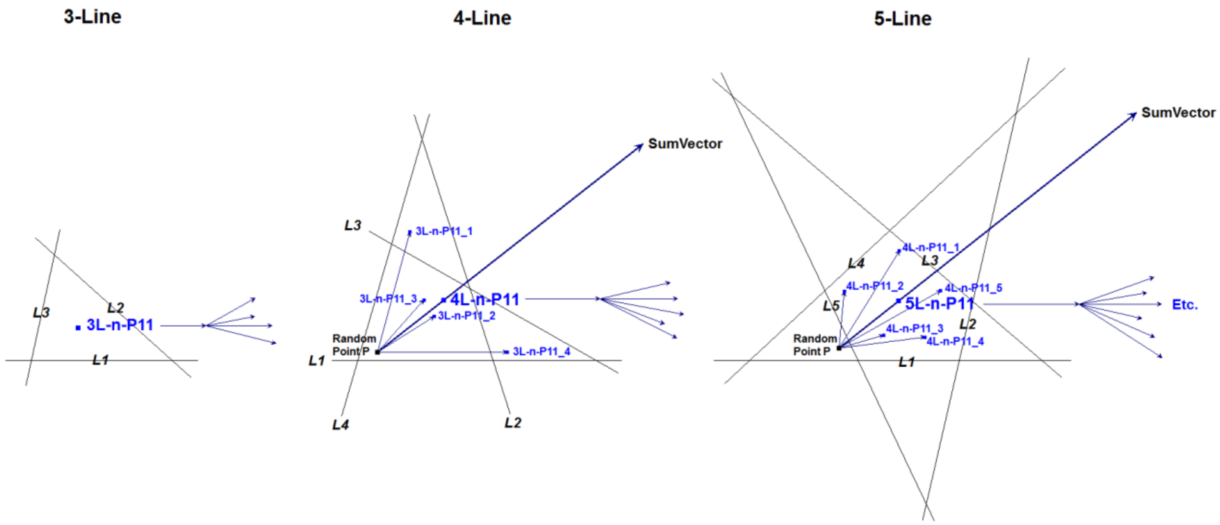
Correspondence with ETC/EQF:
In a 3-Line:
- 3L-n-P8 = 3L-MVP Centroid = X(2)
- 3L-n-P9 = 3L-MVP Circumcenter = X(3)
- 3L-n-P10 = 3L-MVP Orthocenter = X(4)
- 3L-n-P11 = 3L-MVP Nine-point center = X(5)

In a 4-Line we find:
- 4L-n-P8 = 4L-MVP Centroid = QL-P12 (4L-Centroid)
- 4L-n-P9 = 4L-MVP Circumcenter = QL-P6 (Dimidium Point)
- 4L-n-P10 = 4L-MVP Orthocenter = QL-P2 (Morley Point)
- 4L-n-P11 = 4L-MVP Nine-point center = Midpoint (QL-P2,QL-P6)

Properties:
- nL-n-P8, nL-n-P9, nL-n-P10 and nL-n-P11 are collinear. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nL-n-P11: nL-MVP Nine-point Center

nL-n-P11 is the nL-Mean Vector Point (see nL-n-Luc4) of X(5), the Triangle Nine-point Center.



3L-n-P11 is the 3L-Nine-point Center.

P is a random point and the origin of the vectors with endpoints the 4 3L-Nine-point Centers.
4L-n-P11 is the endpoint of the SumVector / 4.
4L-n-P11 is independent of the location of P.

P is a random point and the origin of the vectors with endpoints the 5 4L-Nine-point Centers.
5L-n-P11 is the endpoint of the SumVector / 5.
5L-n-P11 is independent of the location of P.

Correspondence with ETC/EQF:
In a 3-Line:

3L-n-P8 = 3L-MVP Centroid = X(2)
3L-n-P9 = 3L-MVP Circumcenter = X(3)
3L-n-P10 = 3L-MVP Orthocenter = X(4)
3L-n-P11 = 3L-MVP Nine-point center = X(5)

In a 4-Line we find:

4L-n-P8 = 4L-MVP Centroid = QL-P12 (4L-Centroid)
4L-n-P9 = 4L-MVP Circumcenter = QL-P6 (Dimidium Point)
4L-n-P10 = 4L-MVP Orthocenter = QL-P2 (Morley Point)
4L-n-P11 = 4L-MVP Nine-point center = Midpoint (QL-P2,QL-P6)

Properties:

- nL-n-P8, nL-n-P9, nL-n-P10 and nL-n-P11 are collinear. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nL-n-P12: nL-QL-P4 Par1/Par2-Homothetic Center

nL-n-P12 is the Par1/Par2-Homothetic Center (nL-n-Luc5e) of **mL**-n-P12, where m=(n-1).
This recursive construction can be rolled up to increasing larger values of n.
Starting value for n is 4, where 4L-n-P12=QL-P4.

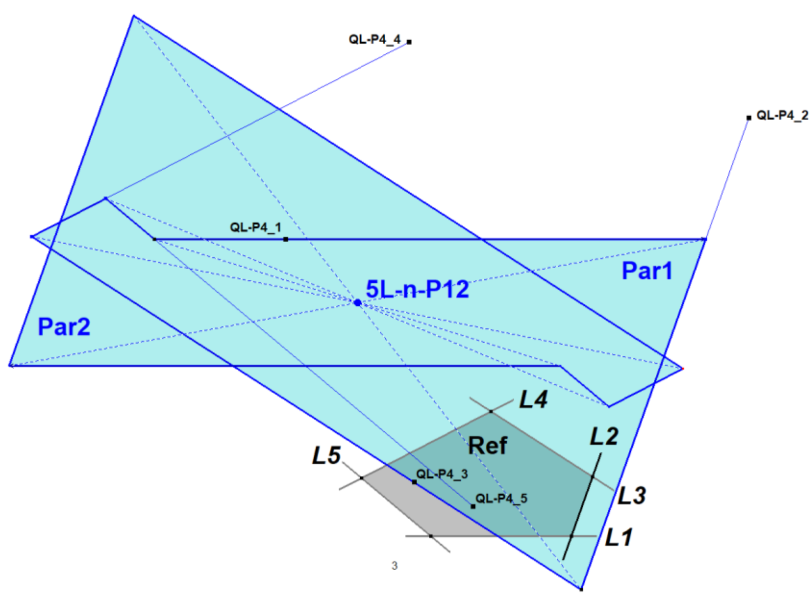
General remarks
With reservations nL-n-P12, nL-n-P13, nL-n-P14 are mentioned as n-points in the neos-system (see nL-1). There are no contra-indications and all Mathematica calculations with numeric examples until reasonable depth confirm they are recursive n-points indeed, however there is no synthetic proof of it yet. So they are mentioned as n-points because their existence is a fact (to a certain n-level) and therefore they deserve registration.
nL-n-P12, nL-n-P13, nL-n-P14 have starting points at n=4, where they represent QL-P4, QL-P28, QL-P29, which are centers related to Hofstadter points, resp. H(2)=X(3), H(3)=X(186), H(-2)=X(265).
Probably there will be corresponding nL-n-points for centers related to other Hofstadter points H(i).

Construction
The construction of nL-n-P12 is as follows.

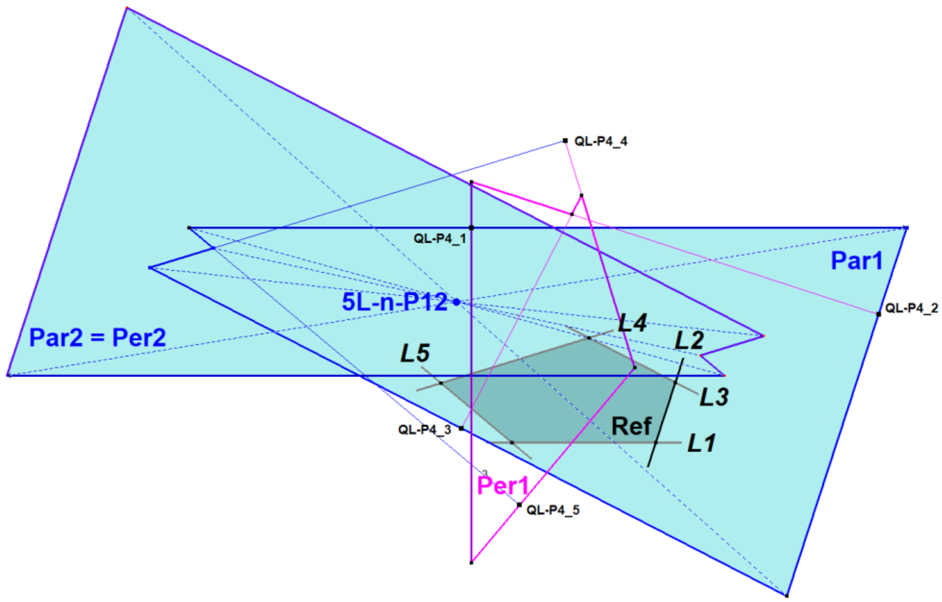
1. Every n-Line contains n (n-1)-Lines leaving one line being called the omitted line.
2. Construct lines through the n versions of (n-1)L-n-P12 parallel to the omitted line.
3. These n lines form a new n-Line being called Par1.
4. In the same way a 2nd generation n-Line Par2 can be constructed.
5. Par1 is homothetic to Par2 and so there will be a Homothetic Center, which is nL-n-P12.
6. This recursive construction can be rolled up to increasing larger values of n. Starting value for n is 4, where 4L-n-P12=QL-P4.

In a corresponding way nL-n-P13 and nL-n-P14 are constructed.

Example in a 5-Line



5L-n-P12 is *also* the Par1/**Per2**-Homothetic Center (see nL-n-Luc5g) of QL-P4 wrt the Reference 5-Line, because Par2 coincides with Per2.



Correspondence with ETC/EQF:
In a 3-Line:
Any 3L-Par1/Par2-predecessor ?
In a 4-Line:
4L-n-P12 = QL-P4
4L-n-P13 = QL-P28
4L-n-P14 = QL-P29

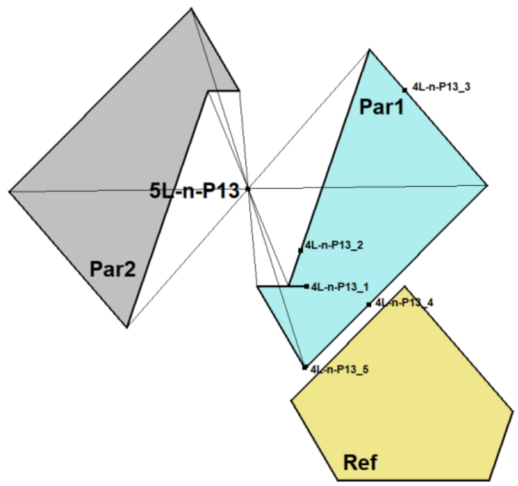
Properties:

- In a 5-Line 5L-n-P12 = Midpoint (5L-n-P7.5L-n-P3).
- In a 5-Line 5L-n-P14 = 5L-n-P7. 5L-n-P5 (2:-1)
- It looks like that for all n Par2 will coincide with Per2.
- It looks like that for all n the lengths of the line segments of Par1 are equal to the corresponding line segments of Par2 as well as Per2.

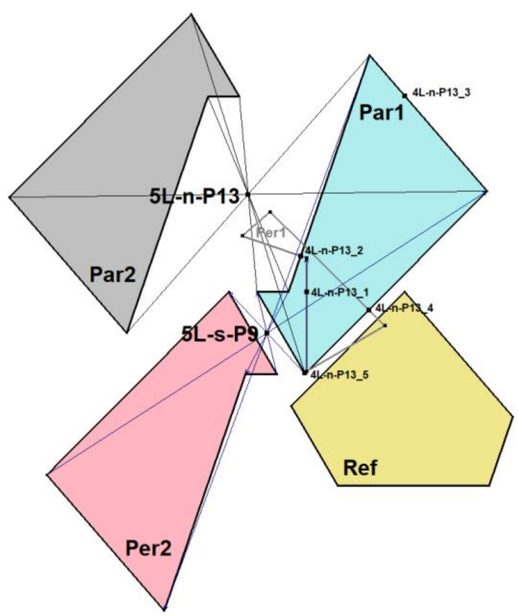
nL-n-P13: nL-QL-P28 Par1/Par2-Homothetic Center

nL-n-P13 is the Par1/Par2-Homothetic Center (nL-n-Luc5e) of **mL**-n-P13, where m=(n-1).
This recursive construction can be rolled up to increasing larger values of n.
Starting value for n is 4, where 4L-n-P13=QL-P28.
See also general remarks and construction at nL-n-P12.

Example of 5L-n-P13:



Example of 5L-n-P13 in relationship to 5L-s-P9:



Correspondence with ETC/EQF:
In a 3-Line:

Any 3L-Par1/Par2-predecessor ?

In a 4-Line:

4L-n-P13 = QL-P28

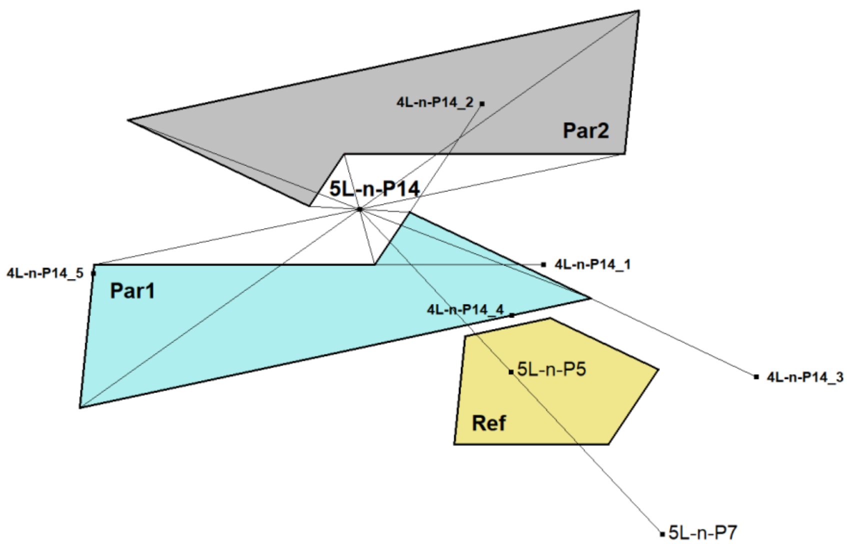
Properties:

- It looks like that for all n the lengths of the line segments of Par1 are equal to the corresponding line segments of Par2 as well as Per2.

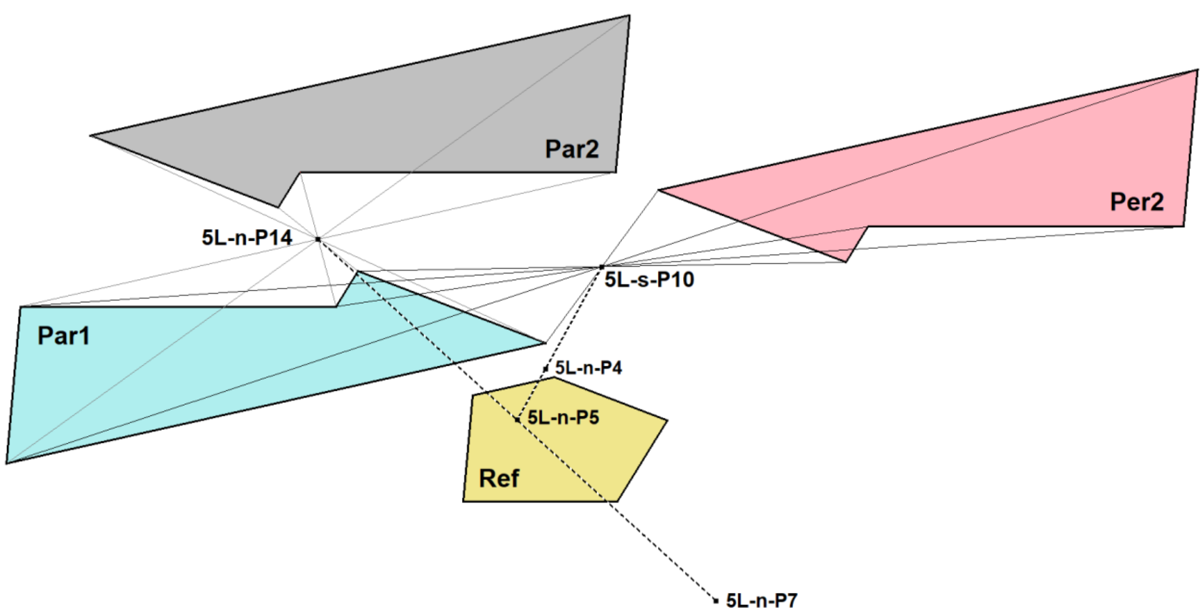
nL-n-P14: nL-QL-P29 Par1/Par2-Homothetic Center

nL-n-P14 is the Par1/Par2-Homothetic Center (nL-n-Luc5e) of **mL**-n-P14, where m=(n-1).
This recursive construction can be rolled up to increasing larger values of n.
Starting value for n is 4, where 4L-n-P14=QL-P29.
See also general remarks and construction at nL-n-P12.

Example of 5L-n-P14:



Example of 5L-n-P14 in relationship to 5L-s-P10:



Note: the Homothetic Center of Par2 and Per2 is the InfinityPoint of 5L-n-P14.5L-s-P10.

Correspondence with ETC/EQF:
In a 3-Line:
Any 3L-Par1/Par2-predecessor ?
In a 4-Line:
4L-n-P14 = QL-P29

Properties:

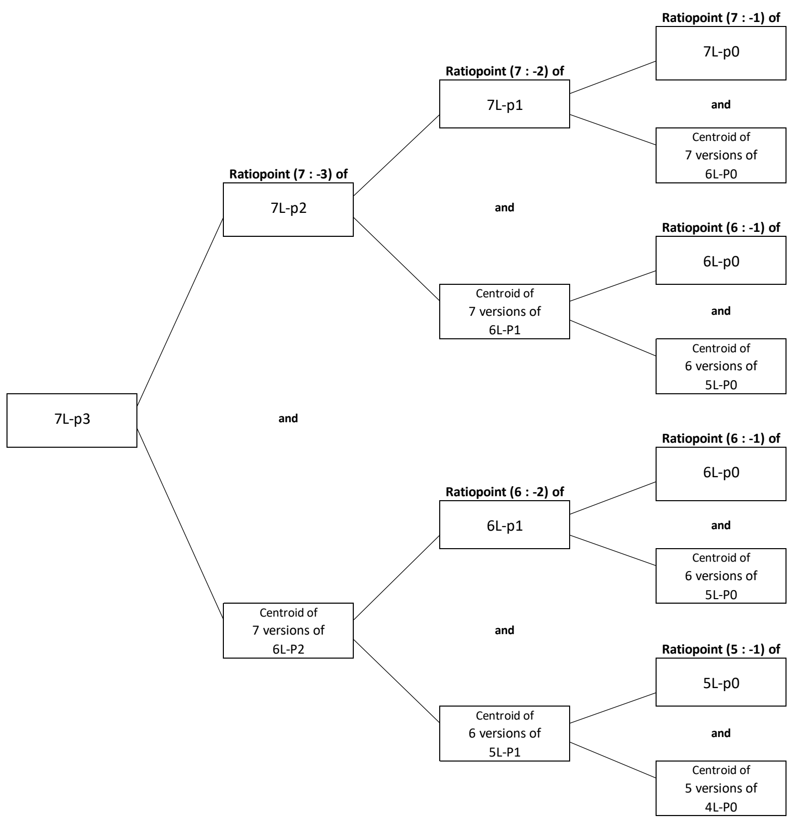
- In a 5-Line 5L-n-P14 = 5L-n-P7. 5L-n-P5 (2:-1)
- It looks like that for all n the lengths of the line segments of Par1 are equal to the corresponding line segments of Par2 as well as Per2.

nL-n-pi: nL-Morley's intermediate recursive pi points

nL-n-pi is a point defined by Morley in Ref-49.
Note that the letter “p” is in lower case. In Morley’s document it is denoted as “p_i”.
Since it can occur in all n-Lines for n>1 it is named here nL-n-pi.
The existence of nL-n-pi as well as nL-n-gi are purely algebraically indicated by Morley.
Morley uses it as intermediate point(s) to make it possible to construct his so called first Orthocenter (nL-o-P1) as well as his so called Ortho Directrix (nL-e-L1).
nL-n-pi (i = 1, .. ,(n-1)/2) is defined in a recursive way:
nL-n-pi = Ratiopoint of

    nL-n-p(i-1) and
    nL-n-g(i-1) being the centroid of n points (n-1)L-n-p(i-1)
    with ratio n : (i-n).

By applying this formula to an increasingly lower level finally the level is reached unto some nL-n-p0, and nL-n-p0 is defined as follows:
nL-n-p0 = Center of the nL-Centric Circle of an n-Line, also called the nL-Centric Center being nL-n-P3.
The nL-Centric Center now in turn is defined as the center of the circle through the n (n-1)L-Centric Centers.
By applying this definition to an increasingly lower level finally the level is reached of the 3L-Centric Center, which simply is the center of the circumcircle of a triangle.
This is the only known item and can be rolled up (wherever it appears) to the required point nL-n-pi. Basically nL-n-pi is built from large quantities of triangle circumcenters. These triangles being formed by the different combinations of the basic lines of the n-Line.
Example:



See nL-o-P1 (Morley's first Orthocenter) for examples of the use of nL-n-p1.
The first two points for i=0,1 are well known points:
nL-n-p0 = nL-n-P3 = nL-Center Circle Center
nL-n-p1 = nL-n-P7 = nL-Hervey Point

Behavior nL-n-pi points for different values of i

These properties are valid for Morley's nL-pi points **when n = odd** (denote mL = (n-1)L):

- nL-n-pi has a fixed distance ratio with the n versions of mL-n-pi & mL-n-p(i-1), when i=0, ... , (n+1)/2, but not for higher values.
- For n > 5 there are two orthogonal reflective axes for i=(n-1)/2, bisecting angles with lower level mL-n-pi.X.mL-n-p(i-1), where X=nL-o-P1 and m=(n-1). See : nL-o-2L1.
- nL-n-pi for i = 0,1,2,3, ... ,n culminates in a point nL-n-pn, which will be the same point as the centroid of the n lower level points mL-n-p(n-1). That is because, when i=n then nL-g(n-1) =nL-n-pn (Ratiopoint (n:0)).
- When i > n the outcome will produce the same point nL-n-pn because the end of iteration has been reached.

These properties are valid for nL-pi points **when n = even** (denote mL = (n-1)L):

- nL-n-pi has a fixed distance ratio with the n versions of mL-n-pi & mL-n-p(i-1), when i=0, ... , n/2, but not for higher values. This distance ratio =1, when i = n/2 - 1. As a consequence nL-n-p(n/2-1) will be the common point of the perpendicular bisectors of the n pairs mL-n-p(n/2-1), mL-n-p(n/2-2).
- When n=even there are no orthogonal reflective axes at any nL-pi bisecting angles with lower level mL-n-pi and mL-n-p(i-1).
- nL-n-pi for i = 0,1,2,3, ... ,n culminates in a point nL-n-pn, which will be the same point as the centroid of the lower level points mL-n-p(n-1). That is because, when i=n then nL-g(n-1) =nL-n-pn (Ratiopoint (n:0)).
- When i > n the outcome will produce the same point nL-n-pn because the end of iteration has been reached.

In general:	nL-n-p0	= nL-n-P3 = nL-Center Circle Center
	nL-n-p1	= nL-n-P7 = nL-Hervey Point
for n=odd:	nL-n-p((n-1)/2)	= nL-o-P1 = nL-Morley's 1st Orthocenter
for n=even:	nL-n-p((n/2)-1)	= nL-e-P1 = nL-Morley's EnnaDeltoid Center

See also the notes at nL-n-gi.

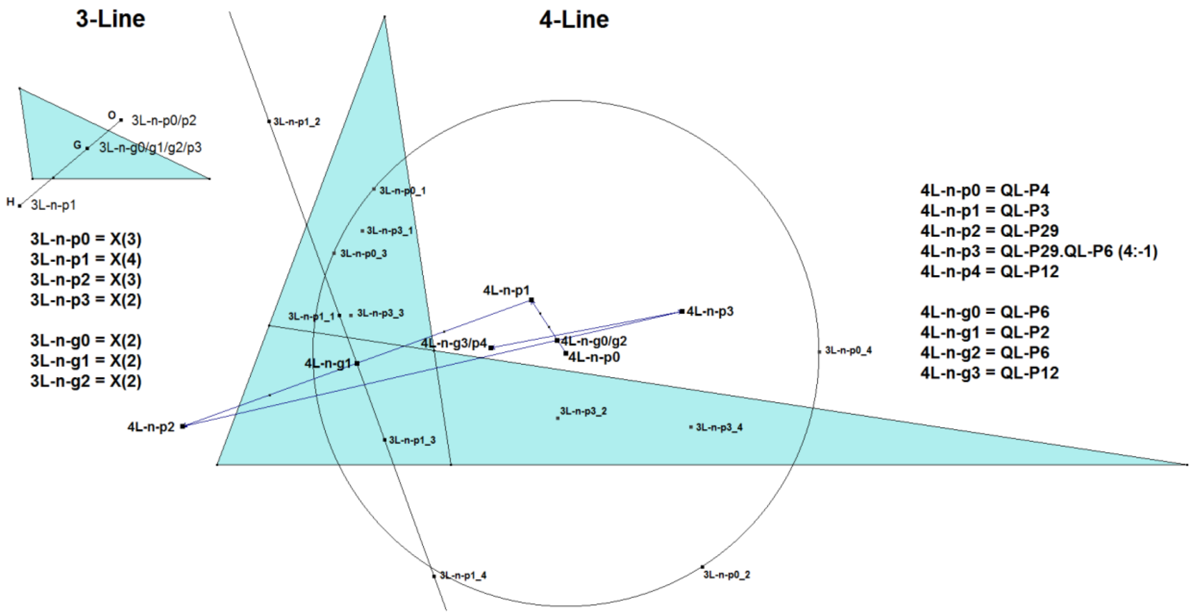
Correspondence with ETC/EQF:
When n=3, then:

3L-n-p0 = X(3)
3L-n-p1 = X(4)
3L-n-p2 = X(3)
3L-n-p3 = X(2)

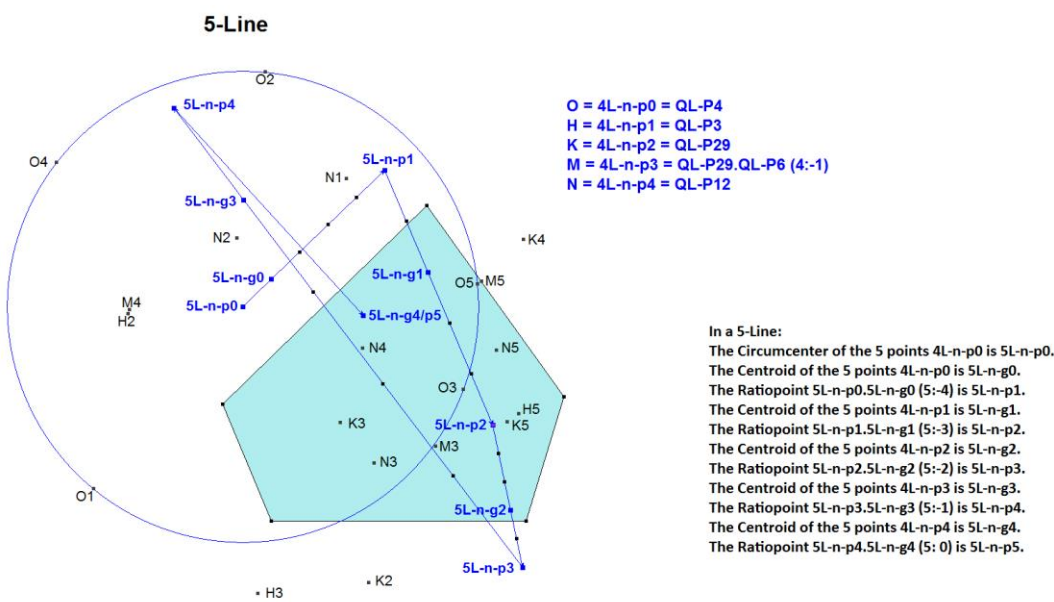
When n=4, then:

4L-n-p0 = QL-P4
4L-n-p1 = QL-P3
4L-n-p2 = QL-P29
4L-n-p3 = QL-P29.QL-P6 (4:-1)
4L-n-p4 = QL-P12

Behavior of nL-n-pi and nL-n-gi in a 3-Line and a 4-Line:
(using that in a 2-Line every outcome of 2L-n-pi=2L-n-gi=L1^L2)



Behavior of nL-n-pi and nL-n-gi in a 5-Line:



Note that the results of the lower levels are used in the higher levels.
The results of nL-n-pi are shown for i=1, … ,n, although they are used for lower values of i:

in odd cases nL-o-P1 = nL-n-pi for i = (n-1)/2, and

in even cases nL-n-P1 = nL-n-pi for i = (n/2) -1 .

nL-n-gi: nL-Morley's intermediate recursive gi points

nL-n-gi is defined as the Centroid of n points (n-1)L-n-p(i-1).
Note that the letter "p" is in lower case. In Morley's document Ref-49 it is denoted as "p_i".
Morley uses it as intermediate point(s) to make it possible to construct his so called first Orthocenter (nL-o-P1) as well as his so called Ortho Directrix (nL-e-L1).
It is used in the definition of nL-n-p1.
nL-n-pi = Ratiopoint of
nL-n-p(i-1) and
nL-n-g(i-1) being the Centroid of n points (n-1)L-n-p(i-1)
with ratio n : (i-n).

Serial steps of construction
The meaning of Morley's intermediate recursive pi- and gi-points can best be understood in writing down the first serial steps for increasing n.

In a 3-Line:		
The Circumcenter of the 3 vertices is 3L-n-p0.	= 3L-n-P3	
The Centroid of the 3 points 2L-n-p0 is 3L-n-g0.		
The Ratiopoint 3L-n-p0.3L-n-g0 (3:-2) is 3L-n-p1.	= 3L-n-P7	= 3L-o-P1
In a 4-Line:		
The Circumcenter of the 4 points 3L-n-p0 is 4L-n-p0.	= 4L-n-P3	
The Centroid of the 4 points 3L-n-p0 is 4L-n-g0.		
The Ratiopoint 4L-n-p0.4L-n-g0 (4:-3) is 4L-n-p1.	= 4L-n-P7	= 4L-e-P1
In a 5-Line:		
The Circumcenter of the 5 points 4L-n-p0 is 5L-n-p0.	= 5L-n-P3	
The Centroid of the 5 points 4L-n-p0 is 5L-n-g0.		
The Ratiopoint 5L-n-p0.5L-n-g0 (5:-4) is 5L-n-p1.	= 5L-n-P7	
The Centroid of the 5 points 4L-n-p1 is 5L-n-g1.		
The Ratiopoint 5L-n-p1.5L-n-g1 (5:-3) is 5L-n-p2.		= 5L-o-P1
In a 6-Line:		
The Circumcenter of the 6 points 5L-n-p0 is 6L-n-p0.	= 6L-n-P3	
The Centroid of the 6 points 5L-n-p0 is 6L-n-g0.		
The Ratiopoint 6L-n-p0.6L-n-g0 (6:-5) is 6L-n-p1.	= 6L-n-P7	
The Centroid of the 6 points 5L-n-p1 is 6L-n-g1.		
The Ratiopoint 6L-n-p1.6L-n-g1 (6:-4) is 6L-n-p2.		= 6L-e-P1
In a 7-Line:		
The Circumcenter of the 7 points 6L-n-p0 is 7L-n-p0.	= 7L-n-P3	
The Centroid of the 7 points 6L-n-p0 is 7L-n-g0.		
The Ratiopoint 7L-n-p0.7L-n-g0 (7:-6) is 7L-n-p1.	= 7L-n-P7	
The Centroid of the 7 points 6L-n-p1 is 7L-n-g1.		
The Ratiopoint 7L-n-p1.7L-n-g1 (7:-5) is 7L-n-p2.		
The Centroid of the 7 points 6L-n-p2 is 7L-n-g2.		
The Ratiopoint 7L-n-p2.7L-n-g2 (7:-4) is 7L-n-p3.		= 7L-o-P1
In a 8-Line:		
The Circumcenter of the 8 points 7L-n-p0 is 8L-n-p0.	= 8L-n-P3	
The Centroid of the 8 points 7L-n-p0 is 8L-n-g0.		
The Ratiopoint 8L-n-p0.8L-n-g0 (8:-7) is 8L-n-p1.	= 8L-n-P7	
The Centroid of the 8 points 7L-n-p1 is 8L-n-g1.		
The Ratiopoint 8L-n-p1.8L-n-g1 (8:-6) is 8L-n-p2.		

The Centroid of the 8 points 7L-n-p2 is 8L-n-g2.
The Ratiopoint 8L-n-p2.8L-n-g2 (8:-5) is 8L-n-p3. = 8L-e-P1

As can be seen always nL-n-p0 = nL-n-P3 and nL-n-p1 = nL-n-P7.
For even n, nL-n-p((n/2)-1) = nL-e-P1.
For odd n, nL-n-p((n-1)/2) = nL-o-P1.
After all the whole circus with pi- and gi-points is developed by Morley for constructing nL-o-P1 and nL-e-P1. A bycatch is that nL-n-p1 = nL-n-P7, but nL-n-P7 also can be constructed as a vectorsum (see nL-n-Luc3 and nL-n-P7).

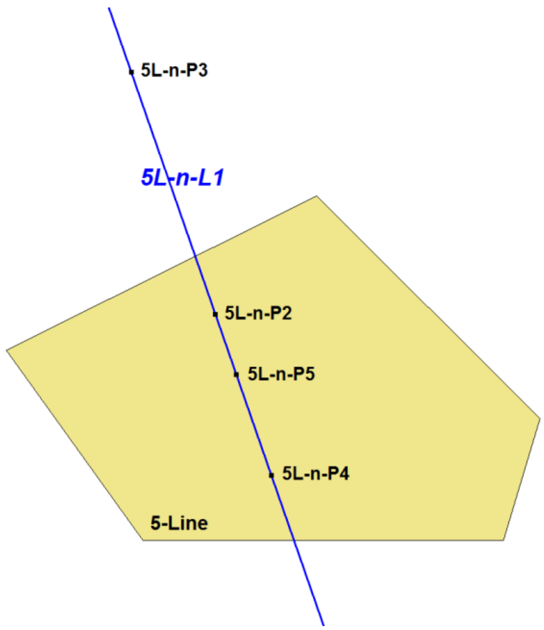
See also the notes at nL-n-pi.

Correspondence with ETC/EQF:

- In a 3-Line:
  3L-n-g0 = 3L-n-g1 = 3L-n-g2 = 3L-n-g3 = X(2).
- In a 4-Line:
  4L-n-g0 = QL-P6
  4L-n-g1 = QL-P2
  4L-n-g2 = QL-P6
  4L-n-g3 = QL-P12

nL-n-L1: nL-Morley's Eulerline

Since Morley described the equivalents of a circumcenter (nL-n-P3), an orthocenter (nL-n-P4) and a Nine-point-center (nL-n-P5) in a general n-Line which also happen to be collinear it is evident that the connecting line of these points will be Morley's Eulerline here coded nL-n-L1.
For the allocation of the centroid, circumcenter, orthocenter and nine-point center on the nL-Morley's Eulerline see nL-n-P2.
Next figure gives an example of nL-n-L1 in a 5-Line.



Correspondence with ETC/EQF:
When n=3, then nL-n-L1 = Triangle Eulerline X(3).X(4), with

- 3L-n-P2 = Centroid X(2)
- 3L-n-P3 = Circumcenter X(3)
- 3L-n-P4 = Orthocenter X(4)
- 3L-n-P5 = Nine-point Center X(5)

When n=4, then nL-n-L1 = Quadrilateral Eulerline QL-P2.QL-P4, with

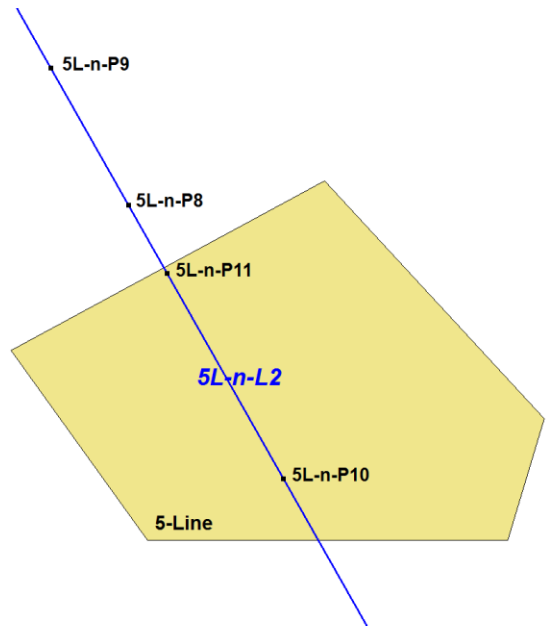
- 4L-n-P2 = Centroid-equivalent QL-P22
- 4L-n-P3 = Circumcenter-equivalent QL-P4
- 4L-n-P4 = Orthocenter-equivalent QL-P2
- 4L-n-P5 = Nine-point Center-equivalent QL-P30

Properties:

- nL-n-P2, nL-n-P3, nL-n-P4, nL-n-P5 lie on nL-n-L1.

nL-n-L2: nL-MVP Eulerline

The nL-MVP Eulerline is the line connecting collinear points nL-n-P8, nL-n-P9, nL-n-P10, nL-n-P11.
For the allocation of the centroid, circumcenter, orthocenter and nine-point center on the nL-MVP Eulerline see nL-n-P8.
Next figure gives an example of nL-n-L2 in a 5-Line.



Correspondence with ETC/EQF:
When n=3, then nL-n-L2 = Triangle Eulerline X(3).X(4), with

- 3L-n-P8 = Centroid X(2)
- 3L-n-P9 = Circumcenter X(3)
- 3L-n-P10 = Orthocenter X(4)
- 3L-n-P11 = Nine-point Center X(5)

When n=4, then nL-n-L2 = Quadrilateral Eulerline QL-P2.QL-P6, with

- 4L-n-P8 = Centroid-equivalent QL-P12 (4L-Centroid)
- 4L-n-P9 = Circumcenter-equivalent QL-P6 (Dimidium Point)
- 4L-n-P10 = Orthocenter-equivalent QL-P2 (Morley Point)
- 4L-n-P11 = Nine-point Center-equivalent Midpoint (QL-P2,QL-P6)

Properties:

- nL-n-P8, nL-n-P9, nL-n-P10, nL-n-P11 lie on nL-n-L2.

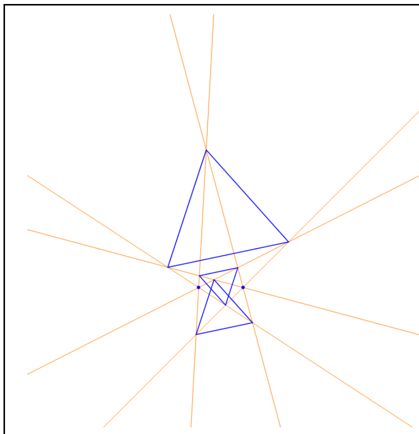
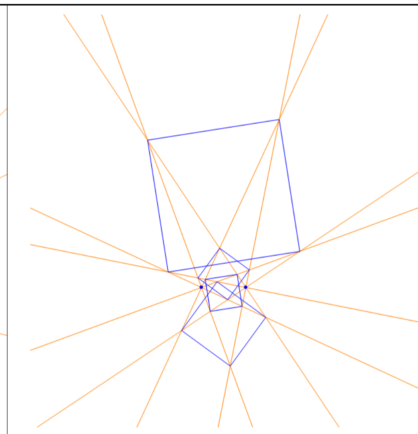
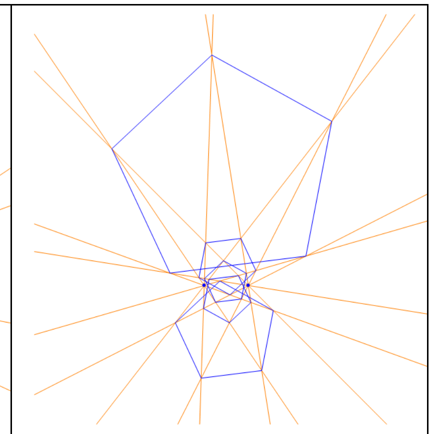
nL-n-iL1: nL-Morley's Axes

Morley describes in his document Ref-37, page 470 that in an n-Line n^{n-1} Axes can be constructed mutually crossing at angles of i.π/n. Moreover the angles of these axes with some random line will be the mean angle of the angles of L1,L2, … ,Ln with that random line eventually corrected with i.π/n.
So in a Triangle (3-Line) there are 3x3 axes mutually crossing at $i.60^0$. These are the sides of the well-known equilateral Morley Triangle complemented with 6 other parallel axes.
In a Quadrilateral (4-Line) there are 4x4x4 axes mutually crossing at $i.45^0$. These 64 axes are hardly known and there is no literature known about its construction.
In the Yahoo Quadri-Figures Group (Ref-34) there was a discussion in the period October 2014-April 2015 on this subject by Bernard Keizer (France), Eckart Schmidt (Germany) and Chris van Tienhoven (Netherlands). Finally Bernard Keizer found a solution for the construction of these axes in a 4-Line. See Ref-34, QFG#1032. The general method for n-Lines was described by Chris van Tienhoven. See Ref-34, QFG#1138.

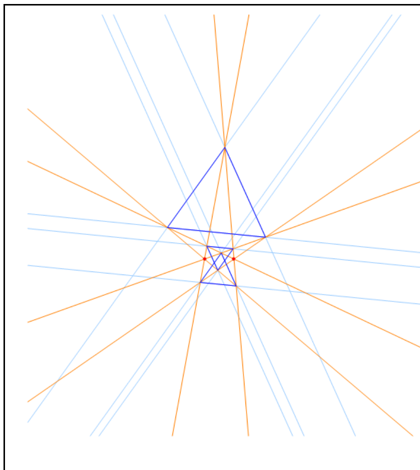
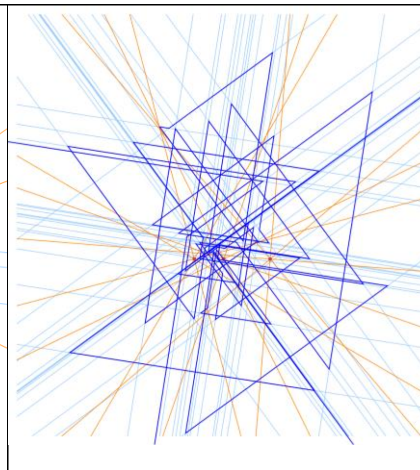
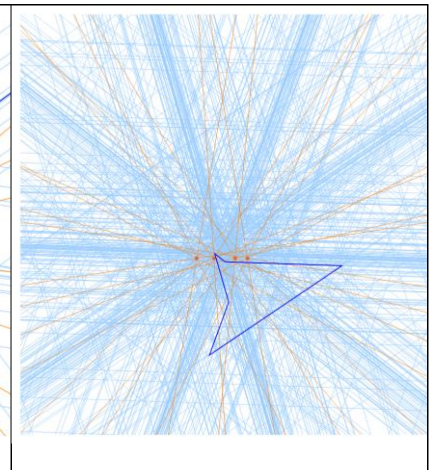
Construction:
The Lighthouse Theorem is needed for constructing the nL-Morley Axes.

1. The Lighthouse Theorem was discovered by R. K. Guy of the University of Calgary (Ref-57).The Lighthouse Theorem describes how regular polygons can be constructed from the intersection points of *regular beams* emanating from *two* Lighthouses.
2. For constructing the nL-Morley Axes an extended version of the Lighthouse Theorem is introduced of n *collinear* Lighthouses also producing (semi-)regular polygons. The Lighthouse Theorem for two Lighthouses can be used for constructing Morley's Trisector Theorem in a triangle. The extended Lighthouse Theorem is useful for constructing Morley's Axes in an n-Line.
3. According to the regular Lighthouse Theorem there should be known two points (Lighthouses) and two initial lines (beams) emanating from these points and a number n describing the angle π/n with which the beams n times will be rotated (n is a natural number). The result is n regular n-gons.
4. According to the extended Lighthouse Theorem when are known m collinear points (Lighthouses) with a set of accompanying initial beams and a number n describing the angle π/n used for rotating the beams n times (m and n are natural numbers) then n^{m-1} semi-regular n-gons (sides crossing at angles i.π/n) can be constructed from intersection points at consecutive levels from these beams. For explanation and details see Ref-34, QFG #1138.
5. Any nL-Morley Axis (being a side of the semi-regular n-gons) is uniquely defined by a constellation of (n-1) Lighthouses with a set of accompanying initial beams and a rotation-angle π/n.

Examples regular Lighthouse Theorem:

		
2 Lighthouses each emanating **3** beams produce 3 regular 3-gons	**2** Lighthouses each emanating **4** beams produce 4 regular 4-gons	**2** Lighthouses each emanating **5** beams produce 5 regular 5-gons

Examples Extended Lighthouse Theorem:

		
2 Lighthouses each emanating **3** beams produce 3^1 regular 3-gons (i=0,1,2) giving 3^2=9 axes at angles $i.60^0$ (i=0,1,2)	**3** Lighthouses each emanating **4** beams produce 4^2 semi-regular 4-gons giving 4^3=64 axes at angles $i.45^0$ (i=0,1,2,3)	**4** Lighthouses each emanating **5** beams produce 5^3 semi-regular 5-gons (1 shown) giving 5^4=625 axes at angles $i.36^0$ (i=0,1,2,3,4)

Relationship with n-Lines:

1. In an n-Line there are n random lines. One of these lines (say L_0) can be chosen as baseline. The other (n-1) lines cross this baseline in just as many intersection points P_1, P_2, ... , P_{n-1} (it will appear that changing the baseline will give the same results).
2. These (n-1) intersection points can be considered as (n-1) collinear Lighthouses.
3. Let α_1, α_2, ..., α_{n-1} be the directed angles between baseline L_0 and lines L_1, L_2, ... , L_{n-1}. The directed angle between L_0 and L_i is defined as the angle needed for rotating L_0 anti-clockwise onto L_i.
4. Rotating L_0 in P_1, P_2, ... , P_{n-1} resp. about angles α_1/n, α_2/n, ..., α_{n-1}/n will produce the n-sectors of the defining lines L_1, L_2, ... , L_{n-1} with L_0 nearest to L_0.

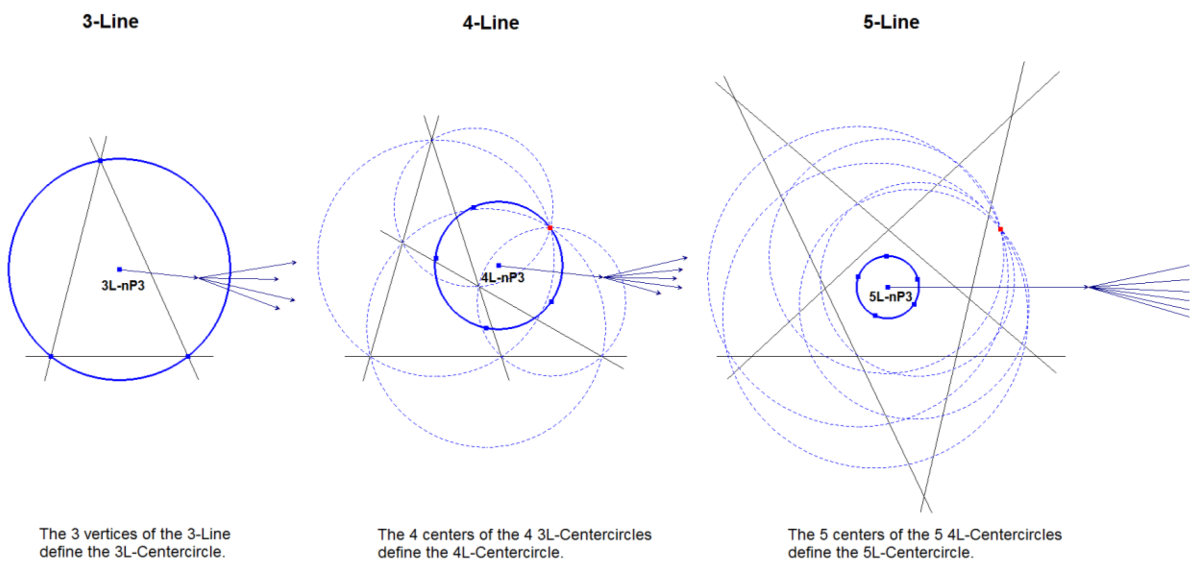
5. By rotating these n-sectors (n-1) times about rotation-angle π/n we end up with the n beams emanated from (n-1) Lighthouses.
6. Now we have (n-1) collinear Lighthouses each emanating n beams.
7. Per Lighthouse we can choose one beam from these n beams per Lighthouse. Together they are the initial beams emanated from (n-1) Lighthouses defining a unique nL-Morley Axis. For details see Ref-34, QFG#1138.
8. Since per Lighthouse of the (n-1) Lighthouses there are n beams to be chosen as initial beam we finally have n^{n-1} Morley Axes.

Properties:

- In an n-Line the n times $(n-1)^{n-2}$ Morley axes of the component (n-1)-Lines meet in $(n-1)^{n-1}$ incenters, there being n axes on a point and (n-1) points on an axis. See Ref-37, page 470. These incenters are the centers of the EnnaCardioids nL-n-Cv1.

nL-n-Ci1: nL-Center Circle

A *Triangle (3-Line)* has a circumcircle. Morley in Ref-49 calls this circle a Centercircle.
In a *Quadrilateral (4-Line)* there are 4 component 3-Lines whose 3L-Centercircle Centers are concyclic on the 4L-Centercircle.
In a *Pentalateral (5-Line)* there are 5 component 4-Lines whose 4L-Centercircle Centers are concyclic on the 5L-Centercircle. Etc.
Morley proved in Ref-49 that there exists a Centercircle in an n-Line for all n, built from the centers of the Centercircles from the Component m-Lines.
The Center of this Centercircle is nL-n-P3.



Correspondence with ETC/EQF:
When n=3, then nL-n-Ci1 = Triangle Circumcircle
When n=4, then nL-n-P3 = Quadrilateral Circumcircle QL-Ci1.

Properties:

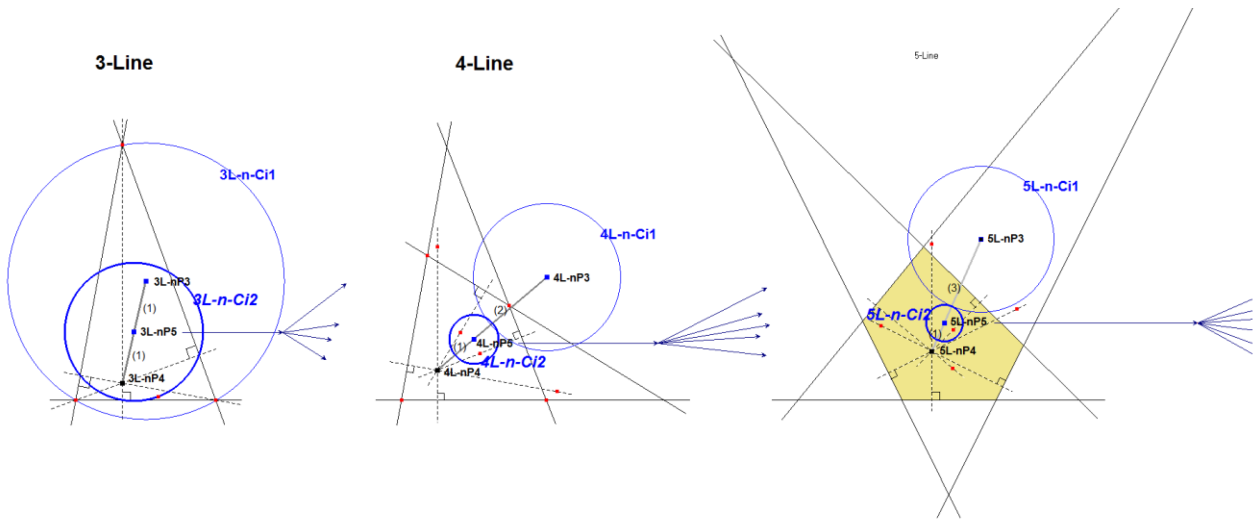
- Each Oi.Oj-intercepted inscribed nL-n-Ci1-angle = Angle(Li,Lj) mod π,
  where (i,j) are different numbers from (1, … ,n).
  Note that intercepted inscribed angles in a circle are twofold: α and π - α.
  When taken mod π the angles are α and -α. Anyway the circle is the locus of points which form inscribed angles (mod π) with a line segment, +α when occurring on one side of the line segment and -α when occurring on the other side.
  The same is true for angles between two intersecting lines. They are twofold and when taken mod π they are +α and -α.
  Example: let V be variable point on nL-n-Ci1, now the twofold angle Oi.V.Oj = twofold angle (Li,Lj). See Ref-34, QFG#1893.
- When n=5 (in a 5-Line) 5L-s-P2 lies on the Centercircle 5L-n-Ci1.

nL-n-Ci2: nL-Second Circle

Morley defines a 2nd circle with radius 1/(n-1) radius of the center circle (1/2 for the triangle and 1/3 for the quadrilateral, etc.). The center of this circle is nL-n-P5.
nL-n-Ci2 can be used for the construction of nL-n-P4.

Construction of nL-Ci2:
nL-n-Ci2 can be constructed in a recursive way (first in a 3-Line, then in a 4-Line, .. up to an n-Line):

- Construct the perpendiculars of the n versions of (n-1)L-n-P5 of the Component (n-1)-Lines to the omitted line. They will concur in nL-n-P4.
- Construct nL-n-P5 = nL-n-P4.nL-n-P3 (1 : n-1).
- Construct circle nL-n-Ci2 with center nL-n-P5 and radius 1/(n-1) times the radius of nL-Ci1.



The perpendiculars from the 3 vertices of the 3-Line coincide in 3L-n-P4.
Define 3L-n-P5 = 3L-n-P4.3L-n-P3 (1:1).
3L-n-Ci2 = circle with center 3L-n-P5 and radius = 3L-n-Ci1-radius / 2.

The perpendiculars from the 4 versions of 3L-n-P5 to the omitted line coincide in 4L-n-P4.
Define 4L-n-P5 = 4L-n-P4.4L-n-P3 (1:2).
4L-n-Ci2 = circle with center 4L-n-P5 and radius = 4L-n-Ci1-radius / 3.

The perpendiculars from the 5 versions of 4L-n-P5 to the omitted line coincide in 5L-n-P4.
Define 5L-n-P5 = 5L-n-P4.5L-n-P3 (1:3).
5L-n-Ci2 = circle with center 5L-n-P5 and radius = 5L-n-Ci1-radius / 4.

Correspondence with ETC/EQF:
When n=3, then nL-n-Ci2 = Euler Circle (or Nine-point Circle or Feuerbach Circle) in a Triangle.
When n=4, then nL-n-Ci2 = QL-Ci2.

Properties:

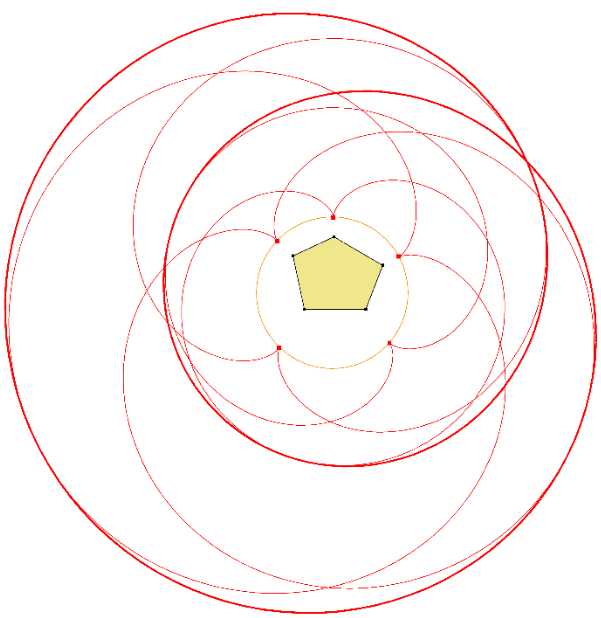
- nL-n-P4 is also the External Homothetic Center of nL-n-Ci1 & nL-n-Ci2. See Ref-49.

nL-n-Cv1: Morley's Mono EnnaCardioid
nL-n-iCv1: Morley's Multiple EnnaCardioids

nL-n-Cv1 and nL-n-iCv1 are curves described by Morley in Ref-37 and Ref-47, with the notation C^n. The name EnnaCardioid is a generic name introduced by Morley in Ref-47, On Reflexive Geometry, page 15. The names Cardioid, TetraCardioid and PentaCardioid are also used by him.
In a 3-Line it is a circle (Morley Code C^2).
In a 4-Line it is a regular Cardioid (Morley Code C^3).
In a 5-line it is called a TetraCardioid (Morley code C^4).
In a 6-line it is called a PentaCardioid (Morley code C^5).
etc.

Morley's pupil Edward C. Philips wrote a dissertation (Ref-56) "On the PentaCardioid" in 1908. In this paper he gives a general description of the different types of PentaCardioids that occur as well as a method of constructing the PentaCardioid.

Each nL-Mono EnnaCardioid nL-n-Cv1 is circumscribing n lower-level (n-1)L-EnnaCardioids.
All nL-Multiple EnnaCardioids nL-n-iCv1 are inscribed in the corresponding n-Line and their centers lie on Morley's Axes nL-n-iL1.
When n=4 it is a regular Cardioid. At higher levels it is a special type of curve often with many cusps.
See also QFG messages #815-#825, #831.



Mono EnnaCardioid of a 5Line
circumscribing the 5 Mono Cardioids
of the Component 4Lines.

Mono EnnaCardioids and Multiple EnnaCardioids
In a **3-Line** there are:

- 1 Mono Circle C^2 here coded 3L-n-Cv1 (circumcircle),
- 4 (=2^2) Multiple Circles C^2 here coded 3L-n-4Cv1 (in-/excircles) touching the defining 3 lines of the 3-Line.

In a **4-Line** there are:

- 1 Mono Cardioid C^3 here coded 4L-n-Cv1 (in EQF QL-Qu1) circumscribing the Mono Circles of the 4 Component 3-Lines of the 4-Line,
- 3^3 Multiple Cardioids C^3 here coded 4L-n-27Cv1 (in EQF QL-27Qu1) touching the defining 4 lines of the 4-Line.

In a **5-Line** there are:

- 1 Mono TetraCardioid C^4 here coded 5L-n-Cv1 circumscribing the Mono Cardioids C^3 of the 5 Component 4-Lines of the 5-Line,
- 4^4 Multiple TetraCardioids C^4 here coded 5L-n-64Cv1 touching the defining 5 lines of the 5-Line.

In a **6-Line** there are:

- 1 Mono PentaCardioid C^5 here coded 6L-n-Cv1 circumscribing the Mono TetraCardioids C^4 of the 6 Component 5-Lines of the 6-Line,
- 5^5 Multiple PentaCardioids C^5 here coded 6L-n-3125Cv1 touching the defining 6 lines of the 6-Line.

etc.

Construction

For a construction of Morley's Mono Cardiod in a 4-Line see EQF, QL-Qu1.
For a construction of Morley's Multiple Cardiods in a 4-Line see EQF, QL-27Qu1.

Construction of 5L-n-Cv1 (example in a 5-Line by Eckart Schmidt, see QFG#815):

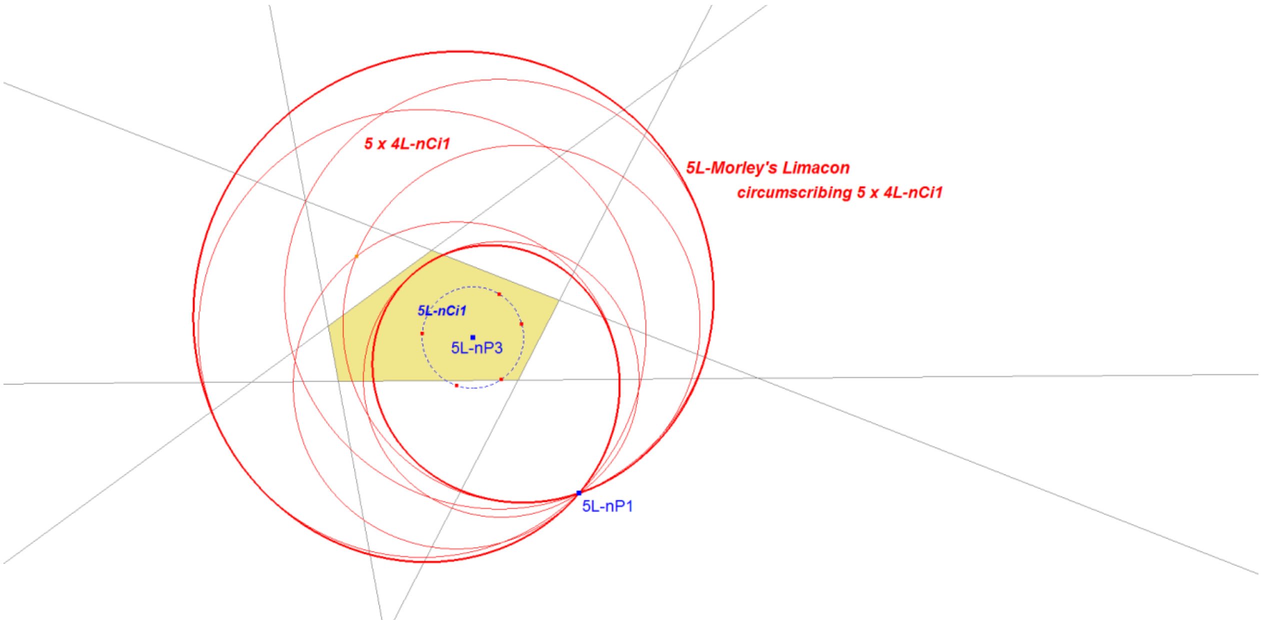
1. Let Q be a variable point on the 5L-n-Ci1-circle,
2. let Ci1 be a circle round Q through 5L-n-P1,
3. let X be the second intersection of Ci1 and the QL-P1-circle,
4. let Y be the second intersection of X.5L-o-P2 and Ci1,
5. let Ci2 be a circle round Y through X,
6. let Z be the second intersection of Ci1 and Ci2,
7. then Z reflected in Y is a point P of Morley's EnnaCardioid.

Correspondence with ETC/EQF:

When n=3, then nL-n-Cv1 = circumcircle of the 3-Line.
When n=3, then nL-n-iCv1 = combination of incircle and excircles of the 3-Line.
When n=4, then nL-n-Cv1 = QL-Qu1.
When n=4, then nL-n-iCv1 = QL-27Qu1.

nL-n-Cv2: nL-Morley's Limacon

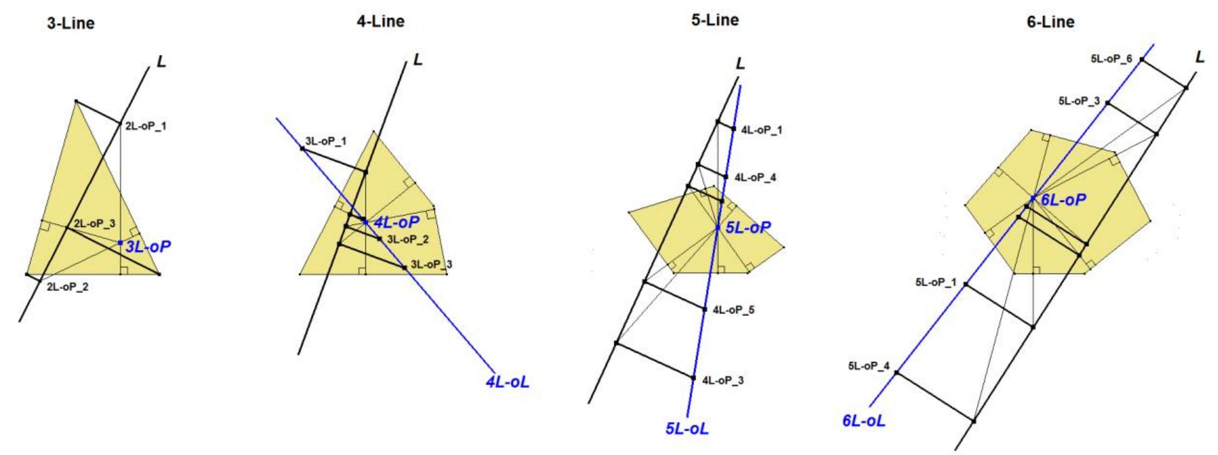
nL-Morley's Limacon is described by Morley in Ref-48, On the metric geometry of the plane n-line. Whilst nL-n-Cv1 is circumscribing (n-1)L-EnnaCardioids this curve is circumscribing the (n-1)L-Centercircles.
It can be constructed as the locus of nL-n-P1 reflected in the tangents at the Centercircle nL-n-Ci1.
More detailed this Limacon is described by Eckart Schmidt in Ref-34, QFG-messages #918, #919.
Following is an example of Morley's Limacon in a 5-Line.



Correspondence with ETC/EQF:
When n=4, then nL-n-Cv2 = QL-Qu1.

nL-n-Tf1 nL-Orthopole

nL-n-Tf1 is the transformation which transforms a random line L into an “nL-Orthopole”, which is a point in an n-Line.
In a 3-Line it is the well-known Orthopole in a Triangle. See Ref-13.
In a 4-Line it is the Orthopole in a Quadrilateral which was introduced by Tran Quang Hung. See Ref-34, QFG#2062, #2064, #2069, #2070.
The method for constructing an Orthopole in an n-Line can be made recursive by using the same method. See figure below. See Ref-34, QFG#2086.



Given a 3-Line and a random line L.
From the projection points of the 3-Line vertices on L perpendiculars are drawn to the opposite sides.
They concur in 3L-oP = 3L-n-Tf1(L).

Given a 4-Line and a random line L.
A 4-Line contains four 3-Lines.
The projection points of the 4 versions of 3L-oP on L are collinear on 4L-oL = 4L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 4L-oP = 4L-n-Tf1(L).
4L-oP lies on 4L-oL.

Given a 5-Line and a random line L.
A 5-Line contains five 4-Lines.
The projection points of the 5 versions of 4L-oP on L are collinear on 5L-oL = 5L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 5L-oP = 5L-n-Tf1(L).
5L-oP lies on 5L-oL.

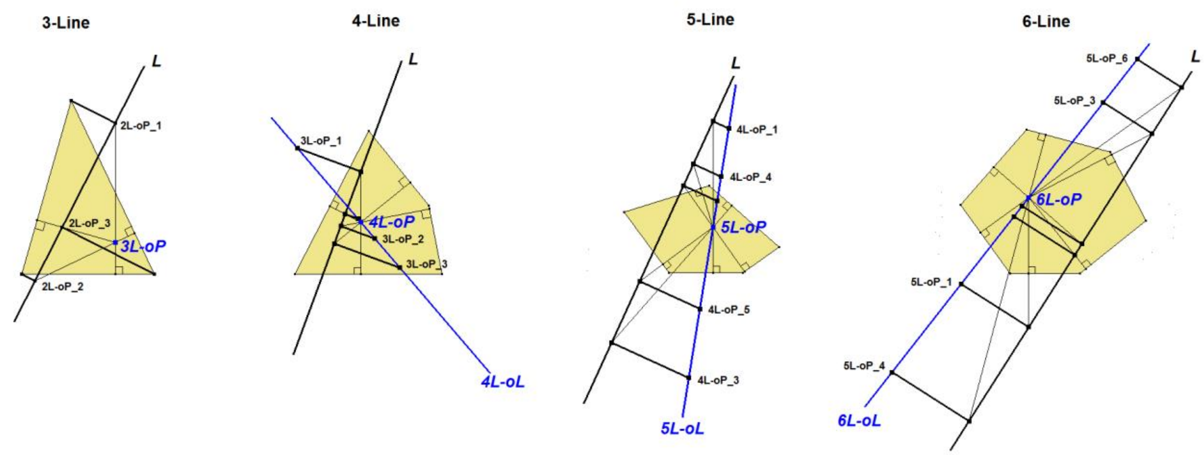
Given a 6-Line and a random line L.
A 6-Line contains six 5-Lines.
The projection points of the 6 versions of 4L-oP on L are collinear on 6L-oL = 6L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 6L-oP = 6L-n-Tf1(L).
6L-oP lies on 6L-oL.

Properties:

- An n-Line contains n (n-1)-Lines. The n versions of the (n-1)L-Orthopole in an n-Line will be collinear on the line nL-n-Tf2(L), whilst nL-n-Tf1(L) will be lying on nL-n-Tf2(L).
- An (n+1)-Line contains (n+1) n-Lines. The (n+1) versions of the nL-Orthopole in an (n+1)-Line will be collinear. See nL-n-Tf2.

nL-n-Tf2 nL-Orthopolar

nL-n-Tf2 is the transformation which transforms a random line L into an "Orthopolar" Line in an n-Line.
In a 3-Line there is no Orthopolar.
In a 4-Line it is the Orthopolar in a Quadrilateral. See Ref-13. This is the line being made of the four 3L-Orthopoles of the Component Triangles of the 4-Line, which are collinear.
The method for constructing an Orthopole in an n-Line can be made recursive by using the same method as in a 4-Line. It is the line being made of the n (n-1)L-Orthopoles (n-1)L-n-Tf1(L) of the Component (n-1)-Lines of the n-Line, which are collinear.
See figure below. See Ref-34, QFG#2086.



Given a 3-Line and a random line L.
From the projection points of the 3-Line vertices on L perpendiculars are drawn to the opposite sides.
They concur in 3L-oP = 3L-n-Tf1(L).

Given a 4-Line and a random line L.
A 4-Line contains four 3-Lines.
The projection points of the 4 versions of 3L-oP on L are collinear on 4L-oL = 4L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 4L-oP = 4L-n-Tf1(L).
4L-oP lies on 4L-oL.

Given a 5-Line and a random line L.
A 5-Line contains five 4-Lines.
The projection points of the 5 versions of 4L-oP on L are collinear on 5L-oL = 5L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 5L-oP = 5L-n-Tf1(L).
5L-oP lies on 5L-oL.

Given a 6-Line and a random line L.
A 6-Line contains six 5-Lines.
The projection points of the 6 versions of 4L-oP on L are collinear on 6L-oL = 6L-n-Tf2(L).
From these projection points perpendiculars are drawn to the omitted line.
They concur in 6L-oP = 6L-n-Tf1(L).
6L-oP lies on 6L-oL.

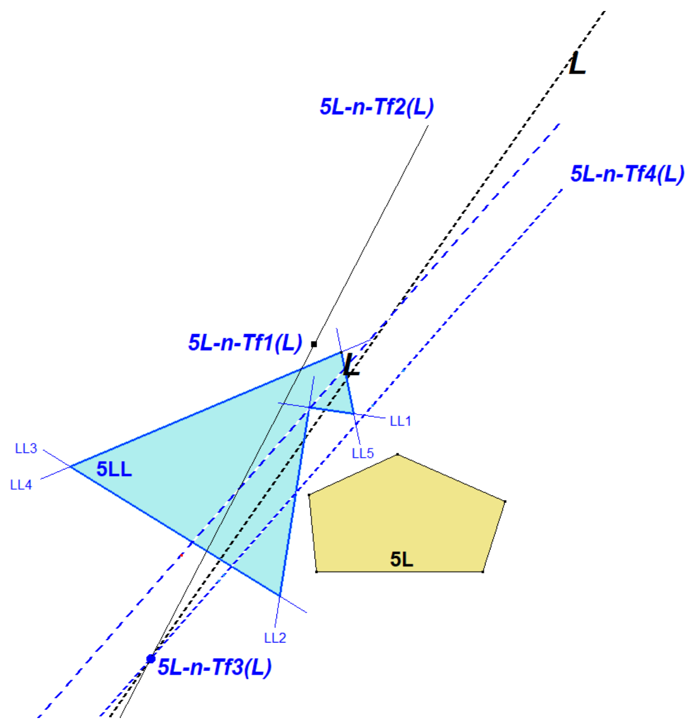
Conjecture:
Let L0 be a random line.
Let nLL be the n-Line made up from the n versions of (n-1)L-n-Tf2(L0).
Let nLL-n-Tf2 be the nL-n-Tf2 transformation wrt nLL.
nLL-n-Tf2 has these special properties:
* nLL-n-Tf2(La) // nLL-n-Tf2(Lb), where La and Lb are two different random lines.
* nLL-n-Tf2(L0) will be a line passing through the intersection point of L0 and nL-n-Tf2(L0) and will be parallel to the lines described in former property.
* the n versions of (n-1)LL-n-Tf2(L0) coincide with nLL-n-Tf2(L0).
* the n versions of (n-1)LL-n-Tf2(LLi) coincide with a line // nLL-n-Tf2(L0), where LLi is the omitted line of nLL and i=1, …, n.

nL-n-Tf3 nL-2nd Generation nL-Orthopole

Let L be a random line.
Let nLL be the 2nd generation n-Line made up from the n versions of (n-1)L-n-Tf2(L).
The 3rd generation n-Line constructed in a similar way upon nLL are n coinciding Lines, being **nL-n-Tf4(L).** The lines L, nL-n-Tf2(L), nL-n-Tf4(L) coincide in one point being **nL-n-Tf3(L).**

5-Line Example

The 2nd Generation nL-Orthopolar best can be understood by the example in a 5-Line.
The 5 versions of 4L-n-Tf2(L) form the 2nd generation Orthopolar 5-Line, being called here 5LL.
The 5 versions of 4L-n-Tf2(L) wrt 5LL delivers 5 times the same line, which is **5L-n-Tf4(L).**
The lines L, 5L-n-Tf2(L), 5L-n-Tf4(L) coincide in one point being **5L-n-Tf3(L).**



NOTES
1. Note that for all L the sides of 5LL have a fixed direction regardless the position of L.
2. Note that for all L in the same direction corresponding 5LL's are congruent.

Properties:

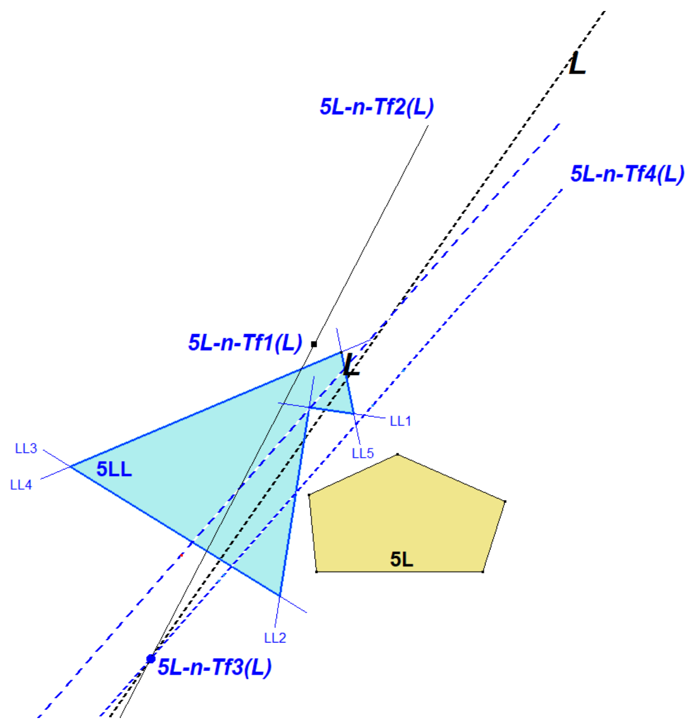
- Let **nLL**-n-Tf2(L) be nL-n-Tf2(L) wrt nLL.
  nLL-n-Tf2 has some very special properties:
  * nLL-n-Tf2(La) // nLL-n-Tf2(Lb), where La and Lb are two different random lines.
  * nLL-n-Tf2(L) will be a line passing through the intersection point of L and nL-n-Tf2(L).
  * the n versions of (n-1)LL-n-Tf2(LLi) coincide with a line // nLL-n-Tf2(L), where LLi is the omitted line of nLL and i=1, …, n.
  * the n versions of (n-1)LL-n-Tf2(L) coincide with nLL-n-Tf2(L).

nL-n-Tf4 2nd Generation nL-Orthopolar

Let L be a random line.
Let nLL be the 2nd generation n-Line made up from the n versions of (n-1)L-n-Tf2(L).
The 3rd generation n-Line constructed in a similar way upon nLL are n coinciding Lines, being **nL-n-Tf4(L).** The lines L, nL-n-Tf2(L), nL-n-Tf4(L) coincide in one point being **nL-n-Tf3(L).**

5-Line Example

The 2nd Generation nL-Orthopolar best can be understood by the example in a 5-Line.
The 5 versions of 4L-n-Tf2(L) form the 2nd generation Orthopolar 5-Line, being called here 5LL.
The 5 versions of 4L-n-Tf2(L) wrt 5LL delivers 5 times the same line, which is **5L-n-Tf4(L).**
The lines L, 5L-n-Tf2(L), 5L-n-Tf4(L) coincide in one point being **5L-n-Tf3(L).**



NOTES
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Properties:

- Let **nLL**-n-Tf2(L) be nL-n-Tf2(L) wrt nLL.
  nLL-n-Tf2 has some very special properties:
  * nLL-n-Tf2(La) // nLL-n-Tf2(Lb), where La and Lb are two different random lines.
  * nLL-n-Tf2(L) will be a line passing through the intersection point of L and nL-n-Tf2(L).
  * the n versions of (n-1)LL-n-Tf2(LLi) coincide with a line // nLL-n-Tf2(L), where LLi is the omitted line of nLL and i=1, …, n.
  * the n versions of (n-1)LL-n-Tf2(L) coincide with nLL-n-Tf2(L).

nL-n-Luc-1: nL-Level-up constructions

Level-up constructions are constructions that transform under given circumstances a Central Point of an n-Line into a Central point of an (n+1)-Line.
All described nL-n-Luc-Transformations are Level-up constructions.
These constructions cannot always be applied to any point because not always there will be an intended result.

nL-n-Luc1 nL-Common Point of Perpendiculars On Omitted Line

From an n-Line n different (n-1)-Lines can be constructed by omitting one line.
Through the (n-1)L-versions of some central point perpendiculars are drawn to the omitted line. In special cases they will concur and so a new central nL-point is created
This method is used for nL-n-P4, nL-o-P1, nL-o-L2.

Examples:

3L-point	*4L-point*	*5L-point*	*6L-point*	
3L-n-P5 =X(5)	4L-n-P4/P10 = QL-P2	None	---	---
	4L-e-P1 = QL-P3	5L-o-P1 (per definition) (lies on 5L-o-L2)	None	---
	QL-P20 = 4L-e-P1.4L-n-P5 (3:-1) = QL-P3.QL-P30 (3:-1)	5L-o-P1.5L-n-P4 (3:-1) (lies on 5L-o-L2) 5L-n-P4.5L-n-P10 (-5:6)	None	---
	4L-n-P5 = QL-P30	5L-n-P4 (per definition) (lies on 5L-o-L2)	None	---
		5L-n-P5	6L-n-P4 (per definition)	None
		Other known 5L-points	None	---

QL-P3.QL-P30.QL-P20 is transformed into 5L-o-P1.5L-n-P4.XX preserving distance ratios
(XX= 5L-o-P1.5L-n-P4 (3:-1))

nL-n-Luc2 nL-Common Point of Perpendicular Bisectors

From an n-Line n different (n-1)-Lines can be constructed by omitting one line.
The (n-1)L-versions of some central point will be connected with another fixed (n-1)L-point (often nL-n-P3) making up line segments from which perpendicular bisectors are drawn. In special cases they will concur and so a new central nL-point is created.
This method is used for nL-e-P1.

nL-n-Luc3 nL-Sum of Vectors Point

Morley brought up this technique: vectors can be made up from a fixed origin (nL-n-P3) to the (n-1)L-versions of another point. The endpoint of the sum of these vectors will be a new central nL-point. This method is used for nL-n-P7.
The construction is sometimes abbreviated as SVP.
Application in a 4-Line:

- There is a QL-point X on the line QL-L4 for which the Sum Vector of X.H1, X.H2, X.H3, X.H4 is QL-P3. It is the point QL-P2.QL-P3 (-1 : 4), as well as QL-P12.QL-P27 (-3:4).
  (H1, H2, H3, H4 being the orthocenters of the 4 Component Triangles of the QL)
  See Ref-34, QFG#850.

nL-n-Luc4 nL-Mean Vector Point

A Mean Vector Point (MVP) is the mean of a bunch of n vectors with identical origin.
It is constructed by adding these vectors and then dividing the Sumvector by n.
The Mean Vector Point is the endpoint of the divided Sumvector.
This method is used for nL-n-P8 to nL-n-P11.

Resemblance with nL-n-Luc3
nL-n-Luc4 looks like nL-n-Luc3. In both cases a Sumvector is used. Only in nL-n-Luc4 the Sumvector is divided by the number of vectors.

Origin independent
It is most special that with the definition of nL-n-Luc4 the location of the origin is unimportant.
In all n-Lines we can use any random point as origin. The endpoint of the resultant vector will be the same for all different origins.

Recursive application
Every Triangle Center can be transferred to a corresponding point in an n-Line by a simple recursive construction. The resulting point which will be called an nL-MVP Center, where MVP is the abbreviation for Mean Vector Point.
When X(i) is a triangle Center we define the nL-MVP X(i)-Center as the Mean Vector Point of the n (n-1)L-MVP X(i)-Centers.
When the (n-1)L-MVP X(i)-Centers aren't known they can be constructed from the MVP X(i)-Centers another level lower, according to the same definition. By applying this definition to an increasingly lower level finally the level is reached of the 3L-MVP X(i)-Center, which simply is the X(i) Triangle Center.
See Ref-34, QFG#869,#873,#878,#881.

Universal Level-up construction
Unlike other Level-up constructions this construction can be applied to all Central Points at all levels.
Consequently all known ETC-points and all known EQF-points will have a related MVP-point in every n-Line (n>3,4).

Another general construction of nL-n-Luc4(X(i)):
An nL-Mean Vector Point of some Triangle Center X(i) also can be constructed as the Centroid of the corresponding (n-1)L-Mean Vector Points of some Triangle Center X(i). Again by applying this definition to an increasingly lower level finally the level is reached of the 3L-MVP X(i)-center, which simply is the X(i) Triangle Center.

Preservation of distance ratios
The Centroid, Circumcenter, Orthocenter and Nine-point Center are when transferred to an n-Line collinear and their mutual distance ratios are preserved. This is deviating from Morley's Centroid, Circumcenter, Orthocenter and Nine-point Center (resp. nL-n-P2, nL-n-P3, nL-n-P4, nL-n-P5) in an n-Line. Clearly they are collinear, but their mutual distance ratios are not preserved. See nL-n-P2.
However when Triangle Centers (other than X(2), X(3), X(4), X(5)) are transferred to higher level n-Lines, usually collinearity of MVP-points will not be preserved. The mentioned triangle centers on the Eulerline are exceptions.

nL-n-Luc5 nL-Ref/Per/Par constructions

nL-n-Luc5 is called a Level-up construction because circumstantially it transforms a Central Point of an n-Line into a Central point of an (n+1)-Line.
nL-n-Luc5 is a class of constructions which will be subdivided later.

nL-n-Luc5 transforms an n-Line into another n-Line by drawing lines through the n versions of some Central Point (n-1)-Px *perpendicular* or *parallel* to the omitted line.

- The reference n-Line is called **Ref.**
- When drawing *parallel* lines through the n versions of (n-1)-Px the result will be an n-Line called **Par**. When drawing more than one generations the resulting n-Lines will be called **Par1**, **Par2**, etc.
- When drawing *perpendicular* lines through the n versions of (n-1)-Px the result will be an n-Line called **Per**. When drawing more than one generations the resulting n-Lines will be called **Per1**, **Per2**, etc.
- When a pair of the occurrences of Ref, Par1, Par2, Per1, Per2 are perspective there will be a *Perspective Center* **XXX/YYY-PC(Px),** where XXX and YYY are different names taken from the group Ref, Par1, Par2, Per1, Per2, etc.
- When the corresponding lines of XXX and YYY are parallel and XXX and YYY are perspective, then this Perspective Center will be called *Homothetic Center* **XXX/YYY-HC(Px),** where XXX and YYY are different names taken from the group Ref, Par1, Par2, Per1, Per2, etc.

More specific:

1. Every n-Line has n Component (n-1)-Lines, each (n-1)-Line constructed by omitting one line of the n-Line.
2. Through the n (n-1)L-versions of some central point ***parallels*** are drawn to the omitted line, thus producing a new n-Line called **Par1**.
3. When this construction is repeated by using Par1 as Reference n-Line the outcome will be a 2^{nd} generation n-Line called **Par2**.
4. Through the (n-1)L-versions of some central point ***perpendiculars*** are drawn to the omitted line, thus producing a new n-Line called **Per1**.
5. When this last construction is repeated by using nL-Per1 as Reference n-Line the outcome will be a 2^{nd} generation n-Line called **Per2**.

It appears that all kind of combinations of nL-Ref, Par1, Par2, Per1, Per2 can be homothetic or perspective, where they give rise to a Homothetic Center(**HC**) / Perspective Center (**PC**).

Examples
Although most of the times there will no perspectivity there are plenty of positive examples:

- nL-n-P5 applied in n (n-1)-Lines gives homothetic Ref / Par1, creating nL-n-P2.
- X(4) applied in 4 3-Lines gives a Ref/Par1-HC, being QL-P20.
- X(4) applied in 4 3-Lines gives a Par1/Per1-PC, being QL-P21.
- 5L-s-P1 applied in 6 5-Lines gives a Ref/Par2-HC, being 6L-s-P2.
- 5L-s-P1 applied in 6 5-Lines gives a Ref/Per2-HC, being 6L-s-P3.
- 5L-s-P1 applied in 6 5-Lines gives a Par2/Per2-HC, being 6L-s-P4.
- etc.

Not always a Perspective Axis

Note that although there is a Perspective Center/Homothetic Center of two n-Lines for n>3 there *not always* is a Perspective Axis. Actually there mostly is no Perspective Axis. There is a Perspective Axis when the intersection points of corresponding lines are collinear on a Perspective Axis.
A nice example is the Perspective Axis of Par1/Per1-Perspective Center QL-P21, being the Steiner Line QL-L2.

Present state of research
There is a huge differentiation in perspective pairs of n-Lines coming from (Ref, Per1, Per2, Per3, Per4, Par1, Par2, Par3, Par4).
Most common are the perspectivities of these pairs of n-Lines:

- Ref/Par1 (consequently also Par1/Par2, etc.)
- Par1/Par2 (without perspectivity of Ref/Par1)
- Ref/Per2
- Par1/Per2

But it has to be said that most of the times there will be no homothetic / perspective pair of n-Lines.
So each occurrence of a Ref-Per-Par-perspectivity for some Px is special.

Examples ETC-points applied in a 4-Line
X(2) in a 4-Line
Perspective/Homothetic Centers:

- Ref/Par1 = QL-P12
- Ref/Per2 = Ref/Per4 = Per1/Per3 = Per2/Per4 = QL-Px =
  Midpoint QL - P5.QL - P29 = Midpoint QL - P2.QL - P20 = Reflection of QL - P6 in QL - P22
- Par1/Per2=QL-P12.QL-Px (4:1)
- Par1/Per4=QL-P12.QL-Px (40:1)

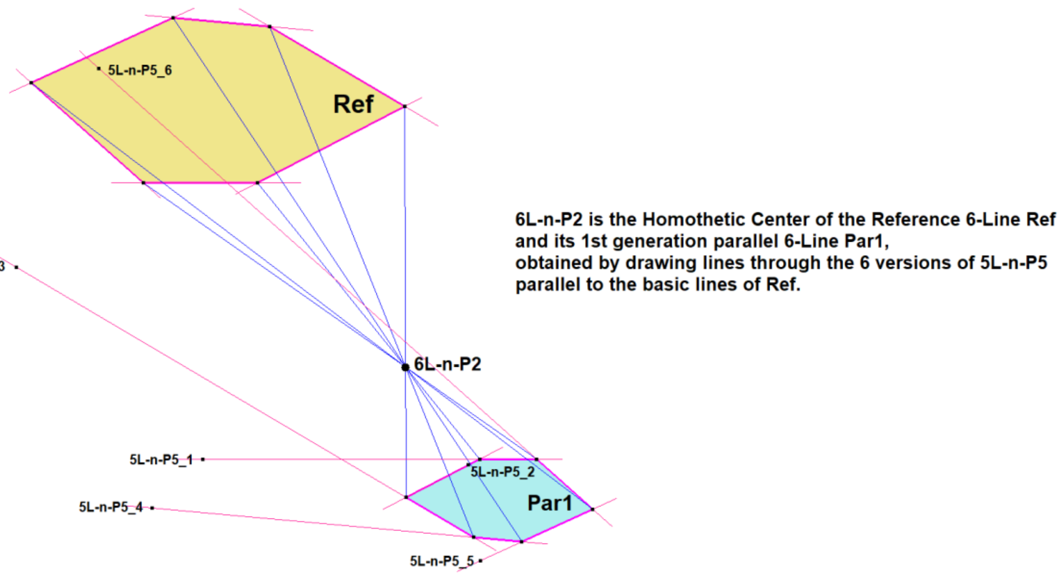
These 4 Perspective/Homothetic Centers are collinear.

X(4) in a 4-Line
Note: Ref=Par2=Par4, Par1=Par3
Perspective/Homothetic Centers:

- Ref/Par1 = QL-P20
- Ref/Per2 = InfinityPoint (QL-P2.QL-P20)
- Par1/Per1 = QL-P21
- Par1/Per2 = QL-P2.QL-P20 (-1:2)

nL-n-Luc5a nL-Ref/Par1 construction

nL-n-Luc5a is a Level-up construction which uses the Reference n-Line **Ref** and the 1st generation parallel n-Line **Par1** wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.



Examples:

3L-point	*4L-point*	*5L-point*	*6L-point*	
3L-n-P8 = X(2)	4L-n-P8 = QL-P12 (Ref/Par1-HC, Ref=Par2)	5L-n-P8 (Ref/Par1/Par2-HC)	6L-n-P8 (Ref/Par1/Par2-HC)	Etc.
3L-n-P3/P9 = X(3)	4L-n-Px = QL-P5 (Ref/Par1-HC, Ref=Par2)	No Ref/Par1-relationship. No Ref/Par2-relationship. No Par1/Par2-relationship.		
3L-n-P4/P10 = X(4)	4L-n-Px = QL-P20 (Ref/Par1-HC, Ref=Par2)	5L-s-P7 (Par1/Par2-HC) No Ref/Par1-relationship. No Ref/Par2-relationship.	No Ref/Par1-relationship. No Ref/Par2-relationship. No Par1/Par2-relationship.	
3L-n-P5/P11 = X(5)	4L-n-P2 = QL-P22 (Ref/Par1-HC, Ref=Par2)	No Ref/Par1-relationship. No Ref/Par2-relationship. No Par1/Par2-relationship.		
	Points on line 4L-n-P8.4L-n-P5 4L-n-P8 = QL-P12 4L-n-P5 = QL-P30 4L-n-P5. 4L-n-P8 (2:-1)= QL-P27	Points on line 5L-n-P8. 5L-n-P2 preserving distance ratio's (Ref/Par1/Par2-HC) 5L-n-P8 5L-n-P2 5L-n-P2. 5L-n-P8 (2:-1)	No Ref/Par1-relationship. No Ref/Par2-relationship. No Par1/Par2-relationship.	

Note that the mutual distance ratios of points X(2), X(3), X(4), X(5) (lying on the 3L-Eulerline) and resp. of QL-P12, QL-P5, QL-P20, QL-P22 (lying on the 4L-Newton Line) are identical.

Summary

General cases

- nL-n-Luc5a (nL-n-P8) = pL-n-P8, where p=(n+1)
- nL-n-Luc5a (nL-n-P5) = pL-n-P2, where p=(n+1)
- nL-n-Luc5a (nL-n-Px) = pL-n-Py, where p=(n+1) and
  nL-n-Px = Ratiopoint nL-n-P8.nL-n-P5 (s:t)

pL-n-Py = Ratiopoint pL-n-P8.pL-n-P2 (s:t),
meaning that the line nL-n-P8.nL-n-P5 is leveled-up into the line pL-n-P8.pL-n-P2 and that distance ratios are preserved.

Specific cases

- nL-n-Luc5a (X(3)) = QL-P5, there is no nL-n-Luc5a (QL-P5)
- nL-n-Luc5a (X(4)) = QL-P20, there is no nL-n-Luc5a (QL-P20)
- nL-n-Luc5a (X(5)) = QL-P22, there is no nL-n-Luc5a (QL-P22)

nL-n-Luc5b nL-Ref/Par2-Construction

nL-n-Luc5b is a Level-up construction which uses the Reference n-Line **Ref** and the 2nd generation parallel n-Line **Par2** wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.
When Par1 is homothetic with Ref, consequently also Par2 will be homothetic with Par1. We have actually a Ref/Par1/Par2 relationship. However it is possible that Par1 is not homothetic or even perspective with Ref, whilst Par2 still is.
There are no examples of this level-up construction found yet, but in principle their existence should be possible.

nL-n-Luc5c nL-Ref/Per1-Construction

nL-n-Luc5c is a Level-up construction which uses the Reference n-Line **Ref** and the 1st generation perpendicular n-Line **Per1** wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.
There are no examples of this level-up construction found yet, but in principle their existence should be possible.

nL-n-Luc5d nL-Ref/Per2 constructions

There are indications that the Ref/Per2 Construction applies for all ETC-points.
After checking several different ETC-points it appeared that all these ETC-points could be Ref-Per2 transformed into 4L-points.
See Ref-34, QFG#1937.
There is no indication that all these 4L-points are Ref/Per2-transferable into 5L-points.
Let X(r) be a point on the 3L-Euler line dividing X(3).X(4) with ratio r, then the Ref-Per2-transformed point will be a point on the line QL-P2.QL-P20.
Other collinear 3L-ETC-points were transformed into 4L-points on a conic.
Possibly it is a transformation of the 2^{nd} degree.
See Ref-34, QFG#1938.
Enough indications for further research.

Ref/Per2-HC constructions

3L-point	*4L-point*	*5L-point*	*6L-point*	
3L-n-P2/P8 = X(2)	4L-n-Px = Midpoint (QL-P2.QL-P20)	No new Ref/Per2-HC		
3L-n-P3/P9 = X(3)	Ref=Per2, so indefinite result	Indefinite Ref/Per2-HC		
3L-n-P4/P10 = X(4)	4L-n-Px InfinityPoint (QL-P2.QL-P20)	Indefinite Ref/Per2-HC		
3L-n-P5/P11 = X(5)	Per1=Point QL-P2, so indefinite result.	Indefinite Ref/Per2-HC		
	4L-n-P8 = QL-P12			
	4L-n-P5 = QL-P30	5L-n-P8		

nL-n-Luc5e   nL-Par1/Par2 constructions

Par1/Par2-Level-up constructions on nL-n-Pi
It appears that :
* nL-n-Luc5e(3L-n-Pi) = 4L-n-Pi for i = 1,...,11
* nL-n-Luc5e(nL-n-P8) = (n+1)L-n-P8 for n = 4,5,6,7,8,9, . . .
* nL-n-Luc5e(nL-n-P3) exists for n = 4,5,6,7 (then possible end of homothecy)
* nL-n-Luc5e(nL-n-P5) exists for n = 4,5,6 (then possible end of homothecy)
* nL-n-Luc5e(nL-n-P7) exists for n = 4,5,6,7,8 (9 gives calculation problems)

Examples

3L-point	*4L-point*	*5L-point*	*6L-point*	
	4L-e-P1 = QL-P3	5L-n-P7. 5L-o-P1 (1:2) (Par1/Par2-HC(4L-e-P1))	None	
	4L-n-P12 = 4L-n-P3 = QL-P4 = CC(H(2))	**5L-n-P12** = 5L-n-P7. 5L-n-P3 (1:1) (Par1/Par2-HC(4L-n-P12))	**6L-n-P12** No linear relation with known 6L-points. (Par1/Par2-HC(5L-n-P12))	Etc.
	4L-n-P13 = QL-P28 = CC(H(3))	**5L-n-P13** = Nonlin. rel. 5L-HC-points (Par1/Par2-HC(4L-n-P13))	**6L-n-P13** = No linear relation with known 6L-points. (Par1/Par2-HC(5L-n-P13))	Etc.
	4L-n-P14 = 4L-n-p2 = QL-P29 = CC(H(-2))	**5L-n-P14** = 5L-n-P7. 5L-n-P5 (2:-1) (Par1/Par2-HC(4L-n-P14))	**6L-n-P14** = No linear relation with known 6L-points. (Par1/Par2-HC(5L-n-P14))	Etc.

CC(H(i)) = Center of the 4L-Centercircle wrt HofstadterPoint(i).

Par1/Par2 constructions on nL-Hofstadter Points
* In a 5-Line starting with 4L- n-P13 (QL-P28) as Central Point for the Component 4-Lines it appears that 5L-Par1 is homothetic with 5L-Par2 giving a Homothetic Center 5L-n-P13.
* In a 6-Line starting with 5L- n-P13 as Central Point for the Component 5-Lines it appears that 6L-Par1 is homothetic with 6L-Par2 giving a Homothetic Center 6L- n-P13.
* In a 7-Line starting with 6L- n-P13 as Central Point for the Component 5-Lines it appears that 7L-Par1 is homothetic with 7L-Par2 giving a Homothetic Center 7L- n-P13.
* etc.

This process can be repeated for all other known QL-points generated from Hofstadter Points X(3), X(186), X(256), X(5961), X(5962), X(5963), X(5964).
Corresponding Central Points in the 4-Line will be QL-P4 (wrt X(3)), QL-P28 (wrt X(186)), QL-P29 (wrt X(256)).

I checked it graphically in Cabri for X(256) up to level n=6. Further drawings for n>6 were impossible because of the many internal calculations for the drawing software.
So I checked them with Mathematica Software.
Again there were limitations wrt the many internal calculations.
However there were no contra indications for:
X(3)
X(186) up to level n=8
X(256) up to level n=7
X(5961) up to level n=7
X(5962) up to level n=7

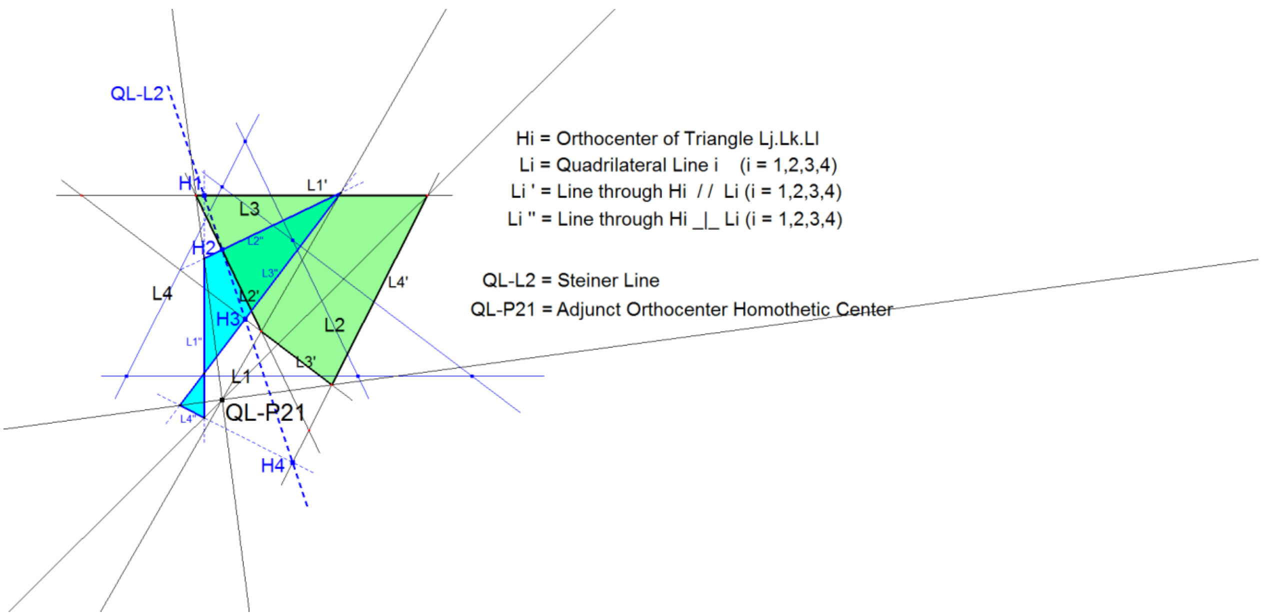
X(5963) up to level n=6
X(5964) up to level n=4

Therefore I feel confident enough for this conjecture :
Let 3L-P(i) be n-Angle Centers P(i) in a Triangle as described in QFG-message #1872, where i <> -1, 0, 1.
Let 4L-Q(i) be the Circumcenter of the 4 versions of 3L-P(i) in a 4-Line.
For these points nL-Par1 will be homothetic with nL-Par2 using (n-1)L-Q(i) as Central Point, producing new Homothetic Center nL-Q(i), for all n > 4.

nL-n-Luc5f   nL-Par1/Per1-construction

nL-n-Luc5f is a Level-up construction which uses the 1st generation parallel n-Line Par1 and the 1st generation perpendicular n-Line Per1 wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.

Example QL-P21.



Note that QL-L2=Steiner line is the Perspective Axis of Par1/Per1.

nL-n-Luc5g nL-Par1/Per2-construction
nL-n-Luc5g is a Level-up construction which uses the 1st generation parallel n-Line Par1 and the 2nd generation perpendicular n-Line Per2 wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.

Examples are 5L-s-P9 and 5L-s-P10 and 5L-n-P12.
Also

- nL-n-Luc4g(X(2)) = QL-P12.Mid(QL-P2,QL-P20) (5:-1)
- nL-n-Luc4g(X(4)) = QL-P20.QL-P2 (2:-1).

nL-n-Luc5h nL-Par2/Per1-Construction

nL-n-Luc5h is a Level-up construction which uses 2nd generation parallel n-Line **Par2** and the 1st generation perpendicular n-Line **Per1** wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.
There are no examples of this level-up construction yet, but in principle their existence should be possible.

nL-n-Luc5i nL-Par2/Per2-construction
nL-n-Luc5i is a Level-up construction which uses the 2nd generation parallel n-Line Par2 and the 2nd generation perpendicular n-Line Per2 wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.

Examples are QL-P28 and QL-P29 used as lower-level-points in a 5-Line.
See pictures 5L-s-P9 and 5L-s-P10.
Possibly there are other incidences with related Hofstadter 4L-Points.

nL-n-Luc5j nL-Per1/Per2-Construction

nL-n-Luc5j is a Level-up construction which uses the 1st generation perpendicular n-Line **Per1** and the 2nd generation perpendicular n-Line **Per2** wrt some lower-level center to construct a Perspective / Homothetic Center. See nL-n-Luc5.
There are no examples of this level-up construction yet, but in principle their existence should be possible.

nL-e: Even recursive Objects in an n-Line

nL-e-P1: nL-Morley's EnnaDeltoid Center

Morley describes this point in his paper: Orthocentric properties of the Plane n-line (Ref-49).
The range of points nL-e-P1 in a 4-Line, 6-Line, 8-Line, 10-Line will be resp. 4L-n-p1, 6L-n-p2, 8L-n-p3, 10L-n-p4, etc.. See nL-n-pi points.
Schematically it shows (note the use of lower cases in items p0, p1, etc.):

In a 4-Line:

The Circumcenter of the 4 points 3L-n-p0 is 4L-n-p0.	= 4L-n-P3	
The Centroid of the 4 points 3L-n-p0 is 4L-n-g0.		
The Ratiopoint 4L-n-p0.4L-n-g0 (4:-3) is 4L-n-p1.	= 4L-n-P7	= 4L-e-P1

In a 6-Line:

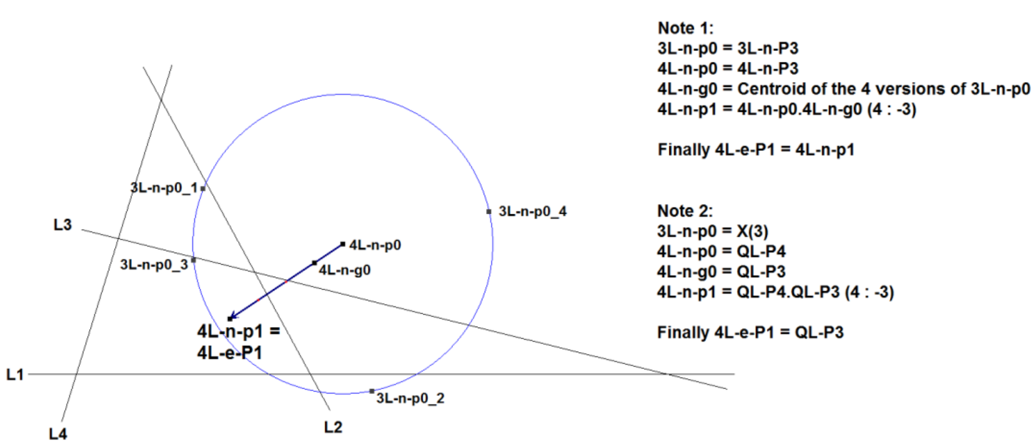
The Circumcenter of the 6 points 5L-n-p0 is 6L-n-p0.	= 6L-n-P3	
The Centroid of the 6 points 5L-n-p0 is 6L-n-g0.		
The Ratiopoint 6L-n-p0.6L-n-g0 (6:-5) is 6L-n-p1.	= 6L-n-P7	
The Centroid of the 6 points 5L-n-p1 is 6L-n-g1.		
The Ratiopoint 6L-n-p1.6L-n-g1 (6:-4) is 6L-n-p2.		= 6L-e-P1

In a 8-Line:

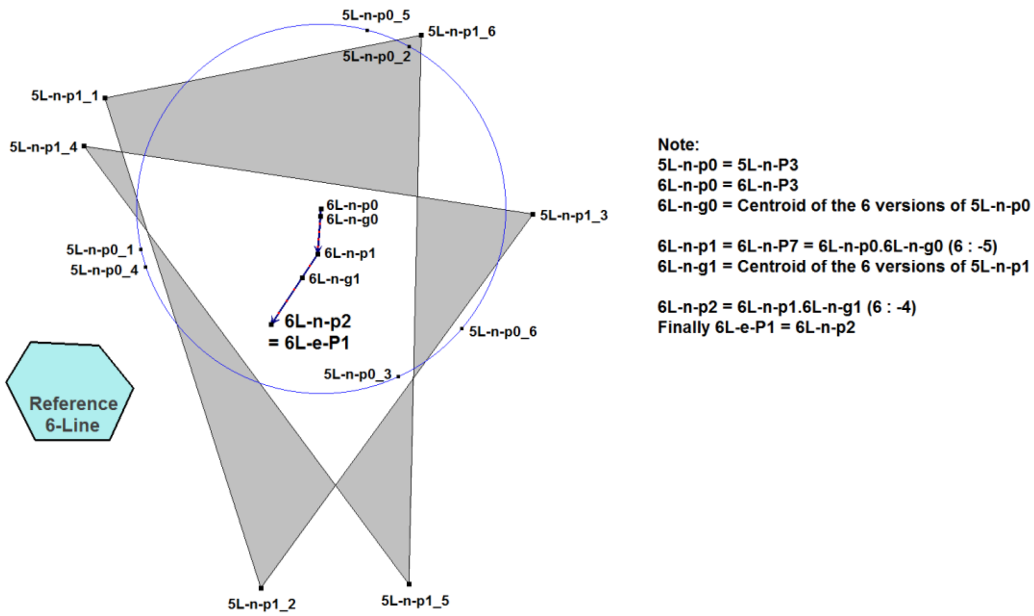
The Circumcenter of the 8 points 7L-n-p0 is 8L-n-p0.	= 8L-n-P3	
The Centroid of the 8 points 7L-n-p0 is 8L-n-g0.		
The Ratiopoint 8L-n-p0.8L-n-g0 (8:-7) is 8L-n-p1.	= 8L-n-P7	
The Centroid of the 8 points 7L-n-p1 is 8L-n-g1.		
The Ratiopoint 8L-n-p1.8L-n-g1 (8:-6) is 8L-n-p2.		
The Centroid of the 8 points 7L-n-p2 is 8L-n-g2.		
The Ratiopoint 8L-n-p2.8L-n-g2 (8:-5) is 8L-n-p3.		= 8L-e-P1

Etc.

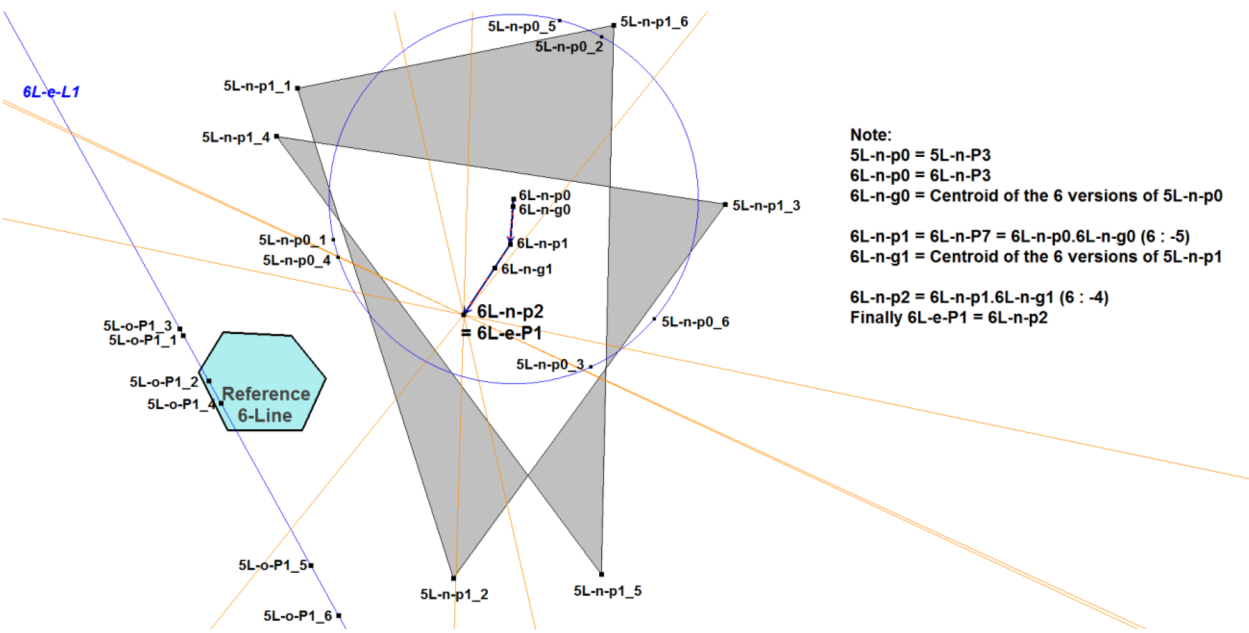
Example of nL-e-P1 in a 4-Line:



Example of nL-e-P1 in a 6-Line:



Example of nL-e-P1 in a 6-Line, where incidentally 6L-e-P1 is the common point of the perpendicular bisectors of all 6 occurrences of 5L-o-P1_i.5L-n-P7_i (i=1, … , 6).



Correspondence with ETC/EQF:
When n=4, then nL-e-P1 = QL-P3.

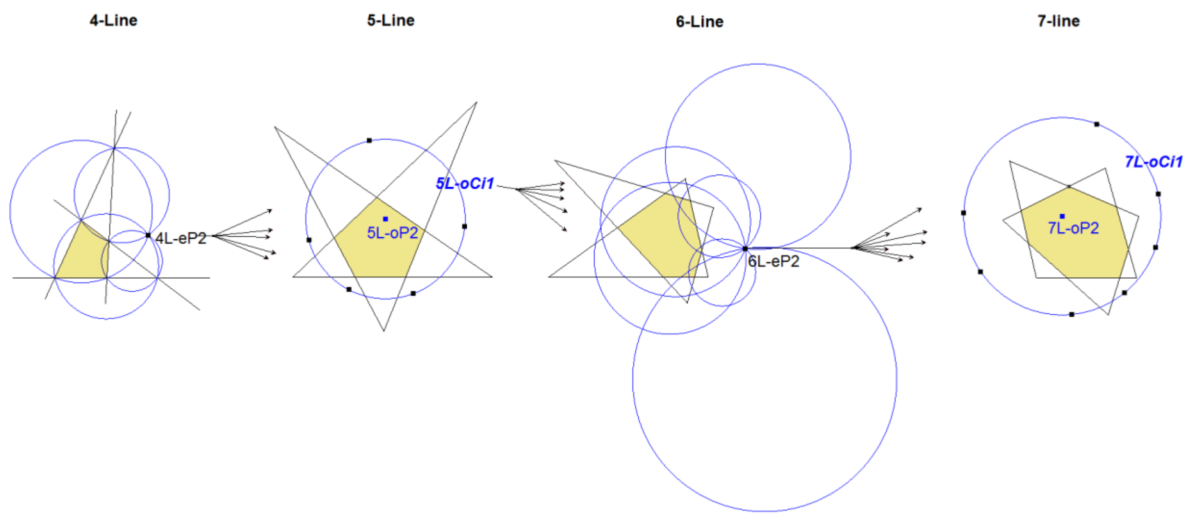
Properties:

- nL-e-P1 can be constructed as the common point of the perpendicular bisectors (Level-up Construction nL-n-Luc2) of (n-1)L-o-P1. (n-1)L-n-pk, where m=n-1, k=(n-4)/2. See nL-n-pi points.
- nL-e-P1 can be constructed as the common point of the perpendicular bisectors (Level-up Construction nL-n-Luc2) of (n-1)L-n-ph. (n-1)L-n-pk, where m=n-1, h=(n-2)/2, k=(n-4)/2. See nL-n-pi points.

nL-e-P2: nL-Clifford's Point

See nL-o-P2.
There is an alternating construction of circles/circle centers (odd case) and common circle points (even case) in the sequence of n-Lines with increasing n.
Morley described this sequence as Clifford's Chain in Ref-48, page 103 and Ref-37.
It is best to understand with numbers:

- *Clifford's Circle* in a 3-Line is supposed to be the circumscribed circle of the triangle.
- 4 *Clifford's Circles* of a 4-Line (Circumscribed circles of the component triangles) intersect at a common point 4L-e-P2: ***4L-Clifford's Point***, which is QL-P1 (Miquel Point) in the case of a 4-Line.
- 5 Clifford's Points of a 5-Line (Miquel Points of the 5 component 4-Lines) lie on a circle 5L-o-Ci1: *Clifford's Circle* with center 5L-o-P2: 5L-Clifford's Circle Center.
- 6 *Clifford's Circles* of a 6-Line (5L-o-Ci1 of the 6 component 5-Lines) intersect at a common point 6L-e-P2: ***6L-Clifford's Point***.
- 7 Clifford's Points of a 7-Line (6L-e-P2 of the 7 component 6-Lines) lie on a circle 7L-o-Ci1: 7L-*Clifford's Circle* with center 7L-o-P2: 7L-Clifford's Circle Center.
- 8 *Clifford's Circles* of a 8-Line (7L-o-Ci1 of the 8 component 7-Lines) intersect at a common point 8L-e-P2: **8L-Clifford's Point**.
- 9 Clifford's Points of a 9-Line (8L-e-P2 of the 9 component 8-Lines) lie on a circle 9L-o-Ci1: 9L-*Clifford's Circle* with center 9L-o-P2: 9L-Clifford's Circle Center.
- etc.

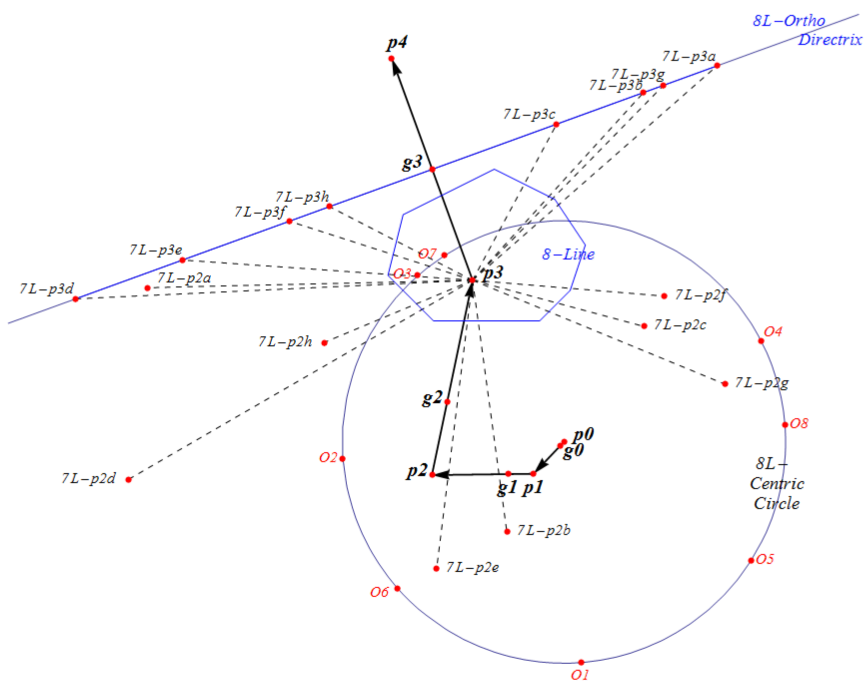


Correspondence with ETC/EQF:
When n=4, then nL-e-P2 = QL-P1.

nL-e-L1: nL-Morley's Ortho Directrix

This line is described in Ref-49, Morley's paper: Orthocentric properties of the Plane n-line. There is no nL-first Orthocenter for n=even but there are n (n-1)L-first Orthocenters (since an n-Line contains n (n-1)-Lines). They will be collinear on the so called nL-Morley's Ortho directrix. When n=4, then QL-L2 (Steiner Line) will be the 4L-Morley's Ortho directrix, containing the Orthocenters X(4) of the 4 Component triangles.

Example Morley's Ortho directrix in an 8-Line
The sides of the blue 8-Gon represent the basic lines of the 8-Line.
8L-n-p3 (p3 in the picture) is Morley's 1st orthocenter of the 8-Line.
It is constructed via g0 to p1 (1:8), via g1 to p2 (2:8), via g2 to p3 (3:8) , via g3 to p4 (4:8).
g0 = centroid of 8 points 7L-n-p0
g1 = centroid of 8 points 7L-n-p1
g2 = centroid of 8 points 7L-n-p2
g3 = centroid of 8 points 7L-n-p3.
In this picture where n=even one of the pi has a prominent place: 8L-n-p3, which also is the common point of the perpendicular bisectors of all component 7-Line segments 7L-n-p2.7L-n-p3.
In general when n=even, then nL-n-p(n-1)=Center of the inscribed EnnaDeltoid. See Ref-49, §4.



O1, ... ,O8 are actually the 8 versions of 7L−p0, being the Centric Centers of the 8 Component 7−Lines.
p0, g0, p1, g1, p2, g2, p3, g3, p4 are actually 8L−p0, 8L−g0, 8L−p1, 8L−g1, 8L−p2, 8L−g2, 8L−p3, 8L−g3, 8L−p4
The line segments 8L−p3.7L−p2x and 8L−p3.7L−p3x (x = a,b, ... ,h) are of equal size.
The 8L−Ortho Directrix is the line through 7L−p3a, ... ,7L−p3h, being the 1st Orthocenters of the 8 Component 7−Lines.
It also passes through 8L−g3, being the Centroid of 7L−p3a, ... ,7L−p3h.
It also is the perpendicular bisector of 8L−p3.8L−p4.

Correspondence with ETC/EQF:
When n=4, then nL-e-L1 = QL-L2 (Steiner Line).

Properties:
- nL-e-L1 is the Perpendicular Bisector of nL-n-p((n-2)/2) and nL-n-p(n/2)). See nL-n-pi.

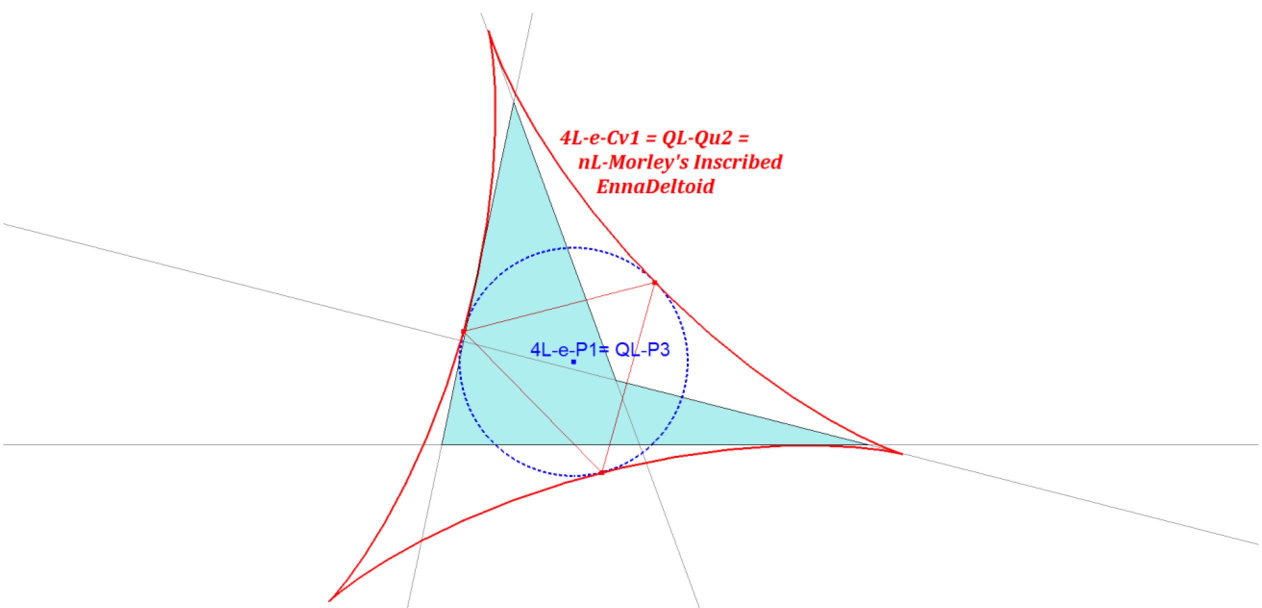
nL-e-L2: nL-Morley's Alternate Line of Orthocenters

nL-e-L2 connects nL-e-P1 (Morley's EnnaDeltoid Center) and nL-n-P5 (Morley's 2nd Circle Center).
Let X be some point on (n-1)L-e-L2 with fixed ratio wrt nL-n-P4 and nL-o-P1. Then nL-o-L2 is the locus of the common intersection point of the perpendiculars through the n lower level versions of X to the omitted line preserving distance ratios.
In this way (n-1)L-e-P1 is transformed into nL-o-P1 and (n-1)L-n-P5 is transformed into nL-n-P4.

Correspondence with ETC/EQF:
When n=4, then nL-e-L2 = QL-P3.QL-P30.

nL-e-Cv1: nL-Morley's Inscribed EnnaDeltoid

Morley describes "The Deltoid" in his paper: Orthocentric properties of the Plane n-line (Ref-49).
He writes at page 7: "The peculiar appropriateness of the deltoid for the metrical theory of four lines makes it desirable to have an analogous curve for 2n-lines". This curve is called Δ^{2n-1}.
Then: "Theorem 6: there are 2n-1 cusp-tangents of Δ^{2n-1}; they touch a concentric Δ^{2n-3}."
Important is that he shows that the inscribed deltoid in a 4-Line is the prelude for similar curves at higher even levels.
Similar to the Cardioids (see nL-nCv1) the higher level deltoids will be named here EnnaDeltoids.
It is not quite clear yet how the EnnaDeltoids looks like for n>4.
According to Morley the center of the EnnaDeltoid's is nL-e-P1.

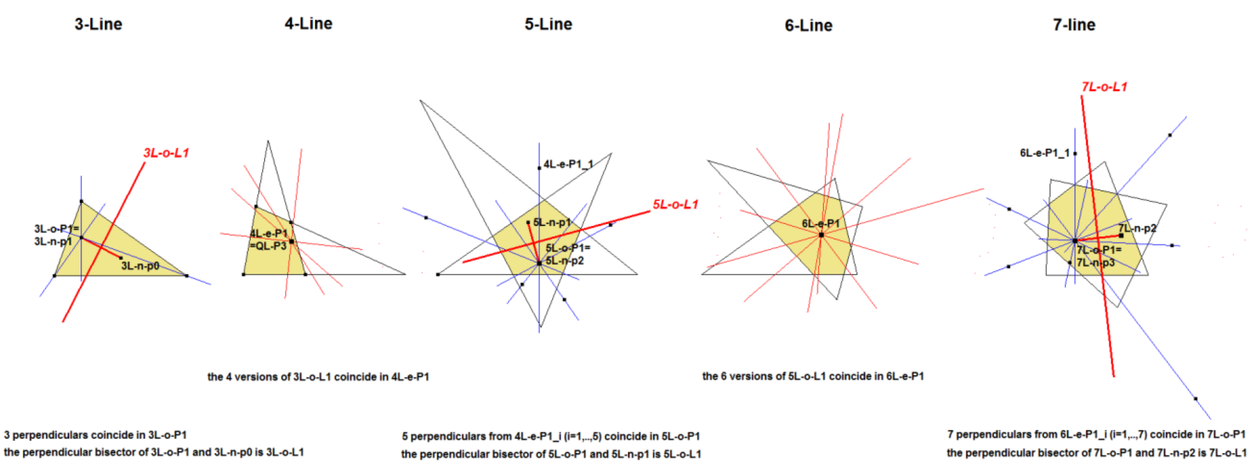


Correspondence with ETC/EQF:
When n=4, then nL-e-Cv1 = QL-Qu2 (Kantor-Hervey Deltoid).

nL-o: Odd recursive Objects in an n-Line

nL-o-P1: nL-Morley's 1st Orthocenter

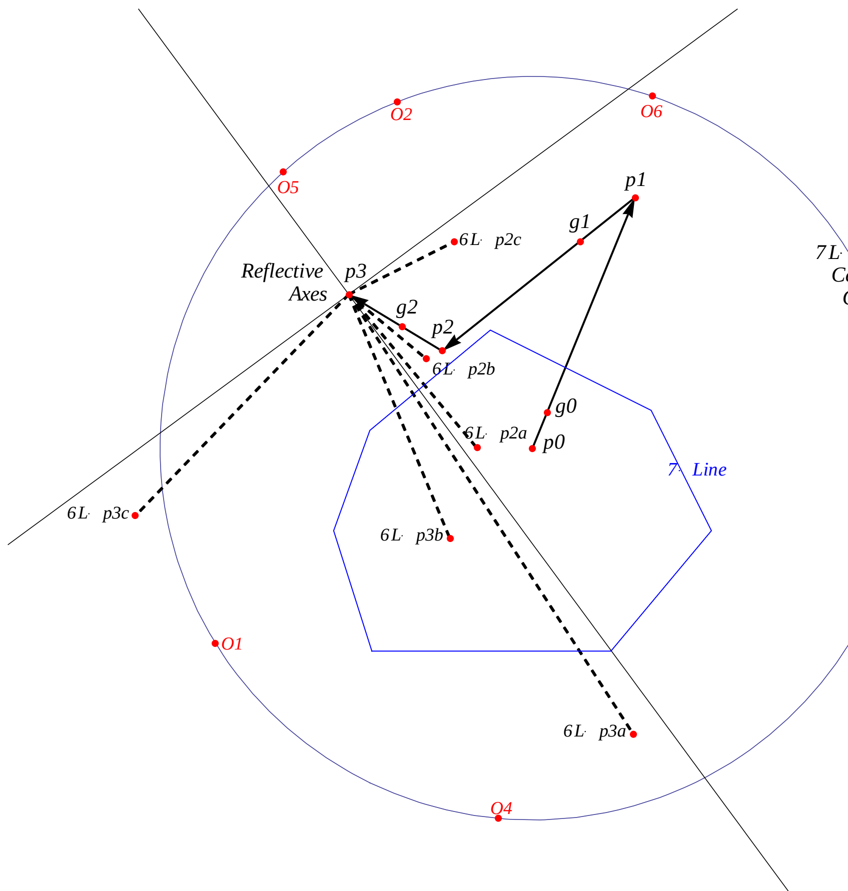
Morley describes a so called ***first Orthocenter*** in his document “Orthocentric properties of the plane n-Line” (Ref-49). Morley’s paper was published in the year 1902. Morley proofs all his results algebraically using calculations in the complex plane. He explains his methods at Ref-48.
See also Ref-34, QFG-messages #910, #912, #913, #917.
In Morley’s description he describes a recursive method of constructing this point, using intermediate points p_i, where ‘i’ is a number in the range 0, … , (n-1)/2 (*note that the letter “p” is in lower case*).
Ultimately pi, when ‘i’ has reached the value (n-1)/2, then it is Morley's 1st Orthocenter.
Morley's 1st Orthocenter in a 3-Line is X(4), the triangle orthocenter.
In general Morley's 1st Orthocenter in an n-Line (n=odd) is nL-n-pi, where i=(n-1)/2.
In a 3-Line Morley's 1st Orthocenter will be 3L-n-p1, in a 5-Line it will be 5L-n-p2, in a 7-Line it will be 7L-n-p3, etc. (see nL-n-pi).
There is also a chain of constructing nL-o-P1 and nL-e-P1 in subsequent alternating odd and even n-Lines:



Other Properties:
To prevent many abstract notations properties in a 9-Line will be mentioned. The general principle works accordingly. Intermediate points nL-n-pi will be mentioned (*note that the letter “p” is in lower case*).

1. Morley's 1st Orthocenters applied for all 10 Component 9-Lines in a 10-Line will be collinear on the so called 10L-*Morley’s Ortho directrix*.
2. Morley's 1st Orthocenter is the *common point of the perpendiculars of 8L-n-p3 on the omitted line* (by regularly omitting a line there are 9 component 8-Lines in a 9-Line).
3. Using 9L-n-p4 as origin, the segments 9L-n-p4.8L-n-p4 and 9L-n-p4.8L-n-p3 have a *fixed ratio* for all 9 occurrences of 8L-n-p4 and 8L-n-p3. In general this property is also valid using lower 9L-level points 9L-n-p3, 9L-n-p2, 9L-n-p1 as origin and connecting them with their lower 8L-level points.
4. There are *two orthogonal axes* (nL-2oL1) at Morley's first Orthocenter X bisecting the 9 versions of angles 8L-n-p3.X.8L-n-p4. In general this property is only valid for n=odd>5 and angles L-n-pi.X.mL-n-pj, where m=(n-1), i=(m-2)/2, j=m/2.

Example Morley's first Orthocenter in a 7-Line
The sides of the blue 7-Gon represent the basic lines of the 7-Line.
7L-n-p3 (p3 in the picture) is Morley's 1st orthocenter of the 7-Line.
It is constructed via g0 to p1 (1:7), via g1 to p2 (2:7), via g2 to p3 (3:7).
g0 = centroid of 7 points 6L-n-p0
g1 = centroid of 7 points 6L-n-p1
g2 = centroid of 7 points 6L-n-p2
Only three of the seven 6L-n-p3i points and 6L-n-p2i points have been drawn for checking the fixed relationship of distances 7L-n-p3 unto 6L-n-p3i and 6L-n-p2i.
In this picture 7L-n-p3.6L-n-p3x : 7L-n-p3.6L-n-p2x = 2.6167 for all x, where x=a,b,c, . . . (total 7).



p0, g0, p1, g1, p2, g2, p3 are actually 7L p0, 7L g0, 7L p1, 7L g1, 7L p2, 7L g2, 7L p3
O1, ... ,O7 are actually the 7 versions of 6L p0

Correspondence with ETC/EQF:
When n=3, then nL-o-P1 = X(4).

Properties:

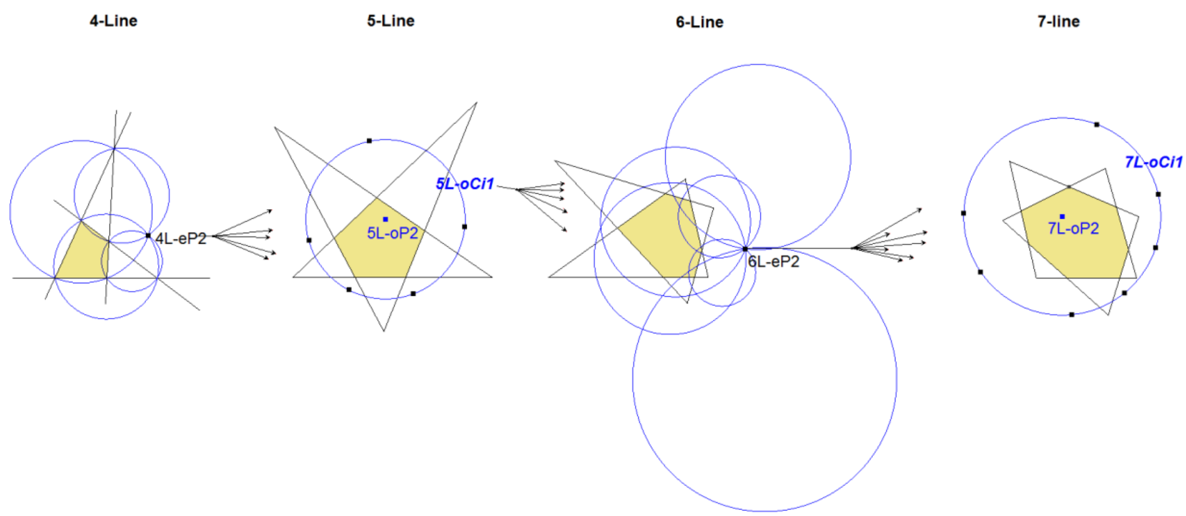
- Morley's 1st Orthocenter nL-o-P1 is the common point of the perpendiculars of all lower level (n-1)L-e-P1 to the omitted line (Level-up Construction nL-n-Luc1).

- Morley's 1st Orthocenters applied for all (n+1) Component n-Lines in a (n+1)-Line will be collinear on the so called (n+1)L-*Morley's Ortho Directrix* (n+1)L-e-L1.
- When n=odd using nL-o-P1 as origin, the segments nL-o-P1.mL-n-p((n-1)/2) and nL-o-P1.mL-n-p((n/2)-1) have a *fixed ratio* for all n occurrences, where m=(n-1). In general this property is also valid using lower nL-level points nL-n-pi (i<(n-1)/2) as origin and connecting them with their corresponding lower (n-1)L-level points.
- When n=odd there are *two Reflective Orthogonal Axes* (nL-o-2L1) at Morley's 1st Orthocenter X bisecting the n versions of angles mL-n-p((n-1)/2).X.mL-n-p((n/2)-1), where m=(n-1).

nL-o-P2: nL-Clifford's Circle Center

There is an alternating construction of circles/circle centers (odd case) and common circle points (even case) in the sequence of n-Lines with increasing n.
Morley described this sequence as Clifford's Chain in Ref-48, page 103 and Ref-37.
It is best to understand with numbers:

- 4 Clifford's Circles of a 4-Line (Circumscribed circles of the component triangles) intersect at a common point 4L-e-P2: Clifford's Point, which is QL-P1 (Miquel Point) in the case of a 4-Line.
- 5 Clifford's Points of a 5-Line (Miquel Points of the 5 component 4-Lines) lie on a circle 5L-o-Ci1: Clifford's Circle with center 5L-o-P2: **5L-**Clifford's Circle Center.
- 6 Clifford's Circles of a 6-Line (5L-o-Ci1 of the 6 component 5-Lines) intersect at a common point 6L-e-P2: 6L-Clifford's Point.
- 7 Clifford's Points of a 7-Line (6L-e-P2 of the 7 component 6-Lines) lie on a circle 7L-o-Ci1: 7L-Clifford's Circle with center 7L-o-P2: **7L-**Clifford's Circle Center.
- 8 Clifford's Circles of a 8-Line (7L-o-Ci1 of the 8 component 7-Lines) intersect at a common point 8L-e-P2: 8L-Clifford's Point.
- 9 Clifford's Points of a 9-Line (8L-e-P2 of the 9 component 8-Lines) lie on a circle 9L-o-Ci1: 9L-Clifford's Circle with center 9L-o-P2: **9L-**Clifford's Circle Center.
- etc.



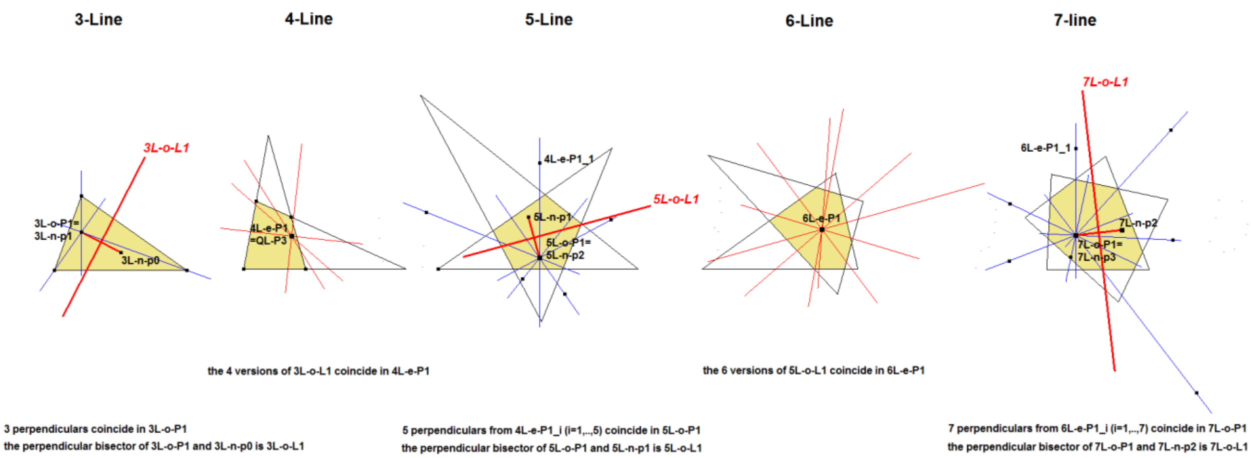
Correspondence with ETC/EQF:
When n=3, then nL-o-P2 = X(3) and nL-o-Ci1 = Triangle circumcircle.
When n=4, then nL-e-P2 = QL-P1.

Properties:

- In a 5-Line 5L-o-Ci1 = Miquels Pentagon Circle and 5L-o-P2 = Center of Miquels Pentagon Circle. See Ref-13, Miquel's Pentagram Theorem. See Ref-34, QFG#1999.

nL-o-L1: nL-Line of Inscribed EnnaDeltoid Centers

Morley describes this line in his paper: Orthocentric properties of the Plane n-line (Ref-49).
It is the locus of all (n+1)L-e-Cv1 centers ((n+1)L-e-P1) and is always a line (Theorem 8) for all odd n. The (n+1)-Line is constructed by adding a random line to the reference n-Line. Every added line contributes a point on nL-o-L1.
nL-o-L1 is the Perpendicular Bisector of nL-n-p((n-3)/2).nL-n-p((n-1)/2). Note that the letter “p” is in lower case. See nL-n-pi.
For example in a 5-Line it is the perpendicular bisector of 5L-n-p1. 5L-n-p2 and in a 7-Line it is the perpendicular bisector of 5L-n-p2. 5L-n-p3.
The merit of nL-o-L1 is that in an (n+1)-Line the n versions of nL-o-L1 concur in (n+1)L-e-P1 being nL-Morley's EnnaDeltoid Center.

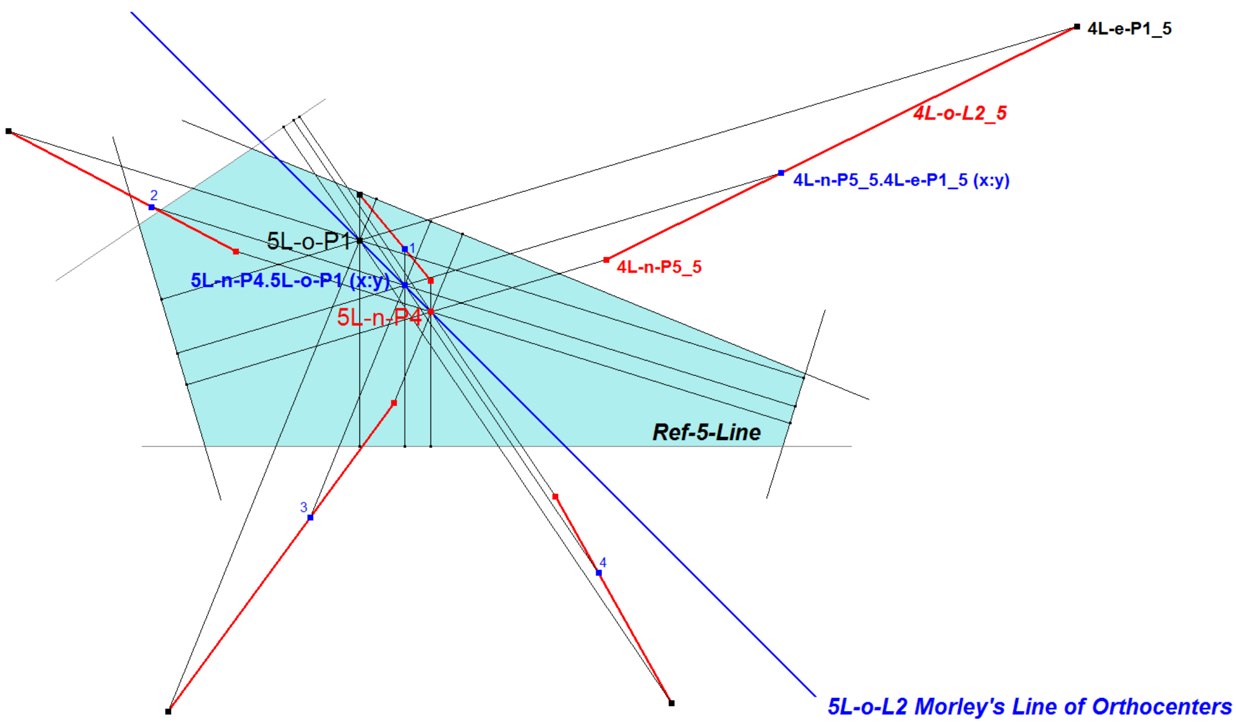


Correspondence with ETC/EQF:
When n=3, then nL-o-L1 = Perpendicular Bisector of line segment X(3).X(4).
When n=4, then the 4 versions of 3L-o-L1 = Perpendicular Bisector of line segment X(3).X(4) concur in 4L-e-P1, which is QL-P3.

nL-o-L2: nL-Morley's Line of Orthocenters

Morley describes this line in his paper: Orthocentric properties of the Plane n-line (Ref-49).
It is the line connecting nL-o-P1 (Morley's 1st Orthocenter) and nL-n-P4 (Morley's 2nd Orthocenter).
Let X be some point on (n-1)L-e-L2 with fixed ratio wrt (n-1)L-n-P5 and (n-1)L-e-P1. Then nL-o-L2 is the locus of the common intersection point of the perpendiculars through the n lower level versions of X to the omitted line (Level-up Construction nL-n-Luc1) preserving distance ratios.
In this way (n-1)L-e-P1 is transformed into nL-o-P1 and (n-1)L-n-P5 is transformed into nL-n-P4.

Correspondence with ETC/EQF:
When n=3, then 3L-o-L2 = X(4).X(4) = undefined line.



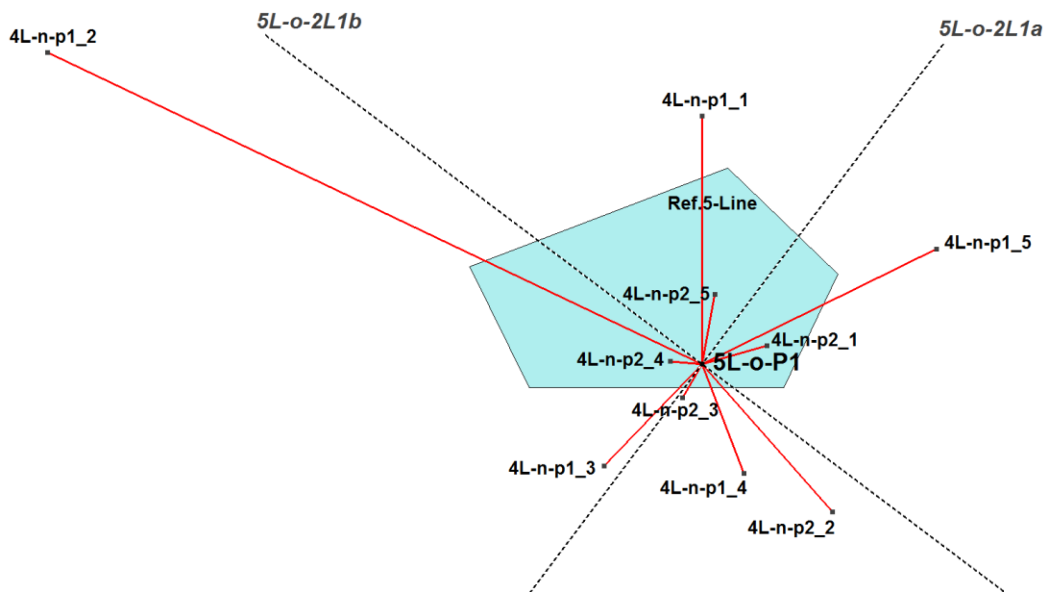
Properties:

- These points lie on nL-o-L2 (all Orthocenters indeed):
  - nL-o-P1 (nL-Morley's 1st Orthocenter
  - nL-n-P4 (nL-Morley's 2nd Orthocenter)
  - nL-n-P10 (nL-MVP Orthocenter)

nL-o-2L1: nL-Orthogonal Reflective Axes

There are n versions of (n-1)-Lines contained in an n-Line.
For n=odd there are *two orthogonal axes* (nL-o-2L1) at Morley's first Orthocenter X=nL-o-P1 being the common angle bisectors of the n versions of angles mL-n-ph.X.mL-n-pk, where m=(n-1), h=(n-1)/2, k=(n-3)/2. Note that the letter "p" is in lower case. See nL-n-pi for definition of these points.
Moreover the distance ratios X.mL-n-ph_i / X.mL-n-pk_i are equal for i=1, … ,n.
The Reflective Axes exist for n=5,7,9,…. (not for n=3).
See nL-o-P1. See also Ref-34, QFG-message #910.

Example:
5L-o-2L1a/b are the common angle bisectors of the 5 versions of angles 4L-n-p2_i.X.4L-n-p1_i (where X=5L-o-P1) and the distance ratios X.4L-n-p2_i / X.4L-n-p1_i are equal for i=1, … ,5.

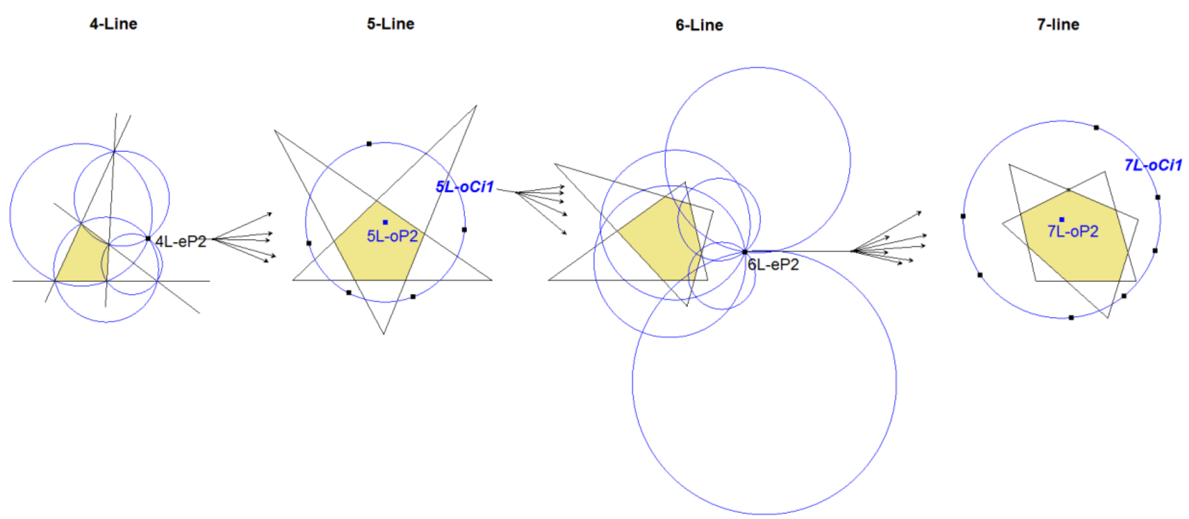


Correspondence with ETC/EQF:
When n=3, then 3L-o-2L1 is not defined.

nL-o-Ci1: nL-Clifford's Circle

See nL-o-P2.
There is an alternating construction of circles/circle centers (odd case) and common circle points (even case) in the sequence of n-Lines with increasing n.
Morley described this sequence as Clifford's Chain in Ref-48, page 103 and Ref-37.
It is best to understand with numbers:

- ***Clifford's Circle*** in a 3-Line is supposed to be the circumscribed circle of the triangle.
- 4 ***Clifford's Circles*** of a 4-Line (Circumscribed circles of the component triangles) intersect at a common point 4L-e-P2: Clifford's Point, which is QL-P1 (Miquel Point) in the case of a 4-Line.
- 5 Clifford's Points of a 5-Line (Miquel Points of the 5 component 4-Lines) lie on a circle 5L-o-Ci1: ***Clifford's Circle*** with center 5L-o-P2: 5L-Clifford's Circle Center.
- 6 ***Clifford's Circles*** of a 6-Line (5L-o-Ci1 of the 6 component 5-Lines) intersect at a common point 6L-e-P2: 6L-Clifford's Point.
- 7 Clifford's Points of a 7-Line (6L-e-P2 of the 7 component 6-Lines) lie on a circle 7L-o-Ci1: 7L-***Clifford's Circle*** with center 7L-o-P2: 7L-Clifford's Circle Center.
- 8 ***Clifford's Circles*** of a 8-Line (7L-o-Ci1 of the 8 component 7-Lines) intersect at a common point 8L-e-P2: 8L-Clifford's Point.
- 9 Clifford's Points of a 9-Line (8L-e-P2 of the 9 component 8-Lines) lie on a circle 9L-o-Ci1: 9L-***Clifford's Circle*** with center 9L-o-P2: 9L-Clifford's Circle Center.
- etc.



Correspondence with ETC/EQF:
When n=3, then nL-o-Ci1 = circumcircle of the triangle.

Properties:

- In a 5-Line 5L-o-Ci1 = Miquels Pentagon Circle and 5L-o-P2 = Center of Miquels Pentagon Circle. See Ref-13, Miquel's Pentagram Theorem. See Ref-34, QFG#1999.

5L-s: Specific Objects in a 5-Line

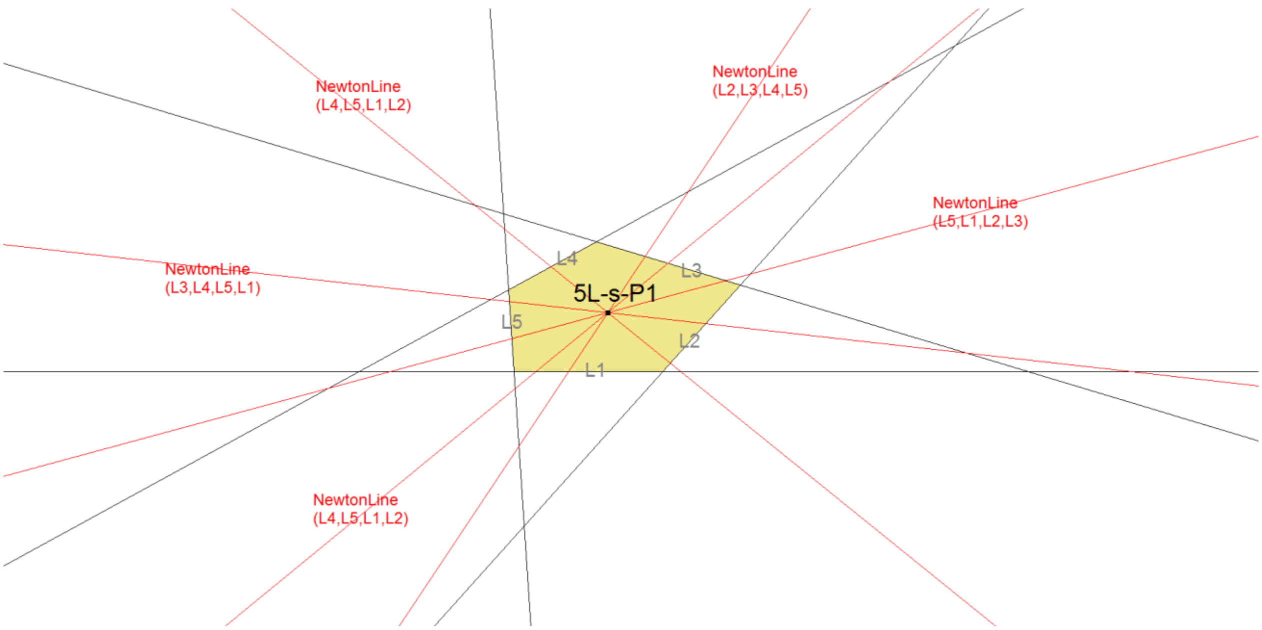
Specific objects in a 5-Line are object that (according to the latest insights) cannot be generalized to recursive objects related to an n-Line.

5L-s-P1: 5L-Inscribed Conic Center

It is well known that in a system of 5 random lines a unique inscribed conic can be constructed. This conic is 5L-s-Co1 and 5L-s-P1 is the center of this conic.
In a 4-Line the Newton Line (QL-L1) is the locus of the centers of all 4L-inscribed conics.
Consequently the Newton Lines of the 5 Component 4-Lines pass through the Center of the 5L-Inscribed Conic.

Construction:

A simple way of construction of 5L-s-P1 is by drawing the Newton Lines (QL-L1) of two Component 4-Lines. The intersection point of these lines will be the center of the 5L-Inscribed Conic.



Coordinates:

When using barycentric coordinates/coefficients:

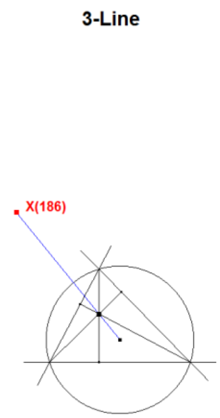
L1=(1:0:0), L2=(0:1:0), L3=(0:0:1), L4=(l:m:n), L5=(L:M:N),

then 5L-s-P1 has coordinates:

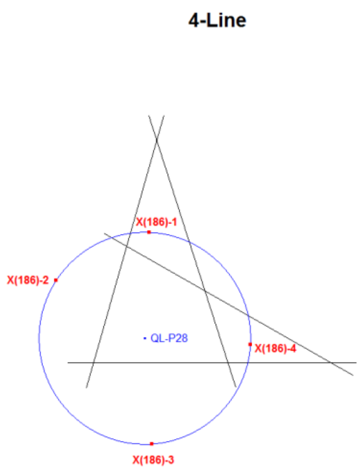
$$(m\,n\,L\,(M - N) - M\,N\;l\,(m - n)\;:\; n\,l\;M\,(N - L) - N\;L\,m\,(n - l)\;:\; l\,m\,N\,(L - M) - L\;M\,n\,(l - m)\,)$$

5L-s-P2: 5L-X(186)-Hofstadter Point

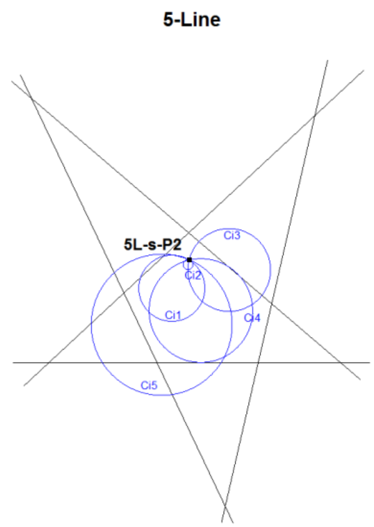
The X(186)-circles of the 4-Lines (described at QL-P28: Circumcenter QL-X(186)-Quadrangle) applied in the 5-Line have one point in common (See Ref-34, QFG#82, Seiichi Kirikami).



X(186) is the Inverse of the Orthocenter X(4) in the Circumcircle.



The versions of X(186) in the 4 Component Triangles of a 5-Line are concyclic on a circle 4L-Ci(186).



The 5 versions of 4L-Ci(186) coincide in 5L-s-P2.

This sequence of *Point --> Circle --> Common-point* is typical for Hofstadter Triangle Points H(n) for n = integer <> -1, 0, +1. See remarks below.

Properties:

- The same procedure exists for X(3), which also is a Hofstadter Point in a 3-Line. In a 4-Line the X(3)-versions are concyclic on 4L-n-Ci1 and the 5 versions of this circle in a 5-Line concur in 5L-n-P1. However this sequel continues infinitely for n>5, which is not the case for X(186).
- 5L-s-P2 lies on the Centercircle of the 5-Line 5L-n-Ci1. See nL-n-Ci1.

Hofstadter Triangle Points

In the beginning of the 1990's Prof. Doug Hofstadter observed a special type of triangle point.
A description of these points can be found at Ref-58 **Clark Kimberling,** "Hofstadter points," *Nieuw Archief voor Wiskunde* 12 (1994) 109-114.
Hofstadter points also are defined in ETC (See Ref-12) at the preamble of X(360):

> *Let r denote a real number, but not 0 or 1. Using vertex B as a pivot, swing line BC toward vertex A through angle rB and swing line BC about C through angle rC. Let A(r) be the point in which the two swung lines meet. Obtain B(r) and C(r) cyclically. Triangle A(r)B(r)C(r) is the r-Hofstadter triangle; its perspector with ABC is called the Hofstadter H(r) point.*

A subset of these points being the *n-Angle Centers* are defined by Ngo Quang Duong in Ref-34, QFG#1843:

> *Pn is "n-angle center" of triangle ABC if (PB,PC)=n(AB,AC)(mod pi); (PC,PA)=n(BC,BA)(mod pi) then of course we have (PA,PB)=n(CA,CB)(mod pi).*

Hofstadter points H(r) are defined for r=real number.

n-Angle Centers P(n) are points defined for n=integer <> 0 and 1.
It appears that the n-Angle Centers P(n) match with the Hofstadter points H(n) provided that n = integer <> 0 and 1.
For another summary and extra properties of these points see QL-P-1 and Ref-34, QFG#1872.
Finally following n-Angle Centers 3L-P(n) are relevant in a 5-Line:
etc.
3L-P(-4) = X(5964)
3L-P(-3) = X(5962)
3L-P(-2) = X(265)
3L-P(-1) = X(4)
3L-P(0) = undefined
3L-P(+1) = undefined
3L-P(+2) = X(3)
3L-P(+3) = X(186)
3L-P(+4) = X(5961)
3L-P(+5) = X(5963)
etc.

Most important is that the n-Angle Centers 3L-P(n) for n<>-1,0,1 have in common that:

1. In a 4-Line the 4 versions of 3L-P(n) of the Component Triangles are concyclic on a circle 4L-Ci(n) with Center 4L-P(n).
2. In a 5-Line the 5 versions of 4L-Ci(n) of the Component 4-Lines concur in a point 5L-P(n).

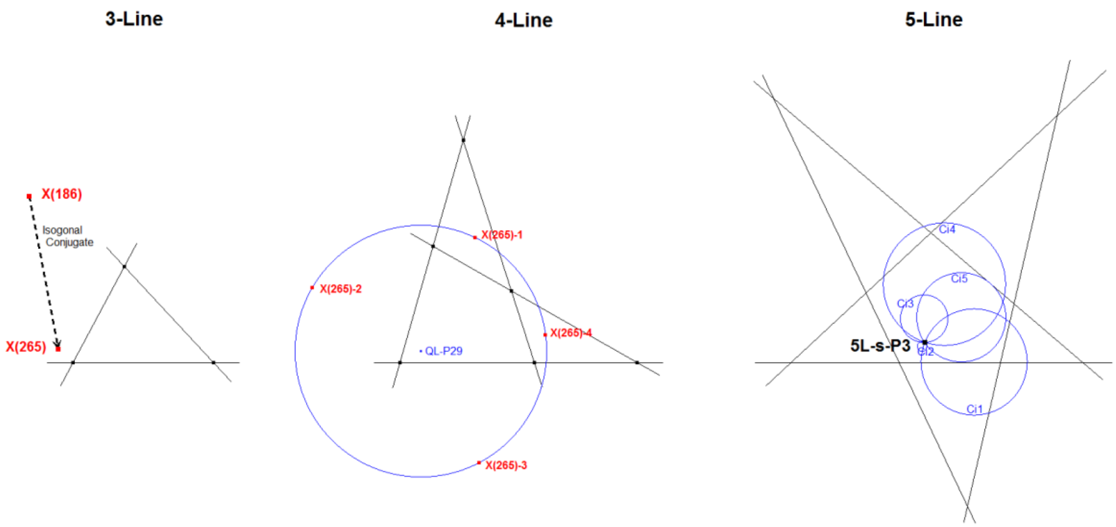
The properties of this subset of the Hofstadter Points were gradually discovered in discussions at the Quadri-Forum (Ref-34) in 2013-2015 by Seiichi Kirikami, Chris van Tienhoven, Ngo Quang Duong, Tsihong Lau, Eckart Schmidt and Bernard Keizer.
It is fair to say that to date no exact proof has been found for the existence of 4L-Ci(n), 4L-P(n) and 5L-P(n) relating to the Hofstadter Points 3L-P(n), though several drawings confirm the validity of the mentioned conjectured properties.

Correspondence with ETC/EQF/EPG:

n	**3L-P(n)**	**4L-Ci(n)**	**4L-P(n)**	**5L-P(n)**
-4	X(5964)	no name	no name	no name
-3	X(5962)	no name	no name	no name
-2	X(265)	no name	QL-P29	5L-s-P3
-1	X(4)	---	---	---
0	undefined	---	---	---
+1	undefined	---	---	---
+2	X(3)	QL-Ci3	QL-P4	5L-n-P1
+3	X(186)	no name	QL-P28	5L-s-P2
+4	X(5961)	no name	no name	no name
+5	X(5963)	no name	no name	no name

5L-s-P3: 5L-X(265)-Hofstadter Point

The X(265)-circles of the 4-Lines (described at QL-P29: Circumcenter QL-X(265)-Quadrangle) in a 5-Line have one point in common (See Ref-34, QFG#82, Seiichi Kirikami).



X(265) is the Isogonal Conjugate of X(186)

The versions of X(256) in the 4 Component Triangles of a 5-Line are concyclic on a circle 4L-Ci(265).

The 5 versions of 4L-Ci(265) coincide in 5L-s-P3.

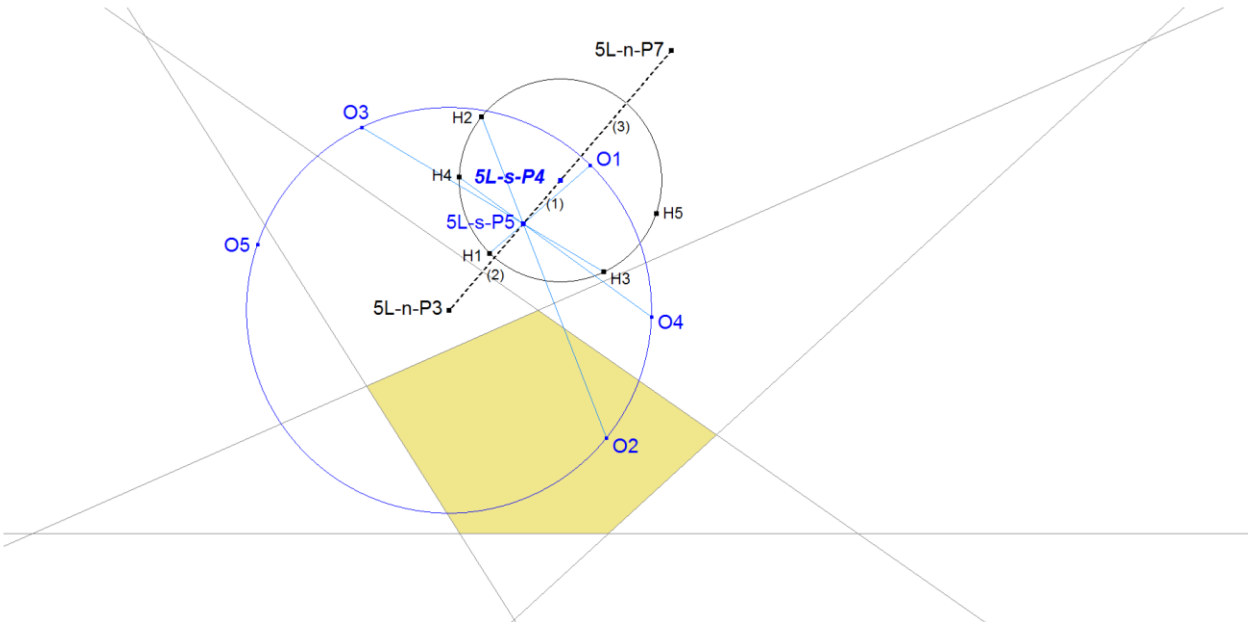
This sequence of *Point --> Circle --> Common-point* is typical for Hofstadter Triangle Points H(n) for n = integer <> -1, 0, +1.
For explanation of Hofstadter Triangle Points see remarks 5L-s-P2.

Properties:

- The same procedure exists for X(3), which also is a Hofstadter Point in a 3-Line. In a 4-Line the X(3)-versions are concyclic on 4L-n-Ci1 and the 5 versions of this circle in a 5-Line concur in 5L-n-P1. However this sequel continues infinitely for n>5, which is not the case for X(265).

5L-s-P4: 5L-Center of the Anticenter Circle

Let Oi (i=1,2,3,4,5) be the concyclic 4L-Circumcenters.
Let Hi be the Anticenter (see Ref-13) of Oj.Ok.Ol.Om, where (i,j,k,l,m) are different numbers from (1,2,3,4,5).
H1, H2, H3, H4, H5 are concyclic on a circle with center 5L-s-P4.
Hi.Oi (i=1,2,3,4,5) have a common point. See 5L-s-P5.
See Ref-34, QFG#1904.

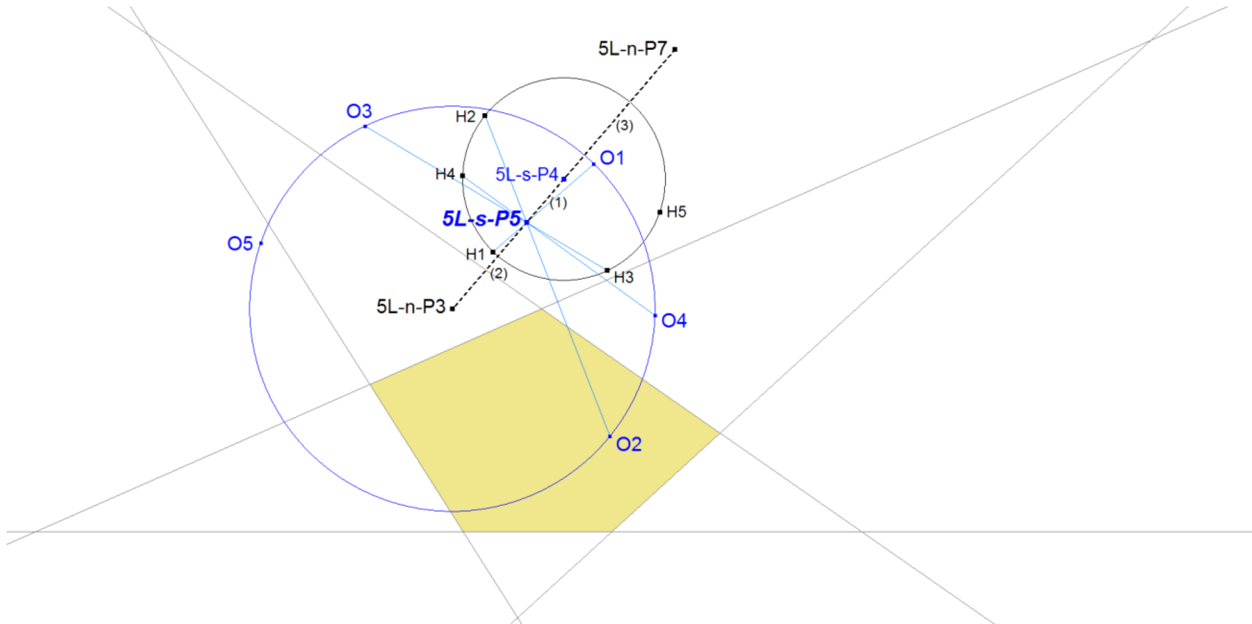


Properties:

- 5L-s-P4 is the midpoint of 5L-n-P3 and 5L-n-P7.
- Radius O-circle = 2 * Radius H-circle.
- The distribution of H-Points on the H-circle is similar to the distribution of O-points on the O-circle. They represent the angles between the defining lines of the 5-Line. See Ref-34, QFG#1893.

5L-s-P5: 5L-OH Division Point

Let Oi (i=1,2,3,4,5) be the concyclic 4L-Circumcenters (4L-n-P3).
Let Hi be the anticenter of Oj.Ok.Ol.Om, where (i,j,k,l,m) are different numbers from (1,2,3,4,5).
H1, H2, H3, H4, H5 are concyclic on a circle with center 5L-s-P4.
The lines Hi.Oi (i=1,2,3,4,5) have a common point 5L-s-P5.
5L-s-P5 divides 5L-n-P3.5L-n-P7 as well as Hi.Oi (i=1,2,3,4,5) in parts (1:2).
There is a remarkable resemblance in a 4-Line where QL-P5 is dividing Hi.Oi (1:1).
See also Ref-34, QFG#1904.

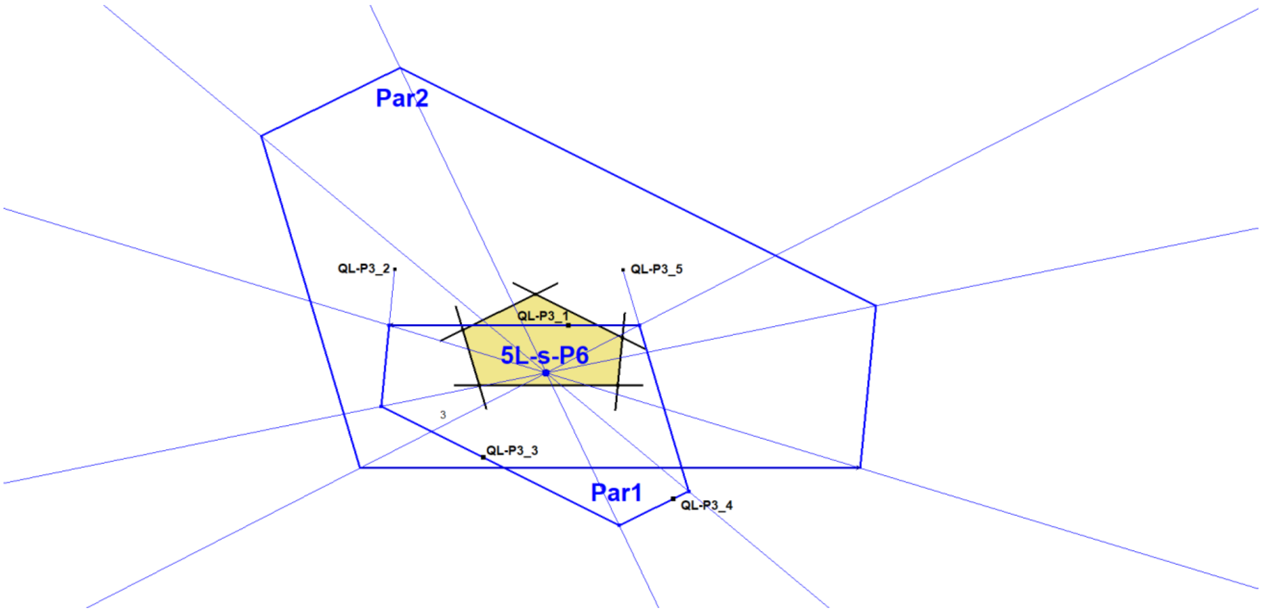


Properties:

- 5L-s-P5 is the Ratiopoint 5L-n-P3.5L-n-P7(2:1).
- Radius O-circle = 2 * Radius H-circle.
- The distribution of H-Points on the H-circle is similar to the distribution of O-points on the O-circle. They represent the angles between the defining lines of the 5-Line. See Ref-34, QFG#1893.

5L-s-P6: 5L-QL-P3 Par1/Par2-Homothetic Center

5L-s-P6 is the Par1/Par2-Homothetic Center (see nL-n-Luc5e) of QL-P3 wrt the Reference 5-Line.

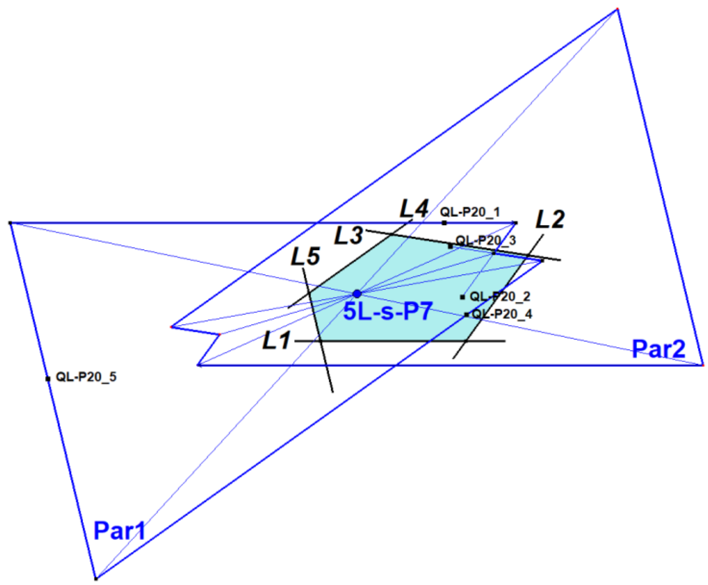


Properties:

- 5L-s-P6 = 5L-n-P7. 5L-o-P1 (1:2).
- The lengths of line segments of Par2 are twice as long as corresponding line segments of Par1.

5L-s-P7:   5L-QL-P20 Par1/Par2-Homothetic Center

5L-s-P7 is the Par1/Par2-Homothetic Center (see nL-n-Luc5e) of QL-P20 wrt the Reference 5-Line.

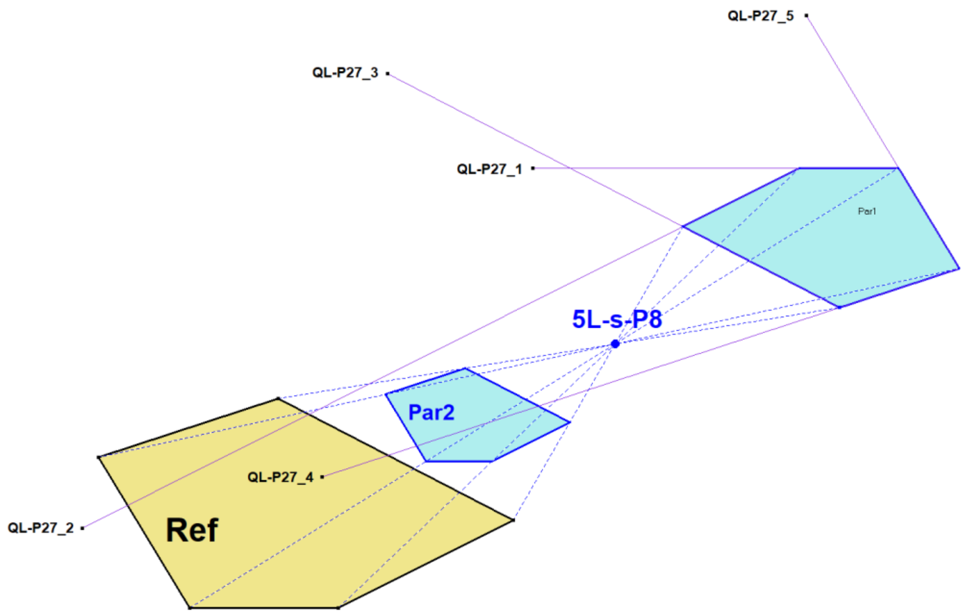


Properties:

- 5L-s-P7 = 5L-n-P2.5L-n-P7 (-1:5)
- The lengths of the line segments of Ref are equal to the corresponding line segments of Par2.

5L-s-P8: 5L-QL-P27 Ref/Par1/Par2-Homothetic Center

5L-s-P8 is the Ref/Par1/Par2-Homothetic Center (see nL-n-Luc5a) of QL-P27 wrt the Reference 5-Line.

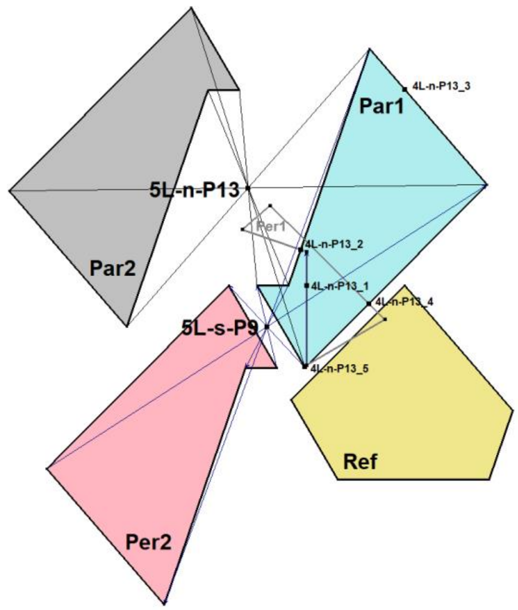


Properties:

- 5L-s-P8 lies on these lines:
  - 5L-n-P2.5L-n-P8 (Reflection of 5L-n-P2 in 5L-n-P8)
  - 5L-n-P4.5L-n-p4 (2 : 3)
  - 5L-n-P9.5L-n-g0.5L-n-g1 (midpoint (g0,g1))
  - 5L-n-P11.5L-n-g2 (-1 : 4)
- The lengths of the line segments of Ref are 1.5 as long as the corresponding line segments of Par1. Consequently the line segments of Par1 are 1.5 as long as the corresponding line segments of Par2.

5L-s-P9: 5L-QL-P28 Par1/Per2-Homothetic Center

5L-s-P9 is the Par1/Per2-Homothetic Center (see nL-n-Luc5g) of QL-P28 wrt the Reference 5-Line.
There is an interesting relationship with 5L-n-P13 (Par1/Par2-Homothetic Center of QL-P28).

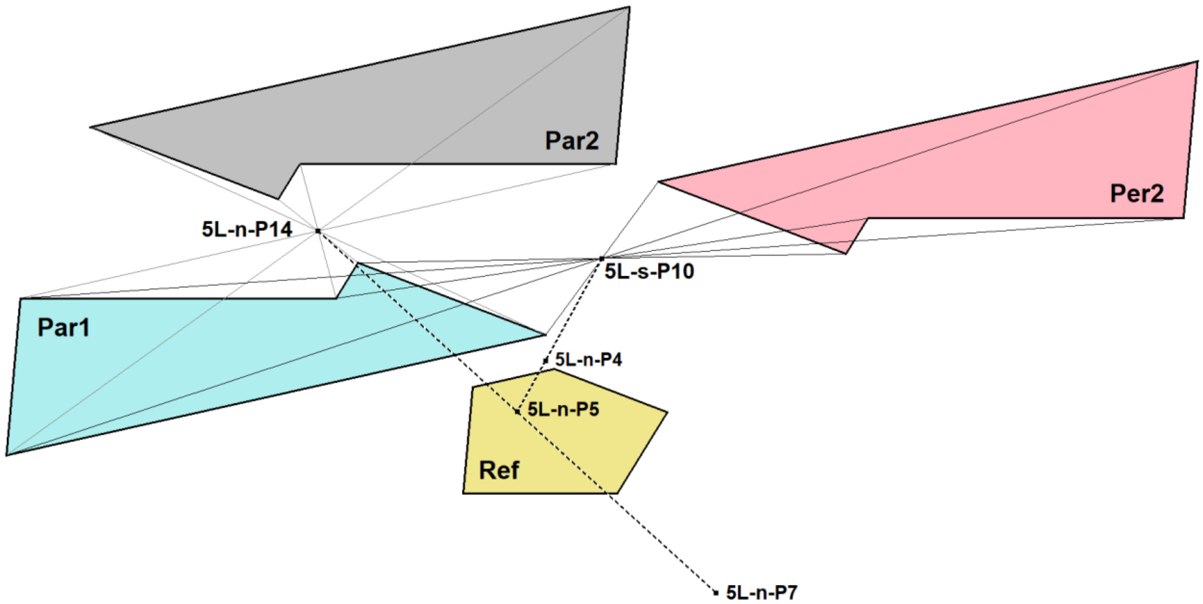


Properties:

- There are no linear relations with other known 5L-points.

5L-s-P10: 5L-QL-P29 Par1/Per2-Homothetic Center

5L-s-P10 is the Par1/Per2-Homothetic Center (see nL-n-Luc5g) of QL-P29 wrt the Reference 5-Line. There is an interesting relationship with 5L-n-P14 (Par1/Par2-Homothetic Center of QL-P29).



Properties:

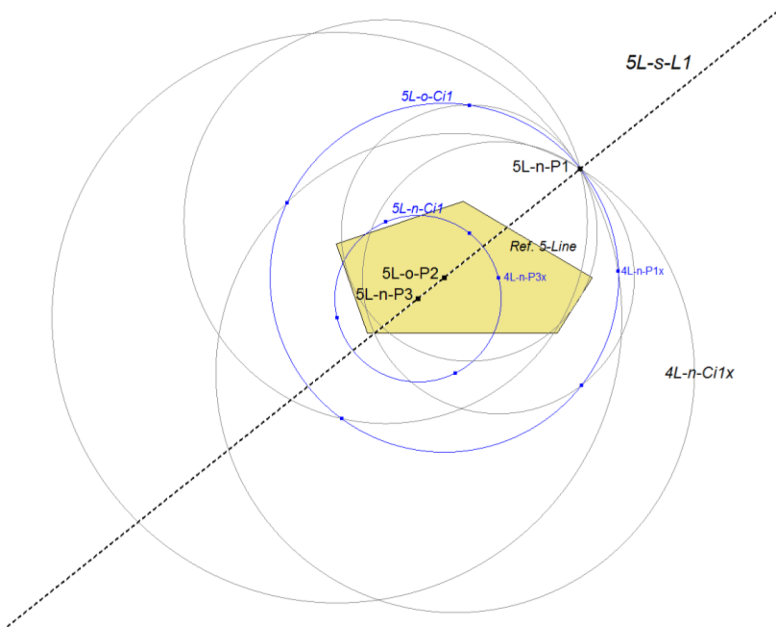
- 5L-s-P10 = 5L-n-P5.5L-n-P7 (-1:2)

5L-s-L1: 5L-Miquel Line

A 5-Line contains 5 Quadrilaterals also called 4-Lines (by Morley).
Each 5-Line has 5 Component 4-Lines.

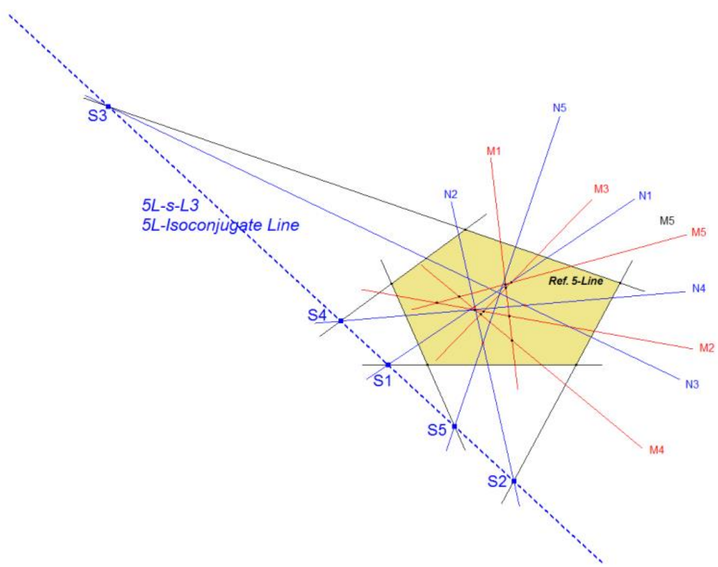
1. So a 5-Line has 5 versions of 4L-n-P1**:** Miquel Points (QL-P1).
   They are concentric on 5L-o-Ci1 and their center is 5L-o-P2.
2. So a 5-Line has 5 versions of 4L-n-P3**:** Miquel Circumcenters (QL-P4).
   They are concentric on 5L-n-Ci1 and their center is 5L-n-P3.
3. So a 5-Line has 5 versions of 4L-n-Ci1**:** Miquel Circles (QL-Ci3).
   They have one common point 5L-n-P1.

5L-o-P2, 5L-n-P3 and 5L-n-P1 are collinear on 5L-s-L1. See also Ref-34, QFG#710.
This collinearity is not a general property and only valid in a 5-Line and for example not in a 7-Line.



5L-s-L2: 5L-Isoconjugate Line

5L-s-L2 is created by using the Quadrilateral transformation QL-Tf2 (Isoconjugation for Lines).
For an explanation of the notion of Isoconjugation see Ref-13.
A 5-Line contains 5 Component Quadrilaterals / Component 4-Lines. When using a Component 4-Line one Line is not used or “omitted”.
Let Mi = QL-Tf2(Li), where Li is the omitted line for i=1,2,3,4,5.
In this way a new 5-Line M1.M2.M3.M4.M5 is created.
We can do the same procedure one level deeper,
Let Ni = QL-Tf2(Mi), where Mi is the omitted line for i=1,2,3,4,5.
In this way a new 5-Line N1.N2.N3.N4.N5 is created.
Now the 5 intersections of lines Li and Ni will be collinear on 5L-s-L2.
Summarized:
By applying the QL-Tf2 transformation (Isoconjugation for Lines) twice a 2nd level 5-Line is created.
The intersection points of corresponding lines of the reference 5-Line and the 2nd level 5-Line will be collinear on 5L-s-L2.
This is the dual case of 5P-s-P2.
See Ref-34, QFG#784.

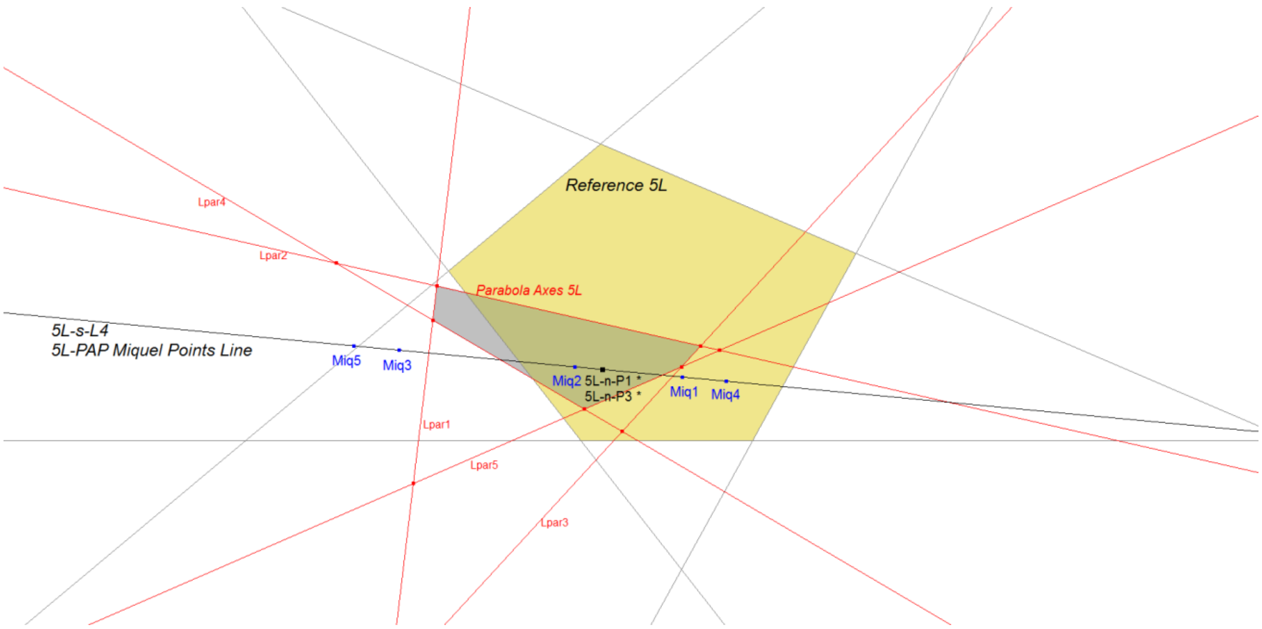


5L-s-L3: 5L-PAP Miquel Points Line

This line was contributed by Eckart Schmidt.
PAP stands for "Parabola Axes Pentalateral".
Consider the 5-Line of the QL-Co1-axes (Axis of the Quadrilateral Inscribed Parabola) of the QL-components:

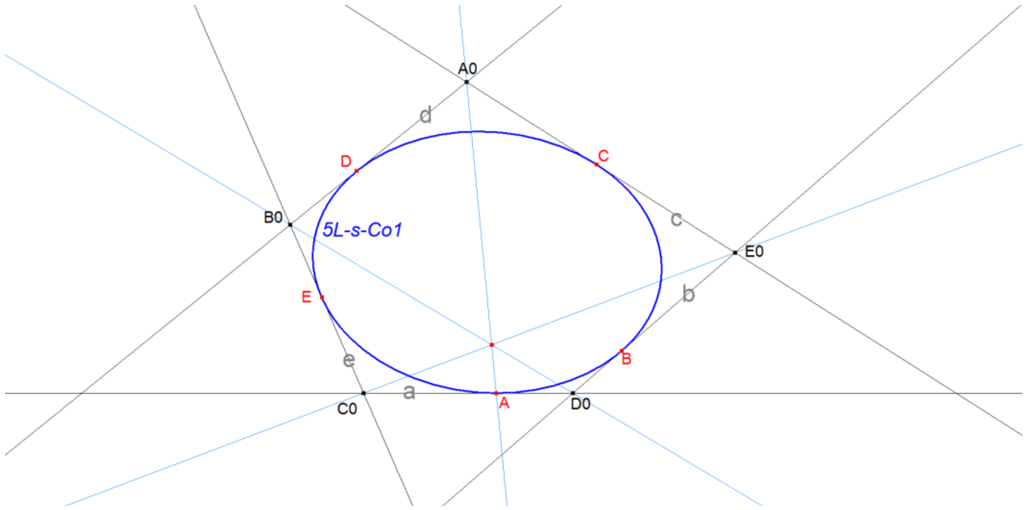
- The Miquel Points of the QL-Components of this 5L are collinear on 5L-s-L3.
- The point 5L-n-P3 of this 5-Line is also a point on this line.
- The points 5L-n-P1 and 5L-n-P3 of this 5-Line are the same.

See also Ref-34, QFG#748.



5L-s-Co1: 5L-Inscribed Conic

It is well known that in a system of 5 random lines a unique inscribed conic can be constructed.



Construction:
See Ref-19.

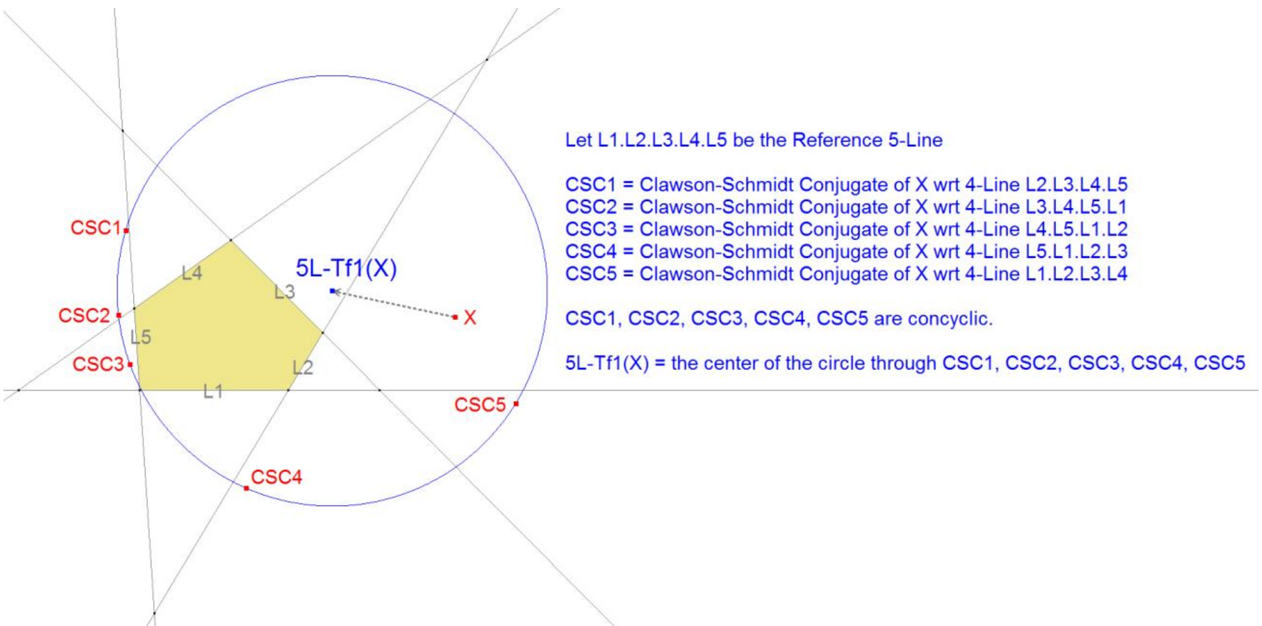
1. Given five lines a, b, c, d, e.
2. Let A0 = cd, B0 = de, C0 = ea, D0 = ab, E0 = bc.
3. The line from A0 through B0D0.C0E0 cuts a in A on the conic, and so on cyclically.
4. We now have five points A, B, C, D, E on the curve.
5. Now go further with construction 5P-s-Co1.

Properties:

- 5L-s-P1 is the center of this conic.

5L-s-Tf1: 5L-Schmidt Transformation

5L-s-Tf1: the 5L-Schmidt Transformation is a sequel to QL-Tf1: the Clawson-Schmidt Conjugate (often abbreviated with CSC).
Consider an arbitrary point X and its Clawson-Schmidt Conjugates wrt the 5 Component Quadrilaterals of the 5-Line. These points are concyclic on a circle Ci with center X', which shall be the 5L-Tf1-image of X.
This transformation was introduced by Eckart Schmidt. See Ref-34, QFG#713.
Unlike QL-Tf1 (Clawson-Schmidt Conjugate) 5L-s-Tf1 is not a reciprocal transformation, meaning that the output not will be re-transferred into the input by the transformation.
Special is that the Ci-circle of an intersection of two 5L-lines is the circumcircle of the triangle of the remaining three 5L-lines.
The transformation is of the 2nd degree. Therefore it maps lines into conics,



Properties:

- the 5L-s-Tf1-image of 5L-o-P2.5L-n-P3 is a conic, centered in 5L-s-P1, containing 5L-o-P2 (Eckart Schmidt, QFG#713).
- the 5L-s-Tf1-image of any line contains 5L-o-P2 (Eckart Schmidt, QFG#713).
- 5L-s-Tf1 swaps the foci of the inscribed conic (Eckart Schmidt, QFG#722).
- the image of 5L-o-P2 is the reflection in 5L-s-P1 (Eckart Schmidt, QFG#722).
- the image of the QL-P1-circle is the line at infinity (Eckart Schmidt, QFG#722).
- the image of a QL-P1-point is the point at infinity of a perpendicular line to the corresponding QL-line (Eckart Schmidt, QFG#722).
- the image of the QL-P4-circle is a hyperbola (Eckart Schmidt, QFG#722).
- the image of the main axis of the inscribed conic is a conic through the foci and 5L-p1, symmetric to the second axis (Eckart Schmidt, QFG#722).
- The CSC-circle of an intersection of 2 5L-lines is the circumcircle of the triangle of the remaining 3 5L-lines (Eckart Schmidt, QFG#754).
- The 5L-s-Tf1-image of the QL-P1-circle is the line at infinity (Eckart Schmidt, QFG#754).

- The CSC-circles of points on the QL-P1-circle degenerate to lines, tangent to the inscribed conic of the 5L (Eckart Schmidt, QFG#754).
- For the QL-P1-points the degenerated CSC-circles are the lines Li of the 5L (Eckart Schmidt, QFG#754).
- For intersections of a line Li and the QL-P1-circle the degenerated CSC-circle contains the 2nd intersection and QL-P1 of the remaining QL (Eckart Schmidt, QFG#754).
- For a point X on the QL-P1-circle the degenerated CSC-circle cuts the QL-P1-circle in 2 points, whose connections with X are tangent to the inscribed conic (Eckart Schmidt, QFG#754).
- For 2 diametral points on the QL-P1-circle the degenerated CSC-circles intersect on a line perpendicular 5L-o-P2.5L-s-P1 (Eckart Schmidt, QFG#754).
- The CSC-circle of 5L-s-P2 contains 5L-s-P3 and the CSC-circle of 5L-s-P3 contains 5L-s-P2 (Eckart Schmidt, QFG#754).
- For 2 points inverse wrt the QL-P1-circle the 5L-s-Tf1-images are symmetric wrt 5L-s-P1 (Eckart Schmidt, QFG#754).
- 5L-s-Tf1(Li^Lj) = Circumcenter (Lk,Ll,Lm), where (i,j,k,l,m) are different numbers from (1,2,3,4,5) (Bernard Keizer, QFG#786).

For those interested in more properties follows here QFG#762 from Eckart Schmidt:

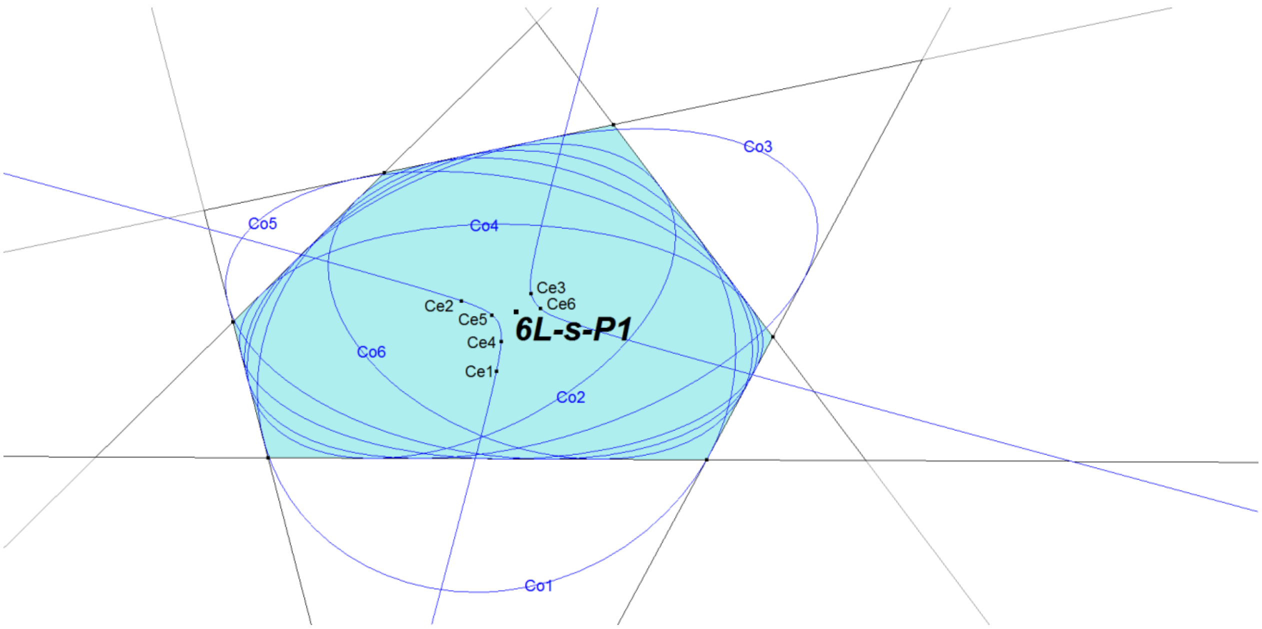
wrt 5L-s-Tf1 (see #713, #722, #754) further properties:
Remember: 5L-s-Tf1 maps a point X to the center of the concyclic 5 CSC-images of X.
... Let F1 and F2 be the foci of the inscribed conic of 5L, swapped by 5L-s-Tf1 (see #722).
... Let F1° and F2° be their inverses wrt the QL-P1-circle, then 5L-s-Tf1(Fi°) = Fi.
... So there are two points F1 and F2° with 5L-s-Tf1-image F2 ...
... Let X be the intersection of F1.F2° and F1°.F2, then holds 5L-s-Tf1(X) = 5L-P1.
... Lines through X have a line through 5L-P1 as 5L-s-Tf1-image...
... with intersections on an orthogonal hyperbola Hy ...
... through X, 5L-P1, F1° and F2° and a tangent in 5L-P1 through 5L-P4.
... (The inverse of Hy wrt the QL-P1-circle is the strophoid of 5L-P1.5L-P4 with pole 5L-P1 and fixed point Y, which is the inverse wrt the QL-P1-circle of the reflection of 5L-P1 in the center of Hy.)
... The transformation 5L-s-Tf1 has three fixed points. They are the intersections of the orthogonal hyperbola Hy and its 5L-s-Tf1-image (a curve of degree 5).
... Generally there are two preimages P' and P'' of a point P wrt 5L-s-Tf1.
... The line P'P'' contains the point X (see above).
... The CSC-circles of P' and P'' are inverse wrt a circle Ci(P) round P with radius sqr(PF1*PF2).
... Construction of the second Point P'', if P and P' are known: Let CSC-Ci(P') be the CSC-circle of P', then the inverse wrt Ci(P) is CSC-Ci(P'') and the common point of the CSC-images of CSC-Ci(P'') is P''.
The properties are only CABRI-observations!

6L-s: Specific Objects in a 6-Line

Specific objects in a 6-Line are objects that (according to the latest insights) cannot be generalized to recursive objects related to an n-Line.

6L-s-P1: 6L-Conical Center

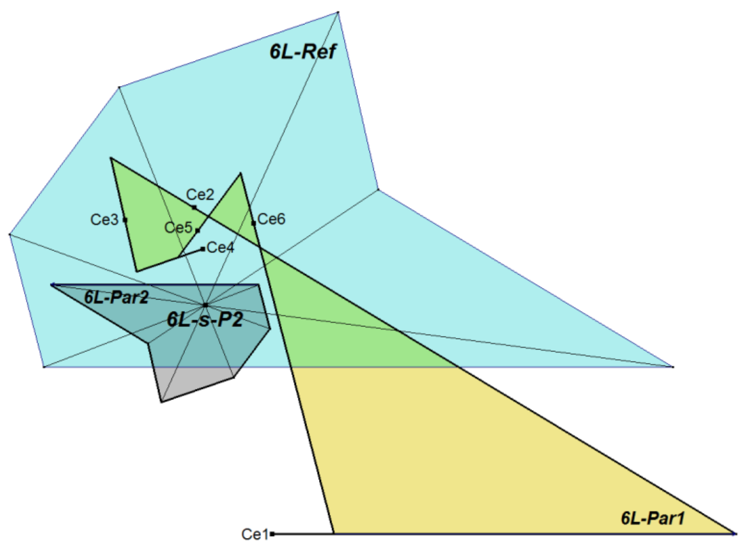
In a 6-Line we have 6 Component 5-Lines.
When constructing 6 times the Inscribed Conic Centers (5L-s-P1) of the Component 5-Lines they are coconic. The center of the conic of these centers is 6L-s-P1.
This feature cannot be extrapolated to a 7-Line.



See also the note at 6L-s-Co1.

6L-s-P2: 6L-Conical Ref-Par2 Homothetic Center

In a 6-Line we have 6 Component 5-Lines.
When constructing 6 times the Inscribed Conic Centers (5L-s-P1) of the Component 5-Lines we get Ce1, … , Ce6. By drawing lines through these centers parallel to the omitted line (not used line of the 5-Line in the 6-Line) we get a 6-Line called 6L-Par1. Doing the same procedure for 6L-Par1 instead of the reference 6-Line we get a 2nd generation 6-Line 6L-Par2. 6L-Par2 is homothetic with the reference 6-Line 6L-Ref, giving rise to a Homothetic Center 6L-s-P2.

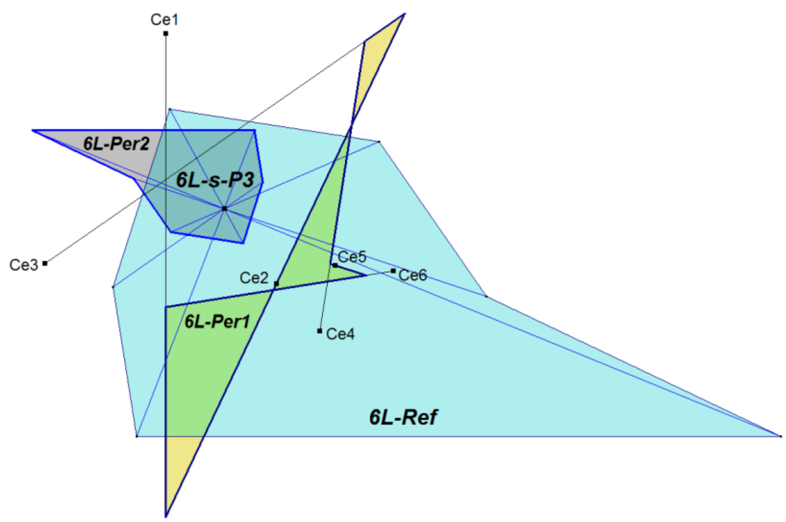


Properties:

- The points
  6L-s-P2 (6L-Conical Ref-Par2 Homothetic Center),
  6L-s-P3 (6L-Conical Ref-Per2 Homothetic Center) and
  6L-s-P4 (6L-Conical Par2-Per2 Homothetic Center) are collinear.

6L-s-P3: 6L-Conical Ref-Per2 Homothetic Center

In a 6-Line we have 6 Component 5-Lines.
When constructing 6 times the Inscribed Conic Centers (5L-s-P1) of the Component 5-Lines we get Ce1, … , Ce6. By drawing lines through these centers perpendicular to the omitted line (not used line of the 5-Line in the 6-Line) we get a 6-Line called 6L-Per1. Doing the same procedure for 6L-Per1 instead of the reference 6-Line we get a 2nd generation 6-Line 6L-Per2. 6L-Per2 is homothetic with the reference 6-Line 6L-Ref, giving rise to a Homothetic Center 6L-s-P3.

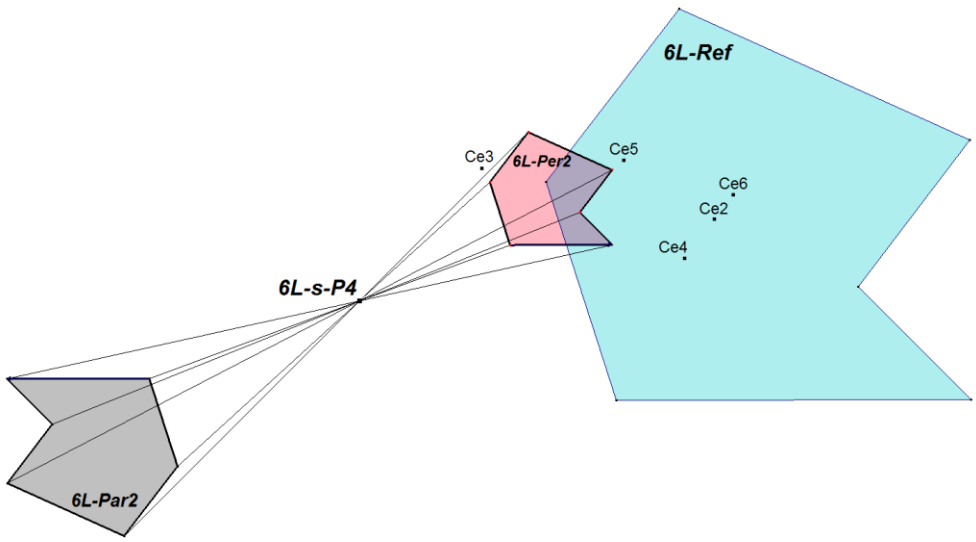


Properties:

- The points
  6L-s-P2 (6L-Conical Ref-Par2 Homothetic Center),
  6L-s-P3 (6L-Conical Ref-Per2 Homothetic Center) and
  6L-s-P4 (6L-Conical Par2-Per2 Homothetic Center) are collinear.

6L-s-P4: 6L-Conical Par2-Per2 Homothetic Center

In a 6-Line we have 6 Component 5-Lines.
When constructing 6 times the Inscribed Conic Centers (5L-s-P1) of the Component 5-Lines we get Ce1, … , Ce6.
By drawing lines through Ce1, … , Ce6 ***parallel*** to the omitted line (not used line of the 5-Line in the 6-Line) we get a 6-Line called 6L-Par1. Doing the same procedure for 6L-Par1 instead of the reference 6-Line 6L-Ref we get a 2nd generation 6-Line 6L-Par2.
By drawing lines through Ce1, … , Ce6 ***perpendicular*** to the omitted line (not used line of the 5-Line in the 6-Line) we get a 6-Line called 6L-Per1. Doing the same procedure for 6L-Per1 instead of the reference 6-Line we get a 2nd generation 6-Line 6L-Per2.
6L-Par2 is homothetic with 6L-Per2, giving rise to a Homothetic Center 6L-s-P4.

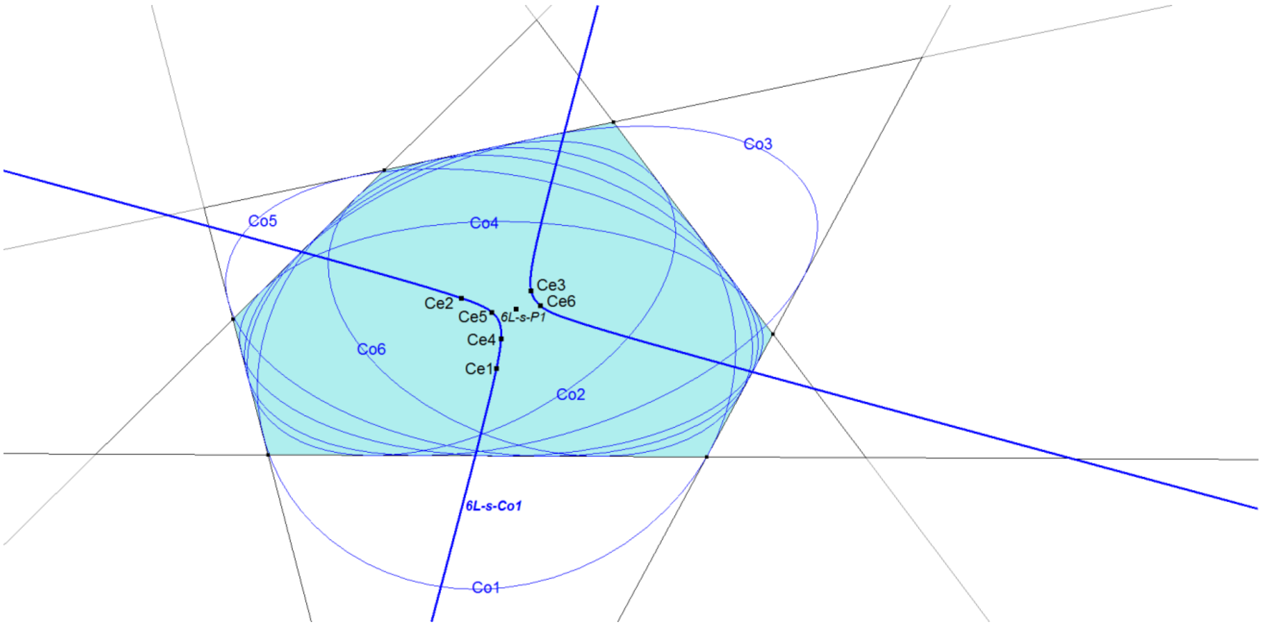


Properties:

- The points
  6L-s-P2 (6L-Conical Ref-Par2 Homothetic Center),
  6L-s-P3 (6L-Conical Ref-Per2 Homothetic Center) and
  6L-s-P4 (6L-Conical Par2-Per2 Homothetic Center) are collinear.

6L-s-Co1: 6L-Conical Center Conic

In a 6-Line we have 6 Component 5-Lines.
When constructing 6 times the Inscribed Conic Centers (5L-s-P1) of the Component 5-Lines they are coconic on 6L-s-Co1.
This feature cannot be extrapolated to a 7-Line.



Relationship with Pascal's Theorem:
Note that for 6L-s-Co1 we have a case of 6 points on a conic *without* order.
Pascal's theorem (see Ref-13) is valid for 6 points on a conic *with* order, stating that if six random points are chosen on a conic and joined by line segments to form a hexagon (6-Gon), then the three pairs of *opposite sides* of the hexagon meet in three points which lie on a straight line, called the *Pascal line* of the hexagon. Since Pascal's theorem is valid for 6 points in a certain order (by stating it is a hexagon (6-Gon)) there are 60 Pascal lines (from the 60 possible 6-Gons) crossing at 20 Points (so-called Steiner Points). These derived 60 lines and 20 points also can be considered as central EPG-objects in 6-Line.

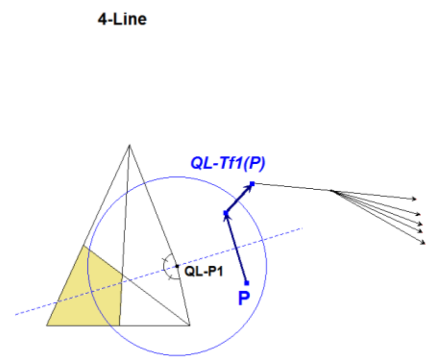
Properties:

- In a 7-Line the 7 versions of 6L-s-Co1 concur in 3 points 7L-s-3P1a/b/c.

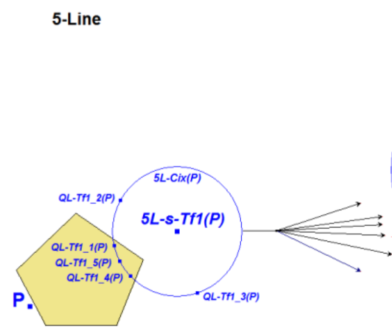
6L-s-Tf1 6L-Schmidt Transformation

6L-s-Tf1 was discovered by Eckart Schmidt in 2014 and described in Ref-34, QFG #784, #786, #861, #864.
The QL-Tf1-images of a point P wrt the 4-lines of a 5-line lie on a circle, these circles for the 5-lines of a 6-line have a common point 6L-s-Tf1(P).
In steps:

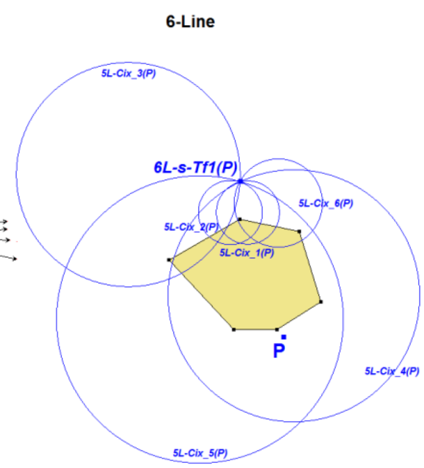
- The 4L-Transformation QL-Tf1 transforms a point P into another point QL-Tf1(p).
- In a 5-Line we have 5 4-Lines and consequently a point P can be transformed into 5 other points which are concyclic on a circle 5L-Cix with center 5L-s-Tf1(P).
- In a 6-Line we have 6 5-Lines and the 6 circles 5L-Cix are concurrent in 6L-s-Tf1(P).



QL-Tf1 is the combination of a reflection and an inversion.



The 5 versions of QL-Tf1 in a 5-Line are concyclic on 5L-Cix(P).



The 6 versions of 5L-Cix in a 6-Line are concurrent in 6L-s-Tf1(P).

In Ref-34, QFG#920, Eckart Schmidt gave this analyses for Morley's notes on a similar transformation in a 2n-Lines *with an inscribed conic* in relationship with the transformations QL-Tf1 (also named CSC), 5L-s-Tf1 (also named 5L-CSC) and 6L-s-Tf1 (also named (6L-CSC):

> *In §6 of his paper Ref-48 "On the metric geometry of the plane n-line" Morley researched 2n-lines* ***with an inscribed conic****. Here are cited his results:*
> *"This involution has the following properties:*
> *(i) Its center is the Clifford point of the 2p-lines.*
> *(ii) The foci are a pair of the involution.*
> *(iii) The Clifford point of 2q lines and that of the remaining 2(p-q) lines are a pair of the involution. The Clifford point of two lines means merely their intersection.*
> *(iv) The Clifford circle of 2q-1 lines and that of the remaining lines are partners. The Clifford circle of a line is merely the line itself."*
> *For a 4-line, which has always an inscribed conic, this involution is the CSC-transformation.*
> *For 6-lines with an inscribed conic this involution is the transformation 6L-CSC, mentioned in QFG #784, #861, #864.*
> *Remember: The CSC-images of a point X wrt the 4-lines of a 5-line lie on a circle, these circles for the 5-lines of a 6-line have a common point, the 6L-CSC-image of the point X.*
> *For an arbitrary 6L the transformation 6L-CSC holds Morley's properties (ii) and (iii):*
> *wrt (ii): 6L-CSC swaps the foci of the inscribed conics of 5 of 6 lines.*

wrt (iii): 6L-CSC swaps the intersection of 2 of 6 lines and the Miquel point of the remaining 4 of 6 lines.
But I don't see any generalization for 2n-CSC!

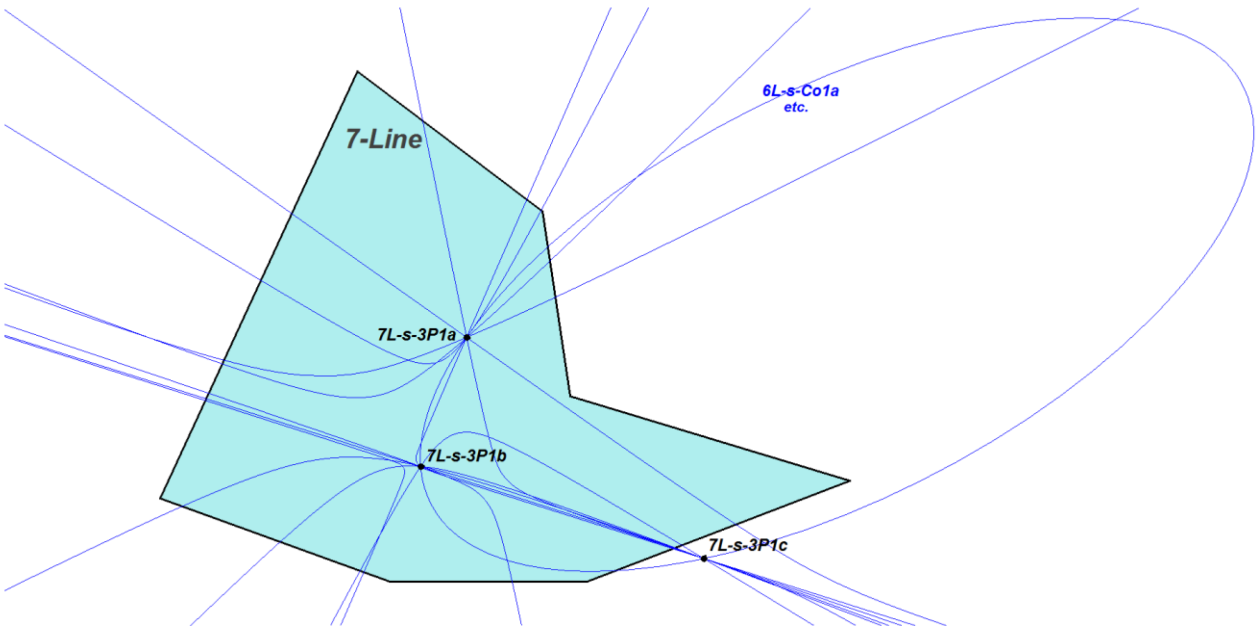
Properties:

- 6L-s-Tf1(P) = 6L-e-P2 for all P on the line at infinity (Cabri-observation).
- 6L-s-Tf1 swaps the foci of the inscribed conics of 5 of 6 lines. See Ref-34, QFG#920.
- 6L-s-Tf1 swaps the intersection of 2 of 6 lines and the Miquel point of the remaining 4 of 6 lines. See Ref-34, QFG#786 and #920.

7L-s-3P1: 7L-Conical Triplet Points

In a 7-Line there are 7 Component 6-Lines.
So 7 6L-s-Co1 conics can be constructed.
They mutually have 4 intersection points, three of which are common points 7L-s-3P1a/b/c.
However 2 of them can be imaginary points.
Finally 7L-s-3P1 is derived as follows:

- in a 5-Line there is the center of the inscribed conic 5L-s-P1,
- in a 6-Line the 6 versions of 5L-s-P1 lie on a conic 6L-s-Co1,
- in a 7-Line the 7 versions of 6L-s-Co1 have 3 common points 7L-s-3P1.



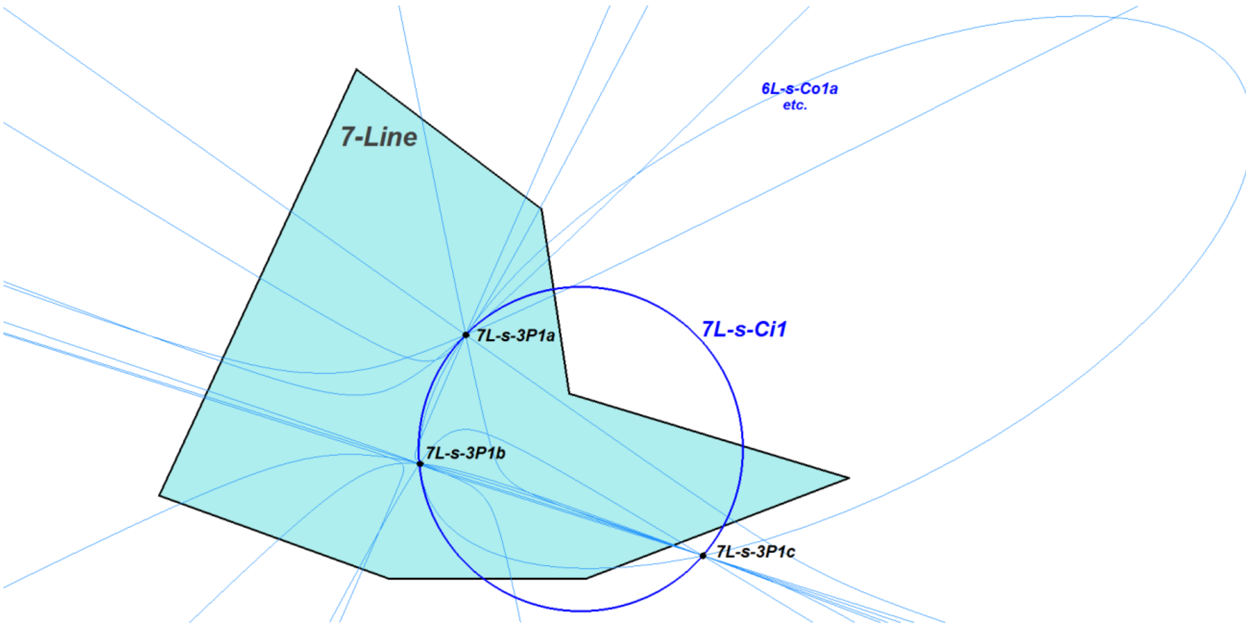
Construction:
The determination of the 3 out of 4 points of intersection is not always easy.
Therefore these steps of construction:

- Construct Co6 (=6L-s-Co1 wrt Ref. 7-Line omitting L6) and Co7 (=6L-s-Co1 wrt Ref. 7-Line omitting L7). See 6L-s-Co1 for the way of construction.
- Because of their definition Co6 and Co7 will meet in the Center of the conic 5L-s-Co1 (L1,L2,L3,L4,L5). Therefore construct 5L-s-P1 (here called ***Ce***) and it will lie on Co6 and Co7.
- Intersect Co6 and Co7 and draw their intersection points. ***Ce*** will be one of these points.
- When 2 points are imaginary the 2 other points will be real and one of them will be Ce. The other real point will be 7L-s-s3P1a, whereas 7L-s-3P1b and 7L-s-3P1c will be imaginary.
- When no points are imaginary construct the QA-Diagonal Triangle ***DT*** of the 4 intersection points. Now the vertices of the Anticevian Triangle of ***Ce*** wrt ***DT*** will be the 3 points 7L-s-3P1a, 7L-s-3P1b and 7L-s-3P1c.

7L-s-Ci1: 7L-Conical Triplet Circle

7L-s-Ci1 is the circumcircle of the 7L-Conical Triplet Points 7L-s-3P1.
7L-s-Ci1 is hierarchically derived as follows:

- in a 5-Line there is the center of the inscribed conic 5L-s-P1,
- in a 6-Line the 6 versions of 5L-s-P1 lie on a conic 6L-s-Co1,
- in a 7-Line the 7 versions of 6L-s-Co1 have 3 common points defining circumcircle 7L-s-Ci1.



Properties:

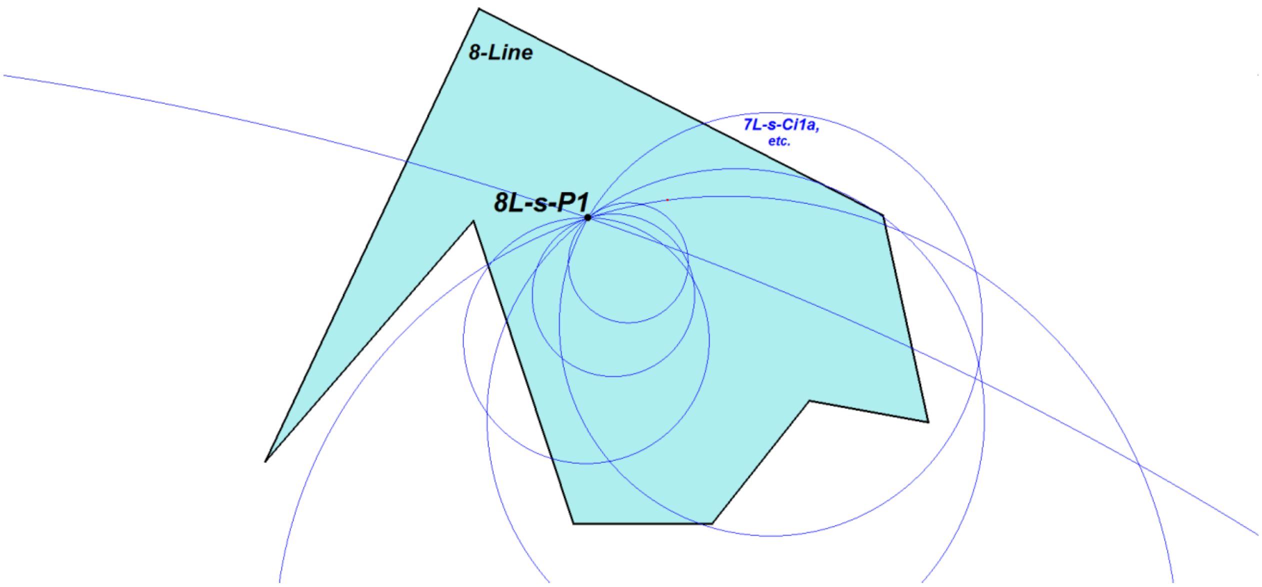
- In an 8-Line the 8 versions of 7L-s-Ci1 are concurrent in one point 8L-s-P1.

8L-s-P1: 8L-Conical Center

Every 8-Line has 8 Component 7-Lines.
The 8 versions of 7L-s-Ci1 in an 8-Line have a common point 8L-s-P1.
At long last 8L-s-P1 is derived as follows:

- in a 5-Line there is the center of the inscribed conic 5L-s-P1,
- in a 6-Line the 6 versions of 5L-s-P1 lie on a conic 6L-s-Co1,
- in a 7-Line the 7 versions of 6L-s-Co1 have 3 common points defining circumcircle 7L-s-Ci1,
- in an 8-Line the 8 versions of 7L-s-Ci1 coincide in 8L-s-P1.

It looks like there is no succession in this range at a higher n-level.



Properties:

- The 9 versions of 8L-s-P1 in a 9-Line are neither concyclic nor coconic.

n-Points

nP-1: General information about Objects in an n-Point

For quick insight pictures of n-Points in EPG often are represented by figures bounded by n line-segments.

How many (n-1)-Points can be made up from an n-Point?
Many of the recursive constructions are based upon the property that from an n-Point exactly n different (n-1)-Points can be made up. This can easily be deduced by omitting one point from the n-Point. This will leave behind an (n-1)-Point. Since exactly n different Points can be omitted there will be n different (n-1)-Points contained in an n-Point. The (n-1)-Points in an n-Point will be called **the Component (n-1)-Points.** The remaining point after choosing an (n-1)-Point in an n-Point will be called **the omitted point**.
In descriptions we say "an n-Point contains n (n-1)-Points" or "an n-Point has n Component (n-1)-Points". When we want to indicate different objects occurring in (n-1)-Points we say that there are n versions of these (n-1)P-objects.
The n versions of an object often will be noted with a suffix consisting of un underscore and a number 1, …, n, indicating the number of the omitted point. For example a 5-Point contains 5 4-Points and therefore has 5 4P-MVP-Centroids (4P-n-P1). They will be noted as 4P-n-P1_1, 4P-n-P1_2, 4P-n-P1_3, 4P-n-P1_4 and 4P-n-P1_5. The suffix number at the end is the number of the omitted point.

Recursive Eulerline situated points in an n-Point
There is a special way of construction of Eulerline points to a higher n-Point level.
This method is based upon the central property that from any n-Point n versions of (n-1)-Points can be constructed. The centroid of these n versions produce a higher level nP-point.

MVP Points: Multi Vector Points:

3P-point	*4P-point*	*5P-point*	*6P-point*	*n-Point*
X(2)	4P-n-P1 = QA-P1	5P-n-P1	6P-n-P1	Etc.
X(3)	4P-n-P2 = QA-P33	5P-n-P2	6P-n-P2	Etc.
X(4)	4P-n-P3 = QA-P32	5P-n-P3	6P-n-P3	Etc.
X(5)	4P-n-P4 = Midpoint (QA-P32,QA-P33)	5P-n-P4	6P-n-P4	Etc.

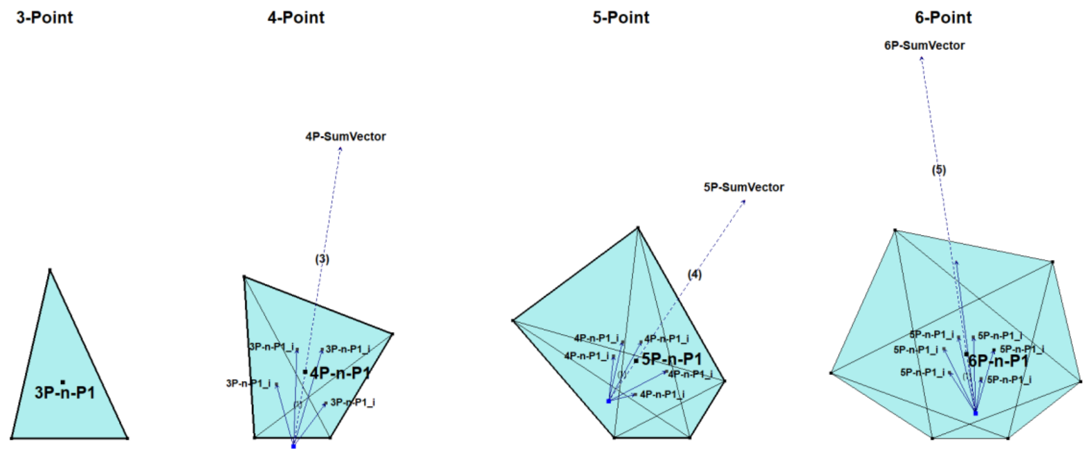
Distance ratios for MVP Eulerline points are preserved at the different n-levels.

X(2)=Centroid, X3=Circumcenter, X(4)=Orthocenter, X(5)=Nine-point Center.

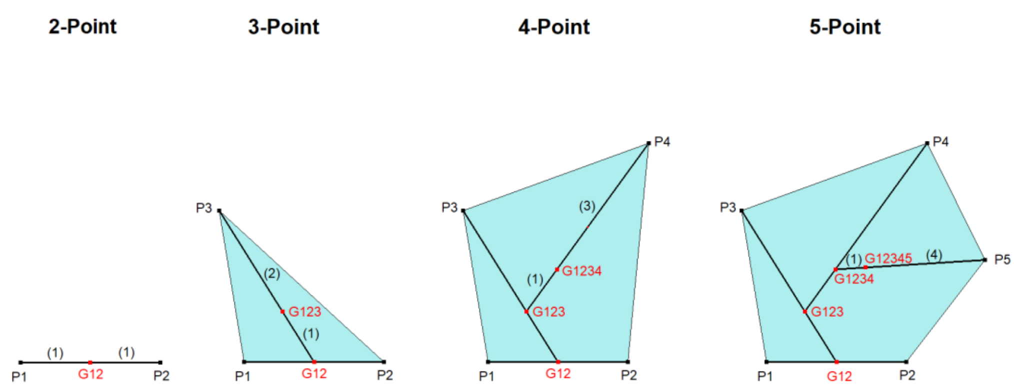
nP-n: General recursive Objects in an n-Point

nP-n-P1 nP-MVP-Centroid

nP-n-P1 is the nL-Mean Vector Point of X(2), the Triangle Centroid.
In this method centroids (or other ETC-points) are successively constructed starting with n=3, then n=4 using the results of n=3, then n=5 using the results of n=4, etc.
See nP-n-Luc1 for a detailed description.



Another construction of nP-n-P1:
There is a simple "chain" for calculating centroids of consecutive n-Point figures.
1-Point Figure: G1 = Point P1
2-Point Figure: G12 = Midpoint (P1,P2) = point dividing G1,P2 with ratio 1 : 1
3-Point Figure: G123 = Centroid (P1,P2,P3) = point dividing G12,P3 with ratio 1 : 2
4-Point Figure: G1234 = Centroid (P1,P2,P3,P4) = point dividing G123,P4 with ratio 1 : 3
5-Point Figure: G12345 = Centroid (P1,P2,P3,P4,P5) = point dividing G1234,P5 with ratio 1 : 4
etc.



The 1st barycentric CT-coordinate is:

When P1=(1:0:0), P2=(0:1:0), P3=(0:0:1), P4=(p:q:r), P5=(P:Q:R), then:
nP-n-P1 = (q + r) (2 P + Q + R) + p (3 P + 2 Q + 2 R)

Correspondence with ETC/EQF:
In a 3-Point:

3P-n-P1 = 3P-MVP Centroid = X(2)
3P-n-P2 = 3P-MVP Circumcenter = X(3)
3P-n-P3 = 3P-MVP Orthocenter = X(4)
3P-n-P4 = 3P-MVP Nine-point center = X(5)

In a 4-Point we find:

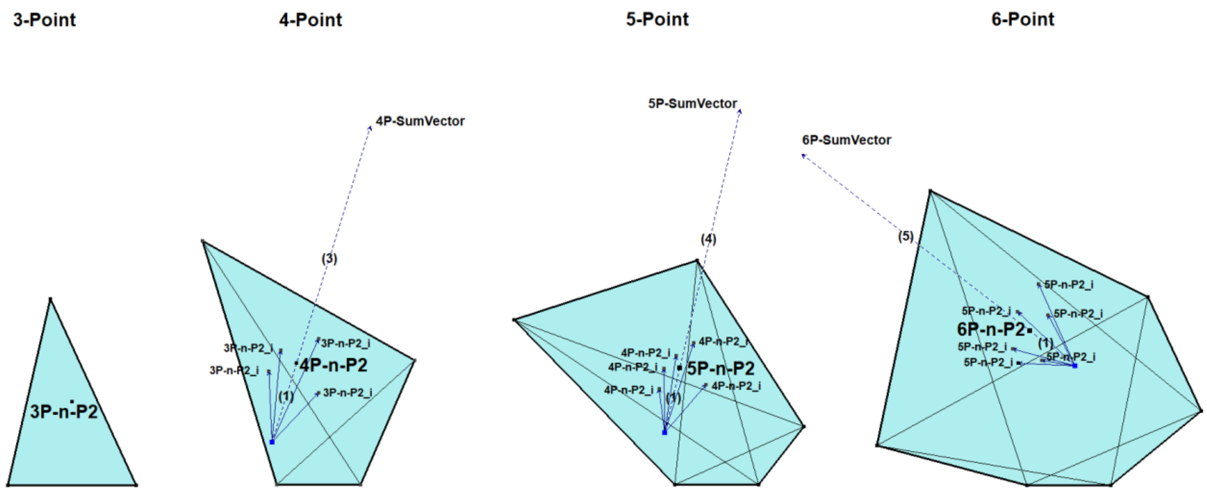
4P-n-P1 = 4P-MVP Centroid = QA-P1 (QA-Centroid)
4P-n-P2 = 4P-MVP Circumcenter = QA-P32 (Centroid Circumcenter Quadrangle)
4P-n-P3 = 4P-MVP Orthocenter = QA-P33 (Centroid Orthocenter Quadrangle)
4P-n-P4 = 4P-MVP Nine-point center = Midpoint (QA-P32,QA-P33)

Properties:

- nP-n-P1, nP-n-P2, nP-n-P3 and nP-n-P4 are collinear on nP-n-L1. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).
- 5P-n-P1 is also the point that minimizes the sum of squared distances to the vertices of the Pentangle. See Ref-34. QFG #730, October 8, 2014 by Seiichi Kirikami.
- 5P-n-P1 is also the point that minimizes the sum of squared distances to all midpoints between vertices of the Pentangle. See Ref-34. QFG #730, October 8, 2014 by Seiichi Kirikami.
- 5P-n-P1 is also the point that minimizes the sum of squared distances to the Centroids of all Component Triangles of the Pentangle. See Ref-34. QFG #730, October 8, 2014 by Seiichi Kirikami.

nP-n-P2 nP-MVP-Circumcenter

nP-n-P2 is the nL-Mean Vector Point of X(3), the Triangle Circumcenter.
In this method central points from ETC are successively constructed in higher level n-Point figures starting with n=3, then n=4 using the results of n=3, then n=5 using the results of n=4, etc.
See nP-n-Luc1 for a detailed description.



Correspondence with ETC/EQF:
In a 3-Point:

3P-n-P1 = 3P-MVP Centroid = X(2)
3P-n-P2 = 3P-MVP Circumcenter = X(3)
3P-n-P3 = 3P-MVP Orthocenter = X(4)
3P-n-P4 = 3P-MVP Nine-point center = X(5)

In a 4-Point we find:

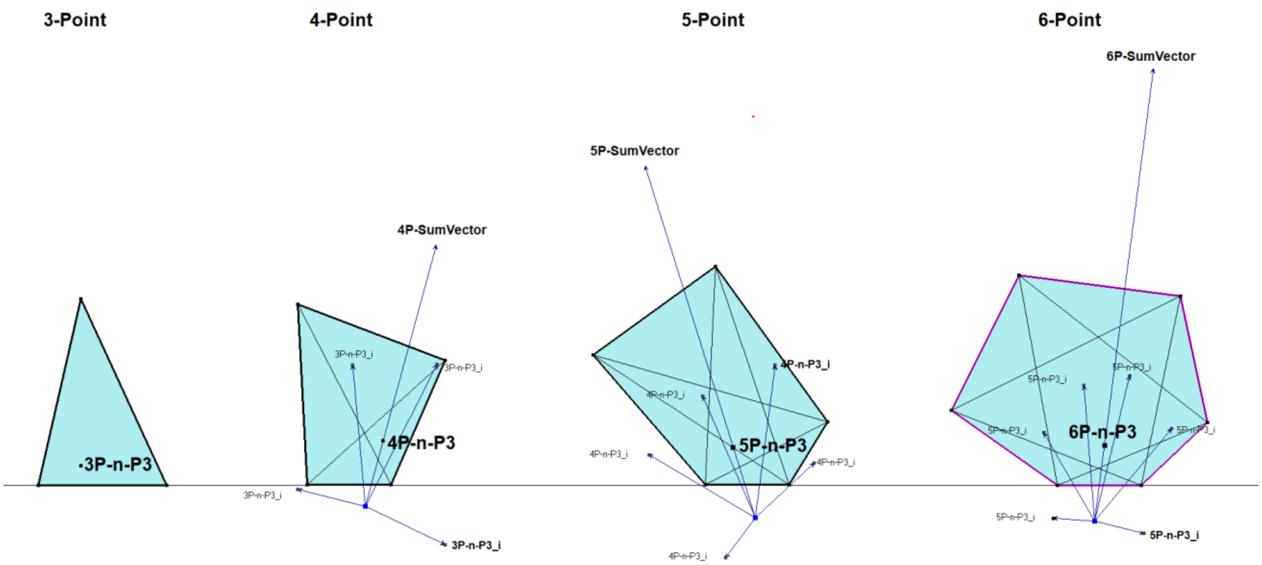
4P-n-P1 = 4P-MVP Centroid = QA-P1 (QA-Centroid)
4P-n-P2 = 4P-MVP Circumcenter = QA-P32 (Centroid Circumcenter Quadrangle)
4P-n-P3 = 4P-MVP Orthocenter = QA-P33 (Centroid Orthocenter Quadrangle)
4P-n-P4 = 4P-MVP Nine-point center = Midpoint (QA-P32,QA-P33)

Properties:

- nP-n-P1, nP-n-P2, nP-n-P3 and nP-n-P4 are collinear on nP-n-L1. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nP-n-P3 nP-MVP-Orthocenter

nP-n-P3 is the nL-Mean Vector Point of X(4), the Triangle Circumcenter.
In this method central points from ETC are successively constructed in higher level n-Point figures starting with n=3, then n=4 using the results of n=3, then n=5 using the results of n=4, etc.
See nP-n-Luc1 for a detailed description.



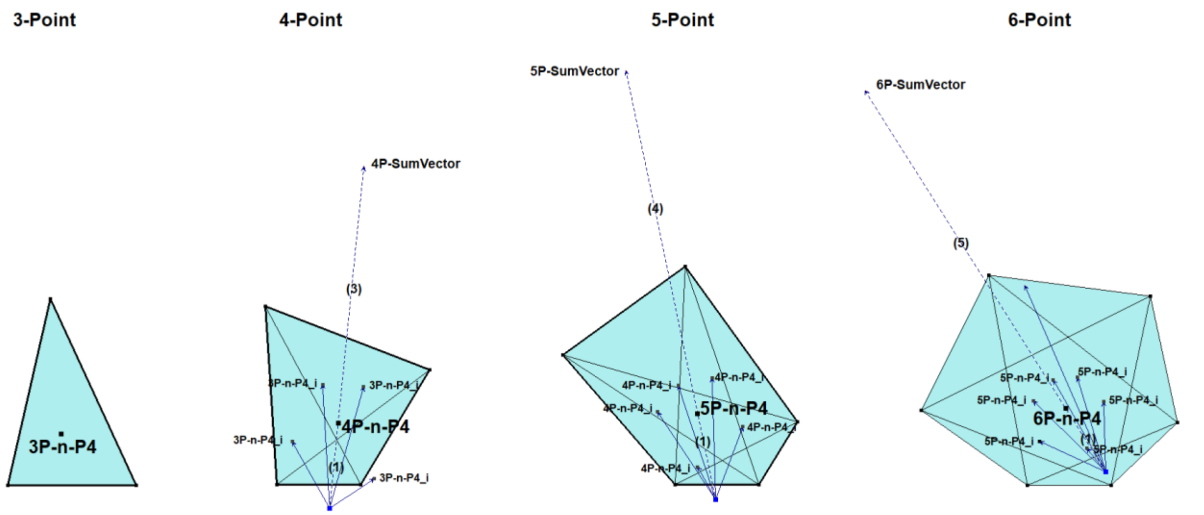
Correspondence with ETC/EQF:
In a 3-Point:
- 3P-n-P1 = 3P-MVP Centroid = X(2)
- 3P-n-P2 = 3P-MVP Circumcenter = X(3)
- 3P-n-P3 = 3P-MVP Orthocenter = X(4)
- 3P-n-P4 = 3P-MVP Nine-point center = X(5)

In a 4-Point we find:
- 4P-n-P1 = 4P-MVP Centroid = QA-P1 (QA-Centroid)
- 4P-n-P2 = 4P-MVP Circumcenter = QA-P32 (Centroid Circumcenter Quadrangle)
- 4P-n-P3 = 4P-MVP Orthocenter = QA-P33 (Centroid Orthocenter Quadrangle)
- 4P-n-P4 = 4P-MVP Nine-point center = Midpoint (QA-P32,QA-P33)

Properties:
- nP-n-P1, nP-n-P2, nP-n-P3 and nP-n-P4 are collinear on nP-n-L1. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nP-n-P4 nP-MVP-Nine-point Center

nP-n-P4 is the nL-Mean Vector Point of X(4), the Triangle Nine-point Center.
In this method central points from ETC are successively constructed in higher level n-Point figures starting with n=3, then n=4 using the results of n=3, then n=5 using the results of n=4, etc.
See nP-n-Luc1 for a detailed description.



Correspondence with ETC/EQF:
In a 3-Point:

3P-n-P1 = 3P-MVP Centroid = X(2)
3P-n-P2 = 3P-MVP Circumcenter = X(3)
3P-n-P3 = 3P-MVP Orthocenter = X(4)
3P-n-P4 = 3P-MVP Nine-point center = X(5)

In a 4-Point we find:

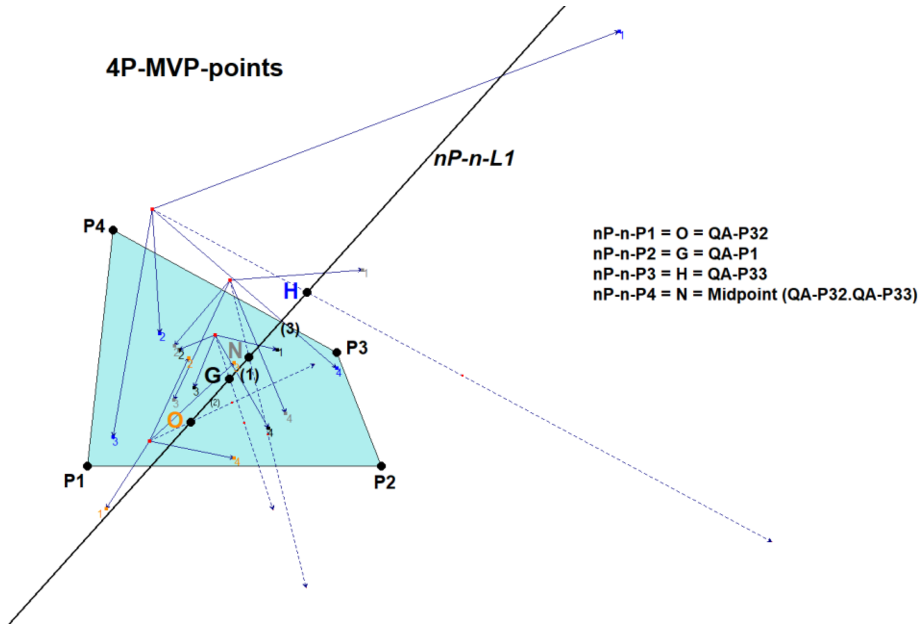
4P-n-P1 = 4P-MVP Centroid = QA-P1 (QA-Centroid)
4P-n-P2 = 4P-MVP Circumcenter = QA-P32 (Centroid Circumcenter Quadrangle)
4P-n-P3 = 4P-MVP Orthocenter = QA-P33 (Centroid Orthocenter Quadrangle)
4P-n-P4 = 4P-MVP Nine-point center = Midpoint (QA-P32,QA-P33)

Properties:

- nP-n-P1, nP-n-P2, nP-n-P3 and nP-n-P4 are collinear on nP-n-L1. Their mutual distance ratios correspond with the mutual distance ratios from triangle centers X(2), X(3), X(4) and X(5).

nP-n-L1: nL-MVP Eulerline

The nL-MVP Eulerline is the line connecting collinear points nP-n-P1, nP-n-P2, nP-n-P3, nP-n-P4. Next figure gives an example of nP-n-L2 in a 4-Point.



Correspondence with ETC/EQF:
When n=3, then nP-n-L1 = Triangle Eulerline X(3).X(4), with

- 3P-n-P1 = 3P-MVP Centroid = X(2)
- 3P-n-P2 = 3P-MVP Circumcenter = X(3)
- 3P-n-P3 = 3P-MVP Orthocenter = X(4)
- 3P-n-P4 = 3P-MVP Nine-point center = X(5)

When n=4, then nP-n-L1 = Quadrilateral Eulerline QA-P1.QA-P32.QA-P33, with

- 4P-n-P1 = 4P-MVP Centroid = QA-P1 (QA-Centroid)
- 4P-n-P2 = 4P-MVP Circumcenter = QA-P32 (Centroid Circumcenter Quadrangle)
- 4P-n-P3 = 4P-MVP Orthocenter = QA-P33 (Centroid Orthocenter Quadrangle)
- 4P-n-P4 = 4P-MVP Nine-point center = Midpoint (QA-P32,QA-P33)

Properties:

- nP-n-P1, nP-n-P2, nP-n-P3, nP-n-P4 lie on nP-n-L1.

nP-n-L2: nP-LSD Line

The nP-LSD line is the line with the Least Sum of Squared Distances of the vertices of an n-Point to this line.

Construction:

Choose a random couple of perpendicular lines intersecting in the Centroid Gp = nP-P1 as axes of the Complex plane.

Let the complex numbers zi = xi + i.yi represent the vertices of the n-Point.

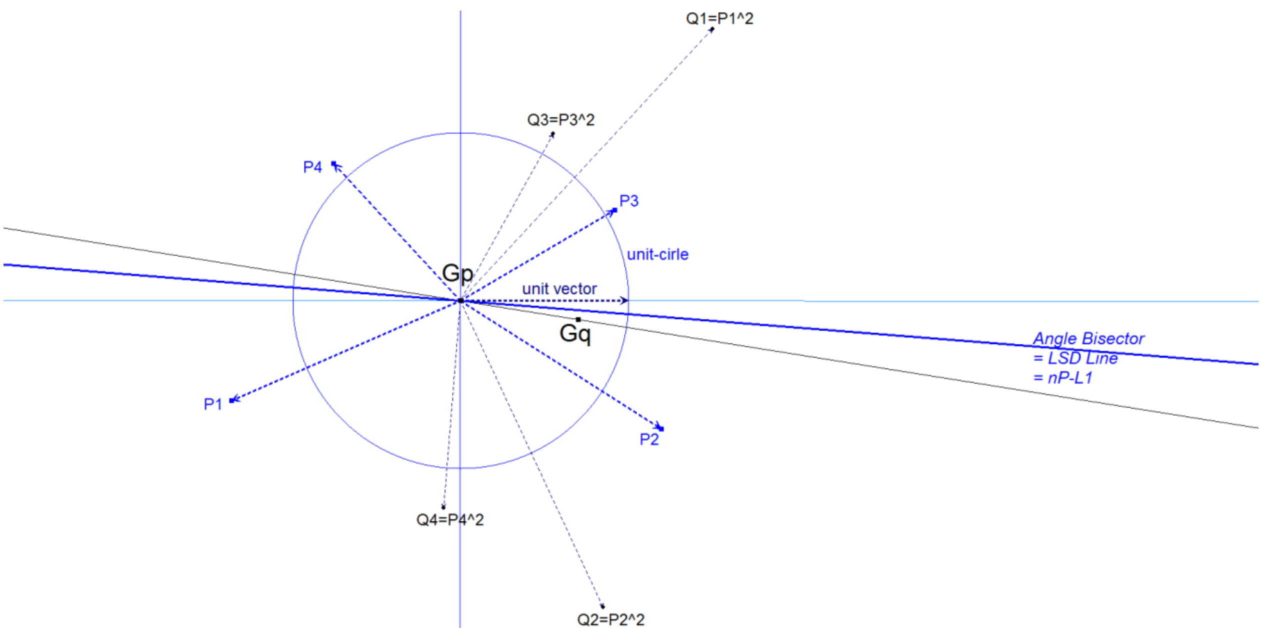
The centroid of the vertices is the point nP-P1 where z = 0.

The centroid of the points represented by zi^2 is a 2nd point Gq = nP-Px.

Draw the line nP-P1.nP-Px (depending of the chosen axes).

The searched LSD Line is the bisector of the angle between this line nP-P1.nP-Px and the x-axis of the xi (this last line is independent of the chosen axes).

See messages Benedetto Scimemi and Bernard Keizer at Ref-34, QFG#1585, #1590, #1593.



Correspondence with ETC/EQF:

In a 3-Point-configuration:

3L-n-L2 = Main Axis of the Steiner CircumEllipse

In a 4-Point-configuration:

4L-n-L2 = QL-L10

nP-n-Luc nP-Level-up Constructions

nP-n-Luc1 nP-Mean Vector Point

A Mean Vector Point (MVP) is the mean of a bunch of n vectors with identical origin.
It is constructed by adding these vectors and then dividing the Sumvector by n.
The Mean Vector Point is the endpoint of the divided Sumvector.
This method is used for nP-n-P1 to nP-n-P4.

Origin independent
It is most special that with the definition of nP-n-Luc1 the location of the origin is unimportant.
In all n-Points we can use any random point as origin. The endpoint of the resultant vector will be the same for all different origins.

Recursive application
Every Triangle Center can be transferred to a corresponding point in an n-Point by a simple recursive construction. The resulting point which will be called an nP-MVP Center, where MVP is the abbreviation for Mean Vector Point.
When X(i) is a triangle Center we define the nP-MVP X(i)-Center as the Mean Vector Point of the n (n-1)P-MVP X(i)-Centers.
When the (n-1)P-MVP X(i)-Centers aren't known they can be constructed from the MVP X(i)-Centers another level lower, according to the same definition. By applying this definition to an increasingly lower level finally the level is reached of the 3P-MVP X(i)-Center, which simply is the X(i) Triangle Center.
See Ref-34, QFG#869,#873,#878,#881.

Universal Level-up construction
Unlike other Level-up constructions this construction probably can be applied to all Central Points at all levels.
Consequently all known ETC-points and all known EQF-points will have a related MVP-point in every n-Line (n>3,4).

Another general construction of nP-n-Luc1(X(i)):
An nL-Mean Vector Point of some Triangle Center X(i) also can be constructed as the Centroid of the corresponding (n-1)P-Mean Vector Points of some Triangle Center X(i). Again by applying this definition to an increasingly lower level finally the level is reached of the 3P-MVP X(i)-center, which simply is the X(i) Triangle Center.

Preservation of distance ratios
The Centroid, Circumcenter, Orthocenter and Nine-point Center are when transferred to an n-Point collinear and their mutual distance ratios are preserved.
However when Triangle Centers (other than X(2), X(3), X(4), X(5)) are transferred to higher level n-Lines, usually collinearity of MVP-points will not be preserved. The mentioned triangle centers on the Eulerline are exceptions.

5-points or Pentangles

5P-s-P1: 5P-Circumscribed Conic Center

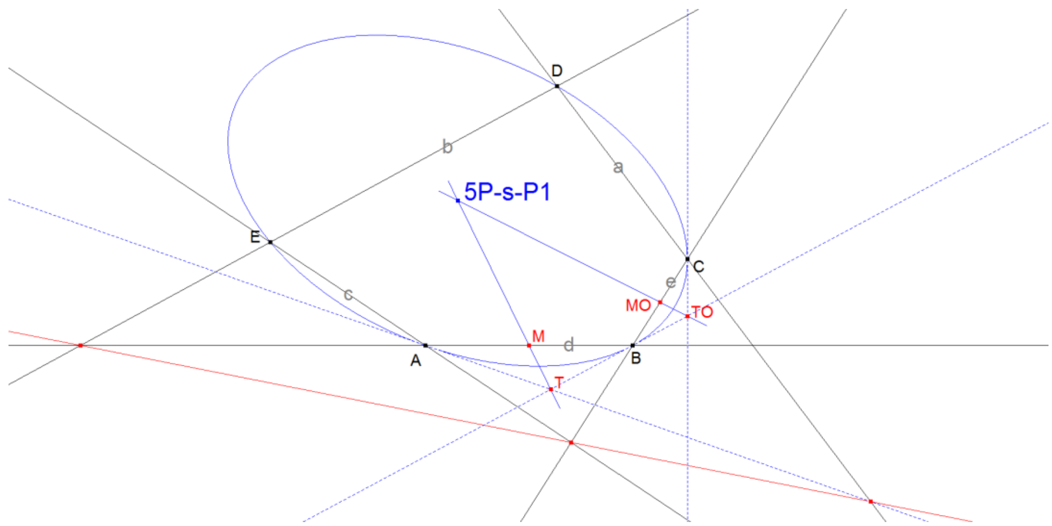
It is well known that in a system of 5 random Points a unique circumscribed conic can be constructed. This conic is 5P-s-Co1 and its center is 5P-s-P1.

Construction (See Ref-19):

1. Let the conic be defined by points A, B, C, D, E.
2. Let the tangents at A, B meet at T, and those at B, C meet at TO.
3. Let M, MO be the midpoints of AB and BC, then the center O is MT.MOTO.

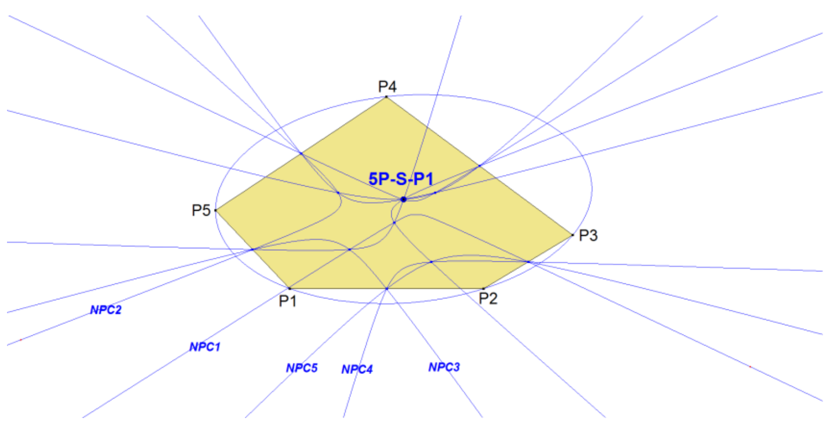
Construction of Conic Tangents:

4. Let d = AB, e = BC, a = CD, b = DE, c = EA, then bd.ce cuts a in a point lying on the tangent at A.



Properties:

- 5P-s-P1 is also the common point of the radical axes of the 5 versions of QA-Ci1 (Circumcircle of the Diagonal Triangle) in the 5-Point.
- 5P-s-P1 is also the common point of the 5 versions of QA-Co1 (Nine Point Conic) in the 5-Point.



- 5P-s-Tf3(5P-s-P1) = 5P-s-P1.

5P-s-P2: 5P-Involutary Center

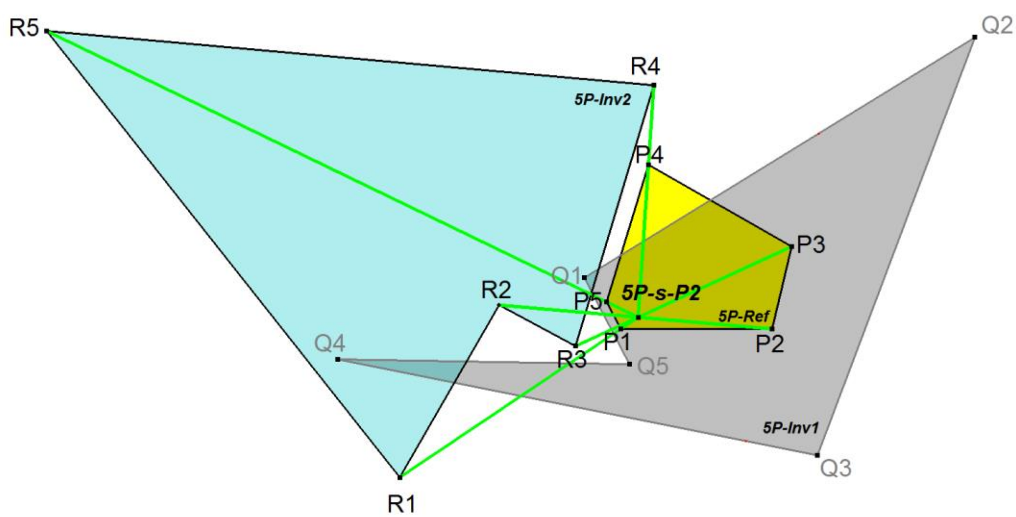
Let P1.P2.P3.P4.P5 be a Pentangle (system of 5 independent random points).

Let Q1 = Involutary Conjugate of P1 wrt Quadrangle P2.P3.P4.P5.
Let Q2 = Involutary Conjugate of P2 wrt Quadrangle P3.P4.P5.P1.
Let Q3 = Involutary Conjugate of P3 wrt Quadrangle P4.P5.P1.P2.
Let Q4 = Involutary Conjugate of P4 wrt Quadrangle P5.P1.P2.P3.
Let Q5 = Involutary Conjugate of P5 wrt Quadrangle P1.P2.P3.P4.

Next:

Let R1 = Involutary Conjugate of Q1 wrt Quadrangle Q2.Q3.Q4.Q5.
Let R2 = Involutary Conjugate of Q2 wrt Quadrangle Q3.Q4.Q5.Q1.
Let R3 = Involutary Conjugate of Q3 wrt Quadrangle Q4.Q5.Q1.Q2.
Let R4 = Involutary Conjugate of Q4 wrt Quadrangle Q5.Q1.Q2.Q3.
Let R5 = Involutary Conjugate of Q5 wrt Quadrangle Q1.Q2.Q3.Q4.

Now Pentangle P1.P2.P3.P4.P5 is point perspective with R1.R2.R3.R4.R5.
The perspector 5P-s-P2 is a regular Pentangle Center.
See Ref-34, QFG#704.



Coordinates:

When using barycentric coordinates: P1=(1:0:0), P2=(0:1:0), P3=(0:0:1), P4=(p:q:r), P5=(x:y:z), then 5P-s-P2 has coordinates:

$$((-r y + q z) (p^2 q^2 r^6 x^6 y^4 - p^3 q r^6 x^5 y^5 + 2 q^5 r^5 x^8 y z - 5 p q^4 r^5 x^7 y^2 z - 2 p^2 q^3 r^5 x^6 y^3 z + 5 p^3 q^2 r^5 x^5 y^4 z + 2 p^4 q r^5 x^4 y^5 z - p^5 r^5 x^3 y^6 z - 5 p q^5 r^4 x^7 y z^2 + 15 p^2 q^4 r^4 x^6 y^2 z^2 - 20 p^4 q^2 r^4 x^4 y^4 z^2 + 5 p^5 q r^4 x^3 y^5 z^2 + p^6 r^4 x^2 y^6 z^2 - 2 p^2 q^5 r^3 x^6 y z^3 + 10 p^4 q^3 r^3 x^4 y^3 z^3 - 2 p^6 q r^3 x^2 y^5 z^3 + p^2 q^6 r^2 x^6 z^4 + 5 p^3 q^5 r^2 x^5 y z^4 - 20 p^4 q^4 r^2 x^4 y^2 z^4 + 15 p^6 q^2 r^2 x^2 y^4 z^4 - 5 p^7 q r^2 x y^5 z^4 - p^3 q^6 r x^5 z^5 + 2 p^4 q^5 r x^4 y z^5 + 5 p^5 q^4 r x^3 y^2 z^5 - 2 p^6 q^3 r x^2 y^3 z^5 - 5 p^7 q^2 r x y^4 z^5 + 2 p^8 q r y^5 z^5 - p^5 q^5 x^3 y z^6 + p^6 q^4 x^2 y^2 z^6 : :)$$

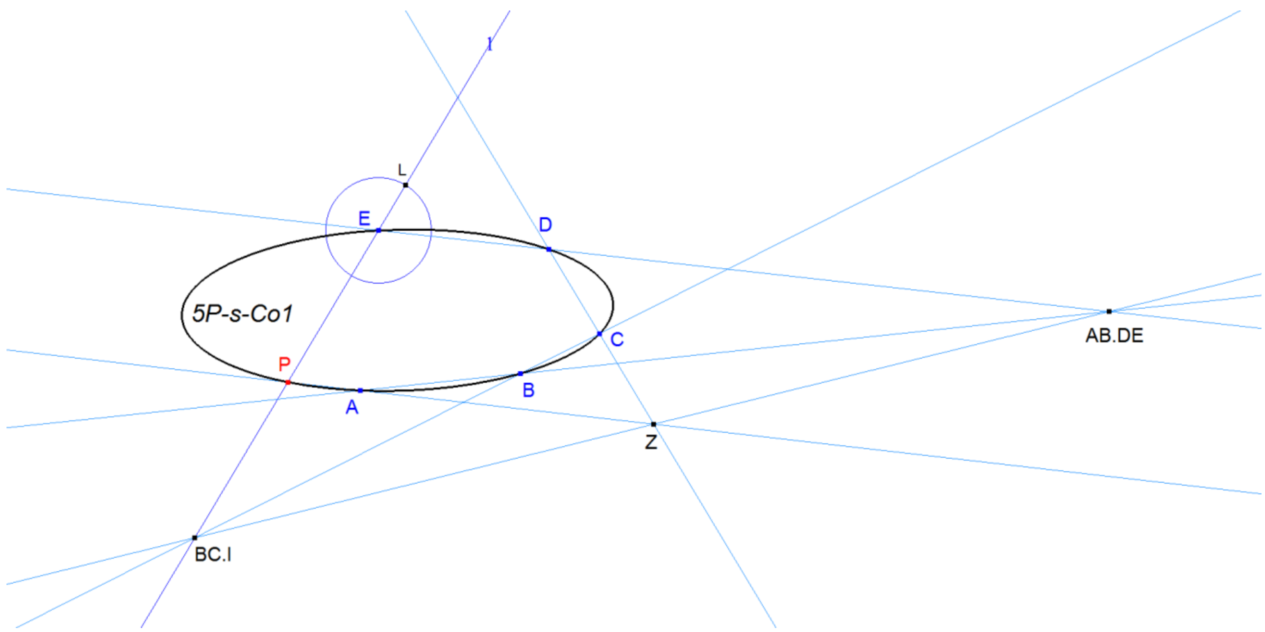
5P-s-Co1 5P-Circumscribed Conic

It is well known that in a system of 5 random Points a unique circumscribed conic can be constructed. This conic is 5P-s-Co1 and its center is 5P-s-P1.

Construction:
See Ref-19.

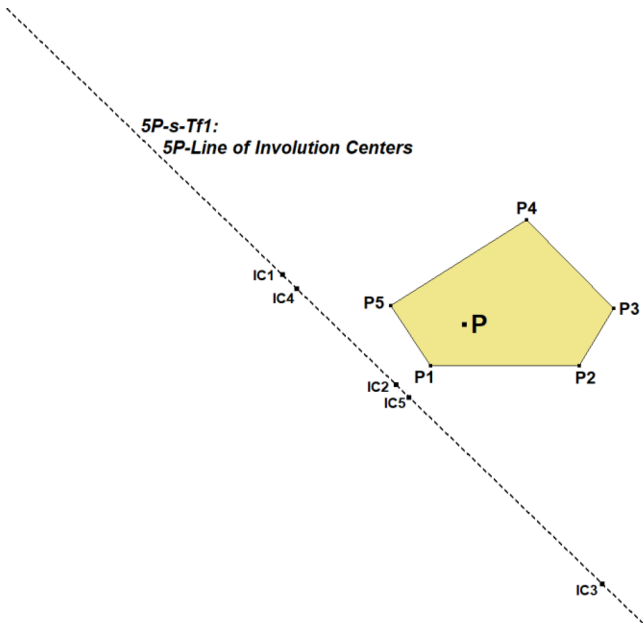
1. Given five points A, B, C, D, E.
2. Let l be a variable line through E. (Draw a circle center E with any radius. Let L be an arbitrary point on the circle, and take l to be the line EL.)
3. The line joining AB.DE and BC.l cuts CD in Z, then P = AZ.l lies on the conic.

As L moves round the circle, P traces the conic. (Select L and P, and Construct/Locus.)



5P-s-Tf1: 5P-Line of Involution Centers

5P-s-Tf1 is a transformation that maps some point P into a line of Involution Centers.
Each 5-Point (Pentangle) contains 5 4-Points (Quadrangles).
In a Quadrangle QA-Tf1(P) is the Involution Center of the tangent line at P to the conic through the vertices of the reference quadrangle and P.
The 5 versions of QA-Tf1(P) are collinear on 5P-s-Tf1(P).

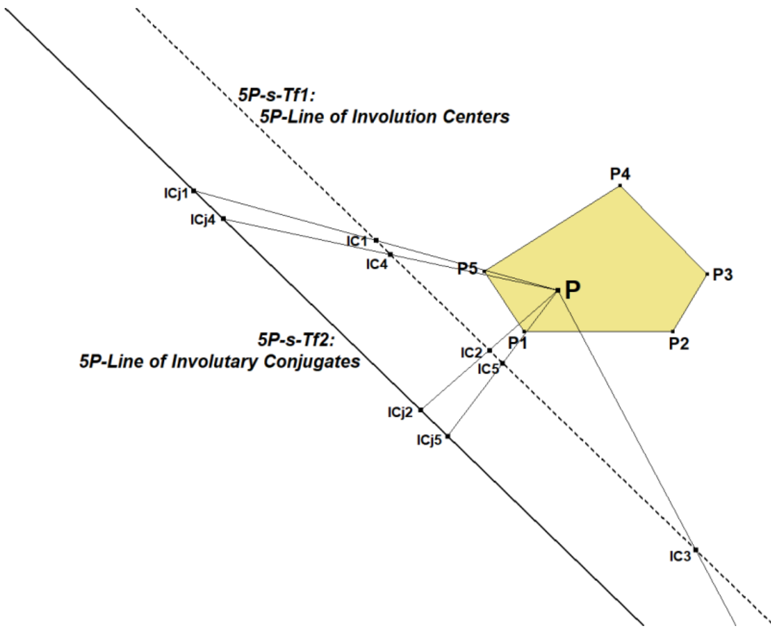


Properties:

- When P = 5P-s-P1, then 5P-s-Tf1(P) = Line at Infinity.
- When P lies on the 5P-circumscribed conic 5P-s-Co1, then 5P-s-Tf1(P) is the tangent at P to 5P-s-Co1.
- 5P-s-Tf1(P) // 5P-s-Tf2(P).
- d(P,5P-s-Tf1(P)) = d(5P-s-Tf1(P),5P-s-Tf2(P)) = d(P,5P-s-Tf2(P)) / 2.
- P is the Railway Watcher of lines 5P-s-Tf1(P) and 5P-s-Tf1(P). See QL-L-1.

5P-s-Tf2: 5P-Line of Involutary Conjugates

5P-s-Tf2 is a transformation that maps some point P into a line of Involutary Conjugates.
Each 5-Point (Pentangle) contains 5 4-Points (Quadrangles).
In a Quadrangle QA-Tf2(P) is the Involutary Conjugate on the tangent line at P to the conic through the vertices of the reference quadrangle and P.
The 5 versions of QA-Tf2(P) are collinear on 5P-s-Tf2(P).

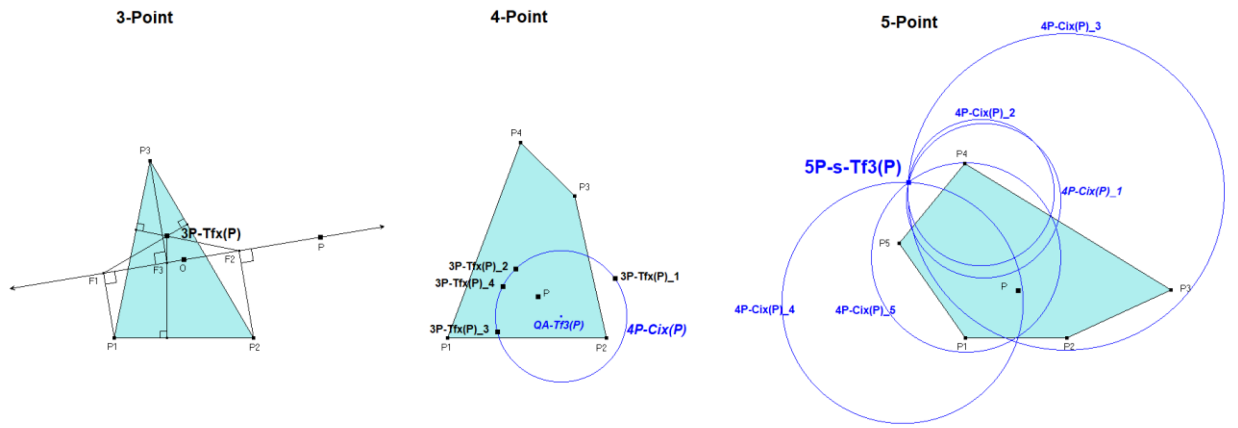


Properties:

- When P = 5P-s-P1, then 5P-s-Tf2(P) = Line at Infinity.
- When P lies on the 5P-circumscribed conic 5P-s-Co1, then 5P-s-Tf2(P) is the tangent at P to 5P-s-Co1.
- 5P-s-P2(P) is the *polar* of P wrt the circumscribed conic 5P-s-Co1 and P is the pole of 5P-s-P2(P) wrt the circumscribed conic 5P-s-Co1.
- 5P-s-Tf1(P) // 5P-s-Tf2(P).
- d(P,5P-s-Tf1(P)) = d(5P-s-Tf1(P),5P-s-Tf2(P)) = d(P,5P-s-Tf2(P)) / 2.
- P is the Railway Watcher of lines 5P-s-Tf1(P) and 5P-s-Tf1(P). See QL-L-1.

5P-s-Tf3: 5P-Orthopole

A Pentangle contains 5 Quadrangles.
Each Quadrangle contains 4 Component Triangles.
Let Oi be the Circumcenters of Component Triangles Tr_i in a Quadrangle (i=1,2,3,4).
Let P be some random point. Let Qi be the Orthopole (see Ref-13) of line P.Oi wrt Triangle Tr_i.
The 4 Orthopoles Qi of P wrt Tr_i will be concyclic on an Orthopole Circle QA-Cix (described in QA-Tf3).
Since a Pentangle contains 5 Quadrangles a random point P will generate 5 Orthopole Circles QA-Cix in a Pentangle.
These 5 Orthopole Circles QA-Cix concur in a single point which is 5P-s-Tf3(P).
This 5P-transformation was found by Telv Cohl. See Ref-33, Anopolis#1986.
As can be seen from the coordinate below 5P-s-Tf3 is a *linear* transformation.



OP-Orthopole Transformation 3P-Tfx

P = random Point, O = Circumcenter P1.P2.P3.
Fi = Perpedicular feet from Pi on line O.P.
3P-Tfx(P) = common point perpendiculars from Fi on Pj.Pk (i,j,k are different numbers 1,2,3)

P = random Point,
3P-Tfx(P)_1 = 3P-Tfx(P) wrt triangle P2.P3.P4
3P-Tfx(P)_2 = 3P-Tfx(P) wrt triangle P3.P4.P1
3P-Tfx(P)_3 = 3P-Tfx(P) wrt triangle P4.P1.P2
3P-Tfx(P)_4 = 3P-Tfx(P) wrt triangle P1.P2.P3
3P-Tfx(P)_1, 3P-Tfx(P)_2, 3P-Tfx(P)_3, 3P-Tfx(P)_4 are concyclic on 4P-Cix(P).

P = random Point,
4P-Cix(P)_1 = 4P-Cix(P) wrt quadrangle P2.P3.P4.P5
4P-Cix(P)_2 = 4P-Cix(P) wrt quadrangle P3.P4.P5.P1
4P-Cix(P)_3 = 4P-Cix(P) wrt quadrangle P4.P5.P1.P2
4P-Cix(P)_4 = 4P-Cix(P) wrt quadrangle P5.P1.P2.P3
4P-Cix(P)_5 = 4P-Cix(P) wrt quadrangle P1.P2.P3.P4
4P-Cix(P)_1 - 4P-Cix(P)_5 concur in 5P-s-Tf1(P).

1st Coordinate mapped point:

Let P1,P2,P3,P4,P5, P have these barycentric coordinates:
P1=(0:1:0), P2=(0:0:1), P3=(1:0:0), P4=(p:q:r), P5=(P:Q:R) and P=(u:v:w).
Then the 1st barycentric coordinate of 5P-s-Tf3(P) will be:

$$p P (-Q r + q R) (-2 b^2 c^2 p P Q r - a^2 c^2 P q Q r - b^2 c^2 P q Q r + c^4 P q Q r + 2 b^2 c^2 p P q R + a^2 c^2 p q Q R + b^2 c^2 p q Q R - c^4 p q Q R + a^2 b^2 P q r R - b^4 P q r R + b^2 c^2 P q r R - a^2 b^2 p Q r R + b^4 p Q r R - b^2 c^2 p Q r R) \mathbf{u}$$

$$+ (a^2 c^2 p^2 P^2 Q^2 r^2 + b^2 c^2 p^2 P^2 Q^2 r^2 - c^4 p^2 P^2 Q^2 r^2 + 2 a^2 c^2 p P^2 q Q^2 r^2 - 2 a^2 c^2 p^2 P^2 q Q r R - 2 b^2 c^2 p^2 P^2 q Q r R + 2 c^4 p^2 P^2 q Q r R - 2 a^2 c^2 p P^2 q^2 Q r R - 2 a^2 c^2 p^2 P q Q^2 r R - a^4 p P^2 q Q r^2 R + b^4 p P^2 q Q r^2 R - 2 b^2 c^2 p P^2 q Q r^2 R + c^4 p P^2 q Q r^2 R - a^4 P^2 q^2 Q r^2 R + a^2 b^2 P^2 q^2 Q r^2 R - a^2 c^2 P^2 q^2 Q r^2 R + a^4 p^2 P Q^2 r^2 R - b^4 p^2 P Q^2 r^2 R + 2 b^2 c^2 p^2 P Q^2 r^2 R - c^4 p^2 P Q^2 r^2 R + a^4 p P q Q^2 r^2 R - a^2 b^2 p P q Q^2 r^2 R + a^2 c^2 p P q Q^2 r^2 R + a^2 c^2 p^2 P^2 q^2 R^2 + b^2 c^2 p^2 P^2 q^2 R^2 - c^4 p^2 P^2 q^2 R^2 + 2 a^2 c^2 p^2 P q^2 Q R^2 + a^4 p P^2 q^2 r R^2 - b^4 p P^2 q^2 r R^2 + 2 b^2 c^2 p P^2 q^2 r R^2 - c^4 p P^2 q^2 r R^2 - a^4 p^2 P q Q r R^2 + b^4 p^2 P q Q r R^2 - 2 b^2 c^2 p^2 P q Q r R^2 + c^4 p^2 P q Q r R^2 + a^4 p P q^2 Q r R^2 - a^2 b^2 p P q^2 Q r R^2 + a^2 c^2 p P q^2 Q r R^2 - a^4 p^2 q Q^2 r R^2 + a^2 b^2 p^2 q Q^2 r R^2 - a^2 c^2 p^2 q Q^2 r R^2 + a^4 P^2 q^2 r^2 R^2 + a^2 b^2 P^2 q^2 r^2 R^2 - a^2 c^2 P^2 q^2 r^2 R^2 - 2 a^4 p P q Q r^2 R^2 - 2 a^2 b^2 p P q Q r^2 R^2 + 2 a^2 c^2 p P q Q r^2 R^2 + a^4 p^2 Q^2 r^2 R^2 + a^2 b^2 p^2 Q^2 r^2 R^2 - a^2 c^2 p^2 Q^2 r^2 R^2) \mathbf{v}$$

$$+ (a^2 b^2 p^2 P^2 Q^2 r^2 - b^4 p^2 P^2 Q^2 r^2 + b^2 c^2 p^2 P^2 Q^2 r^2 + a^4 p P^2 q Q^2 r^2 - b^4 p P^2 q Q^2 r^2 + 2 b^2 c^2 p P^2 q Q^2 r^2 - c^4 p P^2 q Q^2 r^2 + a^4 P^2 q^2 Q^2 r^2 - a^2 b^2 P^2 q^2 Q^2 r^2 + a^2 c^2 P^2 q^2 Q^2 r^2 - 2 a^2 b^2 p^2 P^2 q Q r R + 2 b^4 p^2 P^2 q Q r R - 2 b^2 c^2 p^2 P^2 q Q r R - a^4 p P^2 q^2 Q r R + b^4 p P^2 q^2 Q r R - 2 b^2 c^2 p P^2 q^2 Q r R + c^4 p P^2 q^2 Q r R - a^4 p^2 P q Q^2 r R + b^4 p^2 P q Q^2 r R - 2 b^2 c^2 p^2 P q Q^2 r R + c^4 p^2 P q Q^2 r R - 2 a^4 p P q^2 Q^2 r R + 2 a^2 b^2 p P q^2 Q^2 r R - 2 a^2 c^2 p P q^2 Q^2 r R - 2 a^2 b^2 p P^2 q Q r^2 R - a^4 P^2 q^2 Q r^2 R - a^2 b^2 P^2 q^2 Q r^2 R + a^2 c^2 P^2 q^2 Q r^2 R + 2 a^2 b^2 p^2 P Q^2 r^2 R + a^4 p P q Q^2 r^2 R + a^2 b^2 p P q Q^2 r^2 R - a^2 c^2 p P q Q^2 r^2 R + a^2 b^2 p^2 P^2 q^2 R^2 - b^4 p^2 P^2 q^2 R^2 + b^2 c^2 p^2 P^2 q^2 R^2 + a^4 p^2 P q^2 Q R^2 - b^4 p^2 P q^2 Q R^2 + 2 b^2 c^2 p^2 P q^2 Q R^2 - c^4 p^2 P q^2 Q R^2 + a^4 p^2 q^2 Q^2 R^2 - a^2 b^2 p^2 q^2 Q^2 R^2 + a^2 c^2 p^2 q^2 Q^2 R^2 + 2 a^2 b^2 p P^2 q^2 r R^2 - 2 a^2 b^2 p^2 P q Q r R^2 + a^4 p P q^2 Q r R^2 + a^2 b^2 p P q^2 Q r R^2 - a^2 c^2 p P q^2 Q r R^2 - a^4 p^2 q Q^2 r R^2 - a^2 b^2 p^2 q Q^2 r R^2 + a^2 c^2 p^2 q Q^2 r R^2) \mathbf{w}$$

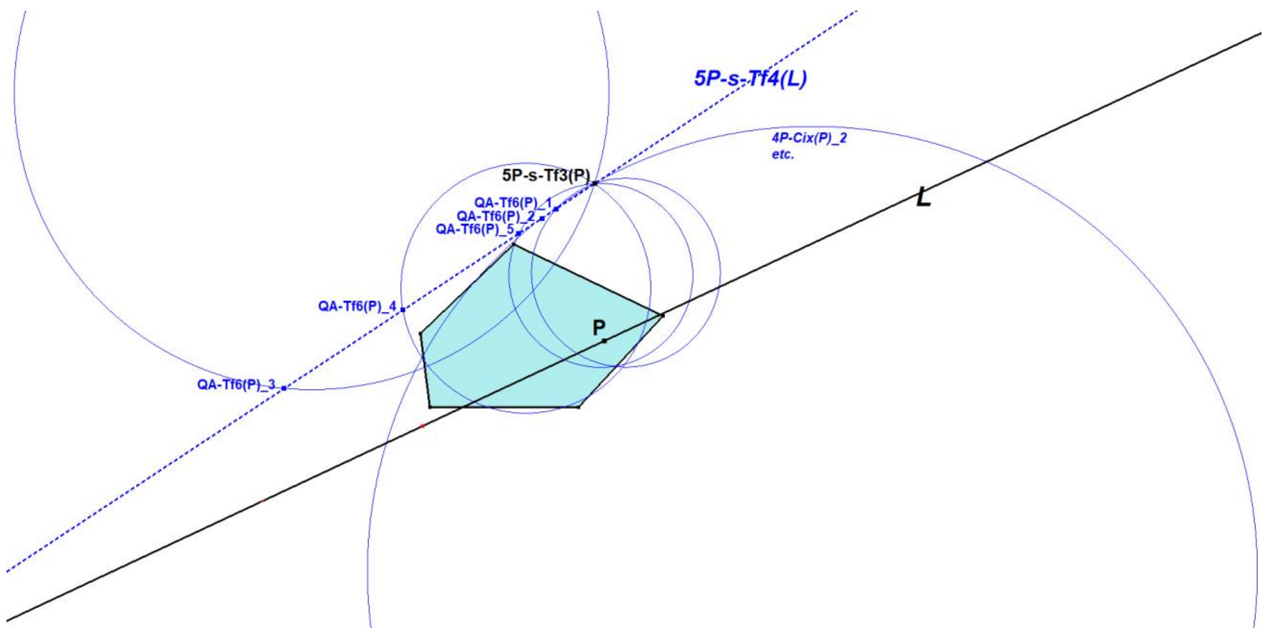
Properties:

- 5P-s-Tf3(5P-s-P1) = 5P-s-P1.
- The locus of QA-Tf6 wrt a pencil of lines through random point P is a circle QA-Tf6a(P). The 5 versions of QA-Tf6a(P) in a 5-Point have as common point 5P-s-Tf3(P).

5P-s-Tf4: 5P-Orthopolar Line

5P-s-Tf4 transforms a line into another line.
Let L be a random line.
5P-s-Tf4(L) is the locus of 5P-s-Tf3(P) with P varying on L.

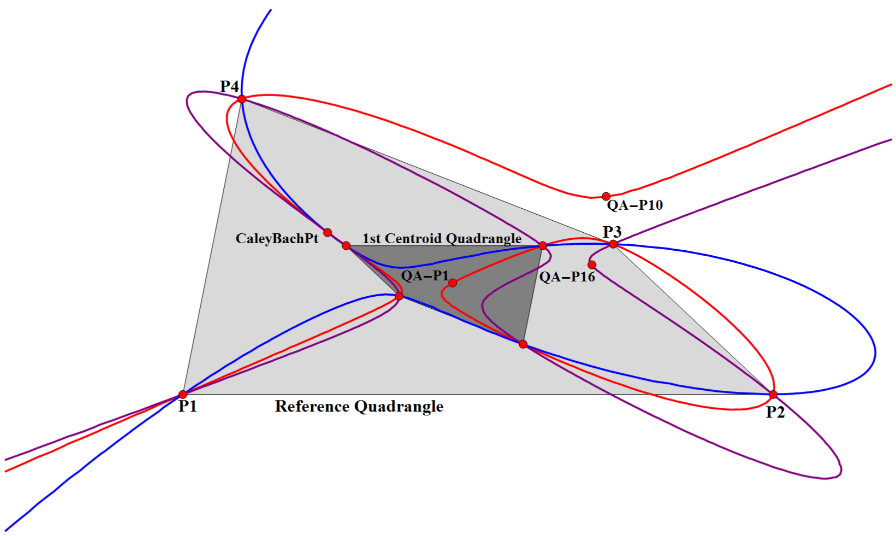
Another construction uses QA-Tf6(L) = Quang Duong's Transformation.
A 5-Point contains 5 4-Points (Quadrangles). The 5 versions of QA-Tf6(L) for these Quadrangles are lying on 5P-s-Tf4(L).



8P-s-P1: Cayley Bacharach Point

The Cayley-Bacharach point in an Octangle is defined as follows:
Let P1,...,P8 be eight distinct points in the plane, no three of which collinear, and no six of them on a conic. There exists a unique ninth point P9 such that every cubic curve through P1,...,P8 also contains P9.

In next picture an example is given of 4 vertices of a Reference Quadrangle combined with the centroids of the 4 component triangles (together 8 points) producing a Cayley Bacharach point common for all cubics through this 8 points.



Many properties can be derived from the Cayley Bacharach Theorem, especially when a cubic is degenerated, for example in a conic and two lines, etc.
See also QFG-messages #732, #733, #736, #737, #1476, #1491, #1517, #1696, #1698-1701, #1733.

In http://arxiv.org/pdf/1405.6438v2.pdf (Cayley-Bacharach Formulas by Qingchun Ren, Jürgen Richter-Gebert and Bernd Sturmfels) is given a method for calculating this point.

Let P1 = (x1:y1:z1), P2 = (x2:y2:z2), etc.

Let C(P1,P2,P3,P4,P5,P6) = Determinant of

$$\begin{pmatrix} x1^2 & x1\,y1 & x1\,z1 & y1^2 & y1\,z1 & z1^2 \\ x2^2 & x2\,y2 & x2\,z2 & y2^2 & y2\,z2 & z2^2 \\ x3^2 & x3\,y3 & x3\,z3 & y3^2 & y3\,z3 & z3^2 \\ x4^2 & x4\,y4 & x4\,z4 & y4^2 & y4\,z4 & z4^2 \\ x5^2 & x5\,y5 & x5\,z5 & y5^2 & y5\,z5 & z5^2 \\ x6^2 & x6\,y6 & x6\,z6 & y6^2 & y6\,z6 & z6^2 \end{pmatrix}$$

Let D(P1;P2,P3,P4,P5,P6,P7,P8) = Determinant of

$$\begin{pmatrix}
x2^3 & x2^2\,y2 & x2^2\,z2 & x2\,y2^2 & x2\,y2\,z2 & x2\,z2^2 & y2^3 & y2^2\,z2 & y2\,z2^2 & z2^3 \\
x3^3 & x3^2\,y3 & x3^2\,z3 & x3\,y3^2 & x3\,y3\,z3 & x3\,z3^2 & y3^3 & y3^2\,z3 & y3\,z3^2 & z3^3 \\
x4^3 & x4^2\,y4 & x4^2\,z4 & x4\,y4^2 & x4\,y4\,z4 & x4\,z4^2 & y4^3 & y4^2\,z4 & y4\,z4^2 & z4^3 \\
x5^3 & x5^2\,y5 & x5^2\,z5 & x5\,y5^2 & x5\,y5\,z5 & x5\,z5^2 & y5^3 & y5^2\,z5 & y5\,z5^2 & z5^3 \\
x6^3 & x6^2\,y6 & x6^2\,z6 & x6\,y6^2 & x6\,y6\,z6 & x6\,z6^2 & y6^3 & y6^2\,z6 & y6\,z6^2 & z6^3 \\
x7^3 & x7^2\,y7 & x7^2\,z7 & x7\,y7^2 & x7\,y7\,z7 & x7\,z7^2 & y7^3 & y7^2\,z7 & y7\,z7^2 & z7^3 \\
x8^3 & x8^2\,y8 & x8^2\,z8 & x8\,y8^2 & x8\,y8\,z8 & x8\,z8^2 & y8^3 & y8^2\,z8 & y8\,z8^2 & z8^3 \\
3\,x1^2 & 2\,x1\,y1 & 2\,x1\,z1 & y1^2 & y1\,z1 & z1^2 & 0 & 0 & 0 & 0 \\
0 & x1^2 & 0 & 2\,x1\,y1 & x1\,z1 & 0 & 3\,y1^2 & 2\,y1\,z1 & z1^2 & 0 \\
0 & 0 & x1^2 & 0 & x1\,y1 & 2\,x1\,z1 & 0 & y1^2 & 2\,y1\,z1 & 3\,z1^2
\end{pmatrix}$$

Cx = C(P1,P4,P5,P6,P7,P8),
Cy = C(P2,P4,P5,P6,P7,P8),
Cz = C(P3,P4,P5,P6,P7,P8),
Dx = D(P1;P2,P3,P4,P5,P6,P7,P8),
Dy = D(P2;P3,P1,P4,P5,P6,P7,P8),
Dz = D(P3;P1,P2,P4,P5,P6,P7,P8).

The Cayley-Bacharach point is given by the formula: P9 = CxDyDz · P1 + DxCyDz · P2 + DxDyCz · P3.

Properties:

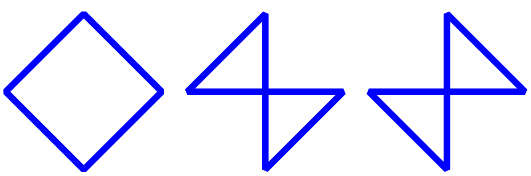
- Let P1P2P3P4 be a quadrangle and when P5P6P7 is its diagonal triangle and P8 is some random point, then P9 = QA-Tf2(P8). See Ref-34, Seiichi Kirikami, QFG#1698.

n-Gons

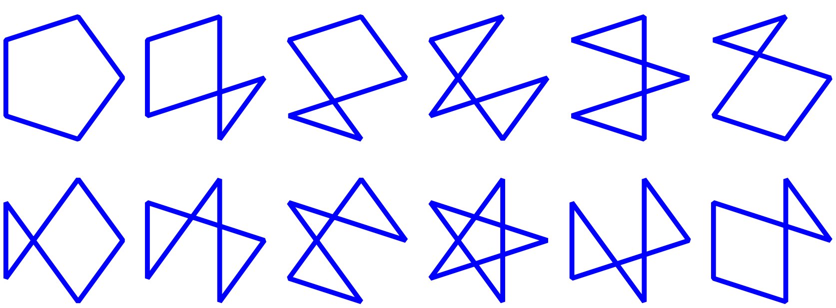
nG-n-1 Systematics for describing n-Gons

There are specific Points/Line/Curves related to n-Gons because n-Gons create a configuration where the order of reference points/reference lines are important and the property of order is used in the construction of these Points/Lines/Curves.

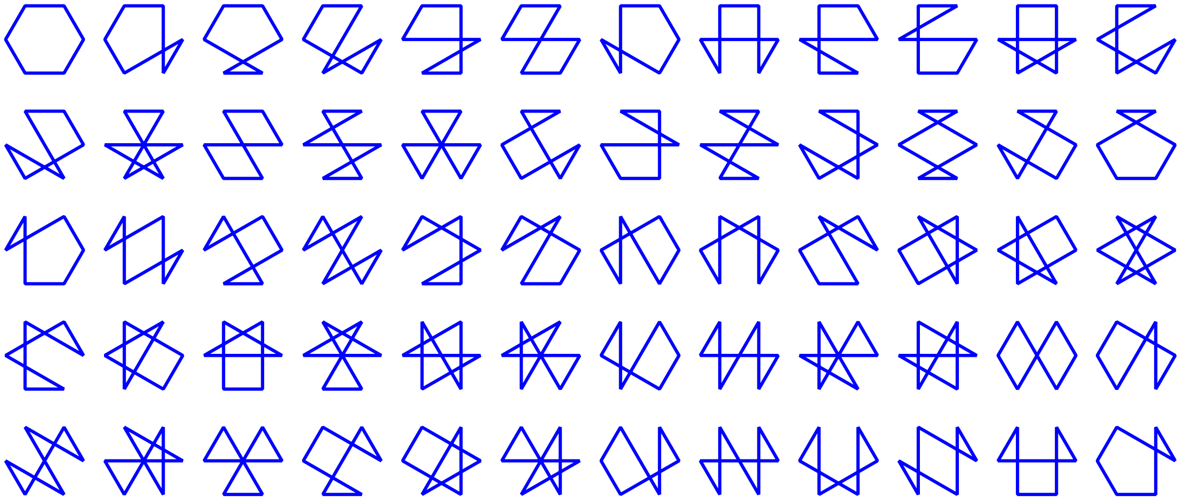
n-Gons are figures made up from n Points cyclically connected by n Line segments (representing n Lines). It is a bounded figure. Contrary to n-Points and n-Lines order does matter.
Dealing with an n-Point/n-Line (where order does not matter) we can imagine that a distinct number of sets of ordered Points/Lines can be formed from the unordered points/lines.
Therefore any n-Point or n-Line contains a certain number n-Gons, just like an n-Point or n-Line contains a certain number of triangles.
In fact the number of n-Gons in an n-Point or n-Line is the number of combinations of numbers 1, …, n being ordered cyclically, where opposite orders are supposed to equalize the normal order.
Here ***combinatorics*** enters the geometry of poly-figures.
The number of n-Gons being contained in an n-Point or n-Line is (n-1)!/2.
For example in a Quadrangle (a 4-Point) we have (4-1)!/2 = 3 Quadrigons (= 4-Gons).
In a Pentangle (a 5-Point) we have (5-1)!/2 = 12 Pentagons (= 5-Gons), etc.
In a Hexangle (a 6-Point) we have (6-1)!/2 = 60 Hexagons (= 5-Gons), etc.
Just to show the difference in n-gons here some regular forms are shown occurring in a regular n-Point, although normally their forms won't be regular. In an n-Line other figures will occur.
The number of 4-Gons (bounded figures) in a 4-Point is 3:



The number of 5-Gons (bounded figures) in a 5-Point is 12:



The number of 6-Gons (bounded figures) in a 6-Point is 60:



Etc.

See also :
http://xahlee.info/MathGraphicsGallery_dir/Combinatorics_dir/loopNPoints.html
http://www.robertdickau.com/cyclicperms.html

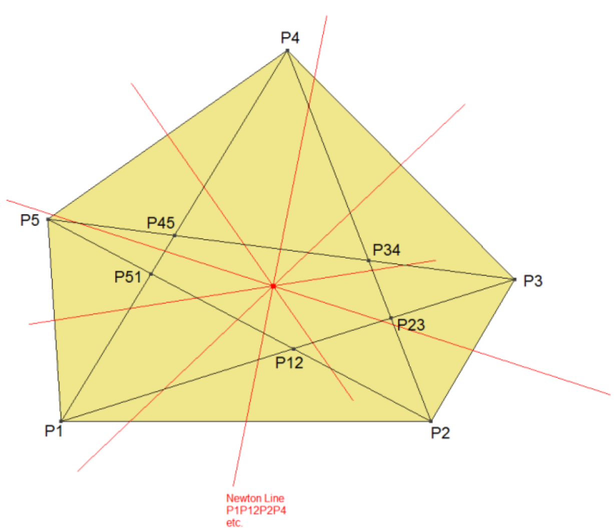
5-Gons

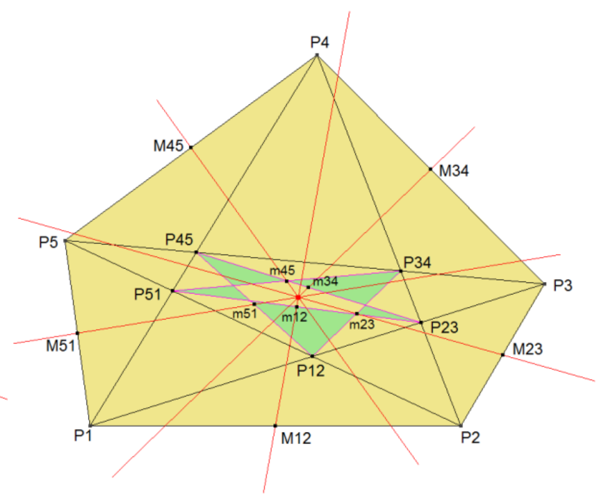
5G-s: Specific Objects in a 5-Gon

5G-s-P1 5G-Common Newton Lines Point

Given a pentagon P1P2P3P4P5.
We denote the intersection of P1P3 and P2P5 by P12.
Similarly P23, P34, P45 and P51 are defined.
The 5 Newton lines of P1P12P2P4, P2P23P3P5, P3P34P4P1, P4P45P5P2 and P5P51P1P3 have a common point 5G-s-P1.
See Ref-34, Seiichi Kirikami, QFG#760.

There is another way to construct this point:
Given a pentagon P1P2P3P4P5.
We denote the intersection of P1P3 and P2P5 by P12.
Similarly P23, P34, P45 and P51 are defined.
We denote the midpoints of PiPi+1 by Mi+3
We denote the midpoints of PiPi+2 by mi. The lines Mimi concur in 5G-s-P1.
See Ref-34, Seiichi Kirikami, QFG#726.





Note that the Newton Line in the left figure coincides with the lines Mimi in the right figure.

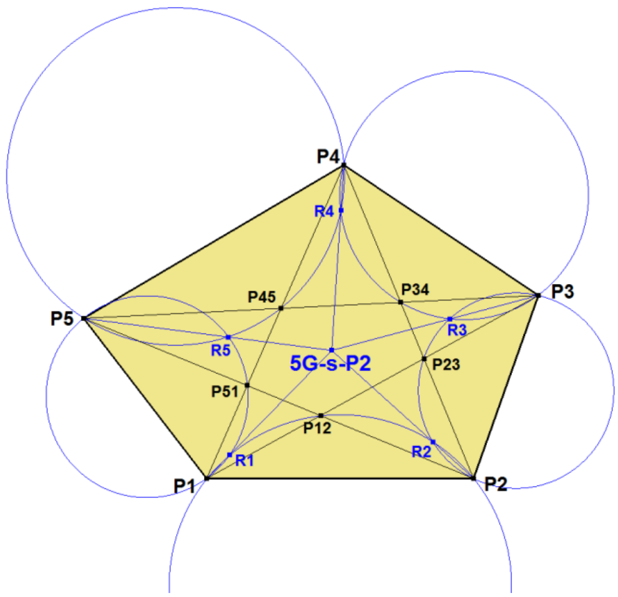
Coordinates:
Let P1,P2,P3,P4,P5 have these barycentric coordinates:
P1=(0:1:0), P2=(0:0:1), P3=(1:0:0), P4=(p:q:r) and P5=(P:Q:R).
Then 5G-s-P1 has coordinates: (p (2 P + Q) + P (q + r) : P q + p Q + q (2 Q + R) : P r + q R)
Calculation Seiichi Kirikami. See Ref-34, QFG#750.

Properties:

- 5G-s-P1, 5G-s-P2 and 5L-s-P1 are collinear.

5G-s-P2 5G-Bicircles Crosspoint

Given a Pentagon P1P2P3P4P5. We denote the intersection of PiPi+2 and Pi-1Pi+1 by Pii+1. We denote the circumcircle of PiPii+1Pi+1 by Cii+1. We denote the second intersection of Ci-1i and Cii+1 by Ri. Then PiRi concur in a point P.
See Ref-34, Seiichi Kirikami, QFG#721.



Coordinates:
Let P1,P2,P3,P4,P5 have these barycentric coordinates:
P1=(0:1:0), P2=(0:0:1), P3=(1:0:0), P4=(p:q:r) and P5=(P:Q:R).
Then 5G-s-P2 has coordinates: p (P+Q) : Q (p+q) : - r R).
Calculation Seiichi Kirikami. See Ref-34, QFG#751.

Properties:

- 5G-s-P1, 5G-s-P2 and 5L-s-P1 are collinear.

5G-s-P3 5G-Inner Miquel Points Center

Given a Pentagon P1P2P3P4P5. We denote the intersection of P1P3 and P2P5 by P12.
Similarly P23, P34, P45 and P51 are defined.
The 5 Miquel points M12, M23, M34, M45, M51of resp. P1P12P2P4, P2P23P3P5, P3P34P4P1, P4P45P5P2 and P5P51P1P3 are concyclic.
See Ref-34, Seiichi Kirikami, QFG#760.