FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 5
2005

http://forumgeom.fau.edu

ISSN 1534-1178

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Forum Geometricorum
Volume 5 (2005) 1–15.

FORUM GEOM
ISSN 1534-1178

Where are the Conjugates?

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Abstract. The positions and properties of a point in relation to its isogonal and isotomic conjugates are discussed. Several families of self-conjugate conics are given. Finally, the topological implications of conjugacy are stated along with their implications for pivotal cubics.

1. Introduction

The edges of a triangle divide the Euclidean plane into seven regions. For the projective plane, these seven regions reduce to four, which we call the central region, the a region, the b region, and the c region (Figure 1). All four of these regions, each distinguished by a different color in the figure, meet at each vertex. Equivalent structures occur in each, making the projective plane a natural background for fundamental triangle symmetries. In the sense that the projective plane can be considered a sphere with opposite points identified, the projective plane divided into four regions by the edges of a triangle can be thought of as an octahedron projected onto this sphere, a remark that will be helpful later.

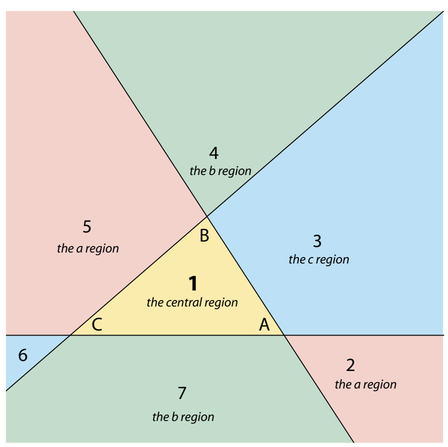

Figure 1. The plane of the triangle, Euclidean and projective views

A point $\mathbf{P}$ in any of the four regions has an harmonic associate in each of the others. Cevian lines through $\mathbf{P}$ and/or its harmonic associates traverse two of the these regions, there being two such possibilities at each vertex, giving 6 Cevian (including exCevian) lines. These lines connect the harmonic associates with the vertices in a natural way.

Publication Date: January 11, 2005. Communicating Editor: Paul Yiu.

Given two points in the plane there are two central points (a non-projective concept), the midpoint and a point at infinity. Given two lines there are two central lines, the angle bisectors. Where there is a sense of center, there is a sense of deviation from that center. For each point not at a vertex of the triangle there is a conjugate point defined using each of these senses of center. The isogonal conjugate is the one defined using angles and the isotomic conjugate is defined using distances. This paper is about the relation of a point to its conjugates.

We shall use the generic term *conjugate* when either type is implied. Other types of conjugacy are possible [2], and our remarks will hold for them as well.

Notation. Points and lines will be identified in bold type. John Conway's notation for points is used. The four incenters (the incenter and the three excenters) are $\mathbf{I}_o$, $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$. The four centroids (the centroid and its harmonic associates) are $\mathbf{G}$, $\mathbf{A}^{\mathbf{G}}$, $\mathbf{B}^{\mathbf{G}}$, $\mathbf{C}^{\mathbf{G}}$. We shall speak of equivalent structures around the four incenters or the four centroids. An angle bisector is identifed by the two incenters on it and a median by the two centroids on it as in "ob", or "ac". $\mathbf{A}_{\mathbf{P}}$ is the Cevian trace of line $\mathbf{AP}$ and $\mathbf{A}^{\mathbf{P}}$ is a vertex of the pre-Cevian triangle of $\mathbf{P}$. We shall often refer to this point as an "ex-"version of $\mathbf{P}$ or as an harmonic associate of $\mathbf{P}$. Coordinates are barycentric. $\mathrm{t}\mathbf{P}$ is the isotomic conjugate of $\mathbf{P}$, $\mathrm{g}\mathbf{P}$ the isogonal conjugate.

The isogonal of a line through a vertex is its reflection across either bisector through that vertex. The isogonal lines of the three Cevian lines of a point $\mathbf{P}$ concur in its conjugate $\mathrm{g}\mathbf{P}$. In the central region of a triangle, the relation of a point to its conjugate is simple. This region of the triangle is divided into 6 smaller regions by the three internal bisectors. If $\mathbf{P}$ is on a bisector, so is $\mathrm{g}\mathbf{P}$, with the incenter between them, making the bisectors fixed lines under isogonal conjugation. If $\mathbf{P}$ is not on a bisector, then $\mathrm{g}\mathbf{P}$ is in the one region of the six that is on the opposite side of each of the three bisectors. This allows us to color the central region with three colors so that a point and its conjugate are in regions of the same color (Figure 2). The isotomic conjugate behaves analogously with the medians serving as fixed or self-conjugate lines.

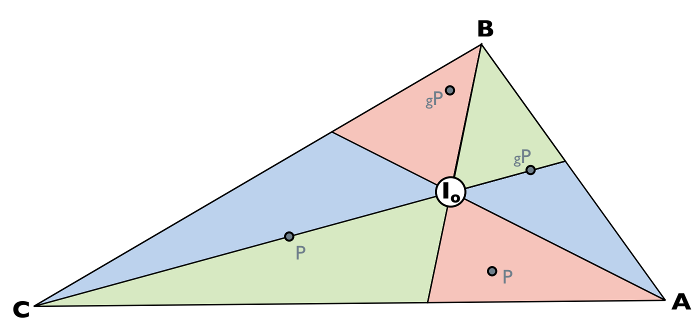

Figure 2. Angle bisctors divide the central region of the triangle into co-isotomic regions. The isogonal conjugate of a point on a bisector is also on that bisector. The conjugate of a point in one of the colored regions is in the other region of the same color.

2. Relation of conjugates to self-conjugate lines

The central region is all well and good, but the other three regions are locally identical in behavior and are to be considered structurally equivalent. Figure 3 shows the triangle with the incentral quadrangle. Each vertex of $\mathbf{ABC}$ hosts two bisectors, traditionally called internal and external. It is important to realize that an isogonal line through any vertex can be created by reflection in either bisector. This means that the three particular bisectors through any of the four incenters (one from each vertex) can be used to define the isogonal conjugate. Hence the behavior of conjugates around $\mathbf{I}_b$, say, is locally identical to that around $\mathbf{I}_o$, as shown in Figure 4.

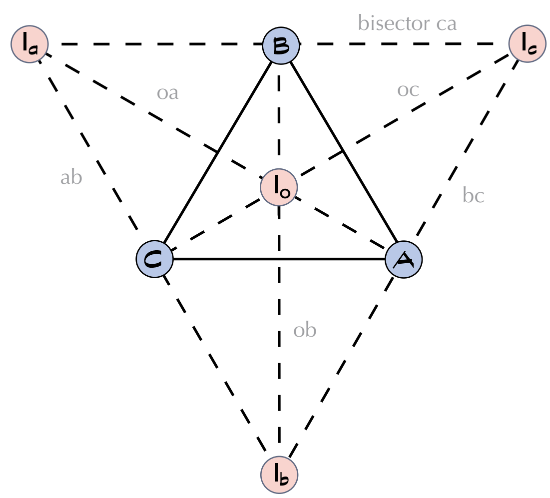

Figure 3. The triangle and its incentral quadrangle

If $\mathbf{P}$ is in the central region, the conjugate $\mathrm{g}\mathbf{P}$ is also; both are on the same side (the interior side) of each of the three external bisectors. So in the central region a point and its conjugate are on opposite sides of three bisectors (the internal ones) and on the same side of three others (the external ones). This is also true in the neighborhood of $\mathbf{I}_b$, although the particular bisectors have changed. No matter where in the plane, a point not on a bisector is on the opposite of three bisectors from its conjugate and on the same side for the other three bisectors. To some extent this statement is justified by the local equivalence of conjugate behavior mentioned above, but this assertion will be fully justified later in §10 on topological properties.

3. Formal properties of the conjugacy operation

Each type of conjugate has special fixed points and lines in the plane. As these properties are generally known, they will be stated without proof. Figures 5 and 8 show the mentioned structures.

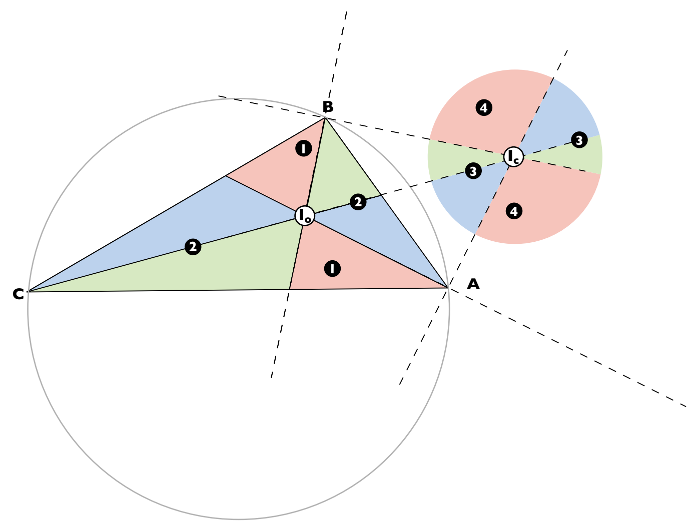

Figure 4. This picture shows the local equivalence of the region around $\mathbf{I}_o$ to that around $\mathbf{I}_c$. This equivalence appears to end at the circumcircle. Numbered points are co-conjugal, each being the conjugate of the other. For each region a pair of points both on and off a bisector is given.

conjugacy	fixed points	fixed lines	special curves	singularities
isotomic	centroid, its harmonic conjugates	medians and ex-medians	line at infinity, Steiner ellipse	vertices
isogonal	incenter, its harmonic conjugates	internal and external bisectors	line at infinity, circumcircle	vertices

For each type of conjugacy there are 4 points in the plane, harmonically related, that are fixed points under conjugacy. For isogonal conjugacy these are the 4 in/excenters. For isotomic conjugacy these are the centroid and its harmonic associates.In each case the six lines that connect the 4 fixed points are the fixed lines.

Special curves: Each point on the Steiner ellipse has the property that its isotomic Cevians are parallel, placing the isotomic conjugate at infinity. Similarly for any point on the circumcircle, its isogonal Cevians are parallel, again placing the isogonal conjugate at infinity. These special curves are very significant in the Euclidean plane, but not at all significant in the projective plane.

The conjugate of a point on an edge of $\mathbf{ABC}$ is at the corresponding vertex, an ∞ to 1 correspondence. This implies that the conjugate at a vertex is not defined, making the vertices the three points in the plane where this is true. This leads to a complicated partition of the Euclidean plane, as the behavior the conjugate of a point inside the Steiner ellipse or the circumcircle is different from that outside. We

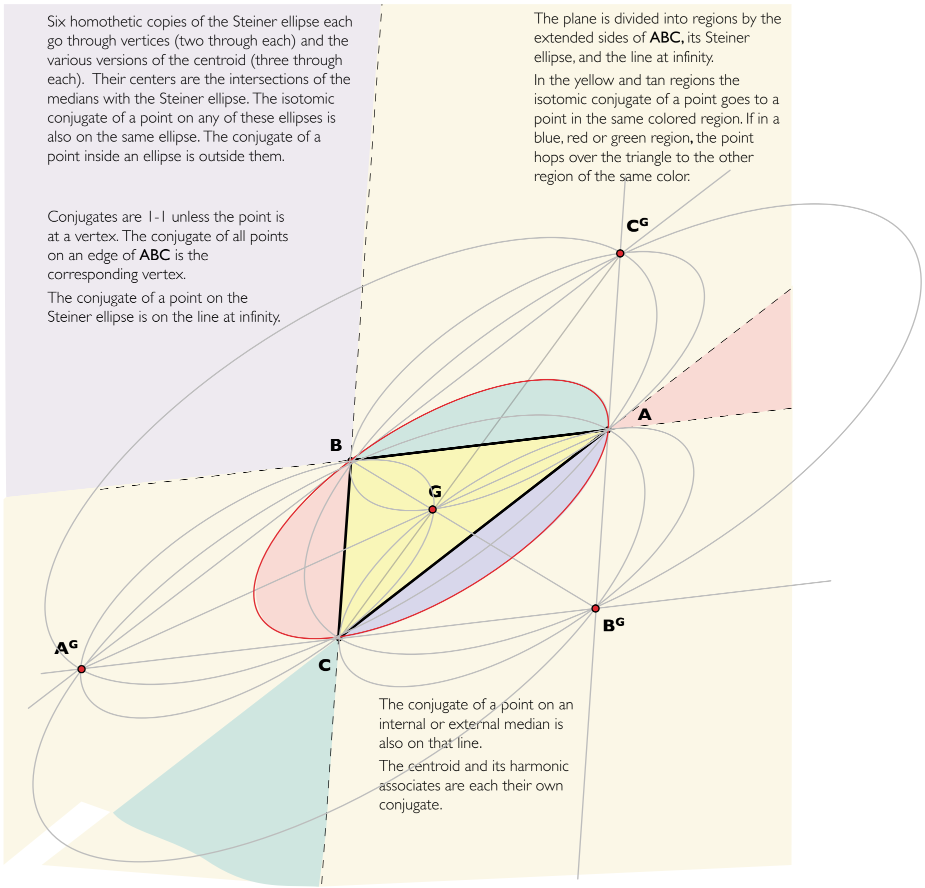

Figure 5. Isotomic conjugates

thus have the pictures of the regions of the plane in terms of conjugates as shown in Figures 5 and 8.

The colors in these two pictures show regions of the plane which are shared by the conjugates. The boundaries of these regions are the sides of the triangle, the circumconic and the line at infinity. The conjugate of a point in a region of a certain color is a region of the same color. For the red, green, and blue regions the conjugate is always in the other region of the same color.

These properties are helpful in locating a point in relation to the position of its conjugate, but there is more to this story.

4. Conjugate curves

4.1. *Lines.* The conjugate of a curve is found by taking the conjugate of each point on the curve. In general the conjugate of a straight line is a circumconic, but there are some exceptions.

Theorem 1. *If a line goes through a vertex of the reference triangle* $\mathbf{ABC}$*, the conjugate of this line is a line through the same vertex.*

Proof. Choose vertex $\mathbf{B}$. A line through this vertex has the form $nz - \ell x = 0$. The isotomic conjugate is $\frac{n}{z} - \frac{\ell}{x} = 0$, which is the same as $nx - \ell z = 0$, a line through the same vertex. The isogonal conjugate works analogously. □

This result is structurally useful. If a point approaches a vertex on a straight line (or a smooth curve, which must approximate one) its conjugate crosses an edge by the conjugate line ([3]).

4.2. *Self conjugate conics (isotomic case).* The isotomic conjugate of the general conic is a quartic curve, but again there are some interesting exceptions.

Theorem 2. *Conics through* $\mathbf{AGCB^G}$ *and* $\mathbf{ACC^GA^G}$ *are self-isotomic.*

Proof. The general conic is $\ell x^2 + my^2 + nz^2 + Lyz + Mzx + Nxy = 0$. Choosing the case $\mathbf{AGCB^G}$, since $\mathbf{A}$ and $\mathbf{C}$ are on the conic, we have that $\ell = n = 0$. From $\mathbf{G}$ and $\mathbf{B^G}$ we get the two equations $m \pm L + M \pm N = 0$, from which we get $M = -m$ and $N = -L$ giving $y^2 - zx + \lambda y(z - x) = 0$ as the family of conics through these two points. Replacing each coordinate with its reciprocal and assuming that $xyz \neq 0$, we see that this equation is self-isotomic.

For the case $\mathbf{CAA^GC^G}$ the equation is $y^2 + zx + \lambda y(z + x) = 0$, also self-isotomic.

□

Each family has one special conic homothetic to the Steiner ellipse and of special interest: $y^2 - zx = 0$, which goes through $\mathbf{AGCB^G}$, and $y^2 + zx + 2y(z+x) = 0$, which goes through $\mathbf{ACC^GA^G}$. Conics homothetic to the Steiner ellipse can be written as $yz + zx + xy + (Lx + My + Nz)(x + y + z) = 0$. Choosing $L = N = 0$ and $M = \pm 1$ gives the two conics of interest. The first of these has striking properties.

Theorem 3. *The ellipse* $y^2 - zx = 0$

(1) *goes through* $\mathbf{C}$, $\mathbf{A}$, $\mathbf{G}$, $\mathbf{A^G}$,

(2) *is tangent to edges* a *and* c,

(3) *contains the isotomic conjugate* $\mathrm{t}\mathbf{P}$ *of every point* $\mathbf{P}$ *on it,* (*and if one of* $\mathbf{P}$ *and* $\mathrm{t}\mathbf{P}$ *is inside, then the other is outside the ellipse; the line connecting a point on the ellipse with its conjugate is parallel to the* b *edge* [3]),

(4) *contains the* $\mathbf{B}$*-harmonic associate of every point on it,*

(5) *has center* $(2 : -1 : 2)$ *which is the intersection of the Steiner ellipse with the b-median,*

(6) *is the translation of the Steiner ellipse by the vector from* $\mathbf{B}$ *to* $\mathbf{G}$,

(7) *contains* $\mathbf{P}^n = (x^n : y^n : z^n)$ *for integer values of* n *if* $\mathbf{P} = (x : y : z)$, $(xyz \neq 0)$*, is on the curve,*

(8) *is the inverse in the Steiner ellipse of the b-edge of* **ABC** .

These last two properties are included for their interest, but have little to do with the topic at hand (other than that $n = -1$ is the isotomic conjugate). A second paper will be devoted to these properties of this curve.

Proof. (1) can be verified by substituting coordinates as done above.

(2) is true by the general principle that if an equation has the form (line 2)^2 = (line 1)$\cdot$(line 3), then the curve has a double intersection at the intersection of line 1 and line 2 and at the intersection of line 3 and line 2 and is tangent to lines 1 and 2 at those points.

For (3) we take the isotomic conjugage of a point on the curve to obtain $\frac{1}{y^2} - \frac{1}{zx} = 0$, which, since this curve only exists where the product zx is positive, is the same as $zx - y^2 = 0$, so that $\mathrm{t}\mathbf{P}$ is on the curve if $\mathbf{P}$ is, which also implies that the point and the conjugate are on different sides of the ellipse. $(yz : zx : xy)$ is the conjugate. If on the ellipse $zx = y^2$ we have $(yz : y^2 : xy) \sim (z : y : x)$. The vector from this point to $(x : y : z)$ is proportional to $(-1 : 0 : 1)$, which is in the direction of the b-edge.

(4) can be verified by noting that if (x, y, z) is on the ellipse, so is its harmonic associate $(x, -y, z)$.

(5) The center is found as the polar of the line at infinity.

(6) is verified by computing the translation $T : \mathbf{B} \to \mathbf{G}$, and computing $S(T^{-1}\mathbf{P})$, where $S(\mathbf{P})$ is the Steiner ellipse in terms of a point $\mathbf{P}$ on the curve.

(7) is verified since $(y^n)^2 - z^n x^n$ has $y^2 - zx$ as a factor, so that $\mathbf{P}^n$ is on the curve if $\mathbf{P}$ is.

(8) $(\cdots : y : \cdots) \to (\cdots : y^2 - zx : \cdots)$ is the Steiner inversion and takes $y = 0$ into $y^2 - zx = 0$. $\square$

5. The isotomic ellipses

Consider the three curves

$$\begin{aligned} &x^2 - yz = 0, \\ &y^2 - zx = 0, \\ &z^2 - xy = 0, \end{aligned}$$

which are translations of the Steiner ellipse, each through two vertices, and tangent to the edges of $\mathbf{ABC}$. Exactly as the three medians are self-isotomic and separate the central region of the triangle, so too do these ellipses. If a point is inside one, its conjugate is outside. The line from a point on one of these curves to its conjugate its parallel to a side of the triangle, or perhaps stated more correctly, to the an ex-median.

Consider the three curves

$$\begin{aligned} &x^2 + yz + 2x(y + z) = 0, \\ &y^2 + zx + 2y(z + x) = 0, \\ &z^2 + xy + 2z(x + y) = 0 \end{aligned}$$

each homothetic to the Steiner ellipse. Each goes through two ex-centroids and two vertices and is centered at the other vertex. These are the exterior versions of the above three, rather as the ex-medians are external versions of the medians. They are self-isotomic and the line from a point to its conjugate is parallel to a median (proved below). These ellipses go through the ex-centroids and serve to define regions about them just as the others do for the central regions. They can also be seen in Figure 5. These six isotomic ellipses are all centered on the Steiner circumellipse of $\mathbf{ABC}$. Their tangents at the vertices are either parallel to the medians or the exmedians. For any point in the plane where the conjugate is defined, the point and its conjugate are on the same side (inside or outside) for three ellipses and on opposite sides for the other three (just as for the medians).

6. $\mathbf{P} - \mathrm{t}\mathbf{P}$ lines

For points on the interior versions (those that pass through $\mathbf{G}$) of these conics, the lines from a point to its conjugate are parallel to the ex-medians (and hence to the sides of $\mathbf{ABC}$). For points on the exterior ellipses, the line joining a point to its conjugate is parallel to a median of $\mathbf{ABC}$. This is illustrated in Figure 6.

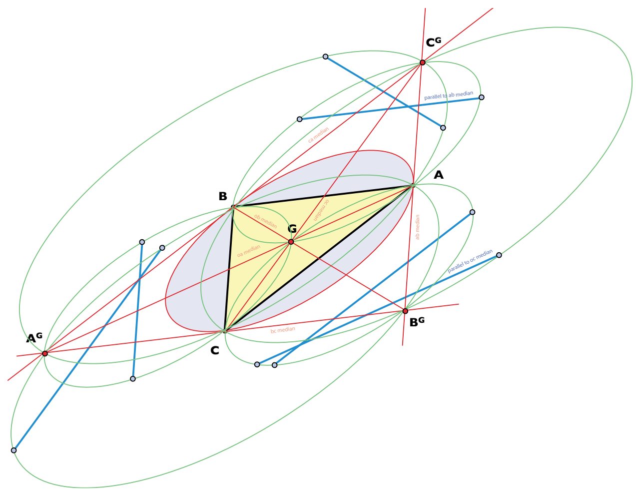

Figure 6. Points paired with their conjugates are connected by blue lines, each of which is parallel to a median or an ex-median of **ABC**. The direction of the lines for the two ellipses through **A** and **B** are noted.

For the interior ellipses, this property has been proved. For the exterior ones the math is a bit harder. Note that a point and its conjugate can be written as $(x : y : z)$ and $(yz : zx : xy)$. The equation of the ellipse can be written as $zx = y^2 + 2y(z + x)$, so that the conjugate becomes

$$(yz : y^2 + 2y(z + x) : xy) \sim (z : y + 2(z + x) : x).$$

The vector between these two (normalized) points is

$$(x + y + z : -2(x + y + z) : x + y + z) \sim (1 : -2 : 1)$$

which is the direction of a median.

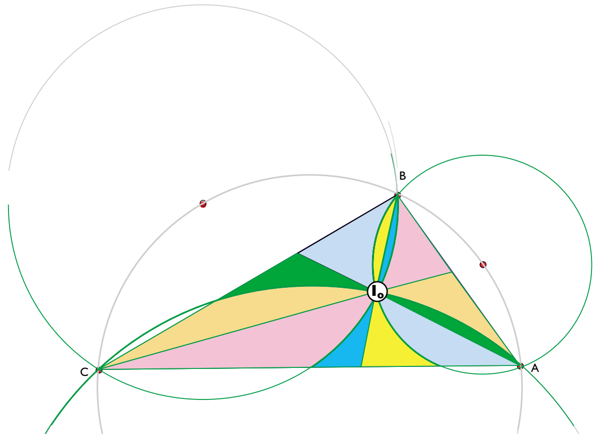

Figure 7. The central region divided by three bisectors and three self-isogonal circles.

7. The self-isogonal circles

Just as the ellipse homothetic to the Steiner ellipse through $\mathbf{CGAG^B}$ is isotomically self-conjugate, the circle through the corresponding set of points $\mathbf{CI}_o\mathbf{AI}_b$ is isogonally self-conjugate, a very pretty result. Just as the there are six versions of the isotomic ellipses, each with a center on the Steiner ellipse, there are 6 isogonal circles, each centered on the circumcircle, also a pretty result (Figure 8).

We note that $\mathbf{I}_o\mathbf{CI}_b\mathbf{A}$ is cyclic because the bisector $\mathbf{AI}_b$ is perpendicular to the bisector $\mathbf{I}_o\mathbf{A}$. The angles at $\mathbf{A}$ and $\mathbf{C}$ are right angles so that opposite angles of the quadrilateral are supplementary. Hence there is a circle through $\mathbf{CI}_o\mathbf{AI}_b$. It is in fact the diametral circle on $\mathbf{I}_o\mathbf{I}_b$.

The equation of a general circle is

$$a^2yz + b^2zx + c^2xy + (\ell x + my + nz)(x + y + z) = 0.$$

Demanding that it go through the above 4 points, we get

$$cay^2 - b^2zx - (a - c)(ayz - cxy) = 0$$

with center $(a(a + c) : -b^2 : c(a + c))$, the midpoint of $\mathbf{I}_o\mathbf{I}_a$. There are six such circles, each through 2 vertices and two incenters. Each pair of incenters determines one of these circles hence there are 6 of them. Just as each bisector goes through 2 incenters, so does each of these circles. Just as the bisectors separate a point from its conjugate, so do these circles, giving an even more detailed view of conjugacy in the neighborhood of an incenter (see Figure 7).

If a point on one of these six circles is connected to its conjugate, the line is parallel to one of the six bisectors, the circles through $\mathbf{I}_o$ pairing with exterior bisectors. The tangent lines at the vertices are also parallel to a bisector. These statements are proved just as for the isotomic ellipses.

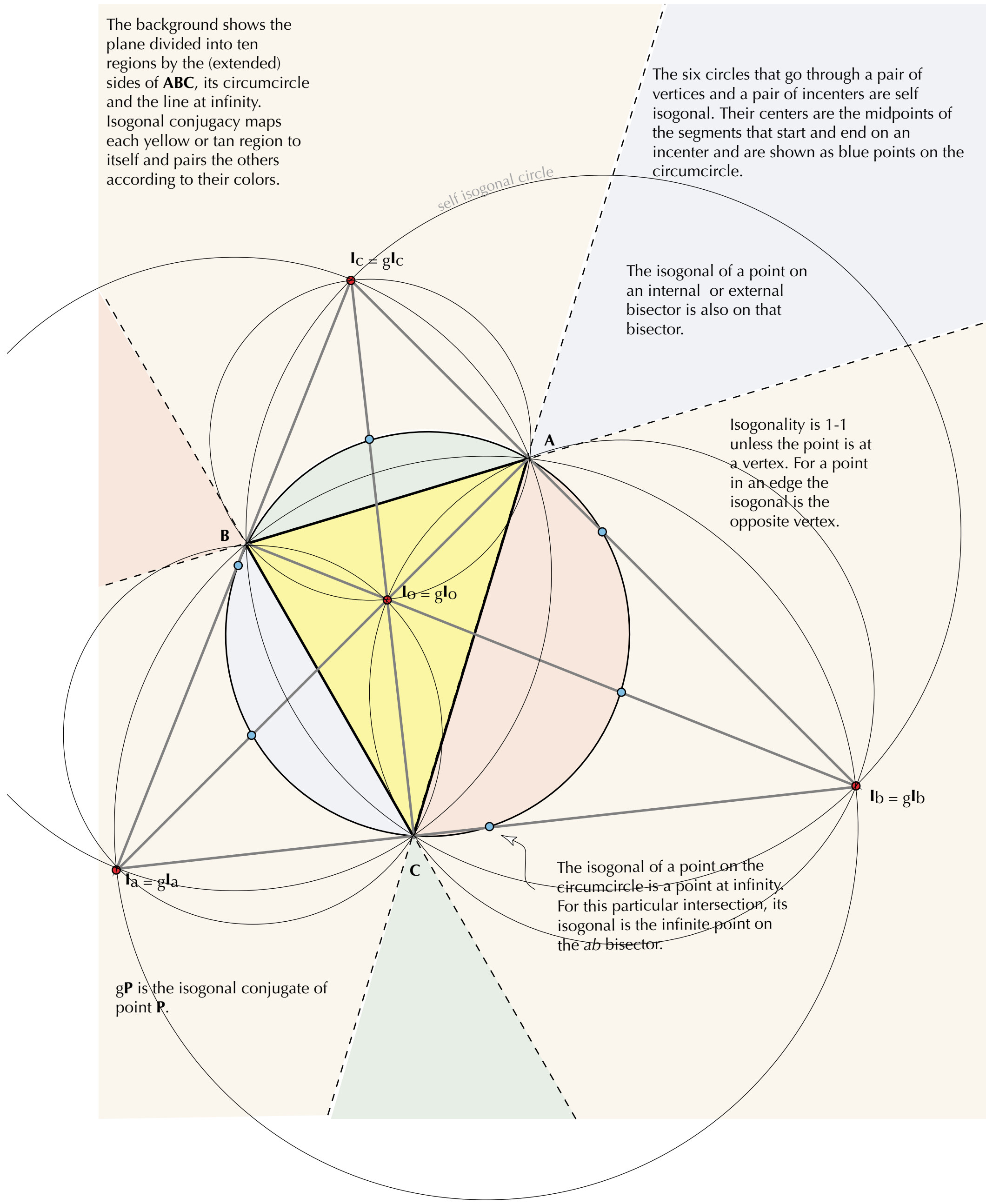

Figure 8. Isogonal conjuates

8. Self-isogonal conics

Demanding a conic go through $\mathbf{CI}_o\mathbf{AI}_b$, we get $cay^2 - b^2zx + \lambda y(az - cx) = 0$, which can be verified to be self- isogonal. Those through $\mathbf{CAI}_a\mathbf{I}_c$ have equation $cay^2 + b^2zx + \lambda y(az + cx) = 0$, and are similarly isogonal.

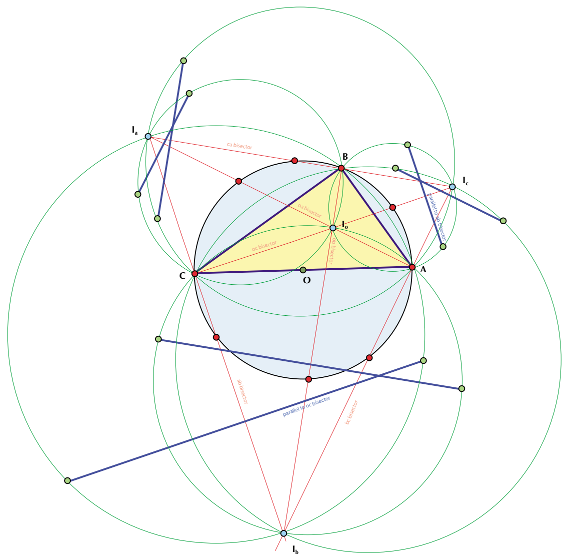

Figure 9. $\mathbf{P} - \mathrm{g}\mathbf{P}$ lines. On each isogonal circle the line from a point to its conjugate is parallel to one of the angle bisectors. If the circle goes through $\mathbf{I}_o$ the line is parallel to the corresponding external bisector. The red points on the circumcircle are the centers of the isogonal circles. For the two circles through $\mathbf{A}$ and $\mathbf{B}$, the directions of the $\mathbf{P} - \mathrm{g}\mathbf{P}$ line is noted.

9. The central region - an enhanced view

These self-conjugate circles thus help us place the isogonal conjugate of $\mathbf{P}$ just as do the median lines. If a point is on one of these circles, then so is its conjugate. If inside, the conjugate is outside and vice versa. This division of the plane into regions is very effective at giving the general location of the conjugate of a point (Figure 7). Of course this behavior around $\mathbf{I}_o$ is mimicked by that around the other incenters.

10. Topological considerations

There is a complication to the above analysis which leads to a very pretty picture of conjugacy in the projective plane. Conjugacy is 1-1 both ways except at the vertices where it blows up. This is in fact a topological blowup. To see this, let $\mathbf{P}$ move out of the central region across the b-edge, say. Near both $\mathbf{I}_o$ and $\mathbf{I}_b$, the behavior of a point to its conjugate is simple and known. In the central region, $\mathbf{P}$ and its conjugate $\mathbf{Q}$ were on opposite sides of the b-bisector; once $\mathbf{P}$ passed through the b-edge, $\mathbf{Q}$ passed through the $\mathbf{B}$-vertex, after which it is on the same side of the b-bisector as $\mathbf{P}$. We say that the plane of the triangle, underwent a Möbius-like twist at the $\mathbf{B}$-vertex. Continuing $\mathbf{P}$'s journey out of the central region through the b-edge towards $\mathbf{I}_b$, we encounter the second problem. As $\mathbf{P}$ nears the circumcircle, $\mathbf{Q}$ goes to infinity. As $\mathbf{P}$ crosses the circumcircle, $\mathbf{Q}$ crosses the line at infinity as well as the bisector, giving another twist to the plane as it passes. As

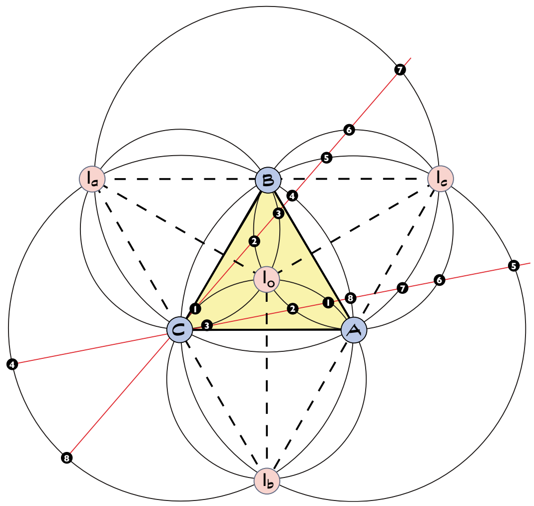

Figure 10. Here points numbered 18 are arranged on a line through **C**. The conjugates, numbered equally, are on the isogonal line through **C**, but are spaced wildly. The isogonal circles show and explain the unusual distribution of the conjugates.

P moves near $\mathbf{I}_b$, the center of the b-excircle, **Q** moves towards it, now again on the opposite side of the bisector. (This emphasis on topological properties is a result of a conversation about conjugacy with John Conway, one of the most interesting conversations about triangle geometry that I have ever had).

The isotomic conjugate behaves analogously at the vertices and at infinity with the Steiner ellipse taking the place of the circumcircle and the six medians replacing the six bisectors.

There is a way to tame the conjugacy operation at the three points in the plane which are not 1-1, and to throw light on the behavior of conjugates at the same time.

As a point approaches a vertex along a line, its conjugate goes to the point on the edge intersected by the isogonal line. Hence although the conjugate at a vertex is undefined, each direction into the vertex corresponds to a point on an edge.We represent this by letting the point “blowup”, becoming a small disc. Each point on the edge of the disc represents a direction with respect to the center. Its antipodal point is on the same line so the disc has opposite points identified. This topological blowup replaces the vertex with a Möbius-like surface (a cross-cap), explaining the shift of the conjugate from the opposite side of a bisector to the same side.

Figure 11 shows the plane of the triangle from this point of view for the isogonal case. It is a very different view indeed. The important lines are the six bisectors and the important points are the three vertices and the four incenters. The edges of the triangle are only shown for orientation and the circumcircle is not relevant to the picture. The colors show co-isogonal regions - if a point is in a region of a certain color, so is its conjugate. The twists of the plane occur at the vertices

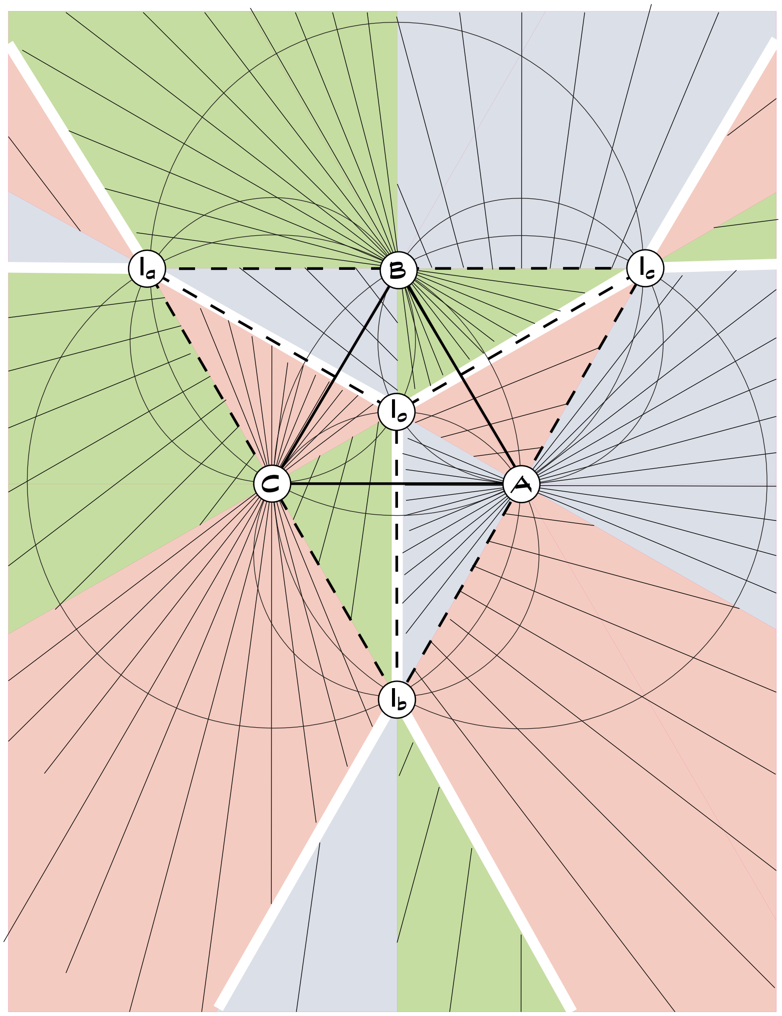

Figure 11. (Drawn with John Conway). Topological view of the location of conjugates The colors show co-isogonal regions. The lines issuing from the vertices show isogonal lines. The isogonal circles are shown. The white lines are the boundries of the three faces of a projective cube.

as shown by the colored regions converging on the vertices. In fact this figure forms a projective cube where the incenters are the four vertices that remain after anitpodes are identified. The view shown is directly toward the "vertex" $\mathbf{I}_o$ with the lines $\mathbf{I}_o\mathbf{I}_a$, $\mathbf{I}_o\mathbf{I}_b$, $\mathbf{I}_o\mathbf{I}_c$ being the three edges from that vertex. $\mathbf{I}_o\mathbf{I}_b\mathbf{I}_c\mathbf{I}_a$ form a face. The white lines are the edges of the cube. In the middle of each face is a cross-cap structure at a vertex. The final picture is of a projective cube with each face containing a crosscap singularity. The triangle $\mathbf{ABC}$ and its sides can be considered the projective octahedron inscribed to the cube with the four regions identified in the introductory paragraph being the four faces.

This leads to a nice view of pivotal cubics which are defined in terms of conjugates. The cubics go through all 7 relevant points.

11. Cubics

We can learn a bit about the shape of pivotal cubics from this topological picture of the conjugates. Pivotal cubics include both a point and its conjugate, so that each branch of the cubic must stay in co-isogonal regions, which are of a definite color on our topological picture.

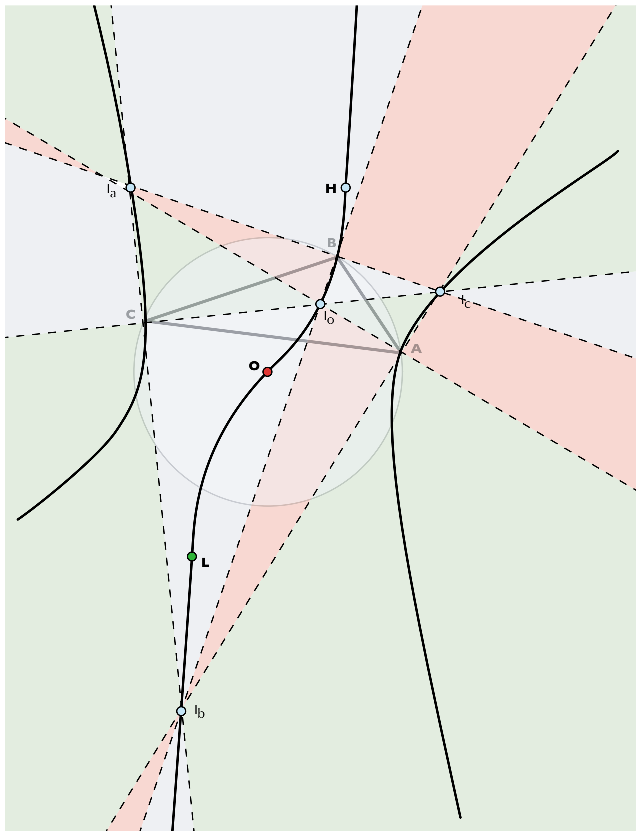

Figure 12. The Darboux cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the deLongchamps point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so is its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

The Darboux cubic (Figure 12) has two branches, one through a single vertex, $\mathbf{I}_o$ and, in the illustration, $\mathbf{I}_b$. The other goes through $\mathbf{I}_c$, $\mathbf{I}_a$ and two vertices, wrapping around through the line at infinity. The Neuberg cubic (Figure 13) does the same. Its "circular component" being more visible since it does not pass through the line at infinity. We can understand the various "wiggles" of these cubics as necessary to stay in a self-conjugal region. Also we can see that a conjugate of a point on one branch cannot be on the other branch.

Geometry is fun.

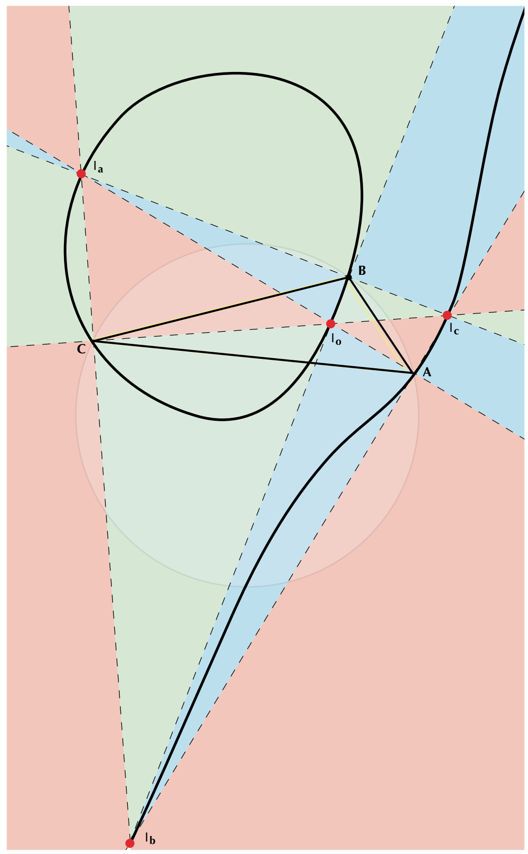

Figure 13. The Neuberg cubic is a pivotal isogonal cubic, meaning that the isogonal conjugate of each point is on the cubic and colinear with the pivot point, which in this case is the Euler infinity point. The colored regions show the pattern of the conjugates. If a point is in a region of a certain color, so its conjugate. This picture shows that the branches of the cubic turn to stay in regions of a particular color.

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Forum Geometricorum
Volume 5 (2005) 17–20.

FORUM GEOM
ISSN 1534-1178

A Synthetic Proof of Goormaghtigh's Generalization of Musselman's Theorem

Khoa Lu Nguyen

Abstract. We give a synthetic proof of a generalization by R. Goormaghtigh of a theorem of J. H. Musselman.

Consider a triangle ABC with circumcenter O and orthocenter H. Denote by A^*, B^*, C^* respectively the reflections of A, B, C in the side BC, CA, AB. The following interesting theorem was due to J. R. Musselman.

Theorem 1 (Musselman [2])**.** *The circles AOA^*, BOB^*, COC^* meet in a point which is the inverse in the circumcircle of the isogonal conjugate point of the nine point center.*

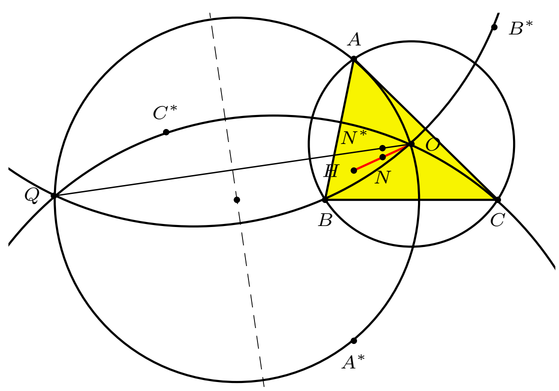

Figure 1

R. Goormaghtigh, in his solution using complex coordinates, gave the following generalization.

Theorem 2 (Goormaghtigh [2])**.** *Let A_1, B_1, C_1 be points on OA, OB, OC such that*

$$\frac{OA_1}{OA} = \frac{OB_1}{OB} = \frac{OC_1}{OC} = t.$$

(1) *The intersections of the perpendiculars to OA at A_1, OB at B_1, and OC at C_1 with the respective sidelines BC, CA, AB are collinear on a line ℓ.*

(2) *If M is the orthogonal projection of O on ℓ, M' the point on OM such that $OM' : OM = 1 : t$, then the inversive image of M' in the circumcircle of ABC*

Publication Date: January 24, 2005. Communicating Editor: Paul Yiu.

The author thanks the communicating editor for his help and also appreciates the great support of his teacher Mr. Timothy Do.

is the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$. See Figure 1.

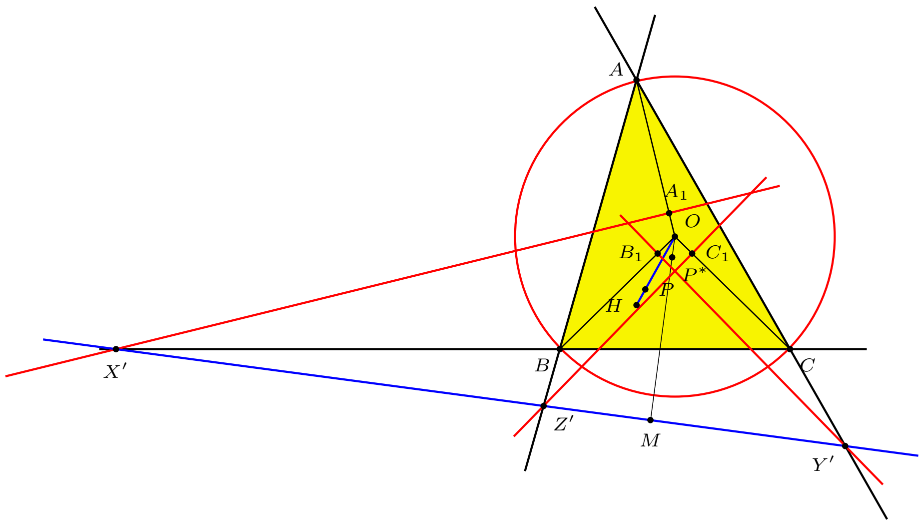

Figure 2

Musselman's Theorem is the case when $t = \frac{1}{2}$. Since the centers of the circles OAA^*, OBB^*, OCC^* are collinear, the three circles have a second common point which is the reflection of O in the line of centers. This is the inversive image of the isogonal conjugate of the nine-point center, the midpoint of OH.

By Desargues' theorem [1, pp.230–231], statement (1) above is equivalent to the perspectivity of ABC and the triangle bounded by the three perpendiculars in question. We prove this as an immediate corollary of Theorem 3 below. In fact, Goormaghtigh [2] remarked that (1) was well known, and was given in J. Neuberg's *Mémoir sur le Tétraèdre*, 1884, where it was also shown that the envelope of ℓ is the inscribed parabola with the Euler line as directrix (Kiepert parabola). He has, however, inadvertently omitted "the isogonal conjugate of " in statement (2).

Theorem 3. *Let $A'B'C'$ be the tangential triangle of ABC. Consider points X, Y, Z dividing OA', OB', OC' respectively in the ratio*

$$\frac{OX}{OA'} = \frac{OY}{OB'} = \frac{OZ}{OC'} = t. \tag{†}$$

The lines AX, BY, CZ are concurrent at the isogonal conjugate of the point P on the Euler line dividing OH in the ratio $OP : PH = 1 : 2t$.

Proof. Let the isogonal line of AX (with respect to angle A) intersect OA' at X'. The triangles OAX and $OX'A$ are similar. It follows that $OX \cdot OX' = OA^2$, and X, X' are inverse in the circumcircle. Note also that A' and M are inverse in the

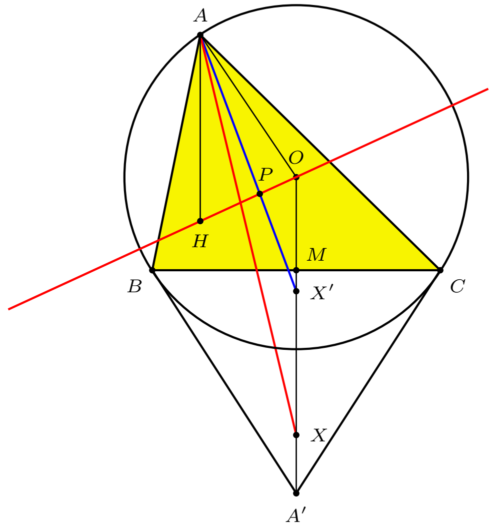

Figure 3

same circumcircle, and $OM \cdot OA' = OA^2$. If the isogonal line of AX intersects the Euler line OH at P, then

$$\frac{OP}{PH} = \frac{OX'}{AH} = \frac{OX'}{2 \cdot OM} = \frac{1}{2} \cdot \frac{OA'}{OX} = \frac{1}{2t}.$$

The same reasoning shows that the isogonal lines of BY and CZ intersect the Euler line at the same point P. From this, we conclude that the lines AX, BY, CZ intersect at the isogonal conjugate of P. $\square$

For $t = \frac{1}{2}$, X, Y, Z are the circumcenters of the triangles OBC, OCA, OAB respectively. The lines AX, BY, CZ intersect at the isogonal conjugate of the midpoint of OH, which is clearly the nine-point center. This is Kosnita's Theorem (see [3]).

Proof of Theorem 2. Since the triangle XYZ bounded by the perpendiculars at A_1, B_1, C_1 is homothetic to the tangential triangle at O, with factor t. Its vertices X, Y, Z are on the lines OA', OB', OC' respectively and satisfy (†). By Theorem 3, the lines AX, BY, CZ intersect at the isogonal conjugate of P dividing OH in the ratio $OP : HP = 1 : 2t$. Statement (1) follows from Desargues' theorem. Denote by X' the intersection of BC and YZ, Y' that of CA and ZX, and Z' that of AB and XY. The points X', Y', Z' lie on a line ℓ.

Consider the inversion Ψ with center O and constant $t \cdot R^2$, where R is the circumradius of triangle ABC. The image of M under Ψ is the same as the inverse of M' (defined in statement (2)) in the circumcircle. The inversion Ψ clearly maps A, B, C into A_1, B_1, C_1 respectively. Let A_2, B_2, C_2 be the midpoints of BC, CA, AB respectively. Since the angles BB_1X and BA_2X are both right angles, the points B, B_1, A_2, X are concyclic, and

$$OA_2 \cdot OX = OB \cdot OB_1 = t \cdot R^2.$$

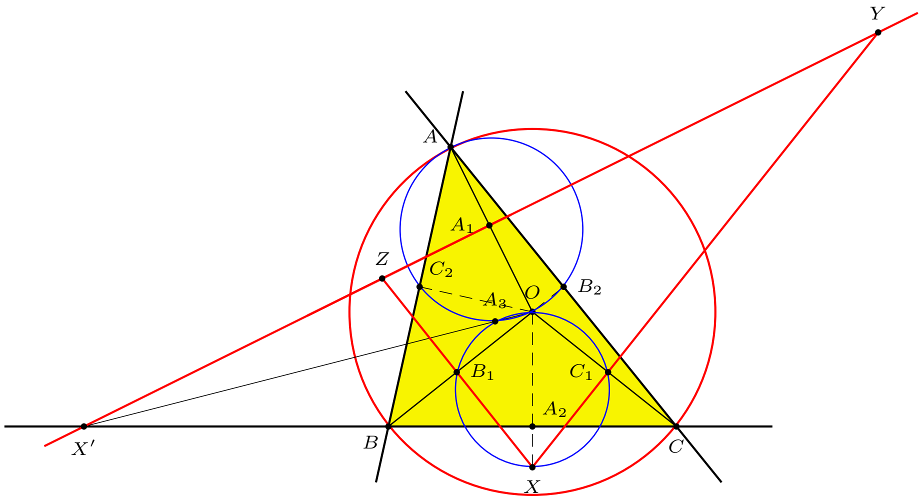

Figure 4

Similarly, $OB_2 \cdot OB_2' = OC_2 \cdot OC_2' = t \cdot R^2$. It follows that the inversion Ψ maps X, Y, Z into A_2, B_2, C_2 respectively.

Therefore, the image of X' under Ψ is the second common point A_3 of the circles OB_1C_1 and OB_2C_2. Likewise, the images of Y' and Z' are respectively the second common points B_3 of the circles OC_1A_1 and OC_2A_2, and C_3 of OA_1B_1 and OA_2B_2. Since X', Y', Z' are collinear on ℓ, the points O, A_3, B_3, C_3 are concyclic on a circle $\mathcal{C}$.

Under Ψ, the image of the line AX is the circle OA_1A_2, which has diameter OX' and contains M, the projection of O on ℓ. Likewise, the images of BY and CZ are the circles with diameters OY' and OZ' respectively, and they both contain the same point M. It follows that the common point of the lines AX, BY, CZ is the image of M under Ψ, which is the intersection of the line OM and $\mathcal{C}$. This is the antipode of O on $\mathcal{C}$.

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Forum Geometricorum
Volume 5 (2005) 21–22.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Lengths of One Side, the Opposite and One Adjacent Angle Bisectors

Victor Oxman

Abstract. We give a necessary and sufficient condition for the existence of a triangle with given lengths of one sides, its opposite angle bisector, and one adjacent angle bisector.

In [1] the problem of existence of a triangle with given lengths of one side and two adjacent angle bisectors was solved. In this note we consider the same problem with one of the adjacent angle bisector replaced by the opposite angle bisector. We prove the following theorem.

Theorem 1. *Given a, ℓ_a, $\ell_b > 0$, there is a unique triangle ABC with $BC = a$ and lengths of bisectors of angles A and B equal to ℓ_a and ℓ_b respectively if and only if $\ell_b \leq a$ or*

$$a < \ell_b < 2a \quad \text{and} \quad \ell_a > \frac{4a\ell_b(\ell_b - a)}{(2a-\ell_b)(3\ell_b - 2a)}.$$

Proof. In a triangle ABC with $BC = a$ and given ℓ_a, ℓ_b, let $y = CA$ and $z = AB$. We have $\ell_b = \frac{2az}{a+z} \cos\frac{B}{2}$ and

$$z = \frac{a\ell_b}{2a\cos\frac{B}{2} - \ell_b}. \tag{1}$$

It follows that $\cos\frac{B}{2} > \frac{\ell_b}{2a}$, $\ell_b < 2a$, and

$$B < 2\arccos\frac{\ell_b}{2a}. \tag{2}$$

Also,

$$y^2 = a^2 + z(z - 2a\cos B), \tag{3}$$

$$\ell_a^2 = yz\left(1 - \frac{a^2}{(y+z)^2}\right). \tag{4}$$

Publication Date: February 10, 2005. Communicating Editor: Paul Yiu.

Case 1: $\ell_b \le a$. Clearly, (1) defines z as an increasing function of B on the open interval $\left(0, 2\arccos\frac{\ell_b}{2a}\right)$. As B increases from 0 to $2\arccos\frac{\ell_b}{2a}$, z increases from $\frac{a\ell_b}{2a-\ell_b}$ to ∞. At the same time, from (3), y increases from $a - \frac{a\cdot\ell_b}{2a-\ell_b} = \frac{2a(a-\ell_b)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) can be any positive number. From the intermediate value theorem, there exists a unique B for which (4) is satisfied. This proves the existence and uniqueness of the triangle.

Case 2: $a < \ell_b < 2a$. In this case, (1) defines the same increasing function z as before, but y increases from $\frac{a\cdot\ell_b}{2a-\ell_b} - a = \frac{2a(\ell_b-a)}{2a-\ell_b}$ to ∞. Correspondingly, the right hand side of (4) increases from

$$\frac{a\ell_b}{2a-\ell_b}\cdot\frac{2a(\ell_b-a)}{2a-\ell_b}\left(1-\frac{a^2}{\left(\frac{a\ell_b}{2a-\ell_b}+\frac{2a(\ell_b-a)}{2a-\ell_b}\right)^2}\right) = \frac{16a^2\ell_b^2(\ell_b-a)^2}{(2a-\ell_b)^2(3\ell_b-2a)^2}$$

to ∞. This means $\ell_a > \frac{4a\ell_b(\ell_b-a)}{(2a-\ell_b)(3\ell_b-2a)}$. Therefore, there is a unique value B for which (4) is satisfied. This proves the existence and uniqueness of the triangle. $\square$

Corollary 2. *For the existence of an isosceles triangle with equal sides a with opposite angle bisectors ℓ_a, it is necessary and sufficient that $\ell_a < \frac{4}{3}a$.*

Reference

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Forum Geometricorum
Volume 5 (2005) 23–31.

FORUM GEOM
ISSN 1534-1178

On the Maximal Inflation of Two Squares

Thierry Gensane and Philippe Ryckelynck

Abstract. We consider two non-overlapping congruent squares q_1, q_2 and the homothetic congruent squares q_1^k, q_2^k obtained from two similitudes centered at the centers of the squares. We study the supremum of the ratios of these similitudes for which q_1^k, q_2^k are non-overlapping. This yields a function $\psi = \psi(q_1, q_2)$ for which the squares q_1^ψ, q_2^ψ are non-overlapping although their boundaries intersect. When the squares q_1 and q_2 are not parallel, we give a 8-step construction using straight edge and compass of the intersection $q_1^\psi \cap q_2^\psi$ and we obtain two formulas for ψ. We also give an angular characterization of a vertex which belongs to $q_1^\psi \cap q_2^\psi$.

1. Introduction and notation

We study here the problem of maximizing the *inflation* of two non-overlapping congruent squares $q_1 = q_{a_1,b_1,\theta_1,c}$ and $q_2 = q_{a_2,b_2,\theta_2,c}$. The square q_i has the four vertices

$$S_j(q_i) = (a_i, b_i) + c \cdot (\cos(\theta_i + j\tfrac{\pi}{2}), \sin(\theta_i + j\tfrac{\pi}{2})).$$

Let $q^k_{a,b,\theta,c} = q_{a,b,\theta,k}$ be the homothetic of ratio k/c of the square $q_{a,b,\theta,c}$. Our problem amounts to determining the supremum $\psi = \psi(q_1, q_2)$ of the numbers $k > 0$ for which q_1^k and q_2^k are disjoint.

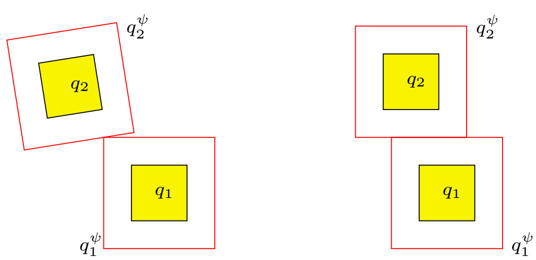

Figure 1

In [3, §4], $\psi = \psi(q_1, q_2)$ is called the *maximum inflation* of a configuration of two squares. It plays a central part in computation of dense packings of squares in a larger square. We refer to the paper of P. Erdős and R. Graham [1] who initiated the problem of maximizing the area sum of packings of an arbitrary square by unit

Publication Date: February 24, 2005. Communicating Editor: Paul Yiu.
We thank the referee for his valuable and helpful suggestions.

squares, see also the survey of E. Friedman [2]. We note that ψ is independent of c and that

$$k \leq \psi \quad \Leftrightarrow \quad \mathrm{int}(q_1^k) \cap \mathrm{int}(q_2^k) = \emptyset, \tag{1}$$

$$k \geq \psi \quad \Leftrightarrow \quad \partial q_1^k \cap \partial q_2^k \neq \emptyset, \tag{2}$$

where as usual, we denote by $\mathrm{int}(q)$ and ∂q the interior and the boundary of a square q. An explicit formula for $\psi = \psi(q_1, q_2)$ is given in [3, Prop.2] as follows. Let us define

$$\psi_0(a,b,\theta) = \min_{i=1,\ldots,4} \left\{ \frac{|a|+|b|}{\left|1 - \sqrt{2}\mathrm{sgn}(ab)\sin(\theta + \frac{\pi}{4} + i\frac{\pi}{2})\right|} \right\},$$

and

$$\rho(q_1,q_2) = \psi_0\left(t\cos\theta_1 + t'\sin\theta_1, -t\sin\theta_1 + t'\cos\theta_1, \theta_2 - \theta_1\right),$$

with $(t,t') = (a_2 - a_1, b_2 - b_1)$. The maximal inflation of two squares q_1 and q_2 is the maximum of $\rho(q_1,q_2)$ and $\rho(q_2,q_1)$. The minimum value, say $k = \rho(q_1,q_2) < \psi$, corresponds to the belongness of a vertex E of q_2^k to a straight line AB when $q_1^k = ABCD$, but without having E between A and B. This expression of ψ gives an efficient tool for doing calculations of maximal inflation of configurations of $n \geq 2$ squares.

In this paper, the two congruent squares q_1, q_2 are such that $q_1 \cap q_2 = \emptyset$ and their centers are denoted by $C_i = C(q_i)$. We say as in [3, §4], that q_2 *strikes* q_1 if the set $q_1^\psi \cap q_2^\psi$ contains a vertex of q_2^ψ. In §§3–5, we suppose that the squares q_1, q_2 are not parallel so that $q_1^\psi \cap q_2^\psi = \{P\}$, where the vertex P of q_1 or q_2 is the *percussion point*. However, at the end of each of these sections, we discuss the parallel case in a final remark. We find in §4 a 8-step construction using straight edge and compass of P. Since P is a vertex of q_1^ψ or q_2^ψ, the construction gives immediately the other vertices of q_1^ψ, q_2^ψ. At the same time, we choose a frame in which we obtain two simpler formulas for ψ. We give in §5 an angular characterization which allows to identify which square q_1 or q_2 strikes the other.

2. Quadrants defined by squares

If $q = q_{a,b,\theta,c}$ is a square, we define the two *axes* $A_1(q)$ and $A_2(q)$ of q as the straight lines through $(a,b) \in \mathbb{R}^2$ which are parallel to the sides of q. We define the four counterclockwise consecutive *rays* $D_i(q)$ as the half-lines with origin (a,b) and which contain the vertices of q; we set $D_0(q) = D_4(q)$. A couple of consecutive rays $D_i(q)$ and $D_{i+1}(q)$ defines the i^{th} quadrant $Q_i(q)$ in $\mathbb{R}^2$ associated to the square q.

If a point M, distinct from the center of q, belongs to $\mathrm{int}(Q_i(q))$, then we note $S(q,M) = Q_i(q)$. If the point M lies on the boundaries of two consecutive quadrants $Q_{i-1}(q)$ and $Q_i(q)$, then we choose indifferently $S(q,M)$ as one of the two quadrants $Q_{i-1}(q)$ or $Q_i(q)$. Note that $M \in \mathrm{int}(S(q,N))$ iff $N \in \mathrm{int}(S(q,M))$.

Lemma 1. *If the intersection set $q_1^\psi \cap q_2^\psi$ contains a vertex P of q_2^ψ, then $P \in S(q_2, C_1)$.*

Proof. Let D be the straight line containing a diagonal of q_2 and which does not contain P. Then the disc with center P and radius ψ contains C_1 and C_2 since $d(C_1, P) \leq d(C_2, P) = \psi$. Hence there is only one half-plane $\mathcal{H}$, bounded by D, which contains this disc. Now, $\mathcal{H}$ is the union $\mathcal{S}_1 \cup \mathcal{S}_2$ of two quadrants associated to q_2 and the ray $D_i(q_2)$ through P is $\mathcal{S}_1 \cap \mathcal{S}_2$. If $C_1 \notin D_i(q_2)$, one of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $S(q_2, C_1)$; but $P \in D_i(q_2) = \mathcal{S}_1 \cap \mathcal{S}_2$ gives $P \in S(q_2, C_1)$. If $C_1 \in D_i(q_2)$, then $P \in D_i(q_2) \subset S(q_2, C_1)$. $\square$

Lemma 2. *We have*

$$q_1^\psi \cap q_2^\psi \subset S(q_1, C_2) \cap S(q_2, C_1). \tag{3}$$

The intersection of the two quadrants is depicted in Figure 2.

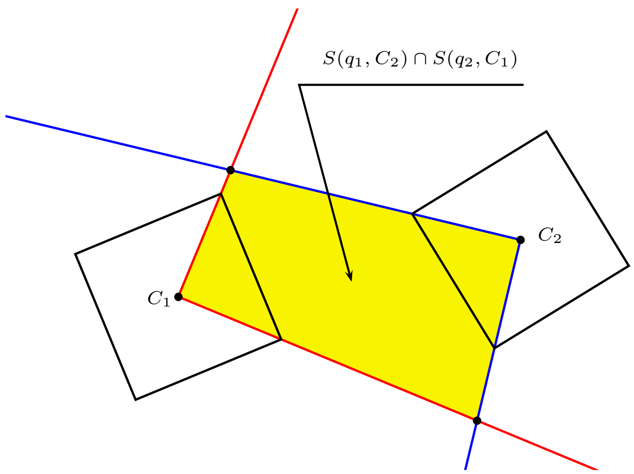

Figure 2

Proof. The proof is divided in three exclusive and exhaustive situations.
(i) First, we suppose that the intersection set $q_1^\psi \cap q_2^\psi = \{P\}$ where P is a common vertex of q_1^ψ and q_2^ψ. We readily obtain $P \in S(q_2, C_1)$ and $P \in S(q_1, C_2)$ from Lemma 1.
(ii) Second, we suppose that $q_1^\psi \cap q_2^\psi$ contains a vertex $P = (x_P, y_P)$ of q_2^ψ and that P is not a vertex of q_1^ψ. We denote by $ABCD$ the square q_1^ψ with $P \in]A, B[$ and let C_1A, C_1B be respectively the x-axis and the y-axis. For the interiors of the two squares to be disjoint, C_2 must be in $\{(x, y) \,:\, x \geq x_p \text{ and } y \geq y_p\}$ since the straight line $x + y = \psi$ separates the two squares. Hence the percussion point P and the center $C_2 = (a, b)$ of q_2 lie in the same quadrant $S(q_1, C_2)$. Due to Lemma 1, P is also in $S(q_2, C_1)$.
(iii) Third, when $q_1^\psi \cap q_2^\psi$ is a common edge of the two squares q_i^ψ, then $S(q_1, C_2) \cap S(q_2, C_1)$ is a square of size ψ and having vertices C_1, P_1, C_2, P_2. Since $q_1^\psi \cap q_2^\psi$ is a diagonal of this square, the inclusion (3) is obvious. $\square$

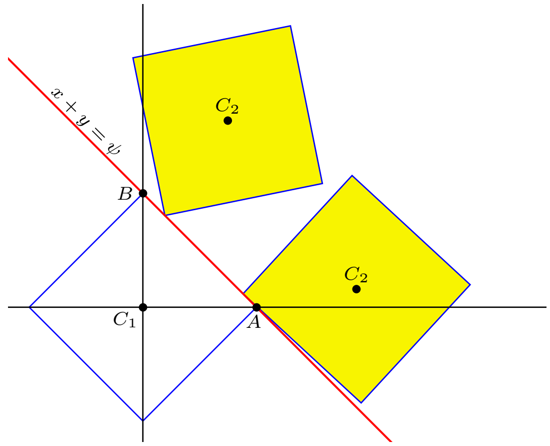

Figure 3

Remark. When the segment $[C_1, C_2]$ contains a vertex of q_1^ψ or q_2^ψ, say A, the statement in (3) can be strengthened: $q_1^\psi \cap q_2^\psi = \{A\}$ is the percussion point.

3. Location of the percussion point

We consider the integer $i_1 \in \{0,1\}$ such that the axis $A_{i_1}(q_1)$ bounds an half-plane containing $S(q_1, C_2)$. Similarly, we consider the axis $A_{i_2}(q_2)$ which bounds an half-plane containing $S(q_2, C_1)$. Since $A_{i_1}(q_1)$, $A_{i_2}(q_2)$ are not parallel, we can set $A_{i_1}(q_1) \cap A_{i_2}(q_2) = \{W\}$. We use in §4 the point V which is the intersection of the axis $A_{j_2}(q_2)$ and WC_1 and where $j_2 \in \{0,1\}$ is the integer different from i_2.

The two straight lines $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ define one dihedral angle which contains both C_1 and C_2, that we denote as $\angle C_1WC_2$. Let $\gamma = \gamma(q_1,q_2) = 2\omega = \widehat{C_1WC_2} \in [0,\pi]$ be the measure of this dihedral angle. We define now $B(q_1,q_2)$ as the half-line which bisects $\angle C_1WC_2$. We also note $\ell_1 = ||\overrightarrow{WC_1}||$ and $\ell_2 = ||\overrightarrow{WC_2}||$.

Lemma 3. *We have* $\gamma = \gamma(q_1,q_2) \in \left]0, \frac{\pi}{2}\right[$.

Proof. If $\gamma = 0$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are equal to some straight line D. The centers C_1 and C_2 lie on D. But by construction $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ have to be perpendicular to the line D, contradiction.

If $\gamma = \frac{\pi}{2}$, the two axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ are perpendicular but this is excluded because the squares are not parallel.

We now suppose that $\frac{\pi}{2} < \gamma < \frac{3\pi}{4}$. The quadrant $S(q_2, C_1)$ intersects the axis $A_{i_1}(q_1)$ at a point M which belongs to the segment $[W, C_1]$ for C_1 lies in $S(q_2, C_1)$. Since the angle $\widehat{WMC_2} = \frac{3}{4}\pi - \gamma$ is strictly less than $\frac{\pi}{4}$, the quadrant $S(q_1, C_2)$ does not contain C_2, contradiction. See Figure 4.

The last case $\frac{3\pi}{4} \leq \gamma \leq \pi$ implies that $S(q_2, C_1)$ does not intersect the boundary of $\angle C_1WC_2$. This is in contradiction with $C_1 \in A_{i_1}(q_1) \cap S(q_2, C_1)$. $\square$

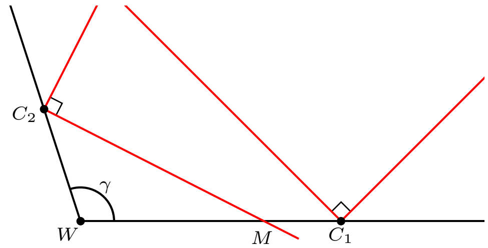

Figure 4

Lemma 4. *We have* $q_1^{\psi} \cap q_2^{\psi} \subset B(q_1, q_2)$.

Proof. Let $0 < k \leq \psi$. The homothetic square q_1^k (resp. q_2^k) has two vertices in $S(q_1, C_2)$ (resp. $S(q_2, C_1)$). The straight line passing through those vertices of q_1 (resp. q_2) is parallel at distance $k/\sqrt{2}$ to the axis $A_{i_1}(q_1)$ (resp. $A_{i_2}(q_2)$). The intersection of those two parallels belongs to $B(q_1, q_2)$ and, according to Lemma 2, allows to localize the point of percussion which is equal to $q_1^k \cap q_2^k$ when $k = \psi$. Thus $P \in B(q_1, q_2)$. □

Remark. When q_1 and q_2 are parallel, Lemma 4 remains true provided $B(q_1, q_2)$ is replaced with the straight line containing the points equidistant from the two parallel axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$.

4. Construction of the percussion point

Two rays $D_i(q_1)$ and $D_{i+1}(q_1)$ intersect $B(q_1, q_2)$ at I_1, I_3. We use the natural order on $B(q_1, q_2)$ and we can suppose that $W < I_1 < I_3$. Similarly, we define $W < I_2 < I_4$ relatively to q_2.

Lemma 5. *We have*
(a) $\ell_1 = \ell_2 \Leftrightarrow I_1 = I_2 < I_3 = I_4$.
(b) $\ell_1 < \ell_2 \Leftrightarrow I_1 < I_2 < I_3 < I_4$.
(c) $\ell_2 < \ell_1 \Leftrightarrow I_2 < I_1 < I_4 < I_3$.

Proof. If $\ell_1 = \ell_2$ then $I_1 = I_2 < I_3 = I_4$. Shifting C_1 along WC_1 towards W causes C_1I_1 and C_1I_3 to slide in a parallel fashion, so that $I_1 < I_2$ and $I_3 < I_4$. Since $C_1 \in S(q_2, C_1)$, the point C_1 cannot pass the intersection C_ℓ of C_2I_2 and WC_1. But when $C_1 = C_\ell$, we have $\widehat{WC_1I_2} = \widehat{WC_\ell C_2} = 3\pi/4 - \gamma$. By Lemma 3, we deduce that $\pi/4 < \widehat{WC_1I_2} < 3\pi/4$ and accordingly $I_2 < I_3$. The remaining implications are straightforward. □

Theorem 6. (i) *Among the four points* $I_1, \ldots, I_4$*, the second one is the percussion point:* $P = q_1^{\psi} \cap q_2^{\psi} = \max\{I_1, I_2\}$. *We have*

$$\psi = \max\{\ell_1, \ell_2\} \frac{\sqrt{2}}{1 + \cot \omega}. \tag{4}$$

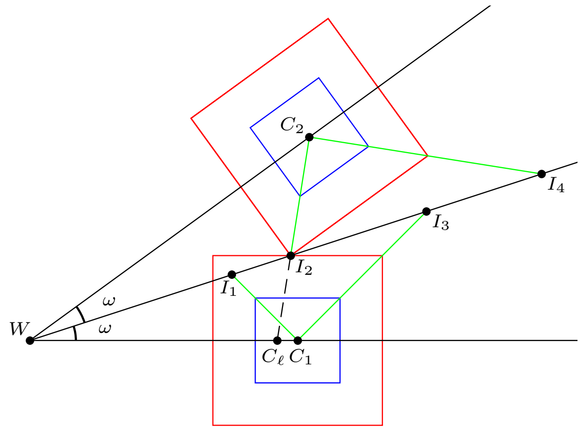

Figure 5

(ii) *If, say $\ell_2 \geq \ell_1$, then q_2 strikes q_1 at the point P which is the incenter of the triangle C_2WV.*

Proof. (i). We suppose first that $\ell_2 > \ell_1$. By Lemma 5 we have

$$d_1 = ||\overrightarrow{C_1I_1}|| < d_2 = ||\overrightarrow{C_2I_2}|| < d_3 = ||\overrightarrow{C_1I_3}|| < d_4 = ||\overrightarrow{C_2I_4}||.$$

We know from Lemma 4 that P is one of the four points $I_1, \dots, I_4$ and thus the percussion occurs at $P = I_i$ if and only if $\psi = d_i$. It is impossible that $P = I_1$ because in that case $\psi = d_1 < d_2$ and then $P \in q_2^{d_1} \cap B(q_1, q_2) = \emptyset$. Hence $\psi > d_1$. If $\psi \geq d_3$ and since $I_2 \in]I_1, I_3[$ by Lemma 5, the point $I_2 \in q_2^{\psi}$ belongs also to the interior of q_1^{ψ} and then the two interiors are not disjoint. We get $\psi = d_2$ and $P = I_2 > I_1$. Easy calculations in the frame centered at $W = (0,0)$ and with x-axis WC_1, give $I_2 = \ell_2(1/(1 + \tan\omega), \tan\omega/(1 + \tan\omega))$ and (4).

The symmetric case $\ell_1 > \ell_2$ gives q_1 strikes q_2 at $P = I_1 > I_2$ and (4) again. Finally, if $\ell_1 = \ell_2$ the point $P = I_1 = I_2$ is effectively the percussion point.

(ii) If $\ell_2 \geq \ell_1$, by Lemma 4, the point $P = I_2$ belongs to the bisector ray $B(q_1, q_2)$ of the geometric angle $\angle C_1WC_2 = \angle VWC_2$. Now, since P is a vertex of q_2, we have $\widehat{VC_2P} = \widehat{PC_2W} = \pi/4$, so that P belongs to the bisector ray of the geometric angle $\angle VC_2W$. We conclude that P is the incenter of the triangle VC_2W. □

Corollary 7. *We have*

$$\begin{aligned}
\ell_1 < \ell_2 \quad &\Leftrightarrow \quad q_2 \text{ strikes } q_1 \text{ and } q_1 \text{ does not strike } q_2,\\
\ell_2 < \ell_1 \quad &\Leftrightarrow \quad q_1 \text{ strikes } q_2 \text{ and } q_2 \text{ does not strike } q_1,\\
\ell_1 = \ell_2 \quad &\Leftrightarrow \quad q_2 \text{ strikes } q_1 \text{ and } q_1 \text{ strikes } q_2.
\end{aligned}$$

Proof. The three implications from left to right are direct consequences of Theorem 6 and its proof. Since the three cases are exclusive and exhaustive, the three converse implications readily follow. □

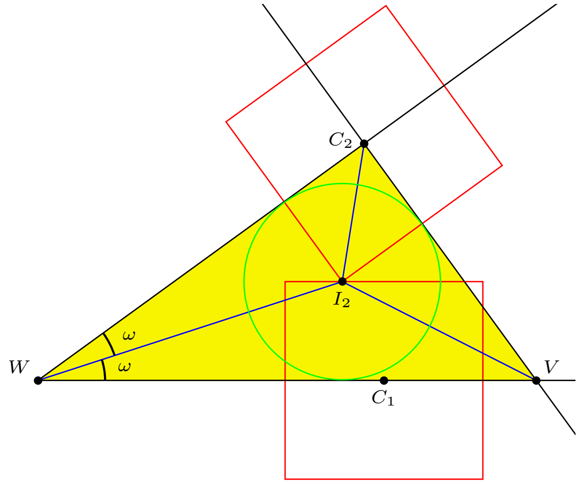

Figure 6

We now synthesize the whole preceding results. For two points M and N, we denote by $\Gamma(M, N)$ the circle with M as center and MN as radius.

Construction of P. Given the eight vertices of two congruent, non parallel and non-overlapping squares q_1 and q_2, construct

(1-2) the two centers C_1, C_2, intersection of the straight lines passing through opposite vertices of q_i, $i = 1, 2$,

(3-4) the axes $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$ (this requires the determination of the quadrants $S(q_1, C_2)$ and $S(q_2, C_1)$ as much as two intermediate points),

(5) the point W, intersection of $A_{i_1}(q_1)$ and $A_{i_2}(q_2)$,

(6) the point C_r, intersection of $\Gamma(W, C_2)$ and the half-line WC_1,

(7) the bisector $B(q_1, q_2)$ through W and $\Gamma(C_2, W) \cap \Gamma(C_r, W)$ (the four points $I_1, \cdots, I_4$ appear at this stage),

(8) the percussion point P, the second among the four points $I_1, \cdots, I_4$ on the oriented half-line $B(q_1, q_2)$.

Remarks. (1) We know that the area of the triangle VWC_2 is equal to $p \cdot r$ where p is the half-perimeter of the triangle and $r = \psi/\sqrt{2}$ the radius of the incircle. Now, we also have the formula

$$\psi = \frac{\sqrt{2}\mathrm{Area}(VWC_2)}{p} = \frac{\sqrt{2}VC_2 \cdot WC_2}{VC_2 + WC_2 + VW} = \ell_2 \frac{\sqrt{2}\sin\gamma}{\sin\gamma + \cos\gamma + 1}.$$

The last value is equal to (4) when $\ell_2 \geq \ell_1$.

(2) Let us suppose that the segment $[C_1, C_2]$ contains a vertex $S_i(q_2)$. This amounts to saying that $C_1 = C_\ell$, so that $S(q_2, C_1)$ has been chosen as one of two quadrants $Q_{i-1}(q_2)$, $Q_i(q_2)$. But these choices lead to consider the two dihedral angles $\angle C_1WC_2$ and $\angle C_1VC_2$. Due to the second part of Theorem 6, P and the formula for ψ are not altered by this choice.

(3) When q_1 and q_2 are parallel, the construction of the four points $I_1, \cdots, I_4$ makes sense using again the straight line $B(q_1, q_2)$ equidistant from the two axes

$A_{i_1}(q_1)$ and $A_{i_2}(q_2)$. We choose an order on $B(q_1, q_2)$ and next we label those four points in such a way that $[I_2, I_3] \subset [I_1, I_4]$ and we have $q_1^\psi \cap q_2^\psi = [I_2, I_3]$. In consequence, the steps (5-8) in the above Construction are replaced with the construction of the midpoint $(C_1+C_2)/2$ (three steps), of the straight line $B(q_1, q_2)$ (three steps) and lastly of the two points I_2, I_3.

5. An angular characterization of the percussion point

We define $\alpha(q_1, q_2)$ as the minimum of $\{S_i(q_1)\widehat{C(q_1)}C(q_2), 0 \leq i \leq 3\}$. This set contains two acute and two obtuse angles. We have $0 \leq \alpha(q_1, q_2) \leq \frac{\pi}{4}$ since $\alpha(q_1, q_2) \leq \frac{\pi}{2} - \alpha(q_1, q_2)$.

Theorem 8. *The square q_2 strikes q_1 if and only if $\alpha(q_2, q_1) \leq \alpha(q_1, q_2)$. The percussion point is the vertex of q_1 or q_2 which realizes the minimum of the eight angles appearing in $\alpha(q_1, q_2)$ and $\alpha(q_2, q_1)$.*

Proof. Suppose that q_2 strikes $q_1 = ABCD$ at P in the interior of side AB, see Figure 7. Let AB be the x-axis and P the origin. Then for the interiors of q_1 and q_2 to be disjoint, the center C_2 of q_2 must be in $\{(x, y) \, : \, y \geq |x|\}$. Also, C_2 lies on the arc $x^2 + y^2 = \psi^2$. Let C_0, C_ℓ, C_r be the three points on this arc which intersect the lines C_1P, $y = -x$ and $y = x$ respectively.

Letting C_2 moving along the arc from C_0 to C_r, the angle $\widehat{PC_2C_1}$ increases from $\widehat{PC_0C_1} = 0$ to $\widehat{PC_rC_1} = \widehat{BC_1C_r}$ and the angle $\widehat{BC_1C_2}$ decreases from $\widehat{BC_1C_0}$ to $\widehat{BC_1C_r}$. Hence throughout the move we have $\widehat{PC_2C_1} \leq \widehat{BC_1C_r} \leq \widehat{BC_1C_2}$. But we have obviously $\widehat{PC_2C_1} < \widehat{AC_1C_2}$ and thus $\widehat{PC_2C_1} \leq \alpha(q_1, q_2)$. The same proof holds when C_2 moves on the arc C_0C_ℓ.

Since $\widehat{PC_2C_1} \leq \pi/4$, we get $\alpha(q_2, q_1) = \widehat{PC_2C_1}$ and next $\alpha(q_2, q_1) \leq \alpha(q_1, q_2)$. The angle $\widehat{PC_2C_1}$ realizes effectively the minimum of the eight angles. The converse implication holds because $\alpha(q_2, q_1) = \alpha(q_1, q_2)$ is equivalent to the fact that q_1 and q_2 strike each other at a common vertex. □

Remark. In case q_1 and q_2 are parallel, q_1 strikes q_2 at P_1 and q_2 strikes q_1 at P_2. We have $\alpha(q_1, q_2) = \widehat{C_2C_1P_1} = \widehat{C_1C_2P_2} = \alpha(q_2, q_1)$. Hence the results in Theorem 8 remain true.

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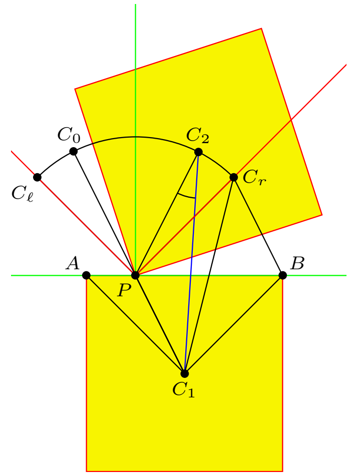

Figure 7

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Forum Geometricorum
Volume 5 (2005) 33–36.

FORUM GEOM
ISSN 1534-1178

Triangle Centers with Linear Intercepts and Linear Subangles

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Abstract. Let ABC be a triangle with side-lengths a, b, and c, and with angles A, B, and C. Let AA', BB', and CC' be the cevians through a point V, let x, y, and z be the lengths of the segments BA', CB', and AC', and let ξ, η, and ζ be the measures of the angles $\angle BAA'$, $\angle CBB'$, and $\angle ACC'$. The centers V for which x, y, and z are linear forms in a, b, and c are characterized. So are the centers for which ξ, η, and ζ are linear forms in A, B, and C.

Let ABC be a non-degenerate triangle with side-lengths a, b, and c, and let V be a point in its plane. Let AA', BB', and CC' be the cevians of ABC through V and let the intercepts x, y, and z be defined to be the directed lengths of the segments BA', CB', and AC', where x is positive or negative according as A' and C lie on the same side or on opposite sides of B, and similarly for y and z; see Figure 1. To avoid infinite intercepts, we assume that V does not lie on any of the three exceptional lines passing through the vertices of ABC and parallel to the opposite sides.

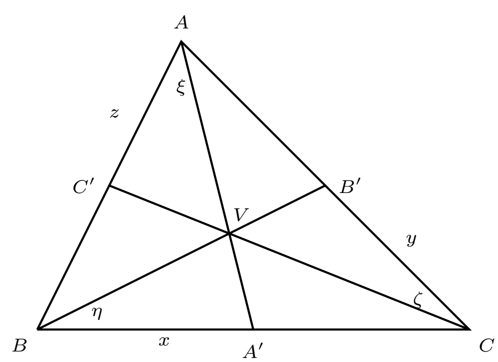

Figure 1

If V is the centroid of ABC, then the intercepts (x, y, z) are clearly given by $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$. It is also easy to see that the triples (x, y, z) determined by the Gergonne and Nagel points are

$$\left(\frac{a-b+c}{2}, \frac{a+b-c}{2}, \frac{-a+b+c}{2}\right), \ \left(\frac{a+b-c}{2}, \frac{-a+b+c}{2}, \frac{a-b+c}{2}\right),$$

Publication Date: March 10, 2005. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.

respectively. We now show that these are the only three centers whose corresponding intercepts (x, y, z) are linear forms in a, b, and c. Here, and in the spirit of [4] and [5], a center is a function that assigns to a triangle, in a family $\mathbf{U}$ of triangles, a point in its plane in a manner that is symmetric and that respects isometries and dilations. It is assumed that $\mathbf{U}$ has a non-empty interior, where $\mathbf{U}$ is thought of as a subset of $\mathbf{R}^3$ by identifying a triangle ABC with the point (a, b, c).

Theorem 1. *The triangle centers for which the intercepts x, y, z are linear forms in a, b, c are the centroid, the Gergonne and the Nagel points.*

Proof. Note first that if (x, y, z) are the intercepts corresponding to a center V, and if

$$x = \alpha a + \beta b + \gamma c,$$

then it follows from reflecting ABC about the perpendicular bisector of the segment BC that

$$a - x = \alpha a + \beta c + \gamma b.$$

Therefore $\alpha = \frac{1}{2}$ and $\beta + \gamma = 0$. Applying the permutation $(A\ B\ C) = (a\ b\ c) = (x\ y\ z)$, we see that

$$x = \alpha a + \beta b + \gamma c,\ y = \alpha b + \beta c + \gamma a,\ z = \alpha c + \beta a + \gamma b.$$

Substituting in the cevian condition $xyz = (a - x)(b - y)(c - z)$, we obtain the equation

$$\begin{aligned} & \left(\frac{a}{2} + \beta(b - c)\right)\left(\frac{b}{2} + \beta(c - a)\right)\left(\frac{c}{2} + \beta(a - b)\right) \\ = \quad & \left(\frac{a}{2} - \beta(b - c)\right)\left(\frac{b}{2} - \beta(c - a)\right)\left(\frac{c}{2} - \beta(a - b)\right) \end{aligned}$$

which simplifies into

$$\beta\left(\beta + \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)(a - b)(b - c)(c - a) = 0.$$

This implies the three possibilities $\beta = 0, -\frac{1}{2}$, or $\frac{1}{2}$ that correspond to the centroid, the Gergonne point and the Nagel point, respectively. □

In the same vein, the cevians through V define the subangles ξ, η, and ζ of the angles A, B, and C of ABC as shown in Figure 1. These are given by

$$\xi = \angle BAV, \qquad \eta = \angle CBV, \qquad \zeta = \angle ACV.$$

Here we temporaily take V to be inside ABC for simplicity, and treat the general case in Note 1 below. It is clear that the subangles (ξ, η, ζ) corresponding to the incenter of ABC are given by $(\frac{A}{2}, \frac{B}{2}, \frac{C}{2})$. Also, if ABC is acute-angled, then the orthocenter and circumcenter lie inside ABC and the triples (ξ, η, ζ) of subangles that they determine are given by

$$\left(\frac{A - B + C}{2}, \frac{A + B - C}{2}, \frac{-A + B + C}{2}\right), \left(\frac{A + B - C}{2}, \frac{-A + B + C}{2}, \frac{A - B + C}{2}\right), \tag{1}$$

or equivalently by

$$\left(\frac{\pi}{2}-B, \frac{\pi}{2}-C, \frac{\pi}{2}-A\right), \ \left(\frac{\pi}{2}-C, \frac{\pi}{2}-A, \frac{\pi}{2}-B\right), \tag{2}$$

respectively. Here again, we prove that these are the only centers whose corresponding subangles (ξ, η, ζ) are linear forms in A, B, and C. As before, we first show that the subangles (ξ, η, ζ) determined by such a center are of the form

$$\xi = \alpha A + \beta B + \gamma C, \ \eta = \alpha B + \beta C + \gamma A, \ \zeta = \alpha C + \beta A + \gamma B,$$

where $\alpha = \frac{1}{2}$ and $\beta + \gamma = 0$. Substituting in the trigonometric cevian condition

$$\sin \xi \ \sin \ \eta \ \sin \ \zeta \ = \ \sin\,(a-\xi) \ \sin\,(b-\eta) \ \sin\,(c-\zeta), \tag{3}$$

we obtain the equation

$$\begin{aligned} &\sin\left(\frac{A}{2}+\beta(B-C)\right) \ \sin\left(\frac{B}{2}+\beta(C-A)\right) \ \sin\left(\frac{C}{2}+\beta(A-B)\right) \\ = \ &\sin\left(\frac{A}{2}-\beta(B-C)\right) \ \sin\left(\frac{B}{2}-\beta(C-A)\right) \ \sin\left(\frac{C}{2}-\beta(A-B)\right). \end{aligned} \tag{4}$$

Using the facts that

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}, \ \beta(B-C)+\beta(C-A)+\beta(A-B)=0,$$

and the facts [3, Formulas 677, 678, page 166] that if $u+v+w=0$, then

$$\begin{aligned} 4 \ \cos \ u \ \cos \ v \ \sin \ w \ &= \ -\sin \ 2u \ - \ \sin \ 2v \ + \ \sin \ 2w, \\ 4 \ \sin \ u \ \sin \ v \ \sin \ w \ &= \ -\sin \ 2u \ - \ \sin \ 2v \ - \ \sin \ 2w, \end{aligned}$$

and that if $u+v+w=\pi/2$, then

$$\begin{aligned} 4 \ \cos \ u \ \cos \ v \ \cos \ w \ &= \ \sin \ 2u \ + \ \sin \ 2v \ + \ \sin \ 2w, \\ 4 \ \sin \ u \ \sin \ v \ \cos \ w \ &= \ \sin \ 2u \ + \ \sin \ 2v \ - \ \sin \ 2w, \end{aligned}$$

(4) simplifies into

$$\sin A \ \sin(2\beta(B-C)) + \sin \ B \ \sin(2\beta(C-A)) + \sin \ C \ \sin(2\beta(A-B)) = 0. \tag{5}$$

It is easy to check that for $\beta = -\frac{1}{2}$, 0, and $\frac{1}{2}$, this equation is satisfied for all triangles. Conversely, since (5) holds on a set **U** having a non-empty interior, it holds for all triangles, and in particular it holds for the triangle $(A, B, C) = (\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$. This implies that

$$\sin \frac{\beta\pi}{3} \ \left(\cos \frac{\beta\pi}{3} - \frac{\sqrt{3}}{2}\right) \ = \ 0.$$

Since $-\frac{3}{2} \leq \beta \leq \frac{3}{2}$ for this particular triangle, it follows that β must be $-\frac{1}{2}$, 0, or $\frac{1}{2}$. Thus the only solutions of (5) are $\beta = -\frac{1}{2}$, 0, and $\frac{1}{2}$. These correspond to the orthocenter, incenter and circumcenter, respectively. We summarize the result in the following theorem.

Theorem 2. *The triangle centers for which the subangles ξ, η, ζ are linear forms in A, B, C are the orthocenter, incenter, and circumcenter.*

Remarks. (1) Although the subangles ξ, η, and ζ of a given point V were defined for points that lie inside ABC only, it is possible to extend this definition to include exterior points also, without violating the trigonometric version (3) of Ceva's concurrence condition or the formulas (1) and (2) for the subangles corresponding to the orthocenter and the circumcenter. To do so, we let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the open half planes determined by the line that is perpendicular at A to the internal angle-bisector of A, where we take $\mathbf{H}_1$ to be the half-plane containing B and C. For $V \in \mathbf{H}_1$, we define the subangle ξ to be the signed angle $\angle BAV$, where $\angle BAV$ is taken to be positive or negative according as the rotation within $\mathbf{H}_1$ that takes AB to AV has the same or opposite handedness as the one that takes AB to AC. For $V \in \mathbf{H}_2$, we stipulate that V and its reflection about A have the same subangle ξ. We define η and ζ similarly. Points on the three exceptional lines that are perpendicular at the vertices of ABC to the respective internal angle-bisectors are excluded.

(2) In terms of the intercepts and subangles, the first (respectively, the second) Brocard point of a triangle is the point whose subangles ξ, η, and ζ satisfy $\xi = \eta = \zeta$ (respectively, $A - \xi = B - \eta = C - \zeta$.) Similarly, the first and the second Brocard-like Yff points are the points whose intercepts x, y, and z satisfy $x = y = z$ and $a - x = b - y = c - z$, respectively. Other Brocard-like points corresponding to features other than intercepts and subangles are being explored by the authors.

(3) The requirement that the intercepts x, y, and z be linear in a, b, and c is quite restrictive, since the cevian condition has to be observed. It is thus tempting to weaken this requirement, which can be written in matrix form as $[x\ y\ z] = [a\ b\ c]L$, where L is a 3×3 matrix, to take the form $[x\ y\ z]M = [a\ b\ c]L$, where M is not necessarily invertible. The family of centers defined by this weaker requirement, together of course with the cevian condition, is studied in detail in [2]. So is the family obtained by considering subangles instead of intercepts.

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Forum Geometricorum
Volume 5 (2005) 37–45.

FORUM GEOM
ISSN 1534-1178

The Arbelos in n-Aliquot Parts

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Abstract. We generalize the classical arbelos to the case divided into many chambers by semicircles and construct embedded patterns of such arbelos.

1. Introduction and preliminaries

Let $\{\alpha,\ \beta,\ \gamma\}$ be an arbelos, that is, $\alpha,\ \beta,\ \gamma$ are semicircles whose centers are collinear and erected on the same side of this line, $\alpha,\ \beta$ are tangent externally, and γ touches α and β internally. In this paper we generalize results on the Archimedean circles of the arbelos. We take the line passing through the centers of $\alpha,\ \beta,\ \gamma$ as the x-axis and the line passing through the tangent point O of α and β and perpendicular to the x-axis as the y-axis. Let $\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta$ be $n+1$ distinct semicircles touching α and β at O, where $\alpha_1, \dots, \alpha_{n-1}$ are erected on the same side as α and β, and intersect with γ. One of them may be the line perpendicular to the x-axis (i.e. y-axis). If the n inscribed circles in the curvilinear triangles bounded by $\alpha_{i-1},\ \alpha_i,\ \gamma$ are congruent we call this configuration of semicircles $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ an arbelos in n-aliquot parts, and the inscribed circles the Archimedean circles in n-aliquot parts. In this paper we calculate the radii of the Archimedean circles in n-aliquot parts and construct embedded patterns of arbelos in aliquot parts.

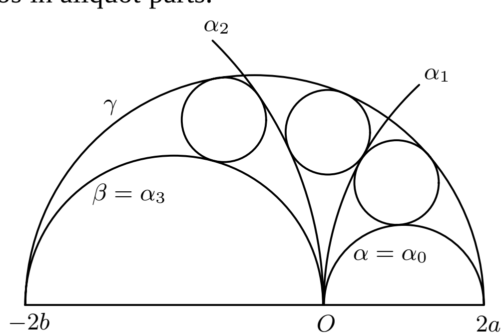

Figure 1. The case $n = 3$

For the arbelos $\{\alpha,\ \beta,\ \gamma\}$ we denote by $\Phi(\alpha,\beta,\gamma)$ the family of semicircles through O, having the common point with γ in the region $y \geq 0$ and with centers on the x-axis, together with the line perpendicular to the x-axis at O. Renaming if necessary we assume α in the region $x \geq 0$. Let $a,\ b$ be the radii of $\alpha,\ \beta$. The semicircle γ meets the x-axis at $-2b$ and $2a$.

For a semicircle $\alpha_i \in \Phi(\alpha,\beta,\gamma)$, let a_i be the x-coordinate of its center. Define $\mu(\alpha_i)$ as follows.

Publication Date: March 22, 2005. Communicating Editor: Paul Yiu.

If $a \neq b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{a_i - a + b}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 1, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

If $a = b$,

$$\mu(\alpha_i) = \begin{cases} \dfrac{1}{a_i}, & \text{if } \alpha_i \text{ is a semi-circle,} \\ 0, & \text{if } \alpha_i \text{ is the line.} \end{cases}$$

In both cases $\mu(\alpha_i)$ depends only on α_i and the center of γ, but not on the radius of γ. For $\alpha_i,\ \alpha_j \in \Phi(\alpha,\beta,\gamma)$, the equality $\mu(\alpha_i) = \mu(\alpha_j)$ holds if and only if $\alpha_i = \alpha_j$. For any $\alpha_i \in \Phi(\alpha,\beta,\gamma)$,

$$\begin{aligned} \frac{b}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = \frac{a}{b} \text{ if } a < b, \\ \frac{1}{a} &= \mu(\alpha) \geq \mu(\alpha_i) \geq \mu(\beta) = -\frac{1}{a} \text{ if } a = b, \\ \frac{b}{a} &= \mu(\alpha) \leq \mu(\alpha_i) \leq \mu(\beta) = \frac{a}{b} \text{ if } a > b. \end{aligned}$$

For $\alpha_i,\ \alpha_j \in \Phi(\alpha,\beta,\gamma)$, define the order

$$\alpha_i < \alpha_j \text{ if and only if } \begin{cases} \mu(\alpha_i) > \mu(\alpha_j) & \text{if } a \leq b, \\ \mu(\alpha_i) < \mu(\alpha_j) & \text{otherwise}. \end{cases}$$

This means that α_i is nearer to α than α_j is. Throughout this paper we shall adopt these notations and assumptions.

2. An arbelos in aliquot parts

Lemma 1. *If α_i and α_j are semicircles in $\Phi(\alpha,\beta,\gamma)$ with $\alpha_i < \alpha_j$, the radius of the inscribed circle in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{ab(a_j - a_i)}{a_i a_j - a a_i + b a_j}\ .$$

Proof. Let $\mathcal{C}$ be the inscribed circle with radius r. First we invert $\{\alpha_i,\ \alpha_j,\ \gamma,\ \mathcal{C}\}$ in the circle with center O and radius k. Then α_i and α_j are inverted to the lines $\overline{\alpha_i}$ and $\overline{\alpha_j}$ perpendicular to the x-axis , γ is inverted to the semicircle $\overline{\gamma}$ erected on the x-axis and $\mathcal{C}$ is inverted to the circle $\overline{\mathcal{C}}$ tangent to $\overline{\gamma}$ externally. We write the x-coordinates of the intersections of $\overline{\alpha_i}$, $\overline{\alpha_j}$ and $\overline{\gamma}$ with the x-axis as $s,\ t$ and $p,\ q$ with $q < p$. Then $t < s$ since $a_i < a_j$.

By the definition of inversion we have

$$s = \frac{k^2}{2a_i}\ ,\ t = \frac{k^2}{2a_j}\ ,\ p = \frac{k^2}{2a}\ ,\ q = -\frac{k^2}{2b}\ . \tag{1}$$

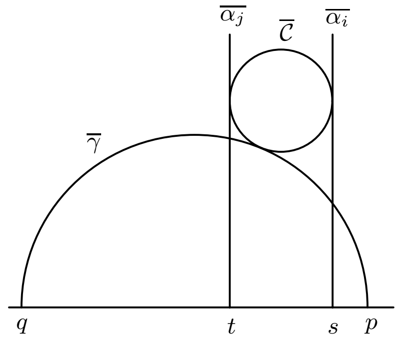

Figure 2

Since the x-coordinates of the center and the radius of $\overline{\mathcal{C}}$ are $\dfrac{s+t}{2}$ and $\dfrac{s-t}{2}$, and those of $\overline{\gamma}$ are $\dfrac{p+q}{2}$ and $\dfrac{p-q}{2}$, we have

$$\left(\frac{s+t}{2}-\frac{p+q}{2}\right)^2+d^2=\left(\frac{s-t}{2}+\frac{p-q}{2}\right)^2,$$

where d is the y-coordinate of the center of $\overline{\mathcal{C}}$. From this,

$$st-sp-tq+pq+d^2=0\,. \tag{2}$$

Since O is outside $\overline{\mathcal{C}}$, we have

$$r=\frac{k^2}{\left|\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2\right|}\cdot\frac{s-t}{2}=\frac{k^2}{\left(\frac{s+t}{2}\right)^2+d^2-\left(\frac{s-t}{2}\right)^2}\cdot\frac{s-t}{2}.$$

By using (1) and (2) we get the conclusion. □

Lemma 2. *If α_i (resp. α_j) is the line, then the radius of the inscribed circle is*

$$\frac{-ab}{a_j-a}(resp.\ \frac{ab}{a_i+b}).$$

Proof. Even in this case (2) in the proof of Lemma 1 holds with $s=0$ (resp. $t=0$), and we get the conclusion. □

Theorem 3. *Assume $a\neq b$, and let $\alpha_i,\ \alpha_j\in\Phi(\alpha,\beta,\gamma)$ with $\alpha_i<\alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by $\alpha_i,\ \alpha_j$ and γ is*

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}.$$

Proof. If α_i and α_j are semicircles, then

$$\frac{ab(\mu(\alpha_i)-\mu(\alpha_j))}{b\mu(\alpha_i)-a\mu(\alpha_j)}=\frac{ab\left(\dfrac{a_i-a+b}{a_i}-\dfrac{a_j-a+b}{a_j}\right)}{b\cdot\dfrac{a_i-a+b}{a_i}-a\cdot\dfrac{a_j-a+b}{a_j}}=\frac{ab(a_j-a_i)}{a_ia_j-aa_i+ba_j}.$$

Hence the theorem follows from Lemma 1. If one of α_i, α_j is the line, the result follows from Lemma 2. □

Similarly we have

Theorem 4. *Assume $a = b$, and let $\alpha_i,\ \alpha_j \in \Phi(\alpha, \beta, \gamma)$ with $\alpha_i < \alpha_j$. The radius of the circle inscribed in the curvilinear triangle bounded by α_i, α_j and γ is*

$$\frac{a^2(\mu(\alpha_j) - \mu(\alpha_i))}{a(\mu(\alpha_j) - \mu(\alpha_i)) - 1}.$$

The functions $x \mapsto \dfrac{ab(1-x)}{b-ax},\ a \neq b$ and $x \mapsto \dfrac{a^2x}{ax-1}, a > 0$ are injective. Therefore, we have

Corollary 5. *Let $\alpha_0,\ \alpha_1, \dots, \alpha_n \in \Phi(\alpha,\ \beta,\ \gamma)$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. The circles inscribed in the curvilinear triangle bounded by α_{i-1}, α_i and γ $(i = 1, 2, \dots n)$ are all congruent if and only if $\mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n)$ is a geometric sequence if $a \neq b$, or an arithmetic sequence if $a = b$.*

Theorem 6. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts. The common radius of the Archimedean circles in n-aliquot parts is*

$$\begin{cases} \dfrac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}, & \text{if } a \neq b, \\[2ex] \dfrac{2a}{n+2}, & \text{if } a = b. \end{cases}$$

Proof. First we consider the case $a \neq b$. We can assume $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ by renaming if necessary. The sequence $\dfrac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n) = \dfrac{a}{b}$ is a geometric sequence by Corollary 5. If we write its common ratio as d, we have $\dfrac{a}{b} = d^n\left(\dfrac{b}{a}\right)$, and then $d = \left(\dfrac{a}{b}\right)^{\frac{2}{n}}$. By Theorem 3 the radius of the Archimedean circle is

$$\frac{ab(1-d)}{b-ad} = \frac{ab\left(1 - \left(\frac{a}{b}\right)^{\frac{2}{n}}\right)}{b - a\left(\frac{a}{b}\right)^{\frac{2}{n}}} = \frac{ab\left(b^{\frac{2}{n}} - a^{\frac{2}{n}}\right)}{b^{\frac{2}{n}+1} - a^{\frac{2}{n}+1}}.$$

Similarly we can get the second assertion. □

Note that the second assertion is the limiting case of the first assertion when $b \to a$.

Theorem 7. *Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then α_i is the line in $\Phi(\alpha, \beta, \gamma)$ if n is even and $i = \frac{n}{2}$.*

Otherwise it is a semicircle with radius

$$\begin{cases} \left| \dfrac{b^{\frac{2i}{n}-1}(a-b)}{a^{\frac{2i}{n}-1} - b^{\frac{2i}{n}-1}} \right|, & \text{if } a \neq b, \\ \\ \left| \dfrac{na}{n-2i} \right|, & \text{if } a = b. \end{cases}$$

Proof. Suppose $a \neq b$. Since $\frac{b}{a} = \mu(\alpha_0),\ \mu(\alpha_1), \dots, \mu(\alpha_n) = \frac{a}{b}$ is a geometric sequence with common ratio $\left(\frac{a}{b}\right)^{\frac{2}{n}}$, we have $\mu(\alpha_i) = \left(\frac{a}{b}\right)^{\frac{2i}{n}} \left(\frac{b}{a}\right) = \left(\frac{a}{b}\right)^{\frac{2i}{n}-1}$. If n is even and $i = \frac{n}{2}$, then $\mu(\alpha_i) = 1$ and α_i is the line. Otherwise, $\mu(\alpha_i) \neq 1$ and α_i is a semicircle. Let a_i be the x-coordinate of its center. The radius of α_i is $|a_i|$ and $\frac{a_i - a + b}{a_i} = \left(\frac{a}{b}\right)^{\frac{2i}{n}-1}$. From this, $a_i = \frac{b^{\frac{2i}{n}-1}(a-b)}{b^{\frac{2i}{n}-1} - a^{\frac{2i}{n}-1}}$.

The proof for the case $a = b$ is similar. □

3. Embedded patterns of the arbelos

Let $\{\alpha_0 = \alpha,\ \alpha_1, \dots, \alpha_n = \beta,\ \gamma\}$ be an arbelos in n-aliquot parts with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. There exists a semicircle γ' which is tangent to all Archimedean circles externally. It is clearly concentric to γ. (If $n = 1$ we will take for γ' the semicircle concentric to γ and tangent to the Archimedean circle externally). Let α', β' be two semicircles in $y \geq 0$, tangent to $\alpha_i s$ at O and also tangent to γ'. We take α' in the region $x \geq 0$ and β' in the region $x \leq 0$. Let a' and b' be the radii of α' and β' respectively. Clearly α', β' are tangent externally at O, and γ' intersects the x-axis at $-2b'$ and $2a'$, and $\Phi(\alpha, \beta, \gamma) \subseteq \Phi(\alpha', \beta', \gamma')$. Moreover, for any $\alpha_i \in \Phi(\alpha, \beta, \gamma)$, $\mu(\alpha_i)$ considered in $\Phi(\alpha, \beta, \gamma)$ is equal to $\mu(\alpha_i)$ considered in $\Phi(\alpha', \beta', \gamma')$ since the centers of γ and γ' coincide.

Lemma 8. (a) *If $a \neq b$, $\left(\frac{a'}{b'}\right)^n = \left(\frac{a}{b}\right)^{n+2}$.*

(b) *If $a = b$, $\frac{a'}{n} = \frac{a}{n+2}$.*

Proof. If $a \neq b$ we have

$$a' = a - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{a^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$
$$b' = b - \frac{ab\left(a^{\frac{2}{n}} - b^{\frac{2}{n}}\right)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}} = \frac{b^{\frac{2}{n}+1}(a-b)}{a^{\frac{2}{n}+1} - b^{\frac{2}{n}+1}},$$

by the definitions of a' and b'. Then the the first assertion follows. The second assertion follows similarly. □

Theorem 9. $\{\alpha', \alpha_0, \alpha_1, \dots, \alpha_n, \beta', \gamma'\}$ *is an arbelos in $(n+2)$-aliquot parts.*

Proof. Let us assume $a \neq b$. By Lemma 8 and the proof of Theorem 6, $\mu(\alpha_0)$, $\mu(\alpha_1), \ldots, \mu(\alpha_n)$ is a geometric sequence with common ratio $\left(\frac{a'}{b'}\right)^{\frac{2}{n+2}}$. Also by Lemma 8 we have

$$\frac{\mu(\alpha_0)}{\mu(\alpha')} = \frac{b}{a}\frac{a'}{b'} = \left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} \frac{a'}{b'} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}},$$

and

$$\frac{\mu(\beta')}{\mu(\alpha_n)} = \frac{a'}{b'}\frac{b}{a} = \frac{a'}{b'}\left(\frac{b'}{a'}\right)^{\frac{n}{n+2}} = \left(\frac{a'}{b'}\right)^{\frac{2}{n+2}}.$$

The case $a = b$ follows similarly. □

Let $\{\alpha,\ \beta,\ \gamma\}$ be an arbelos and all the semicircles be constructed in $y \geq 0$ such that the diameters lie on the x-axis. Let $\alpha_{-1} = \alpha$, $\alpha_1 = \beta$ and $\gamma_1 = \gamma$. If there exists an arbelos in $(2n-1)$-aliquot parts $\{\alpha_{-n}, \alpha_{-(n-1)}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \gamma_{2n-1}\}$ with $\alpha_{-n} < \alpha_{-(n-1)} < \cdots < \alpha_{-1} < \alpha_1 < \cdots < \alpha_n$, we shall construct an arbelos in $(2n+1)$-aliquot parts as follows.

Let γ_{2n+1} be the semicircle concentric to γ and tangent externally to all Archimedean circles of the above arbelos. This meets the x-axis at two points one of which is in the region $x > 0$ and the other in $x < 0$. We write the semicircle passing through O and the former point as $\alpha_{-(n+1)}$ and the semicircle passing through O and the latter point as α_{n+1}. Then $\{\alpha_{-(n+1)}, \alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_{n+1}, \gamma_{2n+1}\}$ is an arbelos in $(2n+1)$-aliquot parts by Theorem 9. Now we get the set of semicircles

$$\{\ldots, \alpha_{-(n+1)}, \alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots \gamma_1, \gamma_3, \ldots, \gamma_{2n-1} \ldots\},$$

where $\{\alpha_{-n}, \ldots, \alpha_{-1}, \alpha_1, \ldots, \alpha_n, \gamma_{2n-1}\}$ form the arbelos in $(2n-1)$-aliquot parts for any positive integer n. We shall call the above configuration the *odd* pattern.

Theorem 10. *Let δ_{2n-1} be one of the Archimedean circles in*

$$\{\alpha_{-n},\ \alpha_{-(n-1)},\ \ldots, \alpha_{-1},\ \alpha_1, \ldots, \alpha_n\ , \gamma_{2n-1}\}.$$

Then the radii of α_{-n} and α_n are

$$\frac{a^{2n-1}(a-b)}{a^{2n-1}-b^{2n-1}} \quad \text{and} \quad \frac{b^{2n-1}(a-b)}{a^{2n-1}-b^{2n-1}},$$

and the radii of γ_{2n-1} and δ_{2n-1} are respectively

$$\frac{(a^{2n-1}+b^{2n-1})(a-b)}{a^{2n-1}-b^{2n-1}} \quad \text{and} \quad \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1}-b^{2n-1})(a^{2n+1}-b^{2n+1})}.$$

Proof. Let $\overline{a_{-n}}$ and $\overline{a_n}$ be the radii of α_{-n} and α_n respectively. By Lemma 8 we have

$$\left(\frac{\overline{a_{-n}}}{\overline{a_n}}\right)^{\frac{1}{2n-1}} = \left(\frac{\overline{a_{-(n-1)}}}{\overline{a_{n-1}}}\right)^{\frac{1}{2n-3}} = \cdots = \frac{\overline{a_{-1}}}{\overline{a_1}} = \frac{a}{b}. \tag{3}$$

Since γ_{2n-1} and γ are concentric, we have

$$\overline{a_{-n}} - \overline{a_n} = a - b\ . \tag{4}$$

By (3) and (4) we have

$$\overline{a_{-n}} = \frac{a^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}}\ ,$$
$$\overline{a_n} = \frac{b^{2n-1}(a-b)}{a^{2n-1} - b^{2n-1}}.$$

It follows that the radius of γ_{2n-1} is

$$\overline{a_{-n}} + \overline{a_n} = \frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}},$$

and that of δ_{2n-1} is

$$\frac{(a^{2n-1} + b^{2n-1})(a-b)}{a^{2n-1} - b^{2n-1}} - \frac{(a^{2n+1} + b^{2n+1})(a-b)}{a^{2n+1} - b^{2n+1}}$$
$$= \frac{a^{2n-1}b^{2n-1}(a-b)(a^2-b^2)}{(a^{2n-1} - b^{2n-1})(a^{2n+1} - b^{2n+1})}\ .$$

$\square$

As in the odd case, we can construct the *even* pattern of arbelos

$\{\dots\beta_{-(n+1)},\ \beta_{-n},\ \dots,\ \beta_{-1},\ \beta_0,\ \beta_1,\ \dots,\ \beta_n,\ \beta_{n+1},\ \dots,\ \gamma_2,\ \gamma_4,\ \dots,\ \gamma_{2n}\dots\}$ inductively by starting with an arbelos in 2-aliquot parts $\{\beta_{-1},\ \beta_0,\ \beta_1, \gamma_2\}$, where $\beta_{-1} = \alpha,\ \beta_1 = \beta$ and $\gamma_2 = \gamma$. By Theorem 9, $\{\beta_{-n}, \dots, \beta_{-1},\ \beta_0,\ \beta_1, \dots, \beta_n, \gamma_{2n}\}$ forms an arbelos in $2n$-aliquot parts for any positive integer n, and β_0 is the line by Theorem 7. Analogous to Theorem 10 we have

Theorem 11. *Let δ_{2n} be one of the Archimedean circles in*

$$\{\beta_{-n},\ \beta_{-(n-1)}, \dots, \beta_{-1},\ \beta_0,\ \beta_1, \dots, \beta_n\ , \gamma_{2n}\}.$$

The radii of β_{-n} and β_n are

$$\frac{a^n(a-b)}{a^n - b^n} \quad \textit{and} \quad \frac{b^n(a-b)}{a^n - b^n},$$

and the radii of γ_{2n} and δ_{2n} are respectively

$$\frac{(a^n + b^n)(a-b)}{a^n - b^n} \quad \text{and} \quad \frac{a^n b^n (a-b)^2}{(a^n - b^n)(a^{n+1} - b^{n+1})}.$$

Corollary 12. *Let c_n and d_n be the radii of γ_n and δ_n respecgtively.*

$$a_n = b_{2n-1},$$
$$a_{-n} = b_{-(2n-1)},$$
$$c_{2n-1} = c_{2(2n-1)},$$
$$d_{2n-1} = d_{4n-2} + d_{4n}.$$

Figure 3 shows the even pattern together with the odd pattern reflected in the x-axis. The trivial case of these patterns can be found in [2].

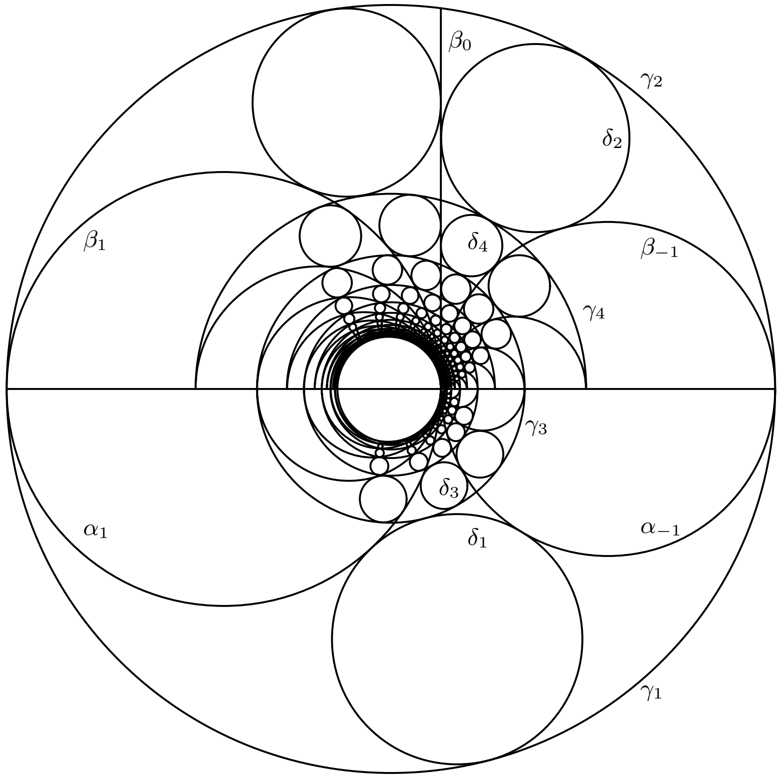

Figure 3

4. Some Applications

We give two applications here, with the same notations as in §3.

Theorem 13. *The external common tangent of β_n and β_{-n} touches γ_{4n} for any positive integer n.*

Proof. The distance between the external common tangents of β_n and β_{-n} and the center of γ_{2n} is $\dfrac{\overline{b_n}^2+\overline{b_{-n}}^2}{\overline{b_n}+\overline{b_{-n}}}$ where $\overline{b_n}$ and $\overline{b_{-n}}$ are the radii of β_n and β_{-n}. By Theorem 11 this is equal to $\dfrac{(a-b)(a^{2n}+b^{2n})}{a^{2n}-b^{2n}}$, the radius of γ_{4n}. $\square$

Theorem 14. *Let BK_n be the circle orthogonal to α, β and δ_{2n-1}, and let AR_n be the inscribed circle of the curvilinear triangle bounded by β_n, β_0 and γ_{2n}. The circles BK_n and AR_n are congruent for every natural number n.*

Proof. Assume $a \neq b$. Since AR_n is the Archimedean circle of the arbelos in 2-aliquot parts $\{\beta_{-n},\ \beta_0,\ \beta_n,\ \gamma_{2n}\}$, the radius of AR_n is

$$\frac{\overline{b_n}\,\overline{b_{-n}}(\overline{b_n}-\overline{b_{-n}})}{\overline{b_n}^2-\overline{b_{-n}}^2}=\frac{a^nb^n(a-b)}{a^{2n}-b^{2n}},$$

by Theorem 6 and Theorem 11.

On the other hand BK_n is the inscribed circle of the triangle bounded by the three centers of α, β, δ_{2n-1}. Since the length of three sides of the triangle are $a+d_{2n-1}$, $b+d_{2n-1}$, $a+b$, the radius of BK_n is

$$\sqrt{\frac{abd_{2n-1}}{a+b+d_{2n-1}}} = \frac{a^n b^n (a-b)}{a^{2n}-b^{2n}},$$

by Theorem 10. □

This theorem is a generalization of Bankoff circle [1]. Bankoff's third circle corresponds to the case $n=1$ in this theorem.

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Forum Geometricorum
Volume 5 (2005) 47–52.

FORUM GEOM
ISSN 1534-1178

On a Problem Regarding the n-Sectors of a Triangle

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Abstract. Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the n-sectors of Δ. In this paper we determine all triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles exist only if $n \in \{2, 3\}$.

1. Introduction

Let Δ be a triangle with vertices A, B, C and angles $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$, $\gamma = \widehat{ACB}$. The $n-1$ lines through A which, together with the lines AB and AC, divide the angle α in $n \geq 2$ equal parts are called the n-*sectors* of Δ. A triangle has $3(n-1)$ n-sectors. The 2-sectors and 3-sectors are also called *bisectors* and *trisectors*. In this paper we study triangles with the property that all three edges and all $3(n-1)$ n-sectors have rational lengths. We show that such triangles can exist only if $n = 2$ or 3. We also determine all triangles with the property that all edges and bisectors (trisectors) have rational lengths. In each of the cases $n = 2$ and $n = 3$, there are infinitely many nonsimilar triangles having that property.

In number theory, there are some open problems of the same type as the above-mentioned problem.

(i) Does there exists a *perfect cuboid*, i.e. a cuboid in which all 12 edges, all 12 face diagonals and all 4 body diagonals are rational? ([3, Problem D18]).

(ii) Does there exist a triangle with integer edges, medians and area? ([3, Problem D21]).

2. Some properties

An elementary proof of the following lemma can also be found in [2, p. 443].

Lemma 1. *The number* $\cos \frac{\pi}{n}$, $n \geq 2$, *is rational if and only if* $n = 2$ *or* $n = 3$.

Proof. Suppose that $\cos \frac{\pi}{n}$ is rational. Put

$$\zeta_{2n} = \cos \frac{2\pi}{2n} + i \sin \frac{2\pi}{2n},$$

Publication Date: March 29, 2005. Communicating Editor: J. Chris Fisher.

then ζ_{2n} is a zero of the polynomial $X^2 - (2 \cdot \cos\frac{\pi}{n}) \cdot X + 1 \in \mathbb{Q}[X]$. So, the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is of the first or second degree. On the other hand, we know that the minimal polynomial of ζ_{2n} over $\mathbb{Q}$ is the $2n$-th cyclotomic polynomial $\Phi_{2n}(x)$, see [4, Theorem 4.17]. The degree of $\Phi_{2n}(x)$ is $\phi(2n)$, where ϕ is the *Euler phi function*. We have $\phi(2n) = 2n \cdot \frac{p_1-1}{p_1} \cdot \frac{p_2-1}{p_2} \cdot \ldots \cdot \frac{p_k-1}{p_k}$, where $p_1, \ldots, p_k$ are the different prime numbers dividing $2n$. From $\phi(2n) \in \{1,2\}$, it easily follows $n \in \{2,3\}$. Obviously, $\cos\frac{\pi}{2}$ and $\cos\frac{\pi}{3}$ are rational. □

Lemma 2. *For every $n \in \mathbb{N} \setminus \{0\}$, there exist polynomials $f_n(x), g_{n-1}(x) \in \mathbb{Q}[x]$ such that*
(i) $\deg(f_n) = n$, $f_n(x) = 2^{n-1}x^n + \cdots$ *and* $\cos(nx) = f_n(\cos x)$ *for every* $x \in \mathbb{R}$*;*
(ii) $\deg(g_{n-1}) = n-1$, $g_{n-1}(x) = 2^{n-1}x^{n-1} + \cdots$ *and* $\frac{\sin(nx)}{\sin x} = g_{n-1}(\cos x)$ *for every* $x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Proof. From $\cos x = \cos x$, $\frac{\sin x}{\sin x} = 1$,

$$\begin{aligned}\cos(k+1)x &= \cos(kx)\cos x - \frac{\sin(kx)}{\sin x}(1-\cos^2 x),\\ \frac{\sin(k+1)x}{\sin x} &= \frac{\sin(kx)}{\sin x}\cos x + \cos(kx)\end{aligned}$$

for $k \geq 1$, it follows that we should make the following choices for the polynomials:

$f_1(x) := x$, $g_0(x) := 1$;
$f_{k+1}(x) := f_k(x) \cdot x - g_{k-1}(x) \cdot (1-x^2)$ for every $k \geq 1$;
$g_k(x) := g_{k-1}(x) \cdot x + f_k(x)$ for every $k \geq 1$.

One easily verifies by induction that f_n and g_{n-1} ($n \geq 1$) have the claimed properties. □

Lemma 3. *Let $n \in \mathbb{N} \setminus \{0\}$, $q \in \mathbb{Q}^+ \setminus \{0\}$ and $x_1, \ldots, x_n \in \mathbb{R}$. If*

$$\cos x_1,\ \sqrt{q}\cdot\sin x_1,\ \ldots,\ \cos x_n,\ \sqrt{q}\cdot\sin x_n$$

are rational, then so are $\cos(x_1 + \cdots + x_n)$ and $\sqrt{q} \cdot \sin(x_1 + \cdots + x_n)$.

Proof. This follows by induction from the following equations ($k \geq 1$).

$$\begin{aligned}\cos(x_1+\cdots+x_{k+1}) &= \cos(x_1+\cdots+x_k)\cdot\cos(x_{k+1})\\ &\quad - \frac{1}{q}\left(\sqrt{q}\cdot\sin(x_1+\cdots+x_k)\right)\cdot\left(\sqrt{q}\cdot\sin(x_{k+1})\right);\\ \sqrt{q}\cdot\sin(x_1+\cdots+x_{k+1}) &= \left(\sqrt{q}\cdot\sin(x_1+\cdots+x_k)\right)\cdot\cos(x_{k+1})\\ &\quad + \cos(x_1+\cdots+x_k)\cdot\left(\sqrt{q}\cdot\sin(x_{k+1})\right).\end{aligned}$$

□

Lemma 4. *Let Δ be a triangle with vertices A, B and C. Put $a = |BC|$, $b = |AC|$, $c = |AB|$, $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{BCA}$. Let $n \in \mathbb{N} \setminus \{0\}$ and suppose that $\cos(\frac{\alpha}{n})$, $\cos(\frac{\beta}{n})$ and $\cos(\frac{\gamma}{n})$ are rational. Then the following are equivalent:*

(i) $\frac{b}{a}$ *and* $\frac{c}{a}$ *are rational numbers.*
(ii) $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ *and* $\frac{\sin\frac{\gamma}{n}}{\sin\frac{\alpha}{n}}$ *are rational numbers.*

Proof. We have

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha} = \frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \cdot \frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}.$$

By Lemma 2, $\frac{\sin\beta}{\sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\alpha}{n}}{\sin\alpha} \in \mathbb{Q}^+ \setminus \{0\}$. So, $\frac{b}{a}$ is rational if and only if $\frac{\sin\frac{\beta}{n}}{\sin\frac{\alpha}{n}}$ is rational. A similar remark holds for the fraction $\frac{c}{a}$. $\square$

3. Necessary and sufficient conditions

Theorem 5. *Let $n \geq 2$ and $0 < \alpha, \beta, \gamma < \pi$ with $\alpha + \beta + \gamma = \pi$. There exists a triangle with angles α, β and γ all whose edges and n-sectors have rational lengths if and only if the following conditions hold:*

(1) $\cos\frac{\pi}{n} \in \mathbb{Q}$,
(2) $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$,
(3) $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$.

Proof. (a) Let Δ be a triangle with the property that all edges and all n-sectors have rational lengths. Let A, B and C be the vertices of Δ. Put $\alpha = \widehat{BAC}$, $\beta = \widehat{ABC}$ and $\gamma = \widehat{ACB}$. Let $A_0, \ldots, A_n$ be the vertices on the edge BC such that $A_0 = B$, $A_n = C$ and $\widehat{A_{i-1}AA_i} = \frac{\alpha}{n}$ for all $i \in \{1, \ldots, n\}$. Put $a_i = |A_{i-1}A_i|$ for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n-1\}$, the line AA_i is a bisector of the triangle with vertices A_{i-1}, A and A_{i+1}. Hence, $\frac{a_i}{a_{i+1}} = \frac{|AA_{i-1}|}{|AA_{i+1}|} \in \mathbb{Q}$. Together with $a_1 + \ldots + a_n = |BC| \in \mathbb{Q}$, it follows that $a_i \in \mathbb{Q}$ for every $i \in \{1, \ldots, n\}$. The cosine rule in the triangle with vertices A, A_0 and A_1 gives

$$\cos\frac{\alpha}{n} = \frac{|AA_0^2| + |AA_1|^2 - a_1^2}{2 \cdot |AA_0| \cdot |AA_1|} \in \mathbb{Q}.$$

In a similar way one shows that $\cos\frac{\beta}{n}, \cos\frac{\gamma}{n} \in \mathbb{Q}$. Put $q := (1 - \cos^2\frac{\alpha}{n})^{-1}$. By Lemma 4, $\sqrt{q} \cdot \sin\frac{\alpha}{n}$, $\sqrt{q} \cdot \sin\frac{\beta}{n}$ and $\sqrt{q} \cdot \sin\frac{\gamma}{n}$ are rational. From Lemma 3, it follows that $\cos\frac{\pi}{n} \in \mathbb{Q}$ and $\sqrt{q} \cdot \sin\frac{\pi}{n} \in \mathbb{Q}$. Hence,

$$\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} = \frac{1 + \cos\frac{\pi}{n}}{\sqrt{q} \cdot \sin\frac{\pi}{n}} \cdot \frac{\sqrt{q} \cdot \sin\frac{\alpha}{n}}{1 + \cos\frac{\alpha}{n}} \in \mathbb{Q}.$$

Similarly, $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\gamma}{2n} \in \mathbb{Q}$.

(b) Conversely, suppose that $\cos\frac{\pi}{n} \in \mathbb{Q}$, $\cot\frac{\pi}{2n} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$ and $\cot\frac{\pi}{2n} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$. Put $q := \sin^2\frac{\pi}{n} = 1 - \cos^2\frac{\pi}{n} \in \mathbb{Q}$. From $\sqrt{q} \cdot \cot\frac{\pi}{2n} = \sqrt{q} \cdot \frac{1+\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}} \in \mathbb{Q}$, it follows that $\sqrt{q} \cdot \tan\frac{\alpha}{2n} \in \mathbb{Q}$, $\sqrt{q} \cdot \tan\frac{\beta}{2n} \in \mathbb{Q}$, $\cos\frac{\alpha}{n} = \frac{1-\tan^2\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\cos\frac{\beta}{n} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\alpha}{n} = \frac{2\sqrt{q}\cdot\tan\frac{\alpha}{2n}}{1+\tan^2\frac{\alpha}{2n}} \in \mathbb{Q}$, $\sqrt{q} \cdot \sin\frac{\beta}{n} \in \mathbb{Q}$. By Lemma 3, also $\cos\frac{\gamma}{n}$, $\sqrt{q} \cdot \sin\frac{\gamma}{n} \in \mathbb{Q}$. Now, choose a triangle Δ with angles α, β and γ such that the edge opposite the angle α has rational length. By Lemma 4, it then follows that also the edges opposite to β and γ have rational lengths. Let A, B and C be the vertices of Δ such that $\widehat{BAC} = \alpha$, $\widehat{ABC} = \beta$ and $\widehat{ACB} = \gamma$. As before, let $A_0, \ldots, A_n$ be vertices on the edge BC such that the $n+1$ lines AA_i, $i \in \{0, \ldots, n\}$, divide the angle α in n equal parts. By the sine rule,

$$|AA_i| = \frac{|AB| \cdot \sin\beta}{\sin(\frac{i\alpha}{n} + \beta)}.$$

Now,

$$\frac{\sin(\frac{i\alpha}{n} + \beta)}{\sin\beta} = \frac{\sin\frac{i\alpha}{n}}{\sin\frac{\alpha}{n}} \cdot \frac{\sqrt{q} \cdot \sin\frac{\alpha}{n}}{\sqrt{q} \cdot \sin\frac{\beta}{n}} \cdot \frac{\sin\frac{\beta}{n}}{\sin\beta} \cdot \cos\beta + \cos\frac{i\alpha}{n}.$$

By Lemma 2, this number is rational. Hence $|AA_i| \in \mathbb{Q}$. By a similar reasoning it follows that the lengths of all other n-sectors are rational as well. $\square$

By Lemma 1 and Theorem 5 (1), we know that the problem can only have a solution in the case of bisectors or trisectors.

4. The case of bisectors

The bisector case has already been solved completely, see e.g. [1] or [5]. Here we present a complete solution based on Theorem 5. Without loss of generality, we may suppose that $\alpha \leq \beta \leq \gamma$. These conditions are equivalent with

$$0 < \alpha \leq \frac{\pi}{3}, \tag{1}$$

$$\alpha \leq \beta \leq \frac{\pi}{2} - \frac{\alpha}{2}. \tag{2}$$

By Theorem 5, $q_\alpha := \tan\frac{\alpha}{4}$ and $q_\beta := \tan\frac{\beta}{4}$ are rational. Equation (1) implies $0 < q_\alpha \leq \tan\frac{\pi}{12}$ and equation (2) implies $q_\alpha \leq q_\beta \leq x$, where $x := \tan(\frac{\pi}{8} - \frac{\alpha}{8})$. Now, $\frac{2x}{1-x^2} = \tan(\frac{\pi}{4} - \frac{\alpha}{4}) = \frac{1-q_\alpha}{1+q_\alpha}$ and hence $x = \frac{\sqrt{2+2q_\alpha^2}-1-q_\alpha}{1-q_\alpha}$. Summarizing, we have the following restrictions for $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$0 < q_\alpha \leq \tan\frac{\pi}{12},$$
$$q_\alpha \leq q_\beta \leq \frac{\sqrt{2+2q_\alpha^2} - 1 - q_\alpha}{1 - q_\alpha}.$$

In Figure 1 we depict the area G corresponding with these inequalities. Every point in G with rational coordinates in G will give rise to a triangle all whose edges and bisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

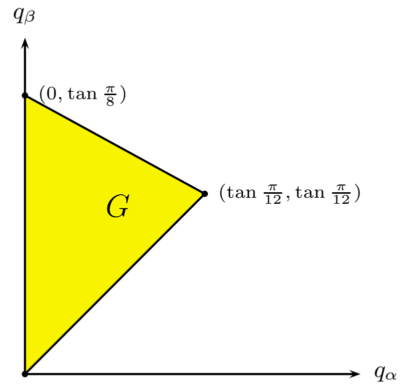

Figure 1

5. The case of trisectors

An infinite but incomplete class of solutions for the trisector case did also occur in the solution booklet of a mathematical competition in the Netherlands (universitaire wiskunde competitie, 1995). Here we present a complete solution based on Theorem 5. Again we may assume that $\alpha \leq \beta \leq \gamma$; so, equations (1) and (2) remain valid here. By Theorem 5, $q_\alpha := \sqrt{3} \cdot \tan\frac{\alpha}{6}$ and $q_\beta := \sqrt{3} \cdot \tan\frac{\beta}{6}$ are rational. As before, one can calculate the inequalities that need to be satisfied by $q_\alpha \in \mathbb{Q}$ and $q_\beta \in \mathbb{Q}$:

$$\begin{aligned} &0 < q_\alpha \leq \sqrt{3} \cdot \tan\frac{\pi}{18}, \\ &q_\alpha \leq q_\beta \leq \frac{\sqrt{12 + 4q_\alpha^2} - 3 - q_\alpha}{1 - q_\alpha}. \end{aligned}$$

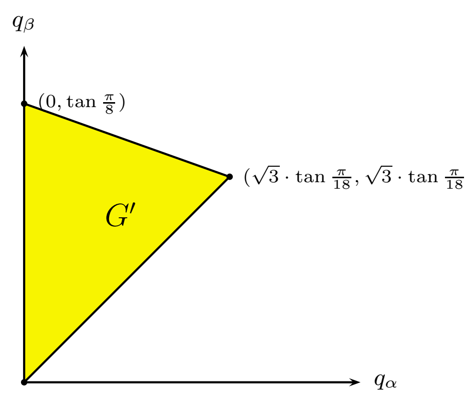

Figure 2

In Figure 2 we depict the area G' corresponding with these inequalities. Every point in G' with rational coordinates will give rise to a triangle all whose edges and trisectors have rational lengths. Two different points in G with rational coefficients correspond with nonsimilar triangles.

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Forum Geometricorum
Volume 5 (2005) 53–56.

FORUM GEOM
ISSN 1534-1178

A Simple Construction of a Triangle from its Centroid, Incenter, and a Vertex

Eric Danneels

Abstract. We give a simple ruler and compass construction of a triangle given its centroid, incenter, and one vertex. An analysis of the number of solutions is also given.

1. Construction

The ruler and compass construction of a triangle from its centroid, incenter, and one vertex was one of the unresolved cases in [3]. An analysis of this problem, including the number of solutions, was given in [1]. In this note we give a very simple construction of triangle ABC with given centroid G, incenter I, and vertex A. The construction depends on the following propositions. For another slightly different construction, see [2].

Proposition 1. *Given triangle ABC with Nagel point N, let D be the midpoint of BC. The lines ID and AN are parallel.*

Proof. The centroid G divides each of the segments AD and NI in the ratio $AG : GD = NG : GI = 2 : 1$. See Figure 1. □

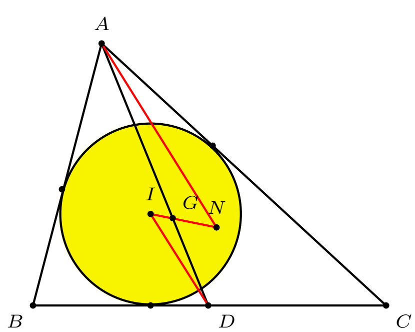

Figure 1

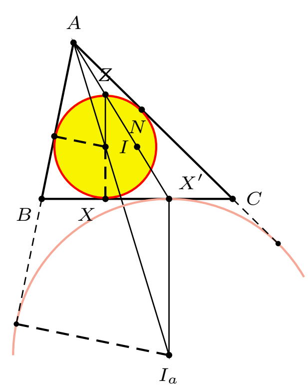

Figure 2

Proposition 2. *Let X be the point of tangency of the incircle with BC. The antipode of X on the circle with diameter ID is a point on AN.*

Publication Date: April 12, 2005. Communicating Editor: Paul Yiu.
The author wishes to thank Paul Yiu for his help in the preparation of this paper.

Proof. This follows from the fact that the antipode of X on the incircle lies on the segment AN. See Figure 2. □

Construction. Given G, I, and A, extend AG to D such that $AG : GD = 2 : 1$. Construct the circle $\mathcal{C}$ with diameter ID, and the line $\mathcal{L}$ through A parallel parallel to ID.

Let Y be an intersection of the circle $\mathcal{C}$ and the line $\mathcal{L}$, and X the antipode of Y on $\mathcal{C}$ such that A is outside the circle $I(X)$. Construct the tangents from A to the circle $I(X)$. Their intersections with the line DX at the remaining vertices B and C of the required triangle. See Figure 3.

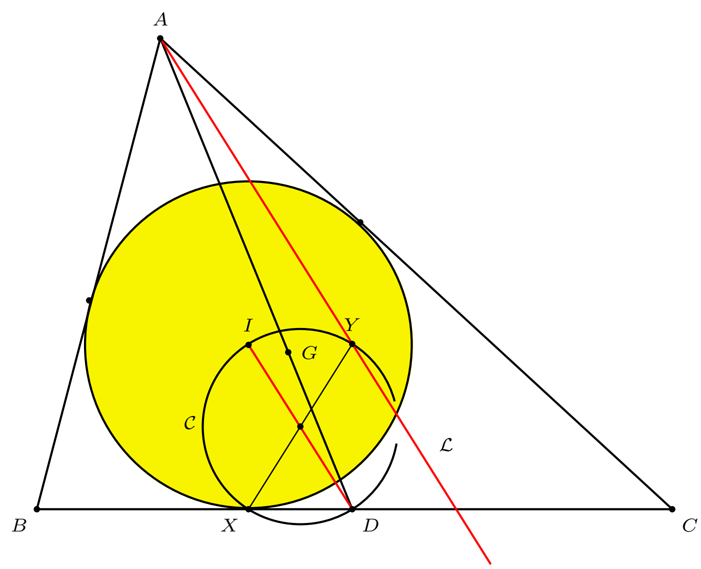

Figure 3

2. Number of solutions

We set up a Cartesian coordinate system such that $A = (0, 2k)$ and $I = (0, -k)$. If $G = (u, v)$, then $D = \frac{1}{2}(3G - A) = \left(\frac{3}{2}u,\ \frac{3}{2}v - k\right)$. The circle $\mathcal{C}$ with diameter ID has equation

$$2(x^2 + y^2) - 3ux - (3v - 4k)y + (2k^2 - 3kv) = 0$$

and the line $\mathcal{L}$ through A parallel to ID has slope $\frac{v}{u}$ and equation

$$vx - uy + 2ku = 0.$$

The line $\mathcal{L}$ and the circle $\mathcal{C}$ intersect at 0, 1, 2 real points according as

$$\Delta := (u^2 + v^2 - 4ku)(u^2 + v^2 + 4ku)$$

is negative, zero, or positive. Since $x^2 + y^2 \pm 4kx = 0$ represent the two circles of radii $2k$ tangent to each other externally and to the y-axis at $(0, 0)$, Δ is negative, zero, or positive according as G lies in the interior, on the boundary, or in the exterior of the union of the two circles.

The intersections of the circle and the line are the points

$$Y_\varepsilon = \left(\frac{3u(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)},\ \frac{8k(u^2+v^2)+3v(u^2+v^2-4kv-\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right)$$

for $\varepsilon = \pm 1$. Their antipodes on $\mathcal{C}$ are the points

$$X_\varepsilon = \left(\frac{3u(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)},\ \frac{-16k(u^2+v^2)+3v(u^2+v^2+4kv+\varepsilon\sqrt{\Delta})}{4(u^2+v^2)}\right).$$

There is a triangle ABC tritangent to the circle $I(X_\varepsilon)$ and with DX_ε as a sideline if and only if the point A lies outside the circle $I(X_\varepsilon)$. Note that $IA = 3k$ and

$$IX_+^2 = \frac{9}{8}(u^2+v^2+\sqrt{\Delta}), \quad IX_-^2 = \frac{9}{8}(u^2+v^2-\sqrt{\Delta}).$$

From these, we make the following conclusions.

(i) If $u^2+v^2-8k^2 \geq \sqrt{\Delta}$, then A lies inside or on $I(X_-)$. In this case, there is no triangle.

(ii) If $-\sqrt{\Delta} \leq u^2+v^2-8k^2 < \sqrt{\Delta}$, then A lies outside $I(X_-)$ but not $I(X_+)$. There is exactly one triangle.

(iii) If $u^2+v^2-8k^2 < -\sqrt{\Delta}$, then A lies outside $I(X_+)$ (and also $I(X_-)$). There are in general two triangles.

It is easy to see that the condition $-\sqrt{\Delta} < u^2+v^2-8k^2 < \sqrt{\Delta}$ is equivalent to $(v-2k)(v+2k) > 0$, *i.e.*, $|v| > 2k$. We also note the following.

(i) When the line D_ε passes through A, the corresponding triangle degenerates. The condition for collinearity leads to

$$u(3u^2+3v^2-4kv\pm\sqrt{\Delta}) = 0.$$

Clearly, $u = 0$ gives the y-axis. The corresponding triangle is isosceles. On the other hand, the condition $3u^2+3v^2-4kv\pm\sqrt{\Delta} = 0$ leads to

$$(u^2+v^2)(u^2+v^2-3kv+2k^2) = 0,$$

i.e., (u, v) lying on the circle tangent to the circles $x^2+y^2\pm 4kx = 0$ at $\left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$ and the line $y = 2k$ at A.

(ii) If $v > 0$, the circle $I(X_\varepsilon)$, instead of being the incircle, is an excircle of the triangle. If G lies inside the region $ATOT'A$ bounded by the circular segments, one of the excircles is the A-excircle. Outside this region, the excircle is always a B/C-excircle.

From these we obtain the distribution of the position of G, summarized in Table 1 and depicted in Figure 4, for the various numbers of solutions of the construction problem. In Figure 4, the number of triangles is

0 if G in an unshaded region, on a dotted line, or at a solid point other than I,

1 if G is in a yellow region or on a solid red line,

2 if G is in a green region.

Table 1. Number N of non-degenerate triangles according to the location of G relative to A and I

N	Location of centroid $G(u,v)$
0	$(0,0),\ (\pm 2k, 2k)$; $\left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$; $v = 2k$; $\lvert u\rvert > 2k - \sqrt{4k^2 - v^2},\ -2k \leq v < 2k$.
1	$u = 0,\ 0 < \lvert v\rvert < 2k$; $-2k < u < 2k,\ v = -2k$; $u = 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$; $\lvert v\rvert > 2k$; $u^2 + v^2 - 3kv + 2k^2 = 0$ except $(0, 2k),\ \left(\pm\frac{2k}{5},\ \frac{6k}{5}\right)$.
2	$\lvert u\rvert < 2k - \sqrt{4k^2 - v^2},\ 0 < \lvert v\rvert < 2k$, but $u^2 + v^2 - 3kv + 2k^2 \neq 0$.

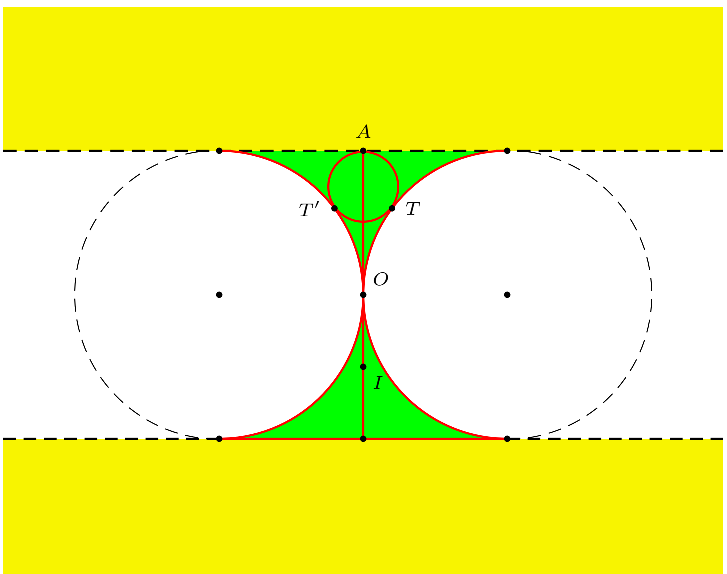

Figure 4

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Forum Geometricorum
Volume 5 (2005) 57–61.

FORUM GEOM
ISSN 1534-1178

Triangle-Conic Porism

Aad Goddijn and Floor van Lamoen

Abstract. We investigate, for a given triangle, inscribed triangles whose sides are tangent to a given conic.

Consider a triangle $A_1B_1C_1$ inscribed in ABC, and a conic $\mathcal{C}$ inscribed in $A_1B_1C_1$. We ask whether there are other inscribed triangles in ABC and tritangent to $\mathcal{C}$. Restricting to circles, Ton Lecluse wrote about this problem in [6]. See also [5]. He suggested after use of dynamic geometry software that in general there is a second triangle tritangent to $\mathcal{C}$ and inscribed in ABC. In this paper we answer Lecluse's question.

Proposition 1. *Let $A'B'C'$ be a variable triangle of which B' and C' lie on CA and AB respectively. If the sidelines of triangle $A'B'C'$ are tangent to a conic $\mathcal{C}$, then the locus of A' is either a conic or a line.*

Proof. Let XYZ be the points on $\mathcal{C}$ and where $C'A'$, $A'B'$, and $B'C'$ respectively meet $\mathcal{C}$. ZX is the polar (with respect to $\mathcal{C}$) of B', which passes through a fixed point P_B, the pole of CA. Similarly XY passes through a fixed point P_C. The mappings $Y \mapsto X$ and $X \mapsto Z$ are thus involutions on $\mathcal{C}$. Hence $Y \mapsto Z$ is a projectivity. That means that the lines YZ form a pencil of lines or envelope a conic according as $Y \mapsto Z$ is an involution or not. Consequently the poles of these lines, the points A', run through a line ℓ_A or a conic $\mathcal{C}_A$. $\square$

Two degenerate triangles $A'B'C'$, corresponding to the tangents from A, arise as limit cases. Hence, when $Y \mapsto Z$ is an involution, the points U_1 and U_2 of contact of tangents from A to $\mathcal{C}$ are its fixed points, and $\ell_A = U_1U_2$ is the polar of A.

The conics $\mathcal{C}$ and $\mathcal{C}_A$ are tangent to each other in U_1 and U_2. We see that $\mathcal{C}$ and $\mathcal{C}_A$ generate a pencil, of which the pair of common tangents, and the polar of A (as double line) are the degenerate elements. In view of this we may consider the line ℓ_A as a conic $\mathcal{C}_A$ degenerated into a "double" line. We do so in the rest of this paper.

Publication Date: April 26, 2005. Communicating Editor: Paul Yiu.

The authors thank Ton Lecluse and Dick Klingens for their inspiring problem and correspondence.

Proposition 1 shows us that if there is one inscribed triangle tritangent to $\mathcal{C}$, there will be in general another such triangle. This answers Lecluse's question for the general case. But it turns out that the other cases lead to interesting configurations as well.

The number of intersections of $\mathcal{C}_A$ with BC gives the number of inscribed triangles tritangent to $\mathcal{C}$. There may be infinitely many, if $\mathcal{C}_A$ degenerates and contains BC. This implies that $BC = \ell_A$. By symmetry it is necessary that ABC is self-polar with respect to $\mathcal{C}$. Of course this applies also when the above A' runs through ℓ_A in the plane of the triangle bounded by AB, CA and ℓ_A.

There are two possibilities for $\mathcal{C}_A$ and BC to intersect in one "double" point. One is that $\mathcal{C}_A$ is nondegenerate and tangent to BC. In this case, by reasons of continuity, the point of tangency belongs to one triangle $A'B'C'$, and similar conics $\mathcal{C}_B$ and $\mathcal{C}_C$ are tangent to the corresponding side as well. The points of tangency form the cevian triangle of the perspector of $\mathcal{C}$.

This can be seen by considering the point M where U_1U_2 meets BC. The polar of M with respect to $\mathcal{C}$ passes through the pole of U_1U_2, and through the intersections of the polars of B and C, hence the pole of BC. So the polar ℓ_M of M is the A-cevian of the perspector[1] of $\mathcal{C}$. The point where U_1U_2 and ℓ_M meet is the harmonic conjugate of M with respect to U_1 and U_2. This all applies to $\mathcal{C}_A$ as well. In case $\mathcal{C}_A$ is tangent to BC, the point of tangency is the pole of BC, and is thus the trace of the perspector of $\mathcal{C}$.

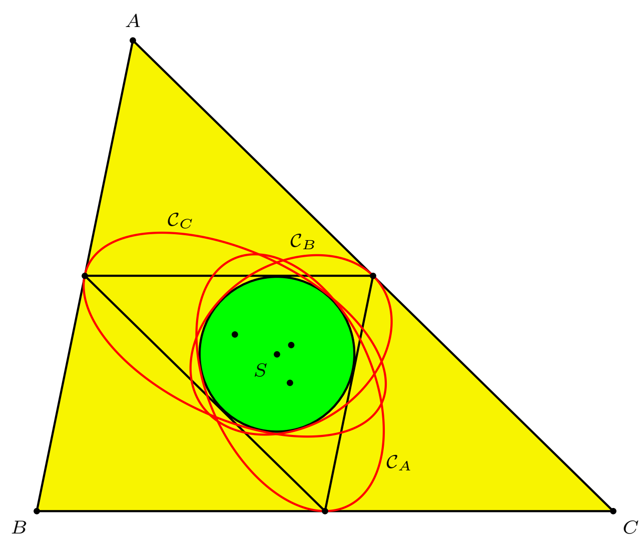

Figure 1

For example, if $\mathcal{C}$ is the incircle of the medial triangle, the conic $\mathcal{C}_A$ is tangent to BC at its midpoint, and contains the points $(s : s-b : b)$, $(s : c : s-c)$,

[1]By Chasles' theorem on polarity [1, 5.61], each triangle is perspective to its polar triangle. The perspector is called the perspector of the conic.

$((a+b+c)(b+c-a) : 2c(c+a-b) : b^2+3c^2-a^2+2ca)$ and $((a+b+c)(b+c-a) : 3b^2+c^2-a^2+2ab : 2b(a+b-c))$. It has center $(s : c+a : a+b)$. See Figure 1.

The other possibility for a double point is when $\mathcal{C}_A$ degenerates into ℓ_A. To investigate this case we prove the following proposition.

Proposition 2. *If $\mathcal{C}_A$ degenerates into a line, the triangle ABC is selfpolar with respect to each conic tangent to the sides of two cevian triangles. The cevian triangle of the trilinear pole of any tangent to such a conic is tritangent to this conic.*

Proof. Let P be a point and $A^PB^PC^P$ its anticevian triangle. ABC is a polar triangle with respect to each conic through $A^PB^PC^P$, as ABC are the diagonal points of the complete quadrilateral $PA^PB^PC^P$. Now consider a second anticevian triangle $A^QB^QC^Q$ of Q. The vertices of $A^PB^PC^P$ and $A^QB^QC^Q$ lie on a conic[2] $\mathcal{K}$. But we also know that triangle PB^PC^P is the anticevian triangle of A^P. So PB^PC^P and $A^QB^QC^Q$ lie on a conic as well, and having 5 common points this must be $\mathcal{K}$. We conclude that ABC is selfpolar with respect to $\mathcal{K}$.

Let R be a point on $\mathcal{K}$. AR intersects $\mathcal{K}$ in a second point R'. Let R_A be the intersection AR and BC, then R and R' are harmonic with respect to A and R_A. But that means that $R' = A^R$ is the $A-$vertex of the anti-cevian triangle of R. Consequently the anti-cevian triangle of R lies on $\mathcal{K}$. Proposition 2 is now proved by duality. □

In the proof B^PC^P is the side of two anticevian triangles inscribed in $\mathcal{K}$ - by duality this means that the vertex of a cevian triangle tangent to $\mathcal{K}$ is a common vertex of two such cevian triangles. In the case of ℓ_A intersecting BC in a double point, clearly the two triangles are cevian triangles with respect to the triangle bounded by AB, AC and ℓ_A. Were they cevian triangles also with respect to ABC, then the four sidelines of these cevian triangles would form the dual of an anticevian triangle, and ABC would be selfpolar with respect to $\mathcal{C}$, and ℓ_A would be BC.

We conclude that two distinct triangles inscribed in ABC and circumscribing $\mathcal{C}$ cannot be cevian triangles.

In the case ABC is selfpolar with respect to $\mathcal{C}$, so that $\mathcal{C}_A$ degenerates into ℓ_A, not each point on ℓ_A belongs to (real) cevian triangles. On the other hand clearly infinitely many points on ℓ_A will lead to two cevian triangles tritangent to $\mathcal{C}$. The perpsectors run through a quartic, the tripoles of the tangents to $\mathcal{C}$.

Theorem 3. *Given a triangle ABC and a conic $\mathcal{C}$, the* triangle-conic poristic *triangles inscribed in ABC and tritangent to $\mathcal{C}$ are as follows.*

(i) *There are be no triangle-conic poristic triangle.*

(ii) *$\mathcal{C}$ is a conic inscribed in a cevian triangle, and ABC is not self-polar with respect to $\mathcal{C}$. In this case the cevian triangle is the only triangle-conic poristic triangle.*

[2] This follows from the dual of the well known theorem that two cevian triangles are circumscribed by and inscribed in a conic.

(iii) *ABC is self-polar with respect to $\mathcal{C}$. In this case there are infinitely many triangle-conic poristic triangles.*

(iv) *There are two distinct triangle-conic poristic triangles, which are not cevian triangles.*

Remarks. (1) In case of a conic with respect to which ABC is self-polar, instead of cevian triangles tritangent to $\mathcal{C}$, we should speak of cevian fourlines quadritangent to $\mathcal{C}$.

(2) When we investigate triangles inscribed in a conic and circumscribed to ABC we get similar results as Theorem 3, simply by duality.

In case $\mathcal{C}$ is a conic with respect to which ABC is selfpolar, we see that each tangent to $\mathcal{C}$ belongs to two cevian triangles tritangent to $\mathcal{C}$ and that each point on $\mathcal{C}$ belongs to two anticevian triangles inscribed in $\mathcal{C}$. In this case speak of *triangle-conic porism* and *conic-triangle porism* in extension of the well known Poncelet porism.

As an example, we consider the *nine-point circle triangles*, hence the medial and orthic triangles. We know that these circumscribe a conic $\mathcal{C}_N$, with respect to which ABC is selfpolar. By Proposition 2 we know that the perspectrices of the medial and orthic triangles are tangent to $\mathcal{C}_N$ as well, hence $\mathcal{C}_N$ must be a parabola tangent to the orthic axis. The barycentric equation of this parabola is

$$\frac{x^2}{a^2(b^2-c^2)}+\frac{y^2}{b^2(c^2-a^2)}+\frac{z^2}{c^2(a^2-b^2)}=0.$$

Its focus is X_{115} of [3, 4], its directrix the Brocard axis, and its axis is the Simson line of X_{98}. See Figure 2. The parabola contains the infinite point X_{512} and passes through X_{661}, X_{647} and X_{2519}. The Brianchon point of the parabola with respect to the medial triangle is $X_{670}(\text{medial})$.

The perspectors of the tangent cevian triangles run through the quartic

$$a^2(b^2-c^2)y^2z^2+b^2(c^2-a^2)z^2x^2+c^2(a^2-b^2)x^2y^2=0,$$

which is the isotomic conjugate of the conic

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0$$

through the vertices of the antimedial triangle, the centroid, and the isotomic conjugates of the incenter and the orthocenter.

This special case leads us to amusing consequences, to which we were pointed by [2]. The sides of every cevian triangle and its perspectrix are tangent to one parabola inscribed in the medial triangle. Consequently the *isotomic conjugates*[3] with respect to to the medial triangle of these are parallel.

In the dual case, we conclude for instance that the isotomic conjugates with respect to the antimedial triangle of the vertices and perspector D of any anticevian triangle are collinear with the centroid G. The line is GD', where D' is the barycentric square of D.

[3]The isotomic conjugate of a line ℓ with respect to a triangle is the line passing through the intercepts of ℓ with the sides reflected through the corresponding midpoints. In [3] this is referred to as *isotomic transversal*.

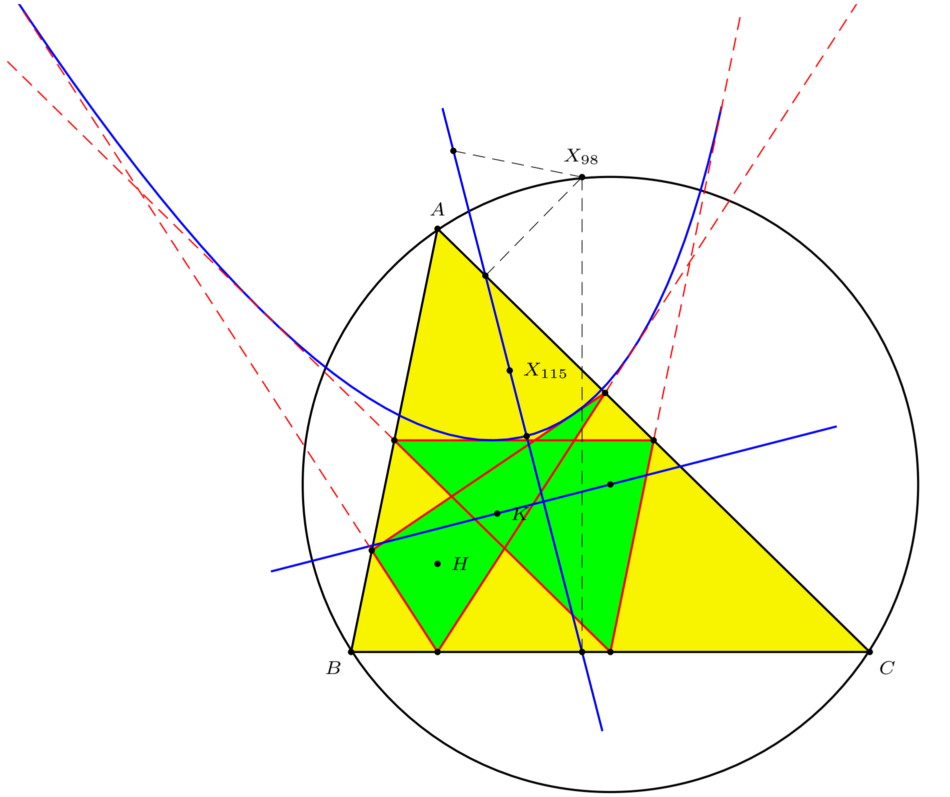

Figure 2.

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Forum Geometricorum
Volume 5 (2005) 63–64.

FORUM GEOM
ISSN 1534-1178

A Maximal Property of Cyclic Quadrilaterals

Antreas Varverakis

Abstract. We give a very simple proof of the well known fact that among all quadrilaterals with given side lengths, the cyclic one has maximal area.

Among all quadrilaterals $ABCD$ be with given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, it is well known that the one with greatest area is the cyclic quadrilateral. All known proofs of this result make use of Brahmagupta formula. See, for example, [1, p.50]. In this note we give a very simple geometric proof.

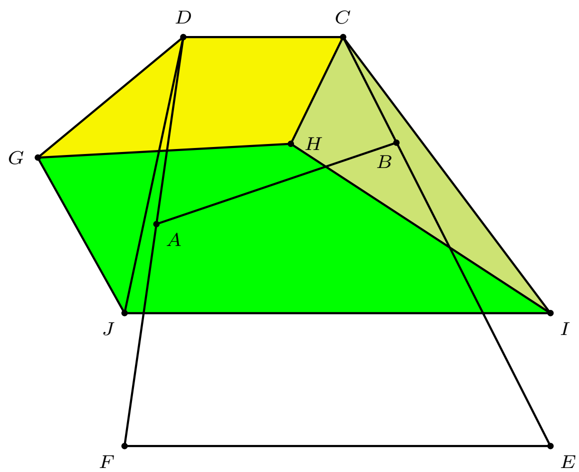

Figure 1

Let $ABCD$ be the cyclic quadrilateral and $GHCD$ an arbitrary one with the same side lengths: $GH = a$, $HC = b$, $CD = c$ and $DG = d$. Construct quadrilaterals $EFAB$ similar to $ABCD$ and $IJGH$ similar to $GHCD$ (in the same order of vertices). Note that

(i) FE is parallel to DC since $ABCD$ is cyclic and DAF, CBE are straight lines;

(ii) JI is also parallel to DC since

Publication Date: May 10, 2005. Communicating Editor: Paul Yiu.

$$\begin{aligned}
\angle CDJ + \angle DJI =&(\angle CDG - \angle JDG) + (\angle GJI - \angle GJD)\\
=&(\angle CDG - \angle JDG) + (\angle CHG - \angle GJD)\\
=&\angle CDG + \angle CHG - (\angle JDG + \angle GJD)\\
=&\angle CDG + \angle CHG - (180^\circ - \angle DGJ)\\
=&\angle CDG + \angle CHG + (\angle DGH + \angle HGJ) - 180^\circ\\
=&\angle CDG + \angle CHG + \angle DGH + \angle HCD - 180^\circ\\
=&180^\circ.
\end{aligned}$$

Since the ratios of similarity of the quadrilaterals are both $\frac{a}{c}$, the areas of $ABEF$ and $GHIJ$ are $\frac{a^2}{c^2}$ times those of $ABCD$ and $GHCD$ respectively. It is enough to prove that

$$\text{area}(DCEF) \geq \text{area}(DCHIJGD).$$

In fact, since $GD \cdot GJ = HC \cdot HI$ and $\angle DGJ = \angle CHI$, it follows that area(DGJ) = area(CHI), and we have

$$\text{area}(DCHIJGD) = \text{area}(DCHG) + \text{area}(GHIJ) = \text{area}(DCIJ).$$

Note that

$$\begin{aligned}
\overrightarrow{CD} \cdot \overrightarrow{DJ} =&\overrightarrow{CD} \cdot (\overrightarrow{DG} + \overrightarrow{GJ})\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} + \overrightarrow{CD} \cdot \overrightarrow{GJ}\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} + \frac{c^2}{a^2}\left(\overrightarrow{IJ} \cdot \overrightarrow{GJ}\right)\\
=&\overrightarrow{CD} \cdot \overrightarrow{DG} - \overrightarrow{CH} \cdot \overrightarrow{HG}\\
=&\frac{1}{2}\left(a^2 + b^2 - CG^2\right) - \frac{1}{2}\left(c^2 + d^2 - CG^2\right)\\
=&\frac{1}{2}(a^2 + b^2 - c^2 - d^2)
\end{aligned}$$

is independent of the position of J. This means that the line JF is perpendicular to DC; so is IE for a similar reason. The vector $\overrightarrow{DJ} = \overrightarrow{DG} + \overrightarrow{GJ}$ has a constant projection on $\overrightarrow{CD}$ (the same holds for $\overrightarrow{CI}$). We conclude that trapezium $DCEF$ has the greatest altitude among all these trapezia constructed the same way as $DCIJ$. Since all these trapezia have the same bases, $DCEF$ has the greatest area. This completes the proof that among quadrilaterals of given side lengths, the cyclic one has greatest area.

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Forum Geometricorum
Volume 5 (2005) 65–74.

FORUM GEOM
ISSN 1534-1178

Some Brocard-like points of a triangle

Sadi Abu-Saymeh and Mowaffaq Hajja

Abstract. In this note, we prove that for every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that $\angle AC'B' = \angle BA'C' = \angle CB'A'$, and a unique interior point M' the cevians AA', BB', and CC' through which have the property that $\angle AB'C' = \angle BC'A' = \angle CA'B'$. We study some properties of these Brocard-like points, and characterize those centers for which the angles $AC'B'$, $BA'C'$, and $CB'A'$ are linear forms in the angles A, B, and C of ABC.

1. Notations

Let ABC be a non-degenerate triangle, with angles A, B, and C. To every point P inside ABC, we associate, as shown in Figure 1, the following angles and lengths.

$$\begin{array}{lll}
\xi = \angle BAA', & \eta = \angle CBB', & \zeta = \angle ACC'; \\
\xi' = \angle CAA', & \eta' = \angle ABB', & \zeta' = \angle BCC'; \\
\alpha = \angle AC'B', & \beta = \angle BA'C', & \gamma = \angle CB'A'; \\
\alpha' = \angle AB'C', & \beta' = \angle BC'A', & \gamma' = \angle CA'B'; \\
x = BA', & y = CB', & z = AC'; \\
x' = A'C, & y' = B'A, & z' = C'B.
\end{array}$$

The well-known Brocard or Crelle-Brocard points are defined by the requirements $\xi = \eta = \zeta$ and $\xi' = \eta' = \zeta'$; see [11]. The angles ω and ω' that satisfy $\xi = \eta = \zeta = \omega$ and $\xi' = \eta' = \zeta' = \omega'$ are equal, and their common value is called the Brocard angle. The points known as Yff's analogues of the Brocard points are defined by the similar requirements $x = y = z$ and $x' = y' = z'$. These were introduced by Peter Yff in [12], and were so named by Clark Kimberling in a talk that later appeared as [8]. For simplicity, we shall refer to these points as *the Yff-Brocard points*.

2. The cevian Brocard points

In this note, we show that each of the requirements $\alpha = \beta = \gamma$ and $\alpha' = \beta' = \gamma'$ defines a unique interior point, and that the angles Ω and Ω' that satisfy $\alpha = \beta = \gamma = \Omega$ and $\alpha' = \beta' = \gamma' = \Omega'$ are equal. We shall call the resulting two points the first and second cevian Brocard points respectively, and the common value of Ω and Ω', the cevian Brocard angle of ABC.

Publication Date: May 24, 2005. Communicating Editor: Paul Yiu.
This work is supported by a research grant from Yarmouk University.

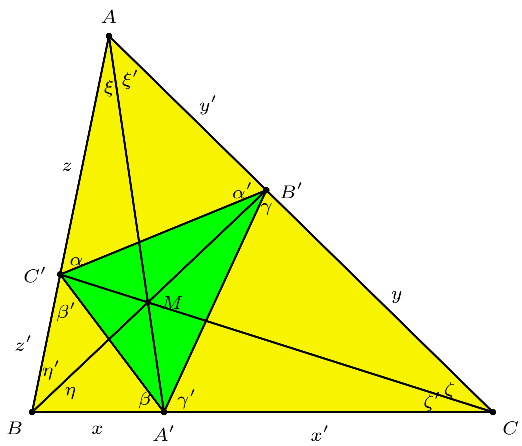

Figure 1.

We shall freely use the trigonometric forms

$$\sin\xi\sin\eta\sin\zeta = \sin\xi'\sin\eta'\sin\zeta' = \sin(A-\xi)\sin(B-\eta)\sin(C-\zeta)$$
$$\sin\alpha\sin\beta\sin\gamma = \sin\alpha'\ \sin\beta'\sin\gamma' = \sin(A+\alpha)\sin(B+\beta)\ \sin(C+\gamma)$$

of the cevian concurrence condition. We shall also freely use a theorem of Seebach stating that for any triangles ABC and UVW, there exists inside ABC a unique point P the cevians AA', BB', and CC' through which have the property that $(A', B', C') = (U, V, W)$, where A', B', and C' are the angles of $A'B'C'$ and U, V, and W are the angles of UVW; see [10] and [7].

Theorem 1. *For every triangle ABC, there exists a unique interior point M the cevians AA', BB', and CC' through which have the property that*

$$\angle AC'B' = \angle BA'C' = \angle CB'A'\ (=\Omega,\ \textit{say}), \tag{1}$$

and a unique interior point M' the cevians AA', BB', and CC' through which have the property that

$$\angle AB'C' = \angle BC'A' = \angle CA'B'\ (=\Omega',\ \textit{say}). \tag{2}$$

Also, the angles Ω and Ω' are equal and acute. See Figures 2A and 2B.

Proof. It is obvious that (1) is equivalent to the condition $(A', B', C') = (C, A, B)$, where A', B', and C' are the angles of the cevian triangle $A'B'C'$. Similarly, (2) is equivalent to the condition $(A', B', C') = (B, C, A)$. According to Seebach's theorem, the existence and uniqueness of M and M' follow by taking $(U, V, W) = (C, A, B)$ and $(U, V, W) = (B, C, A)$.

To prove that Ω is acute, observe that if Ω is obtuse, then the angles Ω, $A+\Omega$, $B+\Omega$, and $C+\Omega$ would all lie in the interval $[\pi/2, \pi]$ where the sine function is positive and decreasing. This would imply that

$$\sin^3\Omega \quad > \quad \sin(A+\Omega)\ \sin(B+\Omega)\ \sin(C+\Omega),$$

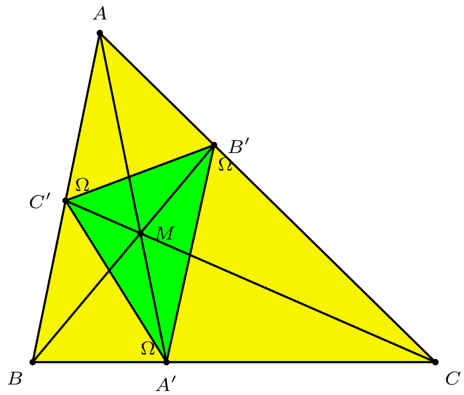

Figure 2A

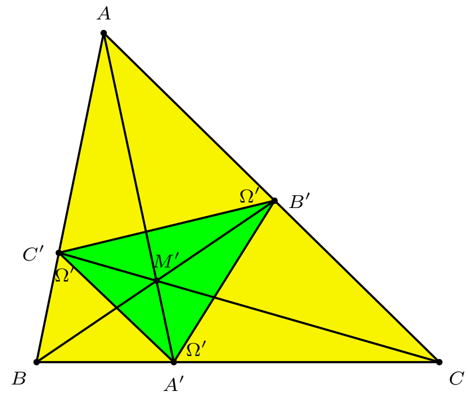

Figure 2B

contradicting the cevian concurrence condition

$$\sin^3 \Omega \quad = \quad \sin(A + \Omega)\ \sin(B + \Omega)\ \sin(C + \Omega). \tag{3}$$

Thus Ω, and similarly Ω', are acute.

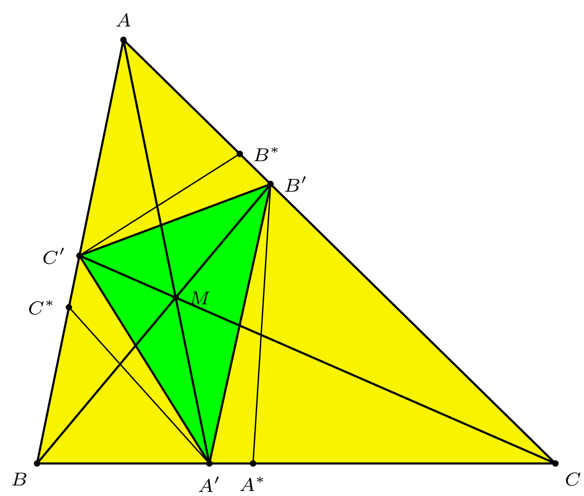

Figure 3.

It remains to prove that $\Omega' = \Omega$. Let $A'B'C'$ be the cevian triangle of M, and suppose that $\Omega' < \Omega$. Then there exist, as shown in Figure 3, points B^*, C^*, and A^* on the line segments $A'C$, $B'A$, and $C'B$, respectively, such that

$$\angle AC'B^* = \angle BA'C^* = \angle CB'A^* = \Omega'.$$

Then

$$\begin{aligned} 1 &= \frac{AB'}{B'C}\cdot\frac{CA'}{A'B}\cdot\frac{BC'}{C'A} > \frac{AB^*}{B'C}\cdot\frac{CA^*}{A'B}\cdot\frac{BC^*}{C'A} = \frac{AB^*}{AC'}\cdot\frac{CA^*}{CB'}\cdot\frac{BC^*}{BA'} \\ &= \frac{\sin\Omega'}{\sin(A+\Omega')}\cdot\frac{\sin\Omega'}{\sin(C+\Omega')}\cdot\frac{\sin\Omega'}{\sin(B+\Omega')}. \end{aligned}$$

This contradicts the cevian concurrence condition

$$\sin^3\Omega' \quad = \quad \sin(A+\Omega')\sin(B+\Omega')\sin(C+\Omega')$$

for M'. $\square$

The points M and M' in Theorem 1 will be called the *first* and *second cevian Brocard points* and the common value of Ω and Ω' the *cevian Brocard angle*.

3. An alternative proof of Theorem 1

An alternative proof of Theorem 1 can be obtained by noting that the existence and uniqueness of M are equivalent to the existence and uniqueness of a positive solution $\Omega < \min\{\pi - A, \pi - B, \pi - C\}$ of (3). Letting $u = \sin\Omega$, $U = \cos\Omega$, and $T = U/u = \cot\Omega$, and setting

$$\begin{aligned} c_0 &= \sin A\sin B\sin C, \\ c_1 &= \cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C, \\ c_2 &= \cos A\cos B\sin C + \cos A\sin B\cos C + \sin A\cos B\cos C, \\ c_3 &= \cos A\cos B\cos C, \end{aligned}$$

(3) simplifies into

$$u^3 = c_0U^3 + c_1U^2u + c_2Uu^2 + c_3u^3. \tag{4}$$

Using the formulas

$$c_2 = c_0 \quad \text{and} \quad c_1 = c_3 + 1 \tag{5}$$

taken from [5, Formulas 674 and 675, page 165], this further simplifies into

$$\begin{aligned} u^3 &= c_0U^3 + (c_3+1)U^2u + c_0Uu^2 + c_3u^3 \\ &= c_0U(U^2+u^2) + c_3u(U^2+u^2) + U^2u \\ &= c_0U + c_3u + U^2u \\ &= u(c_0T + c_3 + U^2). \end{aligned}$$

Since $u^2 = \frac{1}{1+T^2}$ and $U^2 = \frac{T^2}{1+T^2}$, this in turn reduces to $f(T) = 0$, where

$$f(X) = c_0X^3 + (c_3+1)X^2 + c_0X + (c_3 - 1). \tag{6}$$

Arguing as in the proof of Theorem 1 that Ω must be acute, we restrict our search to the interval $\Omega \in [0, \pi/2]$, i.e., to $T \in [0, \infty)$. On this interval, f is clearly increasing. Also, $f(0) < 0$ and $f(\infty) > 0$. Therefore f has a unique zero in $[0, \infty)$. This proves the existence and uniqueness of M. A similar treatment of M' leads to the same f, proving that M' exists and is unique, and that $\Omega = \Omega'$.

This alternative proof of Theorem 1 has the advantage of exhibiting the defining polynomial of $\cot\Omega$, which is needed in proving Theorems 2 and 3.

4. The cevian Brocard angle

Theorem 2. *Let Ω be the cevian Brocard angle of triangle ABC.*
(i) $\cot\Omega$ *satisfies the polynomial f given in* (6), *where $c_0 = \sin A \sin B \sin C$ and $c_3 = \cos A \cos B \cos C$.*
(ii) $\Omega \leq \pi/3$ *for all triangles.*
(iii) Ω *takes all values in* $(0, \pi/3]$.

Proof. (i) follows from the alternative proof of Theorem 1 given in the preceding section.

To prove (ii), it suffices to prove that $f(1/\sqrt{3}) \leq 0$ for all triangles ABC. Let

$$G = f\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}\sin A \sin B \sin C + \frac{4}{3}\cos A \cos B \cos C - \frac{2}{3}.$$

Then $G = 0$ if ABC is equilateral, and hence it is enough to prove that G attains its maximum at such a triangle. To see this, take a non-equilateral triangle ABC. Then we may assume that $A > B$ and $C < \pi/2$. If we replace ABC by the triangle whose angles are $(A+B)/2$, $(A+B)/2$, and C, then G increases. This follows from

$$2\sin A \sin B = \cos(A-B) - \cos(A+B) < 1 - \cos(A+B) = 2\sin^2\frac{A+B}{2},$$
$$2\cos A \cos B = \cos(A-B) + \cos(A+B) < 1 + \cos(A+B) = 2\cos^2\frac{A+B}{2}.$$

Thus G attains its maximal value, 0, at equilateral triangles, and hence $G \leq 0$ for all triangles, as desired.

To prove (iii), we let $S = \tan\Omega = 1/T$ and we see that S is a zero of the polynomial $F(X) = c_0 + (c_3+1)X + c_0X^2 + (c_3-1)X^3$. The non-negative zero of F when ABC is degenerate, i.e., when $c_0 = 0$, is 0. By continuity of the zeros of polynomials, we conclude that $\tan\Omega$ can be made arbitrarily close to 0 by taking a triangle whose c_0 is close enough to 0. Note that $c_3 - 1$ is bounded away from zero since $c_3 \leq 3\sqrt{3}/8$ for all triangles. $\square$

Remarks. (1) Unlike the Brocard angle ω, the cevian Brocard angle Ω is not necessarily Euclidean constructible. To see this, take the triangle ABC with $A = \pi/2$, and $B = C = \pi/4$. Then $c_3 = 0$, $c_0 = 1/2$, and $2f(T) = T^3 + 2T^2 + T - 2$. This is irreducible over $\mathbb{Z}$ since none of ±1 and ±2 is a zero of f, and therefore it is the minimal polynomial of $\cot\ \Omega$. Since it is of degree 3, it follows that $\cot\ \Omega$, and hence the angle Ω, is not constructible.

(2) By the cevian concurrence condition, the Brocard angle ω is defined by

$$\sin^3\omega \quad = \quad \sin(A-\omega)\ \sin(B-\omega)\ \sin(C-\omega). \tag{7}$$

Letting $v = \sin\omega$, $V = \cos\omega$ and $t = \cot\omega$ as before, we obtain

$$v^3 \quad = \quad c_0V^3 - c_1V^2v + c_2Vv^2 - c_3v^3. \tag{8}$$

This reduces to the very simple form $g(t) = 0$, where

$$g(X) \quad = \quad c_0X - c_3 - 1, \tag{9}$$

showing that

$$t = \cot\omega = \frac{1 + c_3}{c_0} = \frac{c_1}{c_0} = \cot A + \cot B + \cot C, \tag{10}$$

as is well known, and exhibiting the trivial constructibility of ω. This heavy contrast with the non-constructiblity of Ω is rather curious in view of the great formal similarity between (3) and (4) on the one hand and (7) and (8) on the other.

The next theorem shows that a triangle is completely determined, up to similarity, by its Brocard and cevian Brocard angles. This implies, in particular, that Ω and ω are independent of each other, since neither of them is sufficient for determining the shape of the triangle.

Theorem 3. *If two triangles have equal Brocard angles and equal cevian Brocard angles, then they are similar.*

Proof. Let ω and Ω be the Brocard and cevian Brocard angles of triangle ABC, and let $t = \cot\omega$ and $T = \cot\Omega$. From (10) it follows that $t = c_1/c_0$ and therefore $c_1 = tc_0$. Substituting this in (6), we see that $c_0(T + t)(T^2 + 1) = 2$, and therefore

$$c_0 = \frac{2}{(T + t)(T^2 + 1)}, \quad \text{and} \quad c_1 = \frac{2t}{(T + t)(T^2 + 1)}.$$

Letting s_1, s_2, and s_3 be the elementary symmetric polynomials in $\cot A$, $\cot B$, and $\cot C$, we see that

$$\begin{aligned}
s_1 &= \cot A + \cot B + \cot C = t,\\
s_2 &= \cot A \cot B + \cot B \cot C + \cot C \cot A = \frac{c_2}{c_0} = 1,\\
s_3 &= \cot A \cot B \cot C = \frac{c_3}{c_1} = \frac{c_1 - 1}{c_1} = 1 - \frac{(T + t)(T^2 + 1)}{2t}.
\end{aligned}$$

Since the angles of ABC are completely determined by their cotangents, which in turn are nothing but the zeros of $X^3 - s_1X^2 + s_2X - s_3$, it follows that the angles of ABC are determined by t and T, as claimed. □

5. Some properties of the cevian Brocard points

It is easy to see that the first and second Brocard points coincide if and only if the triangle is equilateral. The same holds for the cevian Brocard points. The next theorem deals with the cases when a Brocard point and a cevian Brocard point coincide. We use the following simple theorem.

Theorem 4. *If the cevians AA', BB', and CC' through a point P inside triangle ABC have the property that two of the quadrilaterals $AC'PB'$, $BA'PC'$, $CB'PA'$, $ABA'B'$, $BCB'C'$, and $CAC'A'$ are cyclic, then P is the orthocenter of ABC. If, in addition, P is a Brocard point, then ABC is equilateral.*

Proof. The first part is nothing but [4, Theorem 4] and is easy to prove. The second part follows from $\omega = \pi/2 - A = \pi/2 - B = \pi/2 - C$. $\square$

Theorem 5. *If any of the Brocard points L and L' of triangle ABC coincides with any of its cevian Brocard points M and M', then ABC is equilateral.*

Proof. Let AA', BB', and CC' be the cevians through L, and let ω and Ω be the Brocard and cevian Brocard angles of ABC; see Figure 4A. By the exterior angle theorem, $\angle ALB' = \omega + (B - \omega) = B$. Similarly, $\angle BLC' = C$ and $\angle CLA' = A$.

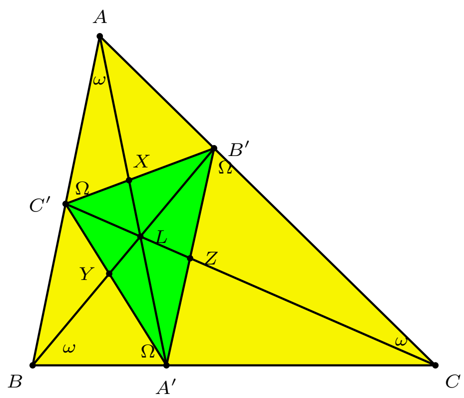

Figure 4A

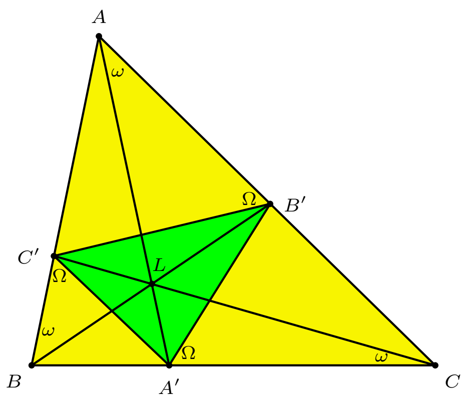

Figure 4B

Suppose that $L = M$. Then $(A', B', C') = (C, A, B)$. Referring to Figure 4A, let X, Y, and Z be the points where AA', BB', and CC' meet $B'C'$, $C'A'$, and $A'B'$, respectively. It follows from $\angle ALB' = B = C'$ and its iterates that the quadrilaterals $XC'YL$, $YA'ZL$, and $ZB'XL$ are cyclic. By Theorem 4, L is the orthocenter of $A'B'C'$. Therefore $\omega + \Omega = \pi/2$. Since $\omega \le \pi/6$ and $\Omega \le \pi/3$, it follows that $\omega = \pi/6$ and $\Omega = \pi/3$. Thus the Brocard and cevian Brocard angles of ABC coincide with those for an equilateral triangle. By Theorem 3, ABC is equilateral.

Suppose next that $L = M'$. Referring to Figure 4B, we see that $\angle AB'C' = \angle ACC' + \angle B'C'C$, and therefore $\angle B'C'C = \Omega - \omega$. Similarly $\angle C'A'A = \angle A'B'B = \Omega - \omega$. Therefore L is the second Brocard point of $A'B'C'$. Since $(A', B', C') = (B, C, A)$, it follows that ABC and $A'B'C'$ have the same Brocard angles. Therefore $\angle BAA' = \angle BB'A'$ and $ABA'B'$ is cyclic. The same holds for the quadrilaterals $BCB'C'$ and $CAC'A'$. By Theorem 4, ABC is equilateral. $\square$

The following theorem answers questions that are raised naturally in the proof of Theorem 5. It also restates Theorem 5 in terms of the Brocard points without reference to the cevian Brocard points.

Theorem 6. *Let L be the first Brocard point of ABC, and let AA', BB', and CC' be the cevians through L. Then L coincides with one of the two Brocard points N and N' of $A'B'C'$ if and only if ABC is equilateral. The same holds for the second Brocard point L'.*

Proof. Let the angles of $A'B'C'$ be denoted by A', B', and C'. The proof of Theorem 5 shows that the condition $L = N'$ is equivalent to $L = M'$, which in turn implies that ABC is equilateral. This leaves us with the case $L = N$. In this case, let ω and μ be the Brocard angles of ABC and $A'B'C'$, respectively, as shown in Figure 5. The exterior angle theorem shows that

$$A = \pi - \angle AC'B' - \angle AB'C' = \pi - (\mu + B - \omega) - (\omega + C' - \mu) = \pi - B - C'.$$

Thus $C = C'$. Similarly, $A = A'$ and $B = B'$. Therefore $\mu = \omega$, and the quadrilaterals $AC'LB'$ and $BA'LC'$ are cyclic. By Theorem 4, ABC is equilateral. $\square$

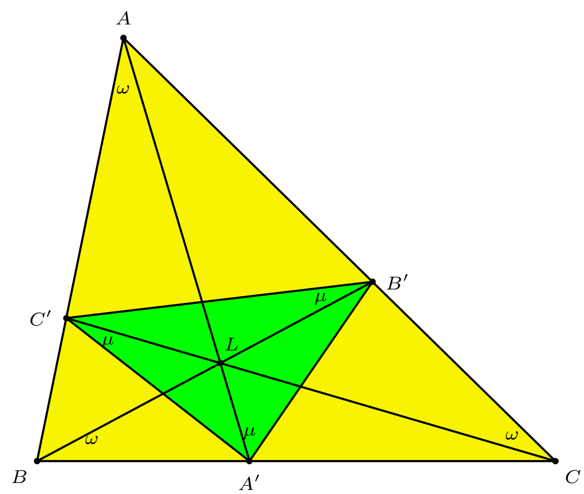

Figure 5

Remark. (3) It would be interesting to investigate whether the many inequalities involving the Brocard angle, such as Yff's inequality [1], have analogues for the cevian Brocard angles, and whether there are inequalities that involve both the Brocard and cevian Brocard angles. Similar questions can be asked about other properties of the Brocard points. For inequalities involving the Brocard angle, we refer the reader to [2] and [9, pp.329-333] and the references therein.

6. A characterization of some common triangle centers

We close with a theorem that complements Theorems 1 and 2 of [3].

Theorem 7. *The triangle centers for which the angles α, β, γ are linear forms in A, B, C are the centroid, the orthocenter, and the Gergonne point.*

Proof. Arguing as in Theorems 1 and 2 of [3], we see that α, β, γ are of the form

$$\alpha = \frac{\pi - A}{2} + t(B - C), \quad \beta = \frac{\pi - B}{2} + t(C - A), \quad \gamma = \frac{\pi - C}{2} + t(A - B).$$

In particular, $\alpha + \beta + \gamma = \pi$, and therefore

$$4 \sin\alpha \sin\beta \sin\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma;$$

see [5, Formula 681, p. 166]. Thus the Ceva's concurrence relation takes the form

$$\begin{aligned} &\sin(A-2t(B-C))+\sin(B-2t(C-A))+\sin(C-2t(A-B)) \\ =&\sin(A+2t(B-C))+\sin(B+2t(C-A))+\sin(C+2t(A-B)), \end{aligned}$$

which reduces to

$$\cos A\sin(2t(B-C))+\cos B\sin(2t(C-A))+\cos C\sin(2t(A-B))=0.$$

Following word by word the way equation (5) of [3] was treated, we conclude that $t=-1/2$, $t=0$, or $t=1/2$.

If $t=0$, then $\alpha=(\pi-A)/2$, and therefore $\alpha=\alpha'$ and $AB'=AC'$. Thus A', B', and C' are the points of contact of the incircle, and the point of intersection of AA', BB', and CC' is the Gergonne point.

If $t=1/2$, then $(\alpha,\beta,\gamma)=(B,C,A)$, and $(A',B',C')=(A,B,C)$. This clearly corresponds to the centroid.

If $t=-1/2$, then $(\alpha,\beta,\gamma)=(C,A,B)$, and $(A',B',C')=(\pi-A,\pi-B,\pi-C)$. This clearly corresponds to the orthocenter. $\square$

Remarks. (4) In establishing the parts pertaining to the centroid and the orthocenter in Theorem 7, we have used the uniqueness component of Seebach's theorem. Alternative proofs that do not use Seebach's theorem follow from [4, Theorems 4 and 7].

(5) In view of the proof of Theorem 7, it is worth mentioning that the proof of Theorem 2 of [3] can be simplified by noting that $\xi+\eta+\zeta=\pi/2$ and using the identity

$$1+4\sin\xi\sin\eta\sin\zeta=\cos 2\xi+\cos 2\eta+\cos 2\zeta$$

given in [5, Formula 678, p. 166].

(6) It is clear that the first and second cevian Brocard points of triangle ABC can be equivalently defined as the points whose cevian triangles $A'B'C'$ have the properties that $(A',B',C')=(C,A,B)$ and $(A',B',C')=(B,C,A)$, respectively. The point corresponding to the requirement that $(A',B',C')=(A,B,C)$ is the centroid; see [6] and [4, Theorem 7]. It would be interesting to explore the point defined by the condition $(A',B',C')=(A,C,B)$.

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Forum Geometricorum
Volume 5 (2005) 75–96.

FORUM GEOM
ISSN 1534-1178

Elegant Geometric Constructions

Paul Yiu

Dedicated to Professor M. K. Siu

Abstract. With the availability of computer software on dynamic geometry, beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry, and focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®.

After a half century of curriculum reforms, it is fair to say that mathematicians and educators have come full circle in recognizing the relevance of Euclidean geometry in the teaching and learning of mathematics. For example, in [15], J. E. McClure reasoned that "Euclidean geometry is the only mathematical subject that is really in a position to provide the grounds for its own axiomatic procedures". See also [19]. Apart from its traditional role as the training ground for logical reasoning, Euclidean geometry, with its construction problems, provides a stimulating milieu of learning mathematics *constructivistically*. One century ago, D. E. Smith [17, p.95] explained that the teaching of constructions using ruler and compass serves several purposes: "it excites [students'] interest, it guards against the slovenly figures that so often lead them to erroneous conclusions, it has a genuine value for the future artisan, and its shows that geometry is something besides mere theory". Around the same time, the British Mathematical Association [16] recommended teaching school geometry as two parallel courses of *Theorems* and *Constructions*. "The course of constructions should be regarded as a *practical*

Publication Date: June 18, 2005. Quest Editor: Ngai Ying Wong.
This paper also appears in N. Y. Wong et al (ed.), *Revisiting Mathematics Education in Hong Kong for the New Millennium*, pp.173–203, Hong Kong Association for Mathematics Education, 2005.

course, the constructions being accurately made with instruments, and no construction, or proof of a construction, should be deemed invalid by reason of its being different from that given in Euclid, or by reason of its being based on theorems which Euclid placed after it".

A good picture is worth more than a thousand words. This is especially true for students and teachers of geometry. With good illustrations, concepts and problems in geometry become transparent and more understandable. However, the difficulty of drawing good blackboard geometric sketches is well appreciated by every teacher of mathematics. It is also true that many interesting problems on constructions with ruler and compass are genuinely difficult and demand great insights for solution, as in the case of geometrical proofs. Like handling difficult problems in synthetic geometry with analytic geometry, one analyzes construction problems by the use of algebra. It is well known that historically analysis of such ancient construction problems as the trisection of an angle and the duplication of the cube gave rise to the modern algebraic concept of field extension. A geometric construction can be effected with ruler and compass if and only if the corresponding algebraic problem is reducible to a sequence of linear and quadratic equations with constructible coefficients. For all the strength and power of such algebraic analysis of geometric problems, it is often impractical to carry out detailed constructions with paper and pencil, so much so that in many cases one is forced to settle for mere constructibility. For example, Howard Eves, in his solution [6] of the problem of construction of a triangle given the lengths of a side and the median and angle bisector on the same side, made the following remark after proving constructibility.

> The devotee of the game of Euclidean constructions is not really interested in the actual mechanical construction of the sought triangle, but merely in the assurance that the construction is possible. To use a phrase of Jacob Steiner, the devotee performs his construction "simply by means of the tongue" rather than with actual instruments on paper.

Now, the availability in recent years of computer software on dynamic geometry has brought about a change of attitude. Beautiful and accurate geometric diagrams can be drawn, edited, and organized efficiently on computer screens. This new technological capability stimulates the desire to strive for elegance in actual geometric constructions. The present paper advocates a closer examination of the geometric meaning of the algebraic expressions in the analysis of a construction problem to actually effect a construction as elegantly and efficiently as possible on the computer screen. [1] We present a fantasia of euclidean constructions the analysis of which make use of elementary algebra and very basic knowledge of euclidean geometry. [2] We focus on incorporating simple algebraic expressions into actual constructions using the Geometer's Sketchpad®. The tremendous improvement

[1]See §6.1 for an explicit construction of the triangle above with a given side, median, and angle bisector.

[2]The Geometer's Sketchpad® files for the diagrams in this paper are available from the author's website http://www.math.fau.edu/yiu/Geometry.html.

on the economy of time and effort is hard to exaggerate. The most remarkable feature of the Geometer's Sketchpad® is the capability of customizing a tool folder to make constructions as efficiently as one would like. Common, basic constructions need only be performed once, and saved as tools for future use. We shall use the Geometer's Sketchpad® simply as ruler and compass, assuming a tool folder containing at least the following tools[3] for ready use:
(i) basic shapes such as equilateral triangle and square,
(ii) tangents to a circle from a given point,
(iii) circumcircle and incircle of a triangle.

Sitting in front of the computer screen trying to perform geometric constructions is a most ideal constructivistic learning environment: a student is to bring his geometric knowledge and algebraic skill to bear on natural, concrete but challenging problems, experimenting with various geometric interpretations of concrete algebraic expressions. Such analysis and explicit constructions provide a fruitful alternative to the traditional emphasis of the deductive method in the learning and teaching of geometry.

1. Some examples

We present a few examples of constructions whose elegance is suggested by an analysis a little more detailed than is necessary for constructibility or routine constructions. A number of constructions in this paper are based on diagrams in the interesting book [9]. We adopt the following notation for circles:
(i) $A(r)$ denotes the circle with center A, radius r;
(ii) $A(B)$ denotes the circle with center A, passing through the point B, and
(iii) (A) denotes a circle with center A and unspecified radius, but unambiguous in context.

1.1. Construct a regular octagon by cutting corners from a square.

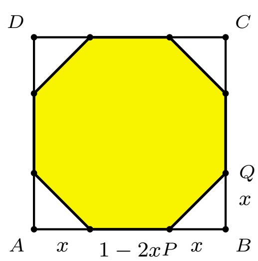

Figure 1A

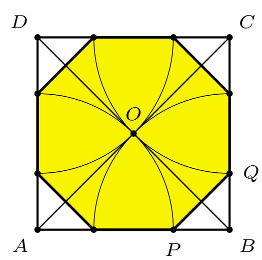

Figure 1B

Suppose an isosceles right triangle of (shorter) side x is to be cut from each corner of a unit square to make a regular octagon. See Figure 1A. A simple calculation shows that $x = 1 - \frac{\sqrt{2}}{2}$. This means $AP = 1 - x = \frac{\sqrt{2}}{2}$. The point P, and the

[3] A construction appearing in sans serif is assumed to be one readily performable with a customized tool.

other vertices, can be easily constructed by intersecting the sides of the square with quadrants of circles with centers at the vertices of the square and passing through the center O. See Figure 1B.

1.2. The centers A and B of two circles lie on the other circle. Construct a circle tangent to the line AB, to the circle (A) internally, and to the circle (B) externally.

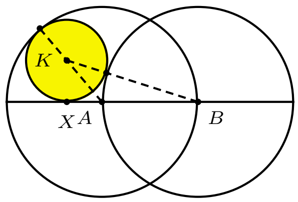

Figure 2A

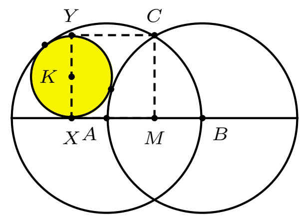

Figure 2B

Suppose $AB = a$. Let r = radius of the required circle (K), and $x = AX$, where X is the projection of the center K on the line AB. We have

$$(a+r)^2 = r^2 + (a+x)^2, \qquad (a-r)^2 = r^2 + x^2.$$

Subtraction gives $4ar = a^2 + 2ax$ or $x + \frac{a}{2} = 2r$. This means that in Figure 2B, $CMXY$ is a square, where M is the midpoint of AB. The circle can now be easily constructed by first erecting a square on CM.

1.3. *Equilateral triangle in a rectangle.* Given a rectangle $ABCD$, construct points P and Q on BC and CD respectively such that triangle APQ is equilateral.

Construction 1. *Construct* equilateral triangles CDX *and* BCY, *with* X *and* Y *inside the rectangle. Extend* AX *to intersect* BC *at* P *and* AY *to intersect* CD *at* Q.

The triangle APQ *is equilateral.* See Figure 3B.

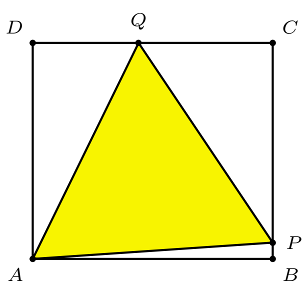

Figure 3A

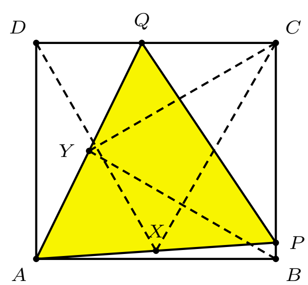

Figure 3B

This construction did not come from a lucky insight. It was found by an analysis. Let $AB = DC = a$, $BC = AD = b$. If $BP = y$, $DQ = x$ and APQ is equilateral, then a calculation shows that $x = 2a - \sqrt{3}b$ and $y = 2b - \sqrt{3}a$. From these expressions of x and y the above construction was devised.

1.4. *Partition of an equilateral triangle into 4 triangles with congruent incircles.* Given an equilateral triangle, construct three lines each through a vertex so that the incircles of the four triangles formed are congruent. See Figure 4A and [9, Problem 2.1.7] and [10, Problem 5.1.3], where it is shown that if each side of the equilateral triangle has length a, then the small circles all have radii $\frac{1}{8}(\sqrt{7}-\sqrt{3})a$. Here is a calculation that leads to a very easy construction of these lines.

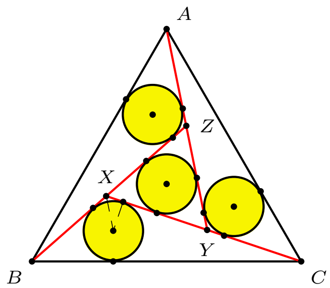

Figure 4A

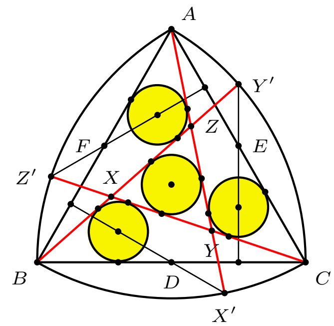

Figure 4B

In Figure 4A, let $CX = AY = BZ = a$ and $BX = CY = AZ = b$. The equilateral triangle XYZ has sidelength $a-b$ and inradius $\frac{\sqrt{3}}{6}(a-b)$. Since $\angle BXC = 120°$, $BC = \sqrt{a^2+ab+b^2}$, and the inradius of triangle BXC is

$$\frac{1}{2}(a+b-\sqrt{a^2+ab+b^2})\tan 60° = \frac{\sqrt{3}}{2}(a+b-\sqrt{a^2+ab+b^2}).$$

These two inradii are equal if and only if $3\sqrt{a^2+ab+b^2} = 2(a+2b)$. Applying the law of cosines to triangle XBC, we obtain

$$\cos XBC = \frac{(a^2+ab+b^2)+b^2-a^2}{2b\sqrt{a^2+ab+b^2}} = \frac{a+2b}{2\sqrt{a^2+ab+b^2}} = \frac{3}{4}.$$

In Figure 4B, Y' is the intersection of the arc $B(C)$ and the perpendicular from the midpoint E of CA to BC. The line BY' makes an angle $\arccos\frac{3}{4}$ with BC. The other two lines AX' and CZ' are similarly constructed. These lines bound the equilateral triangle XYZ, and the four incircles can be easily constructed. Their centers are simply the reflections of X' in D, Y' in E, and Z' in F.

2. Some basic constructions

2.1. *Geometric mean and the solution of quadratic equations.* The following constructions of the geometric mean of two lengths are well known.

Construction 2. (a) *Given two segments of length a, b, mark three points A, P, B on a line (P between A and B) such that $PA = a$ and $PB = b$. Describe a semicircle with AB as diameter, and let the perpendicular through P intersect the semicircle at Q. Then $PQ^2 = AP \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5A.

(b) *Given two segments of length $a < b$, mark three points P, A, B on a line such that $PA = a$, $PB = b$, and A, B are on the* same *side of P. Describe a semicircle with PB as diameter, and let the perpendicular through A intersect the semicircle at Q. Then $PQ^2 = PA \cdot PB$, so that the length of PQ is the geometric mean of a and b.* See Figure 5B.

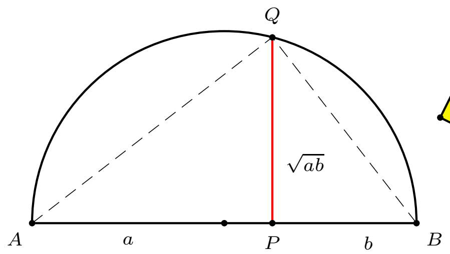

Figure 5A

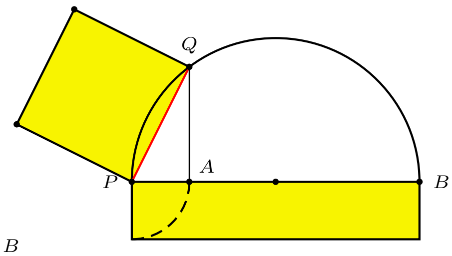

Figure 5B

More generally, a quadratic equation can be solved by applying the theorem of intersecting chords: *If a line through P intersects a circle $O(r)$ at X and Y, then the product $PX \cdot PY$ (of signed lengths) is equal to $OP^2 - r^2$.* Thus, if two chords AB and XY intersect at P, then $PA \cdot PB = PX \cdot PY$. See Figure 6A. In particular, if P is outside the circle, and if PT is a tangent to the circle, then $PT^2 = PX \cdot PY$ for any line intersecting the circle at X and Y. See Figure 6B.

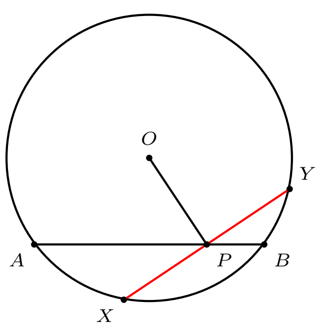

Figure 6A

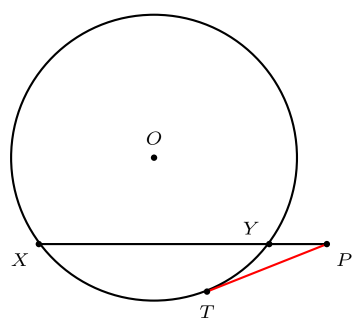

Figure 6B

A quadratic equation can be put in the form $x(x \pm a) = b^2$ or $x(a - x) = b^2$. In the latter case, for real solutions, we require $b \leq \frac{a}{2}$. If we arrange a and b as the legs of a right triangle, then the positive roots of the equation can be easily constructed as in Figures 6C and 6D respectively.

The algebraic method of the solution of a quadratic equation by completing squares can be easily incorporated geometrically by using the Pythagorean theorem. We present an example.

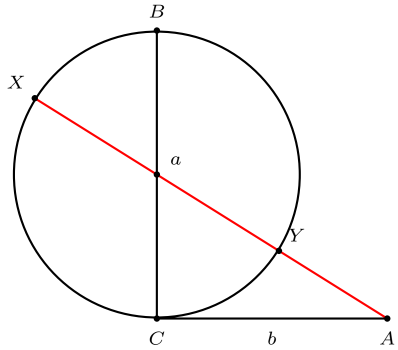

Figure 6C

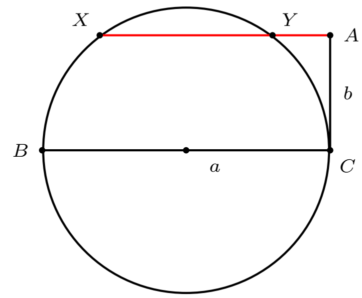

Figure 6D

2.1.1. Given a chord BC perpendicular to a diameter XY of circle (O), to construct a line through X which intersects the circle at A and BC at T such that AT has a given length t. Clearly, $t \leq YM$, where M is the midpoint of BC.

Let $AX = x$. Since $\angle CAX = \angle CYX = \angle TCX$, the line CX is tangent to the circle ACT. It follows from the theorem of intersecting chords that $x(x-t) = CX^2$. The method of completing squares leads to

$$x = \frac{t}{2} + \sqrt{CX^2 + \left(\frac{t}{2}\right)^2}.$$

This suggests the following construction.[4]

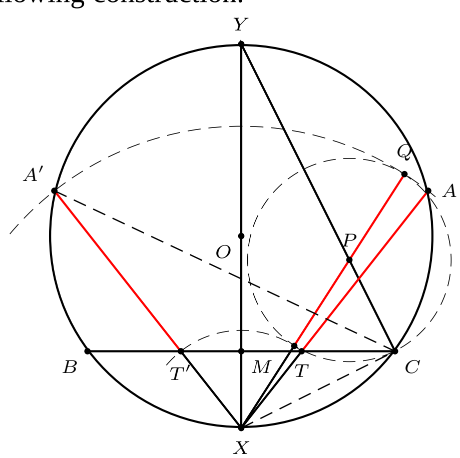

Figure 7

Construction 3. *On the segment CY, choose a point P such that $CP = \frac{t}{2}$. Extend XP to Q such that $PQ = PC$. Let A be an intersection of $X(Q)$ and (O). If the line XA intersects BC at T, then $AT = t$.* See Figure 7.

[4] This also solves the construction problem of triangle ABC with given angle A, the lengths a of its opposite side, and of the bisector of angle A.

2.2. *Harmonic mean and the equation* $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. The harmonic mean of two quantities a and b is $\frac{2ab}{a+b}$. In a trapezoid of parallel sides a and b, the parallel through the intersection of the diagonals intercepts a segment whose length is the harmonic mean of a and b. See Figure 8A. We shall write this harmonic mean as $2t$, so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$. See Figure 8B.

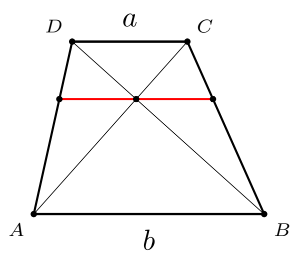

Figure 8A

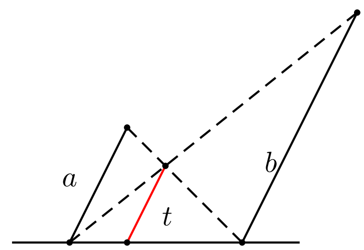

Figure 8B

Here is another construction of t, making use of the formula for the length of an angle bisector in a triangle. If $BC = a$, $AC = b$, then the angle bisector CZ has length

$$t_c = \frac{2ab}{a+b} \cos \frac{C}{2} = 2t \cos \frac{A}{2}.$$

The length t can therefore be constructed by completing the rhombus $CXZY$ (by constructing the perpendicular bisector of CZ to intersect BC at X and AC at Y). See Figure 9A. In particular, if the triangle contains a right angle, this trapezoid is a square. See Figure 9B.

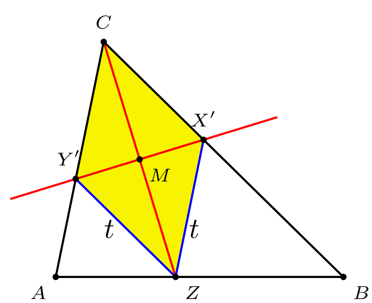

Figure 9A

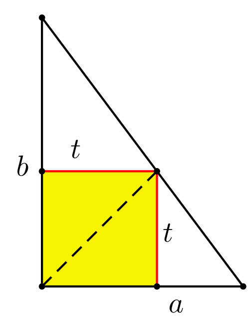

Figure 9B

3. The shoemaker's knife

3.1. *Archimedes' Theorem.* A shoemaker's knife (or arbelos) is the region obtained by cutting out from a semicircle with diameter AB the two smaller semicircles with diameters AP and PB. Let $AP = 2a$, $PB = 2b$, and the common tangent of the smaller semicircles intersect the large semicircle at Q. The following remarkable theorem is due to Archimedes. See [12].

Theorem 1 (Archimedes)**.** (1) *The two circles each tangent to PQ, the large semicircle and one of the smaller semicircles have equal radii $t = \frac{ab}{a+b}$.* See Figure 10A.

(2) *The circle tangent to each of the three semicircles has radius*

$$\rho = \frac{ab(a+b)}{a^2+ab+b^2}. \tag{1}$$

See Figure 10B.

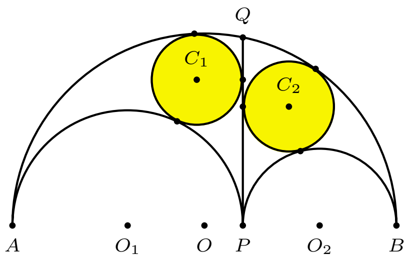

Figure 10A

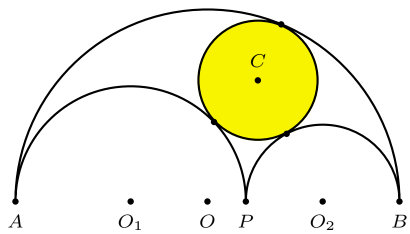

Figure 10B

Here is a simple construction of the Archimedean "twin circles". Let Q_1 and Q_2 be the "highest" points of the semicircles $O_1(a)$ and $O_2(b)$ respectively. The intersection $C_3 = O_1Q_2 \cap O_2Q_1$ is a point "above" P, and $C_3P = t = \frac{ab}{a+b}$.

Construction 4. *Construct the circle $P(C_3)$ to intersect the diameter AB at P_1 and P_2 (so that P_1 is on AP and P_2 is on PB).*

The center C_1 (respectively C_2) is the intersection of the circle $O_1(P_2)$ (respectively $O_2(P_1)$) and the perpendicular to AB at P_1 (respectively P_2). See Figure 11.

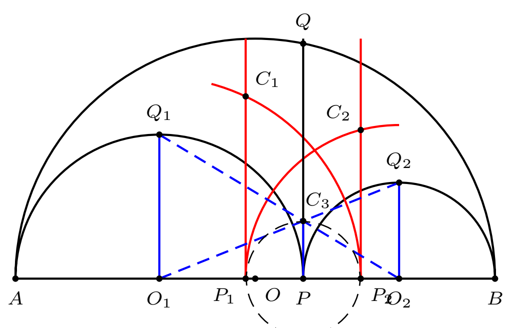

Figure 11

Theorem 2 (Bankoff [3])**.** *If the incircle $C(\rho)$ of the shoemaker's knife touches the smaller semicircles at X and Y, then the circle through the points P, X, Y has the same radius t as the Archimedean circles.* See Figure 12.

This gives a very simple construction of the incircle of the shoemaker's knife.

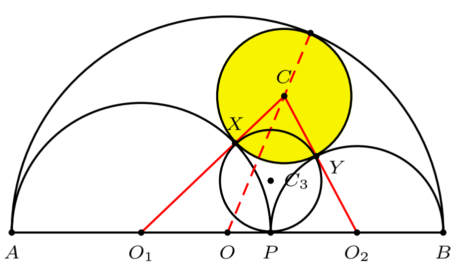

Figure 12

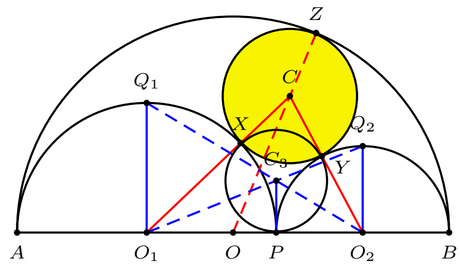

Figure 13

Construction 5. *Let $X = C_3(P) \cap O_1(a)$, $Y = C_3(P) \cap O_2(b)$, and $C = O_1X \cap O_2Y$. The circle $C(X)$ is the incircle of the shoemaker's knife. It touches the large semicircle at $Z = OC \cap O(a + b)$.* See Figure 13.

A rearrangement of (1) in the form

$$\frac{1}{a+b} + \frac{1}{\rho} = \frac{1}{t}$$

leads to another construction of the incircle (C) by directly locating the center and one point on the circle. See Figure 14.

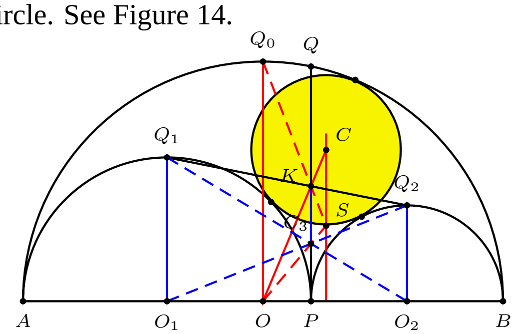

Figure 14

Construction 6. *Let Q_0 be the "highest" point of the semicircle $O(a + b)$. Construct*
(i) *$K = Q_1Q_2 \cap PQ$,*
(ii) *$S = OC_3 \cap Q_0K$, and*
(iii) *the perpendicular from S to AB to intersect the line OK at C.*

The circle $C(S)$ is the incircle of the shoemaker's knife.

3.2. *Other simple constructions of the incircle of the shoemaker's knife.* We give four more simple constructions of the incircle of the shoemaker's knife. The first is by Leon Bankoff [1]. The remaining three are by Peter Woo [21].

Construction 7 (Bankoff)**.** (1) *Construct the circle $Q_1(A)$ to intersect the semicircles $O_2(b)$ and $O(a + b)$ at X and Z respectively.*

(2) *Construct the circle $Q_2(B)$ to intersect the semicircles $O_1(a)$ and $O(a + b)$ at Y and the same point Z in* (1) *above.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 15.

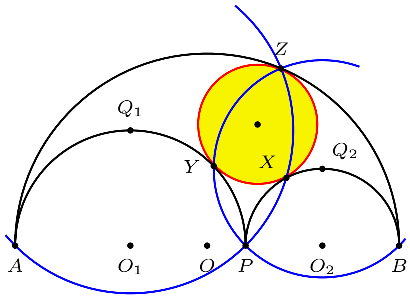

Figure 15

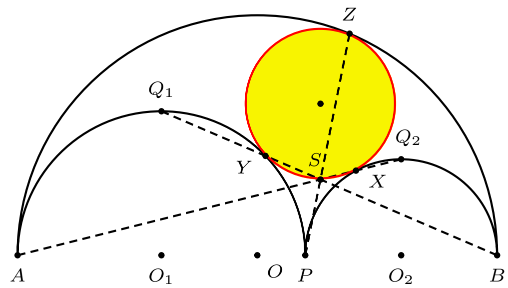

Figure 16

Construction 8 (Woo)**.** (1) *Construct the line AQ_2 to intersect the semicircle $O_2(b)$ at X.*

(2) *Construct the line BQ_1 to intersect the semicircle $O_1(a)$ at Y.*

(3) *Let $S = AQ_2 \cap BQ_1$. Construct the line PS to intersect the semicircle $O(a+b)$ at Z.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 16.

Construction 9 (Woo)**.** *Let M be the "lowest" point of the circle $O(a+b)$. Construct*

(i) *the circle $M(A)$ to intersect $O_1(a)$ at Y and $O_2(b)$ at X,*

(ii) *the line MP to intersect the semicircle $O(a+b)$ at Z.*

The circle through X, Y, Z *is the incircle of the shoemaker's knife.* See Figure 17.

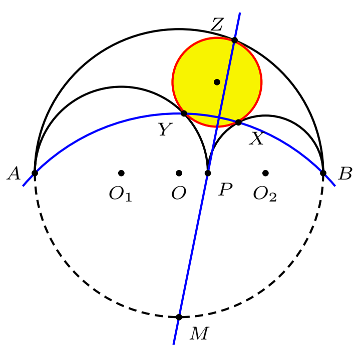

Figure 17

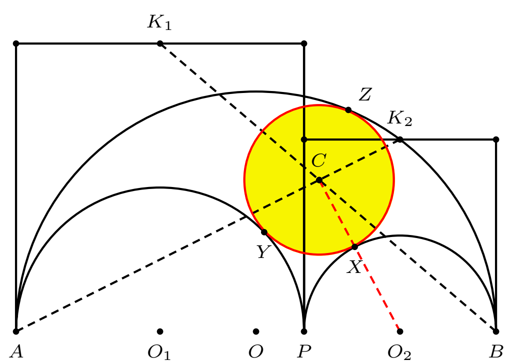

Figure 18

Construction 10 (Woo)**.** *Construct* squares *on AP and PB on the same side of the shoemaker knife. Let K_1 and K_2 be the midpoints of the opposite sides of AP and PB respectively. Let $C = AK_2 \cap BK_1$, and $X = CO_2 \cap O_2(b)$.*

The circle $C(X)$ is the incircle of the shoemaker's knife. See Figure 18.

4. Animation of bicentric polygons

A famous theorem of J. V. Poncelet states that if between two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ there is a polygon of n sides with vertices on $\mathcal{C}_1$ and sides tangent to $\mathcal{C}_2$, then there is one such polygon of n sides with a vertex at an arbitrary point on $\mathcal{C}_1$. See, for example, [5]. For circles $\mathcal{C}_1$ and $\mathcal{C}_2$ and for $n = 3, 4$, we illustrate this theorem by constructing animation pictures based on simple metrical relations.

4.1. *Euler's formula.* Consider the construction of a triangle given its circumcenter O, incenter I and a vertex A. The circumcircle is $O(A)$. If the line AI intersects this circle again at X, then the vertices B and C are simply the intersections of the circles $X(I)$ and $O(A)$. See Figure 19A. This leads to the famous Euler formula

$$d^2 = R^2 - 2Rr, \tag{2}$$

where d is the distance between the circumcenter and the incenter.[5]

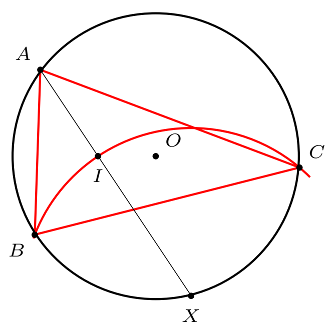

Figure 19A

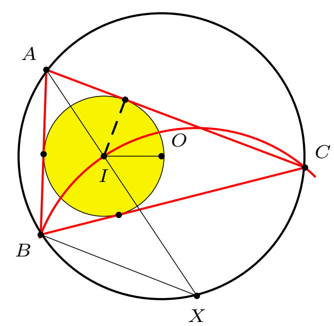

Figure 19B

4.1.1. Given a circle $O(R)$ and $r < \frac{R}{2}$, to construct a point I such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 11. *Let $P(r)$ be a circle tangent to (O) internally. Construct a line through O* tangent to the circle *$P(r)$ at a point I.*

The circle $I(r)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 20.

---

[5]*Proof*: If I is the incenter, then $AI = \frac{r}{\sin\frac{A}{2}}$ and $IX = IB = \frac{2R}{\sin\frac{A}{2}}$. See Figure 19B. The power of I with respect to the circumcircle is $d^2 - R^2 = IA \cdot IX = -r \sin\frac{A}{2} \cdot \frac{2R}{\sin\frac{A}{2}} = -2Rr$.

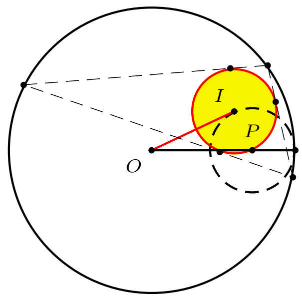

Figure 20

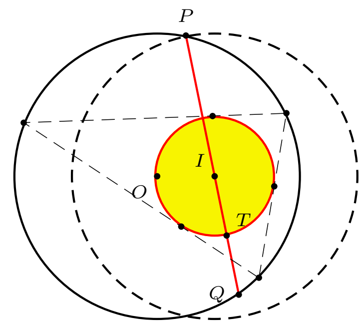

Figure 21

4.1.2. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a triangle.

Construction 12. *Construct the circle $I(R)$ to intersect $O(R)$ at a point P, and construct the line PI to intersect $O(R)$ again at Q. Let T be the midpoint of IQ.*

The circle $I(T)$ is the incircle of triangles which have $O(R)$ as circumcircle. See Figure 21.

4.1.3. Given a circle $I(r)$ and a point O, to construct a circle $O(R)$ which is the circumcircle of triangles with $I(r)$ as incircle. Since $R = r + \sqrt{r^2 + d^2}$ by the Euler formula (2), we have the following construction. See Figure 22.

Construction 13. *Let IP be a radius of $I(r)$ perpendicular to IO. Extend OP to a point A such that $PA = r$.*

The circle $O(A)$ is the circumcircle of triangles which have $I(r)$ as incircle.

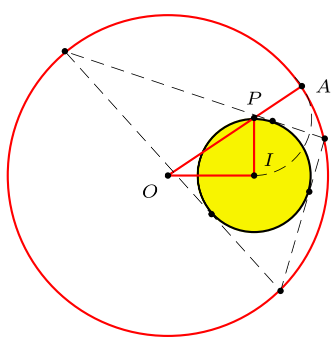

Figure 22

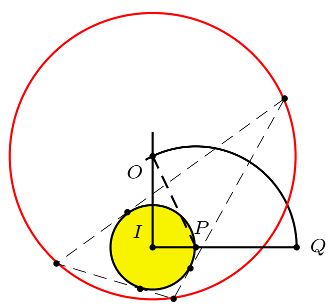

Figure 23

4.1.4. Given $I(r)$ and $R > 2r$, to construct a point O such that $O(R)$ is the circumcircle of triangles with $I(r)$ as incircle.

Construction 14. *Extend a radius IP to Q such that $IQ = R$. Construct the perpendicular to IP at I to intersect the circle $P(Q)$ at O.*

The circle $O(R)$ is the circumcircle of triangles which have $I(r)$ as incircle. See Figure 23.

4.2. *Bicentric quadrilaterals.* A bicentric quadrilateral is one which admits a circumcircle and an incircle. The construction of bicentric quadrilaterals is based on the Fuss formula

$$2r^2(R^2 + d^2) = (R^2 - d^2)^2, \tag{3}$$

where d is the distance between the circumcenter and incenter of the quadrilateral. See [7, §39].

4.2.1. Given a circle $O(R)$ and a point I, to construct a circle $I(r)$ such that $O(R)$ and $I(r)$ are the circumcircle and incircle of a quadrilateral.

The Fuss formula (3) can be rewritten as

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

In this form it admits a very simple interpretation: r can be taken as the altitude on the hypotenuse of a right triangle whose shorter sides have lengths $R \pm d$. See Figure 24.

Construction 15. *Extend IO to intersect $O(R)$ at a point A. On the perpendicular to IA at I construct a point K such that $IK = R - d$. Construct the altitude IP of the right triange AIK.*

The circles $O(R)$ and $I(P)$ are the circumcircle and incircle of bicentric quadrilaterals.

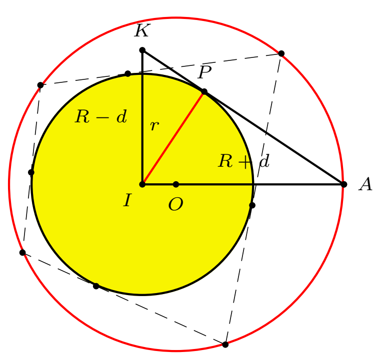

Figure 24

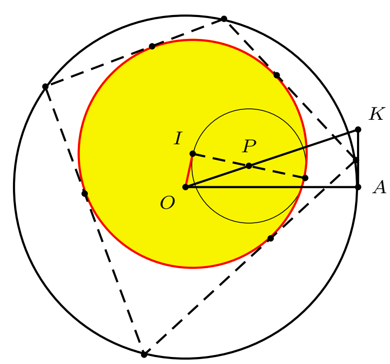

Figure 25

4.2.2. Given a circle $O(R)$ and a radius $r \leq \frac{R}{\sqrt{2}}$, to construct a point I such that $I(r)$ is the incircle of quadrilaterals inscribed in $O(R)$, we rewrite the Fuss formula (3) in the form

$$d^2 = \left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{r}{2}\right)\left(\sqrt{R^2 + \frac{r^2}{4}} - \frac{3r}{2}\right).$$

This leads to the following construction. See Figure 25.

Construction 16. *Construct a right triangle OAK with a right angle at A, $OA = R$ and $AK = \frac{r}{2}$. On the hypotenuse OK choose a point P such that $KP = r$. Construct a* tangent from O to the circle $P(\frac{r}{2})$. *Let I be the point of tangency.*

The circles $O(R)$ and $I(r)$ are the circumcircle and incircle of bicentric quadrilaterals.

4.2.3. Given a circle $I(r)$ and a point O, to construct a circle (O) such that these two circles are respectively the incircle and circumcircle of a quadrilateral. Again, from the Fuss formula (3),

$$R^2 = \left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{r}{2}\right)\left(\sqrt{d^2 + \frac{r^2}{4}} + \frac{3r}{2}\right).$$

Construction 17. *Let E be the midpoint of a radius IB perpendicular to OI. Extend the ray OE to a point F such that $EF = r$. Construct a* tangent OT to the circle $F\left(\frac{r}{2}\right)$. *Then OT is a circumradius.*

5. Some circle constructions

5.1. *Circles tangent to a chord at a given point.* Given a point P on a chord BC of a circle (O), there are two circles tangent to BC at P, and to (O) internally. The radii of these two circles are $\dfrac{BP \cdot PC}{2(R \pm h)}$, where h is the distance from O to BC. They can be constructed as follows.

Construction 18. *Let M be the midpoint of BC, and XY be the diameter perpendicular to BC. Construct*
(i) *the circle center P, radius MX to intersect the arc BXC at a point Q,*
(ii) *the line PQ to intersect the circle (O) at a point H,*
(iii) *the circle $P(H)$ to intersect the line perpendicular to BC at P at K* (*so that H and K are on the same side of BC*).

The circle with diameter PK is tangent to the circle (O). See Figure 26A.

Replacing X by Y in (i) *above we obtain the other circle tangent to BC at P and internally to (O).* See Figure 26B.

5.2. *Chain of circles tangent to a chord.* Given a circle (Q) tangent internally to a circle (O) and to a chord BC at a given point P, there are two neighbouring circles tangent to (O) and to the same chord. These can be constructed easily by observing that in Figure 27, the common tangent of the two circles cuts out a segment whose

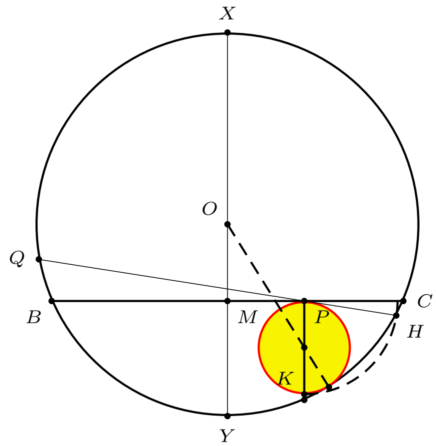

Figure 26A

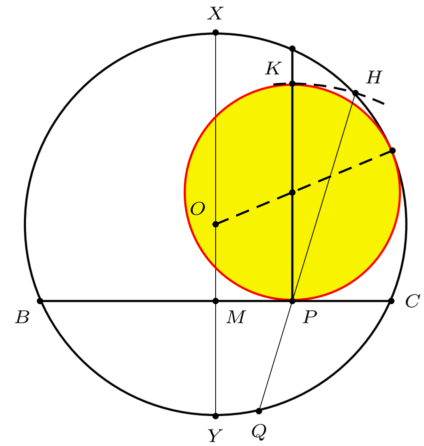

Figure 26B

midpoint is B. If (Q') is a neighbour of (Q), their common tangent passes through the midpoint M of the arc BC complementary to (Q). See Figure 28.

Construction 19. *Given a circle (Q) tangent to (O) and to the chord BC, construct*
(i) *the circle $M(B)$ to intersect (Q) at T_1 and T_2, MT_1 and MT_2 being tangents to (Q),*
(ii) *the bisector of the angle between MT_1 and BC to intersect the line QT_1 at Q_1.*

The circle $Q_1(T_1)$ is tangent to (O) and to BC.

Replacing T_1 by T_2 in (ii) *we obtain Q_2. The circle $Q_2(T_2)$ is also tangent to (O) and BC.*

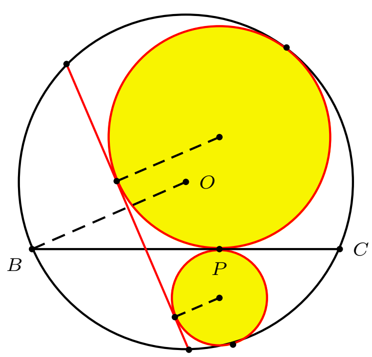

Figure 27

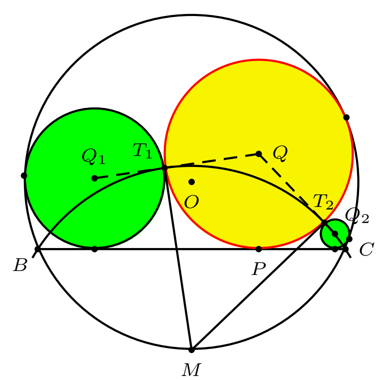

Figure 28

5.3. *Mixtilinear incircles.* Given a triangle ABC, we construct the circle tangent to the sides AB, AC, and also to the circumcircle internally. Leon Bankoff [4] called this the A- mixtilinear incircle of the triangle. Its center is clearly on the bisector of angle A. Its radius is $r\sec^2\frac{A}{2}$, where r is the inradius of the triangle. The mixtilinear incircle can be constructed as follows. See Figure 29.

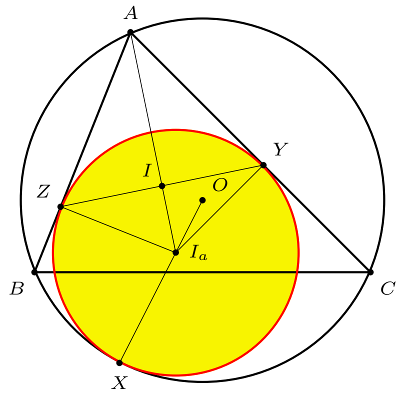

Figure 29

Construction 20 (Mixtilinear incircle)**.** *Let I be the* incenter *of triangle ABC. Construct*
(i) *the perpendicular to IA at I to intersect AC at Y,*
(ii) *the perpendicular to AY at Y to intersect the line AI at I_a.*
The circle $I_a(Y)$ is the A-mixtilinear incircle of ABC.

The other two mixtilinear incircles can be constructed in a similar way. For another construction, see [23].

5.4. *Ajima's construction.* The interesting book [10] by Fukagawa and Rigby contains a very useful formula which helps perform easily many constructions of inscribed circles which are otherwise quite difficult.

Theorem 3 (Ajima)**.** *Given triangles ABC with circumcircle (O) and a point P such that A and P are on the same side of BC, the circle tangent to the lines PB, PC, and to the circle (O) internally is the image of the incircle of triangle PBC under the homothety with center P and ratio $1+\tan\frac{A}{2}\tan\frac{BPC}{2}$.*

Construction 21 (Ajima)**.** *Given two points B and C on a circle (O) and an arbitrary point P, construct*
(i) *a point A on (O) on the same side of BC as P,* (*for example, by taking the midpoint M of BC, and intersecting the ray MP with the circle (O)*)*,*
(ii) *the* incenter *I of triangle ABC,*
(iii) *the* incenter *I' of triangle PBC,*
(iv) *the perpendicular to $I'P$ at I' to intersect PC at Z.*
(v) *Rotate the ray ZI' about Z through an* (*oriented*) *angle equal to angle BAI to intersect the line AP at Q.*
Then the circle with center Q, tangent to the lines PB and PC, is also tangent to (O) internally. See Figure 30.

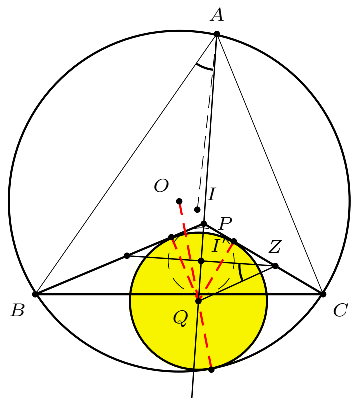

Figure 30

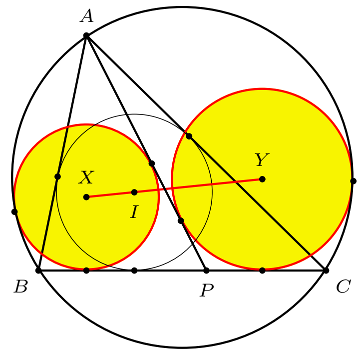

Figure 31

5.4.1. *Thébault's theorem.* With Ajima's construction, we can easily illustrate the famous Thébault theorem. See [18, 2] and Figure 31.

Theorem 4 (Thébault)**.** *Let P be a point on the side BC of triangle ABC. If the circles (X) and (Y) are tangent to AP, BC, and also internally to the circumcirle of the triangle, then the line XY passes through the incenter of the triangle.*

5.4.2. *Another example.* We construct an animation picture based on Figure 32 below. Given a segment AB and a point P, construct the squares $APX'X$ and $BPY'Y$ on the segments AP and BP. The locus of P for which A, B, X, Y are concyclic is the union of the perpendicular bisector of AB and the two quadrants of circles with A and B as endpoints. Consider P on one of these quadrants. The center of the circle $ABYX$ is the center of the other quadrant. Applying Ajima's construction to the triangle XAB and the point P, we easily obtain the circle tangent to AP, BP, and (O). Since $\angle APB = 135°$ and $\angle AXB = 45°$, the radius of this circle is twice the inradius of triangle APB.

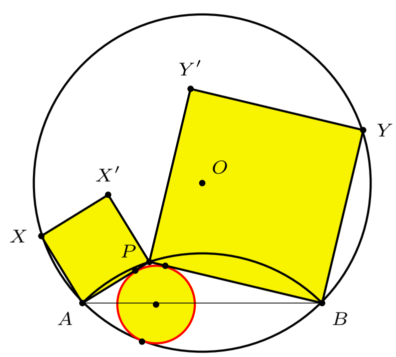

Figure 32

6. Some examples of triangle constructions

There is an extensive literature on construction problems of triangles with certain given elements such as angles, lengths, or specified points. Wernick [20] outlines a project of such with three given specific points. Lopes [14], on the other hand, treats extensively the construction problems with three given lengths such as sides, medians, bisectors, or others. We give three examples admitting elegant constructions. [6]

6.1. *Construction from a sidelength and the corresponding median and angle bisector.* Given the length $2a$ of a side of a triangle, and the lengths m and t of the median and the angle bisector on the same side, to construct the triangle. This is Problem 1054(a) of the *Mathematics Magazine* [6]. In his solution, Howard Eves denotes by z the distance between the midpoint and the foot of the angle bisector on the side $2a$, and obtains the equation

$$z^4 - (m^2 + t^2 + a^2)z^2 + a^2(m^2 - t^2) = 0,$$

from which he concludes constructibility (by ruler and compass). We devise a simple construction, assuming the data given in the form of a triangle $AM'T$ with $AT = t$, $AM' = m$ and $M'T = a$. See Figure 33. Writing $a^2 = m^2 + t^2 - 2tu$, and $z^2 = m^2 + t^2 - 2tw$, we simplify the above equation into

$$w(w - u) = \frac{1}{2}a^2. \tag{4}$$

Note that u is length of the projection of AM' on the line AT, and w is the length of the median AM on the bisector AT of the sought triangle ABC. The length w can be easily constructed, from this it is easy to complete the triangle ABC.

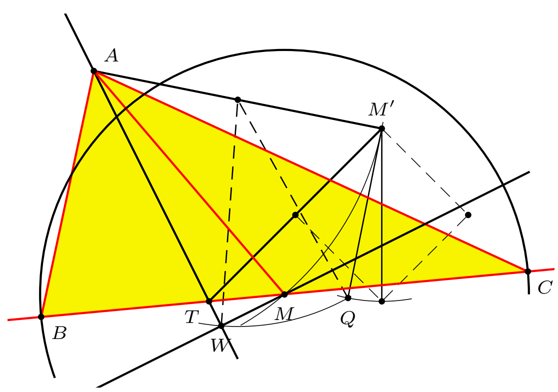

Figure 33

[6] Construction 3 (Figure 7) solves the construction problem of triangle ABC given angle A, side a, and the length t of the bisector of angle A. See Footnote 4.

Construction 22. (1) *On the perpendicular to AM' at M', choose a point Q such that $M'Q = \frac{M'T}{\sqrt{2}} = \frac{a}{\sqrt{2}}$.*

(2) *Construct the circle with center the midpoint of AM' to pass through Q and to intersect the line AT at W so that T and W are on the same side of A. (The length w of AW satisfies (4) above).*

(3) *Construct the perpendicular at W to AW to intersect the circle $A(M')$ at M.*

(4) *Construct the circle $M(a)$ to intersect the line MT at two points B and C.*

The triangle ABC has AT as bisector of angle A.

6.2. *Construction from an angle and the corresponding median and angle bisector.* This is Problem 1054(b) of the *Mathematics Magazine*. See [6]. It also appeared earlier as Problem E1375 of the *American Mathematical Monthly*. See [11]. We give a construction based on Thébault's solution.

Suppose the data are given in the form of a right triangle OAM, where $\angle AOM = A$ or $180° - A$, $\angle M = 90°$, $AM = m$, along with a point T on AM such that $AT = t$. See Figure 34.

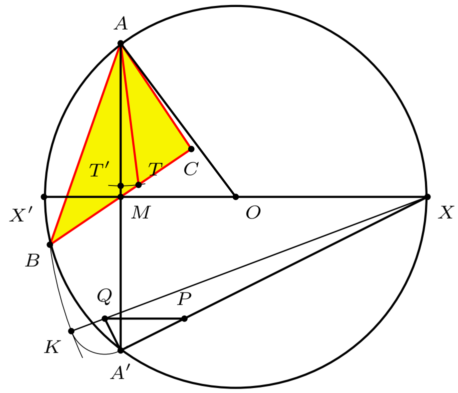

Figure 34

Construction 23. (1) *Construct the circle $O(A)$. Let A' be the mirror image of A in M. Construct the diameter XY perpendicular to AA', X the point for which $\angle AXA' = A$.*

(2) *On the segment $A'X$ choose a point P such that $A'P = \frac{t}{2}$. and construct the parallel through P to XY to intersect $A'Y$ at Q.*

(3) *Extend XQ to K such that $QK = QA'$.*

(4) *Construct a point B on $O(A)$ such that $XB = XK$, and its mirror image C in M.*

Triangle ABC has given angle A, median m and bisector t on the side BC.

6.3. *Construction from the incenter, orthocenter and one vertex.* This is one of the unsolved cases in Wernick [20]. See also [22]. Suppose we put the incenter I at the origin, $A = (a, b)$ and $H = (a, c)$ for $b > 0$. Let r be the inradius of the triangle.

A fairly straightforward calculation gives

$$r^2 - \frac{b-c}{2}r - \frac{1}{2}(a^2+bc) = 0. \tag{5}$$

If M is the midpoint of IA and P the orthogonal projection of H on the line IA, then $\frac{1}{2}(a^2+bc)$, being the dot product of IM and IH, is the (signed) product $IM \cdot IP$. Note that if angle AIH does not exceed a right angle, equation (5) admits a unique positive root. In the construction below we assume H closer than A to the perpendicular to AH through I.

Construction 24. *Given triangle AIH in which the angle AIH does not exceed a right angle, let M be the midpoint of IA, K the midpoint of AH, and P the orthogonal projection of H on the line IA.*
(1) *Construct the circle $\mathcal{C}$ through P, M and K. Let O be the center of $\mathcal{C}$ and Q the midpoint of PK.*
(2) *Construct a tangent from I to the circle $O(Q)$ intersecting $\mathcal{C}$ at T, with T farther from I than the point of tangency.*

The circle $I(T)$ is the incircle of the required triangle, which can be completed by constructing the tangents from A to $I(T)$, and the tangent perpendicular to AH through the "lowest" point of $I(T)$. See Figure 35.

If H is farther than A to the perpendicular from I to the line AH, the same construction applies, except that in (2) T is the intersection with $\mathcal{C}$ closer to I than the point of tangency.

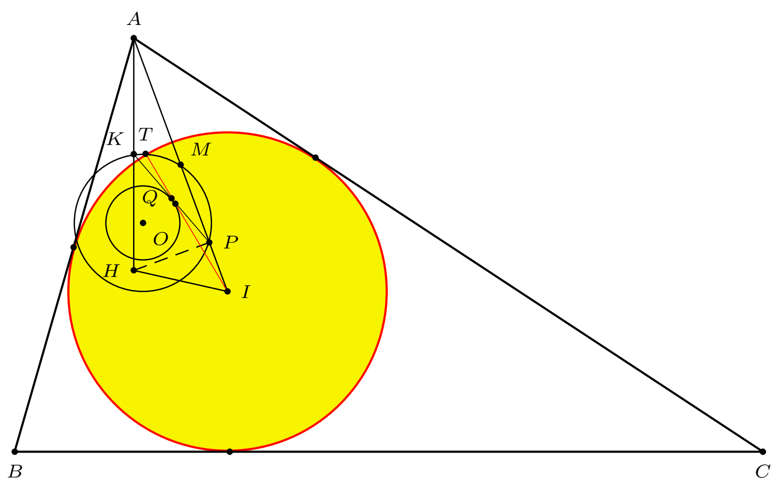

Figure 35

Remark. The construction of a triangle from its circumcircle, incenter, orthocenter was studied by Leonhard Euler [8], who reduced it to the problem of trisection of an angle. In Euler's time, the impossibility of angle trisection by ruler and compass was not yet confirmed.

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Forum Geometricorum
Volume 5 (2005) 97–106.

FORUM GEOM
ISSN 1534-1178

Circles and Triangle Centers Associated with the Lucas Circles

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Abstract. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex. In this paper we present some further interesting circles and triangle centers associated with the Lucas circles.

1. Introduction

In this paper we study circles and triangle centers associated with the three Lucas circles of a triangle. The Lucas circles of a triangle are the three circles mutually tangent to each other externally, and each tangent internally to the circumcircle of the triangle at a vertex.

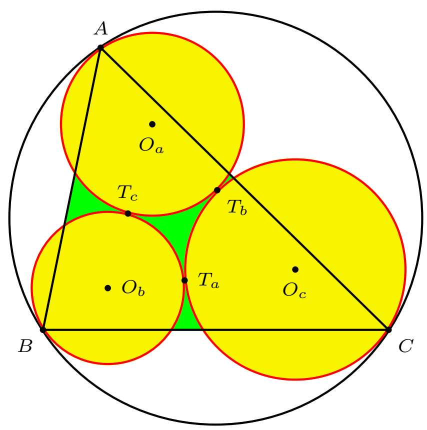

Figure 1

We work with homogeneous barycentric coordinates and make use of John H. Conway's notation in triangle geometry. The indexing of triangle centers follows Kimberling's *Encyclopedia of Triangle Centers* [2]. Many of the triangle centers in this paper are related to the Kiepert perspectors. We recall that given a triangle ABC, the Kiepert perspector $K(\theta)$ is the perspector of the triangle formed by the apices of similar isosceles triangles with base angles θ on the sides of ABC.

Publication Date: July 5, 2005. Communicating Editor: Paul Yiu.
The author thanks Clark Kimberling and Paul Yiu for their helps in the preparation of this paper.

In barycentric coordinates,

$$K(\theta) = \left(\frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta}\right).$$

Its isogonal conjugate is the point

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta))$$

on the Brocard axis joining the circumcenter O and the symmedian point K.

2. The centers and points of tangency of the Lucas circles

The Lucas circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ of triangle ABC are the images of the circumcircle under the homotheties with centers A, B, C, and ratios $\frac{S}{a^2+S}$, $\frac{S}{b^2+S}$, $\frac{S}{c^2+S}$ respectively. As such they have centers

$$\begin{aligned}
O_a =&(a^2(S_A + 2S) : b^2 S_B : c^2 S_C),\\
O_b =&(a^2 S_A : b^2(S_B + 2S) : c^2 S_C),\\
O_c =&(a^2 S_A : b^2 S_B : c^2(S_C + 2S)),
\end{aligned}$$

and equations

$$\mathcal{C}_A : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{a^2+S} \cdot (x+y+z)\left(\frac{y}{b^2} + \frac{z}{c^2}\right) = 0,$$

$$\mathcal{C}_B : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{b^2+S} \cdot (x+y+z)\left(\frac{z}{c^2} + \frac{x}{a^2}\right) = 0,$$

$$\mathcal{C}_C : \qquad a^2yz + b^2zx + c^2xy - \frac{a^2b^2c^2}{c^2+S} \cdot (x+y+z)\left(\frac{x}{a^2} + \frac{y}{b^2}\right) = 0.$$

The Lucas circles are mutually tangent to each other, externally, at

$$\begin{aligned}
T_a = \mathcal{C}_B \cap \mathcal{C}_C =&(a^2 S_A : b^2(S_B + S) : c^2(S_C + S)),\\
T_b = \mathcal{C}_C \cap \mathcal{C}_A =&(a^2(S_A + S) : b^2 S_B : c^2(S_C + S)),\\
T_c = \mathcal{C}_A \cap \mathcal{C}_B =&(a^2(S_A + S) : b^2(S_B + S) : c^2 S_C).
\end{aligned}$$

See Figure 1. These points of tangency form a triangle perspective with ABC at

$$K^*(\frac{\pi}{4}) = (a^2(S_A + S) : b^2(S_B + S) : c^2(S_C + C)),$$

which is X_{371} of [2].

By Desargues' theorem, the triangles $O_aO_bO_c$ and $T_aT_bT_c$ are perspective. Their perspector is clearly the Gergonne point of triangle $O_aO_bO_c$; it has coordinates

$$(a^2(3S_A + 2S) : b^2(3S_B + 2S) : c^2(3S_C + 2S)).$$

This is the point $K^*(\arctan \frac{3}{2})$.

The exsimilicenter (external center of similitude) of $\mathcal{C}_B$ and $\mathcal{C}_C$ is the point $(0 : b^2 : -c^2)$. Likewise, those of the pairs $\mathcal{C}_C$, $\mathcal{C}_A$ and $\mathcal{C}_A$, $\mathcal{C}_B$ are $(-a^2 : 0 : c^2)$ and $(a^2 - b^2 : 0)$. These three exsimilicenters all lie on the Lemoine axis,

$$\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 0.$$

Proposition 1. *The pedals of O_a on BC, O_b on CA, and O_c on AB form the cevian triangle of the Kiepert perspector $K(\arctan 2)$.*[1]

Proof. These pedals are the points $(0 : 2S_C + S : 2S_B + S)$, $(2S_C + S : 0 : 2S_A + S)$, and $(2S_B + S : 2S_A + S : 0)$. □

Proposition 2. *The pedals of T_a on BC, T_b on CA, and T_c on AB form the cevian triangle of the point $(a^2 + S : b^2 + S : c^2 + S)$.*

Proof. These pedals are the points $(0 : b^2 + S : c^2 + S)$, $(a^2 + S : 0 : c^2)$, and $(a^2 + S : b^2 + S : 0)$. □

3. The radical circle of the Lucas circles

From the equations of the Lucas circles, the radical center of these circles is the point $(x : y : z)$ satisfying

$$\frac{\frac{y}{b^2} + \frac{z}{c^2}}{a^2 + S} = \frac{\frac{z}{c^2} + \frac{x}{a^2}}{b^2 + S} = \frac{\frac{x}{a^2} + \frac{y}{b^2}}{c^2 + S}.$$

This means that $\left(\frac{x}{a^2} : \frac{y}{b^2} : \frac{z}{c^2}\right)$ is the anticomplement of $(a^2 + S : b^2 + S : c^2 + S)$, namely, $(2S_A + S : 2S_B + S : 2S_C + S)$, and the radical center is the point

$$K^*(\arctan 2) = (a^2(2S_A + S) : b^2(2S_B + S) : c^2(2S_C + S)) = X_{1151}$$

on the Brocard axis. Since the Lucas circles are tangent to each other, their radical circle is simply the circle through the tangent points T_a, T_b and T_c. It is also the incircle of triangle $O_aO_bO_c$. As such, it has radius $\frac{2S}{a^2+b^2+c^2+4S} \cdot R$, where R is the circumradius of triangle ABC. Its equation is

$$a^2yz + b^2zx + c^2xy - \frac{2a^2b^2c^2(x+y+z)}{a^2+b^2+c^2+4S}\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right).$$

4. The inner Soddy circle of the Lucas circles

There are two nonintersecting circles which are tangent to all three Lucas circles. These are the outer and inner Soddy circles of triangle $O_aO_bO_c$. Since the outer Soddy circle is the circumcircle of ABC, the inner Soddy circle is the inverse of this circumcircle with respect to the radical circle. Indeed the points of tangency are the inverses of A, B, C in the radical circle. They are simply the second

[1]This is X_{1131} of [2].

intersections of the lines AT with $\mathcal{C}_a$, BT with $\mathcal{C}_b$, and CT with $\mathcal{C}_c$, where $T = K^*(\arctan 2)$. These are the points

$$(a^2(4S_A+3S) : 2b^2(2S_B+S) : 2c^2(2S_C+S)),$$
$$(2a^2(2S_A+S) : b^2(4S_B+3S) : 2c^2(2S_C+S)),$$
$$(2a^2(2S_A+S) : 2b^2(2S_B+S) : c^2(4S_C+3S)).$$

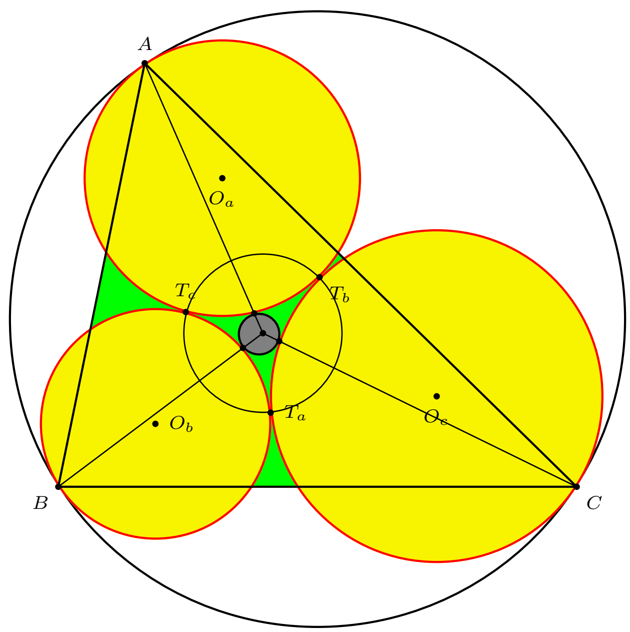

Figure 2

The circle through these points has center $K^*\left(\arctan\frac{7}{4}\right)$ and radius $\frac{2S\cdot R}{4(a^2+b^2+c^2)+14S}$. It has equation

$$a^2yz+b^2zx+c^2xy-\frac{4a^2b^2c^2(x+y+z)}{2(a^2+b^2+c^2)+7S}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)=0.$$

Proposition 3. *The circumcircle, the radical circle, the inner Soddy circle, and the Brocard circles are coaxal, with the Lemoine axis as radical axis.*

The Brocard circle has equation

$$a^2yz+b^2zx+c^2xy-\frac{a^2b^2c^2(x+y+z)}{a^2+b^2+c^2}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)=0.$$

The radical trace of these circles, namely, the intersection of the radical axis and the line of centers, is the point

$$(a^2(b^2+c^2-2a^2):\cdots:\cdots)=K^*(-\arctan(\frac{6S}{a^2+b^2+c^2})).$$

This is X_{187}, the inverse of K in the circumcircle.

5. The Schoute coaxal system

According to [5], the coaxal system of circles containing the circles in Proposition 3 is called the Schoute coaxal system. It has the two isodynamic points as limit points. Indeed, the circle with center X_{187} passing through the isodynamic point $X_{15} = K^*(\frac{\pi}{3})$ is the radical circle of these circles.

Proposition 4. *The circles of the Schoute coaxal system have centers $K^*(\theta)$ where $|\theta| \geq \frac{\pi}{3}$, and radius $\left|\frac{\sqrt{\tan^2\theta-3S}}{2(S_\omega+S\cdot\tan\theta)}\right| \cdot R$. It has equation*

$$\mathcal{C}_{\rm s}(\theta): \qquad a^2yz+b^2zx+c^2xy-\frac{a^2b^2c^2(x+y+z)}{S_\omega+S\cdot\tan\theta}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)=0.$$

Therefore, a circle with center $(a^2(pS_A+qS):b^2(pS_B+qS):c^2(pS_C+qS))$ and square radius $\frac{(p^2-3q^2)a^2b^2c^2}{(2pS+q(a^2+b^2+c^2))^2}$ is the circle $\mathcal{C}_{\rm s}(\arctan\frac{p}{q})$.

circle	$\mathcal{C}_{\rm s}(\theta)$with $\tan\theta =$
circumcircle	∞
Brocard circle	$\cot\omega$
Lemoine axis	$-\cot\omega$
radical circle of Lucas circles	2
inner Soddy circle of Lucas circles	$\frac{7}{4}$

$\theta = \frac{\pi}{3}$ yields the limit point X_{15}.

Proposition 5. *The inversive image of $\mathcal{C}_{\rm s}(\theta)$ in $\mathcal{C}_{\rm s}(\varphi)$ is the circle $\mathcal{C}_{\rm s}(\psi)$, where*

$$\tan\psi = \frac{\tan\theta(\tan^2\varphi+3)-6\tan\varphi}{2\tan\theta\tan\varphi-(\tan^2\varphi+3)}.$$

Corollary 6. (a) *The inverse of $\mathcal{C}_{\rm s}(\theta)$ in the circumcircle is $\mathcal{C}_{\rm s}(-\theta)$.*

(b) *The inverse of the circumcircle in $\mathcal{C}_{\rm s}(\varphi)$ is the circle $\mathcal{C}_{\rm s}\left(\arctan\frac{\tan^2\varphi+3}{2\tan\varphi}\right)$.*

6. Three infinite families of circles

Let $A'B'C'$ be the circumcevian triangle of the symmedian point K, and $K' = K^*(\frac{\pi}{4})$. The line OA' intersects O_aK' at

$$O_1^a = (a^2(S_A-2S):b^2(S_B+4S):c^2(S_C+4S)).$$

This is the center of the circle tangent to the B- and C-Lucas circles, and the circumcircle. It touches the circumcircle at K_0^a. We label this circle $\mathcal{C}_1^a$. The points of tangency with the B- and C-Lucas circles are

$$(a^2(S_A-S):b^2(S_B+3S):c^2(S_C+2S)),$$
$$(a^2(S_A-S):b^2(S_B+2S):c^2(S_C+3S))$$

respectively.

Similarly, there are circles $\mathcal{C}_1^b$ and $\mathcal{C}_1^c$ each tangent internally to the circumcircle and externally to two Lucas circles. The centers of the three circles $\mathcal{C}_1^a$, $\mathcal{C}_1^b$, $\mathcal{C}_1^c$ are perspective with ABC at $K^*(\arctan \frac{1}{4})$.

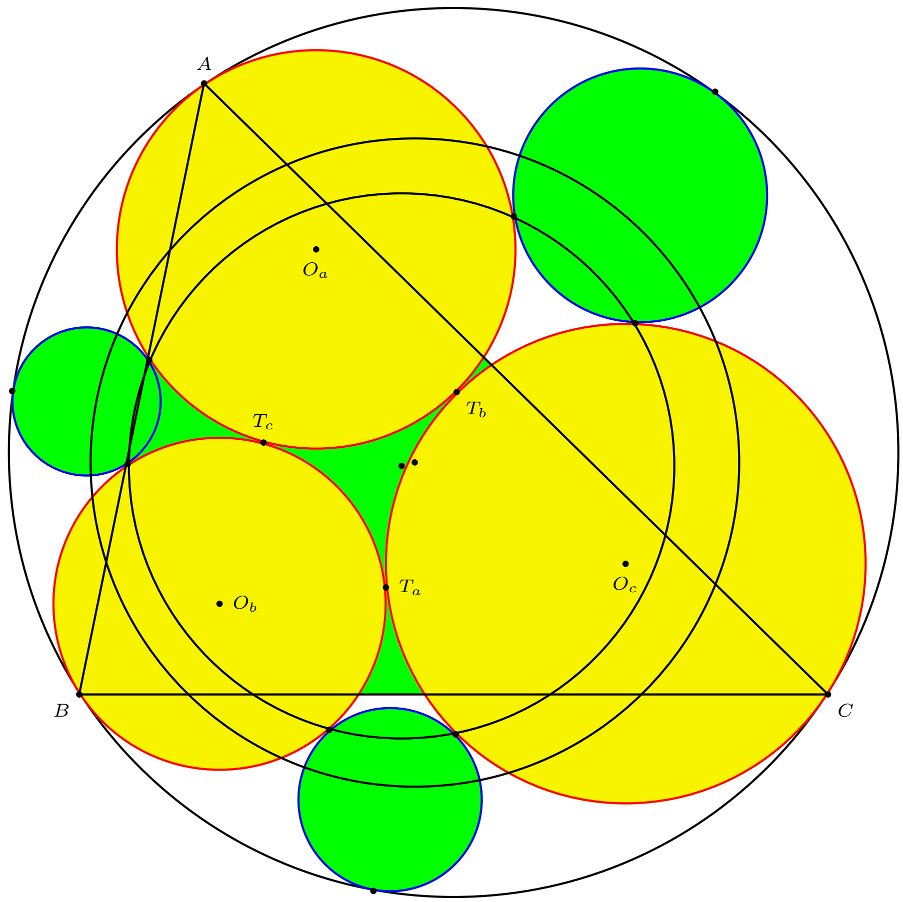

Figure 3

Remarks. (1) The 6 points of tangency with the Lucas circles lie on $\mathcal{C}_s(\arctan 4)$.
(2) The radical circle of these circles is $\mathcal{C}_s(\arctan 6)$. See Figure 3.

The Lucas circles lend themselves to the creation of more and more circle tangencies. There is, for example, an infinite sequence of circles $\mathcal{C}_n^a$ each tangent externally to the B- and C-Lucas circles, so that $\mathcal{C}_n^a$ touches $\mathcal{C}_{n-1}^a$ externally at T_n^a. (We treat $\mathcal{C}_0^a$ as the circumcircle of ABC so that $T_1^a = A'$.

$$O_n^a = (a^2((2n^2-1)S_A - 2nS) : b^2((2n^2-1)S_B + 2n(n+1)S) : c^2((2n^2-1)S_C + 2n(n+1)S)),$$
$$T_n^a = (a^2(2n(n-1)S_A - (2n-1)S) : 2nb^2((n-1)S_B + nS) : 2nc^2((n-1)S_C + nS)).$$

The centers O_n^a of these circles lie on the hyperbola through O_a with foci O_b and O_c. It also contains O and T_a. This is the inner Soddy hyperbola of triangle $O_aO_bO_c$. The points of tangency T_n^a lie on the A-Apollonian circle.

Similarly, we have two analogous families of circles $\mathcal{C}_n^b$ and $\mathcal{C}_n^c$, respectively with centers O_n^b, O_n^c and points of tangency T_n^b, T_n^c.

Remarks. (1) The centers of $\mathcal{C}_n^a, \mathcal{C}_n^b, \mathcal{C}_n^c$ lie on the circle $\mathcal{C}_{\rm s}\left(\arctan \frac{4n^2-2n+1}{2n(n-1)}\right)$.

(2) The six points of tangency with the Lucas circles lie on the circle $\mathcal{C}_{\rm s}\left(\arctan \frac{2n^2+n+1}{n^2}\right)$.

(3) The radical circle of $\mathcal{C}_n^a, \mathcal{C}_n^b, \mathcal{C}_n^c$ is the circle $\mathcal{C}_{\rm s}\left(\arctan \frac{2n(2n+1)}{2n^2-1}\right)$.

Proposition 7. *The following pairs of triangles are perspective. The perspectors are all on the Brocard axis.*

Triangle	Triangle	Perspector $= K^*(\theta)$ with $\tan\theta =$
$O_n^aO_n^bO_n^c$	ABC	$\frac{2n^2-1}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_aO_bO_c$	$\frac{3n-1}{2n}$
$O_n^aO_n^bO_n^c$	$T_aT_bT_c$	$\frac{4n+1}{2n}$
$O_n^aO_n^bO_n^c$	circumcevian triangle of K	$\frac{6n^2-3}{2n(n-1)}$
$O_n^aO_n^bO_n^c$	$O_1^aO_1^bO_1^c$	$\frac{5n+3}{2n}$
$O_n^aO_n^bO_n^c$	$O_{n+1}^aO_{n+1}^bO_{n+1}^c$	$\frac{4n^2+6n+3}{2n(n+1)}$
$O_n^aO_n^bO_n^c$	$O_m^aO_m^bO_m^c$	$\frac{4mn+m+n+2}{2mn}$
$T_n^aT_n^bT_n^c$	ABC	$\frac{n-1}{n}$
$T_n^aT_n^bT_n^c$	$O_aO_bO_c$	$\frac{6n^2-2n-1}{4n^2}$
$T_n^aT_n^bT_n^c$	$T_aT_bT_c$	$\frac{4n-1}{2n-1}$
$T_n^aT_n^bT_n^c$	$T_m^aT_m^bT_m^c$	$\frac{4mn-m-n+1}{2mn-m-n}$

7. Centers of similitude

Since the Lucas radical circle, the inner Soddy circle and the circumcircle all belong to the Schoute family, their centers of similitude are all on the Brocard axis.

		Internal	External
inner Soddy circle	circumcircle	$K^*(\arctan 2)$	$K^*(\arctan \frac{3}{2})$
inner Soddy circle	radical circle	$K^*(\arctan \frac{9}{5})$	$K^*(\arctan \frac{5}{3})$

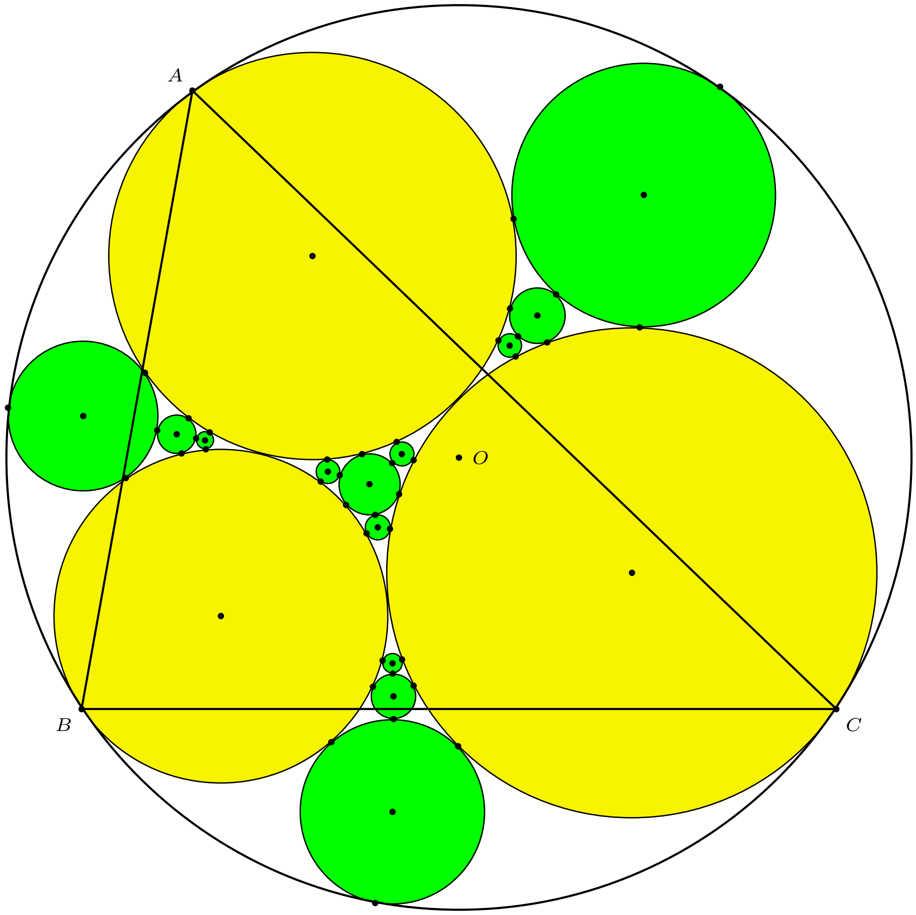

Figure 4

Proposition 8. (a) *The insimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with* ABC *at* $K^*(\arctan 3)$.

(b) *The exsimilicenters of the Lucas radical circle and the individual Lucas circles form a triangle perspective with* ABC *at* $K^*(\frac{\pi}{4})$.

Proof. These insimilicenters are the points

$$\begin{array}{ccccc} (3a^2(S_A+S) & : & b^2(3S_B+S) & : & c^2(3S_C+S)), \\ (a^2(3S_A+S) & : & 3b^2(S_B+S) & : & c^2(3S_C+S)), \\ (a^2(3S_A+S) & : & b^2(3S_B+S) & : & 3c^2(S_C+S)). \end{array}$$

Likewise, the exsimilicenters are the points

$$\begin{array}{ccccc} (a^2(S_A-S) & : & b^2(S_B+S) & : & c^2(S_C+S)), \\ (a^2(S_A+S) & : & b^2(S_B-S) & : & c^2(S_C+S)), \\ (a^2(S_A+S) & : & b^2(S_B+S) & : & c^2(S_C-S)). \end{array}$$

□

8. Two conics

As explained in [1], the Lucas circles of a triangle are also associated with the inscribed squares of the triangle. We present two interesting conics associated with these inscribed squares. Given a triangle ABC, the A-inscribed square $X_1X_2X_3X_4$ has vertices

$$X_1 = (0 : S_C + S : S_B), \quad \text{and} \quad X_2 = (0 : S_C : S_B + S)$$

on the line BC and

$$X_3 = (a^2 : 0 : S) \quad \text{and} \quad X_4 = (a^2 : S : 0)$$

on AC and AB respectively. It has center $(a^2 : S_C + S : S_B + S)$. Similarly, the coordinates of the $B-$ and C-inscribed squares, and their centers, can be easily written down. It is clear that the centers of these squares form a triangle perspective with ABC at the Kiepert perspector

$$K(\frac{\pi}{4}) = \left(\frac{1}{S_A + S} : \frac{1}{S_B + S} : \frac{1}{S_C + S}\right).$$

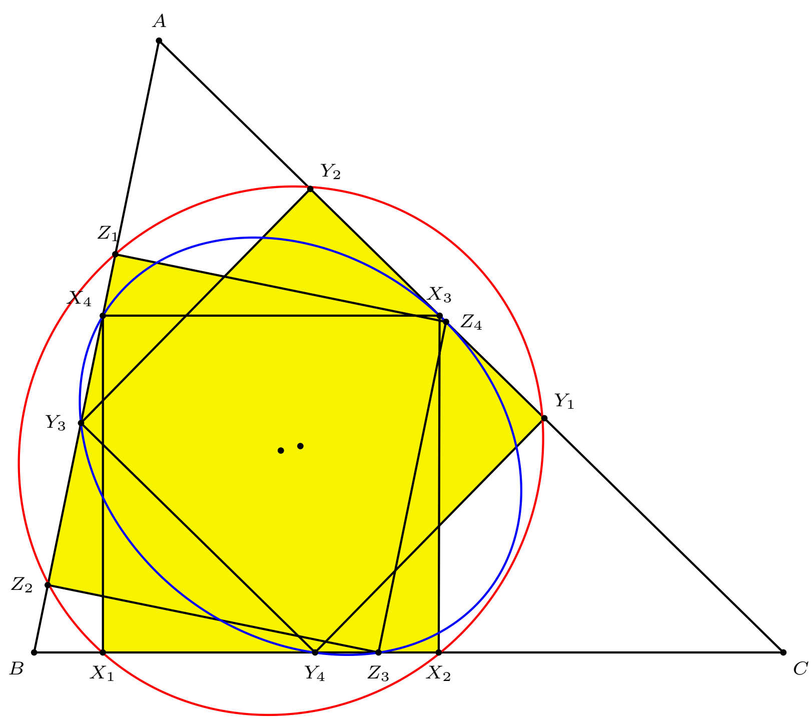

Figure 5.

Proposition 9. *The six points* X_1, X_2, Y_1, Y_2, Z_1, Z_2 *lie on the conic*

$$\sum_{\text{cyclic}}(a^2+S)^2yz=(x+y+z)\sum_{\text{cyclic}}S_A(S_A+S)x.$$

This conic has center $(a^2+S:b^2+S:c^2+S)$.

Proposition 10. *The six points* X_3, X_4, Y_3, Y_4, Z_3, Z_4 *lie on the conic*

$$\sum_{\text{cyclic}}\frac{a^2}{a^2+S}yz=\frac{a^2b^2c^2S(x+y+z)}{(a^2+S)(b^2+S)(c^2+S)}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right).$$

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Forum Geometricorum
Volume 5 (2005) 107–117.

FORUM GEOM
ISSN 1534-1178

On the Geometry of Equilateral Triangles

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Dedicated to the memory of Angela Vasiu (1941-2005)

Abstract. By studying the distances of a point to the sides, respectively the vertices of an equilateral triangle, certain new identities and inequalities are deduced. Some inequalities for the elements of the Pompeiu triangle are also established.

1. Introduction

The equilateral (or regular) triangle has some special properties, generally not valid in an arbitrary triangle. Such surprising properties have been studied by many famous mathematicians, including Viviani, Gergonne, Leibnitz, Van Schooten, Toricelli, Pompeiu, Goormaghtigh, Morley, etc. ([2], [3], [4], [7]). Our aim in this paper is the study of certain identities and inequalities involving the distances of a point to the sides or the vertices of an equilateral triangle. For the sake of completeness, we shall recall some well-known results.

1.1. Let ABC be an equilateral triangle of side length $AB = BC = CA = l$, and height h. Let P be any point in the plane of the triangle. If O is the center of the triangle, then the Leibnitz relation (valid in fact for any triangle) implies that

$$\sum PA^2 = 3PO^2 + \sum OA^2. \tag{1}$$

Let $PO = d$ in what follows. Since in our case $OA = OB = OC = R = \dfrac{l\sqrt{3}}{3}$, we have $\sum OA^2 = l^2$, and (1) gives

$$\sum PA^2 = 3d^2 + l^2. \tag{2}$$

Therefore, $\sum PA^2 =$ constant if and only if $d =$ constant, *i.e.*, when P is on a circle with center O. For a proof by L. Moser via analytical geometry, see [12]. For a proof using Stewart's theorem, see [13].

1.2. Now, let P be in the interior of triangle ABC, and denote by p_a, p_b, p_c its distances from the sides. Viviani's theorem says that

$$\sum p_a = p_a + p_b + p_c = h = \frac{l\sqrt{3}}{2}.$$

Publication Date: July 20, 2005. Communicating Editor: Paul Yiu.
The author thanks the referee for some useful remarks, which have improved the presentation of the paper.

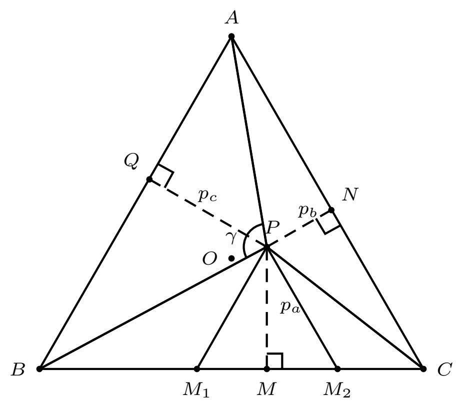

Figure 1

This follows by area considerations, since

$$S(BPC) + S(CPA) + S(APB) = S(ABC),$$

where S denotes area. Thus,

$$\sum p_a = \frac{l\sqrt{3}}{2}. \tag{3}$$

1.3. By Gergonne's theorem one has $\sum p_a^2 =$ constant, when P is on the circle of center O. For such related constants, see for example [13]. We shall obtain more general relations, by expressing $\sum p_a^2$ in terms of l and $d = OP$.

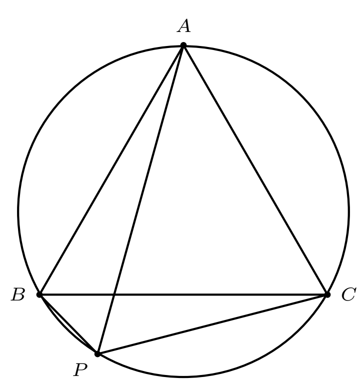

Figure 2

1.4. Another famous theorem, attributed to Pompeiu, states that for any point P in the plane of an equilateral triangle ABC, the distances PA, PB, PC can be the sides of a triangle ([9]-[10], [7], [12], [6]). (See also [1], [4], [11], [15], [16], where extensions of this theorem are considered, too.) This triangle is degenerate if P is on the circle circumscribed to ABC, since if for example P is on the interior or arc BC, then by Van Schooten's theorem,

$$PA = PA + PC. \tag{4}$$

Indeed, by Ptolemy's theorem on $ABPC$ one can write

$$PA \cdot BC = PC \cdot AB + PB \cdot AC,$$

so that $BC = AB = AC = l$ implies (4). For any other positions of P (*i.e.*, P **not** on this circle), by Ptolemy's inequality in quadrilaterals one obtains

$$PA < PB + PC, \quad PB < PA + PC, \text{ and } PC < PA + PB,$$

so that PA, PB, PC are the sides of a triangle. See [13] for many proofs. We shall call a triangle with sides PA, PB, PC a **Pompeiu triangle**. When P is in the interior, the Pompeiu triangle can be explicitly constructed. Indeed, by rotating the triangle ABP with center A through an angle of $60°$, one obtains a triangle $AB'C$ which is congruent to ABP. Then, since $AP = AB' = PB'$, $BP = CB'$, the Pompeiu triangle will be PCB'. Such a rotation will enable us also to compute the area of the Pompeiu triangle.

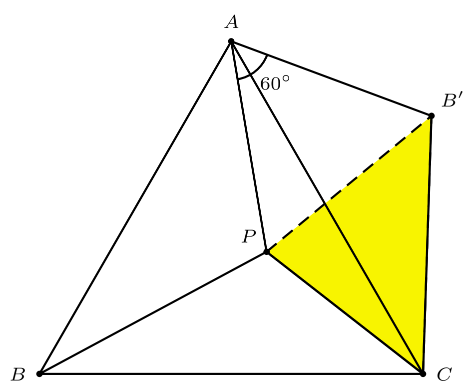

Figure 3

1.5. There exist many known inequalities for the distances of a point to the vertices of a triangle. For example, for any point P and any triangle ABC,

$$\sum PA \geq 6r, \tag{5}$$

where r is the radius of incircle (due to M. Schreiber (1935), see [7], [13]). Now, in our case $6r = l\sqrt{3}$, (5) gives

$$\sum PA \geq l\sqrt{3} \tag{6}$$

for any point P in the plane of equilateral triangle ABC. For an independent proof see [12, p.52]. This is based on the following idea: let M_1 be the midpoint of BC. By the triangle inequality one has $AP + PM_1 \geq AM_1$. Now, it is well known that

$PM_1 \le \dfrac{PB+PC}{2}$. From this, we get $l\sqrt{3} \le 2PA + PB + PC$, and by writing two similar relations, the relation (6) follows after addition. We note that already (2) implies $\sum PA^2 \ge l^2$, but (6) offers an improvement, since

$$\sum PA^2 \ge \frac{1}{3}\left(\sum PA\right)^2 \ge l^2 \tag{7}$$

by the classical inequality $x^2+y^2+z^2 \ge \dfrac{1}{3}(x+y+z)^2$. As in (7), equality holds in (6) when $P \equiv O$.

2. Identities for p_a, p_b, p_c

Our aim in this section is to deduce certain identities for the distances of an interior point to the sides of an equilateral triangle ABC.

Let P be in the interior of triangle ABC (see Figure 1). Let $PM \perp BC$, etc., where $PM = p_a$, etc. Let $PM_1 \| AB$, $PM_2 \| AC$. Then triangle PM_1M_2 is equilateral, giving $\overrightarrow{PM} = \dfrac{\overrightarrow{PM}_1 + \overrightarrow{PM}_2}{2}$. By writing two similar relations for $\overrightarrow{PQ}$ and $\overrightarrow{PN}$, and using $\overrightarrow{PO} = \dfrac{\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}}{3}$, one easily can deduce the following vectorial identity:

$$\overrightarrow{PM} + \overrightarrow{PN} + \overrightarrow{PQ} = \frac{3}{2}\overrightarrow{PO}. \tag{8}$$

Since $\overrightarrow{PM} \cdot \overrightarrow{PN} = PM \cdot PN \cdot \cos 120^\circ = -\dfrac{1}{2} PM \cdot PN$ (in the cyclic quadrilateral $CNPM$), by putting $PO = d$, one can deduce from (8)

$$\sum PM^2 + \frac{1}{2}\sum \overrightarrow{PM} \cdot \overrightarrow{PN} = \frac{9}{4}PO^2,$$

so that

$$\sum p_a^2 - \sum p_a p_b = \frac{9}{4}d^2. \tag{9}$$

For similar vectorial arguments, see [12]. On the other hand, from (3), we get

$$\sum p_a^2 + 2\sum p_a p_b = \frac{3l^2}{4}. \tag{10}$$

Solving the system (9), (10) one can deduce the following result.

Proposition 1.

$$\sum p_a^2 = \frac{l^2+6d^2}{4}, \tag{11}$$

$$\sum p_a p_b = \frac{l^2-3d^2}{4}. \tag{12}$$

There are many consequences of (11) and (12). First, $\sum p_a^2 =$ constant if and only if $d =$ constant, *i.e.*, P lying on a circle with center O. This is Gergonne's theorem. Similarly, (12) gives $\sum p_a \cdot p_b =$ constant if and only if $d =$ constant, *i.e.*, P again lying on a circle with center O. Another consequence of (11) and (12) is

$$\sum p_a p_b \le \frac{l^2}{4} \le \sum p_a^2. \tag{13}$$

An interesting connection between $\sum PA^2$ and $\sum p_a^2$ follows from (2) and (11):

$$\sum PA^2 = 2\sum p_a^2 + \frac{l^2}{2}. \tag{14}$$

3. Inequalities connecting p_a, p_b, p_c with PA, PB, PC

This section contains certain new inequalities for PA, p_a, etc. Among others, relation (18) offers an improvement of known results.

By the arithmetic-geometric mean inequality and (3), one has

$$p_a p_b p_c \le \left(\frac{p_a + p_b + p_c}{3}\right)^3 = \left(\frac{l\sqrt{3}}{6}\right)^3 = \frac{l^3\sqrt{3}}{72}.$$

Thus,

$$p_a p_b p_c \le \frac{l^3\sqrt{3}}{72} \tag{15}$$

for any interior point P of equilateral triangle ABC. This is an equality if and only only if $p_a = p_b = p_c$, *i.e.*, $P \equiv O$.

Now, let us denote $\alpha = mes(\sphericalangle BPC)$, etc. Writing the area of triangle BPC in two ways, we obtain

$$BP \cdot CP \cdot \sin\alpha = l \cdot p_a.$$

Similarly,

$$AP \cdot BP \cdot \sin\gamma = l \cdot p_c, \quad AP \cdot CP \cdot \sin\beta = l \cdot p_c.$$

By multiplying these three relations, we have

$$PA^2 \cdot PB^2 \cdot PC^2 = \frac{l^3 p_a p_b p_c}{\sin\alpha \sin\beta \sin\gamma}. \tag{16}$$

We now prove the following result.

Theorem 2. *For an interior point P of an equilateral triangle ABC, one has*

$$\prod PA^2 \ge \frac{8l^3}{3\sqrt{3}} \prod p_a \quad \textit{and} \quad \sum PA \cdot PB \ge l^2.$$

Proof. Let $f(x) = \ln \sin x$, $x \in (0, \pi)$. Since $f''(x) = -\dfrac{1}{\sin^2 x} < 0$, f is concave, and

$$f\left(\frac{\alpha+\beta+\gamma}{3}\right) \geq \frac{f(\alpha)+f(\beta)+f(\gamma)}{3},$$

giving

$$\prod \sin \alpha \leq \frac{3\sqrt{3}}{8}, \tag{17}$$

since $\dfrac{\alpha+\beta+\gamma}{3} = 120^\circ$ and $\sin 120^\circ = \dfrac{\sqrt{3}}{2}$. Thus, (16) implies

$$\prod PA^2 \geq \frac{8l^3}{3\sqrt{3}} \prod p_a. \tag{18}$$

We note that $\dfrac{8l^3}{3\sqrt{3}} \prod p_a \geq 64 \prod p_a^2$, since this is equivalent to $\prod p_a \leq \dfrac{l^3\sqrt{3}}{72}$, i.e. relation (15). Thus (18) improves the inequality

$$\prod PA \geq 8 \prod p_a \tag{19}$$

valid for any triangle (see [2, inequality 12.25], or [12, p.46], where a slightly improvement appears).

On the other hand, since $\dfrac{\alpha}{2} + \dfrac{\beta}{2} + \dfrac{\gamma}{2} = 180^\circ$, one has

$$\begin{aligned}
&\cos\alpha + \cos\beta + \cos\gamma + \frac{3}{2}\\
=&2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} + \frac{1}{2}\\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\right)\\
=&2\left(\cos^2\frac{\gamma}{2} - \cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + \frac{1}{4}\cos^2\frac{\alpha-\beta}{2} + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right)\\
=&2\left[\left(\cos\frac{\gamma}{2} - \frac{1}{2}\cos\frac{\alpha-\beta}{2}\right)^2 + \frac{1}{4}\sin^2\frac{\alpha-\beta}{2}\right] \geq 0,
\end{aligned}$$

with equality only for $\alpha = \beta = \gamma = 120^\circ$. Thus:

$$\cos\alpha + \cos\beta + \cos\gamma \geq -\frac{3}{2} \tag{20}$$

for any α, β, γ satisfying $\alpha + \beta + \gamma = 360^\circ$.

Now, in triangle APB one has, by the law of cosines,

$$l^2 = PA^2 + PB^2 - 2PA \cdot PB \cdot \cos\gamma,$$

giving

$$\cos\gamma = \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB}.$$

By writing two similar relations, one gets, by (20),

$$\frac{PA^2+PC^2-l^2}{2PA\cdot PC}+\frac{PB^2+PC^2-l^2}{2PB\cdot PC}+\frac{PA^2+PB^2-l^2}{2PA\cdot PB}+\frac{3}{2}\geq 0,$$

so that

$$\begin{aligned}&(PA^2\cdot PB+PB^2\cdot PA+PA\cdot PB\cdot PC)\\&+(PC^2\cdot PB+PB^2\cdot PC+PA\cdot PB\cdot PC)\\&+(PA^2\cdot PC+PC^2\cdot PA+PA\cdot PB\cdot PC)\\&-l^2(PA+PB+PC)\\&\geq 0.\end{aligned}$$

This can be rearranged as

$$(PA+PB+PC)\left(\sum PA\cdot PB-l^2\right)\geq 0,$$

and gives the inequality

$$\sum PA\cdot PB\geq l^2, \tag{21}$$

with equality when $P\equiv O$. □

4. The Pompeiu triangle

In this section, we deduce many relations connecting PA, PB, PC, etc by obtaining an identity for the area of Pompeiu triangle. In particular, a new proof of (21) will be given.

4.1. Let P be a point inside the equilateral triangle ABC (see Figure 3). The Pompeiu triangle $PB'C$ has the sides PA, PB, PC. Let R be the radius of circumcircle of this triangle. It is well known that $\sum PA^2\leq 9R^2$ (see [1, p.171], [6, p.52], [9, p.56]). By (2) we get

$$R^2\geq\frac{l^2+3d^2}{9}\geq\frac{l^2}{9}, \tag{22}$$

$$R\geq\frac{l}{3}, \tag{23}$$

with equality only for $d=0$, *i.e.*, $P\equiv O$. Inequality (23) can be proved also by the known relation $s\leq\frac{3R\sqrt{3}}{2}$, where s is the semi-perimeter of the triangle. Thus we obtain the following inequalities.

Proposition 3.

$$3R\sqrt{3}\geq\sum PA\geq l\sqrt{3}, \tag{24}$$

where the last inequality follows by (6).

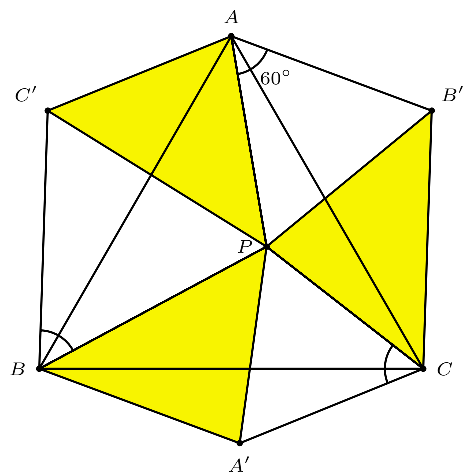

Figure 4

Now, in order to compute the area of the Pompeiu triangle, let us make two similar rotations as in Figure 3, *i.e.*, a rotation of angle 60° with center C of triangle APC, and another with center B of BPC. We shall obtain a hexagon (see Figure 4), $AB'CA'BC'$, where the Pompeiu triangles PBA', PAC', PCB' have equal area T. Since $\triangle APC \equiv \triangle BA'C$, $\triangle APB \equiv \triangle AB'C$, $\triangle AC'B \equiv \triangle BPC$, the area of hexagon $= 2\mathrm{Area}(ABC)$. But $\mathrm{Area}(APB') = \dfrac{AP^2\sqrt{3}}{4}$, APB' being an equilateral triangle. Therefore,

$$\frac{2l^2\sqrt{3}}{4} = 3T + \frac{PA^2\sqrt{3}}{4} + \frac{PB^2\sqrt{3}}{4} + \frac{PC^2\sqrt{3}}{4},$$

which by (2) implies

$$T = \frac{\sqrt{3}}{12}(l^2 - 3d^2). \tag{25}$$

Theorem 4. *The area of the Pompeiu triangle is given by relation (25).*

Corollary 5.

$$T \le \frac{\sqrt{3}}{12} l^2, \tag{26}$$

with equality when $d = 0$, i.e., *when* $P \equiv O$.

Now, since in any triangle of area T, and sides PA, PB, PC one has

$$2\sum PA \cdot PB - \sum PA^2 \ge 4\sqrt{3} \cdot T$$

(see for example [14], relation (8)), by (2) and (25) one can write

$$2\sum PA \cdot PB \ge 3d^2 + l^2 + l^2 - 3d^2 = 2l^2,$$

giving a new proof of (21).

Corollary 6.

$$\sum PA^2 \cdot PB^2 \geq \frac{1}{3}\left(\sum PA \cdot PB\right)^2 \geq \frac{l^4}{3}. \tag{27}$$

4.2. Note that in any triangle, $\sum PA^2 \cdot PB^2 \geq \dfrac{16}{9}S^2$, where $S = \mathrm{Area}(ABC)$ (see [13, pp.31-32]). In the case of equilateral triangles, (27) offers an improvement.

Since $r = \dfrac{T}{s}$, where s is the semi-perimeter and r the radius of inscribed circle to the Pompeiu triangle, by (6) and (26) one can write

$$r \leq \frac{\left(\frac{\sqrt{3}}{12}l^2\right)}{\left(\frac{l\sqrt{3}}{2}\right)} = \frac{l}{6}.$$

Thus, we obtain the following result.

Proposition 7. *For the radii r and R of the Pompeiu triangle one has*

$$r \leq \frac{l}{6} \leq \frac{R}{2}. \tag{28}$$

The last inequality holds true by (23). This gives an improvement of Euler's inequality $r \leq \dfrac{R}{2}$ for the Pompeiu triangle. Since $T = \dfrac{PA \cdot PB \cdot PC}{4R}$, and $r = \dfrac{T}{s}$, we get

$$PA \cdot PB \cdot PC = 2Rr(PA + PB + PC),$$

and the following result.

Proposition 8.

$$PA \cdot PB \cdot PC \geq \frac{2l^2 r\sqrt{3}}{3} \geq 4r^2 l\sqrt{3}. \tag{29}$$

The last inequality is the first one of (28). The following result is a counterpart of (29).

Proposition 9.

$$PA \cdot PB \cdot PC \leq \frac{\sqrt{3}l^2 R}{3}. \tag{30}$$

This follows by $T = \dfrac{PA \cdot PB \cdot PC}{4R}$ and (26).

4.3. The sides PA, PB, PC can be expressed also in terms of p_a, p_b, p_c. Since in triangle PNM (see Figure 1), $\sphericalangle NPM = 120°$, by the Law of cosines one has

$$MN^2 = PM^2 + PN^2 - 2PM \cdot PN \cdot \cos 120°.$$

On the other hand, in triangle NMC, $NM = PC \cdot \sin C$, PC being the diameter of circumscribed circle. Since $\sin C = \sin 60° = \dfrac{\sqrt{3}}{2}$, we have $MN = PC\dfrac{\sqrt{3}}{2}$, and the following result.

Proposition 10.

$$PC^2 = \frac{4}{3}(p_b^2 + p_a^2 + p_a p_b). \tag{31}$$

Similarly,

$$PA^2 = \frac{4}{3}(p_b^2 + p_c^2 + p_b p_c), \quad PB^2 = \frac{4}{3}(p_c^2 + p_a^2 + p_c p_a). \tag{32}$$

In theory, all elements of Pompeiu's triangle can be expressed in terms of p_a, p_b, p_c. We note that by (11) and (12) relation (2) can be proved again. By the arithmetic-geometric mean inequality, we have

$$\prod PA^2 \le \left(\frac{\sum PA^2}{3}\right)^3,$$

and the following result.

Theorem 11.

$$PA \cdot PB \cdot PC \le \left(\frac{l^2 + 3d^2}{3}\right)^{3/2}. \tag{33}$$

On the other hand, by the Pólya-Szegö inequality in a triangle (see [8], or [14]) one has

$$T \le \frac{\sqrt{3}}{4}(PA \cdot PB \cdot PC)^{2/3},$$

so by (25) one can write (using (12)):

Theorem 12.

$$PA \cdot PB \cdot PC \ge \left(\frac{l^2 - 3d^2}{3}\right)^{3/2} = \left(\frac{4\sum p_a p_b}{3}\right)^{3/2}. \tag{34}$$

4.4. Other inequalities may be deduced by noting that by (31),

$$(p_a + p_b)^2 \le PC^2 \le 2(p_a^2 + p_b^2).$$

Since $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 \le 3(x + y + z)$ applied to $x = p_a^2 + p_b^2$, etc., we get

$$\sum PA \le 4\sqrt{3} \cdot \sqrt{p_a^2 + p_b^2 + p_c^2},$$

i.e. by (11) we deduce the following inequality.

Theorem 13.

$$\sum PA \le \sqrt{3(l^2 + 6d^2)}. \tag{35}$$

This is related to (6). In fact, (6) and (35) imply that $\sum PA = l\sqrt{3}$ if and only if $d = 0$, *i.e.*, $P \equiv O$.

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Forum Geometricorum
Volume 5 (2005) 119–126.

FORUM GEOM
ISSN 1534-1178

Construction of Brahmagupta n-gons

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Abstract. The Indian mathematician Brahmagupta's contributions to mathematics and astronomy are well known. His principle of adjoining Pythagorean triangles to construct general Heron triangles and cyclic quadrilaterals having integer sides, diagonals and area can be employed to appropriate Heron triangles themselves to construct any inscribable n-gon, $n \geq 3$, that has integer sides, diagonals and area. To do so we need a different description of Heron triangles by families that contain a common angle. In this paper we describe such a construction.

1. Introduction

A right angled triangle with rational sides is called a rational Pythagorean triangle. This has rational area. When these rationals are integers, it is called a Pythagorean triangle. More generally, an n-gon with rational sides, diagonals and area is called a rational Heron n-gon, $n \geq 3$. When these rationals are converted into integers by a suitable similarity transformation we obtain a Heron n-gon. If a Heron n-gon is cyclic, *i.e.*, inscribable in a circle then we obtain a Brahmagupta n-gon. In this journal and elsewhere a number of articles have appeared on various descriptions of Heron triangles and Brahmagupta quadrilaterals. Some of these are mentioned in the references. Hence we assume familiarity with the basic geometric and trigonometric results. Also, the knowledge of Pythagorean triples is assumed.

We may look upon the family of Pythagorean triangles as the particular family of Heron triangles that contain a right angle. This suggests that the complete set of Heron triangles may be described by families that contain a common Heron angle (A Heron angle has its sine and cosine rational). Once this is done we may look upon the Brahmagupta principle as follows: He took two Heron triangles ABC and $A'B'C'$ that have $\cos A + \cos A' = 0$ and ajoined them along a common side to describe Heron triangles. This enables us to generalize the Brahmagupta principle to members of appropriate families of Heron triangles to construct rational Brahmagupta n-gons, $n \geq 3$. A similarity transformation assures that these rationals can be rendered integers to obtain a Brahmagupta n-gon, $n \geq 3$.

Publication Date: August 4, 2005. Communicating Editor: Paul Yiu.

2. Description of Heron triangles by angle families

In the interest of clarity and simplicity we first take a numerical example and then give the general result [4]. Suppose that we desire the description of the family of Heron triangles ABC each member of which contains the common Heron angle given by $\cos A = \frac{3}{5}$. The cosine rule applied to a member of that family shows that the sides (a, b, c) are related by the equation

$$a^2 = b^2 + c^2 - \frac{6}{5}bc = \left(b - \frac{3}{5}c\right)^2 + \left(\frac{4}{5}c\right)^2.$$

Since a, b, c are natural numbers the triple a, $b - \frac{3}{5}c$, $\frac{4}{5}c$ must be a Pythagorean triple. That is to say

$$a = \lambda(u^2 + v^2), \quad b - \frac{3}{5}c = \lambda(u^2 - v^2), \quad \frac{4}{5}c = \lambda(2uv).$$

In the above, u, v are relatively prime natural numbers and $\lambda = 1, 2, 3, \ldots$. The least value of λ that makes c integral is 2. Hence we have the description

$$(a, b, c) = (2(u^2 + v^2),\ (u + 2v)(2u - v),\ 5uv), \quad (u, v) = 1,\ u > \frac{1}{2}v. \quad (1)$$

A similar procedure determines the Heron triangle family $A'B'C'$ that contains the supplementary angle of A, *i.e.*, $\cos A' = -\frac{3}{5}$:

$$(a, b, c) = (2(u^2 + v^2),\ (u - 2v)(2u + v),\ 5uv), \quad (u, v) = 1,\ u > 2v. \quad (2)$$

The reader is invited to check that the family (1) has $\cos A = \frac{3}{5}$ and that (2) has $\cos A' = -\frac{3}{5}$ independently of u and v.

More generally the Heron triangle family determining the common angle A given by $\cos A = \frac{p^2-q^2}{p^2+q^2}$ and the supplementary angle family generated by $\cos A' = -\frac{p^2-q^2}{p^2+q^2}$ are given respectively by

$$\begin{aligned}(a, b, c) = \quad & (pq(u^2 + v^2),\ (pu - qv)(qu + pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{q}{p}v \quad \text{and } p > q.\end{aligned} \quad (3)$$

$$\begin{aligned}(a', b', c') = \quad & (pq(u^2 + v^2),\ (pu + qv)(qu - pv),\ (p^2 + q^2)uv), \\ & (u, v) = (p, q) = 1,\ u > \tfrac{p}{q}v \quad \text{and } p > q.\end{aligned} \quad (4)$$

Areas of (3) and (4) are given by $\frac{1}{2}bc \sin A$ and $\frac{1}{2}b'c' \sin A'$ respectively. Notice that $p = 2$, $q = 1$ in (3) and (4) yield (1) and (2) and that $\angle BAC$ and $\angle B'A'C'$ are supplementary angles. Hence these triangles themselves can be adjoined when $u > \frac{p}{q}v$. The consequences are better understood by a numerical illustration:

$u = 5$, $v = 1$ in (1) and (2) yield $(a, b, c) = (52, 63, 25)$ and $(a', b', c') = (52, 33, 25)$. These can be adjoined along the common side 25. See Figure 1. The result is the isosceles triangle $(96, 52, 52)$ that reduces to $(24, 13, 13)$. As a matter of fact the families (1) and (2) or (3) and (4) may be adjoined likewise to describe the complete set of isosceles Heron triangles:

$$(a, b, c) = (2(u^2 - v^2),\ u^2 + v^2,\ u^2 + v^2), \quad u > v,\ (u, v) = 1. \tag{5}$$

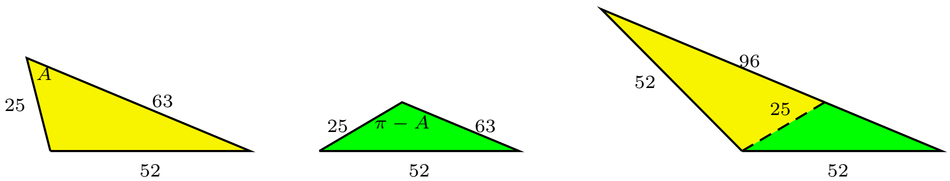

Figure 1

As mentioned in the beginning of this section, the general cases involve routine algebra so the details are left to the reader.

However, the families (1) and (2) or (3) and (4) may be adjoined in another way. This generates the complete set of Heron triangles. Again, we take a numerical illustration.

$u = 3$, $v = 2$ in (1) yields $(a, b, c) = (13, 14, 15)$ (after reduction by the gcd of (a, b, c)). Now we put different values for u, v in (2), say, $u = 4$, $v = 1$. This yields $(a', b', c') = (17, 9, 10)$. It should be remembered that we still have $\angle BAC + \angle B'A'C' = \pi$. As they are, triangles ABC and $A'B'C'$ cannot be adjoined. They must be enlarged suitably by similarity transformations to have $AB = A'B'$, and then adjoined. See Figure 2.

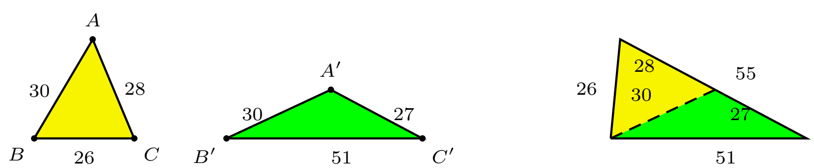

Figure 2

The result is the new Heron triangle $(55, 26, 51)$. More generally, if we put $u = u_1$, $v = v_1$ in (1) or(3) and $u = u_2$, $v = v_2$ in (2) or (4) and after applying the necessary similiarity transformations, the adjoin (after reduction by the gcd) yields

$$(a, b, c) = (u_1 v_1 (u_2^2 + v_2^2),\ (u_1^2 - v_1^2) u_2 v_2 + (u_2^2 - v_2^2) u_1 v_1,\ u_2 v_2 (u_1^2 + v_1^2). \tag{6}$$

This is the same description of Heron triangles that Euler and others obtained [1]. Now we easily see that Brahmagupta took the case of $p = q$ in (3) and (4).

In the next section we extend this remarkable adjoining idea to generate Brahmagupta n-gons, $n > 3$. At this point recall Ptolemy's theorem on convex cyclic quadrilaterals: *The product of the diagonals is equal to the sum of the products of the two pairs of opposite sides*. Here is an important observation: In a convex cyclic quadrilateral with sides a, b, c, d in order and diagonals e, f, Ptolemy's theorem, viz., $ef = ac + bd$ shows that if five of the preceding elements are rational then the sixth one is also rational.

3. Construction of Brahmagupta n-gons, $n > 3$

It is now clear that we can take any number of triangles, either all from one of the families or some from one family and some from the supplementary angle family and place them appropriately to construct a Brahmagupta n-gon. To convince the reader we do illustrate by numerical examples. We extensively deal with the case $n = 4$. This material is different from what has appeared in [5, 6]. The following table shows the primitive (a, b, c) and the suitably enlarged one, also denoted by (a, b, c). T_1 to T_6 are family (1) triangles, and T_7, T_8 are family (2) triangles. These triangles will be used in the illustrations to come later on.

Table 1: Heron triangles

	u	v	Primitive (a, b, c)	Enlarged (a, b, c)
T_1	3	1	$(4, 5, 3)$	$(340, 425, 255)$
T_2	4	1	$(17, 21, 10)$	$(340, 420, 200)$
T_3	5	3	$(68, 77, 75)$	$(340, 385, 375)$
T_4	7	6	$(85, 76, 105)$	$(340, 304, 420)$
T_5	9	2	$(85, 104, 45)$	$(340, 416, 180)$
T_6	13	1	$(68, 75, 13)$	$(340, 375, 65)$
T_7	4	1	$(17, 9, 10)$	$(340, 180, 200)$
T_8	13	1	$(340, 297, 65)$	$(340, 297, 65)$

The same or different Heron triangles can be adjoined in different ways. We first show this in the illustration of the case of quadrilaterals. Once the construction process is clear, the case of $n > 4$ would be analogous to that $n = 4$. Hence we just give one illustration of $n = 5$ and $n = 6$.

3.1. *Brahmagupta quadrilaterals.* The Brahmagupta quadrilateral can be generated in the following ways:

(i) A triangle taken from family (1) (respectively (3)) or family (2) (respectively (4), henceforth this is to be understood) adjoined with itself,

(ii) two different triangles taken from the same family adjoined,

(iii) one triangle taken from family (1) adjoined with a triangle from family (2).

Here are examples of each case.

Example 1. We take the primitive $(a, b, c) = (17, 21, 10)$, *i.e.*, T_2 and adjoin with itself (see Figure 3). Since $\angle CAD = \angle CBD$, $ABCD$ is cyclic. Ptolemy's theorem shows that $AB = \frac{341}{17}$ is rational. By enlarging the sides and diagonals 17 times each we get the Brahmagupta quadrilateral $ABCD$, in fact a trapezoid, with

$$AB = 341,\ BC = AD = 170,\ CD = 289,\ AC = BD = 357.$$

See Figure 3. Rather than calculating the actual area, we give an argument that shows that the area is integral. This is so general that it is applicable to other adjunctions to follow in our discussion.

Since $\angle BAC = \angle BDC$, $\angle ABD = \angle ACD$, and $\angle BAD = \angle BAC + \angle CAD$, $\angle BAD$ is also a Heron angle and that triangle ABD is Heron. (Note:

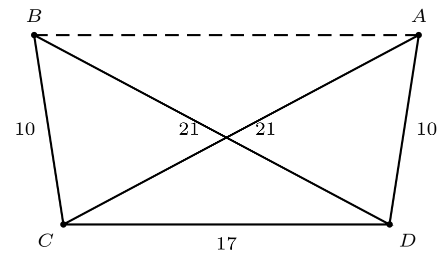

Figure 3

If α and β are Heron angles then $\alpha \pm \beta$ are also Heron angles. To see this consider $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$). $ABCD$ being the disjoint sum of the Heron triangles BCD and BDA, its area must be integral.

This particular adjunction can be done along any side, ***i.e.***, 17, 10, or 21. However, such a liberty is not enjoyed by the remaining constructions which involve adjunction of different Heron triangles. We leave it to the reader to figure out why.

Example 2. We adjoin the primitive triangles T_4, T_5 from Table 1. This can be done in two ways.

(i) Figure 4A illustrates one way. As in Example 1, $AB = \frac{1500}{17}$, so Figure 4A is enlarged 17 times. The area is integral (reasoned as above). Hence the resulting quadrilateral is Brahmagupta.

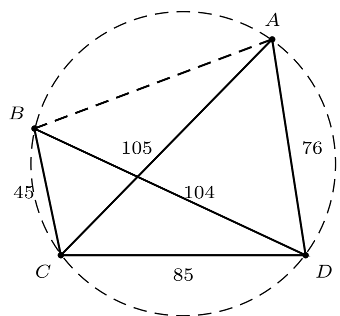

Figure 4A

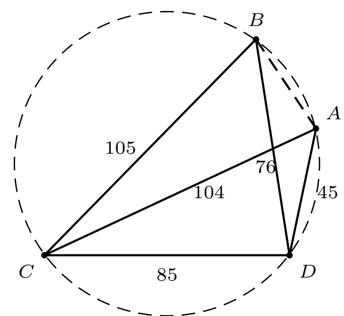

Figure 4B

(ii) Figure 4B illustrates the second adjunction in which the vertices of one base are in reverse order. In this case, $AB = \frac{187}{5}$ hence the figure needs only five times enlargement. Henceforth, we omit the argument to show that the area is integral.

Example 3. We adjoin the primitive triangles T_1 and T_7, which contain supplementary angles A and $\pi - A$. Here, too two ways are possible. In each case no enlargement is necessary. See Figures 5A and 5B.

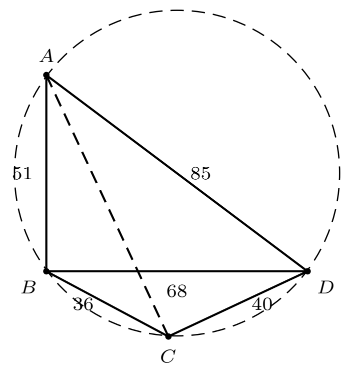

Figure 5A

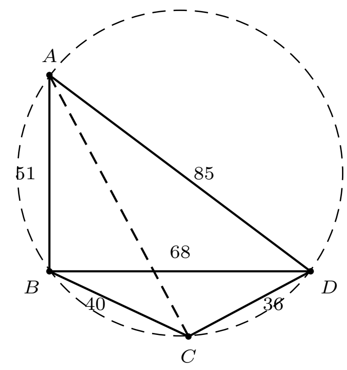

Figure 5B

3.2. *Brahmagupta pentagons.* To construct a Brahmagupta pentagon we need three Heron triangles, in general, taken either all from (1) or some from (1) and the rest from (2) in any combination. Here, too, one triangle can be used twice as in Example 1 above. Hence, a Brahmagupta pentagon can be constructed in more than two ways. We give just one illustration using the (enlarged) triangles T_3, T_4, and T_7. The reader is invited to play the adjuction game using these to consider all possibilities, *i.e.*, T_3, T_3, T_4; T_3, T_4, T_4; T_7, T_7, T_3 etc.

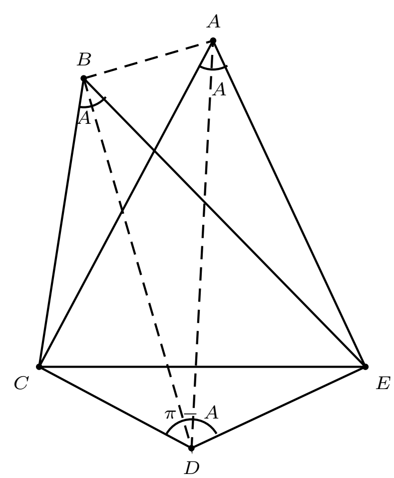

Figure 6

Figure 6 shows one Brahmagupta pentagon. It is easy to see that it must be cyclic. The side AB, the diagonals AD and BD are to be calculated. We apply Ptolemy's theorem successively to $ABCE$, $ACDE$ and $BCDE$. This yields

$$AB = \frac{2023}{17}, \quad AD = \frac{7215}{17}, \quad BD = \frac{6820}{17}.$$

The figure needs 17 times enlargement. The area $ABCDE$ must be integral because it is the disjoint sum of the Brahmagupta quadrilateral $ABCE$ and the Heron triangle ACD.

3.3. *Brahmagupta hexagons.* To construct a Brahmagupta hexagon it is now easy to see that we need at most four Heron triangles taken in any combination from the families (1) and (2). We use the four triangles T_2, T_3, T_5, T_8 to illustrate the hexagon in Figure 7. We leave the calculations to the reader.

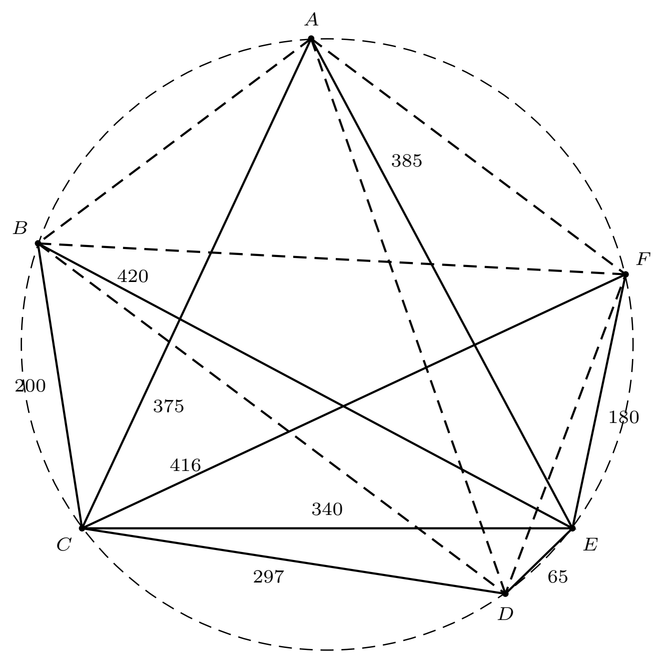

Figure 7

4. Conclusion

In principle the problem of determining Brahmagupta n-gons, $n > 3$, has been solved because all Heron triangle families have been determined by (3) and (4) (in fact by (3) alone). In general to construct a Brahmagupta n-gon, at most $n-2$ Heron triangles taken in any combination from (3) and (4) are needed. They can be adjoined as described in this paper. We pose the following counting problem to the reader.

Given $n-2$ Heron triangles, (i) all from a single family, or (ii) m from one Heron family and the remaining $n-m-2$ from the supplementary angle family, how many Brahmagupta n-gons can be constructed?

It is now natural to conjecture that Heron triangles chosen from appropriate families adjoin to give Heron n-gons. To support this conjecture we give two Heron quadrilaterals generated in this way.

Example 4. From the $\cos\theta = \frac{3}{5}$ family, $7(5,5,6)$ and $6(4,13,15)$ adjoined with $(35,53,24)$ and $6(7,15,20)$ from the supplementary family (with $\cos\theta = -\frac{3}{5}$) to give $ABCD$ with

$$AB = 35,\ BC = 53,\ CD = 78,\ AD = 120,\ AC = 66,\ BD = 125,$$

and area 3300. See Figure 8A.

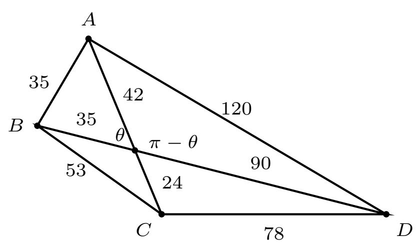

Figure 8A

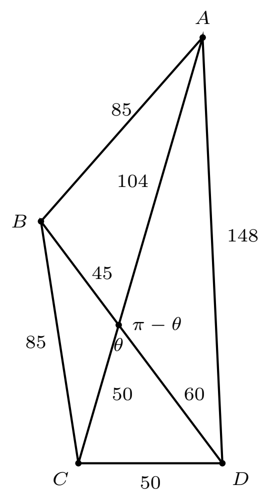

Figure 8B

Example 5. From the same families, the Heron triangles $10(5,5,6)$, $(85,45,104)$ with $5(17,9,10)$ and $4(37,15,26)$ to give a Heron quadrilateral $ABCD$ with

$$AB = 85,\ BC = 85,\ CD = 50,\ AD = 148,\ AC = 154,\ BD = 105,$$

and area 6468. See Figure 8B.

Now, the haunting question is: Which appropriate two members of the θ family adjoin with two appropriate members of the $\pi - \theta$ family to generate Heron quadrilaterals?

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Forum Geometricorum
Volume 5 (2005) 127–132.

FORUM GEOM
ISSN 1534-1178

Another Proof of van Lamoen's Theorem and Its Converse

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Abstract. We give a proof of Floor van Lamoen's theorem and its converse on the circumcenters of the cevasix configuration of a triangle using the notion of directed angle of two lines.

1. Introduction

Let P be a point in the plane of triangle ABC with traces A', B', C' on the sidelines BC, CA, AB respectively. We assume that P does not lie on any of the sidelines. According to Clark Kimberling [1], triangles PCB', $PC'B$, PAC', $PA'C$, PBA', $PB'A$ form the *cevasix configuration* of P. Several years ago, Floor van Lamoen discovered that when P is the centroid of triangle ABC, the six circumcenters of the cevasix configuration are concylic. This was posed as a problem in the *American Mathematical Monthly* and was solved in [2, 3]. In 2003, Alexei Myakishev and Peter Y. Woo [4] gave a proof for the converse, that is, if the six circumcenters of the cevasix configuration are concylic, then P is either the centroid or the orthocenter of the triangle.

In this note we give a new proof, which is quite different from those in [2, 3], of Floor van Lamoen's theorem and its converse, using the directed angle of two lines. Remarkably, both necessity part and sufficiency part in our proof are basically the same. The main results of van Lamoen, Myakishev and Woo are summarized in the following theorem.

Theorem. *Given a triangle ABC and a point P, the six circumcenters of the cevasix configuration of P are concyclic if and only if P is the centroid or the orthocenter of ABC.*

We shall assume the given triangle non-equilateral, and omit the easy case when ABC is equilateral. For convenience, we adopt the following notations used in [4].

Triangle	PCB'	$PC'B$	PAC'	$PA'C$	PBA'	$PB'A$
Notation	$\triangle(A_+)$	$\triangle(A_-)$	$\triangle(B_+)$	$\triangle(B_-)$	$\triangle(C_+)$	$\triangle(C_-)$
Circumcenter	A_+	A_-	B_+	B_-	C_+	C_-

It is easy to see that two of these triangles may possibly share a common circumcenter only when they share a common vertex of triangle ABC.

Publication Date: August 24, 2005. Communicating Editor: Floor van Lamoen.

The author thansk Le Chi Quang of Hanoi, Vietnam for his help in translation and preparation of the article.

2. Preliminary Results

Lemma 1. *Let P be a point not on the sidelines of triangle ABC, with traces B', C' on AC, AB respectively. The circumcenters of triangles APB' and APC' coincide if and only if P lies on the reflection of the circumcircle ABC in the line BC.*

The Proof of Lemma 1 is simple and can be found in [4]. We also omit the proof of the following easy lemma.

Lemma 2. *Given a triangle ABC and M, N on the line BC, we have*

$$\frac{\overline{BC}}{\overline{MN}} = \frac{S[ABC]}{S[AMN]},$$

where $\overline{BC}$ and $\overline{MN}$ denote the signed lengths of the line segments BC and MN, and $S[ABC]$, $S[AMN]$ the signed areas of triangle ABC, and AMN respectively.

Lemma 3. *Let P be a point not on the sidelines of triangle ABC, with traces A', B', C' on BC, AC, AB respectively, and K the second intersection of the circumcircles of triangles PCB' and $PC'B$. The line PK is a symmedian of triangle PBC if and only if A' is the midpoint of BC.*

Proof. Triangles $KB'B$ and KCC' are directly similar (see Figure 1). Therefore,

$$\frac{S[KB'B]}{S[KCC']} = \Big(\frac{\overline{B'B}}{\overline{CC'}}\Big)^2.$$

On the other hand, by Lemma 2 we have

$$\frac{S[KPB]}{S[KPC]} = \frac{\frac{\overline{PB}}{\overline{B'B}} \cdot S[KB'B]}{\frac{\overline{PC}}{\overline{CC'}} \cdot S[KCC']}.$$

Thus,

$$\frac{S[KPB]}{S[KPC]} = \frac{\overline{PB}}{\overline{PC}}.\frac{\overline{B'B}}{\overline{CC'}}.$$

It follows that PK is a symmedian line of triangle PBC, which is equivalent to the following

$$\frac{S[KPB]}{S[KPC]} = -\Big(\frac{\overline{PB}}{\overline{PC}}\Big)^2, \quad \frac{\overline{PB}.\overline{B'B}}{\overline{PC}.\overline{CC'}} = -\Big(\frac{\overline{PB}}{\overline{PC}}\Big)^2, \quad \frac{\overline{B'B}}{\overline{C'C}} = \frac{\overline{PB}}{\overline{PC}}.$$

The last equality is equivalent to $BC \parallel B'C'$, by Thales' theorem, or A' is the midpoint of BC, by Ceva's theorem. □

Remark. Since the lines BC' and CB' intersect at A, the circumcircles of triangles PCB' and $PC'B$ must intersect at two distinct points. This remark confirms the existence of the point K in Lemma 3.

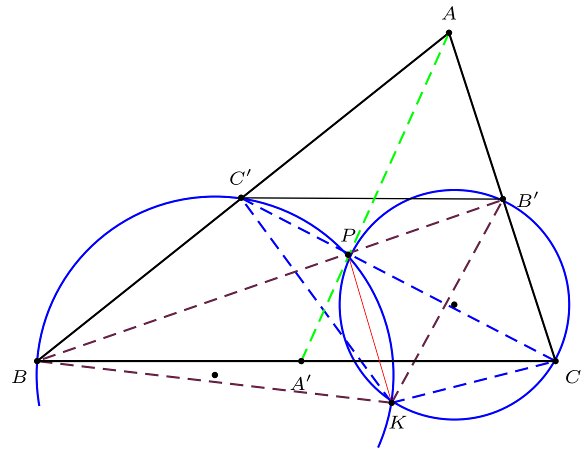

Figure 1

Lemma 4. *Given a triangle XYZ and pairs of points M, N on YZ, P, Q on ZX, and R, S on XY respectively. If the points in each of the quadruples P, Q, R, S; R, S, M, N; M, N, P, Q are concyclic, then all six points M, N, P, Q, R, S are concyclic.*

Proof. Suppose that (O_1), (O_2), (O_3) are the circles passing through the quadruples (P, Q, R, S), (R, S, M, N), and (M, N, P, Q) respectively. If O_1, O_2, O_3 are disctinct points, then YZ, ZX, XY are respectively the radical axis of pairs of circles $(O_2), (O_3)$; $(O_3), (O_1)$; $(O_1), (O_2)$. Hence, YZ, ZX, XY are concurrent, or parallel, or coincident, which is a contradiction. Therefore, two of the three points O_1, O_2, O_3 coincide. It follows that six points M, N, P, Q, R, S are concyclic. □

Remark. In Lemma 4, if $M = N$ and the circumcircles of triangles RSM, MPQ touch YZ at M, then the five points M, P, Q, R, S lie on the same circle that touches YZ at the same point M.

3. Proof of the main theorem

Suppose that perpendicular bisectors of AP, BP, CP bound a triangle XYZ. Evidently, the following pairs of points B_+, C_-; C_+, A_-; A_+, B_- lie on the lines YZ, ZX, XY respectively. Let H and K respectively be the feet of the perpendiculars from P on A_-A_+, B_-B_+ (see Figure 2).

Sufficiency part. If P is the orthocenter of triangle ABC, then $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$. Obviously, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle. If P is the centroid of triangle ABC, then no more than one of the three following possibilities happen: $B_+ = C_-$; $C_+ = A_-$; $A_+ = B_-$, by Lemma 1. Hence, we need to consider two cases.

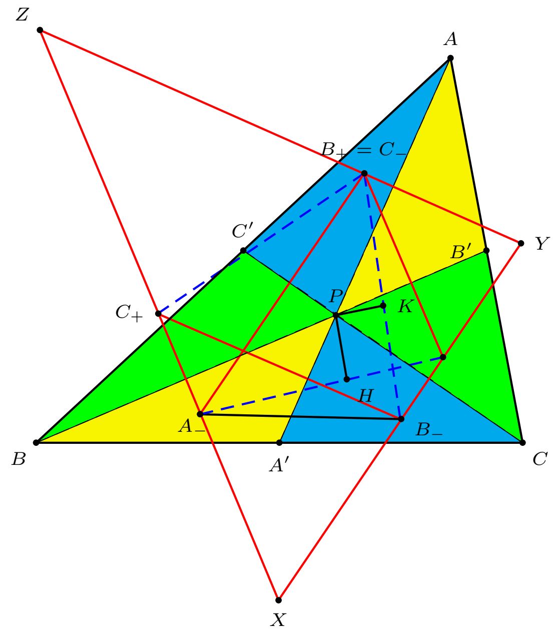

Figure 2

Case 1. Only one of three following possibilities occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Without loss of generality, we may assume that $B_+ = C_-$, $C_+ \neq A_-$ and $A_+ \neq B_-$ (see Figure 2). Since P is the centroid of triangle ABC, A' is the midpoint of the segment BC. By Lemma 3, we have

$$(PH, PB) = (PC, PA') \pmod{\pi}.$$

In addition, since A_-A_+, A_-C_+, B_-A_+, B_-C_+ are respectively perpendicular to PH, PB, PC, PA', we have

$$(A_-A_+, A_-C_+) \equiv (PH, PB) \pmod{\pi}.$$
$$(B_-A_+, B_-C_+) \equiv (PC, PA') \pmod{\pi}.$$

Thus, $(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}$, which implies that four points C_+, A_-, A_+, B_- are concyclic.

Similarly, we have

$$(PK, PC) = (PA, PB') \pmod{\pi}.$$

Moreover, since B_-B_+, B_-A_+, YZ, B_+A_+ are respectively perpendicular to PK, PC, PA, PB', we have

$$(B_-B_+, B_-A_+) \equiv (PK, PC) \pmod{\pi}.$$
$$(YZ, B_+A_+) \equiv (PA, PB') \pmod{\pi}.$$

Thus, $(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}$, which implies that the circumcircle of triangle $B_+B_-A_+$ touches YZ at B_+.

The same reasoning also shows that the circumcircle of triangle $B_+C_+A_-$ touches YZ at B_+.

Therefore, the six points B_+, C_-, C_+, A_-, A_+, B_- lie on the same circle and this circle touches YZ at $B_+ = C_-$ by the remark following Lemma 4.

Case 2. None of the three following possibilities occurs: $B_+ = C_-; C_+ = A_-; A_+ = B_-$.

Similarly to case 1, each quadruple of points (C_+, A_-, A_+, B_-), (A_+, B_-, B_+, C_-), (B_+, C_-, C_+, A_-) are concyclic. Hence, by Lemma 4, the six points B_+, C_-, C_+, A_-, A_+, B_- are concyclic.

Necessity part. There are three cases.

Case 1. No less than two of the following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

By Lemma 1, P is the orthocenter of triangle ABC.

Case 2. Only one of the following possibilites occurs: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$. We assume without loss of generality that $B_+ = C_-$, $C_+ \neq A_-$, $A_+ \neq B_-$.

Since the six points B_+, C_-, C_+, A_-, A_+, B_- are on the same circle, so are the four points C_+, A_-, A_+, B_-. It follows that

$$(A_-A_+, A_-C_+) \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Note that lines PH, PB, PC, PA' are respectively perpendicular to A_-A_+, A_-C_+, B_-A_+, B_-C_+. It follows that

$$(PH, PB) \equiv (A_-A_+, A_-C_+) \pmod{\pi}.$$
$$(PC, PA') \equiv (B_-A_+, B_-C_+) \pmod{\pi}.$$

Therefore, $(PH, PB) \equiv (PC, PA') \pmod{\pi}$. Consequently, A' is the midpoint of BC by Lemma 3.

On the other hand, it is evident that $B_+A_- \parallel B_-A_+$; $B_+A_+ \parallel C_+A_-$, and we note that each quadruple of points (B_+, A_-, B_-, A_+), (B_+, A_+, C_+, A_-) are concyclic. Therefore, we have $B_+B_- = A_+A_- = B_+C_+$. It follows that triangle $B_+B_-C_+$ is isosceles with $C_+B_+ = B_+B_-$. Note that YZ passes B_+ and is parallel to C_+B_-, so that we have YZ touches the circle passing six points $B_+ = C_-$, C_+, A_-, A_+, B_- at $B_+ = C_-$. It follows that

$$(B_-B_+, B_-A_+) \equiv (YZ, B_+A_+) \pmod{\pi}.$$

In addition, since PK, PC, PA, PB' are respectively perpendicular to B_-B_+, B_-A_+, YZ, B_+A_+, we have

$$(PK, PC) \equiv (B_-B_+, B_-A_+) \pmod{\pi}.$$
$$(PA, PB') \equiv (YZ, B_+A_+) \pmod{\pi}.$$

Thus, $(PK, PC) \equiv (PA, PB') \pmod{\pi}$. By Lemma 3, B' is the midpoint of CA. We conclude that P is the centroid of triangle ABC.

Case 3. None of the three following possibilities occur: $B_+ = C_-$, $C_+ = A_-$, $A_+ = B_-$.

Similarly to case 2, we can conclude that A', B' are respectively the midpoints of BC, CA. Thus, P is the centroid of triangle ABC.

This completes the proof of the main theorem.

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Forum Geometricorum
Volume 5 (2005) 133–134.

FORUM GEOM
ISSN 1534-1178

Some More Archimedean Circles in the Arbelos

Frank Power

Abstract. We construct 4 circles in the arbelos which are congruent to the Archimedean twin circles.

Thomas Schoch [2] tells the remarkable story of his discovery in the 1970's of the many Archimedean circles in the arbelos (shoemaker's knife) that were eventually recorded in the paper [1]. In this note, we record four more Archimedean circles which were discovered in the summer of 1998, when the present author took a geometry course ([3]) with one of the authors of [1].

Consider an arbelos with inner semicircles $\mathcal{C}_1$ and $\mathcal{C}_2$ of radii a and b, and outer semicircle $\mathcal{C}$ of radius $a+b$. It is known the Archimedean circles have radius $t=\frac{ab}{a+b}$. Let Q_1 and Q_2 be the "highest" points of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Theorem. *A circle tangent to $\mathcal{C}$ internally and to OQ_1 at Q_1 (or OQ_2 at Q_2) has radius $t=\frac{ab}{a+b}$.*

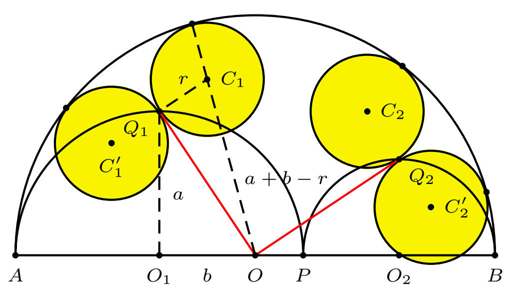

Figure 1

Proof. There are two such circles tangent at Q_1, namely, (C_1) and (C_1') in Figure 1. Consider one such circle (C_1) with radius r. Note that

$$OQ_1^2 = O_1Q_1^2 + OO_1^2 = a^2 + b^2.$$

It follows that

$$(a+b-r)^2 = (a^2+b^2) + r^2,$$

from which $r=\frac{ab}{a+b}=t$. The same calculation shows that (C_1') also has radius t, and similarly for the two circles at Q_2. □

Publication Date: September 2, 2005. Communicating Editor: Paul Yiu.

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Forum Geometricorum
Volume 5 (2005) 135–136.

FORUM GEOM
ISSN 1534-1178

Divison of a Segment in the Golden Section with Ruler and Rusty Compass

Kurt Hofstetter

Abstract. We give a simple 5-step division of a segment into golden section, using ruler and rusty compass.

In [1] we have given a 5-step division of a segment in the golden section with ruler and compass. We modify the construction by using a *rusty* compass, *i.e.*, one when set at a particular opening, is not permitted to change. For a point P and a segment AB, we denote by $P(AB)$ the circle with P as center and AB as radius.

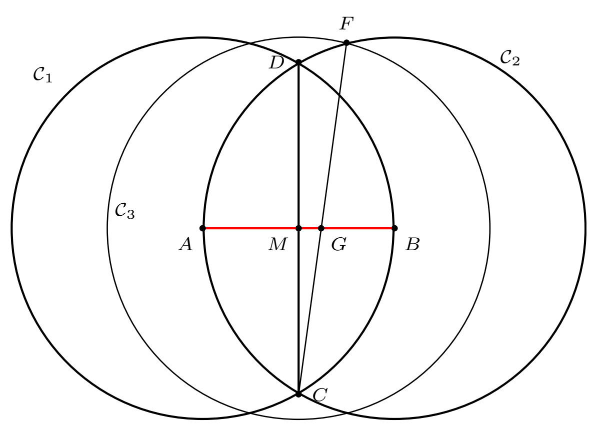

Figure 1

Construction. Given a segment AB, construct

(1) $\mathcal{C}_1 = A(AB)$,
(2) $\mathcal{C}_2 = B(AB)$, intersecting $\mathcal{C}_1$ at C and D,
(3) the line CD to intersect AB at its midpoint M,
(4) $\mathcal{C}_3 = M(AB)$ to intersect $\mathcal{C}_2$ at F (so that C and D are on opposite sides of AB),
(5) the segment CF to intersect AB at G.

The point G divides the segment AB in the golden section.

Publication Date: September 13, 2005. Communicating Editor: Paul Yiu.

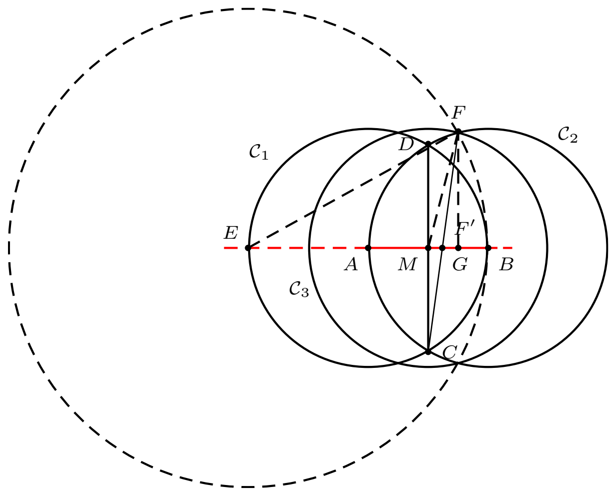

Figure 2

Proof. Extend BA to intersect $\mathcal{C}_1$ at E. According to [1], it is enough to show that $EF = 2 \cdot AB$. Let F' be the orthogonal projection of F on AB. It is the midpoint of MB. Without loss of generality, assume $AB = 4$, so that $MF' = F'B = 1$ and $EF' = 2 \cdot AB - F'B = 7$. Applying the Pythagorean theorem to the right triangles EFF' and MFF', we have

$$\begin{aligned} EF^2 =& EF'^2 + FF'^2 \\ =& EF'^2 + MF^2 - MF'^2 \\ =& 7^2 + 4^2 - 1^2 \\ =& 64. \end{aligned}$$

This shows that $EF = 8 = 2 \cdot AB$. □

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Forum Geometricorum
Volume 5 (2005) 137–141.

FORUM GEOM
ISSN 1534-1178

On an Erdős Inscribed Triangle Inequality

Ricardo M. Torrejón

Abstract. A comparison between the area of a triangle and that of an inscribed triangle is investigated. The result obtained extend a result of Aassila giving insight into an inequality of P. Erdős.

1. Introduction

Consider a triangle ABC divided into four smaller non-degenerate triangles, a central one $C_1A_1B_1$ inscribed in ABC and three others on the sides of this central triangle, as depicted in

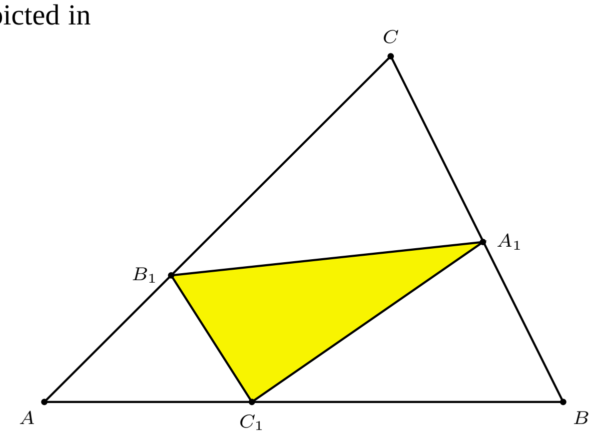

Figure 1

A question with a long history is that of comparing the area of ABC to that of the inscribed triangle $C_1A_1B_1$. In 1956, H. Debrunnner [5] proposed the inequality

$$\mathbf{area}\,(C_1A_1B_1) \geq \min\left\{\mathbf{area}\,(AC_1B_1),\ \mathbf{area}\,(C_1BA_1),\ \mathbf{area}\,(B_1A_1C)\right\}; \tag{1}$$

according to John Rainwater [7], this inequality originated with P. Erdős and was communicated by N. D. Kazarinoff and J. R. Isbell. However, Rainwater was more precise in stating that $C_1A_1B_1$ cannot have the smallest area of the four unless all four are equal with A_1, B_1, and C_1 the midpoints of the sides BC, CA, and AB.

A proof of (1) first appeared in A. Bager [2] and later in A. Bager [3] and P. H. Diananda [6]. Diananda's proof is particularly noteworthy; in addition to proving Erdős' inequality, it also shows that the stronger form of (1) holds

$$\mathbf{area}\,(C_1A_1B_1) \geq \sqrt{\mathbf{area}\,(AC_1B_1) \cdot \mathbf{area}\,(C_1BA_1)} \tag{2}$$

where, without loss of generality, it is assumed that

$$0 < \mathbf{area}\,(AC_1B_1) \leq \mathbf{area}\,(C_1BA_1) \leq \mathbf{area}\,(B_1A_1C)\,.$$

Publication Date: September 28, 2005. Communicating Editor: Paul Yiu.

The purpose of this paper is to show that a sharper inequality is possible when more care is placed in choosing the points $A_1,\ B_1$ and C_1. In so doing we extend Aassila's inequality [1]:

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC),$$

which is valid when these points are chosen so as to partition the perimeter of ABC into equal length segments. Our main result is

Theorem 1. *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\frac{AB+BA_1}{AC+CA_1}=\frac{BC+CB_1}{AB+AB_1}=\frac{AC+AC_1}{BC+BC_1}=\alpha,$$

then

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC)+s^4\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)^{-1}$$

where s is the semi-perimeter of ABC.

When $\alpha=1$ we obtain Aassila's result.

Corollary 2 (Aassila [1])**.** *Let ABC be a triangle, and let $A_1,\ B_1,\ C_1$ be on $BC,\ CA,\ AB$, respectively, with none of $A_1,\ B_1,\ C_1$ coinciding with a vertex of ABC. If*

$$\begin{aligned}AB+BA_1&=AC+CA_1,\\BC+CB_1&=AB+AB_1,\\AC+AC_1&=BC+BC_1,\end{aligned}$$

then

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\le\mathbf{area}\,(ABC)\,.$$

2. Proof of Theorem 1

We shall make use of the following two lemmas.

Lemma 3 (Curry [4])**.** *For any triangle ABC, and standard notation,*

$$4\sqrt{3}\cdot\mathbf{area}\,(ABC)\le\frac{9abc}{a+b+c}. \tag{3}$$

Equality holds if and only if $a=b=c$.

Lemma 4. *For any triangle ABC, and standard notation,*

$$\min\{a^2+b^2+c^2,\ ab+bc+ca\}\ge 4\sqrt{3}\cdot\mathbf{area}\,(ABC)\,. \tag{4}$$

To prove Theorem 1, we begin by computing the area of the corner triangle AC_1B_1:

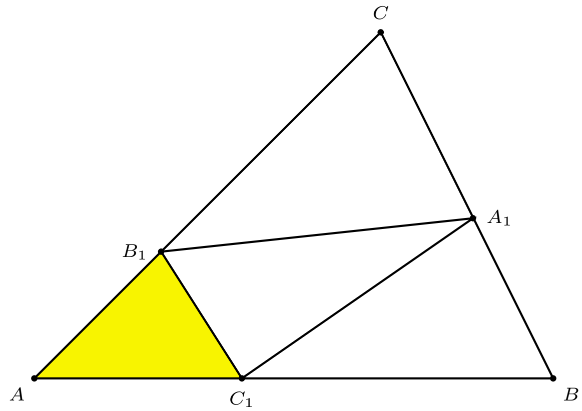

Figure 2

then

$$\begin{aligned}\mathbf{area}\,(AC_1B_1) &= \frac{1}{2}AC_1 \cdot AB_1 \cdot \sin A \\ &= \frac{1}{2}AC_1 \cdot AB_1 \cdot \frac{2 \cdot \mathbf{area}\,(ABC)}{AB \cdot AC} \\ &= \frac{AC_1}{AB} \cdot \frac{AB_1}{AC} \cdot \mathbf{area}\,(ABC)\,.\end{aligned}$$

For the semi-perimeter s of ABC we have

$$\begin{aligned}2s &= AB + BC + AC \\ &= (AB + AB_1) + (BC + CB_1) \\ &= (\alpha + 1)(c + AB_1),\end{aligned}$$

and

$$AB_1 = \frac{2}{\alpha + 1}s - c$$

where $c = AB$. Also,

$$\begin{aligned}2s &= AB + BC + AC \\ &= (AC + AC_1) + (BC + BC_1) \\ &= \left(1 + \frac{1}{\alpha}\right)(AC + AC_1) \\ &= \frac{\alpha + 1}{\alpha}(b + AC_1),\end{aligned}$$

and

$$AC_1 = \frac{2\alpha}{\alpha + 1}s - b$$

with $b = AC$. Hence

$$\mathbf{area}\,(AC_1B_1) = \frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s-b\right)\left(\frac{2}{\alpha+1}s-c\right)\cdot\mathbf{area}\,(ABC)\,. \tag{5}$$

Similar computations yield

$$\mathbf{area}\,(C_1BA_1) = \frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s-c\right)\left(\frac{2}{\alpha+1}s-a\right)\cdot\mathbf{area}\,(ABC), \tag{6}$$

and

$$\mathbf{area}\,(B_1A_1C) = \frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s-a\right)\left(\frac{2}{\alpha+1}s-b\right)\cdot\mathbf{area}\,(ABC)\,. \tag{7}$$

From these formulae,

$$\begin{aligned}
&\mathbf{area}\,(A_1B_1C_1)\\
=&\,\mathbf{area}\,(ABC)-\mathbf{area}\,(AC_1B_1)-\mathbf{area}\,(C_1BA_1)-\mathbf{area}\,(B_1A_1C)\\
=&\left[1-\frac{1}{bc}\left(\frac{2\alpha}{\alpha+1}s-b\right)\left(\frac{2}{\alpha+1}s-c\right)-\frac{1}{ca}\left(\frac{2\alpha}{\alpha+1}s-c\right)\left(\frac{2}{\alpha+1}s-a\right)\right.\\
&\quad\left.-\frac{1}{ab}\left(\frac{2\alpha}{\alpha+1}s-a\right)\left(\frac{2}{\alpha+1}s-b\right)\right]\cdot\mathbf{area}\,(ABC)\\
=&\frac{1}{abc}\left[\left(\frac{2}{\alpha+1}s-a\right)\left(\frac{2}{\alpha+1}s-b\right)\left(\frac{2}{\alpha+1}s-c\right)\right.\\
&\quad\left.+\left(\frac{2\alpha}{\alpha+1}s-a\right)\left(\frac{2\alpha}{\alpha+1}s-b\right)\left(\frac{2\alpha}{\alpha+1}s-c\right)\right]\cdot\mathbf{area}\,(ABC)\,.
\end{aligned}$$

But

$$\begin{aligned}
&\left(\frac{2}{\alpha+1}s-a\right)\left(\frac{2}{\alpha+1}s-b\right)\left(\frac{2}{\alpha+1}s-c\right)\\
+&\left(\frac{2\alpha}{\alpha+1}s-a\right)\left(\frac{2\alpha}{\alpha+1}s-b\right)\left(\frac{2\alpha}{\alpha+1}s-c\right)\\
=&2(s-a)(s-b)(s-c)+2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3\\
=&\frac{2}{s}\left[\mathbf{area}\,(ABC)\right]^2+2\left(\frac{\alpha-1}{\alpha+1}\right)^2 s^3.
\end{aligned}$$

Hence

$$\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1) = \left[\mathbf{area}\,(ABC)\right]^3+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)\,. \tag{8}$$

From (3) and (4)

$$\begin{aligned}\frac{abc\cdot s}{2} &\geq \frac{\sqrt{3}}{9}\cdot(a+b+c)^2\cdot\mathbf{area}\,(ABC)\\ &\geq \frac{\sqrt{3}}{9}[a^2+b^2+c^2+2(ab+bc+ca)]\cdot\mathbf{area}\,(ABC)\\ &\geq \frac{\sqrt{3}}{9}\cdot 12\sqrt{3}\cdot\mathbf{area}\,(ABC)^2\\ &\geq 4\cdot\mathbf{area}\,(ABC)^2.\end{aligned}$$

Finally, from (8)

$$\begin{aligned}&4\cdot\mathbf{area}\,(ABC)^2\cdot\mathbf{area}\,(A_1B_1C_1)\\ \leq\ &\frac{abc\cdot s}{2}\cdot\mathbf{area}\,(A_1B_1C_1)\\ \leq\ &[\mathbf{area}\,(ABC)]^3+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot\mathbf{area}\,(ABC)\end{aligned}$$

and a division by $\mathbf{area}\,(ABC)^2$ produces

$$4\cdot\mathbf{area}\,(A_1B_1C_1)\leq\mathbf{area}\,(ABC)+s^4\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2\cdot[\mathbf{area}\,(ABC)]^{-1}$$

completing the proof of the theorem.

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Forum Geometricorum
Volume 5 (2005) 143–148.

FORUM GEOM
ISSN 1534-1178

Applications of Homogeneous Functions to Geometric Inequalities and Identities in the Euclidean Plane

Wladimir G. Boskoff and Bogdan D. Suceavă

Abstract. We study a class of geometric identities and inequalities that have a common pattern: they are generated by a homogeneous function. We show how to extend some of these homogeneous relations in the geometry of triangle. Then, we study the geometric configuration created by two intersecting lines and a pencil of n lines, where the repeated use of Menelaus's Theorem allows us to emphasize a result on homogeneous functions.

1. Introduction

The purpose of this note is to present an extension of a certain class of geometric identities or inequalities. The idea of this technique is inspired by the study of homogeneous polynomials and has the potential for additional applications besides the ones described here.

First of all, we recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous* if $f(tx_1, tx_2, ..., tx_n) = t^m f(x_1, x_2, ...x_n)$, for $t \in \mathbb{R} - \{0\}$ and $x_i \in \mathbb{R}$, $i = 1, ..., n$, $m, n \in \mathbb{N}$, $m \neq 0$, $n \geq 2$. The natural number m is called the degree of the homogeneous function f.

Remarks. 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function. If for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we have $f(x) \geq 0$, then $f(tx) \geq 0$, for $t > 0$. Furthermore, if m is an even natural number, $f(x) \geq 0$, yields $f(tx) \geq 0$ for any real number t.

2. Any $x > 0$ can be written as $x = \frac{a}{b}$, with $a, b \in (0, 1)$.

2. Application to the geometry of triangle

Consider the homogeneous function $f_\alpha : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f_\alpha(x_1, x_2, x_3) = \alpha x_1 x_2 x_3,$$

with $\alpha \in \mathbb{R} - \{0\}$. Denote by a, b, c the lengths of the sides of a triangle ABC, by R the circumradius and by $\triangle$ the area of this triangle. By the law of sines, we get

$$f_1(a, b, c) = f_1(a, b, 2R \sin C) = 2R f_1(a, b, \sin C) = 4R\triangle.$$

Thus, we obtain $abc = 4R\triangle$.

Publication Date: October 11, 2005. Communicating Editor: Paul Yiu.

Since $f_1(a,b,c) = 8R^3 f_1(\sin A, \sin B, \sin C)$, we get also the equality

$$\triangle = 2R^2 \sin A \sin B \sin C.$$

Heron's formula can be represented by the following setting. The function $f_{\sqrt{r}}(x_1, x_2, x_3)$ for $x_1 = \sqrt{s-a}$, $x_2 = \sqrt{s-b}$, $x_3 = \sqrt{s-c}$, yields

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = \triangle.$$

Furthermore, using $\cot \frac{A}{2} = \frac{s-a}{r}$ and the similar equalities in B and C, we obtain

$$f_{\sqrt{s}}(\sqrt{s-a}, \sqrt{s-b}, \sqrt{s-c}) = r\sqrt{r} f_{\sqrt{s}}\left(\sqrt{\cot \frac{A}{2}}, \sqrt{\cot \frac{B}{2}}, \sqrt{\cot \frac{C}{2}}\right),$$

which yields

$$\triangle = r^2 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

3. Homogeneous polynomials in $a^2, b^2, c^2, \triangle$ and their applications

Consider now a triangle ABC in the Euclidean plane, and denote by a, b, c the length of its sides and by $\triangle$ its area. We prove the following.

Proposition 1. *Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) \geq 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have:*

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle\right) \geq 0. \tag{1}$$

Proof. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. In the triangle ABC we consider A_1 and B_1 on the sides BC and AC, respectively, such that $CA_1 = \alpha a$, $BC = a$, $CB_1 = \beta b$, $AC = b$, with $\alpha, \beta \in (0,1)$. It results that the area of triangle CA_1B_1 is $\sigma[CA_1B_1] = \alpha\beta\triangle$. By the law of cosines we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

and therefore

$$A_1B_1^2 = \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2).$$

Since the given inequality $p(a^2, b^2, c^2, \triangle) \geq 0$ takes place in any triangle, then it must take place also in the triangle CA_1B_1, thus

$$p(\alpha^2 a^2, \beta^2 b^2, \alpha\beta c^2 + (\alpha - \beta)(\alpha a^2 - \beta b^2), \alpha\beta\triangle) = 0.$$

Let us take now $t = \alpha\beta$, and $x = \frac{\alpha}{\beta}$, with $\alpha, \beta \in (0,1)$. For $x \in (0, \infty)$, we have

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + q(x), \triangle\right) \geq 0.$$

$\square$

Remark. In terms of identities, we state the following. Let $p : \mathbb{R}^4 \to \mathbb{R}$ a homogeneous function with the property that $p(a^2, b^2, c^2, \triangle) = 0$, for any triangle in the Euclidean plane. Then for any $x > 0$ we have

$$p\left(xa^2, \frac{1}{x}b^2, c^2 + \left(1 - \frac{1}{x}\right)(xa^2 - b^2), \triangle\right) = 0. \tag{2}$$

The proof is similar to the proof of Proposition 1.

We present now a few applications of Proposition 1.

3.1. In any triangle ABC in the Euclidean plane, for any $x \in (0, \infty)$, we have

$$4\triangle \le \min\left[xa^2 + \frac{1}{x}b^2, xa^2 + c^2 + q(x), \frac{1}{x}b^2 + c^2 + q(x)\right].$$

To prove this inequality, it is sufficient to prove the statement for $x = 1$, then we apply Proposition 1. Let us assume, without losing any generality, that $a \ge b \ge c$. We also use $b^2 + c^2 \ge 2bc$, and $2bc \ge 2bc \sin A = 4\triangle$. Thus, $b^2 + c^2 \ge 4\triangle$, and this means

$$4\triangle \le \min(b^2 + c^2, a^2 + c^2, a^2 + b^2).$$

Applying this result in the triangle CA_1B_1, considered as in the proof of Proposition 1, we obtain the stated inequality.

3.2. Consider $q(x) = \left(1 - \frac{1}{x}\right)(xa^2 - b^2)$, for $x > 0$. Then in any triangle we have the inequality

$$a^2b^2[c^2 + q(x)] \ge \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

This results as a direct consequence of Carlitz' inequality

$$a^2b^2c^2 \ge \left(\frac{4\triangle}{3\sqrt{3}}\right)^3.$$

by applying Proposition 1.

3.3. It is known that in any triangle we have Hadwiger's inequality

$$a^2 + b^2 + c^2 \ge \triangle\sqrt{3}.$$

This inequality can be generalized for any $x \in (0, \infty)$ as follows

$$(2x - 1)a^2 + \left(\frac{2}{x} + 1\right)b^2 + c^2 \ge 4\triangle\sqrt{3}.$$

(This inequality appears in *Matematika v Shkole,* No. 5, 1989.)

Hadwiger's inequality can be proven by using the law of cosines to get

$$a^2 + b^2 + c^2 = 2(b^2 + c^2) - 2bc\cos A.$$

Then, keeping in mind that $2\triangle = bc\sin A$, we get

$$\begin{aligned} a^2+b^2+c^2-4\triangle\sqrt{3} =& 2(b^2+c^2-2bc\cos A-2bc\sqrt{3}\sin A) \\ =& 2\left(b^2+c^2-4bc\cos\left(\frac{\pi}{3}-A\right)\right) \\ \geq& 2\left(b^2+c^2-4bc\cos\frac{\pi}{3}\right) \\ =& 2(b-c)^2 \\ \geq& 0. \end{aligned}$$

The equality holds when $b=c$ and $A=\frac{\pi}{3}$, i.e. when triangle ABC is equilateral.

Applying Hadwiger's inequality to the triangle CA_1B_1 constructed in Proposition 1, we get

$$\alpha^2a^2+\beta^2b^2+\alpha\beta c^2+(\alpha-\beta)(\alpha a^2-\beta b^2)\geq 4\alpha\beta\triangle\sqrt{3}.$$

Dividing by $\alpha\beta$ and denoting, as before, $x=\frac{\alpha}{\beta}$, we obtain

$$xa^2+\frac{1}{x}b^2+c^2+q(x)\geq 4\triangle\sqrt{3}.$$

After grouping the factors, we get the inequality that we wanted to prove in the first place. $\square$

3.4. Consider Goldner's inequality

$$b^2c^2+c^2a^2+a^2b^2\geq 16\triangle^2.$$

This inequality can be extended by using the technique presented here to the following relation:

$$a^2b^2+\left(xa^2+\frac{1}{x}b^2\right)\left[c^2+\left(1-\frac{1}{x}\right)(xa^2-b^2)\right]\geq 16\triangle^2.$$

To remind here the proof of Goldner's inequality, we use an argument based on a consequence of Heron's formula:

$$2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4)=16\triangle^2,$$

and the inequality

$$a^4+b^4+c^4\geq a^2b^2+a^2c^2+b^2c^2.$$

This proves Goldner's inequality. For its extension, we apply Goldner's inequality to triangle CA_1B_1, as in Proposition 1.

4. Menelaus' Theorem and homogeneous polynomials

In this section we prove the following result.

Proposition 2. *Let $p:\mathbb{R}^n\to\mathbb{R}$ be a homogeneous function of degree m, and consider n collinear points $A_1, A_2, ..., A_n$ lying on the line d. Let S be a point exterior to the line $\mathcal{L}$ and a secant $\mathcal{L}'$ whose intersection with each of the segments*

(SA_i) *is denoted* A'_i, *with* $i = 1, ..., n$. *Denote by* K *the intersection point of* $\mathcal{L}$ *and* $\mathcal{L}'$. *Then,*

$$p(KA_1, KA_2, ..., KA_n) = 0$$

if and only if

$$p\left(\frac{A_1A'_1}{A'_1S}, \frac{A_2A'_2}{A'_2S}, ..., \frac{A_nA'_n}{A'_nS}\right) = 0.$$

Proof. Denote $a_i = \frac{A_iA'_i}{A'_iS}$, for $i = 1, ..., n$. Applying Menelaus' Theorem in each of the triangles $SA_1A_2, SA_2A_3, \ldots, SA_{n-1}A_n$ we have, for all $i = 1, ..., n-1$,

$$\frac{1}{a_i} \cdot \frac{A_iK}{A_{i+1}K} \cdot a_{i+1} = 1.$$

This yields

$$\frac{A_1K}{a_1} = \frac{A_2K}{a_2} = ... = \frac{A_nK}{a_n} = t,$$

where $t > 0$. The fact that $p(KA_1, KA_2, ..., KA_n) = 0$ is equivalent, by Remark 1, with

$$p(ta_1, ta_2, ..., ta_n) = 0,$$

or, furthermore

$$t^m p(a_1, a_2, ..., a_n) = 0.$$

Since $t > 0$, the conclusion follows immediately. □

Remark. 3. As in the case of Proposition 1, we can discuss this result in terms of inequalities. For example, the Proposition 2 is still true if we claim that

$$p(KA_1, KA_2, ..., KA_n) \geq 0$$

if and only if

$$p\left(\frac{A_1A'_1}{A'_1S}, \frac{A_2A'_2}{A'_2S}, ..., \frac{A_nA'_n}{A'_nS}\right) \geq 0.$$

We present now an application.

4.1. A line intersects the sides AC and BC and the median CM_0 of an arbitrary triangle in the points B_1, A_1, and M_3, respectively. Then,

$$\frac{1}{2}\left(\frac{AB_1}{B_1C} + \frac{BA_1}{A_1C}\right) = \frac{M_3M_0}{M_3C}, \tag{3}$$

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}. \tag{4}$$

Furthermore, (3) is still true if we apply to this configuration a projective transformation that maps K into ∞.

We use Proposition 2 to prove (3). Let $\{K\} = AB \cap A_1B_1$. Then, the relation we need to prove is equivalent to $KA + KB = 2KM_0$, which is obvious, since M_0 is the midpoint of (AB).

To prove (4), remark that the anharmonic ratios $[KM_3B_1A_1]$ and $[KM_0AB]$ are equal, since they are obtained by intersecting the pencil of lines CK, CA, CM_0, CB with the lines KA and KB. Therefore, we have

$$\frac{M_3B_1}{M_3A_1} : \frac{KB_1}{KA_1} = \frac{M_0A}{M_0B} : \frac{KA}{KB}.$$

Since $M_0A = M_0B$, we have

$$\frac{M_3B_1}{M_3A_1} = \frac{KB_1}{KA_1} \cdot \frac{KB}{KA}.$$

Finally, by mapping M into the point at infinity, the lines B_1A_1 and BA become parallel. By Thales Theorem, we have

$$\frac{B_1A}{B_1C} = \frac{BA_1}{A_1C} = \frac{M_3M_0}{M_3C},$$

therefore the relation is still true.

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Forum Geometricorum
Volume 5 (2005) 149–164.

FORUM GEOM
ISSN 1534-1178

On the Complement of the Schiffler Point

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Abstract. Consider a triangle ABC with excircles (I_a), (I_b), (I_c), tangent to the nine-point circle respectively at F_a, F_b, F_c. Consider also the polars of A, B, C with respect to the corresponding excircles, bounding a triangle XYZ. We present, among other results, synthetic proofs of (i) the perspectivity of XYZ and $F_aF_bF_c$ at the complement of the Schiffler point of ABC, (ii) the concurrency at the same point of the radical axes of the nine-point circles of triangles I_aBC, I_bCA, and I_cAB.

1. Introduction

Consider a triangle ABC with excircles (I_a), (I_b), (I_c). It is well known that the nine-point circle (W) is tangent externally to the each of the excircles. Denote by F_a, F_b, and F_c the points of tangency. Consider also the polars of the vertices A with respect to (I_a), B with respect to (I_b), and C with respect to (I_c). These are the lines B_aC_a, C_bA_b, and A_cB_c joining the points of tangency of the excircles with the sidelines of triangle ABC. Let these polars bound a triangle XYZ. See Figure 1. Juan Carlos Salazar [12] has given the following interesting theorem.

Theorem 1 (Salazar). *The triangles XYZ and $F_aF_bF_c$ are perspective at a point on the Euler line.*

Darij Grinberg [3] has identified the perspector as the triangle center X_{442} of [6], the complement of the Schiffler point. Recall that the Schiffler point S is the common point of the Euler lines of the four triangles IBC, ICA, IAB, and ABC, where I is the incenter of ABC. Denote by A', B', C' the midpoints of the sides BC, CA, AB respectively, so that $A'B'C'$ is the medial triangle of ABC, with incenter I' which is the complement of I. Grinberg suggested that the lines XF_a, YF_b and ZF_c are the Euler lines of triangles $I'B'C'$, $I'C'A'$ and $I'A'B'$ respectively. The present author, in [10], conjectured the following result.

Theorem 2. *The radical center of the nine-point circles of triangles I_aBC, I_bCA and I_cAB is a point on the Euler line of triangle ABC.*

Subsequently, Jean-Pierre Ehrmann [1] and Paul Yiu [13] pointed out that this radical center is the same point S', the complement of the Schiffler point S. In this paper, we present synthetic proofs of these results, along with a few more interesting results.

Publication Date: October 18, 2005. Communicating Editor: Paul Yiu.

The author is extremely grateful to Professor Paul Yiu for his helps in the preparation of this paper.

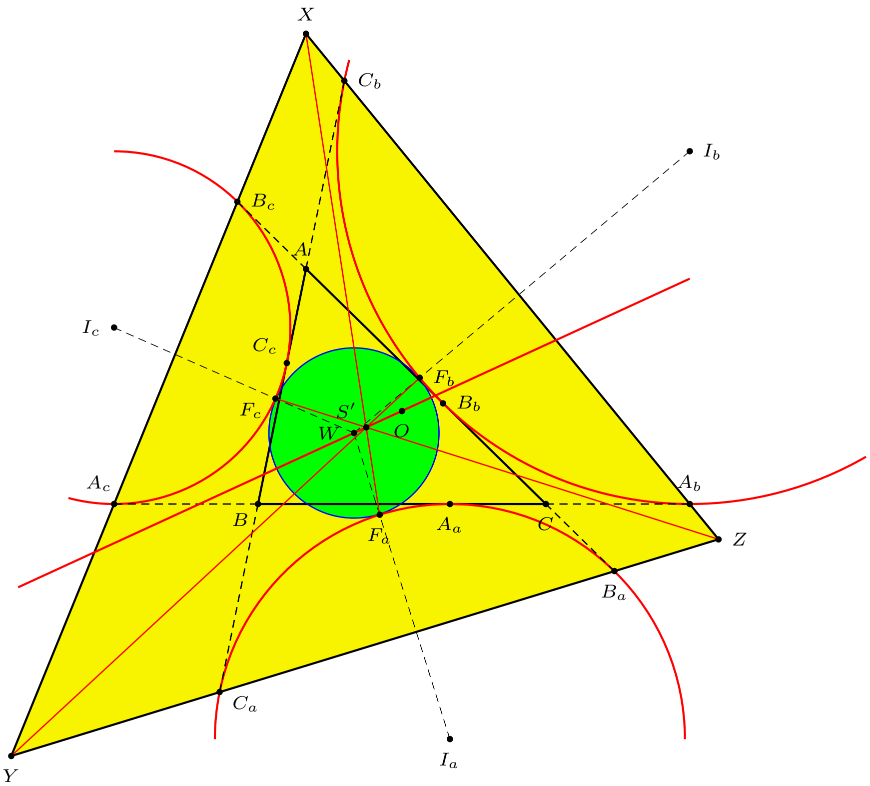

Figure 1.

2. Notations

a, b, c	Lengths of sides BC, CA, AB
R, r, s	Circumradius, inradius, semiperimeter
r_a, r_b, r_c	Exradii
O, G, W, H,	Circumcenter, centroid, nine-point center, orthocenter
I, F, S, M	Incenter, Feuerbach point, Schiffler point, Mittenpunkt
P'	Complement of P in triangle ABC
A', B', C'	Midpoints of BC, CA, AB
A_1, B_1, C_1	Points of tangency of incircle with BC, CA, AB
I_a, I_b, I_c	Excenters
F_a, F_b, F_c	Points of tangency of the nine-point circle with the excircles
A_a, B_a, C_a	Points of tangency of the A-excircle with the lines BC, CA, AB; similarly for A_b, B_b, C_b and A_c, B_c, C_c
W_a, W_b, W_c	Nine-point centers of I_aBC, I_bCA, I_cAB
M_a, M_b, M_c	Midpoints of AI_a, BI_b, CI_c
X	$A_bC_b \cap A_cB_c$; similarly for Y, Z
X_b, X_c	Orthogonal projections of B on CI_a and C on BI_a; similarly for Y_c, Y_a, Z_a, Z_b
J_a	Midpoint of arc BC of circumcircle not containing A; similarly for J_b, J_c
K_a	$A_bF_b \cap A_cF_c$; similarly for K_b, K_c

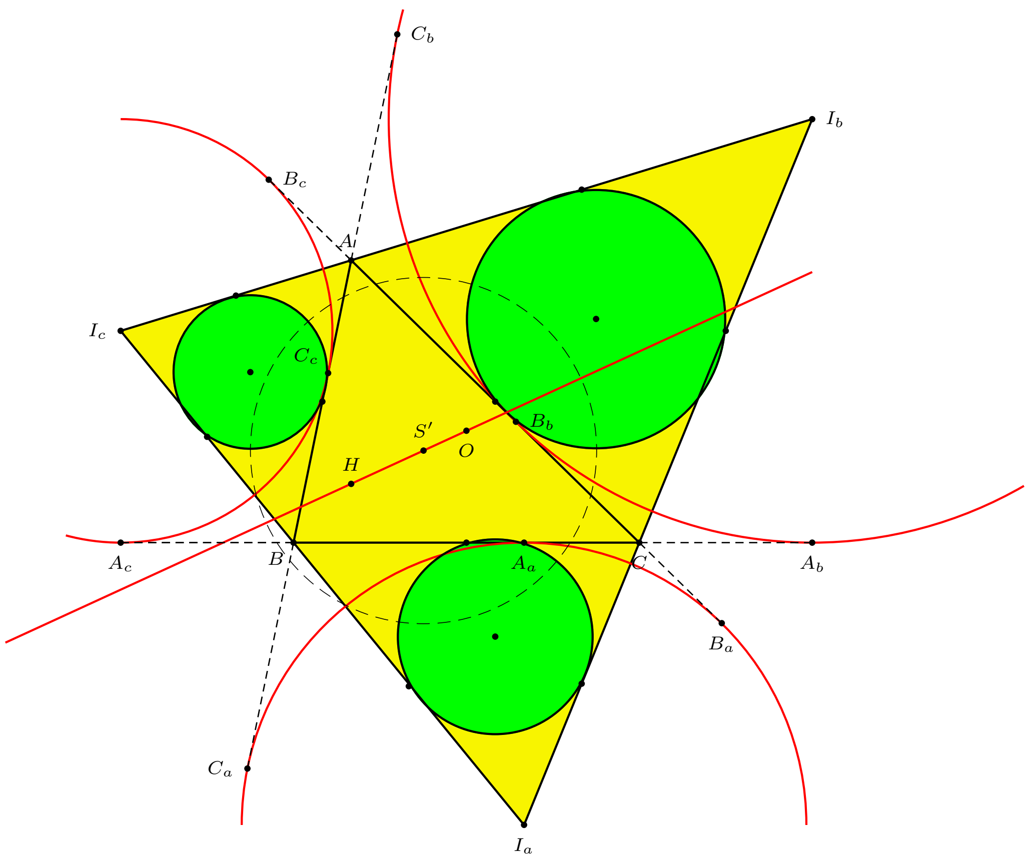

Figure 2.

3. Some preliminary results

We shall make use of the notion of directed angle between two lines. Given two lines a and b, the directed angle (a, b) is the angle of counterclockwise rotation from a to b. It is defined modulo $180°$. We shall make use of the following basic properties of directed angles. For further properties of directed angles, see [7].

Lemma 3. (i) *For arbitrary lines* a, b, c,

$$(\mathsf{a}, \mathsf{b}) + (\mathsf{b}, \mathsf{c}) \equiv (\mathsf{a}, \mathsf{c}) \bmod 180°.$$

(ii) *Four points A, B, C, D are concyclic if and only if* $(AC, CB) = (AD, DB)$.

Lemma 4. *Let (O) be a circle tangent externally to two circles (O_a) and (O_b) respectively at A and B. If PQ is a common external tangent of (O_a) and (O_b), then the quadrilateral $APQB$ is cyclic, and the lines AP, BQ intersect on the circle (O).*

Proof. Let PA intersect (O) at K. Since (O) and (O_a) touch each other externally at A, OK is parallel to O_aP. On the other hand, O_aP is also parallel to O_bQ as they are both perpendicular to the common tangent PQ. Therefore KO is parallel

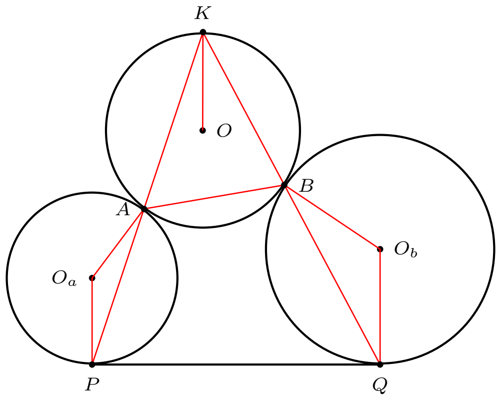

Figure 3

to O_bQ in the same direction. This implies that K, B, Q are collinear since (O_b) and (O) touch each other at externally at B. Therefore

$$(PQ, QB) = \frac{1}{2}(QO_b, O_bB) = \frac{1}{2}(KO, OB) = (KA, AB) = (PA, AB),$$

and $APQB$ is cyclic. □

We shall make use of the following results.

Lemma 5. *Let ABC be a triangle inscribed in a circle (O), and points M and N lying on AB and AC respectively. The quadrilateral $BNMC$ is cyclic if and only if MN is perpendicular to OA.*

Theorem 6. *The nine-point circles of ABC, I_aBC, I_aCA, and I_aAB intersect at the point F_a.*

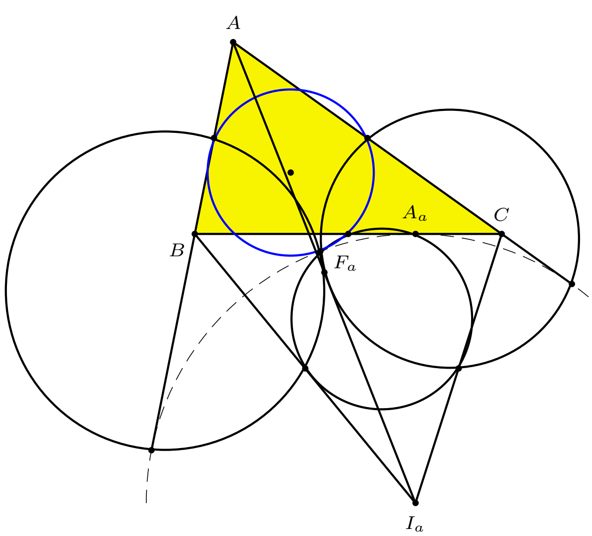

Figure 4.

Proposition 7. *The circle with diameter A_aM_a contains the point F_a.*

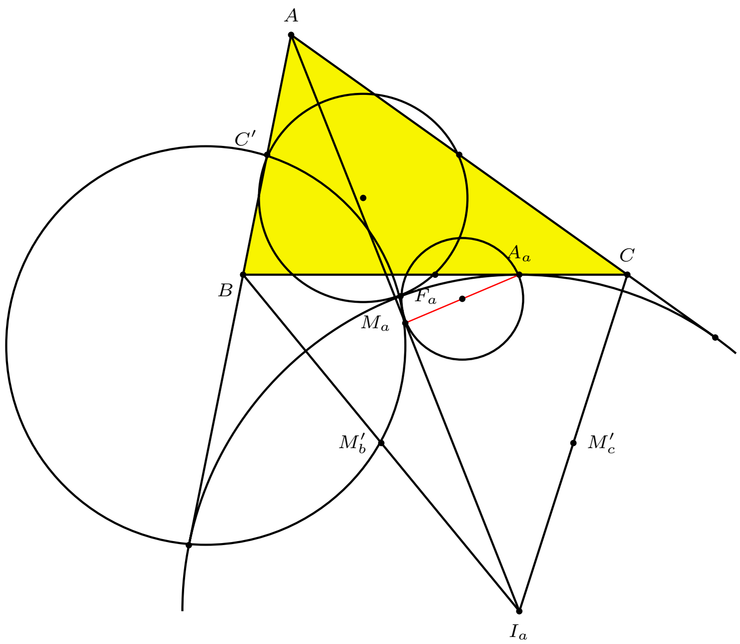

Figure 5.

Proof. Denote by M_b' and M_c' the midpoints of I_aB and I_aC respectively. The point F_a is common to the nine-point circles of I_aBC, I_aCA and I_aAB. See Figure 5. We show that $(A_aF_a, F_aM_a) = 90^\circ$.

$$\begin{aligned}(A_aF_a, F_aM_a) =&(A_aF_a, F_aM_b') + (M_b'F_a, F_aM_a)\\ =&(A_aM_c', M_c'M_b) + (M_b'C', C'M_a)\\ =&-(I_aM_c', M_c'M_b') - (BI_a, I_aA)\\ =&-((I_aC, BC) + (BI_a, I_aA)) = 90^\circ.\end{aligned}$$

□

4. Some properties of triangle XYZ

In this section we present some important properties of the triangle XYZ.

4.1. *Homothety with the excentral triangle.* Since YZ and I_bI_c are both perpendicular to the bisector of angle A, they are parallel. Similarly, ZX and XY are parallel to I_cI_a and I_aI_b respectively. The triangle XYZ is therefore homothetic to the excentral triangle $I_aI_bI_c$. See Figure 7. We shall determine the homothetic center in Theorem 11 below.

4.2. *Perspectivity with ABC.* Consider the orthogonal projections P and P' of A and X on the line BC. We have

$$A_cP : PA_b = (s-c) + c\cos B : (s-b) + b\cos C = s-b : s-c$$

by a straightforward calculation.

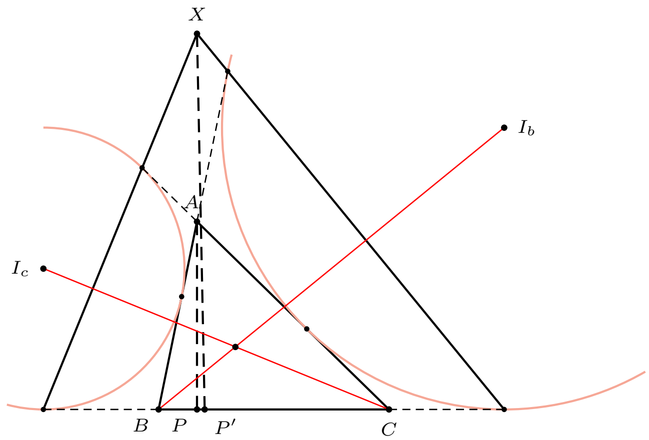

Figure 6.

On the other hand,

$$\begin{aligned}A_cP' : P'A_b =& \cot XA_cA_b : \cot XA_bA_c\\ =& \cot\left(90^\circ - \frac{C}{2}\right) : \cot\left(90^\circ - \frac{B}{2}\right)\\ =& \tan\frac{C}{2} : \tan\frac{B}{2}\\ =& \frac{1}{s-c} : \frac{1}{s-b}\\ =& s-b : s-c.\end{aligned}$$

It follows that P and P' are the same point. This shows that the line XA is perpendicular to BC and contains the orthocenter H of triangle ABC. The same is true for the lines YB and ZX. The triangles XYZ and ABC are perspective at H.

4.3. *The circumcircle of XYZ.* Applying the law of sines to triangle AXB_c, we have

$$XA = (s-b)\cdot\frac{\sin\left(90^\circ - \frac{C}{2}\right)}{\sin\frac{C}{2}} = (s-b)\cot\frac{C}{2} = r_a.$$

It follows that $HX = 2R\cos A + r_a = 2R + r$. See Figure 4. Similarly, $HY = HZ = 2R + r$. Therefore, triangle XYZ has circumcenter H and circumradius $2R + r$.

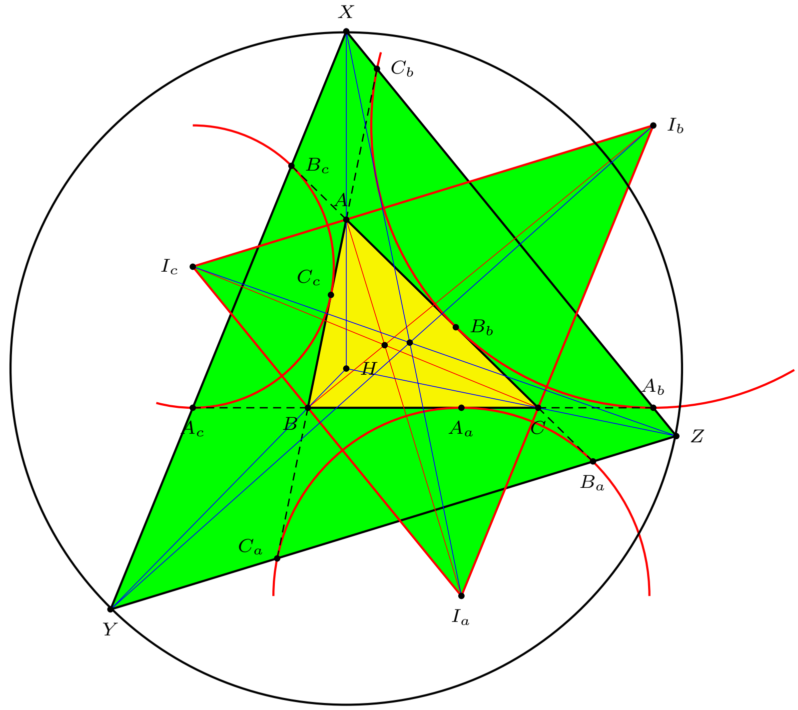

Figure 7.

5. The Taylor circle of the excentral triangle

Consider the excentral triangle $I_aI_bI_c$ with its orthic triangle ABC. The orthogonal projections Y_a and Z_a of A on I_aI_c and I_aI_b, Z_b and X_b of B on I_bI_c and I_aI_b, together with X_c and Y_c of C on I_bI_c and I_cI_a are on a circle called the Taylor circle of the excentral triangle. See Figure 8.

Proposition 8. *The points X_b, X_c lie on the line YZ.*

Proof. The collinearity of C_a, X_b, X_c follows from

$$\begin{aligned}(C_aX_b, X_bB) =&(C_aI_a, I_aB)\\ =&(C_aI_a, AB) + (AB, I_aB)\\ =&90^\circ + (I_aB, BC)\\ =&(X_cC, I_aB) + (I_aB, BC)\\ =&(X_cC, CB)\\ =&(X_cX_b, X_bB).\end{aligned}$$

Similarly, X_b is also on the line YZ, and Z_a, Z_b are on the line XY, Y_c, Y_a are on the line XZ. □

Proposition 9. *The line Y_aZ_a contains the midpoints B', C' of CA, AB, and is parallel to BC.*

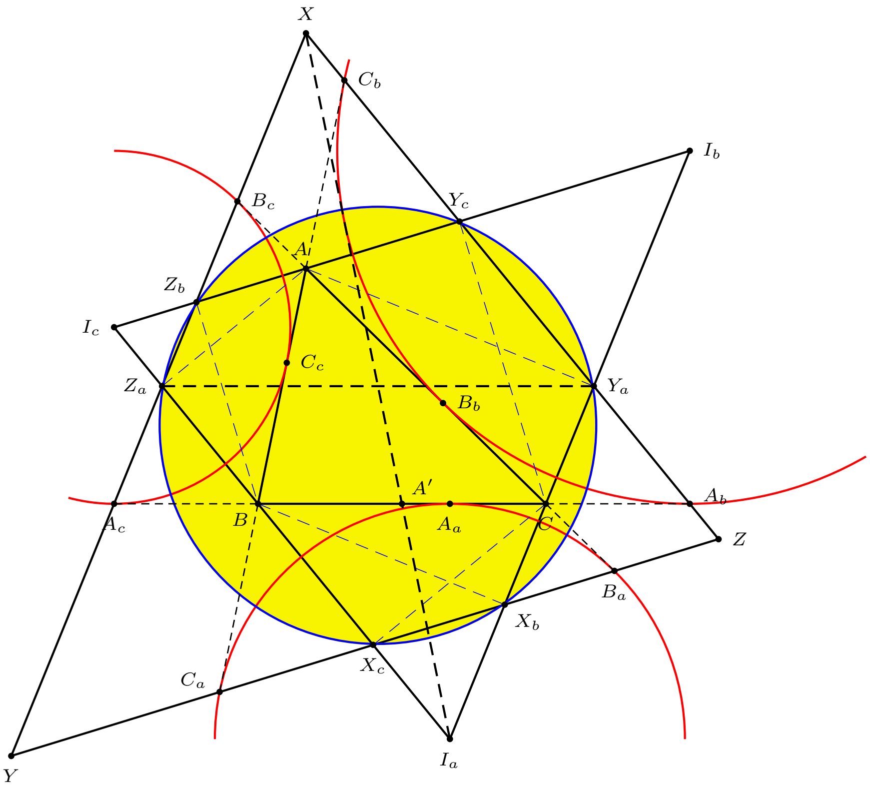

Figure 8.

Proof. Since A, Y_a, I_a, Z_a are concyclic,

$$(AY_a, Y_aZ_a) = (AI_a, I_aZ_a) = \frac{C}{2} = (CA, AY_a).$$

Therefore, the intersection of AC and Y_aZ_a is the circumcenter of the right triangle ACY_a, and is the midpoint B' of CA. Similarly, the intersection of AB and Y_aZ_a is the midpoint C' of AB. □

Proposition 10. *The line I_aX contains the midpoint A' of BC.*

Proof. Since the diagonals of the parallelogram $I_aY_aXZ_a$ bisect each other, the line I_aX passes through the midpoint of the segment Y_aZ_a. Since Y_aZ_a and BC are parallel, with B on I_aZ_a and C on I_aY_a, the same line I_aX also passes through the midpoint of the segment BC. □

Theorem 11. *The triangles XYZ and $I_aI_bI_c$ are homothetic at the Mittenpunkt M of triangle ABC, the ratio of homothety being $2R + r : -2R$.*

Proof. The lines I_aX, I_bY, I_cZ contain respectively the midpoints of A', B', C' of BC, CA, AB. They intersect at the common point of I_aA', I_bB', I_cC', the Mittenpunkt M of triangle ABC. This is the homothetic center of the triangles XYZ and $I_aI_bI_c$. The ratio of homothety of the two triangle is the same as the ratio of their circumradii. □

Theorem 12. *The Taylor circle of the excentral triangle is the radical circle of the excircles.*

Proof. The perpendicular bisector of Y_cZ_b is a line parallel to the bisector of angle A and passing through the midpoint A' of BC. This is the A'-bisector of the medial triangle $A'B'C'$. Similarly, the perpendicular bisectors of Z_aX_c and X_bY_a are the other two angle bisectors of the medial triangle. These three intersect at the incenter of the medial triangle, the Spieker center of ABC.

It is well known that S_{p} is also the center of the radical circle of the excircles. To show that the Taylor circle coincides with the radical circle, we show that they have equal radii. This follows easily from

$$I_aX_c \cdot I_aZ_a = \frac{r_a \sin\frac{A}{2}}{\cos\frac{C}{2}} \cdot I_aA\cos\frac{C}{2} = r_a \cdot I_aA\sin\frac{A}{2} = r_a^2.$$

□

6. Proofs of Theorems 1 and 2

We give a combined proof of the two theorems, by showing that the line XF_a is the radical axis of the nine-point circles (W_b) and (W_c) of triangles I_bCA and I_cAB. In fact, we shall identify some interesting points on this line to show that it is also the Euler line of triangle $I'B'C'$.

6.1. XF_a *as the radical axis of* (W_b) *and* (W_c).

Proposition 13. X *lies on the radical axis of the circles* (W_b) *and* (W_c).

Proof. By Theorem 12, $XZ_a \cdot XZ_b = XY_a \cdot XY_c$. Since Y_c, Y_a are on the nine-point circle (W_b) and Z_a, Z_b on the the circle (W_c), X lies on the radical axis of these two nine-point circles. □

Since AZ_a and AY_a are perpendicular to I_aI_c and I_aI_b, and $I_aI_bI_c$ and XYZ are homothetic, A is the orthocenter of triangle XY_aZ_a. It follows that X is the orthocenter of AY_aZ_a. Since $(AY_a, Y_aI_a) = (AZ_a, Z_aI_a) = 90°$, the triangle AY_aZ_a has circumcenter the midpoint M_a of AI_a. It follows that XM_a is the Euler line of triangle AY_aZ_a.

Proposition 14. M_a *lies on the radical axis of the circles* (W_b) *and* (W_c).

Proof. Let M_b'' and M_c'' be the midpoints of AI_b and AI_c respectively. See Figure 9. Note that these lie on the nine-point circles (W_b) and (W_c) respectively. Since C, I_b, I_c, B are concyclic, we have $I_aB \cdot I_aI_c = I_aC \cdot I_aI_b$. Applying the homthety $\mathsf{h}(A, \frac{1}{2})$, we have the collinearity of M_a, C', M_c'', and of M_a, B', M_b'', Furthermore, $M_aC' \cdot M_aM_c'' = M_aB' \cdot M_aM_b''$. This shows that M_a lies on the radical axis of (W_b) and (W_c). □

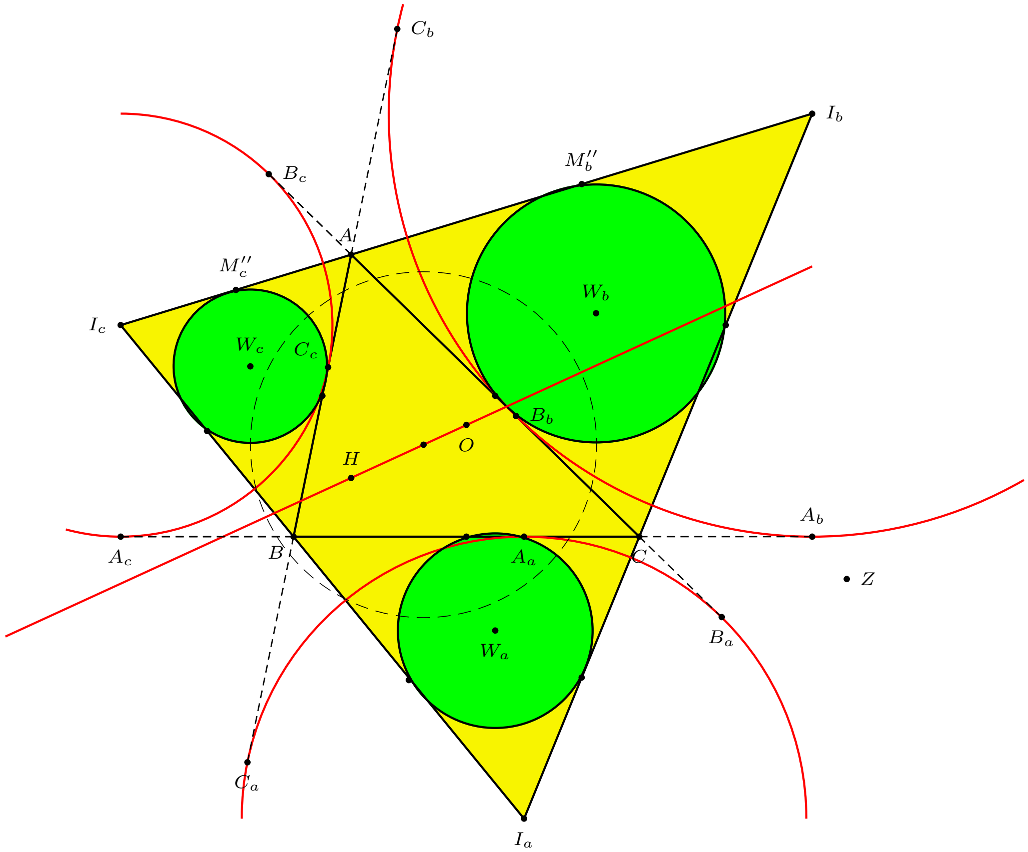

Figure 9.

Proposition 15. *X, F_a, and M_a are collinear.*

Proof. We prove that the Euler line of triangle AY_aZ_a contains the point F_a. The points X and M_a are respectively the orthocenter and circumcenter of the triangle.

Let A'_a be the antipode of A_a on the A-excircle. Since AX has length r_a and is perpendicular to BC, XAA'_aI_a is a parallelogram. Therefore, XA'_a contains the midpoint M_a of AI_a.

By Proposition 7, $(A_aF_a, F_aM_a) = 90^\circ$. Clearly, $(A_aF_a, F_aA'_a) = 90^\circ$. This means that F_a, M_a, and A'_a are collinear. The line containing them also contains X. □

Proposition 16. *XF_a is also the Euler line of triangle AY_aZ_a.*

Proof. The circumcenter of AY_aZ_a is clearly M_a. On the other hand, since A is the orthocenter of triangle XY_aZ_a, X is the orthocenter of triangle AY_aZ_a. Therefore the line XM_a, which also contains F_a, is the Euler line of triangle AY_aZ_a. □

6.2. *XF_a as the Euler line of triangle $I'B'C'$.*

Proposition 17. *M_a is the orthocenter of triangle $I'B'C'$.*

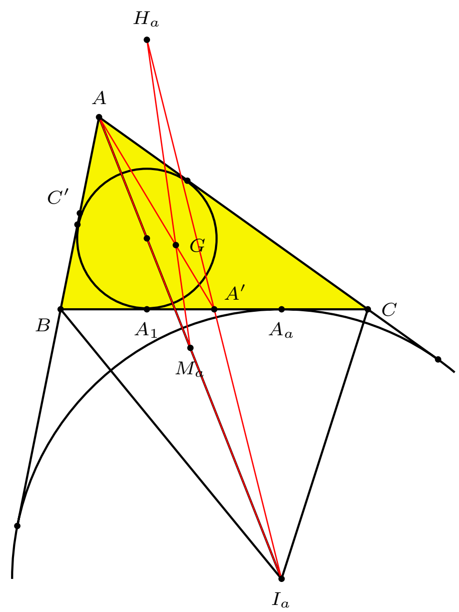

Figure 10.

Proof. Let H_a be the orthocenter of IBC. Since BH_a is perpendicular to IC, it is parallel to I_aC. Similarly, CH_a is parallel to I_aB. Thus, BH_aCI_a is a parallelogram, and A' is the midpoint of I_aH_a. Consider triangle AI_aH_a which has M_a and A' for the midpoints of two sides. The intersection of M_aH_a and AA' is the centroid of the triangle, which coincides with G. Furthermore,

$$GH_a : GM_a = GA : GA' = 2 : -1.$$

Hence, M_a is the orthocenter of $I'B'C'$. $\square$

Proposition 18. *K_a is the circumcenter of $I'B'C'$.*

Proof. By Lemma 4, the points F_b, F_c, A_b and A_c are concyclic, and the lines A_bF_b and A_cF_c intersect at a point K_a on the nine-point circle, which is the midpoint of the arc $B'C'$ not containing A'. See Figure 11. The image of K_a under $\mathsf{h}(G, -2)$ is J_a, the circumcenter of IBC. It follows that K_a is the circumcenter of $I'B'C'$. $\square$

Proposition 19. *K_a lies on the radical axis of (W_b) and (W_c).*

Proof. Let D and E be the second intersections of K_aF_b with (W_b) and K_aF_c with (W_c) respectively. We shall show that $K_aF_b \cdot K_aD = K_aF_c \cdot K_aE$.

Since A_c, F_c, F_b, A_b are concyclic, we have $K_aF_c \cdot K_aA_c = K_aF_b \cdot K_aA_b = k$, say. Note that

$$A_cE \cdot A_cF_c = A_cZ_a \cdot A_cZ_b = \frac{(s-a)^2 \sin\left(B + \frac{A}{2}\right)}{\tan\frac{B}{2}\cos\frac{A}{2}}.$$

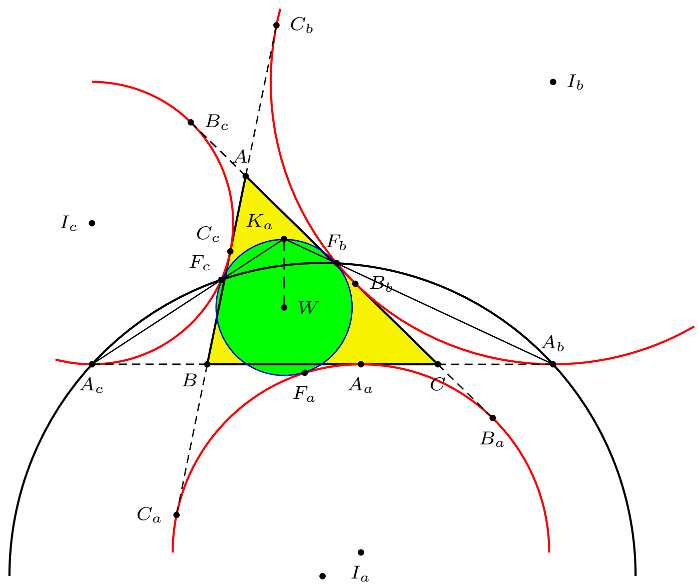

Figure 11.

Since (I_c) and (W) extouch at F_c, we have $\frac{K_aF_c}{A_cF_c} = -\frac{R}{2r_a}$. Therefore,

$$\begin{aligned}\frac{A_cE}{K_aA_c} &= \frac{K_aF_c}{A_cF_c} \cdot \frac{A_cE \cdot A_cF_c}{K_aF_c \cdot K_aA_c} \\ &= -\frac{R}{2r_a} \cdot \frac{(s-a)^2 \sin\left(B + \frac{A}{2}\right)}{k \cdot \tan\frac{B}{2} \cos\frac{A}{2}} \\ &= -\frac{R(s-a)^2 \sin\left(B + \frac{A}{2}\right)}{k \cdot s \tan\frac{B}{2} \tan\frac{C}{2} \cos\frac{A}{2}}.\end{aligned}$$

Similarly,

$$\frac{A_bD}{K_aA_b} = -\frac{R(s-a)^2 \sin\left(C + \frac{A}{2}\right)}{k \cdot s \tan\frac{B}{2} \tan\frac{C}{2} \cos\frac{A}{2}}.$$

Since $\sin\left(B + \frac{A}{2}\right) = \sin\left(C + \frac{A}{2}\right)$, it follows that $\frac{A_bD}{K_aA_b} = \frac{A_cE}{K_aA_c}$. Hence, DE is parallel to A_bA_c. From $K_aF_b \cdot K_aA_b = K_aF_c \cdot K_aA_c$, we have $K_aF_b \cdot K_aD = K_aF_c \cdot K_aE$. This shows that K_a lies on the radical axis of (W_b) and (W_c).

□

Corollary 20. *K_a lies on the line XF_a.*

6.3. *Proof of Theorems 1 and 2.* We have shown that the line XF_a is the radical axis of (W_b) and (W_c). Likewise, YF_b is that of (W_c), (W_a), and ZF_c that of (W_a), (W_b). It follows that the three lines are concurrent at the radical center of the three circles. This proves Theorem 1.

We have also shown that the line XF_a is the image of the Euler line of IBC under the homothety $\mathsf{h}(G, -\frac{1}{2})$; similarly for the lines YF_b and ZF_c. Since the Euler lines of IBC, ICA, and IAB intersect at the Schiffler point S on the Euler line of ABC, the lines XF_a, YF_b, ZF_c intersect at the complement of the Schiffler point S, also on the same Euler line. This proves Theorem 2.

7. Some further results

Theorem 21. *The six points Y, Z, A_b, A_c, F_b, F_c are concyclic.*

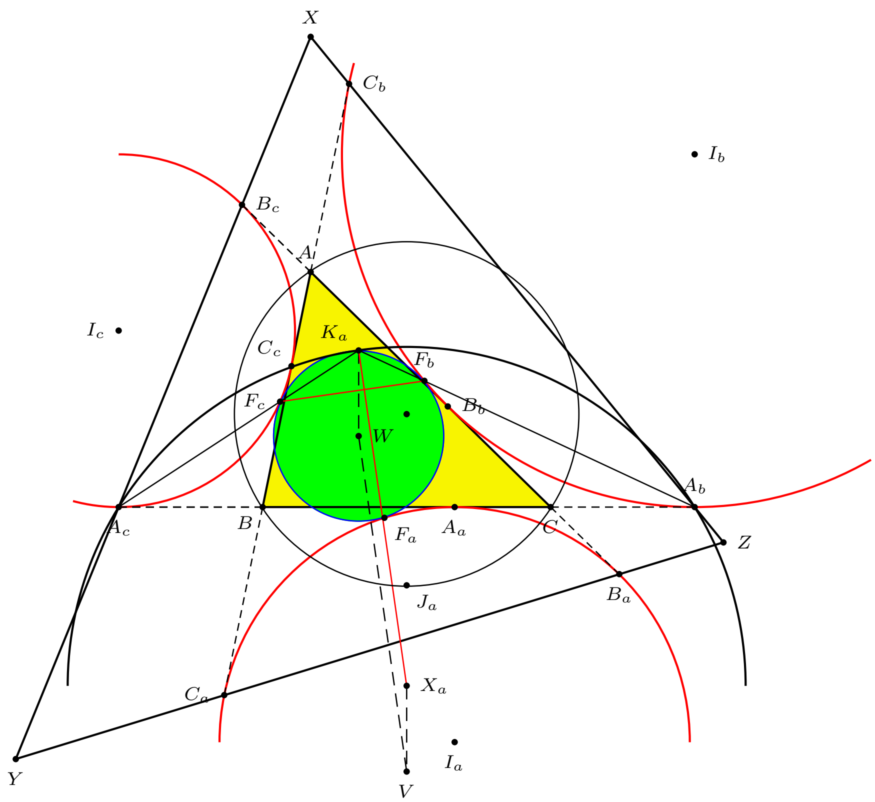

Figure 12.

Proof. (i) The points A_b, A_c, F_b, F_c are concyclic and the lines A_bF_b, A_cF_c meet at K_a. Let X_a be the circumcenter of $K_aA_bA_c$. Since F_b and F_c are points on K_aA_b and K_aA_c, and $F_bA_bA_cF_c$ is cyclic, it follows from Lemma 5 that K_aX_a is perpendicular to F_bF_c. Hence X_a is the intersection of the perpendicular from K_a to F_bF_c and the perpendicular bisector of BC. Since triangle $K_aA_bA_c$ is similar to $K_aF_cF_b$, and $A_bA_c = b + c$, its circumradius is

$$\frac{b+c}{F_bF_c} \cdot \frac{R}{2} = \frac{1}{2}\sqrt{(R+2r_b)(R+2r_c)}.$$

Here, we have made use of the formula

$$F_bF_c = \frac{b+c}{\sqrt{(R+2r_b)(R+2r_c)}} \cdot R$$

from [2].

(ii) A simple angle calculation shows that the points Y, Z, A_b, A_c are also concyclic. Its center is the intersection of the perpendicular bisectors of A_bA_c and YZ. The perpendicular bisector of A_bA_c is clearly the same as that of BC. Since YZ is parallel to I_bI_c, its perpendicular is the parallel through H (the circumcenter of XYZ) to the bisector of angle A.

(iii) Therefore, if this circumcenter is V, then $J_aV = AH = 2R\cos A$.

(iv) To show that the two circle $F_bA_bA_cF_c$ is the same as the circle in (ii), it is enough to show that V lies on the perpendicular bisector of F_bF_c. This is equivalent to showing that VW is perpendicular to F_bF_c. To prove this, we show that K_aWVX_a is a parallelogram. Applying the Pythagorean theorem to triangle $A'A_bX_a$, we have

$$\begin{aligned}
4A'X_a^2 =&(R+2r_b)(R+2r_c)-(b+c)^2\\
=&R^2+4R(r_b+r_c)+4r_br_c-(b+c)^2\\
=&R^2+4R\cdot R(1+\cos A)+4s(s-a)-(b+c)^2\\
=&R^2(1+4(1+\cos A))-a^2\\
=&R^2(1+4(1+\cos A)-4\sin^2 A)\\
=&R^2(1+2\cos A)^2.
\end{aligned}$$

This means that $A'X_a = \frac{R}{2}(1+2\cos A)$, and it follows that

$$\begin{aligned}
X_aV =&A'V - A'X_a = A'J + JV - A'X_a\\
=&R(1-\cos A) + 2R\cos A - \frac{R}{2}(1+2\cos A)\\
=&\frac{R}{2} = K_aW.
\end{aligned}$$

Therefore, VW, being parallel to K_aX_a, is perpendicular to F_bF_c. □

Denote by $\mathcal{C}_a$ the circle through these 6 points. Similarly define $\mathcal{C}_b$ and $\mathcal{C}_c$.

Corollary 22. *The radical center of the circles $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$ is S'.*

Proof. The points X and F_a are common to the circles $\mathcal{C}_b$ and $\mathcal{C}_c$. The line XF_a is the radical axis of the two circles. Similarly the radical axes of the two other two pairs of circles are YF_b and ZF_c. The radical center is therefore S'. □

Proposition 23. *The line XA_a is perpendicular to YZ.*

Proof. With reference to Figure 8, note that

$$\begin{aligned}
A_bY_a : A_bX =& A_bC \cdot \frac{\sin\left(C+\frac{A}{2}\right)}{\sin\frac{C}{2}} : A_bA_c \cdot \frac{\sin\frac{A+B}{2}}{\sin\frac{B+C}{2}} \\
=& A_bC : (b+c) \cdot \frac{\sin\frac{C}{2}\sin\frac{A+B}{2}}{\sin\left(C+\frac{A}{2}\right)\sin\frac{B+C}{2}} \\
=& A_bC : (b+c) \cdot \frac{\sin C}{\sin(C+A)+\sin C} \\
=& A_bC : c \\
=& A_bC : A_bA_a.
\end{aligned}$$

This means that XA_a is parallel to Y_cC, which is perpendicularto I_bI_c and YZ. □

Corollary 24. *XYZ is perspective with the extouch triangle $A_aB_bC_c$, and the perspector is the orthocenter of XYZ.*

Remark. This is the triangle center X_{72} of [6].

Proposition 25. *The complement of the Schiffler point is the point S' which divides HW in the ratio*

$$HS' : S'W = 2(2R+r) : -R.$$

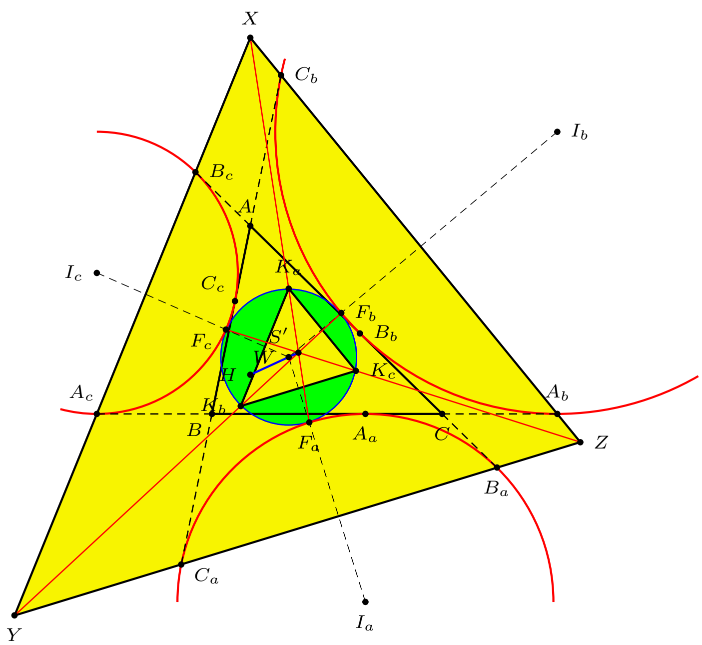

Figure 13.

Proof. We define K_b and K_c similarly as K_a. Since K_b and K_c are the midpoints of the arcs $C'A'$ and $A'B'$, K_bK_c is perpendicular to the A'-bisector of $A'B'C'$, and hence parallel to YZ. The triangle $K_aK_bK_c$ is homothetic to XYZ. The homothetic center is the common point of the lines XK_a, YK_b, and ZK_c, which are XF_a, YF_b, ZF_c. This is the complement of the Schiffler point. Since triangles $K_aK_bK_c$ and XYZ have circumcenters W, H, and circumradii $\frac{R}{2}$ and $2R+r$, this homothetic center S' divides the segment HW in the ratio given above. □

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Forum Geometricorum
Volume 5 (2005) 165–171.

FORUM GEOM
ISSN 1534-1178

On the Existence of Triangles with Given Circumcircle, Incircle, and One Additional Element

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Abstract. We give necesssary and sufficient conditions for the existence of poristic triangles with two given circles as circumcircle and incircle, and (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector. We also consider the question of construction of such triangles.

1. Introduction

It is well known that the distance d between the circumcenter and incenter of a triangle is given by the formula:

$$d^2 = R^2 - 2Rr, \tag{1}$$

where R and r are respectively the circumradius and inradius of the triangle ([3, p.29]). Therefore, if we are given two circles on the plane, with radii R and r, $(R \geq 2r)$, a necessary condition for an existence of a triangle, for which the two circles will be the circumcircle and the incircle, is that the distance d between their centers satifies (1). From Poncelet's closure theorem it follows that this condition is also sufficient. Furthermore, each point on the circle with radius R may be one of the triangle vertex, *i.e.*, in general there are infinitely many such triangles. A natural question is on the existence and uniqueness of such a triangle if we specify one additional element. We shall consider this question when this additional element is one of the following: (1) a side length, (2) the semiperimeter (area), (3) an altitude, and (4) an angle bisector.

2. Main results

Throughout this paper, we consider two given circles $O(R)$ and $I(r)$ with distance d between their centers satisfying (1). Following [2], we shall call a triangle with circumcircle $O(R)$ and incircle $I(r)$ a poristic triangle.

Theorem 1. *Let a be a given positive number.* (1). *If $d \leq r$, i.e. $R \leq (\sqrt{2}+1)r$, then there is a unique poristic triangle ABC with $BC = a$ if and only if*

$$4r(2R - r - 2d) \leq a^2 \leq 4r(2R - r + 2d). \tag{2}$$

Publication Date: November 8, 2005. Communicating Editor: Paul Yiu.

(2). *If* $d > r$, *i.e.* $R > (\sqrt{2}+1)r$, *then there is a unique poristic triangle* ABC *with* $BC = a$ *if and only if*

$$4r(2R - r - 2d) \leq a^2 < 4r(2R - r + 2d) \quad \textit{or} \quad a = 2R, \tag{3}$$

and there are two such triangles if and only if

$$4r(2R - r + 2d) \leq a^2 < 4R^2. \tag{4}$$

Theorem 2. *Given* $s > 0$, *there is a unique poristic triangle with semiperimeter* s *if and only if*

$$\sqrt{R+r-d}(\sqrt{2R} + \sqrt{R-r+d}) \leq s \leq \sqrt{R+r+d}(\sqrt{2R} + \sqrt{R-r-d}). \tag{5}$$

Theorem 3. *Given* $h > 0$, *there is a unique poristic triangle with an altitude* h *if and only if*

$$R + r - d \leq h \leq R + r + d. \tag{6}$$

Theorem 4. *Given* $\ell > 0$, *there is a unique poristic triangle with an angle bisector* ℓ *if and only if*

$$R + r - d \leq \ell \leq R + r + d. \tag{7}$$

3. Proof of Theorem 1

3.1. *Case 1.* $d \leq r$. The length of $BC = a$ attains its minimal value when the distance from O to BC is maximal, which is $d + r$. See Figure 1. Therefore,

$$a^2_{\min} = 4r(2R - r - 2d).$$

Similarly, a attains its maximum when the distance from O to BC is minimal, *i.e.*, $r - d$. See Figure 2.

$$a^2_{\max} = 4r(2R - r + 2d).$$

This shows that (2) is a necessary condition a to be a side of a poristic triangle.

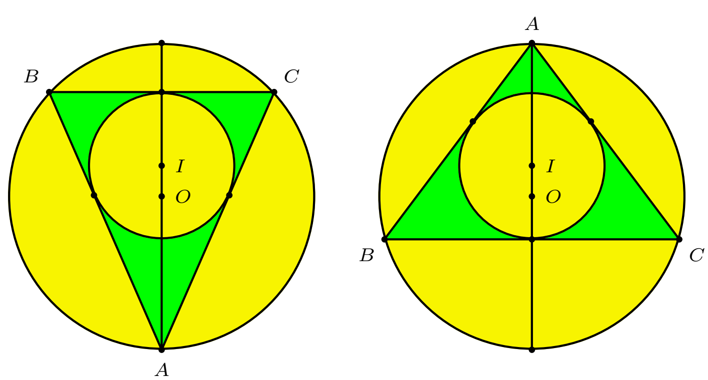

Figure 1 Figure2

We prove the sufficiency part by an explicit construction. If a satisfies (2), we construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$, and a common tangent of this circle and $I(r)$. The segment of this tangent inside the circle $O(R)$ is a side of a poristic triangle with a side of length a. The third vertex is, by Poncelet's closure theorem, the intersection of the tangents from these endpoints to $I(r)$, and it lies on $O(R)$.

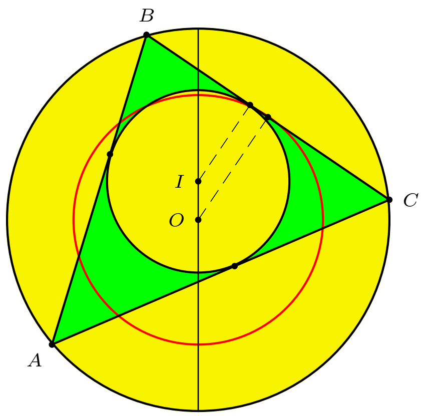

Figure 3

Remark. If $a \neq a_{\max}, a_{\min}$, we can construct two common tangents to the circles $O(R)$ and $I(r)$. These are both external common tangents and are symmetric with respect to the line OI. The resulting triangles are congruent.

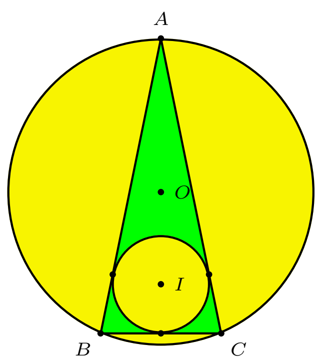

Figure 4

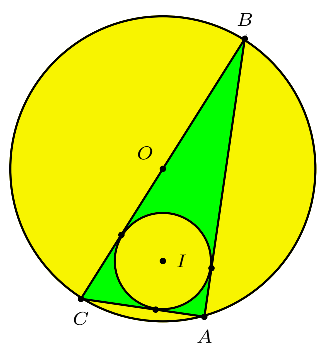

Figure 5

3.2. *Case 2.* $d > r$. In this case by the same way we have

$$a_{\min}^2 = 4r(2R - r - 2d).$$

See Figure 4. On the other hand, the maximum occurs when BC passes through the center O, *i.e.*, $a_{\max} = 2R$. See Figure 5.

For a given $a > 0$, we again construct the circle $O(R_1)$ with $R_1^2 = R^2 - \frac{a^2}{4}$. Chords of the circle (O) which are tangent to $O(R_1)$ have length a. If $R_1 > d - r$, the construction in §3.1 gives a poristic triangle with a side a. Therefore for

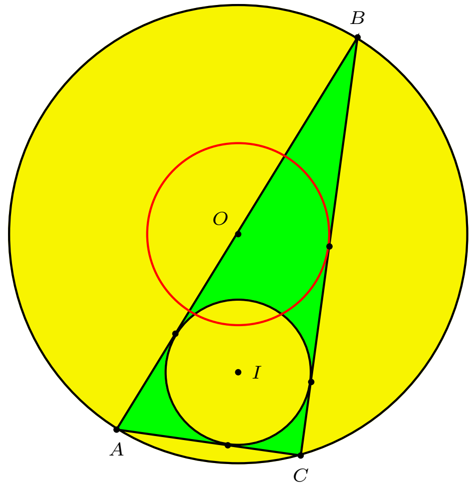

Figure 6

$4r(2R - r - 2d) \leq a^2 < 4r(2R - r + 2d)$, there is a unique poristic triangle with side a. See Figure 6. It is clear that this is also the case if $a = 2R$.

However, if $R_1 \leq d - r$, there are also internal common tangents of the circles $O(R_1)$ and $I(r)$. The internal common tangents give rise to an obtuse angled triangle. See Figures 7 and 8.

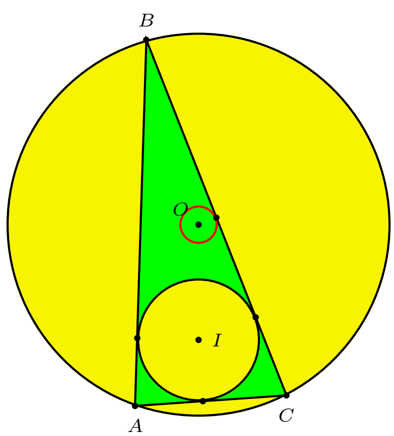

Figure 7

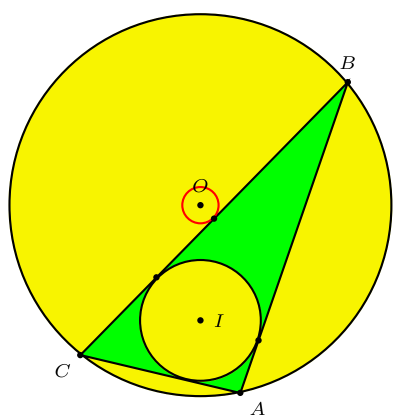

Figure 8

4. Proof of Theorem 2

Let $A_1B_1C_1$ and $A_2B_2C_2$ be the poristic triangles with A_1 and A_2 on the line OI. We assume $\angle A_1 \leq \angle A_2$. If $\angle A_1 = \angle A_2$, the triangle is equilateral and the statement of the theorem is trivial. We shall therefore assume $\angle A_1 < \angle A_2$. Consider an arbitrary poristic triangle ABC with semiperimeter s. According to

[4], s attains its maximum when the triangle coincides with $A_1B_1C_1$ and minimum when it coincides with $A_2B_2C_2$. Therefore,

$$\begin{aligned}
s_{\max} =& \sqrt{R^2-(r+d)^2} + \sqrt{R^2-(r+d)^2+(R+r+d)^2} \\
=& \sqrt{R+r+d}(\sqrt{2R}+\sqrt{R-r-d}), \\
s_{\min} =& \sqrt{R^2-(r-d)^2} + \sqrt{R^2-(r-d)^2+(R+r-d)^2} \\
=& \sqrt{R+r-d}(\sqrt{2R}+\sqrt{R-r+d}).
\end{aligned}$$

This proves (5).

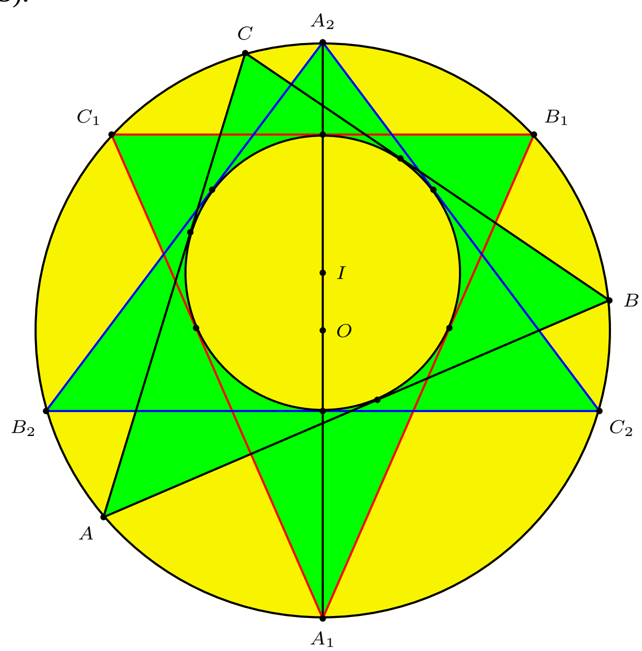

Figure 9

As A traverses a semicircle from position A_1 to A_2, the measure α of angle A is monotonically increasing from $\alpha_{\min} = \angle A_1$ to $\alpha_{\max} = \angle A_2$. For each $\alpha \in [\alpha_{\min},\ \alpha_{\max}]$,

$$s = s(\alpha) = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha.$$

Differentiating with respect to α, we have

$$s'(\alpha) = -\frac{r}{2\sin^2\frac{\alpha}{2}} + 2R\cos\alpha.$$

Clearly, $s'(\alpha) = 0$ if and only if $\sin^2\frac{\alpha}{2} = \frac{R\pm d}{4R}$. Since $\sin\frac{\alpha}{2} > 0$, there are two values of $\alpha \in (\alpha_{\min},\ \alpha_{\max})$ for which $s'(\alpha) = 0$. One of these is $\alpha_1 = \angle B_1$ for which $s(\alpha_1) = s_{\max}$ and the other is $\alpha_2 = \angle C_2$ for which $s(\alpha_2) = s_{\min}$.

Therefore for given real number $s > 0$ satisfying (5), there are three values of α (or two values if $s = s_{\min}$ or $s_{\max}$) for which $s(\alpha) = s$. These values are the values of the three angles of the same triangle that has semiperimeter s. So for such s the triangle is unique up to congruence.

Remark. Generally the ruler and compass construction of the triangle with given R, r and s is impossible. In fact, if $t = \tan\frac{\alpha}{2}$, then from $s = \frac{r}{\tan\frac{\alpha}{2}} + 2R\sin\alpha$ we have

$$st^3 - (4R + r)t^2 + st - r = 0.$$

The triangle is constructible if and only if t is constructible. It is known that the roots of a cubic equation with rational coefficients are constructible if and only if the equation has a rational root [1, p.16]. For $R = 4$, $r = 1$, $s = 8$ (such a triangle exists by Theorem 2) we have

$$8t^3 - 17t^2 + 8t - 1 = 0. \tag{8}$$

It is easy to see that it does not have rational roots. Therefore the roots of (8) are not constructible, and the triangle with given R, r, s is also not constructible.

5. Proof of Theorem 3

Let α be the measure of angle A.

$$h = \frac{2rs}{a} = \frac{\frac{2r^2}{\tan\frac{\alpha}{2}} + 4Rr\sin\alpha}{2R\sin\alpha} = \frac{r^2}{2R\sin^2\frac{\alpha}{2}} + 2r.$$

Since α is monotonically increasing (from $\alpha_{\min}$ to $\alpha_{\max}$ while vertex A moves from A_1 to A_2 along the arc A_1A_2, $h = h(\alpha)$ monotonically decreases from $h_{\max} = h(\alpha_{\min})$ to $h_{\min} = h(\alpha_{\max})$. Furthermore,

$$\begin{aligned} h_{\min} =& R + r - d, \\ h_{\max} =& R + r + d. \end{aligned}$$

This completes the proof of Theorem 3.

Remark. It is easy to construct the triangle by given R, r and h with the help of ruler and compass. Indeed, for a triangle ABC with given altitude $AH = h$ we have

$$AI^2 = \frac{r^2}{\sin^2\frac{\alpha}{2}} = 2R(h - 2r).$$

6. Proof of Theorem 4

The length of the bisector of angle A is given by

$$\ell = \frac{2bc\cos\frac{\alpha}{2}}{b + c}.$$

Since $R = \frac{abc}{4\triangle} = \frac{abc}{4rs}$, we have

$$\ell = \frac{\frac{8Rrs}{a}\cdot\cos\frac{\alpha}{2}}{2s - a} = \frac{r}{\sin\frac{\alpha}{2}} + \frac{2Rr\sin\frac{\alpha}{2}}{r + 2R\sin^2\frac{\alpha}{2}}.$$

Differentiating with respect to α, we have

$$\begin{aligned}\frac{\ell'(\alpha)}{r} &= -\frac{\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}} + \frac{R\cos\frac{\alpha}{2}(r-2R\sin^2\frac{\alpha}{2})}{(r+2R\sin^2\frac{\alpha}{2})^2} \\ &= -\frac{\cos\frac{\alpha}{2}(r^2+2Rr\sin^2\frac{\alpha}{2}+8R^2\sin^4\frac{\alpha}{2})}{2\sin^2\frac{\alpha}{2}(r+2R\sin^2\frac{\alpha}{2})^2} \\ &< 0.\end{aligned}$$

Therefore, $\ell(\alpha)$ monotonically decreases on $[\alpha_{\min}, \alpha_{\max}]$ from $\ell_{\max} = R + r + d$ to $\ell_{\min} = R + r - d$.

Remark. Generally the ruler and compass construction of the triangle with given R, r and ℓ is impossible. Indeed, if $t = \sin\frac{\alpha}{2}$, then

$$2R\ell t^3 - 4Rrt^2 + r\ell t - r^2 = 0.$$

For $R = 3$, $r = 1$ and $\ell = 5$ (such a triangle exists by Theorem 4), we have

$$30t^3 - 12t^2 + 5t - 1 = 0.$$

It can be easily checked that this equation doe not have a rational root. This shows that the ruler and compass construction of the triangle is not possible.

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Forum Geometricorum
Volume 5 (2005) 173–180.

FORUM GEOM
ISSN 1534-1178

The Eppstein Centers and the Kenmotu Points

Eric Danneels

Abstract. The Kenmotu points of a triangle are triangle centers associated with squares each with a pair of opposite vertices on two sides of a triangle. Given a triangle ABC, we prove that the Kenmotu points of the intouch triangle are the same as the Eppstein centers associated with the Soddy circles of ABC.

1. Introduction

D. Eppstein [1] has discovered two interesting triangle centers associated with the Soddy circles of a triangle. Given a triangle ABC, construct three circles with centers at A, B, C, mutually tangent to each other externally at T_a, T_b, T_c respectively. These are indeed the points of tangency of the incircle of triangle ABC, and triangle $T_aT_bT_c$ is the intouch triangle of ABC. The inner (respectively outer) Soddy circle is the circle (S) (respectively (S')) tangent to each of these circles externally at S_a, S_b, S_c (respectively internally at S'_a, S'_b, S'_c).

Theorem 1 (Eppstein [1])**.** (1) *The lines T_aS_a, T_bS_b, and T_cS_c are concurrent at a point M.*
(2) *The lines $T_aS'_a$, $T_bS'_b$, and $T_cS'_c$ are concurrent at a point M'.*

See Figures 1 and 2. In [2], M and M' are the Eppstein centers X_{481} and X_{482}. Eppstein showed that these points are on the line joining the incenter I to the Gergonne point G_{e}. See Figure 1.

The Kenmotu points of a triangle, on the other hand, are associated with triads of congruent squares. Given a triangle ABC, the Kenmotu point K_{e} is the unique point such that there are congruent squares $K_{\text{e}}B_cA_aC_b$, $K_{\text{e}}C_aB_bA_c$, and $K_{\text{e}}A_bC_cB_a$ with the same orientation as triangle ABC, and with A_b, A_c on BC, B_c, B_a on CA, and C_a, C_b on AB respectively. We call K_{e} the positive Kenmotu point. There is another triad of congruent squares with the opposite orientation as ABC, sharing a common vertex at the negative Kenmuto point K'_{e}. See Figure 3. These Kenmotu points lie on the Brocard axis of triangle ABC, which contains the circumcenter O and the symmedian point K.

The intouch triangle $T_aT_bT_c$ has circumcenter I and symmedian point G_{e}. It is immediately clear that the Kenmotu points of the intouch triangle lie on the same

Publication Date: November 15, 2005. Communicating Editor: Paul Yiu.
The author thanks Paul Yiu for his help in the preparation of this paper.

line as do the Soddy and Eppstein centers of triangle ABC. The main result of this note is the following theorem.

Theorem 2. *The positive and negative Kenmotu points of the intouch triangle $T_aT_bT_c$ coincide with the Eppstein centers M and M'.*

We shall give two proofs of this theorem.

2. The Eppstein centers

According to [2], the coordinates of the Eppstein centers were determined by E. Brisse. [1] We shall work with homogeneous barycentric coordinates and make use of standard notations in triangle geometry. In particular, r_a, r_b, r_c denote the radii of the respective excircles, and S stands for twice the area of the triangle.

Theorem 3. *The homogeneous barycentric coordinates of the Eppstein centers are*
(1) $M = (a + 2r_a : b + 2r_b : c + 2r_c)$, *and*
(2) $M' = (a - 2r_a : b - 2r_b : c - 2r_c)$.

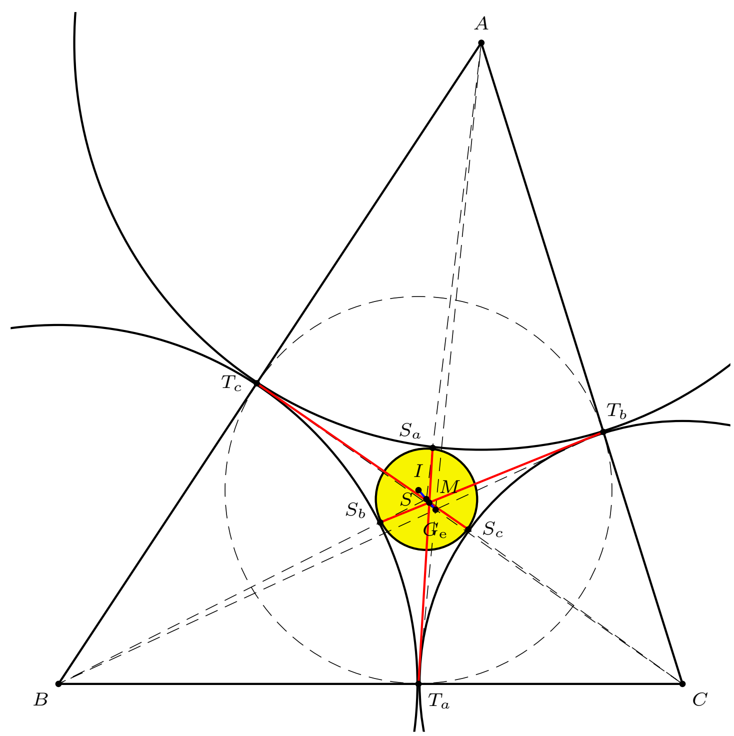

Figure 1. The Soddy center S and the Eppstein center M

[1] The coordinates of X_{481} and X_{482} in [2] (September 2005 edition) should be interchanged.

Remark. In [2], the Soddy centers appear as $X_{175} = S'$ and $X_{176} = S$. In homogeneous barycentric coordinates

$$S = (a + r_a : b + r_b : c + r_c),$$
$$S' = (a - r_a : b - r_b : c - r_c).$$

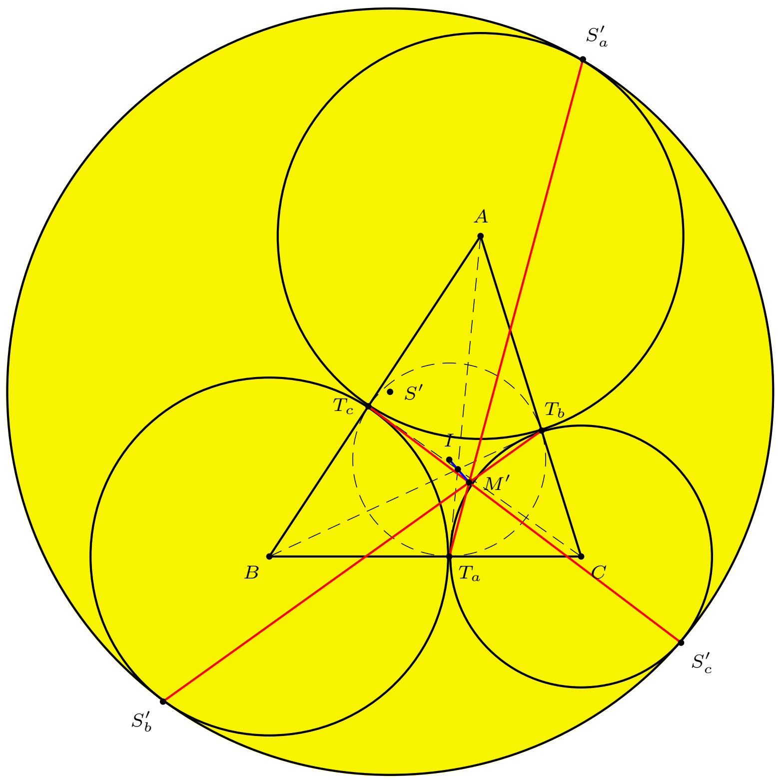

Figure 2. The Soddy center S' and the Eppstein center M'

3. The Kenmotu points

The Kenmotu points K_{e} and K'_{e} have homogeneous barycentric coordinates $(a^2(S_A \pm S) : b^2(S_B \pm S) : c^2(S_C \pm S))$. They are therefore points on the Brocard axis OK. See Figure 3.

Proposition 4. *The Kenmotu points K_{e} and K'_{e} divide the segment OK in the ratio*

$$OK_{\mathrm{e}} : K_{\mathrm{e}}K = a^2 + b^2 + c^2 : 2S,$$
$$OK'_{\mathrm{e}} : K'_{\mathrm{e}}K = a^2 + b^2 + c^2 : -2S.$$

Proof. A typical point on the Brocard axis has coordinates

$$K^*(\theta) = (a^2(S_A + S_\theta) : b^2(S_B + S_\theta) : c^2(S_C + S_\theta)).$$

It divides the segment OK in the ratio

$$OK^*(\theta) : K^*(\theta)K = (a^2 + b^2 + c^2)\sin\theta : 2S \cdot \cos\theta.$$

The Kenmotu points are the points K_{e} and K'_{e} are the points $K^*(\theta)$ for $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$ respectively. $\square$

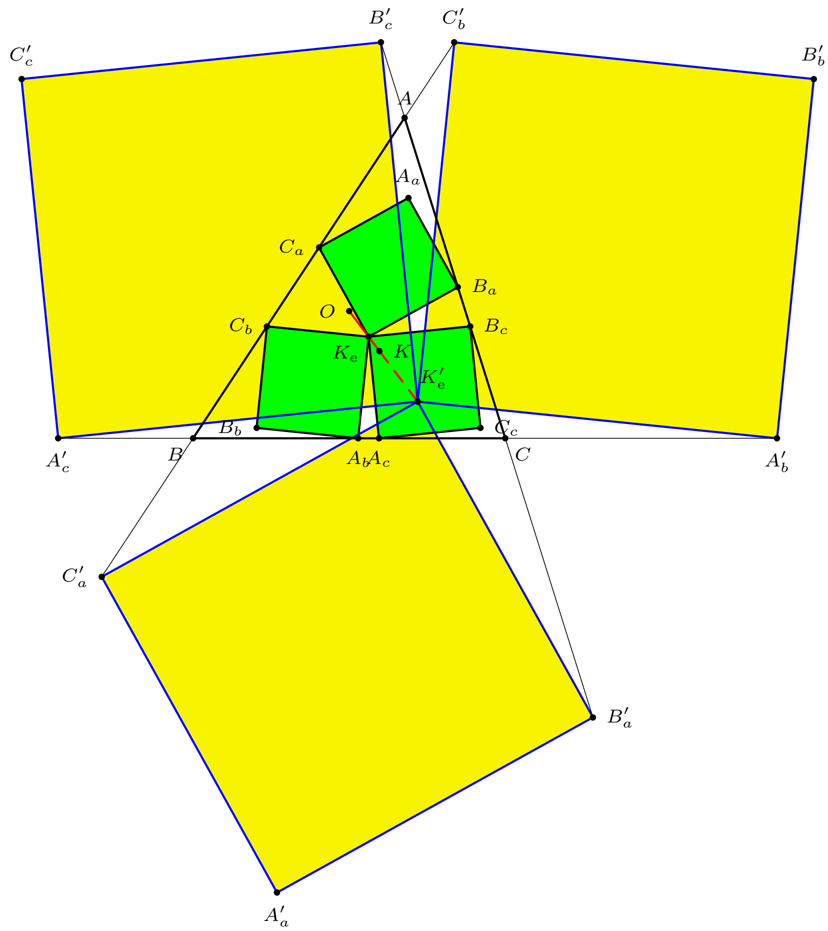

Figure 3. The Kenmotu points K_{e} and K'_{e}

4. First proof of Theorem 2

We shall make use of the following results.

Lemma 5. (1) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$.
(2) $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{4R+r}{2R}$.
(3) $r_a + r_b + r_c = 4R + r$.

The intouch triangle $T_aT_bT_c$ has sidelengths

$$T_bT_c = 2r\cos\frac{A}{2}, \qquad T_cT_a = 2r\cos\frac{B}{2}, \qquad T_aT_b = 2r\cos\frac{C}{2}.$$

The area of the intouch triangle is

$$\frac{1}{2}\overline{S} = \frac{1}{2}T_cT_a\cdot T_aT_b\cdot\sin T_a = 2r^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 2r^2\cdot\frac{s}{4R}.$$

On the other hand,

$$T_bT_c^2 + T_cT_a^2 + T_aT_b^2 = 4r^2\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right) = \frac{2r^2(4R+r)}{R}.$$

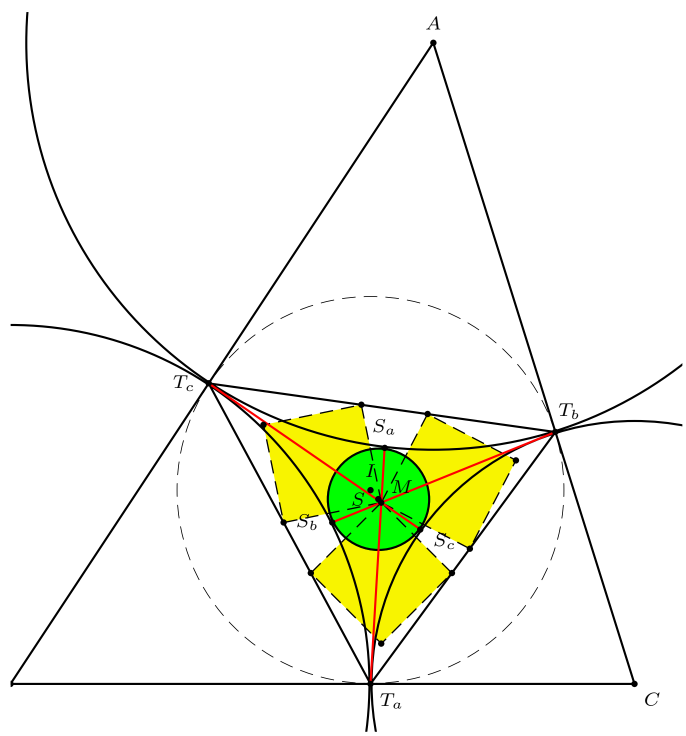

Figure 4. The positive Kenmotu point of the intouch triangle

By Proposition 4, the positive Kenmotu point $\overline{K}_{\rm e}$ of the intouch triangle divides the segment $IG_{\rm e}$ in the ratio

$$\begin{aligned} I\overline{K}_{\rm e} : \overline{K}_{\rm e}G_{\rm e} =& T_bT_c^2 + T_cT_a^2 + T_aT_b^2 : 2\overline{S} \\ =& 4R + r : s \\ =& r_a + r_b + r_c : s. \end{aligned}$$

It has absolute barycentric coordinates

$$\begin{aligned}\overline{K}_{\mathrm{e}} =& \frac{1}{s+r_a+r_b+r_c}\left(s\cdot I+(r_a+r_b+r_c)\cdot G_{\mathrm{e}}\right)\\ =& \frac{1}{s+r_a+r_b+r_c}\left(\frac{1}{2}(a,b,c)+(r_a,r_b,r_c)\right)\\ =& \frac{1}{2(s+r_a+r_b+r_c)}\cdot(a+2r_a,b+2r_b,c+2r_c).\end{aligned}$$

Therefore, $\overline{K}_{\mathrm{e}}$ has homogeneous barycentric coordinates $(a+2r_a : b+2r_b : +2r_c)$. By Theorem 3, it coincides with the Eppstein center M. See Figure 4.

Similar calculations show that the Eppstein center M' coincides with the negative Kenmotu point $\overline{K}'_{\mathrm{e}}$ of the intouch triangle. See Figure 5. The proof of Theorem 2 is now complete.

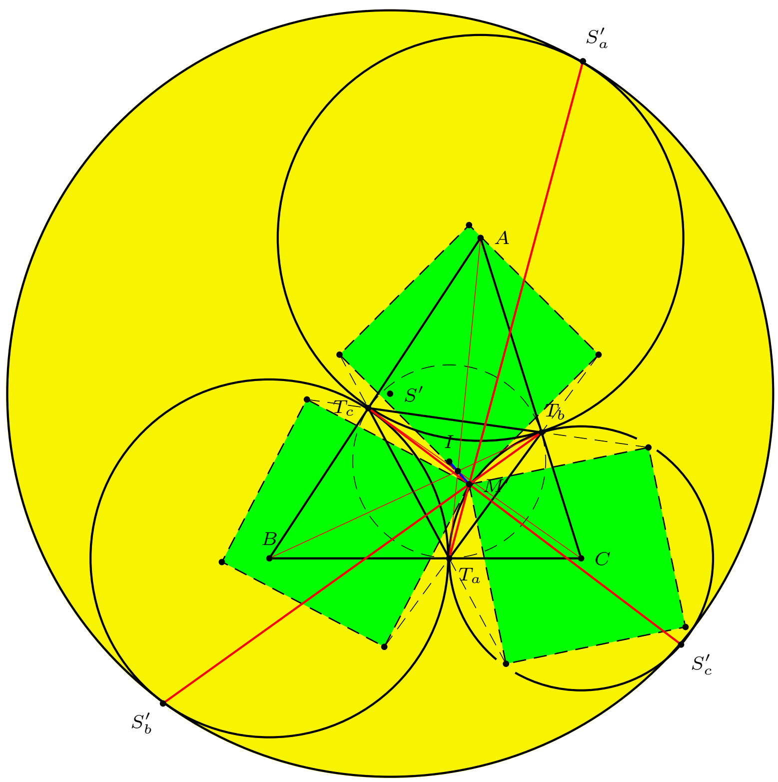

Figure 5. The negative Kenmotu point of the intouch triangle

5. Second proof of Theorem 2

Consider a point P with homogeneous barycentric coordinates $(u' : v' : w')$ with respect to the intouch triangle $T_aT_bT_c$. We determine its coordinates with respect to the triangle ABC. By the definition of barycentric coordinates, a system of three masses u', v' and w' at the points T_a, T_b and T_c will balance at P. The mass u' at T_a can be replaced by a mass $\frac{s-c}{a} \cdot u'$ at B and a mass $\frac{s-b}{a} \cdot u'$ at C. Similarly, the mass v' at T_b can be replaced by a mass $\frac{s-a}{b} \cdot v'$ at C and a mass $\frac{s-c}{b} \cdot v'$ at A, and the mass w' at T_c by a mass $\frac{s-b}{c} \cdot w'$ at A and a mass $\frac{s-a}{c} \cdot w'$ at B. The resulting mass at A is therefore

$$\frac{s-c}{b} \cdot v' + \frac{s-b}{c} \cdot w' = \frac{a(c(s-c)v' + b(s-b)w')}{abc}.$$

From similar expressions for the masses at B and C, we obtain

$$(a(c(s-c)v'+b(s-b)w') : b(a(s-a)w'+c(s-c)u') : c(b(s-b)u'+a(s-a)v'))$$

for the barycentric coordinates of P with respect to ABC.

The Kenmotu point K_{e} appears the triangle center X_{371} in [2]. For the Kenmotu point of the intouch triangle, we may take

$$\begin{aligned} u' =& T_bT_c(\cos T_a + \sin T_a) \\ =& 2(s-a)\sin\frac{A}{2}\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right), \\ v' =& 2(s-b)\sin\frac{B}{2}\left(\sin\frac{B}{2} + \cos\frac{B}{2}\right), \\ w' =& 2(s-c)\sin\frac{C}{2}\left(\sin\frac{C}{2} + \cos\frac{C}{2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} u =& a(c(s-c)v' + b(s-b)w') \\ =& 2a(s-b)(s-c)\left(c\cdot\sin\frac{B}{2}\left(\sin\frac{B}{2}+\cos\frac{B}{2}\right) + b\cdot\sin\frac{C}{2}\left(\sin\frac{C}{2}+\cos\frac{C}{2}\right)\right) \\ =& 2a(s-b)(s-c)\left(c\sin^2\frac{B}{2} + b\sin^2\frac{C}{2} + c\cdot\frac{\sin B}{2} + b\cdot\frac{\sin C}{2}\right) \\ =& 2a(s-b)(s-c)\left(c\cdot\frac{(s-c)(s-a)}{ca} + b\cdot\frac{(s-a)(s-b)}{ab} + \frac{bc}{2R}\right) \\ =& 2(s-a)(s-b)(s-c)\left(a + \frac{abc}{2R(s-a)}\right) \\ =& 2(s-a)(s-b)(s-c)\left(a + \frac{S}{s-a}\right) \\ =& 2(s-a)(s-b)(s-c)(a+2r_a). \end{aligned}$$

Similar expressions for v and w give

$$u : v : w = a + 2r_a : b + 2r_b : c + r_c,$$

which are the coordinates of the Eppstein center M.

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Forum Geometricorum
Volume 5 (2005) 181–190.

FORUM GEOM
ISSN 1534-1178

Statics and the Moduli Space of Triangles

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Abstract. The variance of a weighted collection of points is used to prove classical theorems of geometry concerning homogeneous quadratic functions of length (Apollonius, Feuerbach, Ptolemy, Stewart) and to deduce some of the theory of major triangle centers. We also show how a formula for the distance of the incenter to the reflection of the centroid in the nine-point center enables one to simplify Euler's method for the reconstruction of a triangle from its major centers. We also exhibit a connection between Poncelet's porism and the location of the incenter in the circle on diameter GH (the orthocentroidal or critical circle). The interior of this circle is the moduli (classification) space of triangles.

1. Introduction

There are some theorems of Euclidean geometry which have elegant proofs by means of mechanical principles. For example, if ABC is an acute triangle, one can ask which point P in the plane minimizes $AP + BP + CP$? The answer is the Fermat point, the place where $\angle APB = \angle BPC = \angle CPA = 2\pi/3$. The mechanical solution is to attach three pieces of inextensible massless string to P, and to dangle the three strings over frictionless pulleys at the vertices of the triangle, and attach the same mass to each string. Now hold the triangle flat and dangle the masses in a uniform gravitational field. The forces at P must balance so the angle equality is obtained, and the potential energy of the system is minimized when $AP + BP + CP$ is minimized.

In this article we will develop a geometric technique which involves a notion analogous to the moment of inertia of a mechanical system, but because of an averaging process, this notion is actually more akin to *variance* in statistics. The main result is well known to workers in the analysis of variance. The applications we give will (in the main) not yield new results, but rather give alternative proofs of classical results (Apollonius, Feuerbach, Stewart, Ptolemy) and make possible a systematic statical development of some of the theory of triangle centers. We will conclude with some remarks concerning the problem of reconstructing a triangle from O, G and I which will, we hope, shed more light on the constructions of Euler [3] and Guinand [4].

Publication Date: December 6, 2005. Communicating Editor: Paul Yiu.

I wish to thank Christopher Bradley of Bristol both for helping to rekindle my interest in Euclidean geometry, and for lending plausibility to some of the longer formulas in this article by means of computational experiments.

For geometrical background we recommend [1] and [2].

Definition. Let X and Y be non-empty finite subsets of an inner product space V. We have weight maps $m : X \longrightarrow \mathbb{R}$ and $n : Y \longrightarrow \mathbb{R}$ with the property that $M = \sum_x m(x) \neq 0 \neq \sum_{y\in Y} n(y) = N$. The *mean square distance* between these weighted sets is

$$d^2(X, m, Y, n) = \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - y||^2.$$

Let

$$\overline{x} = \frac{1}{M} \sum_{x\in X} m(x)x$$

be the centroid of X. Ignoring the distinction between $\overline{x}$ and $\{\overline{x}\}$, and assigning the weight 1 to $\overline{x}$, we put

$$\sigma^2(X, m) = d^2(X, m, \overline{x}, 1)$$

and call this the *variance* of X, m. In fact the non-zero weight assigned to $\overline{x}$ is immaterial since it cancels. When the weighting is clear in a particular context, mention of it may be suppressed. We will also be cavalier with the arguments of these functions for economy.

We call the main result the generalized parallel axis theorem (abbreviated to GPAT) because of its relationship to the corresponding result in mechanics.

Theorem 1 (GPAT)**.**

$$d^2(X, m, Y, n) = \sigma^2(X, m) + ||\overline{x} - \overline{y}||^2 + \sigma^2(Y, m).$$

Proof.

$$\frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - y||^2$$

$$= \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - \overline{x} + \overline{x} - \overline{y} + \overline{y} - y||^2$$

$$= \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||x - \overline{x}||^2 + ||\overline{x} - \overline{y}||^2 + \frac{1}{MN} \sum_{x\in X, y\in Y} m(x)n(y)||y - \overline{y}||^2$$

since the averaging process makes the cross terms vanish. We are done. □

Corollary 2. $d^2(X, m, X, m) = 2\sigma^2(X, m)$.

Note that the averaging process ensures that scaling the weights of a given set does not alter mean square distances or variances.

The method of areal co-ordinates involves fixing a reference triangle ABC in the plane, and given a point P in its interior, assigning weights which are the areas of triangles: the weights $[PBC], [PCA]$ and $[PAC]$ are assigned to the points A, B and C respectively. The center of mass of $\{A, B, C\}$ with the given weights is P. With appropriate signed area conventions, this can be extended to define a co-ordinate system for the whole plane. If the weights are scaled by dividing by the area of $\triangle ABC$, then one obtains normalized areal co-ordinates; the co-ordinates of A are then $(1,0,0)$ for example. A similar arrangement works in Euclidean space of any dimension. The GPAT has much to say about these co-ordinate systems.

2. Applications

2.1. *Theorems of Apollonius and Stewart.* Let ABC be a triangle with corresponding sides of length a, b and c. A point D on the directed line CB is such that $CD = m$, $DB = n$ and these quantities may be negative. Let AD have length x. Weighting B with m and C with n, the center of mass of $\{B, C\}$ is at D and the variance of the weighted $\{B, C\}$ is $\sigma^2 = (mn^2 + nm^2)/(m+n) = mn$. The GPAT now asserts that

$$\frac{nb^2 + mc^2}{m+n} = 0 + x^2 + \sigma^2$$

or rather

$$nb^2 + mc^2 = (m+n)(x^2 + mn).$$

This is Stewart's theorem. If $m = n$ we deduce Apollonius's result that $b^2 + c^2 = 2(x^2 + (\frac{a}{2})^2)$.

2.2. *Ptolemy's Theorem.* Let A, B, C and D be four points in Euclidean 3-space. Consider the two sets $\{A, C\}$ and $\{B, D\}$ with weight 1 at each point. The GPAT asserts that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4t^2$$

where t is the distance between the midpoints of the line segments AC and BD. This may be familiar in the context that $t = 0$ and $ABCD$ is a parallelogram.

Recall that Ptolemy's theorem asserts that if $ABCD$ is a cyclic quadrilateral, then

$$AC \cdot BD = AB \cdot CD + BC \cdot DA.$$

We prove this as follows. Let the diagonals AC and BD meet at X. Now weight A, B, C and D so that the centers of mass of both $\{A, C\}$ and $\{B, D\}$ are at X. The GPAT now asserts that

$$\frac{XC \cdot AX^2 + AX \cdot XC^2}{AC} + \frac{XB \cdot DX^2 + DX \cdot BX^2}{BD}$$

$$= \frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{AC \cdot BD}.$$

The left side of this equation tidies to $AX \cdot XC + BX \cdot XD$. One could regard this equation as a generalization of Ptolemy's theorem to quadrilaterals which are not necessarily cyclic.

Now we invoke cyclicity: $AX \cdot XC = BX \cdot XD = x$ by the intersecting chords theorem. Therefore $AC \cdot BD =$

$$\frac{XC \cdot AB^2 \cdot XD + XC \cdot AD^2 \cdot BX + XA \cdot CB^2 \cdot XD + XA \cdot CD^2 \cdot XB}{2x}.$$

However $AB/CD = BX/XC = AX/XD$ and $DA/BC = AX/BX = DX = CX$ (by similarity) so the right side of this equation is $AB \cdot CD + BC \cdot DA$ and Ptolemy's theorem is established.

2.3. *A geometric interpretation of* σ^2. Let ABC be a triangle with circumcenter O and incenter I and the usual side lengths a, b and c. We can arrange that the center of mass of $\{A, B, C\}$ is at I by placing weights a, b and c at A, B and C respectively. By calculating the mean square distance of this set of weighted triangle vertices to itself, we obtain the variance $\sigma_I^2 = \frac{abc}{a+b+c}$. However $abc/4R = [ABC]$, the area of the triangle, and $(a+b+c)r = 2[ABC]$ where R, r are the circumradius and inradius respectively. Therefore

$$\sigma_I^2 = 2Rr = \frac{abc}{a+b+c}. \tag{1}$$

Now calculate the mean square distance from O to the weighted triangle vertices both in the obvious way, and also by the GPAT to obtain Euler's result

$$OI^2 = R^2 - 2Rr. \tag{2}$$

Observation More generally suppose that a finite coplanar set of points Λ is concyclic, and is weighted to have center of mass at L, Let the center of the circle be at X and its radius be ρ. By the GPAT applied to X and the weighted set Λ we obtain

$$LX^2 = \rho^2 - \sigma^2(\Lambda, L)$$

so

$$\sigma^2(\Lambda, L) = \rho^2 - LX^2 = (\rho - LX)(\rho + LX).$$

Thus we conclude that $\sigma^2(\Lambda, L)$ is minus the *power of* L with respect to the circle.

2.4. *The Euler line.* Let ABC be a triangle with circumcenter O, centroid G and orthocenter H. These three points are collinear and this line is called the Euler line. It is easy to show that $OH = 3OG$. It is well known that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2). \tag{3}$$

We derive this formula using the GPAT. Assign unit weights to the vertices of triangle ABC. The center of mass will be at G the intersection of the medians. Calculate the mean square distance of this triangle to itself to obtain the variance σ_G^2 of this triple of points. By the GPAT we have

$$2\sigma_G^2 = \frac{2a^2 + 2b^2 + 2c^2}{9}$$

so $\sigma_G^2 = \frac{a^2+b^2+c^2}{9}$. Now calculate the mean square distance from O to this triangle with unit weight the sensible way, and also by the GPAT to obtain

$$R^2 = OG^2 + \sigma_G^2.$$

Multiply through by 9 and use the fact that $OH = 3OG$ to obtain (3).

2.5. *The Nine-point Circle.* Let ABC be a triangle. The nine-point circle of ABC is the circle which passes through the midpoints of the sides, the feet of the altitudes and the midpoints of the line segments joining the orthocenter H to each vertex. This circle has radius $R/2$ and is tangent to the inscribed circle of triangle ABC (they touch internally to the nine-point circle), and the three escribed circles (externally). We will prove this last result using the GPAT, and calculate the squares of the distances from I to important points on the Euler line.

Proposition 3. *Let p denote the perimeter of the triangle A, B, C. The distance between the incenter I and centroid G satisfies the following equation:*

$$IG^2 = \frac{p^2}{6} - \frac{5}{18}(a^2 + b^2 + c^2) - 4Rr. \tag{4}$$

Proof. Let $\triangle_G$ denote the triangle weighted 1 at each vertex and $\triangle_I$ denote the same triangle with weights attached to the vertices which are the lengths of the opposite sides. We apply the GPAT and a direct calculation:

$$d^2(\triangle_G, \triangle_I) = \sigma_G^2 + IG^2 + \sigma_I^2 = \frac{ab^2 + ba^2 + bc^2 + cb^2 + ca^2 + ac^2}{3(a + b + c)}$$

so

$$\frac{a^2 + b^2 + c^2}{9} + IG^2 + 2Rr = \frac{(ab + bc + ca)(a + b + c) - 3abc}{3(a + b + c)}$$

$$= \frac{ab + bc + ca}{3} - 2Rr.$$

Therefore

$$4Rr + IG^2 + \frac{a^2 + b^2 + c^2}{9} = \frac{(a + b + c)^2}{6} - \frac{a^2 + b^2 + c^2}{6}.$$

This equation can be tidied into the required form.

Corollary Using Euler's inequality $R \geq 2r$ (which follows from $IO^2 \geq 0$) and the condition $|IG|^2 \geq 0$ we obtain that in any triangle we have

$$3p^2 \geq 5(a^2 + b^2 + c^2) + 144r^2$$

with equality exactly when $R = 2r$ and $I = G$. Thus the inequality becomes an equality if and only if the triangle is equilateral. $\square$

Theorem 4 (Feuerbach)**.** *The nine point circle of $\triangle ABC$ is internally tangent to the incircle.*

Proof. (outline) The radius of the nine point circle is $R/2$. The result will established if we show that $|IN| = R/2 - r$. However, in $\triangle INO$ the point G is on the side NO and $NG : GO = 1 : 2$. We know $|IO|, |IG|, |NG|$ and $|GO|$, so Stewart's theorem and some algebra enable us to deduce the result.

Since $OG : GN = 2 : 1$ Stewart's theorem applies and we have

$$IG^2 + \frac{2}{9}ON^2 = \frac{2}{3}IN^2 = \frac{1}{3}IO^2.$$

Rearranging this becomes

$$IN^2 = \frac{3}{2}IG^2 + \frac{3}{4}OG^2 - \frac{1}{2}IO^2.$$

Now we aim to show that this expression is $(R/2 - r)^2$, or rather $R^2/4 - Rr + r^2$. We put in known values in terms of the side lengths, and perform algebraic manipulations, deploying Heron's formula where necessary. Feuerbach's theorem follows. $\square$

It must be admitted that this calculation does little to illuminate Feuerbach's result. We will give a more conceptual statics proof shortly.

2.6. *The location of the incenter.*

Proposition 5. *The incenter of a non-equilateral triangle lies strictly in the interior of the circle on diameter GH.*

This was presumably known to Euler [5], and a stronger version of the result was proved in [4]. Given Feuerbach's theorem, this result almost proves itself. Let N be the nine-point center, the midpoint of the segment OH, Feuerbach's tangency result yields $IN = R/2 - r$. However $OI^2 = R^2 - 2Rr$ so $OI^2 - 4IN^2 = R^2 - 4Rr + 4r^2 - R^2 + 2Rr = 2r(R - 2r)$. However Euler's formula for OI yields $2r < R$ (with equality only for equilateral triangles). Therefore I lies in the interior of the circle of Apollonius consisting of points P such that $OP = 2NP$, which is precisely the circle on diameter GH as required.

We can verify this result by an explicit calculation. Let J be the center of the circle on diameter GH so $OG = GJ = JH$. Using Apollonius's theorem on $\triangle IHO$ we obtain

$$2IN^2 + 2(\frac{3}{2}OG)^2 = OI^2 + IH^2$$

which expands to reveal that

$$HI^2 = \frac{OH^2 - (R^2 - 4r^2)}{2}.$$

Now use Stewart's theorem on $\triangle IHO$ to calculate IJ^2. We have

$$IJ^2 + 2OG^2 = \frac{OI^2 \cdot OG + IH^2 \cdot 2OG}{OH}$$

which after simple manipulation yields that

$$IJ^2 = OG^2 - \frac{2r}{3}(R - 2r) < OG^2. \tag{5}$$

The formulas for the squares of the distances from I to important points on the Euler line can be quite unwieldy, and some care has been taken to calculate these quantities in such a way that the algebraic dependence between the triangle sides and r, R and OG is produces relatively straightforward expressions. More interesting relationship can be found; for example using Stewart's theorem on $\triangle INO$ with Cevian IG we obtain

$$6IG^2 + 3OG^2 = (3R - 2r)(R - 2r).$$

3. Areal co-ordinates and Feuerbach revisited

The use of areal or volumetric co-ordinates is a special but important case of weighted systems of points. The GPAT tells us about the change of co-ordinate frames: given two reference triangles $\triangle_1$ with vertices A, B, C and $\triangle_2$ with vertices A', B', C' and points P and Q in the plane. it is natural to consider the relationship between the areal co-ordinates of a point P in the first frame (x, y, z) and those of Q in the second (x', y', z'). We assume that co-ordinates are normalized. Now GPAT tells us that

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = \sigma_{1,P}^2 + PQ^2 + \sigma_{1,Q}^2.$$

The resulting formulas can be read off. The recipe which determines the square of the distance between two points given in areal co-ordinates with respect to the same reference triangle is straightforward. Suppose that P has areal co-ordinates (p_1, p_2, p_3) and Q has co-ordinates (q_1, q_2, q_3). Let $(x, y, z) = (p_1, p_2, p_3) - (q_1, q_2, q_3)$ (subtraction of 3-tuples) and let $(u, v, w) = (yz, zx, xy)$ (the Cremona transformation) then we deduce that

$$PQ^2 = -(a^2, b^2, c^2) \cdot (u, v, w).$$

Here we are using the ordinary dot product of 3-tuples. Note that (a^2, b^2, c^2) viewed as an areal co-ordinate is the symmedian point, the isogonal conjugate of G. We do not know if this observation has any significance.

A another special situation arises when $\triangle_1$ and $\triangle_2$ have the same circumcircle (perhaps they are the same triangle) and points P and Q are both on the common circle. In this case $\sigma_{1,P}^2 = 0 = \sigma_{2,Q}^2$ and

$$d^2(\triangle_{1,P}, \triangle_{2,Q}) = PQ^2.$$

In the context of areal co-ordinates, we are now in a position to revisit Feuerbach's theorem and give a more conceptual statics proof which yields an interesting corollary.

3.1. *Proof of Feuerbach's theorem.* To prove Feuerbach's theorem it suffices to show that the power of I with respect to the nine-point circle is $-r(R - r)$ or equivalently that $\widehat{\sigma}_I^2 = r(R - r)$ where the hat indicates that we are using the medial triangle (with vertices the midpoints of the sides of $\triangle ABC$) as the triangle of reference. Now the medial triangle is obtained by rotating the original triangle about G through π, and scaling by 1/2. Let I' denote the incenter of the medial triangle with co-ordinates $(a/2, b/2, c/2)$. The co-ordinates of G are $(s/3, s/3, s/3)$. Now I', G, I are collinear and $I'G : GI = 1 : 2$. The co-ordinates of I are therefore $(s-a, s-b, s-c)$, Next we use cyc to indicate a sum over cyclic permutations of a, b and c, and *sym* a sum over all permutations. We calculate

$$\begin{aligned}\widehat{\sigma}_I^2 &= \sum_{cyc} \frac{(s-a)(s-b)c^2}{4s^2} \\ &= \frac{s^2\sum_{cyc} a^2 - s\sum_{sym} a^2b + 2abcs}{4s^2} \\ &= \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2b}{4(a+b+c)} + 2Rr.\end{aligned}$$

However by Heron's formula

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}$$

so

$$rR - r^2 = \frac{2abc}{4(a+b+c)} + \frac{a^3+b^3+c^3}{4(a+b+c)} - \frac{\sum_{sym} a^2b}{4(a+b+c)} + \frac{2abc}{4(a+b+c)} = \widehat{\sigma}_I^2$$

since $abc/(a+b+c) = 2Rr$.

Corollary 6. *The areal co-ordinates of I with respect to the medial triangle are $(s-a, s-b, s-c)$, perhaps better written $(\frac{s}{2}-\frac{a}{2}, \frac{s}{2}-\frac{b}{2}, \frac{s}{2}-\frac{c}{2})$. Therefore the incenter of the reference triangle is the Nagel point of the medial triangle.*

4. The Euler-Guinand problem

In 1765 Euler [3] recovered the sides lengths a, b and c of a non-equilateral triangle from the positions of O, G and I. At the time he did not have access to Feuerbach's formula for IN^2 nor our formula (5). This extra data enables us to make light of Euler's calculations. From (5) we have $r(2R-r)$ and combining with (2) we obtain first R/r and then both R and r. Now (3) yields $a^2+b^2+c^2$ and (4) gives $a+b+c$. Finally (1) yields abc. Thus the polynomial $\triangle(x) = (X-a)(X-b)(X-c)$ can be easily recovered from the positions of O, G and I. We call this the triangle polynomial This may be an irreducible rational cubic so the construction of a, b and c by ruler and compasses may not be possible.

The actual locations of A, B and C may be determined as follows. Note that this addresses the critical remark (3) of [5]. The circumcircle of $\triangle ABC$ is known since O and R are known. Now by the GPAT we obtain the well known formula

$$\frac{0^2+b^2+c^2}{3} = AG^2 + \frac{a^2+b^2+c^2}{9}$$

so

$$AG^2 = \frac{2b^2+2c^2-a^2}{9}$$

and similarly of BG^2 and CG^2. By intersecting circles of appropriate radii centered at G with the circumcircle, we recover at most two candidate locations for each point A, B and C. Now triangle ABC is one of at most $2^3 = 8$ triangles. These can be inspected to see which ones have correct O, G and I. Note that there is only one correct triangle since AG^2, AO^2 and AI^2 are all determined.

In fact every point in the interior of the circle on diameter GH other than the nine-point center N arises as a possible location of an incenter I [4]. We give a new derivation of this result addressing the same question as [4] and [5] but in a different way.

Given any value $k \in (0,1)$ there is a triangle such that $2r/R = k$. Choosing such a triangle, with circumradius R we observe that

$$\left(\frac{IO}{IN}\right)^2 = \frac{R^2 - 2Rr}{\left(\frac{R}{2} - r\right)^2}$$

so

$$\frac{IO}{IN} = 2\sqrt{\frac{R}{R-2r}}. \tag{6}$$

If O and N were fixed, this would force I to lie on a circle of Apollonius with defining ratio $2\sqrt{\frac{R}{R-2r}}$. In what follows we rescale our diagrams (when convenient) so that the distance ON is fixed, so the circle on diameter GH (the orthocentroidal or critical [4] circle) can be deemed to be of fixed diameter.

Consider the configuration of Poncelet's porism for triangle ABC. We draw the circumcircle with radius R and center O, and the incenter I internally tangent to triangle ABC at three points. Now move the point A to A' elsewhere on the circumcircle and generate a new triangle $A'B'C'$ with the same incircle. We move A to A' continuously and monotonically, and observe how the configuration changes; the quantities R and r do not change but in the scaled diagram the corresponding point I' moves continuously on the given circle of Apollonius. When A' reaches B the initial configuration is recovered. Consideration of the largest angle in the moving triangle $A'B'C'$ shows that until the initial configuration is regained, the triangles formed are pairwise dissimilar, so inside the scaled version of the circle on diameter GH, the point I' moves continuously on the circle of Apollonius in a monotonic fashion. Therefore I' makes exactly one rotation round the circle of Apollonius and A' moves to B. Thus all points on this circle of Apollonius arise as possible incenters, and since the defining constant of the circle is arbitrary, all points (other than N) in the interior of the scaled circle on diameter GH arise as possible locations for I and Guinand's result is obtained [4].

Letting the equilateral triangle correspond to N, the open disk becomes a moduli space for direct similarity types of triangle. The boundary makes sense if we allow triangles to have two sides parallel with included angle 0. Some caution should be exercised however. The angles of a triangle are not a continuous function of the side lengths when one of the side lengths approaches 0. Fix A and let B tend to C by spiraling in towards it. The point I in the moduli space will move enthusiastically round and round the disk, ever closer to the boundary.

Isosceles triangles live in the moduli space as the points on the distinguished (Euler line) diameter. If the unequal side is short, I is near H, but if it is long, I is near G.

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Forum Geometricorum
Volume 5 (2005) 191–195.

FORUM GEOM
ISSN 1534-1178

A Gergonne Analogue of the Steiner - Lehmus Theorem

K. R. S. Sastry

Abstract. In this paper we prove an analogue of the famous Steiner - Lehmus theorem from the Gergonne cevian perspective.

1. Introduction

Can a theorem be both famous and infamous simultaneously? Certainly there is one such in Euclidean Geometry if the former is an indicator of a record number of correct proofs and the latter an indicator of a record number of incorrect ones. Most school students must have found it easy to prove the following: The angle bisectors of equal angles of a triangle are equal. However, not many can prove its converse theorem correctly:

Theorem 1 (Steiner-Lehmus)**.** *If two internal angle bisectors of a triangle are equal, then the triangle is isosceles.*

According to available history, in 1840 a Berlin professor named C. L. Lehmus (1780-1863) asked his contemporary Swiss geometer Jacob Steiner for a proof of Theorem 1. Steiner himself found a proof but published it in 1844. Lehmus proved it independently in 1850. The year 1842 found the first proof in print by a French mathematician [3]. Since then mathematicians and amateurs alike have been proving and re-proving the theorem. More than 80 correct proofs of the Steiner - Lehmus theorem are known. Even larger number of incorrect proofs have been offered. References [4, 5] provide extensive bibliographies on the Steiner - Lehmus theorem.

For completeness, we include a proof by M. Descube in 1880 below, recorded in [1, p.235]. The aim of this paper is to prove an analogous theorem in which we consider the equality of two Gergonne cevians. We offer two proofs of it and then consider an extension. Recall that a Gergonne cevian of a triangle is the line segment connecting a vertex to the point of contact of the opposite side with the incircle.

2. Proof of the Steiner - Lehmus theorem

Figure 1 shows the bisectors BE and CF of $\angle ABC$ and $\angle ACB$. We assume $BE = CF$. If $AB \neq AC$, let $AB < AC$, *i.e.*, $\angle ACB < \angle ABC$ or $\frac{C}{2} < \frac{B}{2}$. A

Publication Date: December 20, 2005. Communicating Editor: Paul Yiu.

comparison of triangles BEC with BFC shows that

$$CE > BF. \tag{1}$$

Complete the parallelogram $BFGE$. Since $EG = BF$, $\angle FGE = \frac{B}{2}$, $FG = BE = CF$ implying that $\angle FGC = \angle FCG$. But by assumption $\angle FGE = \frac{B}{2} > \angle FCE = \frac{C}{2}$. So $\angle EGC < \angle ECG$, and $CE < GE = BF$, contradicting (1).

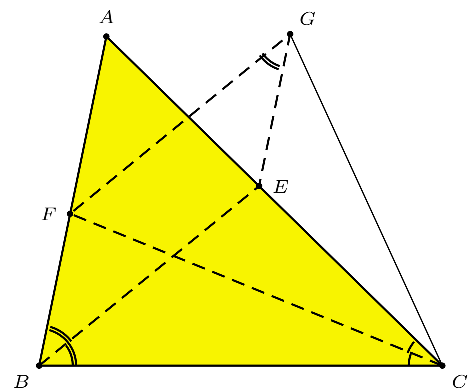

Figure 1.

Likewise, the assumption $AB > AC$ also leads to a contradiction. Hence, $AB = AC$ and $\triangle ABC$ must be isosceles.

3. The Gergonne analogue

We provide two proofs of Theorem 2 below. The first proof equates the expressions for the two Gergonne cevians to establish the result. The second one is modelled on the proof of the Steiner - Lehmus theorem in §2 above.

Theorem 2. *If two Gergonne cevians of a triangle are equal, then the triangle is isosceles.*

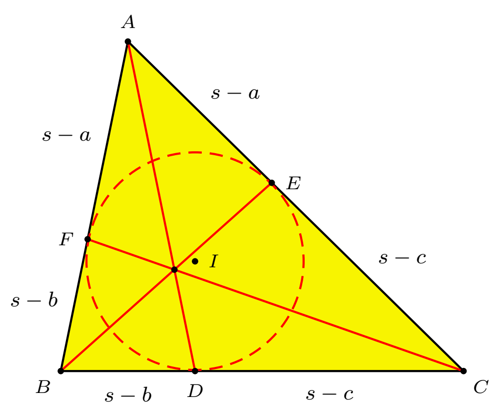

Figure 2.

3.1. *First proof.* Figure 2 shows the equal Gergonne cevians BE, CF of triangle ABC. We consider $\triangle ABE$, $\triangle ACF$ and apply the law of cosines:

$$\begin{aligned} BE^2 =& c^2 + (s-a)^2 - 2c(s-a)\cos A, \\ CF^2 =& b^2 + (s-a)^2 - 2b(s-a)\cos A. \end{aligned}$$

Equating the expressions for BE^2 and CF^2 we see that

$$2(b-c)(s-a)\cos A - (b^2 - c^2) = 0$$

or

$$(b-c)\left[\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c)\right] = 0.$$

There are two cases to consider.

(i) $b - c = 0 \Rightarrow b = c$ and triangle ABC is isosceles.

(ii) $\frac{(-a+b+c)(b^2+c^2-a^2)}{2bc} - (b+c) = 0$. This can be put, after simplification, in the form

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) = 0.$$

This clearly is an impossibility by the triangle inequality.

Therefore (i) must hold and triangle ABC is isosceles.

3.2. *Second proof.* We employ the same construction as in Figure 1 for Theorem 1. Hence we do not repeat the description here for Figure 3.

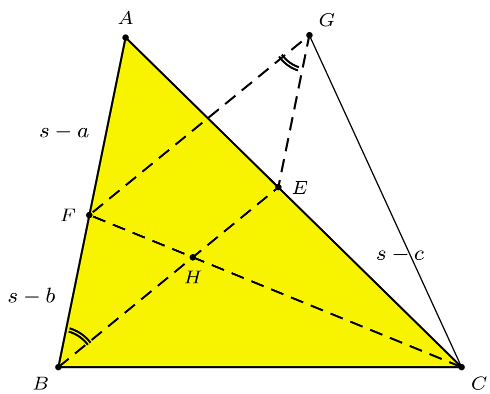

Figure 3.

If $AB \neq AC$, let $AB < AC$, *i.e.*, $c < b$, and $s - c > s - b$. As seen in the proof of Theorem 1, $\angle EBC > \angle FCB \Rightarrow CH > BH$. Since $CF = BE$, we have

$$FH < EH. \tag{2}$$

In triangles ABE and AFC, $AE = AF = s - a$, $BE = CF$ and by assumption $AB < AC$. Hence $\angle AEB < \angle AFC \Rightarrow \angle BEC > \angle BFC$ or

$$\angle HEC > \angle HFB. \tag{3}$$

Therefore, in triangles BFH and EHC, $\angle BHF = \angle EHC$ and from (3) we see that

$$\angle FBH > \angle HCE. \tag{4}$$

Triangle FGC is isosceles by construction, so $\angle FGC = \angle FCG$ or $\angle FGE + \angle EGC = \angle HCE + \angle ECG$. Because of (4) we see that $\angle EGC < \angle ECG$ or $EC < EG$, *i.e.*, $s - c < s - b \Rightarrow b < c$, contradicting the assumption.

Likewise the assumption $b > c$ would lead to a similar contradiction. Hence we must have $b = c$, and triangle ABC is isosceles.

4. An extension

Theorem 3 shows that the equality of the segments of two angle bisectors of a triangle intercepted by a Gergonne cevian itself implies that the triangle is isosceles.

Theorem 3. *The internal angle bisectors of the angles ABC and ACB of triangle ABC meet the Gergonne cevian AD at E and F respectively. If $BE = CF$, then triangle ABC is isosceles.*

Proof. We refer to Figure 4. If $AB \neq AC$, let $AB < AC$. Hence $b > c$, $s - b < s - c$ and E lies below F on AD. A simple calculation with the help of the angle bisector theorem shows that the Gergonne cevian AD lies to the left of the cevian that bisects $\angle BAC$ and hence that $\angle ADC$ is obtuse.

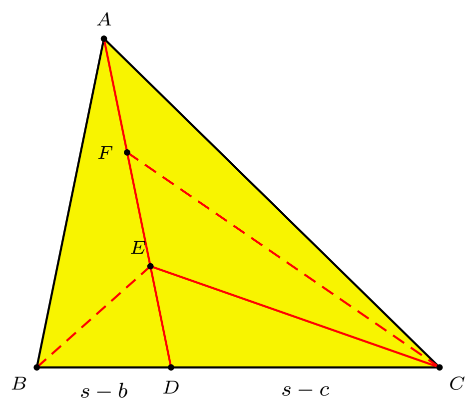

Figure 4.

By assumption, $\angle ABC > \angle ACB \Rightarrow \angle EBC > \angle FCD > \angle ECB$. Therefore,

$$CE > BE \quad \text{or} \quad CE > CF \tag{5}$$

because $BE = CF$. However, $\angle ADC = \angle EDC > \frac{\pi}{2}$ as mentioned above. Hence $\angle FEC = \angle EDC + \angle ECD > \frac{\pi}{2}$ and $\angle EFC < \frac{\pi}{2} \Rightarrow CE < CF$, contradicting (5).

Likewise, the assumption $AB > AC$ also leads to a contradiction. This means that triangle ABC must be isosceles. □

5. Conclusion

The reader is invited to consider other types of analogues or extensions of the Steiner - Lehmus theorem. To conclude the discussion, we pose two problems to the reader.

(1) The external angle bisectors of $\angle ABC$ and $\angle ACB$ meet the extension of the Gergonne cevian AD at the points E and F respectively. If $BE = CF$, prove or disprove that triangle ABC is isosceles.

(2) AD is an internal cevian of triangle ABC. The internal angle bisectors of $\angle ABC$ and $\angle ACB$ meet AD at E and F respectively. Determine a necesssary and sufficient condition so that $BE = CF$ implies that triangle ABC is isosceles.

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Forum Geometricorum
Volume 5 (2005) 197–198.

FORUM GEOM
ISSN 1534-1178

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