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A Pair of Kiepert Hyperbolas

Jean-Pierre Ehrmann

Abstract. The solution of a locus problem of Hatzipolakis can be expressed in terms of a simple relationship concerning points on a pair of Kiepert hyperbolas associated with a triangle. We study a generalization.

Let P be a finite point in the plane of triangle ABC. Denote by a, b, c the lengths of the sides BC, CA, AB respectively, and by A_H, B_H, C_H the feet of the altitudes. We consider rays through P in the directions of the altitudes AA_H, BB_H, CC_H, and, for a nonzero constant k, choose points A', B', C' on these rays such that

$$PA' = ka, \qquad PB' = kb, \qquad PC' = kc. \tag{1}$$

Antreas P. Hatzipolakis [1] has asked, for $k = 1$, for the locus of P for which triangle $A'B'C'$ is perspective with ABC.

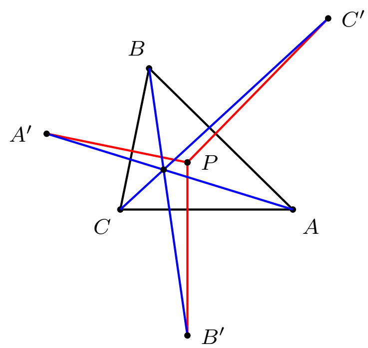

Figure 1

We tackle the general case by making use of homogeneous barycentric coordinates with respect to ABC. Thus, write $P = (u : v : w)$. In the notations introduced by John H. Conway, [1]

$$\begin{aligned}
A' &= (uS - k(u+v+w)a^2 : vS + k(u+v+w)S_C : wS + k(u+v+w)S_B),\\
B' &= (uS + k(u+v+w)S_C : vS - k(u+v+w)b^2 : wS + k(u+v+w)S_A),\\
C' &= (uS + k(u+v+w)S_B : vS + k(u+v+w)S_A : wS - k(u+v+w)c^2).
\end{aligned}$$

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[1]Let ABC be a triangle of side lengths a, b, c, and area $\frac{1}{2}S$. For each ϕ, $S_\phi := S \cdot \cot\phi$. Thus, $S_A = \frac{1}{2}(b^2+c^2-a^2)$, $S_B = \frac{1}{2}(c^2+a^2-b^2)$, and $S_C = \frac{1}{2}(a^2+b^2-c^2)$. These satisfy $S_AS_B + S_BS_C + S_CS_A = S^2$ and other simple relations. For a brief summary, see [3, §1].

The equations of the lines AA', BB', CC' are

$$(wS + k(u+v+w)S_B)y - (vS + k(u+v+w)S_C)z = 0, \quad (2)$$
$$-(wS + k(u+v+w)S_A)x + (uS + k(u+v+w)S_C)z = 0, \quad (3)$$
$$(vS + k(u+v+w)S_A)x - (uS + k(u+v+w)S_B)y = 0. \quad (4)$$

These three lines are concurrent if and only if

$$\begin{vmatrix} 0 & wS + k(u+v+w)S_B & -(vS + k(u+v+w)S_C) \\ -(wS + k(u+v+w)S_A) & 0 & uS + k(u+v+w)S_C \\ vS + k(u+v+w)S_A & -(uS + k(u+v+w)S_B) & 0 \end{vmatrix} = 0.$$

This condition can be rewritten as

$$kS(u+v+w)(S \cdot K(u,v,w) - k(u+v+w)L(u,v,w)) = 0,$$

where

$$K(u,v,w) = (b^2-c^2)vw + (c^2-a^2)wu + (a^2-b^2)uv, \quad (5)$$
$$L(u,v,w) = (b^2-c^2)S_A u + (c^2-a^2)S_B v + (a^2-b^2)S_C w. \quad (6)$$

Note that $K(u,v,w) = 0$ and $L(u,v,w) = 0$ are respectively the equations of the Kiepert hyperbola and the Euler line of triangle ABC. Since P is a finite point and k is nonzero, we conclude, by writing $k = \tan\phi$, that the locus of P for which $A'B'C'$ is perspective with ABC is the rectangular hyperbola

$$S_\phi K(u,v,w) - (u+v+w)L(u,v,w) = 0 \quad (7)$$

in the pencil generated by the Kiepert hyperbola and the Euler line.

Floor van Lamoen [2] has pointed out that this hyperbola (7) is the Kiepert hyperbola of a Kiepert triangle of the dilated (anticomplementary) triangle of ABC. Specifically, let $\mathcal{K}(\phi)$ be the Kiepert triangle whose vertices are the apexes of similar isosceles triangles of base angles ϕ constructed on the sides of ABC. It is shown in [3] that the Kiepert hyperbola of $\mathcal{K}(\phi)$ has equation

$$2S_\phi(\sum_{\text{cyclic}}(b^2-c^2)yz) + (x+y+z)(\sum_{\text{cyclic}}(b^2-c^2)(S_A+S_\phi)x) = 0.$$

If we replace x, y, z respectively by $v+w$, $w+u$, $u+v$, this equation becomes (7) above. This means that the hyperbola (7) is the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC.[2]

The orthocenter H and the centroid G are always on the locus. Trivially, if $P = H$, the perspector is the same point H. For $P = G$, the perspector is the point [3]

$$\left(\frac{1}{3kS_A + S} : \frac{1}{3kS_B + S} : \frac{1}{3kS_C + S}\right),$$

[2] The Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle of ABC is also the dilated triangle of the Kiepert triangle $\mathcal{K}(\phi)$ of triangle ABC.

[3] In the notations of [3], this is the Kiepert perspector $K(\arctan 3k)$.

the second common point of Kiepert hyperbola and the tangent at P to the locus of P, the Kiepert hyperbola of the dilated triangle of $\mathcal{K}(\phi)$.

Now we identify the perspector when P is different from G. Addition of the equations (2,3,4) of the lines AA', BB', CC' gives

$$(v-w)x+(w-u)y+(u-v)z=0.$$

This is the equation of the line joining P to the centroid G, showing that the perspector lies on the line GP.

We can be more precise. Reorganize the equations (2,3,4) as

$$k(S_By-S_Cz)u+(k(S_By-S_Cz)-Sz)v+(k(S_By-S_Cz)+Sy)w=0, \quad (8)$$
$$(k(S_Cz-S_Ax)+Sz)u+\quad (k(S_Cz-S_Ax)v+(k(S_Cz-S_Ax)-Sx)w=0, \quad (9)$$
$$(k(S_Ax-S_By)-Sy)u+(k(S_Ax-S_By)+Sx)v+\quad k(S_Ax-S_By)w=0. \quad (10)$$

Note that the combination x·(8) + y·(9) + z·(10) gives

$$k(u+v+w)(x(S_By-S_Cz)+y(S_Cz-S_Ax)+z(S_Ax-S_By)=0.$$

Since k and $u+v+w$ are nonzero, we have

$$(S_C-S_B)yz+(S_A-S_C)zx+(S_B-S_A)xy=0,$$

or equivalently, $(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0$. It follows that the perspector is also on the Kiepert hyperbola.

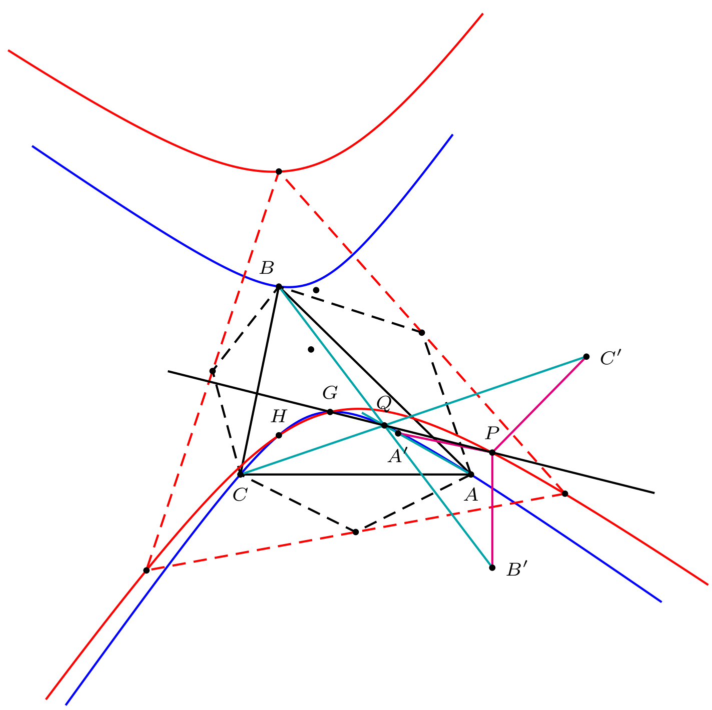

Figure 2

We summarize these results in the following theorem.

Theorem. *Let $k = \tan\phi$ be nonzero, and points A', B', C' be given by (1) along the rays through P parallel to the altitudes. The lines AA', BB', CC' are concurrent if and only if P lies on the Kiepert hyperbola of the Kiepert triangle $\mathcal{K}(\phi)$ of the dilated triangle. The intersection of these lines is the second intersection of the line GP and the Kiepert hyperbola of triangle ABC.*

If we change, for example, the orientation of PA', the locus of P is the rectangular hyperbola with center at the apex of the isosceles triangle on BC of base angle ϕ, [4] asymptotes parallel to the A-bisectors, and passing through the orthocenter H (and also the A-vertex $A^G = (-1 : 1 : 1)$ of the dilated triangle). For $P = A^G$, the perspector is the point $\left(\dfrac{1}{kS_A + S} : \dfrac{1}{kS_B - S} : \dfrac{1}{kS_C - S}\right)$, and for $P \neq A^G$, the second common point of the line PA^G and the rectangular circum-hyperbola with center the midpoint of BC.

We conclude by noting that for a positive k, the locus of P for which we can choose points A', B', C' on the perpendiculars through P to BC, CA, AB such that the lines AA', BB', CC' concur and the distances from P to A', B', C' are respectively k times the lengths of the corresponding side is the union of 8 rectangular hyperbolas.

References

[1] A. P. Hatzipolakis, Hyacinthos message 2510, March 1, 2001.
[2] F. M. van Lamoen, Hyacinthos message 2541, March 6, 2001.
[3] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

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[4] This point has coordinates $(-a^2 : S_C + S_\phi : S_B + S_\phi)$.

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Some Concurrencies from Tucker Hexagons

Floor van Lamoen

Abstract. We present some concurrencies in the figure of Tucker hexagons together with the centers of their Tucker circles. To find the concurrencies we make use of extensions of the sides of the Tucker hexagons, isosceles triangles erected on segments, and special points defined in some triangles.

1. The Tucker hexagon $\mathcal{T}_\phi$ and the Tucker circle $\mathcal{C}_\phi$

Consider a scalene (nondegenerate) reference triangle ABC in the Euclidean plane, with sides $a = BC$, $b = CA$ and $c = AB$. Let B_a be a point on the sideline CA. Let C_a be the point where the line through B_a antiparallel to BC meets AB. Then let A_c be the point where the line through C_a parallel to CA meets BC. Continue successively the construction of parallels and antiparallels to complete a hexagon $B_aC_aA_cB_cC_bA_b$ of which B_aC_a, A_cB_c and C_bA_b are antiparallel to sides BC, CA and AB respectively, while B_cC_b, A_cC_a and A_bB_a are parallel to these respective sides.

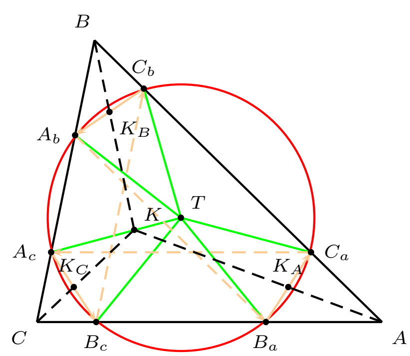

Figure 1

This is the well known way to construct a *Tucker hexagon*. Each Tucker hexagon is circumscribed by a circle, the *Tucker circle*. The three antiparallel sides are congruent; their midpoints K_A, K_B and K_C lie on the symmedians of ABC in such a way that $AK_A : AK = BK_B : BK = CK_C : CK$, where K denotes the symmedian point. See [1, 2, 3].

1.1. *Identification by central angles.* We label by $\mathcal{T}_\phi$ the specific Tucker hexagon in which the congruent central angles on the chords B_aC_a, C_bA_b and A_cB_c have measure 2ϕ. The circumcircle of the Tucker hexagon is denoted by $\mathcal{C}_\phi$, and its radius by r_ϕ. In this paper, the points B_a, C_a, A_b, C_b, A_c and B_c are the vertices of $\mathcal{T}_\phi$, and T denotes the center of the Tucker circle $\mathcal{C}_\phi$.

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The author thanks the Communicating Editor for his assistance with §7.

Let M_a, M_b and M_c be the midpoints of A_bA_c, B_aB_c and C_aC_b respectively. Since

$$\angle M_bTM_c = B + C, \quad \angle M_cTM_a = C + A, \quad \angle M_aTM_b = A + B,$$

the top angles of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b have measures $2(A - \phi)$, $2(B - \phi)$, and $2(C - \phi)$ respectively. [1]

From these top angles, we see that the distances from T to the sidelines of triangle ABC are $r_\phi \cos(A - \phi)$, $r_\phi \cos(B - \phi)$ and $r_\phi \cos(C - \phi)$ respectively, so that in homogeneous barycentric coordinates,

$$T = (a\cos(A - \phi) : b\cos(B - \phi) : c\cos(C - \phi)).$$

For convenience we write $\overline{\phi} := \frac{\pi}{2} - \phi$. In the notations introduced by John H. Conway, [2]

$$T = (a^2(S_A + S_{\overline{\phi}}) : b^2(S_B + S_{\overline{\phi}}) : c^2(S_C + S_{\overline{\phi}})). \tag{1}$$

This shows that T is the isogonal conjugate of the Kiepert perspector $K(\overline{\phi})$.[3] We shall, therefore, write $K^*(\overline{\phi})$ for T. It is clear that $K^*(\overline{\phi})$ lies on the Brocard axis, the line through the circumcenter O and symmedian point K.

Some of the most important $K^*(\overline{\phi})$ are listed in the following table, together with the corresponding number in Kimberling's notation of [4, 5]. We write ω for the Brocard angle.

ϕ	$K^*(\overline{\phi})$	Kimberling's Notation
0	Circumcenter	X_3
ω	Brocard midpoint	X_{39}
$\pm\frac{\pi}{4}$	Kenmotu points	X_{371}, X_{372}
$\pm\frac{\pi}{3}$	Isodynamic centers	X_{15}, X_{16}
$\frac{\pi}{2}$	Symmedian point	X_6

1.2. *Coordinates.* Let K' and C_b' be the feet of the perpendiculars from $K^*(\overline{\phi})$ and C_b to BC. By considering the measures of sides and angles in $C_bC_b'K'K^*(\overline{\phi})$ we find that the (directed) distances α from C_b to BC as

$$\begin{aligned}\alpha &= r_\phi(\cos(A - \phi) - \cos(A + \phi)) \\ &= 2r_\phi \sin A \sin\phi. \end{aligned} \tag{2}$$

In a similar fashion we find the (directed) distance β from C_b to AC as

$$\begin{aligned}\beta &= r_\phi(\cos(B - \phi) + \cos(A - C + \phi)) \\ &= 2r_\phi \sin C \sin(A + \phi). \end{aligned} \tag{3}$$

[1] Here, a negative measure implies a negative orientation for the isosceles triangle.

[2] For an explanation of the notation and a brief summary, see [7, §1].

[3] This is the perspector of the triangle formed by the apexes of isosceles triangles on the sides of ABC with base angles $\overline{\phi}$. See, for instance, [7].

Combining (2) and (3) we obtain the barycentric coordinates of C_b:

$$\begin{aligned} C_b =&(a^2 \sin\phi : bc(\sin(A+\phi) : 0) \\ =&(a^2 : S_A + S_\phi : 0). \end{aligned}$$

In this way we find the coordinates for the vertices of the Tucker hexagon as

$$\begin{array}{ll} B_a = (S_C + S_\phi : 0 : c^2), & C_a = (S_B + S_\phi : b^2 : 0), \\ A_c = (0 : b^2 : S_B + S_\phi), & B_c = (a^2 : 0 : S_A + S_\phi), \\ C_b = (a^2 : S_A + S_\phi : 0), & A_b = (0 : S_C + S_\phi : c^2). \end{array} \tag{4}$$

Remark. The radius of the Tucker circle is $r_\phi = \frac{R \sin\omega}{\sin(\phi+\omega)}$.

2. Triangles of parallels and antiparallels

With the help of (4) we find that the three antiparallels from the Tucker hexagons bound a triangle $A_1B_1C_1$ with coordinates:

$$\begin{aligned} A_1 &= \left(\frac{a^2(S_A - S_\phi)}{S_A + S_\phi} : b^2 : c^2 \right), \\ B_1 &= \left(a^2 : \frac{b^2(S_B - S_\phi)}{S_B + S_\phi} : c^2 \right), \\ C_1 &= \left(a^2 : b^2 : \frac{c^2(S_C - S_\phi)}{S_C + S_\phi} \right). \end{aligned} \tag{5}$$

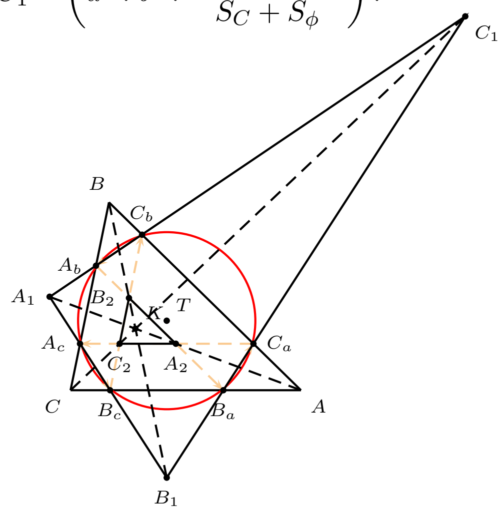

Figure 2

In the same way the parallels bound a triangle $A_2B_2C_2$ with coordinates:

$$\begin{aligned} A_2 =&(-(S_A - S_\phi) : b^2 : c^2), \\ B_2 =&(a^2 : -(S_B - S_\phi) : c^2), \\ C_2 =&(a^2 : b^2 : -(S_C - S_\phi)). \end{aligned} \tag{6}$$

It is clear that the three triangles are perspective at the symmedian point K. See Figure 2. Since ABC and $A_2B_2C_2$ are homothetic, we have a very easy construction of Tucker hexagons without invoking antiparallels: construct a triangle homothetic to ABC through K, and extend the sides of this triangles to meet the sides of ABC in six points. These six points form a Tucker hexagon.

3. Congruent rhombi

Fix ϕ. Recall that K_A, K_B and K_C are the midpoints of the antiparallels B_aC_a, A_bC_b and A_cB_c respectively. With the help of (4) we find

$$\begin{aligned} K_A =&(a^2 + 2S_\phi : b^2 : c^2),\\ K_B =&(a^2 : b^2 + 2S_\phi : c^2),\\ K_C =&(a^2 : b^2 : c^2 + 2S_\phi). \end{aligned} \tag{7}$$

Reflect the point $K^*(\overline{\phi})$ through K_A, K_B and K_C to A_ϕ, B_ϕ and C_ϕ respectively. These three points are the opposite vertices of three congruent rhombi from the point $T = K^*(\overline{\phi})$. Inspired by the figure of the *Kenmotu point* X_{371} in [4, p.268], which goes back to a collection of *Sangaku problems* from 1840, the author studied these rhombi in [6] without mentioning their connection to Tucker hexagons.

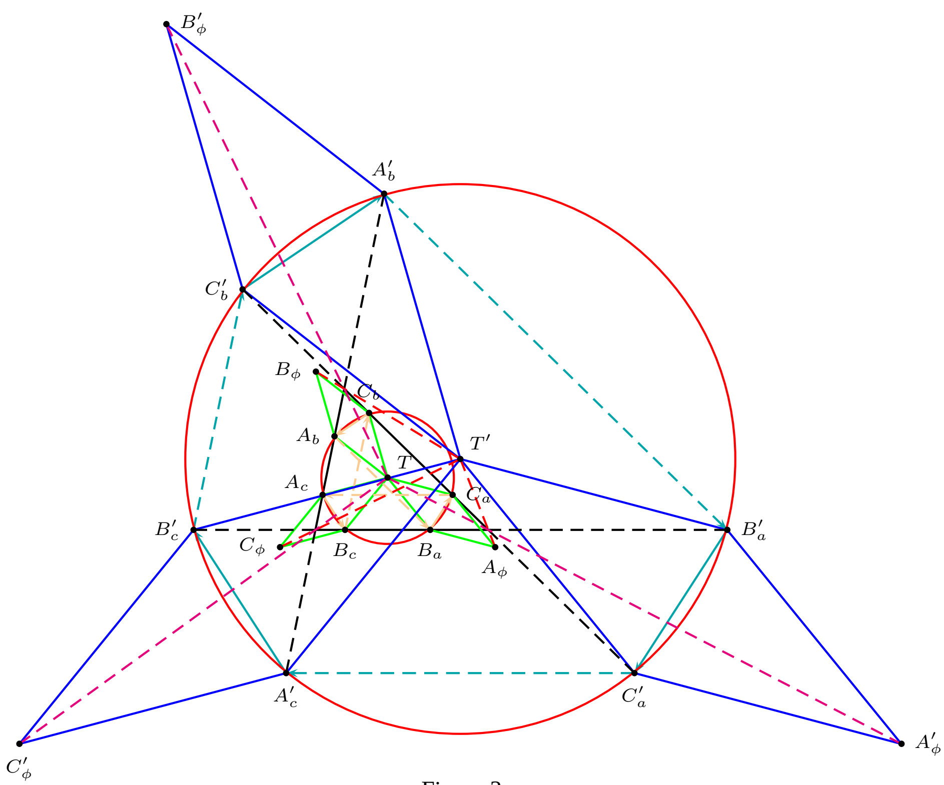

Figure 3

With the help of the coordinates for $K^*(\overline{\phi})$ and K_A found in (1) and (7) we find after some calculations,

$$\begin{aligned} A_\phi =&(a^2(S_A - S_{\overline{\phi}}) - 4S^2 : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}})),\\ B_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) - 4S^2 : c^2(S_C - S_{\overline{\phi}})),\\ C_\phi =&(a^2(S_A - S_{\overline{\phi}}) : b^2(S_B - S_{\overline{\phi}}) : c^2(S_C - S_{\overline{\phi}}) - 4S^2).\end{aligned} \tag{8}$$

From these, it is clear that ABC and $A_\phi B_\phi C_\phi$ are perspective at $K^*(-\overline{\phi})$.

The perspectivity gives spectacular figures, because the rhombi formed from $\mathcal{T}_\phi$ and $\mathcal{T}_{-\phi}$ are parallel. See Figure 3. In addition, it is interesting to note that $K^*(\overline{\phi})$ and $K^*(-\overline{\phi})$ are *harmonic conjugates* with respect to the circumcenter O and the symmedian point K.

4. Isosceles triangles on the sides of $A_bA_cB_cB_aC_aC_b$

Consider the hexagon $A_bA_cB_cB_aC_aC_b$. Define the points A_3, B_3, C_3, A_4, B_4 and C_4 as the apexes of isosceles triangles $A_cA_bA_3$, $B_aB_cB_3$, $C_bC_aC_3$, $B_aC_aA_4$, $C_bA_bB_4$ and $A_cB_cC_4$ of base angle ψ, where all six triangles have positive orientation when $\psi > 0$ and negative orientation when $\psi < 0$. See Figure 4.

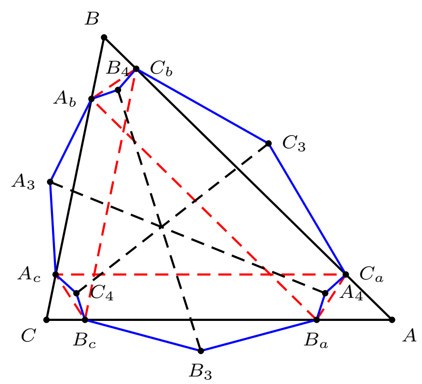

Figure 4

Proposition 1. *The lines A_3A_4, B_3B_4 and C_3C_4 are concurrent.*

Proof. Let $B_aC_a = C_bA_b = A_cB_c = 2t$, where t is given positive sign when C_aB_a and BC have equal directions, and positive sign when these directions are opposite. Note that $K_AK_BK_C$ is homothetic to ABC and that $K^*(\overline{\phi})$ is the circumcenter of $K_AK_BK_C$. Denote the circumradius of $K_AK_BK_C$ by ρ. Then we find the following:

- the signed distance from K_AK_c to AC is $t \sin B = t\frac{|K_AK_C|}{2\rho}$;
- the signed distance from AC to B_3 is $\frac{1}{2}\tan\psi|K_AK_C| - t\tan\psi\cos B$;
- the signed distance from A_4C_4 to K_AK_C is $t\tan\psi\cos B$.

Adding these signed distances we find that the signed distance from A_4C_4 to B_3 is equal to $(\frac{t}{2\rho}+\frac{\tan\psi}{2})|K_AK_C|$. By symmetry we see the signed distances from the sides B_4C_4 and A_4B_4 to A_3 and C_3 respectively are $|K_BK_C|$ and $|K_AK_B|$ multiplied by the same factor. Since triangles $K_AK_BK_C$ and $A_4B_4C_4$ are similar, the three distances are proportional to the sidelengths of $A_4B_4C_4$. Thus, $A_3B_3C_3$ is a Kiepert triangle of $A_4B_4C_4$. From this, we conclude that A_3A_4, B_3B_4 and C_3C_4 are concurrent. □

5. Points defined in *pap* triangles

Let ϕ vary and consider the triangle $A_2C_aB_a$ formed by the lines B_aA_b, B_aC_a and C_aA_c. We call this the A-*pap* triangle, because it consists of a **p**arallel, an **a**ntiparallel and again a **p**arallel. Let the parallels B_aA_b and C_aA_c intersect in A_2. Then, A_2 is the reflection of A through K_A. It clearly lies on the $A-$symmedian. See also §2. The A-*pap* triangle $A_2C_aB_a$ is oppositely similar to ABC. Its vertices are

$$\begin{aligned} A_2 =&(-(S_A-S_\phi):b^2:c^2),\\ C_a =&(S_B+S_\phi:b^2:0),\\ B_a =&(S_C+S_\phi:0:c^2). \end{aligned} \tag{9}$$

Now let $P=(u:v:w)$ be some point given in homogeneous barycentric coordinates with respect to ABC. For $X\in\{A,B,C\}$, the locus of the counterpart of P in the X-*pap* triangles for varying ϕ is a line through X. This can be seen from the fact that the quadrangles $AC_aA_2B_a$ in all Tucker hexagons are similar. Because the sums of coordinates of these points given in (9) are equal, we find that the A-counterpart of P, namely, P evaluated in $A_2C_aB_a$, say P_{A-pap}, has coordinates

$$\begin{aligned} P_{A-pap} \sim& u\cdot A_2+v\cdot C_a+w\cdot B_a\\ \sim& u(-(S_A-S_\phi):b^2:c^2)+v(S_B+S_\phi:b^2:0)+w(S_C+S_\phi:0:c^2)\\ \sim&(-S_Au+S_Bv+S_Cw+(u+v+w)S_\phi:b^2(u+v):c^2(u+w)). \end{aligned}$$

From this, it is clear that P_{A-pap} lies on the line $A\widetilde{P}$ where

$$\widetilde{P}=\left(\frac{a^2}{v+w}:\frac{b^2}{w+u}:\frac{c^2}{u+v}\right).$$

Likewise, we consider the counterparts of P in the B-*pap* and C-*pap* triangles $C_bB_2A_b$ and $B_cA_cC_2$. By symmetry, the loci of P_{B-pap} and P_{C-pap} are the B- and C-cevians of $\widetilde{P}$.

Proposition 2. *For every ϕ, the counterparts of P in the three pap-triangles of the Tucker hexagon $\mathcal{T}_\phi$ form a triangle perspective with ABC at the point $\widetilde{P}$.*

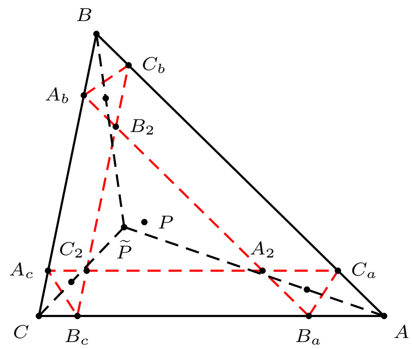

Figure 5

6. Circumcenters of *apa* triangles

As with the *pap*-triangles in the preceding section, we name the triangle $A_1B_cC_b$ formed by the **a**ntiparallel B_cC_b, the **p**arallel A_bC_b, and the **a**ntiparallel A_cB_c the A-*apa* triangle. The other two *apa*-triangles are $A_cB_1C_a$ and $A_bB_aC_1$. Unlike the *pap*-triangles, these are in general not similar to ABC. They are nevertheless isosceles triangles. We have the following interesting results on the circumcenters.

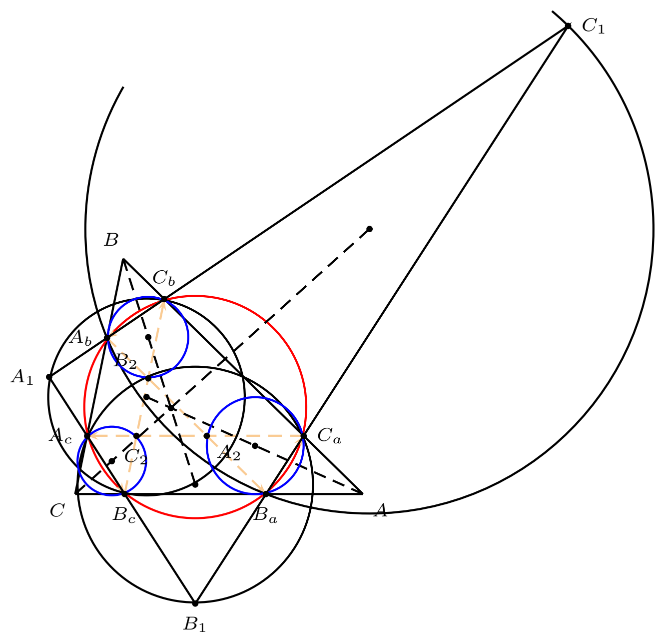

Figure 6

We note that the quadrangles $BA_cO_{B-apa}C_a$ for all possible ϕ are homothetic through B. Therefore, the locus of O_{B-apa} is a line through B. To identify this line, it is sufficient to find O_{B-apa} for one ϕ. Thus, for one special Tucker hexagon, we take the one with $C_a = A$ and $A_c = C$. Then the B-*apa* triangle is the isosceles triangle erected on side b and having a base angle of B, and its circumcenter

O_{B-apa} is the apex of the isosceles triangle erected on the same side with base angle $2B - \frac{\pi}{2}$. Using the identity [4]

$$S^2 = S_{AB} + S_{AC} + S_{BC},$$

we find that

$$\begin{aligned} O_{B-apa} =& (S_C + S_{2B-\frac{\pi}{2}} : -b^2 : S_A + S_{2B-\frac{\pi}{2}}) \\ =& (a^2(a^2 S_A + b^2 S_B) : b^2(S_{BB} - SS) : c^2(b^2 S_B + c^2 S_C)), \end{aligned}$$

after some calculations. From this, we see that the O_{B-apa} lies on the line BN^*, where

$$N^* = \left(\frac{a^2}{b^2 S_B + c^2 S_C} : \frac{b^2}{a^2 S_A + c^2 S_C} : \frac{c^2}{c^2 S_C + b^2 S_B} \right)$$

is the isogonal conjugate of the nine point center N. Therefore, the locus of O_{B-apa} for all Tucker hexagons is the B-cevian of N^*. By symmetry, we see that the loci of O_{A-apa} and O_{C-apa} are the A- and C-cevians of N^* respectively. This, incidentally, is the same as the perspector of the circumcenters of the *pap*-triangles in the previous section.

Proposition 3. *For $X \in \{A, B, C\}$, the line joining the circumcenters of the X-*pap*-triangle and the X-*apa*-triangle passes through X. These three lines intersect at the isogonal conjugate of the nine point center of triangle ABC.*

7. More circumcenters of isosceles triangles

From the center $T = K^*(\overline{\phi})$ of the Tucker circle and the vertices of the Tucker hexagon $\mathcal{T}_\phi$, we obtain six isosceles triangles. Without giving details, we present some interesting results concerning the circumcenters of these isosceles triangles.

(1) The circumcenters of the isosceles triangles TB_aC_a, TC_bA_b and TA_cB_c form a triangle perspective with ABC at

$$K^*(\overline{2\phi}) = (a^2(S_A + S \cdot \tan 2\phi) : b^2(S_B + S \cdot \tan 2\phi) : c^2(S_C + S \cdot \tan 2\phi)).$$

See Figure 7, where the Tucker hexagon $\mathcal{T}_{2\phi}$ and Tucker circle $\mathcal{C}_{2\phi}$ are also indicated.

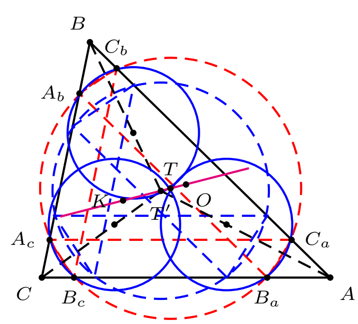

Figure 7

[4]Here, S_{XY} stands for the product $S_X S_Y$.

(2) The circumcenters of the isosceles triangles TA_bA_c, TB_cB_a and TC_aC_b form a triangle perspective with ABC at

$$\left(\frac{a^2}{S^2(3S^2 - S_{BC}) + 2a^2S^2 \cdot S_\phi + (S^2 + S_{BC})S_{\phi\phi}} : \cdots : \cdots\right).$$

See Figure 8.

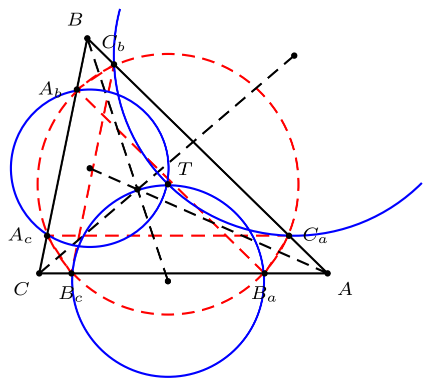

Figure 8

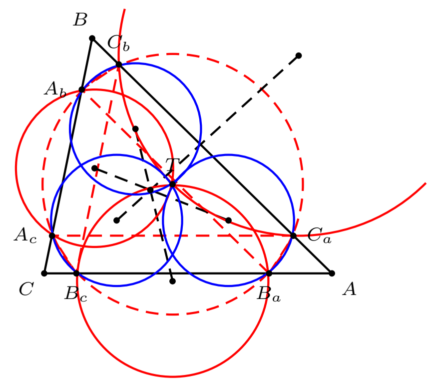

Figure 9

(3) The three lines joining the circumcenters of TB_aC_a, TA_bA_c; ... are concurrent at the point

$$(a^2(S^2(3S^2 - S_{\omega A}) + 2S^2(S_\omega + S_A)S_\phi + (2S^2 - S_{BC} + S_{AA})S_{\phi\phi}) : \cdots : \cdots).$$

See Figure 9.

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Congruent Inscribed Rectangles

Jean-Pierre Ehrmann

Abstract. We solve the construction problem of an interior point P in a given triangle ABC with congruent rectangles inscribed in the subtriangles PBC, PCA and PAB.

1. Congruent inscribed rectangles

Given a triangle with sidelengths a, b, c, let $L_m = \min(a, b, c)$; $L \in (0, L_m)$ and $\mu > 0$. Let P be a point inside ABC with distances d_a, d_b, d_c to the sidelines of ABC. Suppose that a rectangle with lengths of sides L and μL is inscribed in the triangle PBC, with two vertices with distance L on the segment BC, the other vertices on the segments PB and PC. Then, $\dfrac{L}{d_a - \mu L} = \dfrac{a}{d_a}$, or $d_a = \dfrac{\mu a L}{a - L}$.

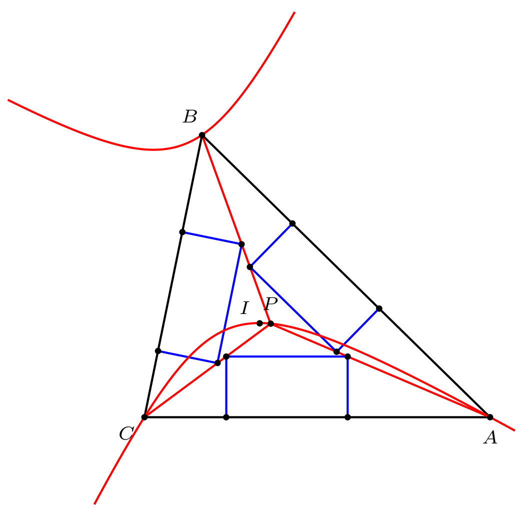

Figure 1

If we can inscribe congruent rectangles with side lengths L and μL in the three triangles PBC, PCA, PAB, we have necessarily

$$f_\mu(L) := \frac{a^2}{a - L} + \frac{b^2}{b - L} + \frac{c^2}{c - L} - \frac{2\Delta}{\mu L} = 0, \tag{1}$$

where Δ is the area of triangle ABC. This is because $ad_a + bd_b + cd_c = 2\Delta$.

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The function $f_\mu(L)$ increases from $-\infty$ to $+\infty$ when L moves on $(0, L_m)$. The equation $f_\mu(L) = 0$ has a unique root L_μ in $(0, L_m)$ and the point

$$P_\mu = \left(\frac{a^2}{a - L_\mu} : \frac{b^2}{b - L_\mu} : \frac{c^2}{c - L_\mu}\right)$$

in homogeneous barycentric coordinates is the only point P inside ABC for which we can inscribe congruent rectangles with side lengths L_μ and μL_μ in the three triangles PBC, PCA, PAB. If $\mathcal{H}_0$ is the circumhyperbola through I (incenter) and K (symmedian point), the locus of P_μ when μ moves on $(0, +\infty)$ is the open arc Ω of $\mathcal{H}_0$ from I to the vertex of ABC opposite to the shortest side. See Figure 1. For $\mu = 1$, the smallest root L_1 of $f_1(L) = 0$ leads to the point P_1 with congruent inscribed squares.

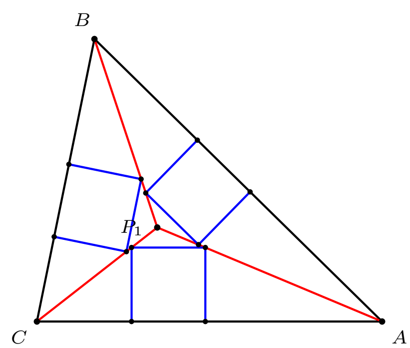

Figure 2

2. Construction of congruent inscribed rectangles

Consider $P \in \Omega$, Q and E the reflections of P and C with respect to the line IB. The parallel to AB through Q intersects BP at F. The lines EF and AP intersect at X. Then the parallel to AB through X is a sideline of the rectangle inscribed in PAB. The reflections of this line with respect to AI and BI will each give a sideline of the two other rectangles. [1]

Proof. We have $\dfrac{\overline{BE}}{\overline{BA}} = \dfrac{a}{c}$, $\dfrac{\overline{BP}}{\overline{BF}} = \dfrac{d_c}{d_a} = \dfrac{c}{a}\dfrac{a - L_\mu}{c - L_\mu}$. Applying the Menelaus theorem to triangle PAB and transversal EFX, we have

$$\frac{\overline{XA}}{\overline{XP}} = \frac{\overline{FB}}{\overline{FP}}\frac{\overline{EA}}{\overline{EB}} = \frac{L_\mu - c}{L_\mu}.$$

More over, the sidelines of the rectangles parallel to BC, CA, AB form a triangle homothetic at I with ABC. □

[1] This construction was given by Bernard Gibert.

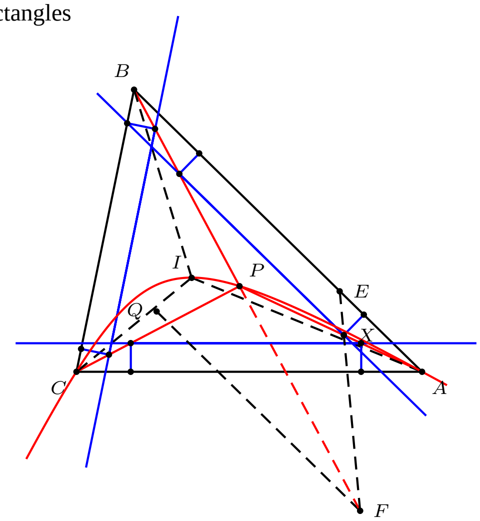

Figure 3

3. Construction of P_μ

The point P_μ is in general not constructible with ruler and compass. We give here a construction as the intersection of the arc Ω with a circle.

Consider the points

$$X_{100} = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right)$$

and

$$X_{106} = \left(\frac{a^2}{b+c-2a} : \frac{b^2}{c+a-2b} : \frac{c^2}{a+b-2c}\right)$$

on the circumcircle. [2] Note that the line $X_{100}X_{106}$ passes through the incenter I. The line joining X_{106} to the symmedian point K intersects the circumcircle again at

$$X_{101} = \left(\frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b}\right).$$

Construction. Draw outwardly a line ℓ parallel to AC at a distance μb from AC, intersecting the line CK at S. The parallel at S to the line CX_{101} intersects the line $KX_{101}X_{106}$ at Y_μ. Then P_μ is the intersection of the arc Ω with the circle through X_{100}, X_{106}, and Y_μ. See Figure 4.

Proof. From

$$L = \frac{2a\Delta x}{2\Delta x + \mu a^2(x+y+z)} = \frac{2b\Delta y}{2\Delta x + \mu b^2(x+y+z)} = \frac{2c\Delta z}{2\Delta z + \mu a^2(x+y+z)},$$

[2]We follow the notations of [1]. Here, X_{100} is the isogonal conjugate of the infinite point of the trilinear polar of the incenter, and X_{106} is the isogonal conjugate of the infinite point of the line GI joining the centroid and the incenter.

we note that P_μ lies on the three hyperbolas $\mathcal{H}_a$, $\mathcal{H}_b$ and $\mathcal{H}_c$ with equations

$$\mu bc(x+y+z)(cy-bz)+2\Delta(b-c)yz=0, \qquad (\mathcal{H}_a)$$
$$\mu ca(x+y+z)(az-cx)+2\Delta(c-a)zx=0, \qquad (\mathcal{H}_b)$$
$$\mu ab(x+y+z)(bx-ay)+2\Delta(a-b)xy=0. \qquad (\mathcal{H}_c)$$

Computing $a^2(a-b)(c-a)(\mathcal{H}_a)+b^2(b-c)(a-b)(\mathcal{H}_b)+c^2(c-a)(b-c)(\mathcal{H}_c)$, we see that P_μ lies on the circle Γ_μ:

$$\mu abc(x+y+z)\Lambda+2\Delta(a-b)(b-c)(c-a)(a^2yz+b^2zx+c^2xy)=0,$$

where

$$\Lambda=bc(b-c)(b+c-2a)x+ca(c-a)(c+a-2b)y+ab(a+b-2c)(a-b)z.$$

As $\Lambda=0$ is the line $X_{100}X_{106}$, the circle Γ_μ passes through X_{100} and X_{106}.

Now, as ℓ is the line $2\Delta y+\mu b^2(x+y+z)=0$, we have

$$S=\left(a^2:b^2:-\left(a^2+b^2+\frac{2\Delta}{\mu}\right)\right).$$

The parallel through S to CX_{101} is the line

$$\mu(b+a-2c)(x+y+z)+2\Delta\left(\frac{(b-c)x}{a^2}+\frac{(a-c)y}{b^2}\right)=0,$$

and KX_{101} is the line

$$b^2c^2(b-c)(b+c-2a)x+c^2a^2(c-a)(c+a-2b)y+a^2b^2(a-b)(a+b-2c)z=0.$$

We can check that these two lines intersect at the point

$$\begin{aligned}Y_\mu=&(a^2(2\Delta(c-a)(a-b)+\mu(-a^2(b^2+c^2)+2abc(b+c)+(b^4-2b^3c-2bc^3+c^4))\\&:b^2(2\Delta(a-b)(b-c)+\mu(-b^2(c^2+a^2)+2abc(c+a)+(c^4-2c^4a-2ca^3+a^4))\\&:c^2(2\Delta(b-c)(c-a)+\mu(-c^2(a^2+b^2)+2abc(a+b)+(a^4-2a^3b-2ab^3+b^4)))\end{aligned}$$

on the circle Γ_μ. □

Remark. The circle through X_{100}, X_{106} and P_μ is the only constructible circle through P_μ, and there is no constructible line through P_μ.

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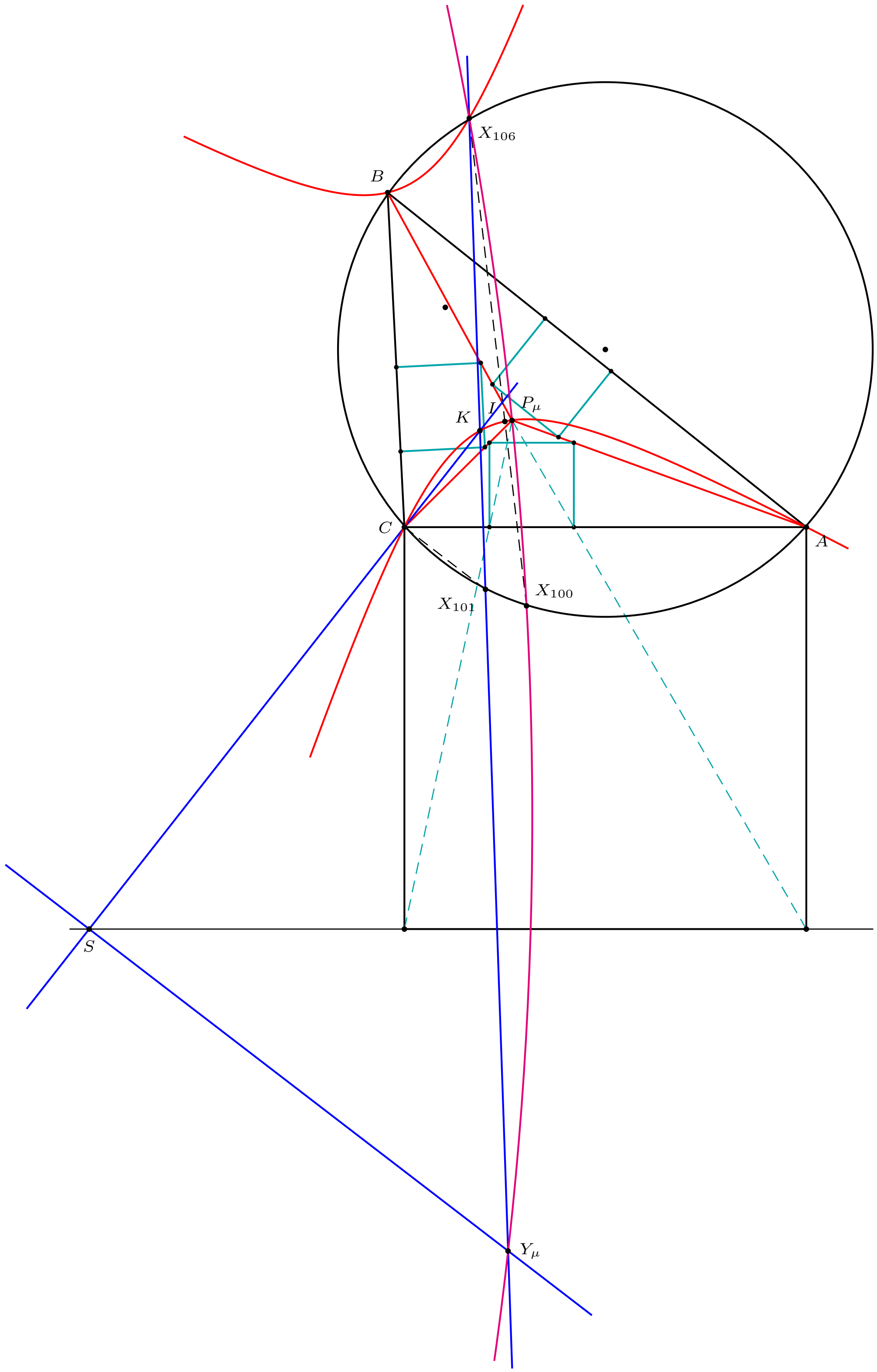

Figure 4

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Collineations, Conjugacies, and Cubics

Clark Kimberling

Abstract. If F is an involution and φ a suitable collineation, then $\varphi \circ F \circ \varphi^{-1}$ is an involution; this form includes well-known conjugacies and new conjugacies, including *aleph, beth, complementary,* and *anticomplementary*. If $Z(U)$ is the self-isogonal cubic with pivot U, then φ carries $Z(U)$ to a pivotal cubic. Particular attention is given to the Darboux and Lucas cubics, D and L, and conjugacy-preserving mappings between D and L are formulated.

1. Introduction

The defining property of the kind of mapping called *collineation* is that it carries lines to lines. Matrix algebra lends itself nicely to collineations as in [1, Chapter XI] and [5]. In order to investigate collineation-induced conjugacies, especially with regard to triangle centers, suppose that an arbitrary point P in the plane of $\triangle ABC$ has homogeneous trilinear coordinates $p : q : r$ relative to $\triangle ABC$, and write

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1,$$

so that

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suppose now that suitably chosen points $P_i = p_i : q_i : r_i$ and $P_i' = p_i' : q_i' : r_i'$ for $i = 1, 2, 3, 4$ are given and that we wish to represent the unique collineation φ that maps each P_i to P_i'. (Precise criteria for "suitably chosen" will be determined soon.) Let

$$\mathbb{P} = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}, \qquad \mathbb{P}' = \begin{pmatrix} p_1' & q_1' & r_1' \\ p_2' & q_2' & r_2' \\ p_3' & q_3' & r_3' \end{pmatrix}.$$

We seek a matrix $\mathbb{M}$ such that $\varphi(X) = X\mathbb{M}$ for every point $X = x : y : z$, where X is represented as a 1×3 matrix:

$$X = \begin{pmatrix} x & y & z \end{pmatrix}$$

In particular, we wish to have

$$\mathbb{P}\mathbb{M} = \mathbb{D}\mathbb{P}' \quad \text{and} \quad P_4\mathbb{M} = \begin{pmatrix} gp_4' & gq_4' & gr_4' \end{pmatrix},$$

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where

$$\mathbb{D} = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$$

for some multipliers d, e, f, g. By homogeneity, we can, and do, put $g = 1$. Then substituting $\mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$ for $\mathbb{M}$ gives $P_4\mathbb{P}^{-1}\mathbb{D} = \mathbb{P}_4'(\mathbb{P}')^{-1}$. Writing out both sides leads to

$$\begin{aligned} d &= \frac{(q_2'r_3' - q_3'r_2')p_4' + (r_2'p_3' - r_3'p_2')q_4' + (p_2'q_3' - p_3'q_2')r_4'}{(q_2r_3 - q_3r_2)p_4 + (r_2p_3 - r_3p_2)q_4 + (p_2q_3 - p_3q_2)r_4}, \\ e &= \frac{(q_3'r_1' - q_1'r_3')p_4' + (r_3'p_1' - r_1'p_3')q_4' + (p_3'q_1' - p_1'q_3')r_4'}{(q_3r_1 - q_1r_3)p_4 + (r_3p_1 - r_1p_3)q_4 + (p_3q_1 - p_1q_3)r_4}, \\ f &= \frac{(q_1'r_2' - q_2'r_1')p_4' + (r_1'p_2' - r_2'p_1')q_4' + (p_1'q_2' - p_2'q_1')r_4'}{(q_1r_2 - q_2r_1)p_4 + (r_1p_2 - r_2p_1)q_4 + (p_1q_2 - p_1q_2)r_4}. \end{aligned}$$

The point $D := d : e : f$ is clearly expressible as quotients of determinants:

$$D = \frac{\begin{vmatrix} p_4' & q_4' & r_4' \\ p_2' & q_2' & r_2' \\ p_3' & q_3' & r_3' \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}} : \frac{\begin{vmatrix} p_4' & q_4' & r_4' \\ p_3' & q_3' & r_3' \\ p_1' & q_1' & r_1' \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_3 & q_3 & r_3 \\ p_1 & q_1 & r_1 \end{vmatrix}} : \frac{\begin{vmatrix} p_4' & q_4' & r_4' \\ p_1' & q_1' & r_1' \\ p_2' & q_2' & r_2' \end{vmatrix}}{\begin{vmatrix} p_4 & q_4 & r_4 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

With $\mathbb{D}$ determined[1], we write

$$\mathbb{M} = \mathbb{P}^{-1}\mathbb{D}\mathbb{P}'$$

and are now in a position to state the conditions to be assumed about the eight initial points:

(i) $\mathbb{P}$ and $\mathbb{P}'$ are nonsingular;

(ii) the denominators in the expressions for d, e, f are nonzero;

(iii) $def \neq 0$.

Conditions (i) and (ii) imply that the collineation φ is given by $\varphi(X) = X\mathbb{M}$, and (iii) ensures that $\varphi^{-1}(X) = X\mathbb{M}^{-1}$. A collineation φ satisfying (i)-(iii) will be called *regular*. If φ is regular then clearly φ^{-1} is regular.

If the eight initial points are centers (*i.e.*, triangle centers) for which no three P_i are collinear and no three P_i' are collinear, then for every center X, the image $\varphi(X)$ is a center. If P_1, P_2, P_3 are respectively the A-, B-, C- vertices of a central triangle [3, pp. 53-57] and P_4 is a center, and if the same is true for P_i' for $i = 1$, 2, 3, 4, then in this case, too, φ carries centers to centers.

---

[1]A geometric realization of D follows. Let $\hat{P}$ denote the circle

$$(p_1\alpha + p_2\beta + p_3\gamma)(a\alpha + b\beta + c\gamma) + p_4(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0,$$

and let $\hat{Q}$, $\hat{R}$, $\hat{P}'$, $\hat{Q}'$, $\hat{R}'$ be the circles likewise formed from the points P_i and P_i'. Following [3, p.225], let Λ and Λ' be the radical centers of circles $\hat{P}, \hat{Q}, \hat{R}$ and $\hat{P}', \hat{Q}', \hat{R}'$, respectively. Then D is the trilinear quotient Λ/Λ'.

The representation $\varphi(X) = X\mathbb{M}$ shows that for $X = x : y : z$, the image $\varphi(X)$ has the form

$$f_1x + g_1y + h_1z : f_2x + g_2y + h_2z : f_3x + g_3y + h_3z.$$

Consequently, if Λ is a curve homogeneous of degree $n \geq 1$ in α, β, γ, then $\varphi(\Lambda)$ is also a curve homogeneous of degree n in α, β, γ. In particular, φ carries a circumconic onto a conic that circumscribes the triangle having vertices $\varphi(A)$, $\varphi(B)$, $\varphi(C)$, and likewise for higher order curves. We shall, in §5, concentrate on cubic curves.

Example 1. Suppose

$$P = p : q : r, \qquad U = u : v : w, \qquad U' = u' : v' : w'$$

are points, none lying on a sideline of $\triangle ABC$, and U' is not on a sideline of the cevian triangle of P (whose vertices are the rows of matrix $\mathbb{P}'$ shown below). Then the collineation φ that carries ABC to $\mathbb{P}'$ and U to U' is regular. We have

$$\mathbb{P}' = \begin{pmatrix} 0 & q & r \\ p & 0 & r \\ p & q & 0 \end{pmatrix}, \quad \text{and} \quad (\mathbb{P}')^{-1} = \frac{1}{|\mathbb{P}'|} \begin{pmatrix} -p & q & r \\ p & -q & r \\ p & q & -r \end{pmatrix},$$

leading to

$$\varphi(X) = X\mathbb{M} = p(ey + fz) : q(fz + dx) : r(dx + ey), \tag{1}$$

where

$$d : e : f = \frac{1}{u}\left(-\frac{u'}{p} + \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{v}\left(\frac{u'}{p} - \frac{v'}{q} + \frac{w'}{r}\right) : \frac{1}{w}\left(\frac{u'}{p} + \frac{v'}{q} - \frac{w'}{r}\right). \tag{2}$$

Example 2. Continuing from Example 1, φ^{-1} is the collineation given by

$$\varphi^{-1}(X) = X\mathbb{M}^{-1} = \frac{1}{d}\left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{e}\left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r}\right) : \frac{1}{f}\left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r}\right). \tag{3}$$

2. Conjugacies induced by collineations

Suppose F is a mapping on the plane of $\triangle ABC$ and φ is a regular collineation, and consider the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \varphi(X) \arrow[d] \\ F(X) \arrow[r] & \varphi(F(X)) \end{tikzcd}$$

On writing $\varphi(X)$ as P, we have $m(P) = \varphi(F(\varphi^{-1}(P)))$. If $F(F(X)) = X$, then $m(m(P)) = P$; in other words, if F is an involution, then m is an involution. We turn now to Examples 3-10, in which F is a well-known involution and φ is the collineation in Example 1 or a special case thereof. In Examples 11 and 12, φ is complementation and anticomplementation, respectively.

Example 3. For any point $X = x : y : z$ not on a sideline of $\triangle ABC$, the isogonal conjugate of X is given by

$$F(X) = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Suppose P, U, φ are as in Example 1. The involution m given by $m(X) = \varphi(F(\varphi^{-1}(X)))$ will be formulated: equation (3) implies

$$F(\varphi^{-1}(X)) = \frac{d}{-\frac{x}{p} + \frac{y}{q} + \frac{z}{r}} : \frac{e}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} : \frac{f}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}},$$

and substituting these coordinates into (1) leads to

$$m(X) = m_1 : m_2 : m_3, \tag{4}$$

where

$$m_1 = m_1(p, q, r, x, y, z) = p \left(\frac{e^2}{\frac{x}{p} - \frac{y}{q} + \frac{z}{r}} + \frac{f^2}{\frac{x}{p} + \frac{y}{q} - \frac{z}{r}} \right) \tag{5}$$

and m_2 and m_3 are determined cyclically from m_1; for example, $m_2(p, q, r, x, y, z) = m_1(q, r, p, y, z, x)$.

In particular, if $U = 1 : 1 : 1$ and $U' = p : q : r$, then from equation (2), we have $d : e : f = 1 : 1 : 1$, and (5) simplifies to

$$m(X) = x \left(-\frac{x}{p} + \frac{y}{q} + \frac{z}{r} \right) : y \left(\frac{x}{p} - \frac{y}{q} + \frac{z}{r} \right) : z \left(\frac{x}{p} + \frac{y}{q} - \frac{z}{r} \right).$$

This is the P-Ceva conjugate of X, constructed [3, p. 57] as the perspector of the cevian triangle of P and the anticevian triangle of X.

Example 4. Continuing with isogonal conjugacy for F and with φ as in Example 3 (with $U = 1 : 1 : 1$ and $U' = p : q : r$), here we use φ^{-1} in place of φ, so that $m(X) = \varphi^{-1}(F(\varphi(X))$. The result is (4), with

$$m_1 = -q^2r^2x^2 + r^2p^2y^2 + p^2q^2z^2 + (-q^2r^2 + r^2p^2 + p^2q^2)(yz + zx + xy).$$

In this case, $m(X)$ is the P-*aleph conjugate of* X.

Let

$$n(X) = \frac{1}{y+z} : \frac{1}{z+x} : \frac{1}{x+y}.$$

Then $X = n(X)$-aleph conjugate of X. Another easily checked property is that a necessary and sufficient condition that

$$X = X\text{-aleph conjugate of the incenter}$$

is that $X =$ incenter or else X lies on the conic $\beta\gamma + \gamma\alpha + \alpha\beta = 0$.

In [4], various triples $(m(X), P, X)$ are listed. A selection of these permuted to $(X, P, m(X))$ appears in Table 1. The notation X_i refers to the indexing of triangle centers in [4]. For example,

$$X_{57} = \frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c} = \tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2},$$

abbreviated in Table 1 and later tables as "57, $\tan\frac{A}{2}$". In Table 1 and the sequel, the area σ of $\triangle ABC$ is given by

$$16\sigma^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

Table 1. Selected aleph conjugates

center, X	P	P-aleph conj. of X
$57, \tan\frac{A}{2}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$63, \cot A$	$2, \frac{1}{a}$	$1, 1$
$57, \tan\frac{A}{2}$	$174, \sec\frac{A}{2}$	$1, 1$
$2, \frac{1}{a}$	$86, \frac{bc}{b+c}$	$2, \frac{1}{a}$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$3, \cos A$
$43, ab+ac-bc$	$1, 1$	$9, b+c-a$
$610, \sigma^2 - a^2\cot A$	$2, \frac{1}{a}$	$19, \tan A$
$165, \tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 5. Here, F is reflection about the circumcenter:

$$F(x:y:z) = 2R\cos A - hx : 2R\cos B - hy : 2R\cos C - hz,$$

where $R =$ circumradius, and h normalizes[2] X. Keeping φ as in Example 4, we find

$$m_1(x,y,z) = 2abc(\cos B + \cos C)\left(\frac{x(b+c-a)}{p} + \frac{y(c+a-b)}{q} + \frac{z(a+b-c)}{r}\right) - 16\sigma^2 x,$$

which, via (4), defines the *P-beth conjugate of* X.

Table 2. Selected beth conjugates

center, X	P	P-beth conj. of X
$110, \frac{a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$	$643, \frac{b+c-a}{b^2-c^2}$
$6, a$	$101, \frac{a}{b-c}$	$6, a$
$4, \sec A$	$8, \csc^2\frac{A}{2}$	$40, \cos B + \cos C - \cos A - 1$
$190, \frac{bc}{b-c}$	$9, b+c-a$	$292, a/(a^2-bc)$
$11, 1-\cos(B-C)$	$11, 1-\cos(B-C)$	$244, (1-\cos(B-C))\sin^2\frac{A}{2}$
$1, 1$	$99, \frac{bc}{b^2-c^2}$	$85, \frac{b^2c^2}{b+c-a}$
$10, \frac{b+c}{a}$	$100, \frac{1}{b-c}$	$73, \cos A(\cos B+\cos C)$
$3, \cos A$	$21, \frac{1}{\cos B+\cos C}$	$56, 1-\cos A$

Among readily verifiable properties of beth-conjugates are these:

(i) $\varphi(X_3)$ is collinear with every pair $\{X, m(X)\}$.

(ii) Since each line $\mathcal{L}$ through X_3 has two points fixed under reflection in X_3, the line $\varphi(\mathcal{L})$ has two points that are fixed by m, namely $\varphi(X_3)$ and $\varphi(\mathcal{L}\cap\mathcal{L}^\infty)$.

---

[2]If $X \notin \mathcal{L}^\infty$, then $h = 2\sigma/(ax+by+cz)$; if $X \in \mathcal{L}^\infty$ and $xyz \neq 0$, then $h = 1/x + 1/y + 1/z$; otherwise, $h = 1$. For $X \notin \mathcal{L}^\infty$, the nonhomogeneous representation for X as the ordered triple (hx, hy, hz) gives the actual directed distances hx, hy, hz from X to sidelines BC, CA, AB, respectively.

(iii) When $P = X_{21}$, φ carries the Euler line $L(3,4,20,30)$ to $L(1,3,56,36)$, on which the m-fixed points are X_1 and X_{36}, and φ carries the line $L(1,3,40,517)$ to $L(21,1,58,1078)$, on which the m-fixed points are X_1 and X_{1078}.

(iv) If X lies on the circumcircle, then the X_{21}-beth conjugate, X', of X lies on the circumcircle. Such pairs (X, X') include (X_i, X_j) for these (i,j): $(99,741)$, $(100,106)$, $(101,105)$, $(102,108)$, $(103,934)$, $(104,109)$, $(110,759)$.

(v) $P = P$-beth conjugate of X if and only if $X = P \cdot X_{56}$ (trilinear product).

Example 6. Continuing Example 5 with φ^{-1} in place of φ leads to the P-*gimel conjugate of* X, defined via (4) by

$$m_1(x,y,z) = 2abc\left(-\frac{\cos A}{p} + \frac{\cos B}{q} + \frac{\cos C}{r}\right)S - 8\sigma^2 x,$$

where $S = x(bq+cr) + y(cr+ap) + z(ap+bq)$.

It is easy to check that if $P \in \mathcal{L}^\infty$, then $m(X_1) = X_1$.

Table 3. Selected gimel conjugates

center, X	P	P-gimel conjugate of X
$1, 1$	$3, \cos A$	$1, 1$
$3, \cos A$	$283, \frac{\cos A}{\cos B + \cos C}$	$3, \cos A$
$30, \cos A - 2\cos B \cos C$	$8, \csc^2 \frac{A}{2}$	$30, \cos A - 2\cos B \cos C$
$4, \sec A$	$21, \frac{1}{\cos B + \cos C}$	$4, \sec A$
$219, \cos A \cot \frac{A}{2}$	$63, \cot A$	$6, a$

Example 7. For distinct points $X' = x' : y' : z'$ and $X = x : y : z$, neither lying on a sideline of $\triangle ABC$, the X'-Hirst inverse of X is defined [4, Glossary] by

$$y'z'x^2 - x'^2 yz : z'x'y^2 - y'^2 zx : x'y'z^2 - z'^2 xy.$$

We choose $X' = U = U' = 1:1:1$. Keeping φ as in Example 4, for $X \neq P$ we obtain m as in expression (4), with

$$m_1(x,y,z) = p\left(\frac{y}{q} - \frac{z}{r}\right)^2 + x\left(\frac{2x}{p} - \frac{y}{q} - \frac{z}{r}\right).$$

In this example, $m(X)$ defines the P-*daleth conjugate of* X. The symbol ω in Table 5 represents the Brocard angle of $\triangle ABC$.

Table 4. Selected daleth conjugates

center, X	P	P-daleth conjugate of X
$518, b^2 + c^2 - a(b+c)$	$1, 1$	$37, b+c$
$1, 1$	$1, 1$	$44, b+c-2a$
$511, \cos(A+\omega)$	$3, \cos A$	$216, \sin 2A \cos(B-C)$
$125, \cos A \sin^2(B-C)$	$4, \sec A$	$125, \cos A \sin^2(B-C)$
$511, \cos(A+\omega)$	$6, a$	$39, a(b^2+c^2)$
$672, a(b^2+c^2-a(b+c))$	$6, a$	$42, a(b+c)$
$396, \cos(B-C) + 2\cos(A - \frac{\pi}{3})$	$13, \csc(A + \frac{\pi}{3})$	$30, \cos A - 2\cos B \cos C$
$395, \cos(B-C) + 2\cos(A + \frac{\pi}{3})$	$14, \csc(A - \frac{\pi}{3})$	$30, \cos A - 2\cos B \cos C$

Among properties of daleth conjugacy that can be straightforwardly demonstrated is that for given P, a point X satisfies the equation

$$P = P\text{-daleth conjugate of } X$$

if and only if X lies on the trilinear polar of P.

Example 8. Continuing Example 7, we use φ^{-1} in place of φ and define the resulting image $m(X)$ as the *P-he conjugate of X*.[3] We have m as in (4) with

$$\begin{aligned} m_1(x,y,z) &= -p(y+z)^2 + q(z+x)^2 + r(x+y)^2 \\ &+ \frac{qr}{p}(x+y)(x+z) - \frac{rp}{q}(y+z)(y+x) - \frac{pq}{r}(z+x)(z+y). \end{aligned}$$

Table 5. Selected he conjugates

center, X	P	P-he conjugate of X
$239, bc(a^2-bc)$	$2, \frac{1}{a}$	$9, b+c-a$
$36, 1-2\cos A$	$6, a$	$43, \csc B + \csc C - \csc A$
$514, \frac{b-c}{a}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$661, \cot B - \cot C$	$21, \frac{1}{\cos B+\cos C}$	$3, \cos A$
$101, \frac{a}{b-c}$	$100, \frac{1}{b-c}$	$101, \frac{a}{b-c}$

Example 9. The X_1-Ceva conjugate of X not lying on a sideline of is $\triangle ABC$ is the point

$$-x(-x+y+z) : y(x-y+z) : z(x+y-z).$$

Taking this for F and keeping φ as in Example 4 leads to

$$m_1(x,y,z) = p(x^2q^2r^2 + 2p^2(ry-qz)^2 - pqr^2xy - pq^2rxz),$$

which via m as in (4) defines the *P-waw conjugate of X*.

Table 6. Selected waw conjugates

center, X	P	P-waw conjugate of X
$37, b+c$	$1, 1$	$354, (b-c)^2 - ab - ac$
$5, \cos(B-C)$	$2, \frac{1}{a}$	$141, bc(b^2+c^2)$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$142, b+c-\frac{(b-c)^2}{a}$
$53, \tan A\cos(B-C)$	$4, \sec A$	$427, (b^2+c^2)\sec A$
$51, a^2\cos(B-C)$	$6, a$	$39, a(b^2+c^2)$

Example 10. Continuing Example 9 with φ^{-1} in place of φ gives

$$m_1(x,y,z) = p(y+z)^2 - ry^2 - qz^2 + (p-r)xy + (p-q)xz,$$

which via m as in (4) defines the *P-zayin conjugate of X*. When $P =$ incenter, this conjugacy is isogonal conjugacy. Other cases are given in Table 7.

[3]The fifth letter of the Hebrew alphabet is *he*, homophonous with *hay*.

Table 7. Selected zayin conjugates

center, X	P	P-zayin conjugate of X
$9, b+c-a$	$2, \frac{1}{a}$	$9, b+c-a$
$101, \frac{a}{b-c}$	$2, \frac{1}{a}$	$661, \cot B - \cot C$
$108, \frac{\sin A}{\sec B - \sec C}$	$3, \cos A$	$656, \tan B - \tan C$
$109, \frac{\sin A}{\cos B - \cos C}$	$4, \sec A$	$656, \tan B - \tan C$
$43, ab+ac-bc$	$6, a$	$43, ab+ac-bc$
$57, \tan\frac{A}{2}$	$7, \sec^2\frac{A}{2}$	$57, \tan\frac{A}{2}$
$40, \cos B + \cos C - \cos A - 1$	$8, \csc^2\frac{A}{2}$	$40, \cos B + \cos C - \cos A - 1$

Example 11. The complement of a point X not on $\mathcal{L}^\infty$ is the point X' satisfying the vector equation

$$\overrightarrow{X'X_2} = \frac{1}{2}\overrightarrow{X_2X}.$$

If $X = x : y : z$, then

$$X' = \frac{by+cz}{a} : \frac{cz+ax}{b} : \frac{ax+by}{c}. \tag{6}$$

If $X \in \mathcal{L}^\infty$, then (6) defines the complement of X. The mapping $\varphi(X) = X'$ is a collineation. Let $P = p : q : r$ be a point not on a sideline of $\triangle ABC$, and let

$$F(X) = \frac{1}{px} : \frac{1}{qy} : \frac{1}{rz},$$

the P-isoconjugate of X. Then m as in (4) is given by

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax-by+cz)} + \frac{c^2}{r(ax+by-cz)}\right)$$

and defines the *P-complementary conjugate of X*. The X_1-complementary conjugate of X_2, for example, is the symmedian point of the medial triangle, X_{141}, and X_{10} is its own X_1-complementary conjugate. Moreover, X_1-complementary conjugacy carries $\mathcal{L}^\infty$ onto the nine-point circle. Further examples follow:

Table 8. Selected complementary conjugates

center X	P	P-complementary conjugate of X
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$141, bc(b^2+c^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$3, \cos A$
$10, \frac{b+c}{a}$	$4, \sec A$	$5, \cos(B-C)$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$141, bc(b^2+c^2)$	$7, \sec^2\frac{A}{2}$	$142, b+c-\frac{(b-c)^2}{a}$
$9, b+c-a$	$9, b+c-a$	$141, bc(b^2+c^2)$
$2, \frac{1}{a}$	$19, \tan A$	$5, \cos(B-C)$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

Example 12. The anticomplement of a point X is the point X'' given by

$$X'' = \frac{-ax+by+cz}{a} : \frac{ax-by+cz}{b} : \frac{ax+by-cz}{c}.$$

Keeping F and φ as in Example 11, we have $\varphi^{-1}(X) = X''$ and define m by $m = \varphi^{-1} \circ F \circ \varphi$, Thus, $m(X)$ is determined as in (4) from

$$m_1(x,y,z) = \frac{1}{a}\left(\frac{b^2}{q(ax+cz)} + \frac{c^2}{r(ax+by)} - \frac{a^2}{p(by+cz)}\right).$$

Here, $m(X)$ defines the *P-anticomplementary conjugate of* X. For example, the centroid is the X_1-anticomplementary conjugate of X_{69} (the symmedian point of the anticomplementary triangle), and the Nagel point, X_8, is its own self X_1-anticomplementary conjugate. Moreover, X_1-anticomplementary conjugacy carries the nine-point circle onto $\mathcal{L}^\infty$. Further examples follow:

Table 9. Selected anticomplementary conjugates

center, X	P	P-anticomplementary conj. of X
$3, \cos A$	$1, 1$	$4, \sec A$
$5, \cos(B-C)$	$1, 1$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$2, \frac{1}{a}$	$69, bc(b^2+c^2-a^2)$
$10, \frac{b+c}{a}$	$3, \cos A$	$20, \cos A - \cos B \cos C$
$10, \frac{b+c}{a}$	$4, \sec A$	$4, \sec A$
$10, \frac{b+c}{a}$	$6, a$	$2, \frac{1}{a}$
$5, \cos(B-C)$	$19, \tan A$	$2, \frac{1}{a}$
$125, \cos A \sin^2(B-C)$	$10, \frac{b+c}{a}$	$513, b-c$

3. The Darboux cubic, D

This section formulates a mapping Ψ on the plane of $\triangle ABC$; this mapping preserves two pivotal properties of the Darboux cubic D. In Section 4, $\Psi(D)$ is recognized as the Lucas cubic. In Section 5, collineations will be applied to D, carrying it to cubics having two pivotal configurations with properties analogous to those of D.

The Darboux cubic is the locus of a point X such that the pedal triangle of X is a cevian triangle. The pedal triangle of X has for its A-vertex the point in which the line through X perpendicular to line BC meets line BC, and likewise for the B- and C- vertices. We denote these three vertices by X_A, X_B, X_C, respectively. To say that their triangle is a cevian triangle means that the lines AX_A, BX_B, CX_C concur. Let $\Psi(P)$ denote the point of concurrence. In order to obtain a formula for Ψ, we begin with the pedal triangle of P:

$$\begin{pmatrix} X_A \\ X_B \\ X_C \end{pmatrix} = \begin{pmatrix} 0 & \beta+\alpha c_1 & \gamma+\alpha b_1 \\ \alpha+\beta c_1 & 0 & \gamma+\beta a_1 \\ \alpha+\gamma b_1 & \beta+\gamma a_1 & 0 \end{pmatrix},$$

where $a_1 = \cos A$, $b_1 = \cos B$, $c_1 = \cos C$. Then

$$\begin{aligned}
BX_B \cap CX_C =& (\alpha + \beta c_1)(\alpha + \gamma b_1) : (\beta + \gamma a_1)(\alpha + \beta c_1) : (\gamma + \beta a_1)(\alpha + \gamma b_1),\\
CX_C \cap AX_A =& (\alpha + \gamma b_1)(\beta + \alpha c_1) : (\beta + \gamma a_1)(\beta + \alpha c_1) : (\gamma + \alpha b_1)(\beta + \gamma a_1),\\
AX_A \cap BX_B =& (\alpha + \beta c_1)(\gamma + \alpha b_1) : (\beta + \alpha c_1)(\gamma + \beta a_1) : (\gamma + \alpha b_1)(\gamma + \beta a_1).
\end{aligned}$$

Each of these three points is $\Psi(X)$. Multiplying and taking the cube root gives the following result:

$$\Psi(X) = \psi(\alpha, \beta, \gamma, a, b, c) : \psi(\beta, \gamma, \alpha, b, c, a) : \psi(\gamma, \alpha, \beta, c, a, b),$$

where

$$\psi(\alpha, \beta, \gamma, a, b, c) = [(\alpha + \beta c_1)^2(\alpha + \gamma b_1)^2(\beta + \alpha c_1)(\gamma + \alpha b_1)]^{1/3}.$$

The Darboux cubic is one of a family of cubics $Z(U)$ given by the form (e.g., [3, p.240])

$$u\alpha(\beta^2 - \gamma^2) + v\beta(\gamma^2 - \alpha^2) + w\gamma(\alpha^2 - \beta^2) = 0, \tag{7}$$

where the point $U = u : v : w$ is called the pivot of $Z(U)$, in accord with the collinearity of U, X, and the isogonal conjugate, X^{-1}, of X, for every point $X = \alpha : \beta : \gamma$ on $Z(U)$. The Darboux cubic is $Z(X_{20})$; that is,

$$(a_1 - b_1c_1)\alpha(\beta^2 - \gamma^2) + (b_1 - c_1a_1)\beta(\gamma^2 - \alpha^2) + (c_1 - a_1b_1)\gamma(\alpha^2 - \beta^2) = 0.$$

This curve has a secondary pivot, the circumcenter, X_3, in the sense that if X lies on D, then so does the reflection of X in X_3. Since X_3 itself lies on D, we have here a second system of collinear triples on D.

The two types of pivoting lead to chains of centers on D:

$$X_1 \xrightarrow{\text{refl}} X_{40} \xrightarrow{\text{isog}} X_{84} \xrightarrow{\text{refl}} \cdots \tag{8}$$

$$X_3 \xrightarrow{\text{isog}} X_4 \xrightarrow{\text{refl}} X_{20} \xrightarrow{\text{isog}} X_{64} \xrightarrow{\text{refl}} \cdots . \tag{9}$$

Each of the centers in (8) and (9) has a trilinear representation in polynomials with all coefficients integers. One wonders if all such centers on D can be generated by a finite collection of chains using reflection and isogonal conjugation as in (8) and (9).

4. The Lucas cubic, L

Transposing the roles of pedal and cevian triangles in the description of D leads to the Lucas cubic, L, i.e., the locus of a point $X = \alpha : \beta : \gamma$ whose cevian triangle is a pedal triangle. Mimicking the steps in Section 3 leads to

$$\Psi^{-1}(X) = \lambda(\alpha, \beta, \gamma, a, b, c) : \lambda(\beta, \gamma, \alpha, b, c, a) : \lambda(\gamma, \alpha, \beta, c, a, b),$$

where $\lambda(\alpha, \beta, \gamma, a, b, c) =$

$$\{[\alpha^2 - (\alpha a_1 - \gamma c_1)(\alpha a_1 - \beta b_1)]([(\alpha\beta + \gamma(\alpha a_1 - \beta b_1)][(\alpha\gamma + \beta(\alpha a_1 - \gamma c_1)]\}^{1/3}.$$

It is well known [1, p.155] that “the feet of the perpendiculars from two isogonally conjugate points lie on a circle; that is, isogonal conjugates have a common pedal circle ...” Consequently, L is self-cyclocevian conjugate [3, p. 226]. Since L is also self-isotomic conjugate, certain centers on L are generated in chains:

$$X_7 \xrightarrow{\text{isot}} X_8 \xrightarrow{\text{cycl}} X_{189} \xrightarrow{\text{isot}} X_{329} \xrightarrow{\text{cycl}} \cdots \tag{10}$$

$$X_2 \xrightarrow{\text{cycl}} X_4 \xrightarrow{\text{isot}} X_{69} \xrightarrow{\text{cycl}} X_{253} \xrightarrow{\text{isot}} X_{20} \xrightarrow{\text{cycl}} \cdots . \tag{11}$$

The mapping Ψ, of course, carries D to L, isogonal conjugate pairs on D to cyclocevian conjugate pairs on L, reflection-in-circumcenter pairs on D to isotomic conjugate pairs on L, and chains (8) and (9) to chains (10) and (11).

5. Cubics of the form $\varphi(Z(U))$

Every line passing through the pivot of the Darboux cubic D meets D in a pair of isogonal conjugates, and every line through the secondary pivot X_3 of D meets D in a reflection-pair. We wish to obtain generalizations of these pivotal properties by applying collineations to D. As a heuristic venture, we apply to D trilinear division by a point $P = p : q : r$ for which $pqr \neq 0$: the set D/P of points X/P as X traverses D is easily seen to be the cubic

$$\begin{aligned}&(a_1 - b_1c_1)px(q^2y^2 - r^2z^2) + (b_1 - c_1a_1)qy(r^2z^2 - p^2x^2)\\ +&(c_1 - a_1b_1)rz(p^2x^2 - q^2y^2) = 0.\end{aligned}$$

This is the self-P-isoconjugate cubic with pivot X_{20}/P and secondary pivot X_3/P. The cubic D/P, for some choices of P, passes through many “known points,” of course, and this is true if for D we substitute any cubic that passes through many “known points”.

The above preliminary venture suggests applying a variety of collineations to various cubics $Z(U)$. To this end, we shall call a regular collineation φ a *tricentral collineation* if there exists a mapping m_1 such that

$$\varphi(\alpha : \beta : \gamma) = m_1(\alpha : \beta : \gamma) : m_1(\beta : \gamma : \alpha) : m_1(\gamma : \alpha : \beta) \tag{12}$$

for all $\alpha : \beta : \gamma$. In this case, φ^{-1} has the form given by

$$n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta),$$

hence is tricentral.

The tricentral collineation (12) carries $Z(U)$ in (7) to the cubic $\varphi(Z(U))$ having equation

$$u\hat{\alpha}(\hat{\beta}^2 - \hat{\gamma}^2) + v\hat{\beta}(\hat{\gamma}^2 - \hat{\alpha}^2) + w\hat{\gamma}(\hat{\alpha}^2 - \hat{\beta}^2) = 0, \tag{13}$$

where

$$\hat{\alpha} : \hat{\beta} : \hat{\gamma} = n_1(\alpha : \beta : \gamma) : n_1(\beta : \gamma : \alpha) : n_1(\gamma : \alpha : \beta).$$

Example 13. Let

$$\varphi(\alpha : \beta : \gamma) = p(\beta + \gamma) : q(\gamma + \alpha) : r(\alpha + \beta),$$

so that

$$\varphi^{-1}(\alpha : \beta : \gamma) = -\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} - \frac{\beta}{q} + \frac{\gamma}{r} : \frac{\alpha}{p} + \frac{\beta}{q} - \frac{\gamma}{r}.$$

In accord with (13), the cubic $\varphi(Z(U))$ has equation

$$\frac{u\alpha}{p}\left(-\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\beta}{q}-\frac{\gamma}{r}\right)+\frac{v\beta}{q}\left(\frac{\alpha}{p}-\frac{\beta}{q}+\frac{\gamma}{r}\right)\left(\frac{\gamma}{r}-\frac{\alpha}{p}\right)$$
$$+\quad\frac{w\gamma}{r}\left(\frac{\alpha}{p}+\frac{\beta}{q}-\frac{\gamma}{r}\right)\left(\frac{\alpha}{p}-\frac{\beta}{q}\right)=0.$$

Isogonic conjugate pairs on $Z(U)$ are carried as in Example 3 to P-Ceva conjugate pairs on $\varphi(Z(U))$. Indeed, each collinear triple X, U, X^{-1} is carried to a collinear triple, so that $\varphi(U)$ is a pivot for $\varphi(Z(U))$.

If $U = X_{20}$, so that $Z(U)$ is the Darboux cubic, then collinear triples X, X_3, $\tilde{X}$, where $\tilde{X}$ denotes the reflection of X in X_3, are carried to collinear triples $\varphi(X)$, $\varphi(X_3)$, $\varphi(\tilde{X})$, where $\varphi(\tilde{X})$ is the P-beth conjugate of X, as in Example 5.

Example 14. Continuing Example 13 with φ^{-1} in place of φ, the cubic $\varphi^{-1}(Z(U))$ is given by

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)+s(v,w,u,q,r,p,\beta,\gamma,\alpha)+s(w,u,v,r,p,q,\gamma,\alpha,\beta)=0,$$

where

$$s(u,v,w,p,q,r,\alpha,\beta,\gamma)=up(\beta+\gamma)(q^2(\gamma+\alpha)^2-r^2(\alpha+\beta)^2).$$

Collinear triples X, U, X^{-1} on $Z(U)$ yield collinear triples on $\varphi^{-1}(Z(U))$, so that $\varphi^{-1}(U)$ is a pivot for $\varphi^{-1}(Z(U))$. The point $\varphi^{-1}(X^{-1})$ is the P-aleph conjugate of X, as in Example 4.

On the Darboux cubic, collinear triples $X, X_3, \tilde{X}$, yield collinear triples $\varphi^{-1}(X)$, $\varphi^{-1}(X_3)$, $\varphi^{-1}(\tilde{X})$, this last point being the P-gimel conjugate of X, as in Example 6.

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Equilateral Chordal Triangles

Floor van Lamoen

Abstract. When a circle intersects each of the sidelines of a triangle in two points, we can pair the intersections in such a way that three chords not along the sidelines bound a triangle, which we call a *chordal triangle*. In this paper we show that equilateral chordal triangles are homothetic to Morley's triangle, and identify all cases.

1. Chordal triangles

Let $T = ABC$ be the triangle of reference, and let a circle γ intersect side a in points B_a and C_a, side b in A_b and C_b and side c in A_c and B_c. The chords $a' = C_bB_c$, $b' = A_cC_a$ and $c' = A_bB_a$ enclose a triangle T', which we call a *chordal triangle*. See Figure 1. We begin with some preliminary results. In writing these the expression (ℓ_1, ℓ_2) denotes the directed angle from ℓ_1 to ℓ_2.

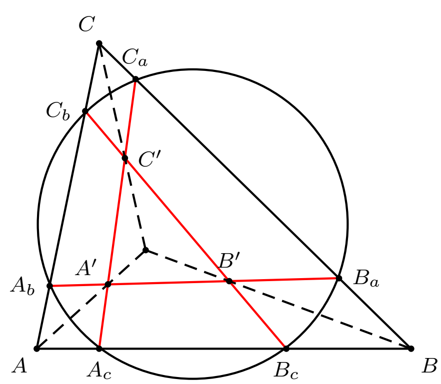

Figure 1

Proposition 1. *The sides of the chordal triangle T' satisfy*

$$(a', a) + (b', b) + (c', c) = 0 \pmod{\pi}.$$

Proof. Note that $(a', c) = (B_cC_b, B_cA)$ and

$$(c', b) = -(A_bA, A_bB_a) = (B_cB_a, B_cC_b) \pmod{\pi}$$

while also

$$(b', a) = -(C_aC, C_aA_c) = (B_cA_c, B_cB_a) \pmod{\pi}.$$

We conclude that $(a', c) + (c', b) + (b', a) = 0 \pmod{\pi}$, and the proposition follows from the fact that the internal directed angles of a triangle have sum π. □

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Proposition 2. *The triangle T' is perspective to ABC.*

Proof. From Pascal's hexagon theorem applied to $C_aB_aA_bC_bB_cA_c$ we see that the points of intersection $C_aB_a \cap C_bB_c$, $B_cA_c \cap B_aA_b$ and $A_bC_b \cap A_cC_a$ are collinear. Therefore, triangles ABC and $A'B'C'$ are line perspective, and by Desargues' two-triangle theorem, they are point perspective as well. □

The triangle T'' enclosed by the lines $a'' = (a \cap b') \cup (a' \cap b)$ and similarly defined b'' and c'' is also a chordal triangle, which we will call the *alternative chordal triangle* of T'. [1]

Proposition 3. *The corresponding sides of T' and T'' are antiparallel with respect to triangle T.*

Proof. From the fact that $B_cA_cA_bC_b$ is a cyclic quadrilateral, immediately we see $\angle AB_cC_b = \angle AA_bA_c$, so that a' and a'' are antiparallel. By symmetry this proves the proposition. □

We now see that there is a family of chordal triangles homothetic to T'. From a starting point on one of the sides of ABC we can construct segments to the next sides alternately parallel to corresponding sides of T' and T''.[2] Extending the segments parallel to T' we get a chordal triangle homothetic to T'. [3]

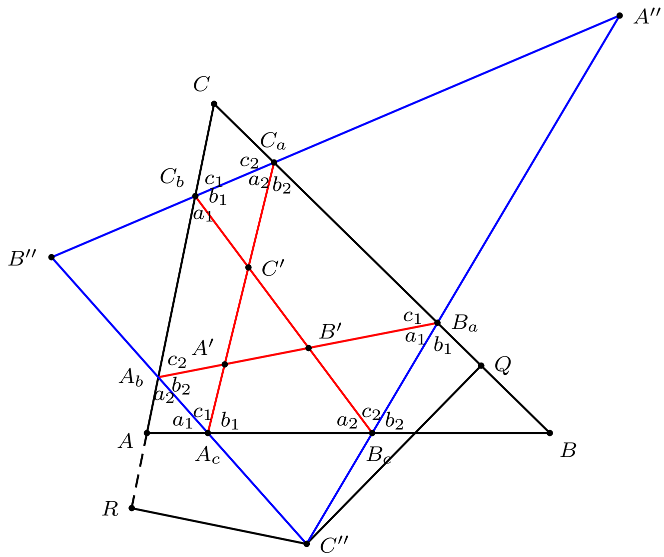

Figure 2

[1] This is the triangle enclosed by the lines A_bA_c, B_cB_a and C_aC_b in Figure 2. The definition of T'' from T and T' is exactly dual to the definition of 'desmic mate' (see [1, §4]). This yields also that T, T' and T'' are perspective through one perspector, which will be shown differently later this section, in order to keep this paper self contained.

[2] This is very similar to the well known construction of the Tucker hexagon.

[3] In fact it is easy to see that starting with any pair of triangles T' and T'' satisfying Propositions 1 and 3, we get a family of chordal triangles with this construction.

With the knowledge of Propositions 1-3 we can indicate angles as in Figure 2. In this figure we have also drawn altitudes $C''Q$ and $C''R$ to BC and AC respectively.

Note that

$$\begin{aligned} C''Q &= \sin(b_1)\sin(b_2)\csc(C'')\cdot A_bB_a, \\ C''R &= \sin(a_1)\sin(a_2)\csc(C'')\cdot A_bB_a. \end{aligned}$$

This shows that the (homogeneous) normal coordinates[4] for C'' are of the form

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \cdots).$$

From this we see that T and T'' have perspector

$$(\csc(a_1)\csc(a_2) : \csc(b_1)\csc(b_2) : \csc(c_1)\csc(c_2)).$$

Clearly this perspector is independent from choice of T' or T'', and depends only on the angles a_1, a_2, b_1, b_2, c_1 and c_2.

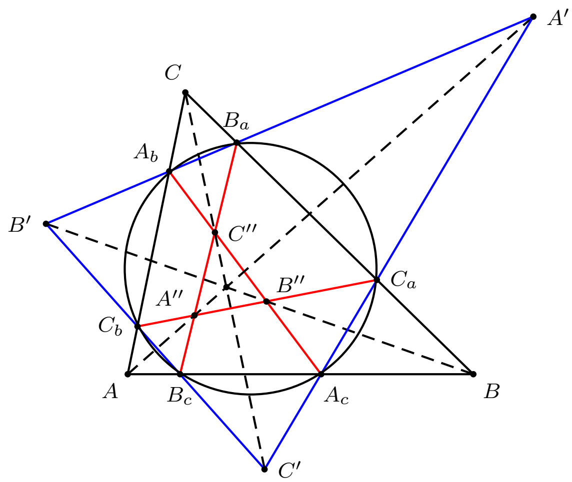

Figure 3

Proposition 4. *All chordal triangles homothetic to a chordal triangle T', as well as all chordal triangles homothetic to the alternative chordal triangle of T', are perspective to T through one perspector.*

2. Equilateral chordal triangles

Jean-Pierre Ehrmann and Bernard Gibert have given a magnificently elegant characterization of lines parallel to sides of Morley's trisector triangle.

Proposition 5. [2, Proposition 5] *A line ℓ is parallel to a side of Morley's trisector triangle if and only if*

$$(\ell, a) + (\ell, b) + (\ell, c) = 0 \pmod{\pi}.$$

[4]These are traditionally called (homogeneous) trilinear coordinates.

An interesting consequence of Proposition 5 in combination with Proposition 1 is that Morley triangles of chordal triangles are homothetic to the Morley triangle of ABC. Furthermore, equilateral chordal triangles themselves are homothetic to Morley's triangle. This means that they are not in general constructible by ruler and compass.

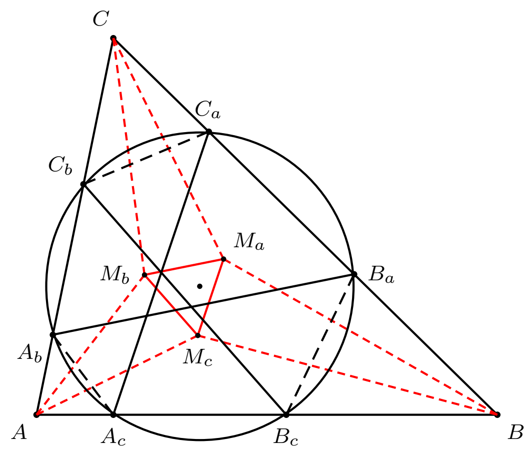

Figure 4

With this knowledge we can identify all equilateral chordal triangles. First we can specify the angles (a', c), (c', b), and (b', a). There are six possibilities. Now we can fix one point, say B_a, and use the specified angles and Proposition 3 to find the other vertices of hexagon $C_aB_aA_bC_bB_cA_c$. The rest is easy.

We can now identify all equilateral chordal triangles by (homogeneous) normal coordinates. As an example we will study the family

$$(a', c) = \frac{2}{3}B + \frac{1}{3}C, \quad (c', b) = \frac{2}{3}A + \frac{1}{3}B, \quad (b', a) = \frac{2}{3}C + \frac{1}{3}A.$$

From the derivation of Proposition 4 we see that the perspector of this family has normal coordinates

$$\left(\csc\frac{2B+C}{3}\csc\frac{B+2C}{3} : \csc\frac{2A+C}{3}\csc\frac{A+2C}{3} : \csc\frac{2A+B}{3}\csc\frac{A+2B}{3}\right).$$

Writing (TU) for the directed arc from T to U, and defining

$$\begin{array}{lll} t_a = (C_bB_c), & t_b = (A_cC_a), & t_c = (B_aA_b), \\ u_a = (C_aB_a), & u_b = (A_bC_b), & u_c = (B_cA_c), \end{array}$$

we find the following system of equations

$$\begin{array}{ll} (C_bA_c) = t_a + u_c = \frac{4}{3}B + \frac{2}{3}C, & (A_bB_c) = u_b + t_a = \frac{4}{3}C + \frac{2}{3}B, \\ (B_aC_b) = t_c + u_b = \frac{4}{3}A + \frac{2}{3}B, & (C_aA_b) = u_a + t_c = \frac{4}{3}B + \frac{2}{3}A, \\ (A_cB_a) = t_b + u_a = \frac{4}{3}C + \frac{2}{3}A, & (B_cC_a) = u_c + t_b = \frac{4}{3}A + \frac{2}{3}C. \end{array}$$

This system can be solved with one parameter τ to be

$$t_a = \frac{4(B+C)}{3} - 2\tau \quad t_b = \frac{4(C+A)}{3} - 2\tau \quad t_c = \frac{4(A+B)}{3} - 2\tau$$

$$u_a = -\frac{2A}{3} + 2\tau \qquad u_b = -\frac{2B}{3} + 2\tau \qquad u_c = -\frac{2C}{3} + 2\tau$$

The coordinates of the centers of these circles are now given by[5]

$$\left(\pm\cos\left(\frac{A}{3}+\tau\right) : \pm\cos\left(\frac{B}{3}+\tau\right) : \pm\cos\left(\frac{C}{3}+\tau\right)\right).$$

Assuming all cosines positive, these centers describe a line, which passes (take $\tau = 0$) through the perspector of the adjoint Morley triangle and ABC, in [3,4] numbered as X_{358}. By taking $\tau = \frac{\pi}{2}$ we see the line also passes through the point

$$\left(\sin\frac{A}{3} : \sin\frac{B}{3} : \sin\frac{C}{3}\right).$$

Hence, the equation of this line through the centers of the circles is

$$\sum_{\text{cyclic}} \left(\sin\frac{B}{3}\cos\frac{C}{3} - \cos\frac{B}{3}\sin\frac{C}{3}\right)x = 0,$$

or simply

$$\sum_{\text{cyclic}} \sin\frac{B-C}{3}x = 0.$$

One can find the other families of equilateral chordal triangles by adding and/or subtracting appropriate multiples of $\frac{\pi}{3}$ to the inclinations of the sides of T' with respect to T. The details are left to the reader.

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[5] We have to be careful with this type of conclusion. We cannot blindly give signs to the coordinates. In particular, we cannot blindly follow the signs of the cosines below - if we would add 360 degrees to u_a, this would yield a change of sign for the first coordinate for the same figure. To establish signs, one can shuffle the hexagon $C_aB_aA_bC_bB_cA_c$ in such a way that the central angles on the segments on the sides are all positive and the sum of central angles is exactly 2π. From this we can draw conclusions on the location of the center with respect to the sides.

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The Napoleon Configuration

Gilles Boutte

Abstract. It is an elementary fact in triangle geometry that the two Napoleon triangles are equilateral and have the same centroid as the reference triangle. We recall some basic properties of the Fermat and Napoleon configurations, and use them to study equilateral triangles bounded by cevians. There are two families of such triangles, the triangles in the two families being oppositely oriented. The locus of the circumcenters of the triangles in each family is one of the two Napoleon circles, and the circumcircles of each family envelope a conchoid of a circle.

1. The Fermat-Napoleon configuration

Consider a reference triangle ABC, with side lengths a, b, c. Let F_a^+ be the point such that CBF_a^+ is equilateral with the same orientation as ABC; similarly for F_b^+ and F_c^+. See Figure 1.

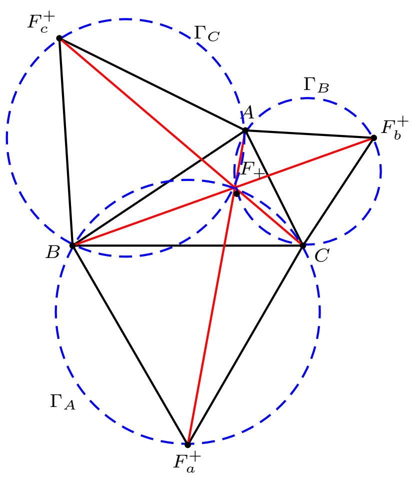

Figure 1. The Fermat configuration

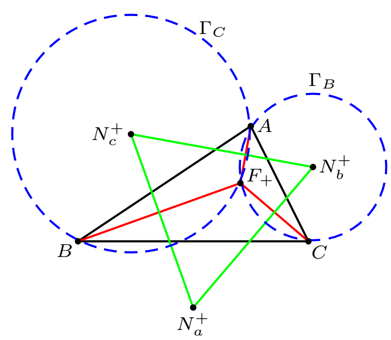

Figure 2. The Napoleon configuration

The triangle $F_a^+F_b^+F_c^+$ is called the *first Fermat triangle*. It is an elementary fact that triangles $F_a^+F_b^+F_c^+$ and ABC are perspective at the *first Fermat point* F_+. We define similarly the *second Fermat triangle* $F_a^-F_b^-F_c^-$ in which CBF_a^-, ACF_b^- and BAF_c^- are equilateral triangles with opposite orientation of ABC. This is perspective with ABC at the *second Fermat point* F_-. [1] Denote by Γ_A the circumcircle of CBF_a^+, and N_a^+ its center; similarly for Γ_B, Γ_C, N_b^+, N_c^+. The

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[1] In [1], these are called the isogonic centers, and are referenced as X_{13} and X_{14}.

triangle $N_a^+N_b^+N_c^+$ is called the *first Napoleon triangle*, and is perspective with ABC at the *first Napoleon point* N_+. Similarly, we define the *second Napoleon triangle* $N_a^-N_b^-N_c^-$ perspective with ABC at the *second Napoleon point* N_-. [2] See Figure 2. Note that N_a^- is the antipode of F_a^+ on Γ_A.

We summarize some of the important properties of the Fermat and Napoleon points.

Theorem 1. *Let ABC be a triangle with side lengths a, b, c and area $\triangle$.*

(1) *The first Fermat point F_+ is the common point to Γ_A, Γ_B and Γ_C.*
(2) *The segments AF_a^+, BF_b^+, CF_c^+ have the same length ℓ given by*

$$\ell^2 = \frac{1}{2}(a^2 + b^2 + c^2 + 4\sqrt{3}\triangle).$$

(3) *The first Napoleon triangle $N_a^+N_b^+N_c^+$ is equilateral with the same orientation as ABC. Its circumradius is $\frac{\ell}{3}$.*
(4) *The Fermat and Napoleon triangles have the same centroid G as ABC.*
(5) *The first Fermat point lies on the circumcircle of the second Napoleon triangle. We shall call this circle the* second Napoleon circle.
(6) *The lines $N_b^+N_c^+$ and AF_+ are respectively the line of centers and the common chord of Γ_B and Γ_C. They are perpendicular.*

Remarks. (i) Similar statements hold for the second Fermat and Napoleon points F_- and N_-, with appropriate changes of signs.

(ii) (4) is an easy corollary of the following important result: Given a triangle $A'B'C'$ with ABC', BCA', CAB' positively similar. Thus ABC and $A'B'C'$ have the same centroid. See, for example, [3, p.462].

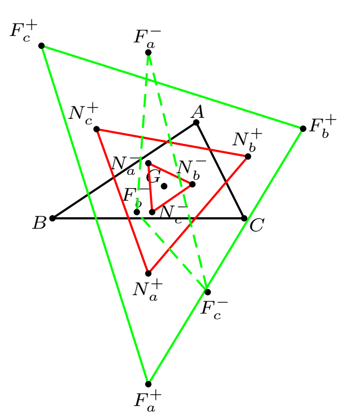

Figure 3. The Fermat and Napoleon triangles

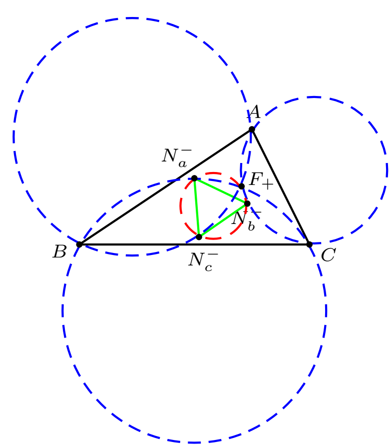

Figure 4. The Fermat point on the second Napoleon circle

[2] In [1], the Napoleon points as X_{17} and X_{18}.

2. Equilateral triangles bounded by cevians

Let $A_1B_1C_1$ be an equilateral triangle, with the same orientation as ABC and whose sides are cevian lines in ABC, i.e. A lies on B_1C_1, B lies on C_1A_1, C lies on A_1B_1. See Figure 5. Thus, CB is seen from A_1 at an angle $\frac{\pi}{3}$, *i.e.*, $\angle CA_1B = \frac{\pi}{3}$, and A_1 lies on Γ_A. Similarly B_1 lies on Γ_B and C_1 lies on Γ_C. Conversely, let A_1 be any point on Γ_A. The line A_1B intersects Γ_C at B and C_1, the line A_1C intersects Γ_B at C and B_1.

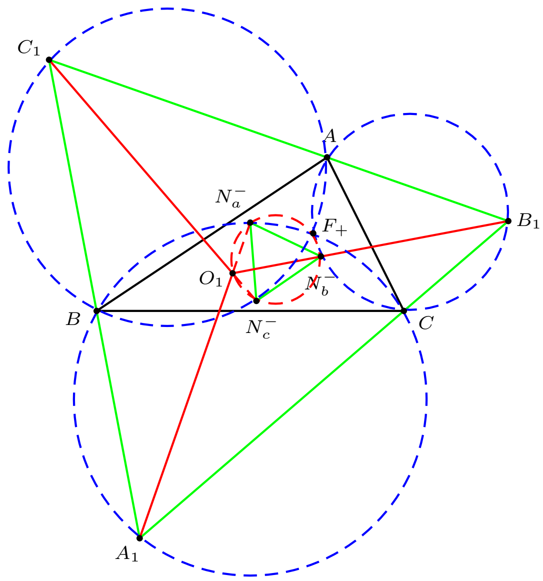

Figure 5. Equilateral triangle bounded by cevians

Three of the angles of the quadrilateral $A_1B_1AC_1$ are $\frac{\pi}{3}$; so A lies on B_1C_1 and $A_1B_1C_1$ is equilateral with the same orientation as ABC. We obtain an equilateral triangle bounded by cevians. There is an infinity of such triangles.

Let O_1 be the center of $A_1B_1C_1$. BO_1 is seen from A_1 at an angle $\frac{\pi}{6} \pmod{\pi}$; similarly for BN_a^-. The line A_1O_1 passes through N_a^-. Similarly the lines B_1O_1 and C_1O_1 pass through N_b^- and N_c^- respectively. It follows that $N_b^-N_c^-$ and B_1C_1 are seen from O_1 at the same angle $\frac{2\pi}{3} = -\frac{\pi}{3} \pmod{\pi}$, and the point O_1 lies on the circumcircle of $N_a^-N_b^-N_c^-$. Thus we have:

Theorem 2. *The locus of the center of equilateral triangles bounded by cevians, and with the same orientation as ABC, is the second Napoleon circle.*

Similarly, the locus of the center of equilateral triangles bounded by cevians, and with the opposite orientation of ABC, is the first Napoleon circle.

3. Pedal curves and conchoids

We recall the definitions of pedal curves and conchoids from [2].

Definitions. Given a curve $\mathcal{C}$ and a point O,

(1) the *pedal curve* of $\mathcal{C}$ with respect to O is the locus of the orthogonal projections of O on the tangent lines of $\mathcal{C}$;

(2) for a positive number k, the *conchoid* of $\mathcal{C}$ with respect to O and with *offset* k is the locus of the points P for which there exists M on $\mathcal{C}$ with O, M, P collinear and $MP = k$.

For the constructions of normal lines, we have

Theorem 3. *Let $\mathcal{P}_O$ be the pedal curve of $\mathcal{C}$ with respect to O. For any point M on $\mathcal{C}$, if P is the projection of O on the tangent to $\mathcal{C}$ at M, and Q is such that $OPMQ$ is a rectangle, then the line PQ is normal to $\mathcal{P}_O$ at P.*

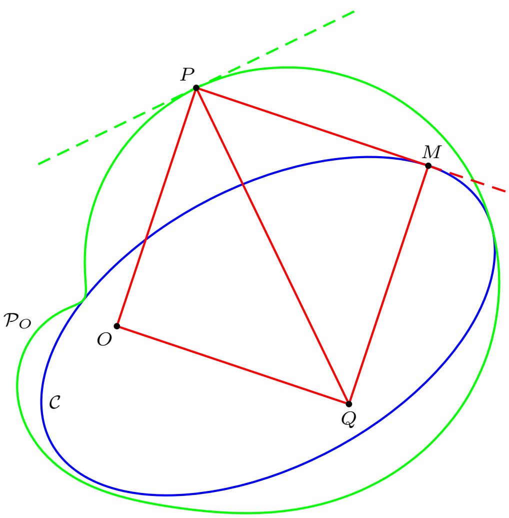

Figure 6. The normal to a pedal curve

Theorem 4. *Let $\mathcal{C}_{O,k}$ be the conchoid of $\mathcal{C}$ with respect to O and offset k. For any point P on $\mathcal{C}_{O,k}$, if M is the intersection of the line OP with $\mathcal{C}$, then the mormal lines to $\mathcal{C}_{O,k}$ at P and to $\mathcal{C}$ at M intersect on the perpendicular to OP at O.*

4. Envelope of the circumcircles

Consider one of the equilateral triangles with the same orientation of ABC. Let $\mathcal{C}_1$ be the circumcircle of $A_1B_1C_1$, R_1 its radius. Its center O_1 lies on the Napoleon circle and the vertex A_1 lies on the circle Γ_A. The latter two circles pass through F_+ and N_a^-. The angles $\angle N_a^- A_1 F_+$ and $\angle N_a^- O_1 F_+$ have constant magnitudes. The shape of triangle $A_1 O_1 F_+$ remains unchanged when O_1 traverses the second Napoleon circle $\mathcal{N}$. The ratio $\dfrac{O_1A_1}{O_1F_+} = \dfrac{R_1}{O_1F_+}$ remains constant, say, λ.

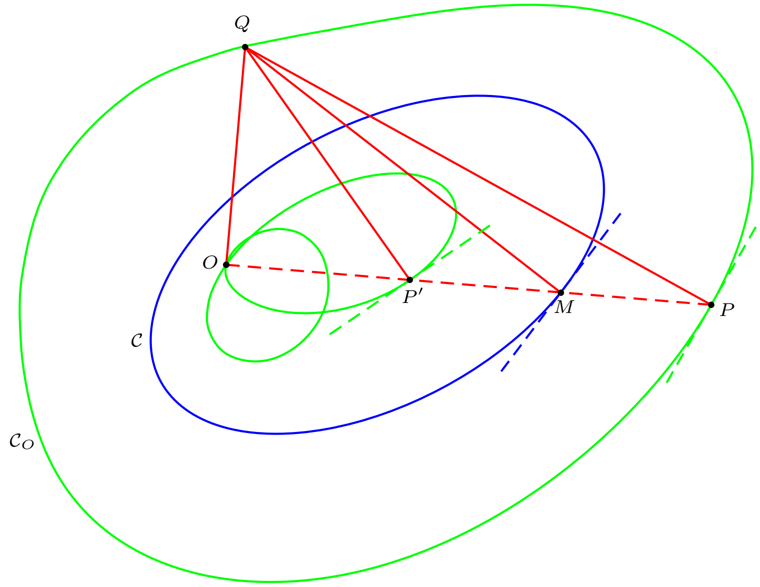

Figure 7. The normal to a conchoid

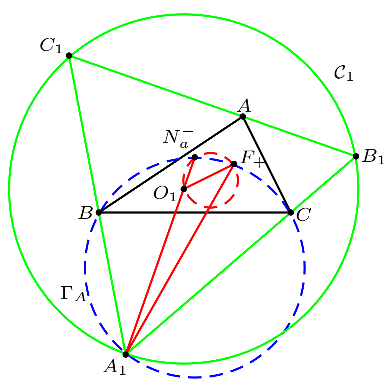

Figure 8. Equilateral triangle bounded by cevians and its circumcircle

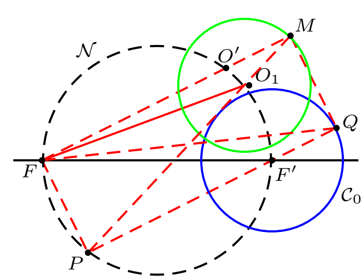

Figure 9. The pedal of $\mathcal{C}_0$ with respect to F

For convenience we denote by $\mathcal{N}$ the Napoleon circle which is the locus of O_1, F the Fermat point lying on this circle, F' the antipode of F on $\mathcal{N}$, and $\mathcal{C}_0$ the particular position of $\mathcal{C}_1$ when O_1 and F' coincide. See Figure 7. Let P be any point on $\mathcal{N}$, the line PF' intersects $\mathcal{C}_0$ at Q and Q' (F' between P and Q), we construct the point M such that $FPQM$ is a rectangle. The locus $\mathcal{P}_F$ of M is the pedal curve of $\mathcal{C}_0$ with respect to F and, by Theorem 3, the line MP is the normal to $\mathcal{P}_F$ at M. The line MP intersects $\mathcal{N}$ at P and O_1 and the circle through M with center O_1 is tangent to $\mathcal{P}_F$ at M.

The triangles FMO_1 and FQF' are similar since $\angle FMO_1 = \angle FQF'$ and $\angle FO_1M = \angle FF'Q$.[3] It follows that $\dfrac{O_1M}{O_1F} = \dfrac{F'Q}{F'F} = \lambda$, and $O_1M = R_1$. The circle through M with center O_1 is one in the family of circle for which we search the envelope.

Furthermore, the line FM intersects $\mathcal{N}$ at F and O', and $O'MQF'$ is a rectangle. Thus, $O'M = F'Q$, the radius of $\mathcal{C}_0$. It follows that M lies on the external branch of the conchoid of $\mathcal{N}$ with respect to F and the length $R =$ radius of $\mathcal{C}_0$.

By the same reasoning for the point Q', we obtain M' on $\mathcal{P}_F$, but on the internal branch of the conchoid. Each circle $\mathcal{C}_1$ touches both branches of the conchoid.

Theorem 5. *Let $\mathcal{F}$ be the family of circumcircles of equilateral triangles bounded by cevians whose locus of centers is the Napoleon circle $\mathcal{N}$ passing through the Fermat point F. The envelope of this family $\mathcal{F}$ is the pedal with respect to F of the circle $\mathcal{C}_0$ of $\mathcal{F}$ whose center is the antipode of F on $\mathcal{N}$, i.e. the conchoid of $\mathcal{N}$ with respect to F and offset the radius of $\mathcal{C}_0$. Each circle of $\mathcal{F}$ is bitangent to the envelope.*

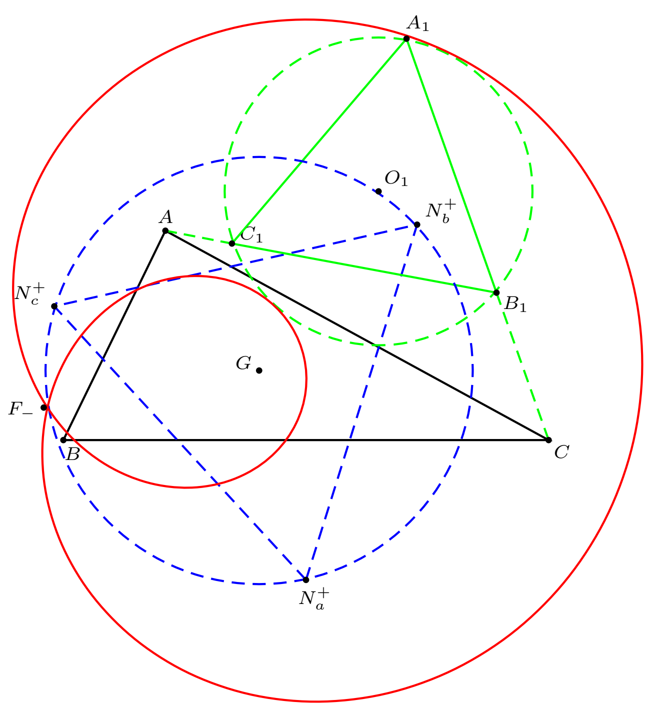

Figure 10. The envelope of the circumcirles ($\lambda < 1$)

[3] FP is seen at the same angle from O_1 and from F'.

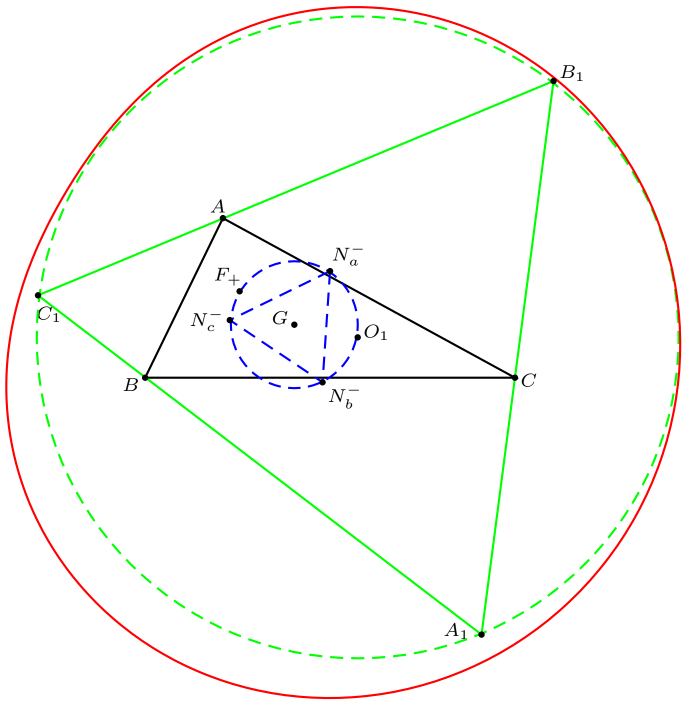

Figure 11. The envelope of the circumcirles ($\lambda > 1$)

Let i be the inversion with respect to a circle $\mathcal{C}$ whose center is F and such that $\mathcal{C}_0$ is invariant under it. The curve $i(\mathcal{P}_F)$ is the image of $\mathcal{C}_0$ by the reciprocal polar transformation with respect to $\mathcal{C}$, i.e., a conic with one focus at F. This conic is :

(1) a hyperbola for $\lambda < 1$ (F is exterior at $\mathcal{C}_0$) ;
(2) a parabola for $\lambda = 1$ (F lies on $\mathcal{C}_0$) ;
(3) an ellipse for $\lambda > 1$ (F is interior at $\mathcal{C}_0$).

So the envelope $\mathcal{P}$ of the circumcircles $\mathcal{C}_1$ is the inverse of this conic with respect to one of its foci, i.e., a conchoid of circle which is :

(1) a limaçon of Pascal for $\lambda < 1$: the hyperbola $i(\mathcal{P})$ as two asymptotes, so F is a node on $\mathcal{P}$;
(2) a cardioid for $\lambda = 1$: the parabola $i(\mathcal{P})$ is tangent to the line at infinity, so F is a cusp on $\mathcal{P}$;
(3) a curve without singularity for $\lambda > 1$: all points of the ellipse $i(\mathcal{P})$ are at finite distance.

We illustrate (1) and (3) in Figures 12 and 13. [4] It should be of great interest to see if always $\lambda > 1$ for F_+ (and < 1 for F_-). We think that the answer is affirmative, and that $\lambda = 1$ is possible if and only if A, B, C are collinear.

[4]Images of inversion of the limaçon of Pascal and the cardioid can also be found in the websites http://www-history.mcs.st-andrews.ac.uk/history/Curves and http://xahlee.org/SpecialPlaneCurves.

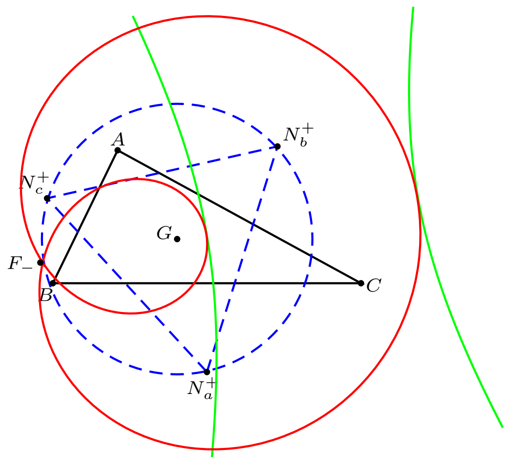

Figure 12. The inverse of the envelope ($\lambda < 1$)

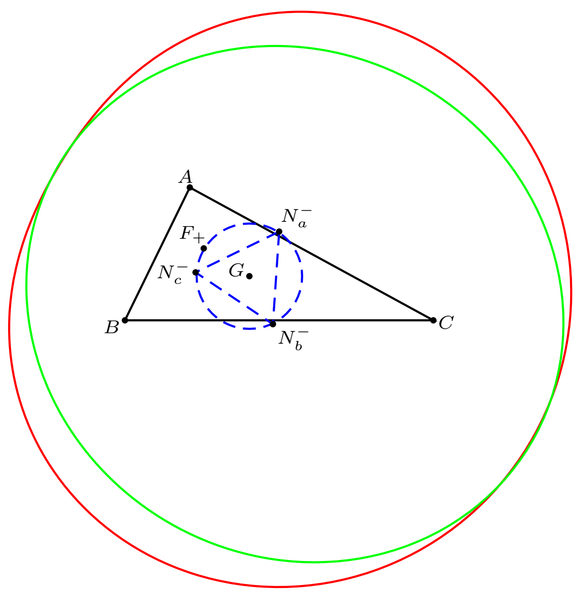

Figure 13. The inverse of the envelope ($\lambda > 1$)

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The Lemoine Cubic and Its Generalizations

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Abstract. For a given triangle, the Lemoine cubic is the locus of points whose cevian lines intersect the perpendicular bisectors of the corresponding sides of the triangle in three collinear points. We give some interesting geometric properties of the Lemoine cubic, and study a number of cubics related to it.

1. The Lemoine cubic and its constructions

In 1891, Lemoine published a note [5] in which he very briefly studied a cubic curve defined as follows. Let M be a point in the plane of triangle ABC. Denote by M_a the intersection of the line MA with the perpendicular bisector of BC and define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is the cubic in question. We shall denote this cubic by $\mathcal{K}(O)$, and follow Neuberg [8] in referring to it as the Lemoine cubic. Lemoine claimed that the circumcenter O of the reference triangle was a triple point of $\mathcal{K}(O)$. As pointed out in [7], this statement is false. The present paper considerably develops and generalizes Lemoine's note.

We use homogeneous barycentric coordinates, and adopt the notations of [4] for triangle centers. Since the second and third coordinates can be obtained from the first by cyclic permutations of a, b, c, we shall simply give the first coordinates. For convenience, we shall also write

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2}.$$

Thus, for example, the circumcenter is $X_3 = [a^2 S_A]$.

Figure 1 shows the Lemoine cubic $\mathcal{K}(O)$ passing through A, B, C, the orthocenter H, the midpoints A', B', C' of the sides of triangle ABC, the circumcenter O, and several other triangle centers such as $X_{32} = [a^4]$, $X_{56} = \left[\frac{a^2}{b+c-a}\right]$ and its extraversions. [1] Contrary to Lemoine's claim, the circumcenter is a node. When M traverses the cubic, the line $\mathcal{L}_M$ envelopes the Kiepert parabola with focus

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[1]The three extraversions of a point are each formed by changing in its homogeneous barycentric coordinates the signs of one of a, b, c. Thus, $X_{56a} = \left(\frac{a^2}{b+c+a} : \frac{b^2}{c-a-b} : \frac{c^2}{-a+b-c}\right)$, and similarly for X_{56b} and X_{56c}.

$F = X_{110} = \left[\frac{a^2}{b^2-c^2}\right]$ and directrix the Euler line. The equation of the Lemoine cubic is

$$\sum_{\text{cyclic}} a^4 S_A yz(y-z) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

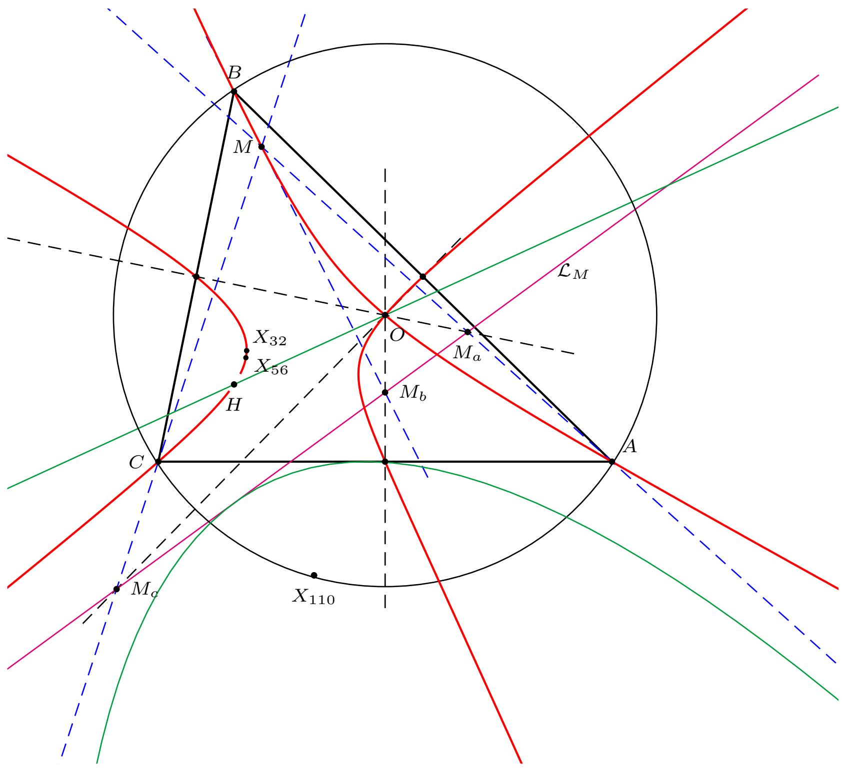

Figure 1. The Lemoine cubic with the Kiepert parabola

We give two equivalent constructions of the Lemoine cubic.

Construction 1. For any point Q on the line GK, the trilinear polar q of Q meets the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. [2] The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

For $Q = (a^2+t : b^2+t : c^2+t)$, this point of concurrency is

$$M = \left(\frac{a^2+t}{b^2c^2+(b^2+c^2-a^2)t} : \frac{b^2+t}{c^2a^2+(c^2+a^2-b^2)t} : \frac{c^2+t}{a^2b^2+(a^2+b^2-c^2)t}\right).$$

[2] The tripolar q envelopes the Kiepert parabola.

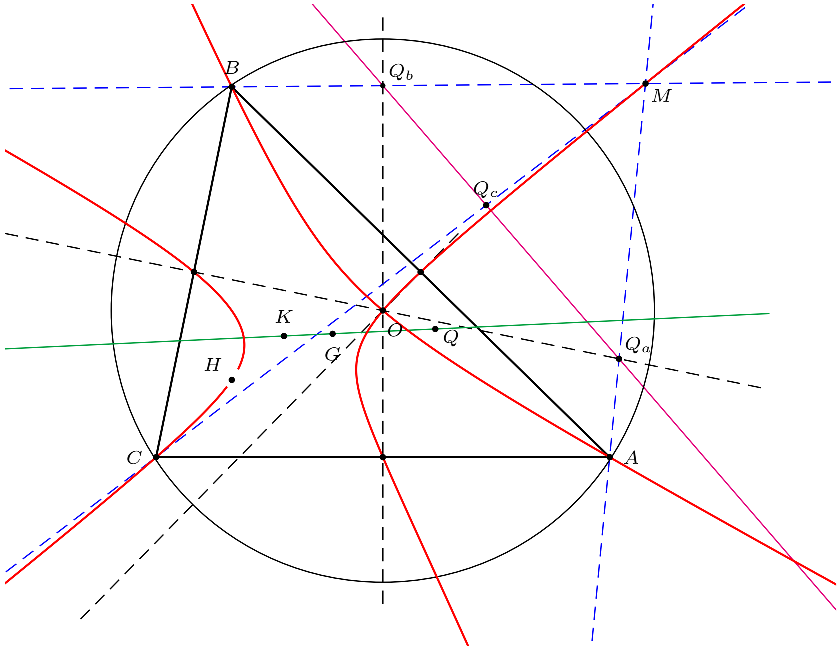

Figure 2. The Lemoine cubic as a locus of perspectors (Construction 1)

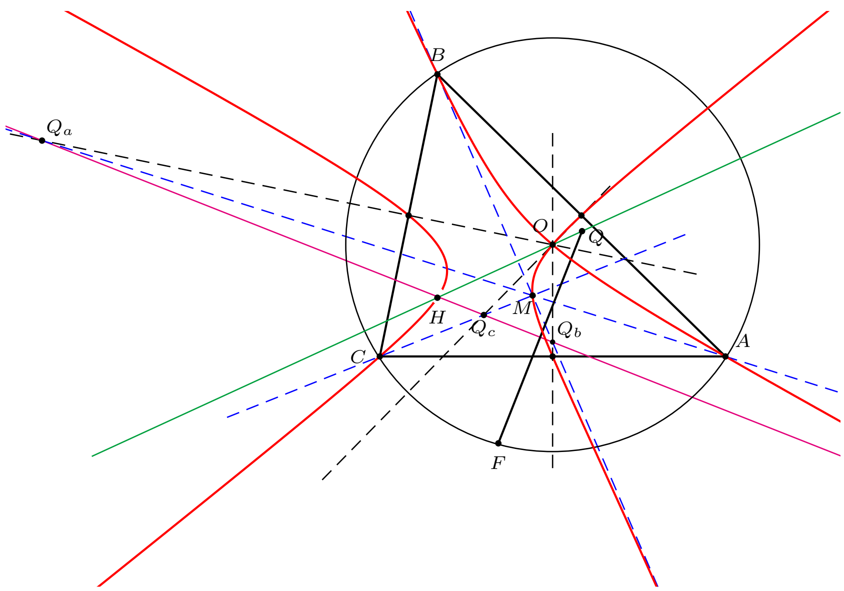

Figure 3. The Lemoine cubic as a locus of perspectors (Construction 2)

This gives a parametrization of the Lemoine cubic. This construction also yields the following points on $\mathcal{K}(O)$, all with very simple coordinates, and are not in [4].

i	$Q = X_i$	$M = M_i$
69	S_A	$\frac{S_A}{b^4+c^4-a^4}$
86	$\frac{1}{b+c}$	$\frac{1}{a(b+c)-(b^2+bc+c^2)}$
141	b^2+c^2	$\frac{b^2+c^2}{b^4+b^2c^2+c^4-a^4}$
193	$b^2+c^2-3a^2$	$S_A(b^2+c^2-3a^2)$

Construction 2. For any point Q on the Euler line, the perpendicular bisector of FQ intersects the perpendicular bisectors OA', OB', OC' at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(O)$.

See Figure 3 and Remark following Construction 4 on the construction of tangents to $\mathcal{K}(O)$.

2. Geometric properties of the Lemoine cubic

Proposition 1. *The Lemoine cubic has the following geometric properties.*

(1) *The two tangents at O are parallel to the asymptotes of the Jerabek hyperbola.*
(2) *The tangent at H passes through the center $X_{125} = [(b^2 - c^2)^2 S_A]$ of the Jerabek hyperbola.* [3]
(3) *The tangents at A, B, C concur at $X_{184} = [a^4 S_A]$, the inverse of X_{125} in the Brocard circle.*
(4) *The asymptotes are parallel to those of the orthocubic,* i.e., *the pivotal isogonal cubic with pivot H.*
(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(O)$ and the altitudes lie on the circle with diameter OH.* [4] *The triangles $A'B'C'$ and $H_AH_BH_C$ are perspective at a point*

$$Z_1 = \left[a^4 S_A(a^4 + b^4 + c^4 - 2a^2(b^2 + c^2))\right]$$

on the cubic. [5]
(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(O)$ and the sidelines of the medial triangle form a triangle perspective with $H_AH_BH_C$ at a point*

$$Z_2 = \left[\frac{a^4 S_A^2}{3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2}\right]$$

on the cubic. [6]
(7) *$\mathcal{K}(O)$ intersects the circumcircle of ABC at the vertices of the circumnormal triangle of ABC.* [7]

---

[3] This is also tangent to the Jerabek hyperbola at H.

[4] In other words, these are the projections of O on the altitudes. The coordinates of H_A are

$$\left(\frac{2a^4 S_A}{a^2(b^2 + c^2) - (b^2 - c^2)^2} : S_C : S_B\right).$$

[5] Z_1 is the isogonal conjugate of X_{847}. It lies on a large number of lines, 13 using only triangle centers from [4], for example, X_2X_{54}, X_3X_{49}, X_4X_{110}, X_5X_{578}, $X_{24}X_{52}$ and others.

[6] This point Z_2 is not in the current edition of [4]. It lies on the lines X_3X_{64}, X_4X_{122} and $X_{95}X_{253}$.

[7] These are the points U, V, W on the circumcircle for which the lines UU^*, VV^*, WW^* (joining each point to its own isogonal conjugate) all pass through O. As such, they are, together with the vertices, the intersections of the circumcircle and the McCay cubic, the isogonal cubic with pivot the circumcenter O. See [3, p.166, §6.29].

We illustrate (1), (2), (3) in Figure 4, (4) in Figure 5, (5), (6) in Figure 6, and (7) in Figure 7 below.

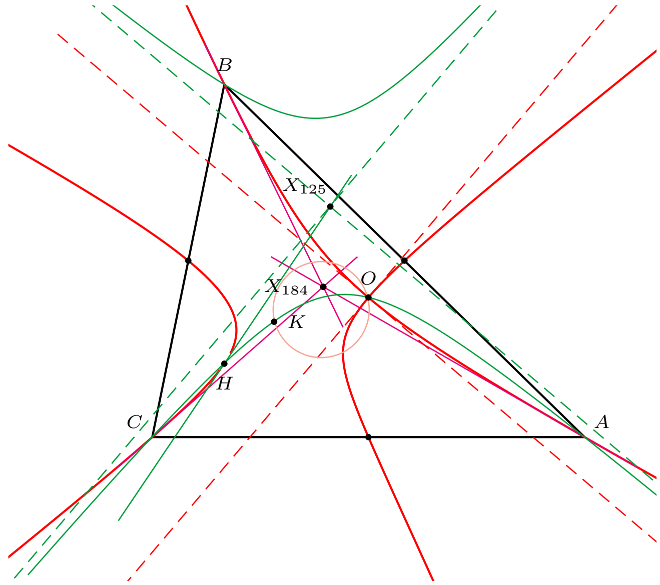

Figure 4. The tangents to the Lemoine cubic at O and the Jerabek hyperbola

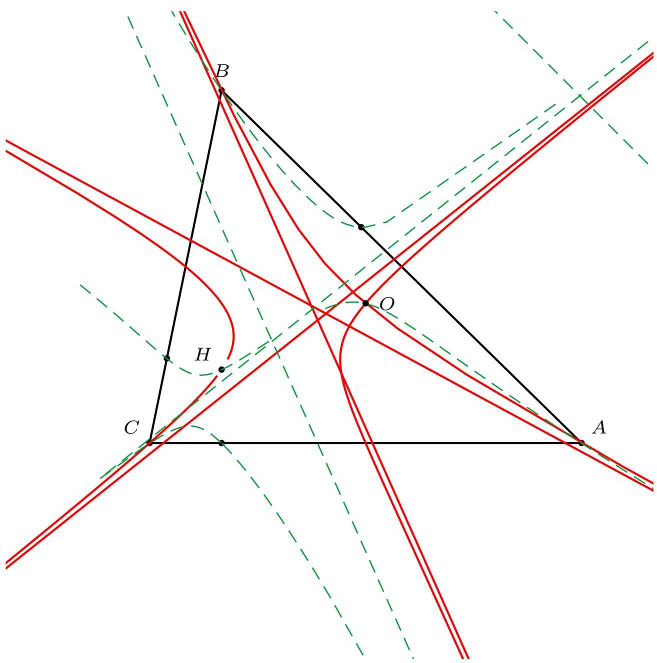

Figure 5. The Lemoine cubic and the orthocubic have parallel asymptotes

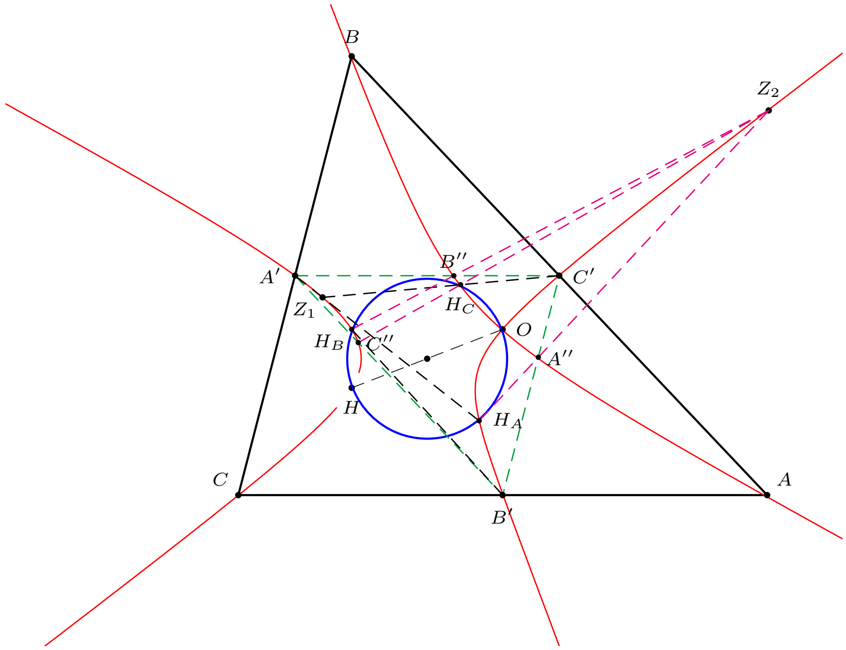

Figure 6. The perspectors Z_1 and Z_2

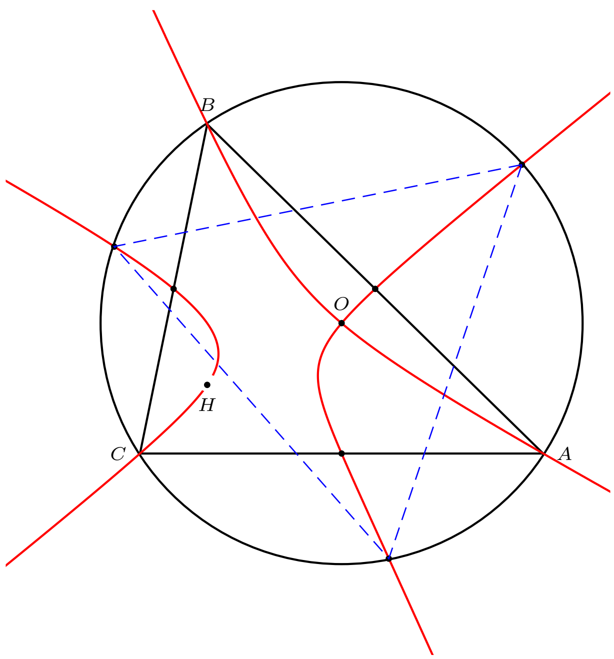

Figure 7. The Lemoine cubic with the circumnormal triangle

3. The generalized Lemoine cubic

Let P be a point distinct from H, not lying on any of the sidelines of triangle ABC. Consider its pedal triangle $P_aP_bP_c$. For every point M in the plane, let $M_a = PP_a \cap AM$. Define M_b and M_c similarly. The locus of M such that the three points M_a, M_b, M_c are collinear on a line $\mathcal{L}_M$ is a cubic $\mathcal{K}(P)$ called the generalized Lemoine cubic associated with P. This cubic passes through A, B, C, H, P_a, P_b, P_c, and P which is a node. Moreover, the line $\mathcal{L}_M$ envelopes the inscribed parabola with directrix the line HP and focus F the antipode (on the circumcircle) of the isogonal conjugate of the infinite point of the line HP.[8] The perspector S is the second intersection of the Steiner circum-ellipse with the line through F and the Steiner point $X_{99} = \left[\frac{1}{b^2-c^2}\right]$.

With $P = (p : q : r)$, the equation of $\mathcal{K}(P)$ is

$$\sum_{\text{cyclic}} x\left(r(c^2p + S_Br)y^2 - q(b^2p + S_Cq)z^2\right) + \left(\sum_{\text{cyclic}} a^2p(q-r)\right) xyz = 0.$$

The two constructions in §1 can easily be adapted to this more general situation.

Construction 3. For any point Q on the trilinear polar of S, the trilinear polar q of Q meets the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Construction 4. For any point Q on the line HP , the perpendicular bisector of FQ intersects the lines PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the cubic $\mathcal{K}(P)$.

Remark. The tangent at M to $\mathcal{K}(P)$ can be constructed as follows: the perpendicular at Q to the line HP intersects the perpendicular bisector of FQ at N, which is the point of tangency of the line through Q_a, Q_b, Q_c with the parabola. The tangent at M to $\mathcal{K}(P)$ is the tangent at M to the circum-conic through M and N. Given a point M on the cubic, first construct $M_a = AM \cap PP_a$ and $M_b = BM \cap PP_b$, then Q the reflection of F in the line M_aM_b, and finally apply the construction above.

Jean-Pierre Ehrmann has noticed that $\mathcal{K}(P)$ can be seen as the locus of point M such that the circum-conic passing through M and the infinite point of the line PM is a rectangular hyperbola. This property gives another very simple construction of $\mathcal{K}(P)$ or the construction of the "second" intersection of $\mathcal{K}(P)$ and any line through P.

Construction 5. A line ℓ_P through P intersects BC at P_1. The parallel to ℓ_P at A intersects HC at P_2. AB and P_1P_2 intersect at P_3. Finally, HP_3 intersects ℓ_P at M on the cubic $\mathcal{K}(P)$.

Most of the properties of the Lemoine cubic $\mathcal{K}(O)$ also hold for $\mathcal{K}(P)$ in general.

[8] Construction of F : draw the perpendicular at A to the line HP and reflect it about a bisector passing through A. This line meets the circumcircle at A and F.

Proposition 2. *Let $\mathcal{K}(P)$ be the generalized Lemoine cubic.*

(1) *The two tangents at P are parallel to the asymptotes of the rectangular circum-hyperbola passing through P.*
(2) *The tangent at H to $\mathcal{K}(P)$ is the tangent at H to the rectangular circum-hyperbola which is the isogonal image of the line OF. The asymptotes of this hyperbola are perpendicular and parallel to the line HP.*
(3) *The tangents at A, B, C concur if and only if P lies on the Darboux cubic.*[9]
(4) *The asymptotes are parallel to those of the pivotal isogonal cubic with pivot the anticomplement of P.*
(5) *The "third" intersections H_A, H_B, H_C of $\mathcal{K}(P)$ with the altitudes are on the circle with diameter HP. The triangles $P_aP_bP_c$ and $H_AH_BH_C$ are perspective at a point on $\mathcal{K}(P)$.*[10]
(6) *The "third" intersections A'', B'', C'' of $\mathcal{K}(P)$ and the sidelines of $P_aP_bP_c$ form a triangle perspective with $H_AH_BH_C$ at a point on the cubic.*

Remarks. (1) The tangent of $\mathcal{K}(P)$ at H passes through the center of the rectangular hyperbola through P if and only if P lies on the isogonal non-pivotal cubic $\mathcal{K}_H$

$$\sum_{\text{cyclic}} x\,(c^2y^2 + b^2z^2) - \Phi\, x\, y\, z = 0$$

where

$$\Phi = \frac{\sum_{\text{cyclic}} \left(2b^2c^2(a^4 + b^2c^2) - a^6(2b^2 + 2c^2 - a^2)\right)}{4S_AS_BS_C}.$$

We shall study this cubic in §6.3 below.

(2) The polar conic of P can be seen as a degenerate rectangular hyperbola. If $P \neq X_5$, the polar conic of a point is a rectangular hyperbola if and only if it lies on the line PX_5. From this, there is only one point (apart from P) on the curve whose polar conic is a rectangular hyperbola. Very obviously, the polar conic of H is a rectangular hyperbola if and only if P lies on the Euler line. If $P = X_5$, all the points in the plane have a polar conic which is a rectangular hyperbola. This very special situation is detailed in §4.2.

4. Special Lemoine cubics

4.1. *$\mathcal{K}(P)$ with concuring asymptotes.* The three asymptotes of $\mathcal{K}(P)$ are concurrent if and only if P lies on the cubic $\mathcal{K}_{conc}$

$$\sum_{\text{cyclic}} \left(S_B \left(c^2(a^2+b^2) - (a^2-b^2)^2\right) y - S_C \left(b^2(a^2+c^2) - (a^2-c^2)^2\right) z\right) x^2$$
$$- 2(a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

[9] The Darboux cubic is the isogonal cubic with pivot the de Longchamps point X_{20}.

[10] The coordinates of this point are $(p^2(-S_Ap + S_Bq + S_Cr) + a^2pqr : \cdots : \cdots)$.

The three asymptotes of $\mathcal{K}(P)$ are all real if and only if P lies inside the Steiner deltoid $\mathcal{H}_3$. [11] For example, the point $X_{76} = \left[\frac{1}{a^2}\right]$ lies on the cubic $\mathcal{K}_{conc}$ and inside the Steiner deltoid. The cubic $\mathcal{K}(X_{76})$ has three real asymptotes concurring at a point on X_5X_{76}. See Figure 8. On the other hand, the de Longchamps point X_{20} also lies on $\mathcal{K}_{conc}$, but it is not always inside $\mathcal{H}_3$. See Figure 10. The three asymptotes of $\mathcal{K}(X_{20})$, however, intersect at the real point X_{376}, the reflection of G in O.

We shall study the cubic $\mathcal{K}_{conc}$ in more detail in §6.1 below.

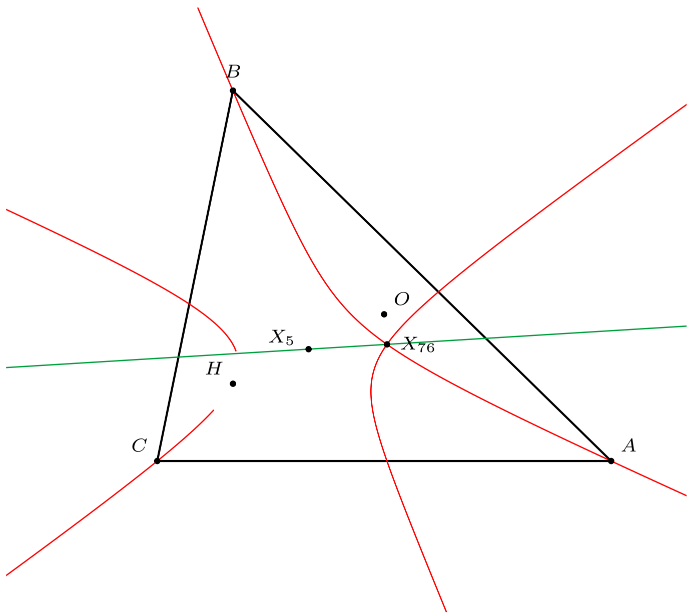

Figure 8. $\mathcal{K}(X_{76})$ with three concurring asymptotes

4.2. $\mathcal{K}(P)$ *with asymptotes making* 60° *angles with one another.* $\mathcal{K}(P)$ has three real asymptotes making 60° angles with one another if and only if P is the nine-point center X_5. See Figure 9. The asymptotes of $\mathcal{K}(X_5)$ are parallel again to those of the McCay cubic and their point of concurrence is [12]

$$Z_3 = \left[a^2((b^2-c^2)^2 - a^2(b^2+c^2))(a^4 - 2a^2(b^2+c^2) + b^4 - 5b^2c^2 + c^4)\right].$$

[11] Cf. Cundy and Parry [1] have shown that for a pivotal isogonal cubic with pivot P, the three asymptotes are all real if and only if P lies inside a certain "critical deltoid" which is the anticomplement of $\mathcal{H}_3$, or equivalently, the envelope of axes of inscribed parabolas.

[12] Z_3 is not in the current edition of [4]. It is the common point of several lines, e.g. X_5X_{51}, $X_{373}X_{549}$ and $X_{511}X_{547}$.

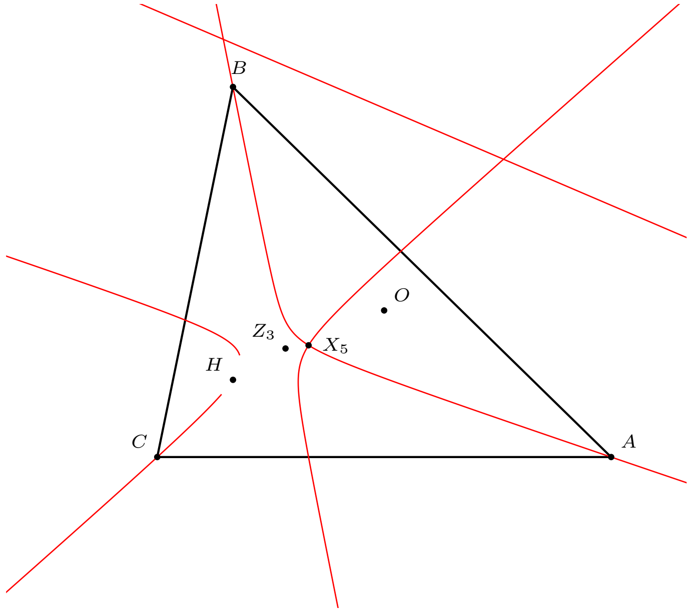

Figure 9. $\mathcal{K}(X_5)$ with three concurring asymptotes making 60° angles

4.3. *Generalized Lemoine isocubics.* $\mathcal{K}(P)$ is an isocubic if and only if the points P_a, P_b, P_c are collinear. It follows that P must lie on the circumcircle. The line through P_a, P_b, P_c is the Simson line of P and its trilinear pole R is the root of the cubic. When P traverses the circumcircle, R traverses the Simson cubic. See [2]. The cubic $\mathcal{K}(P)$ is a conico-pivotal isocubic: for any point M on the curve, its isoconjugate M^* (under the isoconjugation with fixed point P) lies on the curve and the line MM^* envelopes a conic. The points M and M^* are obtained from two points Q and Q' (see Construction 4) on the line HP which are inverse with respect to the circle centered at P going through F, focus of the parabola in §2. (see remark in §5 for more details)

5. The construction of nodal cubics

In §3, we have seen how to construct $\mathcal{K}(P)$ which is a special case of nodal cubic. More generally, we give a very simple construction valid for any nodal circum-cubic with a node at P, intersecting the sidelines again at any three points P_a, P_b, P_c. Let R_a be the trilinear pole of the line passing through the points $AB \cap PP_b$ and $AC \cap PP_c$. Similarly define R_b and R_c. These three points are collinear on a line $\mathcal{L}$ which is the trilinear polar of a point S. For any point Q on the line $\mathcal{L}$, the trilinear polar q of Q meets PP_a, PP_b, PP_c at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur at M on the sought cubic and, as usual, q envelopes the inscribed conic γ with perspector S.

Remarks. (1) The tangents at P to the cubic are those drawn from P to γ. These tangents are

(i) real and distinct when P is outside γ and is a "proper" node,
(ii) imaginary when P is inside γ and is an isolated point, or
(iii) identical when P lies on γ and is a cusp, the cuspidal tangent being the tangent at P to γ.

It can be seen that this situation occurs if and only if P lies on the cubic tangent at P_a, P_b, P_c to the sidelines of ABC and passing through the points $BC \cap P_bP_c$, $CA \cap P_cP_a$, $AB \cap P_aP_b$. In other words and generally speaking, there is no cuspidal circum-cubic with a cusp at P passing through P_a, P_b, P_c.

(2) When P_a, P_b, P_c are collinear on a line ℓ, the cubic becomes a conico-pivotal isocubic invariant under isoconjugation with fixed point P: for any point M on the curve, its isoconjugate M^* lies on the curve and the line MM^* envelopes the conic Γ inscribed in the anticevian triangle of P and in the triangle formed by the lines AP_a, BP_b, CP_c. The tangents at P to the cubic are tangent to both conics γ and Γ.

6. Some cubics related to $\mathcal{K}(P)$

6.1. *The cubic $\mathcal{K}_{conc}$* . The circumcubic $\mathcal{K}_{conc}$ considered in §4.1 above contains a large number of interesting points: the orthocenter H, the nine-point center X_5, the de Longchamps point X_{20}, X_{76}, the point

$$Z_4 = [a^2 S_A^2 (a^2(b^2+c^2) - (b^2-c^2)^2)]$$

which is the anticomplement of X_{389}, the center of the Taylor circle. [13] The cubic $\mathcal{K}_{conc}$ also contains the traces of X_{69} on the sidelines of ABC, the three cusps of the Steiner deltoid, and its contacts with the altitudes of triangle ABC.[14] Z is also the common point of the three lines each joining the trace of X_{69} on a sideline of ABC and the contact of the Steiner deltoid with the corresponding altitude. See Figure 10.

Proposition 3. *The cubic $\mathcal{K}_{conc}$ has the following properties.*

(1) *The tangents at A, B, C concur at X_{53}, the Lemoine point of the orthic triangle.*
(2) *The tangent at H is the line HK.*
(3) *The tangent at X_5 is the Euler line of the orthic triangle, the tangential being the point Z_4.* [15]
(4) *The asymptotes of $\mathcal{K}_{conc}$ are parallel to those of the McCay cubic and concur at a point* [16]

$$Z_5 = \left[a^2(a^2(b^2+c^2) - (b^2-c^2)^2)(2S_A^2 + b^2c^2)\right].$$

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[13] The point Z_4 is therefore the center of the Taylor circle of the antimedial triangle. It lies on the line X_4X_{69}.

[14] The contact with the altitude AH is the reflection of its trace on BC about the midpoint of AH.

[15] This line also contains X_{51}, X_{52} and other points.

[16] Z_5 is not in the current edition of [4]. It is the common point of quite a number of lines, e.g. X_3X_{64}, X_5X_{51}, $X_{113}X_{127}$, $X_{128}X_{130}$, and $X_{140}X_{185}$. The three asymptotes of the McCay cubic are concurrent at the centroid G.

(5) *$\mathcal{K}_{conc}$ intersects the circumcircle at A, B, C and three other points which are the antipodes of the points whose Simson lines pass through X_{389}.*

We illustrate (1), (2), (3) in Figure 11, (4) in Figure 12, and (5) in Figure 13.

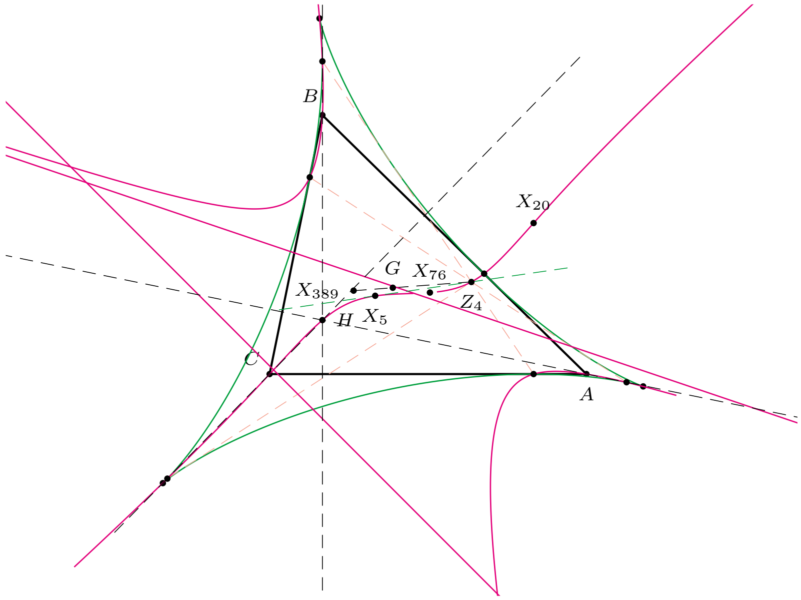

Figure 10. $\mathcal{K}_{conc}$ with the Steiner deltoid

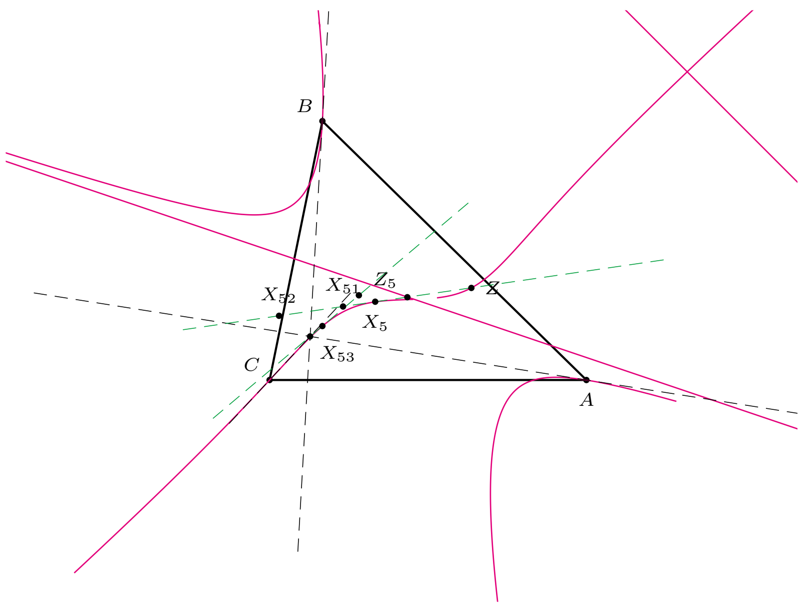

Figure 11. Tangents of $\mathcal{K}_{conc}$

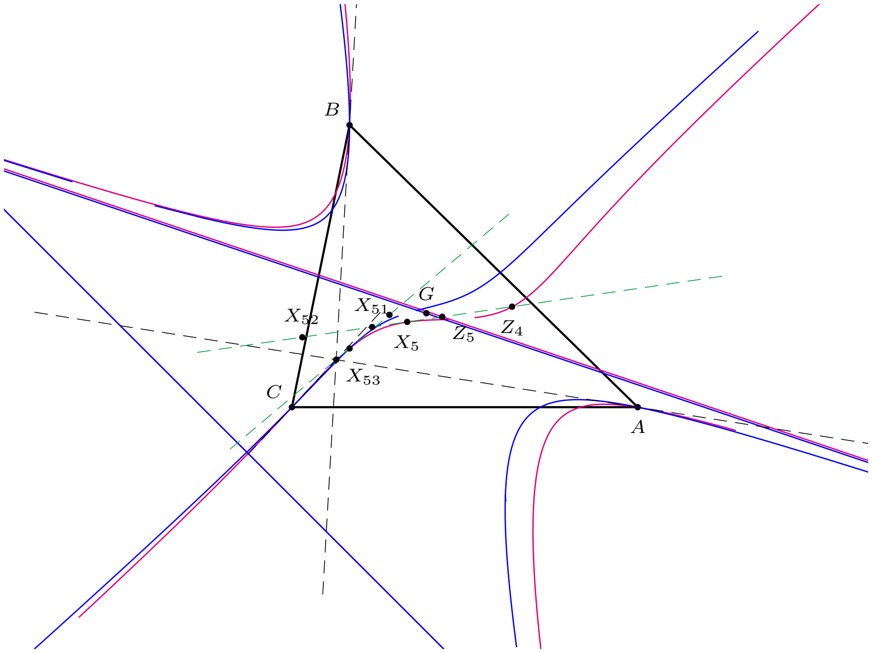

Figure 12. $\mathcal{K}_{conc}$ with the McCay cubic

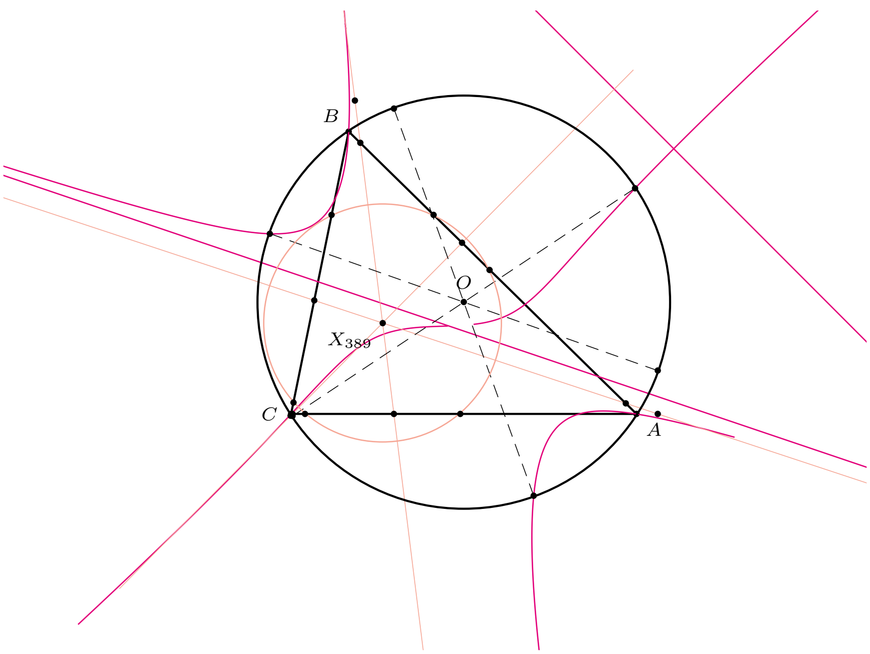

Figure 13. $\mathcal{K}_{conc}$ with the circumcircle and the Taylor circle

6.2. *The isogonal image of* $\mathcal{K}(O)$. Under isogonal conjugation, $\mathcal{K}(O)$ transforms into another nodal circum-cubic

$$\sum_{\text{cyclic}} b^2c^2\,x\,(S_By^2 - S_Cz^2) + (a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0.$$

The node is the orthocenter H. The cubic also passes through O, X_8 (Nagel point) and its extraversions, X_{76}, $X_{847} = Z_1^*$, and the traces of $X_{264} = \left[\frac{1}{a^2 S_A}\right]$. The tangents at H are parallel to the asymptotes of the Stammler rectangular hyperbola[17]. The three asymptotes are concurrent at the midpoint of GH,[18] and are parallel to those of the McCay cubic.

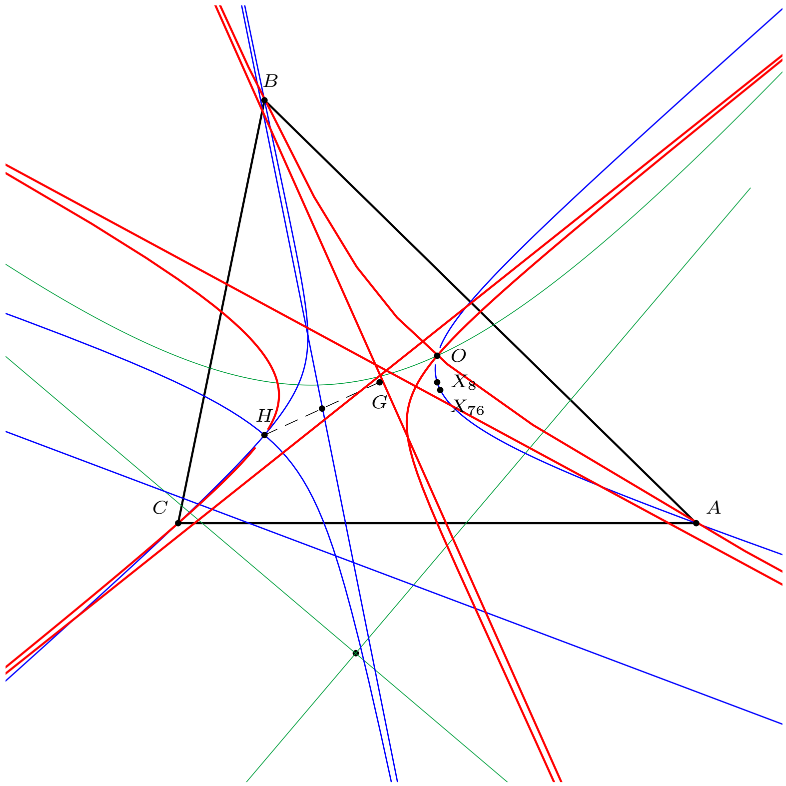

Figure 14. The Lemoine cubic and its isogonal

This cubic was already known by J. R. Musselman [6] although its description is totally different. We find it again in [9] in a different context. Let P be a point on the plane of triangle ABC, and P_1, P_2, P_3 the orthogonal projections of P on the perpendicular bisectors of BC, CA, AB respectively. The locus of P such that the triangle $P_1P_2P_3$ is in perspective with ABC is the Stammler hyperbola and the locus of the perspector is the cubic which is the isogonal transform of $\mathcal{K}(O)$. See Figure 15.

[17]The Stammler hyperbola is the rectangular hyperbola through the circumcenter, incenter, and the three excenters. Its asymptotes are parallel to the lines through X_{110} and the two intersections of the Euler line and the circumcircle

[18]This is $X_{381} = [a^2(a^2 + b^2 + c^2) - 2(b^2 - c^2)^2]$.

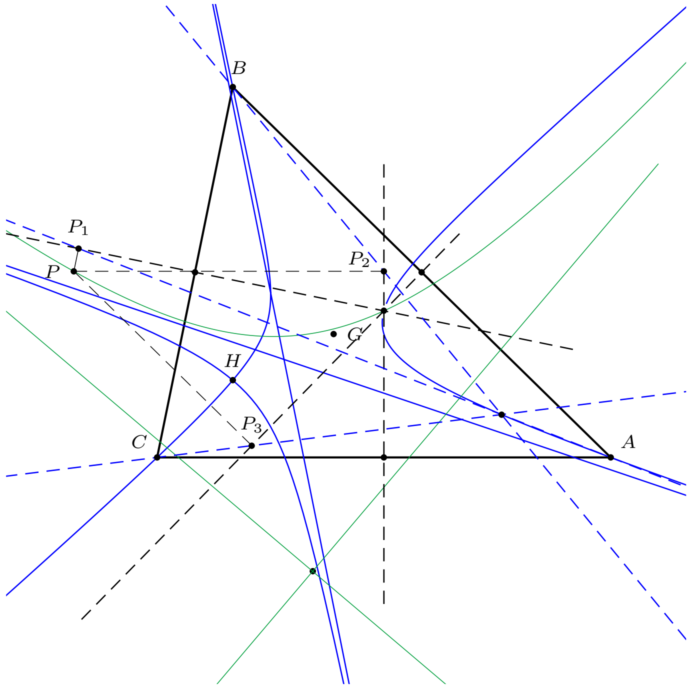

Figure 15. The isogonal of $\mathcal{K}(O)$ with the Stammler hyperbola

6.3. *The cubic* $\mathcal{K}_H$. Recall from Remark (1) following Proposition 2 that the tangent at H to $\mathcal{K}(P)$ passes through the center of the rectangular circum-hyperbola passing through P if and only if P lies on the cubic $\mathcal{K}_H$. This is a non-pivotal isogonal circum-cubic with root at G. See Figure 14.

Proposition 4. *The cubic $\mathcal{K}_H$ has the following geometric properties.*

(1) *$\mathcal{K}_H$ passes through A, B, C, O, H, the three points H_A, H_B, H_C and their isogonal conjugates H_A^*, H_B^*, H_C^*.* [19]
(2) *The three real asymptotes are parallel to the sidelines of ABC.*
(3) *The tangents of $\mathcal{K}_H$ at A, B, C are the sidelines of the tangential triangle. Hence, $\mathcal{K}_H$ is tritangent to the circumcircle at the vertices A, B, C.*
(4) *The tangent at A (respectively B, C) and the asymptote parallel to BC (respectively CA, AB) intersect at a point $\widetilde{A}$ (respectively $\widetilde{B}$, $\widetilde{C}$) on $\mathcal{K}_H$.*
(5) *The three points $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ are collinear on the perpendicular $\mathcal{L}$ to the line OK at the inverse of X_{389} in the circumcircle.*[20]

---

[19]The points H_A, H_B, H_C are on the circle, diameter OH. See Proposition 1(5). Their isogonal conjugates are on the lines OA. OB, OC respectively.

[20]In other words, the line $\mathcal{L}$ is the inversive image of the circle with diameter OX_{389}. Hence, $\widetilde{A}$ is the common point of $\mathcal{L}$ and the tangent at A to the circumcircle, and the parallel through $\widetilde{A}$ to BC is an asymptote of $\mathcal{K}_H$.

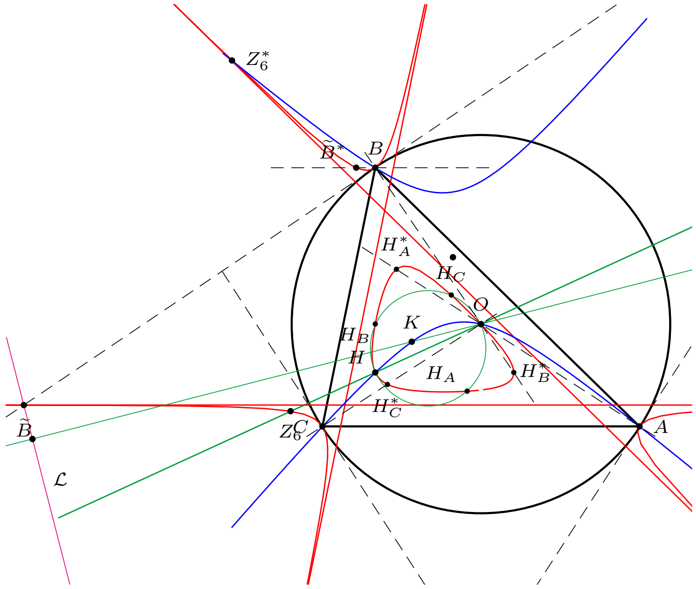

Figure 16. The cubic $\mathcal{K}_H$ with the Jerabek hyperbola

(6) *The isogonal conjugate of* $\widetilde{A}$ *is the "third" intersection of* $\mathcal{K}_H$ *with the parallel to* BC *through* A*; similarly for the isogonal conjugates of* $\widetilde{B}$ *and* $\widetilde{C}$.

(7) *The third intersection with the Euler line, apart from* O *and* H*, is the point* [21]

$$Z_6 = \left[\frac{(b^2 - c^2)^2 + a^2(b^2 + c^2 - 2a^2)}{(b^2 - c^2)\, S_A} \right].$$

(8) *The isogonal conjugate of* Z_6 *is the sixth intersection of* $\mathcal{K}_H$ *with the Jerabek hyperbola.*

We conclude with another interesting property of the cubic $\mathcal{K}_H$. Recall that the polar circle of triangle ABC is the unique circle with respect to which triangle ABC is self-polar. This is in the coaxal system generated by the circumcircle and the nine-point circle. It has center H, radius ρ given by

$$\rho^2 = 4R^2 - \frac{1}{2}(a^2 + b^2 + c^2),$$

and is real only when triangle ABC is obtuse angled. Let $\mathcal{C}$ be the concentric circle with radius $\frac{\rho}{\sqrt{2}}$.

Proposition 5. $\mathcal{K}_H$ *is the locus of point* M *whose pedal circle is orthogonal to circle* $\mathcal{C}$.

---

[21]This is not in [4]. It is the homothetic of X_{402} (Gossard perspector) in the homothety with center G, ratio 4 or, equivalently, the anticomplement of the anticomplement of X_{402}.

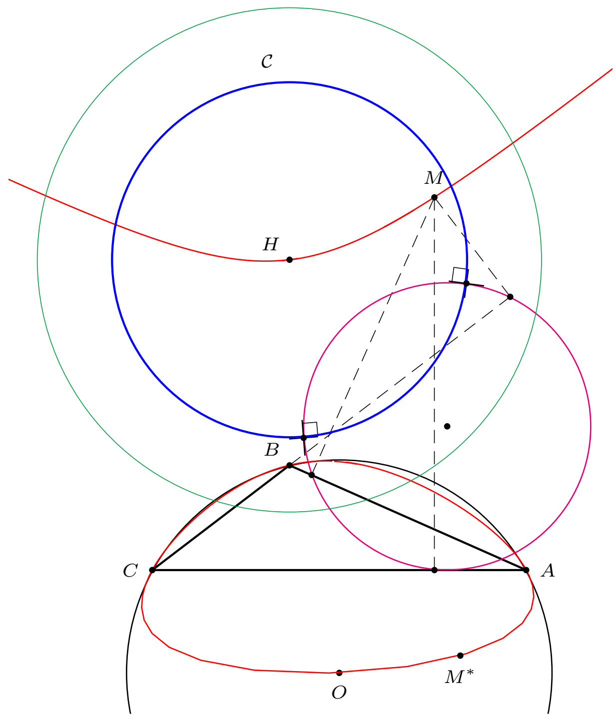

Figure 17. The cubic $\mathcal{K}_H$ for an obtuse angled triangle

In fact, more generally, every non-pivotal isogonal cubic can be seen, in a unique way, as the locus of point M such that the pedal circle of M is orthogonal to a fixed circle, real or imaginary, proper or degenerate.

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A Simple Construction of the Golden Section

Kurt Hofstetter

Abstract. We construct the golden section by drawing 5 circular arcs.

We denote by $P(Q)$ the circle with P as center and PQ as radius. Figure 1 shows two circles $A(B)$ and $B(A)$ intersecting at C and D. The line AB intersects the circles again at E and F. The circles $A(F)$ and $B(E)$ intersect at two points X and Y. It is clear that C, D, X, Y are on a line. It is much more interesting to note that D divides the segment CX in the golden ratio, *i.e.*,

$$\frac{CD}{CX} = \frac{\sqrt{5}-1}{2}.$$

This is easy to verify. If we assume AB of length 2, then $CD = 2\sqrt{3}$ and $CX = \sqrt{15}+\sqrt{3}$. From these,

$$\frac{CD}{CX} = \frac{2\sqrt{3}}{\sqrt{15}+\sqrt{3}} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2}.$$

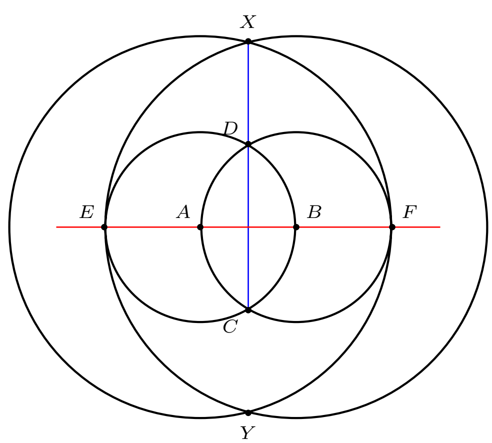

Figure 1

This shows that to construct three collinear points in golden section, we need four circles and one line. It is possible, however, to replace the line AB by a circle, say $C(D)$. See Figure 2. Thus, *the golden section can be constructed with compass only, in 5 steps*.

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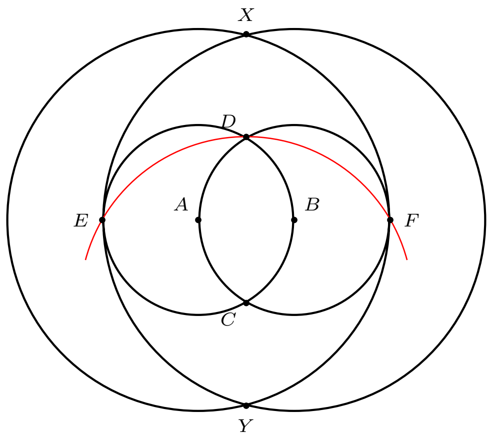

Figure 2

It is interesting to compare this with Figure 3 which also displays the golden section. See [1, p.105, note on 3.5(b)] and [2].[1] Here, ABC is an equilateral triangle. The line joining the midpoints D, E of two sides intersects the circumcircle at F. Then E divides DF in the golden section, *i.e.*,

$$\frac{DE}{DF} = \frac{\sqrt{5}-1}{2}.$$

However, it is unlikely that this diagram can be constructed in fewer than 5 steps, using ruler and compass, or compass alone.

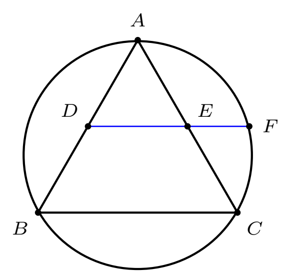

Figure 3

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A Rapid Construction of Some Triangle Centers

Lawrence S. Evans

Abstract. We give a compass and ruler construction of fifteen centers associated with a triangle by drawing 6 circles and 23 lines.

Given triangle T with vertices A, B, and C, draw a red circle centered at A passing through B, another centered at B going through C, and a third centered at C going through A. Now, draw a blue circle centered at A passing through C, one centered at C going through B, and one centered at B going through A. There will be 12 intersections of red circles with blue ones. Three of them are A, B, and C. Three are apices of equilateral triangles erected on the sides of T and pointing outward. Denote such an apex by A_+, B_+, C_+. Three are the apices of equilateral triangles erected on the sides pointing inward. Denote them by A_-, B_-, C_-. The last three are the reflections of the vertices of T in the opposite sides, which we shall call A^*, B^*, C^*.

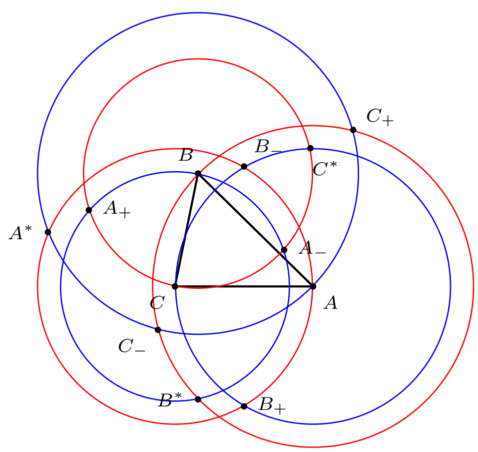

Figure 1. Construction of $A_\pm$, $B_\pm$, $C_\pm$, A^*, B^*, C^*

The four triangles $T = ABC$, $T_+ = A_+B_+C_+$, $T_- = A_-B_-C_-$, and $T^* = A^*B^*C^*$ are pairwise in perspective. The 6 centers of perspectivity are

(1) $[T, T_+] = F_+$, the inner Fermat point,
(2) $[T, T_-] = F_-$, the outer Fermat point,
(3) $[T, T^*] = H$, the orthocenter,

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(4) $[T_+, T_-] = O$, the circumcenter,
(5) $[T_+, T^*] = J_-$, the outer isodynamic point,
(6) $[T_-, T^*] = J_+$, the inner isodynamic point.

Only two lines, AA_+ and BB_+, are needed to determine F_+ by intersection. Likewise, 10 more are necessary to determine the other 5 centers F_-, H, O, J_- and J_+. We have drawn twelve lines so far. [1] See Figure 2, where the green lines only serve to indicate perspectivity; they are not necessary for the constructions of the triangle centers.

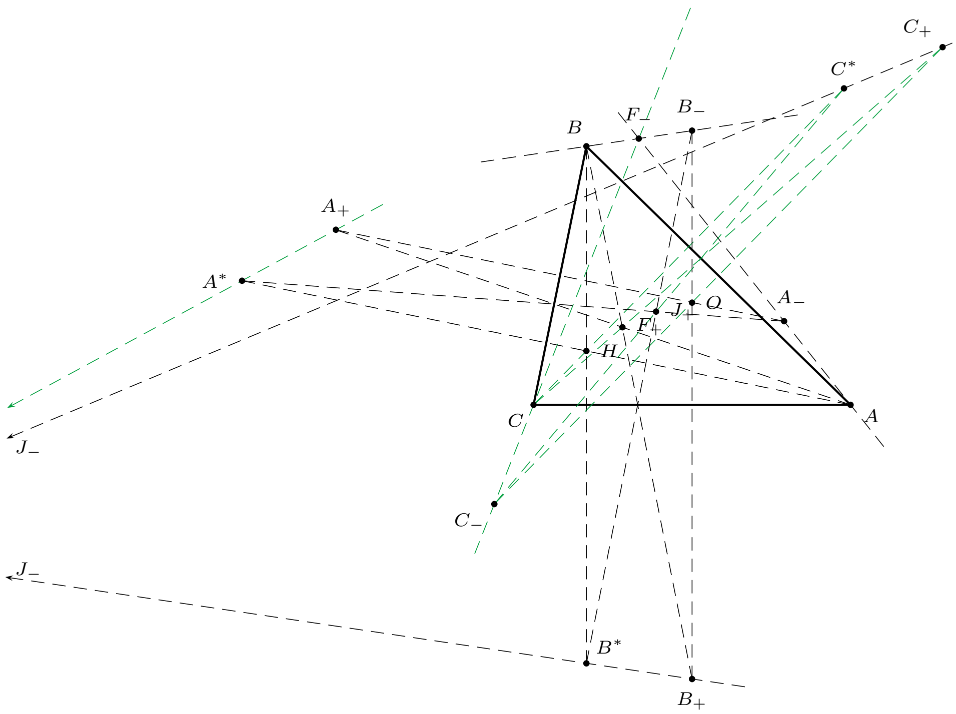

Figure 2. Construction of $F_\pm$, H, O, $J_\pm$

Define three more lines: the Euler line OH, the Fermat line F_+F_-, and the Apollonius line J_+J_-. The Apollonius line J_+J_- is also known as the Brocard axis. It contains the circumcenter O and the (Lemoine) symmedian point K. Then,

(7) $K = J_+J_- \cap F_+F_-$;
(8) $D = OH \cap F_+F_-$ is the center of orthocentroidal circle, the midpoint of between the centroid and the orthocenter.

We construct six more lines to locate four more centers:

(9) the outer Napoleon point is $N_+ = HJ_+ \cap OF_+$,

[1] The 18 points A, $A_\pm$, A^*, B, $B_\pm$, B^*, C, $C_\pm$, C^*, H, O, $F\pm$, $J_\pm$ all lie on a third degree curve called the Neuberg cubic.

(10) the inner Napoleon point is $N_- = HJ_- \cap OF_-$;
(11) the centroid $G = OH \cap J_+F_-$ (or $OH \cap J_-F_+$);
(12) the nine-point center $N_p = OH \cap N_-F_+$ (or $OH \cap N_+F-$).

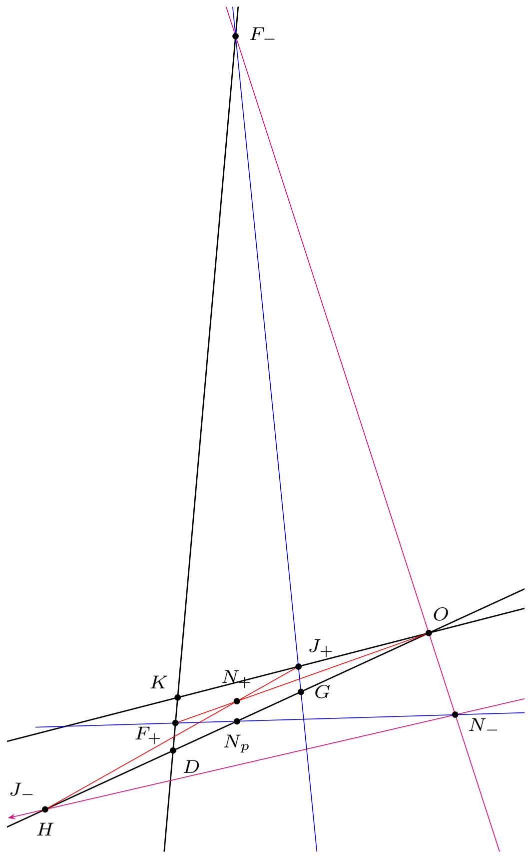

Figure 3. Construction of K, D, $N_\pm$, G, N_p

The line N_-F_+ (used in (12) above to locate N_p) intersects $OK = J_+J_-$ at the isogonal conjugate of N_-. Likewise, the lines N_+F_- and OK intersect at the isogonal conjugate of N_+. We also note that the line J_+N_- intersects the Euler line OH at the nine-point center N'_p of the medial triangle. Thus,

(13) $N_+^* = N_+F_- \cap OK$,
(14) $N_-^* = N_-F_+ \cap OK$, and
(15) $N'_p = J_+N_- \cap OH$ (or $J_-N_+ \cap OH$).

See Figure 4, in which we note that the points G, N_+ and N_-^* are collinear, so are G, N_- and N_+^*.

We have therefore constructed 15 centers with 6 circles and 23 lines: 12 to determine O, H, $F_\pm$, $J_\pm$ as the 6 centers of perspectivity of T, $T_\pm$ and T^*; then 9 to determine K, D, $N_\pm$, G, N_p, N_-^*, and finally 2 more to give N_+^* and N'_p.

Remark. The triangle centers in this note appear in [1, 2] as X_n for n given below.

center	O	H	F_+	F_-	J_+	J_-	K	D	N_+	N_-	G	N_p	N_+^*	N_-^*	N_p'
n	3	4	13	14	15	16	6	381	17	18	2	5	61	62	140

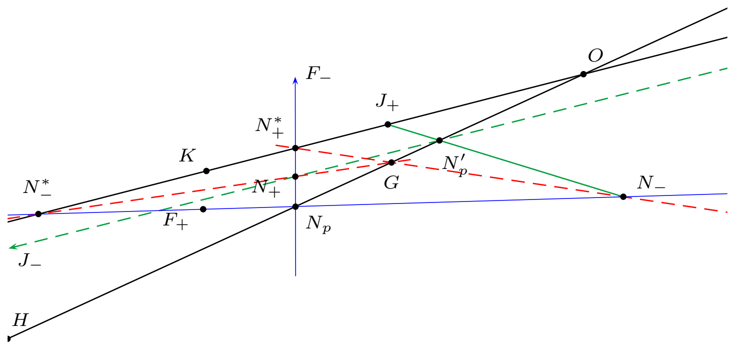

Figure 4. Construction of N_+^*, N_-^*, and N_p'

This construction uses Kimberling's list [1] of collinearities among centers. It can be implemented on a dynamic software like the Geometer's Sketchpad. After hiding the circles and lines, one is left with T and the centers, which can be observed to move in concert as one drags a vertex of T on the computer screen. Some important centers we do not get here are the incenter, the Gergonne and the Nagel points.

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A Generalization of the Tucker Circles

Peter Yff

Abstract. Let hexagon $PQRSTU$ be inscribed in triangle $A_1A_2A_3$ (ordered counterclockwise) such that P and S are on line A_3A_1, Q and T are on line A_1A_2, and R and U are on line A_2A_3. If PQ, RS, and TU are respectively parallel to A_2A_3, A_1A_2, and A_3A_1, while QR, ST, and UP are antiparallel to A_3A_1, A_2A_3, and A_1A_2 respectively, the vertices of the hexagon are on one circle. Now, let hexagon $P'Q'R'S'T'U'$ be described as above, with each of its sides parallel to the corresponding side of $PQRSTU$. Again the six vertices are concyclic, and the process may be repeated indefinitely to form an infinite family of circles (Tucker [3]). This family is a coaxaloid system, and its locus of centers is the Brocard axis of the triangle, passing through the circumcenter and the symmedian point. J. A. Third ([2]) extended this idea by relaxing the conditions for the directions of the sides of the hexagon, thus finding infinitely many coaxaloid systems of circles. The present paper defines a further extension by allowing the directions of the sides to be as arbitrary as possible, resulting in families of homothetic conics with properties analogous to those of the Tucker circles.

1. Circles of Tucker and Third

The system of Tucker circles is a special case of the systems of Third circles. In a Third system the directions of PQ, QR, and RS may be taken arbitrarily, while ST is made antiparallel to PQ (with respect to angle $A_2A_1A_3$). Similarly, TU and UP are made antiparallel to QR and RS respectively. The hexagon may then be inscribed in a circle, and a different starting point P' with the same directions produces another circle. It should be noted that the six vertices need not be confined to the sides of the triangle; each point may lie anywhere on its respective sideline. Thus an infinite family of circles may be obtained, and Third shows that this is a coaxaloid system. That is, it may be derived from a coaxal system of circles by multiplying every radius by a constant. (See Figures 1a and 1b). In particular, the Tucker system is obtained from the coaxal system of circles through the Brocard points Ω and Ω' by multiplying the radius of each circle by $\dfrac{R}{O\Omega}$, R being the circumradius of the triangle and O its circumcenter ([1, p.276]). In general, the line of centers of a Third system is the perpendicular bisector of the segment joining the pair of isogonal conjugate points which are the common points of the corresponding coaxal system. Furthermore, although the coaxal system has no envelope, it

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will be seen later that the envelope of the coaxaloid system is a conic tangent to the sidelines of the triangle, whose foci are the points common to the coaxal circles.

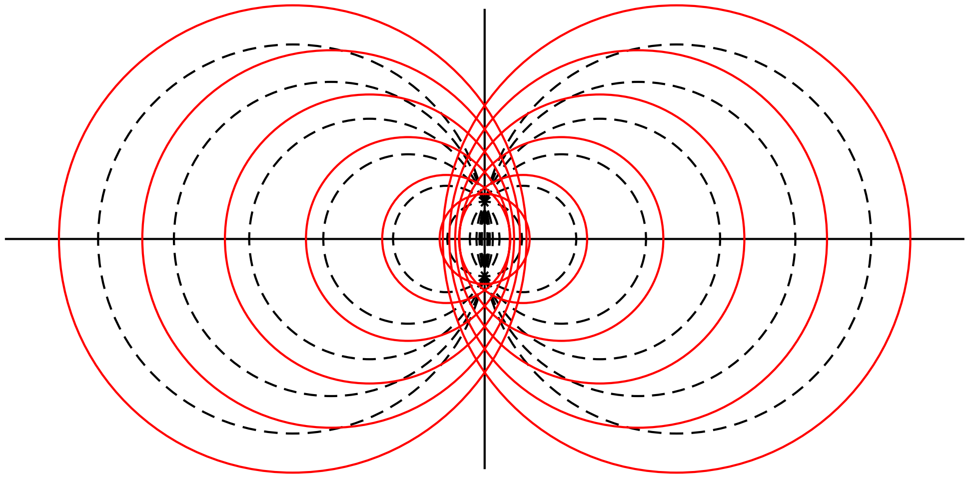

Figure 1a: Coaxaloid system with elliptic envelope, and its corresponding coaxal system

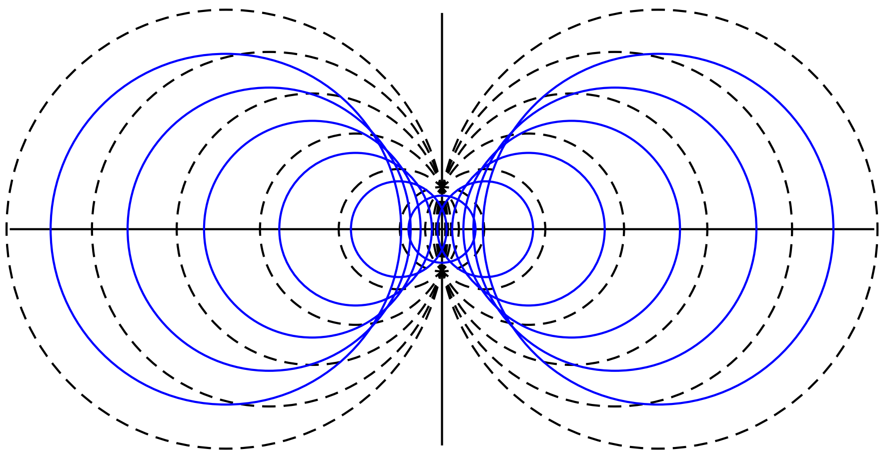

Figure 1b: Coaxaloid system with hyperbolic envelope, and its corresponding coaxal system

2. Two-circuit closed paths in a triangle

2.1. Consider a polygonal path from P on A_3A_1 to Q on A_1A_2 to R on A_2A_3 to S on A_3A_1 to T on A_1A_2 to U on A_2A_3, and back to P. Again the six points may be selected anywhere on their respective sidelines. The vertices of the triangle are numbered counterclockwise, and the lengths of the corresponding sides are denoted by a_1, a_2, a_3. Distances measured along the perimeter of the triangle in the counterclockwise sense are regarded as positive. The length of PA_1 is designated by λ, which is negative in case A_1 is between A_3 and P. Thus, $A_3P = a_2 - \lambda$, and the barycentric coordinates of P are $(a_2 - \lambda : 0 : \lambda)$. Also, six "directions" w_i are defined:

$$w_1 = \frac{PA_1}{A_1Q}, \qquad w_2 = \frac{QA_2}{A_2R}, \qquad w_3 = \frac{RA_3}{A_3S},$$
$$w_4 = \frac{SA_1}{A_1T}, \qquad w_5 = \frac{TA_2}{A_2U}, \qquad w_6 = \frac{UA_3}{A_3P}.$$

Any direction may be positive or negative depending on the signs of the directed segments. Then, $A_1Q = \dfrac{\lambda}{w_1}$, $QA_2 = \dfrac{a_3w_1 - \lambda}{w_1}$, $A_2R = \dfrac{a_3w_1 - \lambda}{w_1w_2}$, and so on.

2.2. A familiar example is that in which PQ and ST are parallel to A_2A_3, QR and TU are parallel to A_3A_1, and RS and UP are parallel to A_1A_2 (Figure 2). Then

$$w_1 = w_4 = \frac{a_2}{a_3}, \qquad w_2 = w_5 = \frac{a_3}{a_1}, \qquad w_3 = w_6 = \frac{a_1}{a_2}.$$

It is easily seen by elementary geometry that this path closes after two circuits around the sidelines of the triangle.

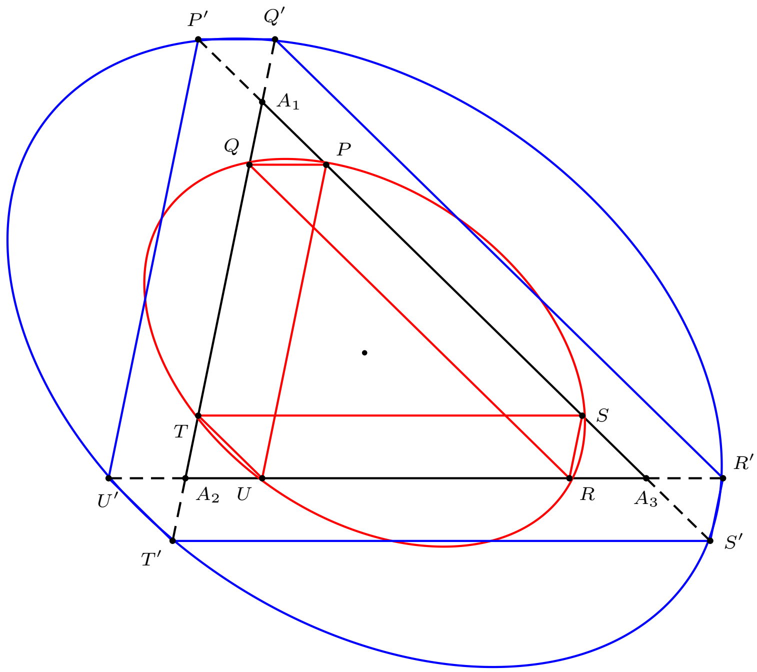

Figure 2. Hexagonal paths formed by parallels

2.3. Closure is less obvious, but still not difficult to prove, when “parallel” in the first example is replaced by “antiparallel” (Figure 3). Here,

$$w_1 = w_4 = \frac{a_3}{a_2}, \qquad w_2 = w_5 = \frac{a_1}{a_3}, \qquad w_3 = w_6 = \frac{a_2}{a_1}.$$

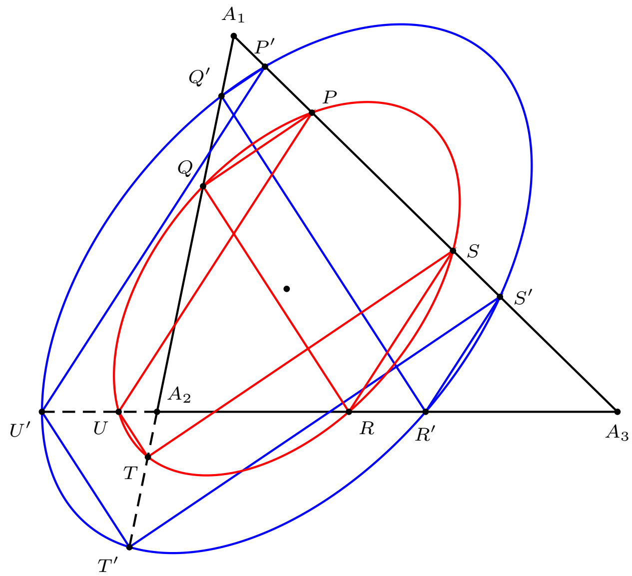

Figure 3. Hexagonal paths formed by antiparallels

2.4. Another positive result is obtained by using isoscelizers ([1, p.93]). That is, $PA_1 = A_1Q$, $QA_2 = A_2R$, $RA_3 = A_3S, \ldots, UA_3 = A_3P$. Therefore,

$$w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 1.$$

2.5. These examples suggest that, if $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the condition $w_1w_2w_3 = 1$ is sufficient to close the path after two circuits. Indeed, by computing lengths of segments around the triangle, one obtains

$$A_3P = \frac{UA_3}{w_3} = \frac{a_1w_1^2w_2^2w_3 - a_3w_1^2w_2w_3 + a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda}{w_1^2w_2^2w_3^2}.$$

But also $A_3P = a_2 - \lambda$, and equating the two expressions yields

$$(1 - w_1w_2w_3)(a_1w_1w_2 - a_2w_1w_2w_3 - a_3w_1 + \lambda(1 + w_1w_2w_3)) = 0. \qquad (1)$$

In order that (1) may be satisfied for all values of λ, the solution is $w_1w_2w_3 = 1$.

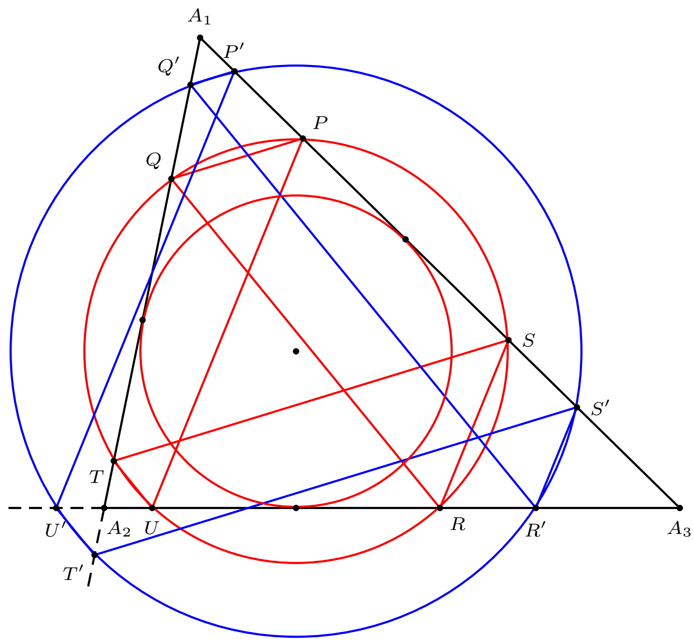

Figure 4. Hexagonal paths formed by isoscelizers

2.6. As a slight digression, the other factor in (1) gives the special solution

$$\lambda = \frac{w_1(a_2w_2w_3 - a_1w_2 + a_3)}{1 + w_1w_2w_3},$$

and calculation shows that this value of λ causes the path to close after only one circuit, that is $S = P$. For example, if antiparallels are used, and if P is the foot of the altitude from A_2, the one-circuit closed path is the orthic triangle of $A_1A_2A_3$.

Furthermore, if also $w_1w_2w_3 = 1$, the special value of λ becomes

$$\frac{a_2 - a_1w_1w_2 + a_3w_1}{2},$$

and the cevians A_1R, A_2P, and A_3Q are concurrent at the point (in barycentric coordinates, as throughout this paper)

$$\left(\frac{1}{-a_1w_1w_2 + a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 - a_2 + a_3w_1} : \frac{1}{a_1w_1w_2 + a_2 - a_3w_1}\right). \tag{2}$$

It follows that there exists a conic tangent to the sidelines of the triangle at P, Q, R. The coordinates of the center of the conic are $(a_1w_1w_2 : a_2 : a_3w_1)$.

2.7. Returning to the conditions $w_1w_2w_3 = 1$, $w_1 = w_4$, $w_2 = w_5$, $w_3 = w_6$, the coordinates of the six points may be found:

$$
\begin{aligned}
P =&(a_2 - \lambda : 0 : \lambda),\\
Q =&(a_3w_1 - \lambda : \lambda : 0),\\
R =&(0 : a_1w_1w_2 - a_3w_1 + \lambda : a_3w_1 - \lambda),\\
S =&(a_1w_1w_2 - a_3w_1 + \lambda : 0 : a_2 - a_1w_1w_2 + a_3w_1 - \lambda),\\
T =&(a_1w_1w_2 - a_2 + \lambda : a_2 - a_1w_1w_2 + a_3w_1 - \lambda : 0),\\
U =&(0 : a_2 - \lambda : a_1w_1w_2 - a_2 + \lambda).
\end{aligned}
$$

These points are on one conic, given by the equation

$$
\begin{aligned}
&\lambda(a_2 - a_1w_1w_2 + a_2w_1 - \lambda)x_1^2\\
&+(a_3w_1 - \lambda)(a_1w_1w_2 - a_2 + \lambda)x_2^2\\
&+(a_2 - \lambda)(a_1w_1w_2 - a_3w_1 + \lambda)x_3^2\\
&-(a_1^2w_1^2w_2^2 + 2a_2a_3w_1 - a_3a_1w_1^2w_2 - a_1a_2w_1w_2\\
&\qquad + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_2x_3\\
&-(a_2^2 + a_2a_3w_1 - a_1a_2w_1w_2 + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_3x_1\\
&-(a_3^2w_1^2 + a_2a_3w_1 - a_3a_1w_1^2w_2 + 2(a_1w_1w_2 - a_2 - a_3w_1)\lambda + 2\lambda^2)x_1x_2\\
&=0.
\end{aligned}
\tag{3}
$$

This equation may also be written in the form

$$
\begin{aligned}
&\lambda(a_2 - a_1w_1w_2 + a_3w_1 - \lambda)(x_1 + x_2 + x_3)^2\\
&+a_3w_1(a_1w_1w_2 - a_2)x_2^2 + a_2w_1(a_1w_2 - a_3)x_3^2\\
&-(a_1^2w_1^2w_2^2 + 2a_2a_3w_1 - a_3a_1w_1^2w_2 - a_1a_2w_1w_2)x_2x_3\\
&-a_2(a_2 - a_1w_1w_2 + a_3w_1)x_3x_1\\
&-a_3w_1(a_2 - a_1w_1w_2 + a_3w_1)x_1x_2\\
&=0.
\end{aligned}
\tag{4}
$$

As λ varies, (3) or (4) represents an infinite family of conics. However, λ appears only when multiplied by $(x_1 + x_2 + x_3)^2$, so it has no effect at infinity, where $x_1 + x_2 + x_3 = 0$. Hence all conics in the system are concurrent at infinity. If they have two real points there, they are hyperbolas with respectively parallel asymptotes. This is not sufficient to make them all homothetic to each other, but it will be shown later that this is indeed the case. If the two points at infinity coincide, all of the conics are tangent to the line at infinity at that point. Therefore they are parabolas with parallel axes, forming a homethetic set. Finally, if the points at infinity are imaginary, the conics are ellipses and their asymptotes are imaginary. As in the hyperbolic case, any two conics have respectively parallel asymptotes and are homothetic to each other.

2.8. The center of (3) may be calculated by the method of [1, p.234], bearing in mind the fact that the author uses trilinear coordinates instead of barycentric. But it is easily shown that the addition of any multiple of $(x_1 + x_2 + x_3)^2$ to the equation of a conic has no effect on its center. Therefore (4) shows that the expression containing λ may be ignored, leaving all conics with the same center. Moreover, this center has already been found, because the conic tangent to the sidelines at $P = S$, $Q = T$, $R = U$ is a special member of (3), obtained when λ has the value $\frac{1}{2}(a_2 - a_1w_1w_2 + a_3w_1)$. Thus, the common center of all the members of (3) is $(a_1w_1w_2 : a_2 : a_3w_1)$; and if they are homothetic, any one of them may be obtained from another by a dilatation about this point. (See Figures 2, 3, 4).

Since the locus of centers is not a line, this system differs from those of Tucker and Third and may be regarded as degenerate in the context of the general thery. One case worthy of mention is that in which the sides of the hexagon are isocelizers, so that

$$w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 1.$$

Exceptionally this is a Third system, because every isocelizer is both parallel and antiparallel to itself. Therefore, the conics are concentric circles, the smallest real one being the incircle of the triangle (Figure 4).

Since the "center" of a parabola is at infinity, (3) consists of parabolas only when $a_1w_1w_2 + a_2 + a_3w_1 = 0$. This can happen if some of the directions are negative, which was seen earlier as a possibility.

2.9. Some perspectivities will now be mentioned. If

$$PU \cap QR = B_1, \quad ST \cap PU = B_2, \quad QR \cap ST = B_3,$$

the three lines A_iB_i are concurrent at (2) for every value of λ. Likewise, if

$$RS \cap TU = C_1, \quad PQ \cap RS = C_2, \quad TU \cap PQ = C_3,$$

the lines A_iC_i also concur at (2). Thus for each i the points B_i andC_i move on a fixed line through A_i.

2.10. Before consideration of the general case it may be noted that whenever the directions w_1, w_2, w_3 lead to a conic circumscribing hexagon $PQRSTU$ (that is, $w_1w_2w_3 = 1$), any permutation of them will do the same. Any permutation of w_1^{-1}, w_2^{-1}, w_3^{-1} will also work. Other such triples may be invented, such as $\frac{w_2}{w_3}$, $\frac{w_3}{w_1}$, $\frac{w_1}{w_2}$.

3. The general case

3.1. Using all six directions w_i, one may derive the following expressions for the lengths of segments:

$$
\begin{aligned}
A_1Q =&w_1^{-1}\lambda,\\
QA_2 =&w_1^{-1}(a_3w_1-\lambda),\\
A_2R =&w_1^{-1}w_2^{-1}(a_3w_1-\lambda),\\
RA_3 =&w_1^{-1}w_2^{-1}(a_1w_1w_2-a_3w_1+\lambda),\\
A_3S =&w_1^{-1}w_2^{-1}w_3^{-1}(a_1w_1w_2-a_3w_1+\lambda),\\
SA_1 =&w_1^{-1}w_2^{-1}w_3^{-1}(a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
A_1T =&w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
TA_2 =&w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1}(a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda).
\end{aligned}
$$

Then working clockwise from P to U to T,

$$
\begin{aligned}
A_3P =&a_2-\lambda,\\
UA_3 =&w_6(a_2-\lambda),\\
A_2U =&a_1-a_2w_6+w_6\lambda,\\
TA_2 =&w_5(a_1-a_2w_6+w_6\lambda).
\end{aligned}
$$

Equating the two expressions for TA_2 shows that, if the equality is to be independent of λ, the product $w_1w_2w_3w_4w_5w_6$ must equal 1. From this it follows that

$$w_5=\frac{a_1w_1w_2+a_2(1-w_1w_2w_3)-a_3w_1(1-w_2w_3w_4)}{a_1w_1w_2w_3w_4}. \tag{5}$$

Hence w_5 and w_6 may be expressed in terms of the other directions. Given P and the first four directions, points Q, R, S, T are determined, and the five points determine a conic. Independence of λ, used above, ensures that U is also on this conic.

Now the coordinates of the six points may be calculated:

$$
\begin{aligned}
P =&(a_2-\lambda:0:\lambda),\\
Q =&(a_3w_1-\lambda:\lambda:0),\\
R =&(0:a_1w_1w_2-a_3w_1+\lambda:a_3w_1-\lambda),\\
S =&(a_1w_1w_2-a_3w_1+\lambda:0:a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda),\\
T =&(a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda\\
&\quad :a_2w_1w_2w_3-a_1w_1w_2+a_3w_1-\lambda:0),\\
U =&(0:a_2-\lambda:a_3w_1w_2w_3w_4-a_2w_1w_2w_3+a_1w_1w_2-a_3w_1+\lambda).
\end{aligned}
$$

3.2. These points are on the conic whose equation may be written in the form

$$
\begin{aligned}
&\lambda(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - \lambda)(x_1 + x_2 + x_3)^2\\
&+a_3w_1(a_3w_1w_2w_3w_4 - a_2w_1w_2w_3 + a_1w_1w_2 - a_3w_1 + (1 - w_2w_3w_4)\lambda)x_2^2\\
&+a_2(a_1w_1w_2 - a_3w_1 + (1 - w_1w_2w_3)\lambda)x_3^2\\
&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1 - w_2w_3w_4) + a_2a_3w_1(1 + w_1w_2w_3)\\
&\qquad - a_3a_1w_1^2w_2(2 - w_2w_3w_4) - a_1a_2w_1^2w_2^2w_3 - (a_2(1 - w_1w_2w_3)\\
&\qquad + a_3w_1(1 - w_2w_3w_4))\lambda)x_2x_3\\
&-a_2(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_1w_2w_3)\lambda)x_3x_1\\
&-a_3w_1(a_2w_1w_2w_3 - a_1w_1w_2 + a_3w_1 - (1 - w_2w_3w_4)\lambda)x_1x_2\\
&=0. \qquad (6)
\end{aligned}
$$

The part of (6) not containing the factor $(x_1 + x_2 + x_3)^2$, being linear in λ, represents a pencil of conics. Each of these conics is transformed by a dilatation about its center, induced by the expression containing $(x_1 + x_2 + x_3)^2$. Thus (6) suggests a system of conics analogous to a coaxaloid system of circles. In order to establish the analogy with the Tucker circles, it will be necessary to find a dilatation which transforms every conic by the same ratio of magnification and also transforms (6) into a pencil of conics.

First, if (6) be solved simultaneously with $x_1 + x_2 + x_3 = 0$, it will be found that all terms containing λ vanish. As in the special case, all conics are concurrent at infinity, and it will be shown that all of them are homothetic to each other.

3.3. It is also expected that the centers of the conics will be on one line. When the coordinates $(y_1 : y_2 : y_3)$ of the center are calculated, the results are too long to be displayed here. Suffice it to say that each coordinate is linear in λ, showing that the locus of $(y_1 : y_2 : y_3)$ is a line. If this line is represented by $c_1x_1+c_2x_2+c_3x_3 = 0$, the coefficients, after a large common factor of degree 4 has been removed, may be written as

$$
\begin{aligned}
c_1 &=a_2a_3w_1w_3(w_1 - w_4)(a_2w_2w_3 - a_1w_2 + a_3),\\
c_2 &=a_3w_1(a_3w_3w_4 - a_2w_3 + a_1)(a_1w_1w_2(1 - y) + a_2(1 - x) - a_3w_1(1 - y)),\\
c_3 &=a_2(a_1w_1w_2 - a_3w_1 + a_2)(-a_1(1 - x) + a_2w_3(1 - x) - a_3w_1w_3(1 - y)).
\end{aligned}
$$

For brevity the products $w_1w_2w_3$ and $w_2w_3w_4$ have been represented by the letters x and y respectively.

3.4. As has been seen, addition of any multiple of $(x_1+x_2+x_3)^2$ to the equation of a conic apparently induces a dilatation of the conic about its center. What must now be done, in order to establish an analogy with the system of Tucker circles, is to select a number σ such that the addition of $\sigma(x_1 + x_2 + x_3)^2$ to (6) dilates every conic by the same ratio ρ and transforms the system of conics into a pencil with two common points besides the two at infinity.

Using a formula for the distance between two points (*e.g.*, [1, p.31]), it may be shown that a dilatation with center $(y_1 : y_2 : y_3)$ sending $(x_1 : x_2 : x_3)$ to $(\overline{x_1} : \overline{x_2} : \overline{x_3})$ with ratio ρ is expressed by $\overline{x_i} \sim y_i + kx_i$, $(i = 1, 2, 3)$, where

$$k = \frac{\pm\rho(y_1 + y_2 + y_3)}{(1 \mp \rho)(x_1 + x_2 + x_3)}$$

or

$$\rho = \frac{\pm(x_1 + x_2 + x_3)}{(y_1 + kx_1) + (y_2 + kx_2) + (y_3 + kx_3)}.$$

In particular, if the conic $\sum a_{ij}x_ix_j = 0$ is dilated about its center $(y_1 : y_2 : y_3)$ with ratio ρ, so that the new equation is

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0,$$

then

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)^2}{\sum a_{ij}y_iy_j}.$$

Here the ambiguous sign is avoided by choosing the a_{ij} so that the denominator of the fraction is positive.

Since it is required that ρ be the same for all conics in (6), it must be free of the parameter λ. For the center $(y_1 : y_2 : y_3)$ of (6), whose coordinates are linear in λ, it may be calculated that $y_1 + y_2 + y_3$ is independent of λ. As for $\sum a_{ij}y_iy_j$, let it first be noted that

$$\begin{aligned}\sum a_{ij}y_jx_i =&(a_{11}y_1 + a_{12}y_2 + a_{13}y_3)x_1\\ &+(a_{12}y_1 + a_{22}y_2 + a_{23}y_3)x_2\\ &+(a_{13}y_1 + a_{23}y_2 + a_{33}y_3)x_3.\end{aligned}$$

(By convention, $a_{ij} = a_{ji}$). Also, $\sum a_{ij}y_jx_i = 0$ is the equation of the polar line of the center with respect to the conic, but this is the line at infinity $x_1 + x_2 + x_3 = 0$. Therefore the coefficients of x_1, x_2, x_3 in the above equation are all equal, and it follows that

$$\sum a_{ij}y_iy_j = (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)(y_1 + y_2 + y_3),$$

and

$$\rho^2 = 1 + \frac{\sigma(y_1 + y_2 + y_3)}{a_{11}y_1 + a_{12}y_2 + a_{13}y_3}.$$

Since the a_{ij} are quadratic in λ, and the y_i are linear, the denominator of the fraction is at most cubic in λ. Calculation shows that

$$a_{11}y_1 + a_{12}y_2 + a_{13}y_3 = M(A\lambda^2 + B\lambda + C),$$

in which

$$\begin{aligned}
M =& a_1w_1(a_2w_2w_3 - a_1w_2 + a_3)\cdot \\
&(-a_1^2w_1w_2 + a_2^2w_3 + a_3^2w_1w_3w_4 - a_2a_3w_3(w_1 + w_4) \\
&\quad + a_3a_1w_1(1-y) - a_1a_2(1-x)), \\
A =& a_1w_1w_2 + a_2 + a_3w_1w_2w_3w_4, \\
B =& w_1(a_1^2w_1w_2^2 - a_2^2w_2w_3 - a_3^2w_1w_2w_3w_4 - 2a_2a_3 \\
&\quad - a_3a_1w_1w_2(1-y) + a_1a_2w_2(1-x)), \\
C =& a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3).
\end{aligned}$$

3.5. If the system (6) is to become a pencil of conics, the equation

$$\sum a_{ij}x_ix_j + \sigma(x_1 + x_2 + x_3)^2 = 0$$

must be linear in λ. Since λ^2 appears in (6) as $-\lambda^2(x_1+x_x+x_3)^2$, this will vanish only if the coefficient of λ^2 in σ is 1. Therefore, to eliminate λ from the fractional part of ρ^2, it follows that

$$\sigma = \lambda^2 + \frac{B}{A}\lambda + \frac{C}{A}.$$

With this value of σ, if $\sigma(x_1 + x_2 + x_3)^2$ be added to (6), the equation becomes

$$\begin{aligned}
&\lambda(-a_2a_3w_1(1-xy)(x_1+x_2+x_3)^2 + (a_1w_1w_2 + a_2 + a_3w_1y)\cdot \\
&\quad (a_3w_1(1-y)x_2^2 + a_2(1-x)x_3^2 + (a_2(1-x) + a_3w_1(1-y))x_2x_3 \\
&\quad + a_2(1-x)x_3x_1 + a_3w_1(1-y)x_1x_2)) \\
&+a_2a_3w_1^2(a_2w_2w_3 - a_1w_2 + a_3)(x_1+x_2+x_3)^2 \\
&+(a_1w_1w_2 + a_2 + a_3w_1y)(a_3w_1(a_1w_1w_2 - a_2x - a_3w_1(1-y))x_2^2 \\
&+a_2w_1(a_1w_2 - a_3)x_3^2 \\
&-(a_1^2w_1^2w_2^2 + a_3^2w_1^2(1-y) + a_2a_3w_1(1+x) \\
&\quad - a_3a_1w_1^2w_2(2-y) - a_1a_2w_1w_2x)x_2x_3 \\
&+a_2(a_1w_1w_2 - a_2x - a_3w_1)x_3x_1 \\
&+a_3w_1(a_1w_1w_2 - a_2x - a_3w_1)x_1x_2) \\
&=0. \qquad (7)
\end{aligned}$$

Since (7) is linear in λ, it represents a pencil of conics. These conics should have four points in common, of which two are known to be at infinity. In order to facilitate finding the other two points, it is noted that a pencil contains three degenerate conics, each one consisting of a line through two of the common points, and the line of the other two points. In this pencil the line at infinity and the line of the other two common points comprise one such degenerate conic. Its equation may be given by setting equal to zero the product of $x_1 + x_2 + x_3$ and a second linear factor. Since it is known that the coefficient of λ vanishes at infinity, the conic represented by $\lambda = \infty$ in (7) must be the required one. The coefficient of λ does indeed factor as follows:

$$\begin{aligned}&(x_1 + x_2 + x_3)(-a_2a_3w_1(1 - xy)x_1\\&\quad + (a_3w_1(1 - y)(a_1w_1w_2 + a_2 + a_3w_1y) - a_2a_3w_1(1 - xy))x_2\\&\quad + (a_2(1 - x)(a_1w_1w_2 + a_2 + a_3w_1y) - a_2a_3w_1(1 - xy))x_3).\end{aligned}$$

Therefore the second linear factor equated to zero must represent the line through the other two fixed points of (7).

3.6. These points may be found as the intersection of this line and any other conic in the system, for example, the conic given by $\lambda = 0$. To solve simultaneously the equations of the line and the conic, x_1 is eliminated, reducing the calculation to

$$a_3^2w_1w_2w_3w_4x_2^2 - a_2a_3(1 + w_1w_2^2w_3^2w_4)x_2x_3 + a_2^2w_2w_3x_3^2 = 0$$

or

$$(a_3x_2 - a_2w_2w_3x_3)(a_3w_1w_2w_3w_4x_2 - a_2x_3) = 0.$$

Therefore,

$$\frac{x_2}{x_3} = \frac{a_2w_2w_3}{a_3} \quad \text{or} \quad \frac{a_2}{a_3w_1w_2w_3w_4}.$$

The first solution gives the point

$$\Lambda = (a_1w_2w_3w_4w_5 : a_2w_2w_3 : a_3)$$

and the second solution gives

$$\Lambda' = (a_1w_1w_2 : a_2 : a_3w_1w_2w_3w_4).$$

Thus the dilatation of every conic of (6) about its center with ratio ρ transforms (6) into pencil (7) with common points Λ and Λ'.

3.7. Returning to the question of whether the conics of (6) are all homothetic to each other, this was settled in the case of parabolas. As for hyperbolas, it was found that they all have respectively parallel asymptotes, but a hyperbola could be enclosed in the acute sectors formed by the asymptotes, or in the obtuse sectors. However, when (6) is transformed to (7), there are at least two hyperbolas in the pencil that are homothetic. Since the equation of any hyperbola in the pencil may be expressed as a linear combination of the equations of these two homothetic ones, it follows that all hyperbolas in the pencil, and therefore in system (6), are homothetic to each other. A similar argument shows that, if (6) consists of ellipses, they must all be homothetic. Figure 5 shows a system (6) of ellipses, with one hexagon left in place. In Figure 6 the same system has been transformed into a pencil with two common points. Figure 7 shows two hyperbolas of a system (6), together with their hexagons. The related pencil is not shown.

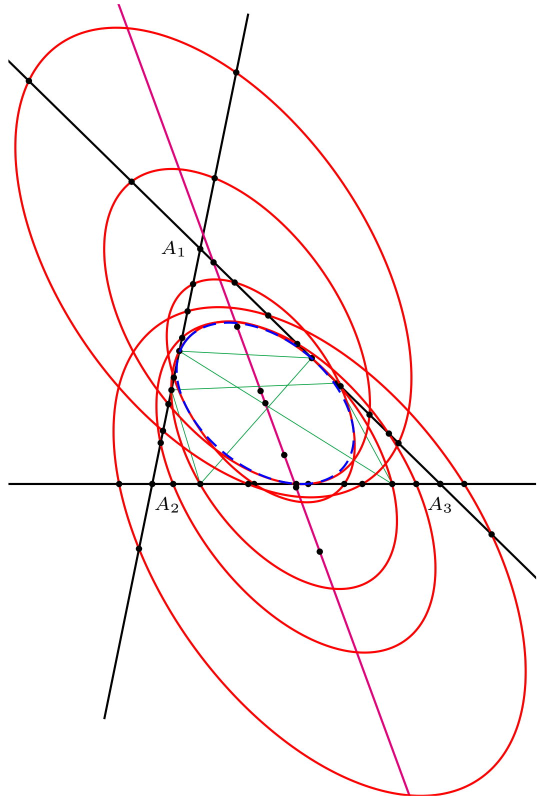

Figure 5

3.8. In the barycentric coordinate system, the midpoint $(v_1 : v_2 : v_3)$ of $(x_1 : x_2 : x_3)$ and $(y_1 : y_2 : y_3)$ is given by

$$v_i \sim \frac{x_i}{x_1 + x_2 + x_3} + \frac{y_i}{y_1 + y_2 + y_3}, \quad i = 1, 2, 3.$$

Thus it may be shown that the coordinates of the midpoint of $\Lambda\Lambda'$ are

$$(2a_1w_1w_2 + a_2(1 - x) - a_3w_1(1 - y) : a_2(1 + x) : a_3w_1(1 + y)).$$

This point is on the line of centers of (6), expressed earlier as

$$c_1x_1 + c_2x_2 + c_3x_3 = 0,$$

so the segment $\Lambda\Lambda'$ is bisected by the line of centers. However, it is not the perpendicular bisector unless (6) consists of circles. This case has already been disposed of, because if a circle cuts the sidelines of $A_1A_2A_3$, PQ and ST must be antiparallel to each other, as must QR and TU, and RS and UP. This would mean that (6) is a Third system.

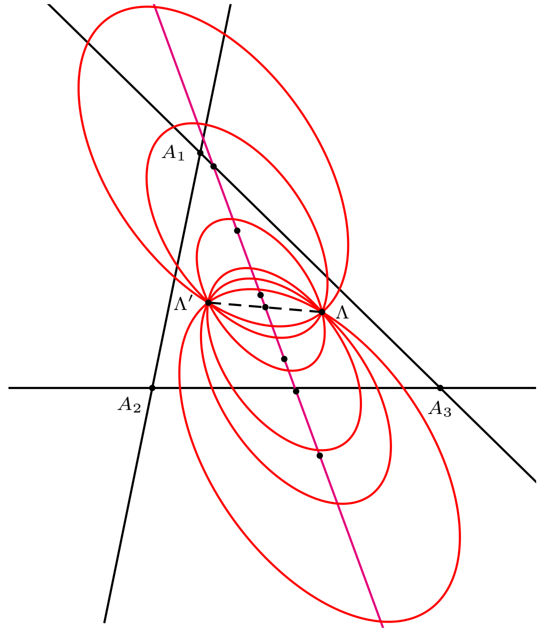

Figure 6

In system (6) the lines PQ, RS, TU are concurrent for a unique value of λ, which has been calculated but will not be written here. The point of concurrence is

$$\left(\frac{1}{-a_1w_1w_2+a_2x+a_3w_1}:\frac{1}{a_1w_1w_2-a_2x+a_3w_1}:\frac{1}{a_1w_1w_2+a_2x-a_3w_1}\right),$$

which is a generalization of (2). The same point is obtained when QR, ST. UP are concurrent. It will also be written as $\left(\frac{1}{F_1}:\frac{1}{F_2}:\frac{1}{F_3}\right)$.

3.9. System (6) has an envelope which may be found by writing (6) as a quadratic equation in λ. Setting its discriminant equal to zero gives an equation of the envelope. The discriminant contains the factor $(x_1+x_2+x_3)^2$, which may be deleted, leaving

$$\sum F_i^2x_i^2-2F_jF_kx_jx_k=0.$$

This is an equation of the conic which touches the sidelines of $A_1A_2A_3$ at $L_1=(0:F_3:F_2)$, $L_2=(F_3:0:F_1)$, and $L_3=(F_2:F_1:0)$. The cevians A_iL_i are concurrent at $\left(\frac{1}{F_1}:\frac{1}{F_2}:\frac{1}{F_3}\right)$.

The center of the envelope is the midpoint of $\Lambda\Lambda'$, but Λ and Λ' are not foci unless they are isogonal conjugates. This happens when $(w_1w_2)(w_2w_3w_4w_5)=(1)(w_2w_3)=(w_1w_2w_3w_4)(1)$, for which the solution is

$$w_1w_4=w_2w_5=w_3w_6=1.$$

Since this defines a Third system, it follows that Λ and Λ' are isogonal conjugates (and foci of the envelope of (6)) if and only if the conics are circles.

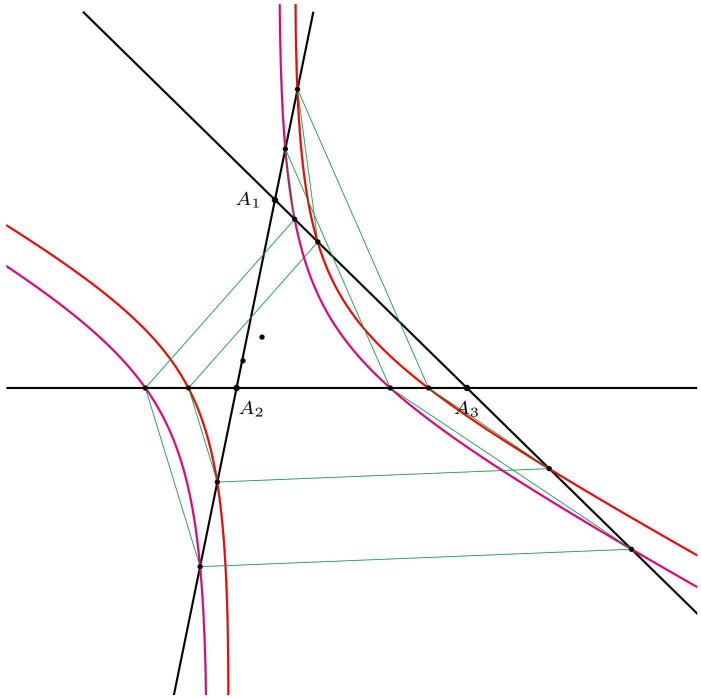

Figure 7

4. The parabolic case

There remains the question of whether (6) can be a system of parabolas. This is because the dilatations used above were made from the centers of the conics, whereas the centers of parabolas may be regarded as being at infinity. If the theory still holds true, the dilatations would have to be translations. That such cases actually exist may be demonstrated by the following example.

Let the triangle have sides $a_1 = 4$, $a_2 = 2$, $a_3 = 3$, and let

$$w_1 = \frac{2}{3}, \quad w_2 = \frac{3}{4}, \quad w_3 = \frac{1}{2}, \quad w_4 = \frac{2}{3}, \quad w_5 = 3, \quad w_6 = 2.$$

Substitution of these values in (6) gives the equation (after multiplication by 2)

$$\begin{aligned} &\lambda(1-2\lambda)(x_1+x_2+x_3)^2 + 3\lambda x_2^2 + 3\lambda x_3^2 \\ &+2(3\lambda-4)x_2x_3 + (3\lambda-2)x_3x_1 + (3\lambda-2)x_1x_2 = 0. \end{aligned} \tag{8}$$

To verify that this is a system of parabolas, solve (8) simultaneously with $x_1 + x_2 + x_3 = 0$, and elimination of x_1 gives the double solution $x_2 = x_3$. This shows that for every λ the conic is tangent to the line at infinity at the point $(-2 : 1 : 1)$. Hence, every nondegenerate conic in the system is a parabola, and all are homothetic to each other. (See Figure 8).

The formulae for σ gives the value $\left(\lambda - \frac{2}{3}\right)^2$, but (8) was obtained after multiplication by 2. Therefore $2\left(\lambda - \frac{2}{3}\right)^2 (x_1 + x_2 + x_3)^2$ is added to (8), yielding the

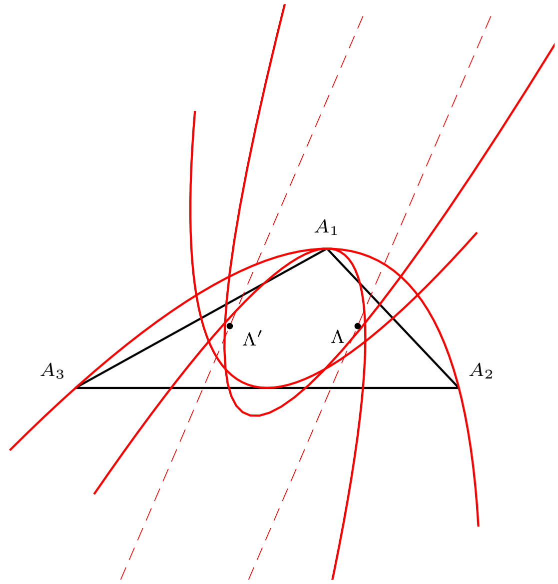

Figure 8. A system of parabolas

equation

$$(8-15\lambda)x_1^2+4(2+3\lambda)(x_2^2+x_3^2)-8(7-3\lambda)x_2x_3-(2+3\lambda)(x_3x_1+x_1x_2)=0,$$

which is linear in λ and represents a pencil of parabolas. The parabola $\lambda=\infty$ is found by using only terms containing λ, which gives the equation

$$-15x_1^2+12x_2^2+12x_3^2+24x_2x_3-3x_3x_1-3x_1x_2=0$$

or

$$3(x_1+x_2+x_3)(-5x_1+4x_2+4x_3)=0.$$

Thus it is the degenerate conic consisting of the line at infinity and the line $-5x_1+4x_2+4x_3=0$. Calculation shows that this line intersects every parabola of the pencil at $\Lambda(4:1:4)$ and $\Lambda'(4:4:1)$. (See Figure 9). The parallel dashed lines in both figures form the degenerate parabola $\lambda=\frac{2}{3}$, which is invariant under the translation which transformed the system into a pencil.

Finally, since all of the "centers" coincide, this is another exception to the rule that the line of centers of (6) bisects the segment $\Lambda\Lambda'$.

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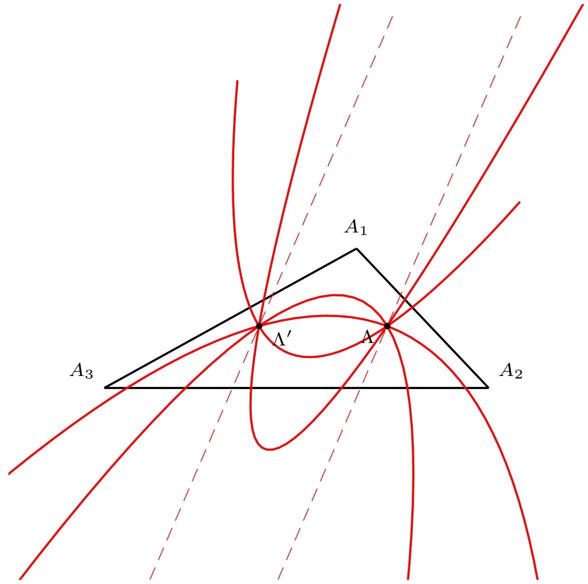

Figure 9. A pencil of parabolas

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A Conic Through Six Triangle Centers

Lawrence S. Evans

Abstract. We show that there is a conic through the two Fermat points, the two Napoleon points, and the two isodynamic points of a triangle.

1. Introduction

It is always interesting when several significant triangle points lie on some sort of familiar curve. One recently found example is June Lester's circle, which passes through the circumcenter, nine-point center, and inner and outer Fermat (isogonic) points. See [8], also [6]. The purpose of this note is to demonstrate that there is a conic, apparently not previously known, which passes through six classical triangle centers.

Clark Kimberling's book [6] lists 400 centers and innumerable collineations among them as well as many conic sections and cubic curves passing through them. The list of centers has been vastly expanded and is now accessible on the internet [7]. Kimberling's definition of triangle center involves trilinear coordinates, and a full explanation would take us far afield. It is discussed both in his book and journal publications, which are readily available [4, 5, 6, 7]. Definitions of the Fermat (isogonic) points, isodynamic points, and Napoleon points, while generally known, are also found in the same references. For an easy construction of centers used in this note, we refer the reader to Evans [3]. Here we shall only require knowledge of certain collinearities involving these points. When points $X, Y, Z, \dots$ are collinear we write $\mathcal{L}(X, Y, Z, \dots)$ to indicate this and to denote their common line.

2. A conic through six centers

Theorem 1. *The inner and outer Fermat, isodynamic, and Napoleon points lie on a conic section.*

Proof. Let O denote the circumcenter of a triangle, H its orthocenter, and G its centroid. Denote the inner Fermat point by F_+, the inner isodynamic point by J_+, and the inner Napoleon point by N_+. Similarly denote the outer Fermat, isodynamic, and Napoleon points by F_-, J_-, and N_-.

Consider the hexagon whose vertices are F_+, N_+, J_+, F_-, N_-, and J_-. Kimberling lists many collineations of triangle centers which are readily verified when

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the centers are given in homogeneous trilinear coordinates. Within the list are these collinearities involving the sides of the hexagon and classical centers on the Euler line: $\mathcal{L}(H, N_+, J_+)$, $\mathcal{L}(H, N_-, J_-)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(G, J_+, F_-)$, and $\mathcal{L}(G, J_-, F_+)$. These six lines pass through opposite sides of the hexagon and concur in pairs at H, O, and G. But we know that H, O, and G are collinear, lying on the Euler line. So, by the converse of Pascal's theorem there is a conic section through the six vertices of the hexagon. □

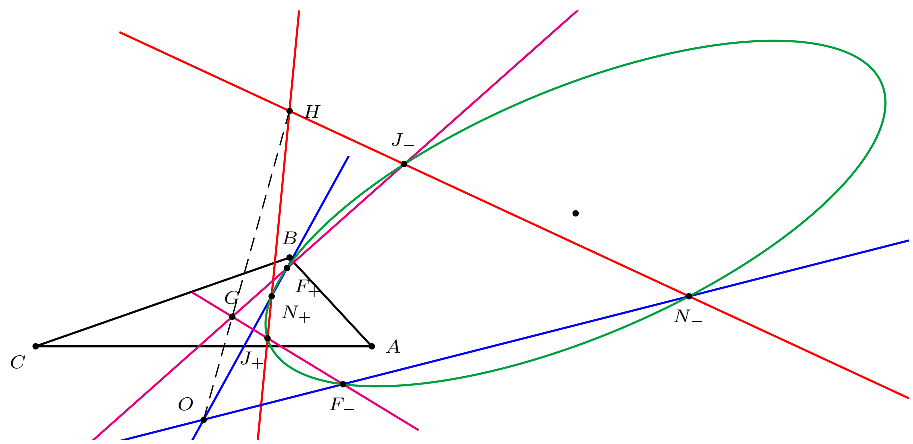

Figure 1. The conic through $F_\pm$, $N_\pm$ and $J_\pm$

Remark. In modern texts one sometimes sees Pascal's theorem stated as an "if and only if" theorem, omitting proper attribution for its converse, first proved independently by Braikenridge and by MacLaurin (See [2]). In the proof above, the Euler line plays the role of the Pascal line for the hexagon.

In Figure 1 the conic is shown as an ellipse, but it can also take the shape of a parabola or hyperbola. Since its announcement, several geometers have contributed knowledge about it. Peter Yff has calculated the equation of this conic [9], Paul Yiu has found criteria for it to be an ellipse, parabola, or a hyperbola [10],[1] and John H. Conway has generalized the conic [1].

3. Another conic

From Kimberling's list of collinearities, there is at least one more set of six points to which similar reasoning applies. We assume the reader is familiar with the concept of isogonal conjugate, fully explained in [6, 7].

Theorem 2. *The inner and outer Fermat (isogonic) and Napoleon points along with the isogonal conjugates of the Napoleon points all lie on a conic consisting of two lines intersecting at the center of the nine-point circle.*

[1]This conic is an ellipse, a parabola, or a hyperbola according as the Brocard angle is less than, equal to, or greater than $\arctan \frac{1}{3}$.

Proof. Denote the isogonal conjugates of the inner and outer Napoleon points by N_+^* and N_-^* respectively. Consider the hexagon with vertices F_+, F_-, N_+, N_-, N_+^*, and N_-^*. Kimberling lists these collinearities: $\mathcal{L}(G, N_+, N_-^*)$, $\mathcal{L}(G, N_-, N_+^*)$, $\mathcal{L}(O, F_+, N_+)$, $\mathcal{L}(O, F_-, N_-)$, $\mathcal{L}(H, F_+, N_+^*)$, $\mathcal{L}(H, F_-, N_-^*)$, so the converse of Pascal's theorem applies with the role of the Pascal line played by the Euler line, $\mathcal{L}(O, G, H)$. The conic is degenerate, consisting of two lines $L(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$, meeting at the nine-point center N_p . □

Second proof. The two collinearities $\mathcal{L}(F_-, N_+, N_+^*, N_p)$ and $\mathcal{L}(F_+, N_-, N_-^*, N_p)$ are in Kimberling's list, which *a fortiori* says that the six points in question lie on the degenerate conic consisting of the two lines. See Figure 2.

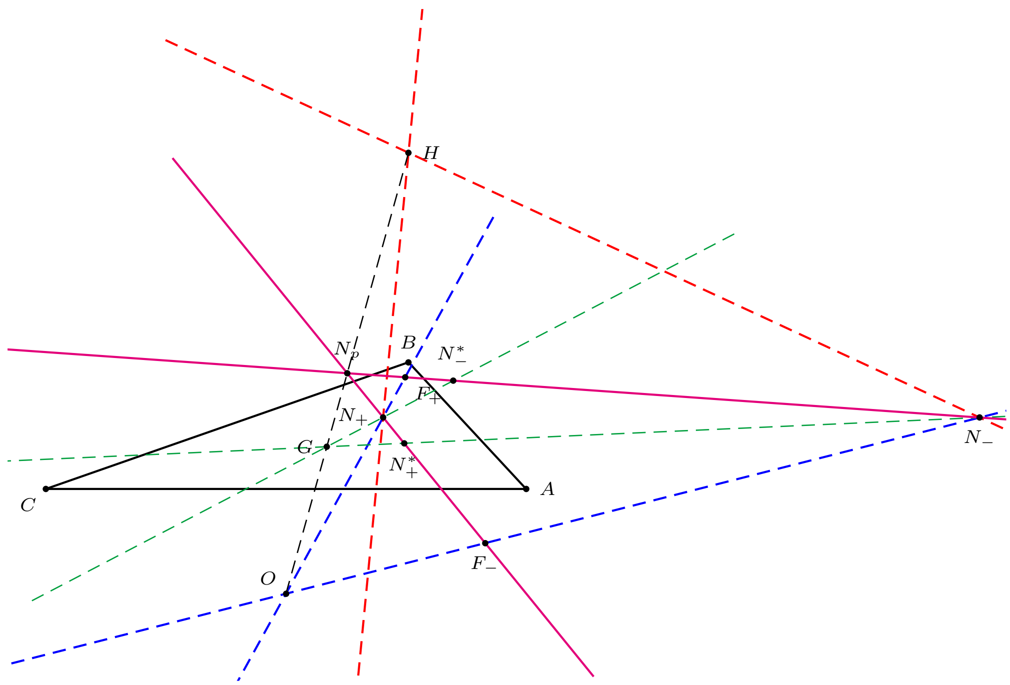

Figure 2. The degenerate conic through $F_\pm$, $N_\pm$ and $N_\pm^*$

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Paper-folding and Euler's Theorem Revisited

Benedetto Scimemi

Abstract. Given three points O, G, I, we give a simple construction by paper-folding for a triangle having these points as circumcenter, centroid, and incenter. If two further points H and N are defined by $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$, we prove that this procedure is successful if and only if I lies inside the circle on GH as diameter and differs from N. This locus for I is also independently derived from a famous paper of Euler, by complementing his calculations and properly discussing the reality of the roots of an algebraic equation of degree 3.

1. Introduction

The so-called *Modern Geometry of the Triangle* can be said to have been founded by Leonhard Euler in 1765, when his article [2] entitled *Easy Solution to some Very Difficult Geometrical Problems* was published in St. Petersburg. In this famous paper the distances between the main notable points of the triangle (centroid G, circumcenter O, orthocenter H, incenter I) are calculated in terms of the side lengths, so that several relationships regarding their mutual positions can be established. Among Euler's results, two have become very popular and officially bear his name: the vector equation $\mathbf{OH} = 3\mathbf{OG}$, implying the collinearity of G, O, H on the Euler line, and the scalar equation $OI^2 = R(R - 2r)$ involving the radii of the circumcircle and the incircle. Less attention has been given to the last part of the paper, though it deals with the problem Euler seems most proud to have solved in a very convenient [1] way, namely, the "determination of the triangle" from its points O, G, H, I. If one wants to avoid the "tedious calculations" which had previously prevented many geometricians from success, says Euler in his introduction, "everything comes down to choosing proper quantities". This understatement hides Euler's masterly use of symmetric polynomials, for which he adopts a cleverly chosen basis and performs complicated algebraic manipulations.

A modern reader, while admiring Euler's far-sightedness and skills, may dare add a few critical comments:

(1) Euler's §31 is inspired by the correct intuition that, given O, G, H, the location of I cannot be free. In fact he establishes the proper algebraic conditions but does not tell what they geometrically imply, namely that I must always lie inside the circle on GH as diameter. Also, a trivial mistake

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[1]Latin: commodissime.

leads Euler to a false conclusion; his late editor's formal correction[2] does not lead any further.

(2) As for the determination of the triangle, Euler reduces the problem of finding the side lengths to solving an algebraic equation of degree 3. However, no attention is given to the crucial requirements that the three roots - in order to be side lengths - be real positive and the triangle inequalities hold. On the other hand, Euler's equation clearly suggests to a modern reader that the problem cannot be solved by ruler and compass.

(3) In Euler's words (§20) the main problem is described as follows: *Given the positions of the four points ..., to construct the triangle*. But finding the side lengths does not imply determining the location of the triangle, given that of its notable points. The word *construct* also seems improperly used, as this term's traditional meaning does not include solving an algebraic equation. It should rather refer, if not to ruler and compass, to some alternative geometrical techniques.

The problem of the locus of the incenter (and the excenters) has been independently settled by Andrew P. Guinand in 1982, who proved in his nice paper [5] that I must lie inside the critical circle on GH as diameter[3] (Theorem 1) and, conversely, any point inside this circle - with a single exception - is eligible for I (Theorem 4). In his introduction, Guinand does mention Euler's paper, but he must have overlooked its final section, as he claims that in all previous researches "the triangle was regarded as given and the properties of the centers were investigated" while in his approach "the process is reversed".

In this paper we give an alternative treatment of Euler's problem, which is independent both of Euler's and Guinand's arguments. Euler's crucial equation, as we said, involves the side lengths, while Guinand discusses the cosines of the angles. We deal, instead, with the coefficients for equations of the sides. But an independent interest in our approach may be found in the role played by the Euler point of the triangle, a less familiar notable point.[4] Its properties are particularly suitable for reflections and suggest a most natural paper-folding reconstruction procedure. Thus, while the first part (locus) of the following theorem is well-known, the construction mentioned in the last statement is new:

Theorem 1. *Let O, G, I be three distinct points. Define two more points H, N on the line OG by letting $\mathbf{OH} = 3\mathbf{OG} = 2\mathbf{ON}$. Then there exists a nondegenerate, nonequilateral triangle $\mathcal{T}$ with centroid G, circumcenter O, orthocenter H, and incenter I, if and only if I lies inside the circle on GH as diameter and differs from N. In this case the triangle $\mathcal{T}$ is unique and can be reconstructed by paper-folding, starting with the points O, G, I.*

[2] A. Speiser in [2, p.155, footnote].

[3] This is also known as the orthocentroidal circle. See [7]. This term is also used by Varilly in [9]. The author thanks the referee for pointing out this paper also treats this subject.

[4] This point is the focus of the Kiepert parabola, indexed as X_{110} in [7], where the notable points of a triangle are called triangle centers.

We shall find the sides of the triangle sides as proper creases, *i.e.*, reflecting lines, which simultaneously superimpose two given points onto two given lines. This can be seen as constructing the common tangents to two parabolas, whose foci and directrices are given. Indeed, the extra power of paper-folding, with respect to ruler-and-compass, consists in the feasibility of constructing such lines. See [4, 8].

The reconstruction of a triangle from three of its points (e.g. one vertex, the foot of an altitude and the centroid G) is the subject of an article of William Wernick [10], who in 1982 listed 139 triplets, among which 41 corresponded to problems still unsolved. Our procedure solves items 73, 80, and 121 of the list, which are obviously equivalent. [5] It would not be difficult to make slight changes in our arguments in order to deal with one of the excenters in the role of the incenter I.

As far as we know, paper-folding, which has been successfully applied to trisecting an angle and constructing regular polygons, has never yet produced any significant contribution to the geometry of the triangle.

This paper is structured as follows: in §2 we reformulate the well-known properties of the Simson line of a triangle in terms of side reflections and apply them to paper-folding. In §3 we introduce the Euler point E and study its properties. The relative positions of E, O, G are described by analytic geometry. This enables us to establish the locus of E and a necessary and sufficient condition for the existence of the triangle. [6] An immediate paper-folding construction of the triangle from E, O, G is then illustrated. In §4 we use complex variables to relate points E and I. In §5 a detailed ruler-and-compass construction of E from I, O, G is described.[7] The expected incenter locus is proved in §6 by reducing the problem to the former results on E, so that the proof of Theorem 1 is complete. In §7 we take up Euler's standpoint and interpret his formulas to find once more the critical circle locus as a necessary condition. Finally, we discuss the discriminant of Euler's equation and complete his arguments by supplying the missing algebraic calculations which imply sufficiency. Thus a third, independent, proof of the first part of Theorem 1 is achieved.

2. Simson lines and reflections

In this section we shall reformulate well-known results on the Simson line in terms of reflections, so that applications to paper-folding constructions will be natural. The following formulation was suggested by a paper of Longuet-Higgins [6].

Theorem 2. *Let H be the orthocenter, $\mathcal{C}$ the circumcircle of a triangle $\mathcal{T} = A_1A_2A_3$.*

(i) *For any point P, let P_i denote the reflection of P across the side A_jA_h of $\mathcal{T}$. (Here, i, j, h is a permutation of 1, 2, 3). Then the points P_i are collinear on a line $r = r(P)$ if and only if P lies on $\mathcal{C}$. In this case H lies on r.*

[5] Given I and two of O, G, H.

[6] Here too, as in the other approaches, the discussion amounts to evaluating the sign of a discriminant.

[7] A ruler and compass construction always entails a paper-folding construction. See [4, 8].

(ii) *For any line r, let r_i denote the reflection of r across the side A_jA_h. Then the lines r_i are concurrent at a point $P = P(r)$ if and only if H lies on r. In this case P lies on $\mathcal{C}$. When P describes an arc of angle α on $\mathcal{C}$, $r(P)$ rotates in the opposite direction around H by an angle $-\frac{\alpha}{2}$.*

All these statements are easy consequences of well-known properties of the Simson line, which is obviously parallel to $r(P)$. See, for example, [1, Theorems 2.5.1, 2.7.1,2]. This theorem defines a bijective mapping $P \mapsto r(P)$. Thus, given any line e through H, there exists a unique point E on $\mathcal{C}$ such that $r(E) = e$.

We now recall the basic assumption of paper-folding constructions, namely the possibility of determining a line, *i.e.*, folding a crease, which simultaneously reflects two given points A, B onto points which lie on two given lines a, b. It is proved in [4, 8] that this problem has either one or three solutions. We shall discuss later how these two cases can be distinguished, depending on the relative positions of the given points and lines. For the time being, we are interested in the case that three such lines (creases) are found. The following result is a direct consequence of Theorem 2.

Corollary 3. *Given two points A, B and two (nonparallel) lines a, b, assume that there exist three different lines r such that A (respectively B) is reflected across r onto a point A' (respectively B') lying on a (respectively b). These lines are the sides of a triangle $\mathcal{T}$ such that*

(i) *a and b intersect at the orthocenter H of $\mathcal{T}$;*
(ii) *A and B lie on the circumcircle of $\mathcal{T}$;*
(iii) *the directed angle $\angle AOB$ is twice the directed angle from b to a. Here, O denotes the circumcenter of $\mathcal{T}$.*

3. The Euler point

We shall now consider a notable point whose behaviour under reflections makes it especially suitable for paper-folding applications. The Euler point E is the unique point which is reflected across the three sides of the triangle onto the Euler line OG. Equivalently, the three reflections of the Euler line across the sides are concurrent at E. [8]

We first prove that for any nonequilateral, nondegenerate triangle with prescribed O and G (hence also H), the Euler point E lies outside a region whose boundary is a cardioid, a closed algebraic curve of degree 4, which is symmetric with respect to the Euler line and has the centroid G as a double-point (a cusp; see Figure 3). If we choose cartesian coordinates such that such that $G = (0, 0)$ and $O = (-1, 0)$ (so that $H = (2, 0)$), then this curve is represented by

$$(x^2 + y^2 + 2x)^2 - 4(x^2 + y^2) = 0 \quad \text{or} \quad \rho = 2(1 - \cos\theta). \tag{1}$$

Since this cardioid is uniquely determined by the choice of the two (different) points G, O, we shall call it the GO-cardioid. As said above, we want to prove that the locus of Euler point E for a triangle is the exterior of the GO-cardioid.

[8] This point can also be described as the Feuerbach point of the tangential triangle.

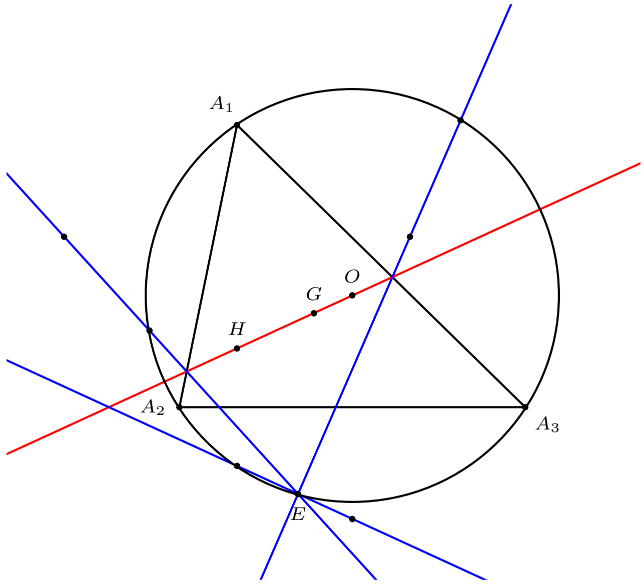

Figure 1. The Euler point of a triangle

Theorem 4. *Let G, O, E be three distinct points. Then there exists a triangle $\mathcal{T}$ whose centroid, circumcenter and Euler point are G, O, E, respectively, if and only if E lies outside the GO-cardioid. In this case the triangle $\mathcal{T}$ is unique and can be constructed by paper-folding, from the points G, O, E .*

Proof. Let us first look at isosceles (nonequilateral) triangles, which can be treated within ruler-and-compass geometry. [9] Here, by symmetry, the Euler point E lies on the Euler line; indeed, by definition, it must be one of the vertices, say $A_3 = E = (e, 0)$. Then being external to the GO-cardioid is equivalent to lying outside the segment GH_O, where $H_O = (-4, 0)$ is the symmetric of H with respect to O. Now the side A_1A_2 must reflect the orthocenter H into the point $E_O = (-2-e, 0)$, symmetric of E with respect to O, and therefore its equation is $x = -\frac{e}{2}$. This line has two intersections with the circumcircle $(x+1)^2 + y^2 = (e+1)^2$ if and only if $e(e+4) > 0$, which is precisely the condition for E to be outside GH_O. Conversely, given any two distinct points O, G, define H and H_O by $\mathbf{GH_O} = -2\mathbf{GH} = 4\mathbf{GO}$. Then for any choice of E on line OG, outside the segment GH_O, we can construct an isosceles triangle having E, O, G, H as its notable points as follows: first construct the (circum)-circle centered at O, through E, and let E_O be diametrically opposite to E. Then, under our assumptions on E, the perpendicular bisector of HE_O intersects the latter circle at two points, say A_1, A_2, and the isosceles triangle $\mathcal{T} = A_1A_2E$ fulfills our requirements.

We now deal with the nonisosceles case. Let $E = (u, v)$, $v \neq 0$ be the Euler point of a triangle $\mathcal{T}$. By definition, E is reflected across the three sides of the triangle into points E' which lie on the line $y = 0$. Now the line which reflects

[9]The case of the isosceles triangle is also studied separately by Euler in [2, §§25–29].

$E(u,v)$ onto a point $E'(t,0)$ has equation $2(u-t)x+2vy-(u^2+v^2-t^2)=0$. If the same line must also reflect (according to Theorem 2) point $E_O=(-u-2,-v)$ onto the line $x=2$ which is orthogonal to the Euler line through $H(2,0)$, then a direct calculation yields the following condition:

$$t^3-3(u^2+v^2)t+2u(u^2+v^2)-4v^2=0. \tag{2}$$

Hence we find three different reflecting lines if and only if this polynomial in t has three different real roots. The discriminant is

$$\Delta(u,v)=108v^2((u^2+v^2+2u)^2-4(u^2+v^2)).$$

Since $v\neq 0$, the inequality $\Delta(u,v)>0$ holds only if and only if E lies outside the cardioid, as we wanted.

The preceding argument can be also used for sufficiency: the assumed locus of E guarantees that (2) has three real roots. Therefore, three different lines exist which simultaneously reflect E onto line $a=OH$ and E_O onto the line b through H, perpendicular to OH. According to Corollary 3, these three lines are the sides of a triangle $\mathcal{T}$ which fulfills our requirements. In fact, H is the intersection of lines a and b and therefore H is the orthocenter of $\mathcal{T}$; a and b are perpendicular, hence E and E_O must be diametrically opposite points on the circumcircle of $\mathcal{T}$, so that their midpoint O is the circumcenter of $\mathcal{T}$. The three sides reflect E onto the x-axis, that is the Euler line of $\mathcal{T}$. Hence, by definition, E is the Euler point of $\mathcal{T}$. Since a polynomial of degree 3 cannot have more than 3 roots, the triangle is uniquely determined. $\square$

Let us summarize the procedure for the reconstruction of the sides from the points O, G, E:

(1) Construct points H and E_O such that $\mathbf{GH}=-2\mathbf{GO}$ and $\mathbf{OE_O}=-\mathbf{OE}$.
(2) Construct line a through O, H and line b through H, perpendicular to a.
(3) Construct three lines that simultaneously reflect E on to a and E_O on to b.

4. Coordinates

The preceding results regarding the Euler point E are essential in dealing with the incenter I. In fact we shall construct E from G, O and I, so that Theorem 1 will be reduced to Theorem 3. To this end, we introduce the Gauss plane and produce complex variable equations relating I and E.[10] The cartesian coordinates will be different from the one we used in §3, but this seems unavoidable if we want to simplify calculations. A point $Z=(x,y)$ will be represented by the complex number $z=x+iy$. We write $Z=z$ and sometimes indicate operations as if they were acting directly on points rather than on their coordinates. We also write $z^*=x-iy$ and $|z|^2=x^2+y^2$.

Let $A_i=a_i$ be the vertices of a nondegenerate, nonequilateral triangle $\mathcal{T}$. Without loss of generality, we can assume for the circumcenter that $O=0$ and $|a_i|=1$, so that $a_i^{-1}=a_i^*$. Now the orthocenter H and the Euler point E have the following simple expressions in terms of elementary symmetric polynomials σ_1, σ_2, σ_3.

[10] A good reference for the use of complex variables in Euclidean geometry is [3].

$$\begin{aligned} H = & a_1 + a_2 + a_3 = \sigma_1, \\ E = & \frac{a_1a_2 + a_2a_3 + a_3a_1}{a_1 + a_2 + a_3} = \frac{\sigma_2}{\sigma_1}. \end{aligned}$$

The first formula is trivial, as $G = \frac{1}{3}\sigma$ and $H = 3G$. As for E, the equation for a side, say A_1A_2, is $z + a_1a_2z^* = a_1 + a_2$, and the reflection across this line takes a point $T = t$ on to $T' = a_1 + a_2 - a_1a_2t^*$. An easy calculation shows that E' lies on the Euler line $z\sigma_1^* - z^*\sigma_1 = 0$. This holds for all sides, and this property characterizes E by Theorem 2. Notice that $\sigma_1 \neq 0$, as we have assumed $G \neq O$.[11]

We now introduce $\sigma_3 = a_1a_2a_3$, $k = |OH|$ and calculate

$$\sigma_1^* = \sigma_2\sigma_3^{-1}, \qquad |\sigma_3|^2 = 1, \quad |\sigma_1|^2 = |\sigma_2|^2 = \sigma_1\sigma_2\sigma_3^{-1} = k^2.$$

Hence,

$$\sigma_3 = \frac{\sigma_1\sigma_2}{k^2} = \left(\frac{\sigma_1}{k}\right)^2 \cdot \frac{\sigma_2}{\sigma_1} = \frac{H}{|H|} \cdot E.$$

In order to deal with the incenter I, let $B_i = b_i$ denote the (second) intersection of the circumcircle with the internal angle bisector of A_i. Then $b_i^{-1} = b_i^*$ and $b_i^2 = a_ja_k$, and $b_1b_2b_3 = -a_1a_2a_3$. Since I is the orthocenter of triangle $B_1B_2B_3$, we have, as above, $I = b_1 + b_2 + b_3$. Likewise, we define

$$\tau_1 = b_1 + b_2 + b_3, \quad \tau_2 = b_1b_2 + b_2b_3 + b_3b_1, \quad \tau_3 = b_1b_2b_3, \quad f = |OI|,$$

and calculate

$$\tau_1^* = \tau_2\tau_3^{-1}, \qquad |\tau_3|^2 = 1, \quad |\tau_1|^2 = |\tau_2|^2 = \tau_1\tau_2\tau_3^{-1} = f^2.$$

From the definition of b_i, we derive

$$\begin{aligned} \tau_2^2 = & \sigma_3(\sigma_1 - 2\tau_1), \\ \tau_3 = & - \sigma_3 = \left(\frac{\tau_3}{\tau_2}\right)^2 \cdot \frac{\tau_2^2}{\tau_3} = -\left(\frac{\tau_1}{f^2}\right)^2 (\sigma_1 - 2\tau_1). \end{aligned}$$

Equivalently,

$$\begin{aligned} \sigma_3 = -\tau_3 = \left(\frac{\tau_1}{f}\right)^2 \cdot \frac{\sigma_1 - 2\tau_1}{f^2}, \\ \left(\frac{H}{|H|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{H - 2I}{|I|^2}, \\ \left(\frac{G}{|G|}\right)^2 \cdot E = \left(\frac{I}{|I|}\right)^2 \cdot \frac{3G - 2I}{|I|^2}, \end{aligned} \tag{3}$$

where the Euler equation $H = 3G$ has been used.

[11]The triangle is equilateral if $G = O$.

5. Construction of the Euler point

The last formulas suggest easy constructions of E from O, G (or H) and I. Since $H - 2I = 3G - 2I = G - 2(I - G)$, our attention moves from $\mathcal{T}$ to its antimedial triangle $\mathcal{T}^*$ (the midpoints of whose sides are the vertices of $\mathcal{T}$) and the homothetic mapping: $Z \mapsto G - 2(Z - G)$. Thus $I^* = 3G - 2I$ is the incenter of $\mathcal{T}^*$. Note that multiplying by a *unit* complex number $\cos\theta + i\sin\theta$ is equivalent to rotating around O by an angle θ. Since $G/|G|$ and $I/|I|$ are unit complex numbers, multiplication by $\dfrac{(I/|I|)^2}{(G/|G|)^2}$ represents a rotation which is the product of two reflections, first across the line OG, then across OI. Since $|I|^2 = f^2$, dividing f^2 by I^* is equivalent to inverting I^* in the circle with center O and radius OI. Altogether, we conclude that E can be constructed from O, G and I by the following procedure. See Figure 2.

(1) Construct lines OG and OI; construct I^* by the equation $\mathbf{GI}^* = -2\mathbf{GI}$.

(2) Construct the circle Ω centered at O through I. By inverting I^* with respect to this circle, construct F^*. Note that this inversion is possible if and only if $I^* \neq O$, or, equivalently, $I \neq N$.[12]

(3) Construct E: first reflect F^* in line OG, and then its image in line OI.

Note that all these steps can be performed by ruler and compass.

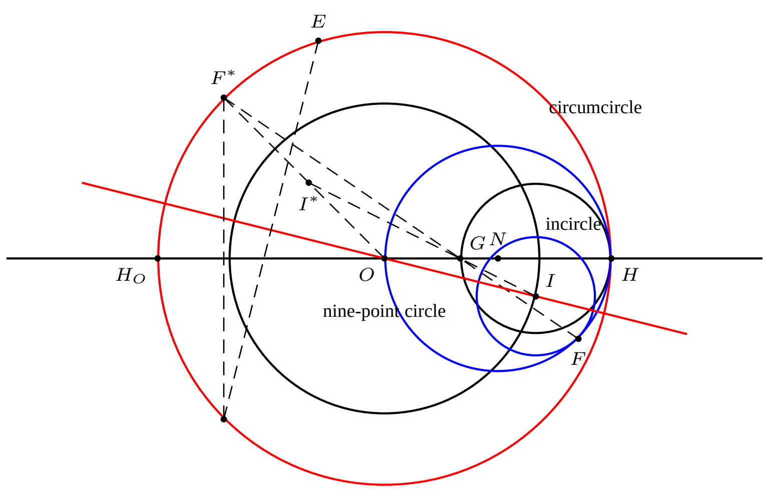

Figure 2. Construction of the Euler point from O, G, I

[12] It will appear that F^* is the Feuerbach point of $\mathcal{T}^*$. Thus, at this stage we have both the circumcircle (center O, through F^*) and the incircle of $\mathcal{T}$ (center I, through F, as defined by $\mathbf{GF}^* = -2\mathbf{GF}$).

6. The locus of incenter

As we know from §3, one can now apply paper-folding to O, G, E and produce the sides of $\mathcal{T}$. But in order to prove Theorem 1 we must show that the critical circle locus for I is equivalent to the existence of three different good creases. To this end we check that I lies inside the orthocentroidal GH-circle if and only if E lies outside the GO-cardioid. If we show that the two borders correspond under the transformation $I \mapsto E$ described by (3) for given O, G, H, then, by continuity, the two ranges, the interior of the circle and the exterior of the cardioid, will also correspond.

We first notice that the right side of (3) can be simplified when I lies on the GH-circle, as $|IO| = 2|IN| = |I^*O|$ implies that the inversion (step 2) does not affect I^*. In order to compare the transformation $I \mapsto E$ with our previous results, we must change scale and return to the cartesian coordinates used in §2, where $G = (0,0)$, $H = (2,0)$. If we set $I = (r,s)$, then $I^* = (-2r,-2s)$. The first reflection (across the Euler line) maps I^* on to $(-2r, 2s)$; the second reflection takes place across line OI: $s(x+1) - (r+1)y = 0$ and yields $E(u,v)$, where

$$(u,v) = \left(\frac{-2(r^3 - 3rs^2 + r + 2r^2 - s^2)}{(r+1)^2 + s^2}, \frac{-2s(3r^2 - s^2 + 3r)}{(r+1)^2 + s^2}\right).$$

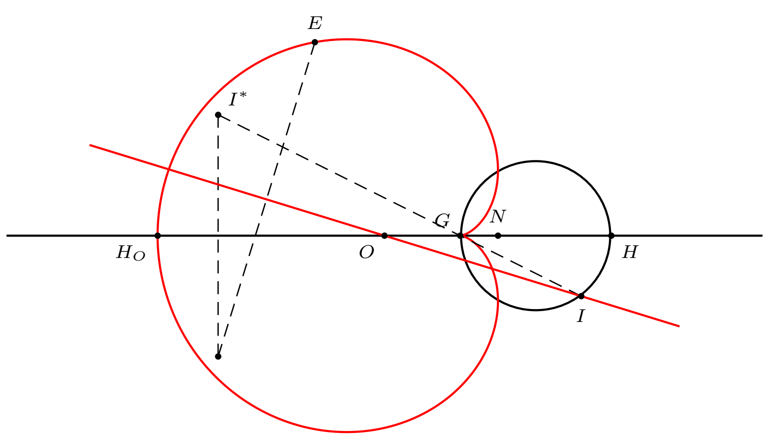

Figure 3. Construction of the GO-cardioid from the GH-circle

Notice that $I \neq O$ implies $(r+1)^2 + s^2 \neq 0$. Then, by direct calculations, we have

$$((u^2+v^2+2u)^2 - 4(u^2+v^2))((r+1)^2+s^2) = 16(r^2+s^2-2r)(r^2+s^2+r)^2$$

and conclude that $I(r,s)$ lies on the GH-circle $x^2+y^2-2x=0$ whenever E lies on the GO-cardioid (1), as we wanted. Thus the proof of Theorem 1 is complete.

7. Euler's theorem revisited

We shall now give a different proof of the first part of Theorem 1 by exploiting Euler's original ideas and complementing his calculations.

Necessity. In [2] Euler begins (§§1-20) with a nonequilateral, nondegenerate triangle and calculates the "notable" lengths

$$|HI| = e, \quad |OI| = f, \quad |OG| = g, \quad |GI| = h, \quad |HO| = k$$

as functions of the side lengths a_1, a_2, a_3. From those expressions he derives a number of algebraic equalities and inequalities, whose geometrical interpretations he only partially studies. [13] In particular, in §31, by observing that some of his quantities can only assume positive values, Euler explicitly states that the two inequalities

$$k^2 < 2e^2 + 2f^2, \tag{4}$$

$$k^2 > 2e^2 + f^2 \tag{5}$$

must hold. However, rather than studying their individual geometrical meaning, he tries to combine them and wrongly concludes, owing to a trivial mistake, that the inequalities $19f^2 > 8e^2$ and $13f^2 < 19e^2$ are also necessary conditions. Speiser's correction of Euler's mistake [2, p.155, footnote] does not produce any interesting result. On the other hand, if one uses the main result $\mathbf{OH} = 3\mathbf{OG}$, defines the nine-point center N (by letting $\mathbf{OH} = 2\mathbf{ON}$) and applies elementary geometry (Carnot's and Apollonius's theorems), it is very easy to check that the two original inequalities (4) and (5) are respectively equivalent to

(4′) I is different from N, and
(5′) I lies inside the GH-circle.

These are precisely the conditions of Theorem 1. It is noteworthy that Euler, unlike Guinand, could not use Feuerbach's theorem.

Sufficiency. In §21 Euler begins with three positive numbers f, g, h and derives a real polynomial of degree 3, whose roots a_1, a_2, a_3 - in case they are sides of a triangle - produce indeed f, g, h for the notable distances. It remains to prove that, under the assumptions of Theorem 1, these roots are real positive and satisfy the triangle inequalities. In order to complete Euler's work, we need a couple of lemmas involving symmetric polynomials.

Lemma 5. *(a) Three real numbers a_1, a_2, a_3 are positive if and only if $\sigma_1 = a_1 + a_2 + a_3$, $\sigma_2 = a_1a_2 + a_2a_3 + a_3a_1$ and $\sigma_3 = a_1a_2a_3$ are all positive.*

[13] The famous result on the collinearity of O, G, H and the equation $\mathbf{OH} = 3\mathbf{OG}$ are explicitly described in [2]. The other famous formula $OI^2 = R(R - 2r)$ is not explicitly given, but can be immediately derived, by applying the well known formulas for the triangle area $\frac{1}{2}r(a_1 + a_2 + a_3) = \frac{a_1a_2a_3}{4R}$.

(b) Three positive real numbers a_1, a_2, a_3 satisfy the triangle inequalities $a_1+a_2 \geq a_3$, $a_2+a_3 \geq a_1$ and $a_3+a_1 \geq a_2$ if and only if [14]

$$\tau(a_1,a_2,a_3) = (a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3) \geq 0.$$

Now suppose we are given three different points I, O, N and define two more points G, H by $3\mathbf{OG} = 2\mathbf{ON} = \mathbf{OH}$. Assume that I is inside the GH-circle. If we let

$$m = |ON|, \quad n = |IN|, \quad f = |OI|,$$

then we have $m > 0$, $n > 0$, $f > 0$, $n+f-m \geq 0$, and also, according to Lemma 5(b), $\tau(f,m,n) \geq 0$. Moreover, the assumed locus of I within the critical circle implies, by Apollonius, $f - 2n > 0$ so that $f^2 - 4n^2 = b^2$ for some real $b > 0$. We now introduce the same quantities p, q, r of Euler,[15] but rewrite their defining relations in terms of the new variables m, n, f as follows:

$$\begin{aligned} n^2 r =& f^4, \\ 4n^2 q =& b^2 f^2, \\ 9h^2 =& (f-2n)^2 + 2((n+f)^2 - m^2), \\ 4n^2 p =& 27b^4 + 128n^2 b^2 + 144h^2 n^2. \end{aligned}$$

Notice that, under our assumptions, all these functions assume positive values, so that we can define three more positive quantities[16]

$$\sigma_1 = \sqrt{p}, \quad \sigma_2 = \frac{p}{4} + 2q + \frac{q^2}{r}, \quad \sigma_3 = q\sqrt{p}.$$

Now let a_1, a_2, a_3 be the (complex) roots of the polynomial $x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$. The crucial point regards the discriminant

$$\begin{aligned} \Delta(a_1,a_2,a_3) =& (a_1-a_2)^2(a_2-a_3)^2(a_3-a_1)^2 \\ =& \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2. \end{aligned}$$

By a tedious but straightforward calculation, involving a polynomial of degree 8 in m, n, f, one finds

$$n^2\Delta(a_1,a_2,a_3) = b^4\tau(f,m,n).$$

Since, by assumption, $n \neq 0$, this implies $\Delta(a_1,a_2,a_3) \geq 0$, so that a_1, a_2, a_3 are real. By Lemma 5(a), since σ_1, σ_2, $\sigma_3 > 0$, we also have a_1, a_2, $a_3 > 0$. A final calculation yields $\tau(a_1,a_2,a_3) = \dfrac{4pq^2}{r} > 0$, ensuring, by Lemma 5(b) again, that the triangle inequalities hold. Therefore, under our assumptions, there exists a triangle with a_1, a_2, a_3 as sides lengths, which is clearly nondegenerate and nonequilateral, and whose notable distances are f, m, n. Thus the alternative proof the first part of Theorem 1 is complete. Of course, the last statement on

[14] The expression $\tau(a_1,a_2,a_3)$ appears under the square root in Heron's formula for the area of a triangle.

[15] These quantities read P, Q, R in [2, p.149].

[16] These quantities read p, q, r in [2, p.144].

construction is missing: the actual location of the triangle, in terms of the location of its notable points, cannot be studied by this approach.

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Loci Related to Variable Flanks

Zvonko Čerin

Abstract. Let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles built on the sides of a triangle ABC such that the oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$, $\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. We explore relationships among the central points of triangles ABC, AR_4R_5, BR_6R_1, and CR_2R_3. Our results extend recent results by Hoehn, van Lamoen, C. Pranesachar and Venkatachala who considered the case when $\lambda = 1$ (with squares erected on sides).

1. Introduction

In recent papers (see [2], [5], and [6]), L. Hoehn, F. van Lamoen, and C. R. Pranesachar and B. J. Venkatachala have considered the classical geometric configuration with squares BS_1S_2C, CS_3S_4A, and AS_5S_6B erected on the sides of a triangle ABC and studied relationships among the central points (see [3]) of the base triangle $\tau = ABC$ and of three interesting triangles $\tau_A = AS_4S_5$, $\tau_B = BS_6S_1$, $\tau_C = CS_2S_3$ (called *flanks* in [5] and *extriangles* in [2]). In order to describe their main results, recall that triangles ABC and XYZ are *homologic* provided that the lines AX, BY, and CZ are concurrent. The point P in which they concur is their *homology center* and the line ℓ containing the intersections of the pairs of lines $(BC,\,YZ)$, $(CA,\,ZX)$, and $(AB,\,XY)$ is their *homology axis*. In this situation we use the notation $ABC \underset{\ell}{\overset{P}{\bowtie}} XYZ$, where ℓ or both ℓ and P may be omitted. Let $X_i = \underline{X}_i(\tau)$, $X_i^j = \underline{X}_i(\tau_j)$ (for $j = A,\,B,\,C$), and $\sigma_i = X_i^A X_i^B X_i^C$, where $\underline{X}_i$ (for $i = 1, \ldots$) is any of the triangle central point functions from Kimberling's lists [3] or [4].

Instead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*. Also, it is customary to use letters I, G, O, H, F, K, and L instead of X_1, X_2, X_3, X_4, X_5, X_6, and X_{20} to denote the incenter, the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian (or Grebe-Lemoine) point, and the de Longchamps point (the reflection of H about O), respectively.

In [2] Hoehn proved $\tau \bowtie \sigma_3$ and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\,j) = (1,\,1),\,(2,\,4),\,(4,\,2)$. In [6] C. R. Pranesachar and B. J. Venkatachala added some new results because they showed that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\,j) = (1,\,1),\,(2,\,4),\,(4,\,2),\,(3,\,6),\,(6,\,3)$. Moreover,

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they observed that if $\tau \overset{X}{\bowtie} X_A X_B X_C$, and Y, Y_A, Y_B, and Y_C are the isogonal conjugates of points X, X_A, X_B, and X_C with respect to triangles τ, τ_A, τ_B, and τ_C respectively, then $\tau \overset{Y}{\bowtie} Y_A Y_B Y_C$. Finally, they also answered negatively the question by Prakash Mulabagal of Pune if $\tau \bowtie XYZ$, where X, Y, and Z are the points of contact of the incircles of triangles τ_A, τ_B, and τ_C with the sides opposite to A, B, and C, respectively.

In [5] van Lamoen said that X_i *befriends* X_j when $\tau \overset{X_j}{\bowtie} \sigma_i$ and showed first that $\tau \overset{X_j}{\bowtie} \sigma_i$ implies $\tau \overset{X_n}{\bowtie} \sigma_m$ where X_m and X_n are the isogonal conjugates of X_i and X_j. Also, he proved that $\tau \overset{X_j}{\bowtie} \sigma_i$ is equivalent to $\tau \overset{X_i}{\bowtie} \sigma_j$, and that $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (1,\, 1),\, (2,\, 4),\, (3,\, 6),\, (4, 2),\, (6,\, 3)$. Then he noted that $\tau \overset{K(\frac{\pi}{2}-\phi)}{\bowtie} K(\phi)$, where $K(\phi)$ denotes the homology center of τ and the Kiepert triangle formed by apexes of similar isosceles triangles with the base angle ϕ erected on the sides of ABC. This result implies that $\tau \overset{X_i}{\bowtie} \sigma_i$ for $i = 485,\, 486$ (Vecten points – for $\phi = \pm\frac{\pi}{4}$), and $\tau \overset{X_j}{\bowtie} \sigma_i$ for $(i,\, j) = (13,\, 17),\, (14,\, 18)$ (isogonic or Fermat points X_{13} and X_{14} – for $\phi = \pm\frac{\pi}{3}$, and Napoleon points X_{17} and X_{18} – for $\phi = \pm\frac{\pi}{6}$). Finally, van Lamoen observed that the Kiepert hyperbola (the locus of $K(\phi)$) befriends itself; so does its isogonal transform, the Brocard axis OK.

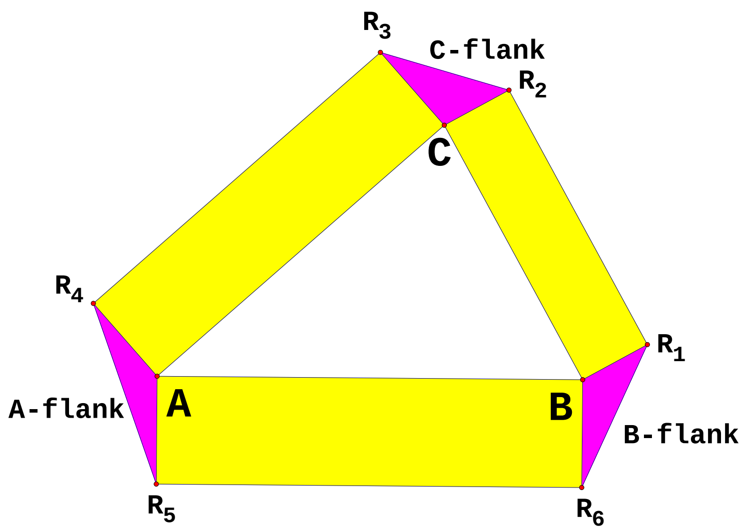

Figure 1. Triangle ABC with three rectangles and three flanks.

The purpose of this paper is to extend and improve the above results by replacing squares with rectangles whose ratio of nonparallel sides is constant. More precisely, let BR_1R_2C, CR_3R_4A, AR_5R_6B be rectangles built on the sides of a triangle ABC such that the oriented distances $|BR_1|$, $|CR_3|$, $|AR_5|$ are $\lambda\,|BC|$,

$\lambda\,|CA|$, $\lambda\,|AB|$ for some real number λ. Let $\tau_A^\lambda = AR_4R_5$, $\tau_B^\lambda = BR_6R_1$, and $\tau_C^\lambda = CR_2R_3$ and let $X_i^j(\lambda)$ and σ_i^λ, for $j = A,\, B,\, C$, have obvious meaning. The most important central points have their traditional notations so that we shall often use these because they might be easier to follow. For example, $H^A(\lambda)$ is the orthocenter of the flank τ_A^λ and σ_G^λ is the triangle $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ of the centroids of flanks.

Since triangles AS_4S_5 and AR_4R_5 are homothetic and the vertex A is the center of this homothety (and similarly for pairs BS_6S_1, BR_6R_1 and CS_2S_3, CR_2R_3), we conclude that $\{A,\, X_i^A,\, X_i^A(\lambda)\}$, $\{B,\, X_i^B,\, X_i^B(\lambda)\}$, and $\{C,\, X_i^C,\, X_i^C(\lambda)\}$ are sets of collinear points so that all statements from [2], [6], and [5] concerning triangles σ_i are also true for triangles σ_i^λ.

However, since in our approach instead of a single square on each side we have a family of rectangles it is possible to get additional information. This is well illustrated in our first theorem.

Theorem 1. *The homology axis of ABC and $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ envelopes the Kiepert parabola of ABC.*

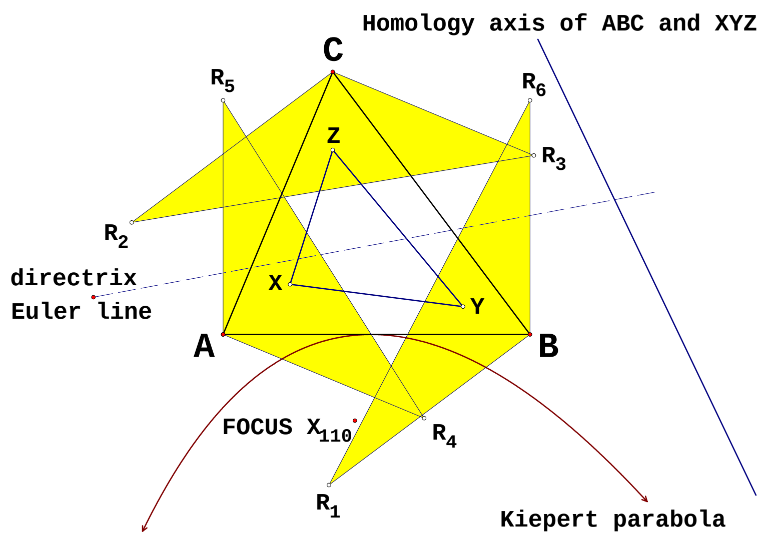

Figure 2. The homology axis of ABC and XYZ envelopes the Kiepert parabola of ABC.

Proof. In our proofs we shall use trilinear coordinates. The advantage of their use is that a high degree of symmetry is present so that it usually suffices to describe part of the information and the rest is self evident. For example, when we write $X_1(1)$ or $I(1)$ or simply say I is 1 this indicates that the incenter has trilinear coordinates $1 : 1 : 1$. We give only the first coordinate while the other two are cyclic permutations of the first. Similarly, $X_2(\frac{1}{a})$, or $G(\frac{1}{a})$, says that the centroid has has trilinears $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, where a, b, c are the lengths of the sides of ABC. The expressions in terms of sides a, b, c can be shortened using the following notation.

$$d_a = b - c, \quad d_b = c - a, \quad d_c = a - b, \quad z_a = b + c, \quad z_b = c + a, \quad z_c = a + b,$$

$$t = a + b + c, \quad t_a = b + c - a, \quad t_b = c + a - b, \quad t_c = a + b - c,$$

$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab, \quad T = \sqrt{t t_a t_b t_c},$$

For an integer n, let $t_n = a^n + b^n + c^n$ and $d_{na} = b^n - c^n$, and similarly for other cases. Instead of t_2, t_{2a}, t_{2b}, and t_{2c} we write k, k_a, k_b, and k_c.

In order to achieve even greater economy in our presentation, we shall describe coordinates or equations of only one object from triples of related objects and use cyclic permutations φ and ψ below to obtain the rest. For example, the first vertex A_a of the anticomplementary triangle $A_a B_a C_a$ of ABC has trilinears $-\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$. Then the trilinears of B_a and C_a need not be described because they are easily figured out and memorized by relations $B_a = \varphi(A_a)$ and $C_a = \psi(A_a)$. One must remember always that transformations φ and ψ are not only permutations of letters but also of positions, *i.e.*,

$$\varphi : \ a,\, b,\, c,\, 1,\, 2,\, 3 \mapsto b,\, c,\, a,\, 2,\, 3,\, 1$$

and

$$\psi : \ a,\, b,\, c,\, 1,\, 2,\, 3 \mapsto c,\, a,\, b,\, 3,\, 1,\, 2.$$

Therefore, the trilinears of B_a and C_a are $\frac{1}{a} : -\frac{1}{b} : \frac{1}{c}$ and $\frac{1}{a} : \frac{1}{b} : -\frac{1}{c}$.

The trilinears of the points R_1 and R_2 are equal to $-2\lambda m : c(T + \lambda k_c) : \lambda b k_b$ and $-2\lambda m : \lambda c k_c : b(T + \lambda k_b)$ (while $R_3 = \varphi(R_1)$, $R_4 = \varphi(R_2)$, $R_5 = \psi(R_1)$, and $R_6 = \psi(R_2)$). It follows that the centroid $X_2^A(\lambda)$ or $G^A(\lambda)$ of the triangle AR_4R_5 is $\frac{3T+2a^2\lambda}{-a} : \frac{k_c\lambda}{b} : \frac{k_b\lambda}{c}$.

Hence, the line $G^B(\lambda)G^C(\lambda)$ has equation

$$a(T\,\lambda^2 + 6\,z_{2a}\,\lambda + 9\,T)\,x + b\,\lambda(T\,\lambda + 3\,k_c)\,y + c\,\lambda(T\,\lambda + 3\,k_b)\,z = 0.$$

It intersects the line BC whose equation is $x = 0$ in the point $0 : \frac{T\,\lambda + 3\,k_b}{b} : \frac{T\,\lambda + 3\,k_c}{-c}$. Joining this point with its related points on lines CA and/or AB we get the homology axis of triangles ABC and $G^A(\lambda)G^B(\lambda)G^C(\lambda)$ whose equation is

$$\sum a(T^2\,\lambda^2 + 6\,a^2\,T\,\lambda + 9\,k_b\,k_c)\,x = 0.$$

When we differentiate this equation with respect to λ and solve for λ we get $\lambda = \frac{-3\,(\sum a^3\,x)}{T\,(\sum a\,x)}$. Substituting this value back into the above equation of the axis we obtain the equation

$$\sum (a^2\,d_{2a}^2\,x^2 - 2\,m_a\,d_{2b}\,d_{2c}\,y\,z) = 0$$

of their envelope. It is well-known (see [1]) that this is in fact the equation of the Kiepert parabola of ABC. □

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars from the vertices of ABC to the sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC,\, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric.

Hence, the perpendiculars from the vertices of XYZ to the sides BC, CA, and AB of ABC are concurrent at a point $[XYZ,\, ABC]$.

Since G (the centroid) befriends H (the orthocenter) it is clear that triangles τ and σ_G^λ are orthologic and $[\sigma_G^\lambda,\, \tau] = H$. Our next result shows that point $[\tau,\, \sigma_G^\lambda]$ traces the Kiepert hyperbola of τ.

Theorem 2. *The locus of the orthology center $[\tau,\, \sigma_G^\lambda]$ of τ and σ_G^λ is the Kiepert hyperbola of ABC.*

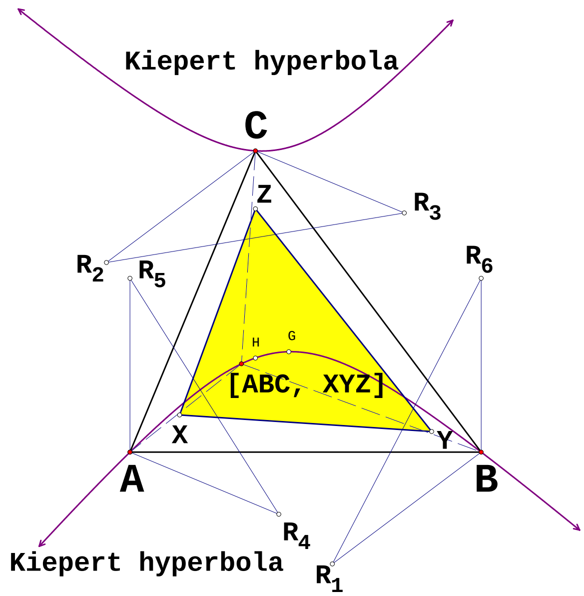

Figure 3. The orthology center $[ABC,\, XYZ]$ of triangles $\tau = ABC$ and $\sigma_G^\lambda = XYZ$ traces the Kiepert hyperbola of ABC.

Proof. The perpendicular from A onto the line $G^B(\lambda)G^C(\lambda)$ has equation

$$b(T\,\lambda + 3\,k_b)y - c(T\,\lambda + 3\,k_c)z = 0.$$

It follows that $[\tau,\, \sigma_G^\lambda]$ is $\frac{1}{a(T\lambda+3k_a)}$. This point traces the conic with equation $\sum m_a\, d_{2a}\, y\, z = 0$. The verification that this is the Kiepert hyperbola is easy because we must only check that it goes through A, B, C, $H(\frac{1}{a\,k_a})$, and $G(\frac{1}{a})$. $\square$

Theorem 3. *For every $\lambda \in \mathbb{R}$, the triangles τ and σ_O^λ are homothetic, with center of homothety at the symmedian point K. Hence, they are homologic with homology center K and their homology axis is the line at infinity.*

Proof. The point $\frac{T+z_{2a}\,\lambda}{-abc} : \frac{\lambda}{c} : \frac{\lambda}{b}$ is the circumcenter $O^A(\lambda)$ of the flank AR_4R_5. Since the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ \frac{T+z_{2a}\,\lambda}{-abc} & \frac{\lambda}{c} & \frac{\lambda}{b} \end{vmatrix}$$

is obviously zero, we conclude that the points A, K, and $O^A(\lambda)$ are collinear. In a similar way it follows that $\{B, K, O^B(\lambda)\}$ and $\{C, K, O^C(\lambda)\}$ are triples of collinear points. Hence, $\tau \overset{K}{\bowtie} \sigma_O^\lambda$. For $\lambda = -\frac{T}{k}$, the points $O^A(\lambda)$, $O^B(\lambda)$, and $O^C(\lambda)$ coincide with the symmedian point K. For $\lambda \neq -\frac{T}{k}$, the line $O^B(\lambda)O^C(\lambda)$ has equation $(a^2\,\lambda + T)x + \lambda\, a\, b\, y + \lambda\, c\, a\, z = 0$ and is therefore parallel to the sideline BC. Hence, the triangles τ and σ_A^λ are homothetic and the center of this homothety is the symmedian point K of τ. □

Theorem 4. *For every $\lambda \in \mathbb{R}$, the triangles τ and σ_O^λ are orthologic. The orthology center $[\,\tau, \sigma_O^\lambda\,]$ is the orthocenter H while the orthology center $[\,\sigma_O^\lambda, \tau\,]$ traces the line HK joining the orthocenter with the symmedian point.*

Proof. Since the triangles τ and σ_O^λ are homothetic and their center of similitude is the symmedian point K, it follows that τ and σ_O^λ are orthologic and that $[\,\tau, \sigma_O^\lambda\,] = H$. On the other hand, the perpendicular $p(O^A(\lambda), BC)$ from $O^A(\lambda)$ onto BC has equation

$$\lambda\, a\, d_{2a}\, k_a\, x + b\,(\lambda\, d_{2a}\, k_a - T\, k_b)\, y + c\,(\lambda\, d_{2a}\, k_a + T\, k_c)\, z = 0.$$

It follows that $[\,\sigma_O^\lambda, \tau\,]$ (= the intersection of $p(O^A(\lambda), BC)$ and $p(O^B(\lambda), CA)$) is the point $\frac{T\, k_b\, k_c + (2\, a^6 - z_{2a}\, a^4 - z_{2a}\, d_{2a}^2)\,\lambda}{a}$. This point traces the line with equation $\sum a\, d_{2a}\, k_a^2\, x = 0$. One can easily check that the points H and K lie on it. □

Theorem 5. *The homology axis of τ and σ_H^λ envelopes the parabola with directrix the line HK and focus the central point X_{112}.*

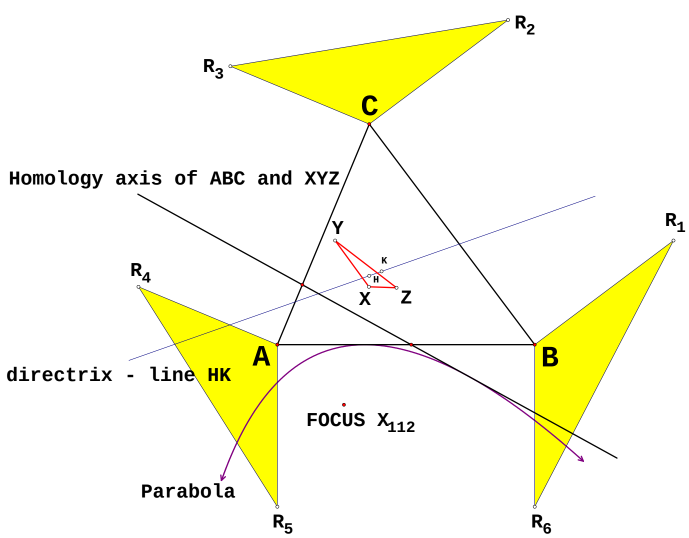

Figure 4. The homology axis of triangles $\sigma_H^\lambda = XYZ$ and $\tau = ABC$ envelopes the parabola with directrix HK and focus X_{112}.

Proof. The orthocenter $H^A(\lambda)$ of the flank AR_4R_5 is $\frac{T-2\lambda k_a}{a\,k_a} : \frac{\lambda}{b} : \frac{\lambda}{c}$. The line $H^B(\lambda)H^C(\lambda)$ has equation

$$a\,(3\,k_b\,k_c\,\lambda^2 - 4\,a^2\,T\,\lambda + T^2)\,x + b\,\lambda\,k_b\,(3\,k_c\,\lambda - T)\,y + c\,\lambda\,k_c\,(3\,k_b\,\lambda - T)\,z = 0.$$

It intersects the sideline BC in the point $0 : \frac{k_c\,(T-3\,k_b\,\lambda)}{b} : \frac{k_b\,(3\,k_c\,\lambda - T)}{c}$. We infer that the homology axis of the triangles τ and σ_H^λ has equation

$$\sum a\,k_a\,(9\,k_b\,k_c\,\lambda^2 - 6\,a^2\,T\,\lambda + T^2)\,x = 0.$$

It envelopes the conic with equation

$$\sum (a^2\,d_{2a}^2\,k_a^2\,x^2 - 2\,m_a\,d_{2b}\,d_{2c}\,k_b\,k_c\,y\,z) = 0.$$

It is easy to check that the above is an equation of a parabola because it intersects the line at infinity $\sum a\,x = 0$ only at the point $\frac{d_{2a}\,k_a}{a}$. On the other hand, we obtain the same equation when we look for the locus of all points P which are at the same distance from the central point $X_{112}(\frac{a}{d_{2a}\,k_a})$ and from the line HK. Hence, the above parabola has the point X_{112} for focus and the line HK for directrix. $\square$

Theorem 6. *For every real number λ the triangles τ and σ_H^λ are orthologic. The locus of the orthology center $[\tau,\,\sigma_H^\lambda]$ is the Kiepert hyperbola of ABC. The locus of the orthology center $[\sigma_H^\lambda,\,\tau]$ is the line HK.*

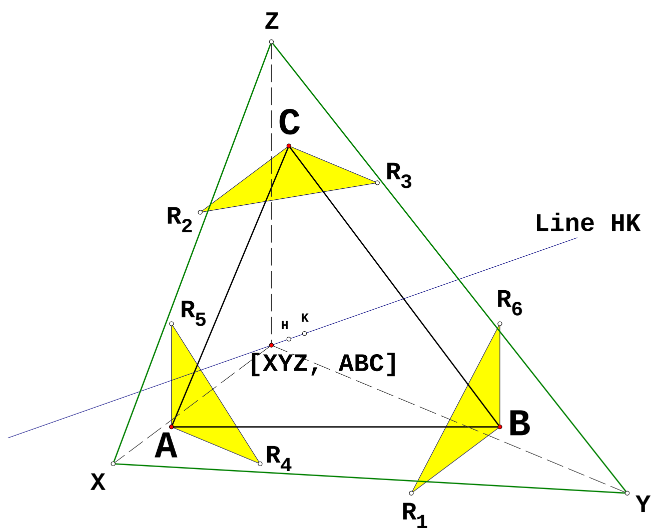

Figure 5. The orthology centers $[\sigma_4^\lambda,\,\tau]$ are on the line HK.

Proof. The perpendicular $p(A,\,H^B(\lambda)H^C(\lambda))$ from A onto the line $H^B(\lambda)H^C(\lambda)$ has equation $M_-(b,\,c)\,y - M_+(c,\,b)\,z = 0$, where

$$M_\pm(b,\,c) = b\,[(3\,a^4 \pm 2\,d_{2a}\,a^2 \pm d_{2a}(b^2 + 3\,c^2))\,\lambda - k_b\,T\,].$$

The lines $p(A,\, H^B(\lambda)H^C(\lambda))$, $p(B,\, H^C(\lambda)H^A(\lambda))$, and $p(C,\, H^A(\lambda)H^B(\lambda))$ concur at the point $\frac{1}{a\,[(a^4+2\,z_{2a}\,a^2-2\,m_{2a}-3\,z_{4a})\,\lambda+k_a\,T\,]}$. Just as in the proof of Theorem 2 we can show that this point traces the Kiepert hyperbola of ABC.

The perpendicular $p(H^A(\lambda),\, BC)$ from $H^A(\lambda)$ onto BC has equation

$$2\,\lambda\,a\,d_{2a}\,k_a\,x+b\,(2\,d_{2a}\,k_a\,\lambda+k_b\,T)\,y+c\,(2\,d_{2a}\,k_a\,\lambda-k_c\,T)\,z=0.$$

The lines $p(H^A(\lambda),\, BC)$, $p(H^B(\lambda),\, CA)$, and $p(H^C(\lambda),\, AB)$ concur at the point $\frac{2\,(2\,a^6-z_{2a}\,a^4-z_{2a}\,d_{2a}^2)\,\lambda-k_b\,k_c\,T}{a}$. We infer that the orthology center $[\sigma_H^\lambda,\,\tau]$ traces the line HK because we get its equation by eliminating the parameter λ from the equations $x=x_0$, $y=y_0$, and $z=z_0$, where x_0, y_0, and z_0 are the trilinears of $[\sigma_H^\lambda,\,\tau]$. □

Theorem 7. *For every $\lambda\in\mathbb{R}\setminus\{0\}$, the triangles ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ are homologic if and only if the triangle ABC is isosceles.*

Proof. The center $F^A(\lambda)$ of the nine-point circle of the flank AR_4R_5 is

$$\frac{(k_a-a^2)\,\lambda-2\,T}{a}\;:\;\frac{\lambda\,d_{2b}}{-b}\;:\;\frac{\lambda\,d_{2c}}{c}.$$

The line $AF^A(\lambda)$ has equation $b\,d_{2c}\,y+c\,d_{2b}\,z=0$. Hence, the condition for these three lines to concur (expressed in terms of the side lengths) is $2\,m\,d_{2a}\,d_{2b}\,d_{2c}=0$, which immediately implies our claim. □

When triangle ABC is scalene and isosceles, one can show easily that the homology center of ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ is the midpoint of the base while the homology axis envelopes again the Kiepert parabola of ABC (which agrees with the line parallel to the base through the opposite vertex).

The following two theorems have the same proofs as Theorem 6 and Theorem 1, respectively.

Theorem 8. *For every real number λ the triangles ABC and $F^A(\lambda)F^B(\lambda)F^C(\lambda)$ are orthologic. The orthology centers $[\,\sigma_F^\lambda,\,\tau\,]$ and $[\,\tau,\,\sigma_F^\lambda\,]$ trace the line HK and the Kiepert hyperbola, respectively.*

Theorem 9. *The homology axis of the triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ envelopes the Kiepert parabola of ABC.*

Theorem 10. *For every $\lambda\in\mathbb{R}\setminus\{0\}$, the triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ are orthologic if and only if the triangle ABC is isosceles.*

Proof. The symmedian point $K^A(\lambda)$ of the flank AR_4R_5 is

$$\frac{(d_{2a}^2-a^2\,z_{2a})\lambda-T\,(3\,k_a+2\,a^2)}{a}\;:\;\lambda\,b\,k_b:\;\lambda\,c\,k_c.$$

It follows that the perpendicular $p(K^A(\lambda),\, BC)$ from $K^A(\lambda)$ to BC has equation

$$\lambda\,a\,d_{2a}\,T\,x+b\,(\lambda\,d_{2a}\,T-k_b\,(3\,k_a+2\,a^2))\,y+c\,(\lambda\,d_{2a}\,T+k_c\,(3\,k_a+2\,a^2))\,z=0.$$

The triangles ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ are orthologic if and only if the coefficient determinant of the equations of the lines $p(K^A(\lambda),\ BC)$, $p(K^B(\lambda),\ CA)$, and $p(K^C(\lambda),\ AB)$ is zero. But, this determinant is equal to $-16\lambda m d_{2a} d_{2b} d_{2c} T^6$, which immediately implies that our claim is true. □

When the triangle ABC is scalene and isosceles one can show easily that the orthology centers of ABC and $K^A(\lambda)K^B(\lambda)K^C(\lambda)$ both trace the perpendicular bisector of the base.

The proofs of the following two theorems are left to the reader because they are analogous to proofs of Theorem 1 and Theorem 6, respectively. However, the expressions that appear in them are considerably more complicated.

Theorem 11. *The homology axis of τ and σ_x^λ envelopes the Kiepert parabola of ABC for $x = 15,\ 16,\ 61,\ 62$.*

Theorem 12. *For every real number λ the triangles τ and σ_L^λ are orthologic. The loci of the orthology centers $[\,\tau,\ \sigma_L^\lambda\,]$ and $[\,\sigma_L^\lambda,\ \tau\,]$ are the Kiepert hyperbola and the line HK, respectively.*

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Napoleon-like Configurations and Sequences of Triangles

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Abstract. We consider the sequences of triangles where each triangle is formed out of the apices of three similar triangles built on the sides of its predecessor. We show under what conditions such sequences converge in shape, or are periodic.

1. Introduction

The well-known geometrical configuration consisting of a given triangle and three equilateral triangles built on its sides, all outwardly or inwardly, has many interesting properties. The most famous is the theorem attributed to Napoleon that the centers of the three equilateral triangles built on the sides are vertices of another equilateral triangle [3, pp. 60–65]. Numerous works have been devoted to this configuration, including various generalizations [6, 7, 8] and converse problems [10].

Some authors [5, 9, 1] considered the iterated configurations where construction of various geometrical objects (e.g. midpoints) on the sides of polygons is repeated an arbitrary number of times. Douglass [5] called such constructions *linear polygon transformations* and showed their relation with circulant matrices. In this paper, we study the sequence of triangles obtained by a modification of such a configuration. Each triangle in the sequence is called a *base* triangle, and is obtained from its predecessor by two successive transformations: (1) the classical construction on the sides of the base triangle triangles similar to a given (*transformation*) triangle and properly oriented, (2) a normalization which is a direct similarity transformation on the apices of these new triangles so that one of the images lies on a fixed circle. The three points thus obtained become the vertices of the new base triangle. The normalization step is the feature that distinguishes the present paper from earlier works, and it gives rise to interesting results. The main result of this study is that under some general conditions the sequence of base triangles converges to an equilateral triangle (in a sense defined at the end of §2). When the limit does not exist, we study the conditions for periodicity. We study two types of sequences of triangles: in the first, the orientation of the transformation triangle is given a priori; in the second, it depends on the orientation of the base triangle.

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The rest of the paper is organized as follows. In §2, we explain the notations and definitions used in the paper. In §3, we give a formal definition of the transformation that generates the sequence. In §4, we study the first type of sequences mentioned above. In §5, we consider the exceptional case when the transformation triangle degenerates into three collinear points. In §6, we consider the second type of sequences mentioned above. In §7, we study a generalization for arbitrary polygons.

2. Terminology and definitions

We adopt the common notations of complex arithmetic. For a complex number z, Re(z) denotes its real part, Im(z) its imaginary part, $|z|$ its modulus, arg(z) its argument (chosen in the interval $(-\pi, \pi]$), and $\overline{z}$ its conjugate. The primitive complex n-th root of unity $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, is denoted by ζ_n. Specifically, we write $\omega = \zeta_3$ and $\eta = \zeta_6$. The important relation between the two is $\omega^2 + \eta = 0$.

A triangle is oriented if an ordering of its vertices is specified. It is positively (negatively) oriented if the vertices are ordered counterclockwise (clockwise). Two oriented triangles are directly (oppositely) similar if they have the same (opposite) orientation and pairs of corresponding vertices may be brought into coincidence by similarity transformations.

Throughout the paper, we coordinatized points in a plane by complex numbers, using the same letter for a point and its complex number coordinate. An oriented triangle is represented by an ordered triple of complex numbers. To obtain the orientation and similarity conditions, we define the following function $z : \mathbb{C}^3 \to \mathbb{C}$ on the set of all vectors $V = (A, B, C)$ by

$$z[V] = z(A, B, C) = \frac{C - A}{B - A}. \tag{1}$$

Triangle ABC is positively or negatively oriented according as arg($z(A, B, C)$) is positive or negative. Furthermore, every complex number z defines a class of directly similar oriented triangles ABC such that $z(A, B, C) = z$. In particular, if ABC is a positively (respectively negatively) oriented equilateral triangle, then $z(A, B, C) = \eta$ (respectively $\overline{\eta}$).

Finally, we define the convergence of triangles. An infinite sequence of triangles $(A_nB_nC_n)$ converges to a triangle ABC if the sequence of complex numbers $z(A_n, B_n, C_n)$ converges to $z(A, B, C)$.

3. The transformation f

We describe the transformations that generate the sequence of triangles we study in the paper. We start with a base triangle $A_0B_0C_0$ and a transformation triangle XYZ. Let G be the centroid of $A_0B_0C_0$, and Γ the circle centered at G and passing through the vertex farthest from G. (Figure 1a). For every $n > 0$, triangle $A_nB_nC_n$ is obtained from its predecessor $A_{n-1}B_{n-1}C_{n-1}$ by $f = f_2 \circ f_1$, where

(i) f_1 maps $A_{n-1}B_{n-1}C_{n-1}$ to $A'_nB'_nC'_n$, by building on the sides of triangle $A_{n-1}B_{n-1}C_{n-1}$, three triangles $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ similar to XYZ and all with the same orientation, [1] (Figure 1b);

(ii) f_2 transforms by similarity with center G the three points A'_n, B'_n, C'_n so that the image of the farthest point lies on the circle Γ, (Figure 1c).

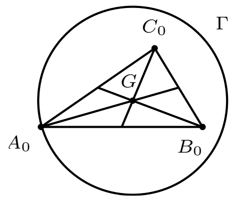

Figure 1a

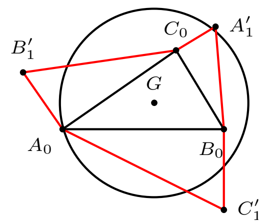

Figure 1b

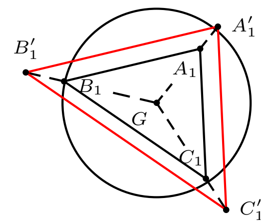

Figure 1c

The three points so obtained are the vertices of the next base triangle $A_nB_nC_n$. We call this the *normalization* of triangle $A'_nB'_nC'_n$. In what follows, it is convenient to coordinatize the vertices of triangle $A_0B_0C_0$ so that its centroid G is at the origin, and Γ is the unit circle. In this setting, normalization is simply division by

$$r_n = \max(|A'_n|, |B'_n|, |C'_n|).$$

It is easy to see that f may lead to a degenerate triangle. Figure 2 depicts an example of a triple of collinear points generated by f_1. Nevertheless, f is well defined, except only when $A_{n-1}B_{n-1}C_{n-1}$ degenerates into the point G. But it is readily verified that this happens only if triangle $A_{n-1}B_{n-1}C_{n-1}$ is equilateral, in which case we stipulate that $A_nB_nC_n$ coincides with $A_{n-1}B_{n-1}C_{n-1}$.

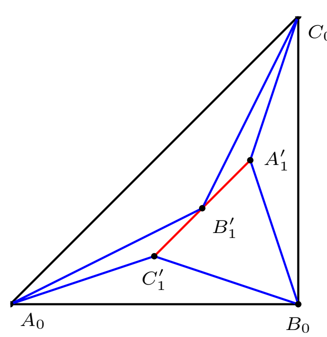

Figure 2

The normalization is a crucial part of this transformation. While preserving direct similarity of the triangles $A'_nB'_nC'_n$ and $A_nB_nC_n$, it prevents the latter from

[1] We deliberately do not specify the orientation of those triangles with respect to the transformation triangle, since they are specific for the different types of sequences we discuss later in this paper.

converging to a single point or diverging to infinity (since every triangle after normalization lies inside a fixed circle, and at least one of its vertices lies on the circle), and the convergence of triangles receives a definite geometrical meaning. Also, since f_1 and f_2 leave G fixed, we have a rather expected important property that G is a fixed point of the transformation.

4. The first sequence

We first keep the orientation of the transformation triangle fixed and independent from the base triangle.

Theorem 1. *Let $A_0B_0C_0$ be an arbitrary base triangle, and XYZ a non-degenerate transformation triangle. The sequence $(A_nB_nC_n)$ of base triangles generated by the transformation f in §3 (with $B_{n-1}C_{n-1}A'_n$, $C_{n-1}A_{n-1}B'_n$, $A_{n-1}B_{n-1}C'_n$ directly similar to XYZ) converges to the equilateral triangle with orientation opposite to XYZ, except when $A_0B_0C_0$, and the whole sequence, is equilateral with the same orientation as XYZ.*

Proof. Without loss of generality let XYZ be positively oriented. We treat the special cases first. The exceptional case stated in the theorem is verified straightforwardly; also it is obvious that we may exclude the cases where $A_nB_nC_n$ is positively oriented equilateral. Hence in what follows it is assumed that $z(A_0, B_0, C_0) \neq \eta$, and $r_n \neq 0$ for every n.

Let $z(X, Y, Z) = t$. Since for every n, triangle $B_{n-1}C_{n-1}A'_n$ is directly similar to XYZ, by (1)

$$A'_n = (1-t)B_{n-1} + tC_{n-1},$$

and similarly for B'_n and C'_n. After normalization,

$$V_n = \frac{1}{r_n} TV_{n-1}, \tag{2}$$

where $V_n = \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}$, T is the circulant matrix $\begin{pmatrix} 0 & 1-t & t \\ t & 0 & 1-t \\ 1-t & t & 0 \end{pmatrix}$, and $r_n = \max(|A'_n|, |B'_n|, |C'_n|)$. By induction,

$$V_n = \frac{1}{r_1 \cdots r_n} T^n V_0.$$

We use the standard diagonalization procedure to compute the powers of T. Since T is circulant, its eigenvectors are the columns of the Fourier matrix ([4, pp.72–73])

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^4 \end{pmatrix},$$

and the corresponding eigenvalues are λ_0, λ_1, λ_2 are [2]

$$\lambda_j = (1-t)\omega^j + t\omega^{2j}, \tag{3}$$

[2] Interestingly enough, ordered triples $(\omega, \omega^2, \lambda_1)$ and $(\omega^2, \omega, \lambda_2)$ form triangles directly similar to XYZ.

for $j = 0, 1, 2$. With these, we have

$$T = F_3 U F_3^{-1},$$

where U is the diagonal matrix $\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

Let $S = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$ be a vector of points in the complex plane that is transformed into V_0 by the Fourier matrix, *i.e.*,

$$V_0 = F_3 S. \tag{4}$$

Since $A_0 + B_0 + C_0 = 3G = 0$, we get $s_0 = 0$, and

$$V_0 = s_1 F_{3,1} + s_2 F_{3,2}, \tag{5}$$

where $F_{3,j}$ is the j-th column of F_3. After n iterations,

$$V_n \sim T^n V_0 = F_3 U^n F_3^{-1}(s_1 F_{3,1} + s_2 F_{3,2}) = s_1 \lambda_1^n F_{3,1} + s_2 \lambda_2^n F_{3,2}. \tag{6}$$

According to (3) and the assumption that XYZ is negatively oriented,

$$|\lambda_2|^2 - |\lambda_1|^2 = \lambda_2 \overline{\lambda_2} - \lambda_1 \overline{\lambda_1} = 2\sqrt{3}\mathrm{Im}(t) < 0,$$

so that $\dfrac{|\lambda_2|}{|\lambda_1|} < 1$, and $\dfrac{|\lambda_2^n|}{|\lambda_1^n|} \to 0$ when $n \to \infty$. Also, expressing $z(A_0, B_0, C_0)$ in terms of s_1, s_2, we get

$$z(A_0, B_0, C_0) = \frac{s_1 \eta + s_2}{s_1 + s_2 \eta}, \tag{7}$$

so that $z(A_0, B_0, C_0) \neq \overline{\eta}$ implies $s_1 \neq 0$. Therefore,

$$\lim_{n\to\infty} z(A_n, B_n, C_n) = \lim_{n\to\infty} z[V_n] = z[F_{3,1}] = \eta.$$

$\square$

Are there cases when the sequence converges after a finite number of iterations? Because the columns of the Fourier matrix F_3 are linearly independent, this may happen if and only if the second term in (6) equals 0. There are two cases:

(i) $s_2 = 0$: this, according to (7), corresponds to a base triangle $A_0 B_0 C_0$ which is equilateral and positively oriented;

(ii) $\lambda_2 = 0$: this, according to (3), corresponds to a transformation triangle XYZ which is isosceles with base angle $\frac{\pi}{6}$. In this case, one easily recognizes the triangle of the Napoleon theorem.

We give a geometric interpretation of the values s_1, s_2. Changing for a while the coordinates of the complex plane so that A_0 is at the origin, we get from (4):

$$|s_1| = |B_0 - C_0 \eta|, \qquad |s_2| = |B_0 - C_0 \overline{\eta}|,$$

and we have the following construction: On the side $A_0 C_0$ of the triangle $A_0 B_0 C_0$ build two oppositely oriented equilateral triangles (Figure 3), then $|s_1| = B_0 B'$,

$|s_2| = B_0B''$. After some computations, we obtain the following symmetric formula for the ratio $\frac{s_1}{s_2}$ in terms of the angles α, β, γ of triangle $A_0B_0C_0$:

$$\left|\frac{s_1}{s_2}\right|^2 = \frac{\sin\alpha\sin(\alpha+\frac{\pi}{3}) + \sin\beta\sin(\beta+\frac{\pi}{3}) + \sin\gamma\sin(\gamma+\frac{\pi}{3})}{\sin\alpha\sin(\alpha-\frac{\pi}{3}) + \sin\beta\sin(\beta-\frac{\pi}{3}) + \sin\gamma\sin(\gamma-\frac{\pi}{3})}.$$

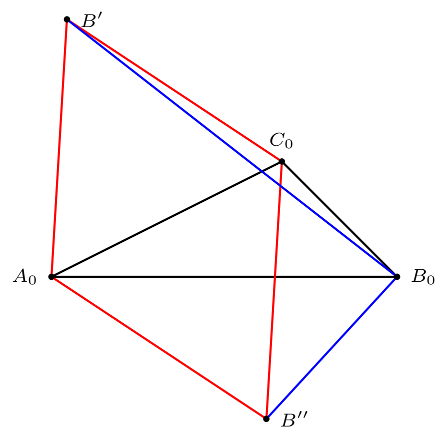

Figure 3

5. An exceptional case

In this section we consider the case t is a real number. Geometrically, it means that the transformation triangle XYZ degenerates into a triple of collinear points, so that A_n', B_n', C_n' divide the corresponding sides of triangle $A_{n-1}B_{n-1}C_{n-1}$ in the ratio $1-t:t$. (Figure 4 depicts an example for $t=\frac{1}{3}$). Can the sequence of triangles still converge in this case? To settle this question, notice that when t is real, λ_1 and λ_2 are complex conjugates, and rewrite (6) as follows:

$$V_n \sim \lambda_1^n\left(s_1F_{3,1} + \frac{\lambda_2^n}{\lambda_1^n}s_2F_{3,2}\right), \tag{8}$$

and because $\dfrac{\lambda_2}{\lambda_1}$ defines a *rotation*, it is clear that it does not have a limit unless $\dfrac{\lambda_2}{\lambda_1} = 1$, in which case the sequence consists of directly similar triangles. Now, $\lambda_1 = \lambda_2$ implies $t = \frac{1}{2}$, so we have the well-known result that the triangle is similar to its medial triangle [3, p. 19].

Next, we find the conditions under which the sequence has a finite period m. Geometrically, it means that m is the least number such that triangles $A_nB_nC_n$ and $A_{n+m}B_{n+m}C_{n+m}$ are directly similar for every $n \geq 0$. The formula (8) shows that it happens when $\dfrac{\lambda_2}{\lambda_1} = \zeta_m^k$, and k, m are co-prime. Plugging this into

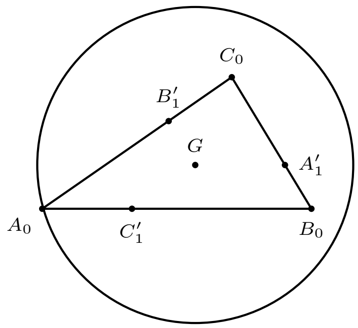

Figure 4a

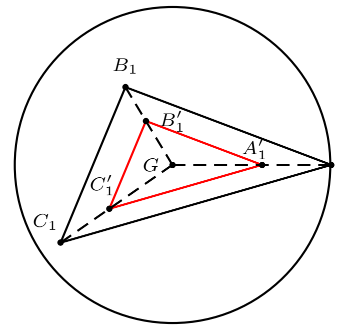

Figure 4b

(3) and solving for t, we conclude that the sequence of triangles with period m exists for t of the form

$$t(m) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan \frac{k\pi}{m}. \tag{9}$$

Several observations may be made from this formula. First, the periodic sequence with finite t exists for every $m \neq 2$. (The case $m = 2$ corresponds to transformation triangle with two coinciding vertices X, Y). The number of different sequences of a given period m is $\phi(m)$, Euler's totient function [2, pp.154–156]. Finally, the case $m = 1$ yields $t = \frac{1}{2}$, which is the case of medial triangles.

Also, several conclusions may be drawn about the position of corresponding triangles in a periodic sequence. Comparing (8) with (5), we see that triangle $A_mB_mC_m$ is obtained from triangle $A_0B_0C_0$ by a rotation about their common centroid G through angle $m \cdot \arg(\lambda_1)$. Because 2arg(λ_1) = 0 $\bmod 2\pi$, it follows that $A_mB_mC_m$ coincides with $A_0B_0C_0$, or is a half-turn. In both cases, the triangle $A_{2m}B_{2m}C_{2m}$ will always coincide with $A_0B_0C_0$. We summarize these results in the following theorem.

Theorem 2. *Let a triangle $A_0B_0C_0$ and a real number t be given. The sequence $(A_nB_nC_n)$ of triangles constructed by first dividing the sides of each triangle in the ratio $1 - t : t$ and then normalizing consists of similar triangles with period m if and only if t satisfies (9) for some k relatively prime to m. In this case, triangles $A_nB_nC_n$ and $A_{n+2m}B_{n+2m}C_{n+2m}$ coincide for every $n \geq 0$. In all other cases the sequence never converges, unless $A_0B_0C_0$, and hence every $A_nB_nC_n$, is equilateral.*

6. The second sequence

In this section we study another type of sequence, where the orientation of transformation triangles depends on the base triangle. More precisely, we consider two cases: when triangles built on the sides of the base triangle are oppositely or equally oriented to it. The main results of this section will be derived using the following important lemma.

Lemma 3. *Let ABC be any triangle, and $V = (A, B, C)$ the corresponding vector of points in the complex plane with the centroid of ABC at the origin. If S is defined as in (4), then ABC is positively (negatively) oriented when $|s_1| > |s_2|$ ($|s_1| < |s_2|$), and A, B, C are collinear if $|s_1| = |s_2|$.*

Proof. According to (7), Im$(z(A, B, C)) \sim s_1\overline{s_1} - s_2\overline{s_2} = |s_1|^2 - |s_2|^2$. □

Before proceeding, we extend notations. As the orientation of the transformation triangle may change throughout the sequence, $z(X, Y, Z)$ equals t or $\overline{t}$, depending on the orientation of the base triangle. So, for the transformation matrix we shall use the notation $T(t)$ or $T(\overline{t})$ accordingly. Note that if the eigenvalues of $T(t)$ are λ_0, λ_1, λ_2, then the eigenvalues of $T(\overline{t})$ are $\overline{\lambda_0}$, $\overline{\lambda_1}$, $\overline{\lambda_2}$. The first result concerning the case of the oppositely oriented triangles is as follows.

Theorem 4. *Let $A_0B_0C_0$ be the base triangle, and XYZ the transformation triangle. If the sequence of triangles $A_nB_nC_n$ is generated as described in §3 with every triangle $B_{n-1}C_{n-1}A'_n$ etc. oppositely oriented to $A_{n-1}B_{n-1}C_{n-1}$, then the sequence converges to the equilateral triangle that has the same orientation as $A_0B_0C_0$.*

Proof. Without loss of generality, we may assume $A_0B_0C_0$ positively oriented. It is sufficient to show that triangle $A_nB_nC_n$ is positively oriented for every n. Then, every triangle $B_{n-1}C_{n-1}A'_n$ etc. is negatively oriented, and the result follows immediately from Theorem 1.

We shall show this by induction. Assume triangles $A_0B_0C_0, \ldots, A_{n-1}B_{n-1}C_{n-1}$ are positively oriented, then they all are the base for the *negatively* oriented directly similar triangles to build their successors, so arg$(t) < 0$, and $|\lambda_1^n| > |\lambda_2^n|$. Also, $|s_1| > |s_2|$, and according to (6) and the above lemma, $A_nB_nC_n$ is positively oriented. □

We proceed with the case when triangles are constructed with the same orientation of the base triangle. In this case, the behavior of the sequence turns out to be much more complicated. Like in the first case, assume $A_0B_0C_0$ positively oriented. If $s_2 = 0$, which corresponds to the equilateral triangle, then all triangles $A_nB_nC_n$ are positively oriented and, of course, equilateral. Otherwise, because arg$(t) > 0$, and therefore $|\lambda_1| < |\lambda_2|$, it follows that $|s_1\lambda_1^n| - |s_2\lambda_2^n|$ eventually becomes negative, and the sequence of triangles changes the orientation. Specifically, it happens exactly after ℓ steps, where

$$\ell = \left\lceil \frac{\ln \frac{s_2}{s_1}}{\ln \frac{\lambda_1}{\lambda_2}} \right\rceil. \tag{10}$$

What happens next? We know that $A_\ell B_\ell C_\ell$ is the first negatively oriented triangle in the sequence, therefore triangles $B_\ell C_\ell A'_{\ell+1}$ etc. built on its sides are also negatively oriented. Thus, $z(B_\ell, C_\ell, A'_{\ell+1}) = \overline{t}$. Therefore, according to (3) and

(6),

$$V_{\ell+1} \sim T(t)^{\ell} T(\overline{t}) V_0 = s_1 \lambda_1^{\ell} \overline{\lambda_2} F_{3,1} + s_2 \lambda_2^{\ell} \overline{\lambda_1} F_{3,2}.$$

Since

$$|s_1 \lambda_1^{\ell} \overline{\lambda_2}| = |s_1 \lambda_1^{\ell-1}||\lambda_1 \lambda_2| > |s_2 \lambda_2^{\ell-1}||\lambda_1 \lambda_2| = |s_2 \lambda_2^{\ell} \overline{\lambda_1}|,$$

triangle $A_{\ell+1} B_{\ell+1} C_{\ell+1}$ is positively oriented. Analogously, we get that for $n \geq 0$, every triangle $A_{\ell+2n} B_{\ell+2n} C_{\ell+2n}$ is negatively oriented, while its successor $A_{\ell+2n+1} B_{\ell+2n+1} C_{\ell+2n+1}$ is positively oriented.

Consider now the sequence $(A_{\ell+2n} B_{\ell+2n} C_{\ell+2n})$ consisting of negatively oriented triangles. Clearly, the transformation matrix for this sequence is the product of $T(t)$ and $T(\overline{t})$, which is a circulant matrix

$$\begin{pmatrix} t + \overline{t} - 2t\overline{t} & t\overline{t} & 1 - t - \overline{t} + t\overline{t} \\ 1 - t - \overline{t} + t\overline{t} & t + \overline{t} - 2t\overline{t} & t\overline{t} \\ t\overline{t} & 1 - t - \overline{t} + t\overline{t} & t + \overline{t} - 2t\overline{t} \end{pmatrix}$$

with eigenvalues

$$\lambda'_j = t + \overline{t} - 2t\overline{t} + t\overline{t}\omega^j + (1 - t - \overline{t} + t\overline{t})\omega^{2j}, \qquad j = 0, 1, 2. \tag{11}$$

Since this matrix is real, the sequence $(A_{\ell+2n} B_{\ell+2n} C_{\ell+2n})$ of triangles does not converge. It follows at once that the sequence $(A_{\ell+2n+1} B_{\ell+2n+1} C_{\ell+2n+1})$ of successors does not converge either.

Finally, we consider the conditions when these two sequences are periodic. Clearly, only even periods $2m$ may exist. In this case, λ'_1 and λ'_2 must satisfy $\dfrac{\lambda'_1}{\lambda'_2} = \zeta_m^k$ for k relatively prime to m. Since λ'_1, λ'_2 are complex conjugates, this is equivalent to arg$(\lambda'_1) = \frac{k\pi}{m}$. Applying (11), we arrive at the following condition:

$$\tan \frac{k\pi}{m} = \frac{1}{\sqrt{3}} \cdot \frac{\mathrm{Re}(t) - \frac{1}{2}}{\mathrm{Re}(t) - |t|^2 - \frac{1}{6}}.$$

Several interesting properties about periodic sequences may be derived from this formula. First, for a given pair of numbers k, m, the locus of t is a *circle* centered at the point O on a real axis, and radius R defined as follows:

$$O(m) = \frac{1}{2} - \frac{1}{2\sqrt{3}} \cot \frac{k\pi}{m}, \qquad R(m) = \frac{1}{2\sqrt{3}} \csc \frac{k\pi}{m}. \tag{12}$$

Furthermore, all the circles have the two points $\frac{1}{3}(1 + \overline{\eta})$ and $\frac{1}{3}(1 + \eta)$ in common. This is clear if we note that they correspond to the cases $\lambda'_1 = 0$ and $\lambda'_2 = 0$ respectively, *i.e.*, when the triangle becomes equilateral after the first iteration (see the discussion following Theorem 1 in §3).

Summarizing, we have the following theorem.

Theorem 5. *Let $A_0 B_0 C_0$ be the base triangle, and XYZ the transformation triangle. The sequence $(A_n B_n C_n)$ of triangles constructed by the transformation f ($B_{n-1} C_{n-1} A'_n$, $C_{n-1} A_{n-1} B'_n$, $A_{n-1} B_{n-1} C'_n$ with the same orientation of $A_{n-1} B_{n-1} C_{n-1}$) converges only if $A_0 B_0 C_0$ is equilateral (and so is the whole sequence). Otherwise the orientation of $A_0 B_0 C_0$ is preserved for first $\ell - 1$ iterations, where ℓ is determined by (10); afterwards, it is reversed in each iteration.*

The sequence consists of similar triangles with an even period $2m$ *if and only if* $t = z(X, Y, Z)$ *lies on a circle* $O(R)$ *defined by (12) for some* k *relatively prime to* m*. In this case, triangles* $A_nB_nC_n$ *and* $A_{n+4m}B_{n+4m}C_{n+4m}$ *coincide for every* $n \geq \ell$*.*

We conclude with a demonstration of the last theorem's results. Setting $m = 1$ in (12), both O and R tend to infinity, and the circle degenerates into line $\mathrm{Re}(t) = \frac{1}{2}$, that corresponds to any isosceles triangle. Figures 5a through 5d illustrate this case when XYZ is the right isosceles triangle, and $A_0B_0C_0$ is also isosceles positively oriented with base angle $\frac{3\pi}{8}$. According to (10), $\ell = 2$. Indeed, $A_2B_2C_2$ is the first negatively oriented triangle in the sequence, $A_3B_3C_3$ is again positively oriented and similar to $A_1B_1C_1$. The next similar triangle $A_5B_5C_5$ will coincide with $A_1B_1C_1$.

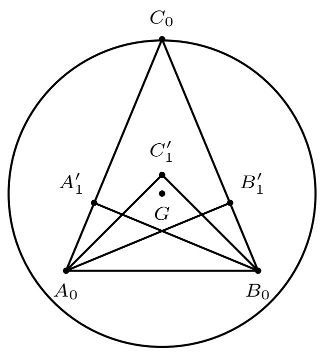

Figure 5a

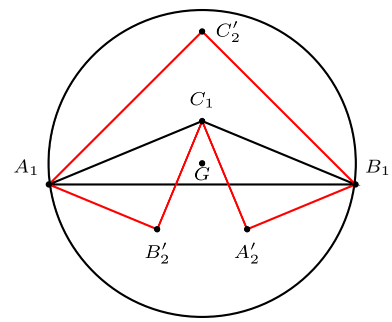

Figure 5b

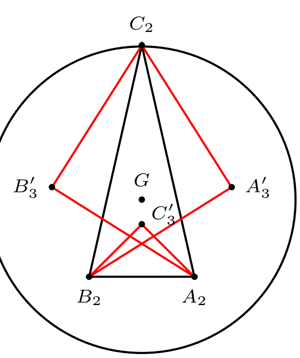

Figure 5c

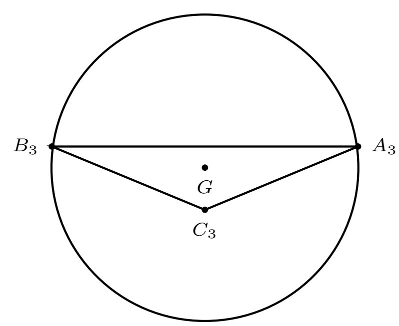

Figure 5d

7. Generalization to polygons

In this section, we generalize the results in §4 by replacing the sequences of triangles by sequences of polygons. The transformation performed at every iteration remains much the same as in §3, with triangles built on every side of the base polygon directly similar to a given transformation triangle. We seek the conditions under which the resulting sequence of polygons converges in shape.

Let the unit circle be divided into k equal parts by the points P_0, P_1, $\ldots$, P_{k-1}. We call the polygon regular k-gon of q-type if it is similar to the polygon $P_0P_q\cdots P_{(k-1)q}$, where the indices are taken modulo k [5, p. 558]. The regular 1-type and $(k-1)$-type polygons are simply the convex regular polygons in an ordinary sense. Other regular k-gons may be further classified into

(i) star-shaped if q, k are co-prime, (for example, a pentagram is a 2-type regular pentagon, Figure 6a), and

(ii) multiply traversed polygons with fewer vertices if q, k have a common divisor, (for example, a regular hexagon of 2-type is an equilateral triangle traversed twice, Figure 6b).

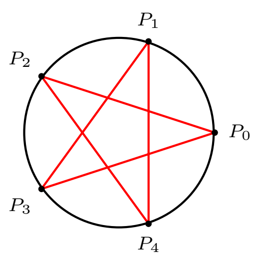

Figure 6a

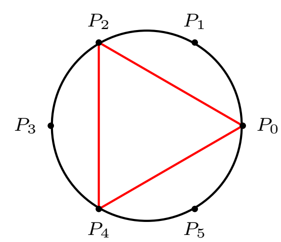

Figure 6b

In general, regular k-gons of q-type and $(k-q)$-type are equally shaped and oppositely oriented. It is also evident that $(-q)$-type and $(k-q)$-type k-gons are identical. We shall show that under certain conditions the sequence of polygons converges to regular polygons so defined.

Let $\Pi_0 = P_{0,0}P_{1,0}\cdots P_{k-1,0}$ be an arbitrary k-gon, and XYZ the non-degenerate transformation triangle, and let the sequence of k-gons $\Pi_n = P_{0,n}P_{1,n}\cdots P_{k-1,n}$ be generated as in §3, with triangles $P_{0,n-1}P_{1,n-1}P'_{0,n}, \ldots P_{k-1,n-1}P_{0,n-1}P'_{k-1,n}$ built on the sides of Π_{n-1} directly similar to XYZ and then normalized. The transformation matrix T_k for such a sequence is a circulant $k\times k$ matrix with the first row

$$\begin{pmatrix}1-t & t & 0 & \cdots & 0\end{pmatrix},$$

whose eigenvectors are columns of Fourier matrix

$$F_k = \frac{1}{\sqrt{k}}(\zeta_k^{ij}), \quad i,j = 0,\ldots,k-1,$$

and the eigenvalues:

$$\lambda_i = (1-t) + t\zeta_k^i, \quad i = 0,\ldots,k-1. \tag{13}$$

Put Π_0 into the complex plane so that its centroid $G = \frac{1}{k}\sum_{i=0}^{k-1} P_{i,0}$ is at the origin, and let V_n be a vector of points corresponding to Π_n. If $S = (s_0, \ldots, s_{k-1})$ is a vector of points that is transformed into V_0 by Fourier matrix, *i.e.*, $S = \overline{F_k} V_0$, then similar to (6), we get:

$$V_n \sim \sum_{i=0}^{k-1} s_i \lambda_i^n F_{k,i}. \tag{14}$$

Noticing that the column vectors $F_{k,q}$ correspond to regular k-gons of q-type, we have the following theorem:

Theorem 6. *The sequence of k-gons Π_n converges to a regular k-gon of q-type, if and only if $|\lambda_q| > |\lambda_i|$ for every $i \neq q$ such that $s_i \neq 0$.*

As in the case of triangles, we proceed to the cases when the sequence converges after a finite number of iterations. As follows immediately from (14), we may distinguish between two possibilities:

(i) $s_q \neq 0$ and $s_i = 0$ for every $i \neq q$. This corresponds to Π_0 - and the whole sequence - being regular of q-type.

(ii) There are two integers q, r such that $\lambda_r = 0$, $s_q, s_r \neq 0$, and $s_i = 0$ for every $i \neq q, r$. In this case, Π_0 turns into regular k-gon of q-type after the first iteration. An example will be in order here. Let $k = 4$, $q = 1$, $\lambda_2 = 0$ and $S = (0, 1, 1, 0)$. Then, $t = \frac{1}{2}$ and Π_0 is a concave kite-shaped quadrilateral; the midpoints of its sides form a square, as depicted in Figure 7.

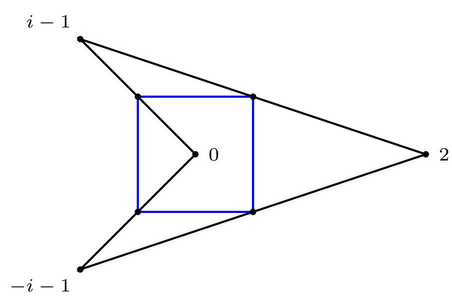

Figure 7

The last theorem shows that the convergence of the sequence of polygons depends on the shapes of both the transformation triangle and the original polygon Π_0. Let us now consider for what transformation triangles the sequence converges for any Π_0? Obviously, this will be the case if no two eigenvalues (13) have equal moduli. That is, for every pair of distinct integers q, r,

$$|(1-t) + t\zeta_k^q| \neq |(1-t) + t\zeta_k^r|.$$

Dividing both sides by $1-t$, we conclude that $\frac{t}{1-t}\zeta_k^q$ and $\frac{t}{1-t}\zeta_k^r$ should not be complex conjugates, that is:

$$\arg\left(\frac{t}{1-t}\right) \neq -\frac{q+r}{k}\pi, \qquad 0 \leq q, r \leq k. \tag{15}$$

Solving for t and designating ℓ for $(q+r) \bmod k$, we get:

$$\frac{\operatorname{Im}(t)}{\operatorname{Re}(t)-|t|^2} \neq \frac{\ell}{k}\pi, \quad 0 \le \ell < k.$$

This last inequality is given a geometric interpretation in the following final theorem.

Theorem 7. *The sequence of k-gons converges to a regular k-gon for every Π_0 if and only if $t = z(X, Y, Z)$ does not lie on any circle $O(R)$ defined as follows:*

$$O = \left(\frac{1}{2},\ \frac{1}{2}\cot\frac{\ell}{k}\pi\right), \quad R = \frac{1}{2}\csc\frac{\ell}{k}\pi, \qquad 0 \le \ell < k.$$

We conclude with a curious application of the last result. Let $k = 5$, and XYZ be a negatively oriented equilateral triangle, *i.e.*, $t = \overline{\eta}$. It follows from (15) that the sequence of pentagons converges for any given Π_0. Let Π_0 be similar to

$$(1+\epsilon,\ \zeta_5,\ \zeta_5^2,\ \zeta_5^3,\ \zeta_5^4).$$

Taking $\epsilon \neq 0$ sufficiently small, Π_0 may be made as close to the regular convex pentagon as we please. The striking fact is that $q = 2$! Figures 8 depict this transforming of an "almost regular" convex pentagon into an "almost regular" pentagram in just 99 iterations.

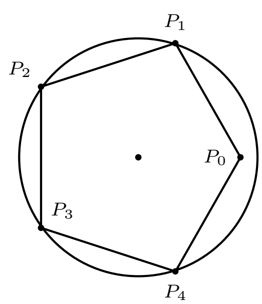

Figure 8a: $n = 0$

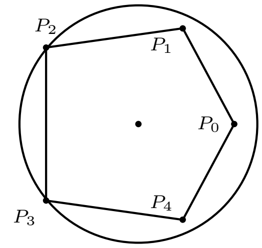

Figure 8b: $n = 20$

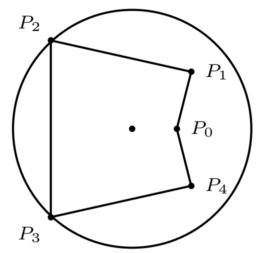

Figure 8c: $n = 40$

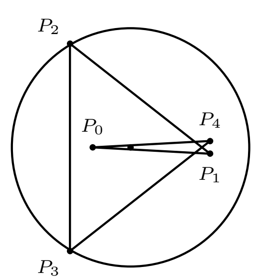

Figure 8d: $n = 60$

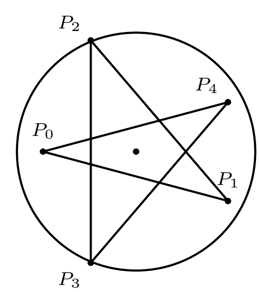

Figure 8e: $n = 80$

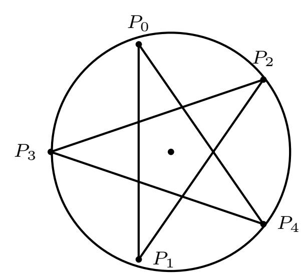

Figure 8f: $n = 99$

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An Elementary Proof of the Isoperimetric Inequality

Nikolaos Dergiades

Abstract. We give an elementary proof of the isoperimetric inequality for polygons, simplifying the proof given by T. Bonnesen.

We present an elementary proof of the known inequality $\mathcal{L}^2 \geq 4\pi\mathcal{A}$, where $\mathcal{L}$ and $\mathcal{A}$ are the perimeter and the area of a polygon. It simplifies the proof given by T. Bonnesen [1, 2].

Theorem. *In every polygon with perimeter $\mathcal{L}$ and area $\mathcal{A}$ we have $\mathcal{L}^2 \geq 4\pi\mathcal{A}$.*

Proof. It is sufficient to prove the inequality for a convex polygon $ABM \cdots Z$. From the vertex A of the polygon we can draw the segment AQ dividing the polygon in two polygons such that

(1) $AB + BM + \cdots + PQ = \frac{\mathcal{L}}{2}$, and
(2) the area $\mathcal{A}_1$ of the polygon $ABM \cdots PQA$ satisfies $\mathcal{A}_1 \geq \frac{\mathcal{A}}{2}$.

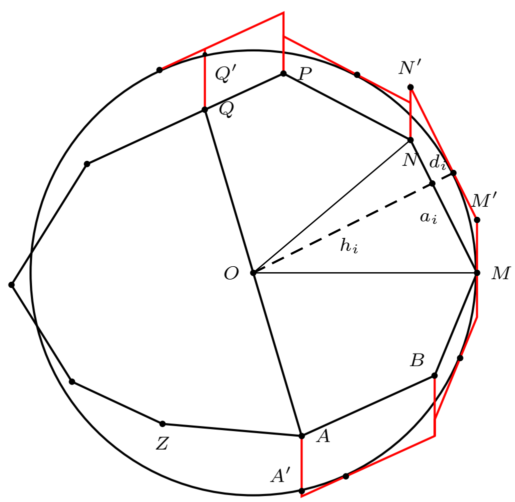

Figure 1

Let O be the mid-point of AQ, and let M be the vertex of $ABM \cdots PQA$ farthest from O, with $OM = \mathcal{R}$. Draw the circle $(O, \mathcal{R})$, and from the points A and Q draw perpendiculars to OM to meet the circle at A', Q' respectively. Because of symmetry, the part of the circle $AA'MQ'QA$ has area $\mathcal{S}$ equal to half of the area of the circle, *i.e.*, $\mathcal{S} = \frac{1}{2}\pi\mathcal{R}^2$. Outside the polygon $ABM \cdots PQ$ construct parallelograms touching the circle, with bases the sides such as $MN = a_i$ and

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other sides parallel to AA'. If h_i is the altitude of triangle OMN and d_i is the height of the parallelogram $MM'N'N$, then $h_i + d_i = \mathcal{R}$. Note that $\mathcal{A}_1$ is the sum of the areas of triangles $OAB, \ldots, OMN, \ldots, OPQ$, *i.e.*,

$$\mathcal{A}_1 = \frac{1}{2}\sum_i a_i h_i.$$

If we denote by $\mathcal{A}_2$ the sum of the areas of the parallelograms, we have

$$\mathcal{A}_2 = \sum_i a_i d_i = \sum_i a_i(\mathcal{R} - h_i) = \mathcal{R} \cdot \frac{\mathcal{L}}{2} - 2\mathcal{A}_1.$$

Since $\mathcal{A}_1 + \mathcal{A}_2 \geq \mathcal{S}$, we have $\mathcal{R} \cdot \frac{\mathcal{L}}{2} - \mathcal{A}_1 \geq \frac{1}{2}\pi\mathcal{R}^2$, and so $\pi\mathcal{R}^2 - \mathcal{L}\mathcal{R} + 2\mathcal{A}_1 \leq 0$. Rewriting this as

$$\pi\left(\mathcal{R} - \frac{\mathcal{L}}{2\pi}\right)^2 - \left(\frac{\mathcal{L}^2}{4\pi} - 2\mathcal{A}_1\right) \leq 0,$$

we conclude that $\mathcal{L}^2 \geq 4\pi \cdot 2\mathcal{A}_1 \geq 4\pi\mathcal{A}$. $\square$

The above inequality, by means of limits can be extended to a closed curve. Since for the circle the inequality becomes equality, we conclude that of all closed curves with constant perimeter $\mathcal{L}$, the curve that contains the maximum area is the circle.

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The Perimeter of a Cevian Triangle

Nikolaos Dergiades

Abstract. We show that the cevian triangles of certain triangle centers have perimeters not exceeding the semiperimeter of the reference triangle. These include the incenter, the centroid, the Gergonne point, and the orthocenter when the given triangle is acute angled.

1. Perimeter of an inscribed triangle

We begin by establishing an inequality for the perimeter of a triangle inscribed in a given triangle ABC.

Proposition 1. *Consider a triangle ABC with $a \leq b \leq c$. Denote by X, Y, Z the midpoints of the sides BC, CA, and AB respectively. Let D, E, F be points on the sides BC, CA, AB satisfying the following two conditions:*

(1.1) *D is between X and C, E is between Y and C, and F is between Z and B.*

(1.2) *$\angle CDE \leq \angle BDF$, $\angle CED \leq \angle AEF$, and $\angle BFD \leq \angle AFE$.*

Then the perimeter of triangle DEF does not exceed the semiperimeter of triangle ABC.

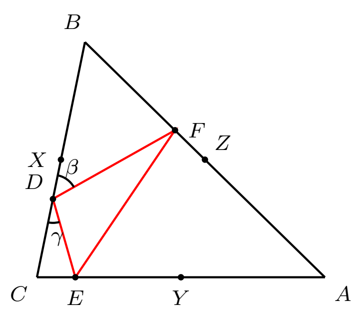

Figure 1

Proof. Denote by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the unit vectors along $\mathbf{EF}$, $\mathbf{FD}$, $\mathbf{DE}$. See Figure 1. Since $\angle BFD \leq \angle AFE$, we have $\mathbf{i} \cdot \mathbf{ZF} \leq \mathbf{j} \cdot \mathbf{ZF}$. Similarly, since $\angle CDE \leq \angle BDF$ and $\angle CED \leq \angle AEF$, we have $\mathbf{j} \cdot \mathbf{XD} \leq \mathbf{k} \cdot \mathbf{XD}$ and $\mathbf{i} \cdot \mathbf{EY} \leq \mathbf{k} \cdot \mathbf{EY}$. Now, we have

$$\begin{aligned} EF + FD + DE &= \mathbf{i} \cdot \mathbf{EF} + \mathbf{j} \cdot \mathbf{FD} + \mathbf{k} \cdot \mathbf{DE} \\ &= \mathbf{i} \cdot (\mathbf{EY} + \mathbf{YZ} + \mathbf{ZF}) + \mathbf{j} \cdot (\mathbf{FZ} + \mathbf{ZX} + \mathbf{XD}) + \mathbf{k} \cdot \mathbf{DE} \\ &\leq (\mathbf{k} \cdot \mathbf{EY} + \mathbf{i} \cdot \mathbf{YZ} + \mathbf{j} \cdot \mathbf{ZF}) \\ &\quad + (\mathbf{j} \cdot \mathbf{FZ} + \mathbf{j} \cdot \mathbf{ZX} + \mathbf{k} \cdot \mathbf{XD}) + \mathbf{k} \cdot \mathbf{DE} \end{aligned}$$

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$$\begin{aligned} &= \mathbf{i}\cdot\mathbf{YZ}+\mathbf{j}\cdot\mathbf{ZX}+\mathbf{k}\cdot\mathbf{XY} \\ &\leq |\mathbf{i}||\mathbf{YZ}|+|\mathbf{j}||\mathbf{ZX}|+|\mathbf{k}||\mathbf{XY}| \qquad (1) \\ &= YZ+ZX+XY \\ &= \frac{1}{2}(AB+BC+CA). \end{aligned}$$

Equality holds in (1) only when the triangles DEF and XYZ have parallel sides, *i.e.*, when the points D, E, F coincide with the midpoints X, Y, Z respectively, as is easily seen. □

2. Cevian triangles

Proposition 2. *Suppose the side lengths of triangle ABC satisfy $a \leq b \leq c$. Let P be an interior point with (positive) homogeneous barycentric coordinates $(x : y : z)$ satisfying*

(2.1) $x \leq y \leq z$,

(2.2) $x \cot A \geq y \cot B \geq z \cot C$.

Then the perimeter of the cevian triangle of P does not exceed the perimeter of the medial triangle of ABC, i.e.*, the cevian triangle of the centroid.*

Proof. In Figure 1, $BD = \frac{az}{y+z}$, $DC = \frac{ay}{y+z}$, and $BF = \frac{cx}{x+y}$. Since $y \leq z$, it is clear that $BD \geq DC$. Similarly, $AE \geq EC$, and $AF \geq FB$. Condition (1.1) is satisfied. Applying the law of sines to triangle BDF, we have $\frac{\sin(B+\beta)}{\sin\beta} = \frac{BD}{BF}$. It follows that

$$\frac{\sin(B+\beta)}{\sin B \sin\beta} = \frac{\sin(B+C)}{\sin B\sin C}\cdot\frac{z(x+y)}{x(y+z)}.$$

From this, $\cot\beta + \cot B = (\cot B + \cot C)\cdot\frac{z(x+y)}{x(y+z)}$. Similarly, $\cot\gamma + \cot C = (\cot B + \cot C)\cdot\frac{y(z+x)}{x(y+z)}$. Consequently,

$$\cot\gamma - \cot\beta = \frac{2(y\cot B - z\cot C)}{y+z},$$

so that $\beta \geq \gamma$ provided $y \cot B \geq z \cot C$. The other two inequalities in (1.2) can be similarly established. The result now follows from Proposition 1. □

This applies, for example, to the following triangle centers. For the case of the orthocenter, we require the triangle to be acute-angled. [1] It is easy to see that the barycentrics of each of these points satisfy condition (2.1).

P	$(x : y : z)$	$x\cot A \geq y\cot B \geq z\cot C$
Incenter	$(a : b : c)$	$\cos A \geq \cos B \geq \cos C$
Centroid	$(1 : 1 : 1)$	$\cot A \geq \cot B \geq \cot C$
Orthocenter	$(\tan A : \tan B : \tan C)$	$1 \geq 1 \geq 1$
Gergonne point	$\left(\tan\frac{A}{2} : \tan\frac{B}{2} : \tan\frac{C}{2}\right)$	$\frac{1}{2}(1-\tan^2\frac{A}{2}) \geq \frac{1}{2}(1-\tan^2\frac{B}{2})$ $\geq \frac{1}{2}(1-\tan^2\frac{C}{2})$

[1] For the homogeneous barycentric coordinates of triangle centers, see [1].

The perimeter of the cevian triangle of each of these points does not exceed the semiperimeter of ABC.[2] The case of the incenter can be found in [2].

3. Another example

The triangle center with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$ provides another example of a point P the perimeter of whose cevian triangle not exceeding the semiperimeter of ABC. It clearly satisfies (2.1). Since $\sin\frac{A}{2}\cot A = \cos\frac{A}{2} - \frac{1}{2\cos\frac{A}{2}}$, it also satisfies condition (2.2). In [1], this point appears as X_{174} and is called the Yff center of congruence. Here is another description of this triangle center [3]:

The tangents to the incircle at the intersections with the angle bisectors farther from the vertices intersect the corresponding sides at the traces of the point with homogeneous barycentric coordinates $(\sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2})$.

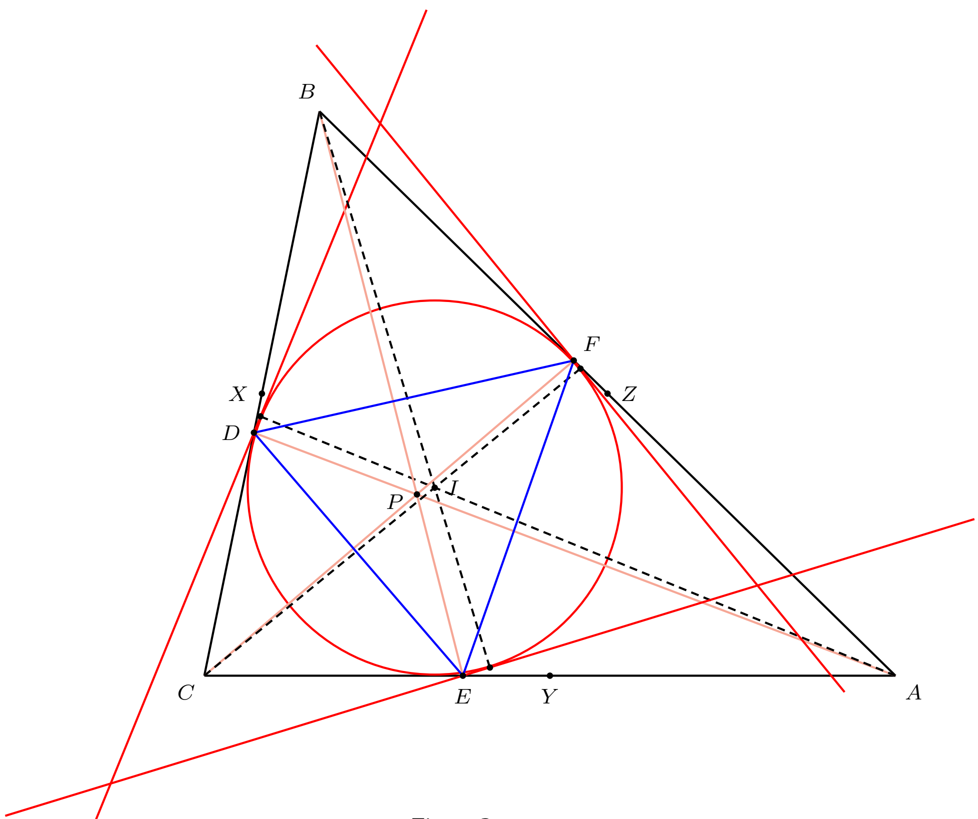

Figure 2

[2]The Nagel point, with homogeneous barycentric coordinates $(\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2})$, also satisfies (2.2). However, it does not satisfy (2.1) so that the conclusion of Proposition 2 does not apply. The same is true for the circumcenter.

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Geometry and Group Structures of Some Cubics

Fred Lang

Abstract. We review the group structure of a cubic in the projective complex plane and give group theoretic formulations of some geometric properties of a cubic. Then, we apply them to pivotal isocubics, in particular to the cubics of Thomson, Darboux and Lucas. We use the group structure to identify different transformations of cubics. We also characterize equivalence of cubics in terms of the Salmon cross ratio.

1. The group structure of a cubic

Let Γ be a nonsingular cubic curve in the complex projective plane, *i.e.*, Γ has no cusp and no node. It is well known that Γ has a group structure, which does not depend on the choice of a neutral element O on the cubic. In other words, the group structures on Γ for various choices of the neutral elements are isomorphic.

If P and Q are points of a cubic Γ, we denote by $P \cdot Q$ the third intersection of the line PQ with Γ. In particular, $P \cdot P := P_t$ is the *tangential* of P, the second intersection of Γ with the tangent at P.

Proposition 1. *The operation $\cdot$ is commutative but not associative. For P, Q, R on Γ,*

(1) $(P \cdot Q) \cdot P = Q$,
(2) $P \cdot Q = R \cdot Q \Longleftrightarrow P = R$,
(3) $P \cdot Q = R \Longleftrightarrow P = R \cdot Q$.

Convention: When we write $P \cdot Q \cdot R$, we mean $(P \cdot Q) \cdot R$.

We choose a point O on Γ as the neutral point,[1] and define a group structure $+$ on Γ by

$$P + Q = (P \cdot Q) \cdot O.$$

We call the tangential of O, the point $N = O_t = O \cdot O$, the *constant point* of Γ. Note that $-N = N_t$, since $N + N_t = N \cdot N_t \cdot O = N \cdot O = O$.

We begin with a fundamental result whose proof can be found in [4, p.392].

Theorem 2. *$3k$ points P_i, $1 \leq i \leq 3k$, of a cubic Γ are on a curve of order k if and only if $\sum P_i = kN$.*

For $k = 1, 2, 3$, we have the following corollary.

Corollary 3. *Let $P, Q, R, S, T, U, V, W, X$ be points of Γ.*

(1) *P, Q, R are collinear if and only if $P + Q + R = N$.*
(2) *P, Q, R, S, T, U are on a conic if and only if $P+Q+R+S+T+U = 2N$.*

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[1] O is not necessarily an inflexion point (a flex).

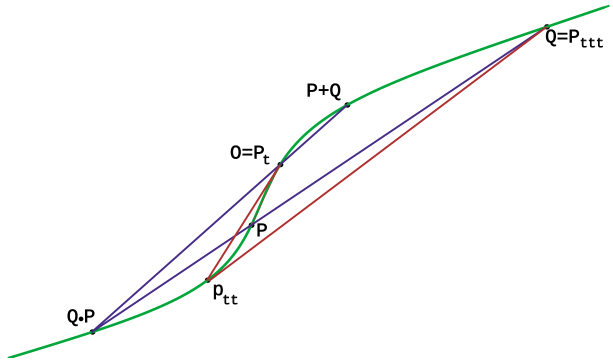

Figure 1. The first three tangentials of P and $P+Q$

(3) *P, Q, R, S, T, U, V, W, X are on a cubic if and only if* $P+Q+R+S+T+U+V+W+X=3N$.

Remark. The case $k=2$ is equivalent to the following property.

Geometric formulation	Group theoretic formulation
Let P, Q, R, S, T, U be six points of a cubic Γ, and let $X=P\cdot Q$, $Y=R\cdot S$, $Z=T\cdot U$, then P, Q, R, S, T, U are on a conic if and only if X, Y, Z are collinear.	Let P, Q, R, S, T, U be six points of a cubic Γ, and let $P+Q+X=N$, $R+S+Y=N$, $T+U+Z=N$, then $P+Q+R+S+T+U=2N$ if and only if $X+Y+Z=N$.

A geometrical proof is given [8, p.135]; an algebraic proof is a straightforward calculation.

We can do normal algebraic calculations in the group, but have to be careful to torsion points: for example $2P=O$ does not imply $P=O$. The group of Γ has non zero torsion points, i.e, points with the property $kP=O$, for $P\neq O$. Indeed the equation $kX=Q$ has k^2 (complex) solutions for the point X. See [10, 17].

The tangential P_t of P is $N-2P$, since P, P, and P_t are collinear. The *second tangential* P_{tt} of P is $N-2(N-2P)=-N+4P$. The *third tangential* is $N-2(-N+4P)=3N-8P$.

2. A sample of theorems on cubics

We give a sample of theorems on cubics, in both geometric and group-theoretic formulations. Most of the theorems can be found in [8, p.135]. In the following table, all points are on a cubic Γ. A point $P\in\Gamma$ is a *sextatic* point if there is a conic through P with contact of order 6 with Γ at P.

	Geometric formulation	Group theoretic formulation
1	P and Q have the same tangential.	$2P = 2Q$ or $2(P-Q) = O$
2	There are four tangents from P.	$2X + P = N$ has four solutions
3	P is a flex	$3P = N$
4	Γ has nine flexes	$3P = N$ has nine solutions
5	If P and Q are flexes, then $R = P \cdot Q$ is another flex. If $P \neq Q$, then $R \neq P,\ Q$.	$3P = N$, $3Q = N$, and $P + Q + R = N$ $\Longrightarrow 3R = N$.
6	Let P_1, P_2, P_3 and P_4 be fixed on Γ. If a variable conic intersects Γ at $P_1 \dots, P_6$, then the line P_5P_6 passes through a fixed point Q on Γ, which we call the *coresidual* of P_1, P_2, P_3, P_4.	$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 2N$ and $P_5 + P_6 + Q = N$ $\Longrightarrow Q = -N + P_1 + P_2 + P_3 + P_4$, which is fixed.
7	If a conic intersects Γ at $P_1, \dots P_6$, then the tangentials $Q_1, \dots, Q_6$ are on another conic	$\sum P_i = 2N$, $2P_i + Q_i = N$ for $i = 1, \dots, 6$ $\Longrightarrow \sum Q_i = 2N$.
8	Let Ω be a conic *tritangent* to Γ at P, Q, R, and let Ψ be another conic which intersects Γ at P, Q, R, P', Q', R', then there exists a conic Λ tangent to Γ at P', Q', R'.	$2P + 2Q + 2R = 2N$ and $P + Q + R + P' + Q' + R' = 2N$ $\Longrightarrow 2P' + 2Q' + 2R' = 2N$.
9	A conic Ω is tritangent to Γ at P, Q, R if and only if the tangentials P', Q', R' of P, Q, R are collinear.	For $2P + P' = N$, $2Q + Q' = N$, and $2R + R' = N$, $2P + 2Q + 2R = 2N$ $\Longleftrightarrow P' + Q' + R' = N$.
10	If Q, R, S are given points, there exist 9 points X such that a conic *osculates* at X and passes through Q, R, S	The equation $3X + Q + R + S = 2N$ has nine solutions.
11	P is sextatic if and only if the tangent at P contains a flex Q different from P.	For $2P + Q = N$, $6P = 2N \Longleftrightarrow 3Q = N$.
12	P is sextatic if and only if P is the tangential of a flex Q.	$6P = 2N \Longleftrightarrow$ $2Q + P = N$ and $3Q = N$.
13	There are 27 sextatic points on a cubic.	$6P = 2N$ has 36 solutions, nine are the flexes, the others 27 are the sextatic points.
14	If P and Q are sextatic, then $R = P \cdot Q$ is sextatic.	$6P = 2N$, $6Q = 2N$ and $P + Q + R = N$ $\Longrightarrow 6R = 2N$.

Remarks. The coresidual in (6) is called the *gegenüberliegende Punkt* in [8, p.140].

3. The group structure of a pivotal isocubic

Let $P \mapsto P^*$ be a given isoconjugation in the plane of the triangle ABC (with trilinear coordinates). See, for example, [5]. For example, $P(x:y:z) \mapsto P^*(\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ is the isogonal transformation and $P(x:y:z) \mapsto (\frac{1}{a^2x} : \frac{1}{b^2y} : \frac{1}{c^2z})$ is the isotomic transformation. We shall also consider the notion of cevian quotient. For any two points P and Q, the cevian triangle of P and the precevian triangle of

Q are always perspective. We call their perspector the *cevian quotient* P/Q. See [11].

Let Γ be a pivotal isocubic with pivot F. See, for example, [6, 7, 14]. Take the pivot F for the neutral element O of the group. The constant point is $N = F_t$.

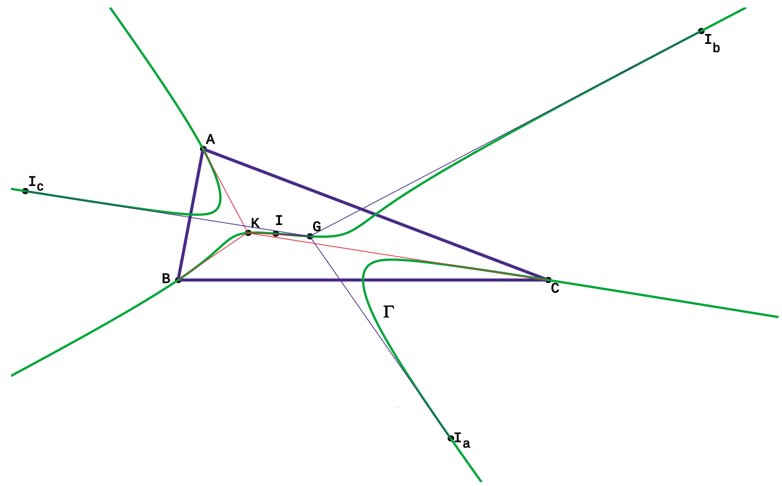

Figure 2. Two tangential quadruples on the Thomson cubic

Definition. Four points of Γ form a *tangential quadruple* if they have the same tangential point.

Theorem 4. *Consider the group structure on a pivotal isocubic with the pivot F as neutral element. The constant point is $N = F_t$.*

(1) $P \cdot P^* = F$, $P \cdot F = P^*$, $P^* \cdot F = P$.
(2) $F_t = F^*$.
(3) $P + P^* = F_t$.
(4) $P + Q = (P \cdot Q)^*$ *or* $P \cdot Q = (P + Q)^*$.
(5) P, Q, R *are collinear if and only if* $P + Q + R = F_t$.
(6) $-P = P \cdot F_t$.
(7) $-P = F/P$.
(8) *If* (P, Q, R, S) *is a tangential quadruple then* (P^*, Q^*, R^*, S^*) *is also a tangential quadruple.*
(9) *Every tangential quadruple is of the form* $(P, P + A, P + B, P + C)$.
(10) A, B, C *are points of order 2,* i.e., $2A = 2B = 2C = F$.

Proof. (1) F is the pivot, so P, P^* and F are collinear.
(2) Put $P = F$ in (1).
(3) $P + P^* = (P \cdot P^*) \cdot F = F \cdot F = F_t$.
(4) $P + Q = (P \cdot Q) \cdot F = (P \cdot Q)^*$. (use (1))
(5) This is Corollary 3.
(6) $P + (P \cdot F_t) = (P \cdot (P \cdot F_t)) \cdot F = F_t \cdot F = F$.

(7) If the pivot F has trilinear coordinates $(u : v : w)$ and $P(x : y : z)$, then the Cevian quotient F/P is the point

$$(x(-vwx + uwy + uvz) : \ y(vwx - uwy + uvz) : \ z(vwx + uwy - uvz)).$$

We can verify that it is on Γ and is collinear with P and F_t.

(8) We have to prove that, if P and Q have a common tangential T, then P^* and Q^* have a common tangential U. Let U be the tangential of P^*, then (5) and (2) give

$$U + 2P^* = F_t = F^*.$$

Since F, P, P^* are collinear, and so are F, Q, Q^*, we have

$$P + P^* = F^* \quad \text{and} \quad Q + Q^* = F^*.$$

Since T is the common tangential of P and Q,

$$2P + T = F^* \quad \text{and} \quad 2Q + T = F^*.$$

From these,

$$\begin{aligned}
U + 2Q^* =&(F^* - 2P^*) + 2Q^* \\
=&F^* - 2(F^* - P) + 2(F^* - Q) \\
=&F^* + 2P - 2Q \\
=&F^* + F^* - T - F^* + T \\
=&F^*,
\end{aligned}$$

and U is the tangential of Q^* too.

(9) We have to prove that, if P is on the cubic, P and $P + A$ have the same tangential. Let Q and Q_a be the tangential of P and $P + A$ respectively. By property (3), $P + P + Q = F^*$ and $(P + A) + (P + A) + Q_a = F^*$. Hence $Q = Q_a \iff 2A = 0 \iff A = -A$. By properties (6) and (2), $-A = A \cdot F^*$, hence we have to prove that $A = A \cdot F^*$, i.e. the tangential of A is F^*. The equation of the tangent to the cubic at A is $r^2vy = q^2wz$, and $F^*(p^2vw : q^2uw : r^2uv)$ is clearly on this line. But F^* is on Γ. Hence it is the tangential point of A.

(10) $2A = A + A = A \cdot A = A_t^* = F^{**} = F$. □

A consequence of (10) is that the cubic is not connected. See, for example, [10, p.20].

4. The Thomson, Darboux and Lucas cubics

These well-known pivotal cubics have for pivots G (centroid), L (de Longchamps point) and K_+ (isotomic of the orthocenter H). Thomson and Darboux are isogonal cubics and Lucas is an isotomic one. We study the subgroups generated by the points G, I, A, B, C for Thomson, L, I', A, B, C for Darboux and K_+, N_o, A, B, C for Lucas. For a generic triangle, [2] these groups are isomorphic to $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

[2] This may be false for some particular triangles. For example, if ABC has a right angle at A, then $H = A$ and for Thomson, $H = 4I$.

Notation. For each point P, we denote by P_a, P_b, P_c the points $P+A$, $P+B$, $P+C$ respectively. We use the notations of [12] for triangle centers, but adopt the following for the common ones.

G	centroid
K	symmedian (Lemoine) point
H	orthocenter
O	circumcenter
I	incenter; I_a, I_b, I_c are the excenters
L	de Longchamps point
M	Mittenpunkt
G_o	Gergonne point
N_o	Nagel point

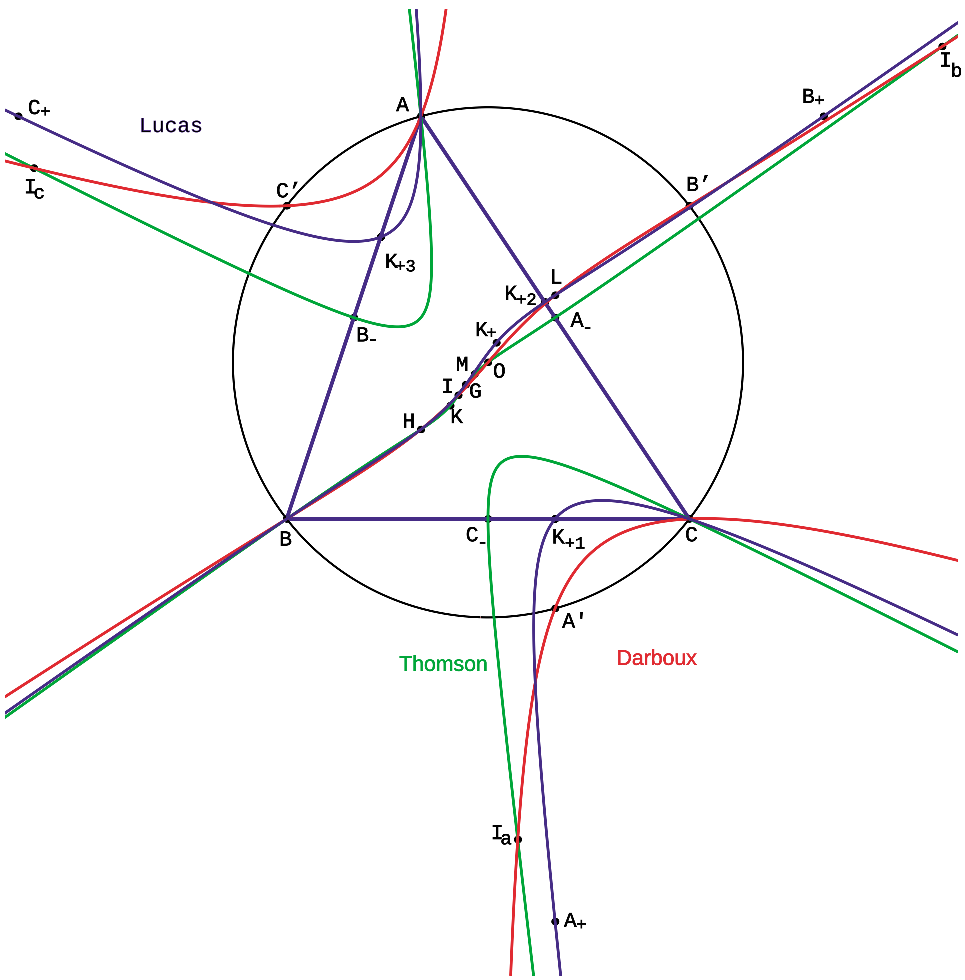

Figure 3. The Thomson, Darboux and Lucas cubics

In the following table, the lines represent the $\mathbb{Z}$-part, and the columns the $\mathbb{Z}_2 \times \mathbb{Z}_2$-part. The last column give the tangential point of the tangential quadruple of the corresponding line. The line number 0 is the subgroup generated by the pivot and A, B, C. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4.1. *The Thomson cubic.*

	P	P_a	P_b	P_c	tangential
-6	H_t	H_{ta}	H_{tb}	H_{tc}	
-5	$-X_{282}$	$-X_{282a}$	$-X_{282b}$	$-X_{282c}$	
-4	O_t^*	O_{ta}^*	O_{tb}^*	O_{tc}^*	
-3	X_{223}	X_{223a}	X_{223b}	X_{223c}	
-2	O	A_-	B_-	C_-	O_t
-1	M	M_a	M_b	M_c	H
0	G	A	B	C	K
1	I	I_a	I_b	I_c	G
2	K	A_-	B_-	C_-	O
3	M^*	M_a^*	M_b^*	M_c^*	O_t^*
4	H	H_a	H_b	H_c	H_t
5	X_{282}	X_{282a}	X_{282b}	X_{282c}	
6	O_t	O_{ta}	O_{tb}	O_{tc}	O_{tt}

(1) Neutral = pivot = G = centroid.
(2) Constant $= G_t = G^* = K$.
(3) Three points are collinear if and only if their sum is 2.
(4) Examples of calculation:
   (a) $I + K = (I \cdot K) \cdot G = M \cdot G = M^*$.
   (b) $A + A = (A \cdot A) \cdot G = K \cdot G = G$.
   (c) To find the intersection X of the line OM with the cubic, we have to solve the equation $x + (-2) + (-1) = 2$. Hence, $x = 5$ and $X = X_{282}$.
(5) A_-, B_-, C_- are the midpoints of the sides of ABC, diagonal triangle of $GABC$.
(6) A^-, B^-, C^- are the midpoints of the altitudes of ABC, diagonal triangle of $KA_-B_-C_-$.
(7) O_t is the isoconjugate of the circumcenter O relative to the pencil of conics through the points K, A_-, B_-, C_-.
(8) O_{tt} is the isoconjugate of O_t relative to the pencil of conics through the points O, A^-, B^-, C^-.
(9) $O_{ta}O_{tb}O_{tc}$ is the diagonal triangle of $OA^-B^-C^-$.
(10) $H_a = A^{-*} = OA \cap A^-G = B^-C \cap C^-B$.
(11) $X_{223} = -(M^*)$ is the third intersection of the line IH and Γ. (Proof: $I + H + X_{223} = 1 + (-3) + 4 = 2 =$ constant).
(12) If a point X has the line number x, then the points X^*, X_t and G/X have line numbers $2 - x$, $2 - 2x$ and $-x$.

4.2. *The Darboux cubic.*

	P	P_a	P_b	P_c	tangential
-6	L_t^*				
-5	$-I^{'*}$				
-4	$-H$				
-3	$I^{'*'*'}$	$I_a^{'*'*'}$	$I_b^{'*'*'}$	$I_c^{'*'*'}$	
-2	$L^{*'}$	L_a'	L_b'	L_c'	
-1	$I^{'*'}$	$I_a^{'*'}$	$I_b^{'*'}$	$I_c^{'*'}$	
0	L	A	B	C	L^*
1	I'	I_a'	I_b'	I_c'	H
2	O	$H_{1\infty}$	$H_{2\infty}$	$H_{3\infty}$	O
3	I	I_a	I_b	I_c	L
4	H	A'	B'	C'	$L_{*'}$
5	$I^{'*}$	$I_a^{'*}$	$I_b^{'*}$	$I_c^{'*}$	
6	L^*	L_1	L_2	L_3	L_t^*
7	$I^{'*'*}$	$I_a^{'*'*}$	$I_b^{'*'*}$	$I_c^{'*'*}$	
8	$L^{*'*}$				

(1) Neutral = pivot = L = de Longchamps point = symmetric of H relative to O; constant point = L^*.
(2) Three points are collinear if and only if their sum is 6.
(3) L_1, L_2, L_3 = Cevian points of L.
(4) $H_{1\infty} =$ infinite point in the direction of the altitude AH.
(5) P' is the symmetric of P relative to O. (Symmetry relative to line 2)
(6) $P^{'*}$ gives the translation of $+2$ and $P^{*'}$ of -2. Three points are collinear if and only if their sum is 6.
(7) If a point X has the line number x, then the points X^*, X_t, X' and G/X have line numbers $6-x$, $6-2x$, $4-x$ and $-x$.

4.3. *The Lucas cubic.*

	P	P_a	P_b	P_c	tangential
-4	K_{+t}				
-3	$-G_o$	$-G_a$	$-G_b$	$-G_c$	
-2	L	L_a	L_b	L_c	X_{1032}
-1	X_{329}	X_{329a}	X_{329b}	X_{329c}	L^r
0	K_+	A	B	C	H
1	N_o	N_a	N_b	N_c	G
2	G	A_+	B_+	C_+	K_+
3	G_o	G_a	G_b	G_c	L
4	H	K_{+1}	K_{+2}	K_{+3}	K_{+t}
5	X_{189}	X_{189a}	X_{189b}	X_{189c}	
6	L^r	L_a^r	L_b^r	L_c^r	
7	X_{1034}	X_{1034a}	X_{1034b}	X_{1034c}	
8	X_{1032}				

(1) Neutral = K_+ = Lemoine point of the precevian triangle $A_+B_+C_+$ of ABC = isotomic of H; Constant point = H. Three points are collinear if and only if their sum is 4.
(2) $P^r =$ isotomic of P (symmetry relative to line 2).
(3) K_{+1}, K_{+2}, K_{+3} = cevian points of K_+ = intersections of Lucas cubic with the sides of ABC.
(4) $X_{329} =$ intersection of the lines N_oH and G_oG with the cubic.
(5) If a point X has line number x, then the points X^*, X_t and G/X have line numbers $4-x$, $4-2x$ and $-x$.
(6) $X_{329}^r = X_{189}$.

5. Transformations of pivotal isocubics

We present here some general results without proofs. See [16, 15, 9, 4].

5.1. *Salmon cross ratio.* The Salmon cross ratio of a cubic is the cross ratio of the four tangents issued from a point P of Γ. It is defined up to permutations of the tangents. We shall therefore take it to be a set of the form

$$\left\{\lambda,\ \lambda-1,\ \frac{1}{\lambda},\ \frac{1}{\lambda-1},\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda}\right\},$$

since if λ is a Salmon cross ratio, then we obtain the remaining five values of permutation of the tangents.

A cubic Γ is *harmonic* if $\lambda = -1$; it is *equiharmonic* if λ satisfies $\lambda^2-\lambda+1=0$.

The Salmon cross ratio is independent of the choice of P.

5.2. *Birational equivalence.* A transformation [9] of Γ is *birational* if the transformation and its inverse are given by rational functions of the coordinates. [3] Two cubics Γ_1 and Γ_2 are equivalent if there is a birational transformation $\Gamma_1 \to \Gamma_2$.

Theorem 5. *A birational transformation of a cubic Γ onto itself induces a transformation of its group of the form $x \mapsto ux+k$, where*

(1) $u^2 = 1$ *for a general cubic,*
(2) $u^4 = 1$ *for a harmonic cubic, and*
(3) $u^6 = 1$ *for an equiharmonic cubic.*

Theorem 6. *Two equivalent cubics have isomorphic groups.*

Examples:

1) The groups of the cubics of Darboux, Thomson and Lucas are isomorphic.

2) The transformation that associate to a point its tangential is given by $X \mapsto N-2X$ and is not birational.

Theorem 7. *Two cubics Γ_1 and Γ_2 are equivalent if and only if their Salmon cross ratios are equal.*

[3]Cautions: Two different transformations of the projective plane may induce the same transformation on curves. see [15].

If the isoconjugation has fixed point $(p:q:r)$, it is easy to prove the following result:

Theorem 8. *A pivotal isocubic of pivot* $(u:v:w)$ *has Salmon cross ratio*

$$\frac{q^2(r^2u^2-p^2w^2)}{r^2(q^2u^2-p^2v^2)}.$$

For example, the cubics of Darboux, Thomson, Lucas all have Salmon cross ratio

$$\frac{b^2(a^2-c^2)}{c^2(a^2-b^2)}.$$

Are Thomson, Darboux and Lucas the only equivalent pivotal cubics? No! Here is a counter-example. Take the isoconjugation with fixed point X_{63}. The pivotal isocubic of pivot X_{69} (the same as Lucas) is equivalent to Thomson.

6. Examples of birational transformations of cubics

We give now a list of birational transformations, with the corresponding effects on the lines of the group table. Recall that N is the tangential of the pivot, *i.e.*, the constant point.

6.1. *Projection:* $\Gamma \to \Gamma$. Let $P \in \Gamma$. A projection of Γ on itself from P gives a transformation $X \mapsto X'$ so that P, X, X' are collinear:

$$x \mapsto n - p - x.$$

6.2. *Cevian quotient:* $\Gamma \to \Gamma$. Let F be the pivot of Γ, then the involution $X \mapsto F/X$ gives the transformation: $x \mapsto -x$.

6.3. *Isoconjugation:* $\Gamma \to \Gamma$. Since F, X, X^* are collinear, the isoconjugation is a projection from the pivot $F : x \mapsto n - x$.

6.4. *Pinkernell's quadratic transformations.* We recall the definition of the $d-$pedal cubics Γ_d and of the $d-$cevian cubics Δ_d. If P has *absolute* trilinear coordinates (x, y, z), then define P_A, P_B, P_C on the perpendiculars from P to the sides such that $PP_A = dx$, etc. The locus of P for which $P_AP_BP_C$ is perspective to ABC is a cubic Γ_d, and the locus of the perspector is another cubic Δ_d. Hence we have a birational transformation $f_d : \Gamma_d \to \Delta_d$.

The $d-$pedal is different from the $(-d)$-pedal, but the d-cevian is the same as the $(-d)$-cevian.

For example: $\Gamma_1 =$ Darboux , $\Gamma_{-1} =$ Thomson, and $\Delta_1 =$ Lucas.

Let L_d be the pivot of Γ_d and X on Γ_d. Since L_d, X and $f_d(X)$ are collinear we can identify f_d as a projection of Γ_d to Δ_d from the pivot L_d.

These transformations are birational. Hence the groups of the cubics Γ_d, Γ_{-d} and Δ_d are isomorphic.

For $d = 1$, X and $f_d(X)$ are on the same line in the group table: $x \mapsto x$.

6.5. *The quadratic transformations* $h_d : \Gamma_d \longrightarrow \Gamma_{-d}$. Let g_d be the inverse of f_d. Define $h_d = g_d \circ f_d$.

$$x \mapsto x.$$

For $d = 1$, we have a map from Darboux to Thomson. In this case, a simple construction of h_1 is given by: Let P be a point on Darboux and P_i the perpendicular projections of P on the sides of ABC, let A^-, B^-, C^- be the midpoint of the altitudes of ABC, then $Q = h_1(P)$ is the intersection of the lines P_1A^-, P_2B^- and P_3C^-.

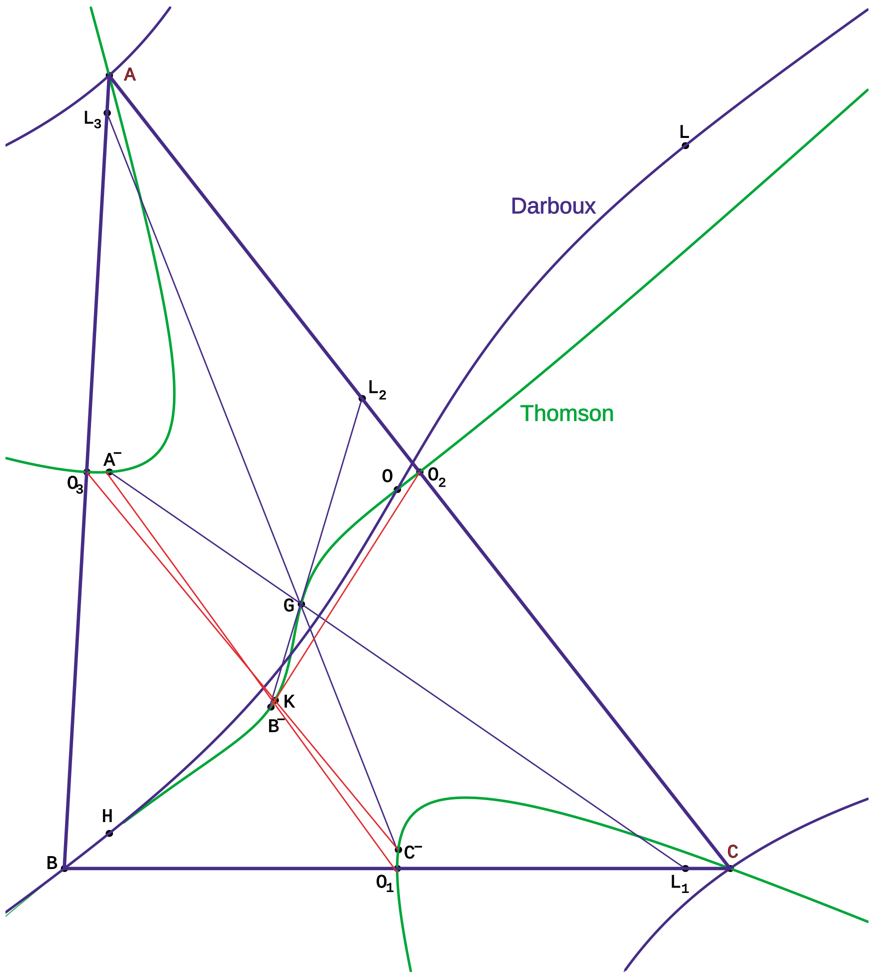

Figure 4. $G = h_1(L)$ and $K = h_1(O)$, h_1 : Darboux $\longrightarrow$ Thomson

6.6. *Cevian, precevian, pedal and prepedal quadratic transformations.* 1. The Lucas cubic is the set of points P such that the cevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is $g_1 : x \mapsto x$.

2. The Lucas cubic is the set of points P such that the cevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the isogonal of $g_1 : x \mapsto 6 - x$.

3. The Thomson cubic is the set of points P such that the precevian triangle of P is the pedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the inverse of $h_1 : x \mapsto x$.

4. The Thomson cubic is the set of points P such that the precevian triangle of P is the prepedal triangle of Q. The locus of Q is the Darboux cubic and the transformation is the symmetric of the inverse of $h_1 : x \mapsto 4 - x$.

This last transformation commutes with isogonality:

Proof: $x \mapsto 4-x \mapsto 6-(4-x) = 2+x$ and $x \mapsto 2-x \mapsto 4-(2-x) = 2+x$.

6.7. *Symmetry of center O of the Darboux cubic and induced transformations on Thomson and Lucas.* The symmetry is a linear transformation of the Darboux cubic: $x \mapsto 4 - x$. It induces via f_d and f_{-d} a quadratic involution of the Thomson cubic: $x \mapsto 4 - x$. And, via f_d and g_d, a quadratic involution of the Lucas cubic: $x \mapsto 4 - x$.

6.8. *Cyclocevian transformation.* The cyclocevian transformation [12] is an involution of the Lucas cubic. It is the symmetry relative to the line 3 of the group table: $x \mapsto 6 - x$.

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On Some Remarkable Concurrences

Charles Thas

Abstract. In [2], Bruce Shawyer proved the following result : "At the midpoint of each side of a triangle, we construct the line such that the product of the slope of this line and the slope of the side of the triangle is a fixed constant. We show that the three lines obtained are always concurrent. Further, the locus of the points of concurrency is a rectangular hyperbola. This hyperbola intersects the sides of the triangle at the midpoints of the sides, and each side at another point. These three other points, when considered with the vertices of the triangle opposite to the point, form a Ceva configuration. Remarkably, the point of concurrency of these Cevians lies on the circumcircle of the original triangle". Here, we extend these results in the projective plane and give a short synthethic proof.

We work in the complex or the real complexified projective plane $\mathcal{P}$. The conic through five points A, B, C, D, E is denoted by $\mathcal{C}(A, B, C, D, E)$ and $(XYZW)$ is the notation for the cross-ratio of four collinear points X, Y, Z, W.

Theorem 1. *Consider a triangle ABC and a line l, not through A, B or C, in $\mathcal{P}$. Put $AB \cap l = C''$, $BC \cap l = A''$, $CA \cap l = B''$ and construct the points A', B' and C' for which $(BCA'A'') = (CAB'B'') = (ABC'C'') = -1$. Then, take two different points I and I' on l (both different from A'', B'', C'') and consider the points A''', B''' and C''' such that $(II'A''A''') = (II'B''B''') = (II'C''C''') = -1$. Then the lines $A'A'''$, $B'B'''$ and $C'C'''$ are concurrent at a point L.*

Proof. The line $A'A'''$ is clearly the polar line of A'' with respect to the conic $\mathcal{C}(A, B, C, I, I')$ and likewise for the line $B'B'''$ and B'', and for the line $C'C'''$ and C''. Thus, $A'A'''$, $B'B'''$ and $C'C'''$ concur at the polar point L of l with respect to $\mathcal{C}(A, B, C, I, I')$. □

Theorem 2. *If I, I' are variable conjugate points in an involution Ω on the line l with double (or fixed) points D and D', then the locus of the point L is the conic $\mathcal{L} = \mathcal{C}(A', B', C', D, D')$. Moreover, putting $\mathcal{L} \cap AB = \{C', Z\}$, $\mathcal{L} \cap BC = \{A', X\}$ and $\mathcal{L} \cap CA = \{B', Y\}$, the triangles ABC and XYZ form a Ceva configuration. The point K of concurrency of the Cevians AX, BY, CZ is the fourth basis point (besides A, B, C) of the pencil of conics $\mathcal{C}(A, B, C, I, I')$.*

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Proof. Since the conics $\mathcal{C}(A, B, C, I, I')$ intersect the line l in the variable conjugate points I, I' of an involution on l, these conics must belong to a pencil with basis points A, B, C and a fourth point K : this follows from the Theorem of Desargues-Sturm (see [1], page 63). So, the locus $\mathcal{L}$ is the locus of the polar point L of the line l with respect to the conics of this pencil. Now, it is not difficult to prove (or even well known) that such locus is the conic through the points $A', B,' C', D, D'$ and through the points K', K'', K''' which are determined by $(AKK'K_1) = (BKK''K_2) = (CKK'''K_3) = -1$, where $K_1 = l \cap KA, K_2 = l \cap KB$ and $K_3 = l \cap KC$, and finally, through the singular points $X = KA \cap BC$, $Y = KB \cap CA, Z = KC \cap AB$ of the degenerate conics of the pencil. This completes the proof. □

Next, let us consider a special case of the foregoing theorems in the Euclidean plane Π. Take a triangle ABC in Π and let $l = l_\infty$ be the line at infinity, while the points D and D' of theorem 2 are the points at infinity of the X-axis and the Y-axis of the rectangular coordinate system in Π, respectively.
Homogeneous coordinates in Π are (x, y, z) and $z = 0$ is the line l_∞; the points D and D' have coordinates $(1, 0, 0)$ and $(0, 1, 0)$, respectively. A line with slope a has an equation $y = ax + bz$ and point at infinity $(1, a, 0)$. Now, if (in Theorem 1) the product of the slopes of the lines BC and $A'A'''$, CA and $B'B'''$, AB and $C'C'''$ is a fixed constant $\lambda (\neq 0)$, then the points at infinity of these lines (i.e. A'' and A''', B'' and B''', C'' and C''') have coordinates of the form $(1, t, 0)$ and $(1, t', 0)$, with $tt' = \lambda$. This means that A'' and A''', B'' and B''', C'' and C''' are conjugate points in the involution on l_∞ with double points $I(1, -\sqrt{\lambda}, 0)$ and $I'(1, \sqrt{\lambda}, 0)$ and thus $(II'A''A''') = (II'B''B''') = (II'C''C''') = -1$. If we let λ be variable, the points I and I' are variable conjugate points in the involution on l_∞ with double points D and D', the latter occuring for $t = 0$ and $t' = \infty$ respectively.

Now all the results of [2], given in the abstract, easily follow from Theorems 1 and 2. For instance, the locus $\mathcal{L}$ is the rectangular hyperbola $\mathcal{C}(A', B', C', D, D')$ (also) through the points K', K'', K''', X, Y, Z. Remark that the basis point K belongs to any conic $\mathcal{C}(A, B, C, I, I')$ and for $\lambda = -1$, we get that $I(1, i, 0)$ and $I'(1, -i, 0)$ are the cyclic points, so that $\mathcal{C}(A, B, C, I, I')$ becomes the circumcircle of ΔABC. For $\lambda = -1$, we have $A'A''' \perp BC, B'B''' \perp CA$ and $C'C''' \perp AB$, and $A'A''', B'B''', C'C'''$ concur at the center O of the circumcircle of ABC.

Remark also that O is the orthocenter of $\Delta A'B'C'$ and that any conic (like $\mathcal{L}$) through the vertices of a triangle and through its orthocenter is always a rectangular hyperbola.

At the end of his paper, B. Shawyer asks the following question : Does the Cevian intersection point K have any particular significance? It follows from the foregoing that K is a point of the parabolas through A, B, C and with centers $D(1, 0, 0)$ and $D'(0, 1, 0)$, the points at infinity of the X-axis and the Y-axis. And from this it follows that the circumcircle of any triangle ABC is the locus of the fourth common point of the two parabolas through A, B, C with variable orthogonal axes.

Next, we look for an (other) extension of the results of B. Shawyer : At the midpoint of each side of a triangle, construct the line such that the slope of this line and the slope of the side of the triangle satisfy the equation $ctt' - a(t+t') - b = 0$, with a, b and c constant and $a^2 + bc \neq 0$. Then these three lines are concurrent. This follows from Theorem 1, since the given equation determines a general non-singular involution. Shawyer's results correspond with $a = 0$ (and $\frac{b}{c} = \lambda$ and λ variable). Now, consider the special case where $c = 0$ and put $-\frac{b}{a} = \lambda$; the sum of the slopes is a constant λ or $t + t' = \lambda$. On the line l_∞ at infinity we get the corresponding points $(1, t, 0)$ and $(1, t', 0)$ and the fixed points of the involution on l_∞ determined by $t + t' = \lambda$ are $I(0, 1, 0)$ (or the point at infinity of the Y-axis) and $I'(1, \frac{\lambda}{2}, 0)$. In this case, the locus $\mathcal{L}$ of the point of concurrency L is the locus of the polar point L of the line l_∞ with respect to the conics of the pencil with basis points A, B, C and $I(0, 1, 0)$. A straightforward calculation shows that this locus $\mathcal{L}$ is the parabola through the midpoints A', B', C' of BC, CA, AB, respectively, and with center I. The second intersection points of this parabola $\mathcal{L}$ with the sides of the triangle are $X = IA \cap BC, Y = IB \cap CA$ and $Z = IC \cap AB$. Remark that IA, IB, IC are the lines parallel with the Y-axis through A, B, C, respectively.

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The Stammler Circles

Jean-Pierre Ehrmann and Floor van Lamoen

Abstract. We investigate circles intercepting chords of specified lengths on the sidelines of a triangle, a theme initiated by L. Stammler [6, 7]. We generalize his results, and concentrate specifically on the Stammler circles, for which the intercepts have lengths equal to the sidelengths of the given triangle.

1. Introduction

Ludwig Stammler [6, 7] has investigated, for a triangle with sidelengths a, b, c, circles that intercept chords of lengths μa, μb, μc ($\mu > 0$) on the sidelines BC, CA and AB respectively. He called these circles *proportionally cutting circles*,[1] and proved that their centers lie on the rectangular hyperbola through the circumcenter, the incenter, and the excenters. He also showed that, depending on μ, there are 2, 3 or 4 circles cutting chords of such lengths.

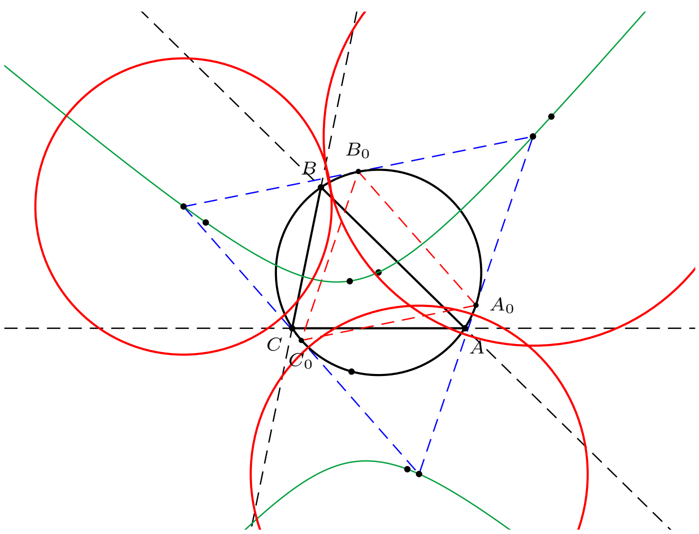

Figure 1. The three Stammler circles with the circumtangential triangle

As a special case Stammler investigated, for $\mu = 1$, the three proportionally cutting circles apart from the circumcircle. We call these the *Stammler circles*. Stammler proved that the centers of these circles form an equilateral triangle, circumscribed to the circumcircle and homothetic to Morley's (equilateral) trisector

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[1] Proportionalschnittkreise in [6].

triangle. In fact this triangle is tangent to the circumcircle at the vertices of the circumtangential triangle. [2] See Figure 1.

In this paper we investigate the circles that cut chords of specified lengths on the sidelines of ABC, and obtain generalizations of results in [6, 7], together with some further results on the Stammler circles.

2. The cutting circles

We define a (u, v, w)-cutting circle as one that cuts chords of lengths u, v, w on the sidelines BC, CA, AB of ABC respectively. This is to be distinguished from a $(u : v : w)$-cutting circle, which cuts out chords of lengths in the proportion $u : v : w$.

2.1. Consider a $(\mu u, \mu v, \mu w)$-cutting circle with center P, whose (signed) distances to the sidelines of ABC are respectively X, Y, Z. [3] It is clear that

$$Y^2 - Z^2 = \left(\frac{\mu}{2}\right)^2 (w^2 - v^2). \tag{1}$$

If $v \neq w$, this equation describes a rectangular hyperbola with center A and asymptotes the bisectors of angle A. In the same way, P also lies on the conics (generally rectangular hyperbolas)

$$Z^2 - X^2 = \left(\frac{\mu}{2}\right)^2 \left(u^2 - w^2\right) \tag{2}$$

and

$$X^2 - Y^2 = \left(\frac{\mu}{2}\right)^2 \left(v^2 - u^2\right). \tag{3}$$

These three hyperbolas generate a pencil which contains the conic with barycentric equation

$$\frac{(v^2 - w^2)x^2}{a^2} + \frac{(w^2 - u^2)y^2}{b^2} + \frac{(u^2 - v^2)z^2}{c^2} = 0. \tag{4}$$

This is a rectangular hyperbola through the incenter, excenters and the points $(\pm au : \pm bv : \pm cw)$.

Theorem 1. *The centers of the* $(u : v : w)$*-cutting circles lie on the rectangular hyperbola through the incenter and the excenters and the points with homogeneous barycentric coordinates* $(\pm au : \pm bv : \pm cw)$.

Remarks. 1. When $u = v = w$, the centers of $(u : v : w)$-cutting circles are the incenter and excenters themselves.

2. Triangle ABC is self polar with respect to the hyperbola (4).

[2] The vertices of the circumtangential triangle are the triple of points X on the circumcircle for which the line through X and its isogonal conjugate is tangent to the circumcircle. These are the isogonal conjugates of the infinite points of the sidelines of the Morley trisector triangle. See [4] for more on the circumtangential triangle.

[3] We say that the point P has *absolute* normal coordinates (X, Y, Z) with respect to triangle ABC.

2.2. Since (1) and (2) represent two rectangular hyperbolas with distinct asymptote directions, these hyperbolas intersect in four points, of which at least two are real points. Such are the centers of $(\mu u, \mu v, \mu w)$-cutting circles. The limiting case $\mu = 0$ always yields four real intersections, the incenter and excenters. As μ increases, there is some $\mu = \mu_0$ for which the hyperbolas (1) and (2) are tangent, yielding a double point. For $\mu > \mu_0$, the hyperbolas (1, 2, 3) have only two real common points. When there are four real intersections, these form an orthocentric system. From (1), (2) and (3) we conclude that A, B, C must be on the nine point circle of this orthocentric system.

Theorem 2. *Given positive real numbers u, v, w, there are four (u, v, w)-cutting circles, at least two of which are real. When there are four distinct real circles, their centers form an orthocentric system, of which the circumcircle is the nine point circle. When two of these centers coincide, they form a right triangle with its right angle vertex on the circumcircle.*

2.3. Let (O_1) and (O_2) be two (u, v, w)-cutting circles with centers O_1 and O_2. Consider the midpoint M of O_1O_2. The orthogonal projection of M on BC clearly is the midpoint of the orthogonal projections of O_1 and O_2 on the same line. Hence, it has equal powers with respect to the circles (O_1) and (O_2), and lies on the radical axis of these circles. In the same way the orthogonal projections of M on AC and AB lie on this radical axis as well. It follows that M is on the circumcircle of ABC, its Simson-Wallace line being the radical axis of (O_1) and (O_2). See Figure 2.

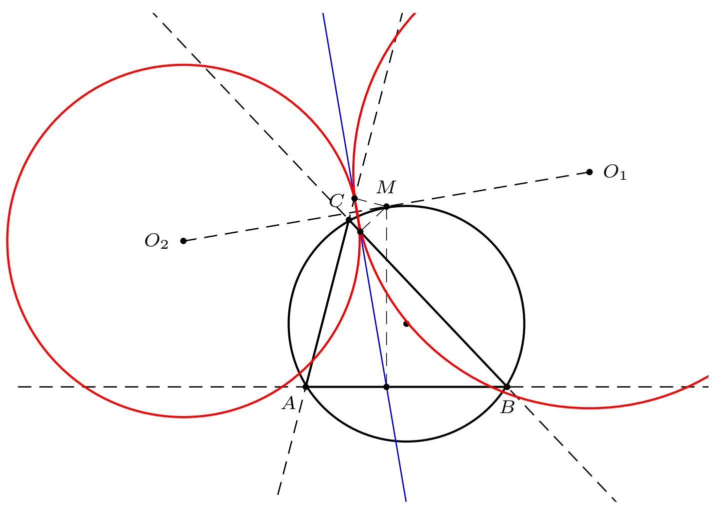

Figure 2. The radical axis of (O_1) and (O_2) is the Simson-Wallace line of M

2.4. Let Q be the reflection of the De Longchamps point L in M.[4] It lies on the circumcircle of the dilated (anticomplementary) triangle. The Simson-Wallace line

[4]The de Longchamps point L is the reflection of the orthocenter H in the circumcenter O. It is also the orthocenter of the dilated (anticomplementary) triangle.

of Q in the dilated triangle passes through M and is perpendicular to the Simson-Wallace line of M in ABC. It is therefore the line O_1O_2, which is also the same as MM^*, where M^* denotes the isogonal conjugate of M (in triangle ABC).

Theorem 3. *The lines connecting centers of (u, v, w)-cutting circles are Simson-Wallace lines of the dilated triangle. The radical axes of (u, v, w)-cutting circles are Simson-Wallace lines of ABC. When there are four real (u, v, w)-cutting circles, their radical axes form the sides of an orthocentric system perpendicular to the orthocentric system formed by the centers of the circles, and half of its size.*

2.5. For the special case of the centers O_1, O_2 and O_3 of the Stammler circles, we immediately see that they must lie on the circle $(O, 2R)$, where R is the circumradius. Since the medial triangle of $O_1O_2O_3$ must be circumscribed by the circumcircle, we see in fact that $O_1O_2O_3$ must be an equilateral triangle circumscribing the circumcircle. The sides of $O_1O_2O_3$ are thus Simson-Wallace lines of the dilated triangle, tangent to the nine point circle of the dilated triangle. See Figure 3.

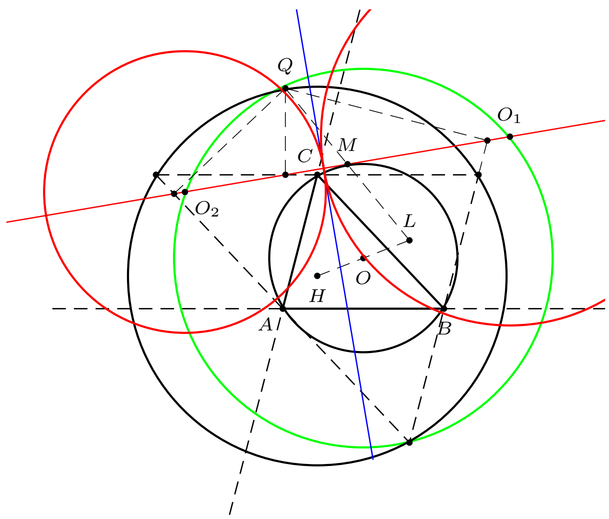

Figure 3. The line O_1O_2 is the dilated Simson-Wallace line of Q

Corollary 4. *The centers of the Stammler circles form an equilateral triangle circumscribing the circumcircle of ABC, and tangent to the circumcircle at the vertices $A_0B_0C_0$ of the circumtangential triangle. The radical axes of the Stammler circles among themselves are the Simson-Wallace lines of A_0, B_0, C_0.* [5] *The radical axes of the Stammler circles with the circumcircle are the sidelines of triangle $A_0B_0C_0$ translated by* $\mathbf{ON}$, *where N is the nine-point center of triangle ABC.*

[5]These are the three Simson-Wallace lines passing through N, *i.e.*, the cevian lines of N in the triangle which is the translation of $A_0B_0C_0$ by $\mathbf{ON}$. They are also the tangents to the Steiner deltoid at the cusps.

Remark. Since the nine-point circle of an equilateral triangle is also its incircle, we see that the centers of the Stammler circles are the only possible equilateral triangle of centers of (u, v, w)-cutting circles.

3. Constructions

3.1. Given a (u, v, w)-cutting circle with center P, let P' be the reflection of P in the circumcenter O. The centers of the other (u, v, w)-cutting circles can be found by intersecting the hyperbola (4) with the circle $P'(2R)$. One of the common points is the reflection of P in the center of the hyperbola.[6] The others are the required centers. This gives a *conic* construction. In general, the points of intersection are not constructible by ruler and compass. See Figure 4.

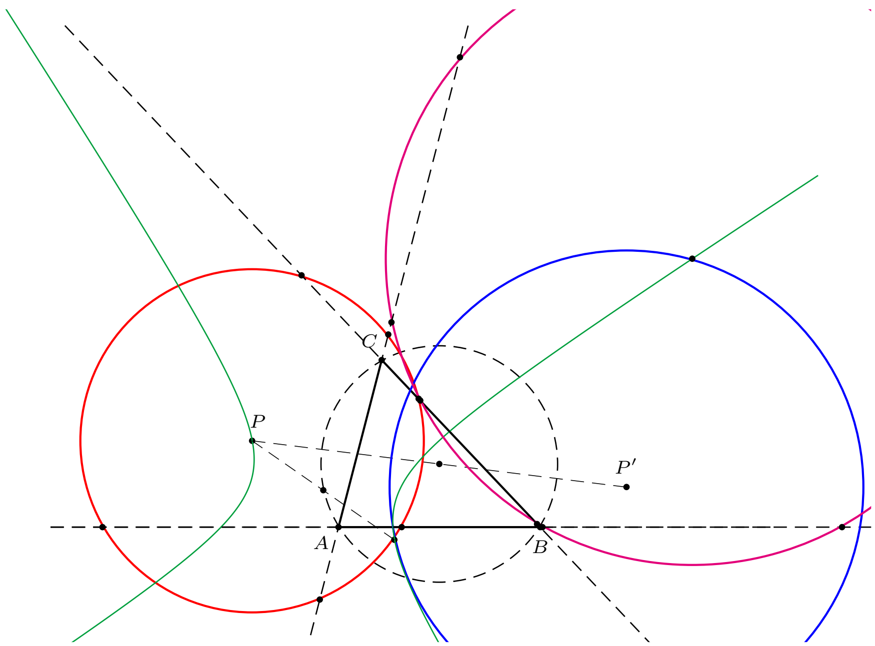

Figure 4. Construction of (u, v, w)-cutting circles

3.2. The same method applies when we are only given the magnitudes u, v, w. The centers of (u, v, w)-cutting circles can be constructed as the common points of the hyperbolas (1), (2), (3) with $\mu = 1$. If we consider two points T_A, T_B lying respectively on the lines CB, CA and such as $CT_A = u$, $CT_B = v$, the hyperbola (3) passes through the intersection M_0 of the perpendicular bisectors of CT_A and CT_B. Its asymptotes being the bisectors of angle C, a variable line through M_0 intersects these asymptotes at D, D'. The reflection of M_0 with respect to the midpoint of DD' lies on the hyperbola.

[6]The center of the hyperbola (4) is the point $\left(\frac{a^2}{v^2-w^2} : \frac{b^2}{w^2-u^2} : \frac{c^2}{u^2-v^2}\right)$ on the circumcircle.

3.3. When two distinct centers P and P' are given, then it is easy to construct the remaining two centers. Intersect the circumcircle and the circle with diameter PP', let the points of intersection be U and U'. Then the points $Q = PU \cap P'U'$ and $Q' = PU' \cap P'U$ are the points desired.

When one center P on the circumcircle is given, then P must in fact be a double point, and thus the right angle vertex of a right triangle containing the three $(u, v, w)-$intercepting circles. As the circumcircle of ABC is the nine point circle of the right triangle, the two remaining vertices must lie on the circle through P with P_r as center, where P_r is the reflection of P through O. By the last sentence before Theorem 3, we also know that the two remaining centers must lie on the line $P_rP_r^*$. Intersection of circle and line give the desired points.

3.4. Let three positive numbers u, v and w be given, and let P be a point on the hyperbola of centers of $(u : v : w)$-cutting circles. We can construct the circle with center P intercepting on the sidelines of ABC chords of lengths μu, μv and μw respectively for some μ.

We start from the point Q with barycentrics $(au : bv : cw)$. Let X, Y and Z be the distances from P to BC, AC and AB respectively. Since P satisfies (4) we have

$$(v^2 - w^2)X^2 + (w^2 - u^2)Y^2 + (u^2 - v^2)Z^2 = 0, \tag{5}$$

which is the equation in normal coordinates of the rectangular hyperbola through Q, the incenter and the excenters.

Now, the parallel through Q to AC (respectively AB) intersects AB (respectively AC) in Q_1 (respectively Q_2). The line perpendicular to Q_1Q_2 through P intersects AQ at U. The power p_a of P with respect to the circle with diameter AU is equal to $\dfrac{w^2Y^2 - v^2Z^2}{w^2 - v^2}$. Similarly we find powers p_b and p_c.

As P lies on the hyperbola given by (5), we have $p_a = p_b = p_c$. Define ρ by $\rho^2 = p_a$. Now, the circle (P, ρ) intercepts chords of with lengths L_a, L_b, L_c respectively on the sidelines of ABC, where

$$\left(\frac{L_a}{L_b}\right)^2 = \frac{\rho^2 - X^2}{\rho^2 - Y^2} = \frac{p_c - X^2}{p_c - Y^2} = \left(\frac{u}{v}\right)^2$$

and similarly

$$\left(\frac{L_b}{L_c}\right)^2 = \left(\frac{v}{w}\right)^2.$$

Hence this circle (P, ρ), if it exists and intersects the side lines, is the required circle. To construct this circle, note that if U' is the midpoint of AU, the circle goes through the common points of the circles with diameters AU and PU'.

4. The Stammler circles

For some particular results on the Stammler circles we use complex number coordinates. Each point is identified with a complex number $\rho \cdot e^{i\theta}$ called its *affix*. Here, (ρ, θ) are the polar coordinates with the circumcenter O as pole, scaled in such a way that points on the circumcircle are represented by unit complex numbers. Specifically, the vertices of the circumtangential triangle are represented by the cube roots of unity, namely,

$$A_0 = 1, \quad B_0 = \omega, \quad C_0 = \omega^2 = \overline{\omega},$$

where $\omega^3 = 1$. In this way, the vertices A, B, C have as affixes unit complex numbers $A = e^{i\theta}$, $B = e^{i\varphi}$, $C = e^{i\psi}$ satisfying $\theta + \varphi + \psi \equiv 0 \pmod{2\pi}$. In fact, we may take

$$\theta = \frac{2}{3}(\beta - \gamma), \quad \varphi = \frac{2}{3}(\beta + 2\gamma), \quad \psi = -\frac{2}{3}(2\beta + \gamma), \tag{6}$$

where α, β, γ are respectively the measures of angles A, B, C. In this setup the centers of the Stammler circles are the points

$$\Omega_A = -2, \qquad \Omega_B = -2\omega, \qquad \Omega_C = -2\overline{\omega}.$$

4.1. The intersections of the A-Stammler circle with the sidelines of ABC are

$$\begin{array}{ll} A_1 = B + \overline{A} - 1, & A_2 = C + \overline{A} - 1, \\ B_1 = C + \overline{B} - 1, & B_2 = A + \overline{B} - 1, \\ C_1 = A + \overline{C} - 1, & C_2 = B + \overline{C} - 1. \end{array}$$

The reflections of A, B, C in the line B_0C_0 are respectively

$$A' = -1 - \overline{A}, \qquad B' = -1 - \overline{B}, \qquad C' = -1 - \overline{C}.$$

The reflections of A', B', C' respectively in BC, CA, AB are

$$A'' = (1 + B)(1 + C), \qquad B'' = (1 + C)(1 + A), \qquad C'' = (1 + A)(1 + B).$$

Now,

$$\begin{aligned} B'' - A'' &= B_2 - A_1 = \frac{2}{\sqrt{3}} (\sin\theta - \sin\varphi)(C_0 - B_0), \\ C'' - B'' &= C_2 - B_1 = \frac{2}{\sqrt{3}}(\sin\varphi - \sin\psi)(C_0 - B_0), \\ A'' - C'' &= A_2 - C_1 = \frac{2}{\sqrt{3}}(\sin\psi - \sin\theta)(C_0 - B_0). \end{aligned}$$

Moreover, as the orthocenter $H = A + B + C$, the points $\Omega_B + H$ and $\Omega_C + H$ are collinear with $A''B''C''$.

4.2. Let R_A be the radius of the A-Stammler circle. It is easy to check that the twelve segments $A'B$, $A'C$, $A''B$, $A''C$, $B'C$, $B'A$, $B''C$, $B''A$, $C'A$, $C'B$, $C''A$, $C''B$ all have length equal to $R_A = \Omega_A A_1$. See Figure 6. Making use of the affixes, we easily obtain

$$R_A^2 = 3 + 2(\cos\theta + \cos\varphi + \cos\psi). \tag{7}$$

Theorem 5. *From the points of intersection of each of the Stammler circles with the sidelines of ABC three chords can be formed, with the condition that each chord is parallel to the side of Morley's triangle corresponding to the Stammler circle. The smaller two of these chords together are as long as the greater one.*

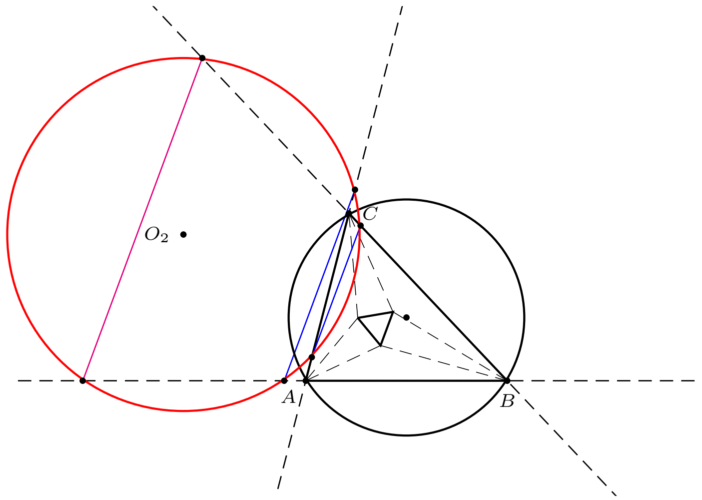

Figure 5. Three parallel chords on the $B-$Stammler circle

Remark. This is indeed true for any conic intercepting chords of lengths a, b, c on the sidelines.

4.3. We investigate the triangles $P_AP_BP_C$ with P_AB, P_AC, P_BA, P_BC, P_CA, P_CB all of length $\rho = \sqrt{\nu}$, which are perspective to ABC through P. Let P have homogeneous barycentric coordinates $(p : q : r)$. The line AP and the perpendicular bisector of BC meet in the point

$$P_A = \left(-(q-r)a^2 : q(b^2-c^2) : r(b^2-c^2)\right).$$

With the distance formula, [7] we have

$$|P_AB|^2 = a^2\frac{(a^2(c^2q^2+b^2r^2)+((b^2-c^2)^2-a^2(b^2+c^2))qr}{((a^2-b^2+c^2)q-(a^2+b^2-c^2)r)^2}$$

Similarly we find expressions for the squared distances $|P_BC|^2$ and $|P_CA|^2$.

Now let $|P_AB|^2 = |P_BC|^2 = |P_CA|^2 = \nu$. From these three equations we can eliminate q and r. When we simplify the equation assuming that ABC is nonisosceles and nondegenerate, this results in

$$p\nu(-16\triangle^2\nu+a^2b^2c^2)(-16\triangle^2\nu^3+a^2b^2c^2(9\nu^2-3(a^2+b^2+c^2)\nu+a^2b^2+b^2c^2+a^2c^2)) = 0. \tag{8}$$

Here, $\triangle$ is the area of triangle ABC. One real solution is clearly $\rho = \frac{a^2b^2c^2}{16\triangle^2} = R^2$. The other nonzero solutions are the roots of the cubic equation

$$\nu^3 - R^2(9\nu^2 - 3(a^2+b^2+c^2)\nu + a^2b^2 + b^2c^2 + a^2c^2) = 0. \tag{9}$$

[7] See for instance [5, Proposition 2].

As $A'B'C'$ is a particular solution of the problem, the roots of this cubic equation are the squares of the radii of the Stammler circles. A simple check of cases shows that the mentioned solutions are indeed the only ones.

Theorem 6. *Reflect the vertices of ABC through one of the sides of the circumtangential triangle to A', B' and C'. Then $A'B'C'$ lie on the perpendicular bisectors. In particular, together with O as a triple point and the reflections of O through the sides of ABC these are the only triangles perspective to ABC with $A'B = A'C = B'A = B'C = C'A = C'B$, for nonisosceles (and nondegenerate) ABC.*

Remark. Theorem 6 answers a question posed by A. P. Hatzipolakis [3].

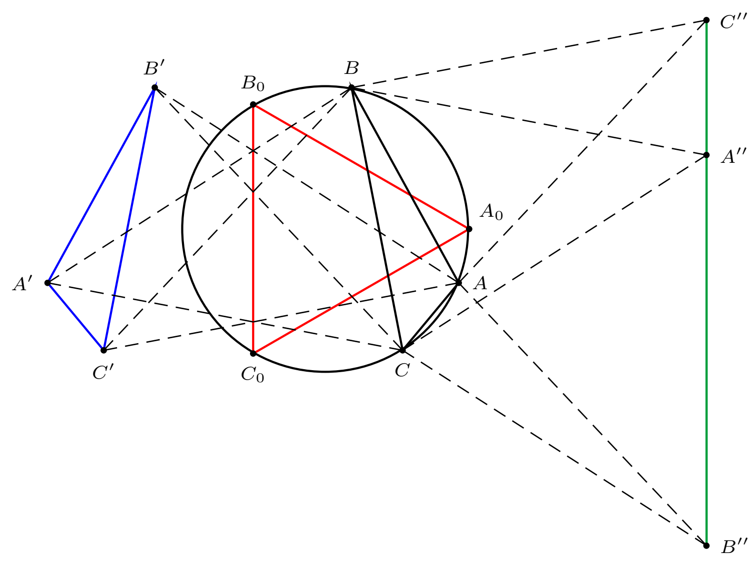

Figure 6. A perspective triangle $A'B'C'$ and the corresponding degenerate $A''B''C''$

4.4. Suppose that three points U, V, W lie on a same line ℓ and that $UB = UC = VC = VA = WA = WB = r \neq R$.

Let z_a the signed distance from A to ℓ. We have $\tan(\ell, BC) = 2 \cdot \dfrac{z_b - z_c}{\overline{VW}}$ and $z_a^2 = r^2 - \frac{1}{4}\overline{VW}^2$. It follows that

$$(\ell, BC) + (\ell, CA) + (\ell, AB) = 0,$$

and ℓ is parallel to a sideline of the Morley triangle of ABC. See [2, Proposition 5]. Now, U, V, W are the intersections of ℓ with the perpendicular bisectors of ABC and, for a fixed direction of ℓ, there is only one position of ℓ for which $VA = WA \neq R$. Hence the degenerate triangles $A''B''C''$, together with O as a triple point, are the only solutions in the collinear cases with $A''B = A''C = B''A = B''C = C''A = C''B$.

Theorem 7. *Reflect $A'B'C'$ through the sides of ABC respectively to A'', B'', C''. Then $A''B''C''$ are contained in the same line ℓ_i parallel to the side L_i of the circumtangential triangle. Together with O as a triple point these are the only degenerate triangles $A''B''C''$ satisfying the condition $A''B = A''C = B''A = B''C = C''A = C''B$. The lines ℓ_A, ℓ_B, ℓ_C bound the triangle which is the translation of $\Omega_A\Omega_B\Omega_C$ through the vector $\mathbf{OH}$.*

The three segments from $A''B''C''$ are congruent to the chords of Theorem 5. See Figure 6.

4.5. With θ, φ, ψ given by (6), we obtain from (7), after some simplifications,

$$\left(\frac{R_A}{R}\right)^2 = 1 + 8\cos\frac{\beta-\gamma}{3}\cos\frac{\beta+2\gamma}{3}\cos\frac{2\beta+\gamma}{3}.$$

Since $\left(\frac{OH}{R}\right)^2 = 1 - 8\sin\alpha\sin\beta\sin\gamma$, (see, for instance, [1, Chapter XI]), this shows that the radius R_A can be constructed, allowing angle trisection. R_A is the distance from O to the orthocenter of the triangle $AB'C'$, where B' is the image of B after rotation through $\frac{2(\beta-\gamma)}{3}$ about O, and C' is the image of A after rotation through $\frac{2(\gamma-\beta)}{3}$ about O.

The barycentric coordinates of Ω_A are

$$\left(a\left(\cos\alpha - 2\cos\frac{\beta-\gamma}{3}\right) : b\left(\cos\beta + 2\cos\frac{\beta+2\gamma}{3}\right) : c\left(\cos\gamma + 2\cos\frac{2\beta+\gamma}{3}\right)\right).$$

We find the distances

$$\begin{array}{ll} B_1C_2 = 2a\cos\frac{\beta-\gamma}{3}, & BA_1 = CA_2 = 2R\sin\frac{|\beta-\gamma|}{3}, \\ C_1A_2 = 2b\cos\frac{\beta+2\gamma}{3}, & CB_1 = AB_2 = 2R\sin\frac{\beta+2\gamma}{3}, \\ A_1B_2 = 2c\cos\frac{2\beta+\gamma}{3}, & AC_1 = BC_2 = 2R\sin\frac{2\beta+\gamma}{3}. \end{array}$$

Finally we mention the following relations of the Stammler radii. These follow easily from the fact that they are the roots of the cubic equation (9).

$$\begin{aligned} &R_A^2 + R_B^2 + R_C^2 = 9R^2; \\ &\frac{1}{R_A^2} + \frac{1}{R_B^2} + \frac{1}{R_C^2} = \frac{3(a^2+b^2+c^2)}{a^2b^2+a^2c^2+b^2c^2}; \\ &R_AR_BR_C = R\sqrt{a^2b^2+b^2c^2+c^2a^2}. \end{aligned}$$

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Some Similarities Associated with Pedals

Jean-Pierre Ehrmann and Floor van Lamoen

Abstract. The pedals of a point divide the sides of a triangle into six segments. We build on these segments six squares and obtain some interesting similarities.

Given a triangle ABC, the pedals of a point P are its orthogonal projections A', B', C' on the sidelines BC, CA, AB of the triangle. We build on the segments AC', $C'B$, BA', $A'C$, CB' and $B'A$ squares with orientation opposite to that of ABC.

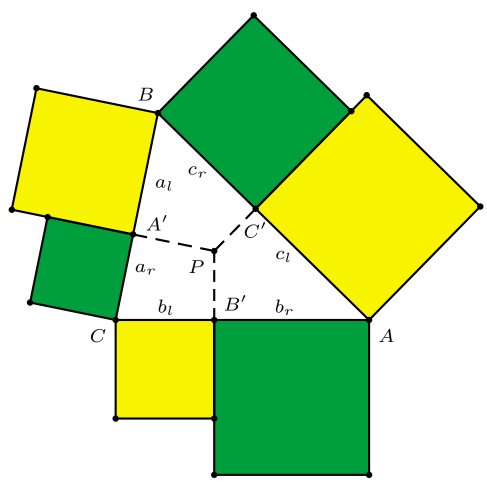

Figure 1

About this figure, O. Bottema [1, §77] showed that the sum of the areas of the squares on BA', CB' and AC' is equal to the sum of the areas of the squares on $A'C$, $B'A$ and $C'B$, namely,

$$a_l^2 + b_l^2 + c_l^2 = a_r^2 + b_r^2 + c_r^2.$$

See also [2, p.112]. Bottema showed conversely that when this equation holds, $A'B'C'$ is indeed a pedal triangle. While this can be easily established by applying the Pythagorean Theorem to the right triangles $AB'P$, $AC'P$, $BA'P$, $BC'P$, $CA'P$ and $CB'P$, we find a few more interesting properties of the figure. We adopt the following notations.

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O	circumcenter	
K	symmedian point	
$\triangle$	area of triangle ABC	
ω	Brocard angle	$\cot\omega = \frac{a^2+b^2+c^2}{4\triangle}$
Ω_1	Brocard point	$\angle BA\Omega_1 = \angle CB\Omega_1 = \angle AC\Omega_1 = \omega$
Ω_2	Brocard point	$\angle AB\Omega_2 = \angle BC\Omega_2 = \angle CA\Omega_2 = \omega$
$\mathsf{h}(P,r)$	homothety with center P and ratio r	
$\rho(P,\theta)$	rotation about P through an angle θ	

Let $A_1B_1C_1$ be the triangle bounded by the lines containing the sides of the squares opposite to BA', CB', AC' respectively. Similarly, let $A_2B_2C_2$ be the one bounded by the lines containing the sides of the squares opposite to $A'C$, $B'A$ and $C'B$.

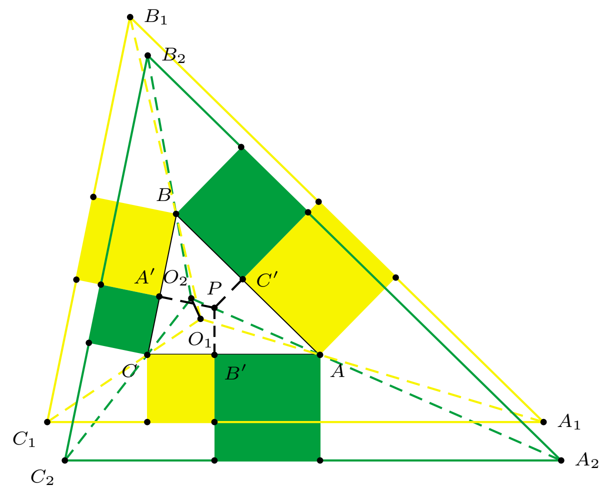

Figure 2

Theorem. *Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are each homothetic to ABC. Let O_1, and O_2 be the respective centers of homothety.*

(1) *The ratio of homothety in each case is $1+\cot\omega$. Therefore, $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic and congruent.*

(2) *The mapping $P \mapsto O_1$ is the direct similarity which is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan\omega)$. Likewise, The mapping $P \mapsto O_2$ is the direct similarity which is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan\omega)$.*

(3) *The midpoint of the segment O_1O_2 is the symmedian point K.*

(4) *The vector of translation $A_1B_1C_1 \mapsto A_2B_2C_2$ is the image of $2\mathbf{OP}$ under the rotation $\rho(O, \frac{\pi}{2})$.*

Proof. We label the directed distances $a_l = BA'$, $a_r = A'C$, $b_l = CB'$, $b_r = B'A$, $c_l = AC'$ and $c_r = C'B$ as in Figure 1. Because ABC and $A_1B_1C_1$ are homothetic through O_1, the distances f, g, h of O_1 to the respective sides of ABC are in the same ratio as the distances between the corresponding sides of ABC and $A_1B_1C_1$. We have $f : g : h = a_l : b_l : c_l$. See Figure 2. Furthermore, the sum of the areas of triangles O_1BC, AO_1C and ABO_1 is equal to the area $\triangle$ of ABC, so that $af + bg + ch = 2\triangle$. But we also have

$$\begin{aligned} a_l^2 + b_l^2 + c_l^2 &= a_r^2 + b_r^2 + c_r^2 \\ &= (a - a_l)^2 + (b - b_l)^2 + (c - c_l)^2, \end{aligned}$$

from which we find

$$aa_l + bb_l + cc_l = \frac{a^2 + b^2 + c^2}{2} = 2\triangle \cot\omega.$$

This shows that $\frac{a_l}{f} = \frac{b_l}{g} = \frac{c_l}{h} = \cot\omega$, and thus that the ratio of homothety of $A_1B_1C_1$ to ABC is $1 + \cot\omega$. By symmetry, we find the same ratio of homothety of $A_2B_2C_2$ to ABC. This proves (1).

Now suppose that $P = O_1$. Then $\tan\angle CBO_1 = \frac{f}{a_l} = \tan\omega$. By symmetry this shows that P must be the Brocard point Ω_1.

To investigate the mapping $P \mapsto O_1$, we imagine that P moves through a line perpendicular to BC. For all points P on this line a_l is the same, so that for all images O_1 the distance f is the same. Therefore, O_1 traverses a line parallel to BC. Now imagine that P travels a distance d in the direction $A'P$. Then $AC' = c_l$ decreases with $d/\sin B$. The distance h of O_1 to AB thus decreases with $\frac{d\tan\omega}{\sin B}$, and O_1 must have travelled in the direction $\mathbf{CB}$ through $d\tan\omega$. Of course we can find similar results by letting P move through a line perpendicular to AC or AB.

Now any point P can be reached from Ω_1 by first going through a certain distance perpendicular to BC and then through another distance perpendicular to AC. Since Ω_1 is a fixed point of $P \mapsto O_1$, we can combine the results of the previous paragraph to conclude that $P \mapsto O_1$ is the rotation $\rho(\Omega_1, \frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_1, \tan\omega)$.

In a similar fashion we see that $P \mapsto O_2$ is the rotation $\rho(\Omega_2, -\frac{\pi}{2})$ followed by the homothety $\mathsf{h}(\Omega_2, \tan\omega)$. This proves (2).

Now note that the pedal triangle of O is the medial triangle, so that the images of O under both mappings are identical. This image must be the point for which the distances to the sides are proportional to the corresponding sides, well known to be the symmedian point K. Now the segment OP is mapped to KO_1 and KO_2 respectively under the above mappings, while the image segments are congruent and make an angle of π. This proves (3).

More precisely the ratio of lengths $|KO_1| : |OP| = \tan\omega : 1$, so that $|O_1O_2| : |OP| = 2\tan\omega : 1$. By (1), we also know that $|O_1O_2| : |A_1A_2| = \tan\omega : 1$. Together with the observation that O_1O_2 and A_1A_2 are oppositely parallel, this proves (4). $\square$

We remark that (1) can be generalized to *inscribed* triangles $A'B'C'$. Since $BA' + A'C = BC$ it is clear that the line midway between B_1C_1 and B_2C_2 is at distance $\frac{a}{2}$ from BC, it is the line passing through the apex of the isosceles right triangle erected outwardly on BC. We conclude that the midpoints of A_1A_2, B_1B_2

and C_1C_2 form a triangle independent from $A'B'C'$, homothetic to ABC through K with ratio $1 + \cot\omega$. But then since $A_1B_1C_1$ and $A_2B_2C_2$ are homothetic to each other, as well as to ABC, it follows that the sum of their homothety ratios is $2(1 + \cot\omega)$.

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Brahmagupta Quadrilaterals

K. R. S. Sastry

Abstract. The Indian mathematician Brahmagupta made valuable contributions to mathematics and astronomy. He used Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals, and area, *i.e.*, Brahmagupta quadrilaterals. In this paper we describe a new numerical construction to generate an infinite family of Brahmagupta quadrilaterals from a Heron triangle.

1. Introduction

A triangle with integer sides and area is called a Heron triangle. If some of these elements are rationals that are not integers then we call it a rational Heron triangle. More generally, a polygon with integer sides, diagonals and area is called a Heron polygon. A rational Heron polygon is analogous to a rational Heron triangle. Brahmagupta's work on Heron triangles and cyclic quadrilaterals intrigued later mathematicians. This resulted in Kummer's complex construction to generate Heron quadrilaterals outlined in [2]. By a Brahmagupta quadrilateral we mean a cyclic Heron quadrilateral. In this paper we give a construction of Brahmagupta quadrilaterals from rational Heron triangles.

We begin with some well known results from circle geometry and trigonometry for later use.

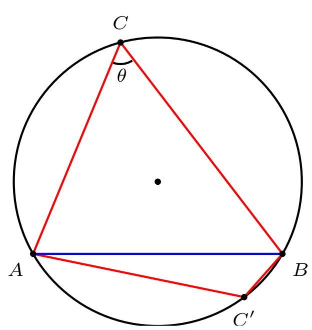

Figure 1

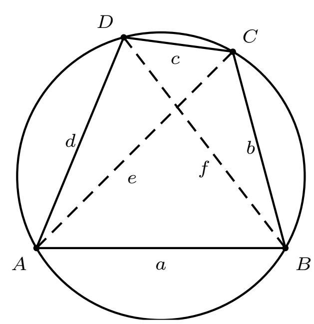

Figure 2

Figure 1 shows a chord AB of a circle of radius R. Let C and C' be points of the circle on opposite sides of AB. Then,

$$\angle ACB + \angle AC'B = \pi;$$
$$AB = 2R\sin\theta. \qquad (1)$$

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Throughout our discussion on Brahmagupta quadrilaterals the following notation remains standard. $ABCD$ is a cyclic quadrilateral with vertices located on a circle in an order. $AB = a$, $BC = b$, $CD = c$, $DA = d$ represent the sides or their lengths. Likewise, $AC = e$, $BD = f$ represent the diagonals. The symbol $\triangle$ represents the area of $ABCD$. Brahmagupta's famous results are

$$e = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}, \tag{2}$$

$$f = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}, \tag{3}$$

$$\triangle = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{4}$$

where $s = \frac{1}{2}(a+b+c+d)$.

We observe that $d = 0$ reduces to Heron's famous formula for the area of triangle in terms of a, b, c. In fact the reader may derive Brahmagupta's expressions in (2), (3), (4) independently and see that they give two characterizations of a cyclic quadrilateral. We also observe that Ptolemy's theorem, viz., *the product of the diagonals of a cyclic quadrilateral equals the sum of the products of the two pairs of opposite sides*, follows from these expressions. In the next section, we give a construction of Brahmagupta quadrilaterals in terms of Heron angles. A Heron angle is one with rational sine and cosine. See [4]. Since

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$

for $t = \tan\frac{\theta}{2}$, the angle θ is Heron if and only $\tan\frac{\theta}{2}$ is rational. Clearly, sums and differences of Heron angles are Heron angles. If we write, for triangle ABC, $t_1 = \tan\frac{A}{2}$, $t_2 = \tan\frac{B}{2}$, and $t_3 = \tan\frac{C}{2}$, then

$$a : b : c = t_1(t_2+t_3) : t_2(t_3+t_1) : t_3(t_1+t_2).$$

It follows that a triangle is rational if and only if its angles are Heron.

2. Construction of Brahmagupta quadrilaterals

Since the opposite angles of a cyclic quadrilateral are supplementary, we can always label the vertices of one such quadrilateral $ABCD$ so that the angles $A,\ B \le \frac{\pi}{2}$ and $C, D \ge \frac{\pi}{2}$. The cyclic quadrilateral $ABCD$ is a rectangle if and only if $A = B = \frac{\pi}{2}$; it is a trapezoid if and only if $A = B$. Let $\angle CAD = \angle CBD = \theta$. The cyclic quadrilateral $ABCD$ is rational if and only if the angles A, B and θ are Heron angles.

If $ABCD$ is a Brahmagupta quadrilateral whose sides AD and BC are not parallel, let E denote their intersection. [1] In Figure 3, let $EC = \alpha$ and $ED = \beta$. The triangles EAB and ECD are similar so that $\dfrac{AB}{CD} = \dfrac{EB}{ED} = \dfrac{EA}{EC} = \lambda$, say.

[1]Under the assumption that $A,\ B \le \frac{\pi}{2}$, these lines are parallel only if the quadrilateral is a rectangle.

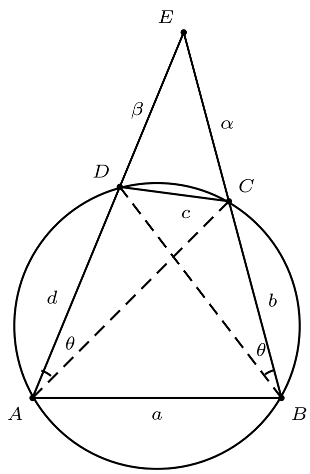

Figure 3

That is,

$$\frac{a}{c} = \frac{\alpha + b}{\beta} = \frac{\beta + d}{\alpha} = \lambda,$$

or

$$a = \lambda c, \quad b = \lambda\beta - \alpha, \quad d = \lambda\alpha - \beta, \qquad \lambda > \max\left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right). \tag{5}$$

Furthermore, from the law of sines, we have

$$e = 2R\sin B = 2R\sin D = \frac{R}{\rho}\cdot\alpha, \quad f = 2R\sin A = 2R\sin C = \frac{R}{\rho}\cdot\beta. \tag{6}$$

where ρ is the circumradius of triangle ECD. Ptolemy's theorem gives $ac + bd = ef$, and

$$\frac{R^2}{\rho^2}\cdot\alpha\beta = c^2\lambda + (\beta\lambda - \alpha)(\alpha\lambda - \beta)$$

This equation can be rewritten as

$$\begin{aligned}\left(\frac{R}{\rho}\right)^2 =& \lambda^2 - \frac{\alpha^2 + \beta^2 - c^2}{\alpha\beta}\lambda + 1 \\ =& \lambda^2 - 2\lambda\cos E + 1 \\ =& (\lambda - \cos E)^2 + \sin^2 E,\end{aligned}$$

or

$$\left(\frac{R}{\rho} - \lambda + \cos E\right)\left(\frac{R}{\rho} + \lambda - \cos E\right) = \sin^2 E.$$

Note that $\sin E$ and $\cos E$ are rational since E is a Heron angle. In order to obtain rational values for R and λ we put

$$\begin{aligned}\frac{R}{\rho} - \lambda - \cos E &= t\sin E,\\ \frac{R}{\rho} + \lambda + \cos E &= \frac{\sin E}{t},\end{aligned}$$

for a rational number t. From these, we have

$$\begin{aligned}R =& \frac{\rho}{2}\sin E\left(t+\frac{1}{t}\right) = \frac{c}{4}\left(t+\frac{1}{t}\right),\\ \lambda =& \frac{1}{2}\sin E\left(\frac{1}{t}-t\right) - \cos E.\end{aligned}$$

From the expression for R, it is clear that $t = \tan\frac{\theta}{2}$. If we set

$$t_1 = \tan\frac{D}{2} \quad \text{and} \quad t_2 = \tan\frac{C}{2}$$

for the Heron angles C and D, then

$$\cos E = \frac{(t_1+t_2)^2 - (1-t_1t_2)^2}{(1+t_1^2)(1+t_2^2)}$$

and

$$\sin E = \frac{2(t_1+t_2)(1-t_1t_2)}{(1+t_1^2)(1+t_2^2)}.$$

By choosing $c = t(1+t_1^2)(1+t_2^2)$, we obtain from (6)

$$\alpha = \frac{tt_1(1+t_1^2)(1+t_2^2)^2}{(t_1+t_2)(1-t_1t_2)}, \quad \beta = \frac{tt_2(1+t_1^2)^2(1+t_2^2)}{(t_1+t_2)(1-t_1t_2)},$$

and from (5) the following simple rational parametrization of the sides and diagonals of the cyclic quadrilateral:

$$\begin{aligned}a =& (t(t_1+t_2)+(1-t_1t_2))(t_1+t_2-t(1-t_1t_2)),\\ b =& (1+t_1^2)(t_2-t)(1+tt_2),\\ c =& t(1+t_1^2)(1+t_2^2),\\ d =& (1+t_2^2)(t_1-t)(1+tt_1),\\ e =& t_1(1+t^2)(1+t_2^2),\\ f =& t_2(1+t^2)(1+t_1^2).\end{aligned}$$

This has area

$$\triangle = t_1t_2(2t(1-t_1t_2)-(t_1+t_2)(1-t^2))(2(t_1+t_2)t+(1-t_1t_2)(1-t^2)),$$

and is inscribed in a circle of diameter

$$2R = \frac{(1+t_1^2)(1+t_2^2)(1+t^2)}{2}.$$

Replacing $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{v}{u}$ for integers m, n, p, q, u, v in these expressions, and clearing denominators in the sides and diagonals, we obtain Brahmagupta quadrilaterals. Every Brahmagupta quadrilateral arises in this way.

3. Examples

Example 1. By choosing $t_1 = t_2 = \frac{n}{m}$ and putting $t = \frac{v}{u}$, we obtain a generic Brahmagupta trapezoid:

$$\begin{aligned}
a =& (m^2u - n^2u + 2mnv)(2mnu - m^2v + n^2v),\\
b = d =& (m^2 + n^2)(nu - mv)(mu + nv),\\
c =& (m^2 + n^2)^2uv,\\
e = f =& mn(m^2 + n^2)(u^2 + v^2),
\end{aligned}$$

This has area

$$\triangle = 2m^2n^2(nu-mv)(mu+nv)((m+n)u-(m-n)v)((m+n)v-(m-n)u),$$

and is inscribed in a circle of diameter

$$2R = \frac{(m^2 + n^2)^2(u^2 + v^2)}{2}.$$

The following Brahmagupta trapezoids are obtained from simple values of t_1 and t, and clearing common divisors.

t_1	t	a	$b = d$	c	$e = f$	$\triangle$	$2R$
$1/2$	$1/7$	25	15	7	20	192	25
$1/2$	$2/9$	21	10	9	17	120	41
$1/3$	$3/14$	52	15	28	41	360	197
$1/3$	$3/19$	51	20	19	37	420	181
$2/3$	$1/8$	14	13	4	15	108	$65/4$
$2/3$	$3/11$	21	13	11	20	192	61
$2/3$	$9/20$	40	13	30	37	420	$1203/4$
$3/4$	$2/11$	25	25	11	30	432	61
$3/4$	$1/18$	17	25	3	26	240	$325/12$
$3/5$	$2/9$	28	17	12	25	300	$164/3$

Example 2. Let ECD be the rational Heron triangle with $c : \alpha : \beta = 14 : 15 : 13$. Here, $t_1 = \frac{2}{3}$, $t_2 = \frac{1}{2}$ (and $t_3 = \frac{4}{7}$). By putting $t = \frac{v}{u}$ and clearing denominators, we obtain Brahmagupta quadrilaterals with sides

$$a = (7u-4v)(4u+7v),\ b = 13(u-2v)(2u+v),\ c = 65uv,\ d = 5(2u-3v)(3u+2v),$$

diagonals

$$e = 30(u^2 + v^2), \quad f = 26(u^2 + v^2),$$

and area

$$\triangle = 24(2u^2 + 7uv - 2v^2)(7u^2 - 8uv - 7v^2).$$

If we put $u = 3$, $v = 1$, we generate the particular one:

$$(a, b, c, d, e, f; \triangle) = (323, 91, 195, 165, 300, 260; 28416).$$

On the other hand, with $u = 11$, $v = 3$, we obtain a quadrilateral whose sides and diagonals are multiples of 65. Reduction by this factor leads to

$$(a, b, c, d, e, f; \triangle) = (65, 39, 33, 25, 52, 60; 1344).$$

This is inscribed in a circle of diameter 65. This latter Brahmagupta quadrilateral also appears in Example 4 below.

Example 3. If we take ECD to be a right triangle with sides $CD : EC : ED = m^2 + n^2 : 2mn : m^2 - n^2$, we obtain

$$\begin{aligned}
a =&(m^2 + n^2)(u^2 - v^2),\\
b =&((m - n)u - (m + n)v)((m + n)u + (m - n)v),\\
c =&2(m^2 + n^2)uv,\\
d =&2(nu - mv)(mu + nv),\\
e =&2mn(u^2 + v^2),\\
f =&(m^2 - n^2)(u^2 + v^2);\\
\triangle =&mn(m^2 - n^2)(u^2 + 2uv - v^2)(u^2 - 2uv - v^2).
\end{aligned}$$

Here, $\frac{u}{v} > \frac{m}{n}, \frac{m+n}{m-n}$. We give two very small Brahmagupta quadrilaterals from this construction.

n/m	v/u	a	b	c	d	e	f	$\triangle$	$2R$
1/2	1/4	75	13	40	36	68	51	966	85
1/2	1/5	60	16	25	33	52	39	714	65

Example 4. If the angle θ is chosen such that $A + B - \theta = \frac{\pi}{2}$, then the side BC is a diameter of the circumcircle of $ABCD$. In this case,

$$t = \tan\frac{\theta}{2} = \frac{1 - t_3}{1 + t_3} = \frac{t_1 + t_2 - 1 + t_1t_2}{t_1 + t_2 + 1 - t_1t_2}.$$

Putting $t_1 = \frac{n}{m}$, $t_2 = \frac{q}{p}$, and $t = \frac{(m+n)q-(m-n)p}{(m+n)p-(m-n)q}$, we obtain the following Brahmagupta quadrilaterals.

$$\begin{aligned}
a =&(m^2 + n^2)(p^2 + q^2),\\
b =&(m^2 - n^2)(p^2 + q^2),\\
c =&((m + n)p - (m - n)q)((m + n)q - (m - n)p),\\
d =&(m^2 + n^2)(p^2 - q^2),\\
e =&2mn(p^2 + q^2),\\
f =&2pq(m^2 + n^2).
\end{aligned}$$

Here are some examples with relatively small sides.

t_1	t_2	t	a	b	c	d	e	f	$\triangle$
$2/3$	$1/2$	$3/11$	65	25	33	39	60	52	1344
$3/4$	$1/2$	$1/3$	25	7	15	15	24	20	192
$3/4$	$1/3$	$2/11$	125	35	44	100	120	75	4212
$6/7$	$1/3$	$1/4$	85	13	40	68	84	51	1890
$7/9$	$1/3$	$1/5$	65	16	25	52	63	39	1134
$8/9$	$1/2$	$3/7$	145	17	105	87	144	116	5760
$7/11$	$1/2$	$1/4$	85	36	40	51	77	68	2310
$8/11$	$1/3$	$1/6$	185	57	60	148	176	111	9240
$11/13$	$1/2$	$2/5$	145	24	100	87	143	116	6006

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The Apollonius Circle as a Tucker Circle

Darij Grinberg and Paul Yiu

Abstract. We give a simple construction of the circular hull of the excircles of a triangle as a Tucker circle.

1. Introduction

The Apollonius circle of a triangle is the circular hull of the excircles, the circle internally tangent to each of the excircles. This circle can be constructed by making use of the famous Feuerbach theorem that the nine-point circle is tangent *externally* to each of the excircles, and that the radical center of the excircles is the Spieker point X_{10}, the incenter of the medial triangle. If we perform an inversion with respect to the radical circle of the excircles, which is the circle orthogonal to each of them, the excircles remain invariant, while the nine-point circle is inverted into the Apollonius circle. The points of tangency of the Apollonius circle, being the inversive images of the points of tangency of the nine-point circle, can be constructed by joining to these latter points to Spieker point to intersect the respective excircles again. [1] See Figure 1. In this paper, we give another simple construction of the Apollonius circle by identifying it as a Tucker circle.

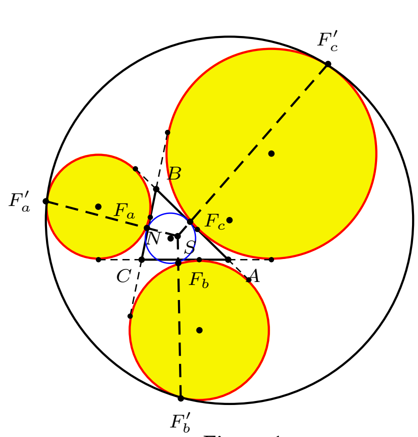

Figure 1

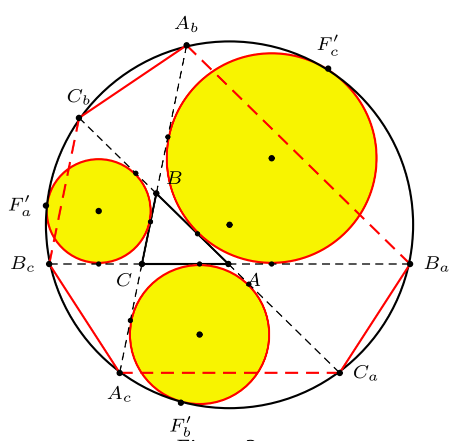

Figure 2

Theorem 1. *Let B_a and C_a be points respectively on the extensions of CA and BA beyond A such that B_aC_a is antiparallel to BC and has length s, the semiperimeter of triangle ABC. Likewise, let C_b, A_b be on the extensions of AB and CB beyond*

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[1] The tangency of this circle with each of the excircles is internal because the Spieker point, the center of inversion, is contained in nine-point circle.

B, with C_bA_b antiparallel to CA and of length s, A_c, B_c on the extensions of BC and AC beyond C, with A_cB_c is antiparallel to AB and of length s. Then the six points A_b, B_a, C_a, A_c, B_c, C_b are concyclic, and the circle containing them is the Apollonius circle of triangle ABC.

The vertices of the Tucker hexagon can be constructed as follows. Let X_b and X_c be the points of tangency of BC with excircles (I_b) and (I_c) respectively. Since BX_b and CX_c each has length s, the parallel of AB through X_b intersects AC at C', and that of AC through X_c intersects AB at B' such that the segment $B'C'$ is parallel to BC and has length s. The reflections of B' and C' in the line I_bI_c are the points B_a and C_a such that triangle AB_aC_a is similar to ABC, with $B_aC_a = s$. See Figure 3. The other vertices can be similarly constructed. In fact, the Tucker circle can be constructed by locating A_c as the intersection of BC and the parallel through C_a to AC.

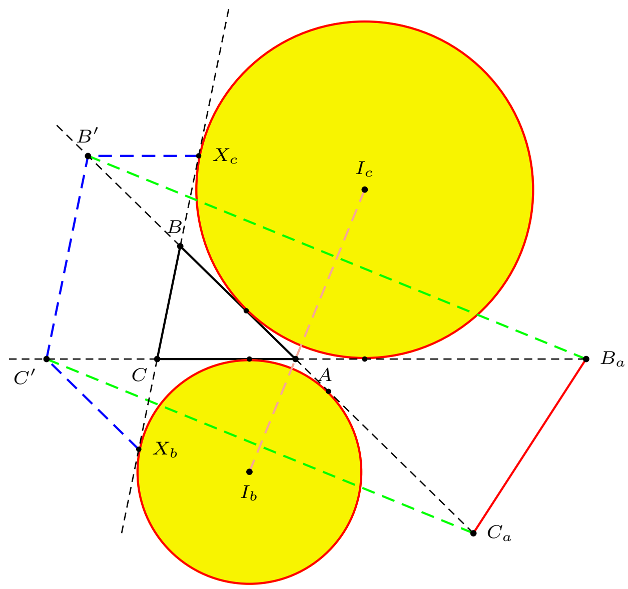

Figure 3

2. Some basic results

We shall denote the side lengths of triangle ABC by a, b, c.

R	circumradius
r	inradius
s	semiperimeter
$\triangle$	area
ω	Brocard angle

The Brocard angle is given by

$$\cot\omega = \frac{a^2+b^2+c^2}{4\triangle}.$$

Lemma 2. (1) $abc = 4Rrs$;
(2) $ab + bc + ca = r^2 + s^2 + 4Rr$;
(3) $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$;
(4) $(a + b)(b + c)(c + a) = 2s(r^2 + s^2 + 2Rr)$.

Proof. (1) follows from the formulae $\triangle = rs$ and $R = \frac{abc}{4\triangle}$.

(2) follows from the Heron formula $\triangle^2 = s(s - a)(s - b)(s - c)$ and

$$s^3 - (s - a)(s - b)(s - c) = (ab + bc + ca)s + abc.$$

(3) follows from (2) and $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$.

(4) follows from $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$. □

Unless explicitly stated, all coordinates we use in this paper are *homogeneous barycentric coordinates*. Here are the coordinates of some basic triangle centers.

circumcenter	O	$(a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$
incenter	I	$(a : b : c)$
Spieker point	S	$(b + c : c + a : a + b)$
symmedian point	K	$(a^2 : b^2 : c^2)$

Note that the sum of the coordinates of O is $16\triangle^2 = 16r^2s^2$. [2] We shall also make use of the following basic result on circles, whose proof we omit.

Proposition 3. *Let p_1, p_2, p_3 be the powers of A, B, C with respect to a circle $\mathcal{C}$. The power of a point with homogeneous barycentric coordinates $(x : y : z)$ with respect to the same circle is*

$$\frac{(x + y + z)(p_1x + p_2y + p_3z) - (a^2yz + b^2zx + c^2xy)}{(x + y + z)^2}.$$

Hence, the equation of the circle is

$$a^2yz + b^2zx + c^2xy = (x + y + z)(p_1x + p_2y + p_3z).$$

3. The Spieker radical circle

The fact that the radical center of the excircles is the Spieker point S is well known. See, for example, [3]. We verify this fact by computing the power of S with respect to the excircles. This computation also gives the radius of the radical circle.

Theorem 4. *The radical circle of the excircles has center at the Spieker point $S = (b + c : c + a : a + b)$, and radius $\frac{1}{2}\sqrt{r^2 + s^2}$.*

[2] This is equivalent to the following version of Heron's formula:

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

Proof. We compute the power of $(b+c : c+a : a+b)$ with respect to the A-excircle. The powers of A, B, C with respect to the A-excircle are clearly

$$p_1 = s^2, \qquad p_2 = (s-c)^2, \qquad p_3 = (s-b)^2.$$

With $x = b+c$, $y = c+a$, $z = a+b$, we have $x+y+z = 4s$ and

$$\begin{aligned}
&(x+y+z)(p_1x+p_2y+p_3z) - (a^2yz+b^2zx+c^2xy)\\
=&4s(s^2(b+c)+(s-c)^2(c+a)+(s-b)^2(a+b))\\
&-(a^2(c+a)(a+b)+b^2(a+b)(b+c)+c^2(b+c)(c+a))\\
=&2s(2abc+(a+b+c)(a^2+b^2+c^2)) - 2s(a^3+b^3+c^3+abc)\\
=&2s(abc+a^2(b+c)+b^2(c+a)+c^2(a+b))\\
=&4s^2(r^2+s^2),
\end{aligned}$$

and the power of the Spieker point with respect to the A-excircle is $\frac{1}{4}(r^2+s^2)$. This being symmetric in a, b, c, it is also the power of the same point with respect to the other two excircles. The Spieker point is therefore the radical center of the excircles, and the radius of the radical circle is $\frac{1}{2}\sqrt{r^2+s^2}$. □

We call this circle the Spieker radical circle, and remark that the Spieker point is the inferior of the incenter, namely, the image of the incenter under the homothety $\mathsf{h}(G, -\frac{1}{2})$ at the centroid G.

4. The Apollonius circle

To find the Apollonius circle it is more convenient to consider its superior, *i.e.*, its homothetic image $\mathsf{h}(G, -2)$ in the centroid G with ratio -2. This homothety transforms the nine-point circle and the Spieker radical circle into the circumcircle $O(R)$ and the circle $I(\sqrt{r^2+s^2})$ respectively.

Let d be the distance between O and I. By Euler's theorem, $d^2 = R^2 - 2Rr$. On the line OI we treat I as the origin, and O with coordinate R. The circumcircle intersects the line IO at the points $d \pm R$. The inversive images of these points have coordinates $\frac{r^2+s^2}{d\pm R}$. The inversive image is therefore a circle with radius

$$\frac{1}{2}\left|\frac{r^2+s^2}{d-R} - \frac{r^2+s^2}{d+R}\right| = \left|\frac{R(r^2+s^2)}{d^2-R^2}\right| = \frac{r^2+s^2}{2r}.$$

The center is the point Q' with coordinate

$$\frac{1}{2}\left(\frac{r^2+s^2}{d-R} + \frac{r^2+s^2}{d+R}\right) = \frac{d(r^2+s^2)}{d^2-R^2} = -\frac{r^2+s^2}{2Rr}\cdot d.$$

In other words,

$$IQ' : IO = -(r^2+s^2) : 2Rr.$$

Explicitly,

$$Q' = I - \frac{r^2+s^2}{2Rr}(O-I) = \frac{(r^2+s^2+2Rr)I - (r^2+s^2)O}{2Rr}.$$

From this calculation we make the following conclusions.

(1) The radius of the Apollonius circle is $\rho = \frac{r^2+s^2}{4r}$.
(2) The Apollonius center, being the homothetic image of Q' under $\mathsf{h}(G, -\frac{1}{2})$, is the point [3]

$$Q = \frac{1}{2}(3G - Q') = \frac{6Rr \cdot G + (r^2+s^2)O - (r^2+s^2+2Rr)I}{4Rr}.$$

Various authors have noted that Q lies on the Brocard axis OK, where the centers of Tucker circles lie. See, for example, [1, 9, 2, 7]. In [1], Aeppli states that if d_A, d_B, d_C are the distances of the vertices A, B, C to the line joining the center of the Apollonius circle with the circumcenter of ABC, then

$$d_A : d_B : d_C = \frac{b^2-c^2}{a^2} : \frac{c^2-a^2}{b^2} : \frac{a^2-b^2}{c^2}.$$

It follows that the barycentric equation of the line is

$$\frac{b^2-c^2}{a^2}x + \frac{c^2-a^2}{b^2}y + \frac{a^2-b^2}{c^2}z = 0.$$

This is the well known barycentric equation of the Brocard axis. Thus, the Apollonius center lies on the Brocard axis. Here, we write Q explicitly in terms of O and K.

Proposition 5. $Q = \frac{1}{4Rr}\left((s^2-r^2)O - \frac{1}{2}(a^2+b^2+c^2)K\right)$.

Proof.

$$\begin{aligned}
Q =& \frac{1}{4Rr}\left((r^2+s^2)O + 6Rr \cdot G - (r^2+s^2+2Rr)I\right)\\
=& \frac{1}{4Rr}\left((s^2-r^2)O + 2r^2 \cdot O + 6Rr \cdot G - (r^2+s^2+2Rr)I\right)\\
=& \frac{1}{16Rrs^2}\left(4s^2(s^2-r^2)O + 8r^2s^2 \cdot O + 24Rrs^2 \cdot G - 4s^2(r^2+s^2+2Rr)I\right).
\end{aligned}$$

Consider the sum of the last three terms. By Lemma 2, we have

$$\begin{aligned}
& 8r^2s^2 \cdot O + 24Rrs^2 \cdot G - 4s^2(r^2+s^2+2Rr)I\\
=& 8r^2s^2 \cdot O + abc \cdot 2s \cdot 3G - 2s(a+b)(b+c)(c+a)I\\
=& \frac{1}{2}(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2))\\
& + (a+b+c)abc(1,1,1) - (a+b)(b+c)(c+a)(a,b,c).
\end{aligned}$$

[3] This point is X_{970} of [7].

Consider the first component.

$$\begin{aligned}
&\frac{1}{2}\left(a^2(b^2+c^2-a^2)+2abc(a+b+c)-2(a+b)(b+c)(c+a)a\right)\\
=&\frac{1}{2}\left(a^2(b^2+2bc+c^2-a^2)+2abc(a+b+c)-2a((a+b)(b+c)(c+a)+abc)\right)\\
=&\frac{1}{2}\left(a^2(a+b+c)(b+c-a)+2abc(a+b+c)-2a(a+b+c)(ab+bc+ca)\right)\\
=&s(a^2(b+c-a)+2abc-2a(ab+bc+ca))\\
=&s(a^2(b+c-a)-2a(ab+ca))\\
=&a^2s(b+c-a-2(b+c))\\
=&-a^2\cdot 2s^2.
\end{aligned}$$

Similarly, the other two components are $-b^2\cdot 2s^2$ and $-c^2\cdot 2s^2$. It follows that

$$\begin{aligned}
Q=&\frac{1}{16Rrs^2}\left(4s^2(s^2-r^2)O-2s^2(a^2,b^2,c^2)\right)\\
=&\frac{1}{4Rr}\left((s^2-r^2)O-\frac{1}{2}(a^2+b^2+c^2)K\right). \qquad (1)
\end{aligned}$$

□

5. The Apollonius circle as a Tucker circle

It is well known that the centers of Tucker circles also lie on the Brocard axis. According to [8], a Tucker hexagon/circle has three principal parameters:

- the chordal angle $\phi\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right]$,
- the radius of the Tucker circle

$$r_\phi=\left|\frac{R}{\cos\phi+\cot\omega\sin\phi}\right|,$$

- the length of the equal antiparallels

$$d_\phi=2r_\phi\cdot\sin\phi.$$

This length d_ϕ is negative for $\phi<0$. In this way, for a given d_ϕ, there is one and only one Tucker hexagon with d_ϕ as the length of the antiparallel segments. In other words, a Tucker circle can be uniquely identified by d_ϕ. The center of the Tucker circle is the isogonal conjugate of the Kiepert perspector $K\left(\frac{\pi}{2}-\phi\right)$. Explicitly, this is the point

$$\frac{4\triangle\cot\phi\cdot O+(a^2+b^2+c^2)K}{4\triangle\cot\phi+(a^2+b^2+c^2)}.$$

Comparison with (1) shows that $4\triangle\cot\phi=-2(s^2-r^2)$. Equivalently,

$$\tan\phi=-\frac{2rs}{s^2-r^2}.$$

This means that $\phi = -2\arctan\frac{r}{s}$. Clearly, since $s > r$,

$$\cos\phi = \frac{s^2 - r^2}{r^2 + s^2}, \qquad \sin\phi = -\frac{2rs}{r^2 + s^2}.$$

Now, the radius of the Tucker circle with chordal angle $\phi = -2\arctan\frac{r}{s}$ is given by

$$r_\phi = \left|\frac{R}{\cos\phi + \cot\omega\sin\phi}\right| = \frac{r^2 + s^2}{4r}.$$

This is exactly the radius of the Apollonius circle. We therefore conclude that the Apollonius circle is the Tucker circle with chordal angle $-2\arctan\frac{r}{s}$. The common length of the antiparallels is

$$d_\phi = 2r_\phi \cdot \sin\phi = 2 \cdot \frac{r^2 + s^2}{4r} \cdot \frac{-2rs}{r^2 + s^2} = -s.$$

This proves Theorem 1 and justifies the construction in Figure 3.

6. Concluding remarks

We record the coordinates of the vertices of the Tucker hexagon.[4]

$$\begin{array}{ll} B_c = (-as : 0 : as + bc), & C_b = (-as : as + bc : 0), \\ A_b = (0 : cs + ab : -cs), & B_a = (cs + ab : 0 : -cs), \\ C_a = (bs + ca : -bs : 0), & A_c = (0 : -bs : bs + ca). \end{array}$$

From these, the power of A with respect to the Apollonian circle is

$$-\frac{cs}{a}\left(b + \frac{as}{c}\right) = \frac{-s(bc + as)}{a}.$$

Similarly, by computing the powers of B and C, we obtain the equation of the Apollonius circle as

$$a^2yz + b^2zx + c^2xy + s(x + y + z)\sum_{\text{cyclic}}\frac{bc + as}{a}x = 0.$$

Finally, with reference to Figure 1, Iwata and Fukagawa [5] have shown that triangles $F'_aF'_bF'_c$ and ABC are perspective at a point P on the line IQ with $IP : PQ = -r : \rho$.[5] They also remarked without proof that according to a Japanese wooden tablet dating from 1797,

$$\rho = \frac{1}{4}\left(\frac{s^4}{r_ar_br_c} + \frac{r_ar_br_c}{s^2}\right),$$

which is equivalent to $\rho = \frac{r^2+s^2}{4r}$ established above.

---

[4]These coordinates are also given by Jean-Pierre Ehrmann [2].

[5]This perspector is the Apollonius point $X_{181} = \left(\frac{a^2(b+c)^2}{s-a} : \frac{b^2(c+a)^2}{s-b} : \frac{c^2(a+b)^2}{s-c}\right)$ in [7]. In fact, the coordinates of F'_a are $(-a^2(a(b+c)+(b^2+c^2))^2 : 4b^2(c+a)^2s(s-c) : 4c^2(a+b)^2s(s-b))$; similarly for F'_b and F'_c.

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An Application of Thébault's Theorem

Wilfred Reyes

Abstract. We prove the "Japanese theorem" as a very simple corollary of Thébault's theorem.

Theorem 1 below is due to the French geometer Victor Thébault [8]. See Figure 1. It had been a long standing problem, but a number of proofs have appeared since the early 1980's. See, for example, [7, 6, 1], and also [5] for a list of proofs in Dutch published in the 1970's. A very natural and understandable proof based on Ptolemy's theorem can be found in [3].

Theorem 1 (Thébault)**.** *Let E be a point on the side of triangle ABC such that $\angle AEB = \theta$. Let $O_1(r_1)$ be a circle tangent to the circumcircle and to the segments EA, EB. Let $O_2(r_2)$ be also tangent to the circumcircle and to EA, EC. If $I(\rho)$ is the incircle of ABC, then*

(1.1) *I lies on the segment O_1O_2 and* $\dfrac{O_1I}{IO_2} = \tan^2 \frac{\theta}{2}$,

(1.2) $\rho = r_1 \cos^2 \frac{\theta}{2} + r_2 \sin^2 \frac{\theta}{2}$.

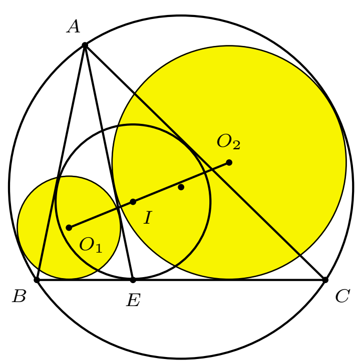

Figure 1

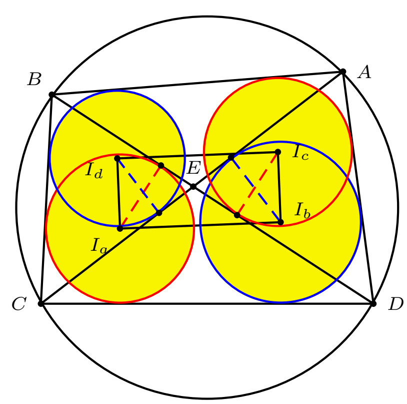

Figure 2

Theorem 2 below is called the "Japanese Theorem" in [4, p.193]. See Figure 2. A very long proof can be found in [2, pp.125–128]. In this note we deduce the Japanese Theorem as a very simple corollary of Thébault's Theorem.

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Theorem 2. *Let $ABCD$ be a convex quadrilateral inscribed in a circle. Denote by $I_a(\rho_a)$, $I_b(\rho_b)$, $I_c(\rho_c)$, $I_d(\rho_d)$ the incircles of the triangles BCD, CDA, DAB, and ABC.*

(2.1) *The incenters form a rectangle.*
(2.2) $\rho_a + \rho_c = \rho_b + \rho_d$.

Proof. In $ABCD$ we have the following circles: $O_{cd}(r_{cd})$, $O_{da}(r_{da})$, $O_{ab}(r_{ab})$, and $O_{bc}(r_{bc})$ inscribed respectively in angles AEB, BEC, CED, and DEA, each tangent internally to the circumcircle. Let $\angle AEB = \angle CED = \theta$ and $\angle BEC = \angle DEA = \pi - \theta$.

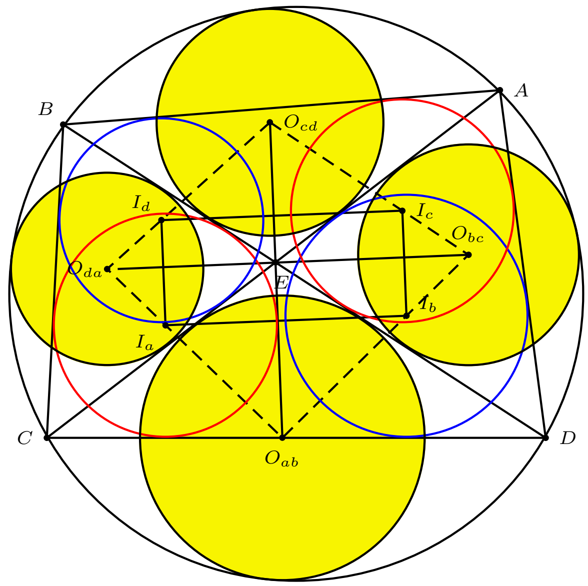

Figure 3

Now, by Theorem 1, the centers I_a, I_b, I_c, I_d lie on the lines $O_{da}O_{ab}$, $O_{ab}O_{bc}$, $O_{bc}O_{cd}$, $O_{cd}O_{da}$ respectively. Furthermore,

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_c}{I_cO_{cd}} = \tan^2\left(\frac{\pi-\theta}{2}\right) = \cot^2\frac{\theta}{2},$$
$$\frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_d}{I_dO_{da}} = \tan^2\frac{\theta}{2}.$$

From these, we have

$$\frac{O_{da}I_a}{I_aO_{ab}} = \frac{O_{bc}I_b}{I_bO_{ab}}, \qquad \frac{O_{ab}I_b}{I_bO_{bc}} = \frac{O_{cd}I_c}{I_cO_{bc}},$$
$$\frac{O_{bc}I_c}{I_cO_{cd}} = \frac{O_{da}I_d}{I_dO_{cd}}, \qquad \frac{O_{cd}I_d}{I_dO_{da}} = \frac{O_{ab}I_a}{I_aO_{da}}.$$

These proportions imply the following parallelism:

$$I_aI_b//O_{da}O_{bc}, \quad I_bI_c//O_{ab}O_{cd}, \quad I_cI_d//O_{bc}O_{da}, \quad I_dI_a//O_{cd}O_{ab}.$$

As the segments $O_{cd}O_{ab}$ and $O_{da}O_{bc}$ are perpendicular because they are along the bisectors of the angles at E, $I_aI_bI_cI_d$ is an inscribed rectangle in $O_{ab}O_{bc}O_{cd}O_{da}$, and this proves (2.1).

Also, the following relation results from (1.2):

$$\rho_a + \rho_c = (r_{ab} + r_{cd})\cos^2\frac{\theta}{2} + (r_{da} + r_{bc})\sin^2\frac{\theta}{2}.$$

This same expression is readily seen to be equal to $\rho_b + \rho_d$ as well. This proves (2.2). □

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