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On Mixtilinear Incircles and Excircles

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Abstract. A mixtilinear incircle (respectively excircle) of a triangle is tangent to two sides and to the circumcircle internally (respectively externally). We study the configuration of the three mixtilinear incircles (respectively excircles). In particular, we give easy constructions of the circle (apart from the circumcircle) tangent the three mixtilinear incircles (respectively excircles). We also obtain a number of interesting triangle centers on the line joining the circumcenter and the incenter of a triangle.

1. Preliminaries

In this paper we study two triads of circles associated with a triangle, the mixtilinear incircles and the mixtilinear excircles. For an introduction to these circles, see [4] and §§2, 3 below. In this section we collect some important basic results used in this paper.

Proposition 1 (d'Alembert's Theorem [1]). *Let $O_1(r_1)$, $O_2(r_2)$, $O_3(r_3)$ be three circles with distinct centers. According as $\varepsilon = +1$ or -1, denote by $A_{1\varepsilon}$, $A_{2\varepsilon}$, $A_{3\varepsilon}$ respectively the insimilicenters or exsimilicenters of the pairs of circles $((O_2),\ (O_3))$, $((O_3),\ (O_1))$, and $((O_1),\ (O_2))$. For $\varepsilon_i = \pm 1$, $i = 1, 2, 3$, the points $A_{1\varepsilon_1}$, $A_{2\varepsilon_2}$ and $A_{3\varepsilon_3}$ are collinear if and only if $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$.* See Figure 1.

The insimilicenter and exsimilicenter of two circles are respectively their internal and external centers of similitude. In terms of one-dimensional barycentric coordinates, these are the points

$$\mathrm{ins}(O_1(r_1),\ O_2(r_2)) = \frac{r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 + r_2}, \tag{1}$$

$$\mathrm{exs}(O_1(r_1),\ O_2(r_2)) = \frac{-r_2 \cdot O_1 + r_1 \cdot O_2}{r_1 - r_2}. \tag{2}$$

Proposition 2. *Let $O_1(r_1)$, $O_2(r_2)$, $O_3(r_3)$ be three circles with noncollinears centers. For $\varepsilon = \pm 1$, let $O_\varepsilon(r_\varepsilon)$ be the Apollonian circle tangent to the three circles, all externally or internally according as $\varepsilon = +1$ or -1. Then the Monge line containing the three exsimilicenters $\mathrm{exs}(O_2(r_2), O_3(r_3))$, $\mathrm{exs}(O_3(r_3), O_1(r_1))$, and $\mathrm{exs}(O_1(r_1), O_2(r_2))$ is the radical axis of the Apollonian circles (O_+) and (O_-).* See Figure 1.

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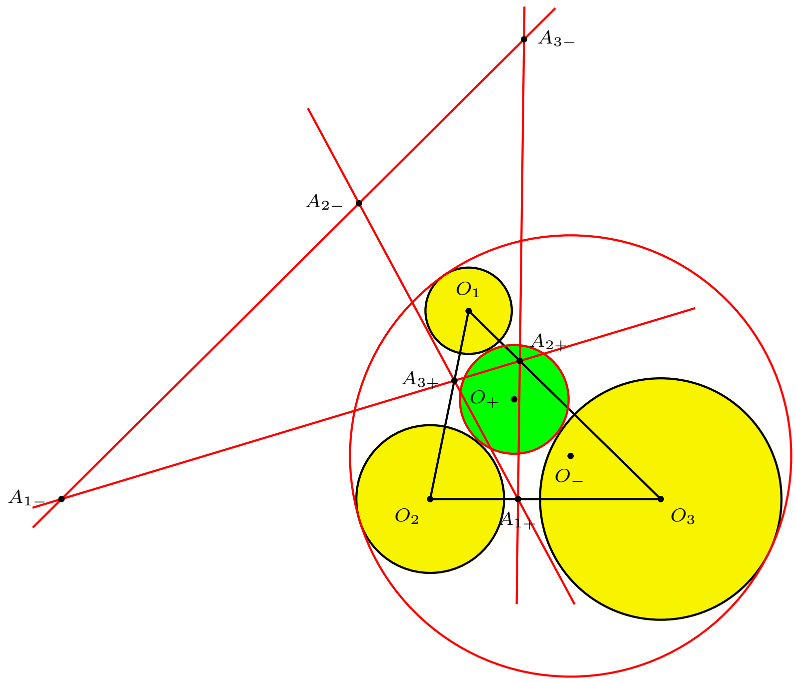

Figure 1.

Lemma 3. *Let BC be a chord of a circle $O(r)$. Let $O_1(r_1)$ be a circle that touches BC at E and intouches the circle (O) at D. The line DE passes through the midpoint A of the arc BC that does not contain the point D. Furthermore, $AD \cdot AE = AB^2 = AC^2$.*

Proposition 4. *The perspectrix of the circumcevian triangle of P is the polar of P with respect to the circumcircle.*

Let ABC be a triangle with circumcenter O and incenter I. For the circumcircle and the incircle,

$$\begin{aligned} \text{ins}((O),\ (I)) =& \frac{r \cdot O + R \cdot I}{R + r} = X_{55}, \\ \text{exs}((O),\ (I)) =& \frac{-r \cdot O + R \cdot I}{R - r} = X_{56}. \end{aligned}$$

in the notations of [3]. We also adopt the following notations.

A_0	point of tangency of incircle with BC
A_1	intersection of AI with the circumcircle
A_2	antipode of A_1 on the circumcircle

Similarly define B_0, B_1, B_2, C_0, C_1 and C_2. Note that
(i) $A_0B_0C_0$ is the intouch triangle of ABC,
(ii) $A_1B_1C_1$ is the circumcevian triangle of the incenter I,

(iii) $A_2B_2C_2$ is the medial triangle of the excentral triangle, *i.e.*, A_2 is the midpoint between the excenters I_b and I_c. It is also the midpoint of the arc BAC of the circumcircle.

2. Mixtilinear incircles

The A-mixtilinear incircle is the circle (O_a) that touches the rays AB and AC at C_a and B_a and the circumcircle (O) internally at X. See Figure 2. Define the B- and C-mixtilinear incircles (O_b) and (O_c) analogously, with points of tangency Y and Z with the circumcircle. See [4]. We begin with an alternative proof of the main result of [4].

Proposition 5. *The lines AX, BY, CZ are concurrent at* $\text{exs}((O),\ (I))$.

Proof. Since $A = \text{exs}((O_a),\ (I))$ and $X = \text{exs}((O),\ (O_a))$, the line AX passes through $\text{exs}((O),\ (I))$ by d'Alembert's Theorem. For the same reason, BY and CZ also pass through the same point. □

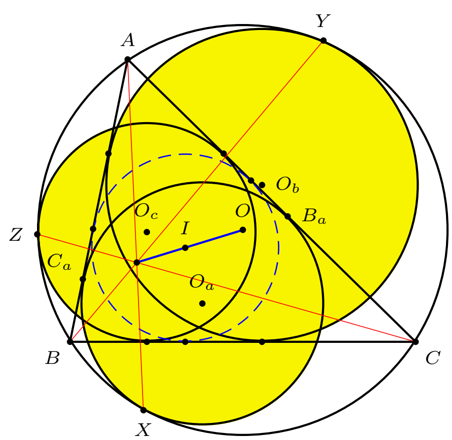

Figure 2

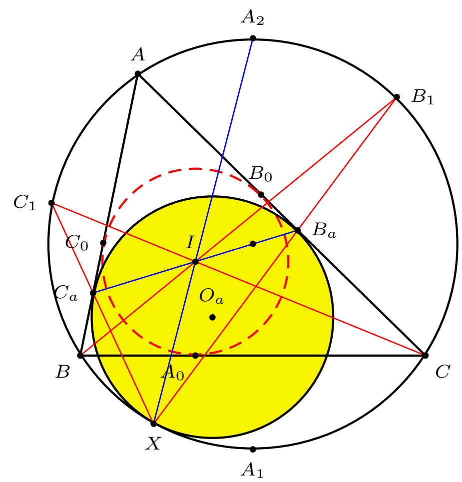

Figure 3

Lemma 6. (1) *I is the midpoint of B_aC_a.*
(2) *The A-mixtilinear incircle has radius* $r_a = \frac{r}{\cos^2 \frac{A}{2}}$.
(3) *XI bisects angle BXC.*
See Figure 3.

Consider the radical axis ℓ_a of the mixtilinear incircles (O_b) and (O_c).

Proposition 7. *The radical axis ℓ_a contains*
(1) *the midpoint A_1 of the arc BC of (O) not containing the vertex A,*
(2) *the midpoint M_a of IA_0, where A_0 is the point of tangency of the incircle with the side BC.*

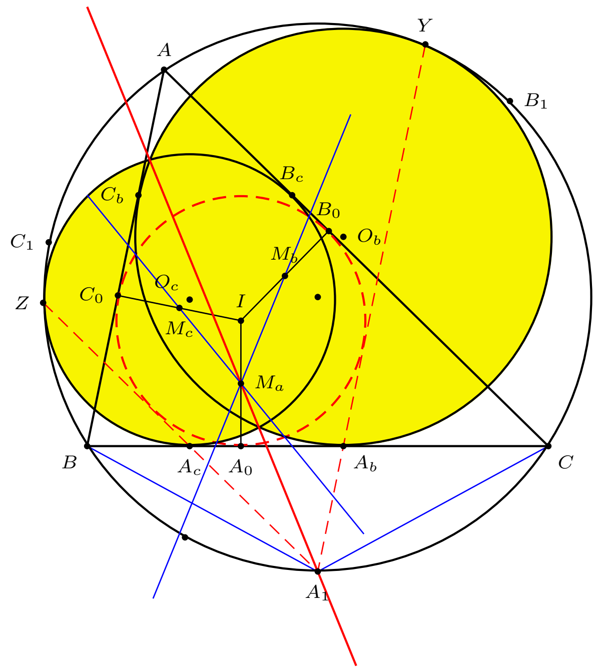

Figure 4.

Proof. (1) By Lemma 3, Z, A_c and A_1 are collinear, so are Y, A_b, A_1. Also, $A_1A_c \cdot A_1Z = A_1B^2 = A_1C^2 = A_1A_b \cdot A_1Y$. This shows that A_1 is on the radical axis of (O_b) and (O_c).

(2) Consider the incircle (I) and the B-mixtilinear incircle (O_b) with common ex-tangents BA and BC. Since the circle (I) touches BA and BC at C_0 and A_0, and the circle (O_b) touches the same two lines at C_b and A_b, the radical axis of these two circles is the line joining the midpoints of C_bC_0 and A_bA_0. Since A_b, I, C_b are collinear, the radical axis of (I) and (O_b) passes through the midpoints of IA_0 and IC_0. Similarly, the radical axis of (I) and (O_c) passes through the midpoints of IA_0 and IB_0. It follows that the midpoint of IA_0 is the common point of these two radical axes, and is therefore a point on the radical axis of (O_b) and (O_c). □

Theorem 8. *The radical center of (O_a), (O_b), (O_c) is the point J which divides OI in the ratio*

$$OJ : JI = 2R : -r.$$

Proof. By Proposition 7, the radical axis of (O_b) and (O_c) is the line A_1M_a. Let M_b and M_c be the midpoints of IB_0 and IC_0 respectively. Then the radical axes of (O_c) and (O_a) is the line B_1M_b, and that of (O_a) and (O_b) is the line C_1M_c. Note that the triangles $A_1B_1C_1$ and $M_aM_bM_c$ are directly homothetic. Since $A_1B_1C_1$ is inscribed in the circle $O(R)$ and $M_aM_bM_c$ in inscribed in the circle $I(\frac{r}{2})$, the homothetic center of the triangles is the point J which divides the segment OI in

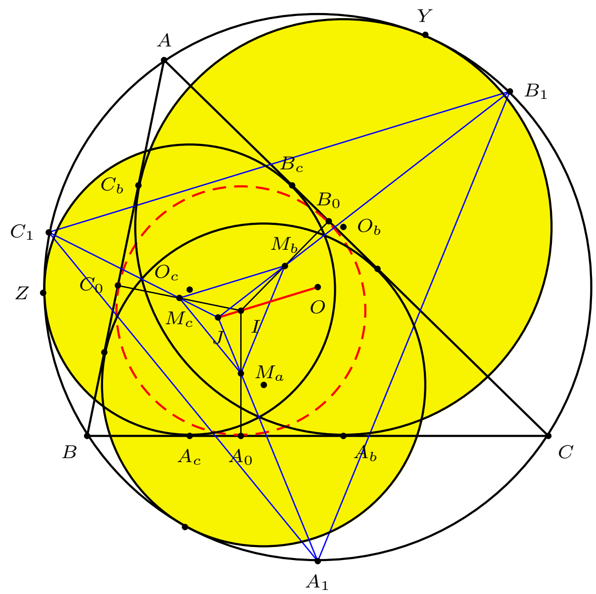

Figure 5.

the ratio

$$OJ : JI = R : -\frac{r}{2} = 2R : -r. \tag{3}$$

See Figure 5. $\square$

Remark. Let T be the homothetic center of the excentral triangle $I_aI_bI_c$ and the intouch triangle $A_0B_0C_0$. This is the triangle center X_{57} in [3]. Since the excentral triangle has circumcenter I', the reflection of I in O,

$$OT : TI' = 2R : -r.$$

Comparison with (3) shows that J is the reflection of T in O.

3. The mixtilinear excircles

The mixtilinear excircles are defined analogously to the mixtilinear incircles, except that the tangencies with the circumcircle are external. The A-mixtilinear excircle (O'_a) can be easily constructed by noting that the polar of A passes through the excenter I_a; similarly for the other two mixtilinear excircles. See Figure 6.

Theorem 9. *If the mixtilinear excircles touch the circumcircle at X', Y', Z' respectively, the lines AX', BY', CZ' are concurrent at* $\mathrm{ins}((O), (I))$.

Theorem 10. *The radical center of the mixtilinear excircles is the reflection of J in O, where J is the radical center of the mixtilinear incircles.*

Proof. The polar of A with respect to (O'_a) passes through the excenter I_a. Similarly for the other two polars of B with respect to (O'_b) and C with respect to (O'_c).

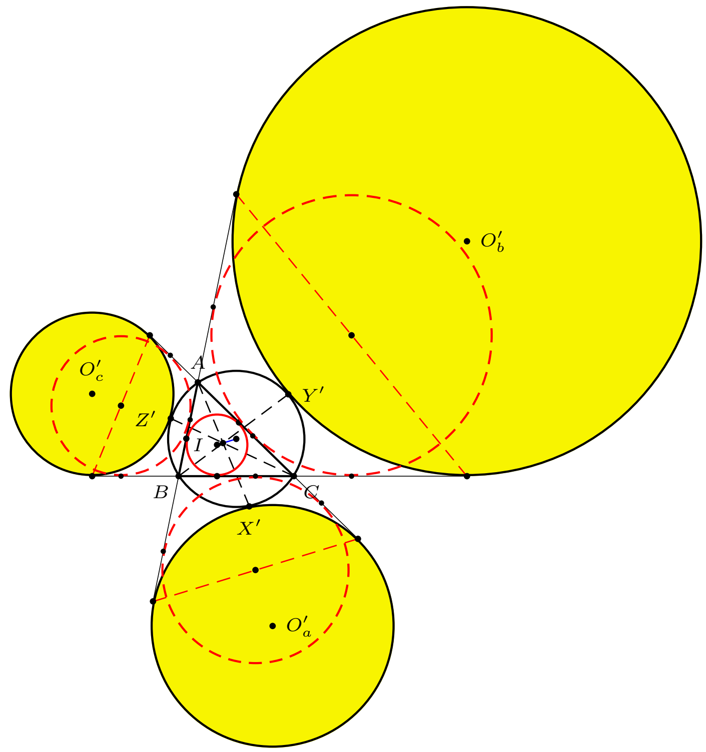

Figure 6.

Let $A_3B_3C_3$ be the triangle bounded by these three polars. Let A_4, B_4, C_4 be the midpoints of A_0A_3, B_0B_3, C_0C_3 respectively. See Figure 7.

Since $I_aI_bA_3I_c$ is a parallelogram, and A_2 is the midpoint of I_bI_c, it is also the midpoint of A_3I_a. Since B_3C_3 is parallel to I_bI_c (both being perpendicular to the bisector AA_1), I_a is the midpoint of B_3C_3. Similarly, I_b, I_c are the midpoints of C_3A_3 and A_3B_3, and the excentral triangle is the medial triangle of $A_3B_3C_3$. Note also that I is the circumcenter of $A_3B_3C_3$ (since it lies on the perpendicular bisectors of its three sides). This is homothetic to the intouch triangle $A_0B_0C_0$ at I, with ratio of homothety $-\frac{r}{4R}$.

If A_4 is the midpoint of A_0A_3, similarly for B_4 and C_4, then $A_4B_4C_4$ is homothetic to $A_3B_3C_3$ with ratio $\frac{4R-r}{4R}$.

We claim that $A_4B_4C_4$ is homothetic to $A_2B_2C_2$ at a point J', which is the radical center of the mixtilinear excircles. The ratio of homothety is clearly $\frac{4R-r}{R}$.

Consider the isosceles trapezoid $B_0C_0B_3C_3$. Since B_4 and C_4 are the midpoints of the diagonals B_0B_3 and C_0C_3, and B_3C_3 contains the points of tangency B_a, C_a of the circle (O'_a) with AC and AB, the line B_4C_4 also contains the midpoints of B_0B_a and C_0C_a, which are on the radical axis of (I) and (O'_a). This means that the line B_4C_4 is the radical axis of (I) and (O'_a).

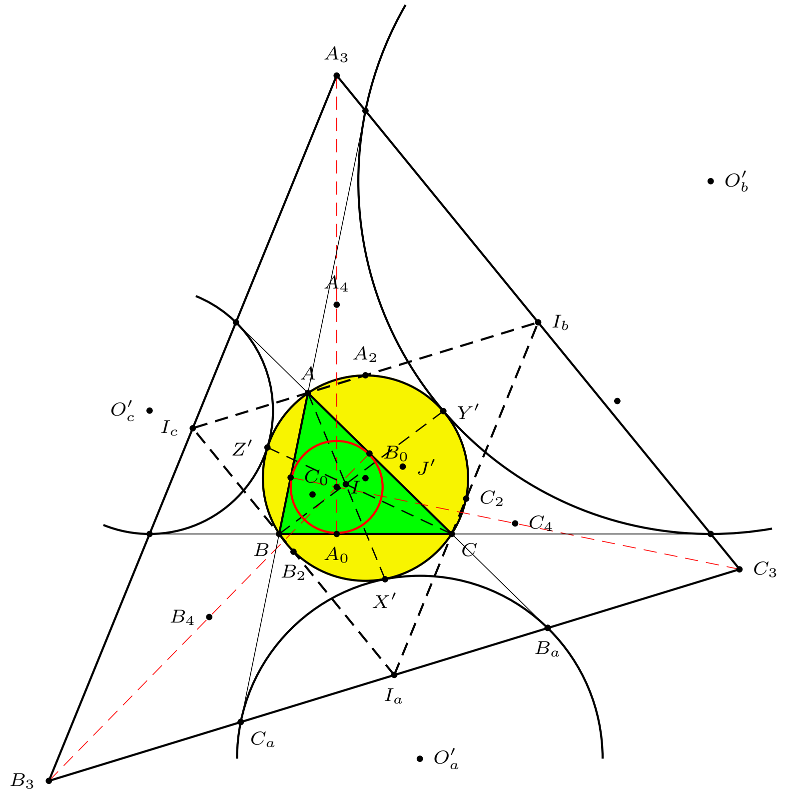

Figure 7.

It follows that A_4 is on the radical axis of (O'_b) and (O'_c). Clearly, A_2 also lies on the same radical axis. This means that the radical center of the mixtilinear excircles is the homothetic center of the triangles $A_2B_2C_2$ and $A_4B_4C_4$. Since these two triangles have circumcenters O and I, and circumradii R and $\frac{4R-r}{2}$, the homothetic center is the point J' which divides IO in the ratio

$$J'I : J'O = 4R - r : 2R. \tag{4}$$

Equivalently, $OJ' : J'I = -2R : 4R - r$. The reflection of J' in O divides OI in the ratio $2R : -r$. This is the radical center J of the mixtilinear incircles. $\square$

4. Apollonian circles

Consider the circle $O_5(r_5)$ tangent internally to the mixtilinear incircles at A_5, B_5, C_5 respectively. We call this the inner Apollonian circle of the mixtilinear incircles. It can be constructed from J since A_5 is the second intersection of the line JX with the A-mixtilinear incircle, and similarly for B_5 and C_5. See Figure 8. Theorem 11 below gives further details of this circle, and an easier construction.

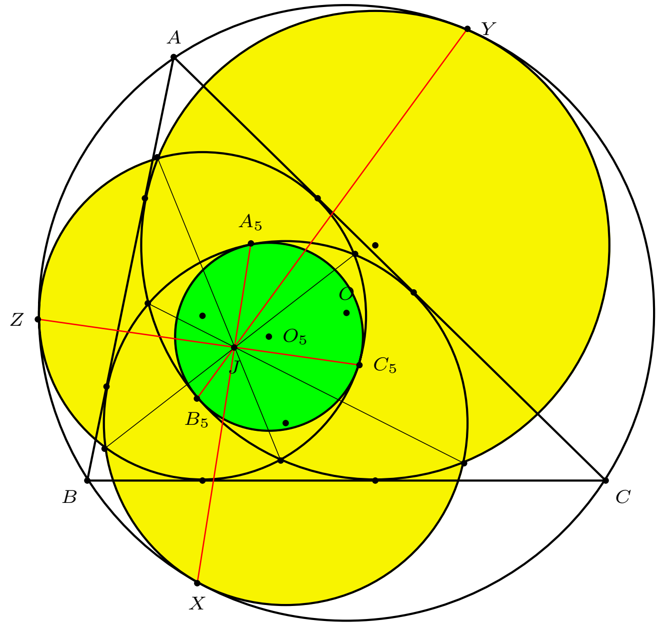

Figure 8.

Theorem 11. (1) *Triangles $A_5B_5C_5$ and ABC are perspective at* $\mathrm{ins}((O),\ (I))$.

(2) *The inner Apollonian circle of the mixtilinear incircles has center O_5 dividing the segment OI in the ratio $4R : r$ and radius $r_5 = \frac{3Rr}{4R+r}$.*

Proof. (1) Let $P = \mathrm{exs}((O_5), (I))$, and $Q_a = \mathrm{exs}((O_5), (O'_a))$. The following triples of points are collinear by d'Alembert's Theorem:
(1) A, A_5, P from the circles (O_5), (I), (O_a);
(2) A, Q_a, A_5 from the circles (O_5), (O_a), (O'_a);
(3) A, Q_a, P from the circles (O_5), (I), (O'_a);
(4) A, X', $\mathrm{ins}((O), (I))$ from the circles (O), (I), (O'_a).

See Figure 9. Therefore the lines AA_5 contains the points P and $\mathrm{ins}((O), (I))$ (along with Q_a, X'). For the same reason, the lines BB_5 and CC_5 contain the same two points. It follows that P and $\mathrm{ins}((O), (I))$ are the same point, which is common to AA_5, BB_5 and CC_5.

(2) Now we compute the radius r_5 of the circle (O_5). From Theorem 8, $OJ : JI = 2R : -r$. As $J = \mathrm{exs}((O), (O_5))$, we have $OJ : JO_5 = R : -r_5$. It follows that $OJ : JI : JO_5 = 2R : -r : -2r_5$, and

$$\frac{OO_5}{OI} = \frac{2(R - r_5)}{2R - r}. \tag{5}$$

Since $P = \mathrm{exs}((O_5), (I)) = \mathrm{ins}((O), (I))$, it is also $\mathrm{ins}((O), (O_5))$. Thus, $OP : PO_5 : PI = R : r_5 : r$, and

$$\frac{OO_5}{OI} = \frac{R + r_5}{R + r}. \tag{6}$$

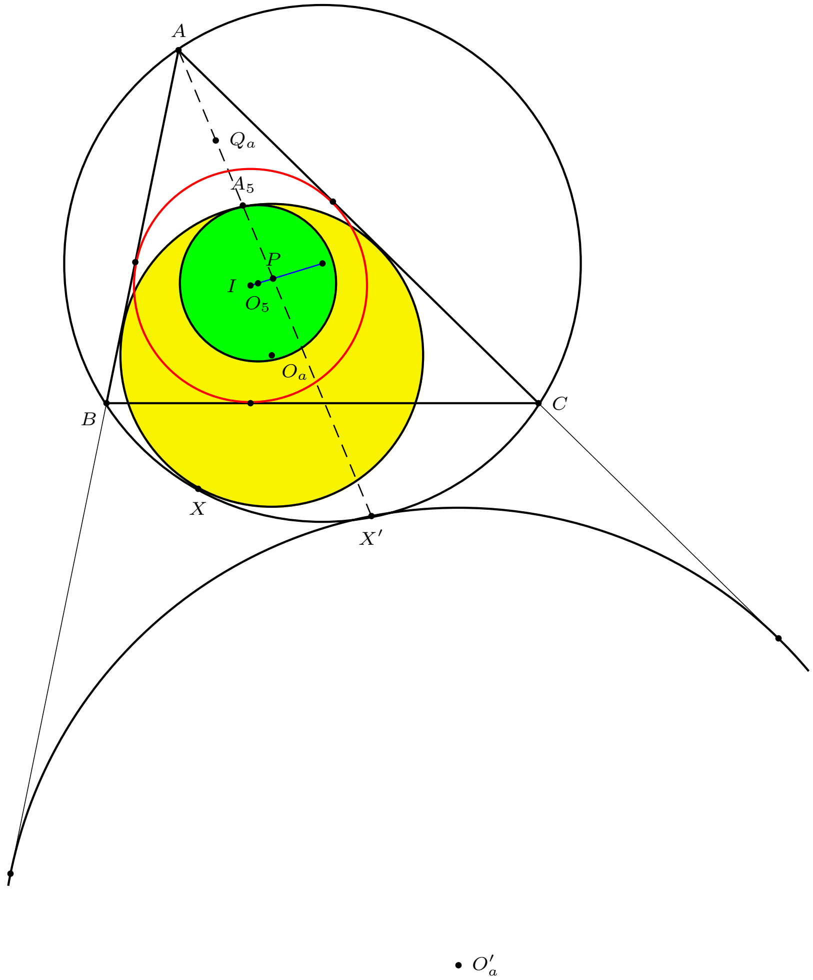

Figure 9.

Comparing (5) and (6), we easily obtain $r_5 = \frac{3Rr}{4R+r}$. Consequently, $\frac{OO_5}{OI} = \frac{4R}{4R+r}$ and O_5 divides OI in the ratio $OO_5 : O_5I = 4R : r$. $\square$

The outer Apollonian circle of the mixtilinear excircles can also be constructed easily. If the lines $J'X'$, $J'Y'$, $J'Z'$ intersect the mixtilinear excircles again at A_6, B_6, C_6 respectively, then the circle $A_6B_6C_6$ is tangent internally to each of the mixtilinear excircles. Theorem 12 below gives an easier construction without locating the radical center.

Theorem 12. (1) *Triangles $A_6B_6C_6$ and ABC are perspective at* $\mathrm{exs}((O),(I))$.

(2) *The outer Apollonian circle of the mixtilinear excircles has center O_6 dividing the segment OI in the ratio $-4R : 4R + r$ and radius $r_6 = \frac{R(4R-3r)}{r}$.*

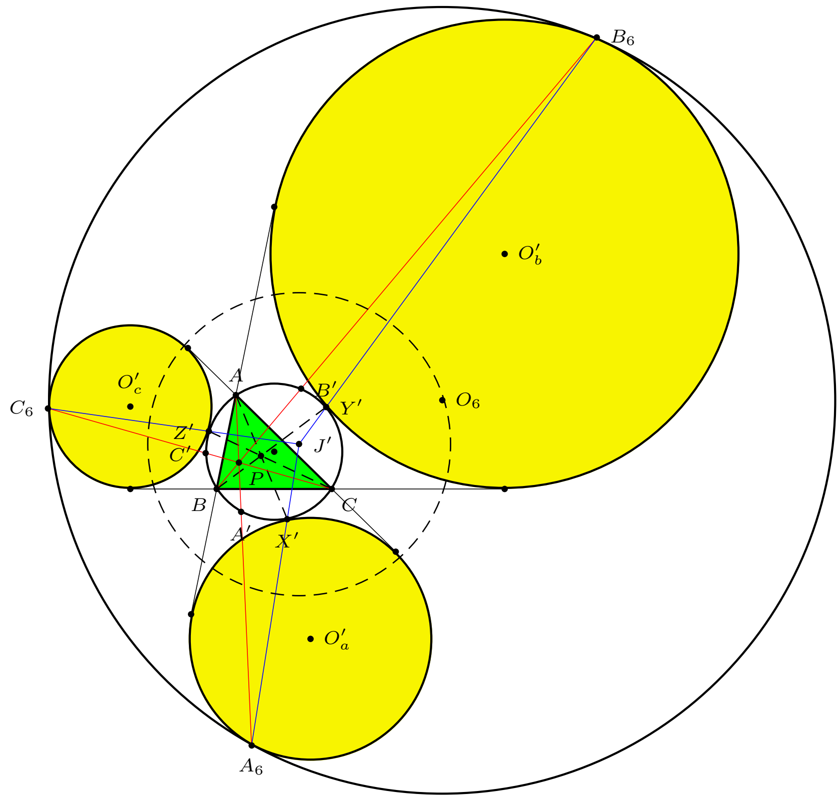

Figure 10.

Proof. (1) Since $A_6 = \operatorname{exs}((O'_a), (O_6))$ and $A = \operatorname{exs}((I), (O'_a))$, by d'Alembert's Theorem, the line AA_6 passes through $P = \operatorname{exs}((O_6), (I))$. For the same reason BB_6 and CC_6 pass through the same point, the triangles $A_6B_6C_6$ and ABC are perspective at $P = \operatorname{exs}((I), (O_6))$. See Figure 10.

By Proposition 2, AB, $X'Y'$, A_6B_6, and $O'_aO'_b$ concur at $S = \operatorname{exs}((O'_a), (O'_b))$ on the radical axis of (O) and (O_6). Now: $SA \cdot SB = SX' \cdot SY' = SA_6 \cdot SB_6$. Let AA_6, BB_6, CC_6 intersect the circumcircle (O) at A', B', C' respectively. Since $SA \cdot SB = SA_6 \cdot SB_6$, ABA_6B_6 is cyclic. Since $\angle BAA_6 = \angle BB'A' = \angle BB_6A_6$, $A'B'$ is parallel to A_6B_6. Similarly, $B'C'$ and $C'A'$ are parallel to B_6C_6 and C_6A_6 respectively. Therefore the triangles $A'B'C'$ and $A_6B_6C_6$ are directly homothetic, and the center of homothety is $P = \operatorname{exs}((O), (O_6))$.

Since $P = \operatorname{exs}((O), (O_6)) = \operatorname{exs}((I), (O_6))$, it is also $\operatorname{exs}((O), (I))$, and $PI : PO = r : R$.

(2) Since $A_6 = \operatorname{exs}((O_6), (O'_a))$ and $X' = \operatorname{ins}((O'_a), (O))$, by d'Alembert's Theorem, the line A_6X' passes through $K = \operatorname{ins}((O), (O_6))$. For the same reason, B_6Y' and C_6Z' pass through the same point K.

We claim that K is the radical center J' of the mixtilinear excircles. Since $SX' \cdot SY' = SA_6 \cdot SB_6$, we conclude that $X'A_6Y'B_6$ is cyclic, and $KX' \cdot KA_6 = KY' \cdot KB_6$. Also, $Y'B_6C_6Z'$ is cyclic, and $KY' \cdot KB_6 = KZ' \cdot KC_6$. It follows that

$$KX' \cdot KA_6 = KY' \cdot KB_6 = KZ' \cdot KC_6,$$

showing that $K = \text{ins}((O),(O_6))$ is the radical center J' of the mixtilinear excircles. Hence, $J'O : J'O_6 = R : -r_6$. Note also $PO : PO_6 = R : r_6$. Then, we have the following relations.

$$\begin{aligned} OJ' : IO =& 2R : 2R - r, \\ J'O_6 : IO =& 2r_6 : 2R - r, \\ OO_6 : IO =& r_6 - R : R - r. \end{aligned}$$

Since $OJ' + J'O_6 = OO_6$, we have

$$\frac{2R}{2R-r} + \frac{2r_6}{2R-r} = \frac{r_6 - R}{R-r}.$$

This gives: $r_6 = \frac{R(4R-3r)}{r}$. Since $K = \text{ins}((O),(O_6)) = \frac{r_6 \cdot O + R \cdot O_6}{R+r_6}$ and $J' = \frac{(4R-r)O-2R\cdot I}{2R-r}$ are the same point, we obtain $O_6 = \frac{(4R+r)O-4R\cdot I}{r}$. □

Remark. The radical circle of the mixtilinear excircles has center J' and radius $\frac{R}{2R-r}\sqrt{(4R+r)(4R-3r)}$.

Corollary 13. $IO_5 \cdot IO_6 = IO^2$.

5. The cyclocevian conjugate

Let P be a point in the plane of triangle ABC, with traces X, Y, Z on the sidelines BC, CA, AB respectively. Construct the circle through X, Y, Z. This circle intersects the sidelines BC, CA, AB again at points X', Y', Z'. A simple application of Ceva's Theorem shows that the AX', BY', CZ' are concurrent. The intersection point of these three lines is called the cyclocevian conjugate of P. See, for example, [2, p.226]. We denote this point by P°. Clearly, $(P^\circ)^\circ = P$. For example, the centroid and the orthocenters are cyclocevian conjugates, and Gergonne point is the cyclocevian conjugate of itself.

We prove two interesting locus theorems.

Theorem 14. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to $X'Y'Z'$ is the line PP°. For Q on PP°, the perspector is also on the same line.*

Proof. Let Q be a point on the line PP°. By Pascal's Theorem for the six points, X', B', X, Y', A', Y, the intersections of the lines $X'A'$ and $Y'B'$ lies on the line connecting Q to the intersection of XY' and $X'Y$, which according to Pappus' theorem (for Y, Z, Z' and C, Y, Y'), lies on PP°. Since Q lies on PP°, it follows that $X'A'$, $Y'B'$, and PP° are concurrent. Similarly, $A'B'C'$ and $X'Y'Z'$ are perspective at a point on PP°. The same reasoning shows that if $A'B'C'$ and $X'Y'Z'$ are perspective at a point S, then both Q and S lie on the line connecting the intersections $XY' \cap X'Y$ and $YZ' \cap Y'Z$, which is the line PP°. □

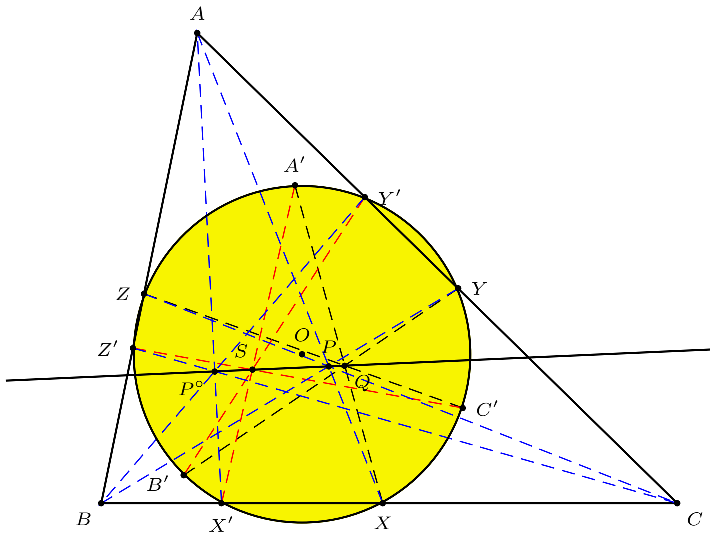

Figure 11.

For example, if $P = G$, then $P^\circ = H$. The line PP° is the Euler line. If $Q = O$, the circumcenter, then the circumcevian triangle of O (with respect to the medial triangle) is perspective with the orthic triangle at the nine-point center N.

Theorem 15. *The locus of Q whose circumcevian triangle with respect to XYZ is perspective to ABC is the line PP°.*

Proof. First note that

$$\begin{aligned}\frac{\sin Z'X'A'}{\sin A'X'Y'} &= \frac{\sin Z'XA'}{\sin A'YY'} = \frac{\sin Z'ZA'}{\sin ZAA'} \cdot \frac{\sin A'AY}{\sin A'YY'} \cdot \frac{\sin ZAA'}{\sin A'AY} \\ &= \frac{AA'}{ZA'} \cdot \frac{A'Y}{AA'} \cdot \frac{\sin BAX}{\sin XAC} = \frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin BAA'}{\sin A'AC}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{\sin Z'X'A'}{\sin A'X'Y'} \cdot \frac{\sin X'Y'B'}{\sin B'Y'Z'} \cdot \frac{\sin Y'Z'C'}{\sin C'Z'X'} \\ =&\left(\frac{\sin YXA'}{\sin A'XZ} \cdot \frac{\sin ZYB'}{\sin B'YX} \cdot \frac{\sin XZC'}{\sin C'ZY}\right)\left(\frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}\right) \\ =&\frac{\sin BAA'}{\sin A'AC} \cdot \frac{\sin ACC'}{\sin C'CB} \cdot \frac{\sin CBB'}{\sin B'BA}.\end{aligned}$$

Therefore, $A'B'C'$ is perspective with ABC if and only if it is perspective with $X'Y'Z'$. By Theorem 14, the locus of Q is the line PP°. □

6. Some further results

We establish some further results on the mixtilinear incircles and excircles relating to points on the line OI.

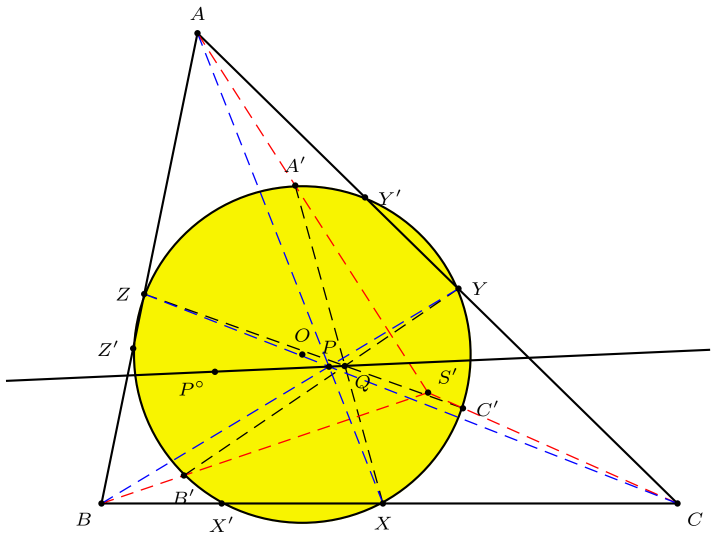

Figure 12.

Theorem 16. *The line OI is the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ.*

Proof. We first show that $D = B_1Y \cap C_1Z$ lies on the line AA_1. Applying Pascal's Theorem to the six points Z, B_1, B_2, Y, C_1, C_2 on the circumcircle, the points $D = B_1Y \cap C_1Z$, $I = B_2Y \cap C_2Z$, and $B_1C_2 \cap B_2C_1$ are collinear. Since B_1C_2 and B_2C_1 are parallel to the bisector AA_1, it follows that D lies on AA_1. See Figure 13.

Now, if $E = C_1Z \cap A_1X$ and $F = A_1X \cap B_1Y$, the triangle DEF is perspective with $A_1B_1C_1$ at I. Equivalently, $A_1B_1C_1$ is the circumcevian triangle of I with respect to triangle DEF. Triangle XYZ is formed by the second intersections of the circumcircle of $A_1B_1C_1$ with the side lines of DEF. By Theorem 14, the locus of P whose circumcevian triangle with respect to $A_1B_1C_1$ is perspective with XYZ is a line through I. This is indeed the line IO, since O is one such point. (The circumcevian triangle of O with respect to $A_1B_1C_1$ is perspective with XYZ at I). □

Remark. If P divides OI in the ratio $OP : PI = t : 1-t$, then the perspector Q divides the same segment in the ratio $OQ : QI = (1+t)R : -2tr$. In particular, if $P = \text{ins}((O),\ (I))$, this perspector is T, the homothetic center of the excentral and intouch triangles.

Corollary 17. *The line OI is the locus of Q whose circumcevian triangle with respect to $A_1B_1C_1$ (or XYZ) is perspective with DEF.*

Proposition 18. *The triangle $A_2B_2C_2$ is perspective*
(1) *with XYZ at the incenter I,*
(2) *with $X'Y'Z'$ at the centroid of the excentral triangle.*

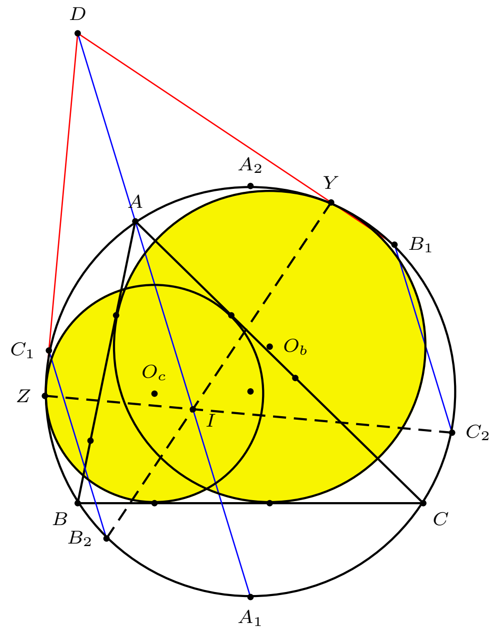

Figure 13.

Proof. (1) follows from Lemma 6(b).

(2) Referring to Figure 7, the excenter I_a is the midpoint B_aC_a. Therefore $X'I_a$ is a median of triangle $X'B_aC_a$, and it intersects B_2C_2 at its midpoint X''. Since $A_2B_2I_aC_2$ is a parallelogram, A_2, X', X'' and I_a are collinear. In other words, the line A_2X' contains a median, hence the centroid, of the excentral triangle. So do B_2Y' and C_2Z. □

Let A_7 be the second intersection of the circumcircle with the line ℓ_a, the radical axis of the mixtilinear incircles (O_b) and (O_c). Similarly define B_7 and C_7. See Figure 14.

Theorem 19. *The triangles $A_7B_7C_7$ and XYZ are perspective at a point on the line OI.*

Remark. This point divides OI in the ratio $4R - r : -4r$ and has homogeneous barycentric coordinates

$$\left(\frac{a(b+c-5a)}{b+c-a} : \frac{b(c+a-5b)}{c+a-b} : \frac{c(a+b-5c)}{a+b-c}\right).$$

7. Summary

We summarize the triangle centers on the OI-line associated with mixtilinear incircles and excircles by listing, for various values of t, the points which divide OI in the ratio $R : tr$. The last column gives the indexing of the triangle centers in [2, 3].

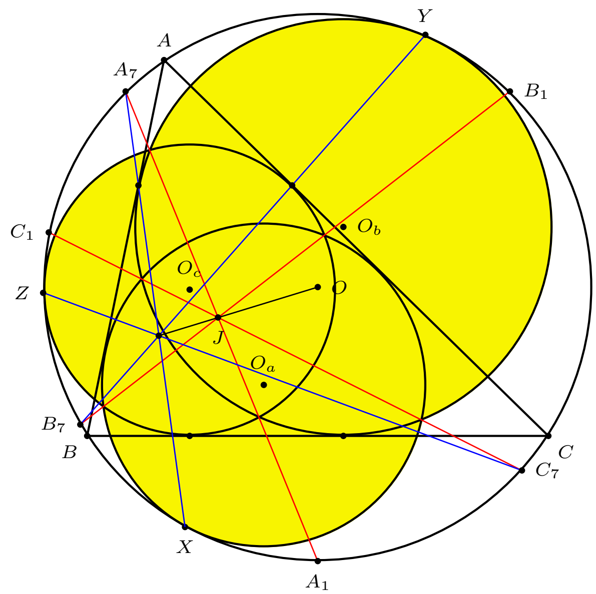

Figure 14.

t	first barycentric coordinate		X_n
1	$a^2(s-a)$	$\text{ins}((O),\ (I))$ perspector of ABC and $X'Y'Z'$ perspector of ABC and $A_5B_5C_5$	X_{55}
-1	$\frac{a^2}{s-a}$	$\text{exs}((O),\ (I))$ perspector of ABC and XYZ perspector of ABC and $A_6B_6C_6$	X_{56}
$-\frac{2R}{2R+r}$	$\frac{a}{s-a}$	homothetic center of excentral and intouch triangles	X_{57}
$-\frac{1}{2}$	$a^2(b^2+c^2-a^2-4bc)$	radical center of mixtilinear incircles	X_{999}
$\frac{1}{4}$	$a^2(b^2+c^2-a^2+8bc)$	center of Apollonian circle of mixtilinear incircles	
$-\frac{4R-r}{2r}$	$a^2f(a,b,c)$	radical center of mixtilinear excircles	
$-\frac{4R+r}{4r}$	$a^2g(a,b,c)$	center of Apollonian circle of mixtilinear excircles	
$-\frac{4R}{r}$	$a(3a^2-2a(b+c)-(b-c)^2)$	centroid of excentral triangle	X_{165}
$-\frac{4R}{4R-r}$	$\frac{a(b+c-5a)}{b+c-a}$	perspector of $A_7B_7C_7$ and XYZ	

The functions f and g are given by

$$\begin{aligned}f(a,b,c) =&a^4-2a^3(b+c)+10a^2bc+2a(b+c)(b^2-4bc+c^2)\\&-(b-c)^2(b^2+4bc+c^2),\\g(a,b,c) =&a^5-a^4(b+c)-2a^3(b^2-bc+c^2)+2a^2(b+c)(b^2-5bc+c^2)\\&+a(b^4-2b^3c+18b^2c^2-2bc^3+c^4)-(b-c)^2(b+c)(b^2-8bc+c^2).\end{aligned}$$

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A Conic Associated with Euler Lines

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Abstract. We study the locus of a point C for which the Euler line of triangle ABC with given A and B has a given slope m. This is a conic through A and B, and with center (if it exists) at the midpoint of AB. The main properties of such an Euler conic are described. We also give a construction of a point C for which triangle ABC, with A and B fixed, has a prescribed Euler line.

1. The Euler conic

Given two points A and B and a real number m, we study the locus of a point C for which the Euler line of triangle ABC has slope m. We show that this locus is a conic through A and B. Without loss of generality, we assume a Cartesian coordinate system in which

$$A = (-1, 0) \quad \text{and} \quad B = (1, 0),$$

and write $C = (x, y)$. The centroid G and the orthocenter H of a triangle can be determined from the coordinates of its vertices. They are the points

$$G = \left(\frac{x}{3}, \frac{y}{3}\right) \quad \text{and} \quad H = \left(x, \frac{-x^2+1}{y}\right). \tag{1}$$

See, for example, [2]. The vector

$$\overrightarrow{GH} = \left(\frac{2x}{3}, \frac{-3x^2-y^2+3}{3y}\right), \tag{2}$$

is parallel to the Euler line. When the Euler line is not vertical, its slope is given by:

$$m = \frac{-3x^2-y^2+3}{2xy}, \quad x, y \neq 0.$$

Therefore, the coordinates of the vertex C satisfy the equation

$$3x^2 + 2mxy + y^2 = 3. \tag{3}$$

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This clearly represents a conic. We call this the Euler conic associated with A, B and slope m. It clearly has center at the origin, the midpoint M of AB. Its axes are the eigenvectors of the matrix

$$\begin{pmatrix} 3 & m \\ m & 1 \end{pmatrix}.$$

The characteristic polynomial being $\lambda^2 - 4\lambda - (m^2 - 3)$, its eigenvectors with corresponding eigenvalues are as follows.

eigenvector	eigenvalue
$(\sqrt{m^2+1}+1,\ m)$	$2+\sqrt{m^2+1}$
$(\sqrt{m^2+1}-1,\ -m)$	$2-\sqrt{m^2+1}$

Thus, equation (3) can be rewritten in the form
(1) $(mx+y)^2 = 3$, if $m = \pm\sqrt{3}$, or
(2) $(2+\sqrt{m^2+1})(x\cos\alpha - y\sin\alpha)^2 + (2-\sqrt{m^2+1})(x\sin\alpha + y\cos\alpha)^2 = 3$, if $m \neq \pm\sqrt{3}$, where

$$\cos\alpha = \sqrt{\frac{\sqrt{m^2+1}+1}{2\sqrt{m^2+1}}}, \qquad \sin\alpha = \sqrt{\frac{\sqrt{m^2+1}-1}{2\sqrt{m^2+1}}}.$$

Remarks. (1) The pairs $(\pm1, 0)$ are always solutions of (3) and correspond to the singular cases in which the vertex C coincides, respectively, with A or B, and consequently, it is not possible to define the triangle ABC.

(2) The pairs $\left(0, \pm\sqrt{3}\right)$ are also solutions of (3) and correspond to the trivial case when the triangle ABC is equilateral. In this case, the centroid, the orthocenter, and the circumcenter coincide.

2. Classification of the Euler conic

The Euler conic is an ellipse or a hyperbola according as $m^2 < 3$ or $m^2 > 3$. It degenerates into a pair of straight lines when $m^2 = 3$.

Proposition 1. *Suppose $m^2 < 3$. The Euler conic is an ellipse with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}+2}}.$$

The foci are the points

$$\pm\left(-\mathrm{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}-1)}{3-m^2}},\ \sqrt{\frac{3(\sqrt{m^2+1}+1)}{3-m^2}}\right),$$

where $\mathrm{sgn}(m) = +1$, 0, *or* -1 *according as* $m >, =,$ *or* < 0.

Figure 1 shows the Euler ellipse for $m = \frac{3}{4}$, a triangle ABC with C on the ellipse, and its Euler line of slope m.

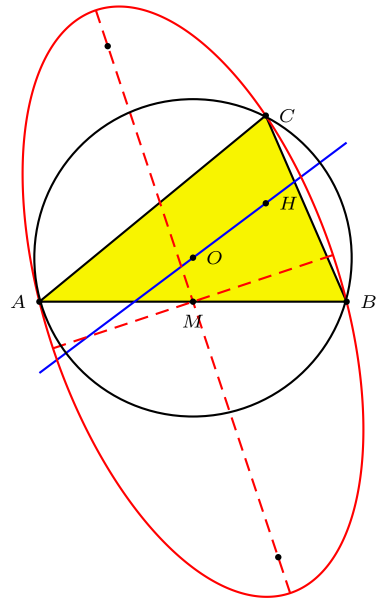

Figure 1

Proposition 2. *Suppose* $m^2 > 3$*. The Euler conic is a hyperbola with eccentricity*

$$\varepsilon = \sqrt{\frac{2\sqrt{m^2+1}}{\sqrt{m^2+1}-2}}.$$

The foci are the points

$$\pm\left(\mathrm{sgn}(m)\cdot\sqrt{\frac{3(\sqrt{m^2+1}+1)}{m^2-3}}, \sqrt{\frac{3(\sqrt{m^2+1}-1)}{m^2-3}}\right),$$

where $\mathrm{sgn}(m) = +1$ *or* -1 *according as* $m > 0$ *or* < 0*. The asymptotes are the lines*

$$y = (-m \pm \sqrt{m^2-3})x.$$

Figure 2 shows the Euler hyperbola for $m = \frac{12}{5}$, a triangle ABC with C on the hyperbola, and its Euler line of slope m.

When $|m| = \sqrt{3}$, the Euler conic degenerates into a pair of parallel lines, whose equations are:

$$y = -mx \pm \sqrt{3}.$$

Examples of triangles for $m = \sqrt{3}$ are shown in Figures 3A and 3B, and for $m = -\sqrt{3}$ in Figures 4A and 4B.

Corollary 3. *The slope of the Euler line of the triangle* ABC *is*

$$m = \pm\sqrt{3},$$

if and only if, one of angles A *and* B *is* $60°$ *or* $120°$*.*

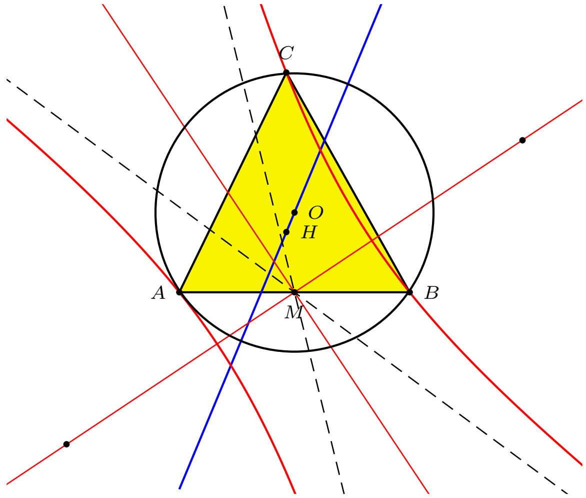

Figure 2

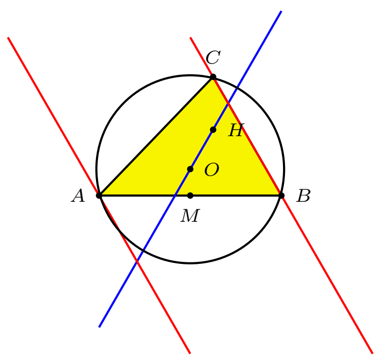

Figure 3A

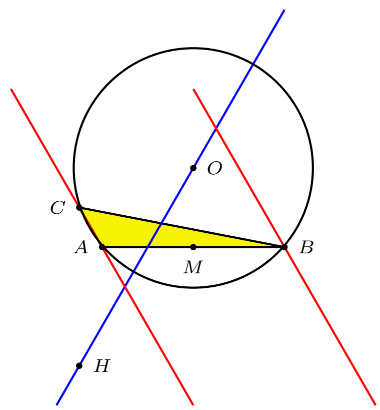

Figure 3B

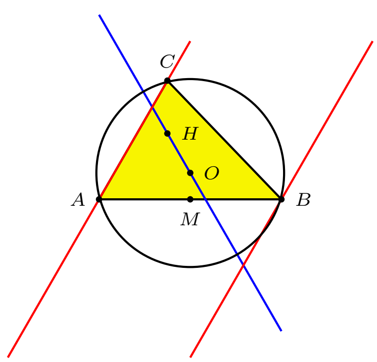

Figure 4A

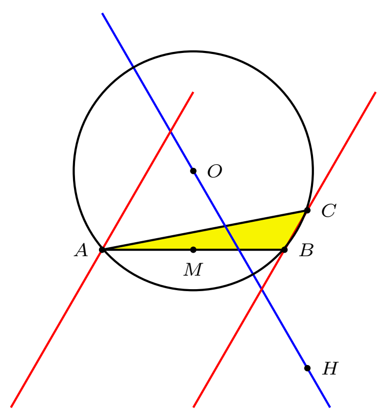

Figure 4B

3. Triangles with given Euler line

In this section we find the cartesian coordinates of the third vertex C in order that a given line be the Euler line of the triangle ABC with vertices $A = (-1, 0)$ and $B = (1, 0)$.

Lemma 4. *The Euler line of triangle ABC is perpendicular to AB if and only if $AB = AC$. In this case, the Euler line is the perpendicular bisector of AB.*

We shall henceforth assume that the Euler line is not perpendicular to AB. It therefore has an equation of the form

$$y = mx + k.$$

The circumcenter is the intersection of the Euler line with the line $x = 0$, the perpendicular bisector of AB. It is the point $O = (0, k)$. The circumcircle is

$$x^2 + (y - k)^2 = k^2 + 1$$

or

$$x^2 + y^2 - 2ky - 1 = 0. \tag{4}$$

Let M be the midpoint of AB; it is the origin of the Cartesian system. If G is the centroid, the vertex C is such that $MC : MG = 3 : 1$. Since G lies on the line $y = mx + k$, C lies on the line $y = mx + 3k$. It can therefore be constructed as the intersection of this line with the circle (4).

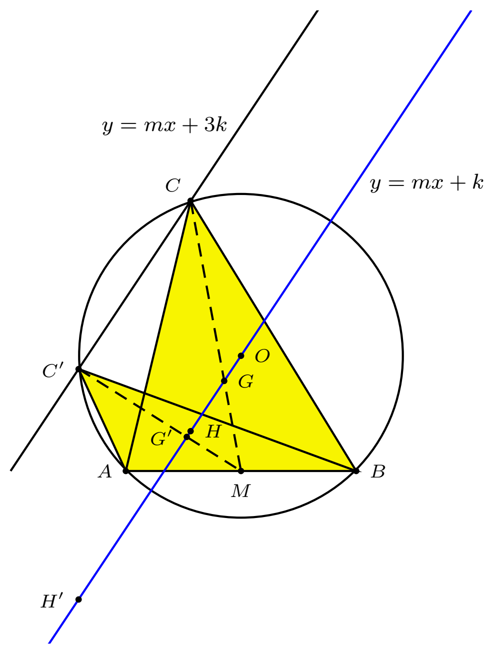

Figure 5

Proposition 5. *The number of points C for which triangle ABC has Euler line $y = mx + k$ is 0, 1, or 2 according as $(m^2 - 3)(k^2 + 1) <, =,$ or > -4.*

In the hyperbolic and degenerate cases $m^2 \geq 3$, there are always two such triangles. In the elliptic case, $m^2 < 3$. There are two such triangles if and only if $k^2 < \frac{m^2+1}{3-m^2}$.

Corollary 6. *For $m^2 < 3$ and $k = \pm\sqrt{\frac{m^2+1}{3-m^2}}$, there is a unique triangle ABC whose Euler line is the line $y = mx + k$. The lines $y = mx + 3k$ are tangent to the Euler ellipse* (3) *at the points*

$$\pm\left(\frac{-2m}{\sqrt{(m^2+1)(3-m^2)}}, \frac{3+m^2}{\sqrt{(m^2+1)(3-m^2)}}\right).$$

Figure 6 shows the configuration corresponding to $k = \sqrt{\frac{m^2+1}{3-m^2}}$. The other one can be obtained by reflection in M, the midpoint of AB.

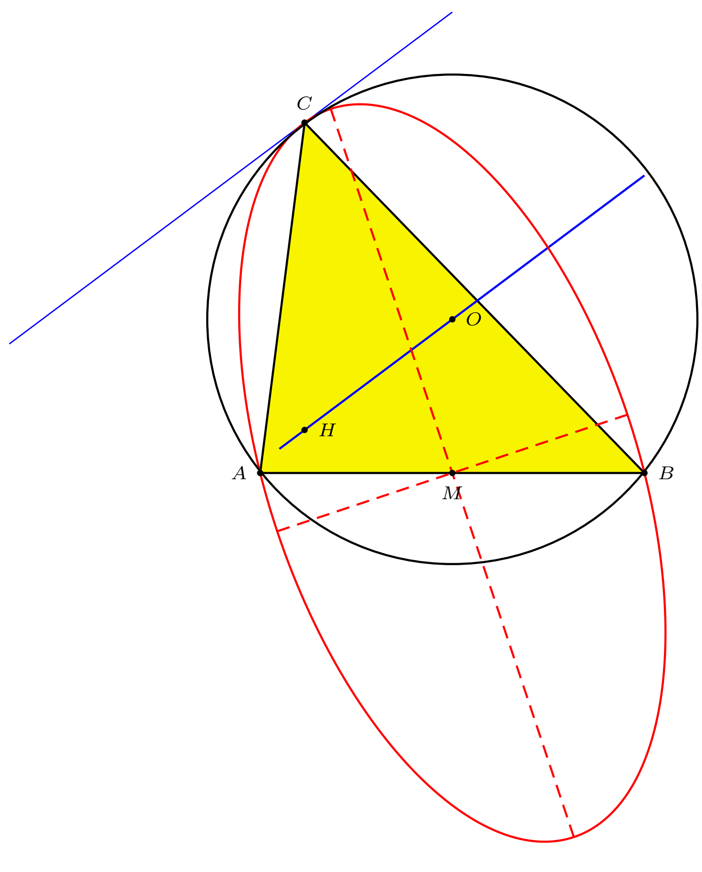

Figure 6

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A Note on the Droz-Farny Theorem

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Abstract. We give a simple characterization of the Droz-Farny pairs of lines through a point of the plane.

In [3] J-P. Erhmann and F. van Lamoen prove a projective generalization of the Droz-Farny line theorem. They say that a pair of lines (l, l') is a *pair of DF-lines through a point P with respect to a given triangle ABC* if they intercept the line BC in the points X and X', CA in Y and Y', and AB in Z and Z' in such a way that the midpoints of the segments XX', YY', and ZZ' are collinear. They then prove that (l, l') is a pair of DF-lines if and only if l and l' are tangent lines of a parabola inscribed in ABC (see also [5]). Thus, the DF-lines through P are the pairs of conjugate lines in the involution $\mathcal{I}$ determined by the lines through P that are tangent to the parabolas of the pencil of parabolas inscribed in ABC. Through a general point P, there passes just one orthogonal pair of DF-lines with respect to ABC; call this pair the ODF-lines through P with respect to ABC.

Considering the tangent lines through P at the three degenerate inscribed parabolas of ABC, it also follows that (PA, line through P parallel with BC), (PB, line parallel through P with CA), and (PC, line parallel through P with AB), are three conjugate pairs of lines of the involution $\mathcal{I}$.

Recall that the medial triangle $A'B'C'$ of ABC is the triangle whose vertices are the midpoints of $BC, CA,$ and AB, and that the anticomplementary triangle $A''B''C''$ of ABC is the triangle whose medial triangle is ABC ([4]).

Theorem. *A pair (l, l') of lines is a pair of DF-lines through P with respect to ABC, if and only if (l, l') is a pair of conjugate diameters of the conic $\mathcal{C}_P$ with center P, circumscribed at the anticomplementary triangle $A''B''C''$ of ABC. In particular, the ODF-lines through P are the axes of this conic.*

Proof. Since A is the midpoint of $B''C''$, and $B''C''$ is parallel with BC, it follows immediately that PA, and the line through P, parallel with BC, are conjugate diameters of the conic $\mathcal{C}_P$. In the same way, PB and the line through P parallel with CA (and PC and the line trough P parallel with AB) are also conjugate

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diameters of $\mathcal{C}_P$. Since two pairs of corresponding lines determine an involution, this completes the proof. $\square$

Remark that the orthocenter H of ABC is the center of the circumcircle of the anticomplementary triangle $A''B''C''$. Since any two orthogonal diameters of a circle are conjugate, we find by this special case the classical Droz-Farny theorem: Perpendicular lines through H are DF-pairs with respect to triangle ABC.

As a corollary of this theorem, we can characterize the axes of any circumscribed ellipse or hyperbola of ABC as the ODF-lines through its center with regard to the medial triangle $A'B'C'$ of ABC. And in the same way we can construct the axes of any circumscribed ellipse or hyperbola of any triangle, associated with ABC.

Examples

1. The Jerabek hyperbola of ABC (the isogonal conjugate of the Euler line of ABC) is the rectangular hyperbola through A, B, C, the orthocenter H, the circumcenter O and the Lemoine (or symmedian) point K of ABC, and its center is Kimberling center X_{125} with trilinear coordinates $(bc(b^2+c^2-a^2)(b^2-c^2)^2, .., ..)$, which is a point of the nine-point circle of ABC (the center of any circumscribed rectangular hyperbola is on the nine-point circle). The axes of this hyperbola are the ODF-lines through X_{125}, with respect to the medial triangle $A'B'C'$ of ABC.

2. The Kiepert hyperbola of ABC is the rectangular hyperbola through A, B, C, H, the centroid G of ABC, and through the Spieker center (the incenter of the medial triangle of ABC). It has center X_{115} with trilinear coordinates $(bc(b^2-c^2)^2, .., ..)$ on the nine-point circle. Its axes are the ODF-lines through X_{115} with respect to the medial triangle $A'B'C'$.

3. The Steiner ellipse of ABC is the circumscribed ellipse with center the centroid G of ABC. It is homothetic to (and has the same axes of) the Steiner ellipses of the medial triangle $A'B'C'$ and of the anticomplementary triangle $A''B''C''$ of ABC. These axes are the ODF-lines through G with respect to ABC (and to $A'B'C'$, and to $A''B''C''$).

4. The Feuerbach hyperbola is the rectangular hyperbola through A, B, C, H, the incenter I of ABC, the Mittenpunkt (the symmedian point of the excentral triangle $I_AI_BI_C$, where I_A, I_B, I_C are the excenters of ABC), with center the Feuerbach point F (at which the incircle and the nine-point circle are tangent; trilinear coordinates $(bc(b-c)^2(b+c-a), .., ..)$) . Its axes are the ODF-lines through F, with respect to the medial triangle $A'B'C'$ of ABC.

5. The Stammler hyperbola of ABC has trilinear equation

$$(b^2-c^2)x_1^2 + (c^2-a^2)x_2^2 + (a^2-b^2)x_3^2 = 0.$$

It is the rectangular hyperbola through the incenter I, the excenters I_A, I_B, I_C, the circumcenter O, and the symmedian point K. It is also the Feuerbach hyperbola of the tangential triangle of ABC, and its center is the focus of the Kiepert parabola (inscribed parabola with directrix the Euler line of ABC), which is Kimberling center X_{110} with trilinear coordinates $(\frac{a}{b^2-c^2}, .., ..)$, on the circumcircle of ABC, which is the nine-point circle of the excentral triangle $I_AI_BI_C$. The axes of this Stammler hyperbola are the ODF-lines through X_{110}, with regard to the medial triangle of $I_AI_BI_C$.

Remark that center X_{110} is the fourth common point (apart from A, B, and C) of the conic through A, B, C, and with center the symmedian point K of ABC, which has trilinear equation

$$a(-a^2+b^2+c^2)x_2x_3 + b(a^2-b^2+c^2)x_3x_1 + c(a^2+b^2-c^2)x_1x_2 = 0,$$

and the circumcircle of ABC.

6. The conic with trilinear equation

$$a^2(b^2-c^2)x_1^2 + b^2(c^2-a^2)x_2^2 + c^2(a^2-b^2)x_3^2 = 0$$

is the rectangular hyperbola through the incenter I, through the excenters I_A, I_B, I_C, and through the centroid G of ABC. It is also circumscribed to the anticomplementary triangle $A''B''C''$ (recall that the trilinear coordinates of A'', B'', and C'' are $(-bc, ac, ab), (bc, -ac, ab)$, and $(bc, ac, -ab)$, respectively). Its center is the Steiner point X_{99} with trilinear coordinates $(\frac{bc}{b^2-c^2}, \frac{ca}{c^2-a^2}, \frac{ab}{a^2-b^2})$, a point of intersection of the Steiner ellipse and the circumcircle of ABC.

Remark that the circumcircle of ABC is the nine-point circle of $A''B''C''$ and also of $I_AI_BI_C$. It follows that the axes of this hyperbola, which is often called the Wallace or the Steiner hyperbola, are the ODF-lines through the Steiner point X_{99} with regard to ABC, and also with regard to the medial triangle of the excentral triangle $I_AI_BI_C$.

Remarks. (1) A biographical note on Arnold Droz-Farny can be found in [1].

(2) A generalization of the Droz-Farny theorem in the three-dimensional Euclidean space was given in an article by J. Bilo [2].

(3) Finally, we give a construction for the ODF-lines through a point P with respect to the triangle ABC, *i.e.*, for the orthogonal conjugate pair of lines through P of the involution $\mathcal{I}$ in the pencil of lines through P, determined by the conjugate pairs (PA, line l_a through P parallel to BC) and (PB, line l_b through P parallel to CA): intersect a circle $\mathcal{C}$ through P with these conjugate pairs:

$$\begin{aligned} &\mathcal{C} \cap PA = Q, &\qquad &\mathcal{C} \cap l_a = Q', \\ &\mathcal{C} \cap PB = R, &\qquad &\mathcal{C} \cap l_b = R', \end{aligned}$$

then (Q, Q') and (R, R') determine an involution $\mathcal{I}'$ on $\mathcal{C}$, with center $QQ' \cap RR' = T$. Each line through T intersect the circle $\mathcal{C}$ in two conjugate points of $\mathcal{I}'$. In particular, the line through T and through the center of $\mathcal{C}$ intersects $\mathcal{C}$ in two points S and S', such that PS and PS' are the orthogonal conjugate pair of lines of the involution $\mathcal{I}$.

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On the Cyclic Complex of a Cyclic Quadrilateral

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Abstract. To every cyclic quadrilateral corresponds naturally a complex of sixteen cyclic quadrilaterals. The radical axes of the various pairs of circumcircles, the various circumcenters and anticenters combine to interesting configurations. Here are studied some of these, considered to be basic in the study of the whole complex.

1. Introduction

Consider a generic convex cyclic quadrilateral $q = ABCD$. Here we consider a simple figure, resulting by constructing other quadrangles on the sides of q, similar to q. This construction was used in a recent simple proof, of the minimal area property of cyclic quadrilaterals, by Antreas Varverakis [1]. It seems though that the figure is interesting for its own. The principle is to construct the quadrilateral $q' = CDEF$, on a side of and similar to q, but with reversed orientation.

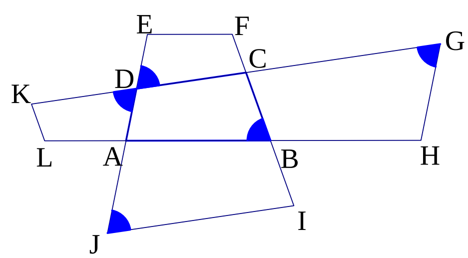

Figure 1. $CDEF$: top-flank of $ABCD$

The sides of the new align with those of the old. Besides, repeating the procedure three more times with the other sides of q, gives the previous basic Figure 1. For convenience I call the quadrilaterals: top-flank $t = CDEF$, right-flank $r = CGHB$, bottom-flank $b = BIJA$ and left-flank $l = DKLA$ of q respectively. In addition to these *main flanks*, there are some other flanks, created by the extensions of the sides of q and the extensions of sides of its four main flanks. Later create the *big flank* denoted below by q^*. To spare words, I drew in Figure 2 these sixteen quadrilaterals together with their names. All these quadrilaterals are cyclic and share the same angles with q. In general, though, only the main flanks are similar to q. More precisely, from their construction, flanks l, r are homothetic, t, b are also homothetic and two adjacent flanks, like r, t are antihomothetic with respect

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to their common vertex, here C. The symbol in braces, $\{rt\}$, will denote the circumcircle, the symbol in parentheses, (rt), will denote the circumcenter, and the symbol in brackets, $[rt]$, will denote the anticenter of the corresponding flank. Finally, a pair of symbols in parentheses, like (rt, rb), will denote the radical axis of the circumcircles of the corresponding flanks. I call this figure the *cyclic complex* associated to the cyclic quadrilateral.

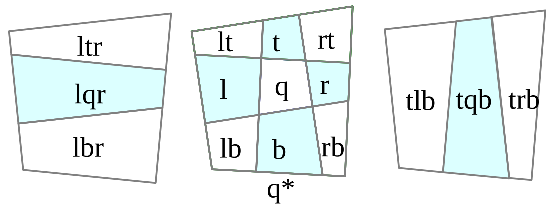

Figure 2. The cyclic complex of q

2. Radical Axes

By taking all pairs of circles, the total number of radical axes, involved, appears to be 120. Not all of them are different though. The various sides are radical axes of appropriate pairs of circles and there are lots of coincidences. For example the radical axes $(t, rt) = (q, r) = (b, rb) = (tqb, rqb)$ coincide with line BC. The same happens with every side out of the eight involved in the complex. Each side coincides with the radical axis of four pairs of circles of the complex. In order to study other identifications of radical axes we need the following:

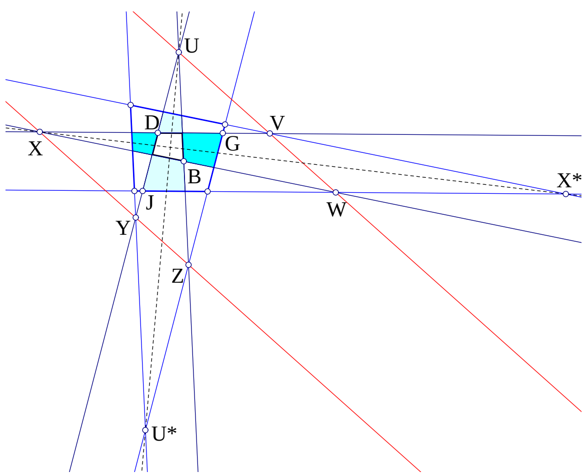

Figure 3. Intersections of lines of the complex

Lemma 1. *Referring to Figure 3, points X, U are intersections of opposite sides of q. U^*, X^* are intersections of opposite sides of $q*$. V, W, Y, Z are intersections of opposite sides of other flanks of the complex. Points X, Y, Z and U, V, W are aligned on two parallel lines.*

The proof is a trivial consequence of the similarity of opposite located main flanks. Thus, l, r are similar and their similarity center is X. Analogously t, b are similar and their similarity center is U. Besides triangles XDY, VDU are anti-homothetic with respect to D and triangles XGZ, WJU are similar to the previous two and also anti-homothetic with respect to B. This implies easily the properties of the lemma.

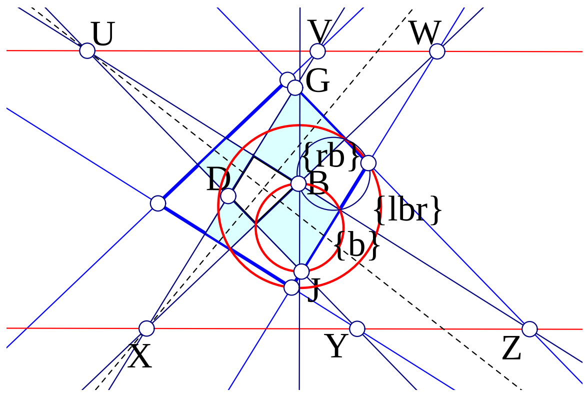

Figure 4. Other radical axes

Proposition 2. *The following lines are common radical axes of the circle pairs:*
(1) *Line XX^* coincides with* $(lt, lb) = (t, b) = (rt, rb) = (ltr, lbr)$.
(2) *Line UU^* coincides with* $(lt, rt) = (l, r) = (lb, rb) = (tlb, trb)$.
(3) *Line XYZ coincides with* $(lbr, b) = (lqr, q) = (ltr, t) = (q^*, tqb)$.
(4) *Line UVW coincides with* $(tlb, l) = (tqb, q) = (trb, r) = (q^*, lqr)$.

Referring to Figure 4, I show that line XYZ is identical with (lbr, b). Indeed, from the intersection of the two circles $\{lbr\}$ and $\{b\}$ with circle $\{rb\}$ we see that Z is on their radical axis. Similarly, from the intersection of these two circles with $\{lb\}$ we see that Y is on their radical axis. Hence line ZYX coincides with the radical axis (lbr, b). The other statements are proved analogously.

3. Centers

The centers of the cyclic complex form various parallelograms. The first of the next two figures shows the centers of the small flanks, and certain parallelograms created by them. Namely those that have sides the medial lines of the sides of the flanks. The second gives a panorama of all the sixteen centers together with a parallelogramic pattern created by them.

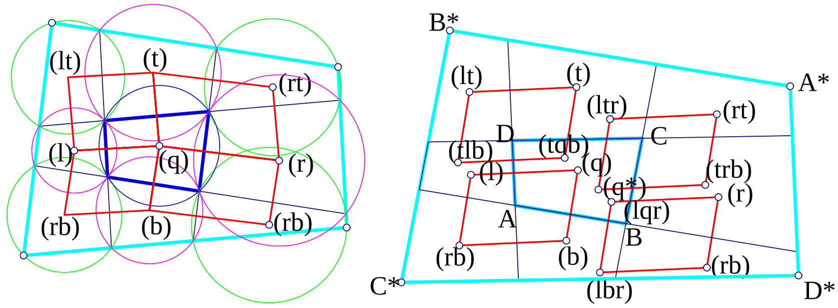

Figure 5. Panorama of centers of flanks

Proposition 3. *Referring to Figure 5, the centers of the flanks build equal parallelograms with parallel sides:* $(lt)(t)(tqb)(tlb)$, $(ltr)(rt)(trb)(q*)$, $(l)(q)(b)(rb)$ *and* $(lqr)(r)(rb)(lbr)$. *The parallelogramic pattern is symmetric with respect to the middle* M *of segment* $(q)(q*)$ *and the centers* $(tqb), (q), (q^*), (lqr)$ *are collinear.*

The proof of the various parallelities is a consequence of the coincidences of radical axes. For example, in the first figure, sides $(t)(rt)$, $(q)(r)$, $(b)(rb)$ are parallel because, all, are orthogonal to the corresponding radical axis, coinciding with line BC. In the second figure $(lt)(t), (tlb)(tqb), (l)(q), (rb)(b)$ are all orthogonal to AD. Similarly $(ltr)(rt)$, $(q*)(trb)$, $(lqr)(r)$, $(lbr)(rb)$ are orthogonal to A^*D^*. The parallelity of the other sides is proved analogously. The equality of the parallelograms results by considering other implied parallelograms, as, for example, $(rb)(b)(t)(lt)$, implying the equality of horizontal sides of the two left parallelograms. Since the labeling is arbitrary, any main flank can be considered to be the left flank of the complex, and the previous remarks imply that all parallelograms shown are equal. An important case, in the second figure, is that of the collinearity of the centers $(tqb), (q), (q^*), (lqr)$. Both lines $(tqb)(q)$ and $(q^*)(lqr)$ are orthogonal to the axis XYZ of the previous paragraph. $(tqb)(q^*)$ is orthogonal to the axis UVW, which, after lemma-1, is parallel to XYZ, hence the collinearity. In addition, from the parallelograms, follows that the lengths are equal: $|(tqb)(q)|$=$|(q^*)(lqr)|$. The symmetry about M is a simple consequence of the previous considerations.

There are other interesting quadrilaterals with vertices at the centers of the flanks, related directly to q. For example the next proposition relates the centers of the main flanks to the anticenter of the original quadrilateral q. Recall that the *anticenter* is the symmetric of the circumcenter with respect to the centroid of the quadrilateral. Characteristically it is the common intersection point of the orthogonals from the middles of the sides to their opposites. Some properties of the anticenter are discussed in Honsberger [2]. See also Court [3] and the miscelanea (remark after Proposition 11) below.

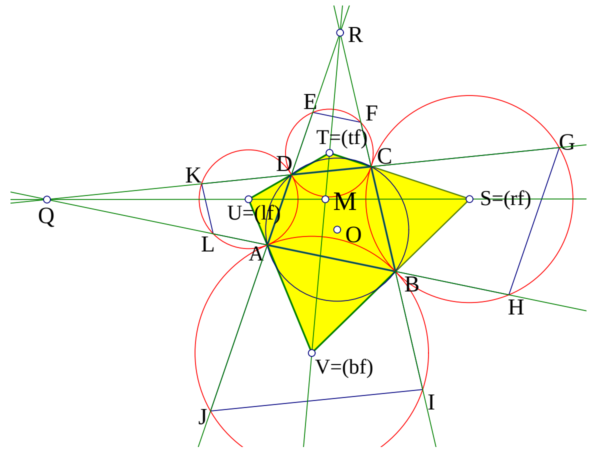

Figure 6. Anticenter through the flanks

Proposition 4. *Referring to Figure 6, the following properties are valid:*
(1) *The circumcircles of adjacent main flanks are tangent at the vertices of* q.
(2) *The intersection point of the diagonals* US, TV *of the quadrilateral* $TSUV$, *formed by the centers of the main flanks, coincides with the anticenter* M *of* q.
(3) *The intersection point of the diagonals of the quadrilateral formed by the anticenters of the main flanks coincides with the circumcenter* O *of* q.

The properties are immediate consequences of the definitions. (1) follows from the fact that triangles FTC and CSB are similar isosceli. To see that property (2) is valid, consider the parallelograms $p = OYMX$, $p^{**} = UXM^{**}Y$ and $p^* = SY^*M^*Y$, tightly related to the anticenters of q and its left and right flanks (Figure 7). X, Y, X^* and Y^* being the middles of the respective sides. One sees easily that triangles UXM and MYS are similar and points U, M, S are aligned. Thus the anticenter M of q lies on line US, passing through Q. Analogously it must lie also on the line joining the two other circumcenters. Thus, it coincides with their intersection.

The last assertion follows along the same arguments, from the similarity of parallelograms $UX^*M^{**}X$ and SYM^*Y^* of Figure 7. O is on the line M^*M^{**}, which is a diagonal of the quadrangle with vertices at the anticenters of the flanks.

Proposition 5. *Referring to the previous figure, the lines* QM *and* QO *are symmetric with respect to the bisector of angle* AQD. *The same is true for lines* QX *and* QY.

This is again obvious, since the trapezia $DAHG$ and $CBLK$ are similar and inversely oriented with respect to the sides of the angle AQD.

Remark. One could construct further flanks, left from the left and right from the right flank. Then repeat the procedure and continuing that way fill all the area of the angle AQD with flanks. All these having alternatively their anticenters and

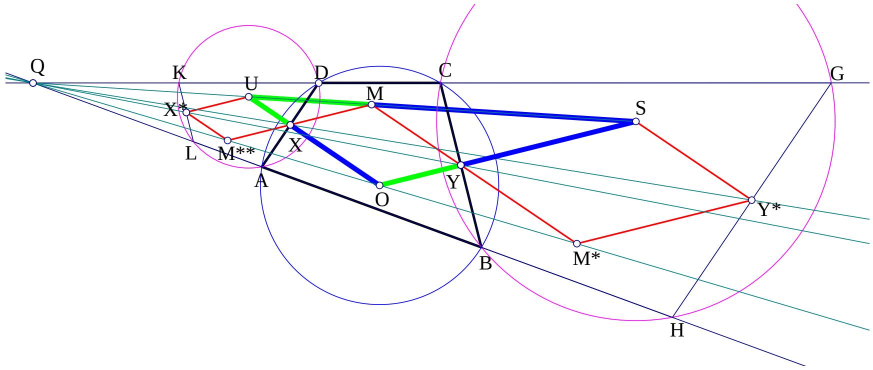

Figure 7. Anticenters M, M^*, M^{**} of the flanks

circumcenters on the two lines QO and QM and the centers of their sides on the two lines OX and OY.

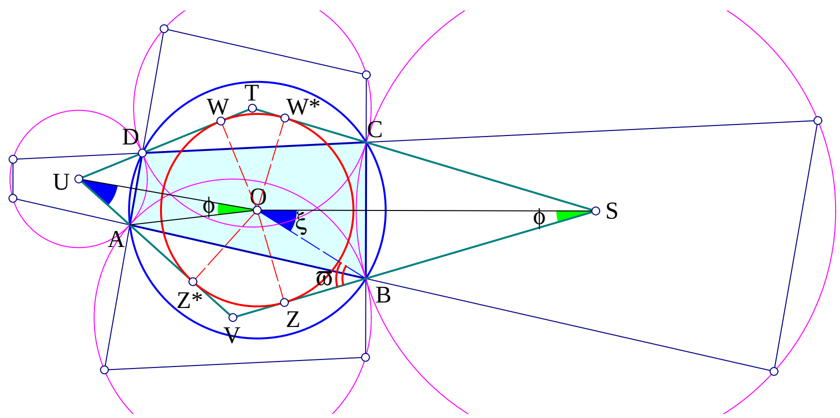

Figure 8. The circumscriptible quadrilateral $STUV$

Proposition 6. *The quadrilateral $STUV$, of the centers of the main flanks, is circumscriptible, its incenter coincides with the circumcenter O of q and its radius is $r \cdot sin(\phi + \xi)$. r being the circumcradius of q and $2\phi, 2\xi$ being the measures of two angles at O viewing two opposite sides of q.*

The proof follows immediately from the similarity of triangles UAO and OBS in Figure 8. The angle $\omega = \phi + \xi$, gives for $|OZ| = r \cdot sin(\omega)$. Z, Z^*, W, W^* being the projections of O on the sides of $STUV$. Analogous formulas hold for the other segments $|OZ^*| = |OW| = |OW^*| = r \cdot sin(\omega)$.

Remarks. (1) Referring to Figure 8, Z, Z^*, W, W^* are vertices of a cyclic quadrilateral q', whose sides are parallel to those of $ABCD$.

(2) The distances of the vertices of q and q' are equal: $|ZB| = |AZ^*| = |DW| = |CW^*| = r \cdot cos(\phi + \xi)$.

(3) Given an arbitrary circumscriptible quadrilateral $STUV$, one can construct the cyclic quadrangle $ABCD$, having centers of its flanks the vertices of $STUV$. Simply take on the sides of $STUV$ segments $|ZB| = |AZ^*| = |DW| = |CW^*|$ equal to the above measure. Then it is an easy exercise to show that the circles centered at S, U and passing from B, C and A, D respectively, define with their intersections on lines AB and CD the right and left flank of $ABCD$.

4. Anticenters

The anticenters of the cyclic complex form a parallelogramic pattern, similar to the previous one for the centers. The next figure gives a panoramic view of the sixteen anticenters (in blue), together with the centers (in red) and the centroids of flanks (white).

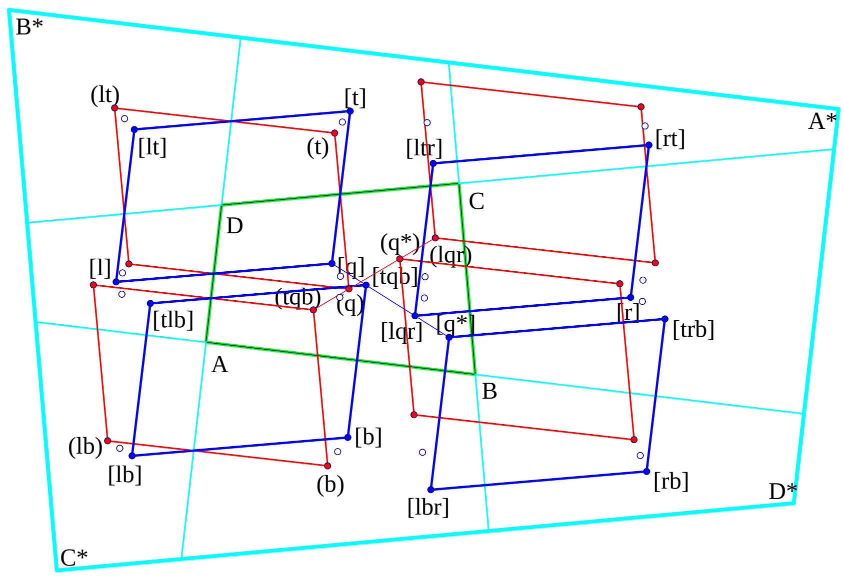

Figure 9. Anticenters of the cyclic complex

Proposition 7. *Referring to Figure 9, the anticenters of the flanks build equal parallelograms with parallel sides:* $[lt][t][q][l]$, $[tlb][tqb][b][lb]$, $[q*][trb][rb][lbr]$ *and* $[ltr][rt][r][lqr]$. *The parallelogramic pattern is symmetric with respect to the middle* M *of segment* $[q][q*]$ *and the anticenters* $[tqb], [q], [q^*], [lqr]$ *are collinear. Besides the angles of the parallelograms are the same with the corresponding of the parallelogramic pattern of the centers.*

The proof is similar to the one of Proposition 2. For example, segments $[lt][t]$, $[l][q]$, $[tlb][tqb]$, $[lb][b]$, $[ltr][rt]$, $[lqr][r]$, $[q*][trb]$, $[lbr][rb]$ are all parallel since they are orthogonal to BC or its parallel B^*C^*.

A similar argument shows that the other sides are also parallel and also proves the statement about the angles. To prove the equality of parallelograms one can

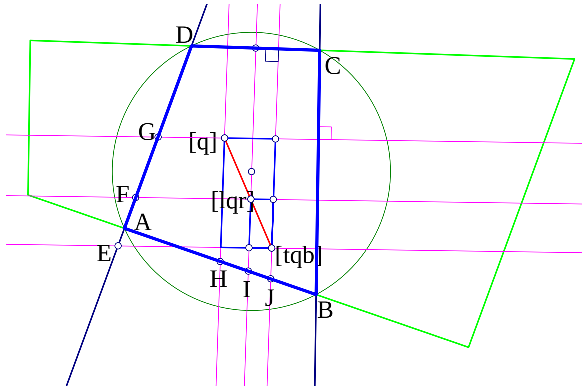

Figure 10. Collinearity of $[tqb], [q], [q^*], [lqr]$

use again implied parallelograms, as, for example, $[lt][t][tqb][tlb]$, which shows the equality of horizontal sides of the two left parallelograms. The details can be completed as in Proposition 2. The only point where another kind of argument is needed is the collinearity assertion. For this, in view of the parallelities proven so far, it suffices to show that points $[q], [lqr], [tqb]$ are collinear. Figure 10 shows how this can be done. G, F, E, H, I, J are middles of sides of flanks, related to the definition of the three anticenters under consideration. It suffices to calculate the ratios and show that $|EF|/|EG| = |JI|/|JH|$. I omit the calculations.

5. Miscelanea

Here I will mention only a few consequences of the previous considerations and some supplementary properties of the complex, giving short hints for their proofs or simply figures that serve as hints.

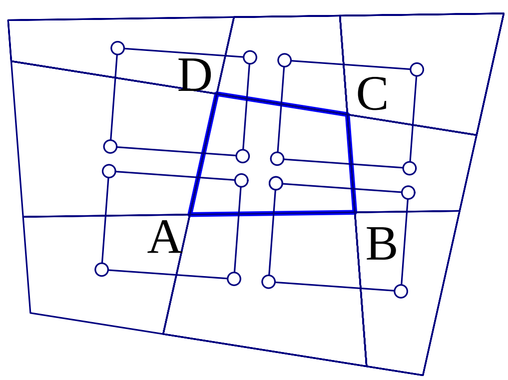

Figure 11. Barycenters of the flanks

Proposition 8. *The barycenters of the flanks build the pattern of equal parallelograms of Figure 11.*

Indeed, this is a consequence of the corresponding results for centers and anticenters of the flanks and the fact that *linear combinations* of parallelograms $c_i = (1-t)a_i + tb_i$, where a_i, b_j denote the vertices of parallelograms, are again parallelograms. Here $t = 1/2$, since the corresponding barycenter is the middle between center and anticenter. The equality of the parallelograms follows from the equality of corresponding parallelograms of centers and anticenters.

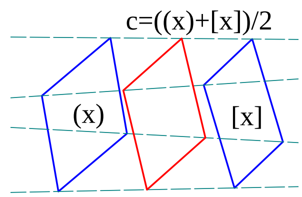

Figure 12. Linear combinations of parallelograms

Proposition 9. *Referring to Figure 13, the centers of the sides of the main flanks are aligned as shown and the corresponding lines intersect at the outer diagonal of q i.e. the line joining the intersection points of opposite sides of q.*

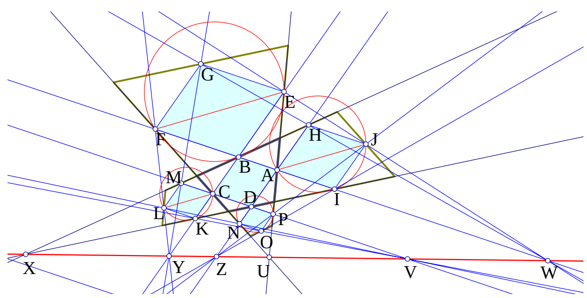

Figure 13. Lines of middles of main flanks

This is due to the fact that the main flanks are antihomothetic with respect to the vertices of q. Thus, the parallelograms of the main flanks are homothetic to each other and their homothety centers are aligned by three on a line. Later assertion can be reduced to the well known one for similarity centers of three circles, by considering the circumcircles of appropriate triangles, formed by parallel diagonals of the four parallelograms. The alignement of the four middles along the sides of $ABCD$ is due to the equality of angles of cyclic quadrilaterals shown in Figure 14.

Proposition 10. *Referring to Figure 15, the quadrilateral $(t)(r)(b)(l)$ of the centers of the main flanks is symmetric to the quadrilateral of the centers of the peripheral flanks $(tlb)(ltr)(trb)(lbr)$. The symmetry center is the middle of the line of $(q)(q^*)$.*

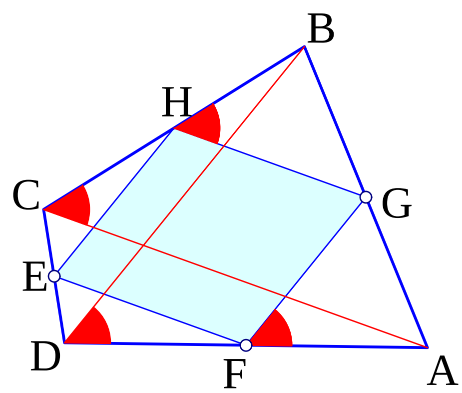

Figure 14. Equal angles in cyclic quadrilaterals

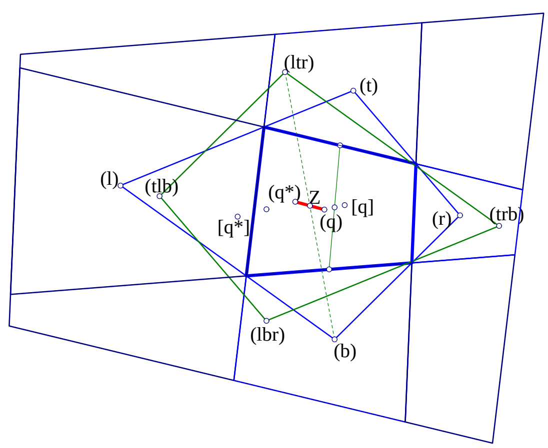

Figure 15. Symmetric quadrilaterals of centers

There is also the corresponding sort of dual for the anticenters, resulting by replacing the symbols (x) with $[x]$:

Proposition 11. *Referring to Figure 16, the quadrilateral* $[t][r][b][l]$ *of the anticenters of the main flanks is symmetric to the quadrilateral of the anticenters of the peripheral flanks* $[tlb][ltr][trb][lbr]$*. The symmetry center is the middle of the line of* $[q][q^*]$.

By the way, the symmetry of center and anticenter about the barycenter leads to a simple proof of the chararacteristic property of the anticenter. Indeed, consider the symmetric $A^*B^*C^*D^*$ of q with respect to the barycenter of q. The orthogonal from the middle of one side of q to the opposite one, to AD say, is also orthogonal to its symmetric A^*D^*, which is parallel to AD (Figure 17). Since A^*D^* is a chord

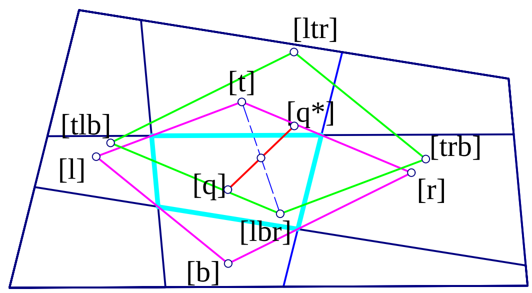

Figure 16. Symmetric quadrilaterals of anticenters

of the symmetric of the circumcircle, the orthogonal to its middle passes through the corresponding circumcenter, which is the anticenter.

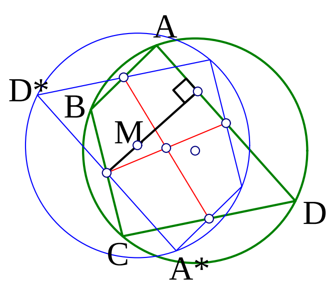

Figure 17. Anticenter's characteristic property

The following two propositions concern the radical axes of two particular pairs of circles of the complex:

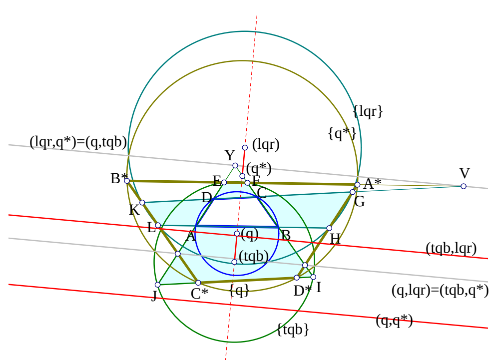

Figure 18. Harmonic bundle of radical axes

Proposition 12. *Referring to Figure 18, the radical axes* $(lqr, q^*) = (q, tqb)$ *and* $(q, lqr) = (tqb, q^*)$. *Besides the radical axes* (tqb, lqr) *and* (q, q^*) *are parallel to the previous two and define with them a harmonic bundle of parallel lines.*

Proposition 13. *Referring to Figure 19, the common tangent* (t, r) *is parallel to the radical axis* (tlb, lbr). *Analogous statements hold for the common tangents of the other pairs of adjacent main flanks.*

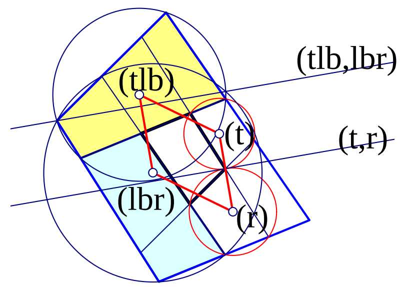

Figure 19. Common tangents of main flanks

6. Generalized complexes

There is a figure, similar to the cyclic complex, resulting in another context. Namely, when considering two arbitrary circles a, b and two other circles c, d tangent to the first two. This is shown in Figure 20. The figure generates a complex of quadrilaterals which I call a *generalized complex* of the cyclic quadrilateral. There are many similarities to the cyclic complex and one substantial difference, which prepares us for the discussion in the next paragraph. The similarities are:

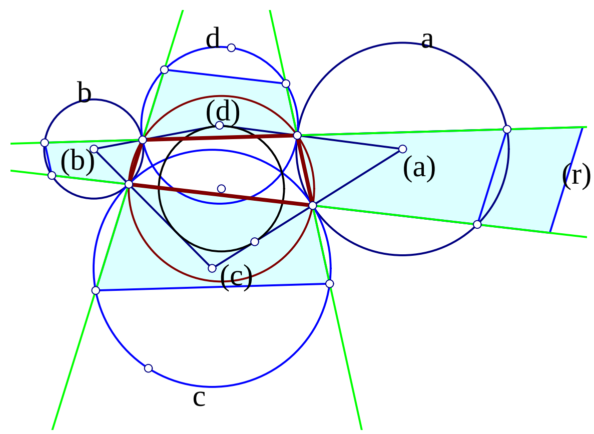

Figure 20. A complex similar to the cyclic one

(1) The points of tangency of the four circles define a cyclic quadrilateral q.
(2) The centers of the circles form a circumscriptible quadrilateral with center at the circumcenter of q.
(3) There are defined flanks, created by the other intersection points of the sides of q with the circles.
(4) Adjacent flanks are antihomothetic with homothety centers at the vertices of q.
(5) The same parallelogramic patterns appear for circumcenters, anticenters and barycenters.

Figure 21 depicts the parallelogrammic pattern for the circumcenters (in red) and the anticenters (in blue). Thus, the properties of the complex, discussed so far, could have been proved in this more general setting. The only difference is that the central cyclic quadrilateral q is not similar, in general, to the flanks, created in this way. In Figure 21, for example, the right cyclic-complex-flank r of q has been also constructed and it is different from the flank created by the general procedure.

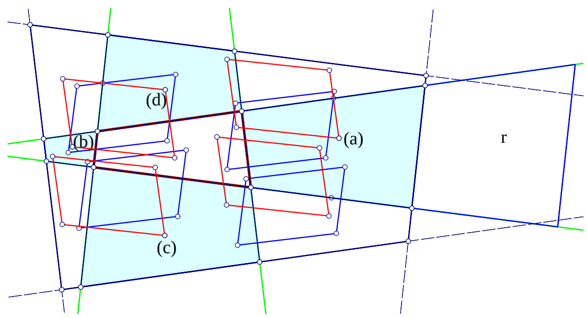

Figure 21. Circumcenters and anticenters of the general complex

Having a cyclic quadrilateral q, one could use the above remarks to construct infinite many generalized complexes (Figure 22) having q as their *central quadrilateral*. In fact, start with a point, F say, on the medial line of side AB of $q = ABCD$. Join it to B, extend FB and define its intersection point G with the medial of BC. Join G with C extend and define the intersection point H with the medial of CD. Finally, join H with D extend it and define I on the medial line of side DA. q being cyclic, implies that there are four circles centered, correspondingly, at points F, G, H and I, tangent at the vertices of q, hence defining the configuration of the previous remark.

From our discussion so far, it is clear, that the *cyclic complex* is a well defined complex, uniquely distinguished between the various generalized complexes, by the property of having its flanks similar to the original quadrilateral q.

7. The inverse problem

The inverse problem asks for the determination of q, departing from the big flank q^*. The answer is in the affirmative but, in general, it is not possible to construct q by elementary means. The following lemma deals with a completion of the figure handled in lemma-1.

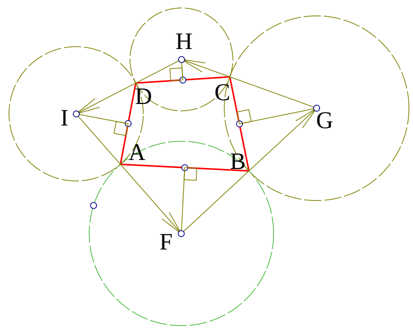

Figure 22. The generalized setting

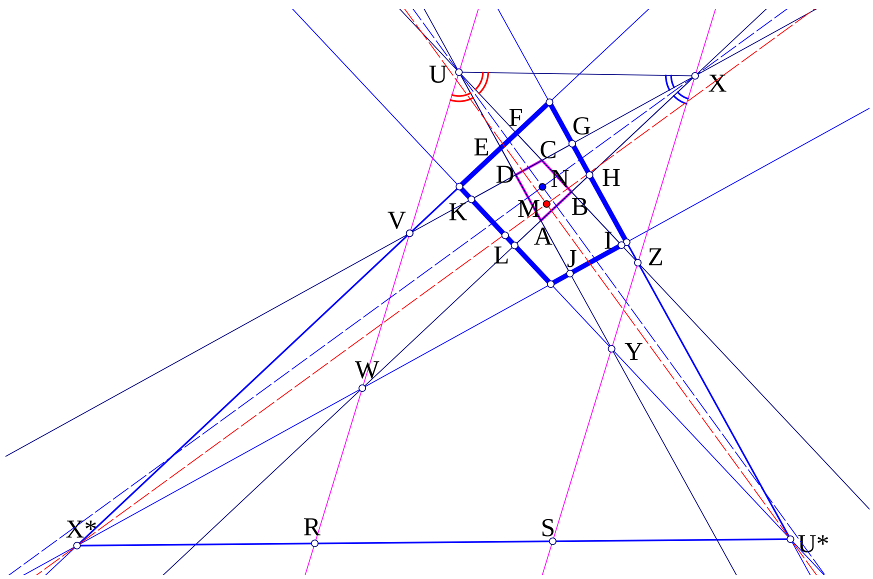

Figure 23. Lines XZY and UVW

Lemma 14. *Referring to Figure 23, lines XZY and UVW are defined by the intersection points of the opposite sides of main flanks of q. Line X^*U^* is defined by the intersection points of the opposite sides of q^*. The figure has the properties:*
(1) *Lines XZY and UVW are parallel and intersect line X^*U^* at points S, R trisecting segment X^*U^*.*
(2) *Triangles YKX, ZCX, UCV, UAX, UIV are similar.*
(3) *Angles $\widehat{VUD} = \widehat{CUX}$ and $\widehat{ZXH} = \widehat{GXU}$.*
(4) *The bisectors of angles $\widehat{VUX}, \widehat{UXZ}$ are respectively identical with those of $\widehat{DUC}, \widehat{DXA}$.*

(5) *The bisectors of the previous angles intersect orthogonaly and are parallel to the bisectors* X^*M, U^*M *of angles* $\widehat{VX^*W}$ *and* $\widehat{YU^*Z}$.

(1) is obvious, since lines UVW and XZY are diagonals of the parallelograms X^*WXV and U^*YUZ. (2) is also trivial since these triangles result from the extension of sides of similar quadrilaterals, namely q and its main flanks. (3) and (4) is a consequence of (2). The orthogonality of (5) is a general property of cyclic quadrilaterals and the parallelity is due to the fact that the angles mentioned are opposite in parallelograms.

The lemma suggests a solution of the inverse problem: Draw from points R, S two parallel lines, so that the parallelograms X^*WXV and U^*YUZ, with their sides intersections, create $ABCD$ with the required properties. Next proposition investigates a similar configuration for a general, not necessarily cyclic, quadrangle.

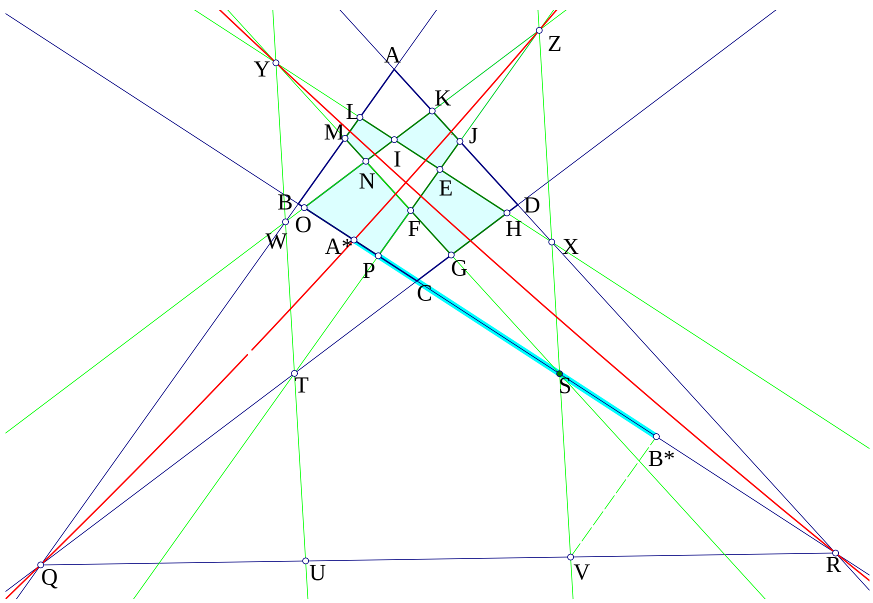

Figure 24. Inverse problem

Proposition 15. *Referring to Figure 24, consider a quadrilateral* $q^* = ABCD$ *and trisect the segmet* QR*, with end-points the intersections of opposite sides of* q^*. *From trisecting points* U, V *draw two arbitrary parallels* UY, VX *intersecting the sides of* q^* *at* W, T *and* S, X *respectively. Define the parallelograms* $QWZT, RSYX$ *and through their intersections and the intersections with* q^* *define the central quadrilateral* $q = EFNI$ *and its flanks* $FEHG, FPON, NMLI, IKJE$ *as shown.*

(1) *The central quadrilateral* q *has angles equal to* q^*. *The angles of the flanks are complementary to those of* q^*.

(2) *The flanks are always similar to each other, two adjacent being anti-homothetic*

with respect to their common vertex. (3) *There is a particular direction of the parallels, for which the corresponding central quadrilateral* q *has side-lengths-ratio* $|EF|/|FN| = |ON|/|OP|$.

In fact, (1) is trivial and (2) follows from (1) and an easy calculation of the ratios of the sides of the flanks. To prove (3) consider point S varying on side BC of q^*. Define the two parallels and in particular VX by joining V to S. Thus, the two parallels and the whole configuration, defined through them, becomes dependent on the location of point S on BC. For S varying on BC, a simple calculation shows that points Y, Z vary on two hyperbolas (red), the hyperbola containing Z intersecting BC at point A^*. Draw VB^* parallel to AB, B^* being the intersection point with BC. As point S moves from A^* towards B^* on segment A^*B^*, point Z moves on the hyperbola from A^* to infinity and the cross ratio $r(S) = \frac{|EF|}{|FN|} : \frac{|ON|}{|OP|}$ varies increasing continuously from 0 to infinity. Thus, by continuity it passes through 1.

Proposition 16. *Given a circular quadrilateral* $q^* = A^*B^*C^*D^*$ *there is another circular quadrilateral* $q = ABCD$*, whose cyclic complex has corresponding big flank the given one.*

The proof follows immediately by applying (3) of the previous proposition to the given cyclic quadrilateral q^*. In that case, the condition of the equality of ratios implies that the constructed by the proposition central quadrilateral q is similar to the main flanks. Thus the given q^* is identical with the big flank of q as required.

Remarks. (1) Figure 24 and the related Proposition 14 deserve some comments. First, they show a way to produce a complex out of any quadrilateral, not necessary a cyclic one. In particular, condition (3) of the aforementioned proposition suggests a unified approach for general quadrilaterals that produces the cyclic complex, when applied to cyclic quadrilaterals. The suggested procedure can be carried out as follows (Figure 25): (a) Start from the given general quadrilateral $q = ABCD$ and construct the first flank $ABFE$ using the restriction $|AE|/|EF| = |BC|/|AB| = k_1$. (b) Use appropriate anti-homotheties centered at the vertices of q to transplant the flank to the other sides of q. These are defined inductively. More precisely, having flank-1, use the anti-homothety $(B, \frac{|BC|}{|FB|})$ to construct flank-2 $BGHC$. Then repeat with analogous constructions for the two remaining flanks. It is easy to see that this procedure, applied to a cyclic quadrilateral, produces its cyclic complex, and this independently from the pair of adjacent sides of q, defining the ratio k_1. For general quadrilaterals though the complex depends on the initial choice of sides defining k_1. Thus defining $k_1 = |DC|/|BC|$ and starting with flank-2, constructed through the condition $|BG|/|GH| = k_1$ etc. we land, in general, to another complex, different from the previous one. In other words, the procedure has an element of arbitrariness, producing four complexes in general, depending on which pair of adjacent sides of q we start it.

(2) The second remark is about the results of Proposition 8, on the centroids or barycenters of the various flanks. They remain valid for the complexes defined

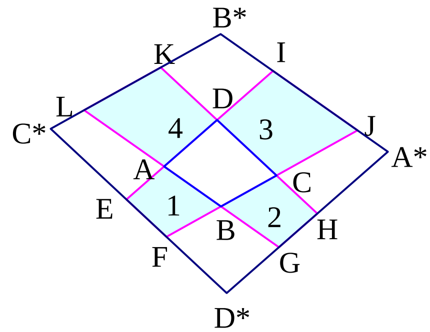

Figure 25. Flanks in arbitrary quadrilaterals

through the previously described procedure. The proof though has to be modified and given more generally, since circumcenters and anticenters are not available in the general case. The figure below shows the barycenters for a general complex, constructed with the procedure described in (1).

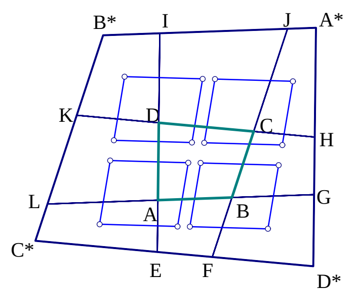

Figure 26. Barycenters for general complexes

An easy approach is to use vectors. Proposition-9, with some minor changes, can also be carried over to the general case. I leave the details as an exercise.
(3) Although Proposition 15 gives an answer to the existence of a sort of *soul* (q) of a given cyclic quadrilateral (q^*), a more elementary constuction of it is desirable. Proposition-14, in combination with the first remark, shows that even general quadrilaterals have *souls*.
(4) One is tempted to look after the soul of a soul, or, stepping inversely, the complex and corresponding big flank of the big flank etc.. Several questions arise in this context, such as (a) are there repetitions or periodicity, producing something similar to the original after a finite number of repetitions? (b) which are the limit points, for the sequence of souls?

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Isocubics with Concurrent Normals

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Abstract. It is well known that the tangents at A, B, C to a pivotal isocubic concur. This paper studies the situation where the normals at the same points concur. The case of non-pivotal isocubics is also considered.

1. Pivotal isocubics

Consider a pivotal isocubic $\mathrm{p}\mathcal{K} = \mathrm{p}\mathcal{K}(\Omega, P)$ with pole $\Omega = p : q : r$ and pivot P, *i.e.*, the locus of point M such as P, M and its $\Omega-$isoconjugate M^* are collinear. This has equation

$$ux(ry^2 - qz^2) + vy(pz^2 - rx^2) + wz(qx^2 - py^2) = 0.$$

It is well known that the tangents at A, B, C and P to $\mathrm{p}\mathcal{K}$, being respectively the lines $-\frac{v}{q}y + \frac{w}{r}z = 0$, $\frac{u}{p}x - \frac{w}{r}z = 0$, $-\frac{u}{p}x + \frac{v}{q}y = 0$, concur at $P^* = \frac{p}{u} : \frac{q}{v} : \frac{r}{w}$. [1]
We characterize the pivotal cubics whose normals at the vertices A, B, C concur at a point. These normals are the lines

$$\begin{array}{ll} \mathsf{n}A: & (S_Arv + (S_A + S_B)qw)y + (S_Aqw + (S_C + S_A)rv)z = 0, \\ \mathsf{n}B: & (S_Bru + (S_A + S_B)pw)x + (S_Bpw + (S_B + S_C)ru)z = 0, \\ \mathsf{n}C: & (S_Cqu + (S_C + S_A)pv)x + (S_Cpv + (S_B + S_C)qu)y = 0. \end{array}$$

These three normals are concurrent if and only if

$$(pvw + qwu + ruv)(a^2qru + b^2rpv + c^2pqw) = 0.$$

Let us denote by $\mathcal{C}_\Omega$ the circumconic with perspector Ω, and by $\mathcal{L}_\Omega$ the line which is the $\Omega-$isoconjugate of the circumcircle. [2] These have barycentric equations

$$C_\Omega: \quad pyz + qzx + rxy = 0,$$

and

$$\mathcal{L}_\Omega: \quad \frac{a^2}{p}x + \frac{b^2}{q}y + \frac{c^2}{r}z = 0.$$

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[1] The tangent at P, namely, $u(rv^2 - qw^2)x + v(pw^2 - ru^2)y + w(qu^2 - pv^2)z = 0$, also passes through the same point.

[2] This line is also the trilinear polar of the isotomic conjugate of the isogonal conjugate of Ω.

Theorem 1. *The pivotal cubic* $\mathrm{p}\mathcal{K}(\Omega, P)$ *has normals at* A, B, C *concurrent if and only if*
(1) P *lies on* $\mathcal{C}_\Omega$, *equivalently,* P^* *lies on the line at infinity, or*
(2) P *lies on* $\mathcal{L}_\Omega$, *equivalently,* P^* *lies on the circumcircle.*

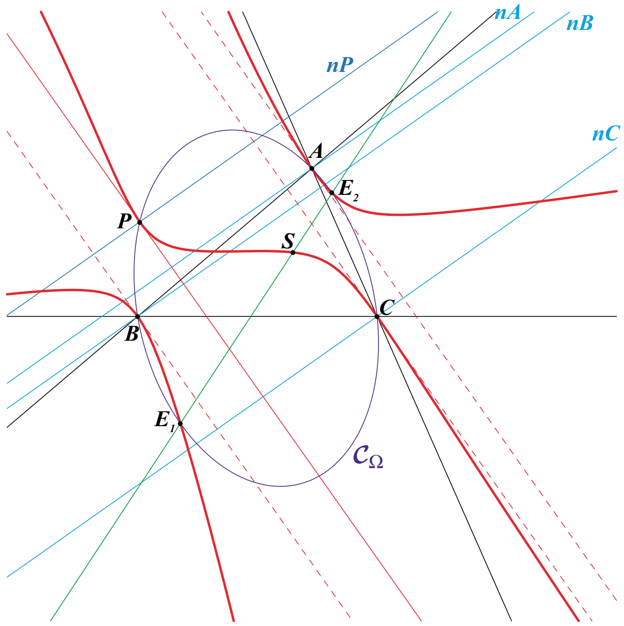

Figure 1. Theorem 1(1): $\mathrm{p}\mathcal{K}$ with concurring normals

More precisely, in (1), the tangents at A, B, C are parallel since P^* lies on the line at infinity. Hence the normals are also parallel and "concur" at X on the line at infinity. The cubic $\mathrm{p}\mathcal{K}$ meets $\mathcal{C}_\Omega$ at A, B, C, P and two other points E_1, E_2 lying on the polar line of P^* in $\mathcal{C}_\Omega$, *i.e.*, the conjugate diameter of the line PP^* in $\mathcal{C}_\Omega$. Obviously, the normal at P is parallel to these three normals. See Figure 1.

In (2), P^* lies on the circumcircle and the normals concur at X, antipode of P^* on the circumcircle. $\mathrm{p}\mathcal{K}$ passes through the (not always real) common points E_1, E_2 of $\mathcal{L}_\Omega$ and the circumcircle. These two points are isoconjugates. See Figure 2.

2. The orthopolar

The tangent $\mathtt{t}M$ at any non-singular point M to any curve is the polar line (or first polar) of M with respect to the curve and naturally the normal $\mathtt{n}M$ at M is the perpendicular at M to $\mathtt{t}M$. For any point M not necessarily on the curve, we define the *orthopolar* of M with respect to the curve as the perpendicular at M to the polar line of M.

In Theorem 1(1) above, we may ask whether there are other points on $\mathrm{p}\mathcal{K}$ such that the normal passes through X. We find that the locus of point Q such that the orthopolar of Q contains X is the union of the line at infinity and the circumconic

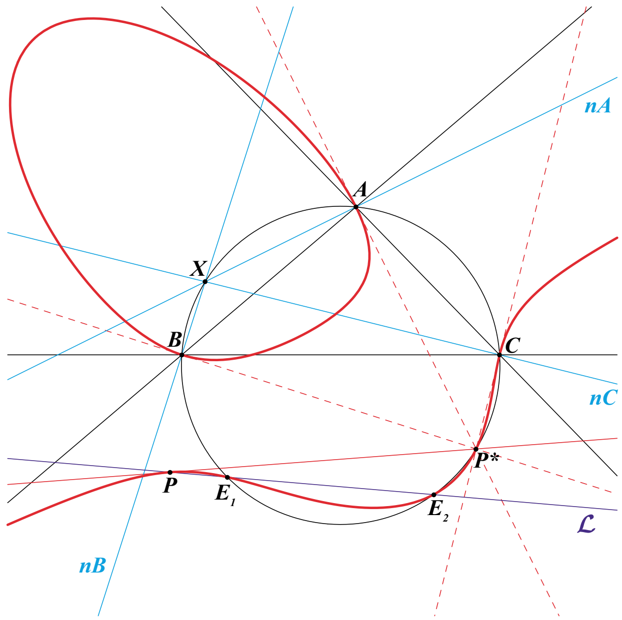

Figure 2. Theorem 1(2): p$\mathcal{K}$ with concurring normals

passing through P and P^*, the isoconjugate of the line PP^*. Hence, there are no other points on the cubic with normals passing through X.

In Theorem 1(2), the locus of point Q such that the orthopolar of Q contains X is now a circum-cubic (K) passing through P^* and therefore having six other (not necessarily real) common points with p$\mathcal{K}$. Figure 3 shows $\mathrm{p}\mathcal{K}(X_2, X_{523})$ where four real normals are drawn from the Tarry point X_{98} to the curve.

3. Non-pivotal isocubics

Lemma 2. *Let M be a point and m its trilinear polar meeting the sidelines of ABC at U, V, W. The perpendiculars at A, B, C to the lines AU, BV, CW concur if and only if M lies on the Thomson cubic. The locus of the point of concurrence is the Darboux cubic.*

Let us now consider a non-pivotal isocubic n$\mathcal{K}$ with pole $\Omega = p : q : r$ and root[3] $P = u : v : w$. This cubic has equation :

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) + kxyz = 0.$$

Denote by $\mathrm{n}\mathcal{K}_0$ the corresponding cubic without xyz term, *i.e.*,

$$ux(ry^2 + qz^2) + vy(pz^2 + rx^2) + wz(qx^2 + py^2) = 0.$$

[3]An n$\mathcal{K}$ meets again the sidelines of triangle ABC at three collinear points U, V, W lying on the trilinear polar of the root.

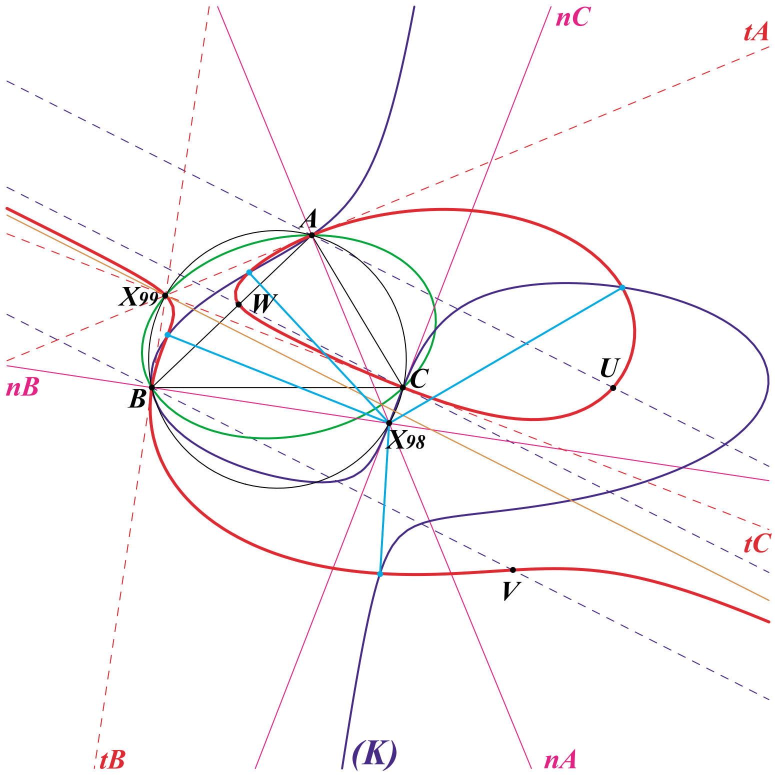

Figure 3. Theorem 1(2): Other normals to $\mathrm{p}\mathcal{K}$

It can easily be seen that the tangents $\mathtt{t}A$, $\mathtt{t}B$, $\mathtt{t}C$ do not depend of k and pass through the feet U', V', W' of the trilinear polar of P^* [4]. Hence it is enough to take the cubic $\mathsf{n}\mathcal{K}_0$ to study the normals at A, B, C.

Theorem 3. *The normals of* $\mathsf{n}\mathcal{K}_0$ *at* A, B, C *are concurrent if and only if*
(1) Ω *lies on the pivotal isocubic* $\mathrm{p}\mathcal{K}_1$ *with pole* $\Omega_1 = a^2u^2 : b^2v^2 : c^2w^2$ *and pivot* P, *or*
(2) P *lies on the pivotal isocubic* $\mathrm{p}\mathcal{K}_2$ *with pole* $\Omega_2 = \frac{p^2}{a^2} : \frac{q^2}{b^2} : \frac{r^2}{c^2}$ *and pivot* $P_2 = \frac{p}{a^2} : \frac{q}{b^2} : \frac{r}{c^2}$.

$\mathrm{p}\mathcal{K}_1$ is the $\mathrm{p}\mathcal{K}$ with pivot the root P of the $\mathsf{n}\mathcal{K}_0$ which is invariant in the isoconjugation which swaps P and the isogonal conjugate of the isotomic conjugate of P.

By Lemma 2, it is clear that $\mathrm{p}\mathcal{K}_2$ is the $\Omega-$isoconjugate of the Thomson cubic.

The following table gives a selection of such cubics $\mathrm{p}\mathcal{K}_2$. Each line of the table gives a selection of $\mathrm{n}\mathcal{K}_0(\Omega, X_i)$ with concurring normals at A, B, C.

[4] In other words, these tangents form a triangle perspective to ABC whose perspector is P^*. Its vertices are the harmonic associates of P^*.

Cubic	Ω	Ω_2	P_2	X_i on the curve for $i =$
$K034$	X_1	X_2	X_{75}	1, 2, 7, 8, 63, 75, 92, 280, 347, 1895
$K184$	X_2	X_{76}	X_{76}	2, 69, 75, 76, 85, 264, 312
$K099$	X_3	X_{394}	X_{69}	2, 3, 20, 63, 69, 77, 78, 271, 394
	X_4	X_{2052}	X_{264}	2, 4, 92, 253, 264, 273, 318, 342
	X_9	X_{346}	X_{312}	2, 8, 9, 78, 312, 318, 329, 346
	X_{25}	X_{2207}	X_4	4, 6, 19, 25, 33, 34, 64, 208, 393
$K175$	X_{31}	X_{32}	X_1	1, 6, 19, 31, 48, 55, 56, 204, 221, 2192
$K346$	X_{32}	X_{1501}	X_6	6, 25, 31, 32, 41, 184, 604, 2199
	X_{55}	X_{220}	X_8	1, 8, 9, 40, 55, 200, 219, 281
	X_{56}	X_{1407}	X_7	1, 7, 56, 57, 84, 222, 269, 278
	X_{57}	X_{279}	X_{85}	2, 7, 57, 77, 85, 189, 273, 279
	X_{58}	X_{593}	X_{86}	21, 27, 58, 81, 86, 285, 1014, 1790
	X_{75}	X_{1502}	X_{561}	75, 76, 304, 561, 1969

For example, all the isogonal $\mathrm{n}\mathcal{K}_0$ with concurring normals must have their root on the Thomson cubic. Similarly, all the isotomic $\mathrm{n}\mathcal{K}_0$ with concurring normals must have their root on $K184 = \mathrm{p}\mathcal{K}(X_{76}, X_{76})$.

Figure 4 shows $\mathrm{n}\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O. It is possible to draw from O six other (not necessarily all real) normals to the curve. The feet of these normals lie on another circum-cubic labeled (K) in the figure.

In the special case where the non-pivotal cubic is a singular cubic $\mathrm{c}\mathcal{K}$ with singularity F and root P, the normals at A, B, C concur at F if and only if F lies on the Darboux cubic. Furthermore, the locus of M whose orthopolar passes through F being also a nodal circumcubic with node F, there are two other points on $\mathrm{c}\mathcal{K}$ with normals passing through F. In Figure 5, $\mathrm{c}\mathcal{K}$ has singularity at O and its root is X_{394}. The corresponding nodal cubic passes through the points O, X_{25}, X_{1073}, X_{1384}, X_{1617}. The two other normals are labelled OR and OS.

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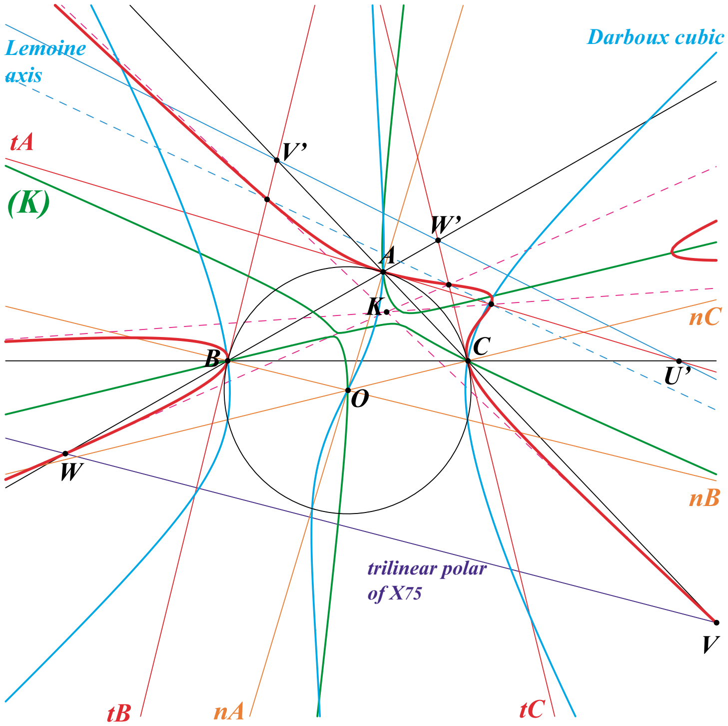

Figure 4. $\mathsf{n}\mathcal{K}_0(X_1, X_{75})$ with normals concurring at O

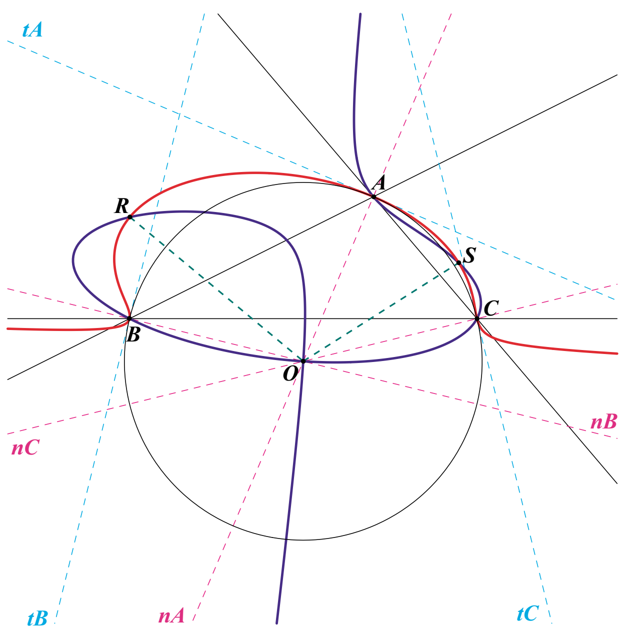

Figure 5. A $c\mathcal{K}$ with normals concurring at O

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A Characterization of the Centroid Using June Lester's Shape Function

Mowaffaq Hajja and Margarita Spirova

Abstract. The notion of triangle shape is used to give another proof of the fact that if P is a point inside triangle ABC and if the cevian triangle of P is similar to ABC in the natural order, then P is the centroid.

Identifying the Euclidean plane with the plane of complex numbers, we define a (non-degenerate) triangle to be any ordered triple (A, B, C) of distinct complex numbers, and we write it as ABC if no ambiguity may arise. According to this definition, there are in general six different triangles having the same set of vertices. We say that triangles ABC and $A'B'C'$ are *similar* if

$$\|A-B\| : \|A'-B'\| = \|B-C\| : \|B'-C'\| = \|C-A\| : \|C'-A'\|.$$

By the SAS similarity theorem and by the geometric interpretation of the quotient of two complex numbers, this is equivalent to the requirement that

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}.$$

June A. Lester called the quantity $\dfrac{A-B}{A-C}$ the *shape* of triangle ABC and she studied properties and applications of this shape function in great detail in [4], [5], and [6].

In this note, we use this shape function to prove that if P is a point inside triangle ABC, and if AA', BB', and CC' are the cevians through P, then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC. This has already appeared as Theorem 7 in [1], where three different proofs are given, and as a problem in the Problem Section of the *Mathematics Magazine* [2]. A generalization to d-simplices for all d is being considered in [3].

Our proof is an easy consequence of two lemmas that may prove useful in other contexts.

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Lemma 1. *Let ABC be a non-degenerate triangle, and let x, y, and z be real numbers such that*

$$x(A-B)^2 + y(B-C)^2 + z(C-A)^2 = 0. \tag{1}$$

Then either $x = y = z = 0$, or $xy + yz + zx > 0$.

Proof. Let $S = \frac{A-B}{A-C}$ be the shape of ABC. Dividing (1) by $(A-C)^2$, we obtain

$$(x+y)S^2 - 2yS + (y+z) = 0. \tag{2}$$

Since ABC is non-degenerate, S is not real. Thus if $x + y = 0$, then $y = 0$ and hence $x = y = z = 0$. Otherwise, $x + y \neq 0$ and the discriminant

$$4y^2 - 4(x+y)(y+z) = -(xy + yz + zx)$$

of (2) is negative, *i.e.*, $xy + yz + zx > 0$, as desired. □

Lemma 2. *Suppose that the cevians through an interior point P of a triangle divide the sides in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$. Then*
(i) $uvw \leq \frac{1}{8}$*, with equality if and only if $u = v = w = \frac{1}{2}$, i.e., if and only if P is the centroid.*
(ii) $\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \leq 0$*, with equality if and only if $u = v = w = \frac{1}{2}$,* i.e., *if and only if P is the centroid.*

Proof. Let $uvw = p$. Then using the cevian condition $uvw = (1-u)(1-v)(1-w)$, we see that

$$\begin{aligned} p &= \sqrt{u(1-u)}\sqrt{v(1-v)}\sqrt{w(1-w)} \\ &\leq \frac{u+(1-u)}{2}\,\frac{v+(1-v)}{2}\,\frac{w+(1-w)}{2}, \text{ by the AM-GM inequality} \\ &= \frac{1}{8}, \end{aligned}$$

with equality if and only if $u = \frac{1}{2}$, $v = \frac{1}{2}$, and $w = \frac{1}{2}$. This proves (i).

To prove (ii), note that

$$\begin{aligned} &\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) \\ &= (uv + vw + wu) - (u + v + w) + \frac{3}{4} \\ &= 2uvw - \frac{1}{4}, \end{aligned}$$

because $uvw = (1-u)(1-v)(1-w)$. Now use (i). □

We now use Lemmas 1 and 2 and the shape function to prove the main result.

Theorem 3. *Let AA', BB', and CC' be the cevians through an interior point P of triangle ABC. Then triangles ABC and $A'B'C'$ are similar if and only if P is the centroid of ABC.*

Proof. One direction being trivial, we assume that $A'B'C'$ and ABC are similar, and we prove that P is the centroid.

Suppose that the cevians AA', BB', and CC' through P divide the sides BC, CA, and AB in the ratios $u : 1-u$, $v : 1-v$, and $w : 1-w$, respectively. Since ABC and $A'B'C'$ are similar, it follows that they have equal shapes, *i.e.*,

$$\frac{A-B}{A-C} = \frac{A'-B'}{A'-C'}. \tag{3}$$

Substituting the values

$$A' = (1-u)B + uC, \quad B' = (1-v)C + vA, \quad C' = (1-w)A + wB$$

in (3) and simplifying, we obtain

$$\left(u - \frac{1}{2}\right)(A-B)^2 + \left(v - \frac{1}{2}\right)(B-C)^2 + \left(w - \frac{1}{2}\right)(C-A)^2 = 0.$$

By Lemma 1, either $u = v = w = \frac{1}{2}$, in which case P is the centroid, or

$$\left(u - \frac{1}{2}\right)\left(v - \frac{1}{2}\right) + \left(v - \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \left(w - \frac{1}{2}\right)\left(u - \frac{1}{2}\right) > 0,$$

in which case Lemma 2(ii) is contradicted. This completes the proof. $\square$

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The Locations of Triangle Centers

Christopher J. Bradley and Geoff C. Smith

Abstract. The orthocentroidal circle of a non-equilateral triangle has diameter GH where G is the centroid and H is the orthocenter. We show that the Fermat, Gergonne and symmedian points are confined to, and range freely over the interior disk punctured at its center. The Mittenpunkt is also confined to and ranges freely over another punctured disk, and the second Fermat point is confined to and ranges freely over the exterior of the orthocentroidal circle. We also show that the circumcenter, centroid and symmedian point determine the sides of the reference triangle ABC.

1. Introduction

All results concern non-equilateral non-degenerate triangles. The orthocentroidal circle $\mathcal{S}_{GH}$ has diameter GH, where G is the centroid and H is the orthocenter of triangle ABC. Euler showed [3] that O, G and I determine the sides a, b and c of triangle ABC. Here O denotes the circumcenter and I the incenter. Later Guinand [4] showed that I ranges freely over the open disk $\mathcal{D}_{GH}$ (the interior of $\mathcal{S}_{GH}$) punctured at the nine-point center N. This work involved showing that certain cubic equations have real roots. Recently Smith [9] showed that both results can be achieved in a straightforward way; that I can be anywhere in the punctured disk follows from Poncelet's porism, and a formula for IG^2 means that the position of I in $\mathcal{D}_{GH}$ enables one to write down a cubic polynomial which has the side lengths a, b and c as roots. As the triangle ABC varies, the Euler line may rotate and the distance GH may change. In order to say that I ranges freely over all points of this punctured open disk, it is helpful to rescale by insisting that the distance GH is constant; this can be readily achieved by dividing by the distance GH or OG as convenient. It is also helpful to imagine that the Euler line is fixed.

In this paper we are able to prove similar results for the symmedian (K), Fermat (F) and Gergonne (G_{e}) points, using the same disk $\mathcal{D}_{GH}$ but punctured at its midpoint J rather than at the nine-point center N. We show that, O, G and K determine a, b and c. The Morleys [8] showed that O, G and the first Fermat point F determine the reference triangle by using complex numbers. We are not able to show that O, G and G_{e} determine a, b and c, but we conjecture that they do.

Since I, G, S_{p} and N_{a} are collinear and spaced in the ratio $2:1:3$ it follows from Guinand's theorem [4] that the Spieker center and Nagel point are confined

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to, and range freely over, certain punctured open disks, and each in conjunction with O and G determines the triangle's sides. Since G_e, G and M are collinear and spaced in the ratio $2:1$ it follows that M ranges freely over the open disk on diameter OG with its midpoint deleted. Thus we now know how each of the first ten of Kimberling's triangle centers [6] can vary with respect to the scaled Euler line.

Additionally we observe that the orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle. We give an areal descriptions of the orthocentroidal circle. We show that the Feuerbach point must lie outside the circle $\mathcal{S}_{GH}$, a result foreshadowed by a recent internet announcement. This result, together with assertions that the symmedian and Gergonne points (and others) must lie in or outside the orthocentroidal disk were made in what amount to research announcements on the Yahoo message board Hyacinthos [5] on 27th and 29th November 2004 by M. R. Stevanovic, though his results do not yet seem to be in published form. Our results were found in March 2005 though we were unaware of Stevanovic's announcement at the time.

The two Brocard points enjoy the *Brocard exclusion principle*. If triangle ABC is not isosceles, exactly one of the Brocard points is in $\mathcal{D}_{GH}$. If it is isosceles, then both Brocard points lie on the circle $\mathcal{S}_{GH}$. This last result was also announced by Stevanovic.

The fact that the (first) Fermat point must lie in the punctured disk $\mathcal{D}_{GH}$ was established by Várilly [10] who wrote ... *this suggests that the neighborhood of the Euler line may harbor more secrets than was previously known*. We offer this article as a verification of this remark.

We realize that some of the formulas in the subsequent analysis are a little daunting, and we have had recourse to the use of the computer algebra system DERIVE from time to time. We have also empirically verified our geometric formulas by testing them with the CABRI geometry package; when algebraic formulas and geometric reality co-incide to 9 decimal places it gives confidence that the formulas are correct. We recommend this technique to anyone with reason to doubt the algebra.

We suggest [1], [2] and [7] for general geometric background.

2. The orthocentroidal disk

This is the interior of the circle on diameter GH and a point X lies in the disk if and only if $\angle GXH > \frac{\pi}{2}$. It will lie on the boundary if and only if $\angle GXH = \frac{\pi}{2}$. These conditions may be combined to give

$$\overline{XG} \cdot \overline{XH} \leq 0, \tag{1}$$

with equality if and only if X is on the boundary.

In what follows we initially use Cartesian vectors with origin at the circumcenter O, with $\overline{OA} = \mathbf{x}, \overline{OB} = \mathbf{y}, \overline{OC} = \mathbf{z}$ and, taking the circumcircle to have radius 1, we have

$$|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1 \tag{2}$$

and

$$\mathbf{y}\cdot\mathbf{z} = \cos 2A = \frac{a^4+b^4+c^4-2a^2(b^2+c^2)}{2b^2c^2} \tag{3}$$

with similar expressions for $\mathbf{z}\cdot\mathbf{x}$ and $\mathbf{x}\cdot\mathbf{y}$ by cyclic change of a, b and c. This follows from $\cos 2A = 2\cos^2 A - 1$ and the cosine rule.

We take X to have position vector

$$\frac{u\mathbf{x}+v\mathbf{y}+w\mathbf{z}}{u+v+w},$$

so that the unnormalised areal co-ordinates of X are simply (u,v,w). Now

$$3\overline{XG} = \frac{((v+w-2u),(w+u-2v),(u+v-2w))}{u+v+w},$$

not as areals, but as components in the $\mathbf{x},\mathbf{y},\mathbf{z}$ frame and

$$\overline{XH} = \frac{(v+w,w+u,u+v)}{u+v+w}.$$

Multiplying by $(u+v+w)^2$ we find that condition (1) becomes

$$\begin{aligned}&\sum_{\text{cyclic}}\{(v+w-2u)(v+w)\}\\ &+\sum_{\text{cyclic}}[(w+u-2v)(u+v)+(w+u)(u+v-2w)]\mathbf{y}\cdot\mathbf{z}\le 0,\end{aligned}$$

where we have used (2). The sum is taken over cyclic changes. Next, simplifying and using (3), we obtain

$$\begin{aligned}&\sum_{\text{cyclic}}2(u^2+v^2+w^2-vw-wu-uv)(a^2b^2c^2)\\ &+\sum_{\text{cyclic}}(u^2-v^2-w^2+vw)[a^2(a^4+b^4+c^4)-2a^4(b^2+c^2)].\end{aligned}$$

Dividing by $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$ the condition that $X(u,v,w)$ lies in the disk $\mathcal{D}_{GH}$ is

$$\begin{aligned}&(b^2+c^2-a^2)u^2+(c^2+a^2-b^2)v^2+(a^2+b^2-c^2)w^2\\ &-a^2vw-b^2wu-c^2uv<0\end{aligned} \tag{4}$$

and the equation of the circular boundary is

$$\begin{aligned}S_{GH}\equiv&(b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2\\ &-a^2yz-b^2zx-c^2xy=0.\end{aligned} \tag{5}$$

The polar circle has equation

$$S_P \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2=0.$$

The circumcircle has equation

$$S_C \equiv a^2yz+b^2zx+c^2xy=0.$$

The nine-point circle has equation

$$S_N \equiv (b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2 \\ -2a^2yz-2b^2zx-2c^2xy=0.$$

Evidently $S_{GH} - S_C = S_P$ and $S_N + 2S_C = S_P$. We have established the following result.

Theorem 1. *The orthocentroidal circle forms part of a coaxal system of circles including the circumcircle, the nine-point circle and the polar circle of the triangle.*

It is possible to prove the next result by calculating that $JK < OG$ directly (recall that J is the midpoint of GH), but it is easier to use the equation of the orthocentroidal circle.

Theorem 2. *The symmedian point lies in the disc $\mathcal{D}_{GH}$.*

Proof. Substituting $u = a^2, v = b^2, w = c^2$ in the left hand side of equation (4) we get $a^4b^2 + b^4c^2 + c^4a^2 + b^4a^2 + c^4b^2 + a^4c^2 - 3a^2b^2c^2 - a^6 - b^6 - c^6$ and this quantity is negative for all real a, b, c except $a = b = c$. This follows from the well known inequality for non-negative l, m and n that

$$l^3 + m^3 + n^3 + 3lmn \geq \sum_{sym} l^2m$$

with equality if and only if $l = m = n$. □

We offer a second proof. The line AK with areal equation $c^2y = b^2z$ meets the circumcircle of ABC at D with co-ordinates $(-a^2, 2b^2, 2c^2)$, with similar expressions for points E and F by cyclic change. The reflection D' of D in BC has co-ordinates $(a^2, b^2+c^2-a^2, b^2+c^2-a^2)$ with similar expressions for E' and F'. It is easy to verify that these points lie on the orthocentroidal disk by substituting in (5) (the circle through D', E' and F' is the Hagge circle of K).

Let $\mathbf{d}', \mathbf{e}'$ and $\mathbf{f}'$ denote the vector positions D', E' and F' respectively. It is clear that

$$\mathbf{s} = (2b^2+2c^2-a^2)\mathbf{d}' + (2c^2+2a^2-b^2)\mathbf{e}' + (2a^2+2b^2-c^2)\mathbf{f}'$$

but $2b^2 + 2c^2 - a^2 = b^2 + c^2 + 2bc\cos A \geq (b-c)^2 > 0$ and similar results by cyclic change. Hence relative to triangle $D'E'F'$ all three areal co-ordinates of K are positive so K is in the interior of triangle $D'E'F'$ and hence inside its circumcircle. We are done.

The incenter lies in $\mathcal{D}_{GH}$. Since IGN_{a} are collinear and $IG : GN_{\mathrm{a}} = 1 : 2$ it follows that Nagel's point is outside the disk. However, it is instructive to verify these facts by substituting relevant areal co-ordinates into equation (5), and we invite the interested reader to do so.

Theorem 3. *One Brocard point lies in $\mathcal{D}_{GH}$ and the other lies outside $\mathcal{S}_{GH}$, or they both lie simultaneously on $\mathcal{S}_{GH}$* (which happens if and only if the reference triangle is isosceles).

Proof. Let $f(u,v,w)$ denote the left hand side of equation (4). One Brocard point has unnormalised areal co-ordinates

$$(u,v,w)=(a^2b^2,b^2c^2,c^2a^2)$$

and the other has unnormalised areal co-ordinates

$$(p,q,r)=(a^2c^2,b^2a^2,c^2b^2),$$

but they have the same denominator when normalised. It follows that $f(u,v,w)$ and $f(p,q,r)$ are proportional to the powers of the Brocard points with respect to $\mathcal{S}_{GH}$ with the same constant of proportionality. If the sum of these powers is zero we shall have established the result. This is precisely what happens when the calculation is made. $\square$

The fact that the Fermat point lies in the orthocentroidal disk was established recently [10] by Várilly.

Theorem 4. *Gergonne's point lies in the orthocentroidal disk $\mathcal{D}_{GH}$.*

Proof. Put $u=(c+a-b)(a+b-c)$, $v=(a+b-c)(b+c-a)$, $w=(b+c-a)(c+a-b)$ and the left hand side of (5) becomes

$$-18a^2b^2c^2+\sum_{\text{cyclic}}\left(-a^5(b+c)+4a^4(b^2-bc+b^2)-6b^3c^3+5a^3(b^2c+bc^2)^2\right)$$

which we want to show is negative. This is not immediately recognisable as a known inequality, but performing the usual trick of putting $a=m+n$, $b=n+l$, $c=l+m$ where $l,m,n>0$ we get the required inequality (after division by 8) to be

$$2(m^3n^3+n^3l^3+l^3m^3)>lmn\left(\sum_{sym}m^2n\right)$$

where the final sum is over all possible permutations and l,m,n not all equal. Now $l^3(m^3+n^3)>l^3(m^2n+mn^2)$ and adding two similar inequalities we are done. Equality holds if and only if $a=b=c$, which is excluded. $\square$

3. The determination of the triangle sides.

3.1. *The symmedian point.* We will find a cubic polynomial which has roots a^2,b^2,c^2 given the positions of O,G and K.

The idea is to express the formulas for OK^2, GK^2 and JK^2 in terms of $u=a^2+b^2+c^2$, $v^2=a^2b^2+b^2c^2+c^2a^2$ and $w^3=a^2b^2c^2$.

We first note some equations which are the result of routine calculations.

$$16[ABC]^2=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$
$$=\sum_{\text{cyclic}}(2a^2b^2-a^4)=4v^2-u^2.$$

It is well known that the circumradius R satisfies the equation $R=\frac{abc}{4[ABC]}$ so

$$R^2 = \frac{a^2b^2c^2}{16[ABC]^2} = \frac{w^3}{(4v^2-u^2)}.$$

$$OG^2 = \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2\right)\right] R^2$$

$$= \frac{u^3+9w^3-4uv^2}{9(4v^2-u^2)} = \frac{w^3}{(4v^2-u^2)} - \frac{u}{9} = R^2 - \frac{a^2+b^2+c^2}{9}.$$

By areal calculations one may obtain the formulas

$$OK^2 = \frac{4R^2\sum_{\text{cyclic}}\left(a^4-a^2b^2\right)}{(a^2+b^2+c^2)^2} = \frac{4w^3(u^2-3v^2)}{u^2(4v^2-u^2)},$$

$$GK^2 = \frac{\left(\sum_{\text{cyclic}} 3a^4(b^2+c^2)\right) - 15a^2b^2c^2 - \left(\sum_{\text{cyclic}} a^6\right)}{(a^2+b^2+c^2)^2} = \frac{6uv^2-u^3-27w^3}{9u^2},$$

$$JK^2 = OG^2\left(1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2}\right) = \frac{4(u^3+9w^3-4uv^2)(u^2-3v^2)}{9u^2(4v^2-u^2)}.$$

The full details of the last calculation will be given when justifying (14).

Note that

$$\frac{OK^2}{JK^2} = \frac{9w^3}{(u^3+9w^3-4uv^2)}$$

or

$$\frac{JK^2}{OK^2} = 1 - \frac{u(4v^2-u^2)}{9w^3}.$$

We simplify expressions by putting $u = p$, $4v^2 - u^2 = q$ and $w^3 = r$. We have

$$OG^2 = \frac{r}{q} - \frac{p}{9}. \tag{6}$$

Now $u^2 - 3v^2 = -\frac{3}{4}(4v^2-u^2) + \frac{1}{4}u^2 = -\frac{3}{4}q + \frac{1}{4}p^2$ so

$$OK^2 = 4r\frac{(\frac{1}{4}p^2 - \frac{3}{4}q)}{p^2q} = \frac{(p^2-3q)r}{p^2q} = \frac{r}{q} - \frac{3r}{p^2} = r\left(\frac{1}{q} - \frac{3}{p^2}\right) \tag{7}$$

Also $6v^2 - u^2 = \frac{3}{2}(4v^2-u^2) + \frac{1}{2}u^2 = \frac{3q}{2} + \frac{p^2}{2}$

$$GK^2 = \frac{p(3q/2 + p^2/2) - 27r}{9p^2} = \frac{p}{18} + \frac{q}{6p} - \frac{3r}{p^2} \tag{8}$$

$$\frac{OK^2}{JK^2} = 1 - \frac{pq}{9r} \tag{9}$$

We now have four quantities that are homogeneous of degree 1 in a^2, b^2 and c^2. These are $p, q/p, r/q, r/p^2 = x, y, z, s$ respectively, where $xs = r/p = yz$. We have (6) $OG^2 = z - x/9$, (7) $OK^2 = z - 3s$, (8) $GK^2 = x + 6y - 3s$ and (9) $\frac{OK^2}{JK^2} = 1 - x/(9z)$ or $9zOK^2 = (9z-x)JK^2$. Now u, v and w are known unambiguously and hence the equations determine a^2, b^2 and c^2 and therefore a, b and c.

3.2. *The Fermat point.* We assume that O, G and the (first) Fermat point are given. Then F determines and is determined by the second Fermat point F' since they are inverse in $\mathcal{S}_{GH}$. In pages 206-208 [8] the Morleys show that O, G and F determine triangle ABC using complex numbers.

4. Filling the disk

Following [9], we fix R and r, and consider the configuration of Poncelet's porism for triangles. This diagram contains a fixed circumcircle, a fixed incircle, and a variable triangle ABC which has the given circumcircle and incircle. Moving point A towards the original point B by sliding it round the circumcircle takes us continuously through a family of triangles which are pairwise not directly similar (by angle and orientation considerations) until A reaches the original B, when the starting triangle is recovered, save that its vertices have been relabelled. Moving through triangles by sliding A to B in this fashion we call a *Poncelet cycle*.

We will show shortly that for X the Fermat, Gergonne or symmedian point, passage through a Poncelet cycle takes X round a closed path arbitrarily close to the boundary of the orthocentroidal disk scaled to have constant diameter. By choosing the neighbourhood of the boundary sufficiently small, it follows that X has winding number 1 (with suitable orientation) with respect to J as we move through a Poncelet cycle.

We will show that when r approaches $R/2$ (as we approach the equilateral configuration) a Poncelet cycle will keep X arbitrarily close to, but never reaching, J in the scaled orthocentroidal disk. Moving the ratio r/R from close to 0 to close to $1/2$ induces a homotopy between the 'large' and 'small' closed paths. So the small path also has winding number 1 with respect to J. One might think it obvious that every point in the scaled punctured disk must arise as a possible X on a closed path intermediate between a path sufficiently close to the edge and a path sufficiently close to the deletion. There are technical difficulties for those who seek them, since we have not eliminated the possibility of exotic paths. However, a rigorous argument is available via complex analysis. Embed the scaled disk in the complex plane. Let γ be an anticlockwise path (i.e. winding number $+1$) near the boundary and δ be an anticlockwise path (winding number also 1) close to the puncture. Suppose (for contradiction) that the complex number z_0 represents a point between the wide path γ and the tight path δ which is not a possible location for X.

The function defined by $1/(z - z_0)$ is meromorphic in $\mathcal{D}_{GH}$ and is analytic save for a simple pole at z_0. However by our hypothesis we have a homotopy of paths from γ to δ which does not involve z_0 being on an intermediate path. Therefore

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0} = \frac{1}{2\pi i}\int_\delta \frac{dz}{z - z_0} = 0.$$

Thus $1 = 0$ and we have the required contradiction.

5. Close to the edge

The areal coordinates of the incenter and the symmedian point of triangle ABC are (a, b, c) and (a^2, b^2, c^2) respectively. We consider the mean square distance of the vertex set to itself, weighted once by (a, b, c) and once by (a^2, b^2, c^2). Note that $a, b, c > 0$ so $\sigma_I^2, \sigma_S^2 > 0$. The GPAT [9] asserts that

$$\sigma_I^2 + KI^2 + \sigma_K^2 = 2\frac{abc}{a+b+c}\frac{ab+bc+ca}{a^2+b^2+c^2} \leq 4Rr.$$

It follows that $SI < 2\sqrt{Rr}$.

In what follows we fix R and investigate what we can achieve by choosing r to be sufficiently small.

Since I lies in the critical disk we have $OH > OI$ so

$$OH^2 > OI^2 = R^2 - 2Rr.$$

By choosing $r < R/8$ say, we force $9GJ^2 = OH^2 > 3R^2/4$ so $GJ > R\sqrt{3}/6$. Now we have

$$\frac{KI}{GJ} < \frac{2\sqrt{Rr}}{R\sqrt{3}/6} = 4\sqrt{\frac{3r}{R}}. \tag{10}$$

For any $\varepsilon > 0$, there is $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{KI}{GJ} < \frac{\varepsilon}{2}$. Observe that we are dividing by GJ to scale the orthocentroidal disk so that it has fixed radius.

Recall that a passage round a Poncelet cycle induces a path for I in the scaled critical disk which is a circle of Apollonius with defining points O and N with ratio $IO : IN = 2\sqrt{\frac{R}{R-2r}}$. It is clear from the theory of Apollonius circles that there is $K_2 > 0$ such that if $0 < r < K_2$, then $1 - \frac{IJ}{GJ} < \varepsilon/2$.

Now choosing r such that $0 < r < \min\{R/8, K_1, K_2\}$ we have

$$1 - \frac{IJ}{GJ} < \frac{\varepsilon}{2} \tag{11}$$

and $SI/GJ < \frac{\varepsilon}{2}$ so by the triangle inequality

$$\frac{IJ}{GJ} < \frac{KJ}{GJ} + \frac{\varepsilon}{2}. \tag{12}$$

Adding equations (11) and (12) and rearranging we deduce that

$$1 - \varepsilon < \frac{KJ}{GJ}.$$

This shows that for sufficiently small r, the path of K in the scaled critical disk (as the triangle moves through a Poncelet cycle) will be confined to a region at most ε from the boundary. Moreover, assuming that $\varepsilon < 1$ the winding number of K about J will increase by 1, because that it is what happens to I, and J moves in proximity to I.

A similar result holds for the Gergonne point G_e. This is the intersection of the Cevians joining triangle vertices to the opposite contact point of the incircle. The Gergonne point must therefore be inside the incircle.

As before we consider the case that R is fixed. Now $G_{\mathrm{e}}I < r$. We proceed as in the argument for the symmedian point. We get a new version of equation (10) which is

$$\frac{G_{\mathrm{e}}I}{GJ} < \frac{r}{R\sqrt{3}/6} = \frac{2r\sqrt{3}}{R} \tag{13}$$

For any $\varepsilon > 0$, there is a possibly new $K_1 > 0$ so that if $0 < r < K_1$, then $\frac{G_{\mathrm{e}}I}{GJ} < \frac{\varepsilon}{2}$. The rest of the argument proceeds unchanged.

6. Near the orthocentroidal center

6.1. *The symmedian point.* For the purposes of the following calculation only, we will normalize so that $R = 1$. We have

$$\overline{OK} = \frac{a^2\mathbf{x} + b^2\mathbf{y} + c^2\mathbf{z}}{a^2 + b^2 + c^2}$$

so

$$\begin{aligned}\overline{KJ} =& \sum_{\text{cyclic}} \left(\frac{2}{3} - \frac{a^2}{a^2+b^2+c^2}\right)\mathbf{x} = \frac{\sum_{\text{cyclic}}(2b^2+2c^2-a^2)\mathbf{x}}{3(a^2+b^2+c^2)}\\ =& l\mathbf{x} + m\mathbf{y} + n\mathbf{z}\end{aligned}$$

where l, m and n can be read off.

We have

$$\begin{aligned}a^2b^2c^2KJ^2 =& a^2b^2c^2\left(l^2 + m^2 + n^2 + \sum_{\text{cyclic}} 2mn\mathbf{y}\cdot\mathbf{z}\right)\\ =& (l^2+m^2+n^2)(a^2b^2c^2) + \sum_{\text{cyclic}} mn(a^2(a^4+b^4+c^4) - 2a^4(b^2+c^2))\\ =& \frac{4P_{10}}{9(a^2+b^2+c^2)^2}\end{aligned}$$

where

$$P_{10} = \sum_{\text{cyclic}} a^{10} - 2a^8(b^2+c^2) + a^6(b^4+4b^2c^2+c^4) - 3a^4b^4c^2.$$

Now

$$OG^2 = \frac{1}{9a^2b^2c^2}\left[\left(\sum_{\text{cyclic}} a^6\right) + 3a^2b^2c^2 - \left(\sum_{sym} a^4b^2)\right)\right]$$

so we define Q_6 by

$$OG^2 = \frac{Q_6}{9a^2b^2c^2}.$$

We have $9a^2b^2c^2OG^2 = Q_6$ and

$$9a^2b^2c^2KJ^2 = \frac{4P_{10}}{(a^2+b^2+c^2)^2}$$

so that

$$\frac{OG^2}{KJ^2} = \frac{Q_6(a^2+b^2+c^2)^2}{4P_{10}}.$$

Now a computer algebra (DERIVE) aided calculation reveals that

$$\frac{Q_6(a^2+b^2+c^2) - 4P_{10}}{3(a+b-c)(b+c-a)(c+a-b)(a+b+c)} = Q_6$$

It follows that

$$\frac{KJ^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2}. \tag{14}$$

Our convenient simplification that $R = 1$ can now be dropped, since the ratio on the left hand side of (14) is dimensionless. As the triangle approaches the equilateral, KJ/OG approaches 0. Therefore in the orthocentroidal disk scaled to have diameter 1, the symmedian point approaches the center J of the circle.

6.2. *The Gergonne point.* Fix the circumcircle of a variable triangle ABC. We consider that the case that r approaches $R/2$, so the triangle ABC approaches (but does not reach) the equilateral. Drop a perpendicular ID to BC.

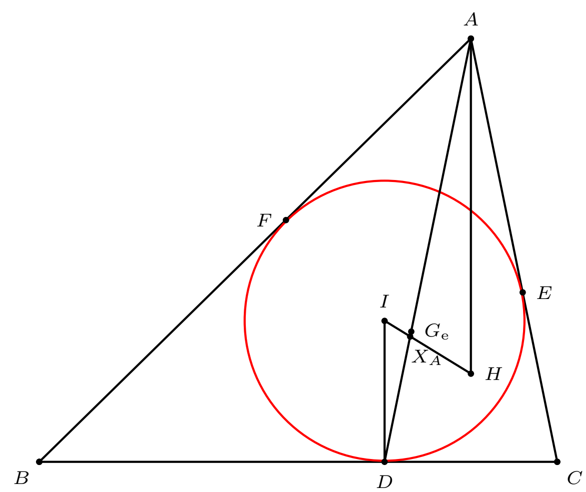

Figure 1

Let AD meet IH at X_A. Triangles IDX_A and HAX_A are similar. When r approaches $R/2$, H approaches O so HA approaches $OA = R$. It follows that $IX_A : X_AH$ approaches $1 : 2$. Similar results hold for corresponding points X_B and X_C.

If we rescale so that points O, G and H are fixed, the points X_A, X_B and X_C all converge to a point X on IH such that $IX : XH = 1 : 2$. Consider the three rays AX_A, BX_B and CX_C which meet at the Gergonne point of the triangle. As ABC approaches the equilateral, these three rays become more and more like the diagonals of a regular hexagon. In particular, if the points X_A, X_B and X_C arise

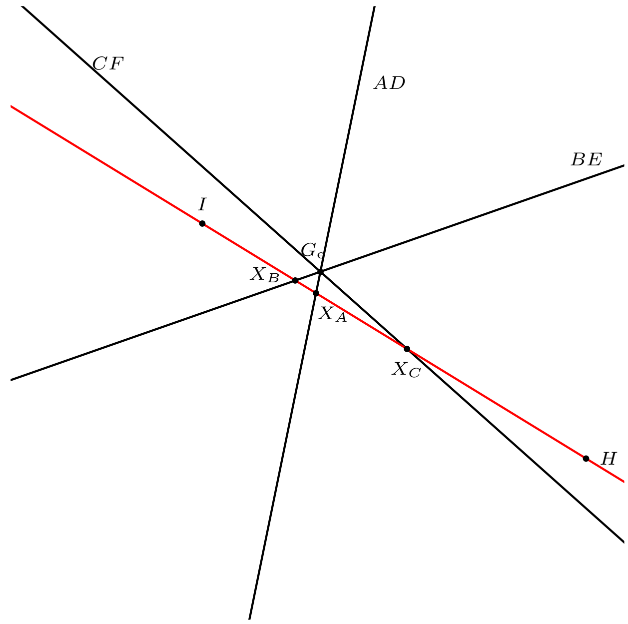

Figure 2

(without loss of generality) in that order on the directed line IH, then $\angle X_A G_e X_C$ is approaching $2\pi/3$, and so we may take this angle to be obtuse. Therefore G_e is inside the circle on diameter $X_A X_C$. Thus in the scaled diagram G_e approaches X, but I approaches N, so G_e approaches the point J which divides NH internally in the ratio $1 : 2$. Thus G_e converges to J, the center of the orthocentroidal circle.

Thus the symmedian and Gergonne points are confined to the orthocentroidal disk, make tight loops around its center J, as well as wide passages arbitrarily near its boundary (as moons of I). Neither of them can be at J for non-equilateral triangles (an easy exercise). By continuity we have proved the following result.

Theorem 5. *Each of the Gergonne and symmedian points are confined to, and range freely over the orthocentroidal disk punctured at its center.*

6.3. *The Fermat point.* An analysis of areal co-ordinates reveals that the Fermat point F lies on the line JK between J and K, and

$$\frac{JF}{FK} = \frac{a^2 + b^2 + c^2}{4\sqrt{3}[ABC]}. \tag{15}$$

From this it follows that as we approach the equilateral limit, F approaches the midpoint of JK. However we have shown that K approaches J in the scaled diagram, so F approaches J.

As r approaches 0 with R fixed, the area $[ABC]$ approaches 0, so in the scaled diagram F can be made arbitrarily close to K (uniformly). It follows that F performs closed paths arbitrarily close to the boundary.

Here is an outline of the areal algebra. The first normalized areal co-ordinate of H is
$$\frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}$$
and the other co-ordinates are obtained by cyclic changes. We suppress this remark in the rest of our explanation. Since J is the midpoint of GH the first areal co-ordinate of J is
$$\frac{a^4-2b^4-2c^4+b^2c^2+c^2a^2+a^2c^2}{3(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$
One can now calculate the areal equation of the line JK as
$$\sum_{\text{cyclic}}(b^2-c^2)(a^2-b^2-bc-c^2)(a^2-b^2+bc-c^2)x=0$$

To calculate the areal co-ordinates of F we first assume that the reference triangle has each angle less that $2\pi/3$. In this case the rays AF, BF and CF meet at equal angles, and one can use trigonometry to obtain a formula for the areal co-ordinates which, when expressed in terms of the reference triangle sides, turns out to be correct for arbitrary triangles. One can either invoke the charlatan's *principle of permanence of algebraic form*, or analyze what happens when a reference angle exceeds $2\pi/3$. In the latter event, the trigonometry involves a sign change dependent on the region in which F lies, but the final formula for the co-ordinates remains unchanged. In such a case, of course, not all of the areal co-ordinates are positive.

The unnormalized first areal co-ordinate of F turns out to be
$$8\sqrt{3}a^2[ABC]+2a^4-4b^4-4c^4+2a^2b^2+2a^2c^2+8b^2c^2.$$

The first areal component K_x of K is
$$\frac{a^2}{a^2+b^2+c^2}$$
and the first component J_x of J is
$$\frac{a^4-2b^4-2c^4+a^2b^2+a^2c^2+4b^2c^2}{48[ABC]^2}.$$
Hence the first co-ordinate of F is proportional to
$$8\sqrt{3}K_x[ABC](a^2+b^2+c^2)+96J_x[ABC]^2.$$
This is linear in J_x and K_x, and the other co-ordinates are obtained by cyclic change. It follows that J, F, K are collinear (as is well known) but also that by the section theorem, JF/FK is given by (15).

The Fermat point cannot be at J in a non-equilateral triangle because the second Fermat point is inverse to the first in the orthocentroidal circle.

We have therefore established the following result.

Theorem 6. *Fermat's point is confined to, and ranges freely over the orthocentroidal disk punctured at its center and the second Fermat point ranges freely over the region external to* $\mathcal{S}_{GH}$.

7. The Feuerbach point

Let F_{e} denote the Feuerbach point.

Theorem 7. *The point F_{e} is always outside the orthocentroidal circle.*

Proof. Let J be the center of the orthocentroidal circle, and N be its nine-point center. In [9] it was established that

$$IJ^2 = OG^2 - \frac{2r}{3}(R-2r).$$

We have $IN = R/2 - r$, $IF_{\mathrm{e}} = r$ and $JN = OG/2$. We may apply Stewart's theorem to triangle $JF_{\mathrm{e}}N$ with Cevian JI to obtain

$$JF_{\mathrm{e}}^2 = OG^2\left(\frac{2R-r}{2R-4r}\right) - \frac{rR}{6}. \tag{16}$$

This leaves the issue in doubt so we press on.

$$JF_{\mathrm{e}}^2 = OG^2 + OG^2\left(\frac{3r}{2R-4r}\right) - \frac{rR}{6}.$$

Now I must lie in the orthocentroidal disk so $IO/3 < OG$ and therefore

$$JF_{\mathrm{e}}^2 > OG^2 + \frac{3rR(R-2r)}{18(R-2r)} - \frac{rR}{6} = OG^2.$$

□

Corollary 8. *The positions of I and F_{e} reveal that the interior of the incircle intersects both $\mathcal{D}_{GH}$ and the region external to $\mathcal{S}_{GH}$ non-trivially.*

Acknowledgements. We thank J. F. Toland for an illuminating conversation about the use of complex analysis in homotopy arguments.

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The Locations of the Brocard Points

Christopher J. Bradley and Geoff C. Smith

Abstract. We fix and scale to constant size the orthocentroidal disk of a variable non-equilateral triangle ABC. We show that the set of points of the plane which can be either type of Brocard point consists of the interior of the orthocentroidal disk. We give the locus of points which can arise as a Brocard point of specified type, and describe this region and its boundary in polar terms. We show that ABC is determined by the locations of the circumcenter, the centroid and the Brocard points. In some circumstances the location of one Brocard point will suffice.

1. Introduction

For geometric background we refer the reader to [1], [3] and [4]. In [2] and [5] we demonstrated that scaling the orthocentroidal circle (on diameter GH) to have fixed diameter and studying where other major triangle centers can lie relative to this circle is a fruitful exercise. We now address the Brocard points. We consider non-equilateral triangles ABC. We will have occasion to use polar co-ordinates with origin the circumcenter O. We use the Euler line as the reference ray, with OG of length 1. We will describe points, curves and regions by means of polar co-ordinates (r, θ). To fix ideas, the equation of the orthocentroidal circle with center $J = (2, 0)$ and radius 1 is

$$r^2 - 4r\cos\theta + 3 = 0. \tag{1}$$

This circle is enclosed by the curve defined by

$$r^2 - 2r(\cos\theta + 1) + 3 = 0. \tag{2}$$

Let Γ_1 denote the region enclosed by the closed curve defined by (2) for $\theta > 0$ and (1) for $\theta \le 0$. Include the boundary when using (2) but exclude it when using (1). Delete the unique point Z in the interior which renders GJZ equilateral. (See Figure 1). Let Γ_2 be the reflection of Γ_1 in the Euler line. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ so Γ consists of the set of points inside or on (2) for any θ, save that G and H are removed from the boundary and two points are deleted from the interior (the points Z such that GJZ is equilateral). It is easy to verify that if the points $(\sqrt{3}, \pm\pi/6)$ are restored to Γ, then it becomes convex, as do each of Γ_1 and Γ_2 if their deleted points are filled in.

2. The main theorem

Theorem. (a) *One Brocard point ranges freely over, and is confined to,* Γ_1*, and the other ranges freely over, and is confined to,* Γ_2*.*
(b) *The set of points which can be occupied by a Brocard point is* Γ*.*

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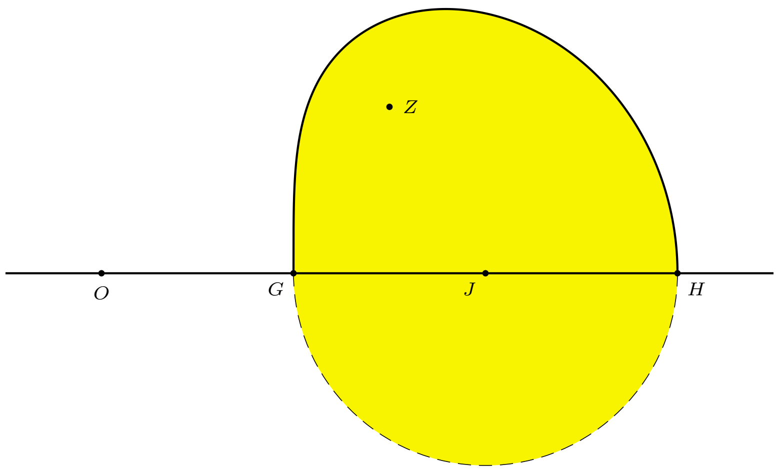

Figure 1

(c) *The points which can be inhabited by either Brocard point form the open orthocentroidal disk.*

(d) *The data consisting of O, G and one of the following items determines the sides of triangle ABC and which of the (generically) two possible orientations it has.*

(1) *the locations of the Brocard points without specifying which is which*;

(2) *the location of one Brocard point of specified type provided that it lies in the orthocentroidal disk*;

(3) *the location of one Brocard point of unspecified type outside the closed orthocentroidal disk together with the information that the other Brocard point lies on the same side or the other side of the Euler line*;

(4) *the location of a Brocard point of unspecified type on the orthocentroidal circle*.

Proof. First we gather some useful information. In [2] we established that

$$\frac{JK^2}{OG^2} = 1 - \frac{48[ABC]^2}{(a^2+b^2+c^2)^2} \tag{3}$$

where $[ABC]$ denotes the area of this triangle. It is well known [6] that

$$\cot\omega = \frac{(a^2+b^2+c^2)}{4[ABC]} \tag{4}$$

so

$$\frac{JK^2}{OG^2} = 1 - 3\tan^2\omega. \tag{5}$$

The sum of the powers of Ω and Ω' with respect to the orthocentroidal circle (with diameter GH) is 0. This result can be obtained by substituting the areal co-ordinates of these points into the areal equation of this circle [2].

$$J\Omega^2 + J\Omega'^2 = 2OG^2. \tag{6}$$

We can immediately conclude that $J\Omega$, $J\Omega' \leq OG\sqrt{2}$.

Now start to address the loci of the Brocard points. Observe that if we specify the locations of O, G and the symmedian point K (at a point within the orthocentroidal circle), then a triangle exists which gives rise to this configuration and its sides are determined [2]. In the subsequent discussion the points O and G will be fixed, and we will be able to conjure up triangles ABC with convenient properties by specifying the location of K. The angle α is just the directed angle $\angle KOG$ and ω can be read off from $JK^2/OG^2 = 1 - 3\tan\omega^2$.

We work with a non-equilateral triangle ABC. We adopt the convention, which seems to have majority support, that when standing at O and viewing K, the point Ω is diagonally to the left and Ω' diagonally to the right.

The Brocard or seven-point circle has diameter OK where K is the symmedian point, and the Brocard points are on this circle, and are mutual refections in the Brocard axis OK. It is well known that the Brocard angle manifests itself as

$$\omega = \angle KO\Omega = \angle KO\Omega'. \tag{7}$$

As the non-equilateral triangle ABC varies, we scale distances so that the length OG is 1 and we rotate as necessary so that the reference ray OG points in a fixed direction. Now let K be at an arbitrary point of the orthocentroidal disk with J deleted. Let $\angle KOJ = \alpha$, so $\angle JO\Omega' = \omega - \alpha$. Viewed as a directed angle the argument of Y in polar terms would be $\angle\Omega' OJ = \alpha - \omega$.

The positions of the Brocard points are determined by (5), (7) and the fact that they are on the Brocard circle. Let $r = O\Omega = O\Omega'$. By the cosine rule

$$J\Omega^2 = 4 + r^2 - 4r\cos(\omega + \alpha) \tag{8}$$

and

$$J\Omega'^2 = 4 + r^2 - 4r\cos(\omega - \alpha) \tag{9}$$

Now add equations (8) and (9) and use (6) so that

$$2OG^2 = 8 + 2r^2 - 8r\cos\omega\cos\alpha.$$

Recalling that the length OG is 1 we obtain

$$r^2 - 4r\cos\omega\cos\alpha + 3 = 0. \tag{10}$$

We focus on the Brocard point Ω with polar co-ordinates $(r, \alpha + \omega)$. The other Brocard point Ω' has co-ordinates $(r, \alpha - \omega)$, but reflection symmetry in the Euler line means that we need not study the region inhabited by Ω' separately.

Consider the possible locations of Ω for a specified $\alpha + \omega$. From (10) we see that its distance from the origin ranges over the interval

$$2\cos\omega\cos\alpha \pm \sqrt{4\cos^2\omega\cos^2\alpha - 3}.$$

This can be written

$$\cos(\alpha + \omega) + \cos(\alpha - \omega) \pm \sqrt{((\cos(\alpha + \omega) + \cos(\alpha - \omega))^2 - 3}. \tag{11}$$

Next suppose that $\alpha > 0$. This expression (11) is maximized when $\alpha = \omega$ and we use the plus sign. Since the product of the roots is 3, we see that the minimum distance also occurs when $\alpha = \omega$ and we use the minus sign.

Let $\theta = \angle\Omega OG$ for Ω on the boundary of the region under discussion, so $\theta = \omega + \alpha = 2\alpha = 2\omega$. Thus locus of the boundary when $\theta > 0$ is given by (10). Using standard trigonometric relations this transforms to

$$r^2 - 2r(\cos\theta + 1) + 3 = 0$$

for $\theta > 0$. This is equation (2). Note that it follows that Ω is on the boundary precisely when Ω' is on the Euler line because the argument of Ω' is $\alpha - \omega$.

Next suppose that $\alpha = 0$. Note that K cannot occupy J. When K is on the Euler line (so ABC is isosceles) equation (10) ensures that Ω is on the orthocentroidal circle. Also $0 < \omega < \pi/6$. Thus the points $(1, 0)$, $(\sqrt{3}, \pi/6)$ and $(3, 0)$ do not arise as possible locations for Ω. The endpoints of the GH interval are on the edge of our region. The more interesting exclusion is that of $(\sqrt{3}, \pi/6)$. We say that this is a *forbidden point* of Ω.

Now suppose that $\alpha < 0$. This time equation (11) tells another story. The expression is maximized (and minimized as before) when $\omega = 0$ which is illegal. In this region the boundary is not attained, and the point Ω' free to range on the axis side of the curve defined by

$$r^2 - 4r\cos\theta + 3 = 0$$

for $\theta = \alpha < 0$. Notice that this is the equation of the boundary of the orthocentroidal circle (1). The reflection of this last curve gives the unattained boundary of Ω' when $\theta > 0$, but it is easy to check that $r^2 - 2(\cos\theta + 1)r + 3 = 0$ encloses the relevant orthocentroidal semicircle and so is the envelope of the places which may be occupied by at least one Brocard point.

Moreover, our construction ensures that every point in Γ_1 arises a possible location for Ω.

We have proved (a). Then (b) follows by symmetry, and (c) is a formality.

Finally we address (d). Suppose that we are given O, G and a Brocard point X. Brocard points come in two flavours. If X is outside the open orthocentroidal disk on the side $\theta > 0$, then the Brocard point must be Ω, and if $\theta < 0$, then it must be Ω'. If X is in the disk, we need to be told which it is. Suppose without loss of generality we know the Brocard point is Ω. We know $J\Omega$ so from (6) we know $J\Omega'$. We also know $O\Omega' = O\Omega$, so by intersecting two circles we determine two candidates for the location of Ω'. Now, if Ω is outside the closed orthocentroidal disk then the two candidates for the location of Ω' are both inside the open orthocentroidal disk and we are stuck unless we know on which side of the Euler line Ω' can be found. If Ω is on the orthocentroidal circle then ABC is isosceles and the position of Ω' is known. If Ω is in the open orthocentroidal disk then only one of the two candidate positions for Ω' lies inside the set of points over which Ω' may range so the location of Ω' is determined. Now the Brocard circle and O are determined, so the antipodal point K to O is known. However, in [2] we showed that the triangle sides and its orientation may be recovered from O, G and K. We are done. $\square$

In Figure 2, we illustrate the loci of the Brocard points for triangles with various Brocard angles ω. These are enveloped by the curve (2). The region Γ_1 is shaded.

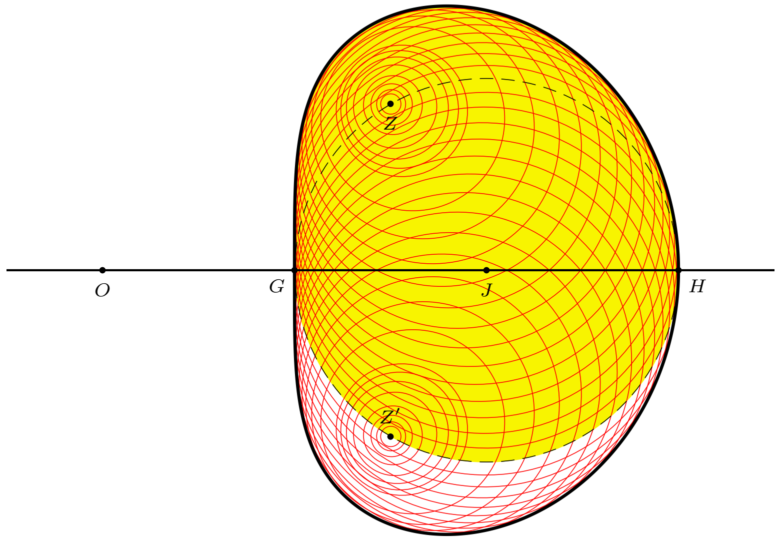

Figure 2

3. A qualitative description

We present an informal and loose qualitative description of the movements of Ω and Ω' as we steer K around the orthocentroidal disk. First consider K near G, with small positive argument. Both Brocard points are close to K, Ω just outside and Ω' just inside orthocentroidal circle. Now let K make one orbit, starting with positive arguments, just inside the circle. All the time ω stays small, and the two Brocard points nestle close to K in roughly the same configuration until K passes H at which point the Brocard points change roles; Ω' dives inside the circle and Ω moves outside. Though their paths cross, the Brocard points do not actually meet of course. For the second half of the passage of K just inside the circle it is Ω which is just inside the circle and Ω' which is just outside. When reaching the Euler line near G, the Brocard points park symmetrically on the circle with Ω having positive argument.

Now move K along the Euler line towards J; the Brocard points move round the circle, mutual reflections in the Euler line and Ω has positive argument. Triangle ABC is isosceles. As K approaches J each Brocard point approaches its forbidden point. Let K make a small swerve round J to rejoin the Euler line on the other side. Suppose that the swerve is on the side $\theta > 0$. In this case Ω swerves round its forbidden point outside the circle, and Ω' swerves inside, both points rejoining the circle almost immediately. Now K sails along the Euler line and the three points come close again together as K approaches H.

Next let K be at an arbitrary legal position on the Euler line. We fix OK and increase the argument of K. Both Brocard points also move in the same general direction; Ω leaves the orthocentroidal disk and heads towards the boundary, and

Ω' chases K. When K reaches a certain critical point, Ω reaches the boundary and at the same instant, Ω' crosses the Euler line. Now K keeps moving towards Ω followed by Ω', but Ω reverses direction and plunges back towards K. The three points come close together as K approaches the unattainable circle boundary. The process will unwrap as K reverses direction until it arrives back on the Euler line.

Another interesting tour which K can take is to move with positive arguments and starting near G along the path defined by the following equation:

$$r^2 - 4r\cos\theta + 3 + 3\tan^2\theta = 0 \qquad (12)$$

This is the path of critical values which has Ω moving on the boundary of Γ_1 and Ω' on the Euler line. Now $S\Omega'O$ is a right angle so in this particular sweep the position of Ω' is the perpendicular projection on the Euler line of the position of K. We derive (12) as follows. Take (2) and express θ in terms of $\theta/2$ and multiply through by $\sec^2\frac{\theta}{2}$. Now $OK = r\sec\frac{\theta}{2}$. Relabel by replacing $r\sec\frac{\theta}{2}$ by r and then replacing the remaining occurence of $\frac{\theta}{2}$ by θ.

A final journey of note for K is obtained by fixing the Brocard angle ω. Then K is free to range over a circle with center J and radius KJ where $KJ^2 = 1 - 3\tan^2\omega$ because of (3). The direct similarity type of triangles $OK\Omega$ and $OK\Omega'$ will not change, so Ω and Ω' will each move round circles. As K takes this circular tour through the moduli space of directed similarity types of triangle, we make the same journey through triangles as when a triangle vertex takes a trip round a Neuberg circle.

4. Discussion

We can obtain an areal equation of the boundary of Γ using the fact that one Brocard point is on the boundary of Γ exactly when the other is on the Euler line. The description is therefore the union of two curves, but the points G and H must be removed by special fiat.

The equation of the Euler line is

$$(b^2+c^2-a^2)(b^2-c^2)x+(c^2+a^2-b^2)(c^2-a^2)y+(a^2+b^2-c^2)((a^2-c^2)z = 0.$$

The Brocard point (a^2b^2, b^2c^2, c^2a^2) lies on the Euler line if and only if

$$a^6c^2 + b^6a^2 + c^6b^2 = a^4b^4 + b^4c^4 + c^4a^4.$$

The locus of the other Brocard point $x = c^2a^2, y = a^2b^2, z = b^2c^2$ is then given by

$$x^3y^2 + y^3z^2 + z^3x^2 = xyz(x^2 + y^2 + z^2).$$

To get the other half of the boundary we must exchange the roles of Ω and Ω' and this yields

$$x^3z^2 + y^3x^2 + z^3y^2 = xyz(x^2 + y^2 + z^2)$$

so the locus is a quintic in areal co-ordinates.

Equation (10) exhibits an intriguing symmetry between $\alpha = \angle KOG$ and $\omega = \angle\Omega OS$ which we will now explain. Suppose that we are given the location Y of a Brocard point within the orthocentroidal circle, but not the information as to whether the Brocard point is Ω or Ω'. If this Brocard point is Ω, we call the location of the other Brocard point Ω_1 and the corresponding symmedian point K_1. On the other hand if the Brocard point at Y is Ω' we call the location of the other Brocard point Ω_2 and the corresponding symmedian point K_2. Let the respective Brocard angles be ω_1 and ω_2.

We have two Brocard circles, so $\angle K_2YO = \angle K_1YO = \pi/2$ and therefore Y lies on the line segment K_1K_2. Using equation (6) we conclude that the lengths $J\Omega_1$ and $J\Omega_2$ are equal. Also $O\Omega_1 = OY = O\Omega_2$. Therefore Ω_1 and Ω_2 are mutual reflections in the Euler line. Now

$$\angle\Omega_2O\Omega_1 = \angle\Omega_2OG + \angle GO\Omega_1 = 2\omega_1 + 2\omega_2.$$

However, the Euler line is the bisector of $\angle\Omega_2OG$ so $\angle K_2OG = \omega_1$ and $\angle GOK_1 = \omega_2$. Thus in the "ω, α" description of Ω_2 which follows from equation (10), we have $\omega = \omega_2$ and $\alpha = \omega_1$. However, exchanging the roles of left and right in the whole discussion (the way in which we have discriminated between the first and second Brocard points), the resulting description of Ω_1 would have $\omega = \omega_1$ and $\alpha = \omega_2$. The symmetry in (10) is explained.

There are a pair of triangles determines by by the quadruple

$$(O, G, \Omega, \Omega')$$

using the values (O, G, Ω_2, Y) and (O, G, Y, Ω_1) which are linked via their common Brocard point in the orthocentroidal disk. We anticipate that there may be interesting geometrical relationships between these pairs of non-isosceles triangles.

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Archimedean Adventures

Floor van Lamoen

Abstract. We explore the arbelos to find more Archimedean circles and several infinite families of Archimedean circles. To define these families we study another shape introduced by Archimedes, the *salinon*.

1. Preliminaries

We consider an arbelos, consisting of three semicircles (O_1), (O_2) and (O), having radii r_1, r_2 and $r = r_1 + r_2$, mutually tangent in the points A, B and C as shown in Figure 1 below.

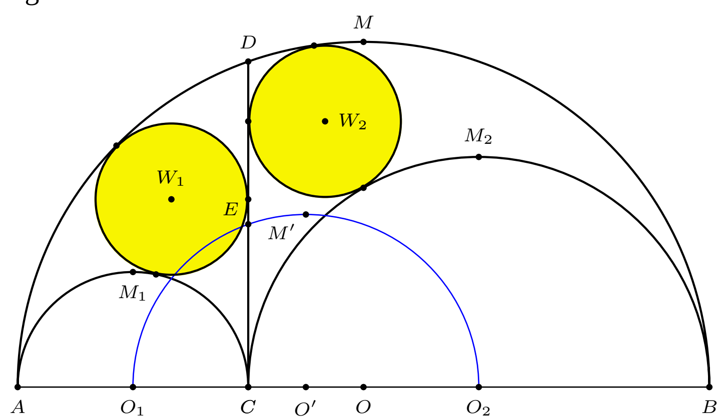

Figure 1. The arbelos with its Archimedean circles

It is well known that Archimedes has shown that the circles tangent to (O), (O_1) and CD and to (O), (O_2) and CD respectively are congruent. Their radius is $r_A = \frac{r_1 r_2}{r}$. See Figure 1. Thanks to [1, 2, 3] we know that these Archimedean twins are not only twins, but that there are many more Archimedean circles to be found in surprisingly beautiful ways. In their overview [4] Dodge et al have expanded the collection of Archimedean circles to huge proportions, 29 individual circles and an infinite family of Woo circles with centers on the Schoch line. Okumura and Watanabe [6] have added a family to the collection, which all pass through O, and have given new characterizations of the circles of Schoch and Woo.[1] Recently Power has added four Archimedean circles in a short note [7]. In this paper

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[1]The circles of Schoch and Woo often make use of tangent lines. The use of tangent lines in the arbelos, being a curvilinear triangle, may seem surprising, its relevance is immediately apparent when we realize that the common tangent of (O_1), (O_2), (O') and $C(2r_A)$ ($\mathcal{E}$ in [6]) and the common tangent of (O), (W_{21}) and the incircle (O_3) of the arbelos meet on AB.

we introduce some new Archimedean circles and some in infinite families. The Archimedean circles (W_n) are those appearing in [4]. New ones are labeled (K_n).

We adopt the following notations.

$P(r)$	circle with center P and radius r
$P(Q)$	circle with center P and passing through Q
(P)	circle with center P and radius clear from context
(PQ)	circle with diameter PQ
(PQR)	circle through P, Q, R

In the context of arbeloi, these are often interpreted as semicircles. Thus, an arbelos consists of three semicircles $(O_1) = (AC)$, $(O_2) = (CB)$, and $(O) = (AB)$ on the same side of the line AB, of radii r_1, r_2, and $r = r_1+r_2$. The common tangent at C to (O_1) and (O_2) meets (O) in D. We shall call the semicircle $(O') = (O_1O_2)$ (on the same side of AB) the *midway semicircle* of the arbelos.

It is convenient to introduce a cartesian coordinate system, with O as origin. Here are the coordinates of some basic points associated with the arbelos.

$A\ (-(r_1+r_2), 0)$	$B\ (r_1+r_2, 0)$	$C\ (r_1-r_2, 0)$
$O_1\ (-r_2, 0)$	$O_2\ (r_1, 0)$	
$M_1\ (-r_2,\ r_1)$	$M_2\ (r_1,\ r_2)$	$M\ (0, r_1+r_2)$
$O'\ \left(\frac{r_1-r_2}{2}, 0\right)$	$M'\ \left(\frac{r_1-r_2}{2},\ \frac{r_1+r_2}{2}\right)$	
$D\ (r_1-r_2,\ 2\sqrt{r_1r_2})$	$E\ (r_1-r_2,\ \sqrt{r_1r_2})$	

2. New Archimedean circles

2.1. (K_1) *and* (K_2). The Archimedean circle (W_8) has C as its center and is tangent to the tangents O_1K_2 and O_2K_1 to (O_1) and (O_2) respectively. By symmetry it is easy to find from these the Archimedean circles (K_1) and (K_2) tangent to CD, see Figure 2. A second characterization of the points K_1 and K_2, clearly equivalent, is that these are the points of intersection of (O_1) and (O_2) with (O'). We note that the tangent to $A(C)$ at the point of intersection of $A(C)$, (W_{13}) and (O) passes through B, and is also tangent to (K_1). A similar statement is true for the tangent to $B(C)$ at its intersection with (W_{14}) and (O).

To prove the correctness, let T be the perpendicular foot of K_1 on AB. Then, making use of the right triangle $O_1O_2K_1$, we find $O_1T = \frac{r_1^2}{r_1+r_2}$ and thus the radius of (K_1) is equal to $O_1C - O_1T = r_A$. By symmetry the radius of (K_2) equals r_A as well.

The circles (K_1) and (K_2) are closely related to (W_{13}) and (W_{14}), which are found by intersecting $A(C)$ and $B(C)$ with (O) and then taking the smallest circles through the respective points of intersection tangent to CD. The relation is clear when we realize that $A(C)$, $B(C)$ and (O) can be found by a homothety with center C, factor 2 applied to (O_1), (O_2) and (O').

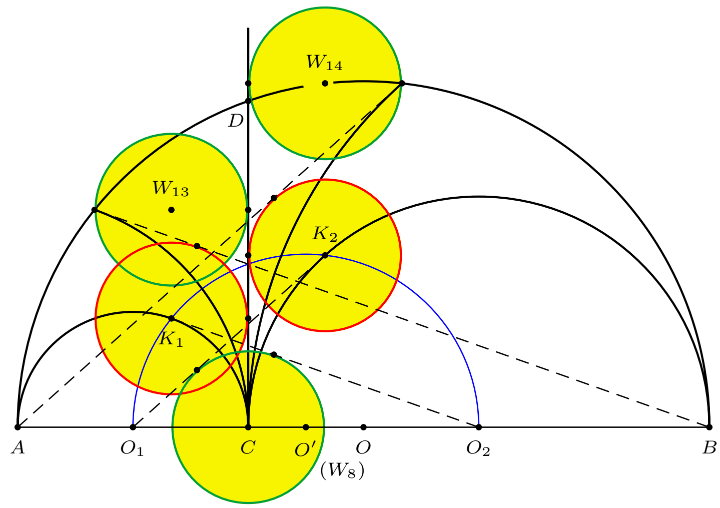

Figure 2. The Archimedean circles (K_1) and (K_2)

2.2. (K_3). In private correspondence [10] about (K_1) and (K_2), Paul Yiu has noted that the circle with center on CD tangent to (O) and (O') is Archimedean. Indeed, if x is the radius of this circle, then we have

$$(r-x)^2-(r_1-r_2)^2=\left(\frac{r}{2}+x\right)^2-\left(\frac{r_1-r_2}{2}\right)^2$$

and that yields $x=r_A$.

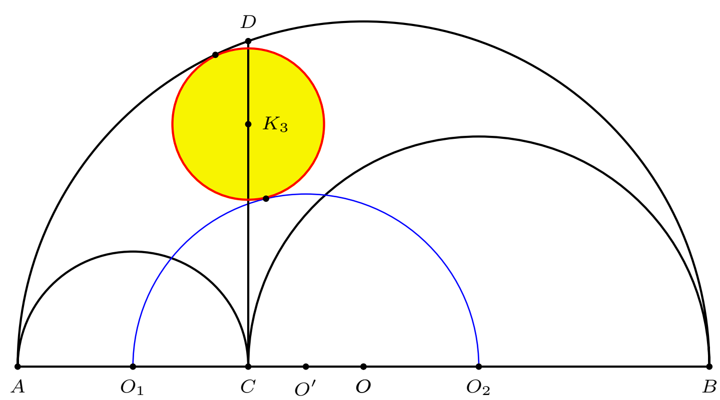

Figure 3. The Archimedean circle (K_3)

2.3. (K_4) *and* (K_5). There is an interesting similarity between (K_3) and Bankoff's triplet circle (W_3). Recall that this is the circle that passes through C and the points of tangency of (O_1) and (O_2) with the incircle (O_3) of the arbelos. [2] Just like (K_3), (W_3) has its center on CD and is tangent to (O'), a fact that seems to have been unnoted so far. The tangency can be shown by using Pythagoras' Theorem in triangle $CO'W_3$, yielding that $O'W_3 = \frac{r_1^2+r_2^2}{2r}$. This leaves for the length of the radius of (O') beyond W_3:

$$\frac{r}{2} - \frac{r_1^2 + r_2^2}{2r} = \frac{r_1 r_2}{r}.$$

This also gives us in a simple way the new Archimedean circle (K_4): the circle tangent to AB at O and to (O') is Archimedean by reflection of (W_3) through the perpendicular to AB in O'.

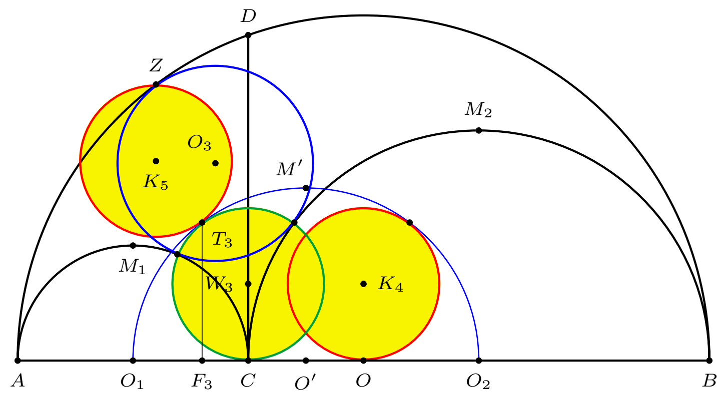

Figure 4. The Archimedean circles (K_4) and (K_5)

Let T_3 be the point of tangency of (O') and (W_3), and F_3 be the perpendicular foot of T_3 on AB. Then

$$O'F_3 = \frac{\frac{r}{2}}{\frac{r}{2} - r_A} \cdot O'C = \frac{(r_2 - r_1)r^2}{2(r_1^2 + r_2^2)},$$

and from this we see that $O_1F_3 : F_3O_2 = r_1^2 : r_2^2$, so that $O_1T_3 : T_3O_2 = r_1 : r_2$, and hence the angle bisector of $\angle O_1T_3O_2$ passes through C. If Z is the point of tangency of (O_3) and (O) then [9, Corollary 3] shows the same for the angle bisector of $\angle AZB$. But this means that the points C, Z and T_3 are collinear. This also gives us the circle (K_5) tangent to (O') and tangent to the parallel to AB through Z is Archimedean by reflection of (W_3) through T_3. See Figure 4.

There are many ways to find the interesting point T_3. Let me give two.

(1) Let M, M', M_1 and M_2 be the midpoints of the semicircular arcs of (O), (O'), (O_1) and (O_2) respectively. T_3 is the second intersection of (O') with line $M_1M'M_2$, apart from M'. This line $M_1M'M_2$ is an angle bisector of the angle formed by lines AB and the common tangent of (O), (O_3) and (W_{21}).

(2) The circle (AM_1O_2) intersects the semicircle (O_2) at Y, and the circle (BM_2O_1) intersects the semicircle (O_1) at X. X and Y are the points of tangency

[2] For simple constructions of (O_3), see [9, 11].

of (O_3) with (O_1) and (O_2) respectively. Now, each of these circles intersects the midway semicircle (O') at T_3. See Figure 5.

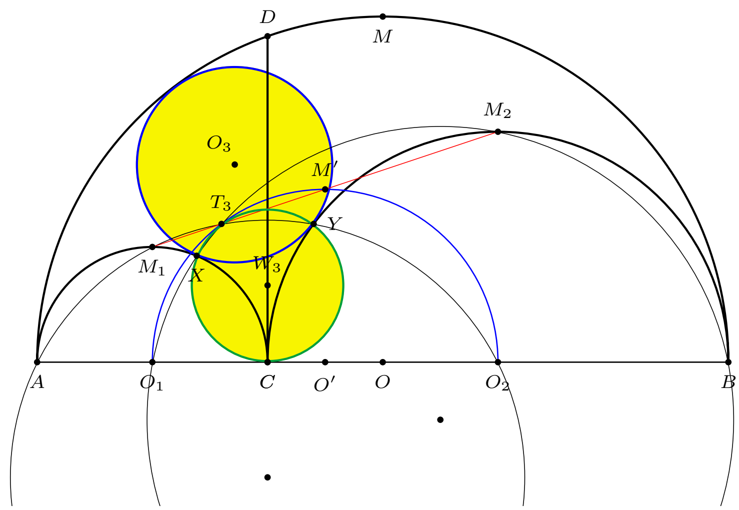

Figure 5. Construction of T_3

2.4. (K_6) *and* (K_7). Consider again the circles $A(C)$ and $B(C)$. The circle with center K_6 on the radical axis of $A(C)$ and (O) and tangent to $A(C)$ as well as (O') is the Archimedean circle (K_6). This circle is easily constructed by noticing that the common tangent of $A(C)$ and (K_6) passes through O_2. Let T_6 be the point of tangency, then K_6 is the intersection of the mentioned radical axis and AT_6. Similarly one finds (K_7). See Figure 6.

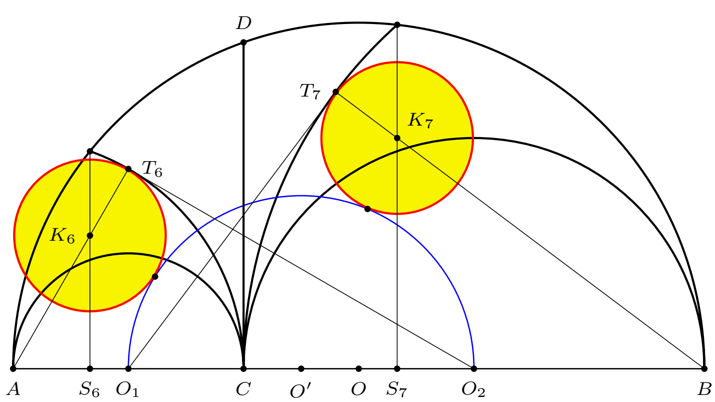

Figure 6. The Archimedean circles (K_6) and (K_7)

To prove that (K_6) is indeed Archimedean, let S_6 be the intersection of AB and the radical axis of $A(C)$ and (O), then $O'S_6 = 2r_A + \frac{r_2 - r_1}{2}$ and $AS_6 = 2(r_1 - r_A)$.

If x is the radius of (K_6), then

$$\left(\frac{r_1+r_2}{2}+x\right)^2-\left(2r_A+\frac{r_2-r_1}{2}\right)^2=(2r_1-x)^2-4(r_1-r_A)^2$$

which yields $x = r_A$.

For justification of the simple construction note that

$$\cos\angle S_6K_6A=\frac{AS_6}{AK_6}=\frac{2r_1-2r_A}{2r_1-r_A}=\frac{2r_1}{2r_1+r_2}=\frac{AT_6}{AO_2}=\cos\angle AO_2T_6.$$

2.5. (K_8) *and* (K_9). Let A' and B' be the reflections of C through A and B respectively. The circle with center on AB, tangent to the tangent from A' to (O_2) and to the radical axis of $A(C)$ and (O) is the circle (K_8). Similarly we find (K_9) from B' and (O_1) respectively.

Let x be the radius of (K_8). We have

$$\frac{4r_1+r_2}{r_2}=\frac{4r_1-2r_A-x}{x}$$

implying indeed that $x = r_A$. And (K_8) is Archimedean, while this follows for (K_9) as well by symmetry. See Figure 7.

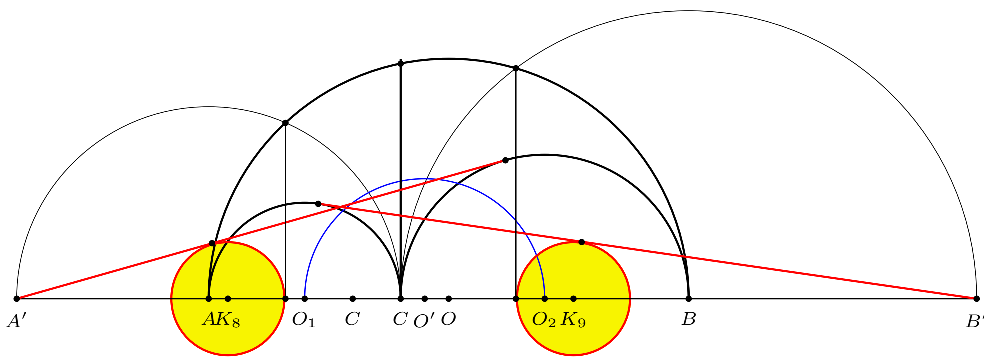

Figure 7. The Archimedean circles (K_8) and (K_9)

2.6. *Four more Archimedean circles from the midway semicircle.* The points M, M_1, M_2, O, C and the point of tangency of (O) and the incircle (O_3) are concyclic, the center of their circle, the *mid-arc circle* is M'. This circle (M') meets (O') in two points K_{10} and K_{11}. The circles with K_{10} and K_{11} as centers and tangent to AB are Archimedean. To see this note that the radius of (MM_1M_2) equals $\sqrt{\frac{r_1^2+r_2^2}{2}}$. Now we know the sides of triangle $O'M'K_{10}$. The altitude from K_{10} has length $\frac{\sqrt{(r_1^2+4r_1r_2+r_2^2)(r_1^2+r_2^2)}}{2r}$ and divides $O'M'$ indeed in segments of $\frac{r_1^2+r_2^2}{2r}$ and r_A. Of course by similar reasoning this holds for (K_{11}) as well.

Now the circle $M(C)$ meets (O) in two points. The smallest circles (K_{12}) and (K_{13}) through these points tangent to AB are Archimedean as well. This can be seen by applying homothety with center C with factor 2 to the points K_{10} and K_{11}.

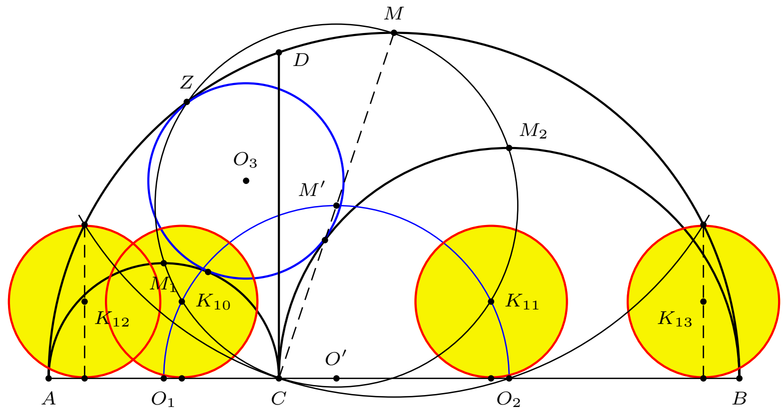

Figure 8. The Archimedean circles (K_{10}), (K_{11}), (K_{12}) and (K_{13})

The image points are the points where $M(C)$ meets (O) and are at distance $2r_A$ from AB. See Figure 8.

2.7. (K_{14}) *and* (K_{15}). It is easy to see that the semicircles $A(C)$, $B(C)$ and (O) are images of (O_1), (O_2) and (O') after homothety through C with factor 2. This shows that $A(C)$, $B(C)$ and (O) have a common tangent parallel to the common tangent d of (O_1), (O_2) and (O'). As a result, the circles (K_{14}) and (K_{15}) tangent internally to $A(C)$ and $B(C)$ respectively and both tangent to d at the opposite of (O_1), (O_2) and (O'), are Archimedean circles, just as is Bankoff's quadruplet circle (W_4).

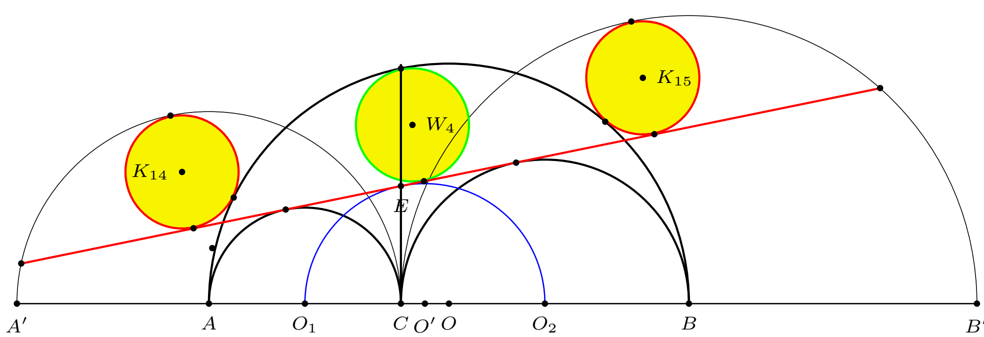

Figure 9. The Archimedean circles (K_{14}) and (K_{15})

An additional property of (K_{14}) and (K_{15}) is that these are tangent to (O) externally. To see this note that, using linearity, the distance from A to d equals $\frac{2r_1^2}{r}$, so $AK_{14} = \frac{r_1(2r_1+r_2)}{r}$. Let F_{14} be the perpendicular foot of K_{14} on AB. In triangle COD we see that $CD = 2\sqrt{r_1 r_2}$ and thus by similarity of COD and $F_{14}AK_{14}$

we have

$$K_{14}F_{14} = \frac{2\sqrt{r_1r_2}}{r}AK_{14} = \frac{2r1\sqrt{r_1r_2}(2r_1+r_2)}{r^2},$$
$$OF_{14} = r + \frac{r_2-r_1}{r}AK_{14} = \frac{r_2^3+4r_1r_2^2+4r_1^2r_2-r_1^3}{r^2},$$

and now we see that $K_{14}F_{14}^2 + OF_{14}^2 = (r+r_A)^2$. In the same way it is shown (K_{15}) is tangent to (O). See Figure 9. Note that O_1K_{15} passes through the point of tangency of d and (O_2), which also lies on $O_1(D)$. We leave details to the reader.

2.8. (K_{16}) *and* (K_{17}). Apply the homothety $\mathsf{h}(A,\lambda)$ to (O) and (O_1) to get the circles (Ω) and (Ω_1). Let $U(\rho)$ be the circle tangent to these two circles and to CD, and U' the perpendicular foot of U on AB. Then $|U'\Omega| = \lambda r - 2r_1 + \rho$ and $|U'\Omega_1| = (\lambda-2)r_1 + \rho$. Using the Pythagorean theorem in triangle $UU'\Omega$ and $UU'\Omega_1$ we find

$$(\lambda r_1+\rho)^2 - ((\lambda-2)r_1+\rho)^2 = (\lambda r-\rho)^2 - (\lambda r - 2r_1+\rho)^2.$$

This yields $\rho = r_A$.

By symmetry, this shows that the twin circles of Archimedes are members of a family of Archimedean twin circles tangent to CD. In particular, (W_6) and (W_7) of [4] are limiting members of this family. As special members of this family we add (K_{16}) as the circle tangent to $C(A)$, $B(A)$, and CD, and (K_{17}) tangent to $C(B)$, $A(B)$, and CD. See Figure 10.

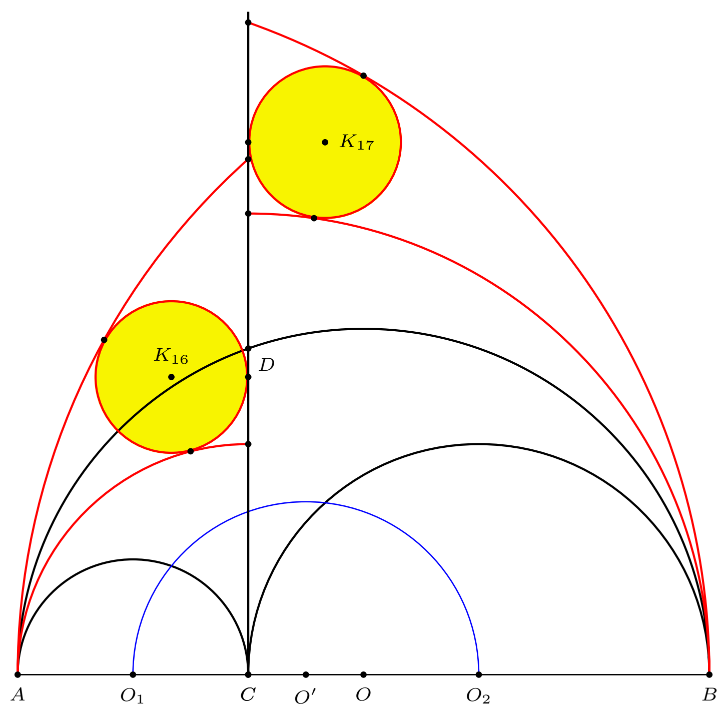

Figure 10. The Archimedean circles (K_{16}) and (K_{17})

3. Extending circles to families

There is a very simple way to turn each Archimedean circle into a member of an infinite Archimedean family, that is by attaching to (O) a semicircle (O') to be the two inner semicircles of a new arbelos. When (O'') is chosen smartly, this new arbelos gives Archimedean circles exactly of the same radii as Archimedean circles of the original arbelos. By repetition this yields infinite families of Archimedean circles. If r'' is the radius of (O''), then we must have

$$\frac{r_1 r_2}{r} = \frac{r r''}{r + r''}$$

which yields

$$r'' = \frac{r r_1 r_2}{r^2 - r_1 r_2},$$

surprisingly equal to r_3, the radius of the incircle (O_3) of the original arbelos, as derived in [4] or in generalized form in [6, Theorem 1]. See Figure 11.

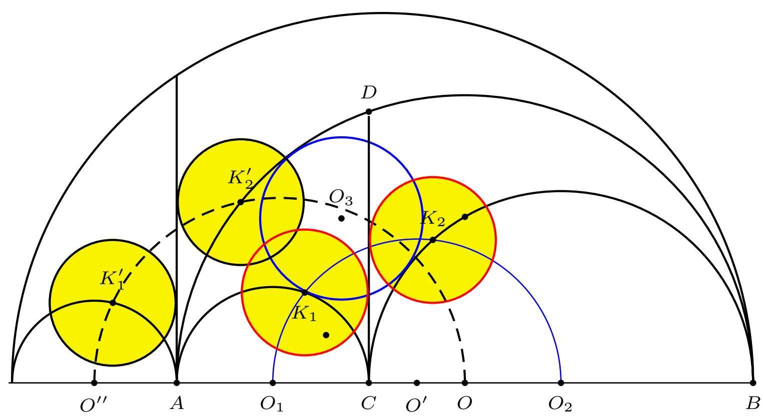

Figure 11

Now let $(O_1(\lambda))$ and $(O_2(\lambda))$ be semicircles with center on AB, passing through C and with radii λr_1 and λr_2 respectively. From the reasoning of §2.7 it is clear that the common tangent of $(O_1(\lambda))$ and $(O_2(\lambda))$ and the semicircles $(O_1(\lambda+1))$, $(O_2(\lambda+1))$ and $(O_1(\lambda+1)O_2(\lambda+1))$ enclose Archimedean circles $(K_{14}(\lambda))$, $(K_{15}(\lambda))$ and $(W_4(\lambda))$. The result is that we have three families. By homothety the point of tangency of $(K_{14}(\lambda))$ and $(O_1(\lambda+1))$ runs through a line through C, so that the centers $K_{14}(\lambda)$ run through a line as well. This line and a similar line containing the centers of $K_{15}(\lambda)$ are perpendicular. This is seen best by the well known observation that the two points of tangency of (O_1) and (O_2) with their common tangent together with C and D are the vertices of a rectangle, one of Bankoff's surprises [2]. Of course the centers $W_4(\lambda)$ lie on a line perpendicular to AB. See Figure 12.

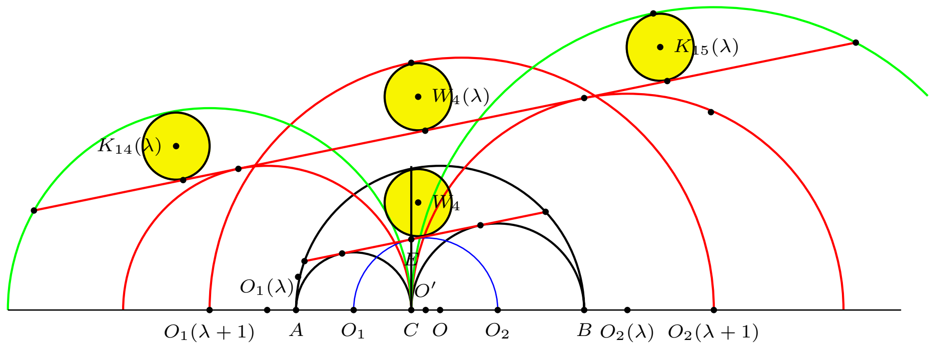

Figure 12

4. A family of salina

We consider another way to generalize the Archimedean circles in infinite families. Our method of generalization is to translate the two basic semicircles (O_1) and (O_2) and build upon them a (skew) *salinon*. We do this in such a way that to each arbelos there is a family of salina that accompanies it. The family of salina and the arbelos are to have common tangents.

This we do by starting with a point O'_t that divides O_1O_2 in the ratio $O_1O'_t$: $O'_tO_2 = t : 1-t$. We create a semicircle $(O'_t) = (O_{t,1}O_{t,2})$ with radius $r'_t = (1-t)r_1 + tr_2$, so that it is tangent to d. This tangent passes through E, see [6, Theorem 8], and meets AB in N, the external center of similitude of (O_1) and (O_2). Then we create semicircles $(O_{t,1})$ and $(O_{t,2})$ with radii r_1 and r_2 respectively. These two semicircles have a semicircular hull $(O_t) = (A_tB_t)$ and meet AB as second points in $C_{t,1}$ and $C_{t,2}$ respectively. Through these we draw a semicircle $(O_{t,4}) = (C_{t,2}C_{t,1})$ opposite to the other semicircles with respect to AB. Assume $r_1 < r_2$. $C_{t,1}$ is on the left of $C_{t,2}$ if and only if $t \geq \frac{1}{2}$.[3] In this case we call the region bounded by the 4 semicircles the t-salinon of the arbelos. See Figure 13.

Here are the coordinates of the various points.

O'_t $(tr_1 + (t-1)r_2, 0)$	
$O_{t,1}$ $((2t-1)r_1 - r_2, 0)$	$O_{t,2}$ $(r_1 + (2t-1)r_2, 0)$
A_t $((2t-2)r_1 - r_2, 0)$	B_t $(r_1 + 2tr_2, 0)$
$C_{t,1}$ $(2tr_1 - r_2, 0)$	$C_{t,2}$ $(r_1 + (2t-2)r_2, 0)$
O_t $\left(\left(t-\frac{1}{2}\right)r,\ 0\right)$	$O_{t,4}$ $\left((t+\frac{1}{2})r_1 + (t-1\frac{1}{2})r_2, 0\right)$
C_t $\left(\frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, 0\right)$	

The radical axis of the circles

$$(O'_t): \qquad (x - tr_1 - (t-1)r_2)^2 + y^2 = ((1-t)r_1 + tr_2)^2$$

[3] If $t < \frac{1}{2}$ we can still find valid results by drawing $(O_{t,4})$ on the same side of AB as the other semicircles. In this paper we will not refer to these results, as the resulting figure is not really like Archimedes' salinon.

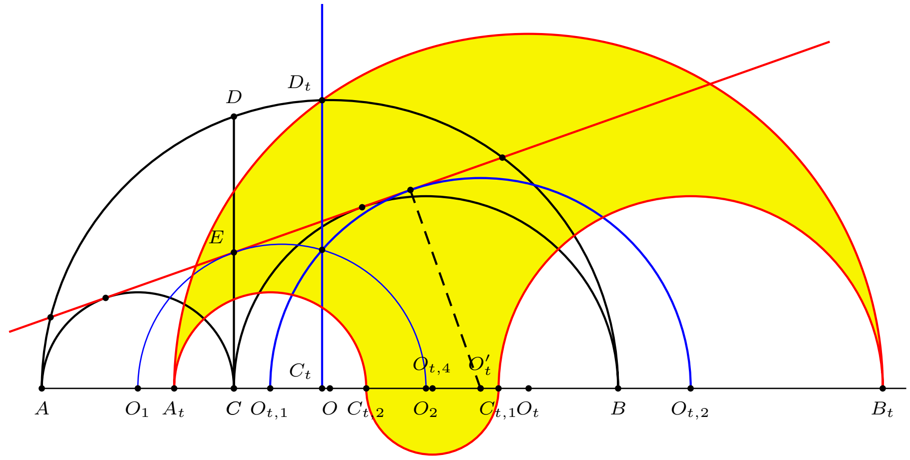

Figure 13. A t-salinon

and

$$(O'): \qquad \left(x - \frac{r_1 - r_2}{2}\right)^2 + y^2 = \frac{r^2}{4}$$

is the line

$$\ell_t: \qquad x = \frac{(2t-1)r_1r_2 + r_1^2 - r_2^2}{r}.$$

So is the radical axis of

$$(O): \qquad x^2 + y^2 = r^2$$

and

$$(O_t): \qquad \left(x - \left(t - \frac{1}{2}\right)r\right)^2 + y^2 = \left(\left(1\frac{1}{2} - t\right)r_1 + \left(t + \frac{1}{2}\right)r_2\right)^2.$$

On this common radical axis ℓ_t, we define points the points C_t on AB and D_t on (O). See Figure 13.

5. Archimedean circles in the t-salinon

5.1. *The twin circles of Archimedes.* We can generalize the well known Adam and Eve of the Archimedean circles to adjoint salina in the following way: The circles $W_{t,1}$ and $W_{t,2}$ tangent to both (O_t) and ℓ_t and to (O_1) and (O_2) respectively are Archimedean.

This can be proven with the above coordinates: The semicircles $O_t((1\frac{1}{2}-t)r_1 + (t+\frac{1}{2})r_2 - r_A))$ and $O_1(r_1 + r_A)$ intersect in the point

$$W_{t,1} \quad \left(\frac{(2t-2)r_1r_2 + r_1^2 - r_2^2}{r},\ r_A\sqrt{(3-2t)(2t+1+\frac{2r_1}{r_2})}\right),$$

which lies indeed r_A left of ℓ_t. Similarly we find for the intersection of $O_t((1\frac{1}{2} - t)r_1 + (t + \frac{1}{2})r_2 - r_A))$ and $O_2(r_2 + r_A)$

$$W_{t,2}\ \left(\frac{r_1^2 + 2r_1r_2t - r_2^2}{r},\ r_A\sqrt{(2t+1)(3-2t+\frac{2r_2}{r_1})}\right),$$

which lies r_A right of ℓ_t. See Figure 14.

These circles are real if and only if $\frac{1}{2} \le t \le \frac{3}{2}$.

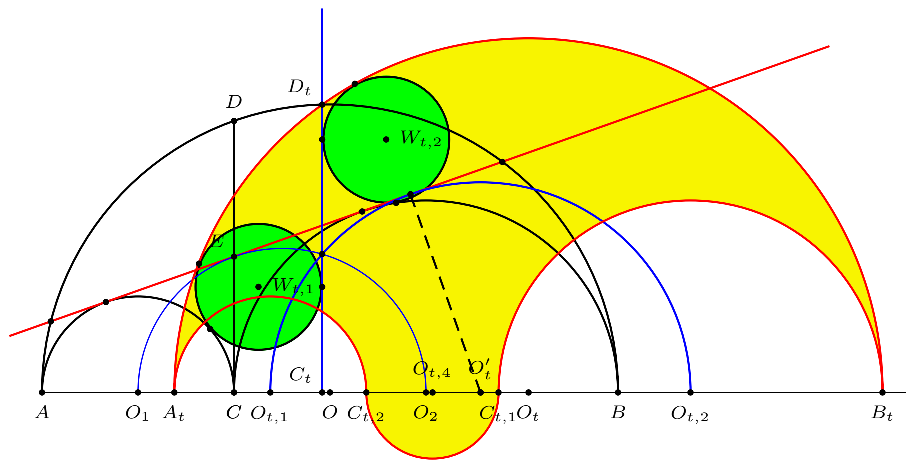

Figure 14. The Archimedean circles $(W_{t,1})$ and $(W_{t,2})$

Two properties of the twin circles of Archimedes can be generalized as well. Dodge et al [4] state that the circle $A(D)$ passes through the point of tangency of (O_2) and (W_2), while Wendijk [8] and d'Ignazio and Suppa in [5, p. 236] ask in a problem to show that the point N_2 where O_2W_2 meets CD lies on $O_2(A)$ (reworded). We can generalize these to

- the circle $A(D_t)$ passes through the point of tangency of $(W_{t,2})$ and (O_2);
- the point $N_{t,2}$ where $O_2W_{t,2}$ meets the perpendicular to AB through $C_{t,1}$ lies on $O_2(A)$.

Of course similar properties are found for $W_{t,1}$.

To verify this note that D_t has coordinates

$$D_t\ \left(\frac{r_1^2 - r_2^2 + (2t-1)r_1r_2}{r}, \frac{\sqrt{r_1r_2((3-2t)r_1 + 2r_2)(2r_1 + (2t+1)r_2)}}{r}\right),$$

while the point of contact R_2 of $(W_{t,2})$ and (O_2) is

$$\begin{aligned} &O_2 + \frac{r_2}{r_2 + r_A}(W_{t,2} - O_2) \\ =&\left(\frac{2r_1^2 - r_2^2 + 2tr_1r_2}{2r_1 + r_2}, \frac{\sqrt{(1+2t)r_1r_2^2((3-2t)r_1 + 2r_2)}}{2r_1 + r_2}\right). \end{aligned}$$

Straightforward verification now shows that

$$d(A, D_t)^2 = d(A, R_2)^2 = 2r_1(2r_1 + (1 + 2t)r_2).$$

Furthermore, we see that the point $N_{t,2} = O_2 + \frac{2r_1+r2}{r}(R_2 - O_2)$ has coordinates

$$N_{t,2} = \left(2tr_1 - r_2, \sqrt{(1 + 2t)r_1((3 - 2t)r_1 + 2r_2)}\right)$$

so that this point lies indeed on $O_2W_{t,2}$, on $O_2(A)$ and on the perpendicular to AB through $C_{t,1}$.

5.2. $(W_{t,3})$. Consider the circle through C tangent to (O'_t) and with center on C_tD_t. When u is the radius of this circle we have

$$\begin{aligned} (r't - u)^2 - O'tCt^2 &= u^2 - CCt^2 \\ ((1 - t)r_1 + tr_2 - u)^2 - (\frac{(t - 1)r_1^2 + tr_2^2}{r})^2 &= u^2 - ((2t - 1)r_A)^2 \end{aligned}$$

which yields $u = r_A$. This shows that this circle $(W_{t,3})$ is an Archimedean circle and generalizes the Bankoff triplet circle (W_3), using the tangency of (W_3) to (O') shown above. See Figure 15. These circles are real if and only if $\frac{1}{2} \le t \le 1$.

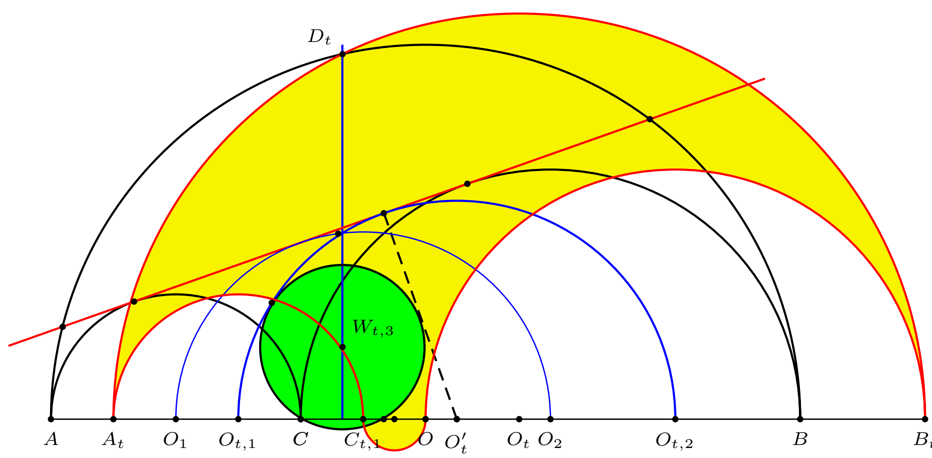

Figure 15. The Archimedean circle $(W_{t,3})$

5.3. $(W_{t,4})$. To generalize Bankoff's quadruplet circle (W_4) we start with a lemma.

Lemma 1. *Let (K) be a circle with center K and ℓ_1 and ℓ_2 be two tangents to (K) meeting in a point P. Let L be a point travelling through the line PK. When L travels linearly, then the radical axis of (K) and the circle (L) tangent to ℓ_1 and ℓ_2 moves linearly as well. The speed relative to the speed of L depends only on the angle of ℓ_1 and ℓ_2.*

Proof. Let P be the origin for Cartesian coordinates. Without loss of generality let ℓ_1 and ℓ_2 be lines making angles $\pm\phi$ with the $x-$axis and let $K(x_1, 0)$, $L(x_2, 0)$. With $v = \sin\phi$ the circles (K) and (L) are given by

$$\begin{aligned}(x - x_1)^2 + y^2 &= v^2 x_1^2,\\ (x - x_2)^2 + y^2 &= v^2 x_2^2.\end{aligned}$$

The radical axis of these is $x = \frac{1-v^2}{2}(x_1 + x_2) = \frac{1}{2}\cos^2\phi(x_1 + x_2)$. □

It is easy to check that the slope of the line through the midpoints of the semicircular arcs (O) and (O_t) is equal to $\frac{r_2 - r_1}{r}$ and thus equal to te slope of the line through M_1 and M_2. This shows that the common tangent of (O) and (O_t) is parallel to the common tangent of (O_1) and (O_2), i.e. d, also the common tangent of (O') and (O'_t). As a result of Lemma 1, it is now clear why the the radical axes of (O) and (O_t) and of (O') and (O'_t) coincide for all t. Another consequence is that the greatest circle tangent to d at the opposite side of (O_1) and (O_2) and to (O_t) internally is, just as the famous example of Bankoff's quadruplet circle (W_4), an Archimedean circle $(W_{t,4})$. See Figure 16. Of course this means that the circles (K_{12}) and (K_{13}) are found as members of the family $(W_{t,4})$.

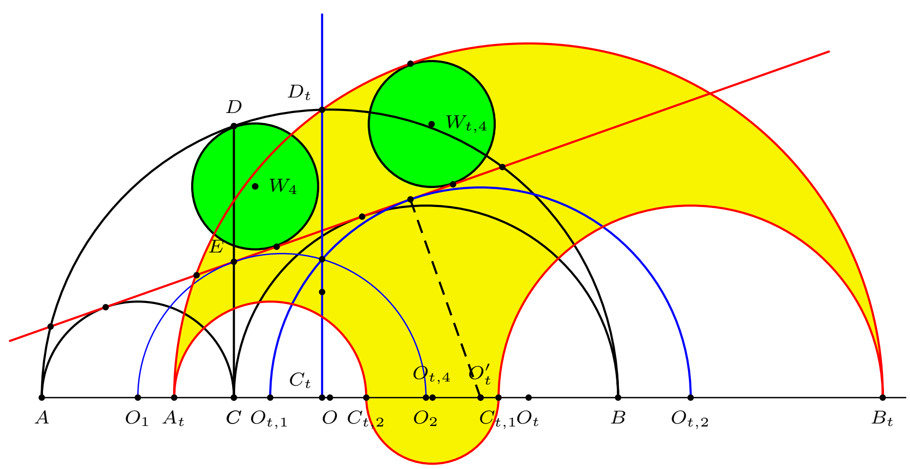

Figure 16. The Archimedean circle $(W_{t,4})$

We note that the the Archimedean circles $(W_{t,4})$ can be constructed easily in any (skew) salinon without seeing the salinon as adjoint to an arbelos and without reconstructing (parts of) this arbelos. To see this we note that N is external center of similitude of $(O_{t,1})$ and $(O_{t,2})$, and d is the tangent from N to (O'_t).

5.4. $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ *and* $(W_{t,14})$. Whereas (W_{13}), (W_{14}), (K_1) and (K_2) are defined in terms of intersections of (semi-)circles or radical axes, with Lemma 1 their generalizations are obvious: CD, (O) and (O') can be replaced by C_tD_t, (O_t) and (O'_t). The generalization of (W_6) and (W_7) as Archimedean circles with center on AB and tangent to $CtDt$ keep having some interest as well. The tangents from A to $(O_{t,2})$ and from B to $(O_{t,1})$ are tangent to $(W_{t,6})$ and $(W_{t,7})$ respectively. This is seen by straightforward calculation, which is left to the reader. As a result we have two stacks of three families. See Figure 17.

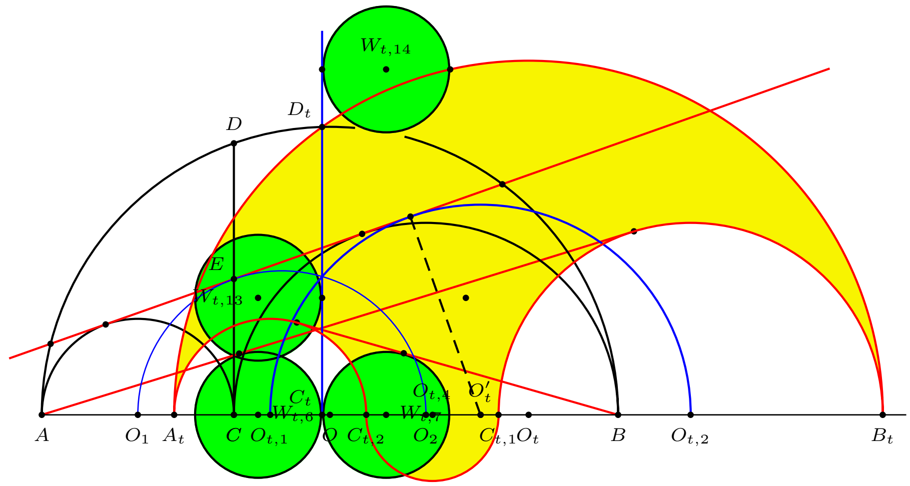

Figure 17. The Archimedean circles $(W_{t,6})$, $(W_{t,7})$, $(W_{t,13})$ and $(W_{t,14})$

We zoom in on the families $(W_{t,13})$ and $(W_{t,14})$. Let $T_{t,13}$ be the point where $(W_{t,13})$, $A(C)$ and (O_t) meet and similarly define $T_{t,14}$. We find that

$$\begin{array}{ll} T_{t,13} & \left(\frac{r_1^2+(2t-3)r_1r_2-r_2^2}{r}, \frac{r_1\sqrt{-r_2((8t-12)r_1+(4t^2-4t-3)r_2)}}{r}\right), \\ T_{t,14} & \left(\frac{r_1^2+(2t+1)r_1r_2-r_2^2}{r}, \frac{r_2\sqrt{-r_1((4t^2-4t-3)r_1-(8t+4)r_2)}}{r}\right). \end{array}$$

The slope of the tangent to $A(C)$ in $T_{t,13}$ with respect to the $x-$axis is equal to

$$s_A = -\frac{x_A - x_{T_{t,13}}}{0 - y_{T_{t,13}}}$$

and the $x-$coordinate of the point R_t where this tangent meets AB is equal to

$$x_{R_t} = x_{T_{t,13}} - \frac{y_{T_{t,13}}}{s_A} = \frac{r(2r_1 + (1-2t)r_2)}{2r_1 + (2t-1)r_2}.$$

Similarly we find for the point L_t where the tangent to $B(C)$ in $T_{t,14}$ meets AB

$$x_{L_t} = \frac{r((2t-1)r_1 + 2r_2)}{(2t-1)r_1 - 2r_2}.$$

The coordinates of Z are given by

$$\left(\frac{r^2(r_1 - r_2)}{r_1^2 + r_2^2}, \frac{2rr_1r_2}{r_1^2 + r_2^2}\right).$$

By straightforward calculation we can now verify that $ZL_t^2 + ZR_t^2 = L_tR_t^2$ and we have a new characterization of the families $(W_{t,13})$ and $(W_{t,14})$.

Theorem 2. *Let $\mathcal{K}$ be a circle through Z with center on AB. This circle meets AB in two points L on the left hand side and R on the right hand side. Let the tangent to $A(C)$ through R meet $A(C)$ in P_1 and similarly find P_2 on $B(C)$. Let k be the line through the midpoint of P_1P_2 perpendicular to AB. Then the smallest circles through P_1 and P_2 tangent to k are Archimedean.*

Remark. Similar characterizations can be found for $(K_{t,1})$ and $(K_{t,2})$.

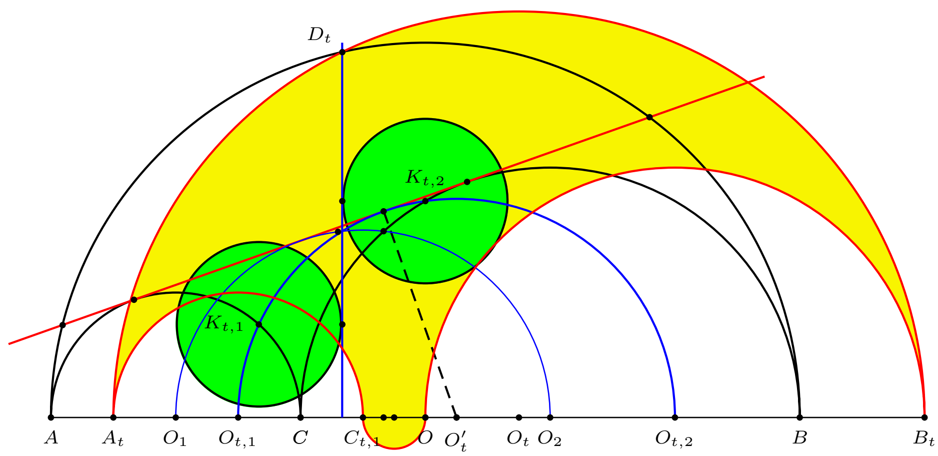

Figure 18. The Archimedean circles $(K_{t,1})$ and $(K_{t,2})$

5.5. *More corollaries.* We go back to Lemma 1. Since the distance between d and the common tangent of (O), (O_t), $A(C)$ and $B(C)$ is equal to $2r_A$, we notice that for instance $A(2r_1 - r_A)$ and $O'_t(r'_t + r_A)$ have a common tangent parallel to d as well. But that implies that we can use Lemma 1 on their intersection (and radical axis). This leads for instance to easy generalizations of (K_3) and (K_7). See Figure 19. The lemma can also help to generalize for instance (K_8) and (K_9), but then some more work has to be done. We leave this to the reader.

5.6. *Archimedean circles from the mid-arc circle of the salinon.* We end the adventures with a sole salinon. We note that the midpoints M_t, $M_{t,1}$, $M_{t,2}$ and $M_{t,4}$ of the semicircular arcs (O_t), $(O_{t,1})$, $(O_{t,2})$ and $(O_{t,4})$ are concyclic. More precisely they are vertices of a rectangle. Their circle, the mid-arc circle (O_5) of the salinon and the circle $(O_{t,1}O_{t,2})$ meet in two points, that are centers of Archimedean circles $(K_{t,10})$ and $(K_{t,11})$. (Of course the parameter t does not really play a role here, but for reasons of uniformity we still use it in the naming).

To verify this, denote by u the distance between $O_{t,1}$ and $O_{t,2}$. Then the distance between $M_{t,1}$ and $M_{t,2}$ equals $\sqrt{u^2 + (r_2 - r_1)^2}$ and the distance from O'' to AB

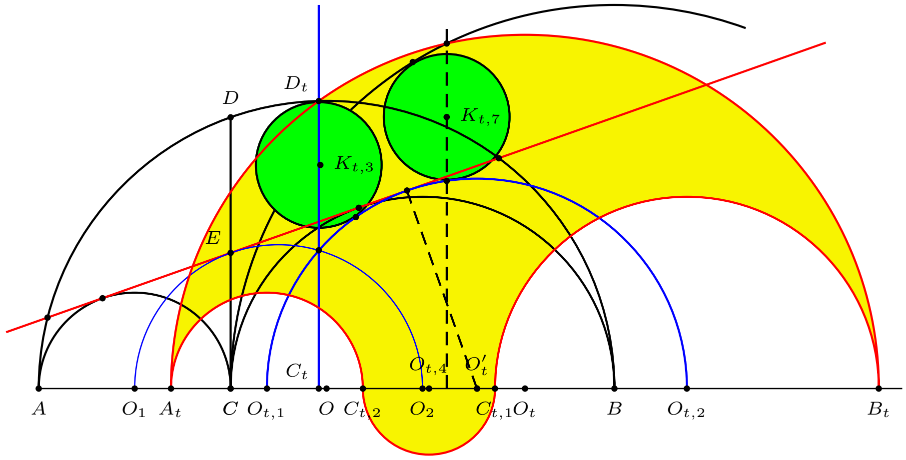

Figure 19. The Archimedean circles $(K_{t,3})$ and $(K_{t,7})$

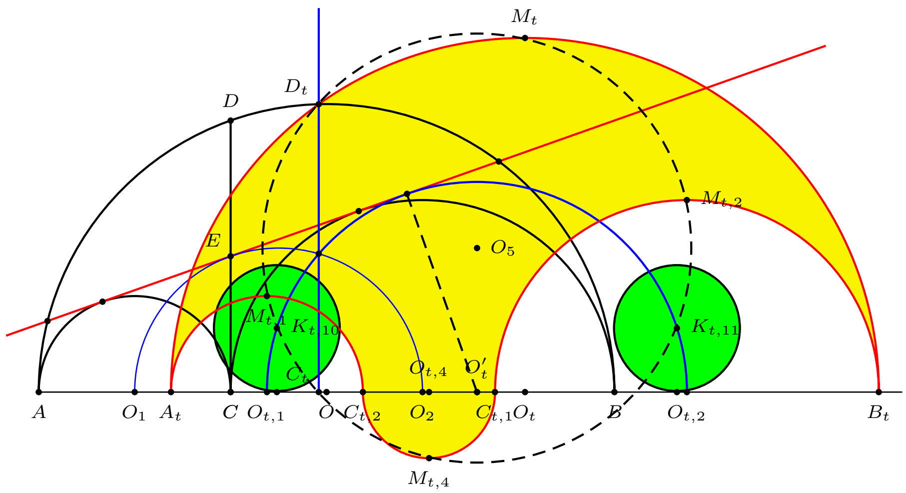

Figure 20. The Archimedean circles $(K_{t,10})$ and $(K_{t,11})$

equals $\frac{r_1+r_2}{2}$. If x is the distance from the intersections of $(O_{t,1}O_{t,2})$ and (O_5) to AB, then

$$x^2 - \frac{u^2}{4} = \left(\frac{r_1+r_2}{2} - x\right)^2 - \frac{u^2+(r_2-r_1)^2}{4}$$

which leads to $x = \frac{r_1r_2}{r_1+r_2} = r_A$. We can generalize (K_{12}) and (K_{13}) in a similar way. To see this we note that the circle with center on O_tM_t in the pencil generated by $(M_{t,4})$ and (O_5) intersects (O_t) in two points at a distance of $2r_A$ from AB. See Figure 21.

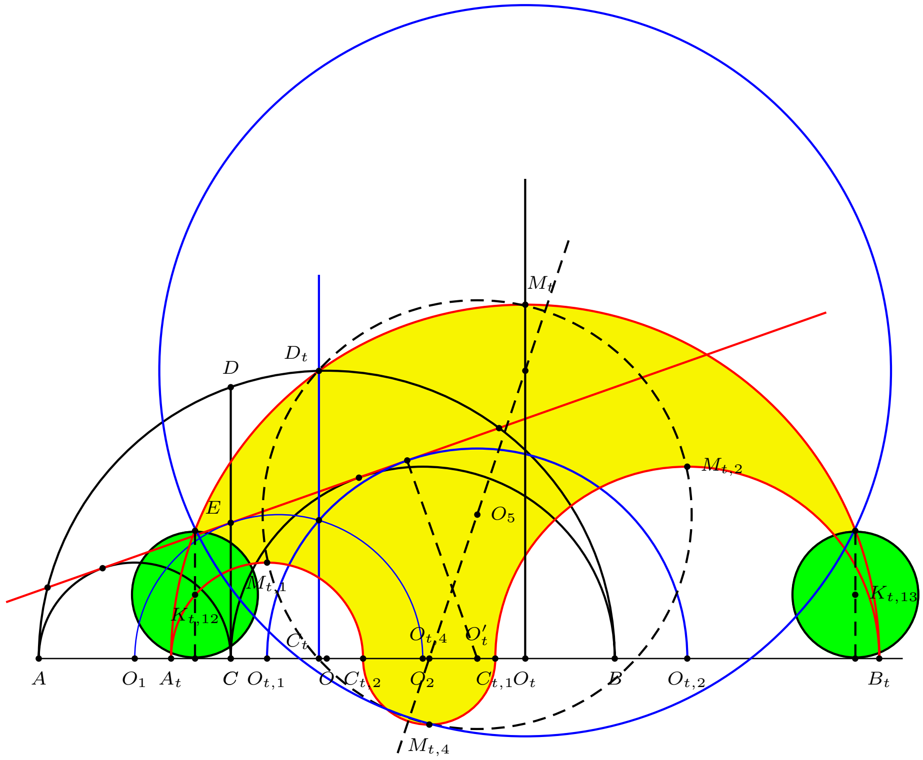

Figure 21. The Archimedean circles $(K_{t,12})$ and $(K_{t,13})$

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Proof by Picture: Products and Reciprocals of Diagonal Length Ratios in the Regular Polygon

Anne Fontaine and Susan Hurley

Abstract. These "proofs by picture" link the geometry of the regular n-gon to formulae concerning the arithmetic of real cyclotomic fields. We illustrate the formula for the product of diagonal length ratios

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

and that for the reciprocal of diagonal length ratios when $\gcd(n,k)=1$,

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j>0 : kj \equiv \pm 1 \bmod n\}.$$

1. Introduction

Consider a regular n-gon. Number the diagonals $d_1, d_2, \ldots, d_{n-1}$ (as shown in Figure 1.1 for $n=9$) including the sides of the polygon as d_1 and d_{n-1}. Although the length of d_i equals that of d_{n-i}, we shall use all $n-1$ subscripts since this simplifies our formulae concerning the diagonal lengths.

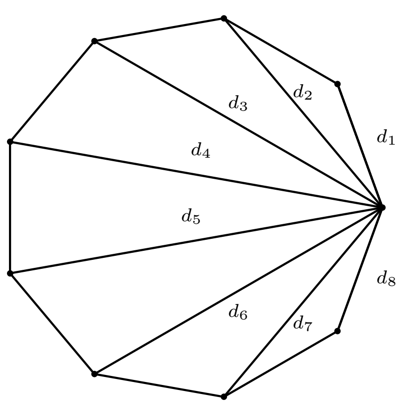

Figure 1.1

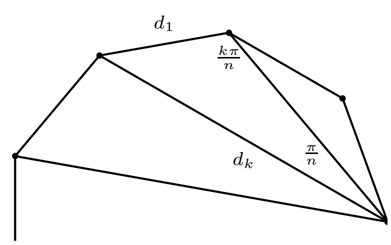

Figure 1.2

Ratios of the lengths of the diagonals are given by the law of sines. The ratio of the length of d_k to that of d_j is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{j\pi}{n}}$. In particular, Figure 1.2 shows us that the

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ratio of the length of d_k to that of d_1 is $\dfrac{\sin\frac{k\pi}{n}}{\sin\frac{\pi}{n}}$. This ratio of sines will be denoted r_k. Note that if the length of the side $d_1 = 1$, r_k is simply the length of the k-th diagonal.

2. Products of diagonal length ratios

It is an exercise in the algebra of cyclotomic polynomials to show that

$$r_h r_k = \sum_{i=1}^{\min[k,h,n-k,n-h]} r_{|k-h|+2i-1}.$$

This formula appears in Steinbach [1] for the case where $h + k \leq n$. Steinbach names it the *diagonal product formula* and makes use of it to derive a number of interesting properties of the diagonal lengths of a regular polygon. It is not hard to extend the formula to cover all $n-1$ values of h and k.

In order to understand the geometry of the *diagonal product formula*, consider two regular n-gons with the side of the larger equal to some diagonal of the smaller. Denote the diagonal lengths by $\{s_i\}_{i=1,\ \ldots,n-1}$ for the smaller polygon and $\{l_i\}_{i=1,\ \ldots,n-1}$ for the larger, so that $l_1 = s_k$ for some k. In this case, since $r_k = \frac{s_k}{s_1} = \frac{l_1}{s_1}$ and $r_h = \frac{l_h}{l_1}$, the product $r_k r_h$ becomes $\frac{l_h}{s_1}$ and when we multiply through by s_1 the diagonal product formula becomes

$$l_h = \sum_{i=1}^{\min[k,h,n-k,n-h]} s_{|k-h|+2i-1}.$$

In other words, each of the larger diagonal lengths is expressible as a sum of the smaller ones.

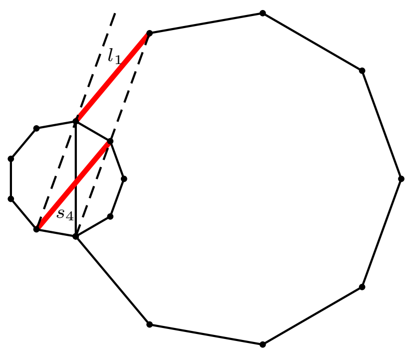

Figure 2.1

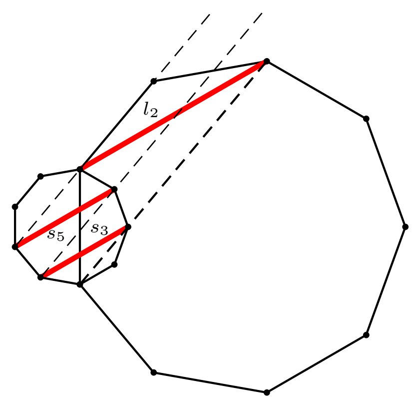

Figure 2.2

Figures 2.1-2.4 illustrate what happens when the nonagon is enlarged so that $l_1 = s_4$. The summation formula for the diagonals can be visualized by projecting the left edge of the larger polygon onto each of its other edges in turn. We observe from the first pair of nonagons that $l_1 = s_4 s_1 = s_4$, from the second that $l_2 =$

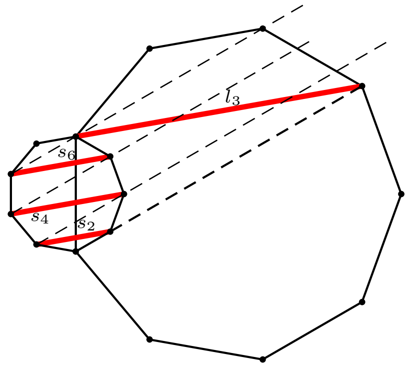

Figure 2.3

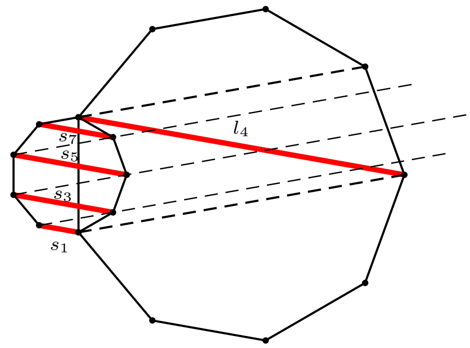

Figure 2.4

$s_4 s_2 = s_3 + s_5$, from the third that $l_3 = s_4 s_3 = s_2 + s_4 + s_6$, and from the last that $l_4 = s_4 s_4 = s_1 + s_3 + s_5 + s_7$. This is exactly what the *diagonal product formula* predicts.

When n and k are both even, the polygons do not have the same orientation, but the same strategy of projecting onto each side of the larger polygon in turn still works. Figure 3 shows the case $(n, k) = (6, 2)$. The sums are

$$\begin{aligned} l_1 &= s_2 s_1 = s_2, \\ l_2 &= s_2 s_2 = s_1 + s_3, \\ l_3 &= s_2 s_3 = s_3 + s_4. \end{aligned}$$

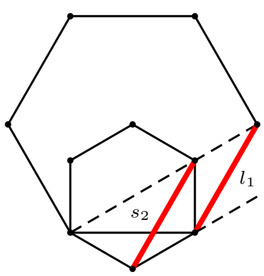

Figure 3.1

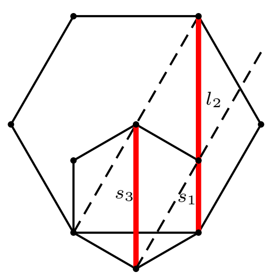

Figure 3.2

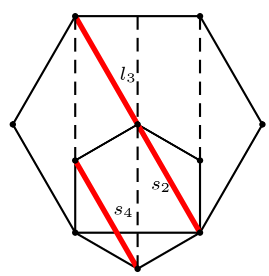

Figure 3.3

3. Reciprocal of diagonal length ratios

Provided $\gcd(k, n) = 1$, the diagonal length ratio r_k is a unit in the ring of integers of the real subfield of $\mathbf{Q}[\xi]$, ξ a primitive n-th root of unity. The set of ratios $\{r_i\}$ with $\gcd(i, n) = 1$ and $i \leq \frac{n}{2}$ forms a basis for this field [2]. Knowing that this was the case, we searched for a formula to express $\frac{1}{r_k}$ as an integral linear combination of diagonal length ratios. This time we found the picture first. We assume that the polygon has unit side, so that $r_i =$ the length of the i-th diagonal.

First note that $\frac{1}{r_k}$ is equal to the length of the line segment obtained when diagonal k intersects diagonal 2 as shown in Figure 4.1 for $(n, k) = (8, 3)$. In order to express this length in terms of the r_i, $i = 1, \ldots, n-1$, notice that, as in Figure 4.2 for $(n, k) = (7, 2)$, one can set off along diagonal k and zigzag back and forth, alternately parallel to diagonal k and in the vertical direction, until one arrives at a vertex adjoining the starting point. Summing the lengths of the diagonals parallel to diagonal k, with positive or negative sign according to the direction of travel, will give the desired reciprocal. For example follow the diagonals shown in Figure 4.2 to see that $\frac{1}{r_2} = r_2 + r_6 - r_4$.

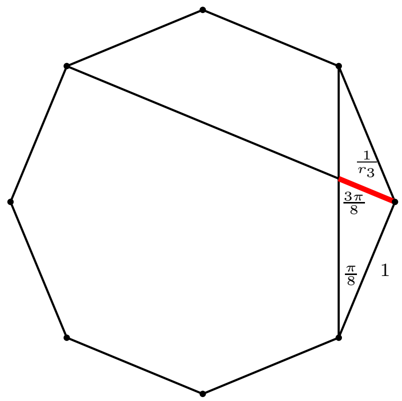

Figure 4.1

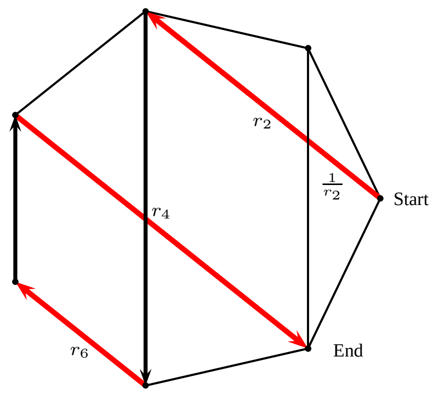

Figure 4.2

Following the procedure outlined above, the head of each directed diagonal used in the sum is k vertices farther around the polygon from the previous one. Also as we zigzag our way through the polygon, the positive contributions to the sum occur as we move in one direction with respect to diagonal k, while the negative contributions occur as we move the other way. In fact, allowing $r_i = \frac{\sin \frac{i\pi}{n}}{\sin \frac{\pi}{n}}$ to be defined for $i > n$, we realized that $r_i = -r_{2n-i}$ would have the correct sign to produce the simplest formula for the reciprocal, which is

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}, \qquad \text{where } s = \min\{j > 0 : kj \equiv \pm 1 \bmod n\}.$$

Once we had discovered this formula, we found it that it was a messy but routine exercise in cyclotomic polynomial algebra to verify its truth.

Although we are almost certain that these formulas for manipulating the diagonal length ratios must be in the classical literature, we have not been able to locate them, and would appreciate any lead in this regard.

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A Generalization of Power's Archimedean Circles

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Abstract. We generalize the Archimedean circles in an arbelos given by Frank Power.

Let three semicircles α, β and γ form an arbelos with inner semicircles α and β with diameters PA and PB respectively. Let a and b be the radii of the circles α and β. Circles with radii $t = \dfrac{ab}{a+b}$ are called Archimedean circles. Frank Power [2] has shown that for "highest" points Q_1 and Q_2 of α and β respectively, the circles touching γ and the line OQ_1 (respectively OQ_2) at Q_1 (respectively Q_2) are Archimedean (see Figure 1). We generalize these Archimedean circles.

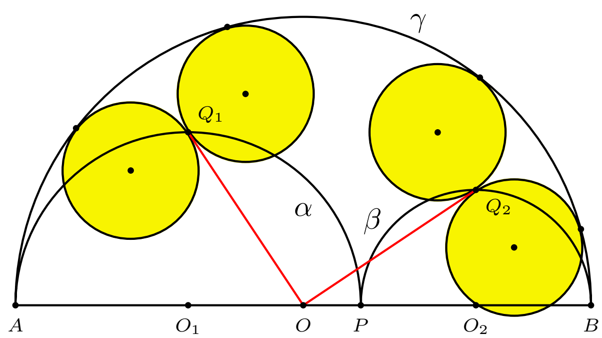

Figure 1

We denote the center of γ by O. Let Q be the intersection of the circle γ and the perpendicular of AB through P, and let δ be a circle touching γ at the point Q from the inside of γ. The radius of δ is expressed by $k(a+b)$ for a real number k satisfying $0 \leq k < 1$. The tangents of δ perpendicular to AB intersect α and β at points Q_1 and Q_2 respectively, and intersect the line AB at points P_1 and P_2 respectively (see Figures 2 and 3).

Theorem. (1) *The radii of the circles touching the circle γ and the line OQ_1 (respectively OQ_2) at the point Q_1 (respectively Q_2) are $2(1-k)t$.*
(2) *The circle touching the circles γ and α at points different from A and the line P_1Q_1 from the opposite side of B and the circle touching the circles γ and β at points different from B and the line P_2Q_2 from the opposite side of A are congruent with common radii $(1-k)t$.*

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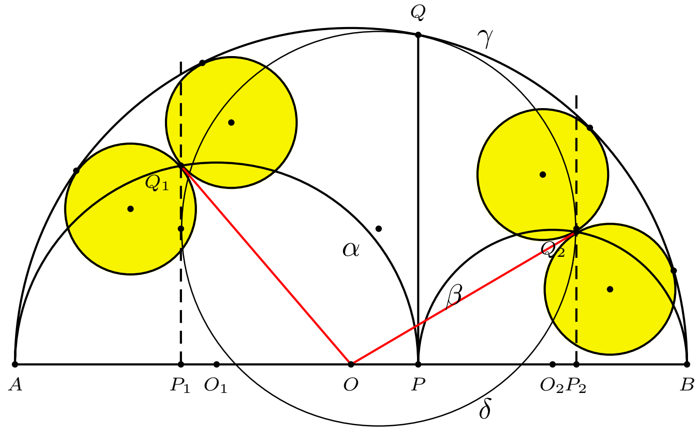

Figure 2

Proof. (1) Since $|PP_1| = 2ka$, $|OP_1| = (b-a) + 2ka$. While

$$|P_1Q_1|^2 = |PP_1||P_1A| = 2ka(2a-2ka) = 4k(1-k)a^2.$$

Hence $|OQ_1|^2 = ((b-a)+2ka)^2 + 4k(1-k)a^2 = (a-b)^2 + 4kab$. Let x be the radius of one of the circles touching γ and the line OQ_1 at Q_1. From the right triangle formed by O, Q_1 and the center of this circle, we get

$$(a+b-x)^2 = x^2 + (a-b)^2 + 4kab$$

Solving the equation for x, we get $x = \frac{2(1-k)ab}{a+b} = 2(1-k)t$. The other case can be proved similarly.

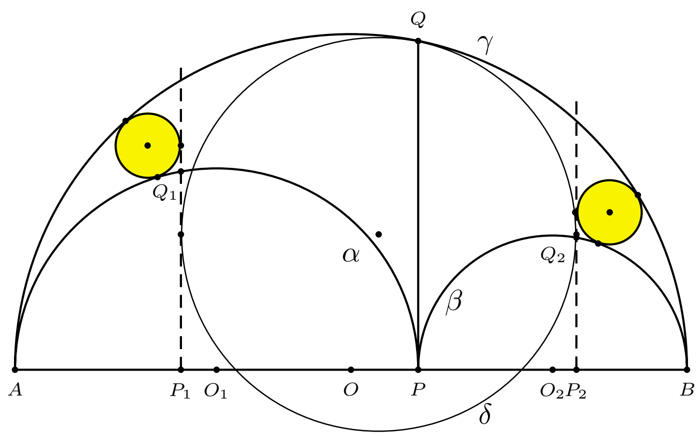

Figure 3

(2) The radius of the circle touching α externally and γ internally is proportional to the distance between the center of this circle and the radical axis of α and γ [1, p. 108]. Hence its radius is $(1-k)$ times of the radii of the twin circles of Archimedes. □

The Archimedean circles of Power are obtained when δ is the circle with a diameter OQ. The twin circles with the half the size of the Archimedean circles in [4] are also obtained in this case. The statement (2) is a generalization of the twin circles of Archimedes, which are obtained when δ is the point circle. In this case the points Q_1, Q_2 and P coincide, and we get the circle with radius $2t$ touching the line AB at P and the circle γ by (1) [3].

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Pseudo-Incircles

Stanley Rabinowitz

Abstract. This paper generalizes properties of mixtilinear incircles. Let (S) be any circle in the plane of triangle ABC. Suppose there are circles (S_a), (S_b), and (S_c) each tangent internally to (S); and (S_a) is inscribed in angle BAC (similarly for (S_b) and (S_c)). Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then it is shown that the lines AX, BY, and CZ meet in a point.

1. Introduction

A mixtilinear incircle of a triangle ABC is a circle tangent to two sides of the triangle and also internally tangent to the circumcircle of that triangle. In 1999, Paul Yiu discovered an interesting property of these mixtilinear incircles.

Proposition 1 (Yiu [8]). *If the points of contact of the mixtilinear incircles of $\triangle ABC$ with the circumcircle are X, Y, and Z, then the lines AX, BY, and CZ are concurrent* (Figure 1)*. The point of concurrence is the external center of similitude of the incircle and the circumcircle.* [1]

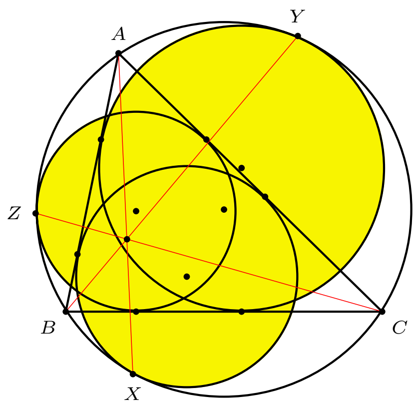

Figure 1

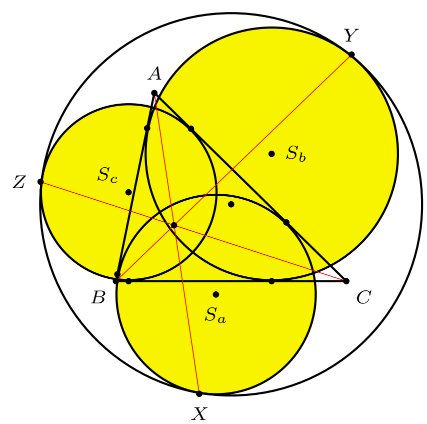

Figure 2

I wondered if there was anything special about the circumcircle in Proposition 1. After a little experimentation, I discovered that the result would remain true if the circumcircle was replaced with any circle in the plane of $\triangle ABC$.

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[1] This is the triangle center X_{56} in Kimberling's list [5].

Theorem 2. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each internally* (*respectively externally*) *tangent to (S). Furthermore, suppose (S_a), (S_b), (S_c) are inscribed in $\angle BAC$, $\angle ABC$, $\angle ACB$ respectively* (Figure 2)*. Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Then the lines AX, BY, and CZ are concurrent at a point P. The point P is the external* (*respectively internal*) *center of similitude of the incircle of $\triangle ABC$ and circle (S).*

When we say that a circle is *inscribed* in an angle ABC, we mean that the circle is tangent to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Definitions. Given a triangle and a circle, a *pseudo-incircle* of the triangle is a circle that is tangent to two sides of the given triangle and internally tangent to the given circle. A *pseudo-excircle* of the triangle is a circle that is tangent to two sides of the given triangle and externally tangent to the given circle.

There are many configurations that meet the requirements of Theorem 2. Figure 2 shows an example where (S) surrounds the triangle and the three circles are all internally tangent to (S). Figure 3a shows an example of pseudo-excircles where (S) lies inside the triangle. Figure 3b shows an example where (S) intersects the triangle. Figure 3c shows an example where (S) surrounds the triangle and the three circles are all externally tangent to (S).

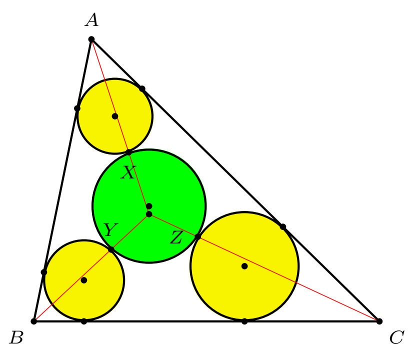

Figure 3a. Pseudo-excircles

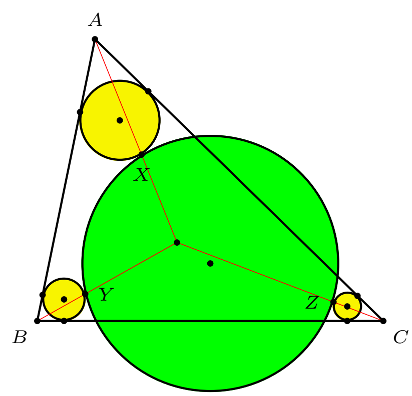

Figure 3b. Pseudo-excircles

There is another way of viewing Figure 3a. Instead of starting with the triangle and the circle (S), we can start with the other three circles. Then we get the following proposition.

Proposition 3. *Let there be given three circles in the plane, each external to the other two. Let triangle ABC be the triangle that circumscribes these three circles* (*that is, the three circles are inside the triangle and each side of the triangle is a common external tangent to two of the circles*)*. Let (S) be the circle that is externally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z* (Figure 3a)*. Then lines AX, BY, and CZ are concurrent.*

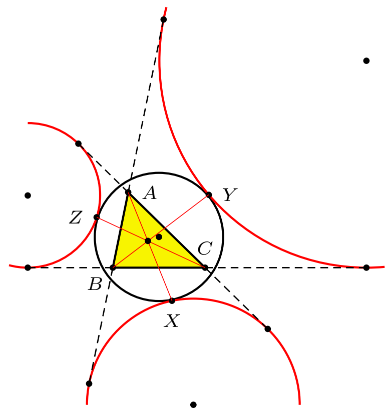

Figure 3c. Pseudo-excircles

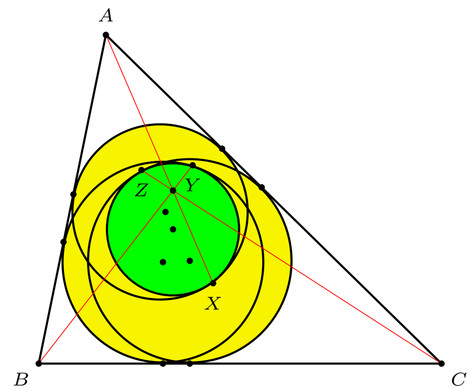

Figure 4. Pseudo-incircles

Figure 4 also shows pseudo-incircles (as in Figure 2), but in this case, the circle (S) lies inside the triangle. The other three circles are internally tangent to (S) and again, AX, BY, and CZ are concurrent. This too can be looked at from the point of view of the circles, giving the following proposition.

Proposition 4. *Let there be given three mutually intersecting circles in the plane. Let triangle ABC be the triangle that circumscribes these three circles* (Figure 4). *Let (S) be the circle that is internally tangent to all three circles. Let the points of tangency of (S) with the three circles be X, Y, and Z. Then lines AX, BY, and CZ are concurrent.*

We need the following result before proving Theorem 2. It is a generalization of Monge's three circle theorem ([7, p.1949]).

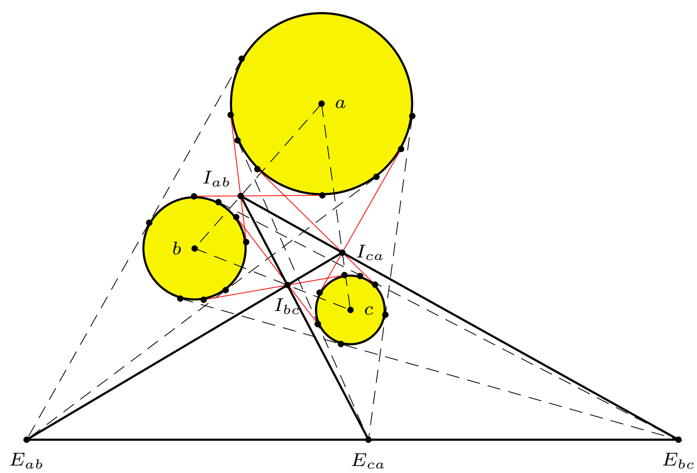

Figure 5. Six centers of similitude

Proposition 5 ([1, p.188], [2, p.151], [6])**.** *The six centers of similitude of three circles taken in pairs lie by threes on four straight lines* (Figure 5)*. In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

2. Proof of Theorem 2

Figure 6 shows an example where the three circles are all externally tangent to (S), but the proof holds for the internally tangent case as well. Let I be the center of the incircle.

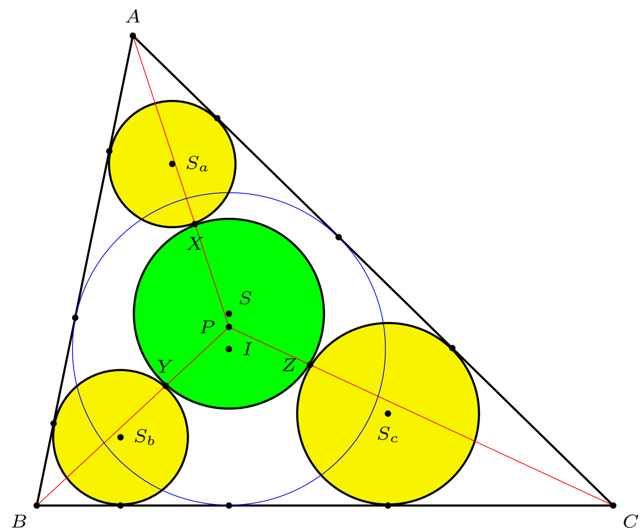

Figure 6

Consider the three circles (S), (I), and (S_c). The external common tangents of circles (J) and (S_c) are sides AC and BC, so C is their external center of similitude. Circles (S) and (S_c) are tangent externally (respectively internally), so their point of contact, Z, is their internal (respectively external) center of similitude. By Proposition 5, points C and Z are collinear with P, the internal (respectively external) center of similitude of circles (S) and (I). That is, line CZ passes through P. Similarly, lines AX and BY also pass through P.

Corollary 6. *In Theorem 2, the points S, P, and I are collinear* (Figure 6).

Proof. Since P is a center of similitude of circles (I) and (S), P must be collinear with the centers of the two circles. $\square$

3. Special Cases

Theorem 2 holds for any circle, (S), in the plane of the triangle. We can get interesting special cases for particular circles. We have already seen a special case in Proposition 1, where (S) is the circumcircle of $\triangle ABC$.

3.1. *Mixtilinear excircles.*

Corollary 7 (Yiu [9])**.** *The circle tangent to sides AB and AC of $\triangle ABC$ and also externally tangent to the circumcircle of $\triangle ABC$ touches the circumcircle at point X. In a similar fashion, points Y and Z are determined* (Figure 7)*. Then the lines AX, BY, and CZ are concurrent. The point of concurrence is the internal center of similitude of the incircle and circumcircle of $\triangle ABC$.*[2]

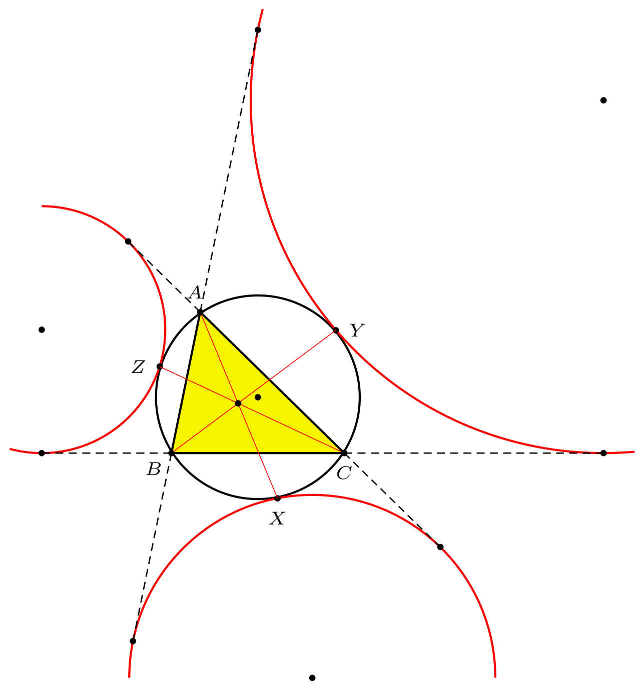

Figure 7. Mixtilinear excircles

3.2. *Malfatti circles.* Consider the Malfatti circles of a triangle ABC. These are three circles that are mutually externally tangent with each circle being tangent to two sides of the triangle.

Corollary 8. *Let (S) be the circle circumscribing the three Malfatti circles,* i.e., *internally tangent to each of them.* (Figure 8a). *Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

Corollary 9. *Let (S) be the circle inscribed in the curvilinear triangle bounded by the three Malfatti circles of triangle ABC* (Figure 8b)*. Let the points of tangency of (S) with the Malfatti circles be X, Y, and Z. Then AX, BY, and CZ are concurrent.*

[2]This is the triangle center X_{55} in Kimberling's list [5]; see also [4, p.75].

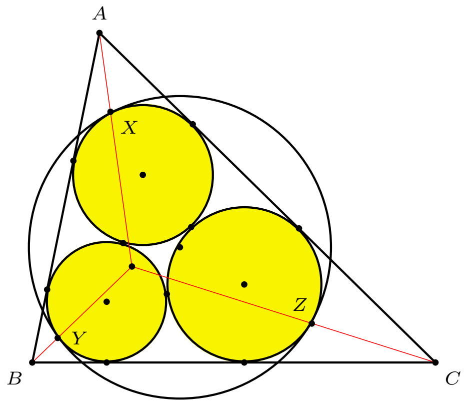

Figure 8a. Malfatti circumcircle

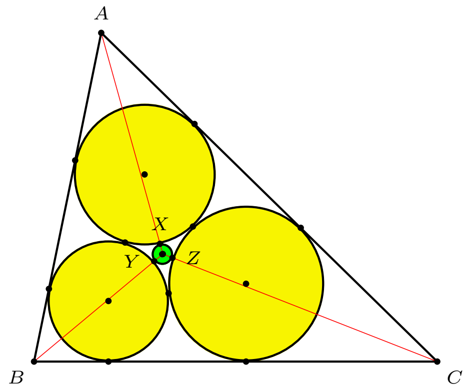

Figure 8b. Malfatti incircle

3.3. *Excircles.*

Corollary 10 (Kimberling [3])**.** *Let (S) be the circle circumscribing the three excircles of triangle ABC,* i.e., *internally tangent to each of them. Let the points of tangency of (S) with the excircles be X, Y, and Z.* (Figure 9). *Then AX, BY, and CZ are concurrent.*

The point of concurrence is known as the Apollonius point of the triangle. It is X_{181} in [5]. See also [4, p.102].

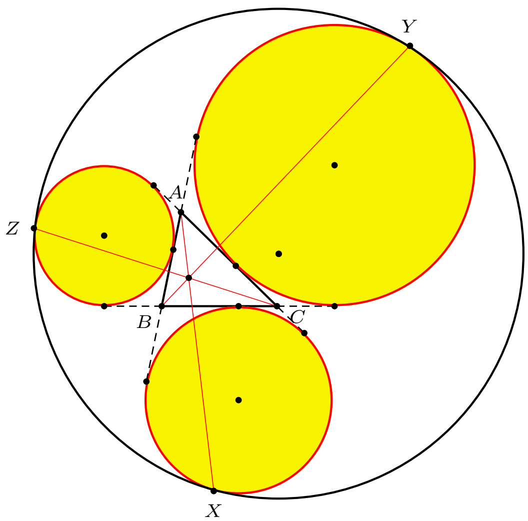

Figure 9. Excircles

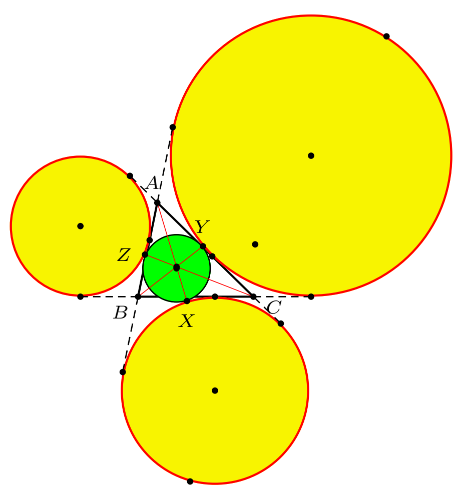

Figure 10. Nine-point circle

If we look at the circle externally tangent to the three excircles, we know by Feuerbach's Theorem, that this circle is the nine-point circle of $\triangle ABC$ (the circle that passes through the midpoints of the sides of the triangle).

Corollary 11 ([4, p.158])**.** *If the nine point circle of $\triangle ABC$ touches the excircles at points X, Y, and Z* (Figure 10)*, then AX, BY, and CZ are concurrent.*

4. Generalizations

In Theorem 2, we required that the three circles (S_a), (S_b), and (S_c) be inscribed in the angles of the triangle. In that case, lines AS_a, BS_b, and CS_c concur at the incenter of the triangle. We can use the exact same proof to handle the case where the three lines AS_a, BS_b, and CS_c meet at an excenter of the triangle. We get the following result.

Theorem 12. *Let (S) be any circle in the plane of $\triangle ABC$. Suppose that there are three circles, (S_a), (S_b), and (S_c), each tangent internally* (*respectively externally*) *to (S). Furthermore, suppose (S_a) is tangent to lines AB and AC; (S_b) is tangent to lines BC and BA; and (S_c) is tangent to lines CA and CB. Let the points of tangency of (S_a), (S_b), and (S_c) with (S) be X, Y, and Z, respectively. Suppose lines AS_a, BS_b, and CS_c meet at the point J, one of the excenters of $\triangle ABC$* (Figure 11)*. Furthermore, assume that sides BA and BC of the triangle are the external common tangents between excircle (J) and circle (S_a); similarly for circles (S_b), and (S_c). Then AX, BY, and CZ are concurrent at a point P. The point P is the external* (*respectively internal*) *center of similitude of circles (J) and (S). The points J, P, and S are collinear.*

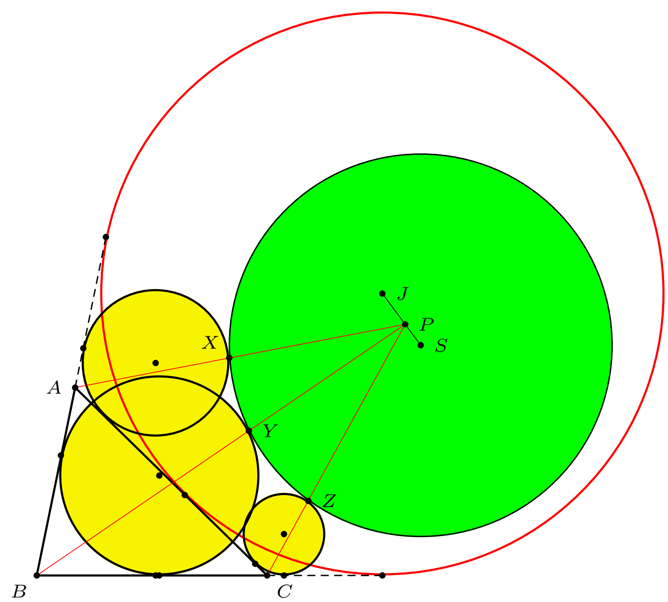

Figure 11. Pseudo-excircles

Figure 12 illustrates the case when (S) is tangent internally to each of (S_a), (S_b), (S_c).

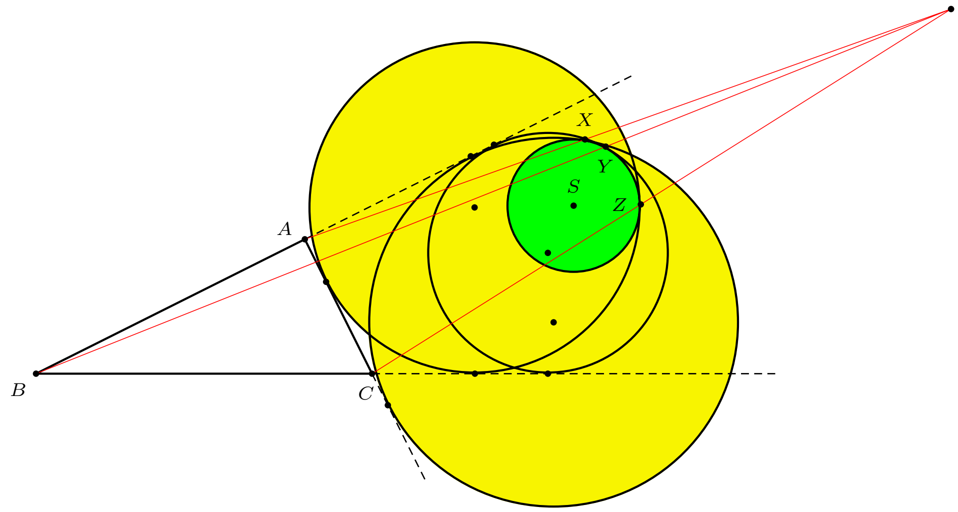

Figure 12. Pseudo-incircles

Examining the proof of Theorem 2, we note that at each vertex of the triangle, we have found that the line from that vertex to the point of contact of the circle inscribed in that angle and circle (S) passes through a fixed point (one of the centers of similitude of (S) and the incircle of the triangle). We note that the result and proof would be the same for a polygon provided that the polygon had an incircle. This gives us the following result.

Theorem 13. *Let $A_1A_2A_3\dots A_n$ be a convex n-gon circumscribed about a circle (J). Let (S) be any circle in the interior of this n-gon. Suppose there are n circles, (S_1), (S_2), …, (S_n) each tangent externally to (S) such that for $i = 1, 2, \dots, n$, circle (S_i) is also inscribed in angle $A_{i-1}A_iA_{i+1}$ (where $A_0 = A_n$ and $A_{n+1} = A_1$). Let X_i be the point of tangency of circles (S_i) and (S)* (Figure 13)*. Then the lines A_iX_i, $i = 1, 2, \dots, n$ are concurrent. The point of concurrence is the internal center of similitude of the circles (S) and (J).*

In order to generalize Theorem 2 to three dimensions, we need to first note that Proposition 5 generalizes to 3 dimensions.

Theorem 14. *The six centers of similitude of three spheres taken in pairs lie by threes on four straight lines. In particular, the three external centers of similitude are collinear; and any two internal centers of similitude are collinear with the third external one.*

Proof. Consider the plane through the centers of the three spheres. This plane passes through all 6 centers of similitude. The plane cuts each sphere in a circle. Thus, on this plane, Proposition 5 applies, thus proving that the result holds for the spheres as well. □

Theorem 15. *Let $T = A_1A_2A_3A_4$ be a tetrahedron. Let (S) be any sphere in the interior of T* (*or let (S) be any sphere surrounding T*)*. Suppose there are four*

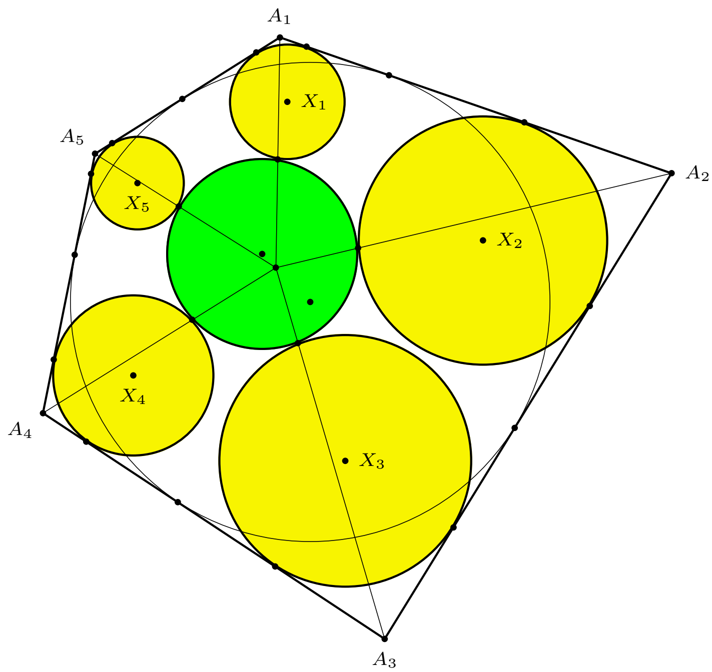

Figure 13. Pseudo-excircles in a pentagon

spheres, (S_1)*,* (S_2)*,* (S_3)*, and* (S_4) *each tangent internally* (*respectively externally*) *to* (S) *such that sphere* (S_i) *is also inscribed in the trihedral angle at vertex* A_i*. Let* X_i *be the point of tangency of spheres* (S_i) *and* (S)*. Then the lines* A_iX_i*,* $i = 1, 2, 3, 4$*, are concurrent. The point of concurrence is the external* (*respectively internal*) *center of similitude of sphere* (S) *and the sphere inscribed in* T*.*

The proof of this theorem is exactly the same as the proof of Theorem 2, replacing the reference to Proposition 5 by Theorem 14.

It is also clear that this result generalizes to E^n.

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Grassmann cubics and Desmic Structures

Wilson Stothers

Abstract. We show that each cubic of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ can be described as a Grassmann cubic. The geometry associates with each such cubic a cubic of type $\mathrm{p}\mathcal{K}$. We call this the parent cubic. On the other hand, each cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many child cubics. The key is the existence of a desmic structure associated with parent and child. This extends work of Wolk by showing that, not only do (some) points of a desmic structure lie on a cubic, but also that they actually generate the cubic as a locus. Along the way, we meet many familiar cubics.

1. Introduction

In Hyacinthos, #3991 and follow-up, Ehrmann and others gave a geometrical description of cubics as loci. They showed that each cubic of type $\mathrm{n}\mathcal{K}_0$ has an associated sister of type $\mathrm{p}\mathcal{K}$, and that each cubic of type $\mathrm{p}\mathcal{K}$ has three sisters of type $\mathrm{n}\mathcal{K}_0$. Here, we show that each cubic of type $\mathrm{n}\mathcal{K}$ but not of type $\mathrm{c}\mathcal{K}$ has a parent of type $\mathrm{p}\mathcal{K}$, and every cubic of type $\mathrm{p}\mathcal{K}$ has infinitely many children, but just three of type $\mathrm{n}\mathcal{K}_0$. Our results do not appear to extend to cubics of type $\mathrm{c}\mathcal{K}$, so the geometry must be rather different. Throughout, we use barycentric coordinates. We write the coordinates of P as $p : q : r$. We are interested in isocubics, that is circumcubics which are invariant under an isoconjugation. The theory of isocubics is beautifully presented in [1]. There we learn that an isocubic has an equation of one of the following forms :

$$\begin{array}{ll} \mathrm{p}\mathcal{K}(W,R): & rx(wy^2-vz^2)+sy(uz^2-wx^2)+tz(vx^2-uy^2)=0. \\ \mathrm{n}\mathcal{K}(W,R,k): & rx(wy^2+vz^2)+sy(uz^2+wx^2)+tz(vx^2+uy^2)+kxyz=0. \end{array}$$

The point $R = r : s : t$ is known as the pivot of the cubic $\mathrm{p}\mathcal{K}$, and the root of the cubic $\mathrm{n}\mathcal{K}$ $W = u : v : w$ is the pole of the isoconjugation. This means that the isoconjuagte of $X = x : y : z$ is $\frac{u}{x} : \frac{v}{y} : \frac{w}{z}$. We may view W as the image of G under the isoconjugation. The constant k in the latter equation is determined by a point on the curve which is not on a sideline. Another interpretation of k appears below. One important subclass of type $\mathrm{n}\mathcal{K}$ occurs when $k = 0$. We have

$$\mathrm{n}\mathcal{K}_0(W,R): \qquad rx(wy^2+vz^2)+sy(uz^2+wx^2)+tz(vx^2+uy^2)=0.$$

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Another subclass consists of the conico-pivotal isocubics. These are defined in terms of the root R and node $F = f : g : h$. The equation has the form

$$\mathrm{c}\mathcal{K}(\#F, R): \qquad rx(hy-gz)^2 + sy(fz-hx)^2 + tz(gx-fy)^2 = 0.$$

It is the $\mathrm{n}\mathcal{K}(W, R, k)$ with $W = f^2 : g^2 : h^2$, so F is a fixed point of the isoconjuation, and $k = -2(rgh + shf + tfg)$. We require a further concept from [1], that of the polar conic of R for $\mathrm{p}\mathcal{K}(W, R)$. In our notation, this has equation

$$\mathrm{p}\mathcal{C}(W, R): \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

Notice that, $\mathrm{p}\mathcal{C}(W, R)$ contains the four fixed points of the isoconjugation, so is determined by one other point, such as R. Hence, as befits a polar object, if S is on $\mathrm{p}\mathcal{C}(W, R)$, then R is on $\mathrm{p}\mathcal{C}(W, S)$. Also, for fixed W, the class of cubics of type $\mathrm{p}\mathcal{K}$ splits into disjoint subclasses, each subclass having a common polar conic. We also remark that the equation for the polar conic vanishes identically when W is the barycentric square of R. In such a case, it is convenient to regard the class of $\mathrm{p}\mathcal{K}(W, R)$ as consisting of just $\mathrm{p}\mathcal{K}(W, R)$ itself.

Definition 1. (a) *For a point X with barycentrics $x : y : z$, the A-harmonic of X is the point X_A with barycentrics $-x : y : z$. The B- and C-harmonics, X_B and X_C, are defined analagously.*
(b) *The harmonic associate of the triangle $\triangle PQR$ is the triangle $\triangle P_A Q_B R_C$.*

Observe that $\triangle X_A X_B X_C$ is the anticevian triangle of X. Also, if X is on $\mathrm{p}\mathcal{C}(W, R)$ then X_A, X_B, X_C also lie on $\mathrm{p}\mathcal{C}(W, R)$. In the preamble to $X(2081)$ in [5] we meet Gibert's PK- and NK-transforms. These are defined in terms of isogonal conjugation. We shall need the general case for W-isoconjugation.

Definition 2. *Suppose that W is a fixed point, let P^* denote the W-isoconjugate of P.*
(a) *The PKW-transform of P is PK(W,P), the intersection of the tripolars of P and P^* (if $P \neq P^*$).*
(b) *The NKW-transform of P is $NK(W, P)$, the crosspoint of P and P^*.*

Note that $PK(W, P)$ is the perspector of the circumconic through P and P^*. $PK(W, P)$ occurs several times in our work, in apparently unrelated contexts. From these definitions, if $W = u : v : w$ and $P = p : q : r$, then

$$\begin{aligned} PK(W,P) &= pu(r^2v - q^2w) : qv(p^2w - r^2u) : rw(q^2u - p^2v), \\ NK(W,P) &= pu(r^2v + q^2w) : qv(p^2w + r^2u) : rw(q^2u + p^2v). \end{aligned}$$

Note that $PK(W, R)$ is undefined if $W = R^2$.

2. Grassmann cubics associated with desmic Structures

Definition 3. *Let $\Delta = \triangle ABC$ be the reference triangle. Let $\Delta' = \triangle A'B'C'$ where A' is not on BC, B' is not on CA and C' is not on AB.*
(a) $GP(\Delta') = \{P :$ *the triangle with vertices $A'P \cap BC$, $B'P \cap CA$, $C'P \cap AB$ is perspective with Δ*$\}$,
(b) $GN(\Delta') = \{P :\ A'P \cap BC,\ B'P \cap CA,\ C'P \cap AB$ *are collinear*$\}$.

We note that $GN(\Delta)$ is a special case of a Grassmann cubic. There are three examples in the current (November 2004) edition of [2]. The Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $K004$ in Gibert's catalogue. Under the properties listed is the fact that it is the locus of points whose pedal triangle is a cevian triangle. This is $GP(\triangle A'B'C')$, where the vertices are the infinite points on the altitudes. The corresponding GN is then the union of the line at infinity and the circumcircle, a degenerate cubic. In the discussion of the cubic $\mathrm{n}\mathcal{K}(K, X(5))$ - Gibert's $K216$ - it is mentioned that it is GN for a certain triangle, and that the corresponding GP is the Neuberg cubic, $\mathrm{p}\mathcal{K}(K, X(30))$ - Gibert's $K001$. As we shall see, there are other examples where suitable triangles are listed, and others where they can be identified.

Lemma 1. *If* $\Delta' = \triangle A'B'C'$ *has* $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, *then,*
(a) $GP(\Delta')$ *has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) + (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0,$$

(b) $GN(\Delta')$ *has equation*

$$(a_2x - a_1y)(b_3y - b_2z)(c_1z - c_3x) - (a_1z - a_3x)(b_1y - b_2x)(c_2z - c_3y) = 0.$$

(c) *Each locus contains* A, B, C *and* A', B', C'.

Proof. It is easy to see that, if $P = x : y : z$, then

$$\begin{aligned} A_P &= A'P \cap BC = 0 : a_2x - a_1y : a_3x - a_1z, \\ B_P &= B'P \cap CA = b_1y - b_2x : 0 : b_3y - b_2z, \\ C_P &= C'P \cap AB = c_1z - c_3x : c_2z - c_3y : 0. \end{aligned}$$

The condition in $GN(\Delta)$ is equivalent to the vanishing of the determinant of the coefficients of these points. This gives (b).

The condition in $GP(\Delta)$ is equivalent to the concurrence of AA_P, BB_P and CC_P. This is another determinant condition. The determinant is formed from the previous one by changing the sign of one entry in each row. This gives (a). Once we have the equations, (c) is clear. □

We will also require a condition for a triangle $\triangle A'B'C'$ to be perspective with $\triangle ABC$.

Lemma 2. *If* $\Delta = \triangle A'B'C'$ *has* $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, *then,*
(a) $\triangle A'B'C'$ *is perspective with* $\triangle ABC$ *if and only if* $a_2b_3c_1 = a_3b_1c_2$.
(b) $\triangle A'B'C'$ *is triply perspective with* $\triangle ABC$ *if and only if* $a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2$.

Proof. We observe that the perspectivity in (a) is equivalent to the concurrence of AA', BB' and CC'. The given equality expresses the condition for the intersection of AA' and BB' to lie on CC'. Part (b) follows by noting that $\triangle A'B'C'$ is triply perspective with $\triangle ABC$ if and only if each of $\triangle A'B'C'$, $\triangle B'C'A'$ and $\triangle C'A'B'$ is perspective with $\triangle ABC$. □

We observe that each of the equations in Lemma 1 has the form

$$x(f_1y^2 + f_2z^2) + y(g_1z^2 + g_2x^2) + z(h_1x^2 + h_2y^2) + kxyz = 0. \qquad (1)$$

This has the correct form for the cubic to be of type $\mathrm{p}\mathcal{K}$ or $\mathrm{n}\mathcal{K}$.

Theorem 3. *For a triangle $\Delta' = \triangle A'B'C'$,*
(a) $GP(\Delta)$ *is of type* $\mathrm{p}\mathcal{K}$ *if and only if* Δ' *is perspective with* $\triangle ABC$.
(b) *If* Δ' *is degenerate, then* $GN(\Delta')$ *is degenerate.*
Suppose that $GN(\Delta')$ *is non-degenerate. Then*
(c) $GN(\Delta')$ *is of type* $\mathrm{n}\mathcal{K}$ *if and only if* Δ' *is perspective with* $\triangle ABC$.
(d) $GN(\Delta')$ *is of type* $\mathrm{n}\mathcal{K}_0$ *if and only if* Δ' *is triply perspective with* $\triangle ABC$.

Proof. Suppose that $A' = a_1 : a_2 : a_3$, $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$, with $a_1b_2c_3 \neq 0$.

(a) We begin by observing that equation (1) gives a cubic of type $\mathrm{p}\mathcal{K}$ if and only if $f_1g_1f_1 + f_2g_2h_2 = 0$ and $k = 0$. The equation in Lemma 1(a) has k = 0 if and only if $a_2b_3c_1 = a_3b_1c_2$. By Lemma 2(a), this is the condition for the triangles to be perspective. A Maple calculation shows that the other condition for a $\mathrm{p}\mathcal{K}$ is satisfied if the triangles are perspective. This establishes (a).

(b) If Δ' is degenerate, then A', B' and C' lie on a line $\mathcal{L}$. For any P on $\mathcal{L}$, the intersection of PA' with BC lies on $\mathcal{L}$, as do those of PB' with CA, and PC' with AB. Thus, these intersections are collinear, so P is on $GN(\Delta')$. Now the locus contains the line $\mathcal{L}$, so that it must be degenerate.

Now suppose that the locus $GN(\Delta')$ is non-degenerate.

(c) Equation (1) gives a cubic of type $\mathrm{n}\mathcal{K}$ if and only if $f_1g_1f_1 - f_2g_2h_2 = 0$. The equation in Lemma 1(b) has this property if and only if

$$(a_2b_3c_1 - a_3b_1c_2)D = 0,$$

where D is the determinant of the matrix whose rows are the coordinates of A', B' and C'. Now, $D = 0$ if and only if A', B' and C' are collinear, so Δ' and hence $GN(\Delta')$ are degenerate. By Lemma 2(a), the other condition is equivalent to the perspectivity of the triangles.

(d) For a cubic of type $\mathrm{n}\mathcal{K}_0$, we require $a_2b_3c_1 - a_3b_1c_2 = 0$, as in (c). We also require that $k = 0$. From the equation in Lemma 1(b), $k = a_2b_3c_1 + a_3b_1c_2 - 2a_1b_2c_3$. These two conditions are equivalent to triple perspectivity by Lemma 2(b). □

Our work so far leads us to consider triangles $\triangle A'B'C'$ perspective to $\triangle ABC$, with A' not on BC, B' not on CA, and C' not on AB. Looking at our loci, we are led to consider a further triangle, also perspective with $\triangle ABC$. This turns out to be the desmic mate of $\triangle A'B'C'$, so we are led to consider desmic structures which include the points A, B and C.

Theorem 4. *Suppose that* $\triangle A'B'C'$ *is perspective to* $\triangle ABC$, *with* A' *not on* BC, B' *not on* CA, *and* C' *not on* AB. *Let the perspector be* $P_1 = p_{11} : p_{12} : p_{13}$.

(a) *Suitably normalized, we have*

$$\begin{aligned} A' &= p_{21} : p_{12} : p_{13}, \\ B' &= p_{11} : p_{22} : p_{13}, \\ C' &= p_{11} : p_{12} : p_{23}. \end{aligned}$$

Let

$$\begin{aligned} W &= p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}, \\ R &= p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}, \\ S &= p_{11} + p_{21} : p_{12} + p_{22} : p_{13} + p_{23}. \end{aligned}$$

(b) $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W, S)$,
(c) $GN(\triangle A'B'C') = \mathrm{n}\mathcal{K}(W, R, 2(p_{21}p_{22}p_{23} - p_{11}p_{12}p_{13}))$.

Let $A'' = p_{11} : p_{22} : p_{23}$, $B'' = p_{21} : p_{12} : p_{23}$, $C'' = p_{21} : p_{22} : p_{13}$. *These are the* W*-isoconjugates of* A', B' *and* C'.
(d) $\triangle A''B''C''$ *is perspective with* $\triangle ABC$, *with perspector* $P_2 = p_{21} : p_{22} : p_{23}$.
(e) $\triangle A''B''C''$ *is perspective with* $\triangle A'B'C'$, *with perspector* S.
(f) (P_1, P_2, R, S) *is a harmonic range.*
(g) $GP(\triangle A''B''C'') = GP(\triangle A'B'C')$. *The common locus includes* P_1, P_2 *and* S.
(h) $GN(\triangle A''B''C'') = GN(\triangle A'B'C')$. *The common locus includes the intersections of the tripolar of* R *with the sidelines.*
(i) $\triangle ABC$, $\triangle A'B'C'$ *and* $\triangle A''B''C''$ *have common perspectrix, the tripolar of* R.
(j) P_1 *and* P_2 *lie on* $\mathrm{p}\mathcal{K}(W, R)$.

Proof. (a) Since we have the perspector, the coordinates of the vertices A', B', C' must be as described.

(b),(c) These are simply verifications using the equations in Lemma 1.

(d) This follows at once from the coordinates of A'', B'', C''.

(e) This requires the calculations that the lines $A'A''$, $B'B''$ and $C'C''$ all pass through S.

(f) The coordinates of the points make this clear.

(g),(h) First, we note that, if we interchange the roles of P_1 and P_2, we get the same equations. The fact that the given points lie on the respective loci are simply verifications.

(i) Once again, this can be checked by calculation. We can also argue geometrically. Suppose that a point X on $B'C'$ lies on the locus. Then XB' and XC' are $B'C'$, so this must be the common line. Then XA' must meet BC on this line. But X is on $B'C'$, so $X = BC \cap B'C'$. If X also lies on $B''C''$, then $X = BC \cap B''C''$. This shows that we must have a common perspectrix. The identification of the perspectrix uses the fact that the cubic meets the each sideline of $\triangle ABC$ in just three points, two vertices and the intersection with the given tripolar.

(j) This is a routine verification. □

In the notation of Theorem 4, we have a desmic structure with the twelve points described as vertices A, B, C, A', B', C', A'', B'', C'', perspectors P_1, P_2, S.

Many authors describe S as the desmon, and R as the harmon of the structure. Some refer to P_1 as the perspector, and P_2 as the coperspector. This description of a desmic structure with vertices including A, B, C is discussed by Barry Wolk in Hyacinthos #462. He observed that the twelve points all lie on $\mathrm{p}\mathcal{K}(W,S)$. What may be new is the fact that the other vertices may be used to generate this cubic as a locus, and the corresponding $\mathrm{n}\mathcal{K}$ as a Grassmann cubic.

Notice that the desmic structure in not determined by its perspectors. If we choose barycentrics for P_1, we need to scale P_2 so that the barycentrics of the desmon and harmon are, respectively, the sum and difference of those of P_1 and P_2. When P_2 is suitably scaled, we say that the perspectors are normalized. The normalization is determined by a single vertex, provided neither perspector is on a sideline.

There are two obvious questions.

(1) Is every cubic of type $\mathrm{p}\mathcal{K}$ a locus of type GP associated with a desmic structure?

(2) Is every cubic of type $\mathrm{n}\mathcal{K}$ a locus of type GN associated with a desmic structure?

The answer to (1) is that, with six exceptions, each point on a $\mathrm{p}\mathcal{K}$ is a perspector of a suitable desmic structure. The answer to (2) is more complicated. There is a class of $\mathrm{n}\mathcal{K}$ which do not possess a suitable desmic structure. This is the class of conico-pivotal isocubics. For each other cubic of type $\mathrm{n}\mathcal{K}$, there is a unique desmic structure.

Theorem 5. *Suppose that P is a point on $\mathrm{p}\mathcal{K}(W,S)$ which is not fixed by W-isoconjugation, and is not S or its W-isoconjugate. Then there is a unique desmic structure with vertices A, B, C and perspector $P_1 = P$ with locus $GP = \mathrm{p}\mathcal{K}(W,S)$.*

Proof. For brevity, we shall write X^* for the W-isoconjugate of a point X. As P is on $\mathrm{p}\mathcal{K}(W,S)$, S is on PP^*. Then $S = mP + nP^*$, for some constants m, n, with $mn \neq 0$. If $P = p : q : r$, $W = u : v : w$, put $A' = nu/p : mq : mr$, $B' = mp : nv/q : mr$, $C' = mp : mq : nw/r$. From Theorem 4, this has locus $GP(\triangle A'B'C') = \mathrm{p}\mathcal{K}(W,S)$. □

Note that the conditions on P are necessary to ensure that S can be expressed in the stated from. For the second question, we proceed in two stages. First, we show that a cubic of type $\mathrm{n}\mathcal{K}$ has at most one suitable desmic structure. This identifies the vertices of the structure. We then show that this choice does lead to a description of the cubic as a Grassmann cubic. The first result uses the idea of A-harmonics introduced in the Introduction.

Theorem 6. *Throughout, we use the notation of Theorem 4.*
(a) *The vertices of the desmic structure on $\mathrm{n}\mathcal{K}(W,R,k)$ are A, B, C and intersections of $\mathrm{n}\mathcal{K}(W,R,k)$ with the cubics $\mathrm{p}\mathcal{K}(W,R_A)$, $\mathrm{p}\mathcal{K}(W,R_B)$, $\mathrm{p}\mathcal{K}(W,R_C)$.*
(b) *$\mathrm{n}\mathcal{K}(W,R,k)$ and $\mathrm{p}\mathcal{K}(W,R_A)$ touch at B and C, intersect at A, at the intersections of the tripolar of R with AB and AC, and at two further points.*

(c) *If* $W = f^2 : g^2 : h^2$*, then* $\mathrm{c}\mathcal{K}(\#F, R)$ *and* $\mathrm{p}\mathcal{K}(W, R_A)$ *touch at* B *and* C*, and meet at* A*, at the intersections of the tripolar of* R *with* AB *and* AC*, and twice at* F.
(d) *If the final two points in* (b) *coincide, then* $\mathrm{n}\mathcal{K}(W, R, k) = \mathrm{c}\mathcal{K}(\#F, R)$*, where* F *is such that* $W = F^2$.
(e) *If either of the final points in* (b) *lie on a sideline, then* $k = 2ust/r$, $2vrt/s$ *or* $2wrs/t$*, where* $W = u : v : w$, $R = r : s : t$.

Proof. We observe that the vertices of the desmic structure can be derived from the normalized versions of the perspectors P_1 and P_2. The normalization is such that $R = P_1 - P_2$ and $S = P_1 + P_2$. Now consider the loci derived from the perspectors P_1 and $-P_2$. From Theorem 4, the cubics are $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{n}\mathcal{K}(W, S, k')$, for some k'. The point $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{n}\mathcal{K}(W, R, k)$, so that $A'_A = -p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R)$. Then A' is on $\mathrm{p}\mathcal{K}(W, R_A)$ as only the first term is affected by the sign change, and this involves the product of the first coordinates. Thus (a) holds.

Part (b) is largely computational. Obviously the cubics meet at A, B and C. The tangents at B and C coincide. The intersections with the sidelines in each case include the stated meetings with the tripolar of R. Since two cubics have a total of nine meets, there are two unaccounted for. These are clearly W-isoconjugate.

For part (c), we use the result of (b) to get seven intersections. Then we need only verify that the cubics meet twice at F. Now F is clearly on $\mathrm{p}\mathcal{K}(W, R_A)$. But F is a double point on $\mathrm{c}\mathcal{K}(\#F, R)$, so there are two intersections here.

Part (d) relies on a Maple calculation. Solving the equations for $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ for $\{y, z\}$, we get the known points and the solutions of a quadratic. The discriminant vanishes precisely when $\mathrm{n}\mathcal{K}(W, R, k)$ is of type $\mathrm{c}\mathcal{K}$.

Part (e) uses the same computation. The quadratic equation in (d) has constant term zero precisely when $k = 2vrt/s$. Looking at other $\mathrm{p}\mathcal{K}(W, R_B)$, $\mathrm{p}\mathcal{K}(W, R_C)$ gives the other cases listed. □

To describe a cubic of type $\mathrm{n}\mathcal{K}$ as a Grassmann cubic as above, we require two perspectors interchanged by W-isoconjugation. Provided W is not on a sideline, we need six vertices not on a sideline. After Theorem 6, there are at most six candidates, with two on each of the associated $\mathrm{p}\mathcal{K}(W, R_X)$, $X = A, B, C$. Thus, there is at most one desmic structure defining the cubic. Also from Theorem 6, there is no structure if the cubic is a $\mathrm{c}\mathcal{K}(\#F, R)$, for then the "six" points and the perspectors are all F. We need to investigate the cases where either of the final solutions in Theorem 6(b) lie on a sideline. Rather than interrupt the general argument, we postpone the discussion of these cases to an Appendix. They still contain a unique structure which can be used to generate the cubics as Grassmann loci. The structure is a degenerate kind of desmic structure. We show that, in any other case, the six points do constitute a suitable desmic structure. In the proof, we assume that we can choose an intersection of $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, R_A)$ not on a sideline, so we need the discussion of the Appendix to tidy up the remaining cases.

Theorem 7. *If a cubic $\mathcal{C}$ is of type* $\mathrm{n}\mathcal{K}$*, but not of type* $\mathrm{c}\mathcal{K}$*, then there is a unique desmic structure which defines $\mathcal{C}$ as a Grassmann cubic.*

Proof. Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}(W, R, k)$ with $W = u : v : w$, and $R = r : s : t$. We require perspectors $P_1 = p_{11} : p_{12} : p_{13}$, and $P_2 = p_{21} : p_{22} : p_{23}$ such that $r : s : t = p_{11} - p_{21} : p_{12} - p_{22} : p_{13} - p_{23}$. This amounts to two linear equations which can be used to solve for p_{22} and p_{23} in terms of p_{11}, p_{12}, p_{13} and p_{21}. We also require that $u : v : w = p_{11}p_{21} : p_{12}p_{22} : p_{13}p_{23}$. This gives three relations in p_{11} and p_{21} in terms of p_{12} and p_{13}. These are consistent provided $A' = p_{21} : p_{12} : p_{13}$ is on $\mathrm{p}\mathcal{K}(W, R_A)$. This uses a Maple calculation. We can solve for p_{11} in terms of p_{12}, p_{13} and p_{21}, provided we do not have (after scaling) $p_{12} = -u/r$, $p_{13} = v/s$ and $p_{21} = w/t$. So far, we have shown that, if A' is on $\mathrm{p}\mathcal{K}(W, R_A)$, then we can reconstruct perspectors which give rise to some $\mathrm{n}\mathcal{K}(W, R, k')$. Provided that we can choose A' also on $\mathcal{C}$, but not on a sideline, then $k' = k$ directly, so we get $\mathcal{C}$ as a Grassmann cubic. As we saw in Theorem 6, there are just two such choices of A', and these are isoconjugate, so we have just one suitable desmic structure. We could equally use a point of intersection of C with $\mathrm{p}\mathcal{K}(W, R_B)$ or with $\mathrm{p}\mathcal{K}(W, R_C)$. It follows that there is a unique desmic structure unless $\mathcal{C}$ has the points $-u/r : v/s : w/t$, $u/r : -v/s : w/t$ and $u/r : v/s : -w/t$. But then $k = 2ust/r = 2vrt/s = 2wrs/t$, so that $W = R_2$, and $k = 2rst$. It follows that C is the degenerate cubic $(ty + sz)(rz + tx)(sx + ry) = 0$. It is easy to check that this is given as a Grassmann cubic by the degenerate desmic structure with $A' = A'' = -r : s : t$, and similarly for B', B'', S' and C''. This has perspectors, desmon and harmon equal to R. The GP locus is the whole plane. □

3. Parents and children

The reader will have noted the resemblance between the equations for the cubics $GP(\Delta')$ and $GN(\Delta')$. In [2, notes on K216], Gibert observes this for $K001$ and $K216$. He refers to $K216$ as a sister of $K001$. In Theorem 7, we saw that each cubic $\mathcal{C}$ of type $\mathrm{n}\mathcal{K}$ which is not of type $\mathrm{c}\mathcal{K}$ is the Grassmann cubic associated with a unique desmic structure, and hence with a unique cubic $\mathcal{C}'$ of type $\mathrm{p}\mathcal{K}$. We call the cubic $\mathcal{C}'$ the parent of $\mathcal{C}$. On the other hand, Theorem 5 shows that a cubic $\mathcal{C}$ of type $\mathrm{p}\mathcal{K}$ contains infinitely many desmic structures, each defining a cubic of type $\mathrm{n}\mathcal{K}$. We call each of these cubics a child of $\mathcal{C}$. Our first task is to describe the children of a cubic $\mathrm{p}\mathcal{K}(W, S)$. This involves the equation of the polar conic of S, see §1. Our calculations also give information on the parents of the family of cubics of type $\mathrm{n}\mathcal{K}$ with fixed pole and root.

Theorem 8. *Suppose that $\mathcal{C} = \mathrm{p}\mathcal{K}(W, S)$ with $W = u : v : w$, and $S = r : s : t$.*
(a) *Any child of $\mathcal{C}$ is of the form $\mathrm{n}\mathcal{K}(W, R, k)$, with R on*

$$\mathrm{p}\mathcal{C}(W, S) : \qquad (vt^2 - ws^2)x^2 + (wr^2 - ut^2)y^2 + (us^2 - vr^2)z^2 = 0.$$

(b) *If R is a point of $\mathrm{p}\mathcal{C}(W, S)$ which is not S and not fixed by W-isoconjugation, then there is a unique child of $\mathcal{C}$ of the form $\mathrm{n}\mathcal{K}(W, R, k)$.*
(c) *Any cubic $\mathrm{n}\mathcal{K}(W, R, k)$ which is not of type $\mathrm{c}\mathcal{K}$ has parent of the form $\mathrm{p}\mathcal{K}(W, S)$ with S on $\mathrm{p}\mathcal{C}(W, R)$.*

(d) *If* $\mathrm{n}\mathcal{K}(W,R,k)$ *has parent* $\mathrm{p}\mathcal{K}(W,S)$*, then the perspectors are the non-trivial intersections of* $\mathrm{p}\mathcal{K}(W,R)$ *and* $\mathrm{p}\mathcal{K}(W,S)$.

Proof. (a) We know from Theorem 7 that a child $\mathrm{n}\mathcal{K}(W,R,k)$ of $\mathcal{C}$ arises from a desmic structure. Suppose the perspectors are P_1 and P_2. From Theorem 4(f) (P_1,P_2,R,S) is a harmonic range. It follows that there are constants m and n with $P_1 = mR+nS$ and $P_2 = -mR+nS$. From Theorem 4(a), W is the barycentric product of P_1 and P_2. Suppose that $R = x:y:z$. Then we have

$$\frac{m^2x^2-n^2r^2}{u} = \frac{m^2y^2-n^2s^2}{v} = \frac{m^2z^2-n^2t^2}{w}.$$

If we eliminate m^2 and n^2 from these, we get $\mathrm{p}\mathcal{C}(W,S)=0$.

(b) Given such an R, we can reverse the process in (a) to obtain a suitable value for $(m/n)^2$. Choosing either root, we get the required perspectors.

(c) is really just the observation that S is on $\mathrm{p}\mathcal{C}(W,R)$ if and only if R is on $\mathrm{p}\mathcal{C}(W,S)$.

(d) In Theorem 4, we noted that the perspectors lie on $\mathrm{p}\mathcal{K}(W,S)$ and on $\mathrm{p}\mathcal{K}(W,R)$. Now these cubics meet at A, B, C and the four points fixed by W-isoconjugation. There must be just two other (non-trivial) intersections. □

Example 1. In terms of triangle centers, the most prolific parent seems to be the Neuberg cubic = $\mathrm{p}\mathcal{K}(K,X(30))$, Gibert's $K001$.

The polar cubic $\mathrm{p}\mathcal{C}(K,X(30))$ is mentioned in [5] in the discussion of its center, the Tixier point, X476. There, it is noted that it is a rectangular hyperbola passing through I, the excenters, and $X(30)$. Of course, being rectangular, the other infinite point must be $X(523)$. Using the information in [5], we see that its asymptotes pass through $X(74)$ and $X(110)$. The perspectors P_1, P_2 of desmic structures on $K001$ must be its isogonal pairs other than $\{X(30),\ X(74)\}$.

By Theorem 4(f), the root of the child cubic must be the mid-point of P_1 and P_2. The pair $\{O,\ H\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(5),k)$. The information in [2] identifies it as $K216$. The pair $\{X(13),\ X(15)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(396),k)$. The pair $\{X(14),\ X(16)\}$ gives a cubic of the form $\mathrm{n}\mathcal{K}(K,X(395),k)$.

As noted above, $X(523)$ is on the polar conic, so we also have a child of the form $\mathrm{n}\mathcal{K}(K,X(523),k)$. Since $X(523)$ is not on the cubic, the perspectors must be at infinity. As they are isogonal conjugates, they must be the infinite circular points. These have already been noted as lying on $K001$. We now have additional centers on $\mathrm{p}\mathcal{C}(K,X(30))$, $X(5)$, $X(395)$, $X(396)$, as well as $X(1)$, $X(30)$, $X(523)$. We also have the harmonic associates of each of these points!

4. Roots and pivots

If we have a cubic $\mathrm{n}\mathcal{K}(W,R,k)$ defined by a desmic structure, then it has a parent cubic $\mathrm{p}\mathcal{K}(W,S)$. From Theorem 8(c), we know that S is on $\mathrm{p}\mathcal{C}(W,R)$. Since R is also on the conic, we can identify S from an equation for RS. Although the results were found by heavy computations, we can establish them quite simply by "guessing" the pole of RS with respect to $\mathrm{p}\mathcal{C}(W,R)$.

Theorem 9. *Suppose that* $W = u : v : w$*,* $R = r : s : t$ *and* k *are such that* $\mathrm{n}\mathcal{K}(W, R, k)$ *is defined by a desmic structure. Let* $\mathrm{p}\mathcal{K}(W, S)$ *be the parent of* $\mathrm{n}\mathcal{K}(W, R, k)$*.*
(a) *The line* RS *is the polar of* $P = 2ust - kr : 2vtr - ks : 2wrs - kt$ *with respect to* $\mathrm{p}\mathcal{C}(W, R)$*.*
(b) *The point* S *has first barycentric coordinate* $4r(-r2vw + s2wu + t2uv) - 4kstu + k2r$*.*

Proof. Suppose that the desmic structure has normalized perspectors $P_1 = f : g : h$ and $P_2 = f' : g' : h'$. Then we have

$$\begin{array}{lll} r = f - f', & s = g - g', & t = h - h'; \\ u = ff', & v = gg', & w = hh', \\ k = 2(f'g'h' - fgh). & & \end{array}$$

(a) The first barycentric of P is then

$$2(ff'(g - g')(h - h') - (f'g'h' - fgh)(f - f')) = 2(fg - f'g')(fh - f'h').$$

The coefficient of x^2 in $\mathrm{p}\mathcal{C}(W, R)$ is

$$vt^2 - ws^2 = (gh - g'h')(hg' - gh').$$

If we discard a symmetric factor the polar of P is the line P_1P_2, *i.e.*, the line RS.

(b) We know that $S = f + f' : g + g' : h + h'$. If we substitute the above values for r, s, t, u, v, w, k, the expression becomes $K(f + f')$, where K is symmetric in the variables. Thus, S is as stated. □

This result allows us to find the parent of a cubic, even if we cannot find the perspectors explicitly. It would also allow us to find the perspectors since these are the W-isoconjugate points on the line RS. Thus the perspectors arise as the intersections of a line and (circum)conic.

Example 2. The second Brocard cubic $\mathrm{n}\mathcal{K}_0(K, X(523))$, Gibert's $K018$. We cannot identify the perspectors of the desmic structure. They are complex. Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, X(5))$ - the Napolean cubic, and Gibert's $K005$.

Example 3. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$, Gibert's $K024$.

Theorem 9(b) gives the parent as $\mathrm{p}\mathcal{K}(K, O)$ - the McCay cubic and Gibert's $K003$. Theorem 9(a) gives RS as the Brocard axis. It follows that the perspectors of the desmic structure are the intersections of the Brocard axis with the Kiepert hyperbola.

Our next result identifies the children of a given $\mathrm{p}\mathcal{K}(W, R)$ which are of the form $\mathrm{n}\mathcal{K}_0(W, R)$. It turns out that the perspectors must lie on another cubic $\mathrm{n}\mathcal{K}_0(W, T)$. The root T is most neatly defined using the generalization of Gibert's PK-transform.

Theorem 10. *Suppose that* $\{P, P^*\} \neq \{S, S^*\}$ *are a pair of* W*-isoconjugates on* $\mathrm{p}\mathcal{K}(W, S)$*. Then they define a desmic structure with associated cubic of the form* $\mathrm{n}\mathcal{K}_0(W, R)$ *if and only if* P*,* P^* *lie on* $\mathrm{n}\mathcal{K}_0(W, PK(W, S))$*.*

Proof. Suppose that $P = x : y : z$, $S = r : s : t$, $W = u : v : w$. Then the point P^* is $u/x : v/y : w/z$. As P is on $\mathrm{p}\mathcal{K}(W,S)$, there exist constants m, n with $P + mP^* = nS$. Then the normalized forms for the perspectors of the desmic structure are P and mP^*, and $R = P - mP^*$. If we look at a pair of coordinates in the expression $P + mP^* = nS$, we get an expression for m in terms of two of x, y, z. These are

$$m_x = \frac{(yt - zt)yx}{ysw - ztv}, \quad m_y = \frac{(zr - xt)xz}{ztu - xrv}, \quad m_z = \frac{(xs - yr)xy}{xrv - ysu}.$$

We can now compute the k such that the cubic is $\mathrm{n}\mathcal{K}(W,R,k)$ as

$$2\left(\frac{m_x m_y m_z uvw}{xyz} - xyz\right).$$

We have an $\mathrm{n}\mathcal{K}_0$ if and only if this vanishes. Maple shows that this happens precisely when P (and hence P^*) is on the cubic $\mathrm{n}\mathcal{K}_0(W,T)$, where

$$T = ur(vt^2 - ws^2) : vs(wr^2 - ut^2) : wt(us^2 - vr^2) = PK(W,S).$$

□

Note that we get an $\mathrm{n}\mathcal{K}_0$ when P, P^* lie on $\mathrm{p}\mathcal{K}(W,S)$ and $\mathrm{n}\mathcal{K}_0(W, PK(W,S))$. Then there are three pairs and three cubics. Also, $PK(W,S) = PK(W,S^*)$ - see §5 - so $\mathrm{p}\mathcal{K}(W,S)$ and $\mathrm{p}\mathcal{K}(W,S^*)$ give rise to the same $\mathrm{n}\mathcal{K}_0$.

Example 4. Applying Theorem 10 to the McCay cubic $\mathrm{p}\mathcal{K}(K,O)$ and the Orthocubic $\mathrm{p}\mathcal{K}(K,H)$ we get the second Brocard cubic $K019 = \mathrm{n}\mathcal{K}_0(K, X(647))$. In the former case, we can identify the perspectors of the desmic structures. From [2, K019], we know that the points of $K019$ are the foci of inconics with centers on the Brocard axis OK. Also, if P is on the McCay cubic, then PP^* passes through O. If PP^* is not OK, then the center is on PP^* and on OK, so must be O. This gives four of the intersections, two of which are real. Otherwise, P and P^* are the unique K-isoconjugates on OK. These are the intersections of the Brocard axis and the Kiepert hyperbola. Again these are complex. They give the cubic $\mathrm{n}\mathcal{K}_0(K,K)$ - see Example 3. In §5, we meet these last two points again.

Example 5. The Thomson cubic $\mathrm{p}\mathcal{K}(K,G)$ and the Grebe cubic $\mathrm{p}\mathcal{K}(K,K)$ give the cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$. We will meet this cubic again in §5.

5. Desmic structures with triply perspective triangles

As we saw in Theorem 3, a cubic of type $\mathrm{n}\mathcal{K}_0$ arises from a desmic structure in which the triangles are triply perspective with the reference triangle $\triangle ABC$. We begin with a discussion of an obvious way of constructing such a structure. It turns out that almost all triply perspective structures arise in this way.

Lemma 11. *Suppose that $P = p : q : r$ and $Q = u : v : w$ are points with distinct cevians.*
(a) *Let*

$$\begin{array}{lll} A' = BP \cap CQ, & B' = CP \cap AQ, & C' = AP \cap BQ, \\ A'' = BQ \cap CP, & B'' = CQ \cap AP, & C'' = AQ \cap BP. \end{array}$$

Then the desmic structure with vertices A, B, C, A', B', C', A'', B'', C'', and perspectors $P_1 = \frac{1}{qw} : \frac{1}{ru} : \frac{1}{pv}$ and $P_2 = \frac{1}{rv} : \frac{1}{pw} : \frac{1}{qu}$ has triangles $\triangle ABC$, $\triangle A'B'C'$, $\triangle A''B''C''$ which are triply perspective.
(b) *Each desmic structure including the vertices A, B, C in which*
(i) *the triangles are triply perspective, and*
(ii) *the perspectors are distinct arises from a unique pair P, Q with $P \neq Q$, and perspectors normalized as in (a).*

Proof. (a) We begin by looking at the desmic structure which is derived from the given P_1 and P_2 as in Theorem 4. Thus, the first three vertices are obtained by replacing a coordinate of P_1 by the corresponding coordinate of P_2. This has the vertices named, for example the first vertex is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$. This is on BP and CQ, so is A'. It is easy to see that the triangles doubly perspective, with perspectors P and Q, and hence triply perspective.

(b) Suppose we are given such a desmic structure. Then the perspectors are $P_1 = f : g : h$ and $P_2 = f' : g' : h'$ with$fgh = f'g'h'$ (see Lemmata 2 and 3). We find points P and Q which give rise to these as in part (a). From the coordinates of P_1, we can solve for u, v and w in terms of p, q, r and the coordinates of P_1. Then, using two of the coordinates of P_2, we can find q and r in terms of p. The equality of the third coordinates follows from the condition $fgh = f'g'h'$. As the perspectors are distinct, $P \neq Q$. $\square$

Note that from the normalized perpsectors, we can recover the vertices, even if some cevians coincide. For example, A' is $\frac{1}{rv} : \frac{1}{ru} : \frac{1}{pv}$.

Definition 4. *The desmic structure defined in Lemma 11 is denoted by $\mathcal{D}(P, Q)$.*

Theorem 12. *If P and Q are triangle centers with functions $p(a, b, c)$ and $q(a, b, c)$, then the desmic structure $\mathcal{D}(P, Q)$ has*
(a) *perspectors P_1, P_2 with functions $h(a, b, c) = \frac{1}{p(b,c,a)q(c,a,b)}$ and $h(a, c, b)$.*
(b) *$\{P_1,\ P_2\}$ is a bicentric pair.*
(c) *The vertices of the triangles are $[h(a, c, b), h(b, c, a), h(c, a, b)]$, and so on.*

The proof requires only the observation that, as P and Q are centers, $p(a, b, c) = p(a, c, b)$ and $q(a, b, c) = q(a, c, b)$.

We leave it as an exercise to the reader that, if $\{P, Q\}$ is a bicentric pair, then P_1, P_2 are centers. It follows that, if P, Q are centers and $\mathcal{D}(P, Q)$ has perspectors P_1, P_2, then $\mathcal{D}(P_1, P_2)$ has triangle centers P', Q' as perspectors. As further exercises, the reader may verify that P', Q' are the Q^2-isoconjugate of P and the P^2-isoconjugate of Q. The desmon of the second structure is the P-Hirst inverse of Q.

We can compute the equations of the cubics from the information in Lemma 11(a). These involve ideas introduced in our Definition 2.

Theorem 13. *Suppose that $P \neq Q$. The cubics associated with the desmic structure $\mathcal{D}(P, Q)$ are $\mathrm{p}\mathcal{K}(W, NK(W, P))$ and $\mathrm{n}\mathcal{K}_0(W, PK(W, P))$, where W is the isoconjugation which interchanges P and Q.*

Proof. We have the perspectors P_1 and P_2 of the desmic structure from Lemma 11(a). These give isoconjugation as that which interchanges P_1 and P_2 - see Theorem 4. Now observe that this also interchanges P and Q, so $Q = P^*$. From Theorem 4, we have coordinates for the desmon S and the harmon R in terms of those of P_1 and P_2. Using our formulae for P_1 and P_2, we get the stated values of S and R. $\square$

Definition 5. *Suppose that $P \neq Q$, and that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is associated with the desmic structure $\mathcal{D}(P, Q)$. Then*
(a) *P, Q are the cevian points for $\mathcal{C}$.*
(b) *The perspectors of $\mathcal{D}(P, Q)$ are the Grassmann points for $\mathcal{C}$.*

Theorem 14. *Suppose that $\mathcal{C} = \mathrm{n}\mathcal{K}_0(W, R)$ is not of type $\mathrm{c}\mathcal{K}$, and does not have $W = R^2$.*
(a) *The cevian points for $\mathcal{C}$ are the W-isoconjugate points on $\mathcal{T}(R^*)$. These are the intersections of $\mathcal{C}(R)$ and $\mathcal{T}(R^*)$.*
(b) *The Grassmann points for $\mathcal{C}$ are the W-isoconjugate points on the polar of $PK(W, R)$ with respect to $\mathcal{C}(R)$.*

Proof. From Theorems 3 and 7, we know that $\mathcal{C}$ is a Grassmann cubic associated with a desmic structure which has triply perspective triangles. If the perspectors of the structure coincide at X, then the equation shows that $R = X$ and $W = X^2$. But we assumed that $W \neq R^2$, so we do not have this case.

(a) From Theorem 13, this structure is $\mathcal{D}(P, Q)$ with $Q = P^*$, and $R = PK(W, P) = PK(W, Q)$. From a remark following Definition 2, P and Q lie on the conic $\mathcal{C}(R)$. Since they are W-isoconjugates, they also lie on the W-isoconjugate of $\mathcal{C}(R)$. This is $\mathcal{T}(R^*)$. The conic and line have just two intersections, so this gives precisely the pair $\{P, Q\}$. These are precisely the pair of W-isoconjugates on $T(R^*)$.

(b) By Theorem 4, the Grassmann points are W-isoconjugate and lie on RS, where S is the desmon of the desmic structure. As $\mathcal{C}$ is an $\mathrm{n}\mathcal{K}_0$, Theorem 9 gives S as a point which we recognize as the pole of $\mathcal{T}(R^*)$ with respect to the conic $\mathcal{C}(R)$. Now R is the pole of $\mathcal{T}(R)$ for this conic, so RS is the polar of $\mathcal{T}(R) \cap \mathcal{T}(R^*) = PK(W, R)$. $\square$

In Theorem 14, we ignored cubics of type $\mathrm{n}\mathcal{K}(R^2, R, k)$. To make the algebra easier, we replace the constant k by $k'rst$, where $R = r : s : t$.

Theorem 15. *If $\mathcal{C} = \mathrm{n}\mathcal{K}(R^2, R, k'rst)$ is not of type $\mathrm{c}\mathcal{K}$, then $\mathcal{C}$ is the Grassmann cubic associated with the desmic structure having perspectors R and aR, where a is a root of*

$$x^2 + \left(1 + \frac{k'}{2}\right) x + 1 = 0.$$

When $k' = 2$, $a = -1$, the desmic structure and $\mathcal{C}$ are degenerate.

When $k' \neq 2,\ -6$, the corresponding $\mathrm{p}\mathcal{K}$ is $\mathrm{p}\mathcal{K}(R^2, R)$, which is the union of the R-cevians.

Proof. When we use Maple to solve the equations to identify A' and A'', we discover them as $r : as : at$, with a as above. This identifies the perspectors as R and aR. It is easy to verify that this choice leads to $\mathcal{C}$. Note that the two roots are inverse, so we get the same vertices from either choice. The quadratic has equal roots when $k' = 2$ or -6. The latter gives the $\mathrm{c}\mathcal{K}$ in the class. In this case, the "cubic" equation is identically zero as $a = 1$. In the former case, $a = -1$, so we do get the $\mathrm{n}\mathcal{K}$ as a Grassmann cubic, but the equation factorizes as $(ty + sz)(rz + tx)(sx + ry) = 0$. In the "desmic structure" $A' = A''$, $B' = B''$, $C' = C''$ as $a = -1$.

Note that here the cubic is not of type $\mathrm{c}\mathcal{K}$ as it contains three fixed points R_A, R_B, R_C of R^2-isoconjugation. When $k' \neq 2, -6$, the desmon is defined, and is again R. Finally, $\mathrm{p}\mathcal{K}(R^2, R)$ is $(ty - sz)(rz - tx)(sx - ry) = 0$. □

Example 6. The third Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(647))$, is $K019$ in Gibert's list. As it is of type $\mathrm{n}\mathcal{K}_0$, but not of type $\mathrm{c}\mathcal{K}$, Theorem 12(b) applies. The conic $\mathcal{C}(R)$ is the Jerabek hyperbola. Also, R^* is $X(648)$, so that the line $\mathcal{T}(R^*)$ is the Euler line. These intersect in (the K-isoconjugates) O and H. This gives some new points on the cubic:

$$\begin{array}{lll} A' = BO \cap CH, & B' = CO \cap AH, & C' = AO \cap BH, \\ A'' = BH \cap CO, & B'' = CH \cap AO, & C'' = AH \cap BO. \end{array}$$

Also, the cubic can be described as the Grassmann cubic $GN(\triangle A'B'C')$ or $GN(\triangle A''B''C'')$. The associated GP has the pivot $NK(K, O) = NK(K, H) = X(185)$. The cubic $\mathrm{p}\mathcal{K}(K, X(185))$ is not (yet) listed in [2], but we know that it has the points A', B', C', A'', B'', C'', the perspectors P_1 and P_2, as well as I, the excenters and $X(185)$.

Example 7. The first Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(385))$, is $K017$ in Gibert's list. Here, the vertices A', B', C', A'', B'', C'' have already been identified. They are the vertices of the first and third Brocard triangles. The points P and Q are the Brocard points. The associated GP is $\mathrm{p}\mathcal{K}(K, X(384))$, the fourth Brocard cubic and $K020$ in Gibert's list. Again, Gibert's website shows the points on $K020$, together with $\{P_1,\ P_2\} = \{X(32),\ X(76)\}$.

Example 8. The cubics $\mathrm{p}\mathcal{K}(K, X(39))$ and $\mathrm{n}\mathcal{K}_0(K, X(512))$. These cubics do not appear in Gibert's list, but the associated desmic structure is well-known. Take $= G$, $Q = K$ in Lemma 11(a). Again we get isogonal cubics. The vertices of the structure are the intersections of medians and symmedians. From the proof of Lemma 11, P_1, P_2 are the Brocard points. This configuration is discussed in, for example, [4], but the proof that the triangles obtained from the intersections are perspective with $\triangle ABC$ uses special properties of G and K. Here, our Lemma 8(a) gives a very simple (geometric) proof of the general case. Here, $PK(K, G) = PK(K, K) = X(512)$, and $NK(K, G) = NK(K, K) = X(39)$.

The points P and Q for a cubic of type $\mathrm{n}\mathcal{K}_0$ are identified as the intersections of a line with a conic. Of course, it is possible that these have complex coordinates. This happens, for example, for the second Brocard cubic, $\mathrm{n}\mathcal{K}_0(K, X(523))$. There, the line is the Brocard axis and the conic is the Kiepert hyperbola. A Cabri sketch shows that these do not have real intersections. We have met these intersections in §4.

Example 9. The kjp cubic $\mathrm{n}\mathcal{K}_0(K, K)$ is $K024$ in Gibert's list. The desmic structure is $\mathcal{D}(P, Q)$, where P, Q are the intersections of the line at infinity and the circumcircle. These are the infinite circular points. We met this cubic in Example 3, where we identified the perspectors of the desmic structure as the intersections of the Brocard axis with the Kiepert hyperbola.

6. Harmonic associates and other cubics

In the Introduction, we introduced the idea of harmonic associates. This gives a pairing of our cubics. We begin with a result which relates desmic structures. This amplifies remarks made in the proof of Theorem 6.

Theorem 16. *Suppose that D is a desmic structure with normalized perspectors P_1, P_2, and cubics $\mathrm{n}\mathcal{K}(W, R, k)$, $\mathrm{p}\mathcal{K}(W, S)$. Then the desmic structure D' with normalized perspectors P_1, $-P_2$ has*
(a) *vertices the harmonic associates of those of D, and*
(b) *cubics $\mathrm{n}\mathcal{K}(W, S, k')$, $\mathrm{p}\mathcal{K}(W, R)$.*

Proof. We refer the reader to the proof of Theorem 6(a), which in turn uses the notation of Theorem 4(a). □

We refer to the cubic $\mathrm{n}\mathcal{K}(W, S, k')$ obtained in this way as the harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$.

Corollary 17. *If $\mathrm{n}\mathcal{K}(W, R, k) = GN(\Delta)$, then $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta')$, where Δ' is the harmonic associate of Δ.*

Example 10. The second Brocard cubic $\mathrm{n}\mathcal{K}(K, X(385)) = K017$ was discussed in Example 7. It is $GN(\Delta)$, where Δ is either the first or third Brocard triangle. From Corollary 17, the cubic $\mathrm{p}\mathcal{K}(K, X(385)) = K128$ is $GP(\Delta')$, where Δ' is the harmonic associate of either of these triangles. This gives us six new points on $K128$. The cubic $\mathrm{n}\mathcal{K}_0(K, X(512))$ was introduced in Example 8. It is $GN(\Delta)$, where Δ is formed from intersections of medians and symmedians. From the Corollary, the fifth Brocard cubic $\mathrm{p}\mathcal{K}(K, X(512)) = K021$ is $GP(\Delta')$. Now the Grassmann points for $\mathrm{n}\mathcal{K}_0(K, X(512))$ are the Brocard points, so these lie on $K021$, as do the vertices of Δ'.

Example 11. Let $\mathcal{C} = K216$ of [2]. This was mentioned in §3. It is of the form $\mathrm{n}\mathcal{K}(K, X(5), k)$ with parent $\mathrm{p}\mathcal{K}(K, X(30)) = K001$. From Theorem 16, the harmonic associate is of the form $\mathrm{n}\mathcal{K}(K, X(30), k')$ with parent $\mathrm{p}\mathcal{K}(K, X(5)) = K005$. Using Theorem 9(b), a calculation shows that $K067 = \mathrm{n}\mathcal{K}(K, X(30), k'')$ has parent $K005$. From Theorem 8(b), $K005$ has a unique child with root $X(30)$. Thus $K067$ is the harmonic associate of $K216$. This gives us six points on $K067$ as harmonic associates of the points identified in [2] as being on $K216$. We will give a geometrical description of these shortly.

Three of the vertices of the desmic structure for $K216$ are the reflections of the vertices A, B, C in BC, CA, AB. These give a triangle with perspector H. We can generalize these to a general X as the result of extending each X-cevian by its own length. If $X = x : y : z$, then, starting from A, we get the point $y+z : -2y : -2z$. The A-harmonic associate is $y+z : 2y : 2z$, the intersection of the H-cevian at A with the parallel to BC through G. This suggests the following definition.

Definition 6. *For $X = x : y : z$, the desmic structure $\mathcal{D}(X)$ is that with normalized perspectors $y + z : z + x : x + y$ and $-2x : -2y : -2z$.*

Note that the first perspector is the complement $\mathsf{c}X$ of X and the second is X. We can summarize our results on such structures as follows.

Theorem 18. *Suppose that $X = x : y : z$. The cubics associated with $\mathcal{D}(X)$ are $\mathrm{n}\mathcal{K}(W, R, k)$ and $\mathrm{p}\mathcal{K}(W, S)$, where*
(i) *$W = x(y + z) : y(z + x) : z(x + y)$, the center of the inconic with perspector X,*
(ii) *$R = 2x + y + z : x + 2y + z : x + y + 2z$, the mid-point of X and $\mathsf{c}X$,*
(iii) *$S = -2x + y + z : x - 2y + z : x + y - 2z$, the infinite point on GX.*

The harmonic associate of $\mathrm{n}\mathcal{K}(W, R, k)$ passes through G, the infinite point of $\mathcal{T}(X)$, and their W-isoconjugates.

Proof. The coordinates of W, R and S follow at once from those of the perspectors and Theorem 4. The final part needs an equation for the harmonic associate. This is given by Theorem 4. The fact that G and $x(y - z) : y(z - x) : z(x - y)$ lie on the cubic is a simple verification using Maple. □

For $X = H$, we get $K216$ and $K001$ and their harmonic associates $K067$ and $K005$. The desmic structures and the points given in Theorem 18 account for most of the known points on $K216$ and $K067$.

There is one further example in [2]. In the notes on $K022 = \mathrm{n}\mathcal{K}(O, X(524), k)$, it is observed that the cubic contains the vertices of the second Brocard triangle, and hence their O-isoconjugates. The latter are the the intersections of the $X(69)$-cevians with lines through G parallel to the corresponding sidelines. These are the harmonic associates of three vertices of $\mathcal{D}(X(69))$. The other perspector is $K = \mathsf{c}X(69)$. The mid-point of $X(69)$ and K is $X(141)$, so $K022$ is the harmonic associate of the cubic $\mathrm{n}\mathcal{K}(O, X(141), k')$ with parent $\mathrm{p}\mathcal{K}(O, X(524))$. These cubics contain the vertices of $\mathcal{D}(X(69))$, including the harmonic associates of the second Brocard triangle. Also, the parent of $K022$ is $\mathrm{p}\mathcal{K}(O, X(141))$.

In the Introduction, we mentioned that the Darboux cubic $\mathrm{p}\mathcal{K}(K, X(20))$ is $GP(\Delta)$, where Δ has vertices the infinite points on the altitudes. Of course, as Δ is degenerate, $GN(\Delta)$ degenerates. It is the union of the circumcircle and the line at infinity. The harmonic associate Δ' has vertices the mid-points of the altitudes, and this leads to an $\mathrm{n}\mathcal{K}(K, X(20), k)$, and its parent which is the Thomson cubic $\mathrm{p}\mathcal{K}(K, G) = K002$. This will follow from our next result. The fact that the mid-points lie on $K002$ is noted in [5], but now we know that those points can be used to generate $K002$ as a locus of type GP. We can replace the vertices of Δ or

Δ by their isogonal conjugates. In the case of Δ the isogonal points lie on the circumcircle. For any point $X = x : y : z$, the mid point of the cevian at A is $y + z : y : z$. We make the following definition.

Definition 7. *For $X = x : y : z$, the desmic structure $\mathcal{E}(X)$ is that with normalized perspectors $y + z : z + x : x + y$ and $x : y : z$.*

Theorem 19. *Suppose that $X = x : y : z$. Let Δ be the triangle $\triangle A'B'C'$ of $\mathcal{E}(X)$. Then $GN(\Delta) = \mathrm{n}\mathcal{K}(W, R, k)$ and $GP(\Delta) = \mathrm{p}\mathcal{K}(W, G)$, where*
$W = x(y + z) : y(z + x) : z(x + y)$, the complement of the isotomic conjugate of X,
$R = -x + y + z : x - y + z : x + y - z$, the anticomplement of X,
$k = 2((y + z)(z + x)(x + y) - xyz)$.

The harmonic associates are $GN(\Delta') = \mathrm{n}\mathcal{K}(W, G, k')$, which degenerates as $\mathcal{C}(W)$ and the line at infinity, and $GP(\Delta') = \mathrm{p}\mathcal{K}(W, R)$, which is a central cubic with center the complement of X.

Most of the result follow from the equations given by Theorem 4. The fact that $\mathrm{p}\mathcal{K}(W, R)$ is central is quite easy to check, but it is a known result. In [6], Yiu shows that the cubic defined by $GP(\Delta')$ has the given center. Yiu derives interesting geometry related to $\mathrm{p}\mathcal{K}(W, R)$, and these are summarized in [1, §3.1.3]. The case $X = H$ gives $W = K$, $R = X(20)$, so we get $GP(\Delta) = K002$, the Thomson cubic, and $GP(\Delta') = K004$, the Darboux cubic.

Remarks. (1) From [1, Theorem 3.1.2], we know that if $W \neq G$, there is a unique central $\mathrm{p}\mathcal{K}$ with pole W. After Theorem 19, this arises from the desmic structure $\mathcal{E}(X)$, where X is the isotomic conjugate of the anticomplement of W. The center is then the complement of X, and hence the G-Ceva conjugate of W. It is also the perspector of $\mathcal{E}(X)$ other than X. The pivot of the central $\mathrm{p}\mathcal{K}$ is then the anticomplement of X, and hence the anticomplement of the isotomic conjugate of the anticomplement of the pole.

(2) The cubic $\mathrm{p}\mathcal{K}(W, G)$ clearly contains G and W. From the previous remark, it contains the G-Ceva conjugate of W and its W-isoconjugate (the point X). It also includes the mid-points of the X-cevians and their W-isoconjugates. The last six are the vertices of a defining desmic structure. Finally, it includes the mid-points of the sides of $\triangle ABC$.

We have seen that there are several pairs of cubics of type $\mathrm{p}\mathcal{K}$ which are loci of type GP from harmonic associate triangles. We can describe when this is possible.

Theorem 20. *For a given W, suppose that R and S are distinct points, neither fixed by W-isoconjugation.*
(a) *There exist harmonic triangles Δ and Δ' with $\mathrm{p}\mathcal{K}(W, R) = GP(\Delta)$ and $\mathrm{p}\mathcal{K}(W, S) = GP(\Delta')$ if and only if $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$.*
(b) *If $\mathrm{p}\mathcal{C}(W, R) = \mathrm{p}\mathcal{C}(W, S)$, then the Grassmann points are*
(i) *the non-trivial intersections of $\mathrm{p}\mathcal{K}(W, R)$ and $\mathrm{p}\mathcal{K}(W, S)$,*
(ii) *the W-isoconjugate points on RS.*

Proof. (a) If $\mathrm{p}\mathcal{K}(W,R)$ and $\mathrm{p}\mathcal{K}(W,S)$ are loci of the given type, then $GN(\Delta') = \mathrm{n}\mathcal{K}(W,R,k)$ by Theorem 16. Thus this $\mathrm{n}\mathcal{K}$ has parent $\mathrm{p}\mathcal{K}(W,S)$. By Theorem 8, $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Now suppose that $\mathrm{p}\mathcal{C}(W,R) = \mathrm{p}\mathcal{C}(W,S)$. Then R is on $\mathrm{p}\mathcal{C}(W,S)$. By Theorem 8, there is a unique child of $\mathrm{p}\mathcal{K}(W,R)$ of the form $\mathrm{n}\mathcal{K}(W,S,k')$. From Theorem 7, $\mathrm{n}\mathcal{K}(W,S,k') = GN(\Delta)$ for a triangle Δ, and $GP(\Delta) = \mathrm{p}\mathcal{K}(W,R)$. By Theorem 16, $GP(\Delta') = \mathrm{p}\mathcal{K}(W,S)$.

(b) The Grassmann points are the same for Δ and Δ', and lie on both cubics. There are just two non-trivial points of intersection, so these are the Grassmann points. The Grassmann points are W-isoconjugate, and lie on RS, giving (ii). □

We already have some examples of pairs of this kind:
$K001$ and $K005$ with Grassmann points O, H. The desmic structures are those for $K067$ and $K216$.
$K002$ and $K004$ with Grassmann points O, H. The desmic structures appear above.
$K020$ and $K128$ with Grassmann points $X(32)$, $X(76)$. See the first of Example 10.

From Example 1, we found children of $K001$ with roots $X(395)$, $X(396)$, $X(523)$. We then have
$K001$ and $K129a = \mathrm{p}\mathcal{K}(K, X(395))$ with Grassmann points $X(14)$, $X(16)$;
$K001$ and $K129b = \mathrm{p}\mathcal{K}(K, X(396))$ with Grassmann points $X(13)$, $X(15)$;
$K001$ and $\mathrm{p}\mathcal{K}(K, X(523))$ with Grassmann points the infinite circular points.

In Example 3, we showed that $K024 = \mathrm{n}\mathcal{K}_0(K,K)$ is a child of $K003$. We therefore have $K003$ and $K102 = \mathrm{p}\mathcal{K}(K,K)$, the Grebe cubic. The Grassmann points are the intersection of the Brocard axis and the Kiepert hyperbola. These are complex.

7. Further examples

So far, almost all of our examples have been isogonal cubics. In this section, we look at some cubics with different poles. We have chosen examples where at least one of the cubics involved is in [2]. In the latest edition of [2], we have the class $CL041$. This includes cubics derived from $W = p : q : r$. In our notation, we have the cubic $\mathrm{n}\mathcal{K}_0(W,R)$, where $R = p^2 - qr : q^2 - pr : r^2 - pq$, the G-Hirst inverse of W. The parent is $\mathrm{p}\mathcal{K}(W,S)$, where $S = p^2 + qr : q^2 + pr : r^2 + pq$. The Grassmann points are the barycentric square and isotomic conjugate of W. The cevian points are the bicentric pair $1/r : 1/p : 1/q$ and $1/q : 1/r : 1/p$. Example 7 is the case where $W = K$, so that the Grassmann points are the centers $X(32)$, $X(76)$, and the cevian points are the Brocard points. Our first two examples come from $CL041$.

Example 12. If we put $W = X(1)$ in construction $CL041$, we get $\mathrm{n}\mathcal{K}_0(X(1), X(239))$ with parent $K132 = \mathrm{p}\mathcal{K}(X(1), X(894))$. The Grassmann points are K, $X(75)$, and the cevian points are those described in [3] as the Jerabek points. A harmonic associate of $K132$ is then $K323 = \mathrm{p}\mathcal{K}(X(1), X(239))$.

Example 13. If we put $W = H$ in construction of $CL041$, we get $\mathrm{n}\mathcal{K}_0(H, X(297))$ with parent $\mathrm{p}\mathcal{K}(H, S)$, where S is as given in the discussion of $CL041$. The Grassmann points are $X(393)$, $X(69)$, and the cevian points are those described in [3] as the cosine orthocenters.

Example 14. The shoemaker's cubics are $K070a = \mathrm{p}\mathcal{K}(H, X(1586))$ and $K070b$ = $\mathrm{p}\mathcal{K}(H, X(1585))$. As stated in [2], these meet in G and H. If we normalize G as $1 : 1 : 1$ and H as $\lambda \tan A : \lambda \tan B : \lambda \tan C$, with $\lambda = \pm 1$, we get an $\mathrm{n}\mathcal{K}(H, X(1585), k)$ with parent $K070a$, and an $\mathrm{n}\mathcal{K}(H, X(1585), k')$ with parent $K070b$. The Grassmann points are G and H. Note that $K070a$ and $K070b$ are therefore harmonic associates.

The next five examples arise from Theorem 19. Further examples may be obtained from the page on central $\mathrm{p}\mathcal{K}$ in [2].

Example 15. The complement of $X(1)$ is $X(10)$, the Spieker center, the anticomplement of $X(1)$ is $X(8)$, the Nagel point. The center of the inconic with perspector $X(1)$ is $X(37)$. If we apply Theorem 19 with $X = X(1)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(37), X(8), k)$ with parent $\mathrm{p}\mathcal{K}(X(37), G)$. The Grassmann points are $X(1)$ and $X(10)$. The cubic $\mathrm{p}\mathcal{K}(X(37), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(10)$ and $X(37)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(37)$, and has parent $K033 = \mathrm{p}\mathcal{K}(X(37), X(8))$.

Example 16. The complement of $X(7)$, the Gergonne point, is $X(9)$, the Mittenpunkt, the anticomplement of $X(7)$ is $X(144)$. The center of the inconic with perspector $X(7)$ is $X(1)$. If we apply Theorem 19 with $X = X(7)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(1), X(144), k)$ with parent $\mathrm{p}\mathcal{K}(X(1), G)$. The Grassmann points are $X(7)$ and $X(9)$. The cubic $\mathrm{p}\mathcal{K}(X(1), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(7)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(1)$, and has parent $K202 = \mathrm{p}\mathcal{K}(X(1), X(144))$.

Example 17. The complement of $X(8)$, the Nagel point, is $X(1)$, the incenter, the anticomplement of $X(8)$ is $X(145)$. The center of the inconic with perspector $X(1)$ is $X(9)$. If we apply Theorem 19 with $X = X(8)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(9), X(145), k)$ with parent $\mathrm{p}\mathcal{K}(X(9), G)$. The Grassmann points are $X(8)$ and $X(1)$. The cubic $\mathrm{p}\mathcal{K}(X(9), G)$ does not appear in the current [2], but contains $X(1)$, $X(2)$, $X(8)$ and $X(9)$. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector $X(9)$, and has parent $K201 = \mathrm{p}\mathcal{K}(X(9), X(145))$.

Example 18. The complement of $X(69)$ is K, the anticomplement of $X(69)$ is $X(193)$. The center of the inconic with perspector $X(69)$ is O. If we apply Theorem 19 with $X = X(69)$, then we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(O, X(193), k)$ with parent $K168 = \mathrm{p}\mathcal{K}(O, G)$. The Grassmann points are $X(69)$ and K. The harmonic associate of $\mathcal{C}$ degenerates as the line at infinity and the circumconic with perspector O, and has parent $\mathrm{p}\mathcal{K}(O, X(193))$.

Example 19. The complement of $X(66)$ is $X(206)$, the anticomplement is not in [5], but appears in [2] as $P161$ - see $K161$. The center of the inconic with perspector $X(66)$ is $X(32)$. From Theorem 19 with $X = X(66)$, we get a cubic $\mathcal{C} = \mathrm{n}\mathcal{K}(X(32), P161, k)$ with parent $K177 = \mathrm{p}\mathcal{K}(X(32), G)$. The Grassmann points are $X(66)$ and $X(206)$. The harmonic associate of C degenerates as the line at infinity and the circumconic with perspector $X(32)$, and has parent $K161 = \mathrm{p}\mathcal{K}(X(32), P161)$.

In Examples 3 and 9, we met the kjp cubic $K024 = \mathrm{n}\mathcal{K}_0(K, K)$. The parent is the McCay cubic $K003 = \mathrm{p}\mathcal{K}(K, O)$. It follows that the harmonic associate of $K024$ is of the form $\mathrm{n}\mathcal{K}(K, O, k)$, and that this has parent the Grebe cubic $K102 = \mathrm{p}\mathcal{K}(K, K)$. From Theorem 9(b), the general $\mathrm{n}\mathcal{K}_0(W, W)$ has parent $\mathrm{p}\mathcal{K}(W, V)$, where V is the G-Ceva conjugate of W. The harmonic associate will be of the form $\mathrm{n}\mathcal{K}(W, V, k)$, with parent $\mathrm{p}\mathcal{K}(W, W)$. This means that the class $CL007$, which contains cubics $\mathrm{p}\mathcal{K}(W, W)$, is related to the class $CL009$, which contains cubics $\mathrm{p}\mathcal{K}(W, V)$, and to the class $CL026$, which contains cubics $\mathrm{n}\mathcal{K}_0(W, W)$. We give four examples. In general, the Grassmann points and cevian points may be complex.

Example 20. As above, the cubic $\mathrm{n}\mathcal{K}_0(X(1), X(1))$ has parent $\mathrm{p}\mathcal{K}(X(1), X(9))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(1), X(9), k)$ has parent $K101 = \mathrm{p}\mathcal{K}(X(1), X(1))$.

Example 21. As above, the cubic $\mathrm{n}\mathcal{K}_0(H, H)$ has parent $K159 = \mathrm{p}\mathcal{K}(H, X(1249))$. The harmonic associate $\mathrm{n}\mathcal{K}(H, X(1249), k)$ has parent $K181 = \mathrm{p}\mathcal{K}(H, H)$.

Example 22. The cubic $\mathrm{n}\mathcal{K}_0(X(9), X(9))$ has parent $K157 = \mathrm{p}\mathcal{K}(X(9), X(1))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(9), X(1), k)$ has parent $\mathrm{p}\mathcal{K}(X(9), X(9))$.

Example 23. The cubic $\mathrm{n}\mathcal{K}_0(X(32), X(32))$ has parent $K160 = \mathrm{p}\mathcal{K}(X(32), X(206))$. The harmonic associate $\mathrm{n}\mathcal{K}(X(32), X(206), k)$ has parent $\mathrm{p}\mathcal{K}(X(32), X(32))$.

8. Gibert's theorem

In private correspondence, Bernard Gibert has noted a further characterization of the vertices of the desmic structures we have used.

Theorem 21 (Gibert)**.** *Suppose that P,and Q are two W-isoconjugate points on the cubic $\mathrm{p}\mathcal{K}(W, S)$. For X on $\mathrm{p}\mathcal{K}(W, S)$, let X^t be the tangential of X, and $\mathrm{p}\mathcal{C}(X)$ be the polar conic of X. Now $\mathrm{p}\mathcal{C}(P^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at P^t (twice), at P, and at three other points A', B', C', and $\mathrm{p}\mathcal{C}(Q^t)$ meets $\mathrm{p}\mathcal{K}(W, S)$ at Q^t (twice), at Q, and at three other points A'', B'', C''. Then the points A', B', C', A'', B'', C'' are the vertices of a desmic structure with perspectors P and Q.*

This can be verified computationally.

Appendix

We observe that the cubic $\mathrm{n}\mathcal{K}(W, R, k)$ meets the sidelines of $\triangle ABC$ at A, B, C and at the intersections with $\mathcal{T}(R)$. This accounts for all three intersections of the cubic with each sideline. The calculation referred to in Theorem 6(e) shows that $\mathrm{n}\mathcal{K}(W,R,k)$ and $\mathrm{p}\mathcal{K}(W,R_A)$ touch at B, C, meet at A, and at the intersections of $\mathcal{T}(R)$ with AB and AC. On algebraic grounds, there are nine intersections, so in the generic case, there are two further intersections. We now look at the Maple results in detail. If we look at the equations for $\mathrm{n}\mathcal{K}(W,R,k)$ and $\mathrm{p}\mathcal{K}(W,R_A)$ and solve for y, z, then we get the expected solutions, and $z = ax$, $y = \frac{-2vx(t+ar)}{2aus+k}$, where a satisfies

$$2u(2vtr-ks)X^2 + (4uvt^2 + 4vwr^2 - 4uvs^2 - k^2)X + 2w(2vtr-ks) = 0.$$

We cannot have $x = 0, or y = z = 0$. Thus we must have $a = 0$, giving $z = 0$, or $a = -\frac{t}{r}$, giving $y = 0$. The equation for a has a nonzero root only when $k = \frac{2vrt}{s}$. If we put $-\frac{t}{r}$ in the equation for a, we get $k = \frac{2ust}{r}$ or $k = \frac{2wrs}{t}$. If we replace $\mathrm{p}\mathcal{K}(W,R_A)$ by $\mathrm{p}\mathcal{K}(W,R_B)$ or $\mathrm{p}\mathcal{K}(W,R_C)$, we clearly get the same results. This establishes the criteria set out in Theorem 6(e).

If we consider the case $k = \frac{2vtr}{s}$, the equation for a becomes $X = 0$, so we can regard the solutions as limits as a tends to 0 or ∞. The former leads to $r : -s : 0$, the latter to $0 : 0 : 1$. We will meet these again below. We now examine the locus $GN(\triangle A'B'C')$ where the vertex $A' = a_1 : a_2 : a_3$ lies on a sideline. If $a_1 = 0$, the equation for the locus has some zero coefficients. This cannot include the cubic $\mathrm{n}\mathcal{K}(W,R,k)$ unless it is the whole plane. Thus we cannot define a cubic as a locus in this case.

Guided by the above discussion, we now examine the case $a_3 = 0$. Let $B' = b_1 : b_2 : b_3$, $C' = c_1 : c_2 : c_3$. The condition for the locus to be an $\mathrm{n}\mathcal{K}$ becomes $a_2b_3c_1 = 0$. Taking $a_2 = 0$ or $b_3 = 0$ leads to an equation with zero coefficients, so we must have $c_1 = 0$. When we equate the coefficients of the equations for the locus and $\mathrm{n}\mathcal{K}(W,R,k)$ other than y^2z and xyz, we find a unique solution. After scaling this is $A' = r : -s : 0$, $B' = u/r : -v/s : w/t$, $C' = 0 : -s : t$. Then the locus is $\mathrm{n}\mathcal{K}(W,R,2vtr/s)$. A little thought shows that for fixed W, R, there are only three such loci, giving $\mathrm{n}\mathcal{K}(W,R,k)$ with $k = 2ust/r$, $2vtr/s$, $2wrs/t$. From our earlier work, we know that there is no other way to express these cubics as loci of type GN.

We should expect to obtain a desmic stucture by taking W-isoconjugates of A', B', C'. If we write the isoconjugate of $X = x : y : z$ as $uyz : vzx : wxy$, then the isconjugates are $A'' = C$, $B'' = r : -s : t = R_B$, $C'' = A$. Then $\triangle ABC$ and $\triangle A'B'C'$ have perspector B, $\triangle ABC$ and $\triangle CR_BA$ have perspector $P = r : 0 : t$, $\triangle CR_BA$ and $\triangle A'B'C'$ have perspector R_B.

It is a moot point whether this should be termed a desmic structure. It satisfies the perspectivity conditions, but has only eight distinct points. If we replace B' by any point, we still get the same perspectors. If we allow this as a desmic, then Theorem 7 holds as stated. If not, we can either add these three cubics to the excluded list or reword in the weaker form.

Theorem 7′. *If a cubic $\mathcal{C}$ is of type $\mathrm{n}\mathcal{K}$, but not of type $\mathrm{c}\mathcal{K}$, then there is a triangle Δ with $\mathcal{C} = GN(\Delta)$, and at most two such triangles.*

To get R as the barycentric difference of perspectors, we need to scale B to $0 : -s : 0$. Then the sum is R_B. A check using Theorem 9(b) shows that the parent is indeed $\mathrm{p}\mathcal{K}(W, R_B) = GP(\triangle A'B'C')$. Replacing each coordinate of P in turn by the corresponding one from B, we get C, R_B, A, as expected. On the other hand, starting from B, we get A', $0 : 0 : 0$, C'. This reflects the fact that the other points do not determine B'. When we compute the equation of $GN(\triangle CR_BA)$, we find that all the coefficients are zero. Thus cubics of this kind are Grassmann cubics for only one triangle rather than the usual two.

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Concurrent Medians of $(2n + 1)$-gons

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Abstract. We exhibit conditions that determine whether a set of $2n + 1$ lines are the medians of a $(2n + 1)$-sided polygon. We describe how to regard certain collections of sets of medians as a linear subspace of related collections of sets of lines, and as a consequence, we show that every set of $2n + 1$ concurrent lines are the medians of some $(2n + 1)$-sided polygon. Also, we derive conditions on $n+1$ points so that they can be consecutive vertices of a $(2n+1)$-sided polygon whose medians intersect at the origin. Each of these constructions demonstrates a procedure that generates $(2n + 4)$-degree of freedom families of median-concurrent polygons. Furthermore, this number of degrees of freedom is maximal.

1. Motivation

It is well-known that the medians of a triangle intersect in a common point. We wish to explore which polygons in general have this property. Necessarily, such polygons must have an odd number of edges. One easy source of such polygons is to begin with a regular $(2n + 1)$-gon centered at the origin and transform the vertices using an affine transformation. This exhausts the triangles as every triangle is an affine image of an origin-centered equilateral triangle. On the other hand, if we begin with either a regular pentagon or a regular pentagram, this process fails to exhaust the median-concurrent pentagons. Consider the pentagon with the sequence of vertices $v_0 = (0, 1)$, $v_1 = (1, 0)$, $v_2 = (2, 1)$, $v_3 = (-2, -2)$, and $v_4 = (6, 2)$, depicted in Figure 1.

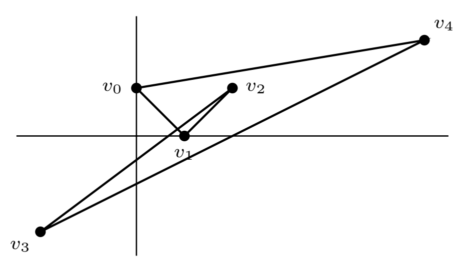

Figure 1. A non-affinely regular median concurrent pentagon

It is easy to check that for each i (all indices modulo 5), the line through v_i and $\frac{1}{2}(v_{i+2} + v_{i+3})$ contains the origin. Alternatively, it suffices to check that v_i and $v_{i+2} + v_{i+3}$ are scalar multiples. On the other hand, this pentagon has a single self-intersection whereas a regular pentagon has none and a regular pentagram has five,

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so this example cannot be the image under an affine map of a regular 5-gon. Thus, we seek alternative and more comprehensive means to construct median concurrent pentagons specifically and $(2n+1)$-gons in general.

We approach this problem by two different means. In the next section, we consider families of lines that serve as the medians of polygons and in the section afterwards, we begin with a collection of $n+1$ consecutive vertices and show how to "complete" the collection with the remaining n vertices; the result will also be a $(2n+1)$-gon whose medians intersect.

2. Families of polygons constructed by medians

2.1. *Oriented lines and the signed law of sines.* In this section, we will be working with oriented lines. Given a line ℓ' in $\mathbb{R}^2$, we associate with it a unit vector $\mathbf{v}$ that is parallel with ℓ'. The oriented line ℓ is defined as the pair $(\ell', \mathbf{v})$. Then given points A and B on ℓ, we can solve $\overrightarrow{AB} = t\mathbf{v}$ for t and say that t is the "signed length" from A to B along ℓ; this quantity is denoted $d_\ell(A, B)$. If $t > 0$, we will say that B is on the "positive side" of A along ℓ; if $t < 0$, we will say that B is on the "negative side" of A along ℓ. Switching the orientation of a line switches the sign of the signed length from one point to another on that line.

Let ℓ_1 and ℓ_2 be two non-parallel oriented lines and C be their intersection point. Let D_i be a point on the positive side of C along ℓ_i. The "signed angle" from ℓ_1 to ℓ_2, denoted θ_{12} is the angle whose magnitude (in the range $(0, \pi)$) is that of $\angle D_1CD_2$ and whose sign is that of the cross product $\mathbf{v}_1 \times \mathbf{v}_2$, the vectors thought of as lying in the $z = 0$ plane of $\mathbb{R}^3$. The signed angle of two parallel lines with the same unit vector is 0, and with opposite unit vectors is π. Switching the orientation of a single line switches the sign of the signed angle between them.

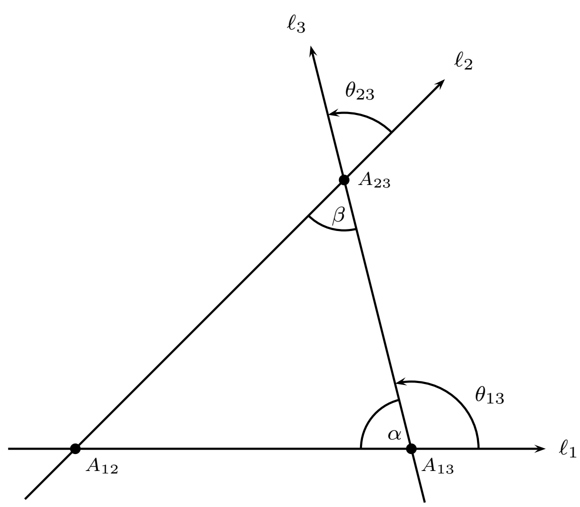

Figure 2. The signed law of sines

In Figure 2 we have three oriented lines ℓ_1, ℓ_2, ℓ_3. Point A_{ij} is at the intersection of ℓ_i and ℓ_j. As drawn, A_{13} is on the positive side of A_{12} along ℓ_1, A_{23} is on the positive side of A_{12} along ℓ_2, and A_{23} is on the positive side of A_{13} along ℓ_3. Letting $\alpha = \angle A_{23}A_{13}A_{12}$ and $\beta = \angle A_{12}A_{23}A_{13}$, we have $\sin\alpha = \sin\theta_{13}$ and $\sin\beta = \sin\theta_{23}$.

By the ordinary law of sines and the above comments about α and β, we have

$$\frac{d_{\ell_1}(A_{12}, A_{13})}{\sin\theta_{23}} = \frac{d_{\ell_2}(A_{12}, A_{23})}{\sin\theta_{13}}.$$

Note that if we switch the orientation of ℓ_1, then the numerator of the LHS and the denominator of the RHS change signs. Switching the orientation of ℓ_2 changes the signs of the numerator of the RHS and the denominator of the LHS. Switching the orientation of ℓ_3 changes the signs of both denominators. Any of these orientation switches leaves the LHS and RHS equal, and so the equation above is true for oriented lines and signed angles as well; we call this equation “the signed law of sines.”

2.2. *Constructing polygons via medians.* Let $\ell_0, \ell_1, \ldots, \ell_{2n}$ be $2n + 1$ oriented lines, no two parallel, in $\mathbb{R}^2$. Let $B_{i,j}$ be the point of intersection of ℓ_i and ℓ_j, and let δ_{i+1} be $d_{\ell_{i+1}}(B_{i,i+1}, B_{i+1,i+2})$. Finally, choose a point A_0 on ℓ_0 and let $S_0 = d_{\ell_0}(B_{0,1}, A_0)$.

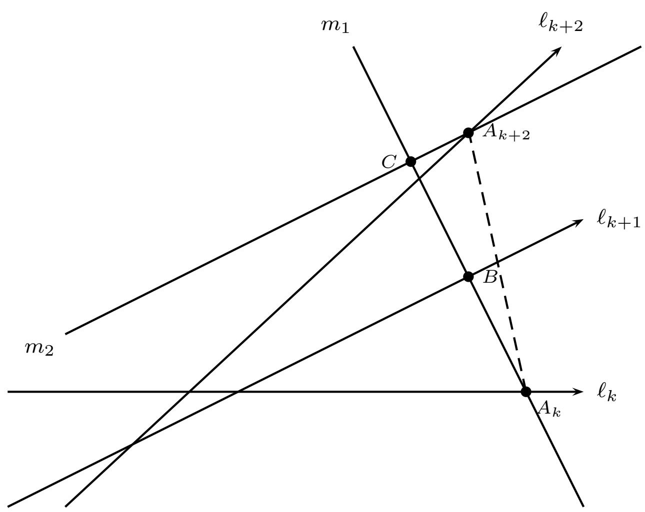

Figure 3. Constructing consecutive points via medians

Given a point A_k on ℓ_k, we construct the point A_{k+2} on ℓ_{k+2} (the indices of lines are modulo $2n + 1$) as follows and as depicted in Figure 3.

Construction 1. *Construct line m_1 through A_k and perpendicular to ℓ_{k+1}. Let point B be the intersection of line ℓ_{k+1} and m_1. Construct point C on line m_1 so that segments $\overline{A_kB}$ and $\overline{BC}$ are congruent. Construct line m_2 through C and perpendicular to m_1. Let point A_{k+2} be the intersection of lines m_2 and ℓ_{k+2}.*

We note that line ℓ_{k+1} intersects segment $\overline{A_k, A_{k+2}}$ at its midpoint.

We define $S_k = d_{\ell_k}(B_{k,k+1}, A_k)$ and θ_{ij} to be the signed angle subtended from ℓ_i to ℓ_j. Letting m be the line through A_k and A_{k+2} and $\varphi = \theta_{\ell_{k+1},m}$, we have by the signed law of sines,

$$\frac{x}{\sin\theta_{k,k+1}} = \frac{S_k}{\sin(\theta_{k,k+1}+\varphi)}$$

and

$$\frac{x}{\sin\theta_{k+1,k+2}} = \frac{S_{k+2}+\delta_{k+1}}{\sin(\theta_{k,k+1}+\varphi)}.$$

Eliminating x, we have

$$S_{k+2} = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}} S_k - \delta_{k+1}.$$

Let

$$g_k = \frac{\sin\theta_{k,k+1}}{\sin\theta_{k+1,k+2}}.$$

Then we have the recurrence

$$S_{k+2} = g_k S_k - \delta_{k+1}.$$

This leads to

$$\begin{aligned} S_{k+4} &= g_{k+2}S_{k+2} - \delta_{k+3} \\ &= g_{k+2}g_k S_k - g_{k+2}\delta_{k+1} - \delta_{k+3}, \end{aligned}$$

and

$$\begin{aligned} S_{k+6} &= g_{k+4}S_{k+4} - \delta_{k+5} \\ &= g_{k+4}g_{k+2}g_k S_k - g_{k+4}g_{k+2}\delta_{k+1} - g_{k+4}\delta_{k+3} - \delta_{k+5}. \end{aligned}$$

In general, if we define

$$h_{k,p,m} = g_{k+2p}g_{k+2p+2}g_{k+2p+4}\cdots g_{k+2(m-1)},$$

we have

$$S_{k+2m} = h_{k,0,m}S_k - h_{k,1,m}\delta_{k+1} - h_{k,2,m}\delta_{k+3} - \cdots - \delta_{k+2m-1}.$$

We are interested in the case when we begin with a point A_0 on ℓ_0 and eventually construct the point $A_{2(2n+1)}$ which will also be on line ℓ_0. When $k = 0$ and $m = 2n+1$, we have

$$S_{2(2n+1)} = h_{0,0,2n+1}S_0 - h_{0,1,2n+1}\delta_1 - h_{0,2,2n+1}\delta_3 - \cdots - \delta_{2(2n+1)-1}.$$

We notice first that

$$h_{0,0,2n+1} = \prod_{k=0}^{2n} g_{2k} = \prod_{k=0}^{2n} \frac{\sin\theta_{2k,2k+1}}{\sin\theta_{2k+1,2k+2}} = \frac{\prod_{k=0}^{2n}\sin\theta_{2k,2k+1}}{\prod_{k=0}^{2n}\sin\theta_{2k+1,2k+2}}.$$

Since the subscripts in the latter products are all modulo $2n+1$, it follows that the terms in the numerator are a permutation of those in the denominator. This means that $h_{0,0,2n+1} = 1$. The second observation is that

$$S_{2(2n+1)} = S_0 + \text{ a linear combination of the } h_{0,i,2n+1} \text{ values}.$$

The coefficients of this linear combination are the δ's. The nullspace of the h values will in fact be a codimension 1 subspace of the space of all possible choices of $(\delta_0, \delta_1, \ldots, \delta_{2n})^T$. An immediate consequence of this is that if for all i we have $\delta_i = 0$ then $S_{4n+2} = S_0$ and so we have closed the polygon A_0, A_2, A_4, $\ldots$, $A_{4n+2} = A_0$. We have shown

Proposition 1. *Any set of $2n+1$ concurrent lines, no two parallel, in $\mathbb{R}^2$ are the medians of some $(2n+1)$-gon.*

Consider choosing a family of $2n+1$ concurrent lines. Each line can be chosen by choosing a unit vector, the choice of each being a single degree of freedom (for instance, the angle that vector makes with the vector $(1,0)^T$). Another degree of freedom is the choice of point A_0 on ℓ_0. Finally, there are two more degrees of freedom in the choice of the point of concurrency. This is a total of $2n+4$ degrees of freedom for constructing $(2n+1)$-gons with concurrent medians.

3. Families of median-concurrent polygons constructed by vertices

Suppose we have three points $(a,b), (c,d)$, and (e,f) in $\mathbb{R}^2$ such that $(a,b) \neq (-c,-d)$. We seek a fourth point (u,v) such that (u,v), $(a+c, b+d)$ and $(0,0)$ are collinear, and (a,b), $(e+u, f+v)$ and $(0,0)$ are also collinear.

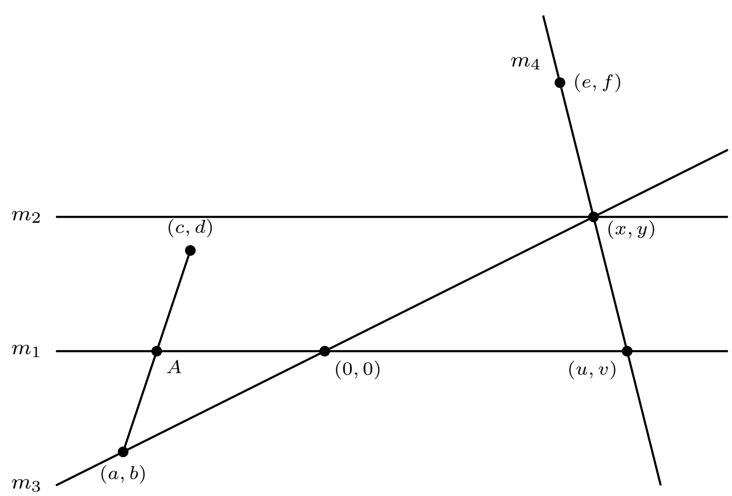

Figure 4. Constructing the fourth point

The point (u,v) can be constructed as follows:

Construction 2. *Let A be the midpoint of the segment joining (a,b) and (c,d) and m_1 be the line through A and the origin. Construct the line m_2 parallel to m_1 that is on the same side of, but half the distance from (e,f) as m_1. Let m_3 be the line through (a,b) and the origin, intersecting m_2 at (x,y), and let m_4 be the line through (e,f) and (x,y). The point (u,v) is the intersection of lines m_1 and m_4.*

It must be the case that $(u, v) = k(a + c, b + d)$ and also

$$u(b + d) = v(a + c)$$
$$b(e + u) = a(f + v).$$

Subtracting, we have

$$ud - be = vc - af$$

or

$$k(a + c)d - be = k(b + d)c - af.$$

Isolating k, we have

$$k = \frac{be - af}{ad - bc}.$$

Notice that the fourth point is uniquely determined by the other three, provided $ad - bc \neq 0$.

We formalize this in the following definition.

Definition 1. *Given point $A = (a, b), C = (c, d)$, and $E = (e, f)$, we define the point $F(A, C, E)$ by the formula*

$$F = F(A, C, E) = \frac{be - af}{ad - bc}(A + C).$$

This point satisfies the property that the lines $\overline{F, (A + C)}$ and $\overline{A, (E + F)}$ intersect at the origin.

Now, suppose we have $n+1$ points in $\mathbb{R}^2$, $x_i = (a_i, b_i)$, $0 \leq i \leq n$, and suppose that no line joining two consecutive points contains the origin. Starting at $i = 0$ we define $x_{i+n+1} = F(x_i, x_{i+1}, x_{i+n})$. We address what happens in the sequence $x_0, x_1, x_2, \ldots$.

We can recast our definition of the point x_{i+n+1} using the following definition.

Definition 2. *For $0 \leq j, k$, $\Delta_{j,k} = a_j b_k - b_j a_k$.*

Armed with this, we have

$$x_{i+n+1} = \frac{\Delta_{i+n,i}}{\Delta_{i,i+1}}(x_i + x_{i+1}).$$

Also, we use induction to prove:

Proposition 2. *For all $j \geq 0$, $\Delta_{j+n,j} = \Delta_{n,0}$.*

Proof. The case $j = 0$ is obvious. Suppose the result is true for $j = k$. Then

$$\begin{aligned}
\Delta_{k+1+n,k+1} &= a_{k+1+n}b_{k+1} - b_{k+1+n}a_{k+1} \\
&= \frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}((a_k + a_{k+1})b_{k+1} - (b_k + b_{k+1})a_{k+1}) \\
&= \frac{\Delta_{k+n,k}}{\Delta_{k,k+1}}(a_k b_{k+1} - b_k a_{k+1}) \\
&= \Delta_{k+n,k} \\
&= \Delta_{n,0}
\end{aligned}$$

which completes the induction. □

As a consequence, we have

$$x_{i+n+1} = \frac{\Delta_{n,0}}{\Delta_{i,i+1}}(x_i + x_{i+1}).$$

We verify a useful property of $\Delta_{j,k}$:

Proposition 3. *For all* j, k, ℓ,

$$\Delta_{j,k}x_\ell + \Delta_{\ell,j}x_k = \Delta_{\ell,k}x_j.$$

Proof. We work component-by-component:

$$\begin{aligned}
\Delta_{j,k}a_\ell + \Delta_{\ell,j}a_k &= (a_jb_k - b_ja_k)a_\ell + (a_\ell b_j - b_\ell a_j)a_k \\
&= a_jb_ka_\ell - b_ja_ka_\ell + a\ell b_ja_k - b_\ell a_ja_k \\
&= (a_\ell b_k - b_\ell a_k)a_j \\
&= \Delta_{\ell,k}a_j,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_{j,k}b_\ell + \Delta_{\ell,j}b_k &= (a_jb_k - b_ja_k)b_\ell + (a_\ell b_j - b_\ell a_j)b_k \\
&= a_jb_kb_\ell - b_ja_kb_\ell + a\ell b_jb_k - b_\ell a_jb_k \\
&= (a_\ell b_k - b_\ell a_k)b_j \\
&= \Delta_{\ell,k}b_j.
\end{aligned}$$

□

We can now prove the following:

Proposition 4. *For all* $0 \leq i \leq 2n$*, there is a* k_i *such that* $x_{i-1} + x_i = k_ix_{n+i}$ (*all subscripts modulo* $2n+1$).

Proof. For $i = 0$, we calculate

$$\begin{aligned}
x_{2n} + x_0 &= \frac{\Delta_{n,0}}{\Delta_{n-1,n}}(x_{n-1} + x_n) + x_0 \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n,0}x_{n-1} + \Delta_{n,0}x_n + \Delta_{n-1,n}x_0) \\
&= \frac{1}{\Delta_{n-1,n}}((\Delta_{n,0}x_{n-1} + \Delta_{n-1,n}x_0) + \Delta_{n,0}x_n) \\
&= \frac{1}{\Delta_{n-1,n}}(\Delta_{n-1,0}x_n + \Delta_{n,0}x_n) \\
&= \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}}x_n
\end{aligned}$$

and so

$$k_0 = \frac{\Delta_{n-1,0} + \Delta_{n,0}}{\Delta_{n-1,n}}.$$

If $1 \leq i \leq n$, then by the definition of x_{n+i}, $k_i = \Delta_{i-1,i}/\Delta_{n,0}$.

To handle the case when $i = n + 1$, we calculate

$$\begin{aligned}
x_n + x_{n+1} &= x_n + \frac{\Delta_{n,0}}{\Delta_{0,1}}(x_0 + x_1) \\
&= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}(x_0 + x_1)) \\
&= \frac{1}{\Delta_{0,1}}(\Delta_{0,1}x_n + \Delta_{n,0}x_1 + \Delta_{n,0}x_0) \\
&= \frac{1}{\Delta_{0,1}}(\Delta_{n,1}x_0 + \Delta_{n,0}x_0) \\
&= \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}x_0
\end{aligned}$$

and so

$$k_{n+1} = \frac{\Delta_{n,1} + \Delta_{n,0}}{\Delta_{0,1}}.$$

For the values of i, $n + 2 \le i \le 2n$, we set $m = i - n$ and we have, using the symbol $\delta = \Delta_{n,0}/(\Delta_{m-2,m-1}\Delta_{m-1,m})$,

$$\begin{aligned}
x_{i-1} + x_i &= x_{n+m-1} + x_{n+m} \\
&= \frac{\Delta_{n,0}}{\Delta_{m-2,m-1}}(x_{m-2} + x_{m-1}) + \frac{\Delta_{n,0}}{\Delta_{m-1,m}}(x_{m-1} + x_m) \\
&= \delta(\Delta_{m-1,m}(x_{m-2} + x_{m-1}) + \Delta_{m-2,m-1}(x_{m-1} + x_m) \\
&= \delta((\Delta_{m-1,m}x_{m-2} + \Delta_{m-2,m-1}x_m) + (\Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1}) \\
&= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{m-1} \\
&= \delta(\Delta_{m-2,m} + \Delta_{m-1,m} + \Delta_{m-2,m-1})x_{i+n}
\end{aligned}$$

noting that $m - 1 = i - n - 1 \equiv i + n$ modulo $2n + 1$, and so for $n + 2 \le i \le 2n$, we have

$$k_i = \frac{\Delta_{n,0}(\Delta_{i-n-2,i-n} + \Delta_{i-n-1,i-n} + \Delta_{i-n-2,i-n-1})}{\Delta_{i-n-2,i-n-1}\Delta_{i-n-1,i-n}}.$$

□

What this proposition says, geometrically, is that the points $x_{i-1} + x_i$, x_{i+n} and the origin are collinear. Alternatively, setting $i = j + n + 1$, we find that the points x_j, $x_{j+n} + x_{j+n+1}$ and the origin are collinear. But this means that $\frac{1}{2}(x_{j+n} + x_{j+n+1})$ is also on the same line, and so the line connecting x_j and the midpoint of the segment joining x_{j+n} and x_{j+n+1} contains the origin.

As a direct consequence, we obtain the following result:

Proposition 5. *Given any sequence of $n + 1$ points, $x_0, x_1, \ldots, x_n$ such that the origin is not on any line $\overline{x_i, x_{i+1}}$ or $\overline{x_n, x_0}$, then these points are $n + 1$ vertices in sequence of a unique $(2n + 1)$-gon whose medians intersect in the origin.*

Here, we can choose $n + 1$ points to determine a $(2n + 1)$-gon whose medians intersect at the origin. Each point contributes two degrees of freedom for a total of

$2n+2$ degrees of freedom. Two more degrees of freedom are obtained for the point of concurrency, for a total of $(2n + 4)$ degrees of freedom. This echoes the final result from the previous section. That we cannot obtain further degrees of freedom follows from the previous section as well. There, *any* set of $2n+1$ concurrent lines (in general position) produced a concurrent-median $(2n+1)$-gon. We cannot hope for more freedom than this.

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On the Generating Motions and the Convexity of a Well-Known Curve in Hyperbolic Geometry

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Abstract. In Euclidean geometry the vertices P of those angles $\angle APB$ of size α that pass through the endpoints A, B of a given segment trace the arc of a circle. In hyperbolic geometry on the other hand a set of equivalently defined points P determines a different kind of curve. In this paper the most basic property of the curve, its convexity, is established. No straight-forward proof could be found. The argument rests on a comparison of the rigid motions that map one of the angles $\angle APB$ into other ones.

1. Introduction

In the hyperbolic plane let AB be a segment and H one of the halfplanes with respect to the line through A and B. What will be established here is the convexity of the locus Ω of the point P which lies in H and which determines together with A and B an angle $\angle APB$ of a given fixed size. In Euclidean geometry this locus is well-known to be an arc of the circle through A and B whose center C determines the (oriented) angle $\angle ACB = 2 \cdot \angle APB$. In hyperbolic geometry, on the other hand, one obtains a wider, flatter curve (see Figure 1; [2, p.79, Exercise 4], [1], and also [6, Section 50], [7, Section 2]). The evidently greater complexity of the non-Euclidean version of this locus shows itself most clearly when one considers the (direct) motion that carries a defining angle $\angle APB$ into another defining angle $\angle AP'B$. Whereas in Euclidean geometry it has to be a rotation, it can in hyperbolic geometry also be a horocyclic rotation about an improper center, or, surprisingly, even a translation. For our convexity proof it appears to be practical to consider the given angle as fixed and the given segment as moving. Then, as will be shown in the *Main Lemma*, the relative position of the centers or axes of our motions can be described in a very simple fashion, with the sought-after convexity proof as an easy consequence. As to proving the Lemma itself, one has to take into account that the motions involved can be rotations, horocyclic rotations, or translations, and it seems that a distinction of cases is the only way to proceed. Still, it would be desirable if the possibility of an overarching but nonetheless elementary argument would be investigated further.

The fact of the convexity of our curve yields at least one often used by-product:

Theorem. *Let AB be a segment, H a halfplane with respect to the line through A and B, and ℓ a line which has points in common with H but avoids segment AB.*

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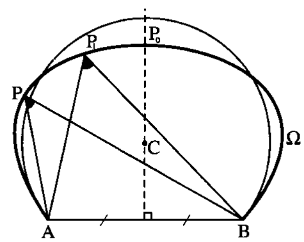

Figure 1

Then the point X, when running through ℓ in H, determines angles $\angle AXB$ that first monotonely increase, and thereafter monotonely decrease in size.

Our approach will be strictly axiomatic and elementary, based on Hilbert's axiom system of Bolyai-Lobachevskian geometry (see [3, Appendix III]). The application of Archimedes' axiom in particular is excluded. Beyond the initial concepts of hyperbolic plane geometry we will only rely on the facts about angle sum, defect, and area of polygons (see e.g. [2, 5, 6, 8]), and on the basic properties of isometries. To facilitate the reading of our presentation we precede it with a list of frequently used abbreviations.

1.1. *Abbreviations.*

1.1.1. $[A_1A_2 \dots A_h \dots A_i \dots A_k \dots A_n]$ for an n-tuple of points with A_i *between* A_h and A_k for $1 \leq h < i < k \leq n$.

1.1.2. $AB, CD, \dots$ for *segments*, and $(AB), (CD), \dots$ for the related *open intervals* $AB - \{A, B\}$, $CD - \{C, D\}, \dots$; $\overrightarrow{AB}, \overrightarrow{CD}, \dots$ for the *rays* from A through B, from C through $D, \dots$, and $\overrightarrow{(A)B}, \overrightarrow{(C)D}, \dots$ for the related *halflines* $\overrightarrow{AB} - \{A\}$, $\overrightarrow{CD} - \{C\}, \dots$; $\ell(AB), \ell(CD), \dots$ for the *lines* through A and B, C and $D, \dots$.

1.1.3. $a, b, c, \dots$ are general abbreviations for lines, $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \dots$ for rays in those lines, and $(\overrightarrow{a}), (\overrightarrow{b}), (\overrightarrow{c}), \dots$ for the related halflines.

1.1.4. $H(a, B)$, where the point B is not on line a, for the *halfplane with respect to a which contains B*, and $\overline{H}(a, B)$ for the *halfplane with respect to a which does not contain B*. The improper ends of rays which enter halfplane H through a are considered as belonging to H.

1.1.5. perp(a, B) for the line which is *perpendicular* to a and incident with B; proj(S, ℓ) for the *orthogonal projection* of the point or pointset S to ℓ.

1.1.6. $ABC\underline{D}$ for the *Lambert quadrilateral* with right angles at A, B, C and an acute angle at D.

1.1.7. $\mathbf{R}$ for the size of a right angle.

1.1.8. $a \times b,\ \ a \times \overrightarrow{p}, \ldots$ for the *intersection point* of the lines a and b, of the line a and the ray $\overrightarrow{p}, \ldots$.

1.1.9. $\cdot, \cdot\cdot, \circ$ (in figures) for specific acute angles with $\cdot\cdot$ denoting a smaller angle than $\cdot$.

Remark. In the figures of Section 3, lines and metric are distorted to better exhibit the betweenness features.

2. Segments that join the legs of an angle

In this section we compile a number of facts about segments whose endpoints move along the legs of a given angle. All statements hold in Euclidean and hyperbolic geometry alike; the easy absolute proofs are for the most part left to the reader.

Let $\angle(\overrightarrow{a}, \overrightarrow{b})$ be an angle with vertex P, and $\mathcal{C}$ be the class of segments $A_\nu B_\nu$ of length s that have endpoint A_ν on leg $(\overrightarrow{a})$ and endpoint B_ν on leg $(\overrightarrow{b})$ of this angle, and satisfy the equivalent conditions

$$\text{(1a)} \quad \angle PA_\nu B_\nu \geq \angle PB_\nu A_\nu, \qquad\qquad \text{(1b)} \quad PA_\nu \leq PB_\nu,$$

(see Figures 2a, b). We will always draw $\overrightarrow{a}, \overrightarrow{b}$ as rays that are *directed downwards* and, to simplify expression, refer to P as the *summit* of $\angle(\overrightarrow{a}, \overrightarrow{b})$. As a result of (1a) the segments $A_\nu B_\nu$ are uniquely determined by their endpoints on $(\overrightarrow{a})$, and $\mathcal{C}$ can be generated by sliding downwards through the points on $(\overrightarrow{a})$ and finding the related points on $(\overrightarrow{b})$.

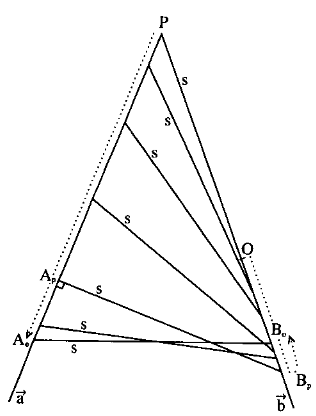

Figure 2a

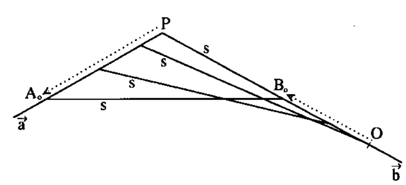

Figure 2b

It is easy to see that during this downwards movement $\angle PA_\nu B_\nu$ decreases and $\angle PB_\nu A_\nu$ increases in size. Due to (1a) the segment A_0B_0 which satisfies $\angle PA_0B_0 \equiv \angle PB_0A_0,\ PA_0 \equiv PB_0$ is the lowest of class $\mathcal{C}$.

If $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ then the class $\mathcal{C}$ contains a segment A_pB_p such that $\angle PA_pB_p = \mathbf{R}$. Note that when A_ν moves downwards from P to A_p, B_ν moves in tandem down from the point O s units below P to B_p, but that when A_ν moves on downwards from A_p to A_0, B_0 moves back upwards from B_p to B_0 (see Figure 2a). If $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ no perpendicular line to $(\overrightarrow{a})$ meets $(\overrightarrow{b})$ and the points B_ν move invariably upwards when the points A_ν move downwards (see Figure 2b).

Now consider three segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ whose endpoints on $(\overrightarrow{a})$ satisfy the order relation $[AA_1A_2P]$, and the direct motions that carry segment AB to segment A_1B_1 and to segment A_2B_2. These motions belong to the inverses of the ones described above and may carry B first downwards and then upwards. As a result there are seven conceivable situations as far as the order of the points B, B_1 and B_2 is concerned (see Figure 3):

(I)	$[B_2B_1BP]$,	(II)	$[B_1BP],\ B_2 = B_1$
(III)	$[B_1B_2BP]$,	(IV)	$[B_1BP],\ B_2 = B$,
(V)	$[B_1BB_2P]$,	(VI)	$[BB_2P],\ B_1 = B$, and
(VII)	$[BB_1B_2P]$.		

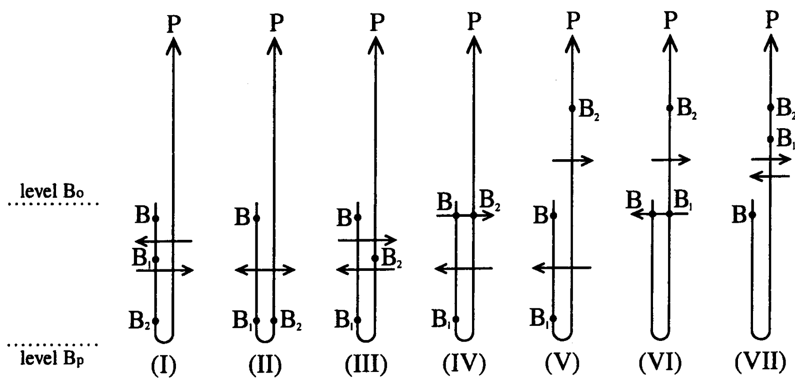

Figure 3

In case $\angle(\overrightarrow{a}, \overrightarrow{b}) \geq \mathbf{R}$ the point B moves solely downwards (see Figure 2b) and we find ourselves automatically in situation (I). On the other hand if $\angle(\overrightarrow{a}, \overrightarrow{b}) < \mathbf{R}$ and A lies on or above A_p both endpoints of segment AB move simultaneously upwards, first to A_1B_1 and then on to A_2B_2 (see Figure 2a); this means that we are dealing with situation (VII).

In Figure 3 the level of the midpoint N_1 of BB_1 is indicated by an arrow to the left, and the level of the midpoint N_2 of BB_2 by an arrow to the right. We recognize at once that we can use N_1 and N_2 instead of B_1 and B_2 to characterize the above seven situations. Set forth explicitly, a triple of segments $AB, A_1B_1, A_2B_2 \in \mathcal{C}$ with $[AA_1A_2P]$ can be classified according to the following conditions on the midpoints N_1, N_2 of BB_1, BB_2:

$$(2)\quad \begin{array}{llll} \text{(I)} & [N_2N_1BP], & \text{(II)} & [N_1BP], N_2 = N_1, \\ \text{(III)} & [N_1N_2BP], & \text{(IV)} & [N_1BP],\ N_2 = B, \\ \text{(V)} & [N_1BN_2P], & \text{(VI)} & [BN_2P],\ N_1 = B, \text{ and} \\ \text{(VII)} & [BN_1N_2P]. & & \end{array}$$

Note that always $N_1N_2 = \dfrac{1}{2}B_1B_2$. The midpoints M_1, M_2 of AA_1, AA_2 similarly satisfy $M_1M_2 = \dfrac{1}{2}A_1A_2$; here the direction $M_1 \to M_2$ like the direction $A_1 \to A_2$ points invariably upwards.

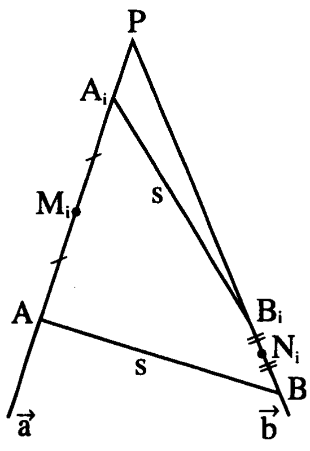

Figure 4

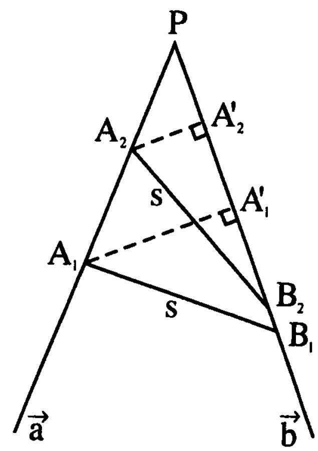

Figure 5

In closing this section we deduce two important inequalities which involve the points M_1, M_2, N_1 and N_2.

From $PA \leq PB,\ PA_i < PB_i$ (see (1b)) follows

$$(3)\qquad PM_i < PN_i \quad (i = 1, 2),$$

(see Figure 4). In addition, for situations (III) - (VII) in which B_2 lies above B_1 and N_2 above N_1 we can establish this. In the right triangles $\triangle A_1'A_1B_1$, $\triangle A_2'A_2B_2$ where $A_1' =$ proj(A_1, b), $A_2' =$ proj(A_2, b), $A_1'A_1 > A_2'A_2$, $A_1B_1 \equiv A_2B_2\ (= s)$, and as a result $A_1'B_1 < A_2'B_2$ (see Figure 5). So $A_1'A_2' > B_1B_2$, and because $A_1A_2 > A_1'A_2'$, $A_1A_2 > B_1B_2$. Noting what was said above we therefore have:

(4) If N_2 lies above N_1 then $M_1M_2 > N_1N_2$.

3. The centers of two key segment motions

In this section we locate the centers of the segment motions described above. Our setting is the hyperbolic plane in which (as is well-known) three kinds of direct motions have to be considered. The Euclidean case could be subsumed with few modifications under the heading of rotations.

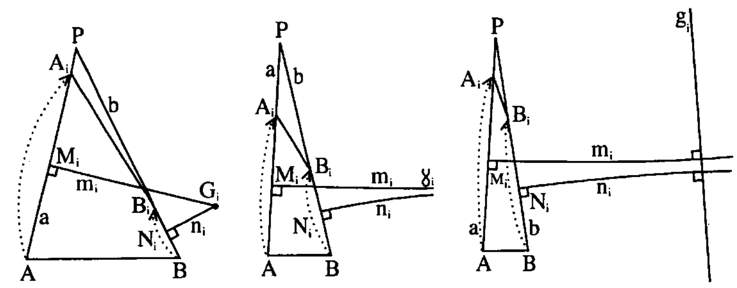

Figure 6

Let μ_i be the rigid, direct motion that carries the segment $AB \in \mathcal{C}$ onto the segment $A_iB_i \in \mathcal{C}$ $(i = 1, 2)$ where A_i lies above A, and let $m_i = \text{perp}(a, M_i)$, $n_i = \text{perp}(b, N_i)$. If lines m_i and n_i meet, μ_i is a *rotation* about their intersection point G_i, if they are boundary parallel, μ_i is an *improper* (horocyclic) *rotation* about their common end γ_i, and if they are hyperparallel, μ_i is a *translation* along their common perpendicular g_i (see e.g. [4, p. 455, Satz 13; Figure 6]). We call G_i, γ_i or g_i the *center* $[G_i]$ of the motion μ_i. For any point X disjoint from the center, $\ell(X[G_i])$ denotes the line joining X to the center of μ_i, namely $\ell(XG_i)$, $\ell(X\gamma_i)$, or $\text{perp}(g_i, X)$. The ray from X contained in this line and in the direction of $[G_i]$ will be referred to as the *ray* $\overrightarrow{X[G_i]}$ *from* X *towards the center of* μ_i; specifically for $X = P, M_i, N_i$ we define $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$, $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ and (if it exists) $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$.

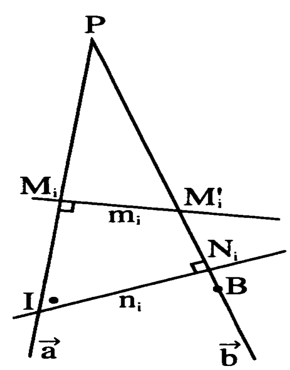

Figure 7

We now show that the center $[G_i]$ of motion μ_i must lie in $H(a, B)$.

If n_i does not intersect $(\overrightarrow{a})$ this is clear; if n_i meets a in a point I (see Figure 7) we verify the statement as follows. Segment PI as the hypothenuse of $\triangle PIN_i$ is larger than PN_i and so (see (3)) larger than PM_i. Consequently the angle $\angle PIN_i = \angle M_i IN_i$ is acute, which indicates that n_i when entering $H(a, B)$ at I, approaches m_i. As a result $[G_i]$ must lie in $H(a, B)$.

Some additional consequences are implied by the fact that the center $[G_i]$ of either motion μ_i is determined by a pair of perpendiculars m_i, n_i to lines a and b (see again Figure 6). If $[G_i] = \gamma_i$ is a common end of m_i, n_i and thus the center of a horocyclic rotation, it cannot be the end of ray $\overrightarrow{b}$. Similarly, if $[G_i] = g_i$ is the common perpendicular of m_i and n_i, and thus the translation axis, it is hyperparallel to both of the intersecting lines a, b and, as a result, has no point in common with either; furthermore, a and b, being connected, must belong to the same halfplane with respect to g_i. On the other hand, if $[G_i] = G_i$ is the common point of m_i and n_i, and thus the rotation center, it is indeed possible that it lies on $(\overrightarrow{b})$. The point G_i then is collinear with B and with its image B_i which means that for $B \neq B_i$ the rotation is a half-turn and G_i coincides with the midpoint N_i of BB_i; in addition G_i should be the midpoint M_i of AA_i which is impossible. So $B = B_i = N_i = G_i$; conversely, one establishes easily that if any two of the three points B, B_i, N_i coincide, μ_i is a rotation with center G_i equal to all three.

We now assume that our plane is furnished with an orientation (see [3, Section 20]), and that without loss of generality P lies to the left of ray $\overrightarrow{AB}$. This ray enters $H(a, B)$ at the point A of $(\overrightarrow{a})$ and $\overline{H}(b, A)$ at the point B of $(\overrightarrow{b})$. Also $\overrightarrow{m_i} = \overrightarrow{M_i[G_i]}$ enters $H(a, B)$ at a point of $(\overrightarrow{a})$ and so has P on its left hand side as well (see Figure 8). As to the ray $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ which (if existing, i.e. for $[G_i] \neq N_i$) originates at the point N_i of $(\overrightarrow{b})$, it has P on its left hand side if and only if it enters $\overline{H}(b, A)$, i.e. if and only if $[G_i]$ belongs to $\overline{H}(b, A)$.

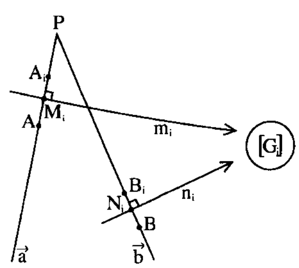

Figure 8a

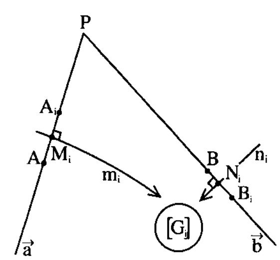

Figure 8b

Because the motion μ_i carries A across $\overrightarrow{m_i}$ to A_i on the side of P, A_i lies to the left and A to the right of $\overrightarrow{m_i}$. Being a direct motion, μ_i consequently also moves B (if $B \neq N_i$) from the right hand side of $\overrightarrow{n_i} = \overrightarrow{N_i[G_i]}$ to B_i on the left hand side which is the side of P iff $[G_i]$ belongs to $\overline{H}(b, A)$. In short, motion μ_i carries B upwards on $(\overrightarrow{b})$ iff $[G_i]$ lies in $\overline{H}(b, A)$.

We gather from the previous two paragraphs that

- $[G_i]$ belongs to $\overline{H}(b, A)$ if B_i and N_i lie above B on $(\overrightarrow{b})$,

• to $(\overrightarrow{b})$ if $B_i = N_i = B$, and

• to $H(b, A)$ if B_i and N_i lie below B on $(\overrightarrow{b})$.

Considering the motions μ_1, μ_2 again together we can tell in each of the seven situations listed in (2) where the two motion centers $[G_1], [G_2]$ (which both belong to $H(a, B)$) lie with respect to b. As we shall see, the relative positions of $[G_1], [G_2]$ can be described in a way that covers all seven situations: rotating ray $\overrightarrow{a} = \overrightarrow{PA}$ about P into $H(a, B)$ we always pass ray $\overrightarrow{P[G_1]}$ first, and ray $\overrightarrow{P[G_2]}$ second. More concisely,

MAIN LEMMA (ML). *Ray* $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ *always enters* $\angle(\overrightarrow{a}, \overrightarrow{p_2}) = \angle AP[G_2]$.

Proof. (The essential steps of the proof are outlined at the end.)

From (2) and the previous paragraph follows that $[G_1]$ lies in $H(b, A)$ in situations (I)-(V), on $\overrightarrow{(b)}$ in situation (VI) and in $\overline{H}(b, A)$ in situation (VII); $[G_2]$ lies in $H(b, A)$ in situations (I)-(III), on $(\overrightarrow{b})$ in situation (IV) and in $\overline{H}(b, A)$ in situations (V)-(VII), (see Figure 9). As a result the Lemma follows trivially for situations (IV)-(VI). The other situations are more complex, and their proofs require that the nature of the motion centers $[G_i]$, $(i = 1, 2)$, which can be a point G_i, end γ_i or axis g_i be taken into account. Thus a pair of motion centers $[G_1], [G_2]$ can be equal to G_1, G_2; G_1, γ_2; G_1, g_2; γ_1, G_2; γ_1, γ_2; γ_1, g_2; g_1, G_2; g_1, γ_2; g_1, g_2.

The arguments to be presented are dependent on the mutual position of P, M_1, M_2 on $\overrightarrow{a}$ and of P, N_1, N_2 on $\overrightarrow{b}$, and are best followed through Figure 9.

We first consider situations (I)-(III) in which $\angle(\overrightarrow{a}, \overrightarrow{b})$ includes $(\overrightarrow{p_1})$ and $(\overrightarrow{p_2})$. To verify (ML) we have to show that $\overrightarrow{p_2}$ does not enter $\angle(\overrightarrow{a}, \overrightarrow{p_1})$, or equivalently that $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$. (This assumes $\overrightarrow{p_1} \neq \overrightarrow{p_2}$ which either follows automatically or as an easy consequence of the arguments below.)

We begin with the special case that $\overrightarrow{p_1}$ meets m_2 in a point I. In this case statement (ML) holds if $\overrightarrow{p_2}$ does not intersect line m_2 at I or in a point between M_2 and I. Obviously this is so if $[G_2] = \gamma_2$ or g_2 because then $\overrightarrow{p_2}$ and m_2 do not intersect. If $[G_2] = G_2$, $\overrightarrow{p_2}$ and m_2 do intersect and we have to show that the intersection point, which is G_2, does not coincide with I or lie between M_2 and I. We first note that line n_1 does not intersect ray $\overrightarrow{p_1}$ in I or between I and P because the intersection point would have to be G_1 and so lie on m_1, a line entirely below m_2. As a consequence I, P, M_2, and, if it would lie between M_2 and I, also G_2, would all belong to the same halfplane with respect to n_1, namely $H(n_1, P)$. However this would entail that line n_2 which runs through G_2 would belong to this halfplane, which is not the case in situations (I) and (II). Thus we have established for those situations that $G_2 \neq I$, and $[M_2 G_2 I]$ does not hold, which means (ML) is true. We will present the proof of the same in situation (III) later on.

Due to the Axiom of Pasch the point I always exists if $\triangle PM_1[G_1]$ is a proper or improper triangle, i.e. if $[G_1] = G_1$, or γ_1. This means that we have so far proved (ML) for the cases G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2 and in addition for G_1, G_2; γ_1, G_2 in situations (I) and (II).

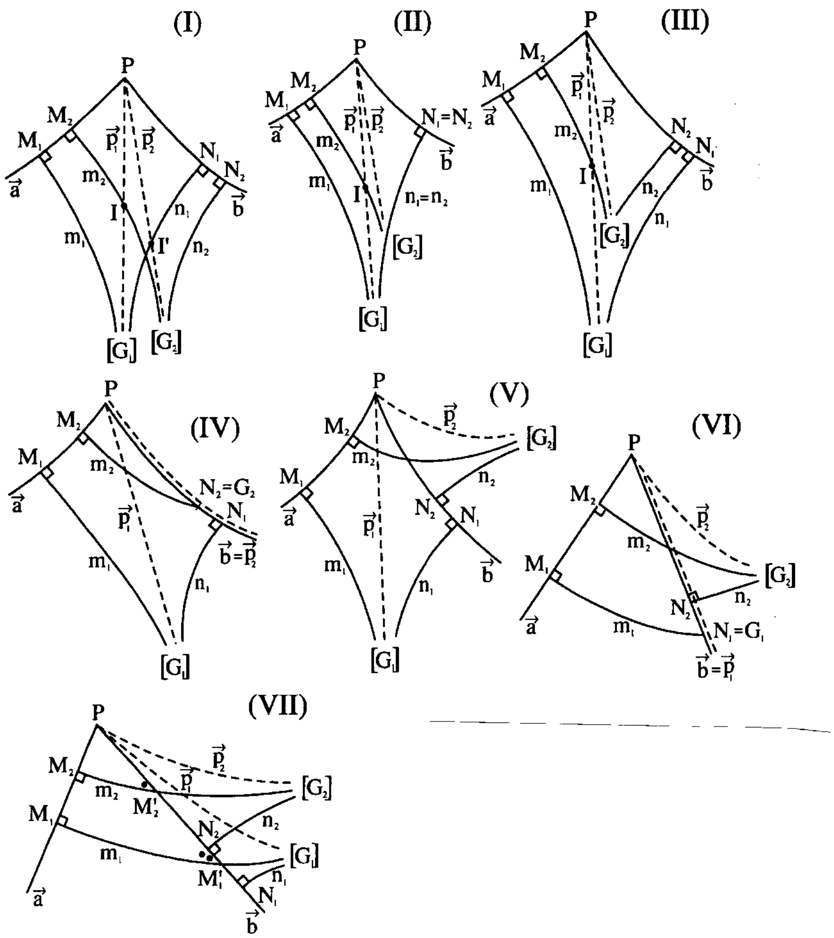

Figure 9

In the opposite special case that $\overrightarrow{p_2}$ meets n_1 in a point I' we can analogously show that for $[G_1] = \gamma_1$ or g_1 ray $\overrightarrow{p_1}$ does not enter $\angle(\overrightarrow{b}, \overrightarrow{p_2})$ and (ML) holds. In fact it is useful to mention here that this statement and its proof can be extended to include configurations in which $\overrightarrow{p_2}$ meets n_1 in an improper point ι'.

In situation (I) the point I' always exists if $[G_2] = G_2$ or γ_2 due to the Axiom of Pasch. In situation (II) with $n_1 = n_2$ I' exists for $[G_2] = G_2$ and ι' exists for $[G_2] = \gamma_2$ because in the first case $G_2 = I'$ and in the second case $\gamma_2 = \iota'$. This means that we have proved (ML) also for g_1, G_2; g_1, γ_2 in situations (I) and (II).

The proofs of the remaining cases, namely g_1, g_2 in situations (I)-(III), and G_1, G_2; γ_1, G_2; g_1, G_2; g_1, γ_2 in situation (III) require metric considerations and will be presented later.

Remark. Taking into account that we have already established (ML) in the case in which $\overrightarrow{p_1}$ and m_2 meet in a point I and $[G_2] = \gamma_2$ or g_2 we will assume when proving (ML) for g_1, g_2 and γ_1, g_2 that $\overrightarrow{p_1}$ and m_2 do not meet. At the same time, taking into account that we have already established (ML) in the case that $\overrightarrow{p_2}$ and n_1 meet in a point I' and $[G_1] = \gamma_1$ or g_1 we will assume that $\overrightarrow{p_2}$ and n_1 do not meet.

Turning to situation (VII) we observe that each of the rays $\overrightarrow{m_i}$ $(i = 1, 2)$ intersects $(\overrightarrow{b})$ in a point M_i' and approaches ray $\overrightarrow{n_i}$ in $\overline{H}(b, A)$, thus causing $\angle N_i M_i'[G_i]$ to be acute. Angle $\angle PM_i'M_i$ of the right triangle $\triangle PM_iM_i'$ is also acute with P above m_i, which means $\angle N_iM_i'[G_i]$ is its vertically opposite angle and N_i lies below m_i. As to the rays $\overrightarrow{p_i} = \overrightarrow{P[G_i]}$ they both enter $\overline{H}(b, A)$ at P which means that the angles $\angle(\overrightarrow{a}, \overrightarrow{p_i})$ have halfline $(\overrightarrow{b})$ in their interior.

If $\overrightarrow{p_2}$ does not intersect m_2, i.e. for $[G_2] = \gamma_2, g_2$, angle $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ includes $(\overrightarrow{m_2}), (\overrightarrow{n_2})$ together with $(\overrightarrow{b})$. So, if in addition $[G_1] = G_1$ or γ_1, halfline $(\overrightarrow{p_1})$ crosses $(\overrightarrow{m_2})$ in order to meet $(\overrightarrow{m_1})$, i.e. runs in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$. Lemma (ML) thus is fulfilled for G_1, γ_2; G_1, g_2; γ_1, γ_2; γ_1, g_2.

The remaining cases of (VII) depend on two metric properties. From $N_1N_2 < M_1M_2$ and $M_1M_2 = \text{proj}(M_1'M_2', a) < M_1'M_2'$ (see (4) and Figure 9, VII) follows $N_1N_2 < M_1'M_2'$ and so

$$N_1M_1' = N_1M_2' - M_1'M_2' < N_1M_2' - N_1N_2 = N_2M_2'. \tag{5}$$

In addition, from the fact that $\triangle PM_2'M_2$ has the smaller area (larger defect) than $\triangle PM_1'M_1$ follows $\angle PM_2'M_2 > \angle PM_1'M_1$, and so

$$\angle N_1M_1'[G_1] < \angle N_2M_2'[G_2]. \tag{6}$$

From (5) and (6) it is clear that if m_2 intersects or is boundary parallel to n_2 then m_1 must intersect n_1, i.e. that the cases $\gamma_1, G_2; g_1, G_2; g_1, \gamma_2$ cannot occur. Also, from (5) and (6) follows that if m_1, n_1 intersect in G_1 and m_2, n_2 intersect in G_2 then side N_1G_1 of $\triangle N_1M_1'G_1$ is shorter than side N_2G_2 of $\triangle N_2M_2'G_2$. This and $PN_1 > PN_2$ applied to $\triangle PN_1G_1, \triangle PN_2G_2$ implies $\angle(\overrightarrow{b}, \overrightarrow{p_1}) < \angle(\overrightarrow{b}, \overrightarrow{p_2})$, and so settles (ML) in the case of G_1, G_2.

The main case left is that of g_1, g_2, both in situation (VII) and situations (I) - (III). For use in the following we define $\text{proj}(M_i, g_i) = R_i$, $\text{proj}(N_i, g_i) = S_i$, $\text{proj}(P, g_i) = P_i$, and, assuming the points exist, $m_2 \curlywedge g_1 = U, n_2 \curlywedge g_1 = V, p_2 \between g_1 = W$.

If in situation (VII) (in which $\overrightarrow{p_2}$ lies above m_2, see Figure 10) the point W does not exist $\overrightarrow{n_1}$ lies with $(\overrightarrow{b})$, g_1 with $\overrightarrow{n_1}$ and $(\overrightarrow{p_1})$ with g_1 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$ thus fulfilling (ML). If W exists, line n_2 which runs between the lines n_1, m_2 and so avoids $\overrightarrow{p_2}$, enters quadrilateral PN_1S_1W and leaves it, defining V, between S_1 and W.

From (6) follows that Lambert quadrilateral $N_1S_1R_1\underline{M_1'}$ has the smaller angle sum and so the larger area than $N_2S_2R_2\underline{M_2'}$, which because of (5) requires that

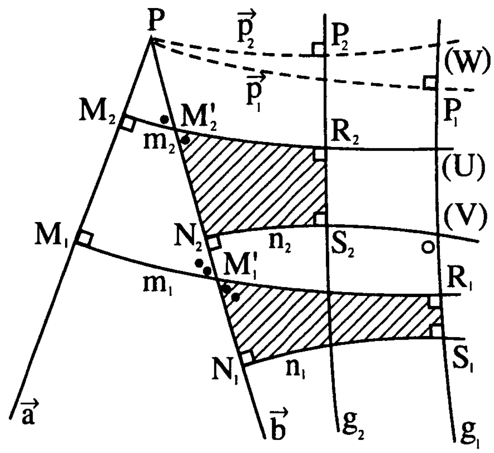

Figure 10

$N_1S_1 > N_2S_2$. As a result V, W on ray $\overrightarrow{S_1R_1}$ satisfy $[N_2S_2V]$, $[PP_2W]$ respectively. As $\angle S_1VN_2 = \angle S_1VS_2$ of $N_2N_1S_1\underline{V}$ is acute, $\angle V$ in P_2S_2VW is obtuse and $\angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$. This means that $\angle W = \angle PWV$ must be acute and identical with $\angle PWP_1$; consequently $(\overrightarrow{p_1}) = (\overrightarrow{PP_1})$ must lie with V, N_2 in the interior of $\angle(\overrightarrow{a}, \overrightarrow{p_2})$, again confirming (ML).

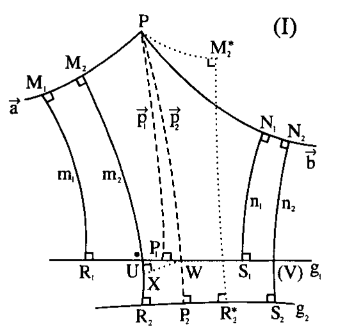

Figure 11a

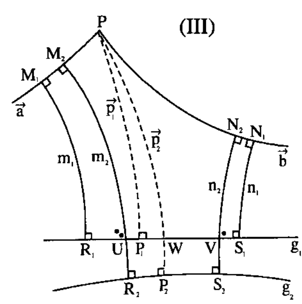

Figure 11b

Each of the Figures 11a, b relating to situations (I), (III) contains two pentagons $PM_iR_iS_iN_i$ $(i = 1, 2)$ with interior altitude PP_i. Adding the images M_i^*, R_i^* of M_i, R_i under reflection in PP_i (as illustrated for $i = 2$ in Figure 11a) we note that $P_iR_i^* < P_iS_i$ because otherwise we would have $PM_i \equiv PM_i^* \geq PN_i$ in contradiction to (3). Moreover $\angle P_iPM_i > \angle P_iPN_i$ as $\angle P_iPM_i \equiv \angle P_iPM_i^* \leq \angle P_iPN_i$ together with $P_iR_i^* < P_iS_i$ would imply that $PM_i^*R_i^*P_i$ would be a part polygon of $PN_iS_iP_i$ while not having a larger angle sum (i.e. smaller defect). So

$$(7a)\quad P_iR_i < P_iS_i, \quad (7b)\quad \angle P_iPM_i > \angle P_iPN_i, \quad (7c)\quad R_iM_i > S_iN_i.$$

In view of an earlier *Remark* we assume that the point U exists and that it satisfies $[R_1UP_1]$; together with $[R_1P_1S_1]$ this extends to $[R_1UP_1S_1]$. In situation (I) we can similarly assume that $\overrightarrow{p_2}$ and n_1 do not meet which means that the point W exists and that it satisfies $[UWS_1]$, a relation that can be extended to $[R_1UWS_1]$. In situation (III) we automatically have V such that $[UVS_1]$ and W such that $[UWV]$ is fulfilled, altogether therefore $[R_1UWVS_1]$.

In both situations m_2 and $\overrightarrow{UR_1}$ include an acute angle which coincides with the fourth angle $\angle R_1UM_2$ of Lambert quadrilateral $M_2M_1R_1\underline{U}$ and so lies on the upper side of g_1. It is congruent to the vertically opposite angle between m_2 and $\overrightarrow{UW}$ which thus lies on the lower side of g_1. In situation (III), for similar reasons, n_2 and $\overrightarrow{VW}$ include an acute angle which is congruent to $\angle N_2VS_1$ and lies on the lower side of g_1. As a result of all this in situation (III) the closest connection R_2S_2 between m_2 and n_2 lies below g_1, and so does the auxiliary point $X = \text{proj}(W, m_2)$ in situation (I).

Statement (ML) holds in both situations if $[UP_1W]$ is fulfilled i.e. if $P_1 = \text{proj}(P, g_1)$ belongs to leg $\overrightarrow{(W)U}$ of $\angle PWU$. We note that this is the case iff $\angle PWU$ is acute.

Now, if in situation (I) R_2, P_2 and S_2 lie below g_1 then the intersection point V of n_2 and g_1 exists and lies between N_2 and S_2, $\angle N_2VS_1$ is acute, $\angle S_2VS_1 = \angle S_2VW$ therefore obtuse and in quadrilateral P_2S_2VW $\angle P_2 + \angle S_2 + \angle V > 3\mathbf{R}$; as a consequence $\angle W = \angle P_2WV$ is acute and so is its vertically opposite angle, $\angle PWU$. This, as we mentioned, proves (ML). If R_2P_2 lies below XW and ray $\overrightarrow{R_2P_2}$ intersects g_1 in a point Y, angle $\angle P_2WY$ in triangle $\triangle P_2WY$ is acute, which leads to the same conclusion. If $R_2P_2 = XW$ then $\angle PWU < \angle PWX = \angle PP_2R_2 = \mathbf{R}$. Finally, if R_2P_2 lies above XW, $\angle PWX$ as the fourth angle of Lambert quadrilateral $XR_2P_2\underline{W}$ is acute, and because $\angle PWU < \angle PWX$, $\angle PWU < \mathbf{R}$. This concludes the proof of (ML) in situation (I).

In situation (III) we have area $M_2M_1R_1\underline{U} > N_2N_1S_1\underline{V}$ because of (4), (7c). Consequently $\angle M_2UR_1 < \angle N_2VS_1$, and so $\angle WUR_2 < \angle WVS_2$ on the other side of g_1. If we also had $\angle P_2WU \leq \angle P_2WV$ then quadrilateral R_2P_2WU would have a smaller angle sum and larger defect than S_2P_2WV. At the same time (7a) and this angle inequality would imply that the former quadrilateral would fit into the latter, i.e. have the smaller area. Since this is contradictory $\angle P_2WU$ must be larger than the adjacent angle $\angle P_2WV$; as a result $\angle P_2WV < \mathbf{R}$ and vertically opposite, $\angle PWU < \mathbf{R}$ which establishes (ML) for g_1, g_2 in situation (III).

The proof of (ML) in situation (III) can be extended with only very minor changes to situation (II). Also closely related is the case of g_1, γ_2 in situation (III). If here, in addition to $\angle WU\gamma_2 < \angle WV\gamma_2$, the inequality $\angle\gamma_2WU \leq \angle\gamma_2WV$ were to hold then line m_2 would have to run farther away from line p_2 than line n_2 in contradiction to (3). An analogous argument applies to the case g_1, G_2 in situation (III) when G_2 lies below g_1. Note that if line m_2 intersects g_1 in a point U between

R_1 and P_1 rather than $\overrightarrow{p_1}$ in a point I between P and P_1 then G must lie below g_1 (see Figure 11b). This is so because according to (4) and (7c) the existence of a Lambert quadrilateral $M_2M_1R_1\underline{U}$ with $R_1U < R_1P_1$ implies the existence of $N_2N_1S_1\underline{V}$ with $S_1V < R_1U < R_1P_1$, and so due to (7a) with $S_1V < S_1P_1$; the point P_1 thus lies between U and V, and M_2 and N_2 meet below g_1.

To conclude the proof of (ML) we still have to settle the cases G_1, G_2; γ_1, G_2 and g_1, G_2 (this with $I = m_2 \wedge \overrightarrow{p_1}$ on or above P_1) of situation (III). We present here the last case (Figure 12b) which is easy and representative also for the proofs of the other two cases (Figure 12a).

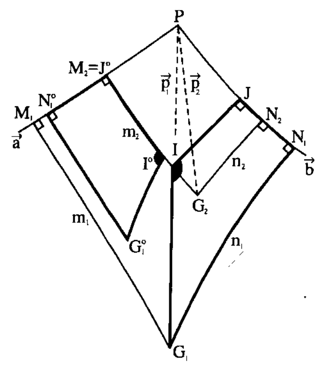

Figure 12a

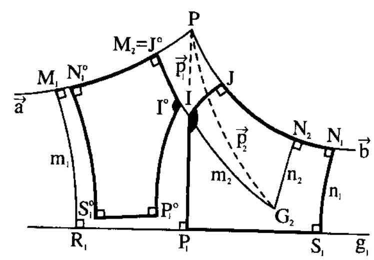

Figure 12b

Call $J = \text{proj}(I, b)$ and note that $\angle IPM_2 > \angle IPJ$ (7b) implies *(i)* $M_2I > JI$, and *(ii)* $\angle PIM_2 < \angle PIJ$, $\angle P_1IM_2 > \angle P_1IJ$. If G_2 would lie in $H(p_1, M_2)$ then the point $N_2 = \text{proj}(G_2, b)$ would determine a segment $N_1N_2 > N_1J$, and due to (4) the inequality *(iii)* $M_1M_2 > N_1J$ would result.

We now carry the pentagon $\mathcal{P}_b = JN_1S_1P_1I$ by an indirect motion to $\mathcal{P}_b^0 = J^0N_1^0S_1^0P_1^0I^0$ where $J_0 = M_2$, N_1^0 lies on $\overrightarrow{M_2M_1}$ and I^0 on $\overrightarrow{M_2I}$. Assuming that G_2 belongs to $H(p_1, M_2)$ we have according to *(i)*, *(iii)* that I^0 lies between M_2 and I, and N_1^0 between M_2 and M_1. Due to (7c) ray $\overrightarrow{S_1^0P_1^0}$ lies in the interior of $\angle M_1R_1P_1$, and due to *(ii)* halfline $\overrightarrow{(I^0)P^0}$ lies in the interior of $\angle P_1IM_2$ which implies that $\mathcal{P}_b^0$ is a proper part of polygon $\mathcal{P}_a = M_2M_1R_1P_1I$ in contradiction to the fact that $\mathcal{P}_b^0$ has the smaller angle sum, i.e. the larger defect. So G_2 and $\overrightarrow{p_2}$ do not lie in $\angle(\overrightarrow{a}, \overrightarrow{p_1})$ and the proof of (ML) is complete. $\square$

Summary of the Proof.
(1) Situations (IV) - (VI) are trivial.
(2) In situations (I) - (III), (ML) holds if $\overrightarrow{p_1} \wedge m_2 = I$ with $[G_2] \neq G_2$, and in situations (I), (II) also with $[G_2] = G_2$.
$\longrightarrow G_1, \gamma_2; G_1, g_2; \gamma_1, \gamma_2; \gamma_1, g_2$ of (I) - (III), $G_1, G_2; \gamma_1, G_2$ of (I), (II).
(3) In situations (I) - (III), (ML) holds if $\overrightarrow{p_2} \wedge n_1 = I' \; (\iota')$ with $[G_1] \neq G_1$.

$\longrightarrow g_1, G_2; g_1, \gamma_2$ of (I), (II).
(4) In situation (VII) a direct comparison of $\triangle N_1 M_1'[G_1]$, $\triangle N_2 M_2'[G_2]$ reveals the relative position of $[G_1], [G_2]$ in all but one case.
$\longrightarrow$ all cases of (VII) except g_1, g_2.
(5) In situation (VII) the area comparison of $N_1 S_1 R_1 \underline{M_1'}$, $N_2 S_2 R_2 \underline{M_2'}$ helps to solve the remaining case.
$\longrightarrow g_1, g_2$ of (VII).
(6) In situations (I), (III) the area comparison of $PM_1R_1S_1N_1$, $PM_2R_2S_2N_2$ helps to solve the same case as in 5.
$\longrightarrow g_1, g_2$ of (I), (III).
(7) The arguments of 6. can be extended to three more cases.
$\longrightarrow g_1, g_2$ of (II); g_1, γ_2 of (III); g_1, G_2 of (III) for G_2 below g_1.
(8) The area comparison between a part polygon of $N_1 S_1 P_1 \underline{P}$, and one of $M_1 R_1 P_1 \underline{P}$, together with two similar comparisons, settle the remaining cases of (III).
$\longrightarrow g_1, G_2$ with G_2 above g_1; G_1, G_2; γ_1, G_2 of (III).

4. Reinterpretation and solution of the posed problem

In the following we formulate, re-formulate and prove a statement which essentially contains the convexity claim of Section 1. Subsequently we discuss the details which make the convexity proof complete.

Theorem 1. *Let AB be a fixed segment and P_2^-, P_1^- and P three points in the same halfplane with respect to the line through A and B such that*

$$\angle AP_2^- B \equiv \angle AP_1^- B \equiv \angle APB \tag{8}$$

and

$$\angle BAP_2^- > \angle BAP_1^- > \angle BAP \geq \angle ABP. \tag{9}$$

Then the line r which joins P_2^- and P separates the point P_1^- from the segment AB (see Figure 13a).

For the purpose of re-formulating this theorem we carry the points A, B, P_1^-, P and the line r of this configuration by a rigid, direct motion μ_1 into the points A_1, B_1, P, P_1 and the line r_1 respectively such that A_1 lies on $\overrightarrow{PA}$ and B_1 on $\overrightarrow{PB}$ (see Figure 13b). This allows us to substitute the following equivalent theorem for Theorem 1.

Theorem 2. *In the configuration of the points A, B, P, A_1, B_1, P_1 and the line r_1 as defined above, the line r_1 separates the point P from segment AB.*

Remark. Note that Theorem 1 amounts to the statement that the intersection point C_1^- of ray $\overrightarrow{AP_1^-}$ and line r lies between A and P_1^-, and Theorem 2 to the statement that the intersection point C_1 of $\overrightarrow{A_1P}$ and r_1 (i.e. the image of C_1^- under the motion μ_1) lies between A_1 and P.

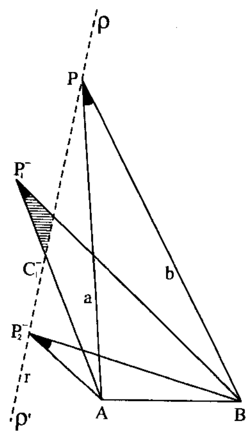

Figure 13a

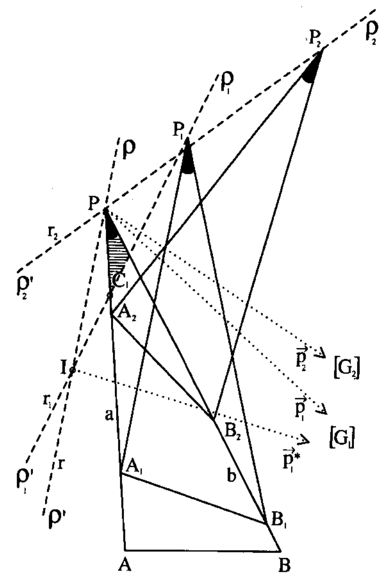

Figure 13b

Proof of Theorem 2. We first augment our configuration by the images of another rigid, direct motion μ_2 which carries the points A, B, P, P_2^- and the line r into A_2, B_2, P_2, P and r_2 respectively where A_2 lies on $\overrightarrow{PA}$ and B_2 on $\overrightarrow{PB}$. We note that because r joins P_2^- and P, r_2 joins P and P_2. From $\angle BAP_2^- > \angle BAP_1^-$ (see (9)) follows $\angle B_2A_2P = \mu_2(\angle BAP_2^-) > \mu_1(\angle BAP_1^-) = \angle B_1A_1P$ and so $[AA_1A_2P]$ according to Section 2. In the following we denote the ends of r by ρ and ρ' (with ρ' on the same side of $a = \ell(AP)$ as C_1^-) and their images on r_1 and r_2 by ρ_1, ρ_1' resp. ρ_2, ρ_2'. Since ρ' lies on the left (right) side of $\overrightarrow{AP_2^-}$ and of $\overrightarrow{AP_1^-}$ if and only if it lies on the left (right) side of $\overrightarrow{AP}$, and since $\overrightarrow{A_2P} = \mu_2(\overrightarrow{AP_2^-})$, $\overrightarrow{A_1P} = \mu_1(\overrightarrow{AP_1^-})$ and $\overrightarrow{AP}$ are equally directed, ρ_2', ρ_1' and ρ' lie together with C_1^- in $\overline{H}(a, B)$. We note that as an exterior angle of $\triangle AP_2^-C_1^-$, $\angle AP_2^-\rho' > \angle AC_1^-\rho'$, and that as an exterior angle of $\triangle AC_1^-P$, $\angle AC_1^-\rho' > \angle AP\rho'$. Applying μ_2 and μ_1 on the two sides of the first and μ_1 on the left hand side of the second inequality we obtain $\angle A_2P\rho_2' > \angle A_1C_1\rho_1'$ and $\angle A_1C_1\rho_1' > \angle AP\rho'$. The supplementary angles consequently satisfy

$$(10) \qquad \angle AP\rho > \angle AC_1\rho_1 > \angle AP\rho_2, \qquad \rho, \rho_1, \rho_2 \in H(a, B).$$

From (10) follows that ρ_2 lies on the same side of line $r = \ell(P\rho)$ as A, and (because ray $\overrightarrow{PA}$ does not enter $\angle\rho P\rho_2$) $\overrightarrow{PA}$ enters $\angle\rho' P\rho_2$.

At this point we augment our figure further by the rays $\overrightarrow{p_1}, \overrightarrow{p_2}$ which connect P to the centers $[G_1], [G_2]$ of the motions μ_1, μ_2 and, if r, r_1 have a point I in common, by the ray $\overrightarrow{p_1}^*$ connecting I to $[G_1]$. Because μ_2 maps r and ρ to r_2 and ρ_2, while μ_1 maps r and ρ to r_1 and ρ_1, the ray $\overrightarrow{p_2}$ is the bisector of angle $\angle\rho' P\rho_2$, and (if existing) the ray $\overrightarrow{p_1}^*$ is the bisector of angle $\angle\rho' I\rho_1$. Since $(\overrightarrow{p_2})$ lies together with ρ_2 in $H(a, B)$ (see Section 3) whereas ρ' lies in $\overline{H}(a, B)$, the ray $\overrightarrow{PA}$ enters $\angle\rho' P[G_2]$.

We now show by indirect proof that C_1 cannot lie on or above P on a.

For $C_1 = P$ (see Figure 14a) formula (10) reads: $\angle AP\rho > \angle AP\rho_1 > \angle AP\rho_2$, $\rho, \rho_1, \rho_2 \in H(a, B)$, and we can add to the sentence following (10) that also ρ_1 and A lie on the same side of r. Thus $\angle\rho' P\rho_1 = \angle\rho' PA + \angle AP\rho_1 > \angle\rho' PA + \angle AP\rho_2 = \angle\rho' P\rho_2$, and $\angle\rho' P[G_1] = \frac{1}{2}\angle\rho' P\rho_1 > \frac{1}{2}\angle\rho' P\rho_2 = \angle\rho' P[G_2]$. This means that $\overrightarrow{p_1}$ does not enter $\angle\rho' P[G_2]$ and so does not enter $\angle AP[G_2]$ in contradiction to Lemma (ML).

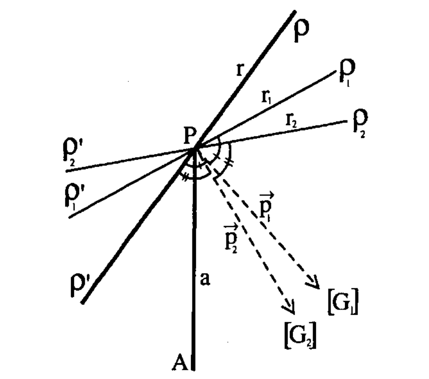

Figure 14a

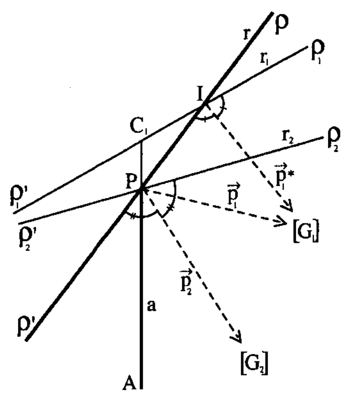

Figure 14b

If C_1 were to lie above P, on a, ray $\overrightarrow{P\rho}$ of r would, according to (10), approach ray $\overrightarrow{C_1\rho_1}$ when entering $H(a, B)$. This means $\overrightarrow{P\rho}$ and $\overrightarrow{P\rho_1}$ either have a point I or the ends ρ, ρ_1 in common, or $r = \ell(P\rho)$ and $r_1 = \ell(P\rho_1)$ share a perpendicular line whose intersection point with r lies in $H(a, B)$. Let us first assume that $\overrightarrow{P\rho}, \overrightarrow{P\rho_1}$ meet in I (see Figure 14b).

In this case line r intersects both segment C_1A and ray $\overrightarrow{C_1\rho_1}$ which means that A and ρ_1 lie on the same side of r. Note that $\angle\rho' I\rho_1$ is equal to the exterior angle $\angle C_1 I\rho$ of $\triangle PC_1 I$ and so satisfies $\angle\rho' I\rho_1 > \angle PC_1 I + \angle C_1 PI$. Because $\angle PC_1 I$ $(= \angle AC_1\rho_1) > \angle AP\rho_2$ (see (10)) and because $\angle C_1 PI \equiv \angle\rho' PA$ we have $\angle\rho' I\rho_1 > \angle AP\rho_2 + \angle\rho' PA = \angle\rho' P\rho_2$. The lower halves of the compared angles consequently satisfy $\angle\rho' I[G_1] > \angle\rho' P[G_2]$ which means that neither $\overrightarrow{p_1}^*$ nor the boundary parallel ray $\overrightarrow{p_1}$ would enter $\angle\rho' P[G_2]$, again in contradiction to Lemma (ML).

Should $\rho = \rho_1$ (see Figure 14c) then this common end is at the same time the center $[G_1]$ of motion μ_1. As a result the ray $\overrightarrow{p_1} = \overrightarrow{P\rho}$, and again fails to enter $\angle AP[G_2]$.

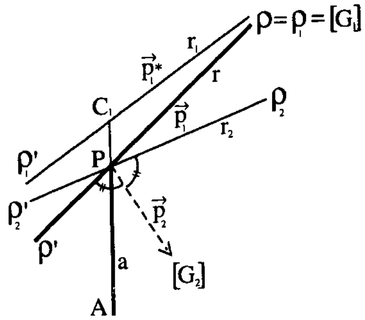

Figure 14c

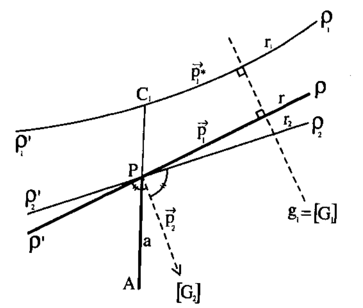

Figure 14d

Finally, if r and r_1 have the perpendicular line g_1 in common (see Figure 14d) then μ_1 which maps r, ρ to r_1, ρ_1 is a translation with axis g_1. As a result $\overrightarrow{p_1} = \overrightarrow{P[G_1]}$ must coincide with $\overrightarrow{P\rho}$ which means it does not enter $\angle \rho' P \rho_2$ and so does not enter $\angle \rho' P[G_2]$ and $\angle AP[G_2]$ in contradiction to Lemma (ML). This completes the proof of Theorem 2 and of Theorem 1.

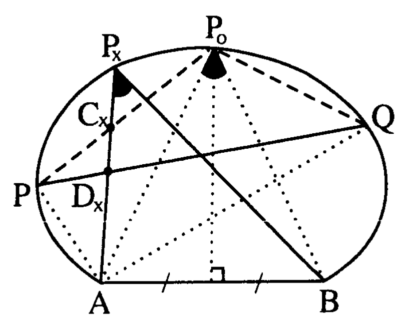

Figure 15

It should be noted that Theorem 1 contains the assumption (9), that P_2^-, P_1^-, P belong to the half-arc of our locus from A to the point P_0 on the perpendicular bisector of segment AB (see Figure 15)). By symmetry the Theorem also shows the convexity of the half-arc from B to P_0. In order to establish the convexity of the whole arc we need to confirm the additional fact that for P a point on the first half-arc and Q a point on the second, line $\ell(PQ)$ separates the points of arc $(\overset{\frown}{PQ})$ from those of segment AB. To do so we choose a point P_X, without loss of generality on half-arc $(\overset{\frown}{PP_0})$, and establish that segment AP_X meets segment PQ between A and P_X. Obviously ray $\overrightarrow{AP_X}$, which enters $\angle PAQ$, meets PQ in a point D_X. Also, by Theorem 1 segment AP_X has a point C_X in common with segment P_0P, which means that our claim follows from $[AD_XC_X]$, a relation which is fulfilled if P_0, and so $(P_0P), (P_0Q)$ belong to $\overline{H}(PQ, A)$. This however is a consequence of the fact that P_0 has a greater distance from $\ell(AB)$ than P and Q, a fact of absolute geometry for which there are many easy proofs.

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A Very Short and Simple Proof of "The Most Elementary Theorem" of Euclidean Geometry

Mowaffaq Hajja

Abstract. We give a very short and simple proof of the fact that if ABB' and $AC'C$ are straight lines with BC and $B'C'$ intersecting at D, then $AB+BD = AC' + C'D$ if and only if $AB' + B'D = AC + CD$. The "only if" part is attributed to Urquhart, and is referred to by Dan Pedoe as "the most elementary theorem of Euclidean geometry".

The theorem referred to in the title states that *if ABB' and $AC'C$ are straight lines with BC and $B'C'$ intersecting at D and if $AB + BD = AC' + C'D$, then $AB' + B'D = AC + CD$* ; see Figure 1. The origin and some history of this theorem are discussed in [9], where Professor Pedoe attributes the theorem to

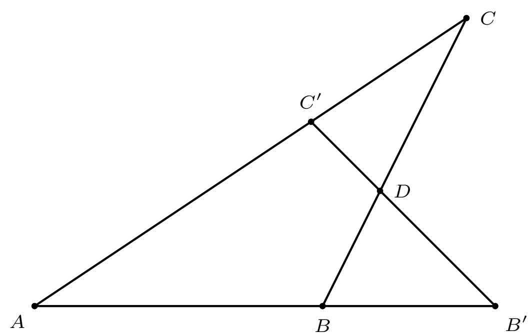

Figure 1

the late L. M. Urquhart (1902-1966) who *discovered it when considering some of the fundamental concepts of the theory of special relativity,* and where Professor Pedoe asserts that *the proof by purely geometric methods is not elementary*. Pedoe calls it *the most "elementary" theorem of Euclidean geometry* and gives variants and equivalent forms of the theorem and cites references where proofs can be found. Unaware of most of the existing proofs of this theorem (e.g., in [3], [4], [13], [14], [8], [10], [11] and [7, Problem 73, pages 23 and 128-129]), the author of this note has published a yet another proof in [5]. In view of all of this, it is interesting to know that De Morgan had published a proof of Urquhart's Theorem in 1841 and that Urquhart's Theorem may be viewed as a limiting case of a result due to Chasles that dates back to 1860; see [2] and [1].

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In this note, we give a much shorter proof based on a very simple and elegant lemma that Robert Breusch had designed for solving a 1961 MONTHLY problem. However, we make no claims that our proof meets the standards set by Professor Pedoe who hoped for a circle-free proof. Clearly our proof does not qualify since it rests heavily on properties of *circular* functions. Breusch's lemma [12] states that if $A_jB_jC_j$ $(j = 1, 2)$, are triangles with angles $A_j = 2\alpha_j$, $B_j = 2\beta_j$, $C_j = 2\gamma_j$, and if $B_1C_1 = B_2C_2$, then the perimeter $p(A_1B_1C_1)$ of $A_1B_1C_1$ is equal to or greater than the perimeter $p(A_2B_2C_2)$ of $A_2B_2C_2$ according as $\tan\beta_1 \tan\gamma_1$ is equal to or greater than $\tan\beta_2 \tan\gamma_2$. This lemma follows immediately from the following sequence of simplifications, where we work with one of the triangles after dropping indices, and where we use the law of sines and the addition formulas for the sine and cosine functions.

$$\begin{aligned}\frac{p(ABC)}{BC} &= 1 + \frac{AB+AC}{BC} = 1 + \frac{\sin 2\gamma + \sin 2\beta}{\sin 2\alpha} = 1 + \frac{\sin 2\gamma + \sin 2\beta}{\sin(2\gamma + 2\beta)} \\ &= 1 + \frac{2\sin(\gamma+\beta)\cos(\gamma-\beta)}{2\sin(\gamma+\beta)\cos(\gamma+\beta)} = 1 + \frac{\cos\gamma\cos\beta + \sin\gamma\sin\beta}{\cos\gamma\cos\beta - \sin\gamma\sin\beta} \\ &= \frac{2\cos\gamma\cos\beta}{\cos\gamma\cos\beta - \sin\gamma\sin\beta} = \frac{2}{1 - \tan\gamma\tan\beta}.\end{aligned}$$

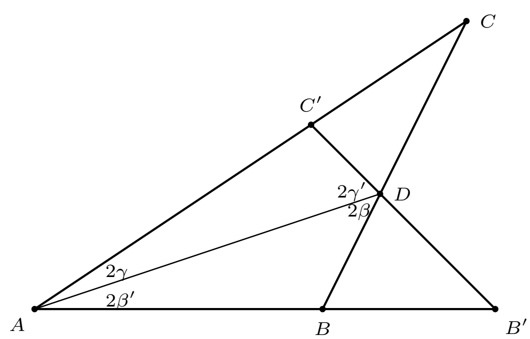

Figure 2

Urquhart's Theorem mentioned at the beginning of this note follows, together with its converse, immediately. Referring to Figure 1, and letting $\angle B'AD = 2\beta'$, $\angle CAD = 2\gamma$, $\angle BDA = 2\beta$, and $\angle C'DA = 2\gamma'$, as shown in Figure 2, we see from Breusch's Lemma that

$$\begin{aligned}p(AB'D) = p(ACD) &\Longleftrightarrow \tan\beta' \tan(90^\circ - \gamma') = \tan\gamma \tan(90^\circ - \beta) \\ &\Longleftrightarrow \tan\beta' \cot\gamma' = \tan\gamma \cot\beta \\ &\Longleftrightarrow \tan\beta' \tan\beta = \tan\gamma \tan\gamma' \\ &\Longleftrightarrow p(ABD) = p(AC'D),\end{aligned}$$

as desired.

The MONTHLY problem that Breusch's lemma was designed to solve appeared also as a conjecture in [6, page 78]. It states that if D, E, and F are points on the sides BC, CA, and AB, respectively, of a triangle ABC, then $p(DEF) \leq \min\{p(AFE), p(BDF), p(CED)\}$ if and only if D, E, and F are the midpoints of the respective sides, in which case the four perimeters are equal. In contrast with the analogous problem obtained by replacing perimeters by areas and the rich literature that this area version has generated, Breusch's solution of the perimeter version is essentially the only solution that the author was able to trace in the literature.

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The Orthic-of-Intouch and Intouch-of-Orthic Triangles

Sándor Kiss

Abstract. Barycentric coordinates are used to prove that the othic of intouch and intouch of orthic triangles are homothetic. Indeed, both triangles are homothetic to the reference triangle. Ratios and centers of homothety are found, and certain collinearities are proved.

1. Introduction

We consider a pair of triangles associated with a given triangle: the orthic triangle of the intouch triangle, and the intouch triangle of the orthic triangle. See Figure 1. Clark Kimberling [1, p. 274] asks if these two triangles are homothetic. We shall show that this is true if the given triangle is acute, and indeed each of them is homothetic to the reference triangle. In this paper, we adopt standard notations of triangle geometry, and denote the side lengths of triangle ABC by a, b, c. Let I denote the incenter, and the incircle (with inradius r) touching the sidelines BC, CA, AB at D, E, F respectively, so that DEF is the intouch triangle of ABC. Let H be the orthocenter of ABC, and let

$$D' = AH \cap BC, \quad E' = BH \cap CA, \quad F' = CH \cap AB,$$

so that $D'E'F'$ is the orthic triangle of ABC. We shall also denote by O the circumcenter of ABC and R the circumradius. In this paper we make use of homogeneous barycentric coordinates. Here are the coordinates of some basic triangle centers in the notations introduced by John H. Conway:

$$\begin{aligned} I =&(a : b : c), \qquad H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right) = (S_{BC} : S_{CA} : S_{AB}), \\ O =&(a^2S_A : b^2S_B : c^2S_C) = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)), \end{aligned}$$

where

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2},$$

and

$$S_{BC} = S_B \cdot S_C, \qquad S_{CA} = S_C \cdot S_A, \qquad S_{AB} = S_A \cdot S_B.$$

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2. Two pairs of homothetic triangles

2.1. *Perspectivity of a cevian triangle and an anticevian triangle.* Let P and Q be arbitrary points not on any of the sidelines of triangle ABC. It is well known that the cevian triangle of $P = (u : v : w)$ is perspective with the anticevian triangle of $Q = (x : y : z)$ at

$$P/Q = \left(x\left(-\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right) : y\left(\frac{x}{u}-\frac{y}{v}+\frac{z}{w}\right) : z\left(\frac{x}{u}+\frac{y}{v}-\frac{z}{w}\right)\right).$$

See, for example, [3, §8.3].

2.2. *The intouch and the excentral triangles.* The intouch and the excentral triangles are homothetic since their corresponding sides are perpendicular to the respective angle bisectors of triangle ABC. The homothetic center is the point

$$\begin{aligned} P_1 =&(a(-a(s-a)+b(s-b)+c(s-c)) : b(a(s-a)-b(s-b)+c(s-c)) \\ &\qquad : c(a(s-a)+b(s-b)-c(s-c))) \\ =&(a(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) \\ =&\left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right). \end{aligned}$$

This is the triangle center X_{57} in [2].

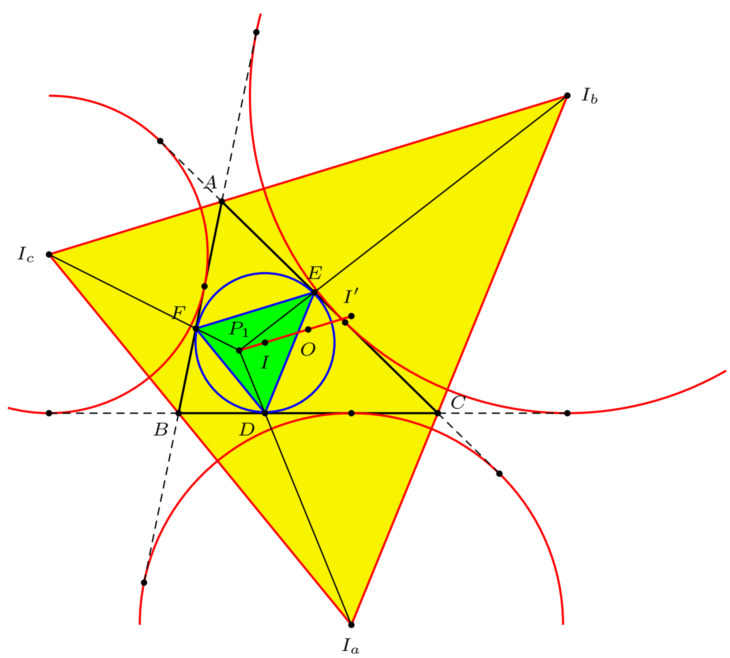

Figure 1.

2.3. *The orthic and the tangential triangle.* The orthic triangle and the tangential triangle are also homothetic since their corresponding sides are perpendicular to the respective circumradii of triangle ABC. The homothetic center is the point

$$\begin{aligned} P_2 = &\big(a^2(-a^2S_A + b^2S_B + c^2S_C) : b^2(-b^2S_B + c^2S_C + a^2S_A) \\ &\quad : c^2(-c^2S_C + a^2S_A + b^2S_B)\big) \\ =&(a^2S_{BC} : b^2S_{CA} : c^2S_{AB}) \\ =&\left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right). \end{aligned}$$

This is the triangle center X_{25} in [2].

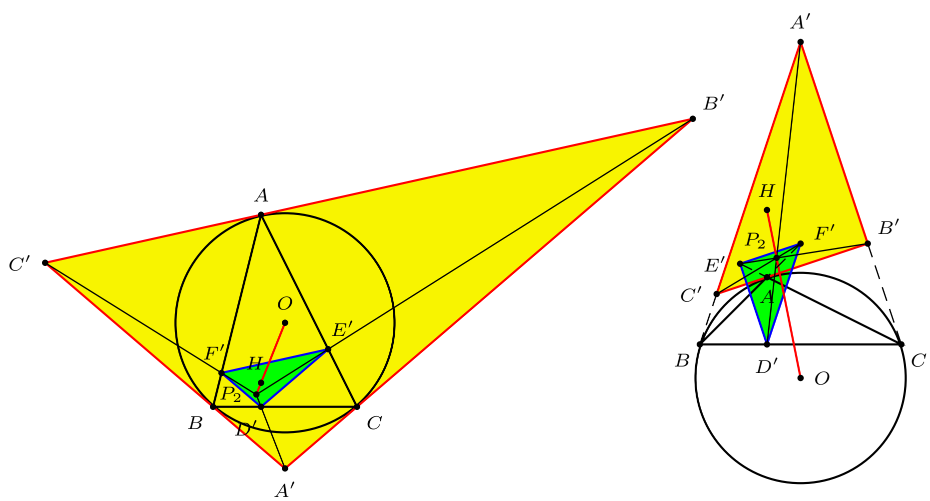

Figure 2A. Figure 2B.

The ratio of homothety is positive or negative according as ABC is acute-angled and obtuse-angled. [1] See Figures 2A and 2B. When ABC is acute-angled, HD', HE' and HF' are the angle bisectors of the orthic triangle, and H is the incenter of the orthic triangle. If ABC is obtuse-angled, the incenter of the orthic triangle is the obtuse angle vertex.

3. The orthic-of-intouch triangle

The orthic-of-intouch triangle of ABC is the orthic triangle UVW of the intouch triangle DEF. Let h_1 be the homothety with center P_1, swapping D, E, F into U, V, W respectively. Consider an altitude DU of DEF. This is the image of the altitude I_aA of the excentral triangle under the homothety h_1. In particular, $U = \mathsf{h}_1(A)$. See Figure 3. Similarly, the same homothety maps B and C

[1]This ratio of homothety is $2\cos A\cos B\cos C$.

into V and W respectively. It follows that UVW is the image of ABC under the homothety h_1.

Since the circumcircle of UVW is the nine-point circle of DEF, it has radius $\frac{r}{2}$. It follows that the ratio of homothety is $\frac{r}{2R}$.

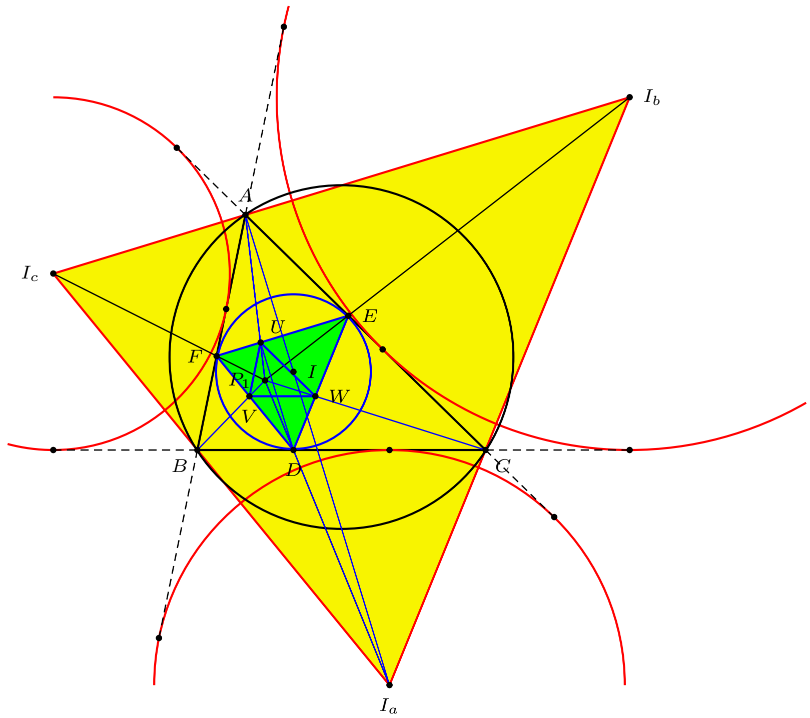

Figure 3.

Proposition 1. *The vertices of the orthic-of-intouch triangle are*

$$U =((b+c)(s-b)(s-c) : b(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{b+c}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right),$$

$$V =(a(s-b)(s-c) : (c+a)(s-c)(s-a) : c(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{c+a}{s-b} : \frac{c}{s-c}\right),$$

$$W =(a(s-b)(s-c) : b(s-c)(s-a) : (a+b)(s-a)(s-b)) = \left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{a+b}{s-c}\right).$$

Proof. The intouch triangle DEF has vertices

$$D = (0 : s-c : s-b), \qquad E = (s-c : 0 : s-a), \qquad F = (s-b : s-a : 0).$$

The sidelines of the intouch triangle have equations

$$\begin{aligned} EF : &\quad -(s-a)x + (s-b)y + (s-c)z = 0, \\ FD : &\quad (s-a)x - (s-b)y + (s-c)z = 0, \\ DE : &\quad (s-a)x + (s-b)y - (s-c)z = 0. \end{aligned}$$

The point U is the intersection of the lines AP_1 and EF. See Figure 3. The line AP_1 has equation

$$-c(s-b)y + b(s-c)z = 0.$$

Solving this with that of EF, we obtain the coordinates of U given above. Those of V and W are computed similarly. □

Corollary 2. *The equations of the sidelines of the orthic-of-intouch triangle are*

$$\begin{aligned}
VW:&\quad -s(s-a)x + (s-b)(s-c)y + (s-b)(s-c)z = 0,\\
WU:&\quad (s-c)(s-a)x - s(s-b)y + (s-c)(s-a)z = 0,\\
UV:&\quad (s-a)(s-b)x + (s-a)(s-b)y - s(s-c)z = 0.
\end{aligned}$$

4. The intouch-of-orthic triangle

Suppose triangle ABC is acute-angled, so that its orthic triangle $D'E'F'$ has incenter H, and is the image of the tangential triangle $A'B'C'$ under a homothety h_2 with center P_2. Consider the intouch triangle XYZ of $D'E'F'$. Under the homothety h_2, the segment $A'A$ is swapped into $D'X$. See Figure 4. In particular, $\mathsf{h}_2(A) = X$. For the same reason, $\mathsf{h}_2(B) = Y$ and $\mathsf{h}_2(C) = Z$. Therefore, the intouch-of-orthic triangle XYZ is homothetic to ABC under h_2.

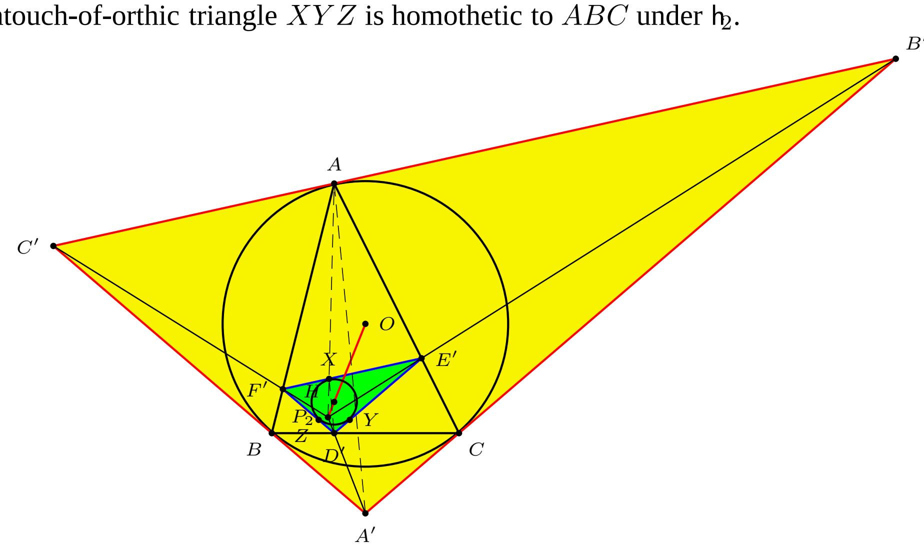

Figure 4

Proposition 3. *If ABC is acute angled, the vertices of the intouch-of-orthic triangle are*

$$\begin{aligned}
X =&((b^2+c^2)S_{BC} : b^2S_{CA} : c^2S_{AB}) = \left(\frac{b^2+c^2}{S_A} : \frac{b^2}{S_B} : \frac{c^2}{S_C}\right),\\
Y =&(a^2S_{BC} : (c^2+a^2)S_{CA} : c^2S_{AB}) = \left(\frac{a^2}{S_A} : \frac{c^2+a^2}{S_B} : \frac{c^2}{S_C}\right),\\
Z =&(a^2S_{BC} : b^2S_{CA} : (a^2+b^2)S_{AB}) = \left(\frac{a^2}{S_A} : \frac{b^2}{S_B} : \frac{a^2+b^2}{S_C}\right).
\end{aligned}$$

Proof. The orthic triangle $D'E'F'$ has vertices

$$D' = (0 : S_C : S_B), \qquad E' = (S_C : 0 : S_A), \qquad F' = (S_B : S_A : 0).$$

The sidelines of the orthic triangle have equations

$$\begin{array}{ll} E'F': & -S_Ax + S_By + S_Cz = 0, \\ F'D': & \quad S_Ax - S_By + S_Cz = 0, \\ D'E': & \quad S_Ax + S_By - S_Cz = 0. \end{array}$$

The point X is the intersection of the lines AP_2 and $E'F'$. See Figure 4. The line AP_2 has equation

$$-c^2S_By + b^2S_Cz = 0.$$

Solving this with that of $E'F'$, we obtain the coordinates of U given above. Those of Y and Z are computed similarly. □

Corollary 4. *If ABC is acute-angled, the equations of the sidelines of the intouch-of-orthic triangle are*

$$\begin{array}{ll} YZ: & -S_A(S_A + S_B + S_C)x + S_{BC}y + S_{BC}z = 0, \\ ZX: & S_{CA}x - S_B(S_A + S_B + S_C)y + S_{CA}z = 0, \\ UV: & S_{AB}x + S_{AB}y - S_C(S_A + S_B + S_C)z = 0. \end{array}$$

5. Homothety of the intouch-of-orthic and orthic-of-intouch triangles

Proposition 5. *If triangle ABC is acute angled, then its intouch-of-orthic and orthic-of-intouch triangles are homothetic at the point*

$$Q = \left(\frac{a(a(b+c) - (b^2+c^2))}{(s-a)S_A} : \frac{b(b(c+a) - (c^2+a^2))}{(s-b)S_B} : \frac{c(c(a+b) - (a^2+b^2))}{(s-c)S_C} \right).$$

Proof. The homothetic center is the intersection of the lines UX, VY, and WZ. See Figure 5. Making use of the coordinates given in Propositions 1 and 3, we obtain the equations of these lines as follows.

$$\begin{array}{ll} UX: & \quad bc(s-a)S_A(c(s-c)S_B - b(s-b)S_C)x \\ & +c(s-b)S_B((b^2+c^2)(s-a)S_C - (b+c)c(s-c)S_A)y \\ & +b(s-c)S_C(b(b+c)(s-b)S_A - (b^2+c^2)(s-a)S_B)z = 0, \\ VY: & \quad c(s-a)S_A(c(c+a)(s-c)S_B - (c^2+a^2)(s-b)S_C)x \\ & +ca(s-b)S_B(a(s-a)S_C - c(s-c)S_A)y \\ & +a(s-c)S_C((c^2+a^2)(s-b)S_A - (c+a)a(s-a)S_B)z = 0, \\ WZ: & \quad b(s-a)S_A((a^2+b^2)(s-c)S_B - (a+b)b(s-b)S_C)x \\ & +a(s-b)S_B(a(a+b)(s-a)S_C - (a^2+b^2)(s-c)S_A)y \\ & +ab(s-c)S_C(b(s-b)S_A - a(s-a)S_B)z = 0. \end{array}$$

It is routine to verify that Q lies on each of these lines. □

Remark. Q is the triangle center X_{1876} in [2].

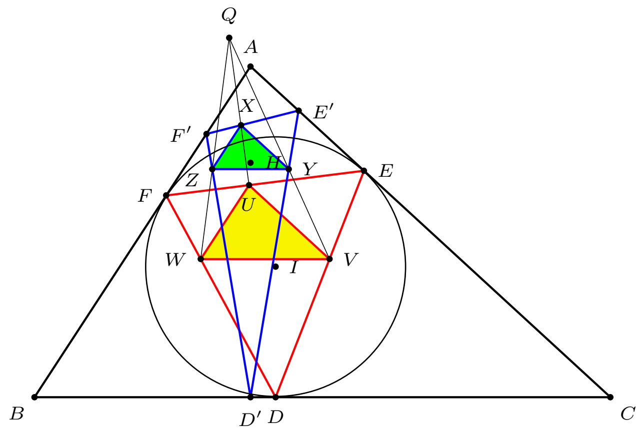

Figure 5

6. Collinearities

Because the circumcenter of XYZ is the orthocenter H of ABC, the center of homothety P_2 of ABC and XYZ lies on the Euler line OH of ABC. See Figure 4. We demonstrate a similar property for the point P_1, namely, that this point lies on the Euler line IF of DEF, where F is the circumcenter of UVW. Clearly, O, F, P_1 are collinear. Therefore, it suffices to prove that the points I, O, P_1 are collinear. This follows from

$$\begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ (s-b)(s-c) & (s-c)(s-a) & (s-a)(s-b) \end{vmatrix} = 0,$$

which is quite easy to check. See Figure 1.

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A 4-Step Construction of the Golden Ratio

Kurt Hofstetter

Abstract. We construct, in 4 steps using ruler and compass, three points two of the distances between which bear the golden ratio.

We present here a 4-step construction of the golden ratio using ruler and compass only. More precisely, we construct, in 4 steps using ruler and compass, three points with two distances bearing the golden ratio. It is fascinating to discover how simple the golden ratio appears. We denote by $P(Q)$ the circle with center P, passing through Q, and by $P(XY)$ that with center P and radius XY.

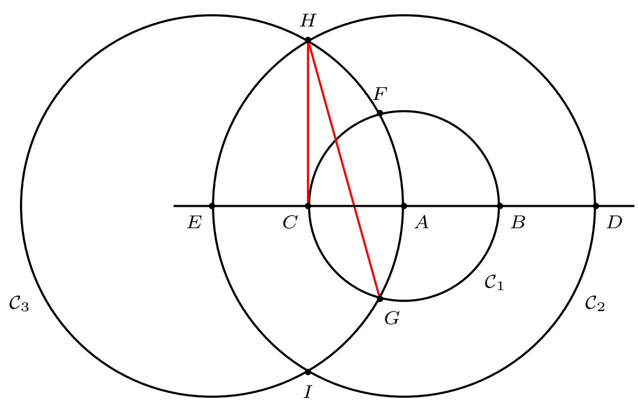

Figure 1

Construction. Given two points A and B, construct

(1) the circle $\mathcal{C}_1 = A(B)$,
(2) the line AB to intersect $\mathcal{C}_1$ again at C and extend it long enough to intersect
(3) the circle $\mathcal{C}_2 = A(BC)$ at D and E,
(4) the circle $\mathcal{C}_3 = E(BC)$ to intersect $\mathcal{C}_1$ at F and G, and $\mathcal{C}_2$ at H and I.

Then $\frac{GH}{CH} = \frac{\sqrt{5}+1}{2}$.

Proof. Without loss of generality let $AB = 1$, so that $BC = AE = AH = EH = 2$. Triangle AEH is equilateral. Let $\mathcal{C}_4 = H(A)$, intersecting $\mathcal{C}_1$ at J. By symmetry, AGJ is an equilateral triangle. Let $\mathcal{C}_5 = J(A) = J(AG) = J(AB)$, intersecting $\mathcal{C}_1$ at K. Finally, let $\mathcal{C}_6 = J(H) = J(BC)$. See Figure 2.

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With $\mathcal{C}_1$, $\mathcal{C}_5$, $\mathcal{C}_2$, $\mathcal{C}_6$, following [1], K divides GH in the golden section. It suffices to prove $CH = GK = \sqrt{3}$. This is clear for GK since the equilateral triangles AJG and AJK have sides of length 1. On the other hand, in the right triangle ACH, $AC = 1$ and $AH = 2$. By the Pythogorean theorem $CH = \sqrt{3}$. $\square$

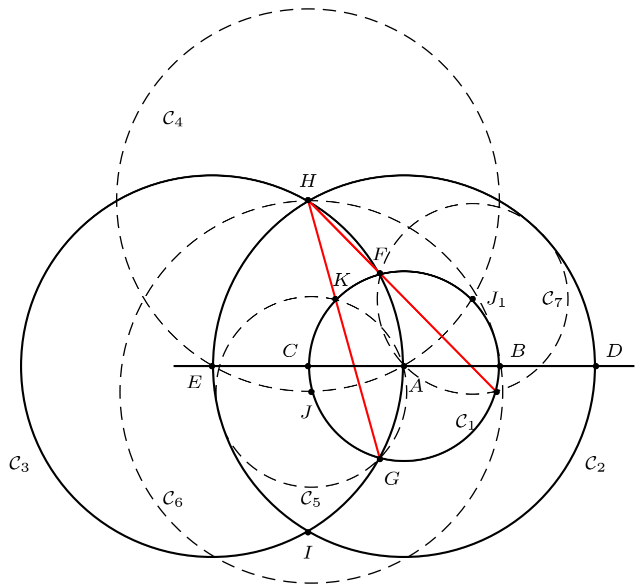

Figure 2

Remark. $\frac{CH}{FH} = \frac{GH}{CH} = \frac{\sqrt{5}+1}{2}$.

Proof. Since $CH^2 = GH \cdot KH$, it is enough to prove that $FH = KH$. Let $\mathcal{C}_4$ intersect $\mathcal{C}_1$ again at J_1. Consider the circle $\mathcal{C}_7 = J_1(A)$. By symmetry, F lies on $\mathcal{C}_7$ and $FH = KH$. $\square$

Reference

[1] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

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A Theorem by Giusto Bellavitis on a Class of Quadrilaterals

Eisso J. Atzema

Abstract. In this note we prove a theorem on quadrilaterals first published by the Italian mathematician Giusto Bellavitis in the 1850s, but that seems to have been overlooked since that time. Where Bellavitis used the functional equivalent of complex numbers to prove the result, we mostly rely on trigonometry. We also prove a converse of the theorem.

1. Introduction

Since antiquity, the properties of various special classes of quadrilaterals have been extensively studied. A class of quadrilaterals that appears to have been little studied is that of those quadrilaterals for which the products of the two pairs of opposite sides are equal. In case a quadrilateral ABCD is cyclic as well, $ABCD$ is usually referred to as a *harmonic* quadrilateral (see [2, pp.90–92], [3, pp.159–160]). Clearly, however, the class of all quadrilaterals $ABCD$ for which $AB \cdot CD = AD \cdot BC$ includes non-cyclic quadrilaterals as well. In particular, all kites are included. As far as we have able to ascertain, no name for this more general class of quadrilaterals has ever been proposed. For the sake of brevity, we will refer to the elements in this class as *balanced* quadrilaterals. In his *Sposizione del metodo delle equipollenze* of 1854, the Italian mathematician Giusto Bellavitis (1803-1880) proved a curious theorem on such balanced quadrilaterals that seems to have been forgotten.[1] In this note, we will give an elementary proof of the theorem. In addition, we will show how the converse of Bellavitis' theorem is (almost) true as well. Our proof of the first theorem is different from that of Bellavitis. The converse is not discussed by Bellavitis at all.

2. Bellavitis' Theorem

Let the lengths of the sides AB, BC, CD and DA of a (convex) quadrilateral $ABCD$ be denoted by a, b, c and d respectively. Similarly, the lengths of the quadrilateral's diagonals AC and BD will be denoted by e and f. Let E be the point of intersection of the two diagonals. The magnitude of $\angle DAB$ will be

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[1]Bellavitis' book is very hard to locate. We actually used Charles-Ange Laisant's 1874 translation into French [1], which is available on-line from the Bibliothèque Nationale. In this translation, the theorem is on p.26 as Corollary III of Bellavitis' derivation of Ptolemy's theorem.

referred to as α, with similar notations for the other angles of the quadrilateral. The magnitudes of $\angle DAC$, $\angle ADB$ etc will be denoted by α_B, δ_C and so on (see Figure 1). Finally, the magnitude of $\angle CED$ will be referred to as ϵ.

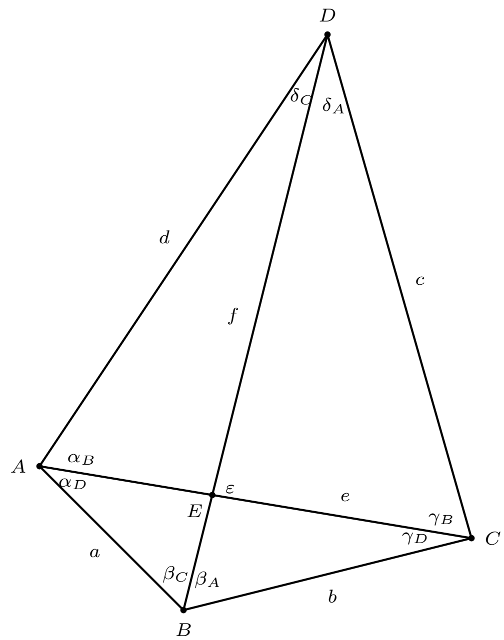

Figure 1. Quadrilateral Notations

With these notations, the following result can be proved.

Theorem 1 (Bellavitis, 1854)**.** *If a* (*convex*) *quadrilateral $ABCD$ is balanced, then*

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ.$$

Note that the convexity condition is a necessary one. The second equality sign does not hold for non-convex quadrilaterals. A trigonometric proof of Bellavitis' Theorem follows from the observation that by the law of sines for any balanced quadrilateral we have

$$\sin\gamma_B \cdot \sin\alpha_D = \sin\alpha_B \cdot \sin\gamma_D,$$

or

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha_D) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma_D).$$

That is,

$$\cos(\gamma_B + \alpha_D) - \cos(\gamma_B - \alpha + \alpha_B) = \cos(\alpha_B + \gamma_D) - \cos(\alpha_B - \gamma + \gamma_B),$$

or

$$\cos(\gamma_B + \alpha_D) + \cos(\delta + \alpha) = \cos(\alpha_B + \gamma_D) + \cos(\delta + \gamma).$$

By cycling through, we also have

$$\cos(\delta_C + \beta_A) + \cos(\alpha + \beta) = \cos(\beta_C + \delta_A) + \cos(\alpha + \delta).$$

Since $\cos(\alpha + \beta) = \cos(\delta + \gamma)$, adding these two equations gives

$$\cos(\gamma_B + \alpha_D) + \cos(\delta_C + \beta_A) = \cos(\alpha_B + \gamma_D) + \cos(\beta_C + \delta_A),$$

or

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos\frac{1}{2}(\gamma_B + \alpha_D - \delta_C - \beta_A) \\ =&\cos\frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) \cdot \cos\frac{1}{2}(\alpha_B + \gamma_D - \beta_C - \delta_A). \end{aligned}$$

Now, note that

$$\gamma_B + \alpha_D - \delta_c - \beta_A = 360 - 2\epsilon - \delta - \beta$$

and, likewise

$$\alpha_B + \gamma_D - \beta_C - \delta_A = 2\epsilon - \beta - \delta.$$

Finally,

$$\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) + \frac{1}{2}(\alpha_B + \beta_C + \gamma_D + \delta_A) = 180^\circ.$$

It follows that

$$\begin{aligned} &\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon + \frac{1}{2}(\beta + \delta)) \\ =& -\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon - \frac{1}{2}(\beta + \delta)), \end{aligned}$$

or

$$\cos\frac{1}{2}(\delta_C + \gamma_B + \beta_A + \alpha_D) \cdot \cos(\epsilon) \cos\frac{1}{2}(\delta + \beta) = 0.$$

This almost concludes our proof. Clearly, if neither of the last two factors are equal to zero, the first factor has to be zero and we are done. The last factor, however, will be zero if and only if $ABCD$ is cyclic. It is easy to see that any such quadrilateral has the angle property of Bellavitis' theorem. Therefore, in the case that $ABCD$ is cyclic, Bellavitis' theorem is true. Consequently, we may assume that $ABCD$ is not cyclic and that the third term does not vanish. Likewise, the second factor only vanishes in case $ABCD$ is orthogonal. For such quadrilaterals, we know that $a^2 + c^2 = b^2 + d^2$. In combination with the initial condition $ac = bd$, this implies that each side has to be congruent to an adjacent side. In other words, $ABCD$ has to be a kite. Again, it is easy to see that in that case Bellavitis' theorem is true. We can safely assume that $ABCD$ is not a kite and that the second term does not vanish either. This proves Bellavitis' theorem.

3. The Converse to Bellavitis' Theorem

Now that we have proved Bellavitis' theorem, it is only natural to wonder for exactly which kinds of (convex) quadrilaterals the angle sums $\delta_C + \gamma_B + \beta_A + \alpha_D$ and $\alpha_A + \beta_C + \gamma_D + \delta_A$ are equal. Assuming that the two angle sums are equal and working our way backward from the preceding proof, we find that

$$\sin\gamma_B \cdot \sin\alpha_D + K = \sin\alpha_B \cdot \sin\gamma_D$$

for some K. Likewise,

$$\sin\delta_C \cdot \sin\beta_A = \sin\beta_C \cdot \sin\delta_A + K.$$

So,

$$\frac{\sin\gamma_B}{\sin\alpha_B} - \frac{\sin\gamma_D}{\sin\alpha_D} = -\frac{K}{\sin\alpha_B \cdot \sin\alpha_D}$$

and

$$\frac{\sin\delta_C}{\sin\beta_C} - \frac{\sin\delta_A}{\sin\beta_A} = \frac{K}{\sin\beta_A \cdot \sin\beta_C}$$

or

$$\frac{d}{c} - \frac{a}{b} = -\frac{K}{\sin\alpha_B \cdot \sin\alpha_D}, \qquad \frac{a}{d} - \frac{b}{c} = \frac{K}{\sin\beta_A \cdot \sin\beta_C}.$$

If $K = 0$, we have $bd = ac$ and $ABCD$ is balanced. If $K \neq 0$, it follows that

$$\frac{d}{b} = \frac{\sin\beta_A \cdot \sin\beta_C}{\sin\alpha_B \cdot \sin\alpha_D}.$$

Cycling through twice also gives us

$$\frac{b}{d} = \frac{\sin\delta_C \cdot \sin\delta_A}{\sin\gamma_D \cdot \sin\gamma_B}.$$

We find

$$\sin\beta_A \cdot \sin\beta_C \cdot \sin\delta_C \cdot \sin\delta_A = \sin\alpha_B \cdot \sin\alpha_D \cdot \sin\gamma_D \cdot \sin\gamma_B.$$

Division of each side by $abcd$ and grouping the factors in the numerators and denominators appropriately shows that this equation is equivalent to the equation

$$R_{ABC} \cdot R_{ADC} = R_{BAC} \cdot R_{BCD},$$

where R_{ABC} denotes the radius of the circumcircle to the triangle ABC etc. Now, the area of ABC is equal to both $abe/4R_{ABC}$ and $\frac{1}{2}e \cdot EB \cdot \sin\epsilon$ with similar expressions for ADC, BAC, and BCD. Consequently, the relation between the four circumradii can be rewritten to the form $EB \cdot EC = EA \cdot EC$. But this means that $ABCD$ has to be cyclic. We have the following result:

Theorem 2. *Any* (*convex*) *quadrilateral* $ABCD$ *for which*

$$\alpha_B + \beta_C + \gamma_D + \delta_A = \beta_A + \gamma_B + \delta_C + \alpha_D = 180^\circ$$

is either cyclic or balanced.

4. Conclusion

We have not been able to find any references to Bellavitis' theorem other than in the *Sposizione*. Bellavitis was clearly mostly interested in the theorem because it allowed him to showcase the power of his method of equipollences.[2] Indeed, the *Sposizione* features a fair number of (minor) results on quadrilaterals that are proved using the method of equipollences. Most of these were definitely well-known at the time. This suggests that perhaps our particular result was reasonably well-known at the time as well. Alternatively, Bellavitis may have derived the theorem in one of the many papers that he published between 1833, when he first published on the method, and 1854. These earlier publications, however, are extremely hard to locate and we have not been able to consult any.[3] Whether the theorem originated with Bellavitis or not, it is not entirely surprising that this result seems to have been forgotten. The sums $\alpha_B+\beta_C+\gamma_D+\delta_A$ and $\beta_A+\gamma_B+\delta_C+\alpha_D$ do not usually show up in plane geometry. We do hope to finish up a paper shortly, however, in which these sums play a role as part of a generalization of Ptolemy's theorem to arbitrary (convex) quadrilaterals.

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[2] This method essentially amounted to a sometimes awkward mix of vector methods and the use of complex numbers in a purely geometrical disguise. In fact, for those interested in the use of complex numbers in plane geometry, it might be a worthwhile exercise to rework Bellavitis' equipollences proof of his theorem to one that uses complex numbers only. This should not be too hard.

[3] See the introduction of [1] for a list of references.

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A Projectivity Characterized by the Pythagorean Relation

Wladimir G. Boskoff and Bogdan D. Suceavă

Abstract. We study an interesting configuration that gives an example of an elliptic projectivity characterized by the Pythagorean relation.

1. A Romanian Olympiad problem

It is known that any projectivity relating two ranges on one line with more than two invariant points is the identity transformation of the line onto itself. Depending on whether the number of invariant points is 0, 1, or 2 the projectivity would be called *elliptic*, *parabolic*, or *hyperbolic* (see Coxeter [1, p.45], or [3, pp.41-43]). This note will point out an interesting and unusual configuration that gives an example of projectivity characterized by a Pythagorean relation.

The configuration appears in a problem introduced in the National Olympiad 2001, in Romania, by Mircea Fianu. The statement of the problem is the following: *Consider the right isosceles triangle* ABC *and the points* M, N *on the hypothenuse* BC *in the order* B, M, N, C *such that* $BM^2 + NC^2 = MN^2$. *Prove that* $\angle MAN = \frac{\pi}{4}$.

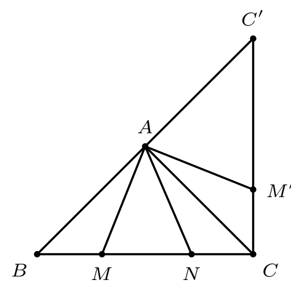

Figure 1

We present first an elementary solution for this problem. Consider a counterclockwise rotation around A of angle $\frac{\pi}{2}$. By applying this rotation, ΔABC becomes $\Delta ACC'$ (see Figure 1) and AM becomes AM'; thus, the angle $\angle MAM'$ is right. The equality $BM^2 + NC^2 = MN^2$ transforms into $CM'^2 + NC^2 = MN^2 = M'N^2$, since $\Delta CNM'$ is right in C. Therefore, $\Delta MAN \equiv \Delta NAM'$ (SSS case), and this means $\angle MAN \equiv \angle NAM'$. Since $\angle MAM' = \frac{\pi}{2}$, we get $\angle MAN = \frac{\pi}{4}$, which is what we wanted to prove.

We shall show that the metric relation introduced in the problem above, similar to the Pythagorean relation, is hiding an elliptic projectivity of focus A. Actually, this is what makes this problem and this geometric structure so special and deserving of our attention. First, we would like to recall a few facts of projective geometry.

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2. Projectivities

Let A, B, C, and D be four points, in this order, on the line $\mathcal{L}$ in the Euclidean plane. Consider a system of coordinates on $\mathcal{L}$ such that A, B, C, and D correspond to x_1, x_2, x_3, and x_4, respectively. The cross ratio of four ordered points A, B, C, D on $\mathcal{L}$, is by definition (see for example [5, p.248]):

$$(ABCD) = \frac{AC}{BC} \div \frac{AD}{BD} = \frac{x_3 - x_1}{x_3 - x_2} \div \frac{x_4 - x_1}{x_4 - x_2}. \tag{1}$$

This definition may be extended to a pencil consisting of four ordered lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$. By definition, the cross ratio of four ordered lines is the cross ratio determined by the points of intersection with a line $\mathcal{L}$. Therefore,

$$(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4) = \frac{A_1A_3}{A_2A_3} \div \frac{A_1A_4}{A_2A_4}$$

where $\{A_i\} = \mathcal{L} \cap \mathcal{L}_i$. The law of sines shows us that the above definition is independent on $\mathcal{L}$.

We call a projectivity on a line $\mathcal{L}$ a map $f : \mathcal{L} \to \mathcal{L}$ with the property that the cross ratio of any four points is preserved, that is

$$(A_1A_2A_3A_4) = (B_1B_2B_3B_4)$$

where $B_i = f(A_i)$, $i = 1, 2, 3, 4$. The points A_i and B_i are called homologous points of the projectivity on $\mathcal{L}$, and the relation $B_i = f(A_i)$ is denoted $A_i \to B_i$.

The following result is presented in many references (see for example [3], Theorem 4.12, p.34).

Theorem 1. *A projectivity on $\mathcal{L}$ is determined by three pairs of homologous points.*

A consequence of this theorem is that two projectivities which have three common pairs of homologous points must coincide. Actually, we will use this consequence in the proof we present below. In fact, the coordinates x and y of the homologous points under a projectivity are related by

$$y = \frac{mx + n}{px + q}\ ,\ mq - np \neq 0,$$

where $m, n, p, q \in \mathbb{R}$.

In formula (1), it is possible that $(ABCD)$ takes the value -1, as for example in the case of the feet of interior and exterior bisectors associated to the side BD of a triangle MBD (see Figure 2).

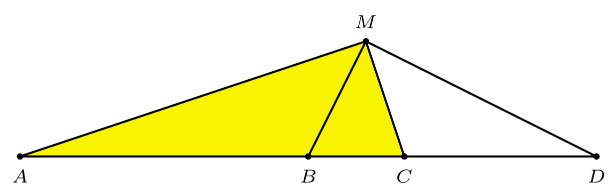

Figure 2

Observe that if C is the midpoint of the segment BD, then point A is not on the line determined by points B and D, since MA becomes parallel to BD. Indeed, for any C on the straight line BD there exist the point M in the plane (not necessarily unique) such that MC is the interior bisector of $\angle BMD$. The point A with the property $(ABCD) = -1$ can be found at the intersection between the external bisector of $\angle BMD$ and the straight line BD. In the particular case when C is the midpoint of BD, we have that ΔMBD is isosceles and the external bisector MA is parallel to BD. To extend the bijectivity of the projectivity presented above, we will say that the homologous of the point C is the point at infinity, denoted ∞, which we attach to the line d. We shall also accept the convention

$$\frac{\infty C}{\infty D} \div \frac{BC}{BD} = -1.$$

For our result, we need the following.

Lemma 2. *A moving angle with vertex in the fixed point A in the plane intersects a fixed line $\mathcal{L}$, A not on $\mathcal{L}$, in a pair of points related by a projectivity.*

Proof. As mentioned in the statement, let A be a fixed point and $\mathcal{L}$ a fixed line such that A is not on $\mathcal{L}$. Consider the rays h and k with origin in A, the moving angle $\angle hk$ with the vertex in A and of constant measure α. Denote by $\{M\} = h \cap \mathcal{L}$ and $\{N\} = k \cap \mathcal{L}$. We have to prove that $f : \mathcal{L} \to \mathcal{L}$ defined by $f(M) = N$ is a projectivity on the line $\mathcal{L}$ determined by the rotation of $\angle hk$. Consider four positions of the angle $\angle hk$, denoted consecutively $\angle h_1k_1$, $\angle h_2k_2$, $\angle h_3k_3$, $\angle h_4k_4$. Their intersections with the line $\mathcal{L}$ yield the points $M_1, N_1; M_2, N_2; M_3, N_3; M_4, N_4$, respectively. It is sufficient to prove that the cross ratio $[M_1M_2M_3M_4]$ and $[N_1N_2N_3N_4]$ are equal. The rotation of the moving angle $\angle hk$ yields, for the pencil of rays h_1, h_2, h_3, h_4 and k_1, k_2, k_3, k_4, respectively, the pairs of equal angles:

$$\begin{aligned}
\angle M_1AM_2 =& \angle N_1AN_2 = \beta_1,\\
\angle M_2AM_3 =& \angle N_2AN_3 = \beta_2,\\
\angle M_3AM_4 =& \angle N_3AN_4 = \beta_3.
\end{aligned}$$

By the law of sines we get that the two cross ratios are equal, both of them having the value

$$\frac{\sin(\beta_1 + \beta_2)}{\sin \beta_2} \div \frac{\sin(\beta_1 + \beta_2 + \beta_3)}{\sin(\beta_2 + \beta_3)}.$$

This proves the claim that f is a projectivity on $\mathcal{L}$ in which the homologous points are M and N. □

3. A projective solution to Romanian Olympiad problem

With these preparations, we are ready to give a projective solution to the initial problem.

Consider a system of coordinates in which the vertices of the right isosceles triangle are $A(0, a)$, $B(-a, 0)$, and $C(a, 0)$. See Figure 3. We consider also $M(x, 0)$ and $N(y, 0)$. The relation $BM^2 + NC^2 = MN^2$ becomes

$$(x + a)^2 + (a - y)^2 = (y - x)^2,$$

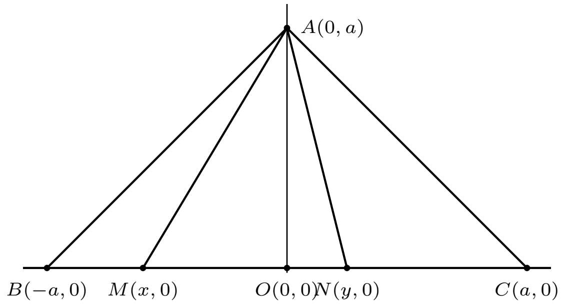

Figure 3

or, solving for y,

$$y = \frac{ax + a^2}{a - x}. \tag{2}$$

This is the equation of a projectivity on the line BC, represented by the homologous points $M \to N$.

Consider now another projectivity on BC determined by the rotation about A by $\frac{\pi}{4}$ (see Lemma 2). This projectivity is completely determined by three pairs of homologous points. First, we see that $B \to O$, since $\angle BAO = \frac{\pi}{4}$. We also have $O \to C$, since $\angle CAO = \frac{\pi}{4}$. Finally, $C \to \infty$, since $\angle CA\infty = \frac{\pi}{4}$.

On the other hand, $B \to O$, since by replacing the $x-$coordinate of B in (2) we get 0, i.e. the $x-$coordinate of O. Similarly, $0 \to a$ and $a \to \infty$ express that $O \to C$ and, respectively, $C \to \infty$. Since a projectivity is completely determined by a triple set of homologous points, the two projectivities must coincide. Therefore, the pair $M \to N$ has the property $\angle MAN = \frac{\pi}{4}$. $\square$

This concludes the proof and the geometric interpretation: the Pythagorean-like metric relation from the original problem reveals a projectivity, which makes this geometric structure remarkable. Furthermore, this solution shows that M and N can be anywhere on the line determined by the points B and C.

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The Feuerbach Point and Euler lines

Bogdan Suceavă and Paul Yiu

Abstract. Given a triangle, we construct three triangles associated its incircle whose Euler lines intersect on the Feuerbach point, the point of tangency of the incircle and the nine-point circle. By studying a generalization, we show that the Feuerbach point in the Euler reflection point of the intouch triangle, namely, the intersection of the reflections of the line joining the circumcenter and incenter in the sidelines of the intouch triangle.

1. A MONTHLY problem

Consider a triangle ABC with incenter I, the incircle touching the sides BC, CA, AB at D, E, F respectively. Let Y (respectively Z) be the intersection of DF (respectively DE) and the line through A parallel to BC. If E' and F' are the midpoints of DZ and DY, then the six points A, E, F, I, E', F' are on the same circle. This is Problem 10710 of the *American Mathematical Monthly* with slightly different notations. See [3].

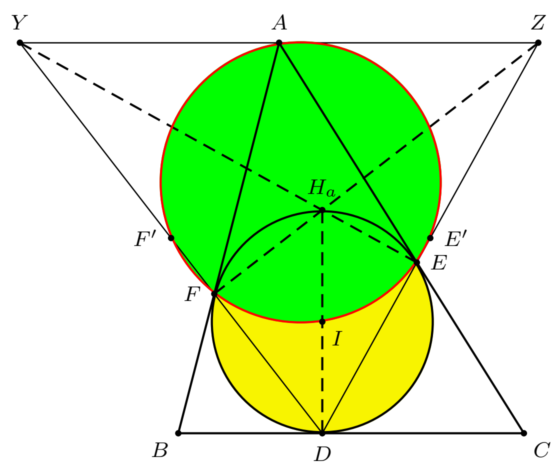

Figure 1. The triangle $\mathbf{T}_a$ and its orthocenter

Here is an alternative solution. The circle in question is indeed the nine-point circle of triangle DYZ. In Figure 1, $\angle AZE = \angle CDE = \angle CED = \angle AEZ$. Therefore $AZ = AE$. Similarly, $AY = AF$. It follows that $AY = AF = AE = AZ$, and A is the midpoint of YZ. The circle through A, E', F', the midpoints of the sides of triangle DYZ, is the nine-point circle of the triangle. Now, since $AY = AZ = AE$, the point E is the foot of the altitude on DZ. Similarly, F

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is the foot of the altitude on DY, and these two points are on the same nine-point circle. The intersection $H_a = EY \cap FZ$ is the orthocenter of triangle DYZ. Since $\angle H_aED = \angle H_aFD$ are right angles, H_a lies on the circle containing D, E, F, which is the incircle of triangle ABC, and has DH_a as a diameter. It follows that I, being the midpoint of the segment DH_a, is also on the nine-point circle. At the same time, note that H_a is the antipodal point of the D on the incircle of triangle ABC.

2. The Feuerbach point on an Euler line

The center of the nine-point circle of DYZ is the midpoint M of IA. The line MH_a is therefore the Euler line of triangle DYZ.

Theorem 1. *The Euler line of triangle DYZ contains the Feuerbach point of triangle ABC, the point of tangency of the incircle and the nine-point circle of the latter triangle.*

Proof. Let O, H, and N be respectively the circumcenter, orthocenter, and nine-point center of triangle ABC. It is well known that N is the midpoint of OH. Denote by ℓ the Euler line MH_a of triangle DYZ. We show that the parallel through N to the line IH_a intersects ℓ at a point N' such that $NN' = \frac{R}{2}$, where R is the circumradius of triangle ABC.

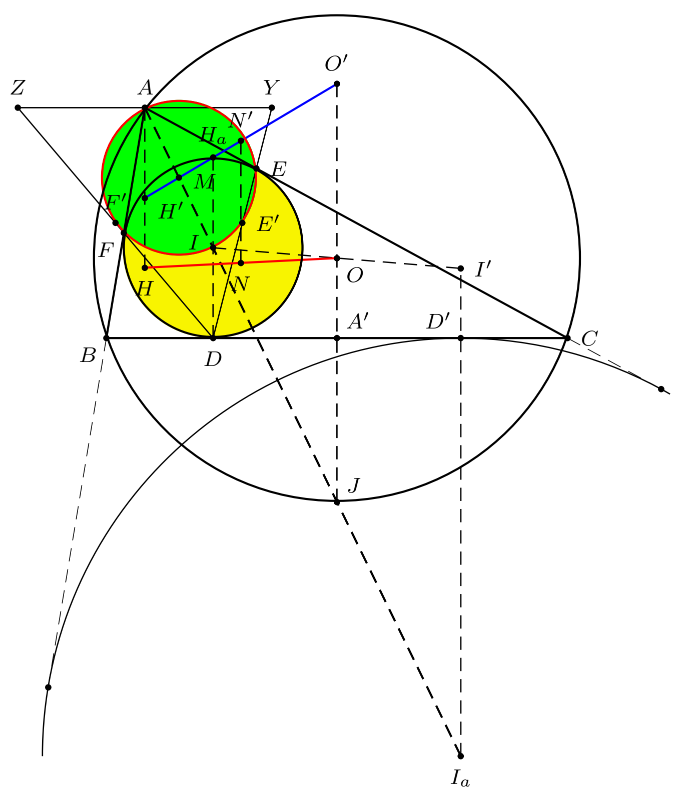

Figure 2. The Euler line of $\mathbf{T}_a$

Clearly, the line HA is parallel to IH_a. Since M is the midpoint of IA, AH intersects ℓ at a point H' such that $AH' = H_aI = r$, the inradius of triangle ABC. See Figure 2. Let the line through O parallel to IH_a intersect ℓ at O'.

If A' is the midpoint of BC, it is well known that $AH = 2 \cdot OA'$.

Consider the excircle (I_a) on the side BC, with radius r_a. The midpoint of II_a is also the midpoint J of the arc BC of the circumcircle (not containing the vertex A). Consider also the reflection I' of I in O, and the excircle (I_a). It is well known that $I'I_a$ passes through the point of tangency D' of (I_a) and BC. We first show that $JO' = r_a$:

$$JO' = \frac{JM}{IM} \cdot IH_a = \frac{I_aA}{IA} \cdot r = \frac{2r_a}{2r} \cdot r = r_a.$$

Since N is the midpoint of OH, and O that of II', we have

$$\begin{aligned}2NN' =& HH' + OO' \\ =& (HA - H'A) + (JO' - R) \\ =& 2 \cdot A'O - r + r_a - R \\ =& DI + D'I' + r_a - (R + r) \\ =& r + (2R - r_a) + r_a - (R + r) \\ =& R.\end{aligned}$$

This means that N' is a point on the nine-point circle of triangle ABC. Since NN' and IH_a are directly parallel, the lines $N'H_a$ and NI intersect at the external center of similitude of the nine-point circle and the incircle. It is well known that the two circles are tangent internally at the Feuerbach point F_e, which is their external center of similitude. See Figure 3. $\square$

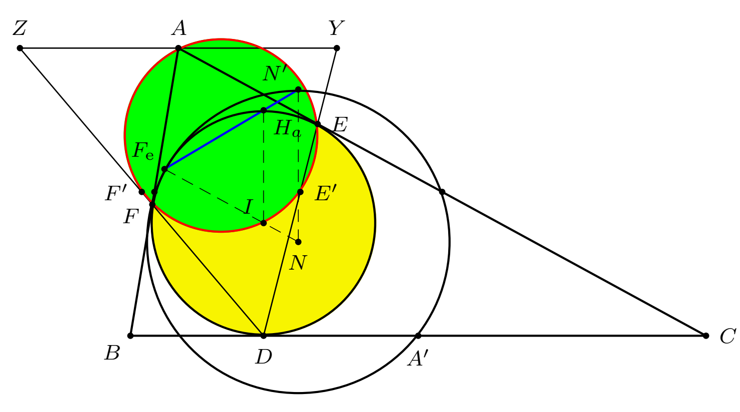

Figure 3. The Euler line of $\mathbf{T}_a$ passes through the Feuerbach point

Remark. Since DH_a is a diameter of the incircle, the Feuerbach point F_e is indeed the pedal of D on the Euler line of triangle DYZ.

Denote the triangle DYZ by $\mathbf{T}_a$. Analogous to $\mathbf{T}_a$, we can also construct the triangles $\mathbf{T}_b$ and $\mathbf{T}_c$ (containing respectively E with a side parallel to CA and F with a side parallel to AB). Theorem 1 also applies to these triangles.

Corollary 2. *The Feuerbach point is the common point of the Euler lines of the three triangles* $\mathbf{T}_a$, $\mathbf{T}_b$, *and* $\mathbf{T}_c$.

3. The excircle case

If, in the construction of $\mathbf{T}_a$, we replace the incircle by the A-excircle (I_a), we obtain another triangle $\mathbf{T}'_a$. More precisely, if the excircle (I_a) touches BC at D', and CA, AB at E', F' respectively, $\mathbf{T}'_a$ is the triangle DYZ bounded by the lines $D'E'$, $D'F'$, and the parallel through A to BC. The method in §2 leads to the following conclusions.

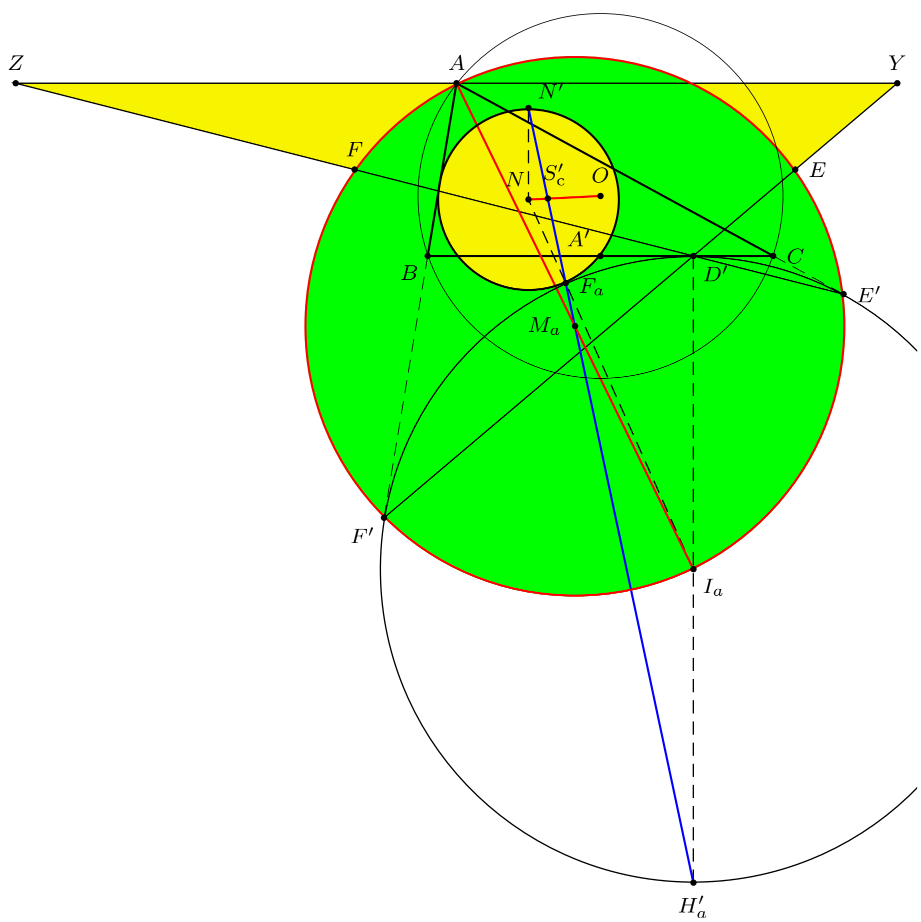

Figure 4. The Euler line of $\mathbf{T}'_a$ passes through $S'_c = X_{442}$

(1) The nine-point circle of $\mathbf{T}'_a$ contains the excenter I_a and the points E', F'; its center is the midpoint M_a of the segment AI_a.

(2) The orthocenter H'_a of $\mathbf{T}'_a$ is the antipode of D' on the excircle (I_a).

(3) The Euler line ℓ'_a of $\mathbf{T}'_a$ contains the point N'.

See Figure 4. Therefore, ℓ'_a also contains the internal center of similitude of the nine-point circle (N) and the excircle (I_a), which is the point of tangency F_a of these two circles. K. L. Nguyen [2] has recently studied the line containing F_a and M_a, and shown that it is the image of the Euler line of triangle IBC under the homothety $\mathsf{h} := \mathsf{h}\left(\mathsf{G}, -\frac{1}{2}\right)$. The same is true for the two analogous triangles $\mathbf{T}'_b$ and $\mathbf{T}'_c$. Their Euler lines are the images of the Euler lines of ICA and IAB under the same homothety. Recall that the Euler lines of triangles IBC, ICA, and IAB intersect at a point on the Euler line, the Schiffler point S_{c}, which is the triangle center X_{21} in [1]. From this we conclude that the Euler lines of $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$ concur at the image of S_{c} under the homothety h. This, again, is a point on the Euler line of triangle ABC. It appears in [1] as the triangle center X_{442}.

4. A generalization

The concurrency of the Euler lines of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$, can be paraphrased as the perspectivity of the "midway triangle" of I with the triangle $H_aH_bH_c$. Here, H_a, H_b, H_c are the orthocenters of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ respectively. They are the antipodes of D, E, F on the incircle. More generally, every homothetic image of ABC in I is perspective with $H_aH_bH_c$. This is clearly equivalent to the following theorem.

Theorem 3. *Every homothetic image of ABC in I is perspective with the intouch triangle DEF.*

Proof. We work with homogeneous barycentric coordinates.

The image of ABC under the homothety $\mathsf{h}(I,t)$ has vertices

$$\begin{aligned} A_t =&(a+t(b+c):(1-t)b:(1-t)c),\\ B_t =&((1-t)a:b+t(c+a):(1-t)c),\\ C_t =&((1-t)a:(1-t)b:c+t(a+b)). \end{aligned}$$

On the other hand, the vertices of the intouch triangle are

$$D=(0:s-c:s-b), \qquad E=(s-c:0:s-a), \qquad F=(s-b:s-a:0).$$

The lines A_tD, B_tE, and C_tF have equations

$$\begin{array}{rcrcrcl} (1-t)(b-c)(s-a)x & + & (s-b)(a+(b+c)t)y & - & (s-c)(a+(b+c)t)z & = & 0,\\ -(s-a)(b+(c+a)t)x & + & (1-t)(c-a)(s-b)y & + & (s-c)(b+(c+a)t)z & = & 0,\\ (s-a)(c+(a+b)t)x & - & (s-b)(c+(a+b)t)y & + & (1-t)(a-b)(s-c)z & = & 0. \end{array}$$

These three lines intersect at the point

$$P_t = \left(\frac{(a+t(b+c))(b+c-a+2at)}{b+c-a} : \cdots : \cdots\right).$$

$\square$

Remark. More generally, for an arbitrary point P, every homothetic image of ABC in $P = (u : v : w)$ is perspective with the cevian triangle of the isotomic conjugate of the superior of P, namely, the point $\left(\frac{1}{v+w-u} : \frac{1}{w+u-v} : \frac{1}{u+v-w}\right)$. With $P = I$, we get the cevian triangle of the Gergonne point which is the intouch triangle.

Proposition 4. *The perspector of $A_tB_tC_t$ and $H_aH_bH_c$ is the reflection of P_{-t} in the incenter.*

It is clear that the perspector P_t traverses a conic Γ as t varies, since its coordinates are quadratic functions of t. The conic Γ clearly contains I and the Gergonne point, corresponding respectively to $t = 0$ and $t = 1$. Note also that $D = P_t$ for $t = -\frac{a}{b+c}$ or $-\frac{s-a}{a}$. Therefore, Γ contains D, and similarly, E and F. It is a cirumconic of the intouch triangle DEF. Now, as $t = \infty$, the line A_tD is parallel to the bisector of angle A, and is therefore perpendicular to EF. Simiarly, B_tE and C_tF are perpendicular to FD and DE respectively. The perspector P_∞ is therefore the orthocenter of triangle DEF, which is the triangle center X_{65} in [1]. It follows that Γ is a rectangular hyperbola. Since it contains also the circumcenter I of DEF, Γ is indeed the Jerabek hyperbola of the intouch triangle. Its center is the point

$$Q = \left(\frac{a(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3))}{b+c-a} : \cdots : \cdots\right).$$

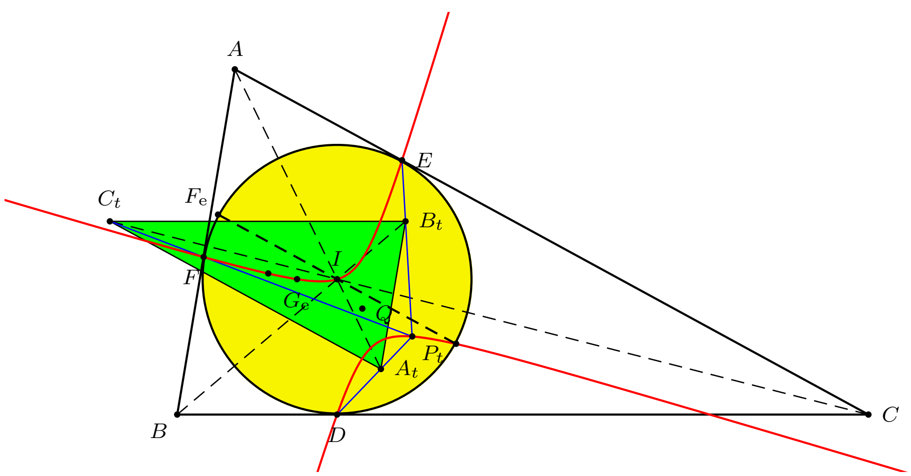

Figure 5. The Jerabak hyperbola of the intouch triangle

The reflection of Γ in the incenter is the conic Γ' which is the locus of the perspectors of $H_aH_bH_c$ and homothetic images of ABC in I.

Note that the fourth intersection of Γ with the incircle is the isogonal conjugate, with respect to the intouch triangle, of the infinite point of its Euler line. Its antipode on the incircle is therefore the Euler reflection point of the intouch triangle.

This must also be the perspector of $H_aH_bH_c$ (the antipode of DEF in the incircle) and a homothetic image of ABC. It must be the Feuerbach point on Γ'.

Theorem 5. *The Feuerbach point is the Euler reflection point of the intouch triangle. This means that the reflections of OI (the Euler line of the intouch triangle) concur at F.*

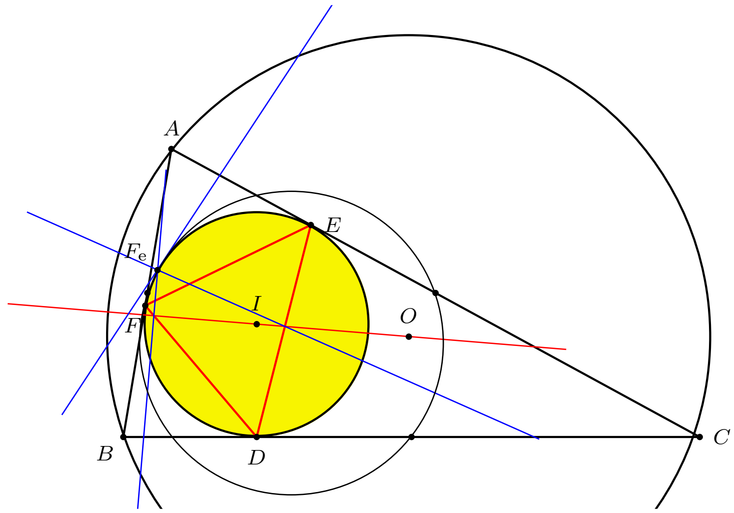

Figure 6. The Feuerbach point as the Euler reflection point of DEF

Remarks. (1) The fourth intersection of Γ with the incircle, being the antipode of the Feuerbach point, is the triangle cente X_{1317}. The conic Γ also contains X_n for the following values of n: 145, 224, and 1537. (Note: X_{145} is the reflection of the Nagel point in the incenter). These are the perspectors for the homothetic images of ABC with ratios $t = -1$, $-\frac{R}{R+r}$, and $-\frac{r}{2(R-r)}$ respectively.

(2) The hyperbola Γ' contains the following triangle centers apart from I and F_e: X_8 and X_{390} (which is the reflection of the Gergonne point in the incenter). These are the perspector for the homothetic images with ratio $+1$ and -1 respectively.

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A Simple Perspectivity

Eric Danneels

Abstract. We construct a simple perspectivity that is invariant under isotomic conjugation.

1. Introduction

In this note we consider a simple transformation of the plane of a given reference triangle ABC. Given a point P with cevian triangle XYZ, construct the parallels through B to XY and through C to XZ to intersect at A'; similarly define B' and C'. Construct

$$A^* = BB' \cap CC', \qquad B^* = CC' \cap AA', \qquad C^* = AA' \cap BB'.$$

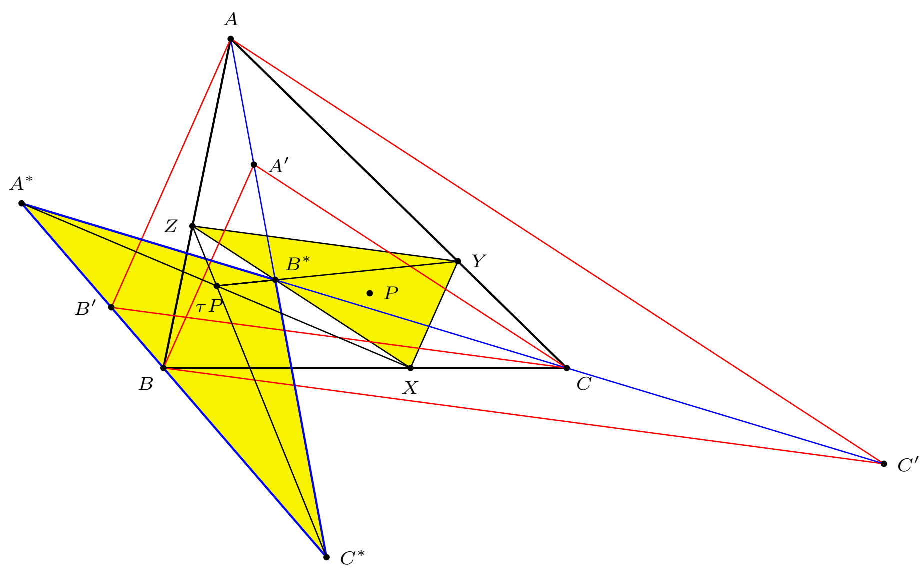

Figure 1

Proposition 1. *Triangle $A^*B^*C^*$ is the anticevian triangle of the infinite point $Q = (u(v-w) : v(w-u) : w(u-v))$ of the trilinear polar of P.*

We shall prove Proposition 1 in §2 below. As an anticevian triangle, $A^*B^*C^*$ is perspective with every cevian triangle. In particular, it is perspective with XYZ at the cevian quotient P/Q, which depends on P only. We write

$$\tau(P) := P/Q = (u(v-w)^2 : v(w-u)^2 : w(u-v)^2).$$

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The author thanks Paul Yiu for his help in the preparation of this paper.

Let $P^\bullet$ denote the isotomic conjugate of P.

Proposition 2. *τ is invariant under isotomic conjugation: $\tau(P^\bullet) = \tau(P)$.*

Proposition 3. *$\tau(P)$ is*
(1) *the center of the circumconic through P and its isotomic conjugate $P^\bullet$,*
(2) *the perspector of the circum-hyperbola with asymptotes the trilinear polars of P and $P^\bullet$.*

Proof. (1) The circumconic through P and $P^\bullet$ has equation

$$\frac{u(v^2-w^2)}{x} + \frac{v(w^2-u^2)}{y} + \frac{w(u^2-v^2)}{z} = 0,$$

with perspector

$$P' = (u(v^2-w^2) : \ w(w^2-u^2) : \ w(u^2-v^2)). \tag{1}$$

Its center is the cevian quotient G/P'. This is $\tau(P)$.

(2) The pencil of hyperbolas with asymptotes the trilinears polars of P and $P^\bullet$ has equation

$$k(x+y+z)^2 + (ux+vy+cz)\left(\frac{x}{u} + \frac{y}{v} + \frac{z}{w}\right) = 0.$$

For $k = -1$, the hyperbola passes through A, B, C, and this circum-hyperbola has equation

$$\frac{u(v-w)^2}{x} + \frac{v(w-u)^2}{y} + \frac{w(u-v)^2}{z} = 0.$$

It has perspector $\tau(P)$, (and center P' given in (1) above). □

Remark. Wilson Stothers [2] has found that one asymptote of a circum-hyperbola determines the other. More precisely, if $ux+vy+wz = 0$ is an asymptote of a circum-hyperbola, then the other is $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0$. This gives a stronger result than (2) above.

Here is a list of triangle centers with their images under τ. The labeling of triangle centers follows Kimberling [1].

$P,\ P^\bullet$	$\tau(P)$		$P,\ P^\bullet$	$\tau(P)$
X_1, X_{75}	X_{244}		$X_3,\ X_{264}$	X_{2972}
$X_4,\ X_{69}$	X_{125}		$X_7,\ X_8$	X_{11}
$X_{20},\ X_{253}$	X_{122}		$X_{30},\ X_{1494}$	X_{1650}
$X_{57},\ X_{312}$	X_{2170}		$X_{88},\ X_{88}^\bullet$	X_{2087}
$X_{94},\ X_{323}$	X_{2088}		$X_{98},\ X_{325}$	X_{868}
$X_{99},\ X_{523}$	X_{1649}		$X_{200},\ X_{1088}$	X_{2310}
$X_{519},\ X_{903}$	X_{1647}		$X_{524},\ X_{671}$	X_{1648}
$X_{536},\ X_{536}^\bullet$	X_{1646}		$X_{538},\ X_{538}^\bullet$	X_{1645}
$X_{694},\ X_{694}^\bullet$	X_{2086}		$X_{1022},\ X_{1022}^\bullet$	X_{1635}
$X_{1026},\ X_{1026}^\bullet$	X_{2254}		$X_{2394},\ X_{2407}$	X_{1637}
$X_{2395},\ X_{2396}$	X_{2491}		$X_{2398},\ X_{2400}$	X_{676}

2. Proof of Proposition 1

The line XY has equation $vwx+wuy-uvz=0$, and infinite point $(-u(v+w):v(w+u):\ w(u-v))$. The parallel through B to XY is the line

$$w(u-v)x+u(v+w)z=0.$$

Similarly, the parallel through C to XZ is the line

$$v(u-w)x+u(v+w)y=0.$$

These two lines intersect at

$$A'=(u(v+w):\ v(w-u):\ w(v-u)).$$

The two analogously defined points are

$$\begin{aligned}B' =&(u(w-v):\ v(w+u):\ w(u-v)),\\ C' =&(u(v-w):\ v(u-w):\ w(u+v)).\end{aligned}$$

Now the lines AA', BB', CC' intersect at the points

$$\begin{aligned}A^* =&BB'\cap CC'=(-u(v-w):\ v(w-u):\ w(u-v)),\\ B^* =&CC'\cap AA'=(u(v-w):\ -v(w-u):\ w(u-v)),\\ C^* =&AA'\cap BB'=(u(v-w):\ v(w-u):\ -w(u-v)).\end{aligned}$$

This is clearly the anticevian triangle of the point

$$Q=(u(v-w):\ v(w-u):\ w(u-v))=\left(\frac{1}{v}-\frac{1}{w}:\ \frac{1}{w}-\frac{1}{u}:\ \frac{1}{u}-\frac{1}{v}\right),$$

which is the infinite point of the trilinear polar $\mathcal{L}$. This completes the proof of Proposition 1.

Remarks. (1) Here is an easy alternative construction of $A^*B^*C^*$. Construct the parallels through A, B, C to the trilinear polar $\mathcal{L}$, intersecting the sidelines BC, CA, AB at A_1, B_1, C_1 respectively. Then, A^*, B^*, C^* are the midpoints of the segments AA_1, BB_1, CC_1. See Figure 2.

(2) The equations of the sidelines of triangle $A^*B^*C^*$ are

$$\begin{aligned}B^*C^*:&\ \frac{y}{v(w-u)}+\frac{z}{w(u-v)}=0,\\ C^*A^*:&\ \frac{x}{u(v-w)}+\frac{z}{w(u-v)}=0,\\ A^*B^*:&\ \frac{x}{u(v-w)}+\frac{y}{v(w-u)}=0.\end{aligned}$$

Proposition 4. *The trilinear polar of $\tau(P)$ with respect to the cevian triangle of P passes through P.*

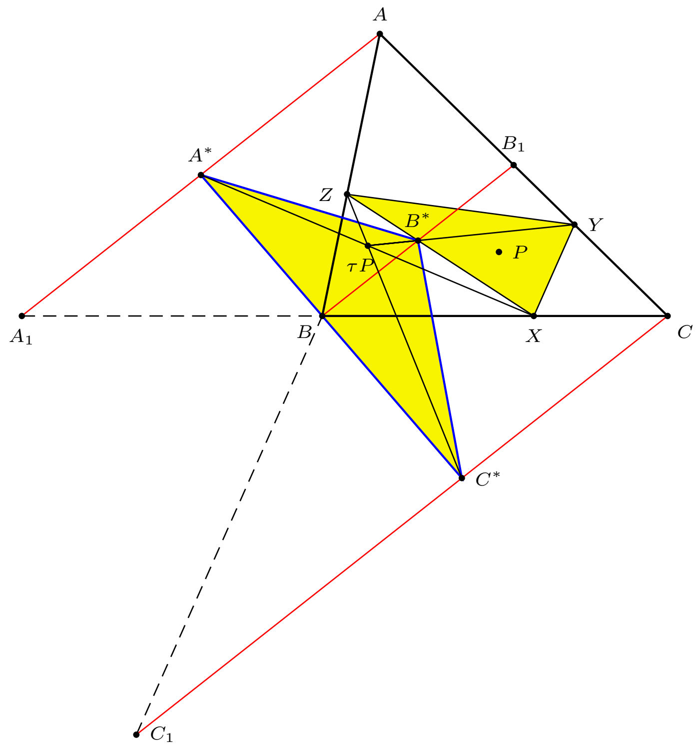

Figure 2

Proof. The trilinear polar of $\tau(P)$ with respect to XYZ is the perspectrix of the triangles XYZ and $A^*B^*C^*$. Now, the sidelines of these triangle intersect at the points

$$\begin{aligned} B^*C^* \cap YZ =&(u(v+w-2u) : \ v(w-u) : \ -w(u-v)), \\ C^*A^* \cap ZX =&(-u(v-w) : \ v(w+u-2v) : \ w(u-v)), \\ A^*B^* \cap XY =&(u(v-w) : \ -v(w-u) : \ w(u+v-2w)). \end{aligned}$$

The line through these three points has equation

$$\frac{v-w}{u}x + \frac{w-u}{v}y + \frac{u-v}{w}z = 0.$$

This clearly contains the point $P = (u : v : w)$. □

3. Generalization

Since the construction in §1 is purely perspective we can replace the line at infinity by an arbitrary line $\ell : px + qy + rz = 0$. The parallel through B to XY becomes the line joining B to the intersection ℓ and XY, etc. The perspector becomes

$$\tau_\ell(P) = (u(qv-rw)^2 : \ v(rw-pu)^2 : \ w(pu-qv)^2).$$

Then $\tau_\ell(P^\ell) = \tau(P)$ where $P^\ell = \left(\frac{1}{p^2u} : \frac{1}{q^2v} : \frac{1}{r^2w}\right)$, and the following remain valid:

(1) $A^*B^*C^*$ is the anticevian triangle of $Q = \mathcal{L} \cap \ell$, where $\mathcal{L}$ is the trilinear polar of P.
(2) The perspectrix of XYZ and $A^*B^*C^*$ contains the point P.

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[2] W. Stothers, Hyacinthos message 8476, October 30, 2003.

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Pedals on Circumradii and the Jerabek Center

Quang Tuan Bui

Abstract. Given a triangle ABC, beginning with the orthogonal projections of the vertices on the circumradii OA, OB, OC, we construct two triangles each with circumcircle tangent to the nine-point circle at the center of the Jerabek hyperbola.

1. Introduction

Given a triangle ABC, with circumcenter O, let A_b and A_c be the pedals (orthogonal projections) of the vertex A on the lines OB and OC respectively. Similarly, define B_c, B_a, C_a and C_b. In this paper we prove some interesting results on triangles associated with these pedals.

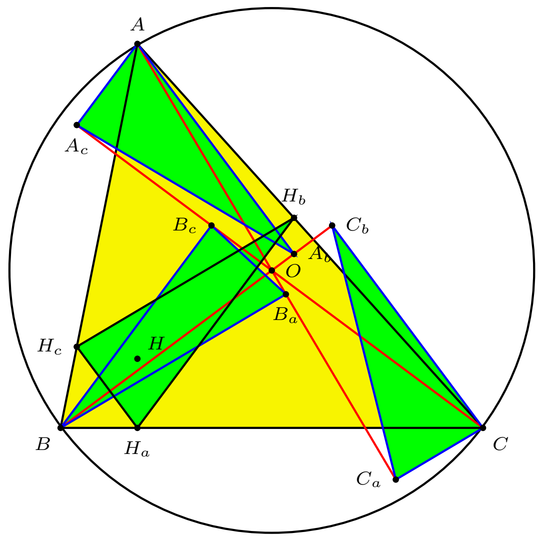

Figure 1

Theorem 1. *The triangles AA_bA_c, B_aBB_c and C_aC_bC are congruent to the orthic triangle $H_aH_bH_c$.* See Figure 1.

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Theorem 2. *The lines B_cC_b, C_aA_c and A_bB_a bound a triangle $\mathbf{T}_1$ homothetic to ABC. The circumcircle of $\mathbf{T}_1$ is tangent to the nine-point circle of ABC at the Jerabek center.*

Recall that the Jerabek center J is the center of the circum-hyperbola through the circumcenter O. This hyperbola is the isogonal conjugate of the Euler line. The Jerabek center J is the triangle center X_{125} in Kimberling's *Encyclopedia of Triangle Centers* [1].

Theorem 3. *The lines A_bA_c, B_cB_a and C_aC_b bound a triangle $\mathbf{T}_2$ whose circumcircle is tangent to the nine-point circle at the Jerabek center.*

Hence, the circumcircles of $\mathbf{T}_1$ and $\mathbf{T}_2$ are also tangent to each other at J. In this paper we work with homogeneous barycentric coordinates and adopt standard notations of triangle geometry. Basic results can be found in [2]. The Jerabek center J, for example, has coordinates

$$(S_A(S_B-S_C)^2 : \ S_B(S_C-S_A)^2 : \ S_C(S_A-S_B)^2). \tag{1}$$

The labeling of triangle centers, except for the common ones, follows [1].

Proposition 4. *The homogeneous barycentric coordinates of the pedals of the vertices of triangle ABC on the circumradii are as follows.*

$$\begin{aligned}
A_b =&(S_A(S_B+S_C) : \ S_C(S_C-S_A) : \ S_C(S_A+S_B)),\\
A_c =&(S_A(S_B+S_C) : \ S_B(S_C+S_A) : \ S_C(S_B-S_A));\\
B_c =&(S_A(S_B+S_C) : \ S_B(S_C+S_A) : \ S_A(S_A-S_B)),\\
B_a =&(S_A(S_C-S_B) : \ S_B(S_C+S_A) : \ S_C(S_A+S_B));\\
C_a =&(S_A(S_B-S_C) : \ S_B(S_C+S_A) : \ S_C(S_A+S_B)),\\
C_b =&S_A((S_B+S_C) : \ S_B(S_A-S_C) : \ S_C(S_A+S_B)).
\end{aligned}$$

Proof. We verify that the point

$$P = (S_A(S_B+S_C) : \ S_C(S_C-S_A) : \ S_C(S_A+S_B))$$

is the pedal A_b of A on the line OB. Since

$$\begin{aligned}
&(S_A(S_B+S_C), \ S_C(S_C-S_A), \ S_C(S_A+S_B))\\
=&(S_A(S_B+S_C), \ S_B(S_C+S_A), \ S_C(S_A+S_B))\\
&+(0, S_C(S_C-S_A)-S_B(S_C+S_A), \ 0),
\end{aligned}$$

this is a point on the line OB. The coordinate sum of P being $(S_B+S_C)(S_C+S_A)$, the infinite point of the line AP is

$$\begin{aligned}
&(S_A(S_B+S_C), \ S_C(S_C-S_A), \ S_C(S_A+S_B))-((S_C+S_A)(S_A+S_B), \ 0, \ 0)\\
=&S_C(-(S_B+S_C), \ (S_C-S_A), \ (S_A+S_B)).
\end{aligned}$$

The infinite point of OB is

$$\begin{aligned}
&(S_A(S_B+S_C), \ S_B(S_C+S_A), \ S_C(S_A+S_B))-(0, \ 2(S_{BC}+S_{CA}+S_{AB}), \ 0)\\
=&(S_A(S_B+S_C), \ -(S_{BC}+2S_{CA}+S_{AB}), \ S_C(S_A+S_B)).
\end{aligned}$$

By the theorem in [2, §4.5], the two lines AP and OB are perpendicular since

$$\begin{aligned}&-S_A\cdot(S_B+S_C)\cdot S_A(S_B+S_C)\\&-S_B\cdot(S_C-S_A)\cdot(S_{BC}+2S_{CA}+S_{AB})\\&+S_C\cdot(S_A+S_B)\cdot S_C(S_A+S_B)\\&=0.\end{aligned}$$

□

2. Proof of Theorem 1

Note that the points A_b and A_c lie on the circle with diameter OA, so do the midpoints of AC and AB. Therefore,

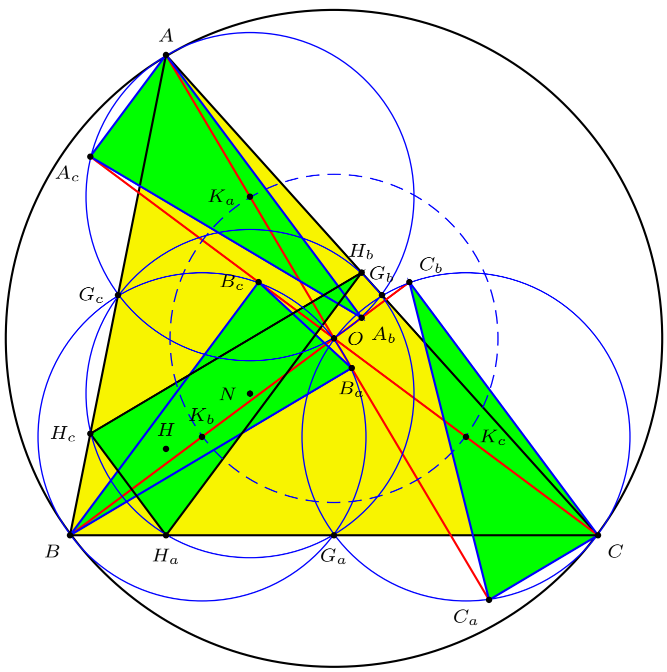

Figure 2

$$\begin{aligned}\angle A_cAA_b=&\pi-\angle A_bOA_c=\pi-\angle BOC=\pi-2A=\angle H_cH_aH_b,\\\angle AA_bA_c=&\angle AOA_c=\pi-\angle COA=\pi-2B=\angle H_aH_bH_c,\\\angle A_bA_cA=&\angle A_bOA=\pi-\angle AOB=\pi-2C=\angle H_bH_cH_a.\end{aligned}$$

Therefore the angles in triangles AA_bA_c and $H_aH_bH_c$ are the same; similarly for triangles B_aBB_c and C_aC_bC. Since these four triangles have equal circumradii $\frac{R}{2}$, they are congruent. This completes the proof of Theorem 1.

Remarks. (1) The side lengths of these triangles are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

(2) If K_a, K_b, K_c are the midpoints of the circumradii OA, OB, OC, triangle $K_aK_bK_c$ is homothetic to
(i) ABC at O, with ratio of homothety $\frac{1}{2}$, and
(ii) the medial triangle $G_aG_bG_c$ at X_{140}, the nine-point center of the medial triangle, with ratio of homothety -1.

(3) The circles (K_b) and (K_c) intersect at the circumcenter O and the midpoint G_a of BC; similarly for the other two pairs (K_c), (K_a) and (K_a), (K_b). The midpoints G_a, G_b, G_c lie on the nine-point circle (N) of triangle ABC. See Figure 2.

3. The triangle $\mathbf{T}_1$

We now consider the triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, and A_bB_a.

Lemma 5. *The quadrilateral B_cC_bCB is an isosceles trapezoid.*

Proof. With reference to Figure 1, we have
(i) $\angle B_cBC = \frac{\pi}{2} - \angle OCB = \frac{\pi}{2} - \angle OBC = \angle C_bCB$,
(ii) $B_cB = C_bC$.
It follows that the quadrilateral B_cC_bCB is an isosceles trapezoid. □

Therefore, the lines B_cC_b and BC are parallel. Similarly, the lines C_aA_c and CA are parallel, as are A_bB_a and AB. The triangle $\mathbf{T}_1$ bounded by the lines B_cC_b, C_aA_c, A_bB_a is homothetic to triangle ABC, and also to the medial triangle $G_aG_bG_c$.

Proposition 6. *Triangle $\mathbf{T}_1$ is homothetic to*
(i) *ABC at the procircumcenter $(a^4S_A : b^4S_B : c^4S_C)$,* [1]
(ii) *the medial triangle $G_aG_bG_c$ at the Jerabek center J.*

Proof. The lines B_cC_b, C_aA_c, and A_bB_a have equations

$$\begin{aligned}
-(S_{AA}+S_{BC})x + S_A(S_B+S_C)y + S_A(S_B+S_C)z =&0,\\
S_B(S_C+S_A)x - (S_{BB}+S_{CA})y + S_B(S_C+S_A)z =&0,\\
S_C(S_A+S_B)x + S_C(S_A+S_B)y - (S_{CC}+S_{AB})z =&0.
\end{aligned}$$

From these, we obtain the coordinates of the vertices of $\mathbf{T}_1$:

$$\begin{aligned}
A_1 =&(S_A(S_B-S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2),\\
B_1 =&(S_A(S_B+S_C)^2 : S_B(S_C-S_A)^2 : S_C(S_A+S_B)^2),\\
C_1 =&(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A-S_B)^2).
\end{aligned}$$

From the coordinates of A_1, B_1, C_1, it is clear that the homothetic center of triangles $A_1B_1C_1$ and ABC is the point

$$(S_A(S_B+S_C)^2 : S_B(S_C+S_A)^2 : S_C(S_A+S_B)^2) = (a^4S_A : b^4S_B : c^4S_C).$$

[1] This is the triangle center X_{184} in [1].

For (ii), the equations of the lines G_aA_1, G_bB_1, G_cC_1 are respectively

$$\begin{aligned}(S_{AA}-S_{BC})x - S_A(S_B-S_C)y + S_A(S_B-S_C)z =&0,\\ S_B(S_C-S_A)x + (S_{BB}-S_{CA})y - S_B(S_C-S_A)z =&0,\\ -S_C(S_A-S_B)x + S_C(S_A-S_B)y + (S_{CC}-S_{AB})z =&0.\end{aligned}$$

It is routine to check that this contains the Jerabek center J whose coordinates are given in (1). $\square$

4. Proof of Theorem 2

Theorem 2 is now an immediate consequence of Proposition 6(ii). Since the homothetic center J lies on the circumcircle of the medial triangle, it must also lie on the circumcircle of the other, and the two circumcircles are tangent at J.

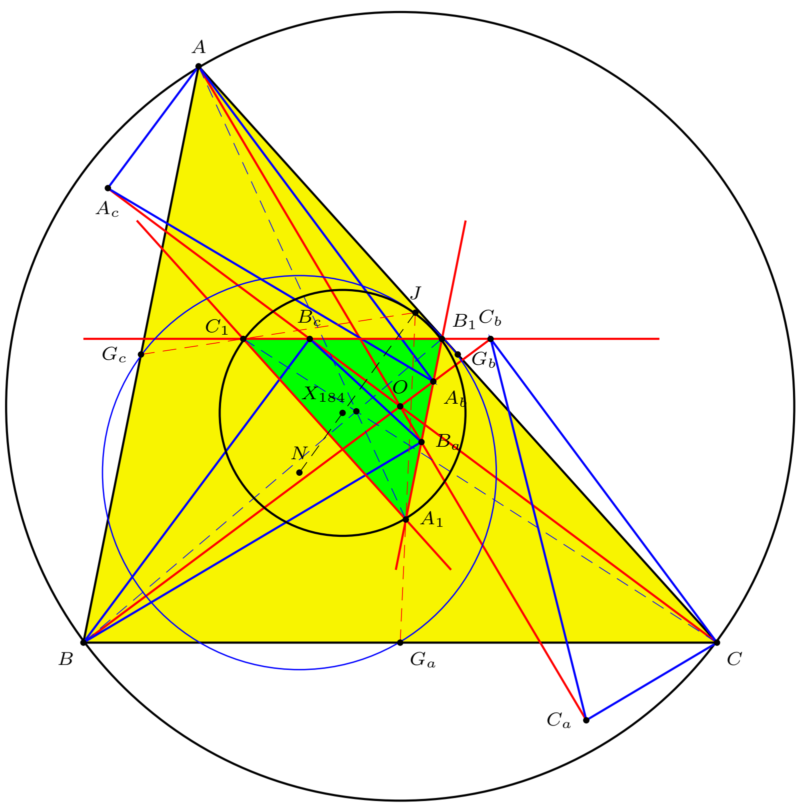

Figure 3

5. The triangle $\mathbf{T}_2$

From the coordinates of the pedals, we obtain the equations of the lines A_bA_c, B_cB_a, and C_aC_b:

$$\begin{aligned}
-2S_{BC}x+(S^2-S_{BB})y+(S^2-S_{CC})z=&0,\\
(S^2-S_{AA})x-2S_{CA}y+(S^2-S_{CC})z=&0,\\
(S^2-S_{AA})x+(S^2-S_{BB})y-2S_{AB}z=&0.
\end{aligned}$$

From these, the vertices of triangle $\mathbf{T}_2$ are the points

$$\begin{aligned}
A_2=&((S_B-S_C)^2:\ 3S_{AB}+S_{BC}+S_{CA}-S_{CC}:\ 3S_{CA}+S_{AB}+S_{BC}-S_{BB}),\\
B_2=&(3S_{AB}+S_{BC}+S_{CA}-S_{CC}:\ (S_C-S_A)^2:\ 3S_{BC}+S_{CA}+S_{AB}-S_{AA}),\\
C_2=&(3S_{CA}+S_{AB}+SBC-S_{CC}:\ 3S_{BC}+S_{CA}+S_{AB}-S_{AA}:\ (S_A-S_B)^2).
\end{aligned}$$

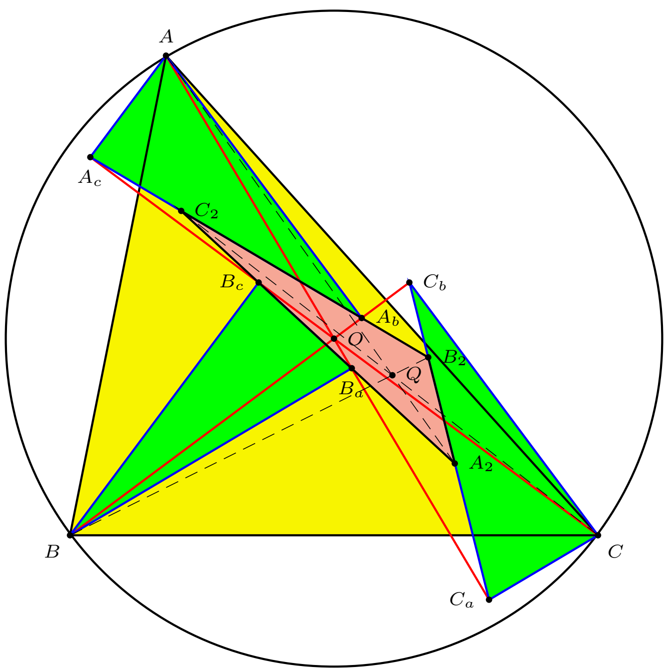

Figure 4

Proposition 7. *Triangles ABC and $A_2B_2C_2$ are perspective at*

$$Q=\left(\frac{1}{a^2b^2+b^2c^2+c^2a^2-b^4-c^4}:\ \frac{1}{a^2b^2+b^2c^2+c^2a^2-c^4-a^4}:\ \frac{1}{a^2b^2+b^2c^2+c^2a^2-a^4-b^4}\right).$$

Proof. From the coordinates of A_2, B_2, C_2 given above, $\mathbf{T_2}$ is perspective with ABC at

$$Q=\left(\frac{1}{3S_{BC}+S_{CA}+S_{AB}-S_{AA}}:\cdots:\cdots\right).$$

These are equivalent to those given above in terms of a, b, c. □

Remark. The triangle center Q does not appear in [1].

6. Proof of Theorem 3

It is easier to work with the image of triangle $\mathbf{T}_2$ under the homothety $\mathsf{h}(H,2)$. The images of the vertices are

$$\begin{aligned}A_2'=&(S_A(S_B+S_C)(S_{BB}-4S_{BC}+S_{CC})+S_{BC}(S_B-S_C)^2\\&:\ (S_C+S_A)(S_A(S_B+S_C)(3S_B-S_C)+S_{BC}(S_B-S_C))\\&:\ (S_A+S_B)(S_A(S_B+S_C)(S_B-3S_C)+S_{BC}(S_B-S_C))),\end{aligned}$$

and B_2', C_2' whose coordinates are obtained by cyclic permutations of S_A, S_B, S_C. The circumcircle of $A_2'B_2'C_2'$ has equation

$$\begin{aligned}&8S^2\cdot S_{ABC}((S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy)\\&+(x+y+z)\left(\sum_{\text{cyclic}}(S_A+S_B)(S_A+S_C)(S_{AB}+S_{CA}-2S_{BC})^2x\right)=0.\end{aligned}$$

To verify that this circle is tangent to the circumcircle

$$(S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy=0,$$

it is enough to consider the pedal of the circumcenter O on the radical axis

$$\sum_{\text{cyclic}}(S_A+S_B)(S_A+S_C)(S_{AB}+S_{CA}-2S_{BC})^2x=0.$$

This is the point

$$Q'=\left(\frac{S_B+S_C}{S_{CA}+S_{AB}-2S_{BC}}:\ \frac{S_C+S_A}{S_{AB}+S_{BC}-2S_{CA}}:\ \frac{S_A+S_B}{S_{BC}+S_{CA}-2S_{AB}}\right),$$

which is clearly on the circumcircle, and also on the Jerabek hyperbola

$$\frac{S_A(S_{BB}-S_{CC})}{x}+\frac{S_B(S_{CC}-S_{AA})}{y}+\frac{S_C(S_{AA}-S_{BB})}{z}=0.$$

This shows that the circle $A_2'B_2'C_2'$ is tangent to the circumcircle at Q'. [2] Under the homothety $\mathsf{h}(H,2)$, Q' is the image of the midpoint of HQ, which is the center of the Jerabek hyperbola. Under the inverse homothety, the circumcircle of $\mathbf{T}_2$ is tangent to the nine-point circle at J. This completes the proof of Theorem 3.

[2] Q' is the triangle center X_{74}

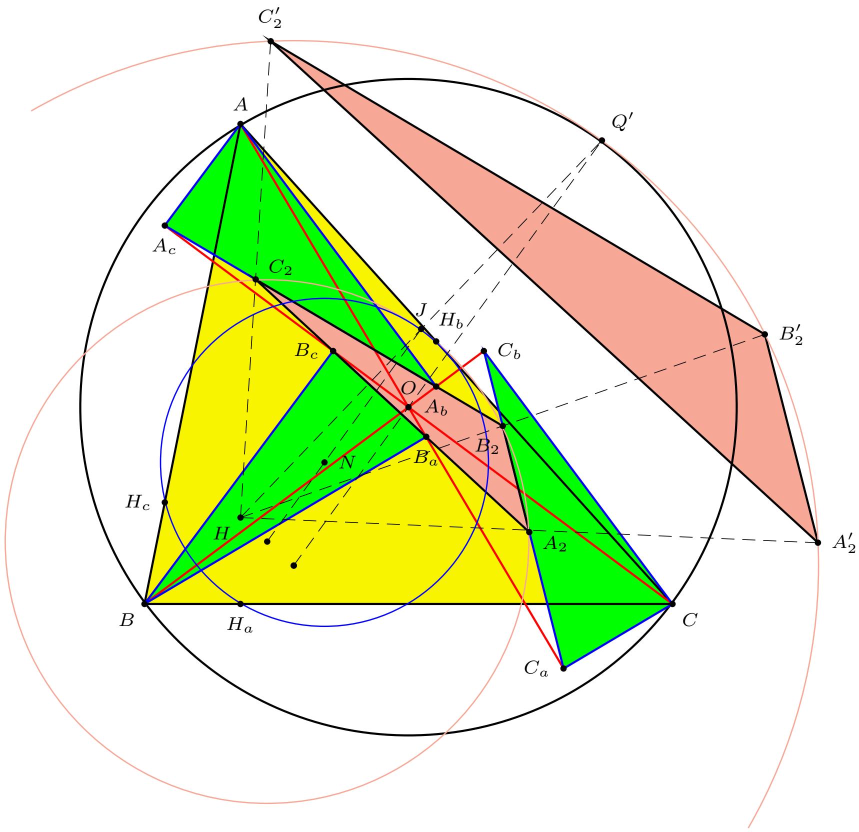

Figure 5

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Simmons Conics

Bernard Gibert

Abstract. We study the conics introduced by T. C. Simmons and generalize some of their properties.

1. Introduction

In [1, Tome 3, p.227], we find a definition of a conic called “ellipse de Simmons” with a reference to E. Vigarié series of papers [8] in 1887-1889. According to Vigarié, this “ellipse” was introduced by T. C. Simmons [7], and has foci the first isogonic center or Fermat point (X_{13} in [5]) and the first isodynamic point (X_{15} in [5]). The contacts of this “ellipse” with the sidelines of reference triangle ABC are the vertices of the cevian triangle of X_{13}. In other words, the perspector of this conic is one of its foci. The given trilinear equation is :

$$\sqrt{\alpha \sin\left(A+\frac{\pi}{3}\right)}+\sqrt{\beta \sin(\left(B+\frac{\pi}{3}\right)}+\sqrt{\gamma \sin\left(C+\frac{\pi}{3}\right)}=0.$$

It appears that this conic is not always an ellipse and, curiously, the corresponding conic with the other Fermat and isodynamic points is not mentioned in [1].

In this paper, working with barycentric coordinates, we generalize the study of inscribed conics and circumconics whose perspector is one focus.

2. Circumconics and inscribed conics

Let $P=(u:v:w)$ be any point in the plane of triangle ABC which does not lie on one sideline of ABC. Denote by $\mathcal{L}(P)$ its trilinear polar.

The locus of the trilinear pole of a line passing through P is a circumconic denoted by $\Gamma_c(P)$ and the envelope of trilinear polar of points of $\mathcal{L}(P)$ is an inscribed conic $\Gamma_i(P)$. In both cases, P is said to be the perspector of the conic and $\mathcal{L}(P)$ its perspectrix. Note that $\mathcal{L}(P)$ is the polar line of P in both conics.

The centers of $\Gamma_c(P)$ and $\Gamma_i(P)$ are

$$\begin{aligned}\Omega_c(P)=&(u(v+w-u)\ :v(w+u-v)\ :w(u+v-w)),\\ \Omega_i(P)=&(u(v+w)\ :v(w+u)\ :w(u+v))\end{aligned}$$

respectively. $\Omega_c(P)$ is also the perspector of the medial triangle and the anticevian triangle $A_PB_PC_P$ of P. $\Omega_i(P)$ is the complement of the isotomic conjugate of P.

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2.1. *Construction of the axes of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. Let X be the fourth intersection of the conic and the circumcircle (X is the trilinear pole of the line KP). The axes of $\Gamma_c(P)$ are the parallels at $\Omega_c(P)$ to the bisectors of the lines BC and AX. A similar construction in the cevian triangle $P_aP_bP_c$ of P gives the axes of $\Gamma_i(P)$.

2.2. *Construction of the foci of* $\Gamma_c(P)$ *and* $\Gamma_i(P)$. The line BC and its perpendicular at P_a meet one axis at two points. The circle with center $\Omega_i(P)$ which is orthogonal to the circle having diameter these two points meets the axis at the requested foci. A similar construction in the anticevian triangle of P gives the foci of $\Gamma_c(P)$.

3. Inscribed conics with focus at the perspector

Theorem 1. *There are two and only two non-degenerate inscribed conics whose perspector P is one focus : they are obtained when P is one of the isogonic centers.*

Proof. If P is one focus of $\Gamma_i(P)$, the other focus is the isogonal conjugate P^* of P and the center is the midpoint of PP^*. This center must be the isotomic conjugate of the anticomplement of P. A computation shows that P must lie on three circum-strophoids with singularity at one vertex of ABC. These strophoids are orthopivotal cubics as seen in [4, p.17]. They are the isogonal transforms of the three Apollonian circles which intersect at the two isodynamic points. Hence, the strophoids intersect at the isogonic centers. □

These conics will be called the (inscribed) *Simmons conics* denoted by $\mathcal{S}_{13} = \Gamma_i(X_{13})$ and $\mathcal{S}_{14} = \Gamma_i(X_{14})$.

Elements of the conics	$\mathcal{S}_{13}$	$\mathcal{S}_{14}$
perspector and focus	X_{13}	X_{14}
other real focus	X_{15}	X_{16}
center	X_{396}	X_{395}
focal axis	parallel to the Euler line	idem
non-focal axis	$\mathcal{L}(X_{14})$	$\mathcal{L}(X_{13})$
directrix	$\mathcal{L}(X_{13})$	$\mathcal{L}(X_{14})$
other directrix	$\mathcal{L}(X_{18})$	$\mathcal{L}(X_{17})$

Remark. The directrix associated to the perspector/focus in both Simmons conics is also the trilinear polar of this same perspector/focus. This will be generalized below.

Theorem 2. *The two (inscribed) Simmons conics generate a pencil of conics which contains the nine-point circle.*

The four (not always real) base points of the pencil form a quadrilateral inscribed in the nine point circle and whose diagonal triangle is the anticevian triangle of X_{523}, the infinite point of the perpendiculars to the Euler line. In Figure 1 we have four real base points on the nine point circle and on two parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

Hence, all the conics of the pencil have axes with the same directions (parallel and perpendicular to the Euler line) and are centered on the rectangular hyperbola

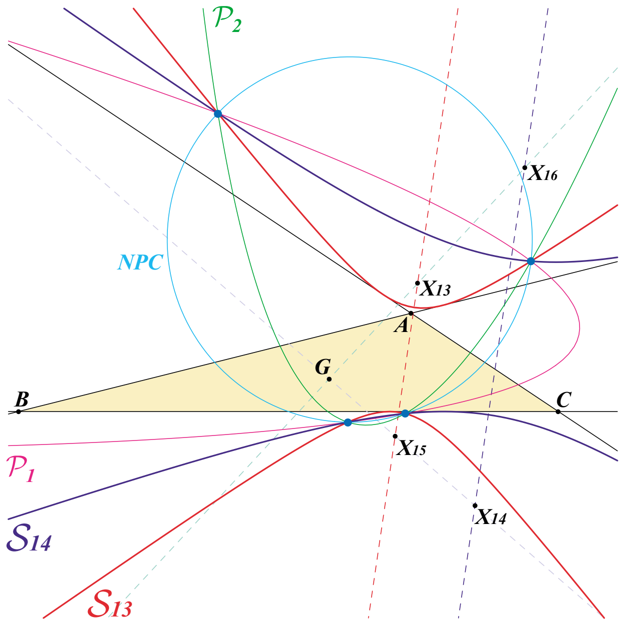

Figure 1. Simmons ponctual pencil of conics

which is the polar conic of X_{30} (point at infinity of the Euler line) in the Neuberg cubic. This hyperbola passes through the in/excenters, X_5, X_{30}, X_{395}, X_{396}, X_{523}, X_{1749} and is centered at X_{476} (Tixier point). See Figure 2. This is the diagonal conic with equation :

$$\sum_{\text{cyclic}} (b^2 - c^2)(4S_A^2 - b^2c^2)x^2 = 0.$$

It must also contain the vertices of the anticevian triangle of any of its points and, in particular, those of the diagonal triangle above. Note that the polar lines of any of its points in both Simmons inconics are parallel.

Theorem 3. *The two (inscribed) Simmons conics generate a tangential pencil of conics which contains the Steiner inellipse.*

Indeed, their centers X_{396} and X_{395} lie on the line GK. The locus of foci of all inconics with center on this line is the (second) Brocard cubic K018 which is $\mathrm{n}\mathcal{K}_0(K, X_{523})$ (See [3]). These conics must be tangent to the trilinear polar of the root X_{523} which is the line through the centers X_{115} and X_{125} of the Kiepert and Jerabek hyperbolas.

Another approach is the following. The fourth common tangent to two inconics is the trilinear polar of the intersection of the trilinear polars of the two perspectors. In the case of the Simmons inconics, the intersection is X_{523} at infinity (the perspector of the Kiepert hyperbola) hence the common tangent must be the trilinear polar of this point. In fact, more generally, any inconic with perspector on the

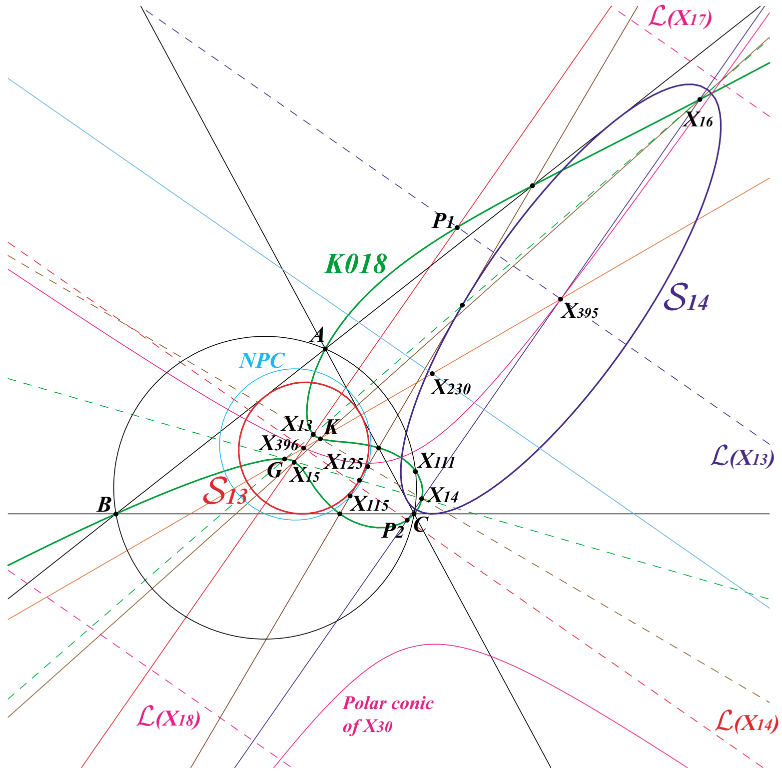

Figure 2. The two Simmons inconics $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$

Kiepert hyperbola must be tangent to this same line (the perpector of each conic must lie on the Kiepert hyperbola since it is the isotomic conjugate of the anticomplement of the center of the conic). In particular, since G lies on the Kiepert hyperbola, the Steiner inellipse must also be tangent to this line. This is also the case of the inconic with center K, perspector H sometimes called K-ellipse (see [1]) although it is not always an ellipse.

Remarks. (1) The contacts of this common tangent with $\mathcal{S}_{13}$ and $\mathcal{S}_{14}$ lie on the lines through G and the corresponding perspector. See Figures 2 and 3.

(2) This line $X_{115}X_{125}$ meets the sidelines of ABC at three points on K018.

(3) The focal axes meet the non-focal axes at the vertices of a rectangle with center X_{230} on the orthic axis and on the line GK. These vertices are X_{396}, X_{395} and two other points P_1, P_2 on the cubic K018 and collinear with X_{111}, the singular focus of the cubic.

(4) The orthic axis is the mediator of the non-focal axes.

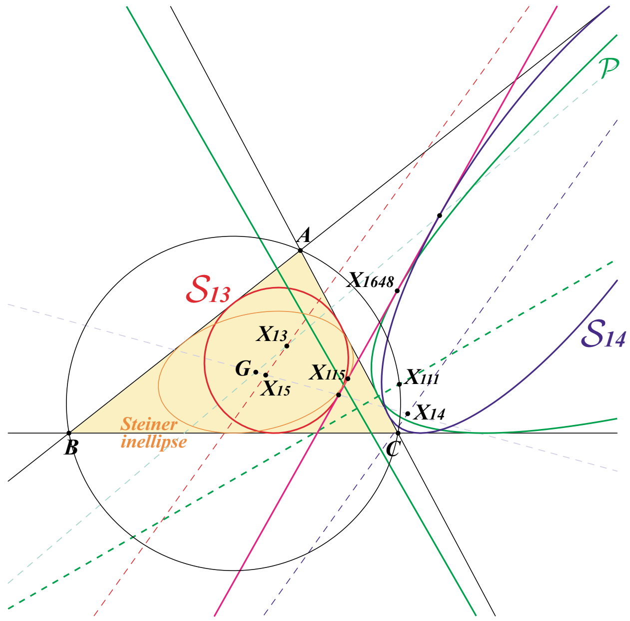

Figure 3. Simmons tangential pencil of conics

(5) The pencil contains one and only one parabola $\mathcal{P}$ we will call *the Simmons parabola*. This is the in-parabola with perspector X_{671} (on the Steiner ellipse), focus X_{111} (Parry point), touching the line $X_{115}X_{125}$ at X_{1648}.[1]

4. Circumconics with focus at the perspector

A circumconic with perspector P is inscribed in the anticevian triangle $P_aP_bP_c$ of P. In other words, it is the inconic with perspector P in $P_aP_bP_c$. Thus, P is a focus of the circumconic if and only if it is a Fermat point of $P_aP_bP_c$. According to a known result [2], it must then be a Fermat point of ABC. Hence,

Theorem 4. *There are two and only two non-degenerate circumconics whose perspector P is one focus : they are obtained when P is one of the isogonic centers.*

They will be called the *Simmons circumconics* denoted by $\Sigma_{13} = \Gamma_c(X_{13})$ and $\Sigma_{14} = \Gamma_c(X_{14})$. See Figure 4.

The fourth common point of these conics is X_{476} (Tixier point) on the circumcircle. The centers and other real foci are not mentioned in the current edition of [6] and their coordinates are rather complicated. The focal axes are those of the Simmons inconics.

[1] X_{1648} is the tripolar centroid of X_{523} i.e. the isobarycenter of the traces of the line $X_{115}X_{125}$. It lies on the line GK.

[2] The angular coordinates of a Fermat point of $P_aP_bP_c$ are the same when they are taken either with respect to $P_aP_bP_c$ or with respect to ABC.

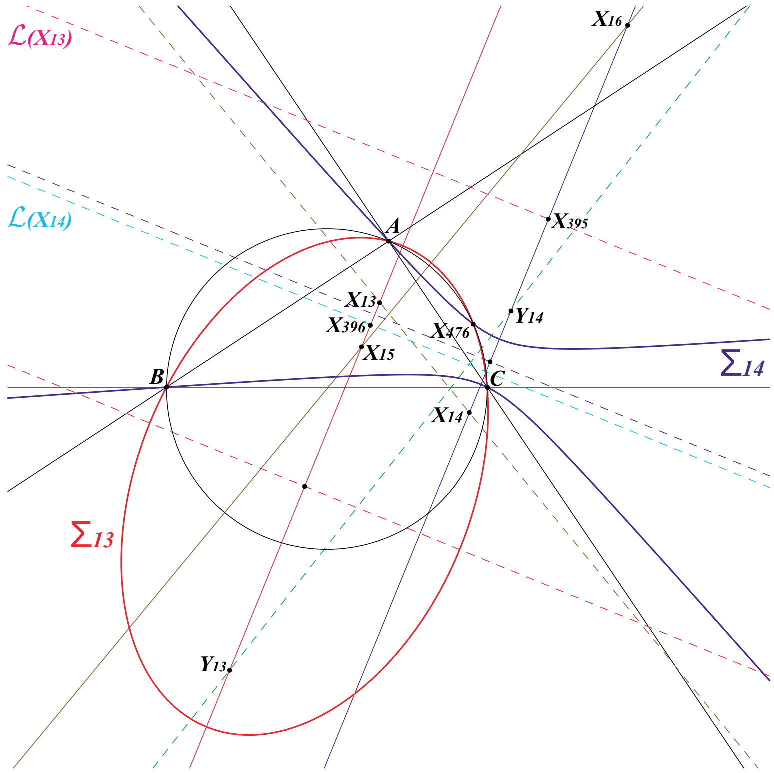

Figure 4. The two Simmons circumconics Σ_{13} and Σ_{14}

A digression: there are in general four circumconics with given focus F. Let $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ the circles passing through F with centers A, B, C. These circles have two by two six centers of homothety and these centers are three by three collinear on four lines. One of these lines is the trilinear polar $\mathcal{L}(Q)$ of the interior point $Q = \frac{1}{AF} : \frac{1}{BF} : \frac{1}{CF}$ and the remaining three are the sidelines of the cevian triangle of Q. These four lines are the directrices of the sought circumconics and their construction is therefore easy to realize. See Figure 5.

This shows that one can find six other circumconics with focus at a Fermat point but, in this case, this focus is not the perspector.

5. Some related loci

We now generalize some of the particularities of the Simmons inconics and present several higher degree curves which all contain the Fermat points.

5.1. *Directrices and trilinear polars.* We have seen that these Simmons inconics are quite remarkable in the sense that the directrix corresponding to the perspector/focus F (which is the polar line of F in the conic) is also the trilinear polar of F. The generalization gives the following

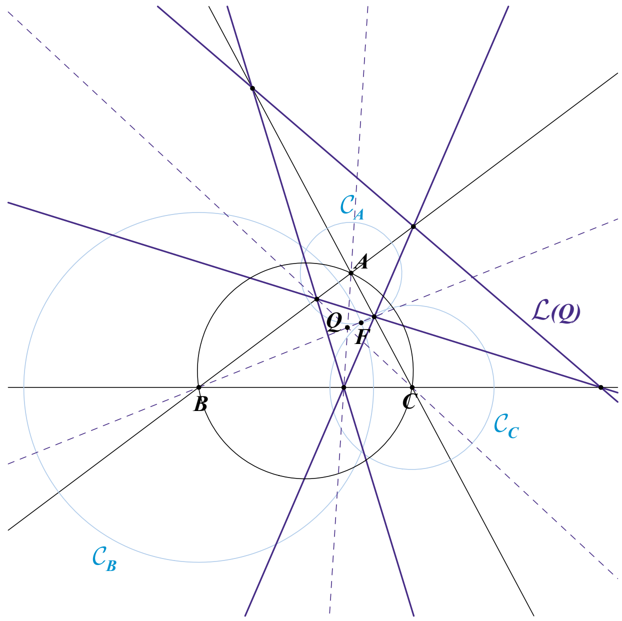

Figure 5. Directrices of circumconics with given focus

Theorem 5. *The locus of the focus F of the inconic such that the corresponding directrix is parallel to the trilinear polar of F is the Euler-Morley quintic Q003.*

Q003 is a very remarkable curve with equation

$$\sum_{\text{cyclic}} a^2(S_B\,y - S_C\,z)y^2z^2 = 0$$

which (at the time this paper is written) contains 70 points of the triangle plane. See [3] and [4].

In Figure 6, we have the inconic with focus F at one of the extraversions of X_{1156} (on the Euler-Morley quintic).

5.2. *Perspector lying on one axis.* The Simmons inconics (or circumconics) have their perspectors at a focus hence on an axis. More generally,

Theorem 6. *The locus of the perspector P of the inconic (or circumconic) such that P lies on one of its axis is the Stothers quintic Q012.*

The Stothers quintic Q012 has equation

$$\sum_{\text{cyclic}} a^2(y-z)(x^2-yz)yz = 0.$$

Q012 is also the locus of point M such that the circumconic and inconic with same perspector M have parallel axes, or equivalently such that the pencil of conics generated by these two conics contains a circle. See [3].

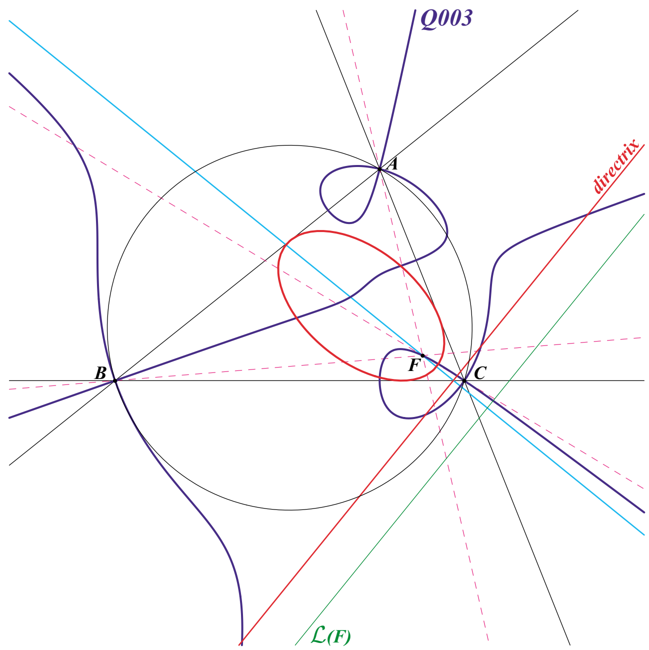

Figure 6. An inconic with directrix parallel to the trilinear polar of the focus

The center of the inconic in Theorem 6 must lie on the complement of the isotomic conjugate of Q012, another quartic with equation

$$\sum_{\text{cyclic}} a^2(y+z-x)(y-z)(y^2+z^2-xy-xz)=0.$$

In Figure 7, we have the inconic with perspector X_{673} (on the Stothers quintic) and center X_{3008}.

The center of the circumconic in Theorem 6 must lie on a septic which is the $G-$Ceva conjugate of Q012.

5.3. *Perspector lying on the focal axis.* The focus F, its isogonal conjugate F^* (the other focus), the center Ω (midpoint of FF^*) and the perspector P (the isotomic conjugate of the anticomplement of Ω) of the inconic may be seen as a special case of collinear points. More generally,

Theorem 7. *The locus of the focus F of the inconic such that F, F^* and P are collinear is the bicircular isogonal sextic* Q039*.*

Q039 is also the locus of point P whose pedal triangle has a Brocard line passing through P. See [3].

Remark. The locus of P such that the polar lines of P and its isogonal conjugate P^* in one of the Simmons inconics are parallel are the two isogonal pivotal cubics K129a and K129b.

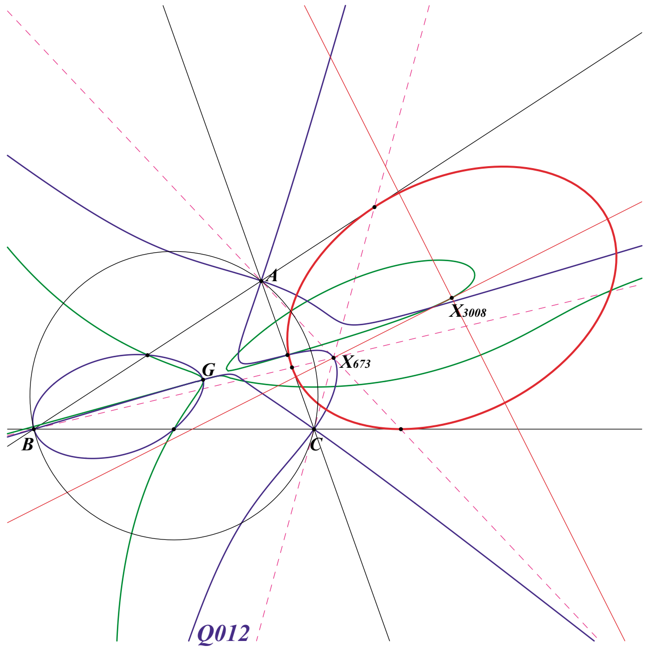

Figure 7. An inconic with perspector on one axis

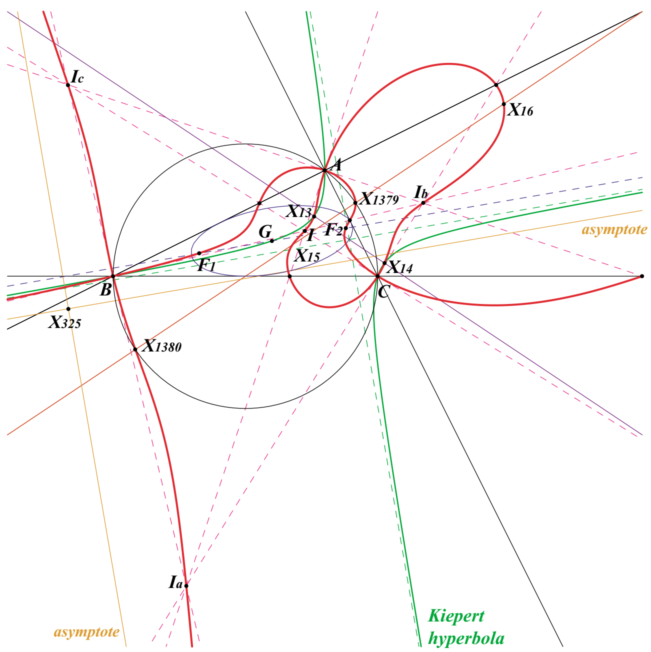

Figure 8. The bicircular isogonal sextic Q039

More precisely, with the conic $\mathcal{S}_{13}$ we obtain K129b = $\mathrm{p}\mathcal{K}(K, X_{396})$ and with the conic $\mathcal{S}_{14}$ we obtain K129a = $\mathrm{p}\mathcal{K}(K, X_{395})$. See [3].

6. Appendices

6.1. In his paper [7], T. C. Simmons has shown that the eccentricity of Σ_{13} is twice that of $\mathcal{S}_{13}$. This is also true for Σ_{14} and $\mathcal{S}_{14}$. The following table gives these eccentricities.

conic	eccentricity
$\mathcal{S}_{13}$	$\dfrac{1}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
$\mathcal{S}_{14}$	$\dfrac{1}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{13}	$\dfrac{2}{\sqrt{2(\cot\omega+\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$
Σ_{14}	$\dfrac{2}{\sqrt{2(\cot\omega-\sqrt{3})}} \times \dfrac{OH}{\sqrt{\Delta}}$

where ω is the Brocard angle, Δ the area of ABC and OH the distance between O and H.

6.2. Since Σ_{13} and $\mathcal{S}_{13}$ (or Σ_{14} and $\mathcal{S}_{14}$) have the same focus and the same directrix, it is possible to find infinitely many homologies (perspectivities) transforming these two conics into concentric circles with center X_{13} (or X_{14}) and the radius of the first circle is twice that of the second circle.

The axis of such homology must be parallel to the directrix and its center must be the common focus. Furthermore, the homology must send the directrix to the line at infinity and, for example, must transform the point P_1 (or P_2, see remark 3 at the end of §3) into the infinite point X_{30} of the Euler line or the line $X_{13}X_{15}$.

Let Δ_1 and Δ_2 be the two lines with equations

$$\sum_{\text{cyclic}} \left(b^2+c^2-2a^2+\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}\right) x = 0$$

and

$$\sum_{\text{cyclic}} \left(b^2+c^2-2a^2-\sqrt{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}\right) x = 0.$$

Δ_1 and Δ_2 are the tangents to the Steiner inellipse which are perpendicular to the Euler line. The contacts lie on the line GK and on the circle with center G passing through X_{115}, the center of the Kiepert hyperbola. Δ_1 and Δ_2 meet the Euler line at two points lying on the circle with center G passing through X_{125}, the center of the Jerabek hyperbola.

If we take one of these lines as an axis of homology, the two Simmons circumconics Σ_{13} and Σ_{14} are transformed into two circles Γ_{13} and Γ_{14} having the same radius. Obviously, the two Simmons inconics are also transformed into two circles having the same radius. See Figure 9.

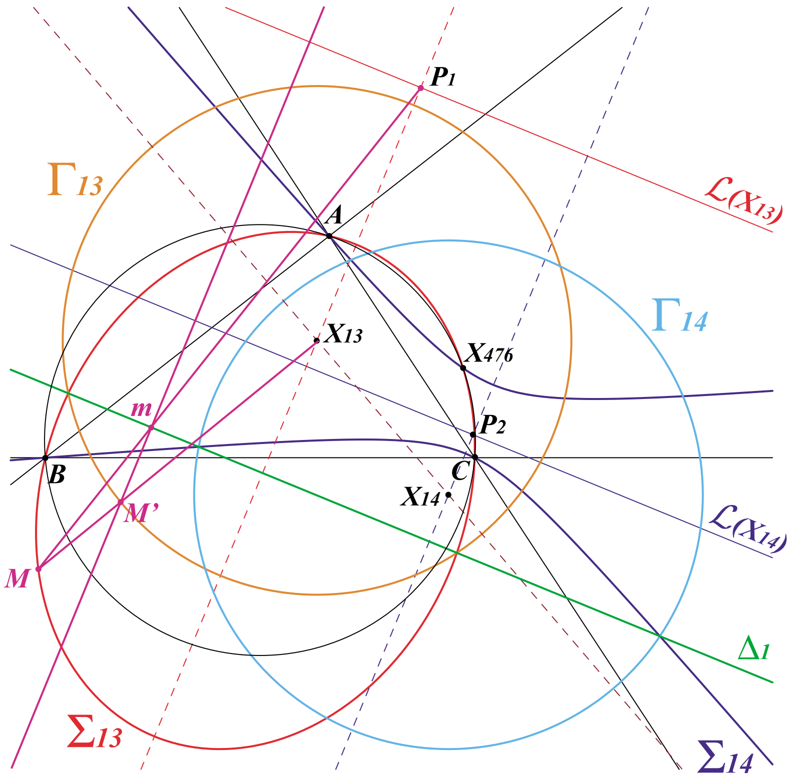

Figure 9. Homologies and circles

For any point M on Σ_{13}, the line MP_1 meets Δ_1 at m. The parallel to the Euler line at m meets the line MX_{13} at M' on Γ_{13}. A similar construction with M on Σ_{14} and P_2 instead of P_1 will give Γ_{14}.

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A Synthetic Proof and Generalization of Bellavitis' Theorem

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Abstract. In this note we give a synthetic proof of Bellavitis' theorem and then generalizing this theorem, for not only convex quadrilaterals, we give a synthetic geometric proof for both theorems direct and converse, as Eisso Atzema proved, by trigonometry, for the convex case [1]. From this approach evolves clearly the connection between hypothesis and conclusion.

1. Bellavitis' theorem

Eisso J. Atzema has recently given a trigonometric proof of Bellavitis' theorem [1]. We present a synthetic proof here. Inside a convex quadrilateral $ABCD$, let the diagonal AC form with one pair of opposite sides angles w_1, w_3. Similarly let the angles inside the quadrilateral that the other diagonal BD forms with the remaining pair of opposite sides be w_2, w_4.

Theorem 1 (Bellavitis, 1854)**.** *If the side lengths of a convex quadrilateral $ABCD$ satisfy $AB \cdot CD = BC \cdot DA$, then $w_1 + w_2 + w_3 + w_4 = 180°$*

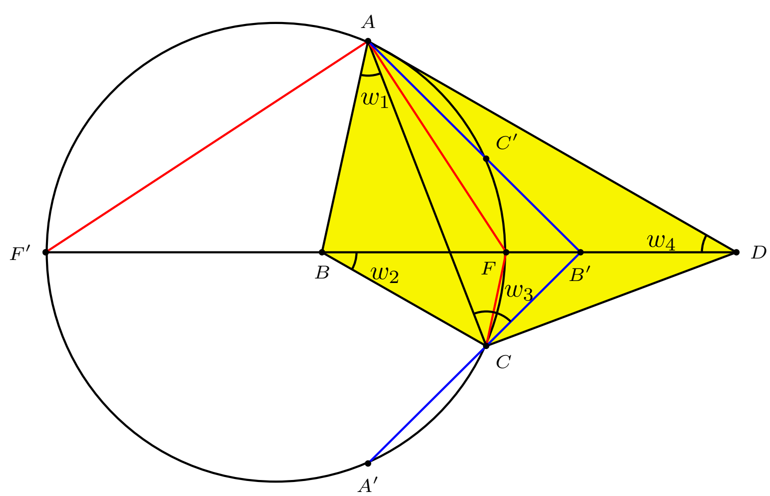

Figure 1

Proof. If $AB = AD$ then $BC = CD$ and AC is the perpendicular bisector of BD. Hence $ABCD$ is a kite, and it is obvious that $w_1 + w_2 + w_3 + w_4 = 180°$.

If $ABCD$ is not a kite, then from $AB \cdot CD = BC \cdot DA$, we have $\frac{AB}{AD} = \frac{CB}{CD}$. Hence, C lies on the A-Apollonius circle of triangle ABD. See Figure 1. This

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circle has diameter FF', where AF and AF' are the internal and external bisectors of angle BAD and CF is the bisector of angle BCD. The reflection of AC in AF meets the Apollonius circle at C'. Since arc CF = arc FC', the point C' is the reflection of C in BD. Similarly the reflection of AC in CF meets the Apollonius circle at A' that is the reflection of A in BD. Hence the lines AC', CA' are reflections of each other in BD and are met at a point B' on BD. So we have

$$w_2 + w_3 = w_2 + \angle BCB' = \angle CB'D = \angle AB'D \tag{1}$$
$$w_1 + w_4 = \angle B'AD + w_4 = \angle BB'A. \tag{2}$$

From (1) and (2) we get

$$w_1 + w_2 + w_3 + w_4 = \angle BB'A + \angle AB'D = 180^\circ.$$

□

2. A generalization

There is actually no need for $ABCD$ to be a convex quadrilateral. Since it is clear that $w_1 + w_2 + w_3 + w_4 = 180^\circ$ for a cyclic quadrilateral, we consider non-cyclic quadrilaterals below. We make use of oriented angles and arcs. Denote by $\theta(XY, XZ)$ the oriented angle from XY to XZ. We continue to use the notation

$$w_1 = \theta(AB, AC), \qquad w_3 = \theta(CD, CA),$$
$$w_2 = \theta(BC, BD), \qquad w_4 = \theta(DA, DB).$$

Theorem 2. *In an arbitrary noncyclic quadrilateral $ABCD$, the side lengths satisfy the equality $AB \cdot CD = BC \cdot DA$ if and only if*

$$w_1 + w_2 + w_3 + w_4 = \pm 180^\circ.$$

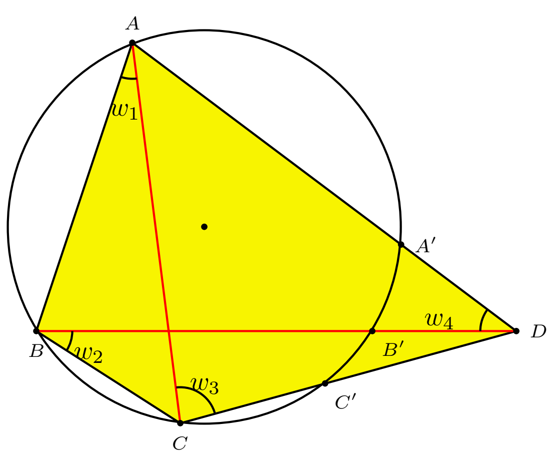

Figure 2

Proof. Since $ABCD$ is not a cyclic quadrilateral the lines DA, DB, DC meet the circumcircle of triangle ABC at the distinct points A', B', C'. The triangle $A'B'C'$ is the circumcevian triangle of D relative to ABC. Note that

$$\begin{aligned} 2w_1 =& \text{arc } BC, \\ 2w_2 =& \text{arc } CC' + \text{arc } C'B', \\ 2w_3 =& \text{arc } C'A, \\ 2w_4 =& \text{arc } AB + \text{arc} A'B'. \end{aligned}$$

From these, $w_1 + w_2 + w_3 + w_4 = \pm 180°$ if and only if

$$(\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB) + \text{arc } C'B' + \text{arc } A'B' = \pm 360°.$$

Since $\text{arc } BC + \text{arc } CC' + \text{arc } C'A + \text{arc } AB = \pm 360°$, the above condition holds if and only if $\text{arc } C'B' = \text{arc } B'A'$. This means that the circumcevian triangle of D is isosceles, *i.e.*,

$$B'A' = B'C'. \tag{3}$$

It is well known that $A'B'C'$ is similar to with the pedal triangle $A''B''C''$ of D. See [2, §7.18] The condition (3) is equivalent to

$$B''A'' = B''C''.$$

This, in turn, is equivalent to the fact that D lies on the B-Apollonius circle of ABC because for a pedal triangle we know that

$$B''A'' = DC \cdot \sin C = B''C'' = DA \cdot \sin A$$

or

$$\frac{DC}{DA} = \frac{\sin A}{\sin C} = \frac{BC}{BA}.$$

From this we have $AB \cdot CD = BC \cdot DA$. □

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On Some Theorems of Poncelet and Carnot

Huub P.M. van Kempen

Abstract. Some relations in a complete quadrilateral are derived. In connection with these relations some special conics related to the angular points and sides of the quadrilateral are discussed. A theorem of Carnot valid for a triangle is extended to a quadrilateral.

1. Introduction

The scope of Euclidean Geometry was substantially extended during the seventeenth century by the introduction of the discipline of Projective Geometry. Until then geometers were mainly concentrating on the *metric* (or *Euclidean*) properties in which the measure of distances and angles is emphasized. Projective Geometry has no distances, no angles, no circles and no parallelism but concentrates on the *descriptive* (or *projective*) properties. These properties have to do with the relative positional connection of the geometric elements in relation to each other; the properties are unaltered when the geometric figure is subjected to a projection.

Projective Geometry was started by the Grecian mathematician Pappus of Alexandria . After more than thirteen centuries it was continued by two Frenchmen, Desargues and his famous pupil Pascal. The latter one published in 1640 his well-known *Essay pour les coniques*. This short study contains the well-known *hexagrammum mysticum*, nowadays known as Pascal's Theorem. Meanwhile, the related subject of perspective had been studied by architects and artists (Leonardo da Vinci). The further development of Projective Geometry was about two hundred years later, mainly by a French group of mathematicians (Poncelet, Chasles, Carnot, Brianchon and others).

An important tool in Projective Geometry is a semi-algebraic instrument, called the *cross ratio*. This topic was introduced, independently of each other, by Möbius (1827) and Chasles (1829).

In this article we present an (almost forgotten) result of Poncelet [4], obtained in an alternative way and we derive some associated relations (Theorem 1). Furthermore, we extend a theorem by Carnot [1] from a triangle to a complete quadrilateral (Theorem 3).

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We take as starting-point the theorems of Ceva and Pappus-Pascal. The first one is a close companion of the theorem of the Grecian mathematician Menelaus. In the analysis we will follow as much as possible the purist/synthetic approach. It will be shown that this approach leads to surprising results derived along unexpected lines.

2. Proof and extension of a Theorem by Poncelet

Theorem 1. *Let the diagonal points of a complete quadrilateral ABCD be P, Q and R. Let the intersections of PQ with AD and BC be H and F respectively and those of PR with CD and AB be G and E respectively* (*Figure 1*). *Then*

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA}=1, \tag{1}$$

$$\frac{AP}{PC}\cdot\frac{CG}{GD}\cdot\frac{DP}{PB}\cdot\frac{BE}{EA}=1, \tag{2}$$

$$\frac{BP}{PD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}\cdot\frac{CF}{FB}=1. \tag{3}$$

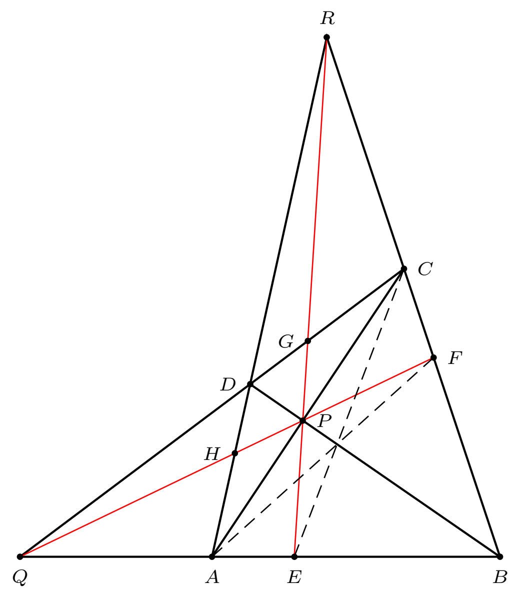

Figure 1

Proof. We apply the Pappus-Pascal theorem to the triples (Q, A, E) and (R, C, F) and find that in triangle ABC the lines AF, BP and CE are concurrent so that by Ceva's theorem

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CP}{PA}=1. \tag{4}$$

Similarly with the triples (Q, G, C) and (R, H, A) we find

$$\frac{CG}{GD}\cdot\frac{DH}{HA}\cdot\frac{AP}{PC}=1. \tag{5}$$

Relation (1) immediately follows from (4) and (5). Again in the same way with triples (E, B, Q) and (H, D, R) we find that

$$\frac{AE}{EB} \cdot \frac{BP}{PD} \cdot \frac{DH}{HA} = 1. \tag{6}$$

(2) follows from (5) and (6), and (3) follows from (4) and (6). □

Poncelet [4] has derived relation (1) by using cross ratios.

We now consider a special case of Theorem 1, taking a convex quadrilateral $ABCD$ in which $AB + CD = BC + DA$, so that it is circumscriptable (Figure 2). Let E', F', G' and H' be the points of tangency of the incircle with AB, BC, CD and DA respectively. Clearly a relation similar to (1) holds:

$$\frac{AE'}{E'B} \cdot \frac{BF'}{F'C} \cdot \frac{CG'}{G'D} \cdot \frac{DH'}{H'A} = 1. \tag{7}$$

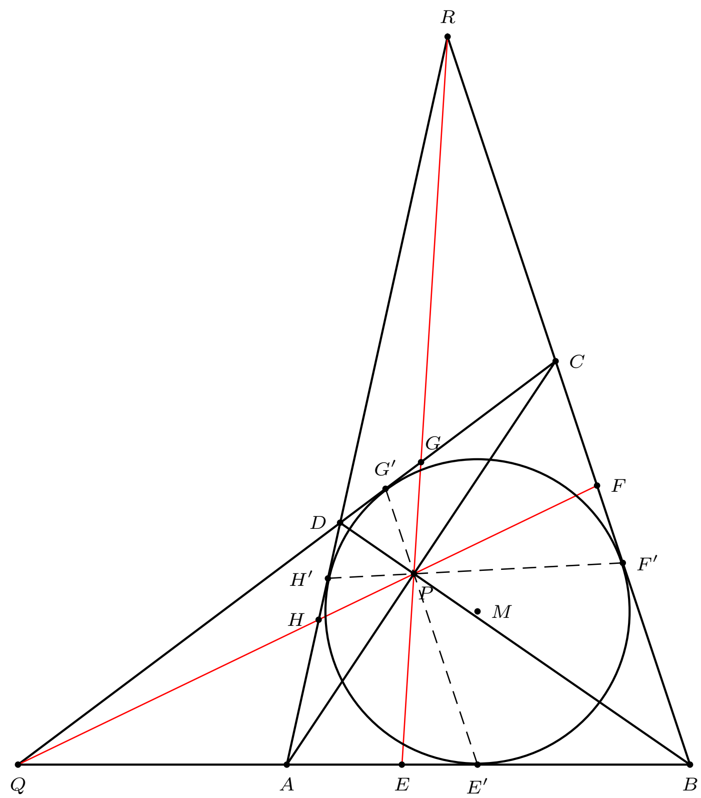

Figure 2

It is well known [5] that the point of intersection of $E'G'$ and $F'H'$ is P. This can be seen for a general quadrilateral with an inscribed conic from subsequent application of Brianchon's theorem to hexagons $AE'BCG'D$ and $BF'CDH'A$. See for instance [2, p.49]. This raises the questions whether or not a relation similar to (1) will hold. We will examine this problem by using Ceva's theorem.

3. Further Analysis

We start with a given quadrilateral $ABCD$ where E and G are arbitrary points on the lines AB and CD respectively. We then construct points F_1 and H_1 on BC and AD respectively such that (1) holds. We can do so by the following construction (Figure 3).

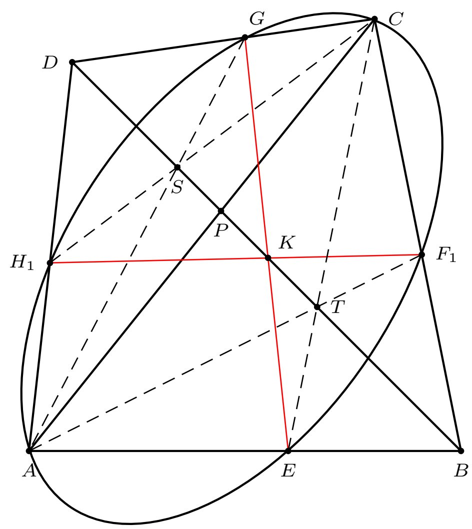

Figure 3

First we consider the triangles ABC and ADC. Let $T = CE \cap BD$ and $F_1 = BC \cap AT$. By Ceva's theorem we have in triangle ABC

$$\frac{AE}{EB} \cdot \frac{BF_1}{F_1C} \cdot \frac{CP}{PA} = 1. \tag{8}$$

Now if $S = AG \cap DB$ and $H_1 = AD \cap CS$, then Ceva's theorem applied to triangle ADC gives

$$\frac{CG}{GD} \cdot \frac{DH_1}{H_1A} \cdot \frac{AP}{PC} = 1. \tag{9}$$

By multiplication of (8) and (9) we find the desired equivalence of (1).

Theorem 2. *If in the quadrilateral $ABCD$ the points E, F_1, G and H_1 lie on AB, BC, CD and DA respectively such that $S = AG \cap CH_1$ and $T = AF_1 \cap CE$ lie on BD, then the points A, E, F_1, C, G and H_1 lie on a conic and $K = EG \cap F_1H_1$ lies on BD.*

Proof. Here we have to switch to the field of Projective Geometry. We will use the cross ratio of pencils in relation to the cross-ratio of ranges. These concepts are extensively described by Eves [3]. Now consider the two pencils (AH_1, AG, AF_1, AE) and (CH_1, CG, CF_1, CE) in Figure 3. We have the cross-ratio equality between ranges and pencils:

$$A(H_1, G; F_1, E) = (D, S; T, B) = (S, D; B, T) = C(H_1, G; F_1, E). \qquad (10)$$

From this equality we see that A, E, F_1, C, G and H_1 lie on a conic. Applying Pascal's theorem to the hexagon AF_1H_1CEG we find that the diagonal BD is the Pascal line and consequently the points S, K and T are collinear. $\square$

By using triangle ABD and triangle CBD instead of triangle ABC and triangle ADC as above, we can also construct F_2 and H_2 such that relation (1) holds (Figure 4). Now we apply Theorem 2, finding that B, F_2, G, D, H_2 and E lie on a conic. Using Pascal's theorem for the hexagon $BGEDF_2H_2$ we find that $L = EG \cap F_2H_2$ lies on AC.

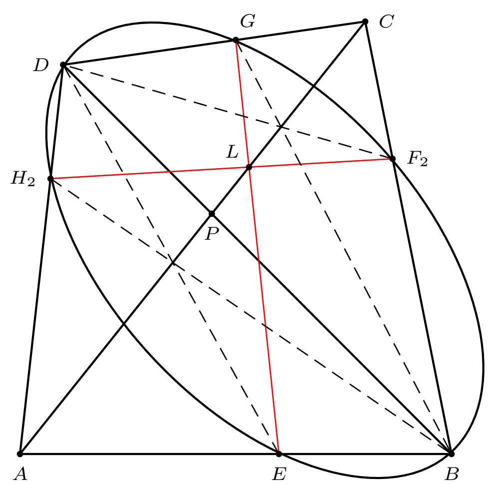

Figure 4

With the help of Theorem 2 we prove an extension of Carnot's theorem in [1] for a triangle to a complete quadrilateral.

Theorem 3. *If in the quadrilateral $ABCD$ the points E, F, G and H lie on AB, BC, CD and DA respectively and EG and FH concur in $P = AC \cap BD$, then* (*1*) *is satisfied if and only if there is a conic inscribed in quadrilateral $ABCD$, which touches its sides in the points E, F, G and H.*

Proof. Assume that relation (1) holds. By Theorem 2 we know that $BFGDHE$ and $AEFCGH$ lie on two conics. Let $V = DE \cap BH$ and $W = DF \cap BG$. First we apply Desargues' theorem to triangle GFC and triangle EHV (Figure 5).

The lines GE, FH and CV concur in P. This means that the intersection points of the corresponding sides are collinear. So $U = GF \cap EH$, $B = FC \cap HV$ and $D = GC \cap EV$ are collinear. Next, consider the unique conic Γ through E, F, G and H which is tangent to CD at G. We examine the direction of the tangent to Γ at the point H. Therefore we consider the hexagon $GGEHHF$. We find that

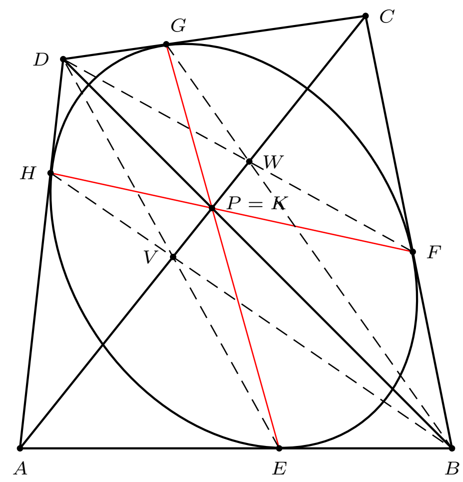

Figure 5

$GE \cap HF = P$ and $GF \cap EH = U$. Since both P and U are collinear with B and D, the line BD is the Pascal line. This means that the tangents to Γ at G and H intersect on BD, which implies that AD is the tangent to Γ at H.

In the same way we prove that the lines AB and BC are tangent to Γ at E and F respectively, which proves the sufficiency part.

Now assume that a conic is tangent to the sides of quadrilateral $ABCD$ at the points E, F, G and H. Note that of course EG and FH intersect in P, as stated earlier. With fixed E, F and G there is exactly one point H^* on AD such that the equivalent version of relation (1) holds. By the sufficiency part this leads to a conic tangent to the sides at E, F, G, and H^*. As these two conics have three double points in common, they must be the same conic. This leads to the conclusion that H and H^* are in fact the same point. This proves the necessity part. □

Applying Theorem 3 to the results of Theorem 2 we find

Corollary 4. *If in the quadrilateral $ABCD$ of Theorem 2 the lines EG and F_1H_1 concur in P, where $P = AC \cap BD$, then F_1H_1 of Figure 3 and F_2H_2 of Figure 4 coincide.*

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The Droz-Farny Theorem and Related Topics

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Abstract. At each point P of the Euclidean plane Π, not on the sidelines of a triangle $A_1A_2A_3$ of Π, there exists an involution in the pencil of lines through P, such that each pair of conjugate lines intersect the sides of $A_1A_2A_3$ in segments with collinear midpoints. If $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution (where orthogonal lines correspond) and we find the well-known Droz-Farny Theorem, which says that any two orthogonal lines through H intersect the sides of the triangle in segments with collinear midpoints. In this paper we investigate two closely related loci that have a strong connection with the Droz-Farny Theorem. Among examples of these loci we find the circumcirle of the anticomplementary triangle and the Steiner ellipse of that triangle.

1. The Droz-Farny Theorem

Many proofs can be found for the original Droz-Farny Theorem (for some recent proofs, see [1], [3]). The proof given in [3] (and [5]) probably is one of the shortest: Consider, in the Euclidean plane Π, the pencil $\mathcal{B}$ of parabola's with tangent lines the sides $a_1 = A_2A_3, a_2 = A_3A_1, a_3 = A_1A_2$ of $A_1A_2A_3$, and the line l at infinity. Let P be any point of Π, not on a sideline of $A_1A_2A_3$, and not on l, and consider the tangent lines r and r' through P to a non-degenerate parabola $\mathcal{P}$ of this pencil $\mathcal{B}$. A variable tangent line of $\mathcal{P}$ intersects r and r' in corresponding points of a projectivity (an affinity, i.e. the points at infinity of r and r' correspond), and from this it follows that the line connecting the midpoints of the segments determined by r and r' on a_1 and a_2, is a tangent line of $\mathcal{P}$, through the midpoint of the segment determined on a_3 by r and r'. Next, by the Sturm-Desargues Theorem, the tangent lines through P to a variable parabola of the pencil $\mathcal{B}$ are conjugate lines in an involution $\mathcal{I}$ of the pencil of lines through P, and this involution $\mathcal{I}$ contains in general just one orthogonal conjugate pair. In the following we call these orthogonal lines through P, the orthogonal Droz-Farny lines through P.

Remark that (PA_i, line through P parallel to a_i), $i = 1, 2, 3$ are the tangent lines through P of the degenerate parabola's of the pencil $\mathcal{B}$, and thus are conjugate pairs in the involution $\mathcal{I}$. From this it follows that in the case where $P = H$, the orthocenter of $A_1A_2A_3$, this involution becomes the orthogonal involution in the pencil of lines through H, and we find the Droz-Farny Theorem.

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Two other characterizations of the orthogonal Droz-Farny lines through P are obtained as follows: Let X and Y be the points at infinity of the orthogonal Droz-Farny lines through P. Since the two triangles $A_1A_2A_3$ and PXY are circumscribed triangles about a conic (a parabola of the pencil $\mathcal{B}$), their vertices are six points of a conic, namely the rectangular hyperbola through A_1, A_2, A_3, and P (and also through H, since any rectangular hyperbola through A_1, A_2, and A_3, passes through H). It follows that the orthogonal Droz-Farny lines through P are the lines through P which are parallel to the (orthogonal) asymptotes of this rectangular hyperbola through A_1, A_2, A_3, P, and H.

Next, since, if $P = H$, the involution $\mathcal{I}$ is the orthogonal involution, the directrix of any parabola of the pencil $\mathcal{B}$ passes through H, and the orthogonal Droz-Farny lines through any point P are the orthogonal tangent lines through P of the parabola, tangent to a_1, a_2, a_3, and with directrix PH.

2. The first locus

Let us recall some basic properties of trilinear (or normal) coordinates (see for instance [4]). Trilinear coordinates (x_1, x_2, x_3), with respect to a triangle A_1A_2A with side-lenghts l_1, l_2, l_3, of any point P of the Euclidean plane, are homogeneous projective coordinates, in the Euclidean plane, for which the vertices A_1, A_2, A_3 are the basepoints and the incenter I of the triangle the unit point. The line at infinity has in trilinear coordinates the equation $l_1x_1 + l_2x_2 + l_3x_3 = 0$. The centroid G of $A_1A_2A_3$ has trilinear coordinates $(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the orthocenter H is $(\frac{1}{\cos A_1}, \frac{1}{\cos A_2}, \frac{1}{\cos A_3})$, the circumcenter O is $(\cos A_1, \cos A_2, \cos A_3)$, the incenter I is $(1, 1, 1)$, and the Lemoine (or symmedian) point K is (l_1, l_2, l_3).

If X has trilinear coordinates (x_1, x_2, x_3) with respect to $A_1A_2A_3$, and if d_i is the "'signed"' distance from X to the side a_i (i.e. d_i is positive or negative, according as X lies on the same or opposite side of a_i as A_i), then, if F is the area of $A_1A_2A_3$, we have $d_i = \frac{2Fx_i}{l_1x_1+l_2x_2+l_3x_3}$, $i = 1, 2, 3$, and (d_1, d_2, d_3) are the *actual trilinear coordinates* of X with respect to $A_1A_2A_3$. Remark that $l_1d_1 + l_2d_2 + l_3d_3 = 2F$.

Our first locus is defined as follows ([5]):

Consider a fixed point P, not on a sideline of $A_1A_2A_3$, and not at infinity, with actual trilinear coordinates $(\delta_1, \delta_2, \delta_3)$ with respect to $A_1A_2A_3$, and suppose that s is a given real number and the set of points of the plane for which the distances d_i from (x_1, x_2, x_3) to a_i are connected by the equation

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = s. \tag{1}$$

Using $d_i = \frac{2F_ix_i}{l_1x_1+l_2x_2+l_3x_3}$, we see that the set is given by the equation

$$4F^2(\frac{l_1}{\delta_1}x_1^2 + \frac{l_2}{\delta_2}x_2^2 + \frac{l_3}{\delta_3}x_3^2) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0, \tag{2}$$

or, if we use general trilinear coordinates (p_1, p_2, p_3) of P:

$$2F(\frac{l_1}{p_1}x_1^2 + \frac{l_2}{p_2}x_2^2 + \frac{l_3}{p_3}x_3^2)(l_1p_1 + l_2p_2 + l_3p_3) - s(l_1x_1 + l_2x_2 + l_3x_3)^2 = 0. \quad (3)$$

We denote this conic by $\mathcal{K}(P, \Delta, s)$: it is the conic determined by (1) and (2), where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with regard to $\Delta = A_1A_2A_3$, and also by (3), where (p_1, p_2, p_3) are any triple of trilinear coordinates of P with regard to Δ, and by the value of s. For P and Δ fixed and s allowed to vary, the conics $\mathcal{K}(\mathcal{P}, \Delta, s)$ belong to a pencil, and a straightforward calculation shows that all conics of this pencil have center P, and have the same points at infinity, which means that they have the same asymptotes and the same axes.

The conics $\mathcal{K}(P, \Delta, s)$ can be (homothetic) ellipses or hyperbola's: this depends on the location of P with regard to Δ, and again a straightforward calculation shows that we find ellipses or hyperbola's, according as the product $\delta_1\delta_2\delta_3 > 0$ or < 0.

Next, the *medial triangle* of $\Delta = A_1A_2A_3$ is the triangle whose vertices are the midpoints of the sides of Δ, and the *anticomplementary triangle* $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ is the triangle whose medial triangle is Δ. An easy calculation shows that the trilinear coordinates of the vertices A_1^{-1}, A_2^{-1}, and A_3^{-1} of this anticomplementary triangle are $(-l_2l_3, l_3l_1, l_1l_2)$, $(l_2l_3, -l_3l_1, l_1l_2)$, and $(l_2l_3, l_3l_1, -l_1l_2)$, respectively.

Lemma 1. *The locus $\mathcal{K}(P, \Delta, S)$ of the points for which the distances d_1, d_2, d_3 to the sides a_1, a_2, a_3 of $\Delta = A_1A_2A_3$ are connected by*

$$\frac{l_1}{\delta_1}d_1^2 + \frac{l_2}{\delta_2}d_2^2 + \frac{l_3}{\delta_3}d_3^2 = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}) \ = S,$$

where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of a given point P, is the conic with center P, and circumscribed about the anticomplementary triangle $A_1^{-1}A_2^{-1}A_3^{-1}$ of Δ.

Proof. Substituting the coordinates $(-\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, -\frac{1}{l_2}, \frac{1}{l_3})$, or $(\frac{1}{l_1}, \frac{1}{l_2}, -\frac{1}{l_3})$ of A_1^{-1}, A_2^{-1}, and A_3^{-1}, in (2), we find immediately that

$$s = S = 4F^2(\frac{1}{l_1\delta_1} + \frac{1}{l_2\delta_2} + \frac{1}{l_3\delta_3}).$$

□

3. The second locus

We work again in the Euclidean plane Π, with trilinear coordinates with respect to $\Delta = A_1A_2A_3$. Assume that $P(p_1, p_2, p_3)$ is a point of Π, not at infinity and not on a sideline of Δ. We look for the locus of the points Q of Π, such that the points $Q_i = q_i \cap a_i, \ i = 1, 2, 3$, where q_i is the line through Q, parallel to PA_i, are collinear. This locus was the subject of the paper [2]. Since $l_1x_1 + l_2x_2 + l_3x_3 = 0$ is the equation of the line at infinity, the point at infinity of PA_1 has coordinates $(l_2p_2 + l_3p_3, -l_1p_2, -l_1p_3)$, and if we give Q the coordinates (x_1, x_2, x_3), we find after an easy calculation that Q_1 has coordinates $(0, l_1p_2x_1 +$

$x_2(l_2p_2+l_3p_3), x_1l_1p_3+x_3(l_2p_2+l_3p_3))$. In the same way, we find for the coordinates of Q_2, and of Q_3: $(x_1(l_1p_1+l_3p_3)+p_1x_2l_2, 0, p_3x_2l_2+x_3(l_1p_1+l_3p_3))$, and $(x_1(l_1p_1+l_2p_2)+x_3p_1l_3, x_2(l_1p_1+l_2p_2)+x_3p_2l_3, 0)$, respectively.

Next, after a rather long calculation, and deleting the singular part $l_1x_1+l_2x_2+l_3x_3=0$, the condition that Q_1, Q_2, and Q_3 are collinear, gives us the following equation for the locus of the point Q:

$$p_3(l_1p_1+l_2p_2)x_1x_2+p_1(l_2p_2+l_3p_3)x_2x_3+p_2(l_3p_3+l_1p_1)x_3x_1=0. \tag{4}$$

This is our second locus, and we denote this conic, circumscribed about $A_1A_2A_3$, by $\mathcal{C}(P,\Delta)$, where $\Delta=A_1A_2A_3$, and where P is the point with trilinear coordinates (p_1,p_2,p_3) with regard to Δ.

Lemma 2. *The center M of $\mathcal{C}(P,\Delta)$ has trilinear coordinates*

$$(l_2l_3(l_2p_2+l_3p_3), l_3l_1(l_3p_3+l_1p_1), l_1l_2(l_1p_1+l_2p_2)).$$

It is the image $f(P)$, where f is the homothety with center G, the centroid of Δ, and homothetic ratio $-\frac{1}{2}$, or, in other words: $2\vec{GM}=-\vec{GP}$.

Proof. An easy calculation shows that the polar point of this point M with regard to the conic (4) is indeed the line at infinity, with equation $l_1x_1+l_2x_2+l_3x_3=0$. Moreover, if P_∞ is the point at infinity of the line PG, the equation $2\vec{GM}=-\vec{GP}$ is equivalent with the equality of the cross-ratio $(MPGP_\infty)$ to $-\frac{1}{2}$. Next, choose on the line PG homogeneous projective coordinates with basepoints $P(1,0)$ and $G(0,1)$, and give P_∞ coordinates (t_1,t_2) (thus $P_\infty=t_1P+t_2G$), then $(t_1,t_2)=(-3, l_1p_1+l_2p_2+l_3p_3)$ and the projective coordinates (t'_1,t'_2) of M follow from

$$(MPGP_\infty)=\frac{\begin{vmatrix} t'_1 & t'_2 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}:\frac{\begin{vmatrix} t'_1 & t'_2 \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -3 & l_1p_1+l_2p_2+l_3p_3 \end{vmatrix}}=-\frac{1}{2},$$

which gives $(t'_1,t'_2)=(-1, l_1p_1+l_2p_2+l_3p_3)$. $\square$

Remark that the second part of the proof also follows from the connection between trilinears (x_1,x_2,x_3) for a point with respect to $A_1A_2A_3$ and trilinears (x'_1,x'_2,x'_3) for the same point with respect to the medial triangle of $A_1A_2A_3$ (see [4, p.207]):

$$\begin{cases} x_1=l_2l_3(l_2x'_2+l_3x'_3) \\ x_2=l_3l_1(l_3x'_3+l_1x'_1) \\ x_3=l_1l_2(l_1x'_1+l_2x'_2). \end{cases}$$

4. The connection between the Droz-Farny -lines and the conics

Recall from §2 that

$$S=4F^2\left(\frac{1}{l_1\delta_1}+\frac{1}{l_2\delta_2}+\frac{1}{l_3\delta_3}\right)=2F(l_1p_1+l_2p_2+l_3p_3)\left(\frac{1}{l_1p_1}+\frac{1}{l_2p_2}+\frac{1}{l_3p_3}\right),$$

where F is the area of $\Delta = A_1A_2A_3$, (p_1, p_2, p_3) are trilinear coordinates of P with regard to $\Delta A_1A_2A_3$, and where $(\delta_1, \delta_2, \delta_3)$ are the actual trilinear coordinates of P with respect to this triangle.
Furthermore, in the foregoing section, f is the homothety with center G and homothetic ratio $-\frac{1}{2}$. Remark that $f^{-1}(\Delta)$ is the anticomplementary triangle Δ^{-1} of Δ. We have:

Theorem 3. (1) *The conics $\mathcal{K}(P, \Delta, S)$ and $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ coincide.*
(2) *The common axes of the conics $\mathcal{K}(P, \Delta, s),\ s \in R$, and of the conic $\mathcal{C}(f^{-1}(P), f^{-1}(\Delta))$ are the orthogonal Droz-Farny -lines through P, with regard to $\Delta = A_1A_2A_3$.*

Proof. (1) Because of Lemma 1 and 2, both conics have center P and are circumscribed about the complementary triangle $f^{-1}(\Delta)$ of $A_1A_2A_3$.

(2) For the conic with center P, circumscribed about the anticomplementary triangle of $A_1A_2A_3$, it is clear that (PA_i, line through P, parallel to a_i), $i = 1, 2, 3$, are conjugate diameters. And the result follows from section 1. □

5. Examples

5.1. If $P = H$, the orthocenter of $\Delta = A_1A_2A_3$, which is also the circumcenter of its anticomplementary triangle Δ^{-1}, the conics $\mathcal{K}(H, \Delta, s)$ are circles with center H, since any two orthogonal lines through H are axes of these conics. In particular, $\mathcal{K}(H, \Delta, S)$, where $S = 2F(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})(\frac{l_1}{\cos A_1} + \frac{l_2}{\cos A_2} + \frac{l_3}{\cos A_3})$, is the circumcircle of Δ^{-1} and it is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by

$$\begin{aligned}&(l_1 \cos A_1)d_1^2 + (l_2 \cos A_2)d_2^2 + (l_3 \cos A_3)d_3^2\\ =&4F^2(\frac{\cos A_1}{l_1} + \frac{\cos A_2}{l_2} + \frac{\cos A_3}{l_3})\\ =&2F^2\frac{l_1^2 + l_2^2 + l_3^2}{l_1l_2l_3},\end{aligned}$$

or equivalently,

$$l_1^2(l_2^2 + l_3^2 - l_1^2)d_1^2 + l_2^2(l_3^2 + l_1^2 - l_2^2)d_2^2 + l_3^2(l_1^2 + l_2^2 - l_3^2)d_3^2 = 4F^2(l_1^2 + l_2^2 + l_3^2).$$

Moreover, $\mathcal{C}(f^{-1}(H), \Delta^{-1})$ is also the circumcirle of Δ^{-1}, which is easily seen from the fact that this circumcircle is the locus of the points for which the feet of the perpendiculars to the sides of Δ^{-1} are collinear. Remark that $f^{-1}(H)$, the orthocenter of Δ^{-1}, is the de Longchamps point $X(20)$ of Δ.

5.2. If $P = K(l_1, l_2, l_3)$, the Lemoine point of $\Delta = A_1A_2A_3$, then $\mathcal{K}(K, \Delta, S)$, with

$$\begin{aligned}S =&2F(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})(l_1^2 + l_2^2 + l_3^2)\\ =&2F^2\frac{(l_2^2l_3^2 + l_3^2l_1^2 + l_1^2l_2^2)(l_1^2 + l_2^2 + l_3^2)}{l_1^2l_2^2l_3^2},\end{aligned}$$

is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $d_1^2 + d_2^2 + d_3^2 = 4F^2(\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2})$, and it is the ellipse with center K, circumscribed about Δ^{-1}. Moreover, the locus $\mathcal{C}(f^{-1}(K), \Delta^{-1})$, where $f^{-1}(K)$ is the Lemoine point of Δ^{-1} (or $X(69)$ with coordinates

$$(l_2l_3(l_2^2 + l_3^2 - l_1^2), l_3l_1(l_3^2 + l_1^2 - l_2^2), l_1l_2(l_1^2 + l_2^2 - l_3^2)),$$

is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through K with respect to Δ.

5.3. If $P = G(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3})$, the centroid of $\Delta = A_1A_2A_3$, then $\mathcal{K}(G, \Delta, S)$, with $S = 18F$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1^2d_1^2 + l_2^2d_2^2 + l_3^2d_3^2 = 12F^2$, and it is the ellipse with center G, circumscribed about Δ^{-1}, i.e., it is the Steiner ellipse of Δ^{-1}, since G is also the centroid of Δ^{-1}. The locus $\mathcal{C}(G, \Delta^{-1})$ is also this Steiner ellipse and its axes are the orthogonal Droz-Farny lines through G with respect to Δ.

5.4. If $P = I(1, 1, 1)$, the incenter of $\Delta = A_1A_2A_3$, then $\mathcal{K}(I, \Delta, S)$, with $S = 2F(l_1 + l_2 + l_3)(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3})$, is the locus of the points for which the distances d_1, d_2, d_3 to the sides of Δ are related by $l_1d_1^2 + l_2d_2^2 + l_3d_3^2 = 4F^2(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3})$, and it is the ellipse with center I, circumscribed about Δ^{-1}. Moreover the locus $\mathcal{C}(f^{-1}(I), \Delta^{-1})$, where $f^{-1}(I)$ is the incenter of Δ^{-1} (which is center $X(8)$ of Δ, the Nagel point with coordinates $(\frac{l_2+l_3-l_1}{l_1}, \frac{l_3+l_1-l_2}{l_2}, \frac{l_1+l_2-l_3}{l_3}))$ is the same ellipse. The axes of this ellipse are the orthogonal Droz-Farny lines through I with respect to Δ.

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On Butterflies Inscribed in a Quadrilateral

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Abstract. We explore a configuration consisting of two quadrilaterals which share the intersection of diagonals. We prove results analogous to the Sidney Kung's Butterfly Theorem for Quadrilaterals in [2].

1. The extended butterfly theorem for quadrilaterals

In this note we consider some properties of pairs of quadrilaterals $ABCD$ and $A'B'C'D'$ with A', B', C' and D' on lines AB, BC, CD, and DA respectively.

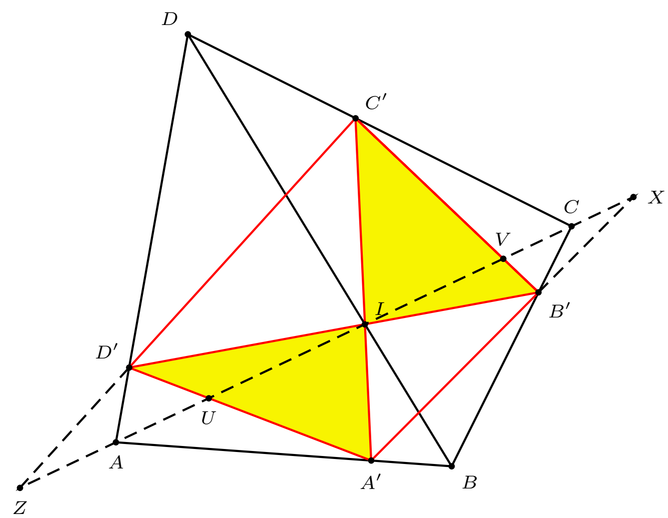

Figure 1

The segments AC, $A'C'$ and $B'D'$ are analogous to the three chords from the classical butterfly theorem (see [1] for an extensive overview of its many proofs and generalizations).

When the intersection of the lines $A'C'$ and $B'D'$ is the intersection I of the lines AC and BD, *i.e.*, when $ABCD$ and $A'B'C'D'$ share the same intersection of diagonals, in [2] the following equality, known as the Butterfly Theorem for Quadrilaterals, was established:

$$\frac{|AU|}{|UI|} \cdot \frac{|IV|}{|VC|} = \frac{|AI|}{|IC|}, \tag{1}$$

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where U and V are intersections of the line AC with the lines $D'A'$ and $B'C'$ (see Figure 1).

For the intersections $X = AC \cap A'B'$ and $Z = AC \cap C'D'$, in this situation, we have similar relations

$$\frac{|XA|}{|AI|} \cdot \frac{|IC|}{|CZ|} = \frac{|XI|}{|IZ|},$$

and

$$\frac{|XU|}{|UI|} \cdot \frac{|IV|}{|VZ|} = \frac{|XI|}{|IZ|}.$$

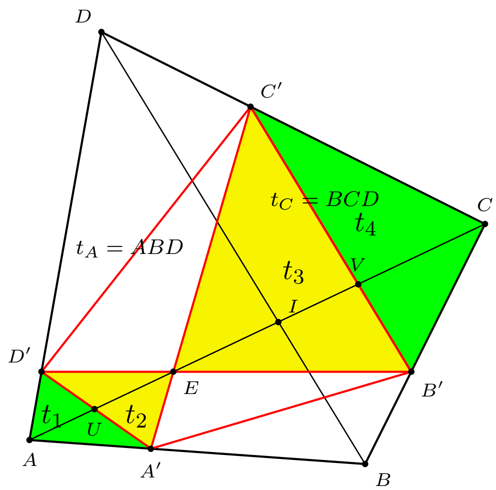

Figure 2

Our first result is the observation that (1) holds when the diagonals of $A'B'C'D'$ intersect at a point E on the diagonal AC of $ABCD$, not necessarily the intersection $I = AC \cap BD$ (see Figure 2).

Theorem 1. *Let $A'B'C'D'$ be an inscribed quadrilateral of $ABCD$, and $E = A'C' \cap B'C'$, $I = AC \cap BD$, $U = AC \cap D'A'$, $V = AC \cap B'C'$. If E lies on the line AC, then*

$$\frac{|AU|}{|UE|} \cdot \frac{|EV|}{|VC|} = \frac{|AI|}{|IC|}. \tag{2}$$

Proof. We shall use analytic geometry of the plane. Without loss of generality we can assume that $A(0,0)$, $B(f,g)$, $C(1,0)$ and $D(p,q)$ for some real numbers f, g, p and q. The points $A'\left(\frac{fu}{u+1}, \frac{gu}{u+1}\right)$ and $B'\left(\frac{f+v}{v+1}, \frac{g}{v+1}\right)$ divide segments AB and BC in ratios u and v, which are real numbers different from -1. Let the rectangular coordinates of the point E be $(h,\ k)$. Then the vertices C' and D' are intersections of the lines CD and DA with the lines $A'E$ and $B'E$, respectively. Their coordinates are a bit more complicated. Next, we determine the points U and

V as intersections of the line AC with the lines $A'D'$ and $B'C'$ and with a small help from Maple V find that the difference

$$\begin{aligned}\mathcal{D} :=& \frac{|AU|^2}{|UE|^2} \cdot \frac{|EV|^2}{|VC|^2} - \frac{|AI|^2}{|IC|^2} \\ =& \frac{(fq-gp)^2 \cdot k \cdot P_5(h,k)}{Q_4(h,k)\cdot(g+fq-q-gp)^2\cdot(guh+(u+1-fu)k-gu)^2},\end{aligned}$$

where $P_5(h,k)$ and $Q_4(h,k)$ are polynomials of degrees 5 and 4 respectively in variables h and k. Both are somewhat impractical to write down explicitly. However, since k is a factor in the numerator we see that when $k = 0$, *i.e.*, when the point E is on the line AC, the difference $\mathcal{D}$ is zero so that the extended Butterfly Theorem for Quadrilaterals holds. □

Remark. There is a version of the above theorem where the points U and V are intersections $U = AE \cap A'D'$ and $V = CE \cap B'C'$. The difference $\mathcal{D}$ in this case is the quotient

$$\frac{-(fq-gp)^2 \cdot k \cdot P_1(h,k)\cdot P_2(h,k)}{(qh-pk)^2\cdot(g+fq-q-gp)^2\cdot(guh+(u+1-fu)k-gu)^2},$$

where

$$\begin{aligned}P_1(h,k) =& (u(q-g)+q(1+uv))h + (uv(1-p)+u(f-p)-p)k \\ & - u(qv-gp+fq), \\ P_2(h,k) =& 2qguh^2 + (gu-2ugp-2fqu+uq-vuq+q)\,hk - \\ & (p+uv+up+uf-pvu-2puf)\,k^2 - 2guqh + u\,(qv+gp+fq)\,k.\end{aligned}$$

Note that these are linear and quadratic polynomials in h and k. In other words, the extended Butterfly Theorem for Quadrilaterals holds not only for points E on the line AC but also when the point E is on a line through D (with the equation $P_1(h,k) = 0$) and on a conic through A, C and D (with the equation $P_2(h,k) = 0$).

Moreover, in this case we can easily prove the following converse of Theorem 1.

If the relation (2) holds when the points A' and B' divide segments AB and BC both in the ratio $1:3$, $1:1$ or $3:1$, then the point E lies on the line AC.

Indeed, if we substitute for $u = v = \frac{1}{3}$, 1 or 3 both into P_1 and P_2 we get three equations whose only common solutions in h and k are coordinates of points A, C, and D (which are definitely excluded as possible solutions).

2. A relation involving areas

Let us introduce shorter notation for six triangles in this configuration: $t_1 = D'AA'$, $t_2 = D'A'E$, $t_3 = EB'C'$, $t_4 = CC'B'$, $t_A = ABD$ and $t_C = CDB$ (see Figure 2).

Our next result shows that the above relationship also holds for areas of these triangles.

Theorem 2. (a) *If E lies on the line AC, then*

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} = \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}. \tag{3}$$

(b) *If the relation (3) holds when the points A' and B' divide AB and BC both in the ratio $1 : 2$ or $2 : 1$, then the point E lies on the line AC.*

Proof. If we keep the same assumptions and notation from the proof of Theorem 1, then

$$\frac{\mathtt{area}(t_1)}{\mathtt{area}(t_2)} \cdot \frac{\mathtt{area}(t_3)}{\mathtt{area}(t_4)} - \frac{\mathtt{area}(t_A)}{\mathtt{area}(t_C)}$$
$$= \frac{(fq-gp)\cdot k \cdot P_1(h,k)}{(qh-pk)\cdot(g+fq-q-gp)\cdot(guh+(u+1-fu)k-gu)}.$$

Hence, (a) is clearly true because $k = 0$ when the point E is on the line AC.

On the other hand, for (b), when $u = \frac{1}{2}$ and $v = \frac{1}{2}$ then

$$\mathcal{E}_1 = 4 \cdot P_1(h,k) = (2g-7q)\,h + (7p-2f-1)\,k + q - 2gp + 2fq,$$

while for $u = 2$ and $v = 2$ then

$$\mathcal{E}_2 = P_1(h,k) = (2g-7q)\,h + (7p-2f-4)\,k + 2fq + 4q - 2gp.$$

The only solution of the system

$$\mathcal{E}_1 = 0, \qquad \mathcal{E}_2 = 0$$

is $(h,h) = (p,q)$, *i.e.*, $E = D$. However, for this solution the triangle t_2 degenerates to a segment so that its area is zero which is unacceptable. □

3. Other relations

Note that the above theorem holds also for (lengths of) the altitudes $h(A,t_1)$, $h(E,t_2)$, $h(E,t_3)$, $h(C,t_4)$, $h(A,t_A)$ and $h(C,t_C)$ because (for example) $\mathtt{area}(t_1) = \frac{1}{2}h(A,t_1)\cdot|D'A'|$.

Let $G(t)$ denote the centroid of the triangle $t = ABC$, and $\varepsilon(A,t)$ the distance of $G(t)$ from the side BC opposite to the vertex A. Since $\varepsilon(A,t) = \frac{2\mathtt{area}(t)}{3|BC|}$ there is a version of the above theorem for the distances $\varepsilon(A,t_1)$, $\varepsilon(E,t_2)$, $\varepsilon(E,t_3)$, $\varepsilon(C,t_4)$, $\varepsilon(A,t_A)$ and $\varepsilon(C,t_C)$.

For a triangle t let $R(t)$ denote the radius of its circumcircle. The following theorem shows that the radii of circumcircles of the six triangles satisfy the same pattern without any restrictions on the point E.

Theorem 3. $\frac{R(t_1)}{R(t_2)} \cdot \frac{R(t_3)}{R(t_4)} = \frac{R(t_A)}{R(t_C)}$.

Proof. Let us keep again the same assumptions and notation from the proof of Theorem 1. Since $R(t) = \frac{\text{product of side lengths}}{4\mathtt{area}(t)}$, we see that the square of the circumradius of a triangle with vertices in the points (x,a), (y,b) and (z,c) is given by

$$\frac{[(y-z)^2+(b-c)^2]\cdot[(z-x)^2+(c-a)^2]\cdot[(x-y)^2+(a-b)^2]}{4(x(b-c)+y(c-a)+z(a-b))^2}.$$

Applying this formula we find $R(t_1)^2$, $R(t_2)^2$, $R(t_3)^2$, $R(t_4)^2$, $R(t_A)^2$ and $R(t_C)^2$. In a few seconds Maple V verifies that the difference $\frac{R(t_1)^2}{R(t_2)^2} \cdot \frac{R(t_3)^2}{R(t_4)^2} - \frac{R(t_A)^2}{R(t_C)^2}$ is equal to zero. □

For a triangle t let $H(t)$ denote its orthocenter. In the next result we look at the distances of a particular vertex of the six triangles from its orthocenter. Again the pattern is independent from the position of the point E.

Theorem 4. $\frac{|AH(t_1)|}{|EH(t_2)|} \cdot \frac{|EH(t_3)|}{|CH(t_4)|} = \frac{|AH(t_A)|}{|CH(t_C)|}$.

Proof. This time one can see that $|AH(t)|^2$ for the triangle $t = ABC$ with the vertices in the points (x, a), (y, b) and (z, c) is given as

$$\frac{[(y-z)^2 + (b-c)^2] \cdot [x^2 + a^2 + yz - x(y+z) + bc - a(b+c)]^2}{(x(b-c) + y(c-a) + z(a-b))^2}.$$

Applying this formula we find $|AH(t_1)|^2$, $|EH(t_2)|^2$, $|EH(t_3)|^2$, $|CH(t_4)|^2$, $|AH(t_A)|^2$, and $|CH(t_C)|^2$. In a few seconds Maple V verifies that the difference $\frac{|AH(t_1)|^2}{|EH(t_2)|^2} \cdot \frac{|EH(t_3)|^2}{|CH(t_4)|^2} - \frac{|AH(t_A)|^2}{|CH(t_C)|^2}$ is equal to zero. □

Let $O(t)$ be the circumcenter of the triangle $t = ABC$. Let $\delta(A, t)$ denote the distance of $O(t)$ from the side BC opposite to the vertex A. Since $|AH(t)| = 2\delta(A, t)$ there is a version of the above theorem for the distances $\delta(A, t_1)$, $\delta(E, t_2)$, $\delta(E, t_3)$, $\delta(C, t_4)$, $\delta(A, t_A)$ and $\delta(C, t_C)$.

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On Triangles with Vertices on the Angle Bisectors

Eric Danneels

Abstract. We study interesting properties of triangles whose vertices are on the three angle bisectors of a given triangle. We show that such a triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle. We present several interesting examples with new triangle centers.

1. Introduction

Let ABC be a given triangle with incenter I. By an I-triangle we mean a triangle UVW whose vertices U, V, W are on the angle bisectors AI, BI, CI respectively. Such triangles are clearly perspective with ABC at the incenter I.

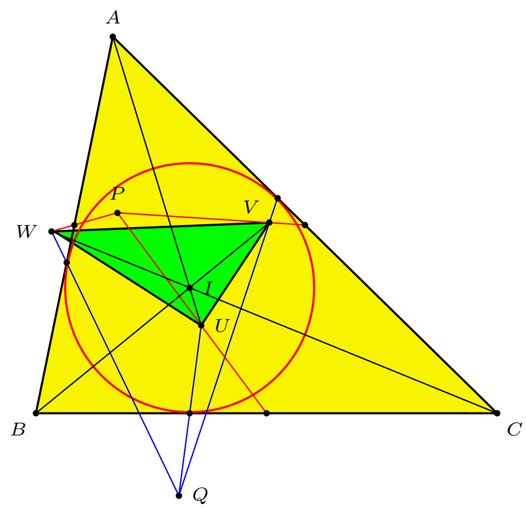

Figure 1.

Theorem 1. *An I-triangle is perspective with the medial triangle if and only if it is perspective with the intouch triangle.*

Proof. The homogeneous barycentric coordinates of the vertices of an I-triangle can be taken as

$$U = (u : b : c), \qquad V = (a : v : c), \qquad W = (a : b : w) \tag{1}$$

for some u, v, w. In each case, the condition for perspectivity is

$$F(u, v, w) := (b-c)vw + (c-a)wu + (a-b)uv + (a-b)(b-c)(c-a) = 0. \tag{2}$$

□

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Let D, E, F be the midpoints of the sides BC, CA, AB of triangle ABC. If $P = (x : y : z)$ is the perspector of an I-triangle UVW with the medial triangle, then U is the intersection of the line DP with the bisector IA. It has coordinates

$$((b-c)x : b(y-z) : c(y-z)).$$

Similarly, the coordinates of V and W can be determined. The triangle UVW is perspective with the intouch triangle at

$$Q = \left(\frac{x(y+z-x)}{s-a} : \frac{y(z+x-y)}{s-b} : \frac{z(x+y-z)}{s-c}\right).$$

Conversely, if an I-triangle is perspective with the intouch triangle at $Q = (x : y : z)$, then it is perspective with the medial triangle at

$$P = ((s-a)x((s-b)y + (s-c)z - (s-a)x) : \cdots : \cdots).$$

Theorem 2. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_1, U_1, W_1 are the inversive images of U, V, W in the incircle, then $U_1V_1W_1$ is also an I-triangle perspective with the medial and intouch triangles.*

Proof. If the coordinates of U, V, W are as given in (1), then

$$U_1 = (u_1 : b : c), \qquad V_1 = (a : v_1 : c), \qquad W_1 = (a : b : w_1),$$

where

$$\begin{aligned} u_1 =& \frac{(a(b+c)-(b-c)^2)u - 2(s-a)(b-c)^2}{2(s-a)u - a(b+c) + (b-c)^2}, \\ v_1 =& \frac{(b(c+a)-(c-a)^2)v - 2(s-b)(c-a)^2}{2(s-b)v - b(c+a) + (c-a)^2}, \\ w_1 =& \frac{(c(a+b)-(a-b)^2)w - 2(s-c)(a-b)^2}{2(s-c)w - c(a+b) + (a-b)^2}. \end{aligned}$$

From these,

$$F(u_1, v_1, w_1) = \frac{64abc(s-a)(s-b)(s-c)}{\prod_{\text{cyclic}}(2(s-a)u - a(b+c) + (b-c)^2)} \cdot F(u, v, w) = 0.$$

It follows from (2) that $U_1V_1W_1$ is perspective to both the medial and the intouch triangles. □

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_1V_1W_1$ is perspective with

(i) the medial triangle at

$$((y+z-x)((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$(a((a(b+c)-(b-c)^2)x - (b+c-a)(b-c)(y-z)) : \cdots : \cdots).$$

Theorem 3. *Let UVW be an I-triangle perspective with the medial and the intouch triangles. If U_2 (respectively V_2 and W_2) is the inversive image of U (respectively V and W) in the A- (respectively B- and C-) excircle, then $U_2V_2W_2$ is also an I-triangle perspective with the medial and intouch triangles.*

If an I-triangle UVW is perspective with the medial triangle at $(x : y : z)$, then $U_2V_2W_2$ is perspective with
(i) the medial triangle at

$$\left((s-a)(y+z-x)((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\right),$$

(ii) the intouch triangle at

$$\left(\frac{a}{s-a}((a(b+c)+(b-c)^2)x+2s(b-c)(y-z)) : \cdots : \cdots\right).$$

2. Some interesting examples

We present some interesting examples of I-triangles perspective with both the medial and intouch triangles. The perspectors in these examples are new triangle centers not in the current edition of [1].

2.1. Let X_a, X_b, X_c be the inversive images of the excenters I_a, I_b, I_c in the incircle. We have $IX_a = \frac{r^2}{II_a}$ and $II_a = AI_a - AI = AI \cdot \left(\frac{s}{s-a} - 1\right) = \frac{a \cdot AI}{s-a}$. Hence,

$$\frac{AI}{IX_a} = \frac{a \cdot AI^2}{r^2(s-a)} = \frac{a}{sin^2(\frac{A}{2})} = \frac{abc}{(s-a)(s-b)(s-c)} = \frac{4R}{r},$$

and by symmetry $\frac{AI}{IX_a} = \frac{BI}{IX_b} = \frac{CI}{IX_c} = \frac{4R}{r}$. Therefore, triangles ABC and $X_aX_bX_c$ are homothetic with ratio $4R : -r$.

$$\begin{aligned}
X_a =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(1,0,0),\\
X_b =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,1,0),\\
X_c =&(a^2+b^2+c^2-2ab-2bc-2ca)(a,b,c)\\
&+(b+c-a)(c+a-b)(a+b-c)(0,0,1).
\end{aligned}$$

Proposition 4. *$X_aX_bX_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_x = (a^2(b+c)+(b+c-2a)(b-c)^2 : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_x = (a(b+c-a)(a^2(b+c)+(b+c-2a)(b-c)^2) : \cdots : \cdots).$$

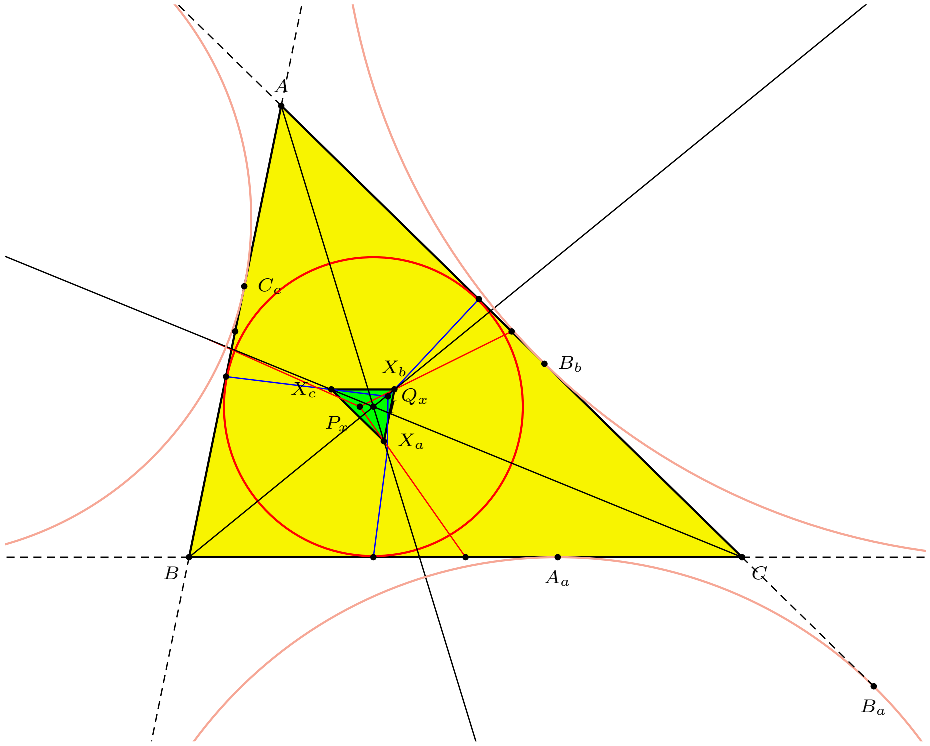

Figure 2.

2.2. Let Y_a, Y_b, Y_c be the inversive images of the incenter I with respect to the A-, B-, C-excircles.

$$\begin{aligned}
Y_a =&(a^2+b^2+c^2-2bc+2ca+2ab)(-a,b,c)\\
&+(a+b+c)(c+a-b)(a+b-c)(1,0,0),\\
Y_b =&(a^2+b^2+c^2+2bc-2ca+2ab)(a,-b,c)\\
&+(a+b+c)(a+b-c)(b+c-a)(0,1,0),\\
Y_c =&(a^2+b^2+c^2+2bc+2ca-2ab)(a,b,-c)\\
&+(a+b+c)(b+c-a)(c+a-b)(1,0,0).
\end{aligned}$$

Proposition 5. *$Y_aY_bY_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_y = ((b+c-a)^2(a^2(b+c)+(2a+b+c)(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_y = \left(\frac{a(a^2(b+c)+(2a+b+c)(b-c)^2)}{b+c-a} : \cdots : \cdots\right).$$

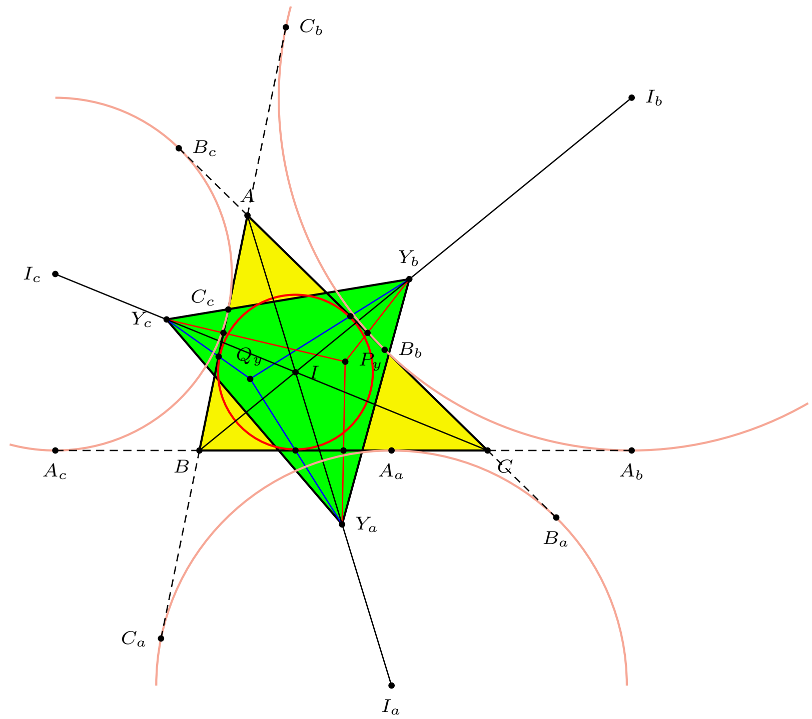

Figure 3.

2.3. Let V_a, V_b, V_c be the inversive images of X_a, X_b, X_c with respect to the A-, B-, C-excircles.

$$\begin{aligned}V_a =&(3a^2 + (b-c)^2)(-a, b, c) + 2a(c + a - b)(a + b - c)(1, 0, 0),\\ V_b =&(3b^2 + (c-a)^2)(a, -b, c) + 2b(a + b - c)(b + c - a)(0, 1, 0),\\ V_c =&(3c^2 + (a-b)^2)(a, b, -c) + 2c(b + c - a)(c + a - b)(0, 0, 1).\end{aligned}$$

Proposition 6. *$V_aV_bV_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_v = (a(b + c - a)^3(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_v = \left(\frac{a(a^2 + 3(b-c)^2)}{b + c - a} : \cdots : \cdots\right).$$

2.4. Let W_a, W_b, W_c be the inversive images of Y_a, Y_b, Y_c with respect to the incircle.

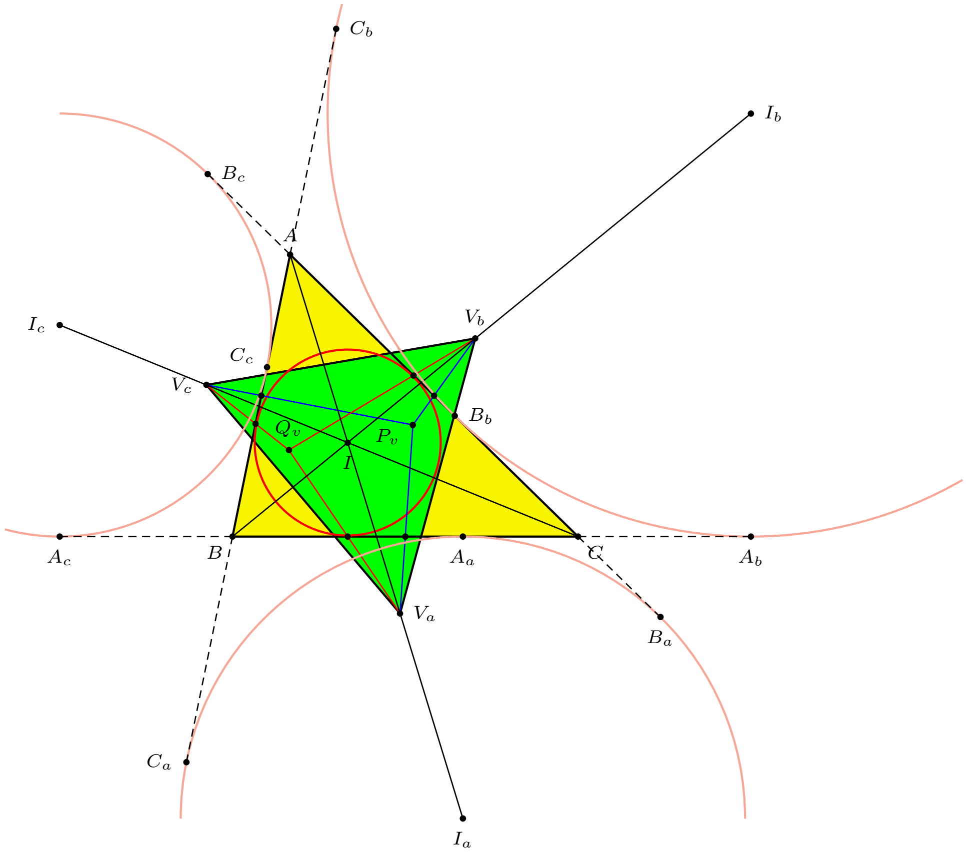

Figure 4.

$$W_a = (3a^2 + (b-c)^2)(a,b,c) - 2a(c+a-b)(a+b-c)(1,0,0),$$
$$W_b = (3b^2 + (c-a)^2)(a,b,c) - 2b(a+b-c)(b+c-a)(0,1,0),$$
$$W_c = (3c^2 + (a-b)^2)(a,b,c) - 2c(b+c-a)(c+a-b)(0,0,1).$$

Proposition 7. *$W_aW_bW_c$ is an I-triangle perspective with*
(i) the medial triangle at

$$P_w = (a(a^2 + 3(b-c)^2) : \cdots : \cdots),$$

(ii) the intouch triangle at

$$Q_w = (a(b+c-a)^2(a^2 + 3(b-c)^2) : \cdots : \cdots).$$

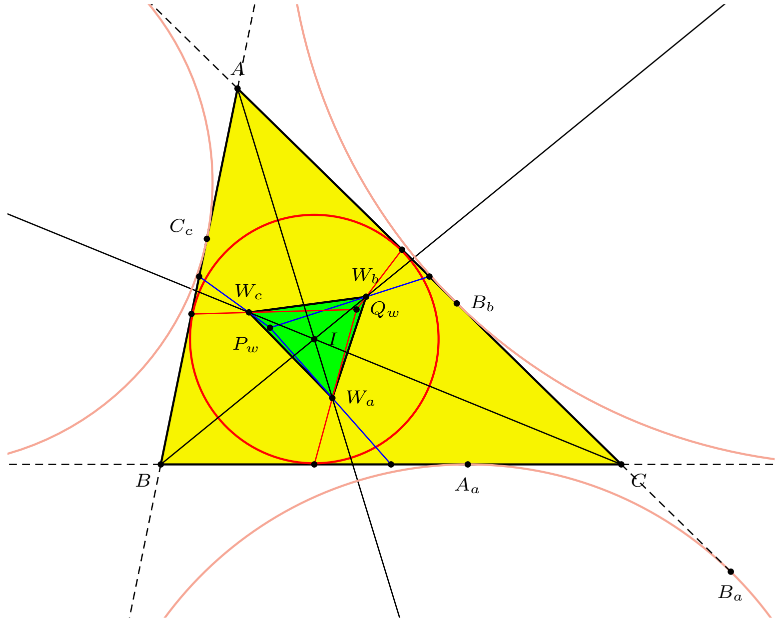

Figure 5.

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Formulas Among Diagonals in the Regular Polygon and the Catalan Numbers

Matthew Hudelson

Abstract. We look at relationships among the lengths of diagonals in the regular polygon. Specifically, we derive formulas for all diagonals in terms of the shortest diagonals and other formulas in terms of the next-to-shortest diagonals, assuming unit side length. These formulas are independent of the number of sides of the regular polygon. We also show that the formulas in terms of the shortest diagonals involve the famous Catalan numbers.

1. Motivation

In [1], Fontaine and Hurley develop formulas that relate the diagonal lengths of a regular n-gon. Specifically, given a regular convex n-gon whose vertices are $P_0, P_1, \ldots, P_{n-1}$, define d_k as the distance between P_0 and P_k. Then the law of sines yields

$$\frac{d_k}{d_j} = \frac{\sin \frac{k\pi}{n}}{\sin \frac{j\pi}{n}}.$$

Defining

$$r_k = \frac{\sin \frac{k\pi}{n}}{\sin \frac{\pi}{n}},$$

the formulas given in [1] are

$$r_h r_k = \sum_{i=1}^{\min\{k,h,n-k,n-h\}} r_{|k-h|+2i-1}$$

and

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where $s = \min\{j > 0 : jk \equiv \pm 1 \bmod n\}$.

Notice that for $1 \leq k \leq n-1$, $r_k = \frac{d_k}{d_1}$, but there is no *a priori* restriction on k in the definition of r_k. Thus, it would make perfect sense to consider $r_0 = 0$ and $r_{-k} = -r_k$ not to mention r_k for non-integer values of k as well. Also, the only restriction on n in the definition of r_n is that n not be zero or the reciprocal of an integer.

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2. Short proofs of r_k formulas

Using the identity $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, we can provide some short proofs of formulas equivalent, and perhaps simpler, to those in [1]:

Proposition 1. *For integers h and k,*

$$r_h r_k = \sum_{j=0}^{k-1} r_{h-k+2j+1}.$$

Proof. Letting h and k be integers, we have

$$\begin{aligned}
r_h r_k &= (\sin\frac{\pi}{n})^{-2} \sin\frac{h\pi}{n} \sin\frac{k\pi}{n} \\
&= (\sin\frac{\pi}{n})^{-2} \frac{1}{2}\left(\cos\frac{(h-k)\pi}{n} - \cos\frac{(h+k)\pi}{n}\right) \\
&= (\sin\frac{\pi}{n})^{-2}\left(\sum_{j=0}^{k-1}\frac{1}{2}\left(\cos\frac{(h-k+2j)\pi}{n} - \cos\frac{(h-k+2j+2)\pi}{n}\right)\right) \\
&= (\sin\frac{\pi}{n})^{-2}\sum_{j=0}^{k-1}\left(\sin\frac{(h-k+2j+1)\pi}{n}\sin\frac{\pi}{n}\right) \\
&= \sum_{j=0}^{k-1} r_{h-k+2j+1}.
\end{aligned}$$

The third equality holds since the sum telescopes. □

Note that we can switch the roles of h and k to arrive at the formula

$$r_k r_h = \sum_{j=0}^{h-1} r_{k-h+2j+1}.$$

To illustrate that this is not contradictory, consider the example when $k = 2$ and $h = 5$. From Proposition 1, we have

$$r_5 r_2 = r_4 + r_6.$$

Reversing the roles of h and k, we have

$$r_2 r_5 = r_{-2} + r_0 + r_2 + r_4 + r_6.$$

Recalling that $r_0 = 0$ and $r_{-j} = r_j$, we see that these two sums are in fact equal.

The reciprocal formula in [1] is proven almost as easily:

Proposition 2. *Given an integer k relatively prime to n,*

$$\frac{1}{r_k} = \sum_{j=1}^{s} r_{k(2j-1)}$$

where s is any (not necessarily the smallest) positive integer such that $ks \equiv \pm 1$ mod n.

Proof. Starting at the right-hand side, we have

$$\begin{aligned}
\sum_{j=1}^{s} r_{k(2j-1)} &= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\sin\frac{k(2j-1)\pi}{n}\sin\frac{k\pi}{n}\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sum_{j=1}^{s}\frac{1}{2}\left(\cos\frac{k(2j-2)\pi}{n}-\cos\frac{2jk\pi}{n}\right)\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\frac{1}{2}\left(\cos 0-\cos\frac{2sk\pi}{n}\right)\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{sk\pi}{n}\\
&= \left(\sin\frac{\pi}{n}\sin\frac{k\pi}{n}\right)^{-1}\sin^2\frac{\pi}{n}\\
&= \left(\sin\frac{k\pi}{n}\right)^{-1}\sin\frac{\pi}{n}\\
&= \frac{1}{r_k}.
\end{aligned}$$

Here, the third equality follows from the telescoping sum, and the fifth follows from the definition of s. □

3. From powers of r_2 to Catalan numbers

We use the special case of Proposition 1 when $h = 2$, namely

$$r_k r_2 = r_{k-1} + r_{k+1},$$

to develop formulas for powers of r_2.

Proposition 3. *For nonnegative integers m,*

$$r_2{}^m = \sum_{i=0}^{m}\binom{m}{i} r_{1-m+2i}.$$

Proof. We proceed by induction on m. When we have $m = 0$, the formula reduces to $r_2{}^0 = r_1$ and both sides equal 1. This establishes the basis step. For the inductive step, we assume the result for $m = n$ and begin with the sum on the right-hand side for $m = n + 1$.

$$\begin{aligned}
\sum_{i=0}^{n+1}\binom{n+1}{i}r_{-n+2i} &= \sum_{i=0}^{n+1}\left(\binom{n}{i}+\binom{n}{i-1}\right)r_{-n+2i}\\
&= \left(\sum_{i=0}^{n+1}\binom{n}{i-1}r_{-n+2i}\right)+\left(\sum_{i=0}^{n+1}\binom{n}{i}r_{-n+2i}\right)\\
&= \left(\sum_{j=0}^{n}\binom{n}{j}r_{2-n+2j}\right)+\left(\sum_{i=0}^{n}\binom{n}{i}r_{-n+2i}\right)\\
&= \sum_{i=0}^{n}\binom{n}{i}(r_{-n+2i}+r_{2-n+2i})\\
&= \sum_{i=0}^{n}\binom{n}{i}r_2 r_{1-n+2i}\\
&= r_2 {r_2}^n\\
&= {r_2}^{n+1}
\end{aligned}$$

which completes the induction. The first equality uses the standard identity for binomial coefficients $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$.

The third equality is by means of the change of index $j=i-1$ and the fact that $\binom{n}{c}=0$ if $c<0$ or $c>n$.

The fifth equality is from Proposition 1 and the sixth is from the induction hypothesis. □

Now, we use Proposition 3 and the identity $r_{-k}=-r_k$ to consider an expression for ${r_2}^m$ as a linear combination of r_k's where $k>0$, i.e., we wish to determine the coefficients α_k in the sum

$${r_2}^m=\sum_{k=1}^{m+1}\alpha_{m,k}r_k.$$

From Proposition 2, the sum is known to end at $k=m+1$. In fact, we can determine α_k directly. One contribution occurs when $k=1-m+2i$, or $i=\frac{1}{2}(m+k-1)$. A second contribution occurs when $-k=1-m+2i$, or $i=\frac{1}{2}(m-k-1)$. Notice that if m and k have the same parity, there is no contribution to $\alpha_{m,k}$. Piecing this information together, we find

$$\alpha_{m,k}=\begin{cases}\binom{m}{\frac{1}{2}(m+k-1)}-\binom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m-k \text{ is odd;}\\ 0, & \text{if } m-k \text{ is even.}\end{cases}$$

Notice that if $m-k$ is odd, then

$$\binom{m}{\frac{1}{2}(m-k+1)}=\binom{m}{\frac{1}{2}(m+k-1)}.$$

Therefore,

$$\alpha_{m,k} = \begin{cases} \binom{m}{\frac{1}{2}(m-k+1)} - \binom{m}{\frac{1}{2}(m-k-1)}, & \text{if } m+k \text{ is odd;} \\ 0, & \text{if } m+k \text{ is even.} \end{cases}$$

Intuitively, if we arrange the coefficients of the original formula in a table, indexed horizontally by $k \in \mathbb{Z}$ and vertically by $m \in \mathbb{N}$, then we obtain Pascal's triangle (with an extra row corresponding to $m = 0$ attached to the top):

	-2	-1	0	1	2	3	4	5	6
0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	0	0	0	0
2	0	0	0	1	0	1	0	0	0
3	0	0	1	0	2	0	1	0	0
4	0	1	0	3	0	3	0	1	0
5	1	0	4	0	6	0	4	0	1

Next, if we subtract the column corresponding to $s = -k$ from that corresponding to $s = k$, we obtain the $\alpha_{m,k}$ array:

	1	2	3	4	5	6
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	1	0	1	0	0	0
3	0	2	0	1	0	0
4	2	0	3	0	1	0
5	0	5	0	4	0	1

As a special case of this formula, consider what happens when $m = 2p$ and $k = 1$. We have

$$\begin{aligned} \alpha_1 &= \binom{2p}{p} - \binom{2p}{p-1} \\ &= \frac{(2p)!}{p!p!} - \frac{(2p)!}{(p-1)!(p+1)!} \\ &= \frac{(2p)!}{p!p!}\left(1 - \frac{p}{p+1}\right) \\ &= \frac{1}{p+1}\binom{2p}{p} \end{aligned}$$

which is the closed form for the p^{th} Catalan number.

4. Inverse formulas, polynomials, and binomial coefficients

We have a formula for the powers of r_2 as linear combinations of the r_k values. We now derive the inverse relationship, writing the r_k values as linear combinations of powers of r_2. We start with Proposition 1: $r_2 r_k = r_{k-1} + r_{k+1}$.

We demonstrate that for natural numbers k, there exist polynomials $P_k(t)$ such that $r_k = P_k(r_2)$. We note that $P_0(t) = 0$ and $P_1(t) = 1$. Now assume the existence of $P_{k-1}(t)$ and $P_k(t)$. Then from the identity

$$r_{k+1} = r_2 r_k - r_{k-1}$$

we have

$$P_{k+1}(t) = tP_k(t) - P_{k-1}(t)$$

which establishes a second-order recurrence for the polynomials $P_k(t)$. Armed with this, we show for $k \geq 0$,

$$P_k(t) = \sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i}.$$

By inspection, this holds for $k = 0$ as the binomial coefficients are all zero in the sum. Also, when $k = 1$, the $i = 0$ term is the only nonzero contributor to the sum. Therefore, it is immediate that this formula holds for $k = 0, 1$. Now, we use the recurrence to establish the induction. Given $k \geq 1$,

$$\begin{aligned} P_{k+1}(t) &= tP_k(t) - P_{k-1}(t) \\ &= t\sum_i \binom{k-1-i}{i} (-1)^i t^{k-1-2i} - \sum_j \binom{k-2-j}{j} (-1)^j t^{k-2-2j} \\ &= \sum_i \binom{k-1-i}{i} (-1)^i t^{k-2i} - \sum_i \binom{k-1-i}{i-1} (-1)^{i-1} t^{k-2i} \\ &= \sum_i \binom{k-i}{i} (-1)^i t^{k-2i} \end{aligned}$$

as desired. The third equality is obtained by replacing j with $i-1$.

As a result, we obtain the desired formula for r_k in terms of powers of r_2:

Proposition 4.

$$r_k = \sum_i \binom{k-1-i}{i} (-1)^i {r_2}^{k-1-2i}.$$

Assembling these coefficients into an array similar to the $\alpha_{m,k}$ array in the previous section, we have

	0	1	2	3	4	5	6
1	1	0	0	0	0	0	0
2	0	1	0	0	0	0	0
3	−1	0	1	0	0	0	0
4	0	−2	0	1	0	0	0
5	1	0	−3	0	1	0	0
6	0	3	0	−4	0	1	0
7	−1	0	6	0	−5	0	1

This array displays the coefficient $\beta(k, m)$ in the formula

$$r_k = \sum_m \beta(k, m) r_2^m,$$

where k is the column number and m is the row number. An interesting observation is that this array and the $\alpha(m,k)$ array are inverses in the sense of "array multiplication":

$$\sum_i \alpha(m,i)\beta(i,p) = \begin{cases} 1, & m = p\,; \\ 0, & \text{otherwise.} \end{cases}$$

Also, the $k^{(th)}$ column of the $\beta(k,m)$ array can be generated using the generating function $x^k(1+x^2)^{-1-k}$. Using machinery in §5.1 of [2], this leads to the conclusion that the columns of the original $\alpha(m,k)$ array can be generated using an inverse function; in this case, the function that generates the $m^{(th)}$ column of the $\alpha(m,k)$ array is

$$x^{m-1}\left(\frac{1-\sqrt{1-4x^2}}{2x^2}\right)^m.$$

This game is similar but slightly more complicated in the case of r_3. Here, we use

$$r_3 r_k = r_{k+2} + r_k + r_{k-2}$$

which leads to

$$r_{k+2} = (r_3 - 1)r_k + r_{k-2}.$$

With this, we show that for $k \geq 0$, there are functions, but not necessarily polynomials, $Q_k(t)$ such that $r_k = Q_k(r_3)$. From the above identity, we have

$$Q_{k+2}(t) = (t-1)Q_k(t) - Q_{k-2}(t).$$

This establishes a fourth-order recurrence relation for the functions $Q_k(t)$ so determining the four functions Q_0, Q_1, Q_2, and Q_3 will establish the recurrence. By inspection, $Q_0(t) = 0$, $Q_1(t) = 1$, and $Q_3(t) = t$ so all that remains is to determine $Q_2(t)$. We have $r_3 = {r_2}^2 - 1$ from Proposition 4. Therefore, $r_2 = \sqrt{r_3+1}$ and so $Q_2(t) = \sqrt{t+1}$.

We now claim

Proposition 5. *For all natural numbers k,*

$$Q_{2k}(t) = \sqrt{t+1}\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}$$

$$Q_{2k+1}(t) = \sum_i \binom{k-i}{i}(-1)^i(t-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}.$$

Proof. We proceed by induction on k. These are easily checked to match the functions Q_0, Q_1, Q_2, and Q_3 for $k = 0, 1$. For $k \geq 2$, we have

$$\begin{aligned}
Q_{2k}(t) &= (t-1)Q_{2(k-1)}(t) - Q_{2(k-2)}(t) \\
&= \sqrt{t+1}\left((t-1)\sum_i \binom{k-2-i}{i}(-1)^i(t-1)^{k-2-2i}\right. \\
&\qquad \left. - \sum_j \binom{k-3-j}{j}(-1)^j(t-1)^{k-3-2j}\right) \\
&= \sqrt{t+1}\left(\sum_i \left(\binom{k-2-i}{i} + \binom{k-2-i}{i-1}\right)(-1)^i(t-1)^{k-1-2i}\right) \\
&= \sqrt{t+1}\left(\sum_i \binom{k-1-i}{i}(-1)^i(t-1)^{k-1-2i}\right)
\end{aligned}$$

as desired. The third equality is obtained by replacing j with $i-1$. The proof for Q_{2k+1} is similar, treating each sum separately, and is omitted. $\square$

As a corollary, we have

Corollary 6.

$$r_{2k} = \sqrt{r_3+1}\sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i}$$

$$r_{2k+1} = \sum_i \binom{k-i}{i}(-1)^i(r_3-1)^{k-2i} + \sum_i \binom{k-1-i}{i}(-1)^i(r_3-1)^{k-1-2i}.$$

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A Note on the Barycentric Square Roots of Kiepert Perspectors

Khoa Lu Nguyen

Abstract. Let P be an interior point of a given triangle ABC. We prove that the orthocenter of the cevian triangle of the barycentric square root of P lies on the Euler line of ABC if and only if P lies on the Kiepert hyperbola.

1. Introduction

In a recent Mathlinks message, the present author proposed the following problem.

Theorem 1. *Given an acute triangle ABC with orthocenter H, the orthocenter H' of the cevian triangle of $\sqrt{H}$, the barycentric square root of H, lies on the Euler line of triangle ABC.*

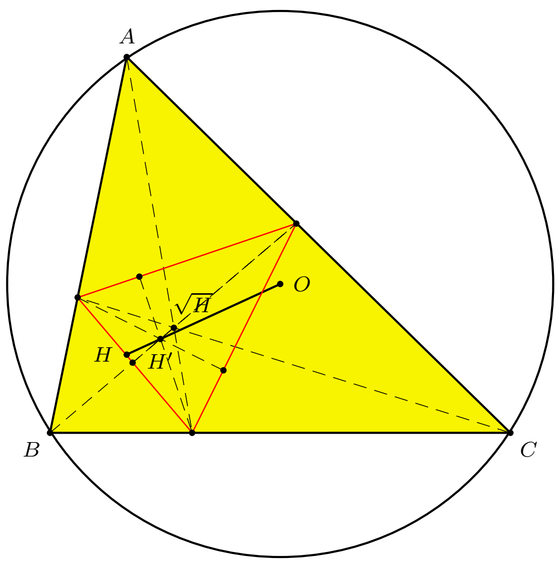

Figure 1.

Paul Yiu has subsequently discovered the following generalization.

Theorem 2. *The locus of point P for which the orthocenter of the cevian triangle of the barycentric square root $\sqrt{P}$ lies on the Euler line is the part of the Kiepert hyperbola which lies inside triangle ABC.*

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The author is grateful to Professor Yiu for his generalization of the problem and his help in the preparation of this paper.

The barycentric square root is defined only for interior points. This is the reason why we restrict to acute angled triangles in Theorem 1 and to the interior points on the Kiepert hyperola in Theorem 2. It is enough to prove Theorem 2.

2. Trilinear polars

Let $A'B'C'$ be the cevian triangle of P, and A_1, B_1, C_1 be respectively the intersections of $B'C'$ and BC, $C'A'$ and CA, $A'B'$ and AB. By Desargues' theorem, the three points A_1, B_1, C_1 lie on a line ℓ_P, the trilinear polar of P.

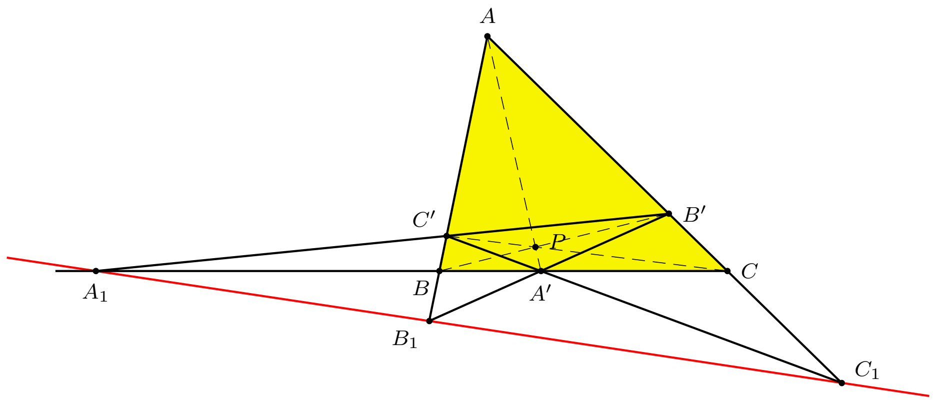

Figure 2.

If P has homogeneous barycentric coordinates $(u : v : w)$, then the trilinear polar is the line

$$\ell_P: \qquad \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.$$

For the orthocenter $H = (S_{BC} : S_{CA} : S_{AB})$, the trilinear polar

$$\ell_H: \qquad S_A x + S_B y + S_C z = 0.$$

is also called the orthic axis.

Proposition 3. *The orthic axis is perpendicular to the Euler line.*

This proposition is very well known. It follows easily, for example, from the fact that the orthic axis ℓ_H is the radical axis of the circumcircle and the nine-point circle. See, for example, [2, §§5.4,5].

The trilinear polar ℓ_P and the orthic axis ℓ_H intersect at the point

$$(u(S_B v - S_C w) : v(S_C w - S_A u) : w(S_A u - S_B v)).$$

In particular, ℓ_P and ℓ_H are parallel, i.e., their intersection is a point at infinity if and only if

$$u(S_B v - S_C w) + v(S_C w - S_A u) + w(S_A u - S_B v) = 0.$$

Equivalently,

$$(S_B - S_C)vw + (S_C - S_A)wu + (S_A - S_B)uv = 0. \tag{1}$$

Note that this equation defines the Kiepert hyperbola. Points on the Kiepert hyperbola are called Kiepert perspectors.

Proposition 4. *The trilinear polar ℓ_P is parallel to the orthic axis if and only if P is a Kiepert perspector.*

3. The barycentric square root of a point

Let P be a point inside triangle ABC, with homogeneous barycentric coordinates $(u : v : w)$. We may assume $u, v, w > 0$, and define the barycentric square root of P to be the point $\sqrt{P}$ with barycentric coordinates $(\sqrt{u} : \sqrt{v} : \sqrt{w})$.

Paul Yiu [2] has given the following construction of $\sqrt{P}$.

(1) Construct the circle $\mathcal{C}_A$ with BC as diameter.

(2) Construct the perpendicular to BC at the trace A' of P to intersect $\mathcal{C}_A$ at X'.

(3) Construct the bisector of angle $BX'C$ to intersect BC at X.

Then X is the trace of $\sqrt{P}$ on BC. Similar constructions on the other two sides give the traces Y and Z of $\sqrt{P}$ on CA and AB respectively. The barycentric square root $\sqrt{P}$ is the common point of AX, BY, CZ.

Proposition 5. *If the trilinear polar ℓ_P intersects BC at A_1, then*

$$A_1X^2 = A_1B \cdot A_1C.$$

Proof. Let M is the midpoint of BC. Since A_1, A' divide B, C harmonically, we have $MB^2 = MC^2 = MA_1 \cdot MA'$ (Newton's theorem). Thus, $MX'^2 = MA_1 \cdot MA'$. It follows that triangles $MX'A_1$ and $MA'X'$ are similar, and $\angle MX'A_1 = \angle MA'X' = 90°$. This means that A_1X' is tangent at X' to the circle with diameter BC. Hence, $A_1X'^2 = A_1B \cdot A_1C$.

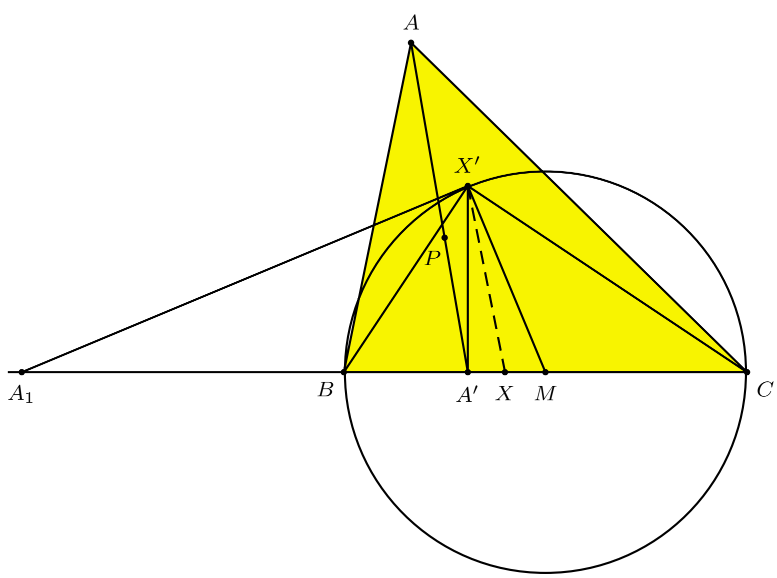

Figure 3.

To complete the proof it is enough to show that $A_1X = A_1X'$, *i.e.*, triangle A_1XX' is isosceles. This follows easily from

$$\begin{aligned}\angle A_1X'X =&\angle A_1X'B + \angle BX'X\\ =&\angle X'CB + \angle XX'C\\ =&\angle X'XA_1.\end{aligned}$$

□

Corollary 6. *If X_1 is the intersection of YZ and BC, then A_1 is the midpoint of XX_1.*

Proof. If X_1 is the intersection of YZ and BC, then X, X_1 divide B, C harmonically. The circle through X, X_1, and with center on BC is orthogonal to the circle $\mathcal{C}_A$. By Proposition 5, this has center A_1, which is therefore the midpoint of XX_1. □

4. Proof of Theorem 2

Let P be an interior point of triangle ABC, and XYZ the cevian triangle of its barycentric square root $\sqrt{P}$.

Proposition 7. *If H' is the orthocenter of XYZ, then the line OH' is perpendicular to the trilinear polar ℓ_P.*

Proof. Consider the orthic triangle DEF of XYZ. Since $DEXY$, $EFYZ$, and $FDZX$ are cyclic, and the common chords DX, EY, FZ intersect at H', H' is the radical center of the three circles, and

$$H'D \cdot H'X = H'E \cdot H'Y = H'F \cdot H'Z. \tag{2}$$

Consider the circles ξ_A, ξ_B, ξ_C, with diameters XX_1, YY_1, ZZ_1. These three circles are coaxial; they are the generalized Apollonian circles of the point $\sqrt{P}$. See [3]. As shown in the previous section, their centers are the points A_1, B_1, C_1 on the trilinear polar ℓ_P. See Figure 4.

Now, since D, E, F lie on the circles ξ_A, ξ_B, ξ_C respectively, it follows from (2) that H' has equal powers with respect to the three circles. It is therefore on the radical axis of the three circles.

We show that the circumcenter O of triangle ABC also has the same power with respect to these circles. Indeed, the power of O with respect to the circle ξ_A is

$$A_1O^2 - A_1X^2 = OA_1^2 - R^2 - A_1X^2 + R^2 = A_1B \cdot A_1C - A_1X^2 + R^2 = R^2$$

by Proposition 5. The same is true for the circles ξ_B and ξ_C. Therefore, O also lies on the radical axis of the three circles. It follows that the line OH' is the radical axis of the three circles, and is perpendicular to the line ℓ_P which contains their centers. □

The orthocenter H' of XYZ lies on the Euler line of triangle ABC if and only if the trilinear polar ℓ_P is parallel to the Euler line, and hence parallel to the orthic axis by Proposition 3. By Proposition 4, this is the case precisely when P lies on the Kiepert hyperbola. This completes the proof of Theorem 2.

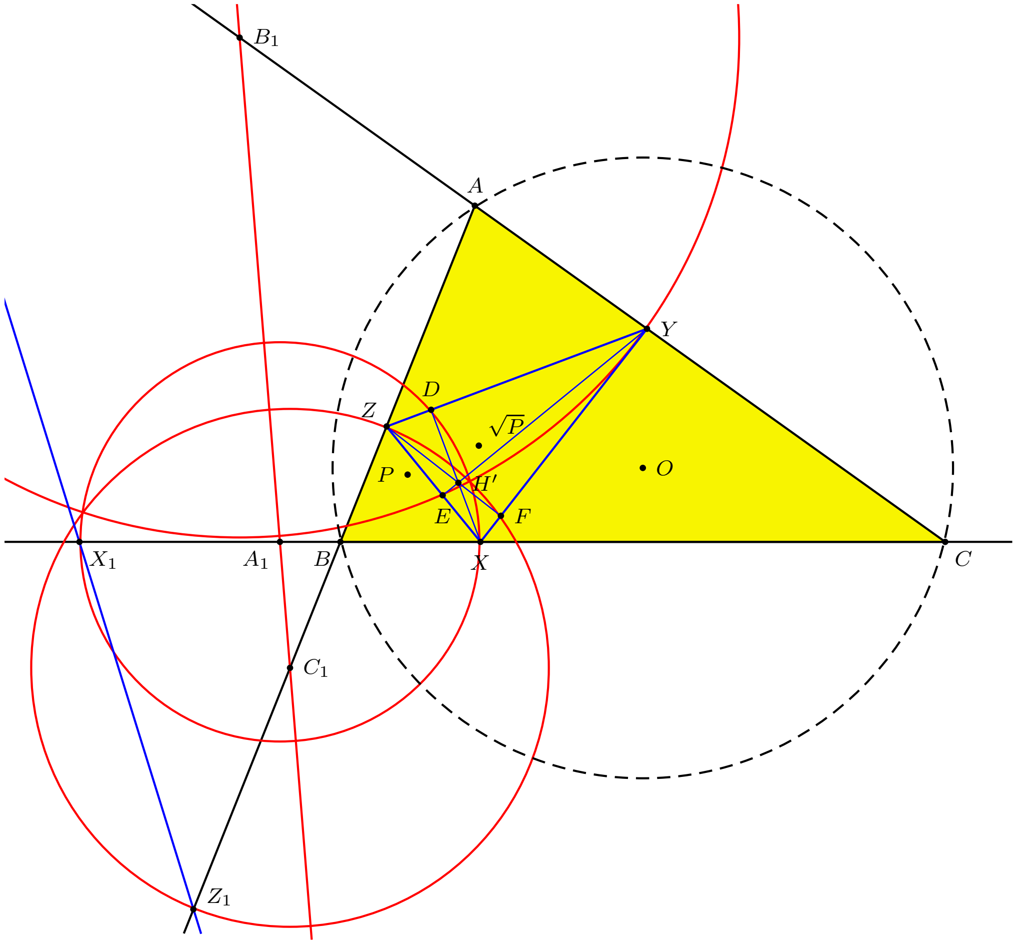

Figure 4.

Theorem 8. *The orthocenter of the cevian triangle of* $\sqrt{P}$ *lies on the Brocard axis if and only if* P *is an interior point on the Jerabek hyperbola.*

Proof. The Brocard axis OK is orthogonal to the Lemoine axis. The locus of points whose trilinear polars are parallel to the Brocard axis is the Jerabek hyperbola. □

5. Coordinates

In homogeneous barycentric coordinates, the orthocenter of the cevian triangle of $(u : v : w)$ is the point

$$\left(\left(S_B\left(\frac{1}{w}+\frac{1}{u}\right)+S_C\left(\frac{1}{u}+\frac{1}{v}\right)\right)\left(-S_A\left(\frac{1}{v}+\frac{1}{w}\right)^2+S_B\left(\frac{1}{u^2}-\frac{1}{w^2}\right)+S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)\right)\right.$$
$$\left.:\cdots:\cdots\right).$$

Applying this to the square root of the orthocenter, with $(u^2 : v^2 : w^2) = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$, we obtain

$$\left(a^2S_A\cdot\sqrt{S_{ABC}}+S_{BC}\sum_{\text{cyclic}}a^2\sqrt{S_A}:\cdots:\cdots\right),$$

which is the point H' in Theorem 1.

More generally, if P is the Kiepert perspector

$$K(\theta)=\left(\frac{1}{S_A+S_\theta}:\frac{1}{S_B+S_\theta}:\frac{1}{S_C+S_\theta}\right),$$

the orthocenter of the cevian triangle of $\sqrt{P}$ is the point

$$\left(a^2S_A\sqrt{(S_A+S_\theta)(S_B+S_\theta)(S_C+S_\theta)}\right.$$
$$\left.+S_{BC}\sum_{\text{cyclic}}a^2\sqrt{S_A+S_\theta}+a^2S_\theta\sum_{\text{cyclic}}S_A\sqrt{S_A+S_\theta}:\cdots:\cdots\right).$$

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Translated Triangles Perspective to a Reference Triangle

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Abstract. Suppose A, B, C, D, E, F are points and L is a line other than the line at infinity. This work examines cases in which a translation $D'E'F'$ of DEF in the direction of L is perspective to ABC, in the sense that the lines AD', BE', CF' concur.

1. Introduction

In the transfigured plane of a triangle ABC, let L^∞ be the line at infinity and L a line other than L^∞. (To say "transfigured plane" means that the sidelengths a, b, c of triangle ABC are variables or indeterminates, and points are defined as functions of a, b, c, so that the "plane" of ABC is infinite dimensional.) Suppose that D, E, F are distinct points, none on L^∞, such that the set $\{A, B, C, D, E, F\}$ consists of at least five distinct points. We wish to translate triangle DEF in the direction of L and to discuss cases in which the translated triangle $D'E'F'$ is perspective to ABC, in the sense that the lines AD', BE', CF' concur. One of these cases is the limiting case that $D' = L \cap L^\infty$; call this point U, and note that $D' = E' = F' = U$.

Points and lines will be given (indeed, are *defined*) by homogeneous trilinear coordinates. The line L^∞ at infinity is given by $a\alpha + b\beta + c\gamma = 0$, and L, by an equation $l\alpha + m\beta + n\gamma = 0$, where $l : m : n$ is a point. Then the point $U = u : v : w$ is given by

$$u = bn - cm, \; v = cl - an, \; w = am - bl. \tag{1}$$

Write the vertices of DEF as

$$D = d_1 : e_1 : f_1, \qquad E = d_2 : e_2 : f_2, \qquad F = d_3 : e_3 : f_3,$$

and let

$$\delta = ad_1 + be_1 + cf_1, \qquad \epsilon = ad_2 + be_2 + cf_2, \qquad \varphi = ad_3 + be_3 + cf_3.$$

The hypothesis that none of D, E, F is on L^∞ implies that none of $\delta, \epsilon, \varphi$ is 0. The line L is given parametrically as the locus of point $D' = D_t = x_1 : y_1 : z_1$ by

$$x_1 = d_1 + \delta tu, \qquad y_1 = e_1 + \delta tv, \qquad z_1 = f_1 + \delta tw.$$

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The point E' traverses the line through E parallel to L, so that $E' = E_t = x_2 : y_2 : z_2$ is given by

$$x_2 = d_2 + \epsilon tu, \qquad y_2 = e_2 + \epsilon tv, \qquad z_2 = f_2 + \epsilon tw.$$

The point F' traverses the line through F parallel to L, so that $F' = F_t = x_3 : y_3 : z_3$ is given by

$$x_3 = d_3 + \varphi tu \quad y_3 = e_3 + \varphi tv \quad z_3 = f_3 + \varphi tw.$$

In these parameterizations, t represents a homogeneous function of a, b, c. The degree of homogeneity of t is that of $(x_1 - d_1)/(\delta u)$.

2. Two basic theorems

Theorem 1. *Suppose ABC and DEF are triangles such that $\{A, B, C, D, E, F\}$ consists of at least five distinct points. Suppose L is a line and $U = L \cap L^\infty$. As D_t traverses the line DU, the triangle $D_tE_tF_t$ of translation of DEF in the direction of L is either perspective to ABC for all t or else perspective to ABC for at most two values of t.*

Proof. The lines AD_t, BE_t, CF_t are given by the equations

$$-z_1\beta + y_1\gamma = 0, \quad z_2\alpha - x_2\gamma = 0, \quad -y_3\alpha + x_3\beta = 0,$$

respectively. Thus, the concurrence determinant,

$$\begin{vmatrix} 0 & -z_1 & y_1 \\ z_2 & 0 & -x_2 \\ -y_3 & x_3 & 0 \end{vmatrix} \tag{2}$$

is a polynomial P, formally of degree 2 in t:

$$P(t) = p_0 + p_1 t + p_2 t^2, \tag{3}$$

where

$$p_0 = d_3e_1f_2 - d_2e_3f_1, \tag{4}$$
$$p_1 = u(\varphi e_1f_2 - \epsilon e_3f_1) + v(\delta d_3f_2 - \varphi d_2f_1) + w(\epsilon e_1d_3 - \delta e_3d_2), \tag{5}$$
$$p_2 = \delta vw(\epsilon d_3 - \varphi d_2) + \epsilon wu(\varphi e_1 - \delta e_3) + \varphi uv(\delta f_2 - \epsilon f_1). \tag{6}$$

Thus, either p_0, p_1, p_0 are all zero, in which case $D_tE_tF_t$ is perspective to ABC for all t, or else $P(t)$ is zero for at most two values of t. □

If triangle DEF is homothetic to ABC, then $D_tE_tF_t$ is homothetic to ABC and hence perspective to ABC, for every t. This is well known in geometry. The geometric theorem, however, does not imply the "same" theorem in the more general setting of triangle algebra, in which the objects are defined in terms of variables or indeterminants. Specifically, perspectivity and parallelism (hence homothety) are defined by zero determinants. When such determinants are "symbolically zero", they are zero not only for Euclidean triangles, for which a, b, c are positive real numbers satisfying $(a > b + c,\ b > c + a,\ c > a + b)$ or $(a \geq b + c,\ b \geq c + a,$

$c \geq a+b$), but also for a, b, c as indeterminates. Among geometric theorems that readily generalize to algebraic theorems are these:

If $L_1 \| L_2$ *and* $L_2 \| L_3$, *then* $L_1 \| L_3$.
If T_1 *is homothetic to* T_2 *and* T_2 *is homothetic to* T_3, *then* T_1 *is homothetic to* T_3.
If T_1 *is homothetic to* T_2, *then* T_1 *is perspective to* T_2.

Theorem 2. *Suppose* ABC *and* DEF *in Theorem 1 are homothetic. Then* $D_tE_tF_t$ *is perspective to* ABC *for all* t.

Proof. ABC and DEF are homothetic, and DEF and $D_tE_tF_t$ are homothetic. Therefore $D_tE_tF_t$ is homothetic to ABC, which implies that $D_tE_tF_t$ is perspective to ABC. □

It is of interest to express the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F. To that end, we shall use cofactors, as defined by the identity

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} D_1 & D_2 & D_3 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix},$$

where

$$\Delta = \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix} = d_1D_1 + e_1E_1 + f_1F_1;$$

that is, $D_1 = e_2f_3 - f_2e_3$, etc. For example, in the case that ABC and DEF are homothetic, line EF is parallel to line BC, as defined by a zero determinant (e.g., [1], p. 29); likewise, the lines FD and CA are parallel, as are DE and AB. The zero determinants yield

$$bF_1 = cE_1, \qquad cD_2 = aF_2, \qquad aE_3 = bD_3. \tag{7}$$

These equations can be used to give a direct but somewhat tedious proof of Theorem 2; we digress to prove only that $p_0 = 0$. Let $\mathcal{L}$ and $\mathcal{R}$ denote the products of the left-hand sides and the right-hand sides in (7). Then $\mathcal{L} - \mathcal{R}$ factors as $abc\Psi\Delta$, where

$$\Psi = e_3d_2f_1 - e_1d_3f_2,$$

and $abc\Psi\Delta = 0$ by (7). It is understood that A, B, C are not collinear, so that D, E, F are not collinear. As the defining equation for collinearity of D, E, F is the determinant equation $\Delta = 0$, we have $\Delta \neq 0$. Therefore, $\Psi = 0$, so that $p_0 = 0$.

Next, substitute from (1) for u, v, w in (5) and (6), getting

$$p_1 = lp_{1l} + mp_{1m} + np_{1n} \qquad \text{and} \qquad p_2 = mnp_{2l} + nlp_{2m} + lm_{2n},$$

where

$$\begin{aligned} p_{1l} =& b^2e_1F_1 + c^2f_1E_1 - abd_2F_2 - cad_3E_3, \\ p_{1m} =& c^2f_2D_2 + a^2d_2F_2 - bce_3D_3 - abe_1F_1, \\ p_{1n} =& a^2d_3E_3 + b^2e_3D_3 - caf_1E_1 - bcf_2D_2, \end{aligned}$$

and

$$\begin{aligned} p_{2l} =& 2a^2bc(e_1d_2f_3 - e_2d_3f_1) - ab^2e_3F_3 + ac^2f_2E_2 \\ & - bc^2f_1D_1 + ba^2d_3F_3 - ca^2d_2E_2 + cb^2e_1D_1, \\ p_{2m} =& 2b^2ca(f_2e_3d_1 - f_3e_1d_2) - bc^2f_1D_1 + ba^2d_3F_3 \\ & - ca^2d_2E_2 + cb^2e_1D_3 - ab^2e_3F_3 + ac^2f_2E_2, \\ p_{2n} =& 2c^2ab(e_3d_1e_2 - d_1f_2e_3) - ca^2d_2E_2 + cb^2e_1D_1 \\ & - ab^2e_3F_3 + ac^2f_2E_1 - bc^2f_1D_1 + ba^2d_3F_3. \end{aligned}$$

The task of expressing the coefficients p_0, p_1, p_2 more directly in terms of a, b, c and the coordinates of D, E, F is now completed.

3. Intersecting conics

We begin with a lemma proved in [7]; see also [2].

Lemma 3. *Suppose a point $P = p : q : r$ is given parametrically by*

$$\begin{aligned} p =& p_1t^2 + q_1t + r_1, \\ q =& p_2t^2 + q_2t + r_2, \\ r =& p_3t^2 + q_3t + r_3, \end{aligned}$$

where the matrix

$$M = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$$

is nonsingular with adjoint (cofactor) matrix

$$M^{\#} = \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{pmatrix}.$$

Then P lies on this conic:

$$(Q_1\alpha + Q_2\beta + Q_3\gamma)^2 = (P_1\alpha + P_2\beta + P_3\gamma)(R_1\alpha + R_2\beta + R_3\gamma). \qquad (8)$$

In Theorem 4, we shall show that the point of concurrence of the lines AD_t, BE_t, CF_t is also the point of concurrence of three conics. Let

$$A_t = BE_t \cap CF_t, \quad B_t = CF_t \cap AD_t, \quad C_t = AD_t \cap BE_t.$$

Theorem 4. *If, in Theorem 1, the line L is not parallel to a sideline of triangle ABC, then the locus of each of the points A_t, B_t, C_t is a conic.*

Proof. The point $A_t = a_t : b_t : c_t$ is given by

$$a_t = x_2x_3, \qquad b_t = x_2y_3, \qquad c_t = z_2x_3,$$

so that

$$a_t = (d_2 + \epsilon tu)\,(d_3 + \varphi tu) = \epsilon\varphi u^2t^2 + (\varphi d_2u + \epsilon d_3u)t + d_2d_3, \tag{9}$$

$$b_t = (d_2 + \epsilon tu)\,(e_3 + \varphi tv) = \epsilon\varphi uvt^2 + (\varphi d_2v + \epsilon e_3u)t + d_2e_3, \tag{10}$$

$$c_t = (f_2 + \epsilon tw)\,(d_3 + \varphi tu) = \epsilon\varphi uwt^2 + (\varphi f_2u + \epsilon d_3w)t + f_2d_3. \tag{11}$$

By Lemma 3, the locus $\{A_t\}$ is a conic unless $\epsilon\varphi uvw = 0$, in which case u, v, or w must be zero. Consider the case $u = 0$; then $u : v : w = 0 : c : -b$, but this is the point in which line BC meets L^∞, contrary to the hypothesis. Likewise, the loci $\{B_t\}$ and $\{C_t\}$ are conics. Note that the conic $\{A_t\}$ passes through B and C, as indicated by Figure 1. [1] □

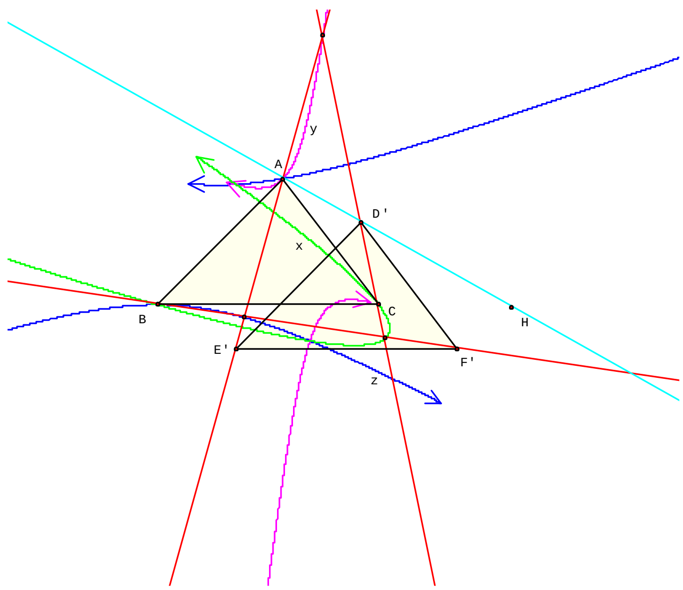

Figure 1. Intersecting conics

Lemma 3 shows how to write out equations of conics starting with a matrix M. As the lemma applies only to nonsingular M, we can, by factoring $|M|$, determine

[1] Figure 1 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. The choice of triangle DEF is given by the equations $D = C$, $E = A$, $F = B$. (The labels D, E, F are not shown.) Point H on line AD' is an independent point, and triangle $D'E'F'$ is a translation of DEF in the direction of line AH. Except for special cases, as D' traverses line AH, points x, y, z traverse conics as in Theorem 4, and the conics meet twice (with $x = y = z$), at the perspectors described in Theorem 1.

criteria for nonsingularity. In connection with $\{A_t\}$ and (9)-(11),

$$|M| = \begin{vmatrix} \epsilon\varphi u^2 & \varphi d_2 u + \epsilon d_3 u & d_2 d_3 \\ \epsilon\varphi uv & \varphi d_2 v + \epsilon e_3 u & d_2 e_3 \\ \epsilon\varphi wu & \varphi f_2 u + \epsilon d_3 w & f_2 d_3 \end{vmatrix}$$
$$= \epsilon\varphi u\,(uf_2 - wd_2)\,(vd_3 - ue_3)\,(be_3 d_2 - be_2 d_3 + cd_2 f_3 - cd_3 f_2)\,.$$

By hypothesis, $\epsilon\varphi u \neq 0$. Also, $(uf_2 - wd_2)\,(vd_3 - ue_3) \neq 0$, as it is assumed that $E \neq U$ and $F \neq U$. Finally, the factor $be_3d_2 - be_2d_3 + cd_2f_3 - cd_3f_2$ is 0 if and only if line EF is parallel to line BC. In conclusion, if EF is not parallel to BC, and FD is not parallel to CA, and DE is not parallel to AB, then the three loci are conics and Theorem 3 applies.

4. Terminology and notation

The main theorem in this paper is Theorem 1. For various choices of DEF and L, the perspectivities indicated by Theorem 1 are of particular interest. Such choices are considered in Sections 3-6; they are, briefly, that DEF is a cevian triangle of a point, or an anticevian triangle, or a rotation of triangle ABC about its circumcenter. In order to describe the configurations, it will be helpful to adopt certain terms and notations.

Unless otherwise noted, the points $U = u : v : w$ and $P = p : q : r$ are arbitrary. If at least one of the products up, vq, wr is not zero, the product $U \cdot P$ is defined by the equation

$$U \cdot P = up : vq : wr.$$

The multiplicative inverse of P, defined if $pqr \neq 0$, is the isogonal conjugate of P, given by

$$P^{-1} = p^{-1} : q^{-1} : r^{-1}.$$

The quotient U/P is defined by

$$U/P = U \cdot P^{-1}.$$

The *isotomic conjugat*e of P is defined if $pqr \neq 0$ by the trilinears

$$a^{-2}p^{-1} : b^{-2}q^{-1} : c^{-2}r^{-1}.$$

Geometric definitions of isogonal and isotomic conjugates are given at *MathWorld* [8]. We shall also employ these terms and notations:

crossdifference of U and $P = CD(U,P) = rv - qw : pw - ru : qu - pv$,

crosssum of U and $P = CS(U,P) = rv + qw : pw + ru : qu + pv$,

crosspoint of U and $P = CP(U,P) = pu(rv + qw) : qv(pw + ru) : rw(qu + pv)$.

Geometric interpretations of these "cross operations" are given at [4].

Notation of the form X_i as in [3] will be used for certain special points, such as

$$\begin{aligned}\text{incenter} =&X_1 = 1:1:1 = \text{ the multiplicative identity},\\ \text{centroid} =&X_2 = 1/a:1/b:1/c,\\ \text{circumcenter} =&X_3 = \cos A : \cos B : \cos C,\\ \text{symmedian point} =&X_6 = a:b:c.\end{aligned}$$

Instead of the trigonometric trilinears for X_3, we shall sometimes use trilinears for X_3 expressed directly in terms of a, b, c. As $\cos A = (b^2 + c^2 - a^2)/(2bc)$, we shall use abbreviations:

$$a_1 = (b^2+c^2-a^2)/(2bc), \quad b_1 = (c^2+a^2-b^2)/(2ca), \quad c_1 = (a^2+b^2-c^2)/(2ab);$$

thus, $X_3 = a_1 : b_1 : c_1$.

The line X_2X_3 is the Euler line, and the line X_3X_6 is the Brocard axis. When working with lines algebraically, it is sometimes helpful to do so with reference to a parameter and the point in which the line meets L^∞. In the case of the Euler line, this point is

$$X_{30} = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

and a parametric representation is given by $x : y : z = x(s) : y(s) : z(s)$, where

$$\begin{aligned}x(s) =&a_1 + s(a_1 - 2b_1c_1),\\ y(s) =&b_1 + s(b_1 - 2c_1a_1),\\ z(s) =&c_1 + s(c_1 - 2a_1b_1).\end{aligned}$$

The point X_{30} will be called the direction of the Euler line. More generally, for any line, its point of intersection with L^∞ will be called the *direction* of the line. The parameter s is not necessarily a numerical variable; rather, it is a function of a, b, c. In this paper, trilinears for any point are homogeneneous functions of a, b, c, all having the same degree of homogeneity; thus in a parametric expression of the form $p + su$, the degree of homogeneity of s is that of p minus that of u.

Two families of cubics will occur in the sequel. The cubic $\mathcal{Z}(U, P)$ is given by

$$(vqy - wrz)px^2 + (wrz - upx)qy^2 + (upx - vqy)rz^2 = 0,$$

and the cubic $\mathcal{ZC}(U, P)$, by

$$L(wy - vz)x^2 + M(uz - wx)y^2 + N(vx - uy)z^2 = 0.$$

For details on these and other families of cubics, see [5].

The remainder of this article is mostly about special translations. It will be helpful to introduce some related terminology. Suppose DEF is a triangle in the transfigured plane of ABC, and U is a direction (i.e., a point on L^∞). A triangle $D'E'F'$, other than DEF itself, such that $D'E'F'$ is a U-translation of DEF and $D'E'F'$ is perspective to ABC (in the sense that the lines AD', BE', CF' concur) will be called a *U-ppt of DEF*. The designation "ppt" means "proper perspective translation".

In view of Theorem 1, except for special cases, each DEF has, for every U, at most two U-ppt's. Thus, if DEF is perspective to ABC, as when DEF is a cevian triangle or an anticevian tiangle, there is "usually" just one ppt. That one ppt is of primary interest in the next three sections; especially in Case 5.4 and Case 6.4.

5. Translated cevian triangles

In this section, DEF is the cevian triangle of a point $X = x : y : z$; thus DEF is given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix},$$

and the perspectivity determinant (2) is given by $-t(\Delta_0 + t\Delta_1)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} a^2ux^2 & b^2vy^2 & c^2wz^2 \\ u & v & w \\ x & y & z \end{vmatrix}.$$

In particular, the equations $\Delta_0 = 0$ and $\Delta_1 = 0$ represent cubics in x, y, z, specifically, $\mathcal{Z}(X_2, X_6 \cdot U^{-1})$ and $\mathcal{Z}(X_{75} \cdot U^{-1}, X_{31})$, respectively. We shall consider four cases:

Case 5.1: $\Delta_0 = 0$ and $\Delta_1 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, let X be the isotomic conjugate of U. Rows 1 and 3 of the determinant Δ_1 are equal, so that $\Delta_1 = 0$. Also,

$$\begin{aligned} \Delta_0 =& bcvw(b^2v^2 - c^2w^2) + cawu(c^2w^2 - a^2u^2) + abuv(a^2u^2 - b^2v^2) \\ =& -(bv - cw)(cw - au)(au - bv)(au + bv + cw) \\ =& 0. \end{aligned}$$

The cevian triangle DEF of X is not homothetic to ABC, yet $D_tE_tF_t$ is perspective to ABC for every t. Another such example is obtained by simply taking X to be U. Further results in Case 1 are given in Theorem 5.

Case 5.2: $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For given U, the point $X = CP(X_2, U)$ satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For quite a different example, let

$$U = X_{511} = \cos(A + \omega) : \cos(B + \omega) : \cos(C + \omega),$$

where ω denotes the Brocard angle. Then the cubic $\Delta_0 = 0$ passes through the following points, X_3 (the circumcenter), X_6 (the symmedian point), X_{297}, X_{325}, X_{694}, X_{2009}, and X_{2010}, none of which lies on the cubic $\Delta_1 = 0$. Other points on the cubic $\Delta_0 = 0$ are given at [5], where the cubic $\Delta_1 = 0$ is classified as $\mathcal{ZC}(511, L(30, 511))$.

Case 5.3: $\Delta_0 \neq 0$ and $\Delta_1 = 0$. In this case, for any U, there is no U-ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_1 = 0$ passes through the two points in which the Euler line meets the circumcircle, these being X_{1113} and X_{1114}, and these points do not also lie on the cubic $\Delta_0 = 0$.

Case 5.4: $\Delta_0 \neq 0$ and $\Delta_1 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_1$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which, after cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{(bv - cw)yz + ax(vz - wy)}, \tag{12}$$

$$y' = \frac{cz - ax}{(cw - au)zx + by(wx - uz)}, \tag{13}$$

$$z' = \frac{ax - by}{(au - bv)xy + cz(uy - vx)}. \tag{14}$$

Note that if X and U are triangle centers for which X' is a point, then X' is a triangle center. For the special case $U = X_{511}$, pairs X and X' are shown here:

X	4	7	54	68	69	99	183	190	385	401	668	670	671	903
X'	3	256	52	52	6	690	262	900	325	297	691	888	690	900

Returning to Case 5.1, in the subcase that X is the isotomic conjugate of U, it is natural to ask about the perspectors, and to find the following theorem.

Theorem 5. *Suppose U is any point on L^∞ but not on a sideline BC, CA, AB. Let X be the isotomic conjugate of U. The locus of the perspector P_t of triangles $D_tE_tF_t$ and ABC is a conic that passes through A, B, C, and the point*

$$X^2 = b^4c^4v^2w^2 : c^4a^4w^2u^2 : a^4b^4u^2v^2. \tag{15}$$

Proof. The perspector is the point $P_t = x_2x_3 : x_2y_3 : z_2x_3$. Substituting and simplifying give

$$\begin{aligned} P_t = & b^3c^3vw(bv - acuwt)(cw - abuvt) \\ & : c^3a^3wu(cw - bavut)(au - bcvwt) \\ & : a^3b^3uv(au - cbwvt)(bv - cawut). \end{aligned}$$

By Theorem 3, the locus of P_t is a conic. Clearly, P_t passes through A, B, C for $t = au/(bcvw), bv/(cawu), cw/(abuv)$, respectively, and P_0 is the point given by (15). See Figure 2. [2] □

An equation for the circumconic described in Theorem 5 is found from (8):

$$b^2c^2(b^2v^2 - c^2w^2)\beta\gamma + c^2a^2(c^2w^2 - a^2u^2)\gamma\alpha + a^2b^2(a^2u^2 - b^2v^2)\alpha\beta = 0.$$

Theorem 6. *Suppose X is the isotomic conjugate of a point U_1 on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 5.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' is on L^∞.*

[2] Figure 2 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. An arbitrary point U on L^∞ is given by $U = Au \cap L^\infty$, where u is an independent point; i.e., the user can vary u freely. The cevian triangle of U is def, the cevian triangle of the isotomic conjugate of U is DEF. Point D' is movable on line DU. Triangle $D'E'F'$ is thus a movable translation of DEF in the direction of U, and $D'E'F'$ stays perspective to ABC. The perspector P traverses a circumconic.

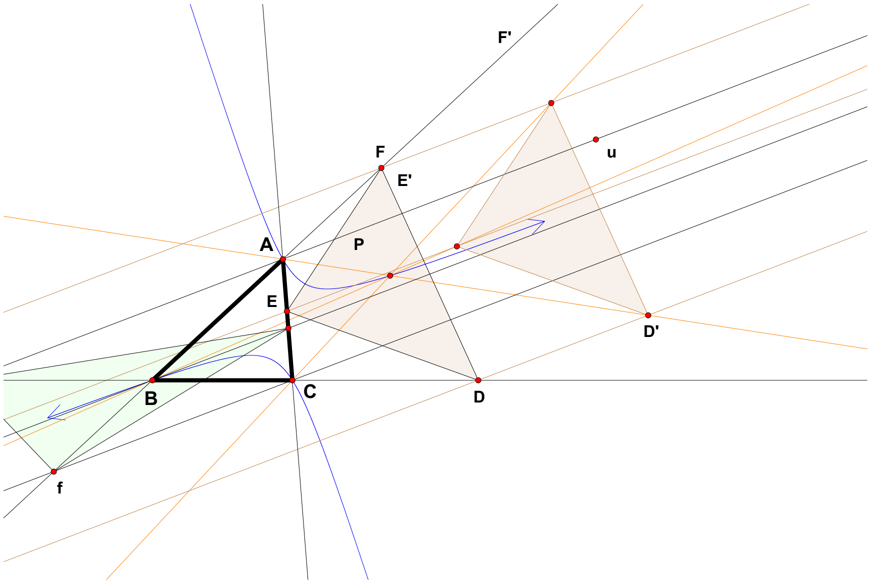

Figure 2. Cevian triangle and circumconic as in Theorem 5.

Proof. Write $X = x : y : z = b^2c^2v_1w_1 : c^2a^2w_1u_1 : a^2b^2u_1v_1$ where the point $U_1 = u_1 : v_1 : w_1$ is on L^∞ and $u_1v_1w_1 \neq 0$. Represent U_1 parametrically by

$$u_1 = (b-c)(1+bcs), \quad v_1 = (c-a)(1+cas), \quad w_1 = (a-b)(1+abs), \quad (16)$$

so that

$$(bv-cw)yz+ax(vz-wy) = \Lambda\left(bv - av - aw + cw + as(b^2v - abv + c^2w - acw)\right),$$

where

$$\Lambda = -a^3b^3c^3(b-c)(c-a)(a-b)(1+bcs)(1+cas)(1+abs).$$

Thus

$$(bv-cw)yz + ax(vz-wy) = \Lambda(-a(u+v+w) + as(a^2u + b^2v + c^2w)),$$

and by (12),

$$\begin{aligned} x' =& \frac{by-cz}{(bv-cw)yz + ax(vz-wy)} \\ =& \frac{(by-cz)/a}{\Lambda(-(u+v+w) + s(a^2u + b^2v + c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$\begin{aligned} x' : y' : z' =& (by-cz)/a : (cz-ax)/b : (ax-by)/c \\ =& u_1(bv_1 - cw_1) : v_1(cw_1 - au_1) : w_1(au_1 - bv_1) \\ =& CD(X_6, U_1^{-1}). \end{aligned}$$

Clearly, $ax' + by' + cz' = 0$, which is to say that the perspector X' is on L^∞. $\square$

Corollary 7. *As X traverses the Steiner circumellipse, the perspector X' traverses the line at infinity.*

Proof. The Steiner circumellipse is given by

$$bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta = 0.$$

The corollary follows from the easy-to-verify fact that the isotomic conjugacy mapping carries the Steiner circumellipse to L^∞, to which Theorem 6 applies. □

Theorem 7 is exemplified by taking $X = X_{190}$; the isotomic conjugate of X is then X_{514}, for which the perspector is $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{99}, X_{690}), (X_{668}, X_{891}), (X_{670}, X_{888}), (X_{671}, X_{690}), (X_{903}, X_{900}). These examples show that the mapping $X \to X'$ is not one-to-one.

6. Translated anticevian triangles

In this section, DEF is the anticevian triangle of a point $X = x : y : z$; given as a matrix by

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

The perspectivity determinant (2) is given by $-2t(\Delta_0 + t\Delta_2)$, where

$$\Delta_0 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ u & v & w \\ x & y & z \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} ax^2 & by^2 & cz^2 \\ (bv+cw)u & (cw+au)v & (au+bv)w \\ x & y & z \end{vmatrix}.$$

The cubic $\Delta_0 = 0$ is already discussed in Section 3. The equation $\Delta_2 = 0$ represents the cubic $\mathcal{Z}(X_2/CS(X_6, U^{-1}))$. We consider four cases as in Section 3.

Case 6.1: $\Delta_0 = 0$ and $\Delta_2 = 0$. In this case, $D_tE_tF_t$ is perspective to ABC for every t. Clearly this holds for $X = X_2$, for all U. Now for any given U, the point $X = U$ is on both cubics. It is easy to prove that the point $CP(X_2, U)$ also lies on both cubics.

Case 6.2: $\Delta_0 = 0$ and $\Delta_2 \neq 0$. For given U, the isotomic conjugate X of U satisfies $\Delta_0 = 0$ and $\Delta_1 \neq 0$. For a diffferent example, let $U = X_{511}$; then the points listed for Case 5.2 in the cevian case are also points for which $\Delta_0 = 0$ and $\Delta_2 \neq 0$.

Case 6.3: $\Delta_0 \neq 0$ and $\Delta_2 = 0$. In this case, for any U, there is no ppt. For example, take $U = X_{523}$. Then the cubic $\Delta_2 = 0$ passes through the points in which the Brocard axis X_3X_6 meets the circumcircle these being X_{1312} and X_{1313}; these points do not also lie on the cubic $\Delta_0 = 0$.

Case 6.4: $\Delta_0 \neq 0$ and $\Delta_2 \neq 0$. In this case, $D_tE_tF_t$ is perspective to ABC for $t = -\Delta_0/\Delta_2$. The perspector is the point $x_2x_3 : x_2y_3 : z_2x_3$, which on cancellation of common factors, is the point $X' = x' : y' : z'$ given by

$$x' = \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)}, \tag{17}$$

$$y' = \frac{cz - ax}{cuz^2 - awx^2 + (cw - au)zx + by(wx - uz)}, \tag{18}$$

$$z' = \frac{ax - by}{avx^2 - buy^2 + (au - bv)xy + cz(uy - vx)}. \tag{19}$$

Theorem 8. *Suppose $X = CP(X_2, U_1)$, where U_1 is a point on L^∞ but not on a sideline BC, CA, AB. Then the perspector X' in Case 6.4 is invariant of the point U. In fact, $X' = CD(X_6, U_1^{-1})$, and X' lies on the circumconic given by*

$$au_1^2(bv_1 - cw_1)\beta\gamma + bv_1^2(cw_1 - au_1)\gamma\alpha + cw_1^2(au_1 - bv_1)\alpha\beta = 0. \tag{20}$$

Proof. Let $X = x : y : z = bu_1v_1 + cu_1w_1 : cv_1w_1 + av_1u_1 : aw_1u_1 + bw_1v_1$. Following the steps of the proof of Theorem 6, we have

$$\begin{aligned} &bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) \\ =&\widehat{\Lambda}(bv - av - aw + cw + as(b^2v - abv + c^2w - acw), \end{aligned}$$

where

$$\widehat{\Lambda} = 2abc(b - c)(c - a)(a - b)(1 + bcs)(1 + cas)(1 + abs).$$

Thus

$$bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy) = \widehat{\Lambda}(-a(u + v + w) + as(a^2u + b^2v + c^2w)),$$

and by (17),

$$\begin{aligned} x' =& \frac{by - cz}{bwy^2 - cvz^2 + (bv - cw)yz + ax(vz - wy)} \\ =& \frac{(by - cz)/a}{\widehat{\Lambda}(-(u + v + w) + s(a^2u + b^2v + c^2w))}. \end{aligned}$$

Coordinates y' and z' are found in the same manner, and multiplying through by the common denominator gives

$$x' : y' : z' = CD(X_6, U_1^{-1}),$$

the same point as at the end of the proof of Theorem 6. It is easy to check that this point satisfies (20). □

Corollary 9. *As X traverses the Steiner inellipse, the perspector X' traverses the circumconic (20).*

Proof. The Steiner inellipse is given by

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\beta\gamma - 2ca\gamma\alpha - 2ab\alpha\beta = 0. \tag{21}$$

First, we note that, using (16), it is easy to show that if U_1 is on L^∞, then the point $X = x : y : z = CP(X_2, U_1)$ satisfies (21). Now, the mapping $U_1 \rightarrow$

$CP(X_2, U_1) = X$ is invertible; specifically, for given $X = x : y : z$ on the Steiner inellipse, the point $U_1 = u_1 : v_1 : w_1$ given by

$$u_1 : v_1 : w_1 = \frac{bc}{by + cz - ax} : \frac{ca}{cz + ax - by} : \frac{ab}{ax + by - cz}$$

is on L^∞, and Theorem 8 applies. □

Corollary 9 is exemplified by taking $X = X_{1086}$, which is $CP(X_2, X_{514})$; the perspector is then $X' = X_{900} = CD(X_6, X_{101})$. Other examples (X, X') are these: (X_{115}, X_{690}), (X_{1015}, X_{891}), (X_{1084}, X_{888}), (X_{2482}, X_{690}). Note that the mapping $X \to X'$ is not one-to-one.

7. Translation along the Euler line

In this section, the perspectivity problem for both families, cevian and anticevian triangles, is discussed for translations in a single direction, namely the direction of the Euler line. Two points on the Euler line are the circumcenter, $a_1 : b_1 : c_1 = \cos A : \cos B : \cos C$ and

$$U = X_{30} = u : v : w = a_1 - 2b_1c_1 : b_1 - 2c_1a_1 : c_1 - 2a_1b_1,$$

the latter being the point in which the Euler line meets L^∞.

Theorem 10. *If X is the isotomic conjugate of a point X' on the Euler line other than X_2, then the perspector, in the case of the cevian triangle of X as given by (12)-(14), is X'.*

Proof. An arbitrary point X' on the Euler line is given parametrically by

$$a_1 + su : b_1 + sv : c_1 + sw,$$

and the isotomic conjugate $X = x : y : z$ by

$$a^{-2}(a_1 + su)^{-1} : b^{-2}(b_1 + sv)^{-1} : c^{-2}(c_1 + sw)^{-1}.$$

Substituting for x, y, z in (12) gives a product of several factors, of which exactly two involve s. The same holds for the results of substituting in (13) and (14). After canceling all common factors that do not contain s, the remaining coordinates for X' have a common factor $3s + 1$. This equals 0 for $s = -1/3$, for which $a_1 + su : b_1 + sv : c_1 + sw = X_2$. As $X' \neq X_2$, we can and do cancel $3s + 1$. The remaining coordinates are equivalent to those given just above for X'. □

Theorem 11. *Suppose P is on the Euler line and $P \neq X_2$. Let $X = CP(X_2, P)$. Then the perspector X', in the case of the anticevian triangle of X, as given by (17)-(19), is the point P.*

Proof. Write

$$p = a_1 + su, \qquad q = b_1 + sv, \qquad r = c_1 + sw,$$

where $(u, v, w) = (a_1 - 2b_1c_1, b_1 - 2c_1a_1, c_1 - 2a_1b_1)$, so that the point $X = CP(X_2, P)$ is given by

$$x = p(bq + cr), \qquad y = q(cr + ap), \qquad z = r(ap + bq).$$

Substituting into (17)-(19) and factoring give expressions with several common factors. Canceling those, including the factor $3s+1$ which corresponds to the disallowed X_2, leaves trilinears for P. □

Theorem 12. *If P is on the circumcircle and $X = CS(X_6, P)$, then the perspector X', in the case of the anticevian triangle of X as given by (17)-(19), is the point $CD(X_6, P)$.*

Proof. Represent an arbitrary point $P = p : q : r$ on the circumcircle parametrically by

$$(p, q, r) = \left(\frac{1}{(b-c)(bc+s)}, \frac{1}{(c-a)(ca+s)}, \frac{1}{(a-b)(ab+s)}\right).$$

Then the point $X = CS(X_6, P)$ is given by

$$x = br + cq, \qquad y = cp + ar, \qquad z = aq + bp.$$

Substituting into (17)-(19) and factoring gives expressions with several common factors. Canceling those leaves trilinears for $CD(X_6, P)$. □

We conclude this section with a pair of examples. First, let $X = X_{618}$, the complement of the Fermat point (or 1st isogonic center), X_{13}. The perspector in the case of the anticevian triangle of X is the point X_{13}. Finally, let $X = X_{619}$, the complement of the 2nd isogonic center, X_{14}. The perspector in this case is the point X_{14}.

8. Translated rotated reference triangle

Let DEF be the rotation of ABC about the circumcenter of ABC. Let $U = u : v : w$ be a point on L^∞. In this section, we wish to translate DEF in the direction of line DU, seeking translations $D'E'F'$ that are perspective to ABC. Except for rotations of 0 and π, triangle DEF is not perspective to ABC, so that by Theorem 1, there are at most two perspective translations.

Yff's parameterization of the circumcircle ([1, p.39]) is used to express the rotation DEF of ABC counterclockwise with angle 2θ as follows:

$$\begin{aligned} D &= \csc\theta : \csc(C-\theta) : -\csc(B+\theta), \\ E &= -\csc(C+\theta) : \csc\theta : \csc(A-\theta), \\ F &= \csc(B-\theta) : -\csc(A+\theta) : \csc\theta. \end{aligned}$$

Let

$$\begin{aligned} r &= ((a+b+c)(b+c-a)(c+a-b)(a+b-c))^{1/2}/(2abc), \\ \theta_1 &= \sin\theta, \\ \theta_2 &= \cos\theta. \end{aligned}$$

Then the vertices D, E, F are given by the rows of the matrices

$$\begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix} = \begin{pmatrix} \theta_1^{-1} & (rc\theta_2 - c_1\theta_1)^{-1} & (rb\theta_2 + b_1\theta_1)^{-1} \\ (rc\theta_2 + c_1\theta_1)^{-1} & \theta_1^{-1} & (ra\theta_2 - a_1\theta_1)^{-1} \\ (rb\theta_2 - b_1\theta_1)^{-1} & (ra\theta_2 + a_1\theta_1)^{-1} & \theta_1^{-1} \end{pmatrix},$$

where

$$(a_1, b_1, c_1) = (\cos A, \cos B, \cos C)$$
$$= ((b^2 + c^2 - a^2)/(2bc), (c^2 + a^2 - b^2)/(2ca), (a^2 + b^2 - c^2)/(2ab)).$$

The perspectivity determinant (2) is factored using a computer. Only one of the factors involves t, and it is a polynomial $P(t)$ as in (3), with coefficients

$$p_0 = 4abc\theta_1^2,$$
$$p_1 = 4\theta_1^2 abc\,(au + bv + cw) = 0,$$
$$p_2 = (a + b - c)\,(a - b + c)\,(b - a + c)\,(a + b + c)\,(avw + buw + cuv)\,,$$

hence roots

$$\pm(\theta_1/r)(-abcs)^{-1/2}, \tag{22}$$

where $s = avw + buw + cuv$.

Conjecture. The perspectors given by (22) are a pair of antipodes on the circumcircle.

See Figure 3. [3] It would perhaps be of interest to study, for fixed H, the loci of D', E', F' as θ varies from 0 to π.

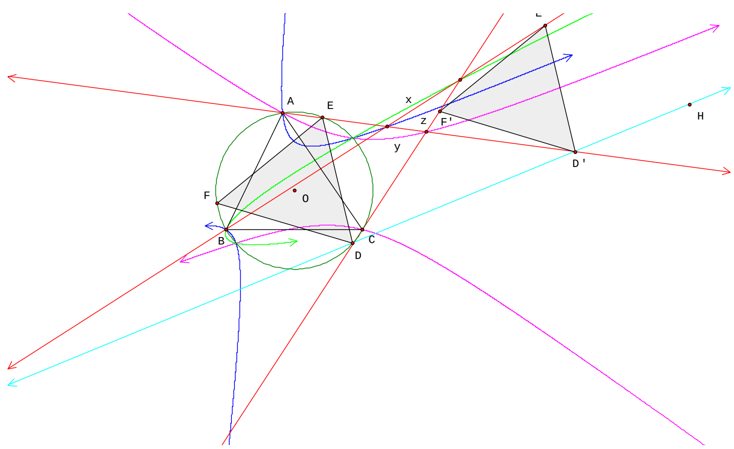

Figure 3. Translated rotation of ABC.

[3]Figure 3 can be viewed dynamically using The Geometer's Sketchpad; see [6] for access. Triangle DEF is a variable rotation of ABC about its circumcenter O. Independent point H determines line DH. Point D' is movable on line DH. Triangle $D'E'F'$is thus a movable translation of DEF in the direction of DH. Three conics as in Theorem 4 meet in two points, which according to the Conjecture are a pair of antipodes on the circumcircle.

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On the Derivative of a Vertex Polynomial

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Abstract. A geometric characterization of the critical points of a complex polynomial f is given, in the special case where the zeros of f are the vertices of a polygon affine-equivalent to a regular polygon.

1. Steiner Polygons

The relationship between the locations of the zeros of a complex polynomial f and those of its derivative has been extensively studied. The best-known theorem in this area is the Gauss-Lucas Theorem, that the zeros of f' lie in the convex hull of the zeros of f. The following theorem [1, p.93], due to Linfield, is also of interest:

Theorem 1. *Let $\lambda_j \in \mathbf{R}\backslash\{0\}$, $j = 1, \ldots, k$, and let z_j, $j = 1, \ldots, k$ be distinct complex numbers. Then the zeros of the rational function $R(z) := \sum_{j=1}^{k} \frac{\lambda_j}{z-z_j}$ are the foci of the curve of class $k-1$ which touches each of the $k(k-1)/2$ line segments $\overline{z_\mu, z_\nu}$ in a point dividing that line segment in the ratio $\lambda_\mu : \lambda_\nu$.*

Since $f' = f \cdot \sum_{j=1}^{k} \frac{1}{z-z_j}$, where the z_j are the zeros of f, Linfield's Theorem can be used to locate the zeros of f' which are not zeros of f.

In this paper, we will consider the case of a polynomial whose zeros form the vertices of a polygon which is affine-equivalent to a regular polygon; the zeros of the derivative can be geometrically characterized in a manner resembling Linfield's Theorem. First, let ζ be a primitive nth root of unity, for some $n \geq 3$. Define $G(\zeta)$ to be the n-gon whose vertices are $\zeta^0, \zeta^1, \ldots, \zeta^{n-1}$.

Proposition 2. *Let $n \geq 3$, and let G be an n-gon with vertices $v_0, \ldots, v_{n-1}$, no three of which are collinear. The following are equivalent.*

(1) *There is an ellipse which is tangent to the edges of G at their midpoints.*
(2) *G is affine-equivalent to $G(\zeta)$ for some primitive nth root of unity ζ.*
(3) *There is a primitive nth root of unity ζ and complex constants g, u, v such that $|u| \neq |v|$ and, for $k = 0, \ldots, n-1$, $v_k = g + u\zeta^k + v\zeta^{-k}$.*

Proof. 1)$\Longrightarrow$2): Applying an affine transformation if necessary, we may assume that the ellipse is a circle centered at 0 and that $v_0 = 1$. Let m_0 be the midpoint of the edge v_0v_1. v_0v_1 is then perpendicular to $0m_0$, and v_0, v_1 are equidistant from m_0; it follows that the right triangles $0m_0v_0$ and $0m_0v_1$ are congruent, and in

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particular that v_1 also lies on the unit circle. Now let m_1 be the midpoint of v_1v_2; since m_0 and m_1 are equidistant from 0 and the triangles $0m_0v_1$, $0m_1v_1$ are right, they are congruent, and m_0, m_1 are equidistant from v_1. It follows that the edges v_0v_1 and v_1v_2 have the same length. Furthermore, the triangles $0v_0v_1$ and $0v_1v_2$ are congruent, whence $v_2 = v_1^2$. Similarly we obtain $v_k = v_1^k$ for all k, and in particular that $v_1^n = v_0 = 1$. $\zeta = v_1$ is a primitive nth root of unity since none of $v_0, \ldots, v_{n-1}$ coincide, and $G = G(\zeta)$.

2)$\Longrightarrow$1): $G(\zeta)$ has an ellipse – indeed, a circle – tangent to its edges at their midpoints; an affine transformation preserves this.

2)$\Longleftrightarrow$3): Any real-linear transformation of $\mathbf{C}$ can be put in the form $z \mapsto uz + v\overline{z}$ for some choice of u, v, and conversely; the transformation is invertible iff $|u| \neq |v|$. □

We will refer to an n-gon satisfying these conditions as a *Steiner n-gon*; when needed, we will say it *has root* ζ. The ellipse is its *Steiner inellipse*. (This is a generalization of the case $n = 3$; every triangle is a Steiner triangle.) The parameters g, u, v are its *Fourier coordinates*. Note that a Steiner n-gon is regular iff either u or v vanishes.

2. The Foci of the Steiner Inellipse

Now, let S_ζ be the set of Steiner n-gons with root ζ for which the constant g, above, is 0. We may use the Fourier coordinates u, v to identify it with an open subset of $\mathbf{C}^2$. Let Φ be the map taking the n-gon with vertices $v_0, v_1, \ldots, v_{n-1}$ to the n-gon with vertices $v_1, \ldots, v_{n-1}, v_0$. If f is a complex-valued function whose domain is a subset of S_ζ which is closed under Φ, write φf for $f \circ \Phi$. Note that $\varphi u = \zeta u$ and $\varphi v = \zeta^{-1} v$; this will prove useful. Note also that special points associated with n-gons may be identified with complex-valued functions on appropriate subsets of S_ζ.

We define several useful fields associated with S_ζ. First, let $F = \mathbf{C}(u, v, \overline{u}, \overline{v})$, where u, v are as in 3) of the above proposition. φ is an automorphism of F. Let $K = \mathbf{C}(x, y, \overline{x}, \overline{y})$ be an extension field of F satisfying $x^2 = u$, $y^2 = v$, $\overline{x}^2 = \overline{u}, \overline{y}^2 = \overline{v}$. Let θ be a fixed square root of ζ; we extend φ to K by setting $\varphi x = \theta x, \varphi y = \theta^{-1} y, \varphi \overline{x} = \theta^{-1} \overline{x}, \varphi \overline{y} = \theta \overline{y}$. Let K_0 be the fixed field of φ and K_1 the fixed field of φ^n. Elements of F may be regarded as complex-valued functions defined on dense open subsets of S_ζ. Functions corresponding to elements of K may only be defined locally; however, given $G \in S_\zeta$ such that $uv \neq 0$ and $f \in K_1$ defined at G, one may choose a small neighborhood U_0 of G which is disjoint from $\Phi^k(U_0)$, $k = 1, \ldots, n-1$ and on which neither u nor v vanish; f may then be defined on $U = \bigcup_{k=0}^{n-1} \Phi^k(U_0)$.

For the remainder of this section, G is a fixed Steiner n-gon with root ζ. The vertices of G are $v_0, \ldots, v_{n-1}$. We have the following.

Proposition 3. *The foci of the Steiner inellipse of G are located at $f_\pm = g \pm (\theta + \theta^{-1})xy$.*

Proof. Translating if necessary, we may assume that $g = 0$, i.e., $G \in S_\zeta$. Note first that $f_\pm \in K_0$. (This is to be expected, since the Steiner inellipse and its foci do not depend on the choice of initial vertex.) For $k = 0, \ldots, n-1$, let $m_k = (v_k + v_{k+1})/2$, the midpoint of the edge $v_k v_{k+1}$. Let $d_\pm$ be the distance from $f_\pm$ to m_0; we will first show that $d_+ + d_-$ is invariant under φ. (This will imply that the sum of the distances from $f_\pm$ to m_k is the same for all k.) Now, $m_0 = (v_0+v_1)/2 = ((1+\zeta)u+(1+\zeta^{-1})v)/2 = (\theta+\theta^{-1})(\theta x^2+\theta^{-1}y^2)/2$. Thus, $m_0-f_+ = (\theta+\theta^{-1})(\theta x^2-2xy+\theta^{-1}y^2)/2 = (\zeta+1)(x-\theta^{-1}y)^2/2$. Hence $d_+ = |m_0-f_+| = |\zeta+1|(x-\theta^{-1}y)(\overline{x}-\theta\overline{y})/2$. Similarly, $d_- = |\zeta+1|(x+\theta^{-1}y)(\overline{x}+\theta\overline{y})/2$, and so $d_+ + d_- = |\zeta+1|(x\overline{x}+y\overline{y})$, which is invariant under φ as claimed. This shows that there is an ellipse with foci $f_\pm$ passing through the midpoints of the edges of G. If $n \geq 5$, this is already enough to show that this ellipse is the Steiner inellipse; however, for $n = 3, 4$ it remains to show that this ellipse is tangent to the sides, or, equivalently, that the side $v_k v_{k+1}$ is the external bisector of the angle $\angle f_+ m_k f_-$. It suffices to show that $A_k = (m_k - v_k)(m_k - v_{k+1})$ is a positive multiple of $B_k = (m_k - f_+)(m_k - f_-)$. Now $A_0 = -(\zeta-1)^2(u-\zeta^{-1}v)^2/4$, and $B_0 = (\zeta+1)^2(x-\theta^{-1}y)^2(x+\theta^{-1}y)^2/4 = (\zeta+1)^2(u-\zeta^{-1}v)^2/4$; thus, $A_0/B_0 = -(\zeta-1)^2/(\zeta+1)^2 = -(\theta-\theta^{-1})^2/(\theta+\theta^{-1})^2$, which is evidently positive. This quantity is invariant under φ; hence A_k/B_k is also positive for all k. □

Corollary 4. *The Steiner inellipse of G is a circle iff G is similar to $G(\zeta)$.*

Proof. $f_+ = f_-$ iff $xy = 0$, i.e., iff one of u and v is zero. (Note that $\theta + \theta^{-1} \neq 0$.) □

Define the *vertex polynomial* $f_G(z)$ of G to be $\prod_{k=0}^{n-1}(z - v_k)$. We have the following.

Proposition 5. *The foci of the Steiner inellipse of G are critical points of f_G.*

Proof. Again, we may assume $G \in S_\zeta$. Since $f_G'/f_G = \sum_{k=0}^{n-1}(z-v_k)^{-1}$, it suffices to show that this sum vanishes at $f_\pm$. Now f_+ is invariant under φ, and $v_k = \varphi^k v_0$; hence $\sum_{k=0}^{n-1}(f_+ - v_k)^{-1} = \sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1}$. $(f_+ - v_0)^{-1} = -\theta/((\theta y - x)(y - \theta x))$ Now let $g = \theta^2/((\theta^2-1)x(\theta y - x))$. Note that $g \in K_1$; that is, $\varphi^n g = g$. A straightforward calculation shows that $(f_+ - v_0)^{-1} = g - \varphi g$; therefore, $\sum_{k=0}^{n-1}\varphi^k(f_+ - v_0)^{-1} = \sum_{k=0}^{n-1}(\varphi^k g - \varphi^{k+1} g) = g - \varphi^n g = 0$, as desired. The proof that f_- is a critical point of f_G is similar. □

3. Holomorphs

Again, we let G be a Steiner n-gon with root ζ and vertices $v_0, \ldots, v_{n-1}$. For any integer m, we set $v_m = v_l$ where $l = 0, \ldots, n-1$ is congruent to m mod n. The following lemma is trivial.

Lemma 6. *Let $k = 1, \ldots, \lfloor n/2 \rfloor$. Then:*

(1) *If k is relatively prime to n, let G^k be the n-gon with vertices $v_0^k, \ldots, v_{n-1}^k$ given by $v_j^k = v_{jk}$. Then G^k is a Steiner n-gon with root ζ^k, and its Fourier coordinates are g, u, v.*

(2) *If* $d = \gcd(k, n)$ *is greater than* 1 *and less than* $n/2$*, set* $m = n/d$*. Then, for* $l = 0, \ldots, d-1$*, let* $G^{k,l}$ *be the* m*-gon with vertices* $v_0^{k,l}, \ldots, v_{m-1}^{k,l}$ *given by* $v_j^{k,l} = v_{kj+l}$*. Then, for each* l*,* $G^{k,l}$ *is a Steiner* m*-gon with root* ζ^k*, and the Fourier coordinates of* $G^{k,l}$ *are* $g, \zeta^l u, \zeta^{-l} v$*. The* $G^{k,l}$ *all have the same Steiner inellipse.*

(3) *If* $k = n/2$*, the line segments* $v_j v_{j+k}$ *all have midpoint* g*.*

In the three given cases, we will say k*-holomorph of* G to refer to G^k, the union of the m-gons $G^{k,l}$, or the union of the line segments $v_j v_{j+k}$. We extend the definition of *Steiner inellipse* to the k-holomorphs in Cases 2 and 3, meaning the common Steiner inellipse of the $G^{k,l}$ or the point g, respectively. The propositions of Section II clearly extend to Case 2; since the foci are critical points of the vertex polynomials of each of the $G^{k,l}$, they are also critical points of their product. In Case 3, taking g as a degenerate ellipse – indeed, circle – with focus at g, the propositions likewise extend; in this case, $\theta = \pm i$, so $\theta + \theta^{-1} = 0$, and the sole critical point of $(z - v_j)(z - v_{j+k})$ is $(v_j + v_{j+k})/2 = g$.

In Cases 1 and 2, it should be noted that the Steiner inellipse is a circle iff the Steiner inellipse of G itself is a circle – i.e., G is similar to $G(\zeta)$. It should also be noted that the vertex polynomials of the holomorphs of G are equal to f_G itself; hence they have the same critical points. Suppose that G is not similar to $G(\zeta)$. If n is odd, G has $(n-1)/2$ holomorphs, each with a noncircular Steiner inellipse and hence two distinct Steiner foci; these account for the $n-1$ critical points of f_G. If n is even, G has $(n-2)/2$ holomorphs in Cases 1 and 2, each with two distinct Steiner foci, and in addition the Case 3 holomorph, providing one more Steiner focus; again, these account for $n-1$ critical points of f_G. On the other hand, if G is similar to $G(\zeta)$, then $f_G = (z-g)^n - r^n$ for some real r; the Steiner foci of the holomorphs of G collapse together, and f_G has an $(n-1)$-fold critical point at g. We have proven the following.

Theorem 7. *If* G *is a Steiner* n*-gon, the critical points of* f_G *are the foci of the Steiner inellipses of the holomorphs of* G*, counted with multiplicities if* G *is regular. They are collinear, lying at the points* $g + (2\cos k\pi/n)xy$*, as* k *ranges from* 0 *to* $n-1$*.*

(For the last statement, note that $\cos(n-k)\pi/n = -\cos k\pi/n$.)

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On Two Remarkable Lines Related to a Quadrilateral

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Abstract. We study the Euler line of an arbitrary quadrilateral and the Nagel line of a circumscriptible quadrilateral.

1. Introduction

Among the various lines related to a triangle the most popular are Euler and Nagel lines. Recall that the Euler line contains the orthocenter H, the centroid G, the circumcenter O and the nine-point center E, so that $HE : EG : GO = 3 : 1 : 2$. On the other hand, the Nagel line contains the Nagel point N, the centroid M, the incenter I and Spieker point S (which is the centroid of the perimeter of the triangle) so that $NS : SG : GI = 3 : 1 : 2$. The aim of this paper is to find some analogies of these lines for quadrilaterals.

It is well known that in a triangle, the following two notions of centroids coincide:
(i) the barycenter of the system of unit masses at the vertices,
(ii) the center of mass of the boundary and interior of the triangle.

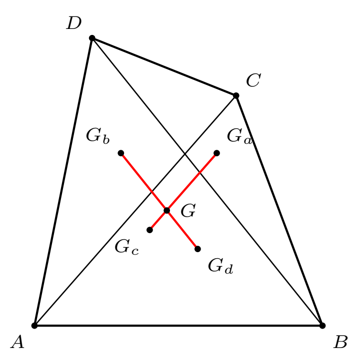

Figure 1.

But for quadrilaterals these are not necessarily the same. We shall show in this note, that to get some fruitful analogies for quadrilateral it is useful to consider the centroid G of quadrilateral as a whole figure. For a quadrilateral $ABCD$, this centroid G can be determined as follows. Let G_a, G_b, G_c, G_d be the centroids of triangles BCD, ACD, ABD, ABC respectively. The centroid G is the intersection of the lines G_aG_c and G_bG_d:

$$G = G_aG_c \cap G_bG_d.$$

See Figure 1.

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2. The Euler line of a quadrilateral

Given a quadrilateral $ABCD$, denote by O_a and H_a the circumcenter and the orthocenter respectively of triangle BCD, and similarly, O_b, H_b for triangle ACD, O_c, H_c for triangle ABD, and O_d, H_d for triangle ABC. Let

$$O = O_aO_c \cap O_bO_d,$$
$$H = H_aH_c \cap H_bH_d.$$

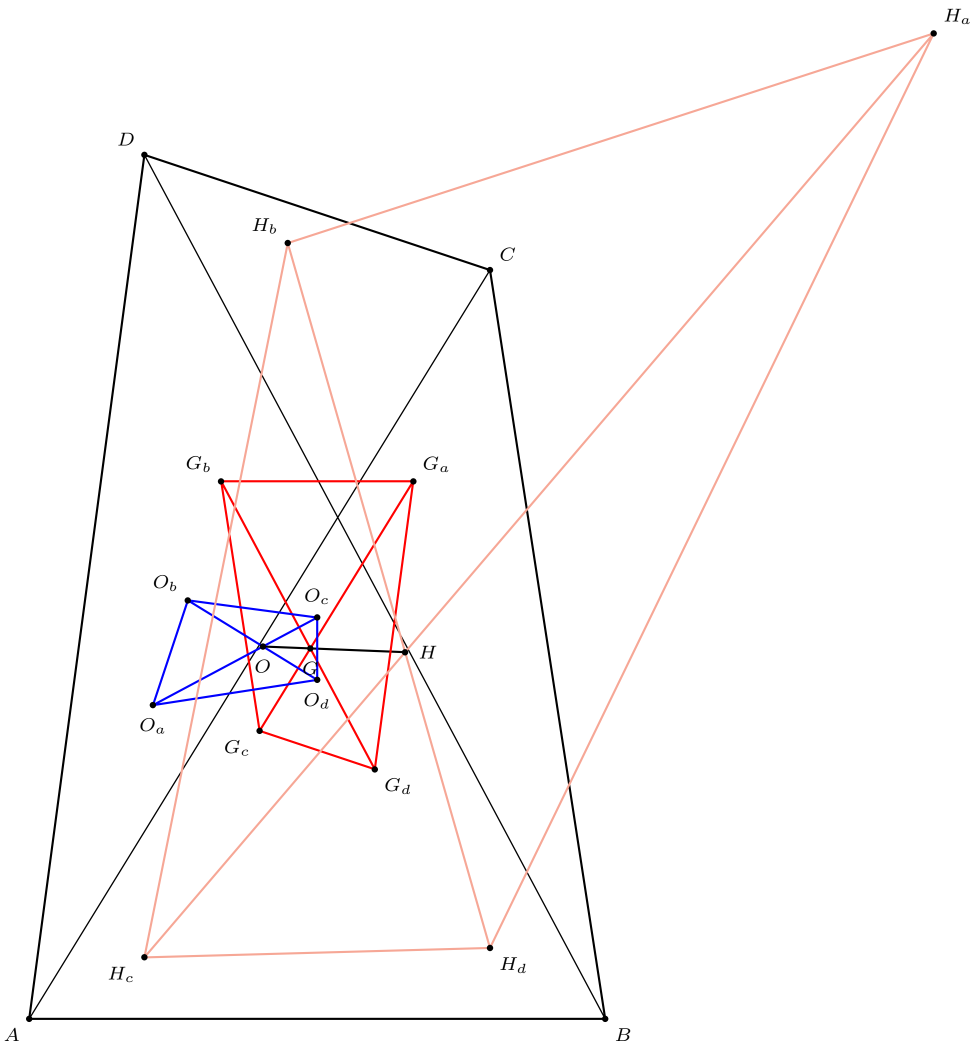

Figure 2

We shall call O the quasicircumcenter and H the quasiorthocenter of the quadrilateral $ABCD$. Clearly, the quasicircumcenter O is the intersection of perpendicular bisectors of the diagonals of $ABCD$. Therefore, if the quadrilateral is cyclic, then O is the center of its circumcircle. Figure 2 shows the three associated quadrilaterals $G_aG_bG_cG_d$, $O_aO_bO_cO_d$, and $H_aH_bH_cH_d$.

The following theorem was discovered by Jaroslav Ganin, (see [2]), and the idea of the proof was due to François Rideau [3].

Theorem 1. *In any arbitrary quadrilateral the quasiorthocenter H, the centroid G, and the quasicircumcenter O are collinear. Furthermore, $OH : HG = 3 : -2$.*

Proof. Consider three affine maps f_G, f_O and f_H transforming the triangle ABC onto triangle $G_aG_bG_c$, $O_aO_bO_c$, and $H_aH_bH_c$ respectively.

In the affine plane, write $D = xA + yB + zC$ with $x + y + z = 1$.

(i) Note that

$$\begin{aligned}
f_G(D) =& f_G(xA + yB + zC)\\
=& xG_a + yG_b + zG_c\\
=& \frac{1}{3}(x(B + C + D) + y(A + C + D) + z(A + B + D))\\
=& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (x + y + z)D)\\
=& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (xA + yB + zC))\\
=& \frac{1}{3}(x + y + z)(A + B + C)\\
=& G_d.
\end{aligned}$$

(ii) It is obvious that triangles ABC and $O_aO_bO_c$ are orthologic with centers D and O_d. See Figure 3. From Theorem 1 of [1], $f_O(D) = O_d$.

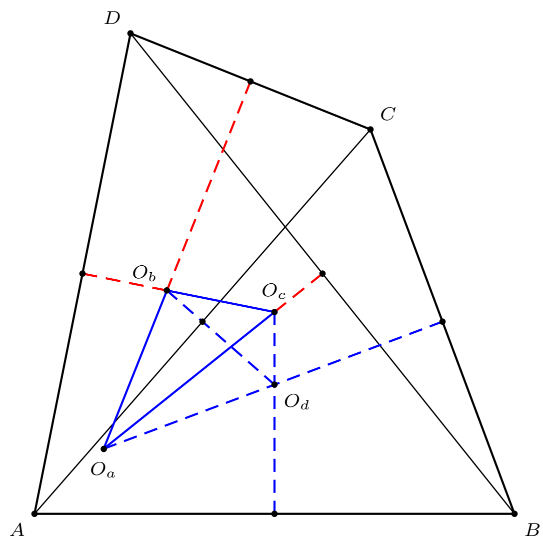

Figure 3

(iii) Since H_a divides O_aG_a in the ratio $O_aH_a : H_aG_a = 3 : -2$, and similarly for H_b and H_c, for $Q = A,\ B,\ C$, the point $f_H(Q)$ divides the segment $f_O(Q)f_G(Q)$ into the ratio $3 : -2$. It follows that for *every* point Q in the plane of ABC, $f_H(Q)$ divides $f_O(Q)f_G(Q)$ in the same ratio. In particular, $f_H(D)$ divides $f_O(D)f_G(D)$, namely, O_dG_d, in the ratio $3 : -2$. This is clearly H_d. We have shown that $f_H(D) = H_d$.

(iv) Let $Q = AC \cap BD$. Applying the affine maps we have

$$\begin{aligned} f_G(Q) =& G_aG_c \cap G_bG_d = G, \\ f_O(Q) =& O_aO_c \cap O_bO_d = O, \\ f_H(Q) =& H_aH_c \cap H_bH_d = H. \end{aligned}$$

From this we conclude that H divides OG in the ratio $3 : -2$. □

Theorem 1 enables one to define the *Euler line* of a quadrilateral $ABCD$ as the line containing the centroid, the quasicircumcenter, and the quasiorthocenter. This line contains also the quasininepoint center E defined as follows. Let E_a, E_b, E_c, E_d be the nine-point centers of the triangles BCD, ACD, ABD, ABC respectively. We define the quasininepoint center to be the point $E = E_aE_c \cap E_bE_d$. The following theorem can be proved in a way similar to Theorem 1 above.

Theorem 2. *E is the midpoint of OH.*

3. The Nagel line of a circumscriptible quadrilateral

A quadrilateral is circumscriptible if it has an incircle. Let $ABCD$ be a circumscriptible quadrilateral with incenter I. Let T_1, T_2, T_3, T_4 be the points of tangency of the incircle with the sides AB, BC, CD and DA respectively. Let N_1 be the isotomic conjugate of T_1 with respect to the segment AB. Similarly define N_2, N_3, N_4 in the same way. We shall refer to the point $N := N_1N_3 \cap N_2N_4$ as the Nagel point of the circumscriptible quadrilateral. Note that both lines divide the perimeter of the quadrilateral into two equal parts.

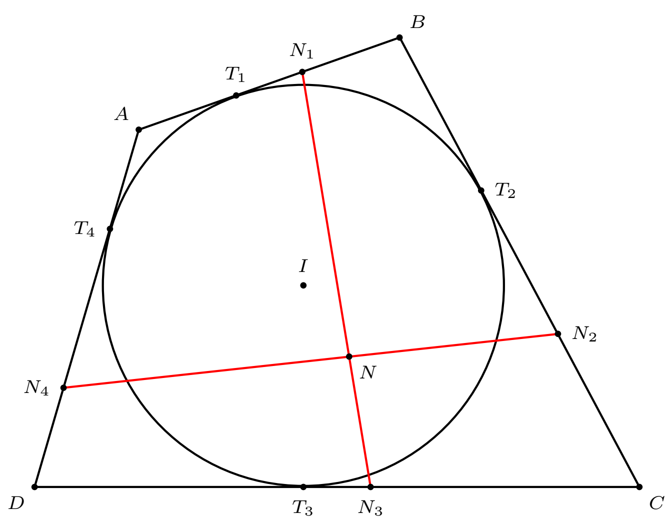

Figure 4.

In Theorem 6 below we shall show that N lies on the line joining I and G. In what follows we shall write

$$P = (x \cdot A,\ y \cdot B,\ z \cdot C,\ w \cdot D)$$

to mean that P is the barycenter of a system of masses x at A, y at B, z at C, and w at D. Clearly, x, y, z, w can be replaced by kx, ky, kz, kw for nonzero k without changing the point P. In Figure 4, assume that $AT_1 = AT_4 = p$, $BT_2 = BT_1 = q$, $CT_3 = CT_2 = r$, and $DT_4 = DT_3 = t$. Then by putting masses p at A, q at B, r at C, and t at D, we see that
(i) $N_1 = (p \cdot A,\ q \cdot B,\ 0 \cdot C,\ 0 \cdot D)$,
(ii) $N_3 = (0 \cdot A,\ 0 \cdot B,\ r \cdot C,\ t \cdot D)$, so that the barycenter $N = (p \cdot A,\ q \cdot B,\ r \cdot C,\ t \cdot D)$ is on the line N_1N_3. Similarly, it is also on the line N_2N_4 since
(iii) $N_2 = (0 \cdot A,\ q \cdot B,\ r \cdot C,\ 0 \cdot D)$, and
(iv) $N_4 = (p \cdot A,\ 0 \cdot B,\ 0 \cdot C,\ t \cdot D)$.

Therefore, we have established the first of the following three lemmas.

Lemma 3. $N = (p \cdot A,\ q \cdot B,\ r \cdot C,\ t \cdot D)$.

Lemma 4. $I = ((q + t)A,\ (p + r)B,\ (q + t)C,\ (p + r)D)$.

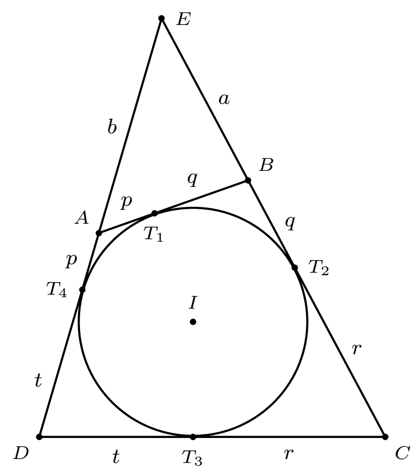

Figure 5.

Proof. Suppose the circumscriptible quadrilateral $ABCD$ has a pair of non-parallel sides AD and BC, which intersect at E. (If not, then $ABCD$ is a rhombus, $p = q = r = s$, and $I = G$; the result is trivial). Let $a = EB$ and $b = EA$.
(i) As the incenter of triangle EDC, $I = ((t+r)E,\ (a+q+r)D,\ (b+p+t)C)$.
(ii) As an excenter of triangle ABE, $I = ((p + q)E,\ -a \cdot A,\ -b \cdot B)$.

Note that $\frac{EC}{EB} = \frac{a+q+r}{a}$ and $\frac{ED}{EA} = \frac{b+p+t}{b}$, so that the system $(p+q+r+t)E$ is equivalent to the system $((a+q+r)B,\ -a \cdot C,\ (b+p+t)A,\ -b \cdot D)$. Therefore,

$$I = ((-a+b+p+t)A,\ (-b+a+q+r)B,\ (-a+b+p+t)C,\ (-b+a+q+r)D).$$

Since $b + p = a + q$, the result follows. □

Lemma 5. $G = ((p+q+t)A,\ (p+q+r)B,\ (q+r+t)C,\ (p+r+t)D)$.

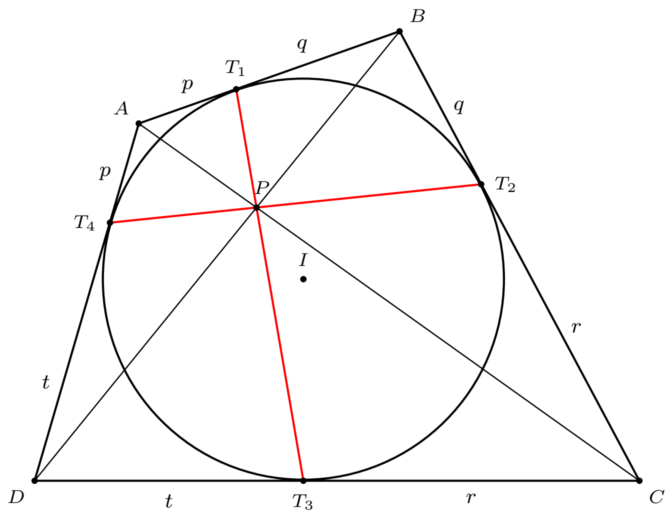

Figure 6.

Proof. Denote the point of intersection of the diagonals by P. Note that $\frac{AP}{CP} = \frac{p}{r}$ and $\frac{BP}{DP} = \frac{q}{t}$. Actually, according to one corollary of Brianchon's theorem, the lines T_1T_3 and T_2T_4 also pass through P. For another proof, see [4, pp.156–157]. Hence,

$$P = \left(\frac{1}{p} \cdot A,\ \frac{1}{q} \cdot B,\ \frac{1}{r} \cdot C,\ \frac{1}{t} \cdot D\right).$$

Consequently, $P = \left(\frac{1}{q} \cdot B,\ \frac{1}{t} \cdot D\right)$ and also $P = \left(\frac{1}{p} \cdot A,\ \frac{1}{r} \cdot C\right)$.

The quadrilateral $G_aG_bG_cG_d$ is homothetic to $ABCD$, with homothetic center $M = (1 \cdot A,\ 1 \cdot B,\ 1 \cdot C,\ 1 \cdot D)$ and ratio $-\frac{1}{3}$. Thus, $\frac{G_aG}{G_cG} = \frac{AP}{CP} = \frac{p}{r}$ and $\frac{G_bG}{G_dG} = \frac{BP}{DP} = \frac{q}{t}$. It follows that $G = (r \cdot G_a,\ p \cdot G_c) = (p \cdot A,\ (r+p)B, r \cdot C,\ (r+p)D)$ and $G = (t \cdot G_b,\ q \cdot G_d) = ((q+t)A,\ q \cdot B,\ (q+t)C,\ t \cdot D)$. To conclude the proof, it is enough to add up the corresponding masses. □

The following theorem follows easily from Lemmas 3, 4, 5.

Theorem 6. *For a circumscriptible quadrilateral, the Nagel point N, centroid G and incenter I are collinear. Furthermore, $NG : GI = 2 : 1$.*

See Figure 7.

Theorem 6 enables us to define the Nagel line of a circumscriptible quadrilateral. This line also contains the Spieker point of the quadrilateral, by which we mean the center of mass S of the perimeter of the quadrilateral, without assuming an incircle.

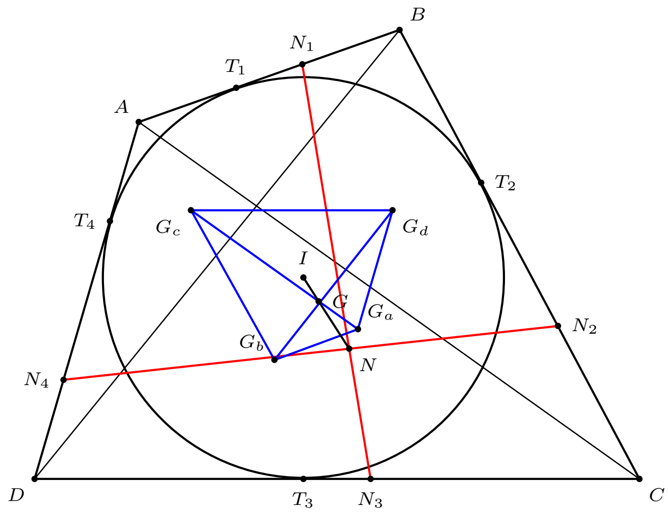

Figure 7.

Theorem 7. *For a circumscriptible quadrilateral, the Spieker point is the midpoint of the incenter and the Nagel point.*

Proof. With reference to Figure 6, each side of the circumscriptible quadrilateral is equivalent to a mass equal to its length located at each of its two vertices. Thus,

$$S = ((2p+q+t)A,\ (p+2q+r)B,\ (q+2r+t)C,\ (p+r+2t)D).$$

Splitting into two systems of equal total masses, we have

$$\begin{aligned} N =& (2pA,\ 2qB,\ 2rC,\ 2tD), \\ I =& ((q+t)A,\ (p+r)B,\ (q+t)C,\ ((p+r)D). \end{aligned}$$

From this the result is clear. $\square$

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Intersecting Circles and their Inner Tangent Circle

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Abstract. We derive the general equation for the radius of the inner tangent circle that is associated with three pairwise intersecting circles. We then look at three special cases of the equation.

It seems to the author that there should be one equation that gives the radius of the inner tangent circle inscribed in a triangular region bounded by either straight lines or circular arcs. As a step toward this goal of a single equation, consider three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ with radii α, β, γ respectively. $\mathcal{C}_A$ intersects $\mathcal{C}_B$ at an angle θ. $\mathcal{C}_B$ intersects $\mathcal{C}_C$ at an angle ρ. And $\mathcal{C}_C$ intersects $\mathcal{C}_A$ at an angle ϕ, with $0 \leqslant \theta, \rho, \phi \leqslant \pi$. We seek the radius of the circle $\mathcal{C}$, tangent externally to each of the given circles. See Figure 1. If the three intersecting cirlces were just touching instead, the inner tangent circle would be the inner Soddy circle. See [1]. The points of tangency of the inner tangent circle form the vertices of an inscribed triangle. We set up a coordinate system with the origin at the center of $\mathcal{C}$. See Figure 1.

Let the points of tangency A, B, C be represented by complex numbers of moduli R, the radius of $\mathcal{C}$. With these labels, the triangle ABC and the inscribed triangle is one and the same. Letting the lengths of the sides BC, CA, AB be a, b, c respectively, then

$$\| A - B \| = c \quad \text{and} \quad \langle A, B \rangle = R^2 - \frac{c^2}{2}. \tag{1}$$

Corresponding relations hold for the pairs B, C and C, A. With the above coordinate system, the centers of the circles $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ are respectively $\frac{R+\alpha}{R}A$, $\frac{R+\beta}{R}B$, $\frac{R+\gamma}{R}C$.

The circles $\mathcal{C}_A$ and $\mathcal{C}_B$ intersect at angle θ if and only if

$$\left\| \frac{R+\alpha}{R}A - \frac{R+\beta}{R}B \right\| = \alpha^2 + \beta^2 + 2\alpha\beta\cos\theta.$$

By an application of (1) and the use of a half angle formula, the above can be shown to be equivalent to

$$c^2 = \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}.$$

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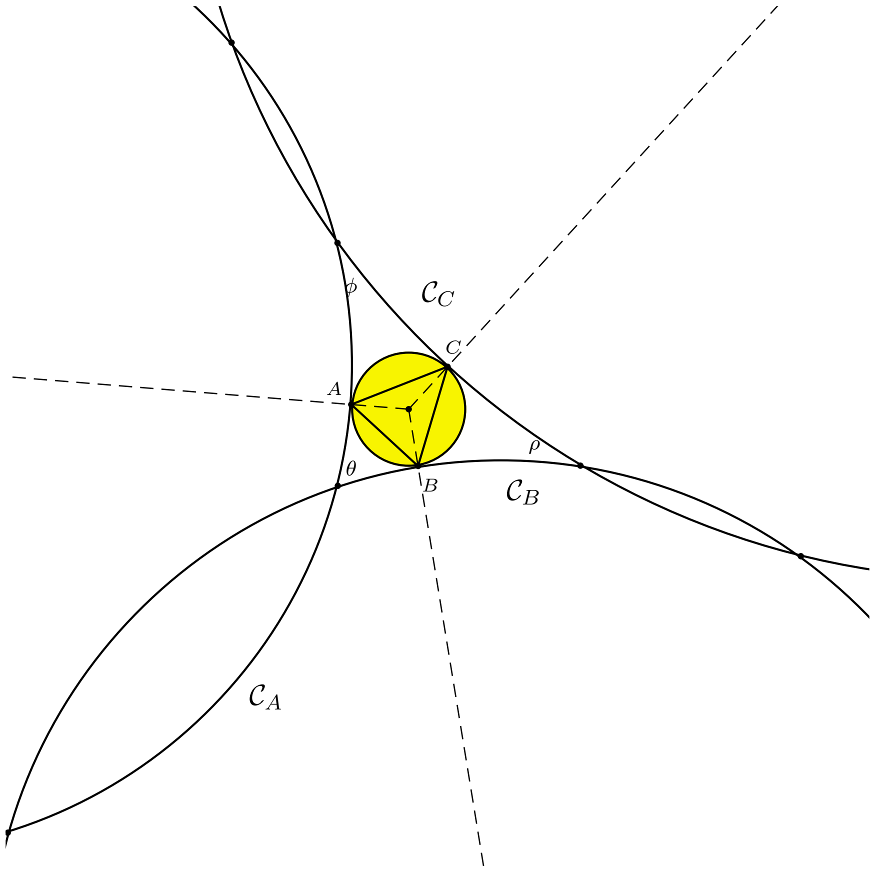

Figure 1

Thus the three circles $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ intersect each other at the given angles if and only if

$$\begin{aligned} a^2 &= \frac{4R^2\beta\gamma\cos^2\frac{\rho}{2}}{(R+\beta)(R+\gamma)}, \\ b^2 &= \frac{4R^2\alpha\gamma\cos^2\frac{\phi}{2}}{(R+\alpha)(R+\gamma)}, \\ c^2 &= \frac{4R^2\alpha\beta\cos^2\frac{\theta}{2}}{(R+\alpha)(R+\beta)}. \end{aligned} \tag{2}$$

These equations are then used to solve for R in terms of α, β, γ, θ, ϕ and ρ. In the first step of this process, we multiply the equations in (2) and take square root to obtain

$$abc = \frac{8\alpha\beta\gamma R^3\cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\rho}{2}}{(R+\alpha)(R+\beta)(R+\gamma)}. \tag{3}$$

Using (3) and (2) we obtain,

$$\begin{aligned}\frac{\alpha}{R+\alpha} &= \frac{bc\cos\frac{\rho}{2}}{2Ra\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ \frac{\beta}{R+\beta} &= \frac{ac\cos\frac{\phi}{2}}{2Rb\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ \frac{\gamma}{R+\gamma} &= \frac{ab\cos\frac{\theta}{2}}{2Rc\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{4}$$

The area, $\triangle$, of the inscribed triangle ABC is given by

$$\triangle = \frac{abc}{4R}. \tag{5}$$

Consequently, equations (4) and (5) lead to

$$\begin{aligned}a^2 &= \frac{(R+\alpha)\triangle\cos\frac{\rho}{2}}{\alpha\cos\frac{\theta}{2}\cos\frac{\phi}{2}},\\ b^2 &= \frac{(R+\beta)\triangle\cos\frac{\phi}{2}}{\beta\cos\frac{\theta}{2}\cos\frac{\rho}{2}},\\ c^2 &= \frac{(R+\gamma)\triangle\cos\frac{\theta}{2}}{\gamma\cos\frac{\rho}{2}\cos\frac{\phi}{2}}.\end{aligned} \tag{6}$$

Now, Heron's formula for the triangle ABC can be written in the form

$$16\triangle^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4.$$

Using the above equation together with equations (6) will enable us to get an equation for R in terms of the parameters of the intersecting circles. This process involves substituting the value of a^2, b^2, c^2 into Heron's formula, dividing by $\triangle^2$, and performing a lengthy algebraic manipulation to yield the equation:

$$\begin{aligned}0 = \frac{1}{R^2}\Big[&4\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}\cos^2\frac{\phi}{2} + \cos^4\frac{\phi}{2} + \cos^4\frac{\rho}{2} + \cos^4\frac{\theta}{2}\\ &-2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2} - 2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2} - 2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}\Big]\\ -\frac{1}{R}\Big[&\frac{2\cos^2\frac{\rho}{2}}{\alpha}(\cos^2\frac{\theta}{2} + \cos^2\frac{\phi}{2} - \cos^2\frac{\rho}{2})\\ &+\frac{2\cos^2\frac{\phi}{2}}{\beta}(\cos^2\frac{\theta}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\phi}{2})\\ &+\frac{2\cos^2\frac{\theta}{2}}{\gamma}(\cos^2\frac{\phi}{2} + \cos^2\frac{\rho}{2} - \cos^2\frac{\theta}{2})\Big]\\ &+\frac{\cos^4\frac{\rho}{2}}{\alpha^2} + \frac{\cos^4\frac{\phi}{2}}{\beta^2} + \frac{\cos^4\frac{\theta}{2}}{\gamma^2}\\ &-\frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\phi}{2}}{\beta\gamma} - \frac{2\cos^2\frac{\theta}{2}\cos^2\frac{\rho}{2}}{\alpha\gamma} - \frac{2\cos^2\frac{\phi}{2}\cos^2\frac{\rho}{2}}{\alpha\beta}.\end{aligned} \tag{7}$$

Although the equation can be formal solved in general, it is rather unwieldy. Let us consider some special cases.

When the three circles $\mathcal{C}_A, \mathcal{C}_B$ and $\mathcal{C}_C$ are mutually tangent, θ, ρ and ϕ equals zero, thus giving the equation:

$$0 = \frac{1}{R^2} - \frac{2}{R}\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{2}{\alpha\beta} - \frac{2}{\beta\gamma} - \frac{2}{\alpha\gamma}.$$

Solving for $\frac{1}{R}$ gives the standard Descartes formula for the Inner Soddy circle. See [2].

When $\mathcal{C}_C$ is a line tangent to $\mathcal{C}_A$ and $\mathcal{C}_B$, we have $\beta = \infty$ and $\theta = \rho = 0$, and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{R}\left[\frac{1}{\alpha} + \frac{1}{\gamma}\right] + \left[\frac{1}{\alpha} - \frac{1}{\gamma}\right]^2.$$

Solving for 1/R, and using the fact that $\frac{1}{R} > \frac{1}{\alpha}$ and $\frac{1}{R} > \frac{1}{\gamma}$, gives the equation

$$\frac{1}{R} = \frac{1}{\cos^2\frac{\phi}{2}}\left[\frac{1}{\alpha} + \frac{1}{\gamma} + 2\sqrt{\frac{1}{\alpha\gamma}}\right].$$

When the circles $\mathcal{C}_A$ and $\mathcal{C}_C$ are lines that intersect at an angle $\phi > 0$ and are both tangent to the circle $\mathcal{C}_B$, we get a cone and equation (7) becomes

$$0 = \frac{1}{R^2}\left[\cos^4\frac{\phi}{2}\right] - \frac{2\cos^2\frac{\phi}{2}}{\gamma R}\left[2 - \cos^2\frac{\phi}{2}\right] + \left[\frac{\cos^2\frac{\phi}{2}}{\gamma}\right]^2.$$

After solving for 1/R, using some trigonometric identities and the fact that $\frac{1}{R} > \frac{1}{\gamma}$, we get the equation

$$\frac{1}{R} = \frac{2}{\gamma}\left[\frac{1 + \sin\frac{\phi}{2}}{1 - \sin\frac{\phi}{2}}\right]^2,$$

the same as obtained from working with the cone directly.

Unfortunately, equation (7) no longer gives any useful result when all three circles, $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$, becomes lines. The inner tangent circle in this case is just the inscribed circle in the triangle.

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Two Brahmagupta Problems

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Abstract. D. E. Smith reproduces two problems from Brahmagupta's work Kutakhādyaka (algebra) in his *History of Mathematics*, Volume 1. One of them involves a broken tree and the other a mountain journey. Normally such objects are represented by vertical line segments. However, it is every day experience that such objects need not be vertical. In this paper, we generalize these situations to include slanted positions and provide integer solutions to these problems.

1. Introduction

School textbooks on geometry and trigonometry contain problems about trees, poles, buildings, hills etc. to be solved using the Pythagorean theorem or trigonometric ratios. The assumption is that such objects are vertical. However, trees grow not only vertically (and tall offering a magestic look) but also assume slanted positions (thereby offering an elegant look). Buildings too need not be vertical in structure, for example the leaning tower of Pisa. Also, a distant planar view of a mountain is more like a scalene triangle than a right one. In this paper we regard the angles formed in such situations as having rational cosines. We solve the following Brahmagupta problems from [5] in the context of rational cosines triangles. In [4] these problems have been given Pythagorean solutions.

Problem 1. A bamboo 18 cubits high was broken by the wind. Its tip touched the ground 6 cubits from the root. Tell the lengths of the segments of the bamboo.

Problem 2. On the top of a certain hill live two ascetics. One of them being a wizard travels through the air. Springing from the summit of the mountain he ascends to a certain elevation and proceeds by an oblique descent diagonally to neighboring town. The other walking down the hill goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill and how high the wizard rose.

We omit the numerical data given in Problem 1 to extend it to an indeterminate one like the second so that an infinity of integer solutions can be found.

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2. Background material

An angle θ is a rational cosine angle if $\cos\theta$ is rational. If both $\cos\theta$ and $\sin\theta$ are rational, then θ is called a Heron angle. If the angles of a triangle are rational cosine (respectively Heron) angles, then the sides are rational in proportion, and they can be rendered integers, by after multiplication by the lcm of the denominators. Thus, in effect, we deal with triangles of integer sides. Given a rational cosine (respectively Heron) angle θ, it is possible to determine the infinite family of integer triangles (respectively Heron triangles) in which each member triangle contains θ. Our discussion depends on such families of triangles, and we give the following description.

2.1. *Integer triangle family containing a given rational cosine angle θ.* Let $\cos\theta = \lambda$ be a rational number. When θ is obtuse, λ is negative. Our discussion requires that $0 < \theta < \frac{\pi}{2}$ so we must have $0 < \lambda < 1$. Let ABC be a member triangle in which $\angle BAC = \theta$. Let $\angle ABC = \phi$ as shown in Figure 1.

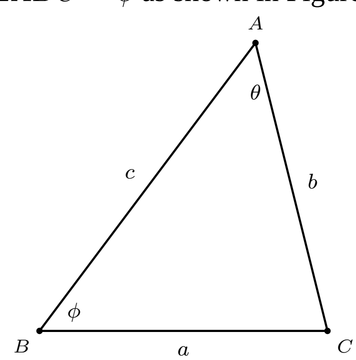

Figure 1

Applying the law of cosines to triangle ABC we have $a^2 = b^2 + c^2 - 2bc\lambda$, or

$$(c-a)(c+a) = b(2\lambda c - b).$$

By the triangle inequality $c - a < b$ so that

$$1 > \frac{c-a}{b} = \frac{2\lambda c - b}{c+a} = \frac{v}{u},$$

say, with $\gcd(u, v) = 1$. We then solve the resulting simultaneous equations

$$c - a = \frac{v}{u}a, \qquad c + a = \frac{u}{v}(2\lambda c - b)$$

for a, b, c in proportional values:

$$\frac{a}{u^2 - 2\lambda uv + v^2} = \frac{b}{2(\lambda u - v)} = \frac{c}{u^2 - v^2}.$$

We replace λ by a rational number $\frac{n}{m}$, $0 < \frac{n}{m} < 1$, and obtain a parametrization of triangles in the $\theta = \arccos\frac{n}{m}$ family:

$$(a,b,c) = (m(u^2+v^2) - 2nuv,\ 2u(nu - mv),\ m(u^2 - v^2)), \qquad \frac{u}{v} > \frac{m}{n}. \quad (\dagger)$$

It is routine to check that

$$\phi = \arccos \frac{mc - nb}{ma},$$

and that $\cos A = \frac{n}{m}$ independently of the parameters u, v of the family described in (†) above. Here are two particular integer triangle families.

(1) The $\frac{\pi}{3}$ integer family is given by (†) with $n = 1$, $m = 2$:

$$(a, b, c) = (u^2 - uv + v^2,\ u(u - 2v),\ u^2 - v^2), \quad u > 2v,\ \gcd(u, v) = 1. \quad (1)$$

It is common practice to list primitive solutions except under special circumstances. In (1) we have removed $\gcd(a, b, c) = 2$.

(2) When θ is a Heron angle, *i.e.*, $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$ for integers p, q with $\gcd(p, q) = 1$, (†) describes a Heron triangle family. For example, with $p = 2$, $q = 1$, we have $\cos\theta = \frac{3}{5}$. Now with $n = 3$, $m = 5$, (†) yields

$$(a, b, c) = (5u^2 - 6uv + 5v^2,\ 2u(3u - 5v),\ 5(u^2 - v^2)), \quad \gcd(u, v) = 1. \quad (2)$$

This has area $\triangle = \frac{1}{2}bc\sin\theta = \frac{2}{5}bc$. We may put $u = 3$, $v = 1$ to obtain the specific Heron triangle $(a, b, c) = (4, 3, 5)$ that is Pythagorean. On the other hand, $u = 4$, $v = 1$ yields the non-Pythagorean triangle $(a, b, c) = (61, 56, 75)$ with area $\triangle = 1680$.

3. Generalization of the first problem

3.1. *Restatement.* Throughout this discussion an integer tree is one with the following properties.
(i) It has an integer length $AB = c$.
(ii) It makes a rational cosine angle ϕ with the horizontal.
(iii) When the wind breaks it at a point D the broken part $AD = d$ and the unbroken part $BD = e$ both have integer lengths.
(iv) The top A of the tree touches the ground at C at an integer length $BC = a$.
All the triangles in the configuration so formed have integer sidelengths. See Figure 2.

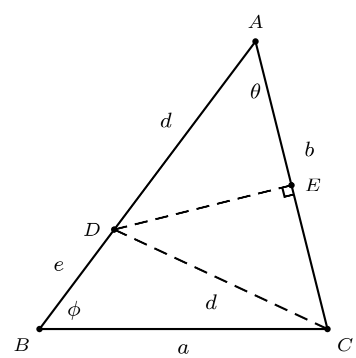

Figure 2

We note from triangle BDC, $BD + DC > BC$, *i.e.*, $AB > BC$. When $AB = BC$ the entire tree falls to the ground. Furthermore, when θ (and therefore

ϕ) is a Heron angle, AB is a Heron tree and Figure 2 represents a Heron triangle configuration.

In the original Problem 1, $c = 18$, $a = 6$ and ϕ is implicitly given (or assumed) to be $\frac{\pi}{2}$. In other words, these elements uniquely determine triangle ABC. Then the breaking point D on AB can be located as the intersection of ED, the perpendicular bisector of AC. Moreover, the present restatement of Problem 1, *i.e.*, the determination of the configuration of Figure 2, gives us an option to use either ϕ or θ as the rational cosine angle to determine triangle ABC and hence the various integer lengths $a, b, \ldots, e$. We achieve this goal with the help of $(\dagger)$. Before dealing with the general solution of Problem 1, we consider some interesting examples.

3.2. *Examples.*

3.2.1. If heavy winds break an integer tree AB at D so that the resulting configuration is an isosceles triangle with $AB = AC$, then the length of the broken part is the cube of an integer.

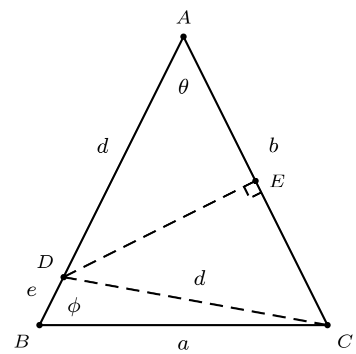

Figure 3

Proof. Suppose $AB = AC = \ell$ and $BC = m$ to begin with. From Figure 3, it follows that $\ell > m$, $\cos\theta = \frac{2\ell^2 - m^2}{2\ell^2}$, $\cos\phi = \frac{m}{2\ell}$, $AD = \frac{\ell}{2\cos\theta} = \frac{\ell^3}{2\ell^2 - m^2}$, $BD = \ell - \frac{\ell^3}{2\ell^2 - m^2} = \frac{\ell(\ell^2 - m^2)}{2\ell^2 - m^2}$. to obtain integer values we multiply each by $2\ell^2 - m^2$. In the notation of Figure 2, the solution is given by

(i) $c =$ the length of the tree $= \ell(2\ell^2 - m^2)$;

(ii) $d =$ the broken part $= \ell^3$, *an integer cube*;

(iii) $e =$ the unbroken part = $\ell(\ell^2 - m^2)$;

(iv) $a =$ the distance between the foot and top of the tree $= m(2\ell^2 - m^2)$;

(v) $\phi =$ the inclination of the tree with the ground $= \arccos\frac{m}{2\ell}$. □

In particular, if $\ell = p^2 + q^2$, and $m = 2(p^2 - q^2)$ for $(\sqrt{2} + 1)q > p > q$, then AB becomes a Heron tree broken by the wind. These yield

$$\begin{aligned}
c = b =& 2(p^2+q^2)(2pq+p^2-q^2)(2pq-p^2+q^2),\\
d =& (p^2+q^2)^3,\\
e =& (p^2+q^2)(-p^2+3q^2)(3p^2-q^2),\\
a =& 4(p^2-q^2)(2pq+p^2-q^2)(2pq-p^2+q^2).
\end{aligned}$$

For a numerical example, we put $p = 3$, $q = 2$. This gives a Heron tree of length 3094 broken by the wind into $d = 2197 = 13^3$ (an integer cube), and $e = 897$ to come down at $a = 2380$. The angle of inclination of the tree with the horizontal is $\phi = \arccos\frac{5}{13}$.

We leave the details of the following two examples to the reader as an exercise.

3.2.2. If the wind breaks an integer tree AB at D in such a way that $AC = BC$, then both the lengths of the tree and the broken part are perfect squares.

3.2.3. If the wind breaks an integer tree AB at D in such a way that $AD = DC = BC$, then the common length is a perfect square.

4. General solution of Problem 1

Ideally, the general solution of Problem 1 involves the use of integral triangles given by (†). For simplicity we first consider a special case of (†) in which $\theta = \frac{\pi}{3}$. The solution in this case is elegant. Then we simply state the general solution leaving the details to the reader.

4.1. *A particular case of Problem 1.* An integral tree AB is broken by the wind at D. The broken part DA comes down so that the top A of the tree touches the ground at C. If $\angle DAC = \frac{\pi}{3}$, determine parametric expressions for the various elements of the configuration formed.

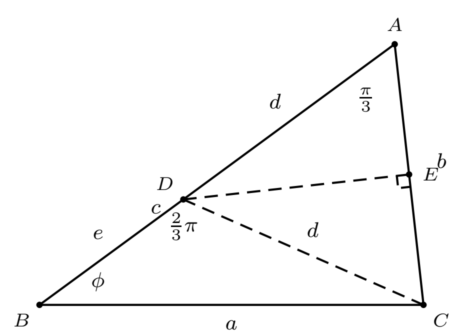

Figure 4

We refer to Figure 4. Since $\angle DAC = \frac{\pi}{3}$, triangle ADC is equilateral and $d = b = DC$. From (1), we have

$$\begin{aligned} a =& u^2 - uv + v^2, \\ d = b =& u(u - 2v), \\ c =& u^2 - v^2, \\ e =& c - d = v(2u - v), \\ \phi =& \arccos \frac{2c - b}{2a} = \arccos \frac{u^2 + 2uv - 2v^2}{2(u^2 - uv + v^2)}, \quad u > 2v. \end{aligned}$$

For a specific numerical example, we take $u = 5$, $v = 2$, and obtain a tree of length $c = 21$, broken into $d = b = 5$, $e = 16$ and $a = 19$, inclined at an angle $\phi = \arccos \frac{37}{38}$.

Remark. No tree in the $\frac{\pi}{3}$ family can be Heron because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ is irrational.

Note also that in Figure 4, $\angle BDC = \frac{2}{3}\pi$. Hence the family of triangles

$$(a, b, e) = (u^2 - uv + v^2,\ u(u - 2v),\ v(2u - v))$$

contains the angle $\frac{2}{3}\pi$ in each member. For example, with $u = 5$, $v = 2$, we have $(a, b, c) = (19, 5, 16)$; $\cos A = -\frac{1}{2}$ and $\angle A = \frac{2}{3}\pi$.

4.2. *General solution of Problem 1.* Let $\cos\theta = \frac{n}{m}$. The corresponding integral trees have
(i) length $c = mn(u^2 - v^2)$,
(ii) broken part $d = mu(nu - mv)$,
(iii) unbroken part $e = mv(mu - nv)$,
(iv) distance between the foot and the top of the tree on the ground

$$a = n(m(u^2 + v^2) - 2nuv),$$

and
(v) the angle of inclination with the ground ϕ where

$$\cos\phi = \frac{(m^2 - 2n^2)u^2 + 2mnuv - m^2v^2}{ma}. \tag{3}$$

Remark. The solution in (3) yields the solution of the Heron tree problem broken by the wind when θ is a Heron angle, *i.e.*, when $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$. Here is a numerical example. Suppose $p = 2$, $q = 1$. Then $\cos\theta = \frac{3}{5}$, *i.e.*, $n = 3$, $m = 5$. To break a specific Heron tree of this family, we put $u = 4$, $v = 1$. Then we find that $c = 225$, $d = 140$, $e = 85$, $a = 183$, and the angle of inclination $\phi = \arccos \frac{207}{305}$, a Heron angle.

5. The second problem

Brahmagupta's second problem does not need any restatement. It is an indeterminate one in its original form.

An integral mountain is one whose planar view is an integral triangle. If the angles of this integral triangle are Heron angles, then the plain view becomes a heron triangle. In such a case we have a Heron mountain. One interesting feature of the second problem is the pair of integral triangles that we are required to generate for its solution. Furthermore, it creates an amusing situation as we shall soon see – in a sense there is more wizardry!

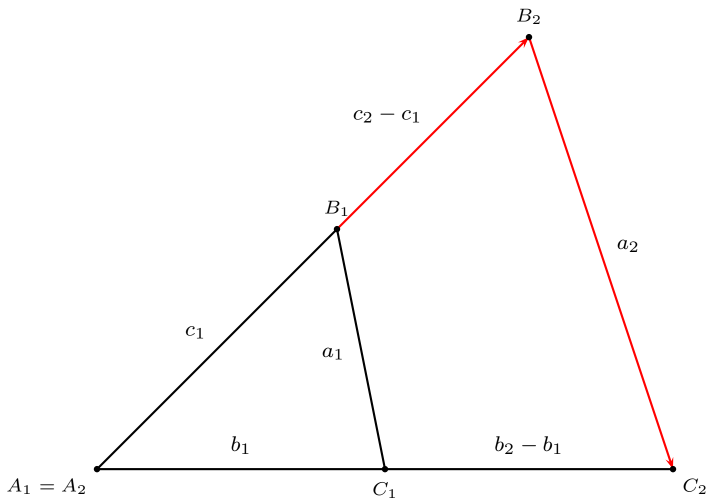

Figure 5

Figure 5 shows an integral mountain $A_1B_1C_1$. At B_1 live two ascetics. The wizard of them flies to B_2 along the direction of A_1B_1, and then reaches the town C_2. The other one walks along the path $B_1C_1C_2$. The hypothesis of the problem is $B_1B_2 + B_2C_2 = B_1C_1 + C_1C_2$, *i.e.*,

$$c_2 + a_2 - b_2 = c_1 + a_1 - b_1. \tag{4}$$

Hence the solution to Problem 2 lies in generating a pair of integral triangles $A_1B_1C_1$ and $A_2B_2C_2$, with
(i) $A_2 = A_1$,
(ii) $\angle B_1A_1C_1 = \angle B_2A_2C_2$,
(iii) $c_2 + a_2 - b_2 = c_1 + a_1 - b_1$. As the referee pointed out, together the two conditions above imply that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have a common excircle opposite to the vertices C_1, C_2. Furthermore, we need integral answers to the questions
(i) the distance between the hill and the town $C_1C_2 = b_2 - b_1$,
(ii) the height the wizard rose, *i.e.*, the altitude $c_2 \sin A_1$ through B_2 of triangle $A_2B_2C_2$.

Now, if $c_2 \sin A_1$ is to be an integer, then $\sin A_1$ should necessarily be rational. Therefore the integral mountain must be a Heron mountain. We may now put $\cos\theta = \frac{p^2-q^2}{p^2+q^2}$, *i.e.*, $n = p^2 - q^2$, $m = p^2 + q^2$ in (†) to find the answers. As it turns out, the solution would not be elegant. Instead, we use the following description of the family of Heron triangles, each member triangle containing θ. This description has previously appeared in this journal [4] so we simply state the description.

Let $\cos A = \frac{p^2-q^2}{p^2+q^2}$. The Heron triangle family determining the common angle A is given by

$$\begin{aligned}(a,b,c) =&(pq(u^2+v^2),\ (pu-qv)(qu+pv),\ (p^2+q^2)uv),\\ &(u,v)=(p,q)=1,\ p\geq 1 \text{ and } pu>qv.\end{aligned} \tag{5}$$

In particular, we note that
(i) $p=q \Rightarrow A=\frac{\pi}{2}$ and $(a,b,c)=(u^2+v^2,\ u^2-v^2,\ 2uv)$, and
(ii) $(p,q)=(u,v) \Rightarrow (a,b,c)=(u^2+v^2,\ 2(u^2-v^2),\ u^2+v^2)$
are respectively the Pythagorean triangle family and the isoceles Heron triangle family.

5.1. *The solution of Problem 2.* We continue to refer to Figure 5. Since $\angle B_1A_1C_1 = \angle B_2A_2C_2$, p and q remain the same in (5). This gives

$$\begin{aligned}(a_1,b_1,c_1) =&(pq(u_1^2+v_1^2),\ (pu_1-qv_1)(qu_1+pv_1),\ (p^2+q^2)u_1v_1),\\ (a_2,b_2,c_2) =&(pq(u_2^2+v_2^2),\ (pu_2-qv_2)(qu_2+pv_2),\ (p^2+q^2)u_2v_2).\end{aligned}$$

Next, $c_2+a_2-b_2=c_1+a_1-b_1$ simplifies to

$$v_2(qu_2+pv_2)=v_1(qu_1+pv_1)=\lambda,\ \text{a constant}. \tag{6}$$

For given p, q, there are four variables u_1, v_1, u_2, v_2, and they generate an infinity of solutions satisfying the equation (6). We now obtain two particular, numerical solutions.

5.2. *Numerical examples.* (1) We put $p=2$, $q=1$, $u_1=3$, $v_1=2$ in (6). This gives $v_2(u_2+2v_2)=14=\lambda$. It is easy to verify that $u_2=12$, $v_2=1$ is a solution. Hence we have

$$(a_1,b_1,c_1)=(13,14,15) \quad \text{and} \quad (a_2,b_2,c_2)=(145,161,30).$$

Note that $\gcd(a_ib_i,c_i)=2$, $i=1,2$, has been divided out. It is easy to verify that $c_i+a_i-b_i=14$, $i=1,2$. The answers to the questions are
(i) the distance between the hill and th twon, $b_2-b_1=147$.
(ii) The wizard rose to a height $c_2\sin A_1 = 30\times\frac{4}{5}=24$.

In fact it is possible to give as many solutions (a_i,b_i,c_i) to (6) as we wish: we have just to take sufficiently large values for λ. This creates an amusing situation as we see below.

(2) Suppose $\lambda=2\times3\times5\times7=210$. Then (6) becomes $v_i(u_i+2v_i)=210$. The indexing of the six solutions below is unconventional in the interest of Figure 6.

i	v_i	u_i	a_i	b_i	c_i
6	1	208	43265	43575	520
5	2	101	10205	10500	505
4	3	64	4105	4375	480
3	5	32	1049	1239	400
2	6	23	565	700	345
1	7	16	305	375	280

It is easy to check that for all six Heron triangles $c_i + a_i - b_i = 210$, $i = 1, 2, \ldots, 6$. From this we deduce that
(i) $B_iC_i + C_iC_{i+1} = B_iB_{i+1} + B_{i+1}C_{i+1}$ and
(ii) $B_iC_i + C_iC_j = B_iB_j + B_jC_j$, $i, j = 1, 2, 3, 4, 5$, $j > i$.

In other words, the two ascetics may choose to live at any of the places B_1, B_2, B_3, B_4, B_5. Then they may choose to reach any next town C_2, C_3, C_4, C_5, C_6. See Figure 6.

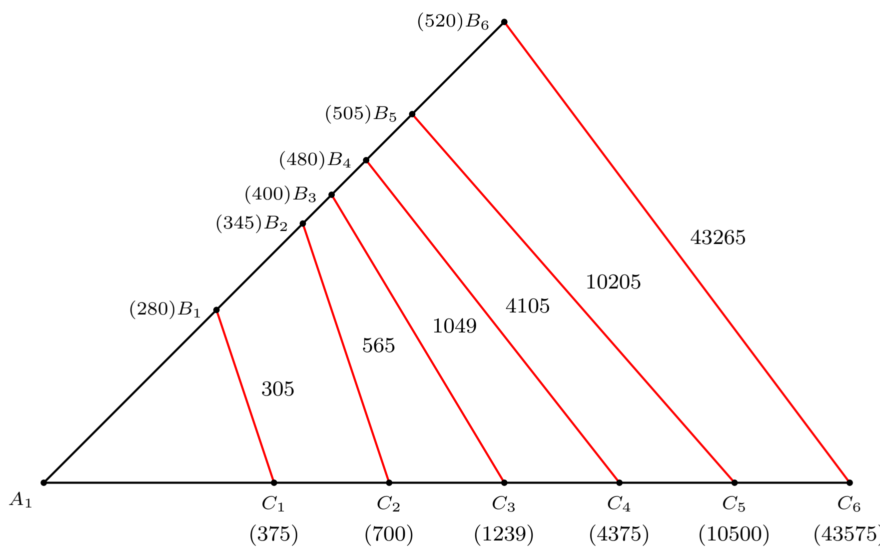

Figure 6

Another famous problem, ladders leaning against vertical walls, has been solved in the context of Heron triangles in [2].

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Square Wreaths Around Hexagons

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Abstract. We investigate the figures that arise when squares are attached to a triple of non-adjacent sides of a hexagon, and this procedure is repeated with alternating choice of the non-adjacent sides. As a special case we investigate the figure that starts with a triangle.

1. Square wreaths around hexagons

Consider a hexagon $\mathcal{H}_1 = H_{1,1}H_{2,1}H_{3,1}H_{4,1}H_{5,1}H_{6,1}$ with counterclockwise orientation. We attach squares externally on the sides $H_{1,1}H_{2,1}$, $H_{3,1}H_{4,1}$ and $H_{5,1}H_{6,1}$, to form a new hexagon $\mathcal{H}_2 = H_{1,2}H_{2,2}H_{3,2}H_{4,2}H_{5,2}H_{6,2}$. Following Nottrot, [8], we say we have made the first *square wreath* around $\mathcal{H}_1$. Then we attach externally squares to the sides $H_{6,2}H_{1,2}$, $H_{2,2}H_{3,2}$ and $H_{4,2}H_{5,2}$, to get a third hexagon $\mathcal{H}_3$, creating the second square wreath. We may repeat this operation to find a sequence of hexagons $\mathcal{H}_n = H_{1,n}H_{2,n}H_{3,n}H_{4,n}H_{5,n}H_{6,n}$ and square wreaths. See Figure 1.

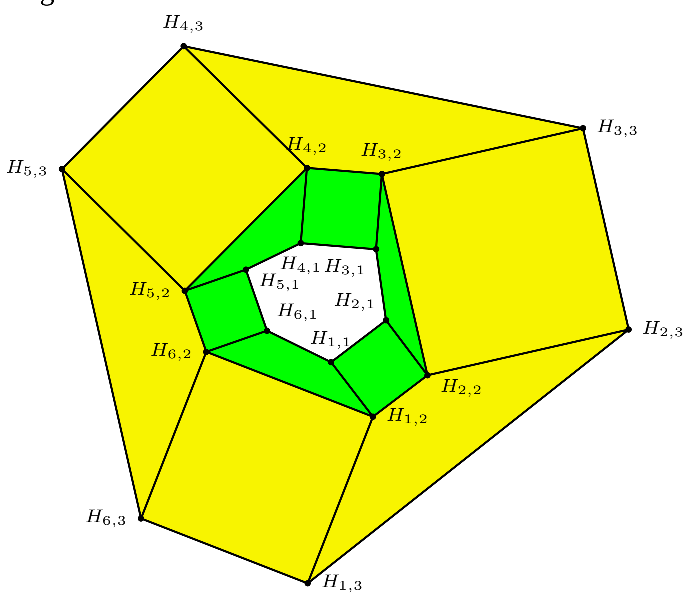

Figure 1

We introduce complex number coordinates, and abuse notations by identifying a point with its affix. Thus, we shall also regard $H_{m,n}$ as a complex number, the first subscript m taken modulo 6.

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Assuming a standard orientation of the given hexagon $\mathcal{H}_1$ in the complex plane, we easily determine the vertices of the hexagons in the above iterations.

If n is even, then $H_{1,n}H_{2,n}$, $H_{3,n}H_{4,n}$ and $H_{5,n}H_{6,n}$ are the opposite sides of the squares erected on $H_{1,n-1}H_{2,n-1}$, $H_{3,n-1}H_{4,n-1}$ and $H_{5,n-1}H_{6,n-1}$ respectively. This means, for $k = 1, 2, 3$,

$$\begin{aligned} H_{2k-1,n} =& H_{2k-1,n-1} - i(H_{2k,n-1} - H_{2k-1,n-1}) \\ =& (1+i)H_{2k-1,n-1} - i \cdot H_{2k,n-1}, \qquad (1) \\ H_{2k,n} =& H_{2k,n-1} + i(H_{2k-1,n-1} - H_{2k,n-1}) \\ =& i \cdot H_{2k-1,n-1} + (1-i)H_{2k,n-1}. \qquad (2) \end{aligned}$$

If n is odd, then $H_{2,n}H_{3,n}$, $H_{4,n}H_{5,n}$, $H_{6,n}H_{1,n}$ are the opposite sides of the squares erected on $H_{2,n-1}H_{3,n-1}$, $H_{4,n-1}H_{5,n-1}$ and $H_{6,n-1}H_{1,n-1}$ respectively. This means, for $k = 1, 2, 3$, reading first subscripts modulo 6, we have

$$\begin{aligned} H_{2k,n} =& H_{2k,n-1} - i(H_{2k+1,n-1} - H_{2k,n-1}) \\ =& (1+i)H_{2k,n-1} - i \cdot H_{2k+1,n-1}, \qquad (3) \\ H_{2k+1,n} =& H_{2k+1,n-1} + i(H_{2k,n-1} - H_{2k+1,n-1}) \\ =& i \cdot H_{2k,n-1} + (1-i)H_{2k+1,n-1}. \qquad (4) \end{aligned}$$

The above recurrence relations (1, 2, 3, 4) may be combined into

$$\begin{aligned} H_{2k,n} =& (1 + (-1)^n i)H_{2k,n-1} + (-1)^n i \cdot H_{2k+(-1)^{n+1},n-1}, \\ H_{2k+1,n} =& (1 + (-1)^n i)H_{2k+1,n-1} + (-1)^{n+1} i \cdot H_{2k+1+(-1)^n,n-1}, \end{aligned}$$

or even more succintly,

$$H_{m,n} = (1 + (-1)^n i)H_{m,n-1} + (-1)^{m+n} i \cdot H_{m+(-1)^{m+n+1},n-1}.$$

Proposition 1. *Triangles $H_{1,n}H_{3,n}H_{5,n}$ and $H_{1,n-2}H_{3,n-2}H_{5,n-2}$ have the same centroid, so do triangles $H_{2,n}H_{4,n}H_{6,n}$ and $H_{2,n-2}H_{4,n-2}H_{6,n-2}$.*

Proof. Applying the relations (1, 2, 3, 4) twice, we have

$$\begin{aligned} H_{1,n} =& -(1-i)H_{6,n-2} + 2H_{1,n-2} + (1-i)H_{2,n-2} - H_{3,n-2}, \\ H_{3,n} =& -(1-i)H_{2,n-2} + 2H_{3,n-2} + (1-i)H_{4,n-2} - H_{5,n-2}, \\ H_{5,n} =& -(1-i)H_{4,n-2} + 2H_{5,n-2} + (1-i)H_{6,n-2} - H_{1,n-2}. \end{aligned}$$

The triangle $H_{1,n}H_{3,n}H_{5,n}$ has centroid

$$\frac{1}{3}\left(H_{1,n} + H_{3,n} + H_{5,n}\right) = \frac{1}{3}\left(H_{1,n-2} + H_{3,n-2} + H_{5,n-2}\right),$$

which is the centroid of triangle $H_{1,n-2}H_{3,n-2}H_{5,n-2}$. The proof for the other pair is similar. □

Theorem 2. *For each* $m = 1, 2, 3, 4, 5, 6$*, the sequence of vertices* $H_{m,n}$ *satisfies the recurrence relation*

$$H_{m,n} = 6H_{m,n-2} - 6H_{m,n-4} + H_{m,n-6}. \tag{5}$$

Proof. By using the recurrence relations (1, 2, 3, 4), we have

$$\begin{aligned}
H_{1,2} =&(1+i)H_{1,1} - iH_{2,1},\\
H_{1,3} =&2H_{1,1} - (1+i)H_{2,1} - H_{5,1} + (1+i)H_{6,1},\\
H_{1,4} =&3(1+i)H_{1,1} - 4iH_{2,1} - (1+i)H_{3,1} + iH_{4,1} - (1+i)H_{5,1} + 2iH_{6,1},\\
H_{1,5} =&8H_{1,1} - 5(1+i)H_{2,1} - H_{3,1} - 6H_{5,1} + 5(1+i)H_{6,1},\\
H_{1,6} =&13(1+i)H_{1,1} - 18iH_{2,1} - 6(1+i)H_{3,1} + 6iH_{4,1} - 6(1+i)H_{5,1} + 11iH_{6,1},\\
H_{1,7} =&37H_{1,1} - 24(1+i)H_{2,1} - 6H_{3,1} - 30H_{5,1} + 24(1+i)H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 2, 3, 4, 5, 6$, from these equations gives

$$H_{1,7} = 6H_{1,5} - 6H_{1,3} + H_{1,1}.$$

The same relations hold if we simultaneously increase each first subscript by 2, or each second subscript by 1. Thus, we have the recurrence relation (5) for $m = 1, 3, 5$. Similarly,

$$\begin{aligned}
H_{2,2} =&iH_{1,1} + (1-i)H_{2,1},\\
H_{2,3} =&-(1-i)H_{1,1} + 2H_{2,1} + (1-i)H_{3,1} - H_{4,1},\\
H_{2,4} =&4iH_{1,1} + 3(1-i)H_{2,1} - 2iH_{3,1} - (1-i)H_{4,1} - iH_{5,1} - (1-i)H_{6,1},\\
H_{2,5} =&-5(1-i)H_{1,1} + 8H_{2,1} + 5(1-i)H_{3,1} - 6H_{4,1} - H_{6,1},\\
H_{2,6} =&18iH_{1,1} + 13(1-i)H_{2,1} - 11iH_{3,1} - 6(1-i)H_{4,1} - 6iH_{5,1} - 6(1-i)H_{6,1},\\
H_{2,7} =&-24(1-i)H_{1,1} + 37H_{2,1} + 24(1-i)H_{3,1} - 30H_{4,1} - 6H_{6,1}.
\end{aligned}$$

Elimination of $H_{m,1}$, $m = 1, 3, 4, 5, 6$, from these equations gives

$$H_{2,7} - 6H_{2,5} + 6H_{2,3} - H_{2,1} = 0.$$

A similar reasoning shows that (5) holds for $m = 2, 4, 6$. □

2. Midpoint triangles

Let M_1, M_2, M_3 be the midpoints of $H_{4,1}H_{5,1}$, $H_{6,1}H_{1,1}$ and $H_{2,1}H_{3,1}$ and M_1', M_2', M_3' the midpoints of $H_{1,3}H_{2,3}$, $H_{3,3}H_{4,3}$ and $H_{5,3}H_{6,3}$ respectively. We have

$$\begin{aligned}
M_1' =&\frac{1}{2}\left(H_{1,3} + H_{2,3}\right)\\
=&\frac{1}{2}\left((1+i)H_{1,1} + (1-i)H_{2,1} + (1-i)H_{3,1} - H_{4,1} - H_{5,1} + (1+i)H_{6,1}\right)\\
=&-M_1 + (1+i)M_2 + (1-i)M_3.
\end{aligned}$$

Similarly,

$$M_2' = (1-i)M_1 - M_2 + (1+i)M_3,$$
$$M_3' = (1+i)M_1 + (1-i)M_2 - M_3.$$

Proposition 3. *For a permutation* (j,k,ℓ) *of the integers* $1,2,3$*, the segments* M_jM_k' *and* M_kM_j' *are perpendicular to each other and equal in length, while* $M_\ell M_\ell'$ *is parallel to an angle bisector of* M_jM_k' *and* M_kM_j'*, and is* $\sqrt{2}$ *times as long as each of these segments.*

Proof. From the above expressions for M_j', $j=1,2,3$, we have

$$M_2' - M_3 = (1-i)M_1 - M_2 + i\cdot M_3, \tag{6}$$
$$\begin{aligned} M_3' - M_2 &= (1+i)M_1 - i\cdot M_2 - M_3,\\ &= i\left((1-i)M_1 - M_2 + iM_3\right),\\ &= i(M_2' - M_3); \end{aligned} \tag{7}$$
$$\begin{aligned} M_1 - M_1' &= 2M_1 - (1+i)M_2 - (1-i)M_3\\ &= (M_2' - M_3) + (M_3' - M_2). \end{aligned} \tag{8}$$

From (6) and (7), M_2M_3' and M_3M_2' are perpendicular and have equal lengths. From (8), we conclude that $M_1'M_1$ is parallel to an angle bisector of M_2M_3' and M_3M_2', and is $\sqrt{2}$ times as long as each of these segments. The same results for $(k,\ell) = (3,1),\ (1,2)$ follow similarly. □

The midpoints of the segments M_jM_j', $j=1,2,3$, are the points

$$N_1 = \frac{1}{2}\left((1+i)M_2 + (1-i)M_3\right),$$
$$N_2 = \frac{1}{2}\left((1+i)M_3 + (1-i)M_1\right),$$
$$N_3 = \frac{1}{2}\left((1+i)M_1 + (1-i)M_2\right).$$

Note that

$$\begin{aligned} N_1 &= \frac{M_2+M_3}{2} + i\cdot\frac{M_2-M_3}{2},\\ &= \frac{M_2'+M_3'}{2} - i\cdot\frac{M_2'-M_3'}{2}. \end{aligned}$$

Thus, N_1 is the center of the square constructed externally on the side M_2M_3 of triangle $M_1M_2M_3$, and also the center of the square constructed internally on $M_1'M_2'M_3'$. Similarly, for N_2 and N_3. From this we deduce the following corollaries. See Figure 2.

Corollary 4. *The triangles $M_1M_2M_3$ and $M_1'M_2'M_3'$ are perspective. The perspector is the outer Vecten point of $M_1M_2M_3$ and the inner Vecten point of $M_1'M_2'M_3'$.* [1]

Corollary 5. *The segments M_jN_ℓ and M_kN_ℓ are equal in length and are perpendicular. The same is true for $M_j'N_\ell$ and $M_k'N_\ell$.*

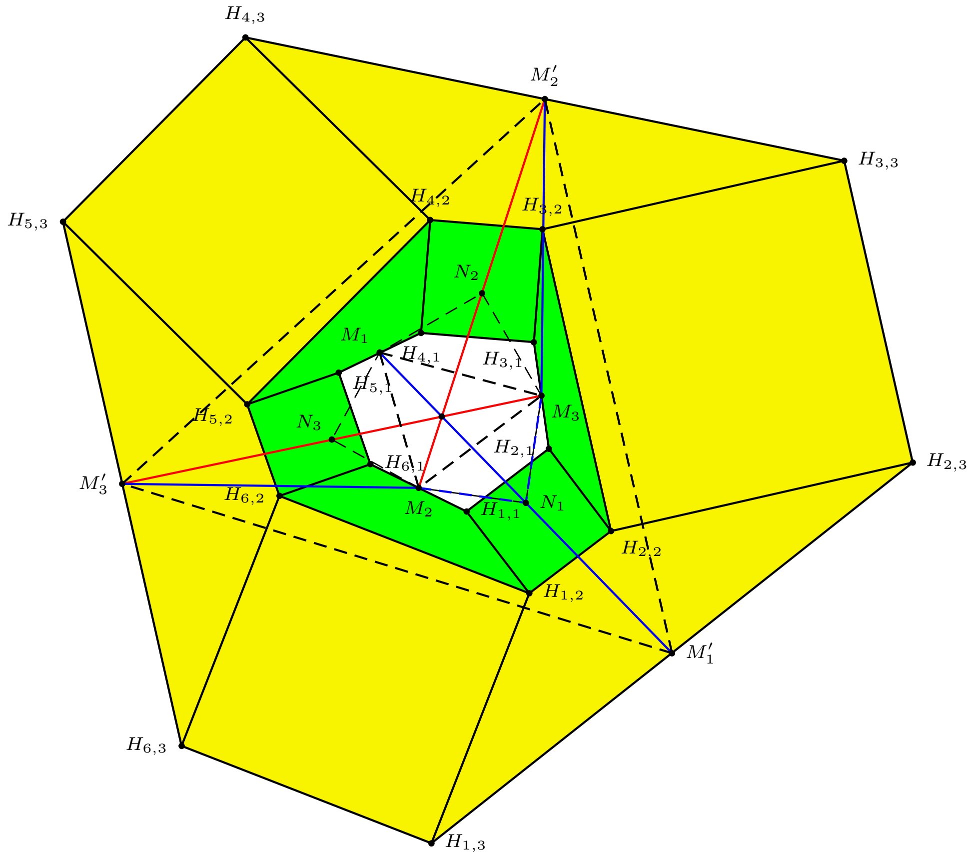

Figure 2

Let $M_1''M_2''M_3''$ be the desmic mate of $M_1M_2M_3$ and $M_1'M_2'M_3'$, *i.e.*, $M_1'' = M_2M_3' \cap M_3M_2'$ etc. By Proposition 3, $\angle M_2M_1''M_3$ is a right angle, and M_1'' lies on the circle with diameter M_2M_3. Since the bisector angle $M_2M_1''M_3$ is parallel to the line N_1M_1, $M_1''N_1$ is perpendicular to this latter line. See Figure 3.

Proposition 6. *If (j, k, ℓ) is a permutation of 1, 2, 3, the lines M_jM_k' and M_kM_j' and the line through N_ℓ perpendicular to $M_\ell M_\ell'$ are concurrent (at M_ℓ'').*

Corollary 7. *The circles $(M_jM_kN_\ell)$, $(M_j'M_k'N_\ell)$, $(M_\ell M_\ell''N_\ell)$ and $(M_\ell'M_\ell''N_\ell)$ are coaxial, so the midpoints of M_jM_k, $M_j'M_k'$, $M_\ell M_\ell''$ and $M_\ell'M_\ell''$ are collinear, the line being parallel to $M_\ell M_\ell'$.*

[1]The outer (respectively inner) Vecten point is the point X_{485} (respectively X_{486}) of [4].

Since the lines M_jM_j', $j = 1, 2, 3$, concur at the outer Vecten point of triangle $M_1M_2M_3$, the intersection of the lines is the inferior (complement) of the outer Vecten point. [2] As such, it is the center of the circle through $N_1N_2N_3$ (see [4]).

Corollary 8. *The three lines joining the midpoints of M_1M_1', M_2M_2', M_3M_3' are concurrent at the center of the circle through N_1, N_2, N_3, which also passes through M_1'', M_2' and M_3''.*

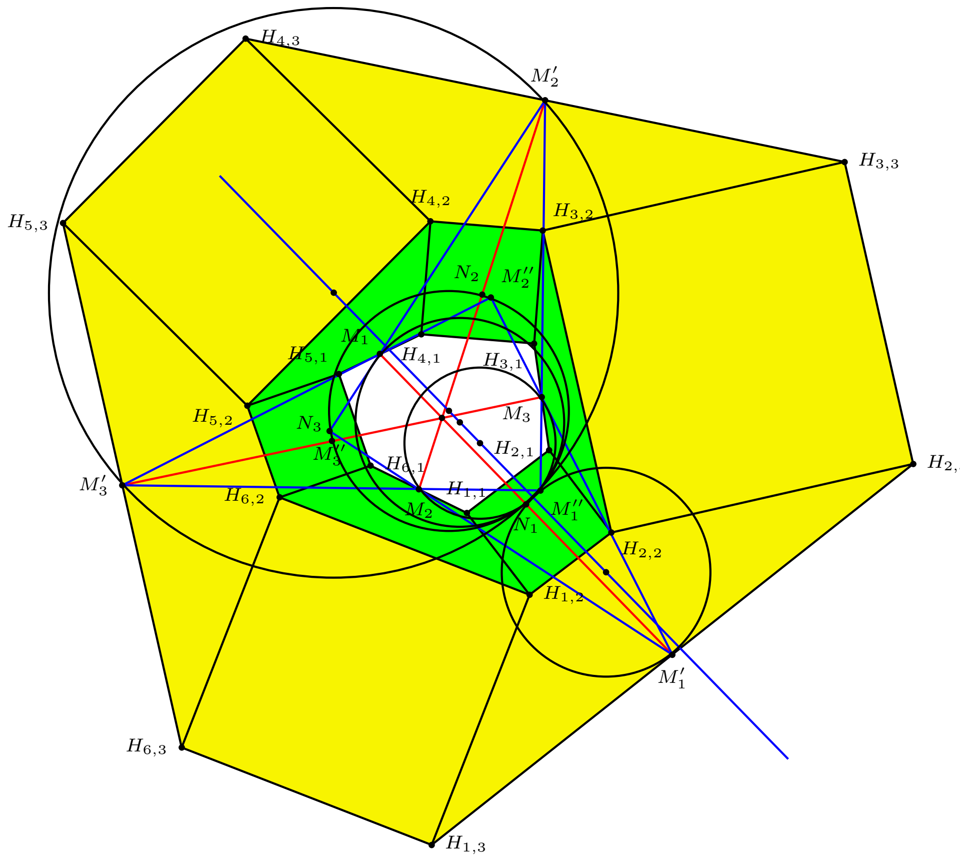

Figure 3

3. Starting with a triangle

An interesting special case occurs when the initial hexagon $\mathcal{H}_1$ degenerates into a triangle with

$$H_{1,1} = H_{6,1} = A, \qquad H_{2,1} = H_{3,1} = B, \qquad H_{4,1} = H_{5,1} = C.$$

This case has been studied before by Haight and Nottrot, who examined especially the side lengths and areas of the squares in each wreath. Under this assumption,

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[2] X_{641} in [4].

between two consecutive hexagons $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ are three squares and three alternating trapezoids of equal areas. The trapzoids between $\mathcal{H}_1$ and $\mathcal{H}_2$ degenerate into triangles. The sides of the squares are parallel and perpendicular to the sides or to the medians of triangle ABC according as n is odd or even. We shall assume the sidelengths of triangle ABC to be a, b, c, and the median lengths m_a, m_b, m_c respectively.

The squares of the first wreath are attached to the triangle sides outwardly.[3] Haight [2] has computed the ratios of the sidelengths of the squares.

If $n = 2k - 1$, the squares have sidelengths a, b, c multiplied by $a_1(k)$, where

$$a_1(k) = 5a_1(k-1) - a_1(k-2),$$
$$a_1(1) = 1,\ a_1(2) = 4.$$

This is sequence A004253 in Sloane's *Online Encyclopedia of Integer sequences* [9]. This also means that

$$H_{m,2k} - H_{m,2k-1} = a_1(k)(H_{m,2} - H_{m,1}). \tag{9}$$

If $n = 2k$, the squares have sidelengths $2m_a$, $2m_b$, $2m_c$ multiplied by $a_2(k)$, where

$$a_2(k) = 5a_2(k-1) - a_2(k-2),$$
$$a_2(1) = 1,\ a_2(2) = 5.$$

This is sequence A004254 in [9]. This also means that

$$H_{m,2k+1} - H_{m,2k} = a_2(k)(H_{m,3} - H_{m,2}). \tag{10}$$

Proposition 9. *Each trapezoid in the wreath bordered by $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ has area $a_2(n) \cdot \triangle ABC$.*

Lemma 10. (1) $\sum_{j=1}^{k} a_1(j) = a_2(k)$.
(2) *The sums $a_3(k) = \sum_{j=1}^{k} a_2(j)$ satisfy the recurrence relation*

$$a_3(k) = 6a_3(k-1) - 6a_3(k-2) + a_3(k-3),$$
$$a_3(1) = 1,\ a_3(2) = 6,\ a_3(3) = 30.$$

The sequence $a_3(k)$ is essentially sequence A089817 in [9].[4]

It follows from (9) and (10) that

$$\begin{aligned} H_{m,2k} =& H_{m,1} + \sum_{j=1}^{k}(H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k-1}(H_{m,2j+1} - H_{m,2j}) \\ =& H_{m,1} + \left(\sum_{j=1}^{k} a_1(j)\right)(H_{m,2} - H_{m,1}) + \left(\sum_{j=1}^{k-1} a_2(j)\right)(H_{m,3} - H_{m,2}) \\ =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k-1)(H_{m,3} - H_{m,2}). \end{aligned}$$

[3] Similar results as those in §§3, 4 can be found if these initial squares are constructed inwardly.

[4] Note that sequences a_1 and a_2 follow this third order recurrence relation as well.

Also,

$$\begin{aligned}H_{m,2k+1} =& H_{m,1} + \sum_{j=1}^{k}(H_{m,2j} - H_{m,2j-1}) + \sum_{j=1}^{k}(H_{m,2j+1} - H_{m,2j})\\ =& H_{m,1} + \left(\sum_{j=1}^{k} a_1(j)\right)(H_{m,2} - H_{m,1}) + \left(\sum_{j=1}^{k} a_2(j)\right)(H_{m,3} - H_{m,2})\\ =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2})\end{aligned}$$

Here is a table of the *absolute* barycentric coordinates (with respect to triangle ABC) of the initial values in the above recurrence relations.

m	$H_{m,1}$	$H_{m,2} - H_{m,1}$	$H_{m,3} - H_{m,2}$
1	$(1,0,0)$	$\frac{1}{S}(S_B, S_A, -c^2)$	$(2,-1,-1)$
2	$(0,1,0)$	$\frac{1}{S}(S_B, S_A, -c^2)$	$(-1,2,-1)$
3	$(0,1,0)$	$\frac{1}{S}(-a^2, S_C, S_B)$	$(-1,2,-1)$
4	$(0,0,1)$	$\frac{1}{S}(-a^2, S_C, S_B)$	$(-1,-1,2)$
5	$(0,0,1)$	$\frac{1}{S}(S_C, -b^2, S_A)$	$(-1,-1,2)$
6	$(1,0,0)$	$\frac{1}{S}(S_C, -b^2, S_A)$	$(2,-1,-1)$

From these we have the homogeneous barycentric coordinates

$$\begin{aligned}H_{m,2k} =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k-1)(H_{m,3} - H_{m,2}),\\ H_{m,2k+1} =& H_{m,1} + a_2(k)(H_{m,2} - H_{m,1}) + a_3(k)(H_{m,3} - H_{m,2}).\end{aligned}$$

These can be combined into a single relation

$$H_{m,n} = H_{m,1} + a_2(n')(H_{m,2} - H_{m,1}) + a_3(n'')(H_{m,3} - H_{m,2}),$$

in which $n' = \lfloor \frac{n}{2} \rfloor$ and $n'' = \lfloor \frac{n-1}{2} \rfloor$.

Here are the coordinates of the points $H_{m,n}$.

m	x	y	z
1	$(2a_3(n'') + 1)S + a_2(n')S_B$	$-a_3(n'')S + a_2(n')S_A$	$-a_3(n'')S - a_2(n')c^2$
2	$-a_3(n'')S + a_2(n')S_B$	$(2a_3(n'') + 1)S + a_2(n')S_A$	$-a_3(n'')S - a_2(n')c^2$
3	$-a_3(n'')S - a_2(n')a^2$	$(2a_3(n'') + 1)S + a_2(n')S_C$	$-a_3(n'')S + a_2(n')S_B$
4	$-a_3(n'')S - a_2(n')a^2$	$-a_3(n'')S + a_2(n')S_C$	$(2a_3(n'') + 1)S + a_2(n')S_B$
5	$-a_3(n'')S + a_2(n')S_C$	$-a_3(n'')S - a_2(n')b^2$	$(2a_3(n'') + 1)S + a_2(n')S_A$
6	$(2a_3(n'') + 1)S + a_2(n')S_C$	$-a_3(n'')S - a_2(n')b^2$	$-a_3(n'')S + a_2(n')S_A$

Note that the coordinate sum of each of the points in the above is equal to S.

Consider the midpoints of the following segments

segment	$H_{1,n}H_{2,n}$	$H_{2,n}H_{3,n}$	$H_{3,n}H_{4,n}$	$H_{4,n}H_{5,n}$	$H_{5,n}H_{6,n}$	$H_{6,n}H_{1,n}$
midpoint	$C_{1,n}$	$B_{2,n}$	$A_{1,n}$	$C_{2,n}$	$B_{1,n}$	$A_{2,n}$

For $j = 1, 2$, denote by $\mathcal{T}_{j,n}$ the triangle with vertices $A_{j,n}B_{j,n}C_{j,n}$.

$A_{1,n}$	$-2a_3(n'')S - 2a_2(n')a^2$	$(a_3(n'') + 1)S + 2a_2(n')S_C$	$(a_3(n'') + 1)S + 2a_2(n')S_B$
$B_{1,n}$	$(a_3(n'') + 1)S + 2a_2(n')S_C$	$-2a_3(n'')S - 2a_2(n')b^2$	$(a_3(n'') + 1)S + 2a_2(n')S_A$
$C_{1,n}$	$(a_3(n'') + 1)S + 2a_2(n')S_B$	$(a_3(n'') + 1)S + 2a_2(n')S_A$	$-2a_3(n'')S - 2a_2(n')c^2$
$A_{2,n}$	$-2(2a_3(n'') + 1)S - a_2(n')a^2$	$2a_3(n'')S + a_2(n')S_C$	$2a_3(n'')S + a_2(n')S_B$
$B_{2,n}$	$2a_3(n'')S + a_2(n')S_C$	$-2(2a_3(n'') + 1)S - a_2(n')b^2$	$2a_3(n'')S + a_2(n')S_A$
$C_{2,n}$	$2a_3(n'')S + a_2(n')S_B$	$2a_3(n'')S + a_2(n')S_A$	$-2(2a_3(n'') + 1)S - a_2(n')c^2$

Proposition 11. *The triangles $\mathcal{T}_{1,n}$ and $\mathcal{T}_{2,n}$ are perspective.*

This is a special case of the following general result.

Theorem 12. *Every triangle of the form*

$$\begin{array}{lclcl} -2fS - ga^2 & : & (f+1)S + gS_C & : & (f+1)S + gS_B \\ (f+1)S + gS_C & : & -2fS - gb^2 & : & (f+1)S + gS_A \\ (f+1)S + gS_B & : & (f+1)S + gS_A & : & -2fS - gc^2 \end{array}$$

where f and g represent real numbers, is perspective with the reference triangle. Any two such triangles are perspective.

Proof. Clearly the triangle given above is perspective with ABC at the point

$$\left(\frac{1}{(f+1)S + gS_A} : \frac{1}{(f+1)S + gS_B} : \frac{1}{(f+1)S + gS_C}\right),$$

which is the Kiepert perspector $K(\phi)$ for $\phi = \cot^{-1}\frac{f+1}{g}$.

Consider a second triangle of the same form, with f and g replaced by p and q respectively. We simply give a description of this perspector P. This perspector is the centroid if and only if $(g - q) + 3(gp - fq) = 0$. Otherwise, the line joining this perspector to the centroid G intersects the Brocard axis at the point

$$Q = (a^2((g-q)S_A - (f-p)S) : b^2((g-q)S_B - (f-p)S) : c^2((g-q)S_C - (f-p)S)),$$

which is the isogonal conjugate of the Kiepert perspector $K\left(-\cot^{-1}\frac{f-p}{g-q}\right)$. The perspector P in question divides GQ in the ratio

$$\begin{aligned} & GP : GQ \\ =& ((g-q) + 3(gp - fq))((g-q)S - (f-p)S_\omega) \\ & : (3(f-p)^2 + (g-q)^2)S + 2(f-p)(g-q)S_\omega. \end{aligned}$$

□

Note that $\mathcal{T}_{j,n}$ for $j \in \{1, 2\}$ and the Kiepert triangles $\mathcal{K}_\phi$[5] are of this form. Also the medial triangle of a triangle of this form is again of the same form. The perspectors of $\mathcal{T}_{j,n}$ and ABC lie on the Kiepert hyperbola. It is the Kiepert perspector $K(\phi_{j,n})$ where

$$\cot\phi_{1,n} = \frac{a_3(n'') + 1}{2a_2(n')},$$

[5] See for instance [6].

and

$$\cot \phi_{2,n} = \frac{2a_3(n'')}{a_2(n')}.$$

In particular the perspectors tend to limits when n tends to infinity. The perspectors and limits are given by

triangle	perspector K_ϕ with	limit for $k \to \infty$
$\mathcal{T}_{1,2k}$	$\phi_{1,2k} = \cot^{-1} \frac{a_3(k-1)+1}{2a_2(k)}$	$\phi_{1,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{12}$
$\mathcal{T}_{1,2k+1}$	$\phi_{1,2k+1} = \cot^{-1} \frac{a_3(k)+1}{2a_2(k)}$	$\phi_{1,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{12}$
$\mathcal{T}_{2,2k}$	$\phi_{2,2k} = \cot^{-1} \frac{2a_3(k-1)}{a_2(k)}$	$\phi_{2,\text{even}} = \cot^{-1} \frac{\sqrt{21}-3}{3}$
$\mathcal{T}_{2,2k+1}$	$\phi_{2,2k+1} = \cot^{-1} \frac{2a_3(k)}{a_2(k)}$	$\phi_{2,\text{odd}} = \cot^{-1} \frac{\sqrt{21}+3}{3}$

Remarks. (1) $\mathcal{T}_{2,2}$ is the medial triangle of $\mathcal{T}_{1,3}$.
(2) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,3}$ is X_{591}.
(3) The perspector of $\mathcal{T}_{2,2}$ and $\mathcal{T}_{2,4}$ is the common circumcenter of $\mathcal{T}_{1,3}$ and $\mathcal{T}_{2,4}$.

Nottrot [8], on the other hand, has found that the sum of the areas of the squares between $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$ as $a_4(n)(a^2+b^2+c^2)$, where wreaths.

$$a_4(n) = 4a_4(n-1) + 4a_4(n-2) - a_4(n-3),$$
$$a_4(1) = 1,\ a_4(2) = 3,\ a_4(3) = 16.$$

This is sequence A005386 in [9]. Note that $a_1(n) = a_4(n) + a_4(n-1)$ for $n \geq 2$.

4. Pairs of congruent triangles

Lemma 13. (a) *If $n \geq 2$ is even, then*

$$\overrightarrow{A_{1,n}A_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}},$$
$$\overrightarrow{B_{1,n}B_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}},$$
$$\overrightarrow{C_{1,n}C_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}}.$$

(b) *If $n \geq 3$ is odd, then*

$$\overrightarrow{A_{2,n}A_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{3,n}H_{3,n+1}} = -\frac{1}{2}\overrightarrow{H_{4,n}H_{4,n+1}},$$
$$\overrightarrow{B_{2,n}B_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{5,n}H_{5,n+1}} = -\frac{1}{2}\overrightarrow{H_{6,n}H_{6,n+1}},$$
$$\overrightarrow{C_{2,n}C_{2,n+1}} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}} = -\frac{1}{2}\overrightarrow{H_{2,n}H_{2,n+1}}.$$

Proof. Consider the case of $A_{1,n}A_{1,n+1}$ for even n. Translate the trapezoid $H_{5,n+1}H_{6,n+1}H_{6,n}H_{5,n}$ by the vector $\overrightarrow{H_{5,n+1}H_{4,n+1}}$ and the trapezoid $H_{2,n+1}H_{2,n}H_{1,n}H_{1,n+1}$ by the vector $\overrightarrow{H_{2,n+1}H_{3,n+1}}$. Together with the trapezoid $H_{3,n}H_{4,n}H_{4,n+1}H_{3,n+1}$, these images form two triangles $XH_{3,n+1}H_{4,n+1}$ and

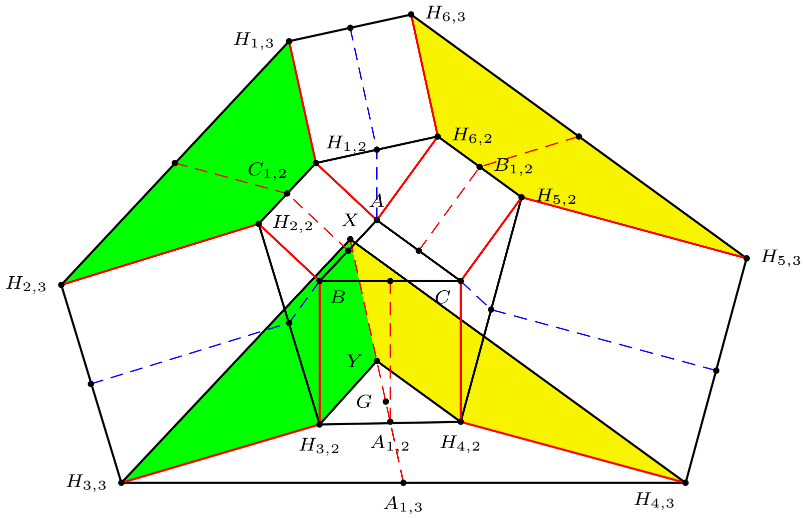

Figure 4.

$YH_{3,n}H_{4,n}$ homothetic at their common centroid G.[6] See Figure 4. It is clear that the points X, Y, $A_{1,n}$, $A_{1,n+1}$ all lie on a line through the centroid G. Furthermore, $\overrightarrow{A_{1,n}A_{1,n+1}} = \frac{1}{2}\overrightarrow{XY} = -\frac{1}{2}\overrightarrow{H_{1,n}H_{1,n+1}}$. The other cases follow similarly. □

Proposition 14. (1) *If $n \geq 3$ is odd, the following pairs of triangles are congruent.*
(i) $H_{2,n}B_{2,n+1}C_{1,n-1}$ *and* $H_{5,n}B_{1,n-1}C_{2,n+1}$,
(ii) $H_{3,n}A_{1,n-1}B_{2,n+1}$ *and* $H_{6,n}A_{2,n+1}B_{1,n-1}$,
(iii) $H_{4,n}C_{2,n+1}A_{1,n-1}$ *and* $H_{1,n}C_{1,n-1}A_{2,n+1}$.
(2) *If $n \geq 2$ is even, the following pairs of triangles are congruent.*
(iv) $H_{2,n}B_{2,n-1}C_{1,n+1}$ *and* $H_{5,n}B_{1,n+1}C_{2,n-1}$,
(v) $H_{3,n}A_{1,n+1}B_{2,n-1}$ *and* $H_{6,n}A_{2,n-1}B_{1,n+1}$,
(vi) $H_{4,n}C_{2,n-1}A_{1,n+1}$ *and* $H_{1,n}C_{1,n+1}A_{2,n-1}$.

Proof. We consider the first of these cases. Let $n \geq 3$ be an odd number. Consider the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$. We show that $H_{2,n}B_{2,n+1}$ and $H_{5,n}B_{1,n-1}$ are perpendicular to each other and equal in length, and the same for $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$.

Consider the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$. By Lemma 13, $B_{2,n}B_{2,n+1}$ is parallel to $H_{5,n}H_{5,n+1}$ and is half its length. It follows that $B_{2,n}B_{2,n+1}$ is perpendicular to and has the same length as $B_{1,n}H_{5,n}$. Similarly, $B_{2,n}H_{2,n}$ is perpendicular to and has the same length as $B_{1,n}B_{1,n-1}$. Therefore, the triangles $H_{2,n}B_{2,n}B_{2,n+1}$ and $B_{1,n-1}B_{1,n}H_{5,n}$ are congruent, and the segments $H_{2,n}B_{2,n+1}$

[6] As noted in the beginning of §3, the sides of the squares are parallel to and perpendicular to the sides of ABC or the medians of ABC according as n is odd or even. In the even case it is thus clear that the homothetic center is the centroid. In the odd case it can be seen as the lines perpendicular to the sides of ABC are parallel to the medians of the triangles in the first wreath (the flank triangles). The centroid and the orthocenter befriend each other. See [5].

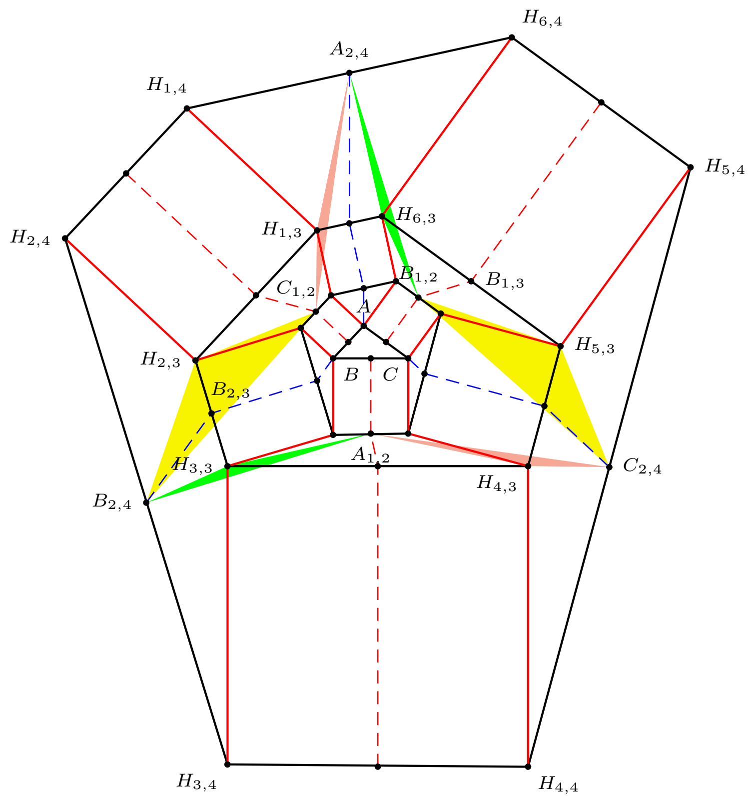

Figure 5.

and $B_{1,n-1}H_{5,n}$ are perpendicular and equal in length. See Figure 5. The same reasoning shows that the segments $H_{2,n}C_{1,n-1}$ and $H_{5,n}C_{2,n+1}$ are perpendicular and equal in length. Therefore, the triangles $H_{2,n}B_{2,n+1}C_{1,n-1}$ and $H_{5,n}B_{1,n-1}C_{2,n+1}$ are congruent.

The other cases can be proved similarly. □

Remark. The fact that the segments $B_{2,n+1}C_{1,n-1}$ and $B_{1,n-1}C_{2,n+1}$ are perpendicular and equal in length has been proved in Proposition 3 for square wreaths arising from an arbitrary hexagon.

5. A pair of Kiepert hyperbolas

The triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ are congruent to ABC. The counterparts of a point P in these triangles are the points with the barycentric coordinates relative to these three triangles as P relative to ABC.

Theorem 15. *The locus of point P in ABC whose counterparts in the triangles $A_{1,3}H_{4,2}H_{3,2}$, $H_{5,2}B_{1,3}H_{6,2}$ and $H_{2,2}H_{1,2}C_{1,3}$ form a triangle $A'B'C'$ perspective to ABC is the union of the line at infinity and the rectangular hyperbola*

$$S\sum_{\text{cyclic}}(S_B-S_C)yz+(x+y+z)\left(\sum_{\text{cyclic}}(S_B-S_C)(S_A+S)x\right)=0. \quad (11)$$

The locus of the perspector is the union of the line at infinity and the Kiepert hyperbola of triangle ABC.

Proof. The counterparts of $P=(x:y:z)$ form a triangle perspective with ABC if and only if the parallels through A, B, C to $A_{1,3}P$, $B_{1,3}P$ and $C_{1,3}P$ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)(x+y+z)-Sz)Y-((S+S_C)(x+y+z)-Sy)Z&=0,\\
-((S+S_A)(x+y+z)-Sz)X+((S+S_C)(x+y+z)-Sx)Z&=0,\\
((S+S_A)(x+y+z)-Sy)X-((S+S_B)(x+y+z)-Sx)Y&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)\left(\sum_{cyclic}(b^2-c^2)((S_A+S)x^2-S_Ayz)\right)=0.$$

The locus therefore consists of the line at infinity and a conic. Rearranging the equation of the conic in the form (11), we see that it is homothetic to the Kiepert hyperbola.

For a point P on the locus (11), let $Q=(x:y:z)$ be the corresponding perspector. This means that the parallels through $A_{1,3}$, $B_{1,3}$, $C_{1,3}$ to AQ, BQ, CQ are concurrent. These parallels have equations

$$\begin{aligned}
((S+S_B)y-(S+S_C)z)X+((S+S_B)y-S_Cz)Y-((S+S_C)z-S_By)Z&=0,\\
-((S+S_A)x-S_Cz)X+((S+S_C)z-(S+S_A)x)Y+((S+S_C)z-S_Ax)Z&=0,\\
((S+S_A)x-S_By)X-((S+S_B)y-S_Ax)Y+((S+S_A)x-(S+S_B)y)Z&=0.
\end{aligned}$$

They are concurrent if and only if

$$(x+y+z)((S_B-S_C)yz+(S_C-S_A)zx+(S_A-S_B)xy)=0.$$

This means the perspector lies on the union of the line at infinity and the Kiepert hyperbola. □

Here are some examples of points on the locus (11) with the corresponding perspectors on the Kiepert hyperbola (see [1, 7]).

Q on Kiepert hyperbola	P on locus
$K\left(\arctan\frac{3}{2}\right)$	centroid
orthocenter	de Longchamps point
centroid	$G' = (-2S_A + a^2 + S : \cdots : \cdots)$
outer Vecten point	outer Vecten point
A	$A' = B_{1,3}H_{5,2} \cap C_{1,3}H_{2,2}$
B	$B' = C_{1,3}H_{1,2} \cap A_{1,3}H_{4,2}$
C	$C' = A_{1,3}H_{3,2} \cap B_{1,3}H_{6,2}$
A_0	$A_{1,3} = (-(S + S_B + S_C) : S + S_C : S + S_B)$
B_0	$B_{1,3} = (S + S_C : -(S + S_C + S_A) : S + S_A)$
C_0	$C_{1,3} = (S + S_B : S + S_A : -(S + S_A + S_B))$

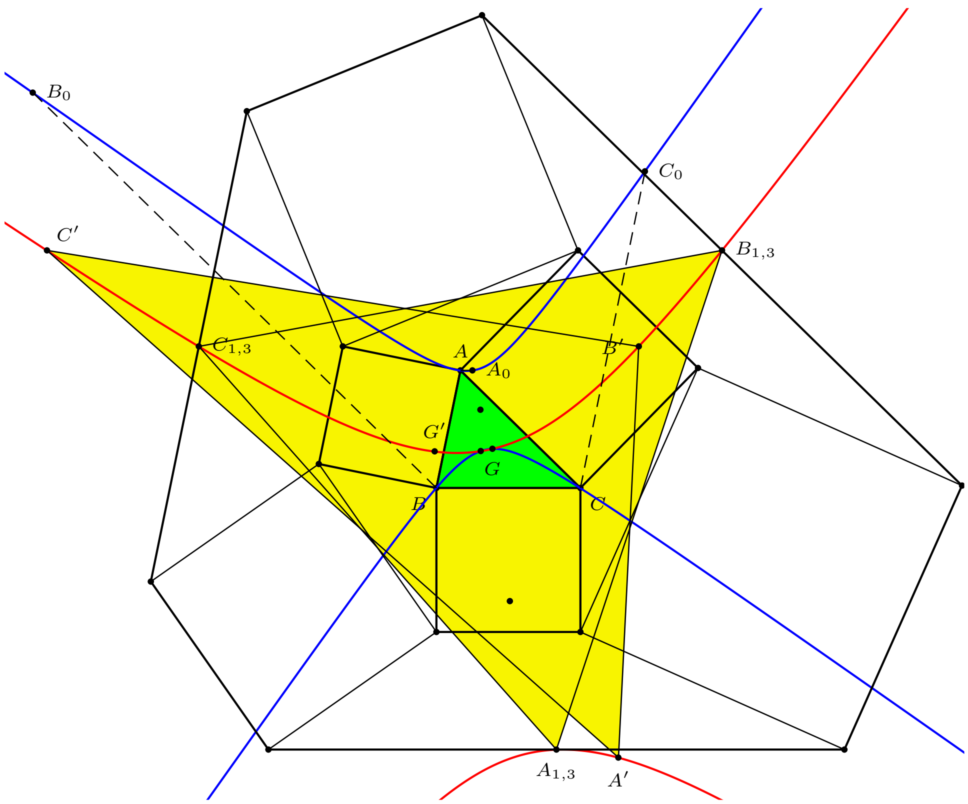

Figure 6.

Here, A_0 is the intersection of the Kiepert hyperbola with the parallel through A to BC; similarly for B_0 and C_0. Since the triangle $A_{1,3}B_{1,3}C_{1,3}$ has centroid G, the rectangular hyperbola (11) is the Kiepert hyperbola of triangle $A_{1,3}B_{1,3}C_{1,3}$. See Figure 6. We show that it is also the Kiepert hyperbola of triangle $A'B'C'$.

This follows from the fact that A', B', C' have coordinates

$$\begin{array}{llllll} A' & -S-2S_A & : & S+S_A & : & S+S_A \\ B' & S+S_B & : & -S-2S_B & : & S+S_B \\ C' & S+S_C & : & S+S_C & : & -S-2S_C \end{array}$$

From these, the centroid of triangle $A'B'C'$ is the point G' in the above table. The rectangular hyperbola (11) is therefore the Kiepert hyperbola of triangle $A'B'C'$.

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Some Geometric Constructions

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Abstract. We solve some problems of geometric construction. Some of them cannot be solved with ruler and compass only and require the drawing of a rectangular hyperbola: (i) construction of the Simson lines passing through a given point, (ii) construction of the lines with a given orthopole, and (iii) a problem of congruent incircles whose analysis leads to some remarkable properties of the internal Soddy center.

1. Simson lines through a given point

1.1. *Problem.* A triangle ABC and a point P are given, $P \neq H$, the orthocenter, and does not lie on the sidelines of the triangle. We want to construct the points of the circumcircle Γ of ABC whose Simson lines pass through P.

1.2. *Analysis.* We make use of the following results of Trajan Lalesco [3].

Proposition 1 (Lalesco)**.** *If the Simson lines of A', B', C' concur at P, then*
(a) *P is the midpoint of HH', where H' is the orthocenter of $A'B'C'$,*
(b) *for any point $M \in \Gamma$, the Simson lines $S(M)$ and $S'(M)$ of M with respect to ABC and $A'B'C'$ are parallel.*

Let h be the rectangular hyperbola through A, B, C, P. If the hyperbola h intersects Γ again at U, the center W of h is the midpoint of HU. Let h' be the rectangular hyperbola through A', B', C', U. The center W' of h' is the midpoint of $H'U$. Hence, by (a) above, $W' = T(W)$, where T is the translation by the vector $\overrightarrow{HP}$.

Let D, D' be the endpoints of the diameter of Γ perpendicular to the Simson line $S(U)$. The asymptotes of h are $S(D)$ and $S(D')$; as, by (b), $S(U)$ and $S'(U)$ are parallel, the asymptotes of h' are $S'(D)$ and $S'(D')$ and, by (b), they are parallel to the asymptotes of h.

It follows that T maps the asymptotes of h to the asymptotes of h'. Moreover, T maps $P \in h$ to $H' \in h'$. As a rectangular hyperbola is determined by a point and the asymptotes, it follows that $h' = T(h)$.

Construction 1. *Given a point $P \neq H$ and not on any of the sidelines of triangle ABC, let the rectangular hyperbola h through A, B, C, P intersect the circumcircle Γ again at U. Let h' be the image of h under the translation T by the vector*

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$\overrightarrow{HP}$. *Then* h' *passes through* U. *The other intersections of* h' *with* Γ *are the points whose Simson lines pass through* P. See Figure 1.

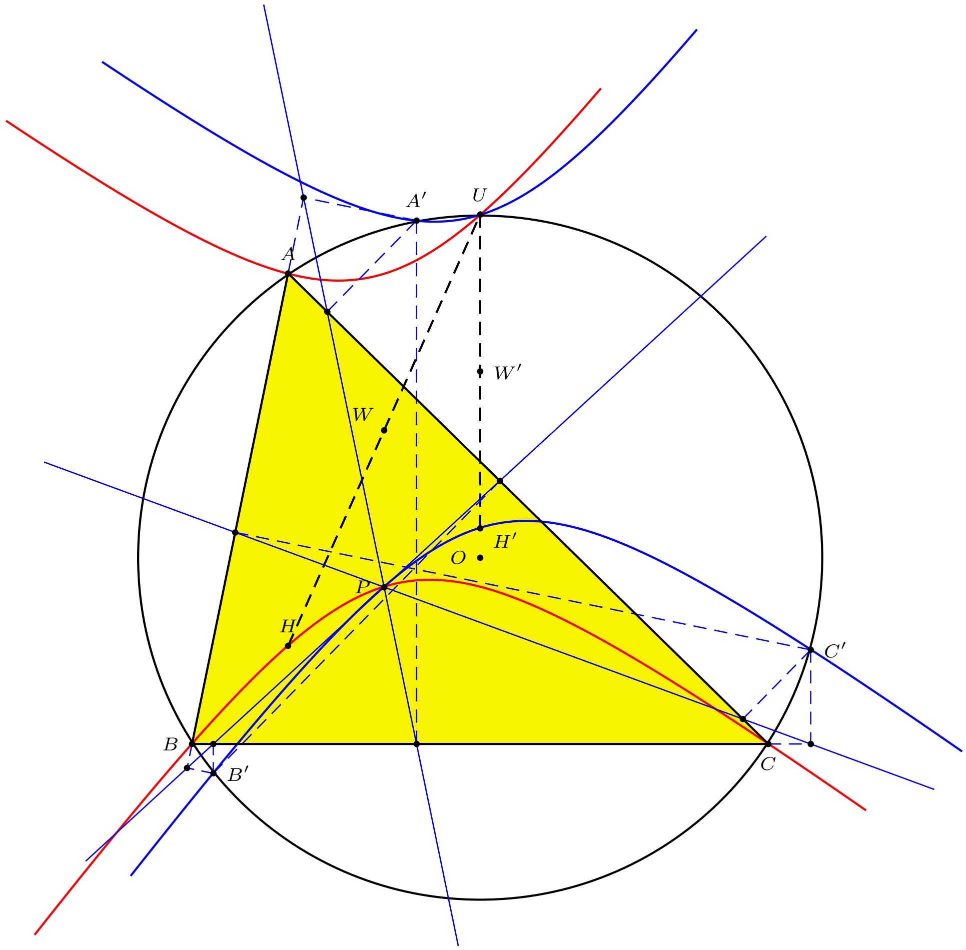

Figure 1.

1.3. *Orthopole.* The above construction leads to a construction of the lines whose orthopole is P. It is well known that, if M and N lie on the circumcircle, the orthopole of the line MN is the common point of the Simson lines of M and N (see [1]). Thus, if we have three real points A', B', C' whose Simson lines pass through P, the lines with orthopole P are the sidelines of $A'B'C'$.

Moreover, the orthopole of a line L lies on the directrix of the insribed parabola touching L (see [1, pp.241–242]). Thus, in any case and in order to avoid imaginary lines, we can proceed this way: for each point M whose Simson line passes through P, let Q be the isogonal conjugate of the infinite point of the direction orthogonal to HP. The line through Q parallel to the Simson line of M intersects the line HP at R. Then P is the orthopole of the perpendicular bisector of QR.

2. Two congruent incircles

2.1. *Problem.* Construct a point P inside ABC such that if B' and C' are the traces of P on AC and AB respectively, the quadrilateral $AB'PC'$ has an incircle congruent to the incircle of PBC.

2.2. *Analysis.* Let h_a be the hyperbola through A with foci B and C, and D_a the projection of the incenter I of triangle ABC upon the side BC.

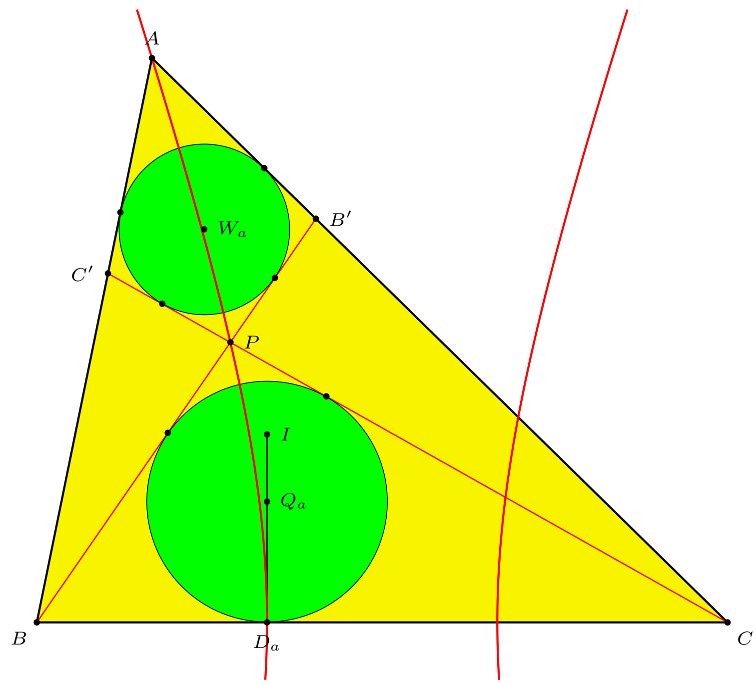

Figure 2.

Proposition 2. *Let P be a point inside ABC and Q_a the incenter of PBC. The following statements are equivalent.*
(a) $PB - PC = AB - AC$.
(b) *P lies on the open arc AD_a of h_a.*
(c) *The quadrilateral $AB'PC'$ has an incircle.*
(d) $IQ_a \perp BC$.
(e) *The incircles of PAB and PAC touch each other.*

Proof. (a) $\Longleftrightarrow$ (b). As $2BD_a = AB+BC-AC$ and $2CD_a = AC+BC-AB$, we have $BD_a-CD_a = AB-AC$ and D_a is the vertex of the branch of h_a through A.

(b)$\Longrightarrow$ (c). AI and PQ_a are the lines tangent to h_a respectively at A and P. If W_a is their common point, BW_a is a bisector of $\angle ABP$. Hence, W_a is equidistant from the four sides of the quadrilateral.

(c)$\Longrightarrow$ (b). If the incircle of $AB'PC'$ touches PB', PC', AC, AB respectively at U, U', V, V', we have $PB - PC = BU - CV = BV' - CU' = AB - AC$.

(a) $\Longleftrightarrow$ (d). If S_a is the projection of Q_a upon BC, we have $2BS_a = PB + BC - PC$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_a = D_a \Longleftrightarrow IQ_a \perp BC$.

(a) $\Longleftrightarrow$ (e). If the incircles of PAC and PAB touch the line AP respectively at S_b and S_c, we have $2AS_b = AC + PA - PC$ and $2AS_c = AB + PA - PB$. Hence, $PB - PC = AB - AC \Longleftrightarrow S_b = S_c$. See Figure 3. $\square$

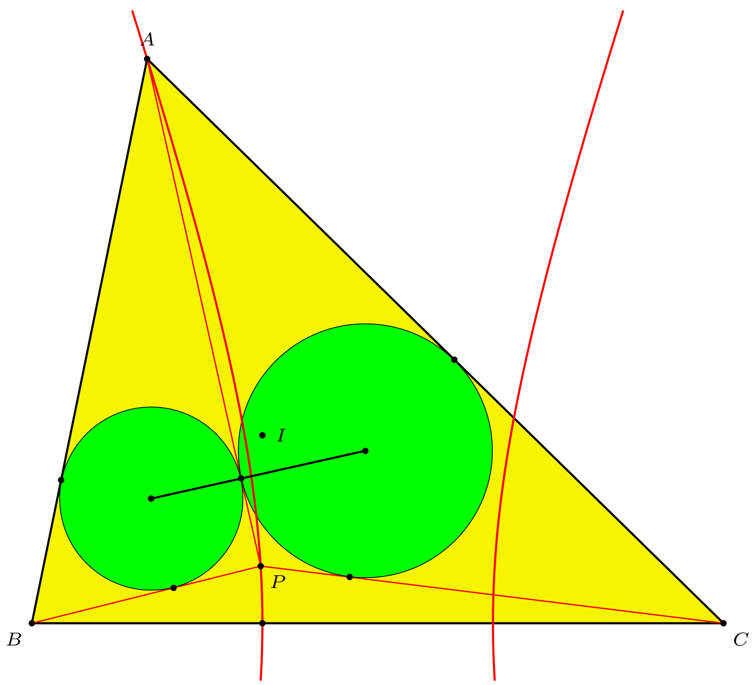

Figure 3.

Proposition 3. *When the conditions of Proposition 2 are satisfied, the following statements are equivalent.*

(a) *The incircles of PBC and $AB'PC'$ are congruent.*

(b) *P is the midpoint of W_aQ_a.*

(c) *W_aQ_a and AD_a are parallel.*

(d) *P lies on the line M_aI where M_a is the midpoint of BC.*

Proof. (a) $\Longleftrightarrow$ (b) is obvious.

Let's notice that, as I is the pole of AD_a with respect to h_a, M_aI is the conjugate diameter of the direction of AD_a with respect to h_a.

So (c) $\Longleftrightarrow$ (d) because W_aQ_a is the tangent to h_a at P.

As the line M_aI passes through the midpoint of AD_a, (b') $\Longleftrightarrow$ (c). $\square$

Now, let us recall the classical construction of an hyperbola knowing the foci and a vertex: For any point M on the circle with center M_a passing through D_a, if L is the line perpendicular at M to BM, and N the reflection of B in M, L touches h_a at $L \cap CN$.

Construction 2. *The perpendicular from B to AD_a and the circle with center M_a passing through D_a have two common points. For one of them M, the perpendicular at M to BM will intersect M_aI at P and the lines D_aI and AI respectively at Q_a and W_a.* See Figure 4.

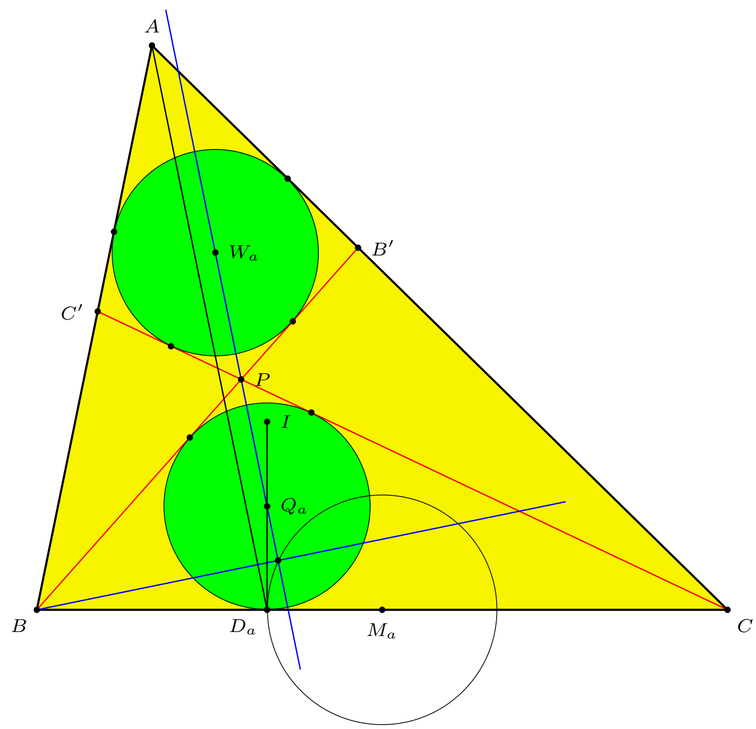

Figure 4.

Remark. We have already known that $PB - PC = c - b$. A further investigation leads to the following results.
(i) $PB + PC = \sqrt{as}$ where s is the semiperimeter of ABC.
(ii) The homogeneous barycentric coordinates of P, Q_a, W_a are as follows.

$$\begin{array}{rl} P: & (a,\ b - s + \sqrt{as},\ c - s + \sqrt{as}) \\ Q_a: & \left(a,\ b + 2(s-c)\sqrt{\dfrac{s}{a}},\ c + 2(s-b)\sqrt{\dfrac{s}{a}}\right) \\ W_a: & (a + 2\sqrt{as},\ b,\ c) \end{array}$$

(iii) The common radius of the two incircles is $r_a\left(1 - \sqrt{\dfrac{a}{s}}\right)$, where r_a is the radius of the $A-$excircle.

3. The internal Soddy center

Let Δ, s, r, and R be respectively the area, the semiperimeter, the inradius, and the circumradius of triangle ABC.

The three circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$ touch each other. The internal Soddy circle is the circle tangent externally to each of these three circles. See Figure 5. Its center is $X(176)$ in [2] with barycentric coordinates

$$\left(a+\frac{\Delta}{s-a},\ b+\frac{\Delta}{s-b},\ c+\frac{\Delta}{s-c}\right)$$

and its radius is

$$\rho=\frac{\Delta}{2s+4R+r}.$$

See [2] for more details and references.

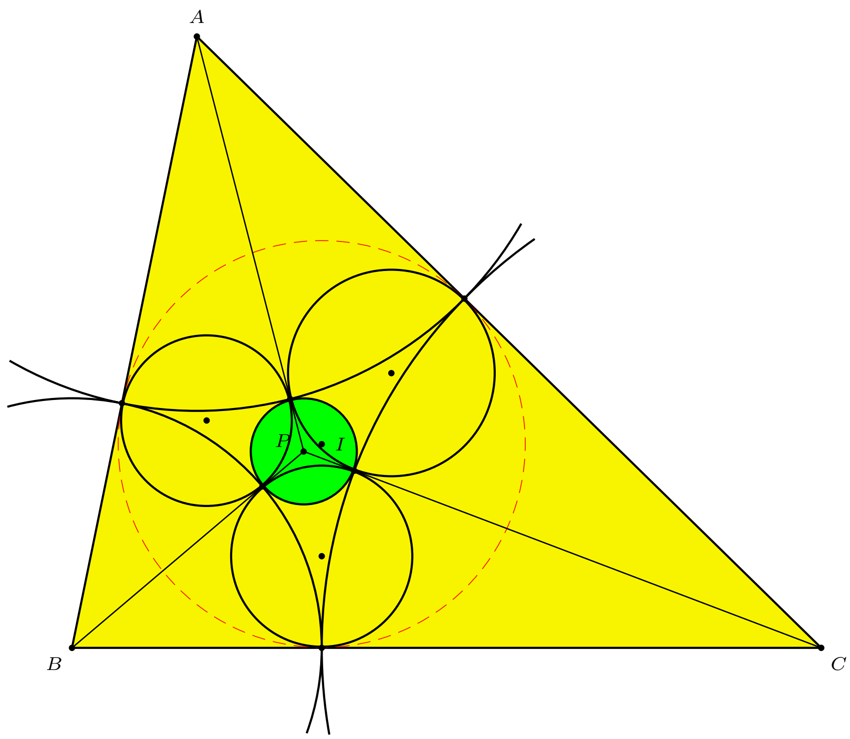

Figure 5.

Proposition 4. *The inner Soddy center $X(176)$ is the only point P inside ABC*
(a) *for which the incircles of PBC, PCA, PAB touch each other;*
(b) *with cevian triangle $A'B'C'$ for which each of the three quadrilaterals $AB'PC'$, $BC'PA'$, $CA'PB'$ have an incircle.* See Figure 6.

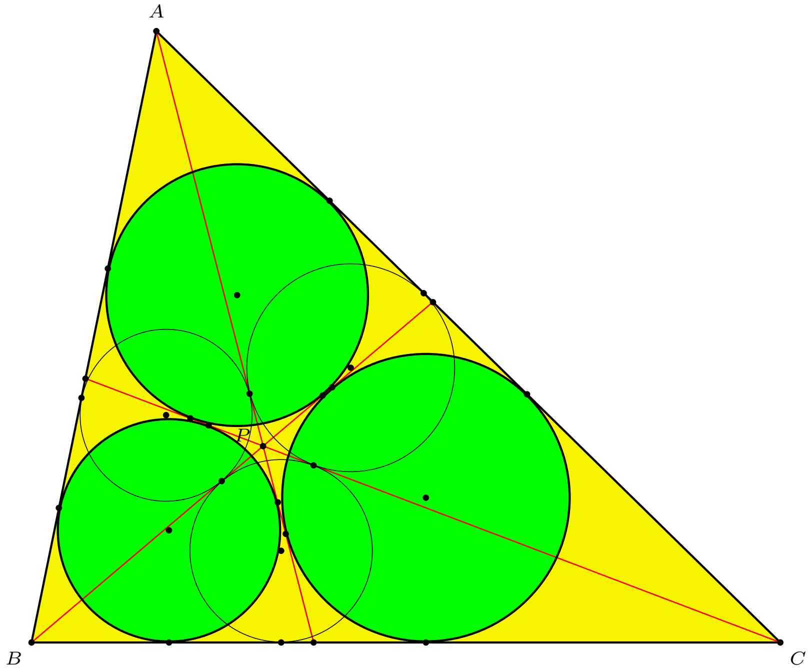

Figure 6.

Proof. Proposition 2 shows that the conditions in (a) and (b) are both equivalent to

$$PB - PC = c - b,\ PC - PA = a - c,\ PA - PB = b - a.$$

As $PA = \rho + s - a$, $PB = \rho + s - b$, $PC = \rho + s - c$, these conditions are satisfied for $P = X(176)$. Moreover, a point P inside ABC verifying these conditions must lie on the open arc AD_a of h_a and on the open arc BD_b of h_b and these arcs cannot have more than a common point. □

Remarks. (1) It follows from Proposition 2(d) that the contact points of the incircles of PBC, PCA, PAB with BC, CA, AB respectively are the same ones D_a, D_b, D_c than the contact points of incircle of ABC.[1]

(2) The incircles of PBC, PCA, PAB touch each other at the points where the internal Soddy circle touches the circles $(A, s-a)$, $(B, s-b)$, $(C, s-c)$.

(3) If Q_a is the incenter of PBC, and W_a the incenter of $AB'PC'$, we have $\dfrac{\overline{Q_aD_a}}{\overline{IQ_a}} = \dfrac{r_a}{a}$, and $\dfrac{\overline{W_aI}}{\overline{AW_a}} = \dfrac{r_a}{s}$, where r_a is the radius of the A-excircle.

(4) The four common tangents of the incircles of $BC'PA'$ and $CA'PB'$ are BC, Q_bQ_c, AP and D_aI.

(5) The lines AQ_a, BQ_b, CQ_c concur at

$$X(482) = \left(a + \frac{2\Delta}{s-a},\ b + \frac{2\Delta}{s-b},\ c + \frac{2\Delta}{s-c}\right).$$

[1]Thanks to François Rideau for this nice remark.

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Hansen's Right Triangle Theorem, Its Converse and a Generalization

Amy Bell

Abstract. We generalize D. W. Hansen's theorem relating the inradius and exradii of a right triangle and its sides to an arbitrary triangle. Specifically, given a triangle, we find two quadruples of segments with equal sums and equal sums of squares. A strong converse of Hansen's theorem is also established.

1. Hansen's right triangle theorem

In an interesting article in *Mathematics Teacher*, D. W. Hansen [2] has found some remarkable identities associated with a right triangle. Let ABC be a triangle with a right angle at C, sidelengths a, b, c. It has an incircle of radius r, and three excircles of radii r_a, r_b, r_c.

Theorem 1 (Hansen)**.** (1) *The sum of the four radii is equal to the perimeter of the triangle:*

$$r_a + r_b + r_c + r = a + b + c.$$

(2) *The sum of the squares of the four radii is equal to the sum of the squares of the sides of the triangle:*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2.$$

We seek to generalize Hansen's theorem to an arbitrary triangle, by replacing a, b, c by appropriate quantities whose sum and sum of squares are respectively equal to those of r_a, r_b, r_c and r. Now, for a right triangle ABC with right angle vertex C, this latter vertex is the orthocenter of the triangle, which we generically denote by H. Note that

$$a = BH \quad \text{and} \quad b = AH.$$

On the other hand, the hypotenuse being a diameter of the circumcircle, $c = 2R$. Note also that $CH = 0$ since C and H coincide. This suggests that a possible generalization of Hansen's theorem is to replace the triple a, b, c by the quadruple AH, BH, CH and $2R$. Since $AH = 2R \cos A$ etc., one of the quantities AH, BH, CH is negative if the triangle contains an obtuse angle.

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The paper is a revision of part of the author's thesis for the degree of Master of Science in Teaching (Mathematics) at Florida Atlantic University, under the direction of Professor Paul Yiu. Thanks are also due to the referee for suggestions leading to improvements on the paper.

We shall establish the following theorem.

Theorem 2. *Let ABC be a triangle with orthocenter H and circumradius R.*
(1) $r_a + r_b + r_c + r = AH + BH + CH + 2R$;
(2) $r_a^2 + r_b^2 + r_c^2 + r^2 = AH^2 + BH^2 + CH^2 + (2R)^2$.

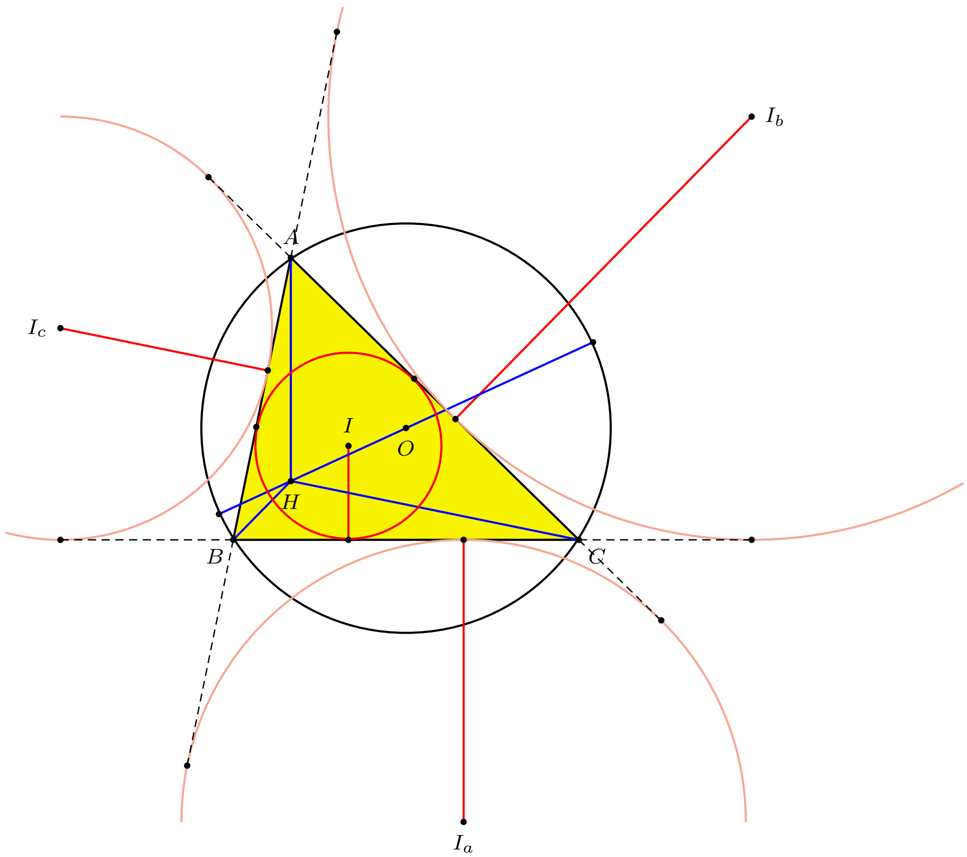

Figure 1. Two quadruples with equal sums and equal sums of squares

2. A characterization of right triangles in terms of inradius and exradii

Proposition 3. *The following statements for a triangle ABC are equivalent.*
(1) $r_c = s$.
(2) $r_a = s - b$.
(3) $r_b = s - a$.
(4) $r = s - c$.
(5) C *is a right angle.*

Proof. By the formulas for the exradii and the Heron formula, each of (1), (2), (3), (4) is equivalent to the condition

$$(s-a)(s-b) = s(s-c). \tag{1}$$

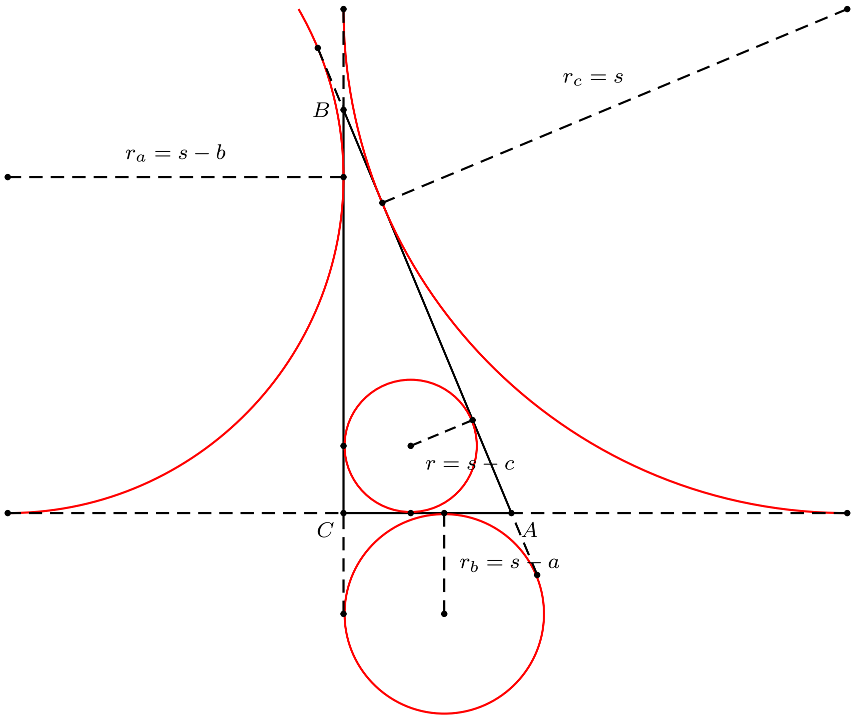

Figure 2. Inradius and exradii of a right triangle

Assuming (1), we have $s^2 - (a+b)s + ab = s^2 - cs$, $(a+b-c)s = ab$, $(a+b-c)(a+b+c) = 2ab$, $(a+b)^2 - c^2 = 2ab$, $a^2 + b^2 = c^2$. This shows that each of (1), (2), (3), (4) implies (5). The converse is clear. See Figure 2. □

3. A formula relating the radii of the various circles

As a preparation for the proof of Theorem 2, we study the excircles in relation to the circumcircle and the incircle. We establish a basic result, Proposition 6, below. Lemma 4 and the statement of Proposition 6 can be found in [3, pp.185–193]. An outline proof of Proposition 5 can be found in [4, §2.4.1]. Propositions 5 and 6 can also be found in [5, §4.6.1]. [1] We present a unified detailed proof of these propositions here, simpler and more geometric than the trigonometric proofs outlined in [3].

Consider triangle ABC with its circumcircle (O). Let the bisector of angle A intersect the circumcircle at M. Clearly, M is the midpoint of the arc BMC. The line BM clearly contains the incenter I and the excenter I_a.

Lemma 4. $MB = MI = MI_a = MC$.

[1]The referee has pointed out that these results had been known earlier, and can be found, for example, in the nineteenth century work of John Casey [1].

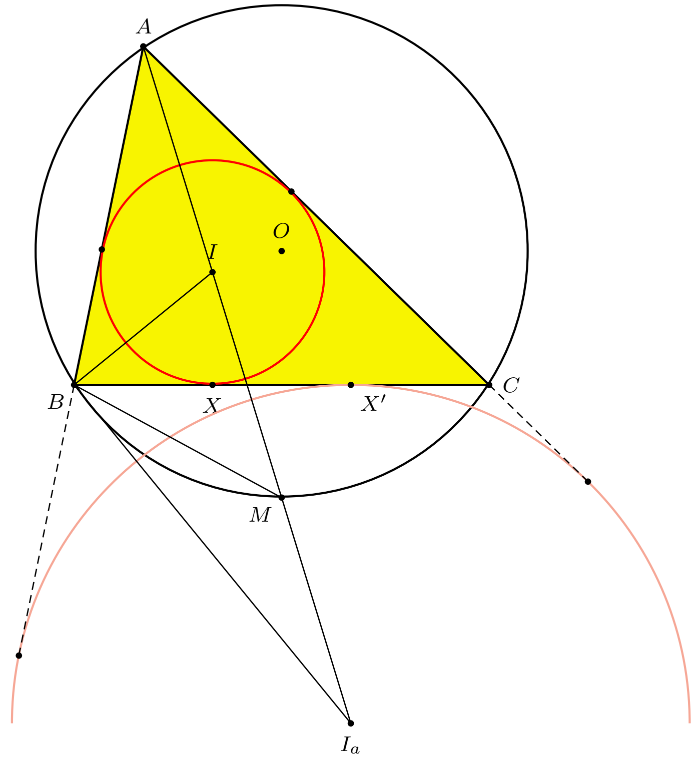

Figure 3. $r_a + r_b + r_c = 4R + r$

Proof. It is enough to prove that $MB = MI$. See Figure 3. This follows by an easy calculation of angles.
(i) $\angle IBI_a = 90°$ since the two bisectors of angle B are perpendicular to each other.
(ii) The midpoint N of I_aI is the circumcenter of triangle IBI_a, so $NB = NI = NI_a$.
(iii) From the circle (IBI_a) we see $\angle BNA = \angle BNI = 2\angle BCI = \angle BCA$, but this means that N lies on the circumcircle (ABC) and thus coincides with M.
It follows that $MI_a = MB = MI$, and M is the midpoint of II_a.

The same reasoning shows that $MC = MI = MI_a$ as well. □

Now, let I' be the intersection of the line IO and the perpendicular from I_a to BC. See Figure 4. Note that this latter line is parallel to OM. Since M is the midpoint of II_a, O is the midpoint of II'. It follows that I' is the reflection of I in O. Also, $I'I_a = 2 \cdot OM = 2R$. Similarly, $I'I_b = I'I_c = 2R$. We summarize this in the following proposition.

Proposition 5. *The circle through the three excenters has radius $2R$ and center I', the reflection of I in O.*

Remark. Proposition 5 also follows from the fact that the circumcircle is the nine point circle of triangle $I_aI_bI_c$, and I is the orthocenter of this triangle.

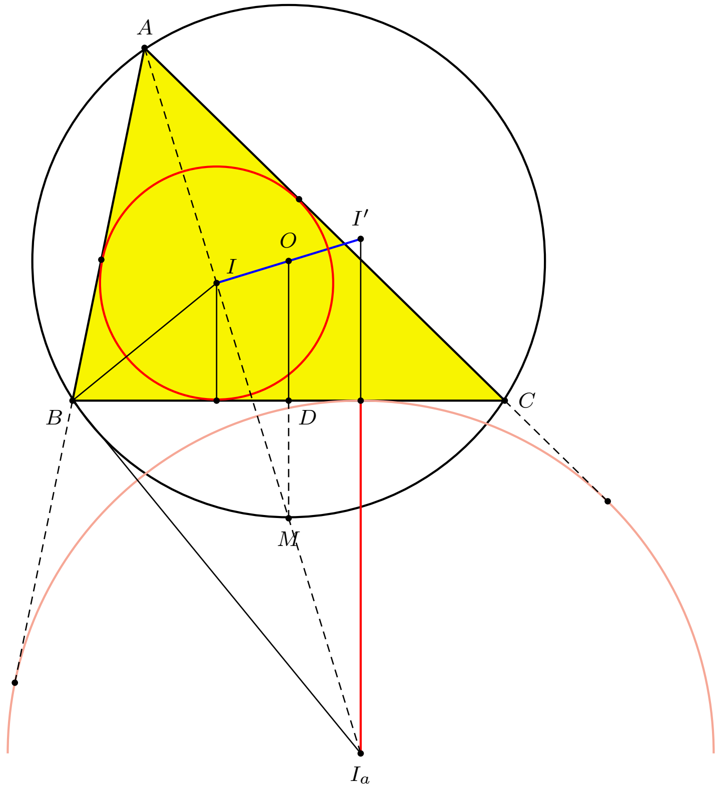

Figure 4. $I'I_a = 2R$

Proposition 6. $r_a + r_b + r_c = 4R + r$.

Proof. The line I_aI' intersects BC at the point X' of tangency with the excircle. Note that $I'X' = 2R - r_a$. Since O is the midpoint of II', we have $IX + I'X' = 2 \cdot OD$. From this, we have

$$2 \cdot OD = r + (2R - r_a). \tag{2}$$

Consider the excenters I_b and I_c. Since the angles I_bBI_c and I_bCI_c are both right angles, the four points I_b, I_c, B, C are on a circle, whose center is the midpoint N of I_bI_c. See Figure 5. The center N must lie on the perpendicular bisector of BC, which is the line OM. Therefore N is the antipodal point of M on the circumcircle, and we have $2ND = r_b + r_c$. Thus, $2(R + OD) = r_b + r_c$. From (2), we have $r_a + r_b + r_c = 4R + r$. □

4. Proof of Theorem 2

We are now ready to prove Theorem 2.

(1) Since $AH = 2 \cdot OD$, by (2) we express this in terms of R, r and r_a; similarly for BH and CH:

$$AH = 2R + r - r_a, \qquad BH = 2R + r - r_b, \qquad CH = 2R + r - r_c.$$

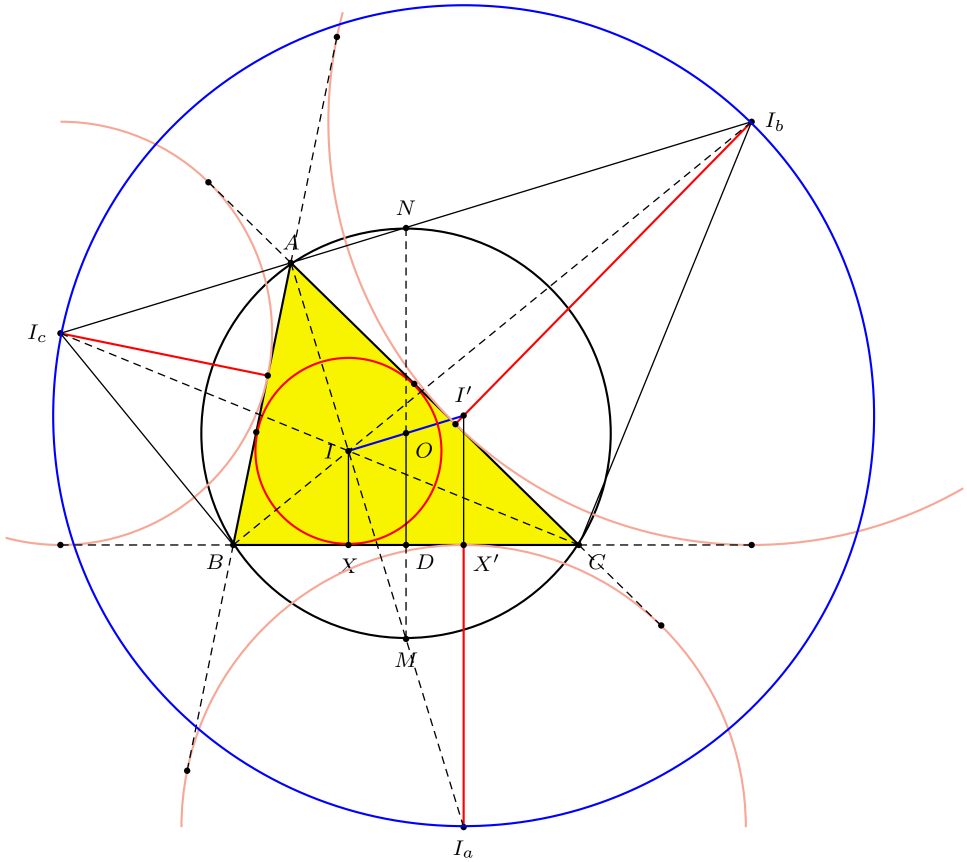

Figure 5. $r_a + r_b + r_c = 4R + r$

From these,

$$\begin{aligned} AH + BH + CH + 2R =& 8R + 3r - (r_a + r_b + r_c) \\ =& 2(4R + r) + r - (r_a + r_b + r_c) \\ =& 2(r_a + r_b + r_c) + r - (r_a + r_b + r_c) \\ =& r_a + r_b + r_c + r. \end{aligned}$$

(2) This follows from simple calculation making use of Proposition 6.

$$\begin{aligned} & AH^2 + BH^2 + CH^2 + (2R)^2 \\ =& (2R + r - r_a)^2 + (2R + r - r_b)^2 + (2R + r - r_c)^2 + 4R^2 \\ =& 3(2R + r)^2 - 2(2R + r)(r_a + r_b + r_c) + r_a^2 + r_b^2 + r_c^2 + 4R^2 \\ =& 3(2R + r)^2 - 2(2R + r)(4R + r) + 4R^2 + r_a^2 + r_b^2 + r_c^2 \\ =& r^2 + r_a^2 + r_b^2 + r_c^2. \end{aligned}$$

This completes the proof of Theorem 2.

5. Converse of Hansen's theorem

We prove a strong converse of Hansen's theorem (Theorem 10 below).

Proposition 7. *A triangle ABC satisfies*

$$r_a^2 + r_b^2 + r_c^2 + r^2 = a^2 + b^2 + c^2 \tag{3}$$

if and only if it contains a right angle.

Proof. Using $AH = 2R\cos A$ and $a = 2R\sin A$, and similar expressions for BH, CH, b, and c, we have

$$\begin{aligned}
&AH^2 + BH^2 + CH^2 + (2R)^2 - (a^2 + b^2 + c^2)\\
=&4R^2(\cos^2 A + \cos^2 B + \cos^2 C + 1 - \sin^2 A - \sin^2 B - \sin^2 C)\\
=&4R^2(2\cos^2 A + \cos 2B + \cos 2C)\\
=&8R^2(\cos^2 A + \cos(B + C)\cos(B - C))\\
=&-8R^2 \cos A(\cos(B + C) + \cos(B - C))\\
=&-16R^2 \cos A \cos B \cos C.
\end{aligned}$$

By Theorem 2(2), the condition (3) holds if and only if $AH^2 + BH^2 + CH^2 + (2R)^2 = a^2 + b^2 + c^2$. One of $\cos A$, $\cos B$, $\cos C$ must be zero from above. This means that triangle ABC contains a right angle. □

In the following lemma we collect some useful and well known results. They can be found more or less directly in [3].

Lemma 8. (1) $r_a r_b + r_b r_c + r_c r_a = s^2$.
(2) $r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2s^2$.
(3) $ab + bc + ca = s^2 + (4R + r)r$.
(4) $a^2 + b^2 + c^2 = 2s^2 - 2(4R + r)r$.

Proof. (1) follows from the formulas for the exradii and the Heron formula.

$$\begin{aligned}
r_a r_b + r_b r_c + r_c r_a =&\frac{\triangle^2}{(s-a)(s-b)} + \frac{\triangle^2}{(s-b)(s-c)} + \frac{\triangle^2}{(s-c)(s-a)}\\
=&s((s-c) + (s-a) + (s-b))\\
=&s^2.
\end{aligned}$$

From this (2) easily follows.

$$\begin{aligned}
r_a^2 + r_b^2 + r_c^2 =&(r_a + r_b + r_c)^2 - 2(r_a r_b + r_b r_c + r_c r_a)\\
=&(4R + r)^2 - 2s^2.
\end{aligned}$$

Again, by Proposition 6,

$$\begin{aligned}
&4R+r\\
=&r_a+r_b+r_c\\
=&\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}\\
=&\frac{\triangle}{(s-a)(s-b)(s-c)}\left((s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)\right)\\
=&\frac{1}{r}\left(3s^2-2(a+b+c)s+(ab+bc+ca)\right)\\
=&\frac{1}{r}\left((ab+bc+ca)-s^2\right).
\end{aligned}$$

An easy rearrangement gives (3).

(4) follows from (3) since $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=4s^2-2(s^2+(4R+r)r)=2s^2-2(4R+r)r$. □

Proposition 9. *$r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ if and only if $2R+r=s$.*

Proof. By Lemma 8(2) and (4), $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$ if and only if $(4R+r)^2-2s^2+r^2=2s^2-2(4R+r)r$; $4s^2=(4R+r)^2+2(4R+r)r+r^2=(4R+2r)^2=4(2R+r)^2$; $s=2R+r$. □

Theorem 10. *The following statements for a triangle ABC are equivalent.*
(1) $r_a+r_b+r_c+r=a+b+c$.
(2) $r_a^2+r_b^2+r_c^2+r^2=a^2+b^2+c^2$.
(3) $R+2r=s$.
(4) *One of the angles is a right angle.*

Proof. (1) $\Longrightarrow$ (3): This follows easily from Proposition 6.

(3) $\Longleftrightarrow$ (2): Proposition 9 above.

(2) $\Longleftrightarrow$ (4): Proposition 7 above.

(4) $\Longrightarrow$ (1): Theorem 1 (1). □

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Some Constructions Related to the Kiepert Hyperbola

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Abstract. Given a reference triangle and its Kiepert hyperbola $\mathcal{K}$, we study several construction problems related to the triangles which have $\mathcal{K}$ as their own Kiepert hyperbolas. Such triangles necessarily have their vertices on $\mathcal{K}$, and are called special Kiepert inscribed triangles. Among other results, we show that the family of special Kiepert inscribed triangles all with the same centroid G form part of a poristic family between $\mathcal{K}$ and an inscribed conic with center which is the inferior of the Kiepert center.

1. Special Kiepert inscribed triangles

Given a triangle ABC and its Kiepert hyperbola $\mathcal{K}$, consisting of the Kiepert perspectors

$$K(t) = \left(\frac{1}{S_A + t} : \frac{1}{S_B + t} : \frac{1}{S_C + t}\right), \qquad t \in \mathbb{R} \cup \{\infty\},$$

we study triangles with vertices on $\mathcal{K}$ having $\mathcal{K}$ as their own Kiepert hyperbolas. We shall work with homogeneous barycentric coordinates and make use of standard notations of triangle geometry as in [2]. Basic results on triangle geometry can be found in [3]. The Kiepert hyperbola has equation

$$K(x, y, z) := (S_B - S_C)yz + (S_C - S_A)zx + (S_A - S_B)xy = 0 \qquad (1)$$

in homogeneous barycentric coordinates. Its center, the Kiepert center

$$K_{\mathrm{i}} = ((S_B - S_C)^2 : (S_C - S_A)^2 : (S_A - S_B)^2),$$

lies on the Steiner inellipse. In this paper we shall mean by a Kiepert inscribed triangle one whose vertices are on the Kiepert hyperbola $\mathcal{K}$. If a Kiepert inscribed triangle is perspective with ABC, it is called the Kiepert cevian triangle of its perspector. Since the Kiepert hyperbola of a triangle can be characterized as the rectangular circum-hyperbola containing the centroid, our objects of interest are Kiepert inscribed triangles whose centroids are Kiepert perspectors. We shall assume the vertices to be finite points on $\mathcal{K}$, and call such triangles special Kiepert inscribed triangles. We shall make frequent use of the following notations.

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$$\begin{aligned}
P(t) &= ((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t)) \\
Q(t) &= ((S_B - S_C)^2(S_A + t) : (S_C - S_A)^2(S_B + t) : (S_A - S_B)^2(S_C + t)) \\
f_2 &= S_{AA} + S_{BB} + S_{CC} - S_{BC} - S_{CA} - S_{AB} \\
f_3 &= S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2 \\
f_4 &= (S_{AA} - S_{BC})S_{BC} + (S_{BB} - S_{CA})S_{CA} + (S_{CC} - S_{AB})S_{AB} \\
g_3 &= (S_A - S_B)(S_B - S_C)(S_C - S_A)
\end{aligned}$$

Here, $P(t)$ is a typical infinite point, and $Q(t)$ is a typical point on the tangent of the Steiner inellipse through $K_{\rm i}$. For $k = 2, 3, 4$, the function f_k, is a symmetric function in S_A, S_B, S_C of degree k.

Proposition 1. *The area of a triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is*

$$\left|\frac{g_3(t_1-t_2)(t_2-t_3)(t_3-t_1)}{\prod(S^2+2(S_A+S_B+S_C)t_i+3t_i^2}\right| \cdot \triangle ABC.$$

Proposition 2. *A Kiepert inscribed triangle with vertices $K(t_i)$, $i = 1, 2, 3$, is special,* i.e.*, with centroid on the Kiepert hyperbola, if and only if*

$$S^2 f_2' + (S_A + S_B + S_C)f_3' - 3f_4' = 0,$$

where f_2', f_3', f_4' are the functions f_2, f_3, f_4 with S_A, S_B, S_C replaced by t_1, t_2, t_3.

We shall make use of the following simple construction.

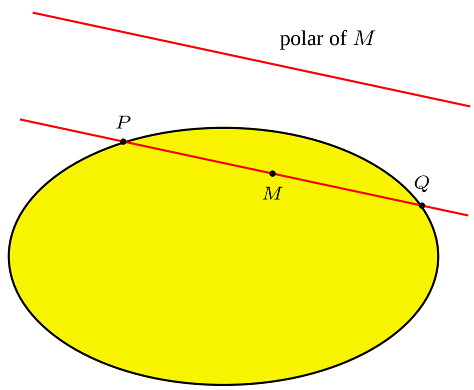

Figure 1. Construction of chord of conic with given midpoint

Construction 3. *Given a conic $\mathcal{C}$ and a point M, to construct the chord of $\mathcal{C}$ with M as midpoint, draw*
(i) *the polar of M with respect to $\mathcal{C}$,*
(ii) *the parallel through M to the line in* (i)*.*
If the line in (ii) *intersects $\mathcal{C}$ at the two real points P and Q, then the midpoint of PQ is M.*

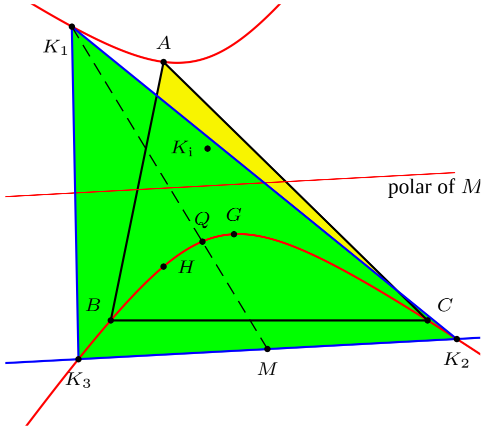

Figure 2. Construction of Kiepert inscribed triangle with prescribed centroid and one vertex

A simple application of Construction 3 gives a Kiepert inscribed triangle with prescribed centroid Q and one vertex K_1: simply take M to be the point dividing K_1Q in the ratio $K_1M : MQ = 3 : -1$. See Figure 2.

Here is an interesting family of Kiepert inscribed triangles with prescribed centroids on $\mathcal{K}$.

Construction 4. *Given a Kiepert perspector $K(t)$, construct*
(i) *K_1 on $\mathcal{K}$ and M such that the segment K_1M is trisected at $K_{\rm i}$ and $K(t)$,*
(ii) *the parallel through M to the tangent of $\mathcal{K}$ at $K(t)$,*
(iii) *the intersections K_2 and K_3 of $\mathcal{K}$ with the line in* (ii).
Then $K_1K_2K_3$ is a special Kiepert inscribed triangle with centroid $K(t)$. See Figure 3.

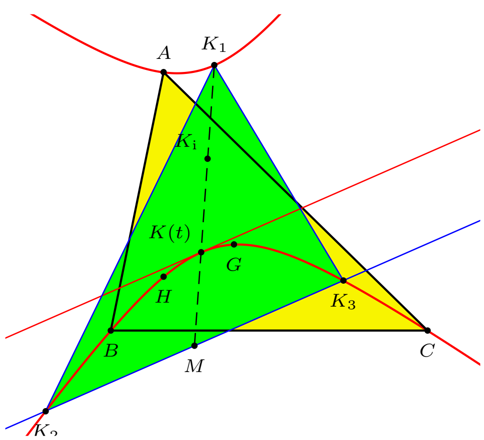

Figure 3. Kiepert inscribed triangle with centroid $K(t)$

It is interesting to note that the area of the Kiepert inscribed triangle is independent of t. It is $\frac{3\sqrt{3}}{2}|g_3|f_2^{-\frac{3}{2}}$ times that of triangle ABC. This result and many others in the present paper are obtained with the help of a computer algebra system.

2. Special Kiepert cevian triangles

Given a point $P = (u : v : w)$, the vertices of its Kiepert cevian triangle are

$$A_P = \left(\frac{-(S_B - S_C)vw}{(S_A - S_B)v + (S_C - S_A)w} : v : w\right),$$
$$B_P = \left(u : \frac{-(S_C - S_A)wu}{(S_B - S_C)w + (S_A - S_B)u} : w\right),$$
$$C_P = \left(u : v : \frac{-(S_A - S_B)uv}{(S_C - S_A)u + (S_B - S_C)v}\right).$$

These are Kiepert perspectors with parameters t_A, t_B, t_C given by

$$t_A = -\frac{S_B v - S_C w}{v - w}, \qquad t_B = -\frac{S_C w - S_A u}{w - u}, \qquad t_C = -\frac{S_A u - S_B v}{u - v}.$$

Clearly, if P is on the Kiepert hyperbola, the Kiepert cevian triangle $A_P B_P C_P$ degenerates into the point P.

Theorem 5. *The centroid of the Kiepert cevian triangle of P lies on the Kiepert hyperbola if and only if P is*
(i) *an infinite point, or*
(ii) *on the tangent at K_{i} to the Steiner inellipse.*

Proof. Let $P = (u : v : w)$ in homogeneous barycentric coordinates. Applying Proposition 2, we find that the centroid of $A_P B_P C_P$ lies on the Kiepert hyperbola if and only if

$$(u + v + w)K(u, v, w)^2 L(u, v, w)P(u, v, w) = 0,$$

where

$$L(u, v, w) = \frac{u}{S_B - S_C} + \frac{v}{S_C - S_A} + \frac{w}{S_A - S_B},$$
$$P(u, v, w) = \prod((S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2).$$

The factors $u + v + w$ and $K(u, v, w)$ clearly define the line at infinity and the Kiepert hyperbola $\mathcal{K}$ respectively. On the other hand, the factor $L(u, v, w)$ defines the line

$$\frac{x}{S_B - S_C} + \frac{y}{S_C - S_A} + \frac{z}{S_A - S_B} = 0, \tag{2}$$

which is the tangent of the Steiner inellipse at K_{i}.

Each factor of $P(u, v, w)$ defines two points on a sideline of triangle ABC. If we set $(x, y, z) = (-(v + w), v, w)$ in (1), the equation reduces to $(S_A - S_B)v^2 - 2(S_B - S_C)vw + (S_C - S_A)w^2$. This shows that the two points on the line BC are the intercepts of lines through A parallel to the asymptotes of $\mathcal{K}$, and the corresponding Kiepert cevian triangles have vertices at infinite points. This is similarly the case for the other two factors of $P(u, v, w)$. $\square$

Remark. Altogether, the six points defined by $P(u, v, w)$ above determine a conic with equation

$$G(x, y, z) = \sum \frac{x^2}{S_B - S_C} - \frac{2(S_B - S_C)yz}{(S_C - S_A)(S_A - S_B)} = 0.$$

Since

$$\begin{aligned} & g_3 \cdot G(x, y, z) \\ = & - f_2(x + y + z)^2 + \sum (S_B - S_C)^2 x^2 - 2(S_C - S_A)(S_A - S_B)yz, \end{aligned}$$

this conic is a translation of the inscribed conic

$$\sum (S_B - S_C)^2 x^2 - 2(S_C - S_A)(S_A - S_B)yz = 0,$$

which is the Kiepert parabola. See Figure 4.

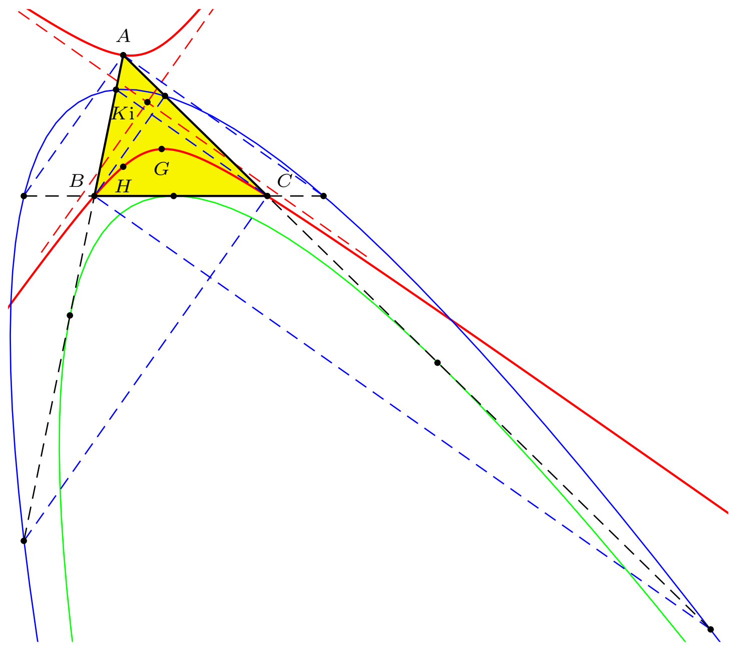

Figure 4. Translation of Kiepert parabola

3. Kiepert cevian triangles of infinite points

Consider a typical infinite point

$$P(t) = ((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t))$$

in homogeneous barycentric coordinates. It can be easily verified that $P(t)$ is the infinite point of perpendiculars to the line joining the Kiepert perspector $K(t)$ to the orthocenter H.[1] The Kiepert cevian triangle of $P(t)$ has vertices

[1]This is the line $\sum S_A(S_B - S_C)(S_A + t)x = 0$.

$$A(t) = \left(\frac{(S_B - S_C)(S_B + t)(S_C + t)}{S_B + S_C + 2t} : (S_C - S_A)(S_B + t) : (S_A - S_B)(S_C + t)\right),$$
$$B(t) = \left((S_B - S_C)(S_A + t) : \frac{(S_C - S_A)(S_C + t)(S_A + t)}{S_C + S_A + 2t} : (S_A - S_B)(S_C + t)\right),$$
$$C(t) = \left((S_B - S_C)(S_A + t) : (S_C - S_A)(S_B + t) : \frac{(S_A - S_B)(S_A + t)(S_B + t)}{S_C + S_A + 2t}\right).$$

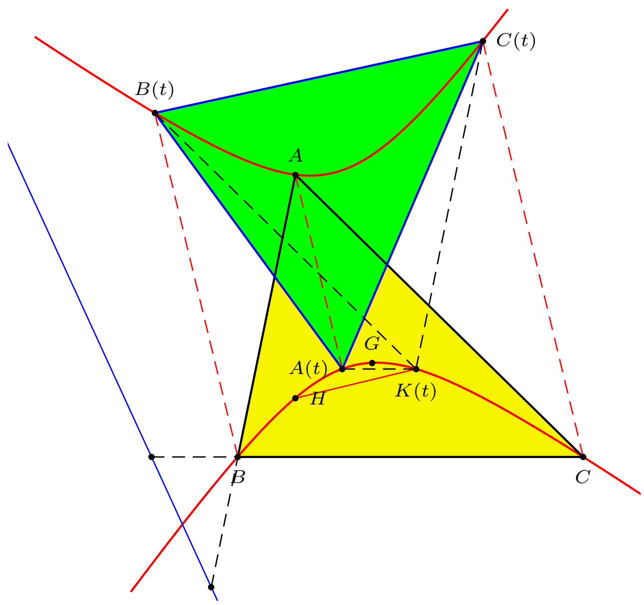

Figure 5. The Kiepert cevian triangle of $P(t)$ is the same as the Kiepert parallelian triangle of $K(t)$

It is also true that the line joining $A(t)$ to $K(t)$ is parallel to BC;[2] similarly for $B(t)$ and $C(t)$. Thus, we say that the Kiepert cevian triangle of the infinite point $P(t)$ is the same as the Kiepert parallelian triangle of the Kiepert perspector $K(t)$. See Figure 5. It is interesting to note that the area of triangle $A(t)B(t)C(t)$ is equal to that of triangle ABC, but the triangles have opposite orientations.

Now, the centroid of triangle $A(t)B(t)C(t)$ is the point

$$\left(\frac{S_B - S_C}{S_{AB} + S_{AC} - 2S_{BC} - (S_B + S_C - 2S_A)t} : \cdots : \cdots\right),$$

which, by Theorem 5, is a Kiepert perspector. It is $K(s)$ where s is given by

$$2f_2 \cdot st + f_3 \cdot (s + t) - 2f_4 = 0. \tag{3}$$

Proposition 6. *Two distinct Kiepert perspectors have parameters satisfying* (3) *if and only if the line joining them is parallel to the orthic axis.*

[2]This is the line $-(S_A + t)(S_B + S_C + 2t)x + (S_B + t)(S_C + t)(y + z) = 0$.

Proof. The orthic axis $S_Ax + S_By + S_Cz = 0$ has infinite point

$$P(\infty) = (S_B - S_C : S_C - S_A : S_A - S_B).$$

The line joining $K(s)$ and $K(t)$ is parallel to the orthic axis if and only if

$$\begin{vmatrix} \frac{1}{S_A+s} & \frac{1}{S_B+s} & \frac{1}{S_C+s} \\ \frac{1}{S_A+t} & \frac{1}{S_B+t} & \frac{1}{S_C+t} \\ S_B - S_C & S_C - S_A & S_A - S_B \end{vmatrix} = 0.$$

For $s \neq t$, this is the same condition as (3). □

This leads to the following construction.

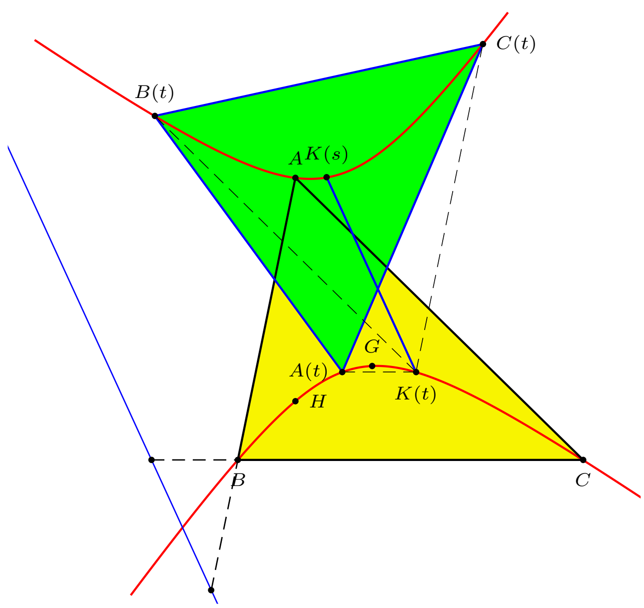

Figure 6. The Kiepert cevian triangle of $P(t)$ has centroid $K(s)$

Construction 7. *Given a Kiepert perspector $K(s)$, to construct a Kiepert cevian triangle with centroid $K(s)$, draw*
(i) *the parallel through $K(s)$ to the orthic axis to intersect the Kiepert hyperbola again at $K(t)$,*
(ii) *the parallels through $K(t)$ to the sidelines of the triangle to intersect $\mathcal{K}$ again at $A(t)$, $B(t)$, $C(t)$ respectively.*
Then, $A(t)B(t)C(t)$ has centroid $K(s)$. See Figure 6.

4. Special Kiepert inscribed triangles with common centroid G

We construct a family of Kiepert inscribed triangles with centroid G, the centroid of the reference triangle ABC. This can be easily accomplished with the help of Construction 3. Beginning with a Kiepert perspector $K_1 = K(t)$ and $Q = G$, we easily determine

$$M = ((S_A+t)(S_B+S_C+2t) : (S_B+t)(S_C+S_A+2t) : (S_C+t)(S_A+S_B+2t)).$$

The line through M parallel to its own polar with respect to $\mathcal{K}$[3] has equation

$$\frac{S_B - S_C}{S_A + t}x + \frac{S_C - S_A}{S_B + t}y + \frac{S_A - S_B}{S_C + t}z = 0. \tag{4}$$

As t varies, this line envelopes the conic

$$\begin{aligned}&(S_B - S_C)^4x^2 + (S_C - S_A)^4y^2 + (S_A - S_B)^4z^2\\ &- 2(S_B - S_C)^2(S_C - S_A)^2xy - 2(S_C - S_A)^2(S_A - S_B)^2yz\\ &- 2(S_A - S_B)^2(S_B - S_C)^2zx = 0,\end{aligned}$$

which is the inscribed ellipse $\mathcal{E}$ tangent to the sidelines of ABC at the traces of

$$\left(\frac{1}{(S_B - S_C)^2} : \frac{1}{(S_C - S_A)^2} : \frac{1}{(S_A - S_B)^2}\right),$$

and to the Kiepert hyperbola at G, and to the line (4) at the point

$$((S_A + t)^2 : (S_B + t)^2 : (S_C + t)^2).$$

It has center

$$((S_C-S_A)^2+(S_A-S_B)^2 : (S_A-S_B)^2+(S_B-S_C)^2 : (S_B-S_C)^2+(S_C-S_A)^2),$$

the inferior of the Kiepert center K_i. See Figure 7.

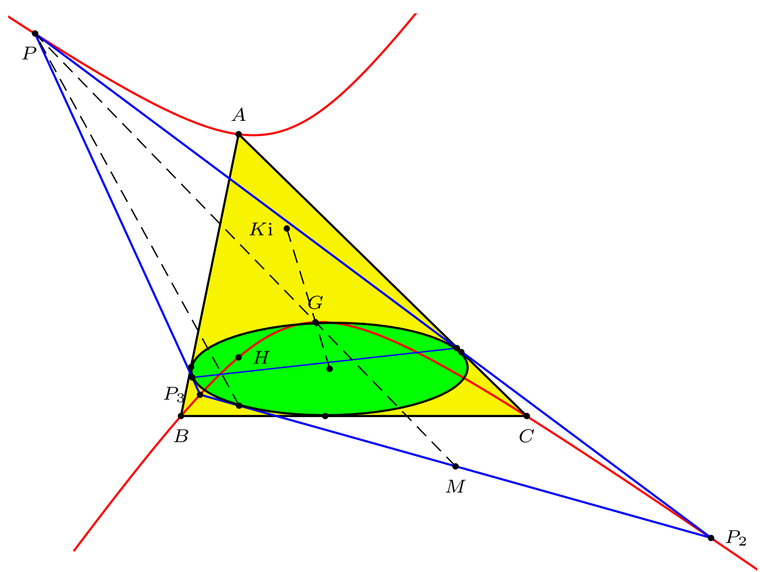

Figure 7. Poristic triangles with common centroid G

[3]The polar of M has equation $\sum(S_B - S_C)(S_{AA} - S^2 - 2(S_B + S_C)t - 2t^2)x = 0$ and has infinite point $((S_A + t)(S_A(S_B + S_C - 2t) - (S_B + S_C)(S_B - S_C + t)) : \cdots : \cdots)$.

Theorem 8. *A poristic triangle completed from a point on the Kiepert hyperbola outside the inscribed ellipse $\mathcal{E}$ (with center the inferior of $K_{\rm i}$) has its center at G and therefore has $\mathcal{K}$ as its Kiepert hyperbola.*

More generally, if we replace G by a Kiepert perspector $K_{\rm g}$, the envelope is a conic with center which divides $K_{\rm i}K_{\rm g}$ in the ratio $3:-1$. It is an ellipse inscribed in the triangle in Construction 4.

5. A family of special Kiepert cevian triangles

5.1. *Triple perspectivity.* According to Theorem 5, there is a family of special Kiepert cevian triangles with perspectors on the line (2) which is the tangent of the Steiner inellipse at $K_{\rm i}$. Since this line also contains the Jerabek center

$$J_{\rm e} = (S_A(S_B-S_C)^2 : S_B(S_C-S_A)^2 : S_C(S_A-S_B)^2),$$

its points can be parametrized as

$$Q(t) = ((S_B-S_C)^2(S_A+t) : (S_C-S_A)^2(S_B+t) : (S_A-S_B)^2(S_C+t)).$$

The Kiepert cevian triangle of $Q(t)$ has vertices

$$A'(t) = \left(\frac{(S_C-S_A)(S_A-S_B)(S_B+t)(S_C+t)}{S_A+t} : (S_C-S_A)^2(S_B+t) : (S_A-S_B)^2(S_C+t)\right),$$
$$B'(t) = \left((S_B-S_C)^2(S_A+t) : \frac{(S_A-S_B)(S_B-S_C)(S_C+t)(S_A+t)}{S_B+t} : (S_A-S_B)^2(S_C+t)\right),$$
$$C'(t) = \left((S_B-S_C)^2(S_A+t) : (S_C-S_A)^2(S_B+t) : \frac{(S_B-S_C)(S_C-S_A)(S_A+t)(S_B+t)}{S_C+t}\right).$$

Theorem 9. *The Kiepert cevian triangle of $Q(t)$ is triply perspective to ABC. The three perspectors are collinear on the tangent of the Steiner inellipse at $K_{\rm i}$.*

Proof. The triangles $B'(t)C'(t)A'(t)$ and $C'(t)A'(t)B'(t)$ are each perspective to ABC, at the points

$$Q'(t) = \left(\frac{S_C+t}{S_C-S_A} : \frac{S_A+t}{S_A-S_B} : \frac{S_B+t}{S_B-S_C}\right),$$

and

$$Q''(t) = \left(\frac{S_B+t}{S_A-S_B} : \frac{S_C+t}{S_B-S_C} : \frac{S_A+t}{S_C-S_A}\right)$$

respectively. These two points are clearly on the line (2). □

5.2. *Special Kiepert cevian triangles with the same area as ABC.* The area of triangle $A'(t)B'(t)C'(t)$ is

$$\frac{(f_2\cdot t^2 + f_3\cdot t - f_4)^3}{\prod(f_2\cdot(S_A+t)^2 - (S_C-S_A)^2(S_A-S_B)^2)}$$

Among these, four have the same area as the reference triangle.

5.2.1. $t = \frac{S_A(S_B+S_C)-2S_{BC}}{S_B+S_C-2S_A}$. The points

$$\begin{aligned} Q(t) =&(-2(S_B - S_C) : S_C - S_A : S_A - S_B),\\ Q'(t) =&(S_B - S_C : -2(S_C - S_A) : S_A - S_B),\\ Q''(t) =&(S_B - S_C : S_C - S_A : -2(S_A - S_B)),\end{aligned}$$

give the Kiepert cevian triangle

$$\begin{aligned} A_1' =&(-(S_B - S_C) : 2(S_C - S_A) : 2(S_A - S_B)),\\ B_1' =&(2(S_B - S_C) : -(S_C - S_A) : 2(S_A - S_B)),\\ C_1' =&(2(S_B - S_C) : 2(S_C - S_A) : -(S_A - S_B)).\end{aligned}$$

This has centroid

$$K\left(-\frac{f_3}{2f_2}\right) = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right).$$

$A'(t)B'(t)C'(t)$ is also the Kiepert cevian triangle of the infinite point $P(\infty)$ (of the orthic axis). See Figure 8.

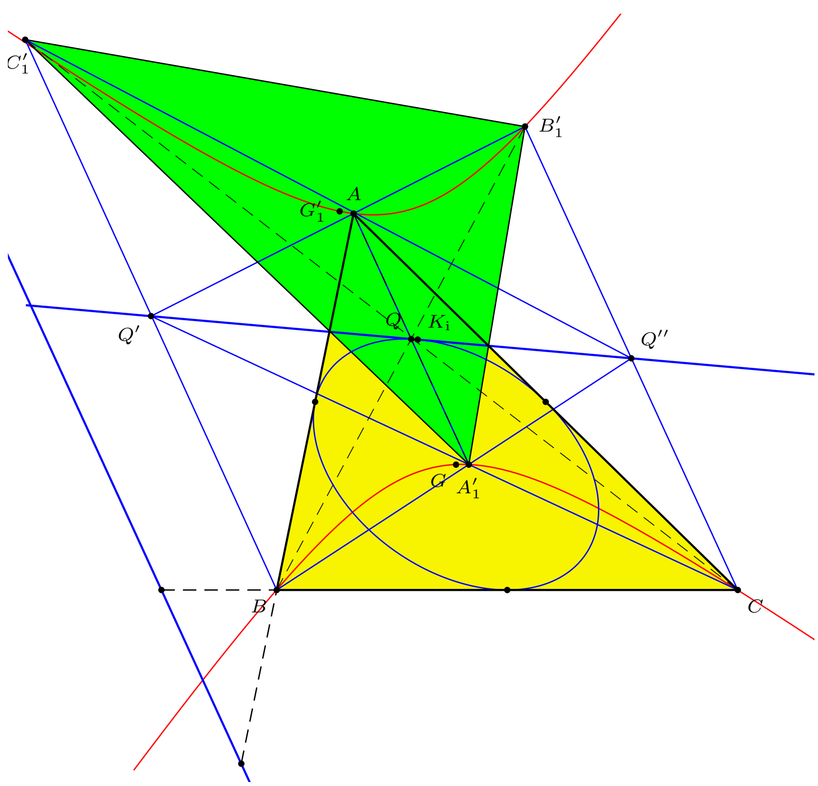

Figure 8. Oppositely oriented triangle triply perspective with ABC at three points on tangent at $K_{\rm i}$

5.2.2. $t=\infty$. With the Kiepert center $K_{\rm i}=Q(\infty)$, we have the points

$$Q(\infty)=((S_B-S_C)^2:(S_C-S_A)^2:(S_A-S_B)^2),$$
$$Q'(\infty)=\left(\frac{1}{S_A-S_B}:\frac{1}{S_B-S_C}:\frac{1}{S_C-S_A}\right),$$
$$Q''(\infty)=\left(\frac{1}{S_C-S_A}:\frac{1}{S_A-S_B}:\frac{1}{S_B-S_C}\right),$$

The points $Q'(\infty)$ and $Q''(\infty)$ are the intersection with the parallels through B, C to the line joining A to the Steiner point $S_{\rm t}=\left(\frac{1}{S_B-S_C}:\frac{1}{S_C-S_A}:\frac{1}{S_A-S_B}\right)$. These points give the Kiepert cevian triangle which is the image of ABC under the homothety $\mathsf{h}(K_{\rm i},-1)$:

$$A_2'=((S_C-S_A)(S_A-S_B):(S_C-S_A)^2:(S_A-S_B)^2),$$
$$B_2'=((S_B-S_C)^2:(S_A-S_B)(S_B-S_C):(S_A-S_B)^2),$$
$$C_2'=((S_B-S_C)^2:(S_C-S_A)^2:(S_C-S_A)(S_B-S_C)),$$

which has centroid

$$K\left(-\frac{S_A+S_B+S_C}{3}\right)=\left(\frac{1}{S_B+S_C-2S_A}:\frac{1}{S_C+S_A-2S_B}:\frac{1}{S_A+S_B-2S_C}\right).$$

The points $Q'(t)$, $Q''(t)$ and G_2' are on the Steiner circum-ellipse. See Figure 9.

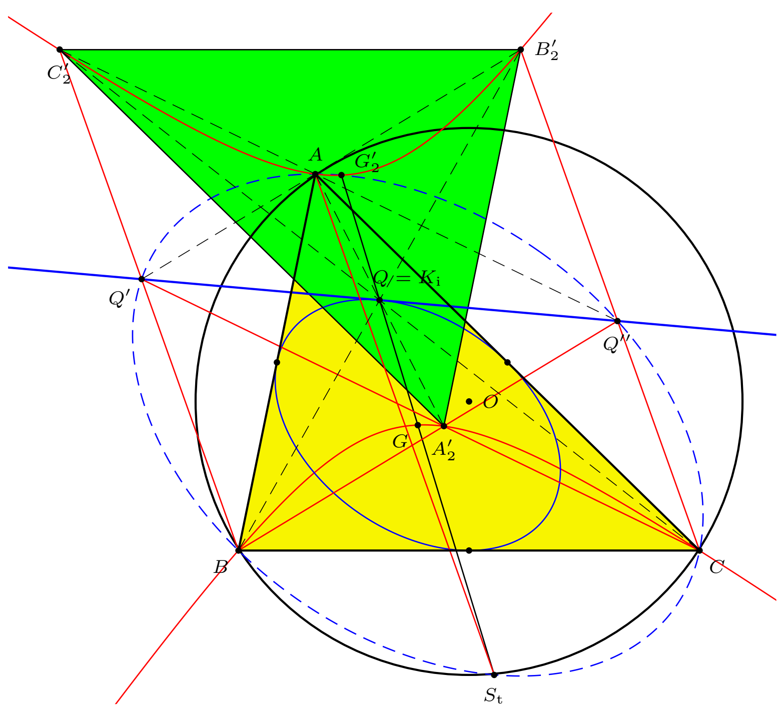

Figure 9. Oppositely congruent triangle triply perspective with ABC at three points on tangent at $K_{\rm i}$

5.2.3. $t = \frac{-f_3}{2f_2}$. $Q(t)$ is the infinite point of the line (2).

$$Q(t) = ((S_B - S_C)(S_B + S_C - 2S_A) : (S_C - S_A)(S_C + S_A - 2S_B) : (S_A - S_B)(S_A + S_B - 2S_C)),$$
$$Q'(t) = ((S_B - S_C)(S_A + S_B - 2S_C) : (S_C - S_A)(S_B + S_C - 2S_A) : (S_A - S_B)(S_C + S_A - 2S_B)),$$
$$Q''(t) = ((S_B - S_C)(S_C + S_A - 2S_B) : (S_C - S_A)(S_A + S_B - 2S_C) : (S_A - S_B)(S_B + S_C - 2S_A)).$$

These give the Kiepert cevian triangle

$$A_3' = \left(\frac{S_B - S_C}{S_B + S_C - 2S_A} : \frac{S_C - S_A}{S_A + S_B - 2S_C} : \frac{S_A - S_B}{S_C + S_A - 2S_B}\right),$$
$$B_3' = \left(\frac{S_B - S_C}{S_A + S_B - 2S_C} : \frac{S_C - S_A}{S_C + S_A - 2S_B} : \frac{S_A - S_B}{S_B + S_C - 2S_A}\right),$$
$$C_3' = \left(\frac{S_B - S_C}{S_C + S_A - 2S_B} : \frac{S_C - S_A}{S_B + S_C - 2S_A} : \frac{S_A - S_B}{S_A + S_B - 2S_C}\right),$$

with centroid

$$\left(\frac{S_B - S_C}{(S_B - S_C)^2 + 2(S_C - S_A)(S_A - S_B)} : \cdots : \cdots\right).$$

See Figure 10.

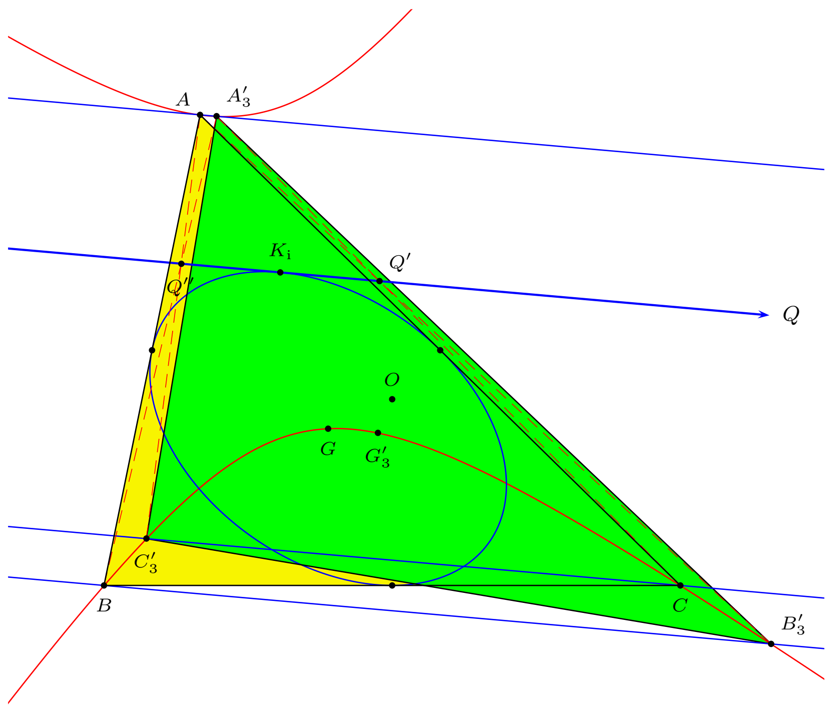

Figure 10. Triangle triply perspective with ABC (with the same orientation) at three points on tangent at K_i

5.2.4. $t = -S_A$. For $t = -S_A$, we have

$$\begin{aligned} Q(t) =&(0 : S_C - S_A : -(S_A - S_B)),\\ Q'(t) =&(-(S_B - S_C) : 0 : S_A - S_B),\\ Q''(t) =&(S_B - S_C : -(S_C - S_A) : 0). \end{aligned}$$

These points are the intercepts Q_a, Q_b, Q_c of the line (2) with the sidelines BC, CA, AB respectively. The lines AQ_a, BQ_b, CQ_c are the tangents to $\mathcal{K}$ at the vertices. The common Kiepert cevian triangle of Q_a, Q_b, Q_c is ABC oppositely oriented as ACB, CBA, BAC, triply perspective with ABC at Q_a, Q_b, Q_c respectively.

6. Special Kiepert inscribed triangles with two given vertices

Construction 10. *Given two points K_1 and K_2 on the Kiepert hyperbola $\mathcal{K}$, construct*
(i) *the midpoint M of K_1K_2,*
(ii) *the polar of M with respect to $\mathcal{K}$,*
(iii) *the reflection of the line K_1K_2 in the polar in* (ii).
If K_3 is a real intersection of $\mathcal{K}$ with the line in (iii), *then the Kiepert inscribed triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$.* See Figure 11.

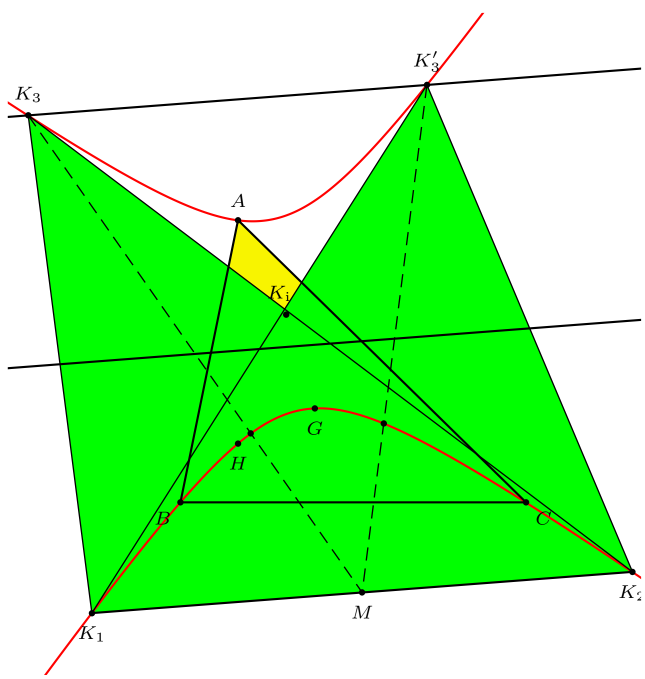

Figure 11. Construction of special Kiepert inscribed triangles given two vertices K_1, K_2

Proof. A point K_3 for which triangle $K_1K_2K_3$ has centroid on $\mathcal{K}$ clearly lies on the image of $\mathcal{K}$ under the homothety $\mathsf{h}(M, 3)$. It is therefore an intersection of $\mathcal{K}$ with this homothetic image. If $M = (u : v : w)$ in homogeneous barycentric coordinates, this homothetic conic has equation

$$\begin{aligned}&(u+v+w)^2K(x,y,z)\\ +&2(x+y+z)\left(\sum((S_B-S_C)vw+(S_C-S_A)(3u+w)w+(S_A-S_B)(3u+v)v)x\right)\\ =&0.\end{aligned}$$

The polar of M in $\mathcal{K}$ is the line

$$\sum((S_A-S_B)v+(S_C-S_A)w)x=0. \tag{5}$$

The parallel through M is the line

$$\sum(3(S_B-S_C)vw+(S_C-S_A)(u-w)w+(S_A-S_B)(u-v)v)x=0. \tag{6}$$

The reflection of (6) in (5) is the radical axis of $\mathcal{K}$ and its homothetic image above. $\square$

If there are two such real intersections K_3 and K_3', then the two triangles $K_1K_2K_3$ and $K_1K_2K_3'$ clearly have equal area. These two intersections coincide if the line in Construction 10 (iii) above is tangent to $\mathcal{K}$. This is the case when K_1K_2 is a tangent to the hyperbola

$$4f_2\cdot K(x,y,z)-3g_3\cdot(x+y+z)^2=0,$$

which is the image of $\mathcal{K}$ under the homothety $\mathsf{h}(K_i, 2)$. See Figure 12.

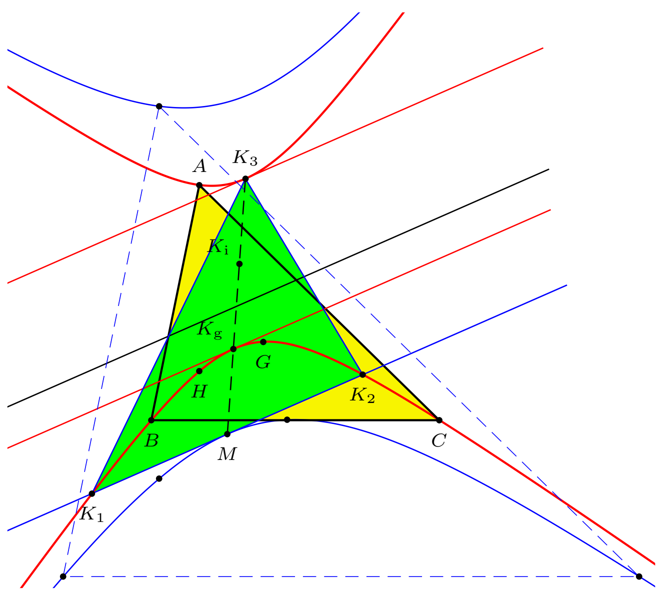

Figure 12. Family of special Kiepert inscribed triangles with K_1, K_2 uniquely determining K_3

The resulting family of special Kiepert inscribed triangles is the same family with centroid $K(t)$ and one vertex its antipode on $\mathcal{K}$, given in Construction 4.

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