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Angle and Circle Characterizations of Tangential Quadrilaterals

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Abstract. We prove five necessary and sufficient conditions for a convex quadrilateral to have an incircle that concerns angles or circles.

1. Introduction

A *tangential quadrilateral* is a convex quadrilateral with an incircle, i.e., a circle inside the quadrilateral that is tangent to all four sides. In [4] and [5] we reviewed and proved a total of 20 different necessary and sufficient conditions for a convex quadrilateral to be tangential. Of these there were 14 dealing with different distances (sides, line segments, radii, altitudes), four were about circles (excluding their radii), and only two were about angles. In this paper we will prove five more such characterizations concerning angles and circles. First we review two that can be found elsewhere.

A characterization involving the four angles and all four sides of a quadrilateral appeared as part of a proof of an inverse altitude characterization of tangential quadrilaterals in [6, p.115]. According to it, a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$ and $d = DA$ is tangential if and only if

$$a \sin A \sin B + c \sin C \sin D = b \sin B \sin C + d \sin D \sin A.$$

In the extensive monograph [9, p.133] on quadrilateral geometry, the following characterization is attributed to Simionescu. A convex quadrilateral is tangential if and only if its consecutive sides a, b, c, d and diagonals p, q satisfies

$$|ac - bd| = pq \cos \theta$$

where θ is the acute angle between the diagonals. The proof is a simple application of the quite well known identity $2pq \cos \theta = \left|b^2 + d^2 - a^2 - c^2\right|$ that holds in all convex quadrilaterals. Rewriting it as

$$2pq \cos \theta = \left|(b + d)^2 - (a + c)^2 + 2(ac - bd)\right|,$$

we see that Simionescu's theorem is equivalent to Pitot's theorem $a + c = b + d$ for tangential quadrilaterals. In Theorem 2 we will prove another characterization for the angle between the diagonals, but it only involves four different distances instead of six.

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2. Angle characterizations of tangential quadrilaterals

It is well known that a convex quadrilateral has an incircle if and only if the four angle bisectors of the internal vertex angles are concurrent. If this point exist, it is the incenter. Here we shall prove a necessary and sufficient condition for an incircle regarding the intersection of two opposite angle bisectors which characterize the incenter in terms of two angles in two different ways. To prove that one of these equalities holds in a tangential quadrilateral (the direct theorem) was a problem in [1, p.67].

Theorem 1. *A convex quadrilateral $ABCD$ is tangential if and only if*

$$\angle AIB + \angle CID = \pi = \angle AID + \angle BIC$$

where I is the intersection of the angle bisectors at A and C.

Proof. ($\Rightarrow$) In a tangential quadrilateral the four angle bisectors intersect at the incenter. Using the sum of angles in a triangle and a quadrilateral, we have

$$\angle AIB + \angle CID = \pi - \left(\frac{A}{2} + \frac{B}{2}\right) + \pi - \left(\frac{C}{2} + \frac{D}{2}\right) = 2\pi - \frac{2\pi}{2} = \pi.$$

The second equality can be proved in the same way, or we can use that the four angles in the theorem make one full circle, so $\angle AID + \angle BIC = 2\pi - \pi = \pi$.

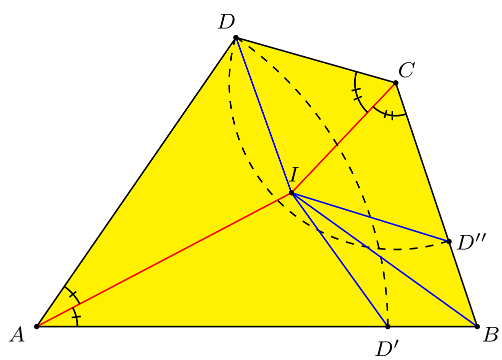

Figure 1. Construction of the points D' and D''

($\Leftarrow$) In a convex quadrilateral where I is the intersection of the angle bisectors at A and C, and the equality

$$\angle AIB + \angle CID = \angle AID + \angle BIC \tag{1}$$

holds, assume without loss of generality that $AB > AD$ and $BC > CD$.[1] Construct points D' and D'' on AB and BC respectively such that $AD' = AD$ and $CD'' = CD$ (see Figure 1). Then triangles AID' and AID are congruent, and so are triangles CID'' and CID. Thus $ID' = ID = ID''$. These two pairs

[1]If instead there is equality in one of these inequalities, then it's easy to see that the quadrilateral is a kite. It's well known that kites have an incircle.

of congruent triangles and (1) yields that $\angle BID' = \angle BID''$, so triangles BID' and BID'' are congruent. Thus $BD' = BD''$. Together with $AD' = AD$ and $CD'' = CD$, we get

$$AD' + D'B + CD = AD + BD'' + D''C \quad \Rightarrow \quad AB + CD = AD + BC.$$

Then $ABCD$ is a tangential quadrilateral according to Pitot's theorem. □

The idea for the proof of the converse comes from [8], where Goutham used this method to prove the converse of a related characterization of tangential quadrilaterals concerning areas. That characterization states that if I is the intersection of the angle bisectors at A and C in a convex quadrilateral $ABCD$, then it has an incircle if and only if

$$S_{AIB} + S_{CID} = S_{AID} + S_{BIC},$$

where S_{XYZ} stands for the area of triangle XYZ. According to [9, p.134], this theorem is due to V. Pop and I. Gavrea. In [6, pp.117–118] a similar characterization concerning the same four areas was proved, but it also includes the four sides. It states that $ABCD$ is a tangential quadrilateral if and only if

$$c \cdot S_{AIB} + a \cdot S_{CID} = b \cdot S_{AID} + d \cdot S_{BIC},$$

where $a = AB$, $b = BC$, $c = CD$ and $d = DA$.

The next characterization is about the angle between the diagonals. We will assume we know the lengths of the four parts that the intersection of the diagonals divide them into. Then the question is, what size the angle between the diagonals shall have for the quadrilateral to have an incircle? This means that the sides of the quadrilateral are not fixed and the lengths of them changes as we vary the angle between the diagonals. See Figure 2. If $\theta \to 0$, then clearly $a + c < b + d$; and if $\theta \to \pi$, then $a + c > b + d$. Hence for some $0 < \theta < \pi$ we have $a + c = b + d$, and the quadrilateral has an incircle.

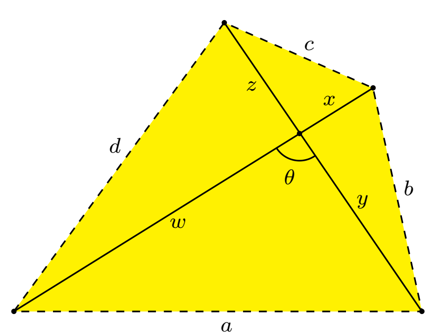

Figure 2. The diagonal parts

Theorem 2. *If the diagonals of a convex quadrilateral are divided into parts w, x and y, z by their point of intersection, then it is a tangential quadrilateral if and only if the angle θ between the diagonals satisfies*

$$\cos\theta = \frac{(w-x)(y-z)\Big(2(wx+yz)-\sqrt{(w+x)^2(y+z)^2+4(wx-yz)^2}\Big)}{(w+x)^2(y+z)^2-16wxyz}.$$

Proof. A convex quadrilateral is tangential if and only if its consecutive sides a, b, c, d satisfies Pitot's theorem

$$a+c=b+d. \tag{2}$$

The sides of the quadrilateral can be expressed in terms of the diagonal parts and the angle between the diagonals using the law of cosines, according to which (see Figure 2)

$$\begin{aligned} a^2 &= w^2+y^2-2wy\cos\theta, \\ b^2 &= x^2+y^2+2xy\cos\theta, \\ c^2 &= x^2+z^2-2xz\cos\theta, \\ d^2 &= w^2+z^2+2wz\cos\theta. \end{aligned}$$

Here we used $\cos(\pi-\theta)=-\cos\theta$ in the second and fourth equation. Inserting these into (2) yields

$$\begin{aligned} &\sqrt{w^2+y^2-2wy\cos\theta}+\sqrt{x^2+z^2-2xz\cos\theta} \\ &= \sqrt{x^2+y^2+2xy\cos\theta}+\sqrt{w^2+z^2+2wz\cos\theta}. \end{aligned}$$

The algebra involved in solving this equation including four square roots is not simple. For this reason we will use a computer calculation to solve it. Squaring both sides results in a new equation with only two square roots. Collecting them alone on one side of the equality sign and squaring again gives another equation, this time with only one square root. The last step in the elimination of the square roots is to separate that last one from the other terms, on one side, and squaring a third time. This results in a polynomial equation in $\cos\theta$ that has 115 terms! Factoring that with the computer, we obtain

$$\begin{aligned} &(w+x)^2(y+z)^2(-1+T)(1+T) \\ &\cdot(-w^2y^2+2wy^2x-y^2x^2+2w^2yz-4wyxz+2yx^2z-w^2z^2+2wxz^2 \\ &-x^2z^2-4w^2yxT+4wyx^2T-4wy^2zT+4w^2xzT+4y^2xzT-4wx^2zT \\ &+4wyz^2T-4yxz^2T+w^2y^2T^2+2wy^2xT^2+y^2x^2T^2+2w^2yzT^2 \\ &-12wyxzT^2+2yx^2zT^2+w^2z^2T^2+2wxz^2T^2+x^2z^2T^2)=0 \end{aligned}$$

where we put $T=\cos\theta$. None of the factors but the last parenthesis gives any valid solutions. Solving the quadratic equation in the last parenthesis with the computer yields

$$T=\frac{4(w-x)(y-z)(wx+yz)\pm\sqrt{4(w-x)^2(y-z)^2P_1}}{2P_2} \tag{3}$$

where

$$P_1 = w^2y^2 + 2wy^2x + 4w^2x^2 + y^2x^2 + 2w^2yz - 4wyxz + 2yx^2z + w^2z^2 + 4y^2z^2 + 2wxz^2 + x^2z^2$$

and

$$\begin{aligned} P_2 &= w^2y^2 + 2wy^2x + y^2x^2 + 2w^2yz - 12wyxz + 2yx^2z + w^2z^2 + 2wxz^2 + x^2z^2 \\ &= (wy + xz)^2 + (wz + yx)^2 + 2(wy + xz)(wz + yx) - 4wxyz - 12wxyz \\ &= (wy + xz + wz + yx)^2 - 16wxyz = (w + x)^2(y + z)^2 - 16wxyz. \end{aligned}$$

Thus

$$\begin{aligned} P_1 &= (w + x)^2(y + z)^2 - 8wxyz + 4w^2x^2 + 4y^2z^2 \\ &= (w + x)^2(y + z)^2 + 4(wx - yz)^2. \end{aligned}$$

Inserting the simplified expressions for P_1 and P_2 into the solutions (3) and factoring them, we get [2]

$$\cos\theta = \frac{(w - x)(y - z)\Big(2(wx + yz) \pm \sqrt{(w + x)^2(y + z)^2 + 4(wx - yz)^2}\Big)}{(w + x)^2(y + z)^2 - 16wxyz}.$$

To determine the correct sign, we study a special case. In an isosceles tangential trapezoid where $w = y = 2u$ and $x = z = u$ (here u is an arbitrary positive number), we have

$$\cos\theta = \frac{u^2\Big(8u^2 \pm \sqrt{9u^2 \cdot 9u^2 + 0}\Big)}{9u^2 \cdot 9u^2 - 16 \cdot 4u^4} = \frac{8 \pm 9}{17}.$$

For the solution with the plus sign, we get $\cos\theta = 1$. Thus $\theta = 0$ which is not a valid solution. Hence the correct solution is the one with the minus sign. □

Corollary 3. *A convex quadrilateral where one diagonal bisect the other has an incircle if and only if it is a kite.*

Proof. ($\Rightarrow$) If in a tangential quadrilateral $w = x$ or $y = z$, then the formula in the theorem indicates that $\cos\theta = 0$. Thus $\theta = \frac{\pi}{2}$, so one diagonal is the perpendicular bisector of the other. Then the quadrilateral must be a kite, since one diagonal is a line of symmetry.

($\Leftarrow$) If the quadrilateral is a kite (they always have the property that one diagonal bisect the other), then it has an incircle according to Pitot's theorem. □

3. Circle characterizations of tangential quadrilaterals

To prove the first circle characterization we need the following theorem concerning the extended sides, which we reviewed in [4] and [5]. Since it is quite rare to find a proof of it in modern literature (particularly the converse), we start by proving it here. It has been known at least since 1846 according to [10].

[2]Here we used that $\sqrt{(w - x)^2(y - z)^2} = (w - x)(y - z)$. We don't have to put absolute values since there is $\pm$ in front of the square root and we don't yet know which sign is correct.

Theorem 4. *In a convex quadrilateral $ABCD$ that is not a trapezoid,*[3] *let the extensions of opposite sides intersect at E and F. If exactly one of the triangles AEF and CEF is outside of the quadrilateral $ABCD$, then it is a tangential quadrilateral if and only if*

$$AE + CF = AF + CE.$$

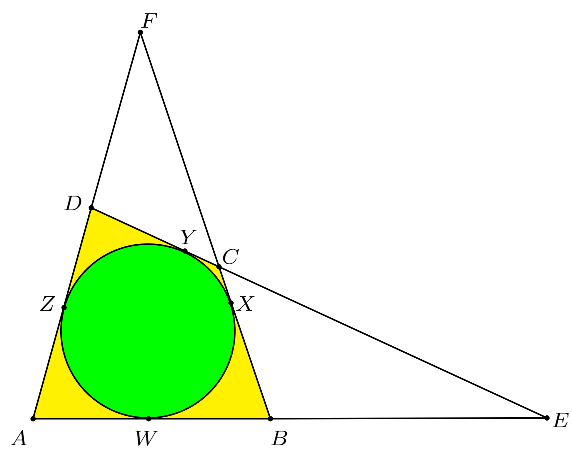

Figure 3. Tangential quadrilateral with extended sides

Proof. ($\Rightarrow$) In a tangential quadrilateral, let the incircle be tangent to the sides AB, BC, CD, DA at W, X, Y, Z respectively. We apply the two tangent theorem (that two tangents to a circle through an external point are congruent) several times to get (see Figure 3)

$$AE + CF = AW + EW + FX - CX = AZ + EY + FZ - CY = AF + CE.$$

($\Leftarrow$) We do an indirect proof of the converse. In a convex quadrilateral where $AE + CF = AF + CE$, we draw a circle tangent to the sides AB, BC, CD. If this circle is not tangent to DA, draw a tangent to the circle parallel to DA. This tangent intersect AB, CD and BF at A', D' and F' respectively (see Figure 4). We assume DA does not cut the circle; the other case can be proved in the same way. Also, let G be a point on DA such that $A'G$ is parallel to (and thus equal to) $F'F$. From the direct part of the theorem we now have

$$A'F' + CE = A'E + CF'.$$

Subtracting this from $AE + CF = AF + CE$, we get

$$AG = AA' + A'G.$$

This equality is a contradiction since it violates the triangle inequality in triangle AGA'. Hence the assumption that DA was not tangent to the circle must be incorrect. Together with a similar argument in the case when DA cuts the circle this completes the proof. $\square$

[3] And thus not a parallelogram, rhombus, rectangle or a square either.

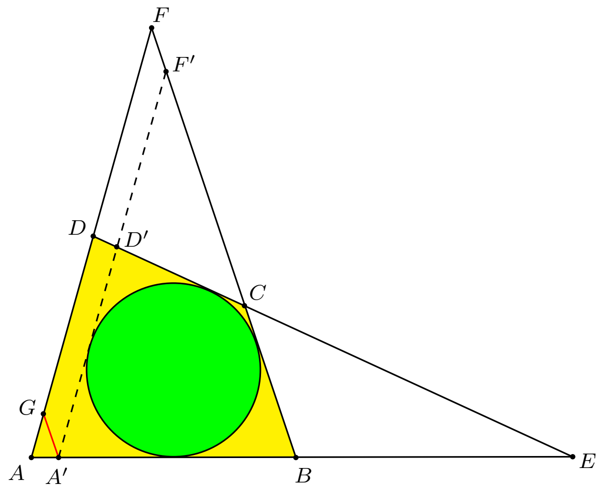

Figure 4. The tangent $A'F'$ is parallel to AF

Remark. If both triangles AEF and CEF are outside of the quadrilateral $ABCD$, then the characterization for a tangential quadrilateral is $BE + DF = BF + DE$. It is obtained by relabeling the vertices according to $A \to B \to C \to D \to A$ in comparison to Theorem 4.

The direct part of the first circle characterization was a problem proposed and solved at [7]. We will use Theorem 4 to give a very short proof including the converse as well.

Theorem 5. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at E and F. If exactly one of the triangles AEF and CEF is outside of the quadrilateral $ABCD$, then it is a tangential quadrilateral if and only if the incircles in triangles AEF and CEF are tangent to EF at the same point.*

Proof. It is well known that in a triangle, the distance from a vertex to the point where the incircle is tangent to a side is equal to the semiperimeter of the triangle subtracted by the side opposite to that vertex [2, p.184]. Now assume the incircles in triangles AEF and CEF are tangent to EF at G and H respectively. Then we have (see Figure 5)

$$2(FG-FH) = (EF+AF-AE)-(EF+CF-CE) = AF+CE-AE-CF.$$

Hence

$$G \equiv H \quad \Leftrightarrow \quad FG = FH \quad \Leftrightarrow \quad AE + CF = AF + CE$$

which proves that the two incircles are tangent at the same point on EF if and only if the quadrilateral is tangential according to Theorem 4. □

Remark. If both triangles AEF and CEF are outside of the quadrilateral $ABCD$ (this happens if F is below AB or E is to the left of AD in Figure 5), then the

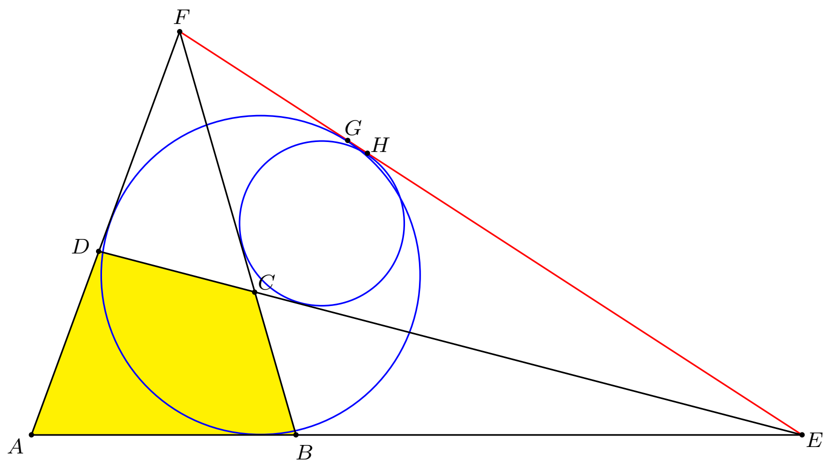

Figure 5. Two tangent points at EF

theorem is not true. In that case the two triangles that shall have tangent incircles at EF are instead BEF and DEF.

The next theorem concerns the same two incircles that we just studied.

Theorem 6. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides AB and DC intersect at E, and the extensions of opposite sides BC and AD intersect at F. Let the incircle in triangle AEF be tangent to AE and AF at K and L respectively, and the incircle in triangle CEF be tangent to BF and DE at M and N respectively. If exactly one of the triangles AEF and CEF is outside of the quadrilateral $ABCD$, then it is a tangential quadrilateral if and only if $KLMN$ is a cyclic quadrilateral.*

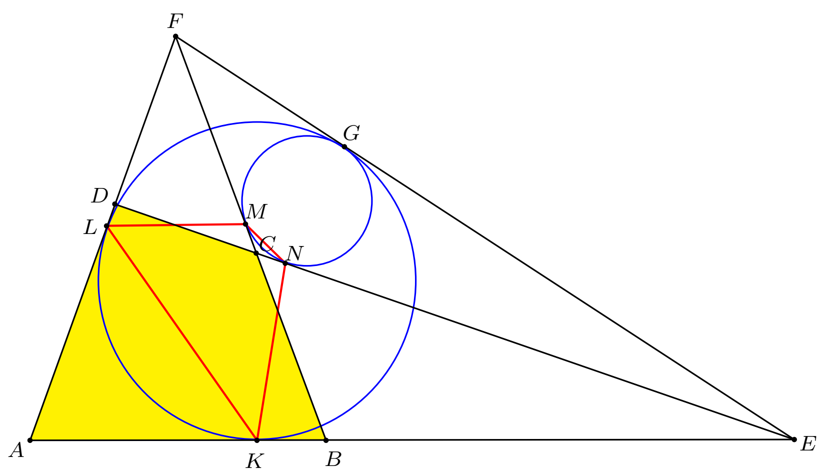

Figure 6. Here $ABCD$ is a tangential quadrilateral

Proof. ($\Rightarrow$) In a tangential quadrilateral $ABCD$, the incircles in triangles AEF and CEF are tangent to EF at the same point G according to Theorem 5. This together with the two tangent theorem yields that $EK = EG = EN$ and $FL = FG = FM$, so the triangles EKN and FLM are isosceles (see Figure 6). Thus $\angle ENK = \frac{A+D}{2}$ and $\angle FLM = \frac{A+B}{2}$. Triangles ALK and CNM are also isosceles, so $\angle ALK = \frac{\pi-A}{2}$ and $\angle CNM = \frac{\pi-C}{2}$. Hence for two opposite angles in quadrilateral $KLMN$, we get

$$\angle KLM + \angle KNM = \left(\pi - \frac{A+B}{2} - \frac{\pi-A}{2}\right) + \left(\frac{\pi-C}{2} + \pi - \frac{A+D}{2}\right)$$
$$= 2\pi - \frac{A+B+C+D}{2} = \pi$$

where we used the sum of angles in a quadrilateral. This means that $KLMN$ is a cyclic quadrilateral according to a well known characterization.

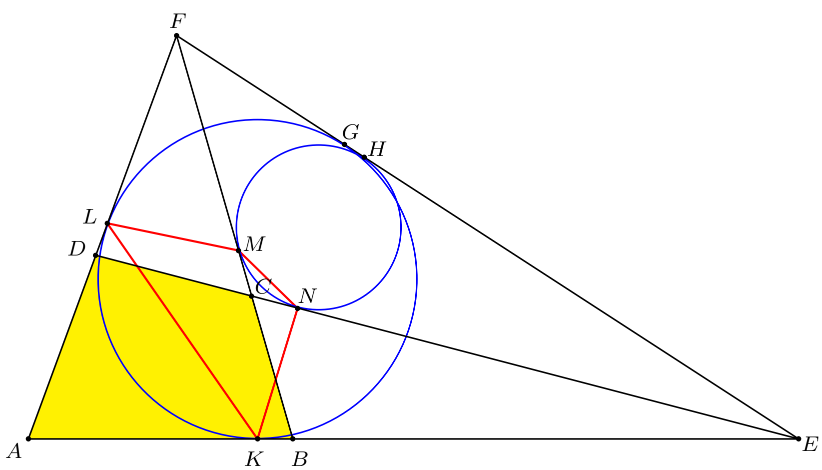

Figure 7. Here $ABCD$ is not a tangential quadrilateral

($\Leftarrow$) If $ABCD$ is not a tangential quadrilateral, we shall prove that $KLMN$ is not a cyclic quadrilateral. When $ABCD$ is not tangential, the incircles in triangles AEF and CEF are tangent to EF at different points G and H respectively. We assume without loss of generality that G is closer to F than H is.[4] Thus $EK = EG > EH = EN$ and $FL = FG < FH = FM$ (see Figure 7). Applying that in a triangle, a longer side is opposite a larger angle, we get $\angle ENK > \frac{A+D}{2}$ and $\angle FLM > \frac{A+B}{2}$. Triangles ALK and CNM are still isosceles. This yields that

$$\angle KLM < \pi - \frac{A+B}{2} - \frac{\pi-A}{2} = \frac{\pi-B}{2}$$

[4]The other case can be dealt with in the same way. What happens is that all inequalities below will be reversed.

and $\angle KND < \pi - \frac{A+D}{2}$. Hence for two opposite angles in $KLMN$,

$$\angle KLM + \angle KNM < \frac{\pi - B}{2} + \left(\frac{\pi - C}{2} + \pi - \frac{A+D}{2}\right) = \pi,$$

again using the sum of angles in a quadrilateral. This proves that if $ABCD$ is not a tangential quadrilateral, then $KLMN$ is not a cyclic quadrilateral. □

Corollary 7. *The incircle in $ABCD$ and the circumcircle to $KLMN$ in Theorem 6 are concentric.*

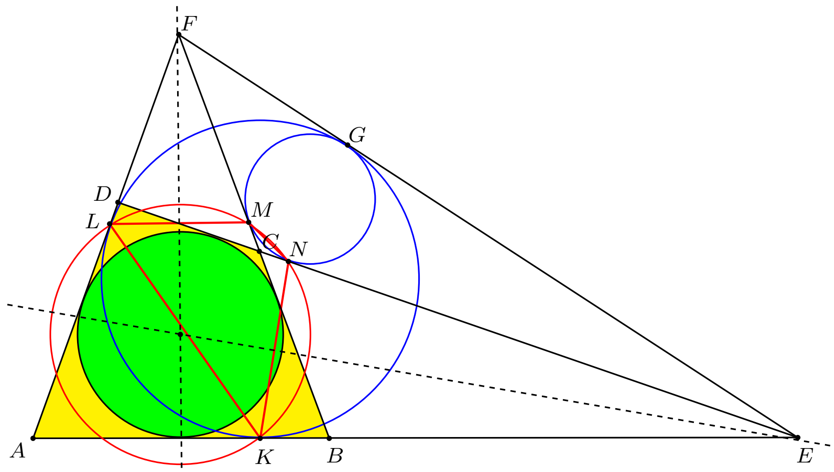

Figure 8. The two concentric circles

Proof. The incircle in $ABCD$ is also an incircle in triangles AED and AFB (see Figure 8). The perpendicular bisectors of the sides KN and LM are also angle bisectors to the angles AED and AFB, hence they intersect at the incenter of $ABCD$. This proves that the two circles are concentric. □

Next we will study a related configuration to the one in Theorem 6, with two other incircles. In [4, pp.66–67] we proved that in a convex quadrilateral $ABCD$, the two incircles in triangles ABD and CBD are tangent to BD at the same point if and only if $ABCD$ is a tangential quadrilateral. These two incircles are also tangent to all four sides of the quadrilateral (two tangency point per circle). In [11, pp.197–198] it was proved that if $ABCD$ is a tangential quadrilateral, then these four tangency points are the vertices of a cyclic quadrilateral that is concentric with the incircle in $ABCD$. Another proof of the concyclic property of the four tangency points was given in [9, pp.272–273]. Now we shall prove that the converse is true as well and thus get another characterization of tangential quadrilaterals.

Theorem 8. *In a convex quadrilateral $ABCD$, let the incircles in triangles ABD and CBD be tangent to the sides of $ABCD$ at K, L, M, N. Then $ABCD$ is a tangential quadrilateral if and only if $KLMN$ is a cyclic quadrilateral.*

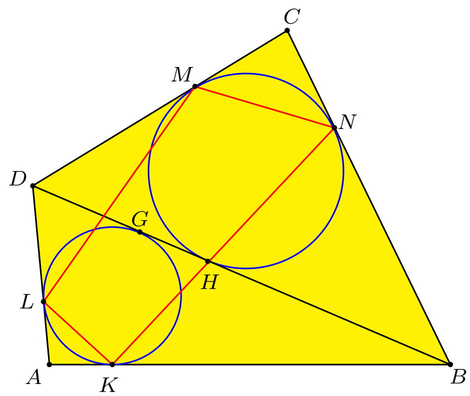

Figure 9. Here $ABCD$ is not a tangential quadrilateral

Proof. Only the proof of the converse is given, but a proof of the direct theorem is obtained by simply changing all the inequalities below to equalities. Thus we prove that if $ABCD$ is not a tangential quadrilateral, then $KLMN$ is not a cyclic quadrilateral.

Let the incircles in triangles ABD and CBD be tangent to BD at G and H respectively, and assume without loss of generality that G is closer to D than H is. If K, L, M, N are the tangency points at AB, AD, CD and CB respectively, then according to the two tangent theorem $BK = BG > BH = BN$ and $DL = DG < DH = DM$ (see Figure 9). Since a larger angle in a triangle is opposite a longer side, we have that $\angle BKN < \frac{\pi - B}{2}$. Also, $\angle AKL = \frac{\pi - A}{2}$ since triangle AKL is isosceles. Thus

$$\angle LKN > \pi - \frac{\pi - B}{2} + \frac{\pi - A}{2} = \frac{A + B}{2}.$$

In the same way we have $\angle DML < \frac{\pi - D}{2}$ and $\angle CMN = \frac{\pi - C}{2}$, so

$$\angle LMN > \pi - \frac{\pi - D}{2} + \frac{\pi - C}{2} = \frac{C + D}{2}.$$

Hence for two opposite angles in $KLMN$,

$$\angle LKN + \angle LMN > \frac{A + B + C + D}{2} = \pi.$$

This proves that if $ABCD$ is not a tangential quadrilateral, then $KLMN$ is not a cyclic quadrilateral. □

4. A related characterization of a bicentric quadrilateral

A *bicentric quadrilateral* is a convex quadrilateral that is both tangential and cyclic, i.e., it has both an incircle and a circumcircle. In a tangential quadrilateral $ABCD$, let the incircle be tangent to the sides AB, BC, CD, DA at W, X, Y, Z respectively. It is well known that the quadrilateral $ABCD$ is also cyclic (and hence bicentric) if and only if the tangency chords WY and XZ are perpendicular

[3, p.124]. Now we will prove a similar characterization concerning the configuration of Theorem 6.

Theorem 9. *In a tangential quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides AB and DC intersect at E, and the extensions of opposite sides BC and AD intersect at F. Let the incircle in triangle AEF be tangent to AE and AF at K and L respectively, and the incircle in triangle CEF be tangent to BF and DE at M and N respectively. If exactly one of the triangles AEF and CEF is outside of the quadrilateral $ABCD$, then it is a bicentric quadrilateral if and only if the extensions of KN and LM are perpendicular.*

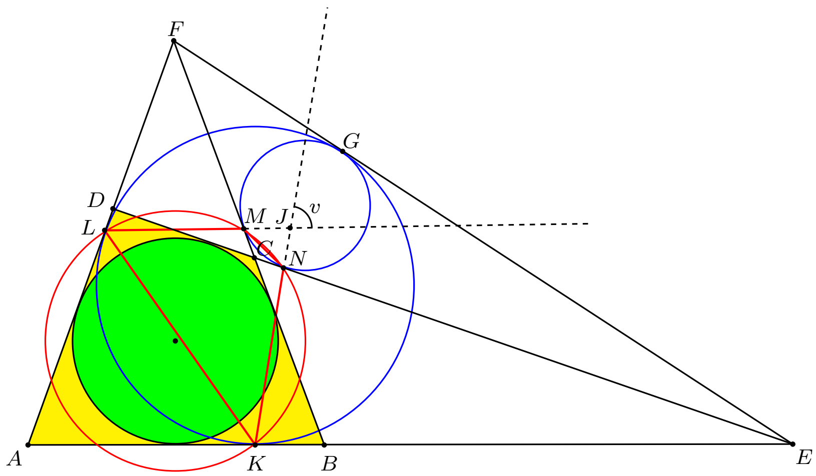

Figure 10. Angle between extensions of opposite sides of $KLMN$

Proof. Let J be the intersection of the extensions of KN and LM, and v the angle between them. Then $\angle JNC = \angle ENK = \frac{A+D}{2}$ and $\angle JMC = \angle FML = \frac{A+B}{2}$ (see Figure 10). Thus, using the sum of angles in quadrilateral $CMJN$, we have

$$v = 2\pi - C - \frac{A+B}{2} - \frac{A+D}{2} = 2\pi - \frac{A+B+C+D}{2} - \frac{A+C}{2} = \pi - \frac{A+C}{2}.$$

Hence

$$v = \frac{\pi}{2} \quad \Leftrightarrow \quad A + C = \pi$$

so the extensions of KN and LM are perpendicular if and only if the tangential quadrilateral $ABCD$ is also cyclic. □

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The Associated Harmonic Quadrilateral

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Abstract. In this article we study a natural association of a harmonic quadrilateral to every non-parallelogrammic quadrilateral. In addition we investigate the corresponding association in the case of cyclic quadrilaterals and the reconstruction of the quadrilateral from its harmonic associated one. Finally, we associate to a generic quadrilateral a cyclic one.

1. Harmonic quadrilaterals

Harmonic quadrilaterals, introduced by Tucker and studied by Neuberg ([1, p.206], [6]) can be defined in various equivalent ways. A simple one is to draw the tangents FA, FC to a circle κ from a point F (can be at infinity) and draw also an additional secant FBD to the circle (see Figure 1(I)). Another definition starts

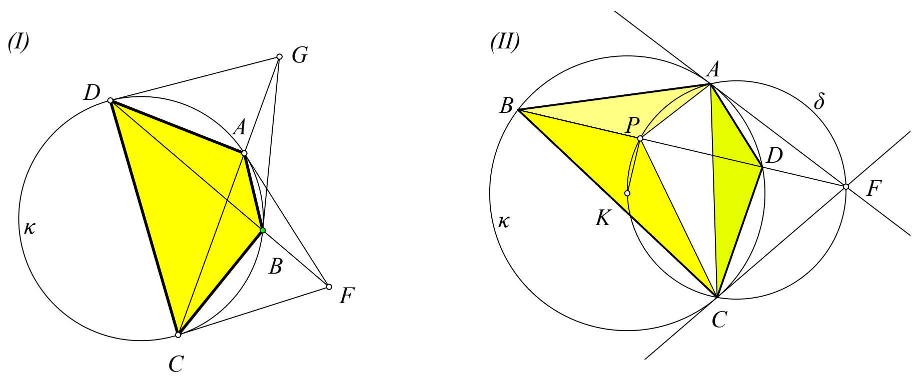

Figure 1. Definition and a basic property

with an arbitrary triangle ABD and its circumcircle κ and defines C as the intersection of κ with the symmedian from A. These convex quadrilaterals have several interesting properties exposed in textbooks and articles ([5, p.100,p.306], [8]). One of them, used in the sequel, is their characterization as convex cyclic quadrilaterals, for which the products of opposite side-lengths are equal $|AB||CD| = |BC||DA|$ or, equivalently, the ratios of adjacent side-lengths are equal $\frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$. Another property, also used below, deals with a dissection of the quadrilateral in similar triangles (see Figure 1(II)), which I formulate as a lemma without a proof.

Lemma 1. *Let $ABCD$ be a harmonic quadrilateral and P be the projection of its circumcenter K onto the diagonal BD. Then triangles ADC, APB and BPC are similar. Furthermore, the tangents of its circumcircle at points A and C intersect*

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at a point F of the diagonal BD and the circumcircle δ of ACF passing through K and P.

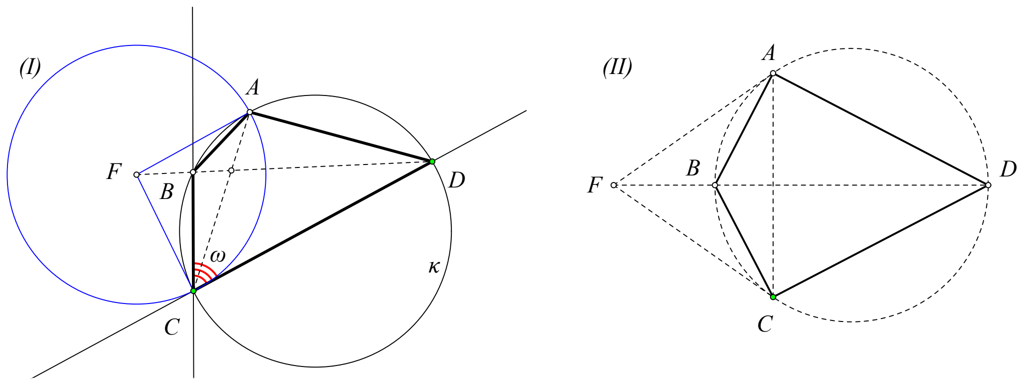

Figure 2. Determination by ω and $r = \frac{|AB|}{|AD|}$ Kite

Note that, up to similarity, a harmonic quadrilateral is uniquely determined by its angle $\omega = \angle BCD$ and the ratio $r = \frac{|AB|}{|AD|} = \frac{|CB|}{|CD|}$ (see Figure 2(I)). In fact, fixing the circle κ and taking an inscribed angle of measure ω, the angle-sides determine a chord BD of length depending only on κ and ω. Then, points A, C on both sides of BD are determined by intersecting κ with the Apollonius circle ([2, p.15]), dividing BD in the given ratio r.

A special class of harmonic quadrilaterals, comprising the squares, is the one of *kites*, which are symmetric with respect to one of their diagonals (see Figure 2(II)). Excluding this special case, for all other harmonic quadrilaterals there is a kind of symmetry with respect to the two diagonals, having the consequence, that in all properties, including one of the diagonals, it is irrelevant which one of the two is actually chosen.

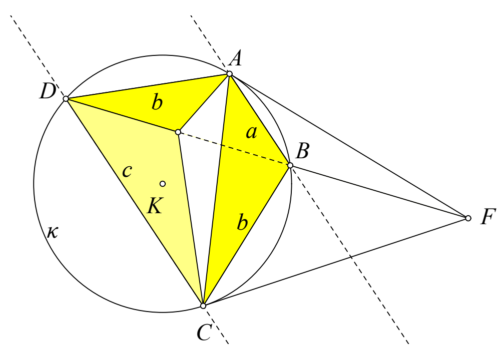

Figure 3. Harmonic trapezia

Another class of special harmonic quadrilaterals is the one of *harmonic trapezia*, comprising all equilateral trapezia with side lengths satisfying $ac = b^2$ (see Figure

3). This, up to similarity, is also a one-parameter family of harmonic quadrilaterals. Given the circle $\kappa(r)$, each harmonic trapezium, inscribed in κ, is determined by the ratio $k = \frac{a}{b} < 1$ of the small parallel to the non-parallel side-length. A short calculation shows that to each such trapezium corresponds a special triangle ABD with data

$$a = k'r, \qquad b = \frac{k'}{k}r, \qquad \cos B = \frac{1-k^2}{2k},$$

where $k' = \sqrt{\frac{4k^2-(1-k^2)^2}{2}}$.

2. The associated harmonic quadrilateral

In the sequel we restrict ourselves to non-parallelogrammic convex quadrilaterals. For every such quadrilateral $p = ABCD$ there is a harmonic quadrilateral q, naturally associated to p. The next theorem shows how to construct it.

Theorem 2. *The two centers Z_1, Z_2 of the similarities f_1, f_2, mapping respectively $f_1(A) = C, f_1(B) = D, f_2(B) = D, f_2(C) = A$, of a non-parallelogramic quadrilateral $p = ABCD$, together with the midpoints M, N of its diagonals AC, BD, are the vertices of a harmonic quadrilateral $q = NZ_1MZ_2$, whose circumcircle κ passes through the intersection point E of the diagonals.*

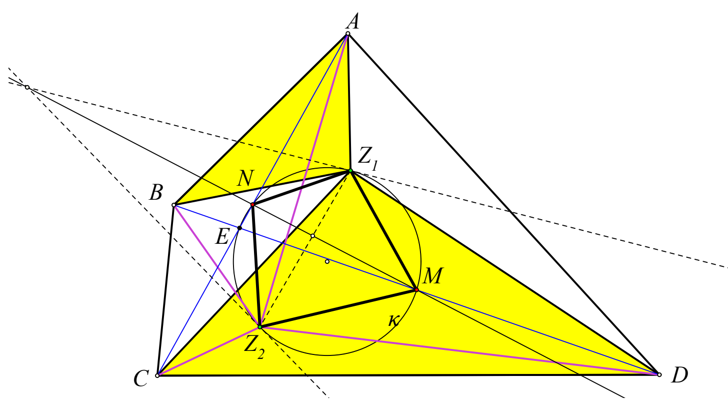

Figure 4. The harmonic quadrilateral associated to a quadrilateral

In fact, let κ be the circle passing through the midpoints M, N of the diagonals and also passing through their intersection point E. Some special cases in which point E is on line MN are handled below. Point Z_1 is the center of similarity f_1 ([3, p.72], [11, II, p.43]) mapping the triangle ABZ_1 correspondingly onto CDZ_1 (see Figure 4). Analogously, point Z_2 is the center of the similarity mapping the triangle BCZ_2 onto DAZ_2. It follows easily, that the triangles based on the diagonals, ACZ_1 and BDZ_1, are also similar, their similarity ratio being equal to those of their medians from Z_1, as well as their corresponding bases coinciding with the diagonals $\lambda = \frac{|Z_1N|}{|Z_1M|} = \frac{|AC|}{|BD|}$. This implies also that the angles formed by corresponding medians of the two similar triangles are equal, i.e., ANZ_1 and

EMZ_1 are equal angles. This implies that Z_1 is on κ. Analogously is seen that Z_2 is also on κ and that the ratio $\frac{|Z_2N|}{|Z_2M|} = \lambda$. Thus,

$$\frac{|Z_1N|}{|Z_1M|} = \frac{|Z_2N|}{|Z_2M|},$$

which means that the cyclic quadrilateral Z_1MZ_2N is harmonic.

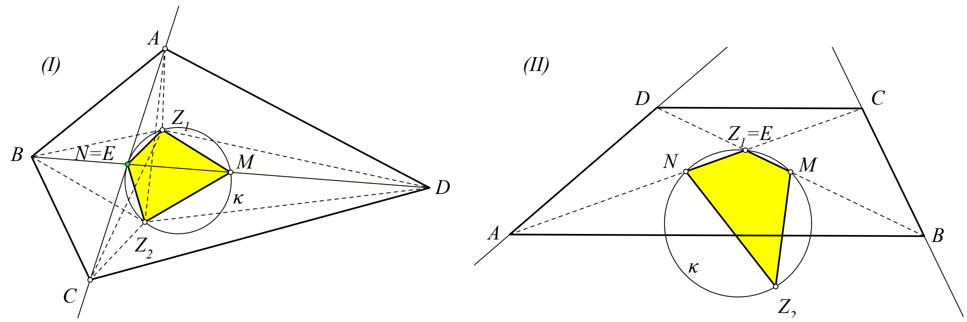

Figure 5. Point E coinciding with N Point Z_1 coinciding with E

In the case one of the midpoints of the diagonals coincides with their intersection point ($N = E$) the circle κ passes through the midpoints M, N of the diagonals and is tangent to the diagonal (AC), whose midpoint coincides with E (see Figure $5(I)$). Another particular class is the one of trapezia, characterized by the fact that one of the similarity centers (Z_1) coincides with the intersection E of the diagonals (see Figure $5(II)$).

3. The inverse construction

Fixing a harmonic quadrilateral q and selecting two opposite vertices Z_1, Z_2 of it, we can easily construct all convex quadrilaterals p having the given one as their associated. This reconstruction is based on the following lemma.

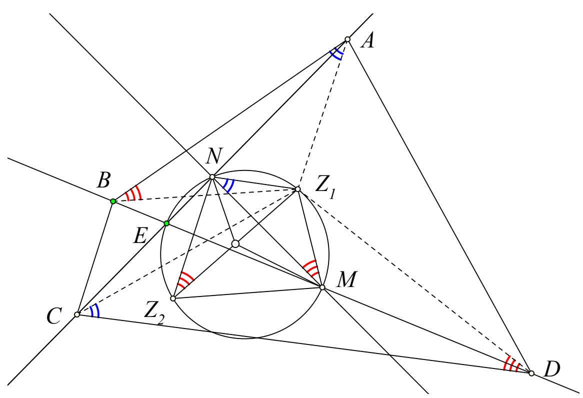

Figure 6. Generating the quadrilateral from its associated harmonic one

Lemma 3. *Let $p = ABCD$ be a convex quadrilateral with associated harmonic one $q = NZ_1MZ_2$, such that Z_1 is the similarity center of triangles ABZ_1, CDZ_1. Then triangle NMZ_1 is also similar to the above triangles.*

In fact, by the Theorem 2, triangles ACZ_1, BDZ_1 are also similar, and N, being the midpoint of side AC, maps, by the similarity sending ACZ_1 to BDZ_1, to the corresponding midpoint M of CD (see Figure 6). This implies that triangles Z_1AN, Z_1BM are also similar, hence $\frac{|Z_1N|}{|Z_1M|} = \frac{|Z_1A|}{|Z_1B|}$. Since the rotation angle, involved in the similarity mapping ACZ_1 to BDZ_1, is the angle AZ_1B, this angle will be also equal to angle NZ_1M, thereby proving the similarity of triangles ABZ_1 and NMZ_1.

Lemma 3 implies that all quadrangles $p = ABCD$, having the given quadrangle $q = NZ_1MZ_2$ as their associated harmonic, are parameterized by the similarities f with center at Z_1. For, each such similarity produces a triangle $ABZ_1 = f(NMZ_1)$ and defines through it the two vertices A, B. The other two vertices C, D of the quadrilateral p are found by taking, correspondingly, the symmetrics of A, B with respect to N and M. Note, that, by reversing the argument in Lemma 3, the diagonals AC, BD of the resulting quadrilateral intersect at a point E of the circumcircle of the harmonic quadrilateral. Hence their angle is the same with angle NZ_1M. Also the ratio of the diagonals of $ABCD$ is equal to the ratio $\frac{|Z_1N|}{|Z_1M|}$, thus it is determined by the harmonic quadrilateral $q = NZ_1MZ_2$. We have proved the following theorem.

Theorem 4. *Given a harmonic quadrilateral $q = NZ_1MZ_2$, there is a double infinity of quadrilaterals $p = ABCD$ having q as their harmonic associate with similarity centers at Z_1 and Z_2 and midpoints of diagonals at M and N. All these quadrilaterals have their diagonals intersecting at the same angle NZ_1M, the same ratio $\frac{|AC|}{|BD|} = \frac{|Z_2N|}{|Z_2M|}$ and their Newton lines coinciding with MN. Each of these quadrilaterals is characterized by a similarity f with center at Z_1, mapping $f(Z_1NM) = Z_1AB$.*

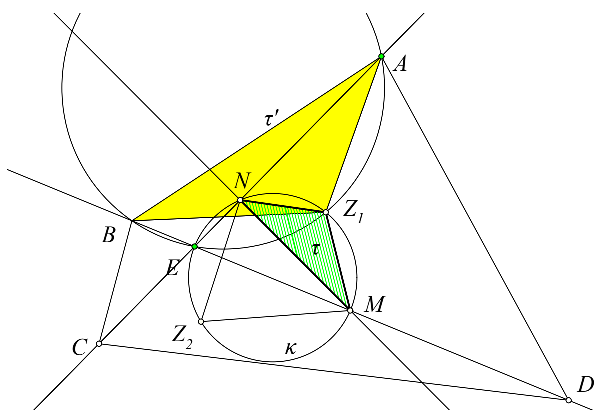

Figure 7. Alternative generation of $ABCD$ from the harmonic quadrilateral

An alternative way to generate all quadrilaterals with given harmonic associate $q = NZ_1MZ_2$ and similarity centers at Z_1, Z_2, is to use a point E on the circumcircle κ of q, draw lines EM, EN, and consider their intersections A, C with the circles passing through E and Z_1. Equivalently, construct all triangles Z_1AB similar to Z_1NM and having the vertex A on line EN. Then the other vertex B moves on line EN ([11, II, p.68]) and C, D are again, respectively, the symmetrics of A, B with respect to N and M. A fourth method is described in §7.

4. Two related similar quadrilaterals

In order to prove some additional properties of our configuration, the following lemma is needed, which, though elementary in character, I could not locate a proof of it in the literature. For the completeness of the exposition I outline a short proof of it.

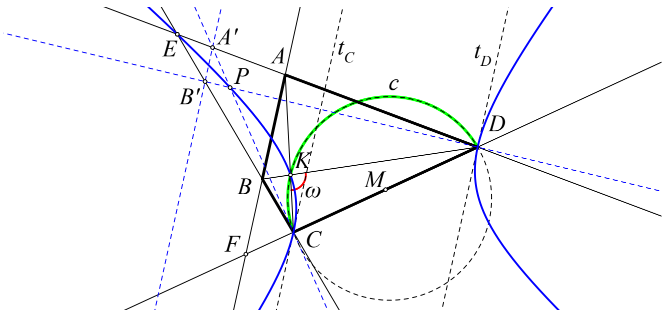

Figure 8. Quadrilateral from angles and angle of diagonals

Lemma 5. *Two quadrilaterals having equal corresponding angles and equal angles between diagonals are similar.*

In fact, let $ABCD$ be a quadrilateral with given angles and the angle ω between its diagonals. The two triangles ECD, FAD, where E, F are the intersection points of opposite sides, have known angles and are constructible up to similarity. Thus, we can fix triangle ECD and move a line parallel to AF intersecting the sides EC, ED correspondingly at B', A'. The quadrilateral with the required data must have the angle formed at the intersection point $P = (A'C, B'D)$ equal to ω. This position K for P is found as follows (see Figure 8). As $A'B'$ moves parallel to itself it creates a homographic correspondence $B' \mapsto A'$ between the points of the lines EC and ED and induces a corresponding homography between the pencils of lines at C and D. Then, according to the Chasles-Steiner theorem, the intersection point P of corresponding rays CA', DB' describes a conic ([9, p.109]). It is easily seen that this conic is a hyperbola passing through the vertices of triangle ECD, whose tangents at C, D are parallel to $A'B'$ and its center is the midpoint M of CD. The intersection point K of the conic with a circular arc c of points viewing CD under the angle ω determines the quadrilateral with the required properties and shows that it is unique, up to similarity.

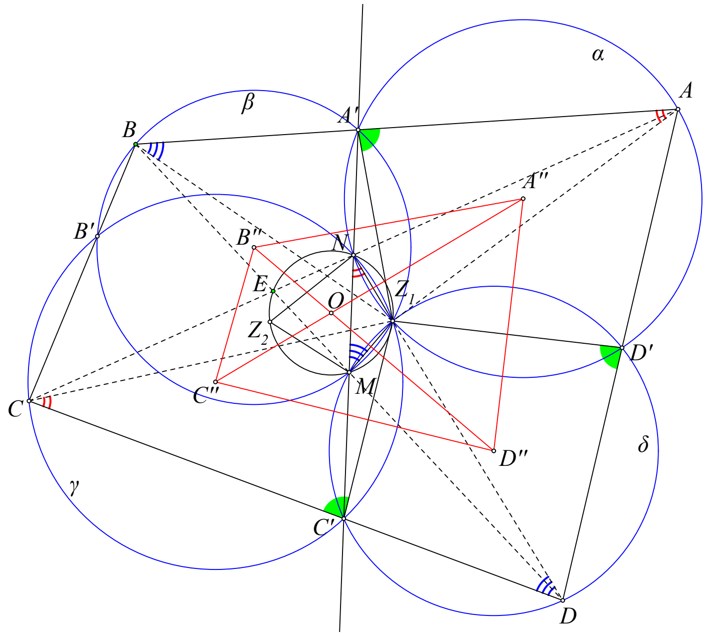

Figure 9. Four circles intersecting on the sides

Theorem 6. (1) *The circles* $\alpha = (Z_1NA), \beta = (Z_1MB)$ *pass through the intersection point* A' *of the Newton line with side* AB. *Analogously, the circles* $\gamma = (Z_1NC), \delta = (Z_1MD)$ *pass through the intersection point* C' *of the Newton line with side* CD.

(2) *Circles* β *and* γ *intersect at a point* B' *of* BC. *Analogously circles* α *and* δ *intersect at a point* D' *of* AD.

(3) *The centers* A'', B'', C'', D'' *of corresponding circles* $\alpha, \beta, \gamma, \delta$ *build a quadrilateral* $A''B''C''D''$ *similar to* $ABCD$, *whose diagonals pass through* O.

(4) *Analogous to the above properties hold by replacing* Z_1 *with* Z_2 *and defining* A', B', C', D' *and circles* $\alpha, \beta, \gamma, \delta$ *properly.*

In fact, (1) and (2) result by a simple angle chasing argument (see Figure 9). (3) follows from the Lemma 5 and the fact that $A''B''C''D''$ has the same angles with $ABCD$ and also the same angle of diagonals, which intersect at O. (4) is proved by the same arguments.

5. The case of cyclic quadrilaterals

The location of the similarity centers Z_1, Z_2 in the case of a cyclic quadrilateral is, in most cases, immediate according to the following.

Theorem 7. *In the case of a cyclic quadrilateral* $p = ABCD$, *whose opposite sides intersect at points* F, G, *the similarity centers* Z_1, Z_2 *are the intersections of the circumcircle* κ *of the associated harmonic quadrilateral with the circle* μ *on diameter* FG.

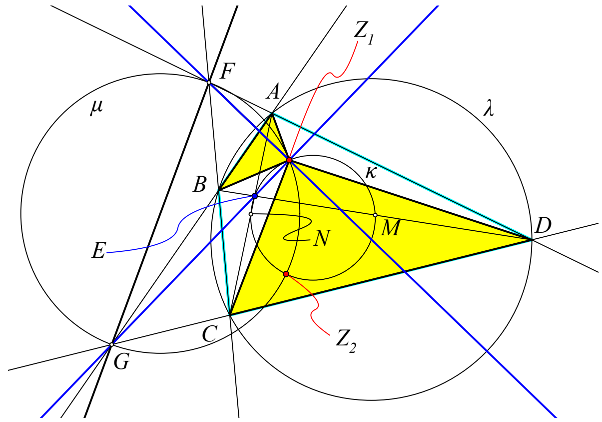

Figure 10. The case of cyclic $ABCD$

In [7] it is proved that a quadrilateral p is cyclic if and only if the circle μ, with diameter FG, is orthogonal to the corresponding circle $\kappa = (MNE)$. Thus, in this case there are indeed two intersection points Z_1, Z_2 on κ (see Figure 10). There is also proved, that in this case line FG is the polar of E and coincides with the radical axis of the pencil of circles generated by κ and the circumcircle λ of $ABCD$. Since angle FZ_1G is a right one and points (B, C, N, F) make a harmonic division, the two lines Z_1G, Z_1F are the bisectors of the angle BZ_1C as well as of angle AZ_1D. Thus, angles AZ_1B and CZ_1D are equal and angles AZ_1C, BZ_1D are also equal. Since G is on the radical axis of κ and λ the quadrilateral CDZ_1E is cyclic, hence the angles ECZ_1 and EDZ_1 are equal. This implies that triangles AZ_1C and BZ_1C are similar and from this follows that triangles AZ_1B, CZ_1D are also similar. This identifies point Z_1 with the center of similarity transforming AB to CD. Analogously is proved the corresponding property for the other intersection point Z_2.

Next theorem explores the possibility to determine a generic cyclic quadrilateral $p = ABCD$ on the basis of its associated harmonic one.

Theorem 8. *A convex cyclic quadrilateral p, whose opposite sides intersect, is uniquely determined from its associated harmonic quadrilateral q and the location of the intersection E of the diagonals of p on the circumcircle κ of q. Point E can be taken arbitrarily on the arc defined by Z_1Z_2, which is less than half the circumference of κ. All cyclic quadrilaterals resulting by such a choice of E have the angle between their diagonals equal to $\angle Z_1MZ_2$ or its complementary and the ratio of diagonal-lengths equal to $\frac{|Z_1M|}{|Z_1N|} = \frac{|Z_2M|}{|Z_2N|}$.*

The first statement follows easily from two facts. The first is that, according to Theorem 7, the circle μ on diameter FG, where F, G are the intersections of opposite sides of $p = ABCD$, is orthogonal to the circumcircle κ of q and its center is at the intersection P of tangents to κ, respectively at Z_1 and Z_2 or the pole of Z_1Z_2 with respect to κ. Hence this circle is constructible from the data of the harmonic

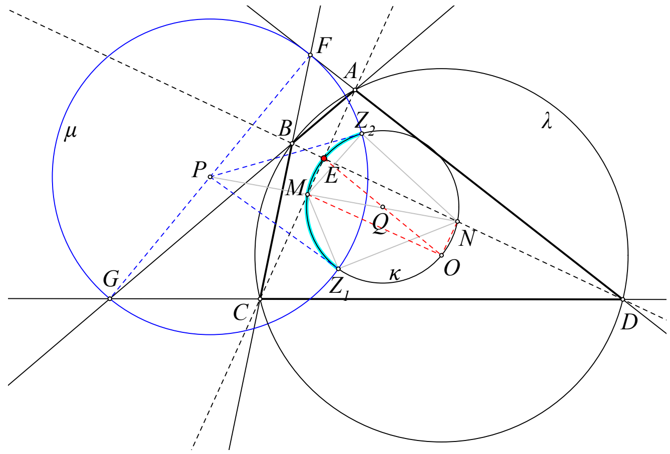

Figure 11. Constructing the cyclic $ABCD$ from its associated harmonic

quadrilateral $q = MZ_1NZ_2$. The second fact, proved in the aforementioned reference, is that the circumcircle λ of the quadrilateral p is orthogonal to μ and its center is the diametral point O of E with respect to circle κ. This implies that λ can be constructed as the circle, which is orthogonal to μ and has its center at O. Having this circle, we obtain the vertices of the quadrilateral p by intersecting it with lines EM and EN. The other statements follow from fundamental properties of the harmonic quadrilateral, such as, for example, the fact, that M, N are separated by Z_1, Z_2 and that generic cyclic convex quadrilaterals have the intersection point E always in the arc Z_1Z_2, which is less than half the circumference of κ.

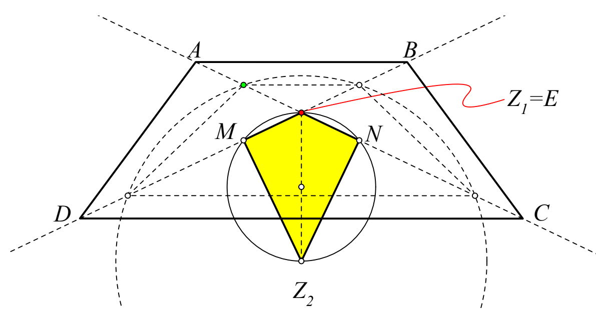

Figure 12. Associated harmonic quadrilateral of an isosceles trapezium

Having excluded from the beginning the parallelogrammic quadrilaterals, which have both pairs of opposite sides intersecting at infinity, the case of cyclic quadrilaterals, not included in both theorems, is the one of equilateral trapezia, having one pair of sides intersecting at infinity. In this case the harmonic associated is found easily, having the similarity centers coinciding correspondingly with the intersection point $E = Z_1$ of the diagonals and the circumcenter $O = Z_2$ (see Figure 12). Theorem 8 is not valid in this case, since, then, there are infinite many cyclic quadrilaterals with the same harmonic associate. In fact, in this case, every circle centered at $Z_2 = O$, with radius $r > |Z_1Z_2|$ defines, through its intersections with lines Z_1M, Z_1N, an equilateral trapezium having the given $q = NZ_1MZ_2$ for harmonic associated. Two other cases, in which the intersection point E of the diagonals of $p = ABCD$ coincides with a particular point, are the quadrilaterals having $E = N$, i.e., coinciding with the midpoint of one diagonal (see Figure 13),

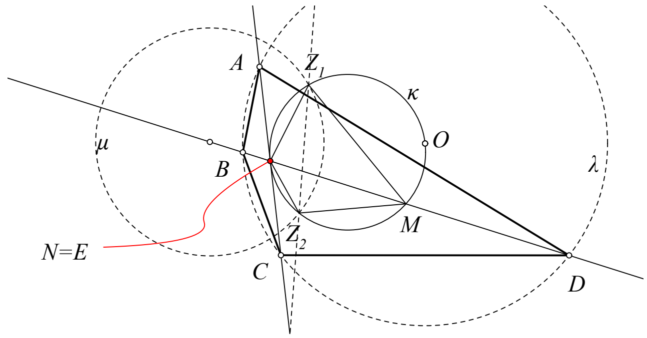

Figure 13. The case $E = N$

and the quadrilaterals $p = ABCD$, which are also themselves harmonic. In this

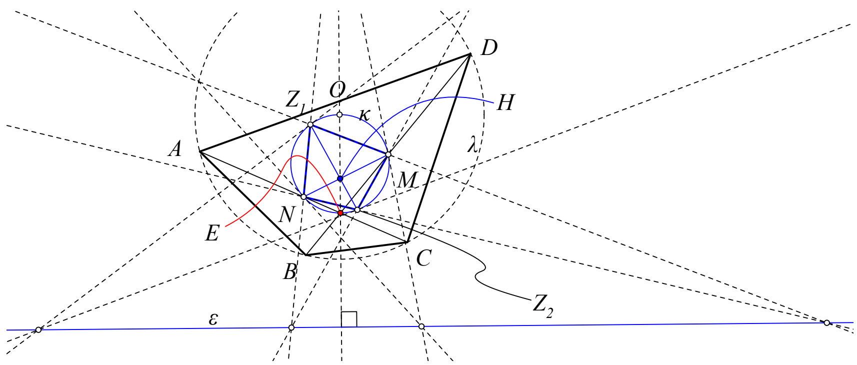

Figure 14. The case $p = ABCD$ is also harmonic

case E is on the diameter of the circumcircle κ of q, which contains the intersection point H of the diagonals of q. Then the polar ε of H with respect to κ coincides with the radical axis of the circle κ and the circumcircle λ of p (see Figure 14).

6. The two lemniscates

Fixing the harmonic quadrilateral $q = NZ_1MZ_2$, as seen in the previous section, all cyclic quadrilaterals p, having q as their associated, are parameterized by a point E varying on an arc Z_1Z_2 of the circumcircle κ of q. The following theorem shows that the vertices of the resulting quadrilaterals $p = ABCD$ vary on two lemniscates of Bernoulli ([10, p.13], [4, p.110]).

Theorem 9. *The vertices of all convex cyclic quadrilaterals $p = ABCD$, having the same harmonic associated quadrilateral $q = NZ_1MZ_2$ are on two Bernoulli lemniscates with nodes, respectively, at M and N. Each pair of opposite vertices lies on the same lemniscate and is symmetric with respect to the corresponding node.*

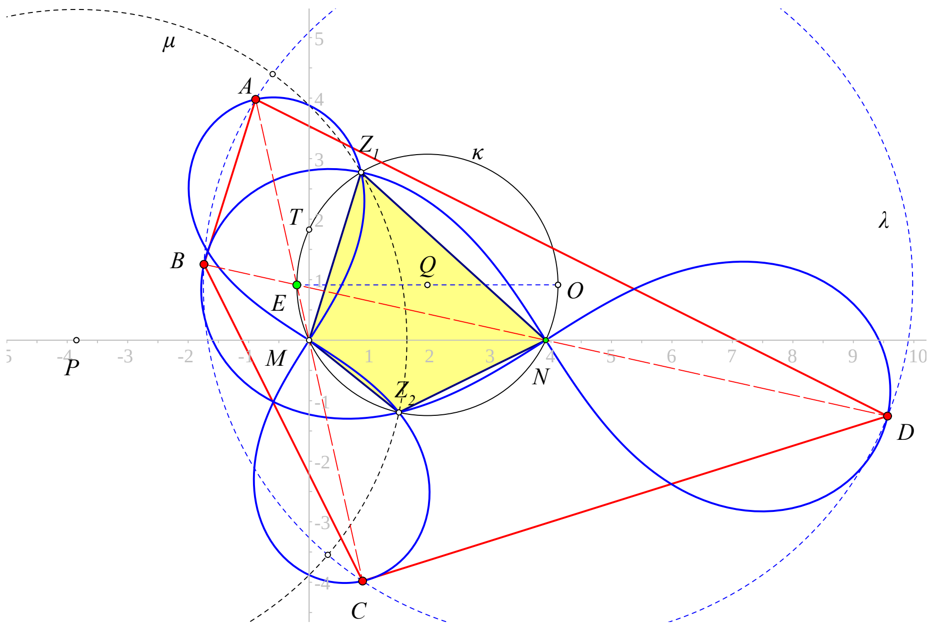

Figure 15. Geometric locus of vertices of $ABCD$ with given harmonic associated

The proof of the theorem follows from a simple calculation, using cartesian coordinates centered at the vertex M of the given harmonic quadrilateral $q = NZ_1MZ_2$. Vertex N is set at $(n, 0)$ and the circumcircle κ of q intersects the y-axis at $(0, t)$. Point $P(p, 0)$ is the center of the circle μ, which passes through Z_1, Z_2 and is orthogonal to κ. The equations can be set in dependence of the parameters n, p and t by following the recipe of reconstruction of p from q, described in Theorem 8. For a variable point $E(u, v)$ on κ, the intersection points of line EM and the circle λ, centered at the diametral O of the point E and orthogonal to μ, are found by eliminating (u, v) from the three equations representing the circle λ, the line ME and the circle κ. These are correspondingly:

$$\begin{aligned} x^2 + y^2 - 2x(n-u) - 2y(t-v) - 2pu + pn &= 0, \\ vx - uy &= 0, \\ u^2 + v^2 - nu - tv &= 0. \end{aligned}$$

Eliminating (u, v) from these equations, leads to an equation of the 8-th degree, which splits into the two quadratics $(x-p)^2 = 0$, $(x-n)^2 + (y-t)^2 - (n^2 + t^2) + np = 0$ and the equation of the fourth degree

$$(x^2 + y^2)^2 + np(y^2 - x^2) - 2ptxy = 0,$$

for the coordinates (x, y) of the points A and C. The first equation represents the line $x = p$ not satisfied by the points A, C. The second represents the circle λ obtained when $E = M$ and satisfied by A, C only when AC is tangent to κ at M. Finally the last equation, by inverting on the unit circle, leads to

$$np(y^2 - x^2) - 2ptxy + 1 = 0,$$

representing a rectangular hyperbola centered at the origin. By the well known property of Bernoulli's lemniscates to be the inverses of such hyperbolas ([4, p.110]), this proves the theorem for the pair of opposite vertices A and C. For the other pair of opposite vertices, B and D, an analogous calculation, leads to a corresponding system of three equations

$$\begin{aligned} x^2 + y^2 - 2x(n-u) - 2y(t-v) - 2pu + pn &= 0, \\ vx + (n-u)y - nv &= 0, \\ u^2 + v^2 - nu - tv &= 0. \end{aligned}$$

Here again, elimination of (u, v), transfer of the origin at N, and inversion on the unit circle centered at N, leads, through the factorization of an equation of th 8-th degree, to the equation of the rectangular hyperbola

$$(n^2 - np)(y^2 - x^2) + 2t(n-p)xy + 1 = 0.$$

This, using the aforementioned property of Bernoulli's lemniscate, proves the theorem for the vertices B and D.

Remarks. (1) Using, for convenience, the corresponding equations of the rectangular hyperbolas, one can easily compute the symmetry axes of the lemniscates and

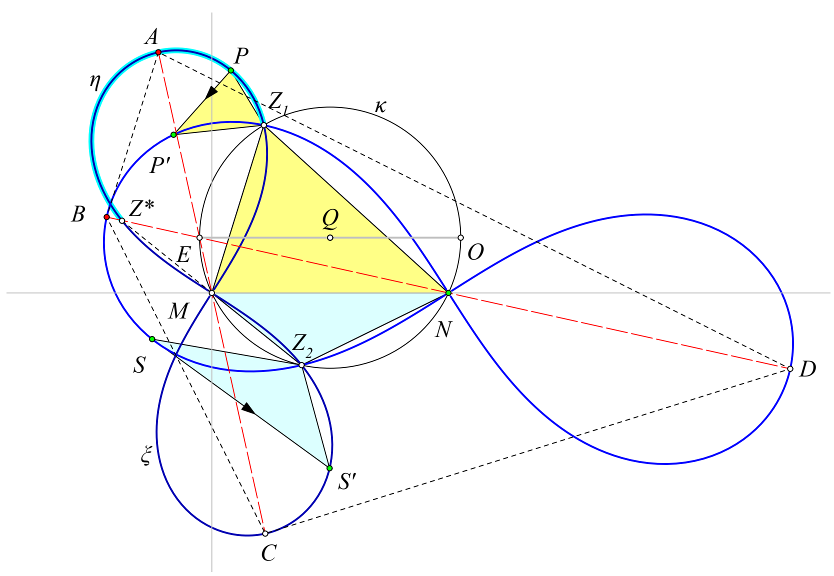

Figure 16. The similarities of the two lemniscates

see that they are obtained, respectively, by lines AC, BD, when their intersection

E is such that EO is parallel to line MN (see Figure 16). A simple computation shows also that the two lemniscates are similar with respect to two similarities. The first one $P' = f_1(P)$ has its center at Z_1, its oriented rotation-angle equals $\angle MZ_1N$ and its ratio is $r_1 = \frac{|Z_1M|}{|Z_1N|}$. The second similarity $S' = f_2(S)$ has its center at Z_2, its oriented rotation-angle equals $\angle NZ_2M$ and its ratio i s $r_2 = \frac{|Z_2N|}{|Z_2M|} = r_1^{-1}$.

(2) Fixing a certain lemniscate ξ, one can use the above results to give a parametrization of all cyclic quadrilaterals, up to similarity, by three points Z_1, Z_2, P properly chosen on the lemniscate. In fact, select first two points Z_1, Z_2, each on a different loop and on the same side of the axis AC of ξ (see Figure 16). This, together with the node M of ξ creates a triangle Z_1MZ_2 with the angle at M greater than a right one. This triangle defines also a unique point N, such that $q = NZ_1MZ_2$ is a harmonic quadrilateral. Excepting the squares, all other harmonic quadrilaterals, up to similarity, are obtained in this way. Having q, one can define the similarity f_1 of the previous remark. Then, every point A on the arc $\eta = Z_1Z^*$, where Z^* the symmetric of Z_2 with respect to M, defines a cyclic quadrilateral $p = ABCD$.

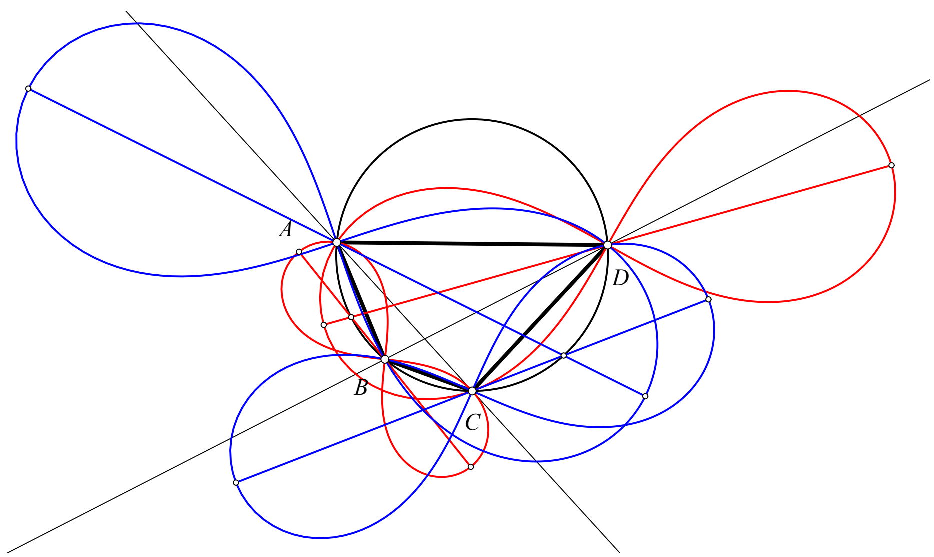

Figure 17. The four lemniscates

Point $B = f_1(A)$, point C is the symmetric of A with respect to M and point D is the symmetric of B with respect to N.

(3) The symbol $q = NZ_1MZ_2$ for the harmonic quadrilateral sets a certain order on its vertices. In the resulting construction of the cyclic quadrilateral $p = ABCD$ it is assumed that Z_1, Z_2 play the role of the similarity centers and M, N are the midpoints of the diagonals. Interchanging these roles, changes also the related cyclic quadrilaterals. Thus, giving q without an ordering for its vertices, produces two families of cyclic quadrilaterals, depending on how we interpret its two pairs of opposite vertices. Figure 17 shows the two pairs of lemniscates

corresponding to the two interpretations of the opposite vertices of the harmonic quadrilateral $q = ABCD$. All cyclic quadrilaterals having q for their associated harmonic, have their vertices on these lemniscates.

7. The associated cyclic quadrilateral

Starting with an arbitrary convex quadrilateral $p = ABCD$ with intersections of opposite sides F and G, we can, through the intermediate construction of its associated harmonic, pass to a natural *associated cyclic* quadrilateral $p' = A'B'C'D'$. In fact, consider the associated harmonic $q = Z_1NZ_2M$ of p and from this, con-

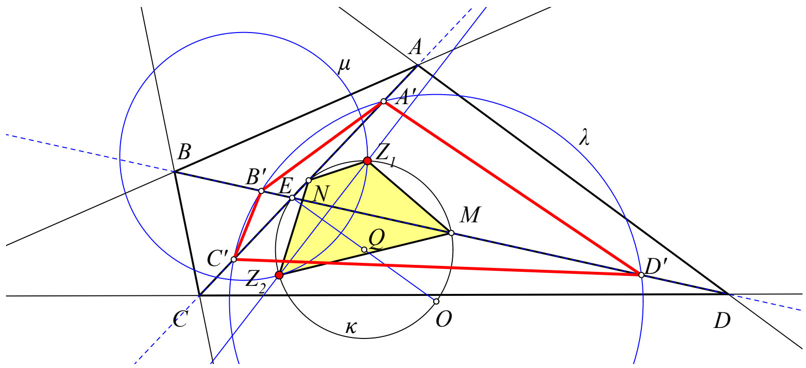

Figure 18. Quadrilateral $p = ABCD$ and its associated cyclic $p' = A'B'C'D'$

struct, following the recipe of Theorem 8, the corresponding cyclic $p' = A'B'C'D'$ (see Figure 18). From its definition, p' has the same harmonic associated q with p. Further it is easy to see that $|AA'| = |CC'|, |BB'| = |DD'|$ and the ratio $\frac{|AA'|}{|BB'|} = \frac{|AC|}{|BD|}$(see Figure 18). If one of the intersection points F, G of the opposite sides is at infinity then p is a trapezium and the corresponding harmonic quadrilateral has one of the similarity centers (Z_1) coinciding with the intersection E of its diagonals. Excluding this case, the procedure described above can be reversed. Starting from the convex cyclic quadrilateral $p' = A'B'C'D'$ and taking on its diagonals segments

$$|AA'| = |CC'|, |BB'| = |DD'| \qquad \text{in ratio} \qquad \frac{|AA'|}{|BB'|} = \frac{|A'C'|}{|B'D'|},$$

we obtain quadrilaterals $p = ABCD$ with the same associated harmonic quadrilateral. This gives an alternative construction of the one exposed in §3. In the excluded case of trapezia $p = ABCD$, the result is different and the procedure must be slightly modified. In fact, in this case there is no proper associated cyclic quadrilateral, the corresponding construction leading to a degenerate cyclic quadrilateral, which coincides with a triangle $Z_1C''D''$ (see Figure 19). In this case the quadrilaterals $p' = A'B'C'D'$, having the same associated harmonic quadrilateral $q = NZ_1MZ_2$ with p are also trapezia and are obtained by taking an arbitrary point A' on Z_1N, on the other halfline than N and projecting it parallel to MN

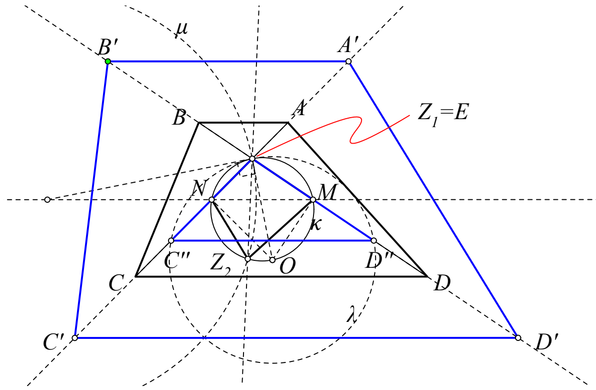

Figure 19. For trapezia the associated cyclic degenerates to a triangle

onto B' on Z_1M. Then taking, respectively, the symmetrics, C', D' with respect to N and M.

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Dynamics of the Nested Triangles Formed by the Tops of the Perpendicular Bisectors

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Abstract. Given a triangle, we construct a new triangle by taking as vertices the tops of the interior perpendicular bisectors. We describe the dynamics of this transformation exhaustively up to similarity. An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic and amounts to a surjective tent map of the interval. An obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.

1. Introduction

The midpoints of the sides of a triangle ABC are the vertices of the *medial* triangle, which is obtained from ABC by a homothety of ratio $-1/2$ about the centroid. By iterating this transformation, one obtains a sequence of directly similar nested triangles that converges to the common centroid. The feet of the medians and of the perpendicular bisectors generate thus a boring sequence! The feet of the angle bisectors are more interesting: the iterated transformation produces a sequence of nested triangles that always converges to an equilateral shape, as shown in [8] for isosceles initial triangles and in [2] for the general case (in 2006). In the 1990s, four papers [3, 4, 9, 1] analyzed the *pedal* or *orthic* sequence defined by the feet of the altitudes: Peter Lax [4] proved the ergodicity of the construction. We considered in [5] reflection triangles and their iterates: the vertices of the new triangle are obtained by reflecting each vertex of ABC in the opposite side. We were able to decrypt the complex fractal structure of this mapping completely: the sequences generated by acute or right-angled triangles behave nicely, as they always converge to an equilateral shape.

The tops of the medians, angle bisectors, and altitudes are the vertices of the original triangle. But what about the tops of the interior perpendicular bisectors of ABC as new vertices? We found no trace of this problem in the literature. The solution presented here offers an elementary and concrete approach to chaos and requires almost no calculations. Note that another kind of triangle and polygon transformations have a long and rich history, those given by circulant linear combinations of the old vertices, like Napoleon's configuration (see the references in

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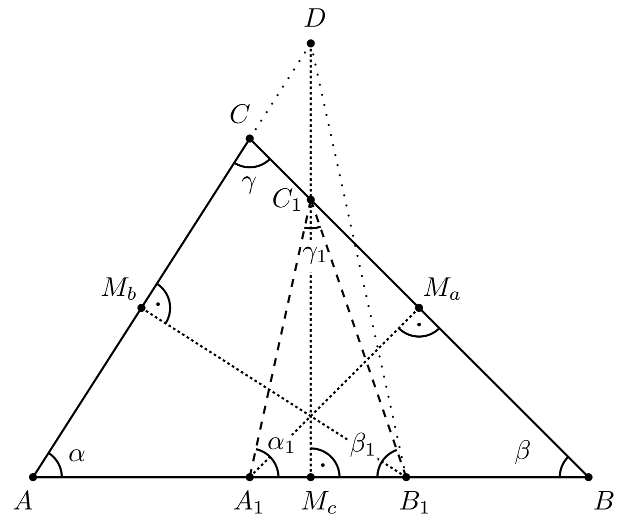

Figure 1. Triangle $A_1B_1C_1$ of the tops of the interior perpendicular bisectors

[6]): their dynamics can be best described by using convolution products and a shape function relying on the discrete Fourier transform [6, 7].

2. Triangle of the tops of the perpendicular bisectors

Let $\Delta = ABC$ be a triangle with corresponding angles α, β, γ and opposite sides a, b, c. The interior perpendicular bisectors issued from the side midpoints M_a, M_b, and M_c end at A_1, B_1, and C_1, respectively (Figure 1). The triangle $\Delta_1 = A_1B_1C_1 = T(\Delta)$ is the *child* of Δ, and Δ a *parent* of Δ_1. We denote the mth iterate of the transformation T by T^m, $m \in \mathbf{Z}$. We are interested in the *shape* of the descendants and ancestors of Δ, *i.e.*, in their angles. In this paper, we provide an exhaustive description of the dynamics of T with respect to shape.

We consider only two types of degenerate triangles, the "isosceles" ones: we assign angles $0°$, $0°$, $180°$ to every nontrivial segment with midpoint and angles $90°$, $90°$, $0°$ to every nontrivial segment with one double endpoint. A nondegenerate triangle is *proper*. We identify the shape of a triangle Δ with the point of the set

$$\mathcal{S} = \left\{(\alpha, \beta) \mid 0° < \beta \leq \alpha \leq 90° - \beta/2\right\} \cup \{(0°, 0°), (90°, 0°)\}$$

given by the two smallest angles of Δ (Figure 2). $\mathcal{S}$ is the disjoint union of the subsets $\mathcal{O}$, $\mathcal{R}$, and $\mathcal{A}$ of the obtuse, right-angled, and acute shapes, respectively. We denote the shape of an isosceles triangle with equal angles α by I_α, $0° \leq \alpha \leq 90°$. The shapes of the isosceles triangles form the *roof* of $\mathcal{S}$, whose top is the equilateral shape $I_{60°}$. The transformation T induces a transformation τ of $\mathcal{S}$. We set $\tau(I_{90°}) = I_{90°}$ by definition. The children of right-angled triangles are degenerate with two vertices at the midpoint of the hypotenuse: τ maps the whole segment $\mathcal{R}$ to $I_{90°}$. There are no other proper triangles with a degenerate child.

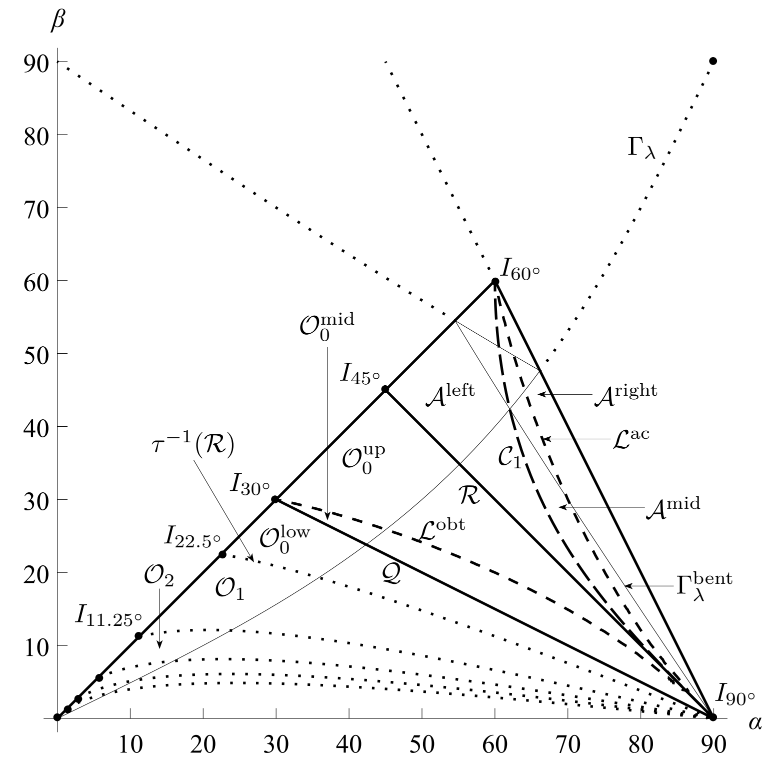

Figure 2. Set $\mathcal{S}$ of the triangle shapes and a curve Γ_λ with corresponding $\Gamma_\lambda^{\mathrm{bent}}$

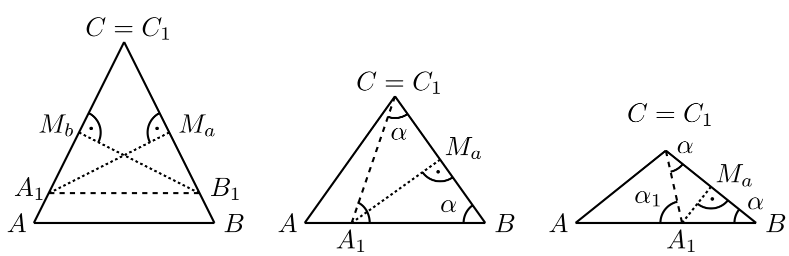

Figure 3. Isosceles parent triangles with angles $\alpha = \beta$ for $\alpha > 60^\circ$, $45^\circ < \alpha < 60^\circ$, and $\alpha < 45^\circ$

The roof is invariant under τ (Figure 3). I_{0° and the points of the right roof side are fixed points, and

$$\tau(I_\alpha) = \begin{cases} I_{2\alpha} & \text{if } 0^\circ \le \alpha \le 45^\circ \\ I_{180^\circ - 2\alpha} & \text{if } 45^\circ \le \alpha \le 60^\circ \\ I_\alpha & \text{if } 60^\circ \le \alpha \le 90^\circ \end{cases} .$$

When α grows, $\tau(I_\alpha)$ travels on the roof as follows: from I_{0° to I_{90° for $0^\circ \leq \alpha \leq 45^\circ$, then back to I_{60° for $45^\circ \leq \alpha \leq 60^\circ$ before descending the right roof side for $60^\circ \leq \alpha \leq 90^\circ$. Each shape I_α has thus one, two, or three isosceles parents according as it lies on the left roof side, at I_{60° or I_{90°, or on the rest of the right roof side, respectively. For $60^\circ < \alpha < 90^\circ$, the three isosceles parents of I_α are itself, the point where the parallel to $\mathcal{R}$ through I_α cuts the left roof side, and the reflection of this parent in the line of $\mathcal{R}$. An equilateral triangle has four isosceles parents: itself and three of shape I_{30°. I_α degenerates eventually to I_{90° (and this after m steps) if and only if $\alpha = 90^\circ/2^m$ for some integer $m \geq 0$. Otherwise, $I_\alpha \neq I_{0^\circ}$ reaches its final nondegenerate state $I_{2^n\alpha}$ or $I_{180^\circ-2^n\alpha}$ after n steps, where $n \geq 0$ is given by $30^\circ/2^{n-1} \leq \alpha < 45^\circ/2^{n-1}$ in the first and by $45^\circ/2^{n-1} < \alpha < 60^\circ/2^{n-1}$ in the second case.

We consider the tangents of the angles of Δ:

$$u = \tan\alpha, \quad v = \tan\beta, \quad w = \tan\gamma = \frac{u+v}{uv-1}.$$

We write $\tan(\alpha,\beta)$ for $(\tan\alpha,\tan\beta)$. In the (u,v)-plane, the transformation induced by τ is essentially the radial stretch $(u,v) \mapsto \dfrac{|w|}{u}(u,v)$. More precisely, we have the following result.

Theorem 1. *When (α,β) is the shape of the proper obtuse or acute triangle Δ, the angles α_1 and β_1 of $T(\Delta)$ are acute with $\alpha_1 \geq \beta_1$ and $\tan(\alpha_1,\beta_1) = \dfrac{|w|}{u}(u,v)$. According as the parent is obtuse or acute, its child has a smaller or the same largest angle.*

Proof. We first look at the acute case $90^\circ > \gamma \geq \alpha \geq \beta > 0^\circ$ (Figure 1). Let D be the intersection of side b with the perpendicular bisector of c. By Thales' theorem, $A_1M_cM_aC_1$ and $M_cB_1DM_b$ are convex cyclic quadrilaterals with circumcircles of diameters A_1C_1 and B_1D, respectively. One has thus $\alpha_1 = \angle M_cM_aB = \gamma = \angle M_cB_1D$ and

$$\tan\beta_1 = \frac{C_1M_c}{M_cB_1} = \frac{C_1M_c}{c/2} \cdot \frac{c/2}{DM_c} \cdot \frac{DM_c}{M_cB_1} = \frac{\tan\beta}{\tan\alpha}\tan\gamma.$$

Since $\alpha_1 = \gamma$, one has $\gamma_1 \leq \angle A_1C_1B = \alpha \leq \gamma$.

In the obtuse case $\gamma > 90^\circ > \alpha \geq \beta > 0^\circ$, A_1 lies on the right of M_c and B_1 on the left. One considers the quadrilaterals $M_cA_1M_aC_1$ and $B_1M_cDM_b$ to get $\alpha_1 = \angle M_cM_aC_1 = 180^\circ - \gamma = \angle M_cB_1D$ and $\tan\beta_1 = \tan\beta \cdot \tan(180^\circ - \gamma)/\tan\alpha$. One has $\gamma_1 = 180^\circ - (\alpha_1 + \beta_1) = \gamma - \beta_1 < \gamma$. □

An acute shape and all its descendants lie thus on a parallel to the line of the right-angled shapes: for every $\varphi \in (0^\circ, 15^\circ]$, the segment of acute shapes $\mathcal{P}_\varphi^{\mathrm{ac}}$ parallel to $\mathcal{R}$ from $I_{45^\circ+\varphi}$ to $I_{90^\circ-2\varphi}$ is invariant under τ (Figure 4).

The shape of an obtuse or right-angled child $\tau(\alpha,\beta)$ is (α_1,β_1) since $\gamma_1 > \alpha_1$ (with equality for $(\alpha,\beta) = I_{90^\circ}$). The shape of an acute child is (α_1,β_1), (γ_1,β_1), or (β_1,γ_1) since $\alpha_1 \geq \beta_1$. We describe below the conditions of each occurrence.

Note that (γ_1, β_1) is the horizontal reflection of (α_1, β_1) in the line of the right roof side and that (β_1, γ_1) is the reflection of (γ_1, β_1) in the line of the left roof side.

Theorem 2. *The acute shape (α, β) and its reflection in the line of $\mathcal{R}$, the obtuse shape $(90° - \beta, 90° - \alpha)$, share the same child.*

Proof. The tangents of the shapes are (u, v) and $(u', v') = (1/v, 1/u)$. Since $w' = -w$ and $v'/u' = v/u$, the children have the same angles. $\square$

As a consequence, the segment $\mathcal{Q}$ of obtuse shapes joining $I_{30°}$ and $I_{90°}$ is reflected in $\mathcal{R}$ by τ to the right roof side consisting of fixed points (Figures 2 and 4). (When an isosceles child triangle has a scalene parent, note that a reflection in the child's axis gives a second parent triangle.) And for $\varphi \in (0°, 15°]$, the action of τ on the segment $\mathcal{P}_\varphi^{\text{obt}}$ of obtuse shapes parallel to $\mathcal{R}$ from $I_{45°-\varphi}$ to $(90° - 4\varphi, 2\varphi) \in \mathcal{Q}$ is the reflection to $\mathcal{P}_\varphi^{\text{ac}}$ followed by τ: since $\mathcal{P}_\varphi^{\text{ac}}$ is invariant under τ, τ maps $\mathcal{P}_\varphi^{\text{obt}}$ to $\mathcal{P}_\varphi^{\text{ac}}$.

3. Dynamics of the transformation

Consider a slope λ with $0 < \lambda \leq 1$. On the segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, $|w|$ is given by $(1 + \lambda)u/(1 - \lambda u^2)$ and is a strictly growing convex function of u with image $[0, +\infty)$. The segment $v = \lambda u$, $0 \leq u < \sqrt{1/\lambda}$, is thus stretched bijectively and continuously to the whole half-line $v = \lambda u$, $u \geq 0$, by the map $(u, v) \mapsto \dfrac{|w|}{u}(u, v)$, whose only fixed point is the origin. The strictly growing curves Γ_λ corresponding to these segments are given by

$$\Gamma_\lambda\colon \quad \beta = \arctan(\lambda \tan \alpha), \quad 0° \leq \alpha \leq 90°,\ 0 < \lambda \leq 1,$$

which is $\beta = \alpha$ for $\lambda = 1$: they link the origin $I_{0°}$ and the point $(90°, 90°)$, are symmetric in the line of $\mathcal{R}$, and provide a simple covering of $\{(\alpha, \beta) \mid 0° < \beta \leq \alpha < 90°\}$ (Figures 2 and 4). The lower parts $\Gamma_\lambda^{\text{obt}}$ of the Γ_λ's for $0° \leq \alpha \leq \arctan\sqrt{1/\lambda}$ cover the set of the obtuse and proper right-angled shapes. The middle parts $\Gamma_\lambda^{\text{ac}}$ for $\arctan\sqrt{1/\lambda} < \alpha \leq \arctan\sqrt{1 + 2/\lambda}$ cover the set of the acute shapes. By Theorem 1, the transformation τ stretches each $\Gamma_\lambda^{\text{obt}}$ to the doubly bent curve $\Gamma_\lambda^{\text{bent}}$ linking $I_{0°}$ and $I_{90°}$ (Figure 2): $\Gamma_\lambda^{\text{bent}}$ is obtained from Γ_λ by a horizontal reflection of its upper part in the line of the right roof side followed by a partial reflection in the left roof side. By Theorem 2, τ maps each path $\Gamma_\lambda^{\text{ac}}$ (joining a right-angled shape and an isosceles fixed point of τ) to the way back along $\Gamma_\lambda^{\text{bent}}$ from $I_{90°}$ to the right roof side.

τ provides thus a simple covering of $\mathcal{O} \cup \mathcal{R} \setminus \{I_{90°}\}$ (by obtuse shapes exclusively), a fivefold covering of $\mathcal{A}$ without roof, a triple covering of the section $\{I_\alpha \mid 45° < \alpha < 60°\}$ of the left roof side, a double covering of $I_{60°}$ and a quadruple covering of the right roof side without endpoints. The covering of $\mathcal{A}$ without roof consists of three layers of obtuse and two layers of acute shapes. More precisely, each region $\mathcal{O}_n \subset \mathcal{O}$ bordered by the curve $\tau^{-n}(\mathcal{R})$ and its parent curve $\tau^{-n-1}(\mathcal{R})$ is mapped by τ bijectively and upwardly (along the curves Γ_λ) to the next less obtuse region $\mathcal{O}_{n-1}$ for all integers $n \geq 1$ (Figure 2). The curve

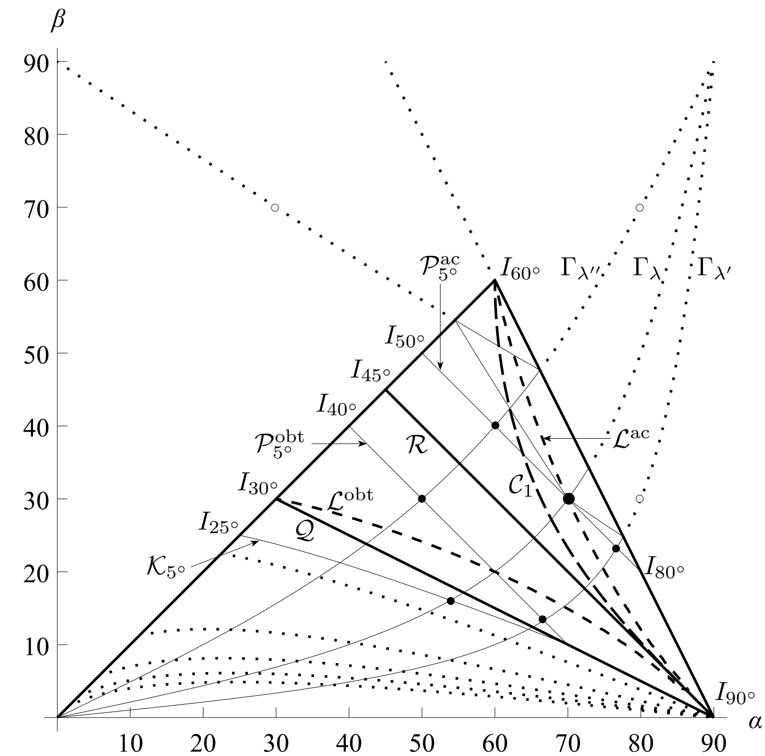

Figure 4. The five parents of the acute shape $(70^\circ, 30^\circ) \in \mathcal{L}^{\mathrm{ac}}$ and the parent curves of the segment $\mathcal{P}^{\mathrm{ac}}_{5^\circ}$

$\tau^{-n}(\mathcal{R})$, $n \geq 0$, joins $I_{45^\circ/2^n}$ and I_{90°. Every neighborhood of the α-axis contains all but finitely many ancestor regions of $\mathcal{O}_0$, the region of the *slightly obtuse shapes*. $\mathcal{O}_0 \setminus \{I_{30^\circ}\}$ is the disjoint union of three layers of the fivefold covering of $\mathcal{A}$ without roof and of a double covering of the acute roof section without I_{60° and I_{90°. Each of the following three subregions of $\mathcal{O}_0$ is mapped bijectively to $\mathcal{A}$ without roof (along the three smooth sections of the curves $\Gamma^{\text{bent}}_{\lambda}$) : the lower subregion $\mathcal{O}_0^{\text{low}}$, the middle lens $\mathcal{O}_0^{\text{mid}}$, and the upper subregion $\mathcal{O}_0^{\text{up}}$ delimited by the curves $\tau^{-1}(\mathcal{R})$, $\mathcal{Q}$, $\mathcal{L}^{\text{obt}}$, and $\mathcal{R}$. The curve $\mathcal{L}^{\text{obt}}$ consists of the obtuse parent shapes mapped to a bend point of some curve $\Gamma^{\text{bent}}_{\lambda}$ on the left roof side. The reflections of $\mathcal{O}_0^{\text{mid}}$ and $\mathcal{O}_0^{\text{up}}$ in the line of $\mathcal{R}$, separated by the curve $\mathcal{L}^{\text{ac}}$, constitute the remaining two layers of acute shapes in the covering of $\mathcal{A}$.

The shape of an acute child $\tau(\alpha, \beta)$ is (α_1, β_1) if the parent (α, β) lies in $\mathcal{O}_0^{\text{low}}$, (γ_1, β_1) if the parent is in $\mathcal{O}_0^{\text{mid}}$ or $\mathcal{A}^{\text{right}}$, and (β_1, γ_1) if the parent is located between $\mathcal{L}^{\text{obt}}$ and $\mathcal{L}^{\text{ac}}$ (Figure 2).

The bijective stretch

$$p^{-1}\colon \{(u,v) \mid u \geq v > 0, uv < 1\} \to \{(u,v) \mid u \geq v > 0\},\ (u,v) \mapsto \frac{|w|}{u}(u,v)$$

is the inverse function of

$$p(u,v) = \frac{\sqrt{(u+v)^2 + 4u^3 v} - (u+v)}{2u^2 v}(u,v).$$

One parent (α,β) of the proper shape (α_1,β_1) is thus the obtuse shape given by $\tan(\alpha,\beta) = p(\tan(\alpha_1,\beta_1))$ (Figure 4). When (α_1,β_1) is an acute shape, the parent (α,β) lies in $\mathcal{O}_0^{\text{low}}$ and the other parents are the obtuse shapes (α',β'), (α'',β'') given by $\tan(\alpha',\beta') = p(\tan(\gamma_1,\beta_1))$, $\tan(\alpha'',\beta'') = p(\tan(\gamma_1,\alpha_1))$ and their acute reflections $(90^\circ - \beta', 90^\circ - \alpha')$ and $(90^\circ - \beta'', 90^\circ - \alpha'')$ on $\mathcal{P}^{\text{ac}}_{(\alpha_1+\beta_1-90^\circ)/2}$. Except (α,β), the parents are the vertices of a rectangle. The parent (α,β) and (α_1,β_1) lie on the same curve Γ_λ ($\lambda = \tan\beta_1/\tan\alpha_1$), whereas (α',β') and (α'',β'') reach their child by traveling on bent curves $\Gamma^{\text{bent}}_{\lambda'}$ and $\Gamma^{\text{bent}}_{\lambda''}$ ($\lambda' = \tan\beta_1/\tan\gamma_1$, $\lambda'' = \tan\alpha_1/\tan\gamma_1$), after the first bend for $(\alpha',\beta') \in \mathcal{O}_0^{\text{mid}}$ and the second for $(\alpha'',\beta'') \in \mathcal{O}_0^{\text{up}}$.

Each acute nonequilateral triangle has exactly five differently placed parents: by axial symmetry, an isosceles child has one or two pairs of inversely congruent scalene parents according as its equal angles are larger or smaller than 60°. We already mentioned the four parents of an equilateral triangle: itself and three of shape I_{30°. The formula for $p(u,v)$ shows that the parents of a given proper triangle are all constructible by straightedge and compass.

The parents (α,β) of the proper right-angled shapes (α_1,β_1) with $\tan(\alpha_1,\beta_1) = (u_1,v_1)$ are the obtuse solutions of $u_1 v_1 = 1$ and constitute thus the curve

$$\tau^{-1}(\mathcal{R}) \setminus \{I_{90^\circ}\}\colon \quad \tan^2(\alpha+\beta) = \tan\alpha/\tan\beta, \quad 0^\circ < \beta \le \alpha,\ \alpha+\beta < 90^\circ$$

(Figure 2). Since $u_1/v_1 = u/v$, the proper shapes of $\tau^{-2}(\mathcal{R})$ are the shapes (α,β) below $\tau^{-1}(\mathcal{R})$ with $\tan^2(\alpha_1+\beta_1) = \tan\alpha/\tan\beta$, and so on. For each integer $n \ge 1$, a parametric representation of the curve $\tau^{-n}(\mathcal{R})$ without I_{90° is given by $\tan(\alpha,\beta) = p^n(\hat{u}, 1/\hat{u})$ with $\tan\beta = \hat{u}^{-2}\tan\alpha$ and $(\hat{u}, 1/\hat{u}) = \tan(\hat{\alpha}, 90^\circ - \hat{\alpha})$, $45^\circ \le \hat{\alpha} < 90^\circ$, where p^n denotes the nth iterate of p. The maximal elevation of $\tau^{-n}(\mathcal{R})$ seems to be approximately $23^\circ/n$.

Because the origin is the unique fixed point of the stretch $(u,v) \mapsto \dfrac{|w|}{u}(u,v)$ in the set $\{(u,v) \mid u \ge v > 0, uv < 1\}$, the only possible fixed point of τ on $\Gamma_\lambda \cap \mathcal{S}$ (besides the endpoints) is the double point of $\Gamma^{\text{bent}}_\lambda$ (Figure 2). But this acute double point is the unique point of $\Gamma^{\text{bent}}_\lambda$ on its parallel to $\mathcal{R}$ and is forced to be its own child. The curve $\mathcal{C}_1$ of the nonisosceles fixed shapes joins I_{60° and I_{90° in $\mathcal{A}$ and lies below $\mathcal{L}^{\text{ac}}$: the shapes $(\alpha,\beta) \in \mathcal{C}_1$ are thus the solutions of the equation $(\alpha,\beta) = \tau(\alpha,\beta) = (\beta_1,\gamma_1)$, *i.e.*, $u = v_1$, which can be transformed into $v^2 + (u - u^3)v + u^2 = 0$. The solution that corresponds to shapes is

$$\mathcal{C}_1\colon \quad v = \frac{u}{2}\left(u^2 - 1 - \sqrt{(u^2-1)^2 - 4}\right), \quad u = \tan\alpha,\ v = \tan\beta,\ u \ge \sqrt{3}.$$

When the shape (α,β) of Δ lies on $\mathcal{C}_1$, $T(\Delta)$ is *directly* similar to Δ, since $\Delta = ABC$ and $A_1B_1C_1$ are then equally oriented with $\gamma = \alpha_1 \ge \alpha = \beta_1 \ge \beta = \gamma_1$.

In general, one finds the child of the acute shape (α, β) as follows. Draw the curve Γ_λ through (α, β) by taking $\lambda = \tan\beta/\tan\alpha$. The child is the point where the parallel to $\mathcal{R}$ through (α, β) cuts one of the two bent sections of $\Gamma_\lambda^{\text{bent}}$ (Figure 2).

A proper scalene shape (α, β) on the curve $\mathcal{L}^{\text{obt}}$, which links I_{90° and I_{30°, is the obtuse parent of some $I_{\hat\alpha}$, $45^\circ < \hat\alpha \le 60^\circ$. $I_{\hat\alpha} = (\hat\alpha, \hat\alpha)$ is the bend of $\Gamma_{\hat\lambda}^{\text{bent}}$ on the left roof side. Before bending, $I_{\hat\alpha}$ was the point $(180^\circ - 2\hat\alpha, \hat\alpha)$ located on the curve $\Gamma_{\hat\lambda}$, like its parent (α, β). One has thus

$$\hat\lambda = \tan\hat\alpha/\tan(180^\circ - 2\hat\alpha) = (\tan^2\hat\alpha - 1)/2$$

and $\tan(\alpha, \beta) = (u, v) = (u, \hat\lambda u)$ with $w < 0$. The condition $\tau(\alpha, \beta) = I_{\hat\alpha}$ is equivalent to $w_1 = v_1$, since the two *smaller* angles of $T(\Delta)$ have to be equal. The equation $w_1 = v_1$ can be transformed into $1 + u/v = \sqrt{1 + w^2}$ and further, since $v = \hat\lambda u$ and $uv < 1$, into

$$\hat\lambda \tan\hat\alpha\, u^2 + \hat\lambda(\hat\lambda + 1)u - \tan\hat\alpha = 0.$$

The positive solution u gives

$$\mathcal{L}^{\text{obt}}: \quad u = \tan\alpha = \sqrt{\frac{(\tan\hat\alpha + \cot\hat\alpha)^2}{16} + \frac{2}{\tan^2\hat\alpha - 1}} - \frac{\tan\hat\alpha + \cot\hat\alpha}{4},$$
$$\beta = 180^\circ - \alpha - 2\hat\alpha, \quad 45^\circ < \hat\alpha \le 60^\circ.$$

One obtains the parametric representation of $\mathcal{L}^{\text{ac}}$ by reflection in $\mathcal{R}$ or (since now $uv > 1$) from the equation $\hat\lambda \tan\hat\alpha\, u^2 - \hat\lambda(\hat\lambda + 1)u - \tan\hat\alpha = 0$, which leads to

$$\mathcal{L}^{\text{ac}}: \quad \tan\alpha = \sqrt{\frac{(\tan\hat\alpha + \cot\hat\alpha)^2}{16} + \frac{2}{\tan^2\hat\alpha - 1}} + \frac{\tan\hat\alpha + \cot\hat\alpha}{4},$$
$$\beta = 2\hat\alpha - \alpha, \quad 45^\circ < \hat\alpha \le 60^\circ.$$

The segment $\mathcal{P}_\varphi^{\text{ac}}$ parallel to $\mathcal{R}$ from $I_{45^\circ+\varphi}$ to $I_{90^\circ-2\varphi}$, $0^\circ < \varphi < 15^\circ$, has five parent curves (Figure 4): $\mathcal{P}_\varphi^{\text{ac}}$ is doubly covered by itself under τ, once by the portion between the left roof side and $\mathcal{L}^{\text{ac}}$, once by the rest. $\mathcal{P}_\varphi^{\text{ac}}$ is covered twice by $\mathcal{P}_\varphi^{\text{obt}}$ through reflection followed by τ. And the curve

$$\mathcal{K}_\varphi: \quad \tan(\alpha, \beta) = p(\tan(\widetilde\alpha, 90^\circ + 2\varphi - \widetilde\alpha)), \quad 45^\circ + \varphi \le \widetilde\alpha \le 90^\circ - 2\varphi,$$

formed by the parents in $\mathcal{O}_0^{\text{low}}$ of some shape of $\mathcal{P}_\varphi^{\text{ac}}$ is mapped bijectively onto $\mathcal{P}_\varphi^{\text{ac}}$ (along the Γ_λ's). $\mathcal{K}_\varphi$ links $I_{22.5^\circ+\varphi/2}$ and the end shape $(90^\circ - 4\varphi, 2\varphi)$ of $\mathcal{P}_\varphi^{\text{obt}}$. The region $\mathcal{O}_0 \setminus \{I_{30^\circ}\}$ is the disjoint union of the nested pointed arches $\mathcal{K}_\varphi \cup \mathcal{P}_\varphi^{\text{obt}}$, $0^\circ < \varphi < 15^\circ$. Such an arch covers $\mathcal{P}_\varphi^{\text{ac}}$ three times under τ: down, and up, and down again.

When restricted to $\mathcal{P}_\varphi^{\text{ac}}$, the transformation τ amounts to the surjective tent map $t \mapsto 2\min(t, 1 - t)$ of the unit interval and is thus ergodic once the appropriate measure has been defined (see Section 4). $\mathcal{P}_\varphi^{\text{ac}}$ is first folded by τ about the point on $\mathcal{L}^{\text{ac}}$ in such a way that its left endpoint coincides with the fixed right endpoint (Figure 4). Both halves, being held firmly at their common right end, are then stretched to the left until each of them covers simply $\mathcal{P}_\varphi^{\text{ac}}$. The subregions of acute shapes $\mathcal{A}^{\text{left}}$, $\mathcal{A}^{\text{mid}}$, and $\mathcal{A}^{\text{right}}$ delimited in $\mathcal{A}$ by the curves $\mathcal{R}$, $\mathcal{C}_1$, and $\mathcal{L}^{\text{ac}}$ (Figure 2) are thus transformed in the following way along the parallels to $\mathcal{R}$: $\mathcal{A}^{\text{left}}$ is mapped bijectively to $\mathcal{A}^{\text{mid}} \cup \mathcal{A}^{\text{right}}$, $\mathcal{A}^{\text{mid}}$ to $\mathcal{A}^{\text{left}}$, and $\mathcal{A}^{\text{right}}$ to $\mathcal{A}$, giving two acute parents to every acute shape not lying on the left roof side. According as the acute shape is on the right or left of $\mathcal{C}_1$, it is located between its acute parents or on their left, respectively (Figure 4).

Take a unit circle and draw a horizontal chord AB below the diameter at distance $\sin 2\varphi$, $0^\circ < \varphi < 15^\circ$. The acute shapes (α, β) of the segment $\mathcal{P}_\varphi^{\text{ac}}$ are then represented by the inscribed triangles ABC for which C lies between the north pole and the limit position $C_{\lim}$ where $\alpha = \gamma = 90^\circ - 2\varphi$. τ moves C on this arc.

The 120 integer-angled shapes I_{n° on the roof and $(2m^\circ, (45-m)^\circ)$ on $\mathcal{Q}$ have integer-angled children. A systematic search shows that there are only eight further such cases: $\tau(36^\circ, 12^\circ) = (48^\circ, 18^\circ)$, $\tau(42^\circ, 12^\circ) = (54^\circ, 18^\circ)$, $\tau(66^\circ, 18^\circ) = \tau(72^\circ, 24^\circ) = (54^\circ, 42^\circ)$, $\tau(50^\circ, 30^\circ) = \tau(60^\circ, 40^\circ) = (70^\circ, 30^\circ)$, $\tau(70^\circ, 30^\circ) = \tau(60^\circ, 20^\circ) = I_{50^\circ}$. These cases verify the identity

$$\tan(60^\circ - \delta)\tan(60^\circ + \delta)\tan\delta = \tan 3\delta$$

for $\delta = 12^\circ$, 18°, 6°, 20°, and 10° in order: $\tan 12^\circ \tan 48^\circ / \tan 36^\circ = \tan 18^\circ$ and so on.

4. The transformation as symbolic dynamics

We use symbolic dynamics to give a short and elementary proof of the ergodicity of τ on every segment $\mathcal{P}_\varphi^{\text{ac}}$. We identify $\mathcal{P}_\varphi^{\text{ac}}$, $0^\circ < \varphi < 15^\circ$, with the interval $[0,1]$, where $0 = I_{90^\circ - 2\varphi}$ and $1 = I_{45^\circ + \varphi}$ are the isosceles endpoints on the *right* and left roof sides, respectively (Figure 4). We represent each shape $s \in \mathcal{P}_\varphi^{\text{ac}}$ by its infinite binary address $x = x_1 x_2 x_3 \dots$ giving the position of s with respect to the fractal subdivision of $\mathcal{P}_\varphi^{\text{ac}}$ induced by the monotonicity intervals of τ and its iterates. The kth digit of the address is 0 or 1 according as τ^k restricted to $\mathcal{P}_\varphi^{\text{ac}}$ is direction-preserving or reversing in a neighborhood of the shape (and addresses of turning points end in a constant sequence of 0s or 1s). If x is eventually periodic, we overline the period's digits. We identify the ends $0\overline{1}$ and $1\overline{0}$. The child $\tau(s)$ is then given by a left shift when $x_1 = 0$ and a left shift with permutation $0 \leftrightarrow 1$ in x when $x_1 = 1$. Note that $\tau^n(x) = x_{n+1}\dots$ or $\tau^n(x) = (x_{n+1}\dots)_{0\leftrightarrow 1}$ according as $x_1 \dots x_n$ contains an even or odd number of 1s.

The shape on $\mathcal{L}^{\text{ac}}$ is $1/2 = 0\overline{1}$ and the nonisosceles fixed point is $2/3 = \overline{10}$. The parents of x are $0x$ and $1x_{0\leftrightarrow 1}$ (these are the *acute* parents of the shape). The only ancestors of $\overline{0}$ are the addresses ending in $\overline{0}$ or $\overline{1}$, *i.e.*, integer multiples of some 2^{-n}.

We prove that the orbit of x becomes eventually periodic if and only if the digits of x are eventually periodic. Consider first an eventually periodic sequence $x = x_1 \dots x_k \overline{p_1 \dots p_n}$: since $\{\overline{p_1 \dots p_n}, (\overline{p_1 \cdots p_n})_{0\leftrightarrow 1}\}$ contains both $\tau^{k+n}(x)$ and $\tau^{k+2n}(x)$, one of them is $\tau^k(x)$, whose orbit is thus periodic. Conversely, if the orbit of $x = x_1 x_2 \dots$ is eventually periodic with $\tau^k(x) = \tau^{k+n}(x)$, $\tau^k(x)$ is a periodic sequence given by $x_{k+1}x_{k+2}\dots$ or $(x_{k+1}x_{k+2}\dots)_{0\leftrightarrow 1}$, *i.e.*, the sequence x is eventually periodic.

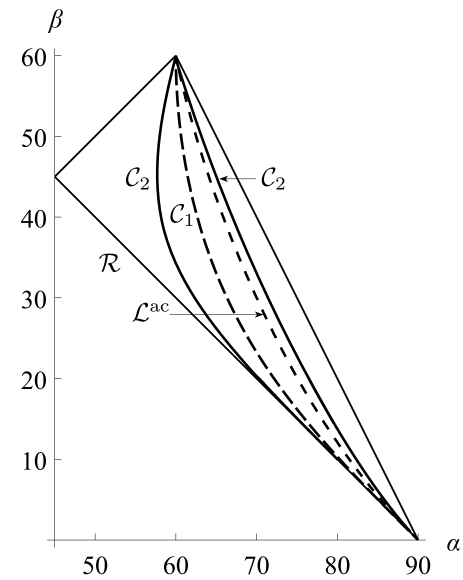

Figure 5. Curves $\mathcal{C}_2$ of the 2-cycles

The only 2-cycle is $2/5 = \overline{0110} \leftrightarrow \overline{1100} = 4/5$. Figure 5 shows the two curves $\mathcal{C}_2$ of these 2-cycles in $\mathcal{A}$. When the shape (α, β) of Δ lies on $\mathcal{C}_2$, $T^2(\Delta)$ is *inversely* similar to Δ, since $\Delta = ABC$ and $A_2B_2C_2$ are then equally oriented with $\gamma = \beta_2 \geq \alpha = \alpha_2 \geq \beta = \gamma_2$ if (α, β) is on the left of $\mathcal{L}^{\mathrm{ac}}$ and $\gamma = \gamma_2 \geq \alpha = \beta_2 \geq \beta = \alpha_2$ if (α, β) is on the right of $\mathcal{L}^{\mathrm{ac}}$. The two 3-cycles are $\overline{010} = 2/7 \mapsto \overline{100} = 4/7 \mapsto \overline{110} = 6/7 \mapsto \overline{010}$ and $00\overline{111000} = 2/9 \mapsto 4/9 \mapsto 8/9 \mapsto 2/9$. The iterated tent map τ^n, $n \geq 1$, has 2^n fixed points in $\mathcal{P}^{\mathrm{ac}}_{\varphi}$: since $2^n > 2 + 2^2 + \cdots + 2^{n-1}$ for $n \geq 2$, τ has n-cycles (of *fundamental* period n) in $\mathcal{P}^{\mathrm{ac}}_{\varphi}$ for all integers $n \geq 1$. It is easy to see that the 2^n fixed points are the fractions $2k/(2^n - 1)$, $2k/(2^n + 1)$ in $[0, 1]$ and that for $n \geq 3$ such a fixed point generates an n-cycle if and only if it cannot be written with a smaller denominator $2^m \pm 1$. One can construct addresses with almost any behavior under iteration of τ. We design for example an address x_{dense} whose forward orbit is dense in $\mathcal{P}^{\mathrm{ac}}_{\varphi}$: concatenate successively all binary words of length 1, 2, 3, and so on to an infinite sequence, and submit if necessary each of them in order to a permutation $0 \leftrightarrow 1$ in such a way that the original word appears as head of the corresponding descendant of x_{dense}.

The measure of an interval of $\mathcal{P}^{\mathrm{ac}}_{\varphi}$ of kth generation, *i.e.*, with an address of length k, is 2^{-k} by definition. τ is then measure-preserving on $\mathcal{P}^{\mathrm{ac}}_{\varphi}$. By using the binomial distribution, it is easy to see that almost all shapes of $\mathcal{P}^{\mathrm{ac}}_{\varphi}$ have a *normal* address [9]. An address x is normal if, for every fixed integer $k \geq 1$, all binary words of k digits appear with the same asymptotic frequency 2^{-k} as heads of the successive $\tau^n(x)$, $n \in \mathbf{N}$. The descendants of a normal address visit thus all intervals of kth generation with equal asymptotic frequency, and this for every k:

the forward orbit of almost every shape of $\mathcal{P}_\varphi^{\mathrm{ac}}$ "goes everywhere in $\mathcal{P}_\varphi^{\mathrm{ac}}$ equally often", τ is an ergodic transformation of $\mathcal{P}_\varphi^{\mathrm{ac}}$.

We conclude by formulating a condensed version of our results.

Theorem 3. *Consider the map that transforms a triangle into the triangle of the tops of its interior perpendicular bisectors.*
(1) *An acute initial triangle generates a sequence with constant largest angle: except for the equilateral case, the transformation is then ergodic in shape and amounts to a surjective tent map of the interval. A proper obtuse initial triangle either becomes acute or degenerates by reaching a right-angled state.*
(2) *A proper triangle is the transform of exactly one, four or five triangles according as it is nonacute, equilateral, or acute but not equilateral, respectively.*

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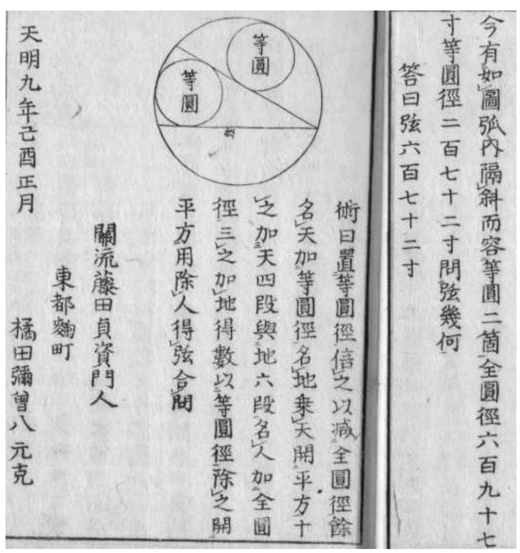
Kitta's Double-Locked Problem

J. Marshall Unger

Abstract. I present strongly contrasting solutions of a remarkable *sangaku*. The general solution requires Fujita's celebrated mixtilinear circle theorem. The data-specific solution shows the premodern Japanese preoccupation with Pythagorean triples.

1. The Problem

Shinpeki sanpō [2] is a collection of problems that FUJITA Sadasuke ordered his son Yoshitoki to compile to show off the prowess of his disciples, although problems of enthusiasts from other schools were also included. The problem in question ([2, 1.42-3]), which was not discussed in [3], [4] or [5], is reproduced in Figure 1 followed by my translation.



今有如圖弧内隔斜而容等圓二箇全圓徑六百九十七
寸等圓徑二百七十二寸問弦幾何
答曰弦六百七十二寸
術曰置等圓徑倍之以減全圓徑餘
名天加等圓徑名地乘天開平方十
之加天四段與地六段名人加全圓
徑三之加地得數以等圓徑除之開
平方用除人得弦合問
關流藤田貞資門人
東都麹町
橋田彌曾八元克
天明九年己酉正月

Figure 1.

A circular segment is split by a line such that its resulting parts contain two congruent circles as shown in the figure. The diameter of the large circle is 697 inches; the diameters of the congruent circles are 272 inches each. What is the length of the chord of the segment?

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> The answer is 672 inches.
> The method is as follows: Substract twice the small diameter (d) from the large diameter (D). Call the difference heaven (h). Add this to d. Call this earth (e). Add 10 times the square root of eh to $4h+6e$. Call this man (m). Now divide $3(m+D)+e$ by d, take the square root, and divide it into m. You get the chord asked for.
>
> By KITTA Yasohachi Motokatsu, a disciple of Fujita Sadasuke of the Seki School, at Kōjimachi in the Eastern capital, in the 9th year of Tenmei (1789), first (lunar) month.

As we shall see, one can analyze the figure and verify that the numerical solution is correct for the given data. Yet the analysis that emerges does not lead to the stated solution procedure, which may be paraphrased in modern notation as follows:

$$h = D-2d,\ e = h+d,\ m = 10\sqrt{eh}+4h+6e,\ w = \sqrt{\frac{3(m+D)+e}{d}},\ x = \frac{m}{w}.$$

2. A data-driven solution

Given $D = 697$ and $d = 272$, we notice that all but the chord length are divisible by 17 and that $(\sqrt{h}, \sqrt{d}, \sqrt{e})$ is a Pythagorean triple.

	D	d	h	e	m	x
n	697	272	153	425	5712	672
$n/17$	41	16	9	25	336	39.52941

Since $w = \frac{17}{2}$ whether one uses the first or second row of numbers, Kitta evidently scaled up the data to ensure that x would be an integer.

Because segment-inscribed circles in many but not all other *sangaku* problems touch the segment's chord at its midpoint, and since that is (approximately) what one sees in Figure 1, Kitta's readers were likely to take this for granted even though it was not included in the statement of the problem. In fact, this is a necessary condition for Kitta's solution, but it would have led readers acquainted with other *sangaku* results into a less than general line of reasoning such as the following.

Lemma 1 ([3, 2.2.7]). *The radius of the mixtilinear incircle touching the legs of a right triangle with sides a, b, c* (hypotenuse) *and its circumcircle is $\rho = a+b-c$.*

Proof. In Figure 2, triangle ABC has a right angle at C, and the circle (J) touches BC, AC at T_a, T_b respectively. P is the orthogonal projection of the circumcenter O on JT_A. In triangle POJ, we have

$$\left(\frac{a}{2}-\rho\right)^2 + \left(\rho-\frac{b}{2}\right)^2 = \left(\frac{c}{2}-\rho\right)^2$$

or

$$\frac{a^2}{4}+\frac{b^2}{4}-a\rho-b\rho+2\rho^2 = \frac{c^2}{4}-c\rho+\rho^2.$$

Since $a^2+b^2=c^2$, this quickly reduces to the result. □

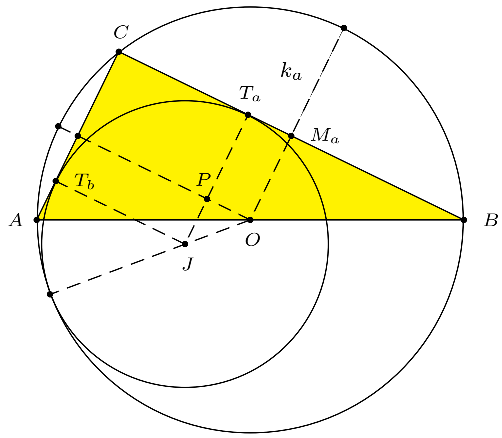

Figure 2.

Corollary 2. *$\rho = 2r$, where r is the inradius of the right triangle ABC.*

Corollary 3 ([5, 254, 275-76])**.** *The difference between each leg and the radius of the mixtilinear circle is twice the sagitta on that leg,* i.e.,

$$2k_a = a - \rho \quad \textit{and} \quad 2k_b = b - \rho.$$

Proof. $a - \rho = a - (a + b - c) = c - b = 2\left(\frac{c}{2} - \frac{b}{2}\right) = 2(R - OM_a) = 2k_a$. The proof for b is analogous. $\square$

If $2\rho = k_a$, then $2(a + b - c) = \frac{1}{2}(c - b)$, $4a + 4b - 4c = c - b$, $4a + 5b = 5c$. From this, $(4a + 5b)^2 = (5c)^2 = 25(a^2 + b^2)$, $16a^2 + 40ab + 25b^2 = 25a^2 + 25b^2$, $9a^2 = 40ab$, and $a : b = 40 : 9$. It follows that $a : b : c : \rho = 40 : 9 : 41 : 8$.

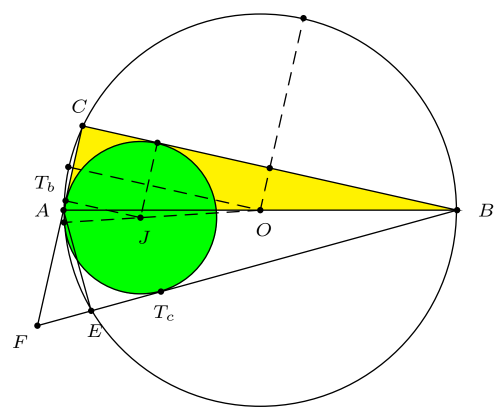

Figure 3.

Assume $BC = 40$, $CA = 9$, and $AB = 41$. Now add E such that BE touches (OJ) at T_c. We seek the length of BE (see Figure 3). Let F be the intersection of CA and BE extended. By equal tangents on (J), let $FT_b = FT_c = x$. Since $BT_c = BT_a = 40 - 8 = 32$, by the Pythagorean relation for the triangle FBC,

$$(8+x)^2 + 40^2 = (32+x)^2 \implies x = \frac{40}{3}.$$

This means that $CF = 8 + \frac{40}{3} = \frac{64}{3}$ and $BF = 32 + \frac{40}{3} = \frac{136}{3}$.

By the intersecting chords theorem, $EF \cdot BF = AF \cdot CF$, and $EF = \frac{AF \cdot CF}{BF} = \frac{37}{3} \cdot \frac{64}{3} \cdot \frac{3}{136} = \frac{296}{51}$. This gives $BE = \frac{136}{3} - \frac{296}{51} = \frac{672}{17}$. Scaling up by 17 gives the answer.

In summary, if one focuses on the data given and applies other well-known *sangaku* results, one can verify the value of the given answer. Kitta does not state that AC touches (J) in Figure 3, although this is true for the given data. But neither does Kitta state that the circle inscribed in the segment touches BC at its midpoint, which is in fact a necessary condition, so it would not be unreasonable for the reader to think that both pieces of information were to be inferred from the data. Yet a little experimentation shows that Kitta's method is valid even when AC does not touch (J).

3. The general case

Let us start afresh with the Figure 4, in which $YJ = \frac{d}{2}$. By an elementary *sangaku* theorem ([3, 1.1]), $EY = 2\sqrt{\frac{d}{2}\left(\frac{D}{2} - d\right)} = \sqrt{d(D-2d)} = \sqrt{dh}$.

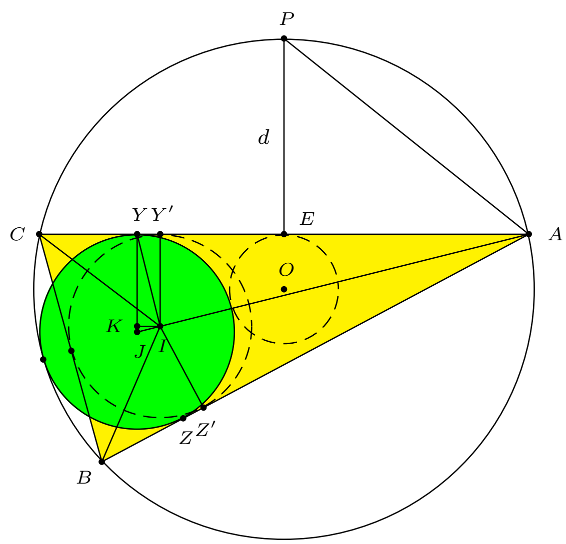

Figure 4.

By the intersecting chords theorem, $EA = \sqrt{d(D-d)} = \sqrt{de}$. Hence $b = 2EA = 2\sqrt{de}$. Let r be the inradius of ABC, $p = \sqrt{de} + \sqrt{dh} = YA$, and $q = Y'A$. Notice that all seven right triangles

$$IJK,\ AIY',\ YIK,\ AJY,\ IYY',\ YJI,\ AYI$$

are similar. In particular, $\frac{KJ}{IK} = \frac{Y'I}{Y'A}$ and $\frac{IK}{KY} = \frac{IY'}{Y'A}$. That is, $\frac{d}{2} - r = \frac{r \cdot IK}{q}$ and $IK = \frac{r^2}{q}$. Hence, $\frac{d}{2} - r = r \cdot \frac{r^2}{q^2}$. But by applying Fujita's mixtilinear circle theorem ([3, 2.2.8], also [7, 8] [1]) with A on (O), $\frac{r}{q} = \frac{d}{2p}$. Therefore, $\frac{d}{2} - r = r \cdot \frac{d^2}{4p^2}$. Solving for r, $r = \frac{2dp^2}{d^2+4p^2}$.

Likewise, since the triangles AEP and $BZ'I$ are similar, $\frac{b}{2d} = \frac{c-q}{r}$. Using $\frac{q}{r} = \frac{2p}{d}$ again, this becomes $\frac{b}{2d} = \frac{c}{r} - \frac{2p}{d}$. Solving for c, $c = \frac{r(b+4p)}{2d}$, or, replacing r,

$$c = \frac{2dp^2}{d^2 + 4p^2} \cdot \frac{b + 4p}{2d} = \frac{p^2(b + 4p)}{d^2 + 4p^2}.$$

Now, $b + 4p = 6\sqrt{de} + 4\sqrt{dh}$, and, because $d = (\sqrt{e} + \sqrt{h})(\sqrt{e} - \sqrt{h})$, we see that $p^2 = (\sqrt{e} + \sqrt{h})^3(\sqrt{e} - \sqrt{h})$ and

$$\begin{aligned} d^2 + 4p^2 &= (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})^2 + 4(\sqrt{e} + \sqrt{h})(\sqrt{e} - \sqrt{h})(\sqrt{e} + \sqrt{h})^2 \\ &= (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})[\sqrt{e} - \sqrt{h} + 4(\sqrt{e} + \sqrt{h})] \\ &= (\sqrt{e} + \sqrt{h})^2(\sqrt{e} - \sqrt{h})(5\sqrt{e} + 3\sqrt{h}). \end{aligned}$$

Hence, $\frac{p^2}{d^2+4p^2} = \frac{\sqrt{e}+\sqrt{h}}{5\sqrt{e}+3\sqrt{h}}$ and

$$c = \frac{2\sqrt{d}(3\sqrt{e} + 2\sqrt{h})(\sqrt{e} + \sqrt{h})}{5\sqrt{e} + 3\sqrt{h}}.$$

This is Kitta's formula because $m = 10\sqrt{eh}+4h+6e = 2(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})$, and

$$3(m+D)+e = 3(m+2e-h)+e = 3m+7e-3h = 25e+30\sqrt{eh}+9h = (5\sqrt{e}+3\sqrt{h})^2.$$

But we are not quite done since Kitta defined w as $\sqrt{\frac{(5\sqrt{e}+3\sqrt{h})^2}{d}}$ and used m to define the numerator of $c = \frac{m}{w}$. Why not set $w_0 = \frac{5\sqrt{e}+3\sqrt{h}}{\sqrt{d}}$ and $m_0 = \frac{dw_0^2-e}{3} - D$ and have $c = \frac{m_0}{w_0}$? I suspect the reason is that Kitta relied on two other *sangaku* results, which took him on a circuitous path to c.

One result was an expression for the diameter of the circumcircle in terms of two sagittae and the inradius of a circumscribed triangle (see Figure 5), $D = \frac{4d\ell(d+\ell+r)}{4d\ell-r^2}$. This follows from the equivalent of Carnot's Theorem stated in terms of the sagittae on the sides of a triangle rather than the signed perpendicular distances from the circumcenter to the sides. This elegant version of the theorem was cited as if common knowledge, with no mention of Carnot, in a Meiji period paper by Y. Sawayama ([6, 153]), and we can reasonably conclude that it was known in premodern Japan.

Proof. If n is the third sagitta in Figure 5, Sawayama's form of Carnot's Theorem states that $d + \ell + n = D - r$. Hence $2D - 2n = 2(d + \ell + r)$. Since the right triangles CPQ and AIY are similar, $\frac{a}{2n} = \frac{q}{r}$; but by the intersecting chords

[1]The proof in [7] was flawed and is superseded by [8].

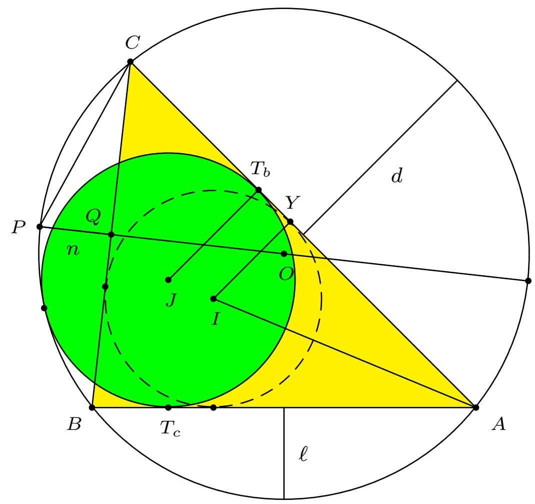

Figure 5.

theorem, $n(D-n) = \left(\frac{a}{2}\right)^2$. Hence $D - n = \frac{q^2}{r^2} \cdot n$, or, $n = \frac{Dr^2}{q^2+r^2}$. Now, another *sangaku* theorem ([3, 2.2]) states that *the square of the distance from the vertex of a triangle to its incenter is four times the product of the sagittae on the adjacent sides*. In the present case, $4d\ell = x^2$, where $x^2 = q^2 + r^2$. Therefore, $2D - 2n = 2D - \frac{D(4d\ell - r^2)}{2d\ell}$ and so $2(d + \ell + r) = \frac{D(4d\ell-r^2)}{2d\ell}$, or $D = \frac{4d\ell(d+\ell+r)}{4d\ell - r^2}$. □

Armed with $D = \frac{4d\ell(d+\ell+r)}{4d\ell-r^2}$ and $\ell = \frac{x^2}{4d}$, one calculates c by the intersecting chords theorem without much difficulty because $D - \ell$ factors nicely into

$$\frac{4d^2\ell + 4d\ell^2 + 4d\ell r}{4d\ell - r^2} - \ell = \frac{4d^2\ell + 4d\ell r + \ell r^2}{4d\ell - r^2} = \frac{\ell(2d+r)^2}{4d\ell - r^2}.$$

Therefore,

$$\begin{aligned}
\left(\frac{c}{2}\right)^2 &= \ell(D-\ell) = \frac{\ell^2(2d+r)^2}{4d\ell - r^2} = \frac{(2d+r)^2x^4}{16d^2(x^2-r^2)} = \frac{(2d+r)^2(q^2+r^2)^2}{16d^2q^2} \\
&= \frac{(d^2+4p^2)^2r^2(2d+r)^2}{64d^4p^2} = \frac{(\sqrt{d}(5\sqrt{e}+3\sqrt{h})+p)^2(d^2+4p^2)^2}{4d^2(5\sqrt{e}+3\sqrt{h})^4} \\
&= \frac{d(d+4(\sqrt{e}+\sqrt{h})^2)^2(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^4} \\
&= \frac{(\sqrt{e}-\sqrt{h})(\sqrt{e}+\sqrt{h})^3(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}.
\end{aligned}$$

Hence,

$$c = 2\sqrt{\frac{(\sqrt{e}-\sqrt{h})(\sqrt{e}+\sqrt{h})^3(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}} = 2\sqrt{d}\sqrt{\frac{(\sqrt{e}+\sqrt{h})^2(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}}.$$

(We get to the same expression if we eliminate D using $D = 2e - h$, but the algebra is lengthier.) Thus we obtain a quadratic in c with no linear term and therefore with symmetric roots, one negative definite and one positive. From this perpsective, $c = \sqrt{\frac{m^2}{w^2}}$ and Kitta's definitions of m and w make sense.

I have recently discovered that this problem is the first one discussed by AIDA Yasuaki in a manuscript criticizing various solutions in *Shinpeki sanpō* ([1, 1.5-6]). Aida's solution is equivalent to the equation $c = \frac{2\sqrt{d}(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})}{5\sqrt{e}+3\sqrt{h}}$ derived above. In terms of the notation we have been using, Aida sets $h_0 = \sqrt{d(D-d)}$ and $e_0 = \sqrt{h_0^2 - d^2} + h_0$ (that is, $e_0 = \sqrt{d(D-2d)} + \sqrt{d(D-d)}$), and writes $c = \frac{e_0^2}{2h_0+3e_0} + e_0$, avoiding Kitta's m and w placeholders altogether.

4. Discussion

Kitta evidently took pains to write up this problem in such a way that even a reader who could verify the numerical result might still be baffled by the general procedure. In this way, it is like a double-locked box. For the sophisticated solver, the unanswered question is why neat Pythagorean relationships emerge in the data.

A connection between the general and special cases of the problem lies in the fact that, if d and p are integers, then h, d, and e are perfect squares and $(\sqrt{h}, \sqrt{d}, \sqrt{e})$ is a Pythagorean triple. To prove this, recall that $dp^2 = (\sqrt{e}+\sqrt{h})^4(\sqrt{e}-\sqrt{h})^2$. Replacing h with $e-d$ and solving for e, $e = \frac{(d^2+p^2)^2}{4dp^2}$; that is, $\frac{e}{d} = \left(\frac{d^2+p^2}{2dp}\right)^2$. Thus, provided d and p are integers, e and d are perfect squares, and, since $h+d=e$ implies $h = (p^2-d^2)^2$, h is a perfect square too.

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Appendix (by the editor): Kitta's configurations with integer diameters

Given a triangle ABC with sidelengths $BC = a$, $CA = b$, $AB = c$, semiperimeter s, and area Δ, the radius of the mixtilinear incircle in angle A is

$$\rho_a = \frac{r}{\cos^2 \frac{A}{2}} = \frac{bc}{s(s-a)} \cdot \frac{\Delta}{s} = \frac{bc\Delta}{s^2(s-a)},$$

where $r = \frac{\Delta}{s}$ is the inradius of the triangle. With reference to Figure 5, $d = R(1 - \cos B) = 2R\sin^2 \frac{B}{2}$. The condition $2\rho_a = d$ becomes $\frac{2bc\Delta}{s^2(s-a)} = \frac{2abc}{4\Delta} \cdot \frac{(s-c)(s-a)}{ca}$. Since $\Delta^2 = s(s-a)(s-b)(s-c)$, this reduces to $4c(s-b) = s(s-a)$, and $a^2 - b^2 + 7c^2 - 10bc + 8ca = 0$. By completing squares, we rewrite this as $(a+4c)^2 = (b+c)(b+9c)$. It follows that

$$a = -4c + \sqrt{(b+c)(b+9c)} = -4c + \sqrt{(b+5c)^2 - (4c)^2}.$$

To obtain integer solutions we put $b + 5c = p^2 + q^2$, $4c = 2pq$ for relatively prime integers p and q. This leads to

$$b = 2p^2 + 2q^2 - 5pq, \quad c = pq, \quad a = 2(p^2 - q^2 - 2pq).$$

These satisfy the triangle inequality if and only if $p > \frac{5q}{2}$. The circumradius of the triangle is rational if and only if the area is an integer. Since $\Delta = 2pq(p-2q)\sqrt{(2p-5q)q}$, we choose p and q such that $2p - 5q : q = u^2 : v^2$ for relatively prime integers u, v. Equivalently, $(p,q) = (\frac{u^2+5v^2}{g}, \frac{2v^2}{g})$ with $g = \gcd(u^2+5v^2, 2v^2)$. The following table shows that with small values of u and v, we exhaust all examples in which a, b, c, after reduction by their gcd, are all integers below 1000.

u	v	p	q	a	b	c	D	$d = 2\rho_a$	ℓ	AT_b
1	1	3	1	4	5	3	5	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{5}{2}$
1	2	21	8	41	85	84	$\frac{697}{8}$	34	32	68
2	1	9	2	41	40	9	41	17	$\frac{1}{2}$	8
1	3	23	9	68	185	207	$\frac{629}{3}$	$\frac{111}{2}$	$\frac{243}{2}$	$\frac{333}{2}$
3	1	7	1	68	65	7	$\frac{221}{3}$	$\frac{39}{2}$	$\frac{1}{6}$	$\frac{13}{2}$
2	3	49	18	313	520	441	$\frac{1565}{3}$	240	$\frac{243}{2}$	360
3	2	29	8	313	325	116	$\frac{7825}{24}$	150	$\frac{32}{3}$	100
4	1	21	2	353	340	21	$\frac{1765}{4}$	80	$\frac{1}{4}$	20
5	1	15	1	388	377	15	$\frac{2813}{5}$	$\frac{145}{2}$	$\frac{1}{10}$	$\frac{29}{2}$

Kitta's example is the one obtained from $(u,v) = (1,2)$, magnified by a factor 8 to make the circumradius an integer. Here are the only configurations with integer values of a, b, c, D, and $2\rho_a = d$, all below 1000:

a	b	c	D	$d = 2\rho_a$	ℓ	AT_b
8	10	6	10	5	1	5
41	40	9	41	17	$\frac{1}{2}$	8
328	680	672	697	272	256	544
408	390	42	442	117	1	39

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Distances Between the Circumcenter of the Extouch Triangle and the Classical Centers of a Triangle

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Abstract. We compute, in a triangle, the distances between the circumcenter of the extouch triangle and the circumcenter, the incenter, and the orthocenter, respectively. For this calculation, we use the absolute barycentric coordinates and obtain relatively simple formulas which seem unknown. To conclude, we compute the barycentric coordinates of the incenter of the extouch triangle.

1. Introduction

We consider a triangle ABC and we denote by O the circumcenter, I the incenter, H the orthocenter, G the centroid, and N the nine-point center. We denote the side-lengths by a, b, c, the semiperimeter by s, R the circumradius, and r the inradius. The distances between the classical centers of the triangle ABC are well known. We recall that

$$\begin{aligned} OI^2 &= R^2 - 2Rr, \\ OH^2 &= R^2 - 8R^2 \cos A \cos B \cos C, \\ HI^2 &= 2r^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

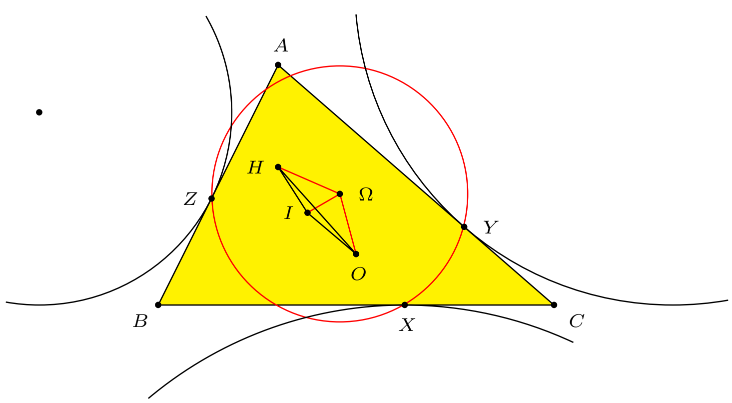

Figure 1.

It is well known that the circle through the excenters of triangle ABC has center I', the reflection of I in O, and that the radii through the excenters are perpendicular to the corresponding sides of ABC. It follows that the extouch triangle XYZ is the pedal triangle of I', and its circumcircle is the common pedal circle of I' and

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its isogonal conjugate I'^*. The circumcenter Ω is the midpoint between I' and I'^*. In this note we compute the distances between Ω and the above classical triangle centers.

Theorem 1.
(a) $\Omega O^2 = R^2 - \frac{4R^3(R-r)}{r^2} \cos A \cos B \cos C$.
(b) $\Omega I^2 = 2R^2 - 4Rr - \frac{4R^3(R-2r)}{r^2} \cos A \cos B \cos C$.
(c) $\Omega H^2 = 2R^2 - 4Rr - 2r^2 - \frac{4R^2(R-r)(R-3r)}{r^2} \cos A \cos B \cos C$.

We collect a number of useful formulas for cyclic sums of trigonometrical expressions involving the angles of a triangle.

Lemma 2.
(a) $\cos A + \cos B + \cos C = \frac{R+r}{R}$.
(b) $\sum_{\text{cyclic}} \cos B \cos C = \frac{(2R+r)r}{2R^2} + \cos A \cos B \cos C$.
(c) $\sum_{\text{cyclic}} \sin A \cos A = \frac{rs}{R^2}$.
(d) $\sum_{\text{cyclic}} (\cos B + \cos C) \sin A = \sin A + \sin B + \sin C = \frac{s}{R}$.
(e) $\sin A \sin B \sin C = \sum_{\text{cyclic}} (\cos B \cos C) \sin A = \frac{rs}{2R^2}$.

2. Homogeneous barycentric coordinates of some centers

In the *Encyclopedia of triangles centers* [1], henceforth referred to as ETC, Kimberling publishes a list of more than 5600 triangle centers with *homogeneous* trilinear and barycentric coordinates. In this paper we consider barycentric coordinates exclusively. An introduction to barycentric coordinates can be found in [3]. Sometimes it is useful to work with *absolute* barycentric coordinates. For a finite point, the absolute barycentric coordinates can be found from a set of homogeneous barycentric coordinates by dividing by its coordinate sum. If the triangle center $X(n)$ in ETC is a finite point, we denote by $(\alpha_n,\ \beta_n,\ \gamma_n)$ its absolute barycentric coordinates.

n	$X(n)$	α_n
1	I	$\frac{R\sin A}{rs} \cdot r$
3	O	$\frac{R\sin A}{rs}(R\cos A)$
4	H	$\frac{R\sin A}{rs}(2R\cos B\cos C)$
8	N_{a}	$\frac{R\sin A}{rs}(2R(\cos A + \cos B\cos C) - 2r)$
20	L	$\frac{R\sin A}{rs} \cdot 2R(\cos A - \cos B\cos C)$
40	I'	$\frac{R\sin A}{rs}(2R\cos A - r)$

The isogonal conjugate of $I' := X(40)$ is the triangle center $X(84)$.

Proposition 3. $\alpha_{84} = \frac{R\sin A}{rs} \cdot \frac{(2R\cos B - r)(2R\cos C - r)}{r}$.

Proof. Since, in homogeneous barycentric coordinates,

$$X(40) = (\sin A(2R\cos A - r) :\ \sin B(2R\cos B - r) :\ \sin C(2R\cos C - r)),$$

we have

$$X(84) = \left(\frac{\sin A}{2R\cos A - r} : \frac{\sin B}{2R\cos B - r} : \frac{\sin C}{2R\cos C - r}\right).$$

Therefore,

$$\begin{aligned}\alpha_{84} =& \frac{\frac{\sin A}{2R\cos A - r}}{\frac{\sin A}{2R\cos A-r} + \frac{\sin B}{2R\cos B-r} + \frac{\sin C}{2R\cos C-r}} \\ =& \frac{\sin A(2R\cos B - r)(2R\cos C - r)}{\sum_{\text{cyclic}} \sin A(2R\cos B - r)(2R\cos C - r)}.\end{aligned}$$

Using the formulas in Lemma 2, we have

$$\begin{aligned}& \sum_{\text{cyclic}} \sin A(2R\cos B - r)(2R\cos C - r) \\ =& 4R^2 \sum_{\text{cyclic}} \sin A\cos B\cos C - 2Rr \sum_{\text{cyclic}} \sin A(\cos B + \cos C) + r^2 \sum_{\text{cyclic}} \sin A \\ =& 4R^2 \cdot \frac{rs}{2R^2} - 2Rr \cdot \frac{s}{R} + r^2 \cdot \frac{s}{R} \\ =& \frac{r^2 s}{R}.\end{aligned}$$

From this the result follows. □

Lemma 4. *The line joining $X(40)$ and $X(84)$ contains the Nagel point $X(8)$.*

Proof. With $t = \frac{r}{2R}$, we have

$$\begin{aligned}& (1-t)\alpha_{40} + t\alpha_{84} \\ =& \frac{R\sin A}{rs}\left(\left(1 - \frac{r}{2R}\right)(2R\cos A - r) + \frac{r}{2R} \cdot \frac{(2R\cos B - r)(2R\cos C - r)}{r}\right) \\ =& \frac{R\sin A}{rs}\left(2R\cos A - r\cos A - r + \frac{r^2}{2R} + 2R\cos B\cos C - r(\cos B + \cos C) + \frac{r^2}{2R}\right) \\ =& \frac{R\sin A}{rs}\left(2R\cos B\cos C + 2R\cos A - r(\cos A + \cos B + \cos C) - r + \frac{r^2}{R}\right) \\ =& \frac{R\sin A}{rs}\left(2R\cos B\cos C + 2R\cos A - r \cdot \frac{R+r}{R} - r + \frac{r^2}{R}\right) \\ =& \frac{R\sin A}{rs}(2R\cos B\cos C + 2R\cos A - 2r) \\ =& \alpha_8.\end{aligned}$$

□

Proposition 5. *The circumcenter Ω of the extouch triangle lies on the line joining $X(40)$ and $X(8)$. It has first absolute barycentric coordinate*

$$\alpha = \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r)\right).$$

Proof. Since Ω is the midpoint of $X(40)$ and $X(84)$, it follows from Lemma 4 that it lies on the line $X(40)X(8)$. Furthermore,

$$\begin{aligned}
\alpha &= \frac{1}{2}(\alpha_{40}+\alpha_{84})\\
&= \frac{R\sin A}{rs}\left(\frac{2R\cos A - r}{2}+\frac{(2R\cos B - r)(2R\cos C - r)}{2r}\right)\\
&= \frac{R\sin A}{rs}\cdot\frac{(2R\cos A - r)r+(2R\cos B - r)(2R\cos C - r)}{2r}\\
&= \frac{R\sin A}{rs}\cdot\frac{4R^2\cos B\cos C+4Rr\cos A-2Rr(\cos A+\cos B+\cos C)}{2r}\\
&= \frac{R\sin A}{rs}\cdot\frac{4R^2\cos B\cos C+4Rr\cos A-2(R+r)r}{2r}\\
&= \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C+2R\cos A-(R+r)\right).
\end{aligned}$$

□

Remark. In ETC, Ω is the triangle center $X(1158)$.

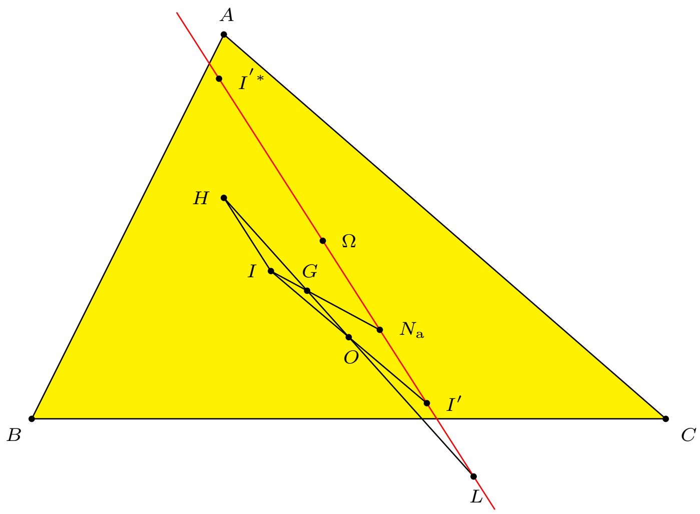

Figure 2.

Figure 2 shows Ω on the line joining I' to N_a. Since the deLongchamps point $L = X(20)$ is the reflection of H in O, O is the common midpoint of II' and HL. From this, IH is parallel to $I'L$. Also, the centroid G divides both segments IN_a and HL in the ratio $1:2$, HI is also parallel to N_a. It follows that L lies on the line $I'N_a$ and I' is the midpoint of LN_a.

Lemma 6. $\Omega I' = \frac{R}{r}\cdot HI$.

Proof. By Proposition 5,

$$\begin{aligned}\alpha - \alpha_{40} = &\ \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C + 2R\cos A - (R+r) - (2R\cos A - r)\right)\\ = &\ \frac{R\sin A}{rs}\left(\frac{2R^2}{r}\cos B\cos C - R\right)\\ = &\ \frac{R}{r}\cdot\frac{R\sin A}{rs}\left(2R\cos B\cos C - r\right)\\ = &\ \frac{R}{r}(\alpha_4 - \alpha_1).\end{aligned}$$

□

3. Proof of Theorem 1

Lemma 7. (a) $2\Omega O^2 - \Omega I^2 = 4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C$.
(b) $2\Omega O^2 - \Omega H^2 = 4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right)\cos A\cos B\cos C$.

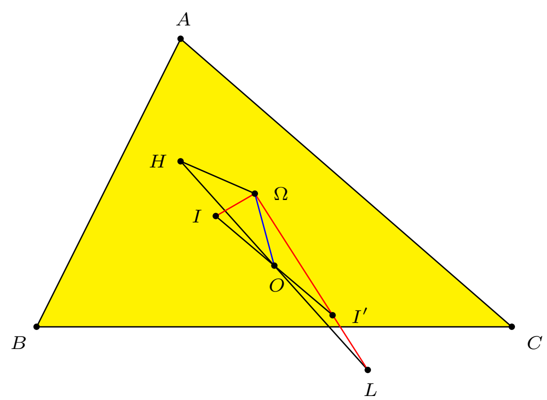

Figure 3.

Proof. (a) Applying Apollonius to the median ΩO of triangle $\Omega II'$, we have

$$\Omega I^2 + \Omega I'^2 = 2(\Omega O^2 + OI^2).$$

From this,

$$\begin{aligned}2\Omega O^2 - \Omega I^2 = &\ \Omega I'^2 - 2OI^2\\ = &\ \frac{R^2}{r^2}HI^2 - 2OI^2\\ = &\ \frac{R^2}{r^2}(2r^2 - 4R^2\cos A\cos B\cos C) - 2R(R-2r)\\ = &\ 4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C.\end{aligned}$$

(b) Applying Apollonius to the median ΩO of triangle ΩHL, we have

$$\Omega H^2 + \Omega L^2 = 2(\Omega O^2 + OH^2).$$

From this,

$$\begin{aligned}
2\Omega O^2 - \Omega H^2 &= \Omega L^2 - 2OH^2 \\
&= \frac{(R+r)^2}{r^2} HI^2 - 2OH^2 \\
&= \frac{(R+r)^2}{r^2}(2r^2 - 4R^2 \cos A \cos B \cos C) \\
&\quad - 2R^2(1 - 8\cos A \cos B \cos C) \\
&= 4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right) \cos A \cos B \cos C.
\end{aligned}$$

□

Lemma 8. $\Omega O^2 - \Omega I^2 = -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C$.

Proof. We begin with $AI^2 = \frac{r^2}{\sin^2 \frac{A}{2}} = \frac{r^2 bc}{(s-b)(s-c)} = \frac{s-a}{s} bc = bc - \frac{abc}{s} = 4R^2 \sin B \sin C - 4Rr$; similarly for BI^2 and CI^2. Therefore,

$$\begin{aligned}
&\alpha AI^2 + \beta AB^2 + \gamma CI^2 \\
&= 4R^2 \sin A \sin B \sin C \left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr(\alpha + \beta + \gamma) \\
&= 2rs \left(\frac{\alpha}{\sin A} + \frac{\beta}{\sin B} + \frac{\gamma}{\sin C}\right) - 4Rr \\
&= 2rs \cdot \frac{R}{rs} \sum_{\text{cyclic}} \left(\frac{2R^2}{r} \cos B \cos C + 2R\cos A - (R+r)\right) - 4Rr \\
&= 2R \sum_{\text{cyclic}} \left(\frac{2R^2}{r} \cos B \cos C + 2R\cos A - (R+r)\right) - 4Rr \\
&= 2R\left(\frac{2R^2}{r}\left(\frac{(2R+r)r}{2R^2} + \cos A \cos B \cos C\right) + 2R \cdot \frac{R+r}{R} - 3(R+r)\right) - 4Rr \\
&= 2R^2 - 4Rr + \frac{4R^3}{r} \cos A \cos B \cos C.
\end{aligned}$$

We make use of a formula of Scheer [2]. For an arbitrary point P,

$$\Omega P^2 = \alpha AP^2 + \beta BP^2 + \gamma CP^2 - (\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2).$$

Applying this to $P = O$ and $P = I$ respectively, we have

$$\begin{aligned}
\Omega O^2 - \Omega I^2 &= (\alpha AO^2 + \beta BO^2 + \gamma CO^2) - (\alpha AI^2 + \beta BI^2 + \gamma CI^2) \\
&= R^2 - (\alpha AI^2 + \beta BI^2 + \gamma CI^2) \\
&= R^2 - (2R^2 - 4Rr + \frac{4R^3}{r} \cos A \cos B \cos C) \\
&= -R^2 + 4Rr - \frac{4R^3}{r} \cos A \cos B \cos C.
\end{aligned}$$

□

Now we complete the proof of Theorem 1.
(a) For the distance from Ω to the circumcenter:

$$\begin{aligned}\Omega O^2 =&\ (2\Omega O^2 - \Omega I^2) - (\Omega O^2 - \Omega I^2)\\ =&\ \left(4Rr - \frac{4R^4}{r^2}\cos A\cos B\cos C\right)\\ &- \left(-R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C\right)\\ =&\ R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C.\end{aligned}$$

(b) For the distance from Ω to the incenter:

$$\begin{aligned}\Omega I^2 =&\ \Omega O^2 - (\Omega O^2 - \Omega I^2)\\ =&\ \left(R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C\right)\\ &- \left(-R^2 + 4Rr - \frac{4R^3}{r}\cos A\cos B\cos C\right)\\ =&\ 2R^2 - 4Rr - \frac{4R^3}{r^2}(R-2r)\cos A\cos B\cos C.\end{aligned}$$

(c) For the distance from Ω to the orthocenter:

$$\begin{aligned}\Omega H^2 =&\ 2\Omega O^2 - (2\Omega O^2 - \Omega H^2)\\ =&\ 2\left(R^2 - \frac{4R^3}{r^2}(R-r)\cos A\cos B\cos C\right)\\ &- \left(4Rr + 2r^2 - \frac{4R^2}{r^2}\left(R^2 + 2Rr - 3r^2\right)\cos A\cos B\cos C\right)\\ =&\ 2R^2 - 4Rr - 2r^2 - \frac{4R^2}{r^2}\left(R^2 - 4Rr + 3r^2\right)\cos A\cos B\cos C.\end{aligned}$$

The proof of Theorem 1 is now complete.

Since the centroid G and the nine-point center N divide the segment OH in the ratio $OG : ON : OH = 2 : 3 : 6$, further applications of the Apollonius theorem yield the distances from Ω to G and N.

Corollary 9.
(a) $\Omega G^2 = \frac{2}{9}(5R^2 - 6Rr - 3r^2) - \frac{4R^2}{9r^2}\left(9R^2 - 18Rr + 5r^2\right)\cos A\cos B\cos C.$
(b) $\Omega N^2 = \frac{5}{4}R^2 - 2Rr - r^2 - \frac{2R^2}{r^2}\left(2R^2 - 5Rr + 2r^2\right)\cos A\cos B\cos C.$

Remarks. (1) Since $4R^2\cos A\cos B\cos C = s^2 - (r+2R)^2$, these distances can all be expressed in terms of R, r, s.

(2) We also note the two simple relations:
(i) $\Omega O^2 + OI^2 = \frac{R(R-r)}{r^2}HI^2$;
(ii) $\Omega I = \frac{OI\times HI}{r}$.

4. The cyclcocevian conjugate of the Nagel point

Since the circumcircle of the extouch triangle is the pedal circle of $I' = X(40)$, it is also the pedal circle of $I'^* = X(84)$. The pedals of $X(84)$ are the vertices of the cyclocevian conjugate of the Nagel point $N_a = X(8)$. It is interesting to note that this is also a point on the line $I'N_a$. In ETC, this is $X(189)$ with homogeneous barycentric coordinates are

$$\left(\frac{1}{\cos B+\cos C-\cos A-1}:\frac{1}{\cos C+\cos A-\cos B-1}:\frac{1}{\cos A+\cos B-\cos C-1}\right).$$

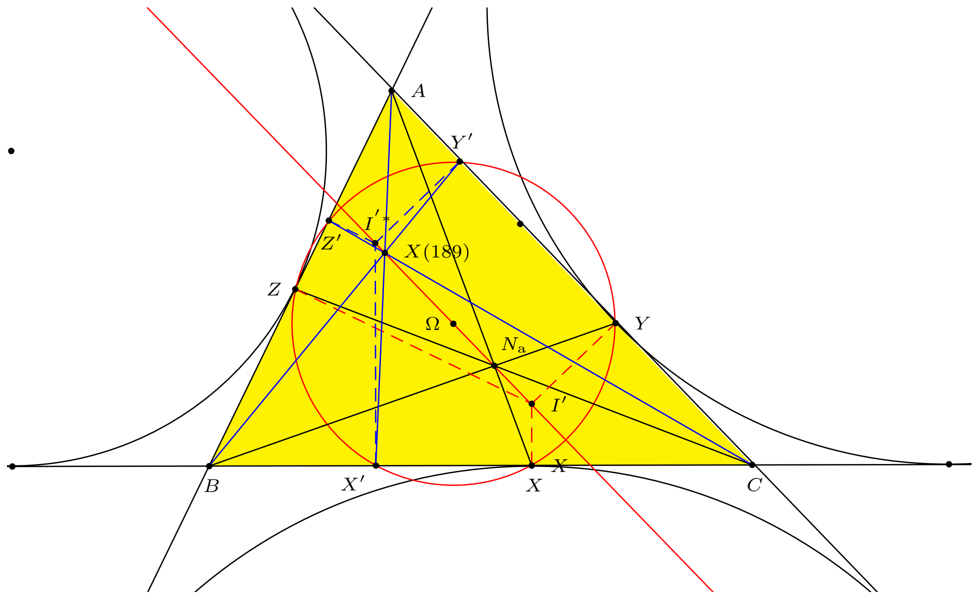

Figure 4

Proposition 10. *The first absolute barycentric coordinate of the cyclocevian conjugate of the Nagel point is*

$$\alpha_{189} = \frac{R\sin A}{rs}\left((2R\cos A - r) + k(2R\cos B\cos C - r)\right),$$

where

$$k = \frac{(4R+r)r + 4R^2\cos A\cos B\cos C}{r^2 + 4R^2\cos A\cos B\cos C}.$$

Proof. The point $X(189)$ divides $I'N_a$ in the ratio

$$N_aX(189) : X(189)I' = t : 1-t$$

for $t = \frac{-4Rr}{r^2+4R^2\cos A\cos B\cos C}$. From this,

$$\begin{aligned}\alpha_{189} =& \; t\alpha_{40} + (1-t)\alpha_8 \\ =& \; \frac{R\sin A}{rs}\left(t(2R\cos A - r) + (1-t)(2R\cos B\cos C + 2R\cos A - 2r)\right) \\ =& \; \frac{R\sin A}{rs}\left((2R\cos A - r) + (1-t)(2R\cos B\cos C - r)\right).\end{aligned}$$

The coefficient $1-t$ is k given in the statement of the proposition. □

5. The centroid and orthocenter of the extouch triangle

The centroid of the extouch triangle is very easy to determine: It is the triangle center

$$\begin{aligned}\frac{X+Y+Z}{3} =& \; \frac{1}{3}\left(\frac{(0, c+a-b, a+b-c)}{2a} + \frac{(b+c-a, 0, a+b-c)}{2b}\right. \\ & \left. + \frac{(b+c-a, c+a-b, 0)}{2c}\right) \\ =& \; \frac{(a(b+c)(b+c-a), b(c+a)(c+a-b), c(a+b)(a+b-c))}{6abc}.\end{aligned}$$

This is the triangle center $X(210)$ in ETC. Clearly,

$$\alpha_{210} = \frac{a(b+c)(b+c-a)}{6abc}.$$

By expressing this in the form

$$\alpha_{210} = \frac{R\sin A}{rs}\left(\frac{R}{3}\cdot(\sin B + \sin C)(\sin B + \sin C - \sin A)\right), \tag{1}$$

we easy determine also the orthocenter of the extouch triangle:

Proposition 11. *The orthocenter of the extouch triangle has first absolute barycentric coordinate*

$$\begin{aligned}\alpha' =& \; \frac{R\sin A}{rs}\left(R((\sin B + \sin C)(\sin B + \sin C - \sin A) - 4\cos A)\right. \\ & \left. - \frac{4R^2}{r}\cos B\cos C + 2(R+r)\right).\end{aligned}$$

Proof. Since the orthocenter divides the centroid $X(210)$ and the circumcenter Ω in the ratio $3 : -2$, we have the first absolute barycentric coordinate equal to $\alpha' = 3\alpha_{210} - 2\alpha$. The result follows from Proposition 5 and (1) above. □

Remark. In terms of a, b, c, the orthocenter of the extouch triangle has homogeneous barycentric coordinates

$$(af(a,b,c)g(a,b,c) : bf(b,c,a)g(b,c,a) : cf(c,a,b)g(c,a,b))$$

where

$$\begin{aligned} f(a,b,c) =\ & a^3(b+c) - a^2(b-c)^2 - a(b+c)(b-c)^2 + (b^2-c^2)^2, \\ g(a,b,c) =\ & a^5 - a^4(b+c) - 2a^3(b-c)^2 + 2a^2(b+c)(b^2+c^2) \\ & + a(b^4 - 4b^3c - 2b^2c^2 - 4bc^3 + c^4) - (b-c)^2(b+c)^3. \end{aligned}$$

It is not in the current edition of ETC and has $(6-9-13)$-search number $52.7618273660\cdots$.

6. The incenter of the extouch triangle

Lemma 12. *Let a', b', c' be the sidelengths of the extouch triangle XYZ.*

$$a'^2 = a^2(1-\sin B\sin C), \quad b'^2 = b^2(1-\sin C\sin A), \quad c'^2 = c^2(1-\sin A\sin B).$$

Proof. It is enough to establish the expression for a'^2. Since $AY = s-c$ and $AZ = s-b$, applying the law of cosines to triangle AYZ, we have

$$\begin{aligned} a'^2 =\ & (s-b)^2 + (s-c)^2 - 2(s-b)(s-c)\cos A \\ =\ & (s-b)^2 + (s-c)^2 - 2(s-b)(s-c)\left(2\cos^2\frac{A}{2} - 1\right) \\ =\ & (s-b)^2 + (s-c)^2 + 2(s-b)(s-c) - 4(s-b)(s-c)\cdot\frac{s(s-a)}{bc} \\ =\ & ((s-b)+(s-c))^2 - \frac{4\Delta^2}{bc} \\ =\ & a^2 - \frac{4\Delta^2}{bc} \\ =\ & a^2(1-\sin B\sin C) \end{aligned}$$

since $\Delta = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$. $\square$

Proposition 13. *The incenter of the extouch triangle has homogeneous barycentric coordinates*

$$(\sin B + \sin C - \sin A)(\sqrt{1-\sin C\sin A} + \sqrt{1-\sin A\sin B}) : \cdots : \cdots).$$

Proof. With reference to triangle XYZ, this incenter has homogeneous barycentric coordinates $(a' : b' : c')$. The absolute barycentric with reference to ABC is therefore

$$\frac{a'X + b'Y + c'Z}{a'+b'+c'}.$$

In homogeneous coordinates, this can be taken as

$$\begin{aligned} & a'X + b'Y + c'Z \\ =\ & \frac{a'(0, c+a-b, a+b-c)}{2a} + \frac{b'(b+c-a, 0, a+b-c)}{2b} + \frac{c'(b+c-a, c+a-b, 0)}{2c} \\ =\ & \frac{1}{2}\left((b+c-a)\left(\frac{b'}{b}+\frac{c'}{c}\right), (c+a-b)\left(\frac{c'}{c}+\frac{a'}{a}\right), (a+b-c)\left(\frac{a'}{a}+\frac{b'}{b}\right)\right). \end{aligned}$$

The result follows from the law of sines and an application of Lemma 12. $\square$

We conclude by giving the coordinates of the incenter of the extouch triangle in terms of a, b, c. Using the Heron formula

$$\Delta^2 = \frac{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}{16},$$

we have

$$\begin{aligned} a'^2 &= \frac{4a^2bc - 16\Delta^2}{4bc} \\ &= \frac{4a^2bc - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 + a^4 + b^4 + c^4}{4bc} \\ &= \frac{a^4 - 2a^2(b-c)^2 + (b^2-c^2)^2}{4bc}. \end{aligned}$$

Therefore,

$$\frac{a'}{a} = \frac{\sqrt{bc(a^4 - 2a^2(b-c)^2 + (b^2-c^2)^2)}}{2abc};$$

similarly for $\frac{b'}{b}$ and $\frac{c'}{c}$. This leads to

$$\Big((b+c-a)\left(\sqrt{ca(b^4 - 2b^2(c-a)^2 + (c^2-a^2)^2)} + \sqrt{ab(c^4 - 2c^2(a-b)^2 + (a^2-b^2)^2)}\right)$$
$$: \cdots : \cdots\Big).$$

This is a triangle center not in the current edition of ETC. It has $(6-9-13)$-search number $4.66290502201\cdots$.

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The Touchpoints Triangles and the Feuerbach Hyperbolas

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Abstract. In this paper we generalize the famous Kariya theorem on the perspectivity of a given triangle with the homothetic images of the intouch triangle from the incenter to the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. We also study in some details the triangle formed by the orthocenters of the touchpoints triangles. An elegant construction is given for the asymptotes of the Feuerbach hyperbolas.

1. Introduction

Consider a triangle ABC with incircle $I(r)$ tangent to the sides BC, CA, AB at X, Y, Z respectively. These form the intouch triangle $\mathbf{T}_{\rm i}$ of ABC. For a real number t, let $\mathbf{T}_{\rm i}(t)$ be the image of the $\mathbf{T}_{\rm i}$ under the homothety $\mathsf{h}\left(I, \frac{t}{r}\right)$. Its vertices are the points $X(t)$, $Y(t)$, $Z(t)$ on the lines IX, IY, IZ respectively, such that

$$IX(t) = IY(t) = IZ(t) = t.$$

The famous Kariya's theorem asserts that the lines $AX(t)$, $BY(t)$, $CY(t)$ are concurrent, i.e., the triangles ABC and $\mathbf{T}_{\rm i}(t)$ are perspective, and that the perspector is a point $Q(t)$ on the Feuerbach hyperbola $\mathscr{F}$, the rectangular circum-hyperbola which is the isogonal conjugate of the line OI joining the circumcenter and the incenter of ABC. This fact was actually known earlier to J. Neuberg and H. Mandart; see [5] and the interesting note in [2, §1242]. We revisit in §3 this theorem with a proof leading to simple relations of the perspectors $Q(t)$ and $Q(-t)$ (Proposition 4 below), and their isogonal conjugates on the line OI. In §5 we obtain analogous results by replacing $\mathbf{T}_{\rm i}(t)$ by homothetic images of the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. Some properties of the triad of ex-Feuerbach hyperbolas are established in §§7–10. Specifically, we give an elegant construction of the asymptotes of the Feuerbach hyperbolas in §10. The final section §11 is devoted to further properties of the touchpoints triangles, in particular, the loci of perspectors of the triangle $H^aH^bH^c$ formed by their orthocenters.

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2. Generalization of Kariya's theorem

Let a, b, c be the sidelengths of triangle ABC, $s = \frac{1}{2}(a+b+c)$ the semiperimeter, and Δ the area of the triangle. It is well known that
(i) $\Delta = rs$,
(ii) $abc = 4R\Delta$ for the circumradius R, and
(iii) $16\Delta^2 = a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)$.

We shall work with homogeneous barycentric coordinates with reference to triangle ABC, and refer to [7] for basic results and formulas.

Let $\mathbf{T}$ be the pedal triangle of a point $P=(u:v:w)$ in homogeneous barycentric coordinates. The vertices of $\mathbf{T}$ are the points

$$\begin{aligned}
X =&\ (0:\ (a^2+b^2-c^2)u+2a^2v:\ (c^2+a^2-b^2)u+2a^2w),\\
Y =&\ ((a^2+b^2-c^2)v+2b^2u:\ 0:\ (b^2+c^2-a^2)v+2b^2w),\\
Z =&\ ((c^2+a^2-b^2)w+2c^2u:\ (b^2+c^2-a^2)w+2c^2v:\ 0).
\end{aligned}$$

For a real number k, let $\mathbf{T}_k$ be the image of $\mathbf{T}$ under the homothety $\mathsf{h}(P,k)$.

Lemma 1. *The vertices of $\mathbf{T}_k$ are the points*

$$X_k = (2a^2(1-k)u:\ (a^2+b^2-c^2)ku+2a^2v:\ (c^2+a^2-b^2)ku+2a^2w), \tag{1}$$
$$Y_k = ((a^2+b^2-c^2)kv+2b^2u:\ 2b^2(1-k)v:\ (b^2+c^2-a^2)kv+2b^2w), \tag{2}$$
$$Z_k = ((c^2+a^2-b^2)kw+2c^2u:\ (b^2+c^2-a^2)kw+2c^2v:\ 2c^2(1-k)w). \tag{3}$$

Proof. The point X_k divides the segment PX in the ratio $PX_k : X_kX = k : 1-k$. In absolute barycentric coordinates,

$$\begin{aligned}
X_k =&\ (1-k)P+kX\\
=&\ \frac{(1-k)(u,v,w)}{u+v+w}+\frac{k(0,(a^2+b^2-c^2)u+2a^2v,(c^2+a^2-b^2)u+2a^2w)}{2a^2(u+v+w)}\\
=&\ \frac{(2a^2(1-k)u,(a^2+b^2-c^2)ku+2a^2v,(c^2+a^2-b^2)ku+2a^2w)}{2a^2(u+v+w)}.
\end{aligned}$$

Ignoring the denominator, we obtain the homogeneous barycentric coordinates of X_k given above; similarly for Y_k and Z_k. □

Proposition 2. *The only points that satisfy Kariya's theorem, as the incenter I does, are the orthocenter H, the circumcenter O, and the three excenters I_a, I_b, I_c.*

Proof. The equations of the lines AX_k, BY_k, CZ_k are

$$\begin{aligned}
((c^2+a^2-b^2)ku+2a^2w)y-((a^2+b^2-c^2)ku+2a^2v)z =&\ 0,\\
-((b^2+c^2-a^2)kv+2b^2w)x+((a^2+b^2-c^2)kv+2b^2u)z =&\ 0,\\
((b^2+c^2-a^2)kw+2c^2v)x-((c^2+a^2-b^2)kw+2c^2u)y =&\ 0.
\end{aligned}$$

They are concurrent if and only if

$$\begin{vmatrix}
0 & (c^2+a^2-b^2)ku+2a^2w & -((a^2+b^2-c^2)ku+2a^2v)\\
-(b^2+c^2-a^2)kv+2b^2w) & 0 & (a^2+b^2-c^2)kv+2b^2u\\
(b^2+c^2-a^2)kw+2c^2v & -((c^2+a^2-b^2)kw+2c^2u) & 0
\end{vmatrix}=0.$$

Equivalently, $2F_1(P) \cdot k - F_2(P) = 0$ for every k, where

$$F_1(P) = \sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)u(c^2v^2 - b^2w^2),$$
$$F_2(P) = \sum_{\text{cyclic}} (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)u(c^2v^2 - b^2w^2).$$

This means that $F_1(P) = F_2(P) = 0$. The point P is common to the McCay cubic p$K(X(6), X(3))$, and the orthocubics p$K(X(6), X(4))$. These appear in [3] as **K003** and **K006** respectively. It is known that the common points of these circumcubics are the vertices of ABC and the points H, O, I, I_a, I_b, I_c, as can be readily verified. □

The case $P = H$ is trivial because the perspector is H for every k.

The case $P = O$ is also trivial because the triangle $\mathbf{T}_k$ is homothetic to ABC, and the perspector is obviously the point Q_k on the Euler line dividing OH in the ratio k : 2. This follows from

$$OQ_k : Q_kH = OX(k) : HA = k : 2.$$

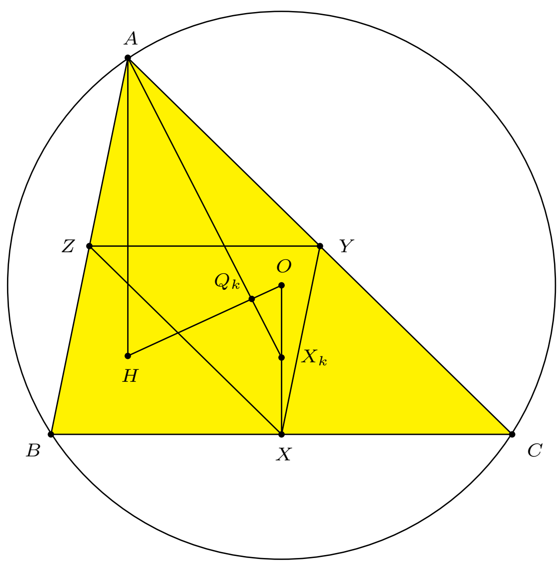

Figure 1

In the remainder of this paper, we study the cases when P is the incenter or an excenter.

3. Kariya's theorem and the Feuerbach hyperbola

The pedal triangle of the incenter $I = (a : b : c)$ has vertices

$$\begin{aligned} X = &\ (0 : a + b - c : c + a - b), \\ Y = &\ (a + b - c : 0 : b + c - a), \\ Z = &\ (c + a - b : b + c - a : 0). \end{aligned}$$

The coordinates of $X(t)$, $Y(t)$, $Z(t)$ can be determined from equations (1), (2), (3) by putting $k = \frac{t}{r}$.

Proposition 3. *The lines $AX(t)$, $BY(t)$, $CZ(t)$ are concurrent at the point*

$$Q(t) = \left(\frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)}\right), \tag{4}$$

which is the isogonal conjugate of the point $P(t)$ dividing OI in the ratio

$$OP(t) : P(t)I = R : t.$$

Proof. Writing the homogeneous barycentric coordinates of $X(t)$, $Y(t)$, $Z(t)$ as

$$\begin{aligned} X(t) = &\ \left(* * * * * : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : \frac{1}{2rab + t(a^2 + b^2 - c^2)}\right), \\ Y(t) = &\ \left(\frac{1}{2rbc + t(b^2 + c^2 - a^2)} : * * * * * : \frac{1}{2rab + t(a^2 + b^2 - c^2)}\right), \\ Z(t) = &\ \left(\frac{1}{2rbc + t(b^2 + c^2 - a^2)} : \frac{1}{2rca + t(c^2 + a^2 - b^2)} : * * * * *\right), \end{aligned}$$

we note that the lines $AX(t)$, $BY(t)$, $CZ(t)$ are concurrent at a point $Q(t)$ with coordinates given in (4) above. This is clearly the isogonal conjugate of the point

$$\begin{aligned} P(t) = &\ (2rabc \cdot a + ta^2(b^2 + c^2 - a^2), 2rabc \cdot b + tb^2(c^2 + a^2 - b^2), \\ &\ \quad 2rabc \cdot c + tc^2(a^2 + b^2 - c^2)) \\ = &\ 2rabc(a, b, c) + t((a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)) \\ = &\ 2r \cdot 4Rrs(a + b + c) \cdot I + t \cdot 16\Delta^2 \cdot O \\ = &\ 16r^2s^2 \cdot R \cdot I + 16\Delta^2 \cdot t \cdot O \\ = &\ 16r^2s^2(R \cdot I + t \cdot O). \end{aligned}$$

In absolute barycentric coordinates, $P(t) = \frac{R \cdot I + t \cdot O}{R + t}$. This is the point dividing OI in the ratio $OP(t) : P(t)I = R : t$. $\square$

It follows that the locus of the point $Q(t)$ is the Feuerbach hyperbola $\mathscr{F}$. The center is the Feuerbach point F_e, the point of tangency of the incircle and the nine-point circle (see Figure 2). Note that for each t, the points $P(t)$ and $P(-t)$ divide OI harmonically.

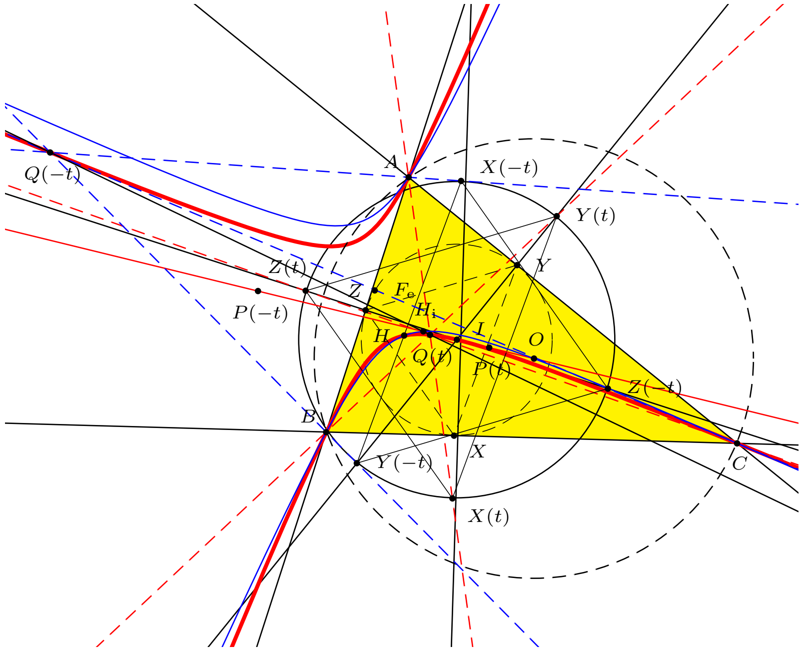

Figure 2.

Proposition 4. *The line joining $Q(t)$ and $Q(-t)$ contains the triangle center*

$$H_\mathrm{i} = \left(\frac{a(b+c)}{b+c-a} : \frac{b(c+a)}{c+a-b} : \frac{c(a+b)}{a+b-c}\right). \tag{5}$$

Proof. The line joining $Q(t)$ to $Q(-t)$ has equation

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)(4b^2c^2r^2 - t^2(b^2+c^2-a^2)^2)x = 0.$$

With $(x : y : z)$ given in (5) we have

$$\begin{aligned}
&\sum_{\text{cyclic}} a^2(b^2-c^2)(4b^2c^2r^2 - t^2(b^2+c^2-a^2)^2) \\
=\ & 4a^2b^2c^2r^2\left(\sum_{\text{cyclic}}(b^2-c^2)\right) - t^2\left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^2+c^2-a^2)^2\right) \\
=\ & 0.
\end{aligned}$$

While the first sum obviously is zero, the second sum vanishes because the Euler line

$$\sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x = 0 \tag{6}$$

contains the circumcenter. □

Remark. The triangle center H_i is the orthocenter of the intouch triangle $\mathbf{T}_\mathrm{i}$; it appears as $X(65)$ in [4]. It divides OI in the ratio $R+r : -r$. Its isogonal conjugate is the Schiffler point

$$S_\mathrm{c} = \left(\frac{a(b+c-a)}{b+c} : \frac{b(c+a-b)}{c+a} : \frac{c(a+b-c)}{a+b}\right), \tag{7}$$

which is the point of concurrency of the Euler lines of the four triangles ABC, IBC, ICA, and IAB. Therefore, H_i is a point on the Jerabek hyperbola $\mathscr{J}$:

$$\frac{a^2(b^2-c^2)(b^2+c^2-a^2)}{x}+\frac{b^2(c^2-a^2)(c^2+a^2-b^2)}{y}+\frac{c^2(a^2-b^2)(a^2+b^2-c^2)}{z}=0,$$

the isogonal conjugate of the Euler line.

4. The touchpoints triangles

Consider the A-excircle $I_a(r_a)$ of triangle ABC, tangent to the sidelines BC at X_a, CA at Y_a, and AB at Z_a respectively. We call triangle $\mathbf{T}_a := X_aY_aZ_a$ the A-touchpoints triangle. Clearly,

$$I_aX_a = I_aY_a = I_aZ_a = r_a = \frac{\Delta}{s-a}.$$

In homogeneous barycentric coordinates these are the points

$$\begin{aligned} I_a &= (-a : b : c),\\ X_a &= (0 : c+a-b : a+b-c),\\ Y_a &= (-(c+a-b) : 0 : a+b+c),\\ Z_a &= (-(a+b-c) : a+b+c : 0). \end{aligned}$$

Similarly, we also have the B-touchpoints triangle $\mathbf{T}_b := X_bY_bZ_b$ from the B-excircle $I_b(r_b)$ and the C-touchpoints triangles $\mathbf{T}_c := X_cY_cZ_c$ from the C-excircle $I_c(r_c)$ (see Figure 3).

Consider the reflection I'_a of I_a in the line Y_aZ_a. Since Y_aZ_a is perpendicular to the line AI_a, I'_a lies on AI_a, and $I_aI'_a = 2r_a \sin\frac{A}{2}$. On the other hand, I_aI, being a diameter of the circle through I, B, I_a, C, has length $\frac{a}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)} = \frac{a}{\cos\frac{A}{2}}$. It follows that

$$I_aI'_a : I_aI = 2r_a\sin\frac{A}{2} : \frac{a}{\cos\frac{A}{2}} = r_a : \frac{a}{\sin A} = r_a : 2R = I_aX_a : I_aI'.$$

Therefore, $X_aI'_a$ and $I'I$ are parallel. Note that the midpoint of $X_aI'_a$ is the nine-point center of $\mathbf{T}_a$.

The same conclusions apply to the other two touchpoints triangles $\mathbf{T}_b := X_bY_bZ_b$ and $\mathbf{T}_c := X_cY_cZ_c$ associated with the B- and C-excircles $I_b(r_b)$ and $I_c(r_c)$.

Corollary 5. (a) *The Euler lines of the touchpoints triangles of the excircles are concurrent at O.*
(b) *The nine-point centers of the touchpoints triangles form a triangle perspective with the extouch triangle $X_aY_bZ_c$ at the infinite point of the OI line.*

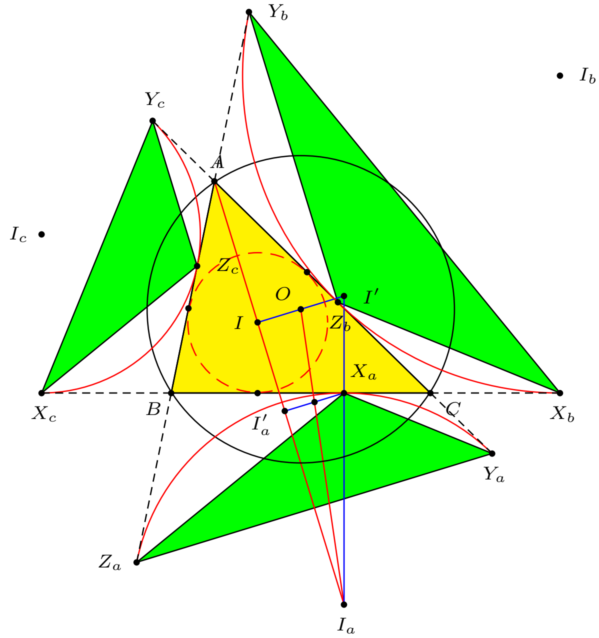

Figure 3.

Let H^a be the orthocenter of $\mathbf{T}_a$. Since the nine-point center N_a is also the midpoint of its circumcenter I_a and H^a, we obtain, from the parallelogram $H^a X_a I_a I'_a$,

$$\begin{aligned}
H^a =&\ X_a + I'_a - I_a \\
=&\ X_a + \frac{r_a}{2R}(I - I_a) \\
=&\ X_a + \frac{2s(s-b)(s-c)}{abc}(I - I_a) \\
=&\ \frac{(s-b)B + (s-c)C}{a} + \frac{2s(s-b)(s-c)}{abc}\left(\frac{aA + bB + cC}{2s} - \frac{-aA + bB + cC}{2(s-a)}\right) \\
=&\ \frac{(b+c)(s-b)(s-c)}{bc(s-a)}A + \frac{s(c-a)(s-b)}{ac(s-a)}B - \frac{s(a-b)(s-c)}{ab(s-a)}C.
\end{aligned}$$

Proposition 6. *In homogeneous barycentric coordinates, the orthocenters of the touchpoints triangles are the points*

$$\begin{aligned}
H^a =& \left(\frac{a(b+c)}{a+b+c} : \frac{b(c-a)}{a+b-c} : \frac{-c(a-b)}{c+a-b}\right), \\
H^b =& \left(\frac{-a(b-c)}{a+b-c} : \frac{b(c+a)}{a+b+c} : \frac{c(a-b)}{b+c-a}\right), \\
H^c =& \left(\frac{a(b-c)}{c+a-b} : \frac{-b(c-a)}{b+c-a} : \frac{c(a+b)}{a+b+c}\right).
\end{aligned}$$

Remark. The orthocenter H^a of $\mathbf{T}_a$ divides OI_a in the ratio $R - r_a : r_a$; similarly for H^b and H^c. These orthocenters lie on the Jerabek hyperbola $\mathscr{J}$ (see Figure 4).

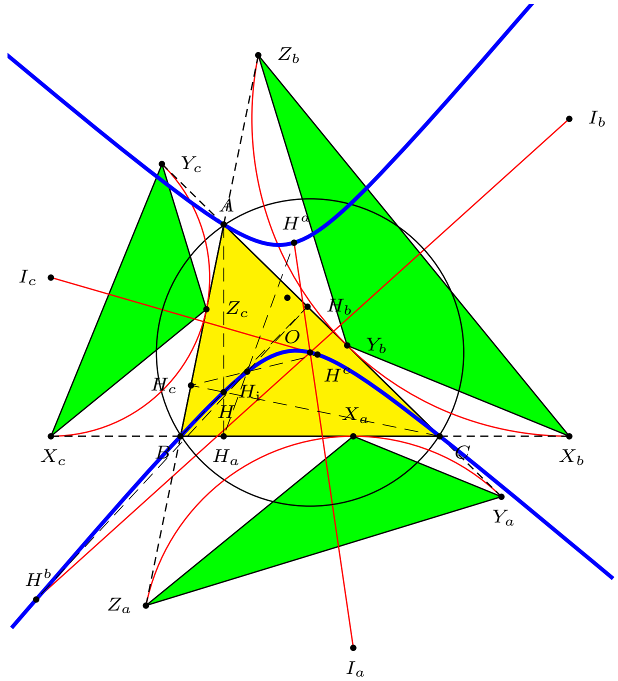

Figure 4.

Proposition 7. *The triangle $H^aH^bH^c$ is perspective with the orthic triangle $H_aH_bH_c$ (of ABC) at H_i.*

5. Kariya's theorem for the A-touchpoints triangle

Consider the image of the A-touchpoints triangle $\mathbf{T}_a$ under the homothety $\mathsf{h}\left(I_a, \frac{t}{r_a}\right)$ for a real number t. This is the triangle $\mathbf{T}_a(t)$ with vertices $X_a(t)$, $Y_a(t)$, $Z_a(t)$ on the lines I_aX_a, I_aY_a, I_aZ_a respectively, such that

$$I_aX_a(t) = I_aY_a(t) = I_aZ_a(t) = t.$$

These points can be determined from Lemma 1 by putting $k = \frac{t}{r_a}$. In homogeneous barycentric coordinates, they are

$$X_a(t) = \ (2(r_a - t)a^2 : -2r_aab + t(a^2 + b^2 - c^2) : -2r_aca + t(c^2 + a^2 - b^2)), \quad (8)$$
$$Y_a(t) = \ (-2r_aab + t(a^2 + b^2 - c^2) : 2(r_a - t)b^2 : 2r_abc + t(b^2 + c^2 - a^2)), \quad (9)$$
$$Z_a(t) = \ (-2r_aca + t(c^2 + a^2 - b^2) : 2r_abc + t(b^2 + c^2 - a^2) : 2(r_a - t)c^2). \quad (10)$$

Proposition 8. *The lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at the point*

$$Q_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right), \tag{11}$$

which is the isogonal conjugate of the point $P_a(t)$ dividing OI_a in the ratio

$$OP_a(t) : P_a(t)I_a = R : -t.$$

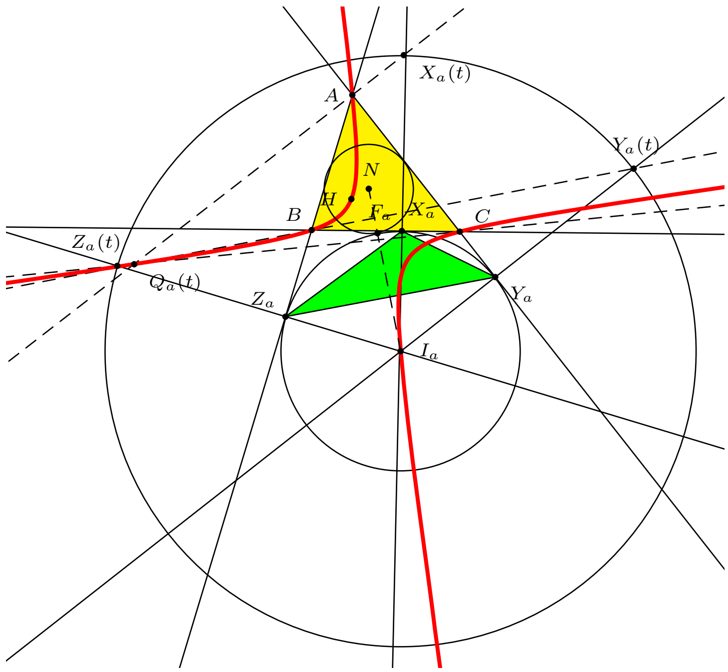

Figure 5.

Proof. Rewrite the homogeneous barycentric coordinates of $X_a(t)$, $Y_a(t)$, $Z_a(t)$ as follows:

$$X_a(t) = \left(*****: \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right),$$

$$Y_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : *****: \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right),$$

$$Z_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : *****\right).$$

It follows easily that the lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at a point with coordinates given in (11) above (see Figure 5). This is clearly the isogonal conjugate of the point

$$\begin{aligned} P_a(t) = &\ (a^2(2r_abc + t(b^2+c^2-a^2)), b^2(-2r_aca + t(c^2+a^2-b^2)), \\ &\qquad\qquad c^2(-2r_aab + t(a^2+b^2-c^2))) \\ =&\ -2r_aabc(-a,b,c) + t((a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)) \\ =&\ -2r_a \cdot 4Rrs(b+c-a)\cdot I_a + t\cdot 16\Delta^2 \cdot O \\ =&\ -16rr_as(s-a)\cdot R\cdot I_a + 16\Delta^2\cdot t\cdot O \\ =&\ -16\Delta^2(R\cdot I_a - t\cdot O). \end{aligned}$$

In absolute barycentric coordinates, $P_a(t) = \frac{R\cdot I_a - t\cdot O}{R-t}$. This is the point dividing OI_a in the ratio $OP_a(t) : P_a(t)I_a = R : -t$. □

Proposition 9. *The locus of the point $Q_a(t)$ is the rectangular circum-hyperbola*

$$\mathscr{F}_a : \ a(b-c)(a+b+c)yz + b(c+a)(a+b-c)zx - c(a+b)(c+a-b)xy = 0$$

with center

$$F_a = (-(b-c)^2(a+b+c) : (c+a)^2(a+b-c) : (a+b)^2(c+a-b)),$$

the point of tangency of the nine-point circle with the A-excircle.

Proof. By Proposition 8, the locus of $Q_a(t)$ is the isogonal conjugate of the line OI_a. It is a rectangular hyperbola since it contains the orthocenter H, the isogonal conjugate of O. The center of the hyperbola is a point on the nine-point circle. The equation of the line OI_a is

$$\begin{vmatrix} x & y & z \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \\ -a & b & c \end{vmatrix} = 0.$$

After simplification, this becomes

$$\begin{aligned} &- bc(b-c)(a+b+c)(b+c-a)x \\ &- ca(c+a)(a+b-c)(b+c-a)y \\ &+ ab(a+b)(b+c-a)(c+a-b)z = 0. \end{aligned}$$

Replacing (x,y,z) by (a^2yz, b^2zx, c^2xy) we obtain the equation of the hyperbola $\mathscr{F}_a$ given above. Since the center of the circumconic $pyz + qzx + rxy = 0$ is the point

$$(p(q+r-p) : q(r+p-q) : r(p+q-r)),$$

with

$$p = bc(b-c)(a+b+c), \quad q = ca(c+a)(a+b-c), \quad r = -ab(a+b)(c+a-b),$$

we obtain the center of the hyperbola as the point

$$\begin{aligned} F_a = &\ p(q+r-p) : q(r+p-q) : r(p+q-r) \\ =&\ -2abc(b-c)^2(a+b+c) : 2abc(c+a)^2(a+b-c) : 2abc(a+b)^2(c+a-b) \\ =&\ -(b-c)^2(a+b+c) : (c+a)^2(a+b-c) : (a+b)^2(c+a-b). \end{aligned}$$

This is indeed a point on the line joining I_a to the nine-point center

$$N=(a^2(b^2+c^2)-(b^2-c^2)^2 : b^2(c^2+a^2)-(c^2-a^2)^2 : c^2(a^2+b^2)-(a^2-b^2)^2). \tag{12}$$

It is routine to verify that

$$\begin{aligned}&2abc(-a,b,c)\\ &+(a^2(b^2+c^2)-(b^2-c^2)^2, b^2(c^2+a^2)-(c^2-a^2)^2, c^2(a^2+b^2)-(a^2-b^2)^2)\\ =&\ (b+c-a)(-(b-c)^2(a+b+c),(c+a)^2(a+b-c),(a+b)^2(c+a-b)).\end{aligned}$$

Since the coordinate sum in (12) is

$$\begin{aligned}&2(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\\ =&\ 2(a+b+c)(b+c-a)(c+a-b)(a+b-c)\\ =&\ 32\Delta^2,\end{aligned}$$

this is the point dividing NI_a in the ratio

$$2abc(b+c-a):32\Delta^2=2\cdot 4R\Delta(b+c-a):32\Delta^2=\frac{R}{2}:\frac{2\Delta}{b+c-a}=\frac{R}{2}:r_a,$$

i.e., the point of tangency of the nine-point circle and the A-excircle. □

Proposition 10. *The line joining $Q_a(t)$ and $Q_a(-t)$ contains the orthocenter H^a of the A-touchpoints triangle $\mathbf{T}_a$.*

Proof. The equation of the line $Q_a(t)Q_a(-t)$ is

$$\begin{aligned}&a(b-c)(a+b+c)(-4b^2c^2r_a^2+t^2(b^2+c^2-a^2)^2)x\\ &+b(c+a)(a+b-c)(-4c^2a^2r_a^2+t^2(c^2+a^2-b^2)^2)y\\ &-c(a+b)(c+a-b)(-4a^2b^2r_a^2+t^2(a^2+b^2-c^2)^2)z\\ &=0.\end{aligned}$$

Substituting the coordinates of the point H^a given in Proposition 6, we obtain

$$\begin{aligned}&-4a^2b^2c^2r_a^2\left((b^2-c^2)+(c^2-a^2)+(a^2-b^2)\right)\\ &+t^2\left(a^2(b^2-c^2)(b^2+c^2-a^2)^2+b^2(c^2-a^2)(c^2+a^2-b^2)^2\right.\\ &\qquad\left.+\ c^2(a^2-b^2)(a^2+b^2-c^2)^2\right)\\ =&\ 0,\end{aligned}$$

as in the proof of Proposition 4. □

6. The triad of ex-Feuerbach hyperbolas

We call the hyperbola $\mathscr{F}_a$ in Proposition 9 the A-ex-Feuerbach hyperbola. We also consider the triangles $\mathbf{T}_b(t):=X_b(t)Y_b(t)Z_b(t)$ and $\mathbf{T}_c(t):=X_c(t)Y_c(t)Z_c(t)$. These vertices are the points on the lines I_bX_b, I_bY_b, I_bZ_b satisfying

$$I_bX_b(t)=I_bY_b(t)=I_bZ_b(t)=t,$$

and $X_c(t)$, $Y_c(t)$, $Z_c(t)$ on I_cX_c, I_cY_c, I_cZ_c satisfying

$$I_cX_c(t) = I_cY_c(t) = I_cZ_c(t) = t.$$

In homogeneous barycentric coordinates,

$$\begin{aligned}
X_b(t) =&\ (2(r_b - t)a^2 : -2r_bab + t(a^2+b^2-c^2) : 2r_bca + t(c^2+a^2-b^2)),\\
Y_b(t) =&\ (-2r_bab + t(a^2+b^2-c^2) : 2(r_b - t)b^2 : -2r_bbc + t(b^2+c^2-a^2)),\\
Z_b(t) =&\ (2r_bca + t(c^2+a^2-b^2) : -2r_bbc + t(b^2+c^2-a^2) : 2(r_b - t)c^2);\\
X_c(t) =&\ (2(r_c - t)a^2 : 2r_cab + t(a^2+b^2-c^2) : -2r_cca + t(c^2+a^2-b^2)),\\
Y_c(t) =&\ (2r_cab + t(a^2+b^2-c^2) : 2(r_c - t)b^2 : -2r_cbc + t(b^2+c^2-a^2)),\\
Z_c(t) =&\ (-2r_cca + t(c^2+a^2-b^2) : -2r_cbc + t(b^2+c^2-a^2) : 2(r_c - t)c^2).
\end{aligned}$$

Clearly there are analogous hyperbolas $\mathscr{F}_b$ and $\mathscr{F}_c$ which are isogonal conjugates of the lines OI_b and OI_c. These hyperbolas have centers

$$\begin{aligned}
F_b =&\ ((b+c)^2(a+b-c) :\ -(c-a)^2(a+b+c) :\ (a+b)^2(b+c-a)),\\
F_c =&\ ((b+c)^2(c+a-b) :\ (c+a)^2(b+c-a) :\ -(a-b)^2(a+b+c)).
\end{aligned}$$

Remark. The centers of the triad of ex-Feuerbach hyperbolas, being the points of tangency of the nine-point circle with the excircles, are perspective with ABC at the outer Feuerbach point

$$X(12) = \left(\frac{(b+c)^2}{b+c-a} : \frac{(c+a)^2}{c+a-b} : \frac{(a+b)^2}{a+b-c}\right).$$

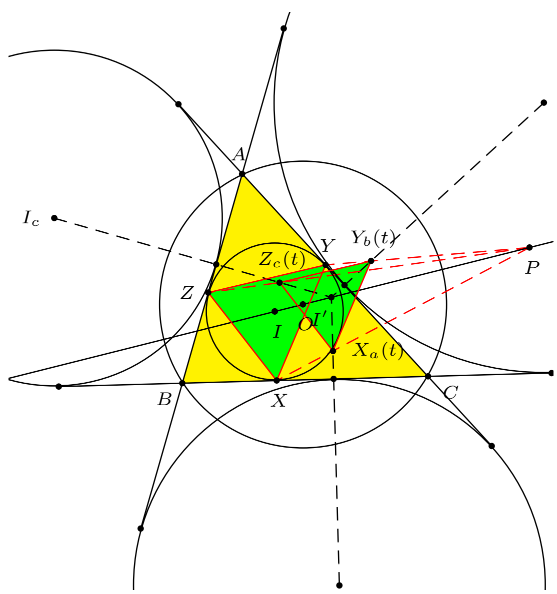

Figure 6.

Proposition 11. *The triangle $X_a(t)Y_b(t)Z_c(t)$ is homothetic to the intouch triangle at the point $(a(r_a - t) : b(r_b - t) : c(r_c - t)$, which divides OI in the ratio* $2R + r - t : -2r$ (see Figure 6).

7. Some special cases

7.1. $t = R$. By Corollary 5(b), the point $P_a(R)$ is the infinite point on the line OI_a. It follows that $Q_a(R)$ is on the circumcircle. It is the (fourth) intersection of the hyperbola $\mathscr{F}_a$ with the circumcircle. These points are the reflections of H in F_a, F_b, F_c respectively (see Figure 7).

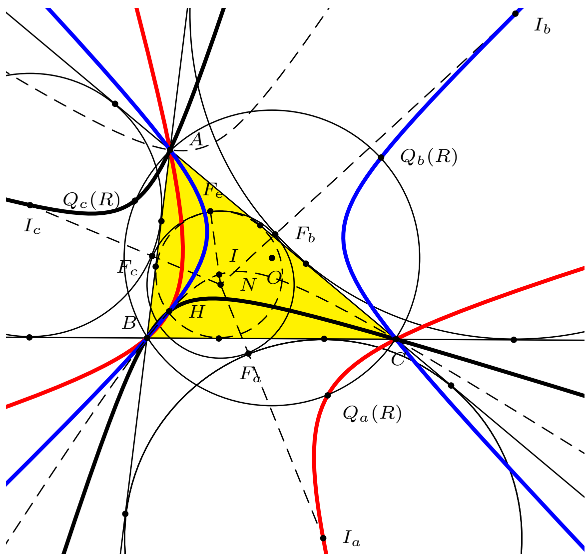

Figure 7.

7.2. $t = 2R$. It is well known that the circumcenter of the excentral triangle is the reflection I' of I in O, and is the point of concurrency of the perpendiculars from the excenters to the respective sidelines of triangle ABC (see Figure 6), and the circumradius is $2R$. It follows that the points $X_a(2R)$, $Y_b(2R)$, $Z_c(2R)$ all coincide with this circumcenter. It follows that the lines $AQ_a(2R)$, $BQ_b(2R)$, $CQ_c(2R)$ are concurrent at this point. $t = 2R$ is the only nonzero value of t for which the triangle $Q_a(t)Q_b(t)Q_c(t)$ is perspective with ABC.

In this case, both $Y_c(2R)$ and $Z_b(2R)$ are the reflection of I' in the line I_bI_c. We call this X'. The line AX' is the reflection of AI' in I_bI_c. Since I_bI_c is the external bisector of angle A of triangle ABC, AX' and AI' are isogonal lines with respect to this angle. Likewise, we have $Y' = Z_a(2R) = X_c(2R)$ with BY', BI' isogonal with respect to B, and and $Z' = Y_a(2R) = X_b(2R)$ with CZ', CI' isogonal with respect to C. It follows that AX', BY', CZ' are concurrent at a point P, which is the isogonal conjugate of I', and lies on the Feuerbach hyperbola $\mathscr{F}$ (see Figure 8).

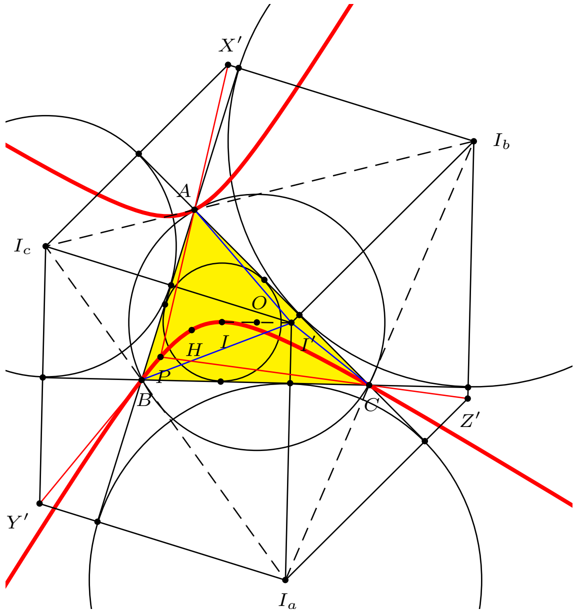

Figure 8.

8. Second tangents from O to the ex-Feuerbach hyperbolas

The hyperbolas $\mathscr{F}_a$, $\mathscr{F}_b$, $\mathscr{F}_c$ also appear in [1], where they are called the ex-central Feuerbach hyperbolas. Neuberg [5] also mentioned these hyperbolas. The A-ex-Feuerbach hyperbola $\mathscr{F}_a$, being the isogonal conjugate of the line OI_a, is tangent to the line at I_a (see Figure 9). If the second tangent from O to $\mathscr{F}_a$ touches it at T_a, then the line I_aT_a is the polar of O with respect to the hyperbola $\mathscr{F}_a$. This is the line

$$\begin{aligned}&bc(b-c)(a+b+c)(a(b^2+c^2-a^2)+(b+c)(c+a-b)(a+b-c))x\\&+ca(c+a)(a+b-c)(c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b))y\\&-ab(a+b)(c+a-b)(b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a))z\\&=0.\end{aligned}$$

Apart from the excenter I_a, this line intersects the hyperbola $\mathscr{F}_a$ again at

$$\begin{aligned}T_a=\bigg(&\frac{a}{a(b^2+c^2-a^2)+(b+c)(c+a-b)(a+b-c)}\\&:\frac{b}{c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b)}\\&:\frac{c}{b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a)}\bigg).\end{aligned}$$

Similarly, the second tangents from O to $\mathscr{F}_b$ and $\mathscr{F}_c$ (apart from OI_b and OI_c) touch these hyperbolas at

$$T_b = \left(\frac{a}{c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b)}\right.$$
$$: \frac{b}{b(c^2+a^2-b^2)+(c+a)(a+b-c)(b+c-a)}$$
$$\left.: \frac{c}{a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c)}\right),$$

and

$$T_c = \left(\frac{a}{b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a)}\right.$$
$$: \frac{b}{a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c)}$$
$$\left.: \frac{c}{c(a^2+b^2-c^2)+(a+b)(b+c-a)(c+a-b)}\right),$$

These three points of tangency form a triangle perspective with ABC at

$$T = (a(a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c))$$
$$: b(b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a))$$
$$: c(c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b))).$$

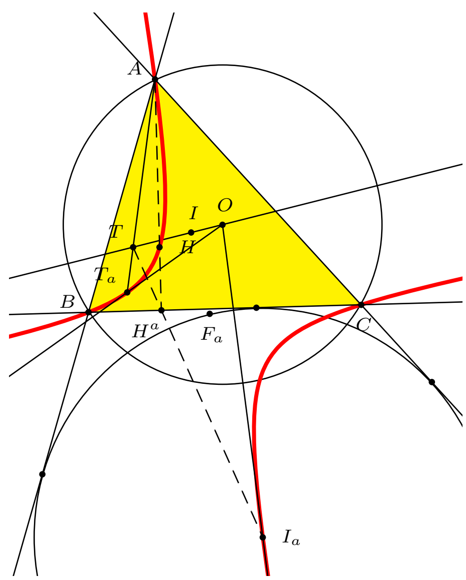

Figure 9.

This is the triangle center $X(46)$ in [4]. It has a number of interesting properties. It divides OI externally in the ratio $R + r : -2r$, and can be constructed as the cevian quotient H/I. In other words, it is the perspector of the orthic triangle and the excentral triangle. Therefore, the point T_a, and similarly T_b and T_c, can be easily constructed as follows.
(1) Join I_a and H_a to intersect the line OI at T.
(2) Join A and T to intersect the hypebola $\mathscr{F}_a$ at T_a (see Figure 9).

9. A correspondence between the Euler line and the Feuerbach hyperbola

Let $P = (u : v : w)$ be an aribitrary point. The lines PI_a, PI_b, PI_c intersect the respective ex-Feuerbach hyperbolas at

$$W_a = \left(\frac{(b-c)(a+b+c)}{cv-bw} : \frac{c+a)(a+b-c)}{aw+cu} : \frac{(a+b)(c+a-b)}{bu+av}\right),$$
$$W_b = \left(\frac{(b+c)(a+b-c)}{cv+bw} : \frac{c-a)(a+b+c)}{aw-cu} : \frac{(a+b)(b+c-a)}{bu+av}\right),$$
$$W_c = \left(\frac{(b+c)(c+a-b)}{cv+bw} : \frac{c+a)(b+c-a)}{aw+cu} : \frac{(a-b)(a+b-c)}{bu-av}\right).$$

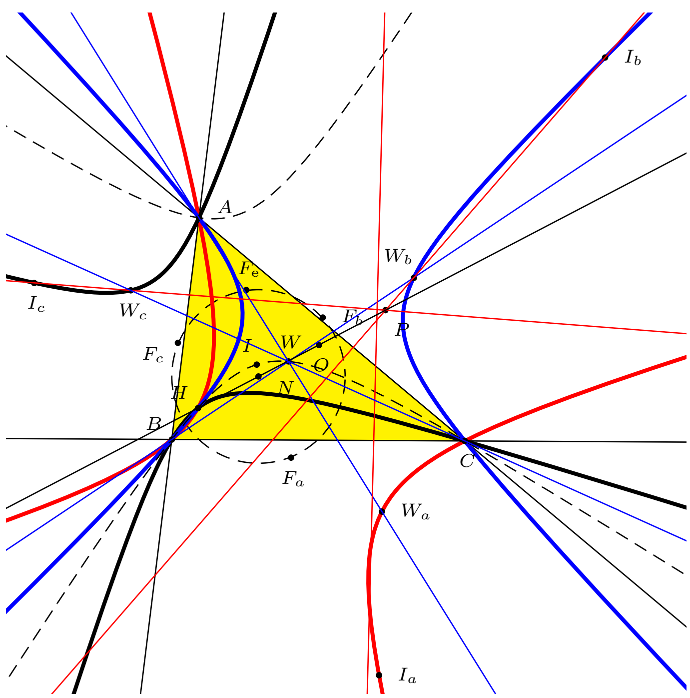

Figure 10.

These form a triangle perspective with ABC. The perspector is the point

$$W = \left(\frac{b+c}{(b+c-a)(cv+bw)} : \frac{c+a}{(c+a-b)(aw+cu)} : \frac{a+b}{(a+b-c)(bu+av)}\right).$$

Proposition 12. *The perspector W is on the Feuerbach hyperbola if and only if P lies on the Euler line.*

Proof. The perspector W is on the Feuerbach hyperbola if and only if its isogonal conjugate

$$W^* = \left(\frac{a^2(b+c-a)(cv+bw)}{b+c} : \frac{b^2(c+a-b)(aw+cu)}{c+a} : \frac{c^2(a+b-c)(bu+av)}{a+b}\right)$$

lies on the line OI with equation

$$\sum_{\text{cyclic}} bc(b-c)(b+c-a)x = 0.$$

By substitution, we have

$$\begin{aligned}
0 =& \sum_{\text{cyclic}} bc(b-c)(b+c-a) \cdot \frac{a^2(b+c-a)(cv+bw)}{b+c} \\
=& \; abc \sum_{\text{cyclic}} \frac{a(b-c)(b+c-a)^2(cv+bw)}{b+c} \\
=& \; \frac{abc}{(b+c)(c+a)(a+b)} \sum_{\text{cyclic}} a(b-c)(c+a)(a+b)(b+c-a)^2(cv+bw).
\end{aligned}$$

Ignoring the nonzero factor, we have

$$\begin{aligned}
0 =& \sum_{\text{cyclic}} a(b-c)(c+a)(a+b)(b+c-a)^2(cv+bw) \\
=& \sum_{\text{cyclic}} (b(c-a)(a+b)(b+c)(c+a-b)^2 \cdot cu \\
& \qquad + c(a-b)(b+c)(c+a)(a+b-c)^2 \cdot bu) \\
=& \sum_{\text{cyclic}} bc(b+c)((c-a)(a+b)(c+a-b)^2 + (a-b)(c+a)(a+b-c)^2)u \\
=& \sum_{\text{cyclic}} bc(b+c) \cdot (-2a(b-c)(b^2+c^2-a^2))u \\
=& -2abc \sum_{\text{cyclic}} (b+c)(b-c)(b^2+c^2-a^2)u.
\end{aligned}$$

This means that $P = (u : v : w)$ lies on the Euler line (with equation given in (6)). $\square$

If P divides OH in the ratio $OP : PH = t : 1-t$, then the isogonal conjugate of W is the point dividing OI in the ratio $OW^* : W^*I = \frac{R}{2}(1-t) : rt$. A simple application of Menelaus' theorem yields the following construction of W^*. Let P' be the inferior of P. Then the line F_eP' intersects OI at W^* (see Figure 11).

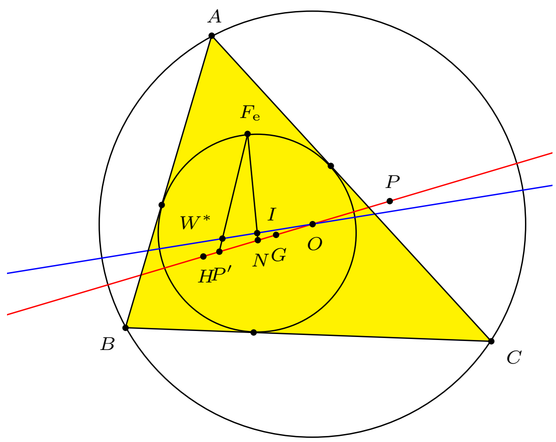

Figure 11.

If we put $W = (x : y : z)$, then P is the point with coordinates

$$\begin{aligned}(u:v:w) = \Bigg(a\left(-\frac{a(b+c)}{(b+c-a)x}+\frac{b(c+a)}{(c+a-b)y}+\frac{c(a+b)}{(a+b-c)z}\right)\\ :\ b\left(\frac{a(b+c)}{(b+c-a)x}-\frac{b(c+a)}{(c+a-b)y}+\frac{c(a+b)}{(a+b-c)z}\right)\\ :\ c\left(\frac{a(b+c)}{(b+c-a)x}+\frac{b(c+a)}{(c+a-b)y}-\frac{c(a+b)}{(a+b-c)z}\right)\Bigg).\end{aligned}$$

10. The asymptotes of the Feuerbach hyperbolas

As is well known, the asymptotes of a rectangular circum-hyperbola which is the isogonal conjugate of a line through O are the Simson lines of the intersections of the line with the circumcircle. For the Feuerbach hyperbola and the ex-Feuerbach hyperbolas, we give an easier construction based on the fact that the lines joining the circumcenter to the incenter and the excenters are tangent to the respective Feuerbach hyperbolas.

Lemma 13. *Let P be a point on a rectangular hyperbola with center O. The tangent to the hyperbola at P intersects the asymptotes at two points on the circle with center P, passing through O.*

Proof. Set up a Cartesian coordinate system with the asymptotes as axes. The equation of the rectangular hyperbola is $xy = c^2$ for some c. If $P\left(ct, \frac{c}{t}\right)$ is a point on the hyperbola, the tangent at P is the line $\frac{1}{2}\left(\frac{c}{t}x + cty\right) = c^2$, or $\frac{x}{t} + yt = 2c$. It intersects the asymptotes (axes) at $X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$. Since P is the midpoint of XY, $PO = PX = PY$. □

Proposition 14. (a) *The lines joining the Feuerbach point $F_{\rm e}$ to the intersections of the incircle with the line OI are the asymptotes of the Feuerbach hyperbola $\mathscr{F}$.*

(b) *The lines joining the point F_a to the intersections of the A-excircle with the line OI_a are the asymptotes of the A-ex-Feuerbach hyperbola $\mathscr{F}_a$; similarly for the hyperbolas $\mathscr{F}_b$ and $\mathscr{F}_c$* (see Figure 12)*.*

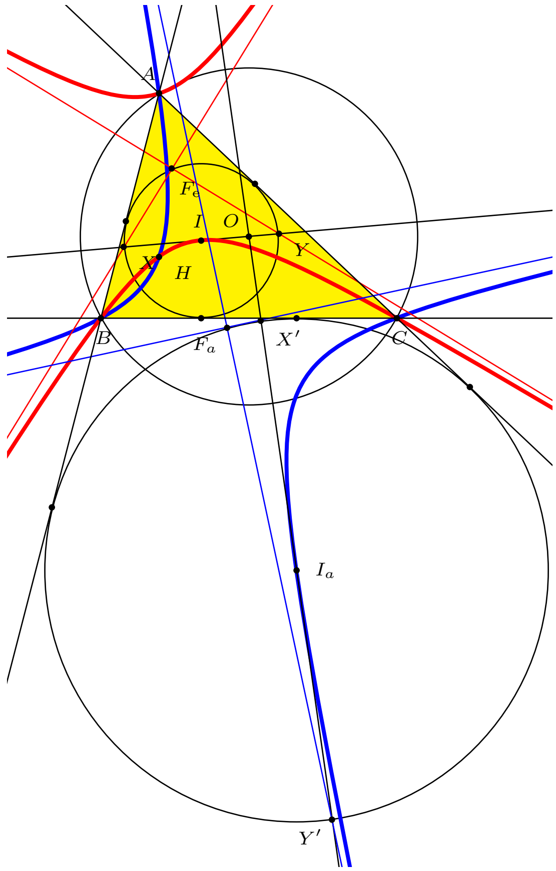

Figure 12.

11. More on the touchpoints triangles

11.1. *The symmedian points of the touchpoints triangles.* Since the A-excircle is the circumcircle of the touchpoints triangle $\mathbf{T}_a$, and the lines BC, CA, AB are the tangents at its vertices, the symmedian point of $\mathbf{T}_a$ is the point of concurrency of BY_a, CZ_a, and AX_a, i.e.,

$$K_a = (-(c+a-b)(a+b-c) : (a+b+c)(c+a-b) : (a+b+c)(a+b-c)).$$

Note that K_a is a point on the A-ex-Feuerbach hyperbola $\mathscr{F}_a$.

The line joining K_a to I_a is the Brocard axis of $\mathbf{T}_a$. It has equation

$$(b-c)(a+b+c)^2 x + (c+a)(a+b-c)^2 y - (a+b)(c+a-b)^2 z = 0.$$

Proposition 15. (a) *The Brocard axes of the touchpoints triangles and the intouch triangle are concurrent at the deLongchamps point L, which is the point on the Euler line of triangle ABC dividing OH in the ratio $-1 : 2$.*

(b) *The van Aubel lines* (joining the orthocenter and the symmedian point) *of the touchpoints triangles and the intouch triangle are concurrent at $H^{\bullet}$, the isotomic conjugate of the orthocenter of ABC* (see Figure 13).

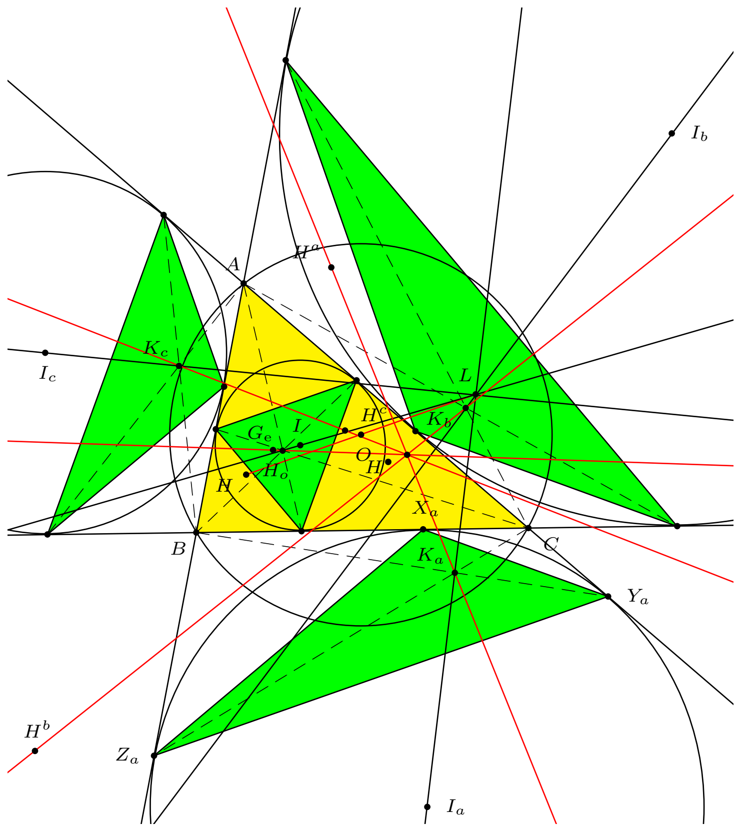

Figure 13.

Remark. The intersection of the Euler line with the line IG_e at the deLongchamps point L is a well known fact. See [6].

Proposition 16. *The triangle $H^aH^bH^c$ is perspective with the cevian triangle of Q if and only if Q lies on the line*

$$\mathscr{L}: \frac{(b+c)(b^2+c^2-a^2)}{b+c-a}x + \frac{(c+a)(c^2+a^2-b^2)}{c+a-b}y + \frac{(a+b)(a^2+b^2-c^2)}{a+b-c}z = 0$$

or the Feuerbach hyperbola $\mathscr{F}$.

(a) *If Q traverses $\mathscr{L}$, the perspector traverses the line* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(b) *If Q is on the Feuerbach hyperbola, the perspector P lies on the Jerabek hyperbola $\mathscr{J}$. The line joining QP passes through the orthocenter H* (see Figure 14).

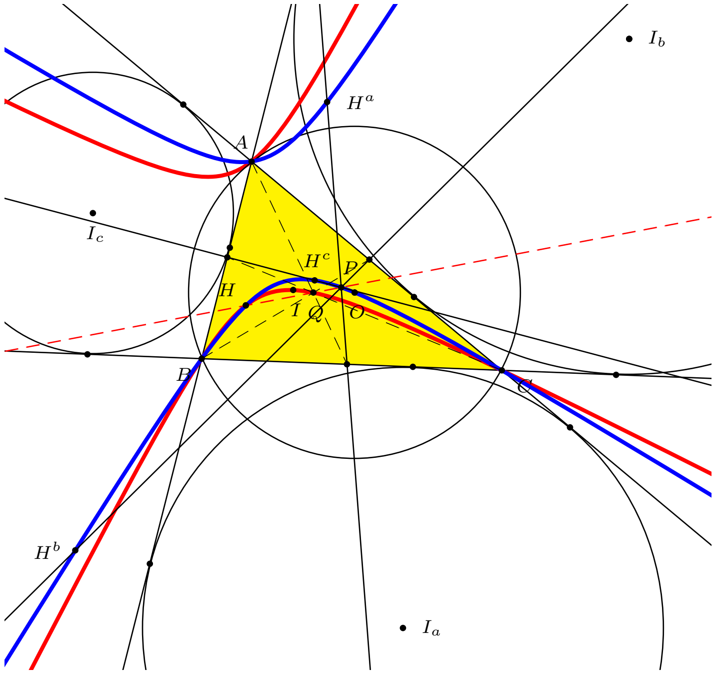

Figure 14.

Proof. Let $Q = (u : v : w)$ with cevian triangle $Q_aQ_bQ_c$ where $Q_a = (0 : v : w)$, $Q_b = (u : 0 : w)$, $Q_c = (u : v : 0)$. The equations of the lines H^aQ_a, H^bQ_b, H^cQ_c are

$$\begin{aligned}&(a+b+c)(c(a-b)(a+b-c)v+b(c-a)(c+a-b)w)x\\ -&a(c+a-b)(a+b-c)(b+c)(wy-vz)=0,\end{aligned} \tag{13}$$

$$\begin{aligned}&(a+b+c)(a(b-c)(b+c-a)w+c(a-b)(a+b-c)u)y\\ -&b(a+b-c)(b+c-a)(c+a)(uz-wx)=0,\end{aligned} \tag{14}$$

$$\begin{aligned}&(a+b+c)(b(c-a)(c+a-b)u+a(b-c)(b+c-a)v)z\\ -&c(b+c-a)(c+a-b)(a+b)(vx-uy)=0.\end{aligned} \tag{15}$$

Eliminating x, y, z from equations (13), (14), (15), we have

$$2abc(a+b+c)\left(\sum_{\text{cyclic}}(b+c)(c+a-b)(a+b-c)(b^2+c^2-a^2)u\right)$$
$$\left(\sum_{\text{cyclic}}a(b-c)(b+c-a)vw\right)=0.$$

Therefore the lines H^aQ_a, H^bQ_b, H^cQ_c are concurrent if and only if $Q = (u : v : w)$ lies on the line $\mathscr{L}$ or the Feuerbach hyperbola $\mathscr{F}$.

Eliminating u, v, w from equations (13), (14), (15), we have

$$32\Delta^2(bcx+cay+abz)\left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^2+c^2-a^2)yz\right)=0.$$

Therefore the locus of the point of concurrency is the union of the line $\mathscr{L}(I)$: $bcx+cay+abz=0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathscr{J}$.

Now the line $\mathscr{L}$ contains the point

$$Q_0=\left(\frac{a(b-c)(b+c-a)^2}{b+c}:\frac{b(c-a)(c+a-b)^2}{c+a}:\frac{c(a-b)(a+b-c)^2}{a+b}\right)$$

as is easily verified. Choosing $Q=(u:v:w)$ to be this point, and solving equations (13), (14), (15), we have the perspector

$$P_0=(a(b^2-c^2):b(c^2-a^2):c(a^2-b^2))$$

on the line $\mathscr{L}(I)$. Therefore, by continuity, when Q traverses the line $\mathscr{L}$, P traverses $\mathscr{L}(I)$.

On the other hand, if Q lies on the Feuerbach hyperbola, then P lies on the Jerabek hyperbola. If we take Q to be the point

$$\left(\frac{1}{bc+t(b^2+c^2-a^2)}:\frac{1}{ca+t(c^2+a^2-b^2)}:\frac{1}{ab+t(a^2+b^2-c^2)}\right)$$

on the Feuerbach hyperbola, then P is the point

$$\left(\frac{a(b+c)(b+c-a)}{(b^2+c^2-a^2)(2rbc+t(b^2+c^2-a^2))}:\cdots:\cdots\right)$$

on the Jerabek hyperbola. The line joining Q and P contains the orthocenter H. $\square$

Remarks. (1) The triangle center Q_0 appears in [4] as $X(1021)$.

(2) The triangle center P_0 is the intersection of the lines $bcx+cay+abz=0$ and $ax+by+cz=0$. It appears in [4] as $X(661)$.

(3) The line $\mathscr{L}$ can be constructed as the line containing the harmonic conjugates of $I_aH\cap BC$ in BC, $I_bH\cap CA$ in CA, and $I_cH\cap AB$ in AB. It is the trilinear polar of the triangle center $X(29)$.

Proposition 17. *The triangle $H^aH^bH^c$ is perspective with the anticevian triangle of Q if and only if Q lies on the orthic axis*

$$\mathscr{L}(H): \qquad (b^2+c^2-a^2)x+(c^2+a^2-b^2)y+(a^2+b^2-c^2)z=0$$

or the circumconic

$$\mathscr{C}: \qquad a(b^2-c^2)yz+b(c^2-a^2)zx+c(a^2-b^2)xy=0$$

passing through I and its isotomic conjugate $I^\bullet=(bc:ca:ab)$.

(a) *If Q traverses the orthic axis, the perspector traverses the line* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (again).

(b) *If Q is on the circumconic $\mathscr{C}$, the perspector P lies on the Jerabek hyperbola* (again) *. The line QP passes through $H_{\rm i} = \left(\frac{a(b+c)}{b+c-a} : \cdots : \cdots\right)$, the common point of the two conics* (see Figure 15).

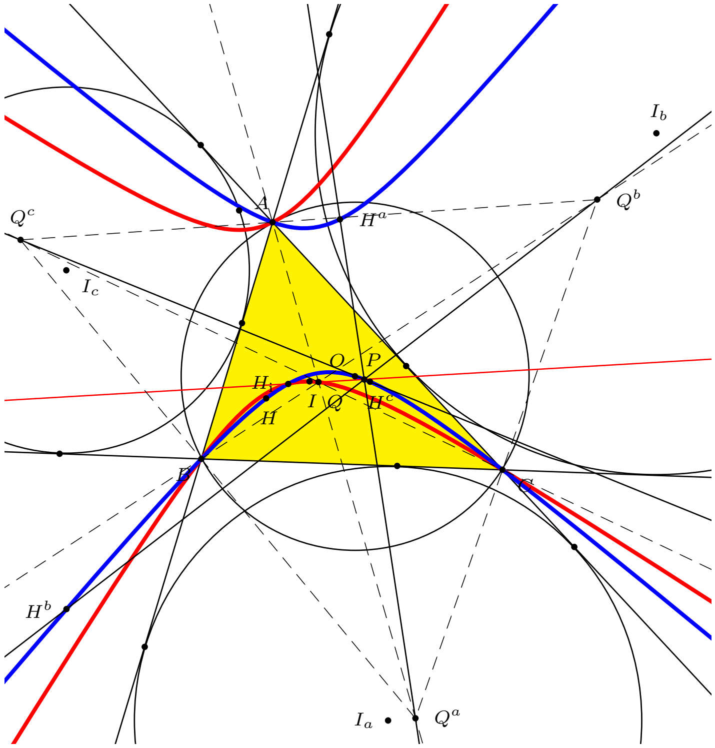

Figure 15.

Proof. Let $Q = (u : v : w)$ with anticevian triangle $Q^aQ^bQ^c$ where $Q^a = (-u : v : w)$, $Q^b = (u : -v : w)$, $Q^c = (u : v : -w)$. The equations of the lines H^aQ^a, H^bQ^b, H^cQ^c are

$$\begin{aligned}
&(a+b+c)(c(a-b)(a+b-c)v + b(c-a)(c+a-b)w)x\\
&+(a+b-c)(-a(b+c)(c+a-b)w + c(a-b)(a+b+c)u)y\\
&+(c+a-b)(b(c-a)(a+b+c)u + a(b+c)(a+b-c)v)z = 0,
\end{aligned}\tag{16}$$

$$\begin{aligned}
&(a+b-c)(c(a-b)(a+b+c)v + b(c+a)(b+c-a)w)x\\
&+(a+b+c)(a(b-c)(b+c-a)w + c(a-b)(a+b-c)u)y\\
&+(b+c-a)(-b(c+a)(a+b-c)u + a(b-c)(a+b+c)v)z = 0,
\end{aligned}\tag{17}$$

$$\begin{aligned}
&(c+a-b)(-c(a+b)(b+c-a)v + b(c-a)(a+b+c)w)x\\
&+(b+c-a)(a(b-c)(a+b+c)w + c(a+b)(c+a-b)u)y\\
&+(a+b+c)(b(c-a)(c+a-b)u + a(b-c)(b+c-a)v)z = 0.
\end{aligned}\tag{18}$$

Eliminating x, y, z, we have

$$64abc\Delta^2\left(\sum_{\text{cyclic}}(b^2+c^2-a^2)u\right)\left(\sum_{\text{cyclic}}a(b^2-c^2)vw\right)=0.$$

Therefore the lines H^aQ^a, H^bQ^b, H^cQ^c are concurrent if and only if $Q=(u:v:w)$ lies on the orthic axis $\mathscr{L}(H)$ or the circumconic $\mathscr{C}$.

Eliminating u, v, w from equations (16), (17), (18), we have

$$64\Delta^2(bcx+cay+abz)\left(\sum_{\text{cyclic}}a^2(b^2-c^2)(b^2+c^2-a^2)yz\right)=0.$$

Therefore the locus of the point of concurrency is again the union of the line $\mathscr{L}(I):bcx+cay+abz=0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathscr{J}$.

Now the circumconic $\mathscr{C}$ is the circum-hyperbola which is the isotomic conjugate of the line joining the incenter I to its isotomic conjugate. Its center is the point

$$(a(b-c)^2:b(c-a)^2:c(a-b)^2).$$

If we choose Q to be the point $\left(\frac{1}{at+bc}:\frac{1}{bt+ca}:\frac{1}{ct+ab}\right)$, then the perspector is the point

$$\left(\frac{a(b^2+c^2-a^2)}{at+bc}:\frac{b(c^2+a^2-b^2)}{bt+ca}:\frac{c(a^2+b^2-c^2)}{ct+ab}\right)$$

on the Jerabek hyperbola. In fact, $\mathscr{C}$ intersects $\mathscr{J}$ at $H_{\text{i}}=\left(\frac{a(b+c)}{b+c-a}:\cdots:\cdots\right)$, and the line joining Q and P passes through H_{i}.

$\square$

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Semi-Similar Complete Quadrangles

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Abstract. Let $\mathcal{A}$ = $A_1A_2A_3A_4$ and $\mathcal{B}$ =$B_1B_2B_3B_4$ be complete quadrangles and assume that each side A_iA_j is parallel to B_hB_k (i, j, h, k is a permutation of $1, 2, 3, 4$). Then $\mathcal{A}$ and $\mathcal{B}$, in general, are not homothetic; they are linked by another strong geometric relation, which we study in this paper. Our main result states that, modulo similarities, the mapping $A_i \to B_i$ is induced by an involutory affinity (an oblique reflection). $\mathcal{A}$ and $\mathcal{B}$ may have quite different aspects, but they share a great number of geometric features and turn out to be similar when $\mathcal{A}$ belongs to the most popular families of quadrangles: cyclic and trapezoids.

1. Introduction

Two triangles whose sides are parallel in pairs are homothetic. A similar statement, trivially, does not apply to quadrilaterals. How about two complete quadrangles with six pairs of parallel sides? An answer cannot be given unless the question is better posed: let $\mathcal{A}$ = $A_1A_2A_3A_4$, $\mathcal{B}$ =$B_1B_2B_3B_4$ be complete quadrangles and assume that each side A_iA_j is parallel to B_iB_j; then $\mathcal{A}$ and $\mathcal{B}$ are indeed homothetic. In fact, the two triangle homotheties, say $A_1A_2A_3 \to B_1B_2B_3$, and $A_4A_2A_3 \to B_4B_2B_3$, must be the same mappings, as they have the same effect on two points. [1] There is, however, another interesting way to relate the six directions. Assume A_iA_j is parallel to B_hB_k (i, j, h, k will always denote a permutation of $1, 2, 3, 4$). Then $\mathcal{A}$ and $\mathcal{B}$ in general are not homothetic. Given any $\mathcal{A}$, here is an elementary construction (Figure 1) producing such a $\mathcal{B}$. Let B_1B_2 be any segment parallel to A_3A_4. Let B_3 be the intersection of the line through B_1 parallel to A_2A_4 with the line through B_2 parallel to A_1A_4; likewise, let B_4 be the intersection of the line through B_1 parallel to A_2A_3 with the line through B_2 parallel to A_1A_3. We claim that the sixth side B_3B_4 is also parallel to A_1A_2. Consider, in fact, the intersections R of A_1A_3 with A_2A_4, and S of B_1B_3 with B_2B_4. The following pairs of triangles have parallel sides: A_1RA_4 and B_2SB_3, A_4RA_3 and B_1SB_2, A_3RA_2 and B_4SB_1. Hence, they are homothetic in pairs. Since a translation $R \to S$ does not affect our statement, we can assume, for simplicity, $R = S$. Now two pairs of sides are on the the same lines: A_1A_3, B_2B_4 and A_2A_4, B_1B_3. Consider the action of the three homotheties: λ: $A_1 \to B_2$, $A_4 \to B_3$, μ: $A_3 \to B_2$, $A_4 \to B_1$,

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[1] In fact, five pairs of parallel sides A_iA_j, B_iB_j suffice to make the sixth pair parallel.

ν: $A_2 \to B_1$, $A_3 \to B_4$. Here $\nu\mu^{-1}\lambda$: $A_1 \to B_4$ and $\lambda\mu^{-1}\nu$: $A_2 \to B_3$. But these products are the same mapping, as all factors have the same fixed point R. Thus the triangles A_1RA_2 , B_4RB_3 are homothetic and B_3B_4, A_1A_2 are parallel, as we wanted.

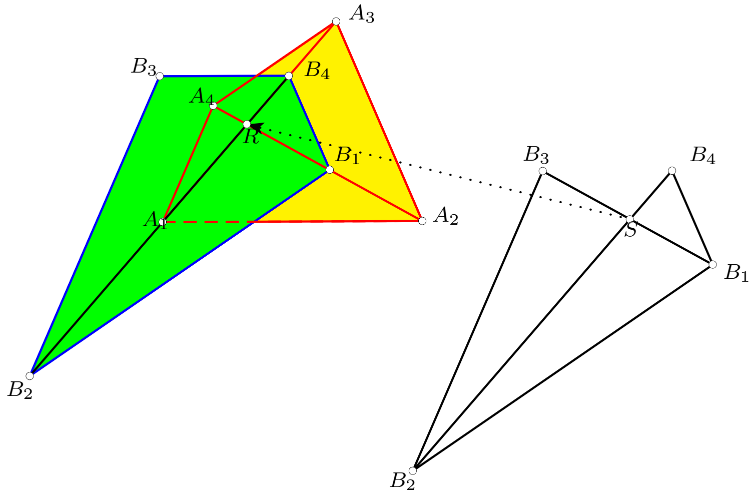

Figure 1. Five pairs of parallel sides imply sixth pair parallel

The purpose of this paper is to examine some geometric relations and invariances connecting $\mathcal{A}$ and $\mathcal{B}$; in particular, $\mathcal{A}$ and $\mathcal{B}$ will have the same pair of *asymptotic directions*, a sort of *central* points at infinity. Some statements, as the somehow unexpected Theorem 1, will be proved by synthetic arguments only; other statements, which involve circumscribed conics, will be proved analytically. Our main result (Theorem 6) states that, modulo similarities, such a mapping $A_i \to B_i$ is induced by an oblique reflection. This involutory affinity depends on $\mathcal{A}$ and can be constructed from $\mathcal{A}$ by ruler and compass. We shall also see that, when $\mathcal{A}$ belongs to the most popular families of quadrangles, namely cyclic quadrangles and trapezoids (including parallelograms), this mapping leaves the shape of the quadrangle unchanged. This may be a reason why this subject, to our knowledge, has not raised previous attention.

2. Notation and terminology

If A, B are points, AB will denote, according to different contexts, the segment or the line through A, B. $|AB|$ is a length. $A^B = C$ means that a half-turn about B maps A onto C; equivalently, we write $B = \frac{1}{2}(A + C)$.

If r, s are lines, $\angle r, s$ denotes the *directed* angle from r to s, to be measured mod π. $\angle ABC$ means $\angle AB, BC$. We shall use the basic properties of directed angles, as described in [3, pp.11–15], for example, $\angle ABC = 0$ is equivalent to A, B, C being collinear, $\angle ABC = \angle ADC$ to A, B, C, D being on a circle. $A - B$ is a vector; $(A-B)\cdot(C-D)$ is a scalar product; $(A-B)\wedge(C-D)$ is a vector product. If two vectors are parallel, we write $\dfrac{A-B}{C-D} = r$ to mean $A - B = r(C - D)$.

In a complete quadrangle $\mathcal{A} = A_1A_2A_3A_4$ the order of the vertices is irrelevant. We shall always assume that three of them are not collinear, so that all the angles $\angle A_iA_jA_h$ are defined and do not vanish. A_iA_j and A_hA_k are a pair of *opposite sides* of $\mathcal{A}$; they meet at the *diagonal point* $A_{ij,hk}$. The diagonal points are vertices of the *diagonal triangle* of $\mathcal{A}$. A quadrangle is a *trapezoid* if there is a pair of parallel opposite sides; then a diagonal point is at infinity. *Parallelograms* have two diagonal points at infinity. $A_{ij} = \frac{1}{2}(A_i + A_j)$ is the *midpoint* of the side A_iA_j. A *bimedian* of $\mathcal{A}$ is the line $A_{ij}A_{hk}$ (or the segment) through the midpoints of a pair of opposite sides. A *complementary triangle* $A_jA_hA_k$ is obtained from $\mathcal{A}$ by ignoring the vertex A_i. A *complementary quadrilateral* is obtained from $\mathcal{A}$ by ignoring a pair of opposite sides. An area is not defined for a complete quadrangle, but its complementary triangles and quadrilaterals do have *oriented areas*, which are mutually related (see §4).

3. Semi-similar and semi-homothetic complete quadrangles

Definition. Two complete quadrangles $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ will be called *directly* (*inversely*) *semi-similar* if there is a mapping $A_i \to B_i$ such that $\angle A_iA_jA_h = \angle B_hB_kB_i$ ($\angle A_iA_jA_h = \angle B_iB_kB_h$, respectively).

In particular, $\mathcal{A}$ and $\mathcal{B}$ will be called *semi-homothetic* if each side A_iA_j is parallel to B_hB_k.

The property of being semi-similar is obviously symmetric but not reflexive (only special classes of quadrangles will be self-semi-similar). Let $\mathcal{A}$ and $\mathcal{B}$ be semi-similar. If $\mathcal{B}$ is similar to $\mathcal{C}$, then $\mathcal{A}$ is semi-similar to $\mathcal{C}$. On the other hand, if $\mathcal{B}$ is semi-similar to $\mathcal{D}$, then $\mathcal{A}$ and $\mathcal{D}$ are similar. This explains the word *semi* (or *half*). *Direct* and *inverse* also follow the usual product rules. Although the relation of semi-similarity is not explicitly defined in the literature, semi-similar quadrangles do appear in classical textbooks; for example, some statements in [3, §399] regard the following case. Given a complete quadrangle $\mathcal{A} = A_1A_2A_3A_4$, let O_i denote the circum-center of the complementary triangle $A_jA_hA_k$. Then O_iO_j is orthogonal to A_hA_k, and this clearly implies that $\mathcal{A}$ is directly semi-similar to the quadrangle $o(\mathcal{A}) = O_1O_2O_3O_4$. [2] It can also be proved that $\mathcal{A}$ is inversely semi-similar to $n(\mathcal{A}) = N_1N_2N_3N_4$, where N_i denotes the nine-point center of the triangle $A_jA_hA_k$.

Semi-homotheties are direct semi-similarities. The construction we gave in §1 confirms that, modulo homotheties, there is a unique $\mathcal{B}$ which is semi-homothetic to a given quadrangle $\mathcal{A}$.

A *cyclic* quadrangle is semi-similar to itself. More precisely, the mapping $A_i \to A_i$ defines an inversely semi-similar quadrangle if and only if all the angle equalities $\angle A_iA_jA_h = \angle A_iA_kA_h$ hold, and this is equivalent for the four points A_i to lie on a circle (see §9). We shall see, however, that, if different mappings $A_i \to A_j$

[2] We shall reconsider the mapping $A_i \to O_i$ in the footnote at the end of §10.

are considered, there exist other families of quadrangles for which semi-similarity implies similarity (see §§10 and 12).

Theorem 1. *If $\mathcal{A}$ and $\mathcal{B}$ are semi-homothetic quadrangles, then the bimedians of $\mathcal{A}$ are parallel to the sides of the diagonal triangle of $\mathcal{B}$.*

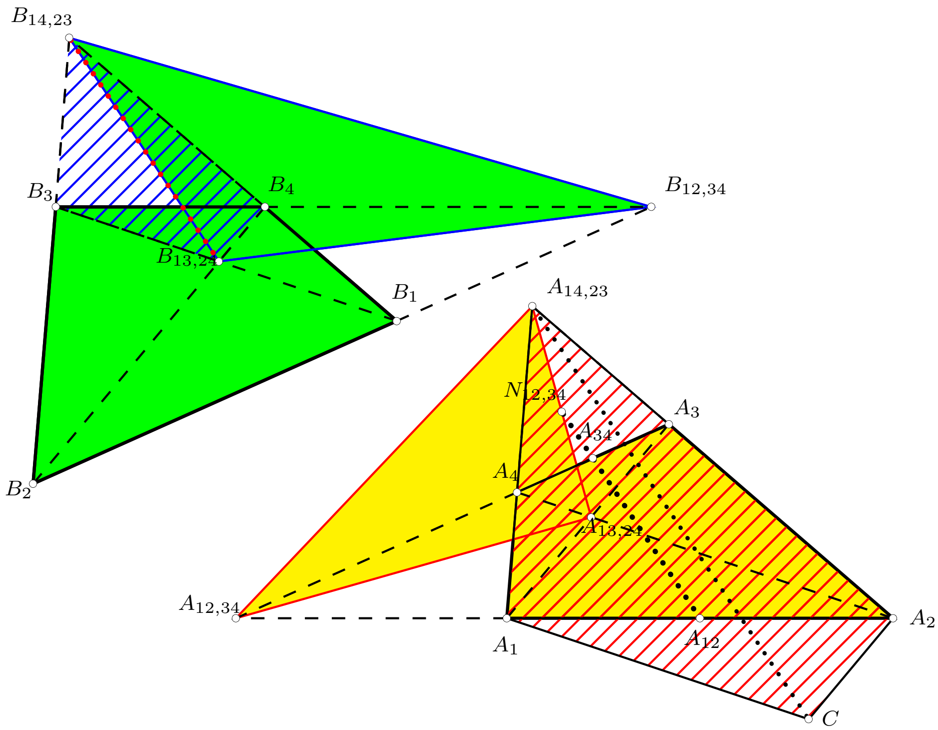

Figure 2. Semi-homothetic quadrangles: the bimedians of $\mathcal{A}$ are parallel to the sides of the diagonal triangle of $\mathcal{B}$

Proof. It is well-known ([3, §91] that a bimedian of $\mathcal{A}$, say $A_{12}A_{34}$, meets a side of the diagonal triangle in its midpoint $N_{12,34} = \frac{1}{2}(A_{13,24} + A_{14,23})$. Let C denote the intersection of the line through A_1 parallel to A_2A_4 with the line through A_2 parallel to A_1A_3. Then $A_{12} = \frac{1}{2}(C + A_{13,24})$, hence $CA_{14,23}$ and $A_{12}N_{12,34}$ are parallel. Now let $\mathcal{B} = B_1B_2B_3B_4$ be semi-homothetic to $\mathcal{A}$ so that each A_iA_j is parallel to B_hB_k. Then in the quadrangles $A_1CA_2A_{14,23}$ and $B_3B_{13,24}B_4B_{14,23}$ (see the striped areas in Figure 2), five pairs of sides are parallel, either by assumption or by construction. Hence the same holds for the sixth pair $A_{14,23}C$, $B_{14,23}B_{13,24}$. But $A_{14,23}C$ is trivially parallel to $A_{34}A_{12}$. Therefore, the side $B_{14,23}B_{13,24}$ of the diagonal triangle of $\mathcal{B}$ is parallel to the bimedian $A_{12}A_{34}$ of $\mathcal{A}$, as we wanted. □

Notice that, by symmetry, the sides of the diagonal triangle of $\mathcal{A}$ are parallel to the bimedians of $\mathcal{B}$.

4. Semi-isometric quadrangles

We shall now introduce a relationship between semi-similar quadrangles which replaces isometry. This notion is based on the following

Theorem 2. *Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be semi-homothetic quadrangles. Then the product $\mu = \dfrac{B_i - B_j}{A_h - A_k} \cdot \dfrac{B_h - B_k}{A_i - A_j}$ is invariant under all permutations of indices.*

Proof. Notice that the factors in the definition of μ are ratios of parallel vectors, hence scalars with their own sign. Now consider, for example, the following triangles: $A_1A_4A_{12,34}$, $A_2A_3A_{12,34}$, $A_1A_3A_{12,34}$, $A_2A_4A_{12,34}$ which are homothetic, respectively, to the triangles $B_3B_2B_{12,34}$, $B_4B_1B_{12,34}$, $B_4B_2B_{12,34}$, $B_3B_1B_{12,34}$. Each of these homotheties implies an equal ratio of parallel vectors:

$$\begin{aligned}
\frac{B_3 - B_2}{A_1 - A_4} &= \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}}, \\
\frac{B_4 - B_1}{A_2 - A_3} &= \frac{B_1 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_4 - B_2}{A_1 - A_3} &= \frac{B_2 - B_{12,34}}{A_3 - A_{12,34}}, \\
\frac{B_3 - B_1}{A_2 - A_4} &= \frac{B_1 - B_{12,34}}{A_4 - A_{12,34}}.
\end{aligned}$$

By multiplication one finds

$$\begin{aligned}
\mu &= \frac{B_3 - B_2}{A_1 - A_4} \cdot \frac{B_4 - B_1}{A_2 - A_3} \\
&= \frac{B_2 - B_{12,34}}{A_4 - A_{12,34}} \cdot \frac{B_1 - B_{12,34}}{A_3 - A_{12,34}} \\
&= \frac{B_1 - B_{12,34}}{A_4 - A_{12,34}} \cdot \frac{B_2 - B_{12,34}}{A_3 - A_{12,34}} \\
&= \frac{B_3 - B_1}{A_2 - A_4} \cdot \frac{B_4 - B_2}{A_1 - A_3}.
\end{aligned}$$

Likewise, $\mu = \dfrac{B_3 - B_4}{A_1 - A_2} \cdot \dfrac{B_1 - B_2}{A_3 - A_4}$, as we wanted. □

Thus a pair $\mathcal{A}, B$ of semi-similar quadrangles defines a scale factor

$$|\mu| = \frac{|B_1B_2||B_3B_4|}{|A_1A_2||A_3A_4|} = \frac{|B_1B_3||B_2B_4|}{|A_1A_3||A_2A_4|} = \frac{|B_1B_4||B_2B_3|}{|A_1A_4||A_2A_3|}.$$

A geometric meaning for the sign of μ will be seen later (§7).

Corollary 3. *Let $\mathcal{A} = A_1A_2A_3A_4$ and $\mathcal{B} = B_1B_2B_3B_4$ be semi-similar quadrangles. Then the products of the lengths of the pairs of opposite sides are proportional:*[3]

$$|A_1A_2||A_3A_4| : |A_1A_3||A_2A_4| : |A_1A_4||A_2A_3|$$
$$= |B_1B_2||B_3B_4| : |B_1B_3||B_2B_4| : |B_1B_4||B_2B_3|.$$

A sort of isometry takes place when $|\mu|$=1:

Definition. Two quadrangles $\mathcal{A}$ and $\mathcal{B}$ will be called semi-isometric if the they are semi-similar and the lengths of two corresponding opposite sides have the same product: $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$.

We have just seen that if this equality holds for a pair of opposite sides of semi-similar quadrangles, then $|\mu| = 1$ and therefore the same happens to the other two pairs.

We shall now derive a number of further relations between semi-isometric quadrangles which are almost immediate consequences of the definition. Some of them are better described if referred to the three complementary quadrilaterals. The oriented areas of these quadrilaterals are given by one half of the cross products $(A_1 - A_2) \wedge (A_3 - A_4), (A_1 - A_4) \wedge (A_2 - A_3), (A_1 - A_3) \wedge (A_2 - A_4)$. Therefore the defining equalities $|A_iA_j||A_hA_k| = |B_iB_j||B_hB_k|$, if combined with the angle equalities $\angle A_iA_j, A_hA_k = -\angle B_iB_j, B_hB_k$, imply that the three areas only change sign when passing from $\mathcal{A}$ to $\mathcal{B}$. On the other hand, it is well-known that the three cross products above, if added or subtracted in the four essentially different ways, produce 4 times the oriented area of the complementary triangles $A_jA_hA_k$. For example, by applying standard properties of vector calculus, one finds

$$(A_1 - A_2) \wedge (A_3 - A_4) + (A_1 - A_4) \wedge (A_2 - A_3) + (A_1 - A_3) \wedge (A_2 - A_4)$$
$$= 2(A_1 - A_4) \wedge (A_1 - A_2),$$
$$(A_1 - A_2) \wedge (A_3 - A_4) + (A_1 - A_4) \wedge (A_2 - A_3) - (A_1 - A_3) \wedge (A_2 - A_4)$$
$$= 2(A_3 - A_4) \wedge (A_2 - A_3),$$

etc. Therefore we have

Theorem 4. *If $\mathcal{A}$ and $\mathcal{B}$ are semi-isometric quadrangles, the four pairs of corresponding complementary triangles $A_iA_jA_h$ and $B_iB_jB_h$ have the same* (absolute) *areas.*

This insures, incidentally, that, when semi-similarity implies similarity, then semi-isometry implies isometry.

Other invariants can be written in terms of perimeters: if one first adds, then subtracts the lengths of the four contiguous sides in a complementary quadrilateral,

[3]The following example shows that the inverse statement does not hold: let A_1A_2 be the diameter of a circle; then, for any choice of a chord A_3A_4 orthogonal to A_1A_2, the three products above are proportional to $2 : 1 : 1$.

then the product is invariant; for example, the product

$$(|A_1A_2| + |A_2A_3| + |A_3A_4| + |A_4A_1|)(|A_1A_2| - |A_2A_3| + |A_3A_4| - |A_4A_1|)$$

is the same if all A_i are changed into B_i. In particular,

$$|A_1A_2| - |A_2A_3| + |A_3A_4| - |A_4A_1| = 0$$

if and only if

$$|B_1B_2| - |B_2B_3| + |B_3B_4| - |B_4B_1| = 0.$$

Since these equalities are well-known to be equivalent to the fact that two pairs of opposite sides are tangent to a same circle, we conclude that semi-similarity of complete quadrangles preserves *inscribability* for a complementary quadrilateral. Another invariance takes place if we subtract the squares of the lengths of two bimedians; for example,

$$|A_{12}A_{34}|^2 - |A_{14}A_{23}|^2 = |B_{12}B_{34}|^2 - |B_{14}B_{23}|^2.$$

These and other similar equalities can be derived by applying the classical formulas for tha area of a quadrilateral (Bretschneider's formula etc., see [5]).

Something more intriguing happens if one considers the circumcircles of the complementary triangles.

Theorem 5. *If two quadrangles are semi-similar, the circumradii of corresponding complementary triangles are inversely proportional* .

Proof. Let R_i and S_i be, respectively, the circumradii of $A_jA_hA_k$ and $B_jB_hB_k$. We claim that R_1, R_2, R_3, R_4 are inversely proportional to S_1, S_2, S_3, S_4. In fact, by the law of sines, we can write, for example, the product $|A_3A_4||B_3B_4|$ in two ways:

$$(2R_1 \sin A_3A_2A_4)(2S_1 \sin B_3B_2B_4) = (2R_2 \sin A_4A_1A_3)(2S_2 \sin B_4B_1B_3).$$

Since $\angle A_3A_2A_4 = \angle B_4B_1B_3$ and $\angle A_4A_1A_3 = \angle B_3B_2B_4$, all the sines can be canceled to conclude $R_1S_1 = R_2S_2$. Thus the product R_iS_i is the same for all indices i. □

5. The principal reference of a quadrangle

The invariants we met in the last section suggest the possible presence of an affinity. In fact, our main result (Theorem 6) will state that for each quadrangle $\mathcal{A}$ there exists an involutory affinity (depending on $\mathcal{A}$) that maps $\mathcal{A}$ into a semi-homothetic, semi-isometric quadrangle $\mathcal{B}$. It will then appear that any semi-similarity of quadrangles $\mathcal{A} \to \mathcal{C}$ is a product of an *oblique reflection* $\mathcal{A} \to \mathcal{B}$ by a similarity $\mathcal{B} \to \mathcal{C}$. Since an oblique reflection is determined, modulo translations, by a pair of directions, in order to identify which reflection properly works for $\mathcal{A}$, we shall describe in the next section how to associate to a quadrangle $\mathcal{A}$ a pair of characteristic directions, that we shall call the *asymptotic directions* of $\mathcal{A}$. As we shall see, these directions also play a basic role in connection with some conics that are canonically defined by four points.[4]

It is well-known that a quadrangle $\mathcal{A} = A_1A_2A_3A_4$ has a unique circumscribed rectangular hyperbola $\Psi = \Psi_A$ [5]. The center of Ψ is a central (synonym: *notable*) point of $\mathcal{A}$ that we denote by $H = H_A$. Several properties and various geometric constructions of H from the points A_i are described in [4] (but this point is defined also in [3, §396-8], and [2, Problem 46]). For example, H is the intersection of the nine-point circles of the four complementary triangles $A_jA_hA_k$. The directions of the asymptotes of $\Psi = \Psi_A$ will be called the *principal directions* of $\mathcal{A}$. The hyperbola Ψ essentially defines what we call the *principal reference* of $\mathcal{A}$, namely an orthogonal Cartesian xy-frame such that the equation for Ψ is $xy = 1$, the ambiguity between x and y not creating substantial difficulties (Figures 3 and 4). Our next proofs will be based on this reference (an approach which was first used by Wood in [6]). The principal reference is not defined when two opposite sides of $\mathcal{A}$ are perpendicular. This class of quadrangles (*orthogonal* quadrangles) requires a different analytic treatment and will be discussed separately in §11.

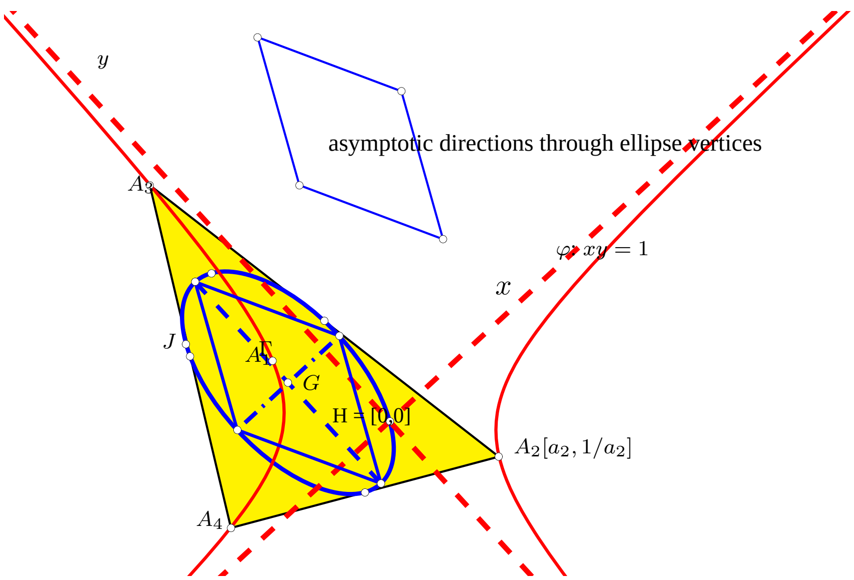

Figure 3. A concave quadrangle and its principal reference. Asymptotic directions derived from the medial ellipse Γ

Within the principal reference of $\mathcal{A}$, the origin is the central point $H_A = [0, 0]$ and the vertices will be denoted by $A_i = [a_i, \frac{1}{a_i}]$, $i = 1, 2, 3, 4$. In view of the next

[4]These directions may be thought of as a pair of *central points* of $\mathcal{A}$ at infinity. In this respect, the present paper may be considered a complement of [4]. Our treatment, however, will be self-contained, not requiring the knowledge of [4].

[5]The only exceptions will be considered in §12.

calculations, it is convenient to introduce the elementary symmetric polynomials:

$$\begin{aligned} s_1 &= a_1 + a_2 + a_3 + a_4, \\ s_2 &= a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4, \\ s_3 &= a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4, \\ s_4 &= a_1a_2a_3a_4. \end{aligned}$$

Notice that the restriction for $\mathcal{A}$ not to be orthogonal not only implies $s_4 \neq 0$ but also excludes $s_4 = -1$, as the scalar product of two opposite sides turns out to be

$$A_iA_j \cdot A_hA_k = (a_i - a_j)(a_h - a_k)\left(1 + \frac{1}{s_4}\right).$$

The sign of $s_4 = a_1a_2a_3a_4$ has the following relevant geometric meaning: s_4 is positive if and only if the number of vertices A_i which lie on a branch of Ψ is even: $4, 2$ or 0. By applying standard arguments to the real convex function $f(x) = \frac{1}{x}$, this condition is found to be equivalent to $\mathcal{A}$ being *convex*. On the other hand, if the branches of Ψ contain 1 and 3 vertices, then $s_4 < 0$ and $\mathcal{A}$ is *concave*, namely, there is a vertex A_i which lies inside the complementary triangle $A_jA_hA_k$.

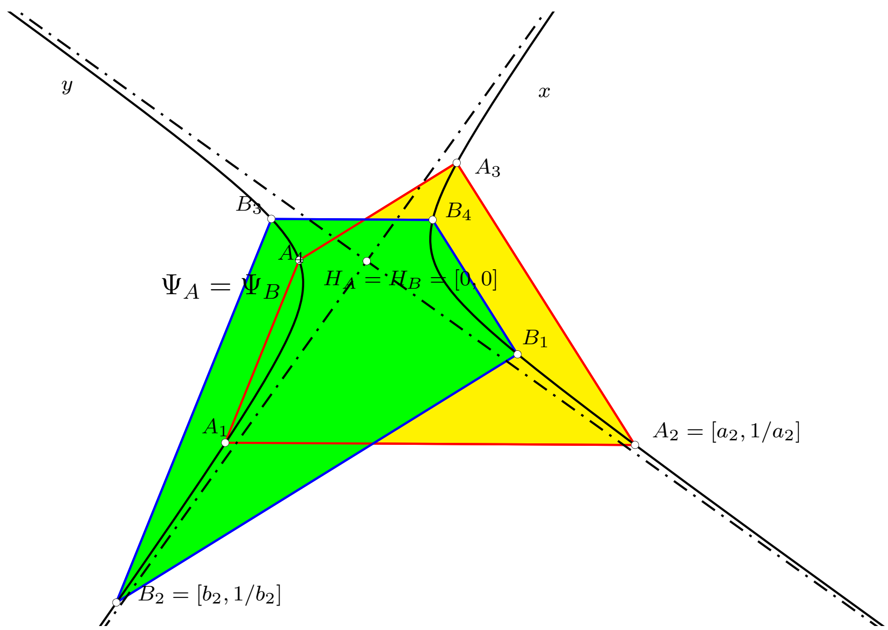

Figure 4. Semi-homothetic semi-isometric quadrangles with the same principal reference

6. Central conics and asymptotic directions

It is well-known that, within the family of the conic sections circumscribed to a given quadrangle $\mathcal{A}$, the locus of the centers is itself a conic $\Gamma = \Gamma_A$. We call it the *medial* or the nine-points conic of $\mathcal{A}$, as Γ contains the six midpoints A_{ij} and the three diagonal points $A_{ij,hk}$ ([1, §16.7.5]). The equation for Γ is calculated to be

$$x^2 - s_4y^2 - \frac{1}{2}s_1x + \frac{1}{2}s_3y = 0,$$

or

$$(x - x_G)^2 - s_4(y - y_G)^2 = \frac{1}{16s_4}(s_4s_1^2 - s_3^2),$$

which confirms that $H = [0, 0]$, the center of Ψ, lies on Γ. On the other hand, the center of Γ is

$$G = G_A = \frac{1}{4}[s_1, \frac{s_3}{s_4}] = \frac{1}{4}(A_1 + A_2 + A_3 + A_4).$$

This is clearly the *centroid* or center of gravity, another central point of $\mathcal{A}$. Two cases must be now distinguished:.

(i) $\mathcal{A}$ is convex: $s_4 > 0$ and Γ is a hyperbola. By definition, the points at infinity of Γ will be called the *asymptotic directions* of $\mathcal{A}$; it appears from the equation that the slopes of the asymptotes are $\pm\frac{1}{\sqrt{s_4}}$ (Figure 5). This proves that the principal directions bisect the asymptotic directions. In particular, when $s_4 = 1$, Γ is a rectangular hyperbola; as we shall soon see, this happens if and only if $\mathcal{A}$ is *cyclic*.

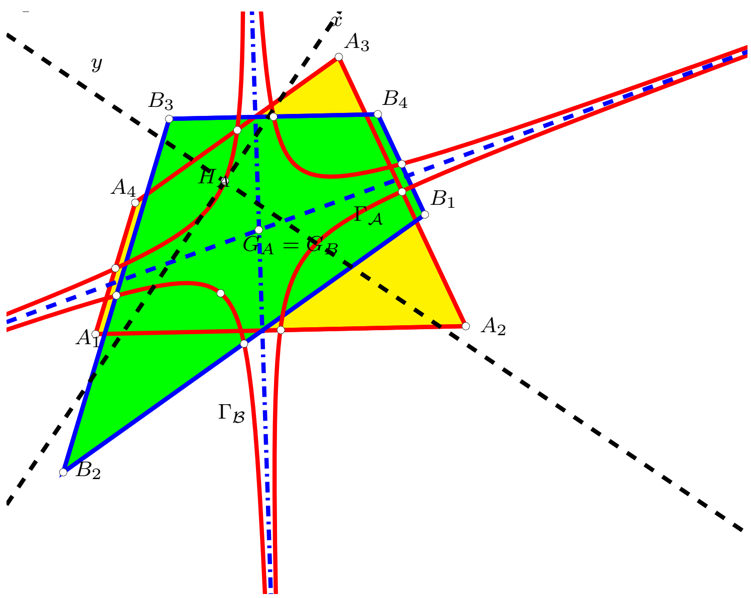

Figure 5. Semi-homothetic semi-isometric convex quadrangles with conjugate medial hyperbolas Γ_A, Γ_B. Asymptotic directions from Γ asymptotes

(ii) $\mathcal{A}$ is concave: $s_4 < 0$ and Γ is an ellipse. The asymptotic directions of $\mathcal{A}$ will be defined by connecting contiguous vertices of the ellipse Γ. Equivalently, we can inscribe Γ in a minimal rectangle and consider the directions of its diagonals. [6]

[6]They can also be defined as the directions of the only pair of *conjugate diameters* of the ellipse Γ_A which have equal lengths; see [2, Problem 54].

Their slopes turn out to be $\pm\dfrac{1}{\sqrt{-s_4}}$ (Figure 3) . Again, the principal directions bisect the asymptotic directions. Notice that Γ cannot be a parabola, as $s_4 \neq 0$. We have also seen in §5 that $s_4 \neq -1$, so that Γ cannot even be a circle. [7]

Next we want to introduce a new central point $J = J_A$, defined as the reflection of H in G: $J = H^G = \frac{1}{2}\left[s_1, \dfrac{s_3}{s_4}\right]$. Since G is the center of Γ and H lies on Γ, the point J also lies on Γ. Therefore, there exists a conic $\Theta = \Theta_A$ circumscribed to $\mathcal{A}$ and centered at J. The equation for Θ is found to be

$$x^2 + s_4 y^2 - s_1 x - s_3 y + s_2 = 0$$

or

$$(x - x_J)^2 + s_4(y - y_J)^2 = \frac{1}{4s_4}(s_4 s_1^2 + s_3^2) - s_2.$$

Looking at the roles of s_4 and $-s_4$ in the equations for Γ and Θ, it appears that Γ is an ellipse when Θ is a hyperbola and conversely. [8]

Moreover, the hyperbola asymptotes are parallel to the ellipse diagonals. We have thus produced two alternative ways for defining the asymptotic directions of any quadrangle $\mathcal{A}$:

if $\mathcal{A}$ is concave, by the asymptotes of Θ or by the vertices of Γ (Figure 3);

if $\mathcal{A}$ is convex, by the asymptotes of Γ or by the vertices of Θ (Figure 5).

A third equivalent definition only applies to the latter case: it is well-known (see, for example, [2, Problem 45]) that a convex quadrangle has two circumscribed parabolas, say Π_+ and Π_-. Their equations are [9]

$$\Pi_+ : \qquad (x + \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 - 2\sqrt{s_4} = 0,$$

$$\Pi_- : \qquad (x - \sqrt{s_4}y)^2 - s_1 x - s_3 y + s_2 + 2\sqrt{s_4} = 0.$$

Therefore the asymptotic directions of a convex quadrangle may be also defined by the axes of symmetry of the two circumscribed parabolas.

If $s_4 = 1$ then Θ is a circle and $\mathcal{A}$ is *cyclic*. In this case the diagonals of Θ are not defined; but Γ, Π_+ , Π_- define the asymptotic directions, which are just the bisectors $y = \pm x$ of the principal directions. [10]

---

[7] See §10 for exceptions.

[8] When $\mathcal{A}$ is convex, Θ_A turns out to be the ellipse that *deviates least from a circle* among all the ellipses circumscribed to $\mathcal{A}$, this meaning that the ratio between the major and the minor axis attains its minimum value. This problem was studied by J. Steiner. One can also prove that each ellipse circumscribed to $\mathcal{A}$ has a pair of conjugate diameters that have the asymptotic directions of $\mathcal{A}$; see [2, Problem 45].

[9] These equations are easily derived from the equation of the generic circumscribed conic, which can be written as $\lambda\Psi + \mu\Theta = 0$.

[10] For a different definition, not involving conics, see §9.

Notice that, whatever choice one makes among the definitions above, there exist classical methods which produce by straight-edge and compass the asymptotic directions of a quadrangle, starting from its vertices[11].

7. Oblique reflections

We are now ready to introduce oblique reflections. We recall this notion by introducing the following

Definition. Given an ordered pair (r, s) of non parallel lines, an (r, s)-reflection is the plane transformation $\phi : P \to P'$, such that $P - P'$ is parallel to r and the midpoint $\frac{1}{2}(P + P')$ lies on s.

An (r, s)-reflection is an involutory affine transformation. Among the well-known properties of affinities, we shall use the fact that they map lines into lines, midpoints into midpoints, conics into conics. Like in a standard reflection (a particular case, when r, s are orthogonal) the line s is the locus of fixed points, the other fixed lines being parallel to r. Replacing r with a parallel line r' does not affect ϕ ; replacing s with a parallel line s' only affects P' by the translation $s \to s'$ parallel to r. Interchanging r with s amounts to letting P' undergo a half turn around the intersection of r and s. An rs-reflection ϕ *preserves* many features of quadrangles; for example, if $\phi(A_i) = B_i$, then the diagonal triangle of $\mathcal{B} = B_1B_2B_3B_4$ is the ϕ-image of the diagonal triangle of $\mathcal{A}$ = $A_1A_2A_3A_4$. Other corresponding elements are the bi-median lines, the centroid, the medial conic, the circumscribed parabolas. As for analytic representations, if, for example, $r : y = rx + p$, $s : y = sx + q$, then ϕ is the bilinear mapping

$$[x, y] \to \frac{1}{(r - s)}[(r + s)x - 2y, 2rsx - (r + s)y] + [x_0, y_0]$$

where $[x_0, y_0]$ is the image of $[0, 0]$. The transformation matrix of ϕ has determinant

$$\frac{1}{(r - s)^2}(-(r + s)^2 + 4rs) = -1.$$

Therefore all oriented areas undergo a change of sign.

Theorem 6. *Let $\mathcal{A}$ be a complete quadrangle. Let ϕ be an (r, s)-reflection, where r, s are parallel to the asymptotic directions of $\mathcal{A}$. Let χ be an* (orthogonal) *reflection in a line p parallel to a principal direction of $\mathcal{A}$. Define a mapping ψ as follows:*

$$\psi = \begin{cases} \phi, & \text{if } \mathcal{A} \text{ is convex,} \\ \phi\chi, & \text{if } \mathcal{A} \text{ is concave.} \end{cases}$$

Then $\mathcal{A}$ and $\mathcal{B} = \psi(\mathcal{A})$ are semi-homothetic, semi-isometric quadrangles.

[11] For example, a celebrated page of Newton describes how to construct the axes of a parabola if four of its points are given.

Proof. Notice that $\mathcal{B}$ is uniquely defined by $\mathcal{A}$, modulo translations and midturns: in fact, a translation parallel to r takes place when s is translated; a midturn takes place if the principal directions or the lines r and s are interchanged.

Along the proof we can assume, without loss of generality, that both s and p pass through $H_A = [0, 0]$, so that $\psi(H_A) = H_A$.

First case: $\mathcal{A}$ is convex ($s_4 > 0$). Then the lines r, s have equations, say $r : y = \dfrac{-x}{\sqrt{s_4}}$ and $s : y = \dfrac{x}{\sqrt{s_4}}$. The rs-reflection maps the point $P[x, y]$ into $\phi(P) = [y\sqrt{s}_4, \frac{x}{\sqrt{s_4}}]$, so that the vertex $A_i = [a_i, \frac{1}{a_i}]$ is mapped into $B_i = \phi(A_i) = [\frac{\sqrt{s}_4}{a_i}, \frac{a_i}{\sqrt{s}_4}]$. Substituting in $xy = 1$ proves that B_i lies on Ψ_A. Since a quadrangle has a unique circumscribed rectangular hyperbola [12], we have $\Psi_B = \Psi_{\phi(A)} = \Psi_A$. In particular, $\mathcal{A}$ and $\mathcal{B}$ have the same principal directions and $H_B = \phi(H_A) = H_{\phi(A)} = H_A$. Now consider the sides $B_i - B_j = [\sqrt{s}_4(\frac{1}{a_i} - \frac{1}{a_j}), \frac{a_i - a_j}{\sqrt{s}_4}]$ and $A_h - A_k = [a_h - a_k, \frac{1}{a_h} - \frac{1}{a_k}]$. If we take into account that $\frac{\sqrt{s}_4}{a_i a_j} = \frac{a_h a_k}{\sqrt{s}_4}$ we find that these vectors are parallel, their ratio being $\dfrac{B_i - B_j}{A_h - A_k} = -\dfrac{\sqrt{s}_4(a_i - a_j)}{(a_h - a_k)a_i a_j} = \dfrac{A_i - A_j}{B_h - B_k}$. This proves that $\mathcal{A}$ and $\mathcal{B}$ are semi-homothetic (Figure 6). Moreover, the following scalar products turn out to be the same $(A_i - A_j).(A_h - A_k) = (a_i - a_j)(a_h - a_k)(1 + \frac{1}{s_4}) = (B_i - B_j).(B_h - B_k)$. Thus $\mathcal{A}$ and $\mathcal{B}$ are also semi-isometric : $|A_i A_j||A_h A_k| = |B_i B_j||B_h B_k|$, as we wanted. Since the matrix for $\phi = \psi$ has determinant -1, the oriented areas of the corresponding complementary triangles and quadrilaterals, as expected, undergo a sign change.

Second case: $\mathcal{A}$ is concave ($s_4 < 0$). The argument is similar: let $r : y = \dfrac{-x}{\sqrt{-s_4}}$ and $s : y = \dfrac{x}{\sqrt{-s_4}}$. Then the reflection χ, for example in the x-axes, takes $[x, y]$ into $[x, -y]$ and $\psi : [x, y] \rightarrow [y\sqrt{-s_4}, \frac{x}{\sqrt{-s_4}}]$. Thus $B_i = \psi(A_i) = [\frac{-\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$. If we take into account that $\frac{\sqrt{-s_4}}{a_i a_j} = \frac{-a_h a_k}{\sqrt{-s_4}}$, we find that the vectors $B_i - B_j, A_h - A_k$ are parallel. For their ratio we find the equality $\dfrac{B_i - B_j}{A_h - A_k} = \dfrac{\sqrt{-s_4}(a_i - a_j)}{(a_h - a_k)a_i a_j} = -\dfrac{A_i - A_j}{B_h - B_k}$. The matrix for ψ has now determinant 1 so that the oriented areas are conserved. □

Incidentally, the foregoing argument also shows that in the semi-homothety of Theorem 2 in §5 the sign of the scalar $\mu = \dfrac{B_i - B_j}{A_h - A_k}.\dfrac{B_h - B_k}{A_i - A_j}$ is respectively 1 or -1 for convex and concave quadrangles. This proves that semisimilarities preserve convexity.

Since the mapping of Theorem 6 is involutory and the principal references for $\mathcal{A}$ and $\mathcal{B}$ have been shown to be the same, we have substantially proved that

[12]The only exceptions will be considered in §12

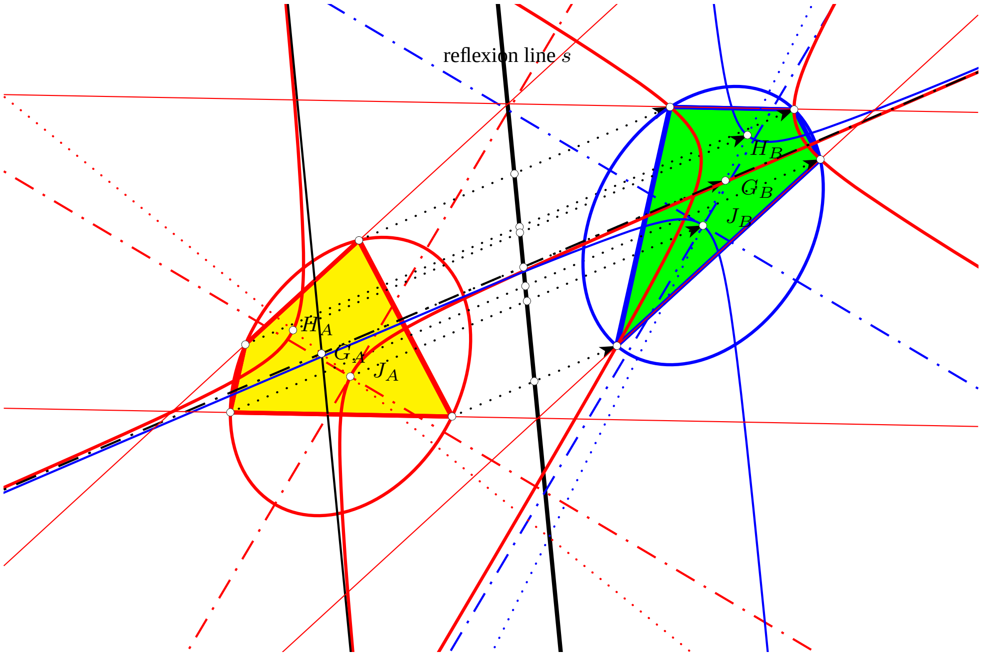

Figure 6. An oblique reflection producing semi-homothetic semi-isometric quadrangles. Pairs of corresponding sides, central lines, central conics etc meet on line s

Theorem 7. *Two semi-homothetic quadrangles have the same asymptotic directions.*

This statement will be confirmed in the next section.

8. Behaviour of central conics

We want now to examine how the central conics Ψ, Γ, Θ (see §4) of two semi-similar quadrangles are related to each other. We claim that, modulo similarities, these conics are either identical ellipses or conjugate hyperbolas.[13]

The problem can obviously be reduced to a pair of semi-homothetic, semi-isometric quadrangles.

Theorem 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-homothetic, semi-isometric quadrangles.*
(1) *Assume Ψ_A and Ψ_B have the same center: $H_A = H_B$. Then either $\Psi_A = \Psi_B$ ($\mathcal{A}$ convex) or Ψ_A and Ψ_B are conjugate ($\mathcal{A}$ concave).*
(2) *Assume Γ_A and Γ_B have the same center: $G_A = G_B$. Then either Γ_A=Γ_B ($\mathcal{A}$ concave) or Γ_A and Γ_B are conjugate ($\mathcal{A}$ convex). In the latter case, the two circumscribed parabolas $\Pi\pm_A$and $\Pi\pm_B$ are either equal or symmetric with respect to G.*

[13] Two hyperbolas are said to be conjugate if their equations, in a convenient orthogonal frame, can be written as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$. Conjugate hyperbolas have the same asymptotes and their foci form a square.

(3) *Assume Θ_A and Θ_B have the same center: $J_A = J_B$. Then either Θ_A=Θ_B ($\mathcal{A}$ convex) or Θ_A and Θ_B are conjugate ($\mathcal{A}$ concave).*

Proof. Without loss of generality, we can assume that $\mathcal{A}$ and $\mathcal{B}$ are linked by a mapping ψ as in Theorem 6. According to the various statements (1), (2), (3), it will be convenient to choose ψ in such a way that a specific point F is fixed. We shall denote by ψ_F this particular mapping: $\psi_F(F)$ =F. For example, in the proof of Theorem 6 we had $\psi = \psi_H$. To obtain ψ_F from ψ_H one may just apply an additional translation $H \to F$.

(1) First assume that $\mathcal{A}$ is convex: $s_4 > 0$. While proving Theorem 6 we have already noticed that the point $B_i = \psi_H(A_i) = \psi_H([a_i, \frac{1}{a_i}]) = [\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}]$ lies on $xy = 1$, hence $H_A = H_B, \Psi_A = \Psi_B$. Now assume $\mathcal{A}$ concave: $s_4 < 0$. A $principal$ reflection, say in the x-axes, takes $[a_i, \frac{1}{a_i}]$ into $[a_i, -\frac{1}{a_i}]$. Then $B_i = \psi_H(A_i) = \psi_H([a_i, -\frac{1}{a_i}]) = [-\frac{\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$, clearly a point of the hyperbola $xy = -1$, the conjugate of Ψ_A, as we wanted.

(2) Since affinities preserve midpoints and conics, for any choice of ψ we have $\Gamma_B = \psi(\Gamma_A)$, and $G_B = \psi(G_A)$. The assumption $G_B = G_A$ suggests the choice $\psi = \psi_G$ in Theorem 6. Assume $\mathcal{A}$ is convex: $s_4 > 0$. Then B_i = $\psi_G(A_i)$ is obtained by applying the translation $\psi_H(G_A) \to G_A$ to the point $\psi_H([a_i, \frac{1}{a_i}]) = [\frac{\sqrt{s_4}}{a_i}, \frac{a_i}{\sqrt{s_4}}]$. Here $G_A = \frac{1}{4}[s_1, \frac{s_3}{s_4}]$, $\psi_H(G_A) = \frac{1}{4\sqrt{s_4}}[s_3, s_1]$. Therefore

$$B_i = [\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{s_4}}].$$

Straightforward calculations prove that the midpoints $\frac{1}{2}(B_i + B_j)$ satisfy the equation

$$(x - x_G)^2 - s_4(y - y_G)^2 = -\frac{1}{16s_4}(s_3^2 + s_4 s_1^2),$$

which is the conjugate hyperbola of Γ_A, as we wanted.

As for the circumscribed parabolas, we already know, again by the general properties of affinities, that $\psi(\Pi_{+A})$ and $\psi(\Pi_{-A})$ will be parabolas circumscribed to $\mathcal{B}$. More precisely, one can verify that the above points $B_i = \psi_H(A_i) + G_A - \psi_H(G_A)$ satisfy the equation for

$$\Pi_{+A}: \qquad (x - y\sqrt{s_4})^2 - s_1 x - s_3 y + s_2 + 2\sqrt{s_4} = 0.$$

A midturn around G_A maps B_i into $2G_A - B_i$. The new points are $-\psi_H(A_i) + G_A + \psi_H(G_A)$ and they are checked to satisfy the equation

$$(x + y\sqrt{s_4})^2 - s_1 x - s_3 y + s_2 - 2\sqrt{s_4} = 0.$$

Therefore $\Pi_{-B} = \Pi_{-A}, \Pi_{+B} = (\Pi_{+A})^G$.

Now assume $\mathcal{A}$ concave: $s_4 < 0$. As before, $\psi_H(A_i) = [-\frac{\sqrt{-s_4}}{a_i}, \frac{a_i}{\sqrt{-s_4}}]$. The mapping ψ_G is again obtained by applying the translation $\psi_H(G_A) \to G_A$, but here $\psi_H(G_A) = \frac{1}{4\sqrt{-s_4}}[s_3, s_1]$. Therefore,

$$B_i = \psi_G(A_i) = [-\frac{\sqrt{-s_4}}{a_i} + \frac{s_1}{4} - \frac{s_3}{4\sqrt{-s_4}}, \frac{a_i}{\sqrt{-s_4}} + \frac{s_3}{4s_4} - \frac{s_1}{4\sqrt{-s_4}}].$$

By direct calculation, one checks that the midpoints $\frac{1}{2}(B_i + B_j)$ lie on

$$\Gamma_A : \qquad (x - x_G)^2 - s_4(y - y_G)^2 = \frac{1}{16}(s_4 s_1^2 - s_3^2).$$

Thus the medial ellipses are the same: $\Gamma_B = \psi_G(\Gamma_A) = \Gamma_A$, as we wanted.

(3) The proof is as above, except that we want $\psi = \psi_J$ and the translation is $\psi_H(J_A) \to J_A$. When $\mathcal{A}$ is convex, one finds that the points

$$B_i = [\frac{\sqrt{s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{s_4}}, \frac{a_i}{\sqrt{s_4}} + \frac{s_3}{2s_4} - \frac{s_1}{2\sqrt{s_4}}]$$

lie on

$$\Theta_A : \qquad (x - x_J)^2 + s_4(y - y_J)^2 = \frac{1}{4s_4}(s_3^2 + s_4 s_1^2) - s_2.$$

Hence $\Theta_B = \psi_J(\Theta_A) = \Theta_A$.

When $\mathcal{A}$ is concave, similar arguments lead to the points

$$B_i = \psi_G(A_i) = [-\frac{\sqrt{-s_4}}{a_i} + \frac{s_1}{2} - \frac{s_3}{2\sqrt{-s_4}}, \frac{a_i}{\sqrt{-s_4}} + \frac{s_3}{2s_4} - \frac{s_1}{2\sqrt{-s_4}}]$$

which satisfy the equation

$$(x - x_J)^2 + s_4(y - y_J)^2 = -\frac{1}{4s_4}(s_3^2 + s_4 s_1^2) + s_2,$$

the conjugate hyperbola of Θ_A. This completes the proof. $\square$

Proof of Theorem 7. By applying proper homotheties, we can reduce the proof to the case that $\mathcal{A}$ and $\mathcal{B}$ are semi-isometric; by further translations, we can even assume that $\psi : \mathcal{A} \to \mathcal{B}$ as in Theorem 6 and the conics centers are the same. Then, according to Theorem 8, the circumscribed conics Γ and Θ are either equal or conjugate. In any case $\mathcal{A}$ and $\mathcal{B}$ have the same asymptotic directions.

9. A special case: cyclic quadrangles

A cyclic (or *circumscriptible*) quadrangle $\mathcal{A}$ is convex and corresponds to $s_4 = 1$. In this case Θ_A is the circumcircle of equation $x^2+y^2-s_1x-s_3y+s_2 = 0$. The center of Θ_A is $J = J_A$ and its radius is $\rho = \frac{1}{2}\sqrt{s_1^2 + s_3^2 - 4s_2}$. Incidentally, since $\rho^2 = |JH|^2 - s_2$, we have discovered for s_2 a geometric interpretation, namely the $power$ of H with respect to the circumcircle Θ. The medial conic Γ is the rectangular hyperbola:

$$x^2 - y^2 - \frac{1}{2}s_1 x + \frac{1}{2}s_3 y = 0$$

and the circumscribed parabolas are

$$(x \pm y)^2 - s_1 x - s_3 y + s_2 = \pm 2.$$

Therefore the lines r and s defining ψ $(= \phi)$ in Theorem 6 are perpendicular with slope ± 1 and ψ is just the orthogonal reflection in the line s . For the asymptotic directions of cyclic quadrangles we have a simple geometric interpretation at finite, not involving conics: they merely bisect the angle formed by any pair of opposite sides of $\mathcal{A}$. In fact, the reflection ψ maps the line A_iA_j into the line B_iB_j, which is parallel to A_hA_k, because of the semi-homothety. We may also think of the asymptotic directions as the mean values of the directions of the radii JA_i, because ψ_J maps the perpendicular bisectors of A_iA_j into the perpendicular bisector of A_hA_k, namely a bisector of $\angle A_iJA_j$ into a bisector of $\angle A_hJA_k$.

For cyclic quadrangles, Theorem 1 states that the oriented angles formed by the sides of the diagonal triangle of $\mathcal{A}$ are just the opposite (as a result of the reflection ψ) of those formed by the bimedians. Using the symbols of Theorem 1 this can be written as $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}G, GN_{12,34} = -\angle A_{12,34}A_{13,24}, A_{13,24}A_{14,23}$. Since a triangle and its medial are obviously homothetic, we have $\angle N_{14,23}GN_{12,34} = \angle N_{14,23}N_{13,24}N_{12,34}$. Hence the four points $N_{ij,hk}$ and G lie on a circle. This suggests the following statement, that we have been unable to find in the literature:

Corollary 9. *A quadrangle is cyclic if and only if its centroid lies on the nine-point circle of its diagonal triangle.*

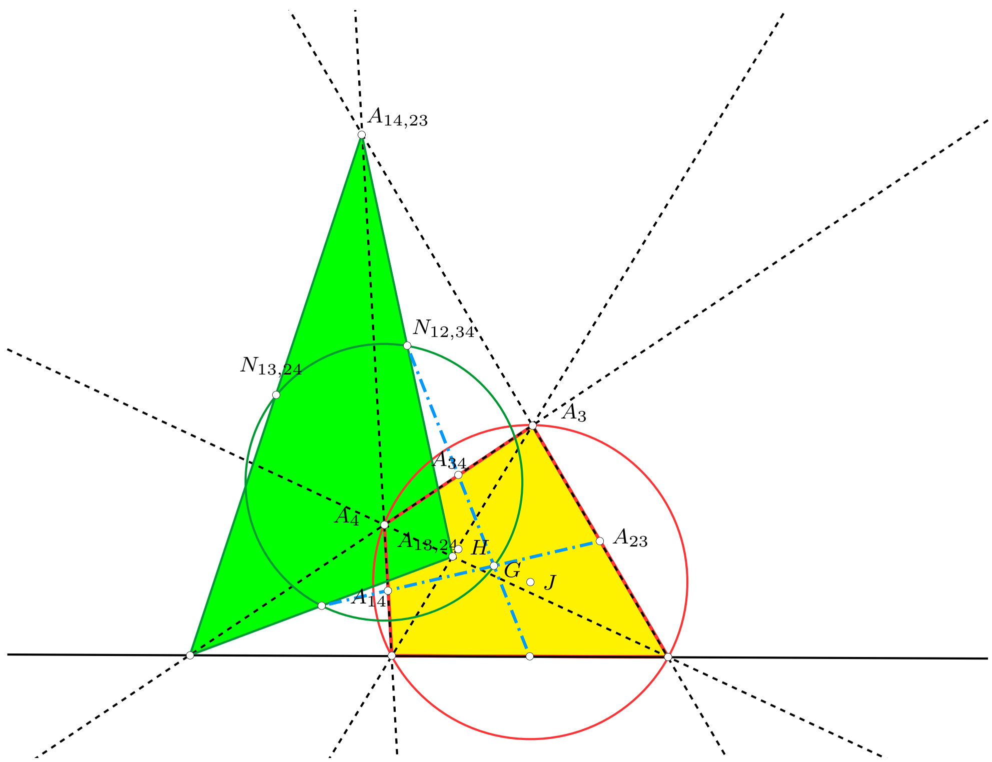

Figure 7. The centroid of a cyclic quadrangle lies on the ninepoint circle of its diagonal triangle

Proof. (Figure 7) It is well-known (see [2, Problem 46], or [1, §17.5.4]) that a conic circumscribed to a triangle $\mathcal{D}$ is a rectangular hyperbola if and only if its center lies on the ninepoint circle of $\mathcal{D}$. Let $\mathcal{A}$ be a quadrangle whose centroid G lies on the nine-point circle of its diagonal triangle $\mathcal{D}$. Since the medial conic

Γ_A is circumscribed to $\mathcal{D}$ and its center is the centroid G_A, we know that Γ_A is a rectangular hyperbola. Then, by previous theorems, we have $s_4 = 1$, Θ_A is a circle and $\mathcal{A}$ is cyclic. The converse argument is similar. □

10. Another special case: trapezoids

Trapezoids form another popular family of convex quadrangles, corresponding to the case $s_4 = \dfrac{s_3^2}{s_1^2}$. In fact, two opposite sides, say A_1A_2, A_3A_4 are parallel if and only if $\dfrac{a_1 - a_2}{\frac{1}{a_1} - \frac{1}{a_2}} = \dfrac{a_3 - a_4}{\frac{1}{a_3} - \frac{1}{a_4}}$, hence $a_1a_2 = a_3a_4$; and a straightforward calculation gives $(a_1a_2 - a_3a_4)(a_1a_3 - a_2a_4)(a_1a_4 - a_2a_3) = s_4s_1^2 - s_3^2$.

For trapezoids the medial conic Γ degenerates into two lines: $(s_1x + s_3y - \frac{1}{2}s_1^2)(s_1x - s_3y) = 0$ which are bimedians for $\mathcal{A}$; their slopes are simply $\pm\dfrac{s_1}{s_3}$: the asymptotic direction $-\dfrac{s_1}{s_3}$ is shared by the parallel sides of $\mathcal{A}$; the other is the direction of the line $s_1x - s_3y = 0$, on which one finds H, G, J, plus the two diagonal points at finite $A_{ih,jk}, A_{ik,jh}$. Π_+ also degenerates into the pair of parallel opposite sides. If two of the differences $a_ia_j - a_ha_k$ vanish, then $\mathcal{A}$ is a parallelogram, $H = J$ is its center and the asymptotic directions are parallel to the sides. For a cyclic trapezoid we have the additional condition $s_1^2 = s_3^2$ and HG is a symmetry line for $\mathcal{A}$. As for the oblique reflection ψ of Theorem 6, if, for example, the parallel sides are A_1A_2 and A_3A_4 , then the lines r and s are the bimedians $A_{12}A_{34}$, $A_{14}A_{23}$ and the oblique reflection interchanges two pairs of vertices: $\psi : A_1 \to A_2, A_2 \to A_1, A_3 \to A_4, A_4 \to A_3$. Thus $\mathcal{A}$ and $\mathcal{B} = \psi(\mathcal{A}$) are the same quadrangle, but ψ is not the identity!

The fact that for both cyclic quadrangles and trapezoids semi-similarities leave the quadrangle shape invariant may perhaps explain why the relation of semi-similarity, to our knowledge, has not been studied. [14]

11. Orthogonal quadrangles

We still have to consider the family of quadrangles which have a pair of orthogonal opposite sides, because in this case the foregoing analytical geometry does not work. We call these quadrangles *orthogonal*. We shall first assume that the other pairs of sides are not perpendicular, leaving still out the subfamily of the so-called *orthocentric* quadrangles, which will be considered as very last. For orthogonal (non orthocentric) quadrangles, the statements of the Theorems of §§3 to 10 remain exactly the same, but a principal reference cannot be defined as before and different analytic proofs must be provided. First notice that for this family the hyperbola

[14] Going back to the semi-similar quadrangles $\mathcal{A}$ and $o(\mathcal{A}) = O_1O_2O_3O_4$ which we mentioned in the introduction and appear in Johnson′s textbook [3], it follows from our previous arguments that $A_i \to O_i$ is induced by an affine transformation which can be thought of as the product of four factors: a rotation of a straight angle, an oblique reflection, a homothety (the three of them fixing H) and a final translation $H \to J$. It can be proved that $A_i \to O_i$ is also induced, modulo an isometry, by a circle inversion centered at the so-called *isoptic* point of $\mathcal{A}$, see [6, 4].

Ψ degenerates into a pair of orthogonal lines. We can represent these lines by the equation $xy = 0$ (replacing $xy = 1$) and use them as xy-axes of a new principal reference (the unit length is arbitrary). Within this frame we can assume without loss of generality $A_1 = [x_1, 0], A_2 = [0, y_2], A_3 = [x_3, 0], A_4 = [0, y_4]$. Notice that the product $x_1y_2x_3y_4$ cannot vanish, as we have excluded quadrangles with three collinear vertices. The role of the elementary symmetric polynomials can be played here by other polynomials, as $s_x = x_1 + x_3, s_y = y_2 + y_4, p_x = x_1x_3, p_y = y_2y_4$. We have $H = [0, 0], G = \frac{1}{4}[s_x, s_y], J = \frac{1}{2}[s_x, s_y]$. One of the diagonal points is H; the remaining two are $\dfrac{1}{x_1y_4 - x_3y_2}[x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)]$ and $\dfrac{1}{x_1y_2 - x_3y_4}[-x_1x_3(y_2 - y_4), y_2y_4(x_1 - x_3)]$. This shows that the xy-axes bisect an angle of the diagonal triangle. The fraction $\dfrac{p_x}{p_y}$ (or the product p_xp_y) plays the role of s_4 . More precisely: convexity and concavity are represented by $p_xp_y > 0$ or $p_xp_y < 0$ respectively ($p_xp_y = 0$ has been already excluded); $\mathcal{A}$ is cyclic if and only if $p_x = p_y$; $\mathcal{A}$ is a non-cyclic trapezoid when $p_xs_y^2 = p_ys_x^2$. Similar conditions can be established for s_x, s_y, p_x, p_y to characterize the various families of quadrangles ($skites$, $diamonds$, $squares$). The equations for the central conics are

$$\Gamma: \qquad p_yx^2 - p_xy^2 - \frac{1}{2}p_ys_xx + \frac{1}{2}p_xs_yy = 0,$$

and

$$\Theta: \qquad p_yx^2 + p_xy^2 - p_ys_xx - p_xs_yy + p_xp_y = 0.$$

The asymptotic directions have slope $\pm\sqrt{\dfrac{p_y}{p_x}}$ and $\pm\sqrt{-\dfrac{p_y}{p_x}}$ for the convex or concave case, respectively; the corresponding affinity of Theorem 6 is $[x, y] \to [y\sqrt{\frac{p_x}{p_y}}, x\sqrt{\frac{p_y}{p_x}}]$ for convex $\mathcal{A}$ etc. Not surprisingly, all statements and proofs of the foregoing theorems remain substantially the same and do not deserve special attention.

12. An extreme case: orthocentric quadrangles

If two pairs of opposite sides of $\mathcal{A}$ are orthogonal, then the same holds for the third pair. Such a concave quadrangle is called *orthocentric*, as each vertex A_i is the orthocenter of the complementary triangle $A_jA_hA_k$. For these quadrangles all the circumscribed conics are rectangular hyperbolas, so that Ψ, H, J, Θ are not defined. On the other hand, the medial conic Γ is defined, being merely the common nine-point circle for all the complementary triangles $A_jA_hA_k$. The asymptotic directions of $\mathcal{A}$ cannot be defined, but any pair of orthogonal directions can be used for defining the affinity of §5, and semi-similar orthocentric quadrangles turn out to be just directly similar. As an example, the elementary construction we gave in the introduction, when applied to an orthocentric quadrangle $\mathcal{A}$, modulo homotheties, just rotates $\mathcal{A}$ by a straight angle. We may also notice that some statements regarding orthocentric quadrangles can be obtained from the general case, as limits for s_4 tending to the value -1.

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Inversions in an Ellipse

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Abstract. In this paper we study the inversion in an ellipse which generalizes the classical inversion with respect to a circle and some properties. We also study the inversive images of lines, ellipses and other curves. Finally, we generalize the Pappus chain theorem to ellipses.

1. Introduction

In this paper we study inversions in an ellipse, which was introduced in [2], and some related properties to the distance of inverse points, cross ratio, harmonic conjugates and the images of various curves. This notion generalizes the classical inversion, which has a lot of properties and applications, see [1, 3, 4].

Definition. Let $\mathcal{E}$ be an ellipse centered at a point O with foci F_1 and F_2 in $\mathbb{R}^2$, the inversion in $\mathcal{E}$ is the mapping $\psi : \mathbb{R}^2 \setminus \{O\} \to \mathbb{R}^2 \setminus \{O\}$ defined by $\psi(P) = P'$, where P' lies on the ray $\overrightarrow{OP}$ and $OP \cdot OP' = OQ^2$, where Q is the intersection of the ray OP with the ellipse.

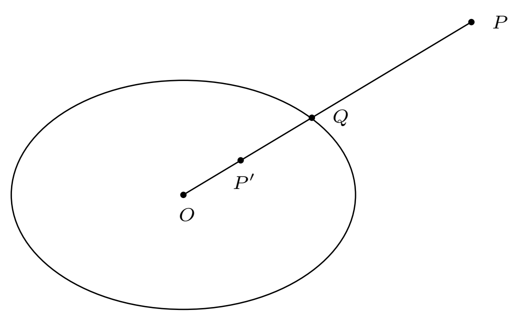

Figure 1

The point P' is said to be the inverse of P in the ellipse $\mathcal{E}$. We call $\mathcal{E}$ the ellipse of inversion, O the center of inversion, and the number $w := OQ$ the radius of inversion (see Figure 1). Unlike the classical case, the radius of inversion is not constant. Clearly, ψ is an involution, i.e., $\psi(\psi(P)) = P$ for every $P \neq O$. The fixed points are the points on the ellipse $\mathcal{E}$. Indeed, P is in the exterior of $\mathcal{E}$ if and only if P' is in the interior of $\mathcal{E}$. By introducing a point at infinity O_∞ as the inversive image of O, we regard ψ as an involution on the extended inversive plane $\mathbb{R}^2_\infty$.

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2. Basic properties

Proposition 1. *The inverse of P in an ellipse $\mathcal{E}$ is the intersection of the line OP and the polar of P with respect to $\mathcal{E}$. More precisely, if $\mathcal{E}$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the inverse of the point (u,v) in the ellipse is the point*

$$\left(\frac{a^2b^2u}{b^2u^2+a^2v^2},\ \frac{a^2b^2v}{b^2u^2+a^2v^2}\right).$$

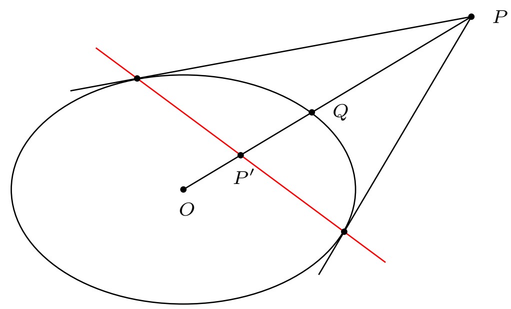

Figure 2

Proof. If $P=(u,v)$, the ray $\overrightarrow{OP}$ intersects $\mathcal{E}$ at $Q=(tu,\ tv)$ for $t>0$ satisfying $t^2\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)=1$. Now, the polar of P is the line $\frac{ux}{a^2}+\frac{vy}{b^2}=1$. This intersects the line OP (with equation $vx-uy=0$) at the point $(u',v')=(ku,kv)$ for k satisfying $k\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)=1$. Comparison gives $k=t^2$. Hence $OP\cdot OP'=OQ^2$, and (u',v') is the inverse of P in $\mathcal{E}$. Explicitly, $u'=\frac{a^2b^2u}{b^2u^2+a^2v^2}$ and $v'=\frac{a^2b^2v}{b^2u^2+a^2v^2}$. □

Theorem 2. *Let P and T be distinct points with inversion radii w and u with respect to $\mathcal{E}$. If P' and T' are the inverses of P and T in $\mathcal{E}$,*

$$P'T'=\begin{cases}\frac{w^2\cdot TP}{OP\cdot OT}, & \text{if } O,\ P,\ T \text{ are collinear},\\[2ex] \frac{\sqrt{(w^2-u^2)(w^2\cdot OT^2-u^2\cdot OP^2)+w^2u^2\cdot PT^2}}{OP\cdot OT}, & \text{otherwise}.\end{cases}$$

Proof. If O, P, T are collinear, the line containing them also contains Q, P' and T'. Clearly,

$$P'T'=OT'-OP'=\frac{OQ^2}{OT}-\frac{OQ^2}{OP}=\frac{w^2(OP-OT)}{OP\cdot OT}=\frac{w^2\cdot TP}{OP\cdot OT}.$$

Now suppose O, P, T are not collinear. Then neither are O, P', T' (see Figure 3). Let α be the measure of angle $P'OT'$. By the law of cosines, we have, in triangle POT,

$$\cos\alpha=\frac{OP^2+OT^2-PT^2}{2\cdot OP\cdot OT}.$$

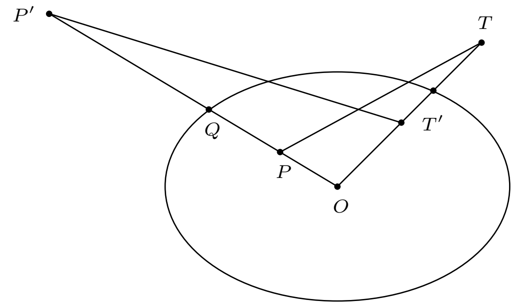

Figure 3.

Also, in triangle $P'OT'$,

$$\begin{aligned}P'T'^2 =&\ OP'^2 + OT'^2 - 2 \cdot OP' \cdot OT' \cdot \cos\alpha \\ =&\ \left(\frac{w^2}{OP}\right)^2 + \left(\frac{u^2}{OT}\right)^2 - 2 \cdot \frac{w^2}{OP} \cdot \frac{u^2}{OT} \cdot \frac{OP^2 + OT^2 - PT^2}{2 \cdot OP \cdot OT} \\ =&\ \frac{w^4 \cdot OT^2 + u^4 \cdot OP^2 - w^2u^2(OP^2 + OT^2 - PT^2)}{OP^2 \cdot OT^2} \\ =&\ \frac{(w^2 - u^2)(w^2 \cdot OT^2 - u^2 \cdot OP^2) + w^2u^2 \cdot PT^2}{OP^2 \cdot OT^2}.\end{aligned}$$

From this the result follows. □

3. Cross ratios and harmonic conjugates

Let A, B, C and D be four distinct points on a line ℓ. We define the cross ratio

$$(AB, CD) := \frac{AC \cdot BD}{AD \cdot BC},$$

where AB denotes the signed distance from A to B. We say that C, D divide A, B harmonically if the cross ratio $(AB, CD) = -1$. In this case we say that C and D are harmonic conjugates with respect to A and B. The cross ratio is an invariant under inversion in a circle whose center is not any of the four points A, B, C, D (see [1]). However, the inversion in an ellipse does not preserve the cross ratio. Nevertheless, in the case of harmonic conjugates, we have the following theorem.

Theorem 3. *Let $\mathcal{E}$ be an ellipse with center O, and Q_1Q_2 a diameter of $\mathscr{E}$. Two points on the line Q_1Q_2 are harmonic conjugates with respect to Q_1 and Q_2 if and only if they are inverse to each other with respect to $\mathcal{E}$.*

Proof. Let P and P' be two points on a diameter Q_1Q_2. Since

$$\begin{aligned}Q_1P \cdot Q_2P' =&\ (Q_1O + OP) \cdot (Q_2O + OP') \\ =&\ (w + OP)(-w + OP') \\ =&\ -w^2 - w(OP - OP') + OP \cdot OP', \\ Q_1P' \cdot Q_2P =&\ -w^2 + w(OP - OP') + OP \cdot OP',\end{aligned}$$

the points P and P' are harmonic conjugates with respect to Q_1 and Q_2 if and only if $OP \cdot OP' = w^2$, i.e., P and P' are inverse with respect to $\mathcal{E}$. $\square$

4. Images of curves under an inversion in an ellipse

Theorem 4. *Consider the inversion ψ in an ellipse $\mathcal{E}$ with center O.*

(a) *Every line containing O is invariant under the inversion.*

(b) *The image of a line not containing O is an ellipse containing O and homothetic to $\mathcal{E}$.*

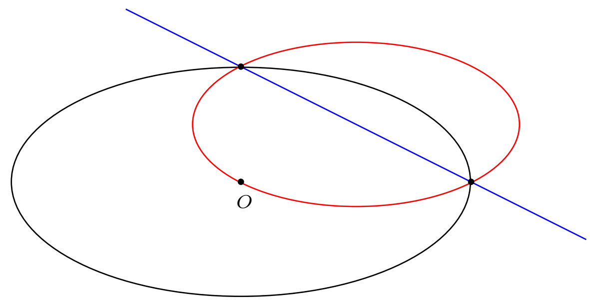

Figure 4

Proof. (a) This is clear from definition.

(b) Consider a line ℓ not containing O, with equation $px + qy + r = 0$ with $r \neq 0$. (x, y) is the inversive image of a point on ℓ, then the image of (x, y) lies on ℓ. In other words,

$$p \cdot \frac{a^2b^2x}{b^2x^2 + a^2y^2} + q \cdot \frac{a^2b^2y}{b^2x^2 + a^2y^2} + r = 0.$$

$$a^2b^2(px + qy) + r(b^2x^2 + a^2y^2) = 0. \tag{1}$$

This is clearly an ellipse containing $O(0, 0)$. Indeed, by rearranging its equation as

$$\frac{\left(x + \frac{a^2p}{2r}\right)^2}{a^2} + \frac{\left(y + \frac{b^2q}{2r}\right)^2}{b^2} = \frac{a^2p^2 + b^2q^2}{4r^2},$$

we note that this is the ellipse with center $\left(-\frac{a^2p}{2r}, -\frac{b^2q}{2r}\right)$, and homothetic to $\mathcal{E}$ with ratio $\frac{2r}{\sqrt{a^2p^2+b^2q^2}}$. $\square$

Corollary 5. *Let ℓ_1 and ℓ_2 be perpendicular lines intersecting at a point P.*

(a) *If $P = O$, then $\psi(\ell_1)$ and $\psi(\ell_2)$ are perpendicular lines.*

(b) *If ℓ_1 does not contain O but ℓ_2 does, then $\psi(\ell_1)$ is an ellipse through O orthogonal to $\psi(\ell_2) = \ell_2$ at O.*

(c) *If none of the lines contains O, then $\psi(\ell_1)$ and $\psi(\ell_2)$ are ellipses containing P' and O, and are orthogonal at O.*

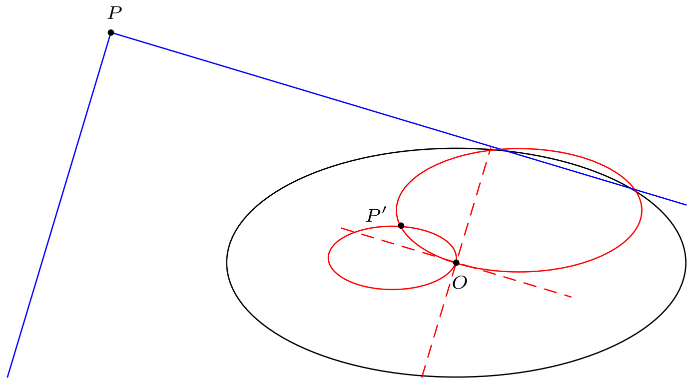

Figure 5

Proof. (a) The lines ℓ_1 and ℓ_2 are invariant.

(b) Let ℓ_1 be the line $px + qy + r = 0$ (with $r \neq 0$). Its image in $\mathcal{E}$ is the ellipse given by (1). The tangent at O is the line whose equation is obtained by suppressing the x^2 and y^2 terms, and replacing x and y by $\frac{1}{2}x$ and $\frac{1}{2}y$. This results in the line $\frac{1}{2}a^2b^2(px + qy) = 0$, or simply $px + qy = 0$, parallel to ℓ_1 and orthogonal to ℓ_2 at O.

(c) Let ℓ_1 and ℓ_2 be the orthogonal lines $p(x - h) + q(y - k) = 0$ and $q(x - h) - p(y - k) = 0$ intersecting at $P = (h, k) \neq O$. Their inverse images in $\mathcal{E}$ are ellipses intersecting at O and P'. By (b) above, the tangents at O are the orthogonal lines $px + qy = 0$ and $qx - py = 0$. $\square$

Remark. In (c), the images are not necessarily orthogonal at P'.

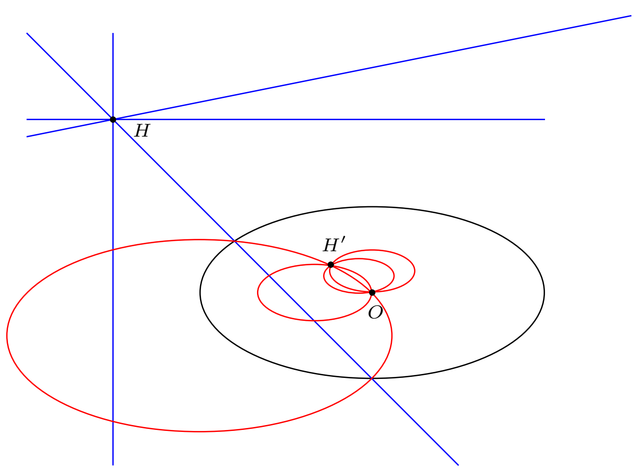

Figure 6.

Corollary 6. (a) *If $P \neq O$, the inverse images of the pencil of lines through P are coaxial ellipses through O and P'* (see Figure 6).

(b) *The inverse images of a system of straight lines parallel to ℓ_0 through O are ellipses homothetic to $\mathcal{E}$ tangent to ℓ_0 at O* (see Figure 7).

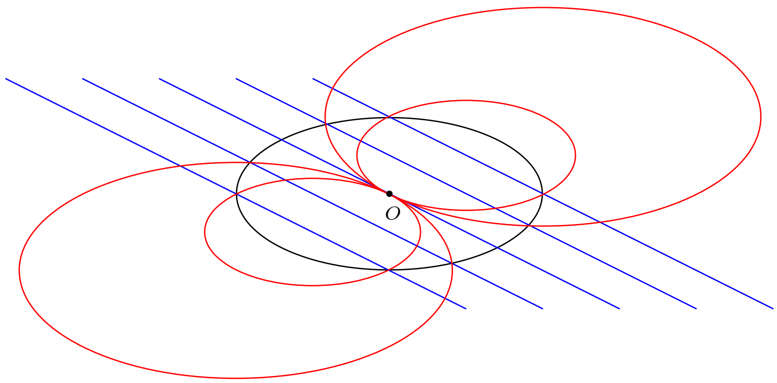

Figure 7.

Theorem 7. *Let $\mathcal{E}$ be the ellipse of inversion with center O, and $\mathcal{E}'$ an ellipse homothetic to $\mathcal{E}$. The image of $\mathcal{E}'$ is*
(a) *an ellipse homothetic to $\mathcal{E}$ if $\mathcal{E}'$ does not pass through O,*
(b) *a line if $\mathcal{E}'$ passes through O.*

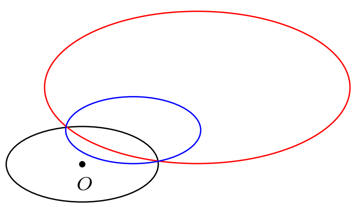

Figure 8

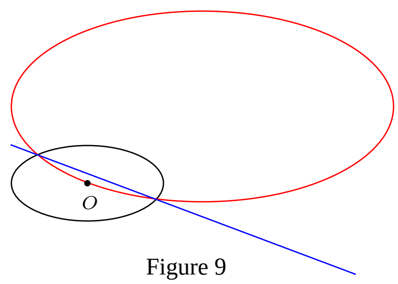

Figure 9

Proof. An ellipse $\mathcal{E}'$ homothetic to $\mathcal{E}$ has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + px + qy + r = 0.$$

The ellipse $\mathcal{E}'$ passes through O if and only if $r = 0$.

(a) If $\mathcal{E}$ does not pass through O, then $r \neq 0$. The inversive image consists of points $P(x, y)$ for which P' lies on the ellipse, i.e.,

$$\frac{\left(\frac{a^2b^2x}{b^2x^2+a^2y^2}\right)^2}{a^2} + \frac{\left(\frac{a^2b^2y}{b^2x^2+a^2y^2}\right)^2}{b^2} + p\left(\frac{a^2b^2x}{b^2x^2+a^2y^2}\right) + q\left(\frac{a^2b^2y}{b^2x^2+a^2y^2}\right) + r = 0. \tag{2}$$

Simplifying, we obtain

$$(b^2x^2+a^2y^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{p}{r}\cdot x+\frac{q}{r}\cdot y+\frac{1}{r}\right)=0.$$

Since $b^2x^2+a^2y^2\neq 0$, we must have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{p}{r}\cdot x+\frac{q}{r}\cdot y+\frac{1}{r}=0.$$

This is an ellipse homothetic to $\mathcal{E}$ (see Figure 8).

(b) If $\mathcal{E}'$ passes through O, then $r=0$. Equation (2) reduces to $px+qy+1=0$ (see Figure 9). $\square$

Corollary 8. *Let $\mathcal{E}'$ be an ellipse with center O' homothetic to $\mathcal{E}$ with center O. If $\mathcal{E}'$ is invariant under inversion in $\mathcal{E}$, and P is a common point of the ellipses, then $O'P$ and OP are tangent to $\mathcal{E}$ and $\mathcal{E}'$ respectively.*

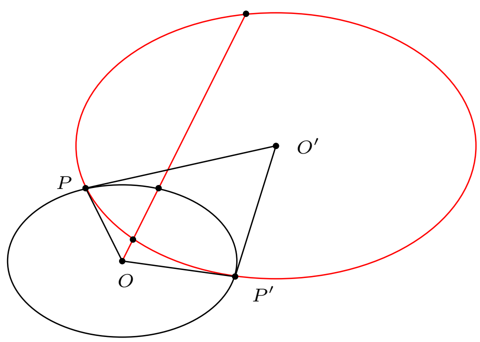

Figure 10

Proof. Comparing the equations of $\mathcal{E}'$ and its image under inversion in $\mathcal{E}$ in the proof of Theorem 7 above, we conclude that the ellipse $\mathcal{E}'$ is invariant if and only if its equation is of the form

$$(\mathcal{E}'):\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+px+qy+1=0.$$

Note that the center O' of $\mathcal{E}'$ has coordinates $\left(-\frac{pa^2}{2},\ -\frac{qb^2}{2}\right)$.

Let $P=(x_0,y_0)$ be a common point of the two ellipses. Clearly,

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1, \tag{3}$$

$$px_0+qy_0+2=0. \tag{4}$$

The tangents to $\mathcal{E}$ and $\mathcal{E}'$ at (x_0,y_0) are the lines

$$(t):\qquad \frac{x_0x}{a^2}+\frac{y_0y}{b^2}-1=0,$$

and

$$(t'): \qquad \frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{1}{2}p(x+x_0)+\frac{1}{2}q(y+y_0)+1=0.$$

Substitution of (x,y) by the coordinates O' into (t) and $(0,0)$ into (t') lead to $\mp\left(\frac{px_0}{2}+\frac{qy_0}{2}+1\right)$ respectively. By (4), this is zero in both cases. This shows that $O'P$ is tangent to $\mathcal{E}$ and OP is tangent to $\mathcal{E}'$. $\square$

Theorem 9. *Given an ellipse $\mathcal{E}$ with center O, the image of a conic $\mathcal{C}$ not homothetic to $\mathcal{E}$ is*
(i) *a cubic curve if $\mathcal{C}$ passes through O,*
(ii) *a quartic curve if $\mathcal{C}$ does not pass through O.*

In Figures 11, 12, 13 below, we show the inversive images of a circle, a parabola, and a hyperbola in an ellipse.

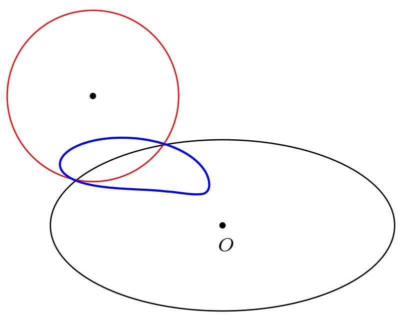

Figure 11

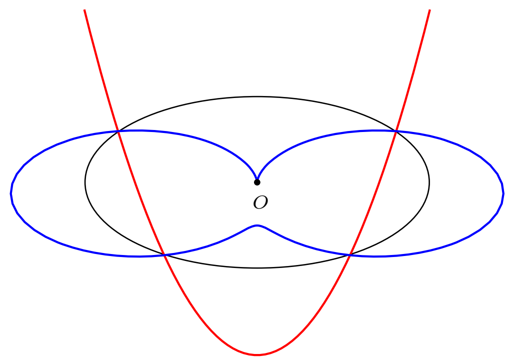

Figure 12

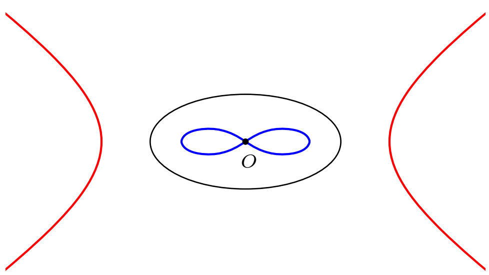

Figure 13

Note that the inversion in an ellipse is not conformal.

5. Pappus chain of ellipses

The classical inversion has a lot of applications, such as the Pappus chain theorem, Feuerbach's Theorem, Steiner Porism, the problem of Apollonius, among others [1, 3, 4]. We conclude this note with a generalization of the Pappus chain theorem to ellipses.

Theorem 10. *Let $\mathcal{E}$ be a semiellipse with principal diameter AB, and $\mathcal{E}'$, $\mathcal{E}_0$ semiellipses on the same side of AB with principal diameters AC and CB respectively, both homothetic to $\mathcal{E}$* (see Figure 14). *Let $\mathcal{E}_1$, $\mathcal{E}_2$, $\ldots$, be a sequence of ellipses tangent to $\mathcal{E}$ and $\mathcal{E}'$, such that $\mathcal{E}_n$ is tangent to $\mathcal{E}_{n-1}$ and $\mathcal{E}_{n+1}$ for all $n \geq 1$. If r_n is the semi-minor axis of $\mathcal{E}_n$ and h_n the distance of the center of $\mathcal{E}_n$ from AB, then $h_n = 2nr_n$.*

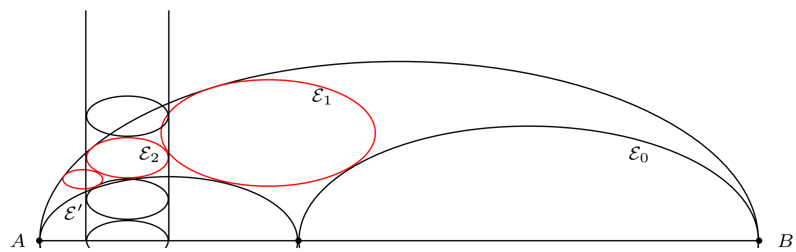

Figure 14.

Proof. Let ψ_i be the inversion in the ellipse $\mathcal{E}_i$. (In Figure 14 we select $i = 2$).

By Theorem 7, $\psi_i(\mathcal{E})$ and $\psi_i(\mathcal{E}_0)$ are lines perpendicular to AB and tangent to the ellipse $\mathcal{E}_i$. Hence, the ellipses $\psi_i(\mathcal{E}_1)$, $\psi_i(\mathcal{E}_2)$, $\ldots$ will be inverted to tangent ellipses to parallel lines to $\psi_i(\mathcal{E})$ and $\psi_i(\mathcal{E}_0)$. Hence, $h_i = 2ir_i$. $\square$

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On a Geometric Locus in Taxicab Geometry

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Abstract. In axiomatic geometry, the taxicab model of geometry is important as an example of a geometry where the SAS Postulate does not hold. Some properties that hold true in Euclidean geometry are not true in taxicab geometry. For this reason, it is important to understand what happens with various classes of geometric loci in taxicab geometry. In the present study, we focus on a geometric locus question inspired by a problem originally posed by Ţiţeica in the Euclidean context; our study presents the solution to this question in the taxicab plane.

1. Introduction

The taxicab geometry is particularly important in foundations of geometry because it provides an example of geometry where the Side Angle Side Postulate does not hold (see e.g. [8]). In the recent decades, several investigations have focused on various properties of taxicab geometry, some of them inspired from questions studied in advanced Euclidean geometry (see e.g. [2, 4, 6]). An introduction in the fundamental properties of taxicab geometry is [3].

A well-known reference in advanced Euclidean geometry is Ţiţeica's problem book [7]. For the historical context in which the problem book [7] was written and on Ţiţeica's research, including his doctoral dissertation written under Gaston Darboux's direction, see [1]. The problem book [7] is cited by many authors and motivated many contemporary problems. In the present work, we will focus on one particular question, namely, Problem 143. We will ask the question not in the Euclidean context, but in the context of taxicab geometry.

The taxicab distance between two points (x_1, y_1) and (x_2, y_2) in the Cartesian plane is defined (see e.g. [8], p. 39) by

$$\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

A direct verification shows that ρ is a metric. The Cartesian plane endowed with the metric induced by the distance ρ yields the taxicab geometry.

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The present study was inspired by the discussions developed around the geometry sessions at the Fullerton Mathematical Circle, a program of the Department of Mathematics at California State University, Fullerton. The author expresses his thanks to all the faculty members who have contributed to this program.

2. Right triangle leg ratio

Many relationships in Euclidean geometry do not hold in taxicab geometry. A well-known example is that SAS congruence fails in the taxicab plane; another is that the area of a triangle cannot simply be expressed in the classic $\frac{1}{2}bh$ (see [2]). Nonetheless, a handful of relationships do remain valid in the taxicab plane. For example, we present the following proposition.

Proposition 1. *The ratio between the two legs of a right triangle in the taxicab plane is equal to the ratio between the same two legs in the Euclidean plane.*

Proof. Let a and b denote the legs of the right triangle. We denote the taxicab lengths of a and b by a_T and b_T, and we denote the Euclidean lengths of a and b by a_E and b_E. If a and b are parallel to the coordinate axes, then $a_T = a_E$ and $b_T = b_E$, so clearly $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. Otherwise, let a have nonzero slope m; this means that b has slope $-\frac{1}{m}$. We have the relations

$$a_T = \frac{1+|m|}{\sqrt{1+m^2}} \cdot a_E,$$

$$b_T = \frac{1+|-\frac{1}{m}|}{\sqrt{1+(-\frac{1}{m})^2}} \cdot b_E = \frac{1+|m|}{\sqrt{1+m^2}} \cdot b_E$$

(see [2]). Dividing these two expressions yields $\frac{a_T}{b_T} = \frac{a_E}{b_E}$. □

3. A novel locus

The problem book [7] is cited by many authors and has motivated many contemporary problems. For example, V. Pambuccian's work [5] incorporates the axiomatic analysis of a problem found originally in Ţiţeica's problem book. Pursuing a similar idea, we now examine what happens to Problem 143, which was originally stated in [7] in the Euclidean context, if we study it in taxicab geometry. Thus, we ask the following:

Question. Consider a circle with center O and radius r in the taxicab plane. Point A is located within the circle. Find the locus of midpoints of all chords of the circle that pass through A.

In Euclidean geometry, the locus is well-known. It is simply a circle with diameter OA. On the other hand, when we consider this same locus problem in the context of taxicab geometry, we quickly see that the locus is not so simple. Figure 1 shows an example of such a locus:

Theorem 2. *In general, the locus of midpoints of chords that pass through a point A consists of two straight line segments and two hyperbolic sections, at least one of which contains A.*

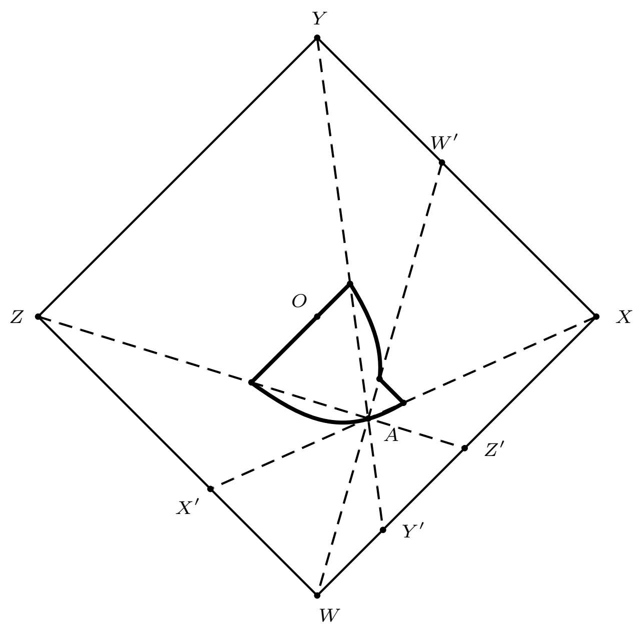

Figure 1

3.1. *Set-Up for proof.* Without loss of generality, we place O at the origin, and let $A = (x_A, y_A)$ such that $x_A \geq 0$ and $y_A \leq -x_A$. We can make that last assumption due to the symmetry of the taxicab plane; reflecting the circle across the axes and the lines $y = \pm x$ essentially preserves the shape of the locus. In other words, given any point A within the circle, we can reflect that point and the specified locus about the axes and the lines $y = \pm x$ until the image of A satisfies $x_A \geq 0$ and $y_A \leq -x_A$.

With the notations specified above, let the vertices of the circle be labeled X, Y, Z, and W, where X lies on the x-axis and the vertices are labeled counter-clockwise. Particularly, $X = (r, 0)$, $Y = (0, r)$, $Z = (-r, 0)$, and $W = (0, -r)$. Furthermore, let XA, YA, ZA, and WA intersect the circle again at X', Y', Z', and W', respectively. Then we claim that the locus consists of four parts:

(1) The locus of midpoints of the chords between XX' and $W'W$ is a straight line from the midpoint of XX' to the midpoint of $W'W$ along the line $y = -x$.

(2) The locus of midpoints of the chords between YY' and ZZ' is a straight line from the midpoint of YY' to the midpoint of ZZ' along the line $y = x$.

(3) The locus of midpoints of the chords between $W'W$ and YY' is a hyperbolic section from the midpoint of $W'W$ to the midpoint of YY' that is centered at $(\frac{x_A+r}{2}, \frac{y_A}{2})$.

(4) The locus of midpoints of the chords between ZZ' and $X'X$ is a hyperbolic section from the midpoint of ZZ' to the midpoint of $X'X$ that is centered at $(\frac{x_A}{2}, \frac{y_A-r}{2})$. Point A lies on this hyperbolic section.

3.2. *Proof of Theorem 2.*

(1) and (2): Consider any chord passing through A between XX' and $W'W$. Since the endpoints of this chord lie on parallel lines, the midpoint of the chord must be halfway between these lines, also forming a parallel line with the two original ones. Particularly, the line that the midpoints fall on is $y = -x$, from the midpoint of XX' to the midpoint of $W'W$. By the same reasoning, the locus of midpoints of any chord passing through A between YY' and $Z'Z$ is on $y = x$, from the midpoint of YY' to the midpoint of $Z'Z$.

(3) and (4): We prove (4); then (3) follows by a similar argument.

The equations of lines ZW and WX are $y = -x-r$ and $y = x-r$, respectively. Consider the endpoint C on ZW of a chord passing through A with coordinates $(x_C, -x_C - r)$. The point D at which CA intersects WX is uniquely determined, so we can calculate D using the slope of CA and hence the equations of lines CA and WX:

$$D = \left(\frac{x_A x_C + r x_C + y_A x_C}{y_A - x_A + 2x_C + r}, \frac{x_A x_C + y_A x_C - r y_A + r x_A - r x_C - r^2}{y_A - x_A + 2x_C + r}\right).$$

The midpoint M of chord CD is simply the average of the points C and D:

$$M = \left(x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r}, \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} - r\right).$$

To find the locus of midpoints with the above expression, we wish to relate the x and y coordinates. We set

$$x = x_C + \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} = \frac{x_C y_A + x_C^2 + r x_C}{y_A - x_A + 2x_C + r},$$
$$y = \frac{x_C(x_A - x_C)}{y_A - x_A + 2x_C + r} - r = \frac{x_C x_A - x_C^2 - 2r x_C - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r}.$$

These equations yield

$$x + y = \frac{x_C y_A + x_C x_A - r x_C - r y_A + r x_A - r^2}{y_A - x_A + 2x_C + r},$$
$$x - y = x_C + r.$$

Multiplying through by the denominator and substituting $x_C = x - y - r$ yields

$$\begin{aligned}
&(x+y)(y_A - x_A + 2(x-y-r) + r) \\
&= (x-y-r)y_A + (x-y-r)x_A - r(x-y-r) - r y_A + r x_A - r^2 \\
&\Longrightarrow x^2 - y^2 - x x_A + y y_A - y r + y_A r = 0 \\
&\Longrightarrow \left(x - \frac{x_A}{2}\right)^2 - \left(y - \frac{y_A - r}{2}\right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}.
\end{aligned}$$

This is precisely the form of a hyperbola, with center $(\frac{x_A}{2}, \frac{y_A - r}{2})$, as desired. Clearly, the point $A = (x_A, y_A)$ lies on this hyperbolic section since

$$\left(x_A - \frac{x_A}{2}\right)^2 - \left(y_A - \frac{y_A - r}{2}\right)^2 = \frac{x_A^2}{4} - \frac{(y_A + r)^2}{4}.$$

A similar argument proves (3) as well. This completes the proof of Theorem 2.

Remarks. (1) In some cases, such as when A lies on the axes or $y = \pm x$ lines, some of these segments or sections may be degenerate. For example, if A lies on either of the coordinate axes, the locus consists of two straight line segments and one hyperbolic section. If A lies on $y = x$ or $y = -x$, the locus consists of one straight line segment and two hyperbolic sections. Clearly, if A is at the origin, the locus is simply a point.

(2) In advanced Euclidean geometry, we work within the axiomatic context given by the postulates of Euclidean geometry, which itself can be viewed in many axiomatic contexts (see [8]). Our present study points out how much a geometric locus can change in an axiomatic framework where the SAS Postulate does not hold any longer.

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A Simple Proof of Gibert's Generalization of the Lester Circle Theorem

Dao Thanh Oai

Abstract. We give a simple proof of Gibert's generalization of the Lester circle theorem.

The famous Lester circle theorem states that for a triangle, the two Fermat points, the nine point center and the circumcenter lie on a circle, the Lester circle of the triangle. Here is Gibert's generalization of the Lester circle theorem, given in [2] and [4, Theorem 6]: *Every circle whose diameter is a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points*. In this note we show that this follows from a property of rectangular hyperbolas.

Lemma 1. *Let F_+ and F_- be two antipodal points on a rectangular hyperbola. For every point H on the hyperbola, the tangent to the circle (F_+F_-H) at H is parallel to the tangents of the hyperbola at F_+ and F_-.*

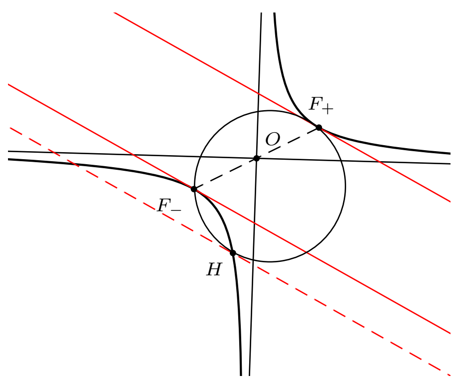

Figure 1

Proof. In a Cartesian coordinate system, let the rectangular hyperbola be represented by $xy = a$, and $F_+\left(x_0, \frac{a}{x_0}\right)$ and $F_-\left(-x_0, \frac{-a}{x_0}\right)$ two antipodal points. The slope of the tangents at $F_\pm$ is $-\frac{a}{x_0^2}$. Let $H\left(x_H, \frac{a}{x_H}\right)$ be a point on the hyperbola. Consider the circle through $F_\pm$ and H. Writing its equation in the form

$$x^2 + y^2 + Ax + By + C = 0,$$

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and substituting the coordinates of $F_\pm$ and H above, we obtain

$$\begin{aligned} x_0^2+y_0^2+Ax_0+By_0+C &= 0,\\ x_0^2+y_0^2-Ax_0-By_0+C &= 0,\\ x_H^2+y_H^2+Ax_H+By_H+C &= 0. \end{aligned}$$

Solving these equations we have

$$A=-x_H+y_H\cdot\frac{a}{x_0^2},\qquad B=-\frac{Ax_0^2}{a},\qquad C=-(x_0^2+y_0^2).$$

The tangent of the circle at H is the line

$$2x_Hx+2y_Hy+A(x+x_H)+B(y+y_H)+2C=0.$$

It has slope

$$-\frac{2x_H+A}{2y_H+B}=-\frac{x_H+y_H\cdot\frac{a}{x_0^2}}{y_H+x_H\cdot\frac{x_0^2}{a}}=-\frac{x_H+\frac{a}{x_H}\cdot\frac{a}{x_0^2}}{\frac{a}{x_H}+x_H\cdot\frac{x_0^2}{a}}=-\frac{x_H^2+\frac{a^2}{x_0^2}}{a+x_H^2\cdot\frac{x_0^2}{a}}=-\frac{a}{x_0^2}.$$

This tangent is parallel to the tangents of the hyperbola at $F_\pm$. □

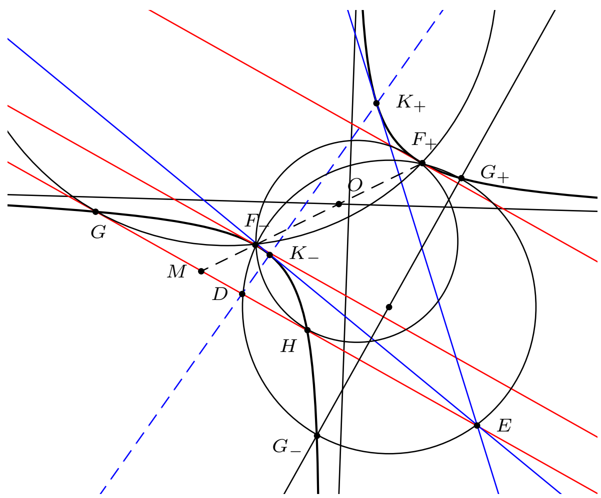

Figure 2

Theorem 2 ([1]). *Let H and G lie on one branch of a rectangular hyperbola, and*
(i) *F_+ and F_- antipodal points on the hyperbola the tangents at which are parallel to the line HG,*
(ii) *K_+ and K_- two points on the hyperbola the tangents at which intersect at a point E on the line HG.*
If the line K_+K_- intersects HG at D, and the perpendicular bisector of DE intersects the hyperbola at G_+ and G_-, then the six points F_+, F_-, D, E, G_+, G_- lie on a circle.

Proof. By Lemma 1, the circle (F_+F_-H) is tangent to HG at H. Similarly, the circle (F_+F_-G) is tangent to the same line HG at G.

Let M be the intersection of F_+F_- and HG. It lies on the radical axis of the circles (F_+F_-H) and (F_+F_-G), and satisfies $MG^2 = MF_+ \cdot MF_- = MH^2$. Therefore, M is the midpoint of HG.

Since the tangents of the hyperbola at K_+ and K_- intersect at E, the line K_+K_- is the polar of E. If it intersects the line HG at D, then $(G, D; H, E)$ is a harmonic range. Since M is the midpoint of HG, by a famous property of harmonic range, we have $MG^2 = MD \cdot ME$. Therefore, $MF_+ \cdot MF_- = MD \cdot ME$, and the four points F_+, F_-, D, E lie on a circle.

Now let the circle (F_+F_-DE) intersect the rectangular hyperbola at two points G_+ and G_-. By Lemma 1, the tangents of the circle at G_+, G_- are parallel to those of the hyperbola at F_+ and F_-, and therefore also to HG. It follows that G_+G_- is a diameter of the circle perpendicular to HG, and G_+, G_- lie on the perpendicular bisector of the chord DE of the circle. The proof of the theorem is complete. □

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A Note on the Fermat-Torricelli Point of a Class of Polygons

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Abstract. The aim of this note is to prove a result related to the Fermat-Torricelli point for a class of polygons.

The French mathematician Pierre Fermat (1601-1665) proposed at the end of his book *Treatise on Minima and Maxima* the search for a point T in the plane of a triangle ΔABC for which the sum $TA+TB+TC$ of the distances from T to the vertices is minimum. As the problem was first proved by the Italian scientist Evangelista Torricelli (1608-1647), the point T is sometimes called the Fermat-Torricelli point. The geometric construction of the Fermat-Torricelli point can be found in many textbooks, the most well known being that which uses the equilateral triangles constructed on the sides of the given triangle. If all angles of the given triangles are smaller than or equal to $2\pi/3$, then

$$\angle ATB = \angle BTC = \angle CTA = \frac{2\pi}{3}.$$

The problem has been studied by Fejes Tóth [2], Kazarinoff [3], and other specialists in geometric inequalities.

A history of Fermat's problem can be find in Boltyanski et al. [1].

We propose here the following new result that can be considered as an extention of Fermat's problem for a particular class of polygons. Remarks on this form can be found in [4]. Moreover, the proof provided is quite elementary.

Let there be given in plane $n+1$ points T, A_1, A_2, $\dots$, A_n. We say that the figure consisting of the union of the segments TA_1, TA_2, $\dots$, TA_n is a *star* if $A_1A_2...A_n$ is a n-sided polygon and

$$\angle A_1TA_2 = \angle A_2TA_3 = \cdots = \angle A_nTA_1 = \frac{2\pi}{n}.$$

Let us denote such a star by $[T; A_1, A_2, \dots, A_n]$. Now we are in the position to give our result.

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Theorem 1. *Let $[T; A_1, A_2, \ldots, A_n]$ be a star. Then for every point M, we have*

$$TA_1 + TA_2 + \cdots + TA_n \leq MA_1 + MA_2 + \cdots + MA_n.$$

Proof. Let us consider the complex plane with the origin T and the positive real axis TA_1. Let $r_k = TA_k$, and assume that $r_k \omega^{k-1}$ is the complex number associated with the point A_k, for every $1 \leq k \leq n$, where

$$\omega = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

For any point M associated with a complex number z, we have

$$\begin{aligned}
& MA_1 + MA_2 + \cdots + MA_n \\
&= |z - r_1| + |z - r_2\omega| + \cdots + \left|z - r_n\omega^{n-1}\right| \\
&= |z - r_1| + \left|\frac{z}{\omega} - r_2\right| + \cdots + \left|\frac{z}{\omega^{n-1}} - r_n\right| \\
&\geq \left|(z - r_1) + \left(\frac{z}{\omega} - r_2\right) + \cdots + \left(\frac{z}{\omega^{n-1}} - r_n\right)\right| \\
&= \left|z\left(1 + \frac{1}{\omega} + \cdots + \frac{1}{\omega^{n-1}}\right) - (r_1 + r_2 + \cdots + r_n)\right| \\
&= |-(r_1 + r_2 + \cdots + r_n)| \\
&= r_1 + r_2 + \cdots + r_n \\
&= TA_1 + TA_2 + \cdots + TA_n,
\end{aligned}$$

and we are done. □

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Properties of Equidiagonal Quadrilaterals

Martin Josefsson

Abstract. We prove eight necessary and sufficient conditions for a convex quadrilateral to have congruent diagonals, and one dual connection between equidiagonal and orthodiagonal quadrilaterals. Quadrilaterals with both congruent and perpendicular diagonals are also discussed, including a proposal for what they may be called and how to calculate their area in several ways. Finally we derive a cubic equation for calculating the lengths of the congruent diagonals.

1. Introduction

One class of quadrilaterals that have received little interest in the geometrical literature are the *equidiagonal quadrilaterals*. They are defined to be quadrilaterals with congruent diagonals. Three well known special cases of them are the isosceles trapezoid, the rectangle and the square, but there are other as well. Furthermore, there exists many equidiagonal quadrilaterals that besides congruent diagonals have no special properties. Take any convex quadrilateral $ABCD$ and move the vertex D along the line BD into a position D' such that $AC = BD'$. Then $ABCD'$ is an equidiagonal quadrilateral (see Figure 1).

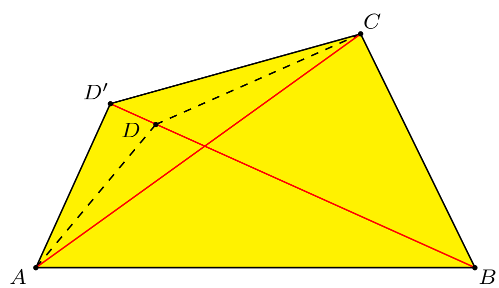

Figure 1. An equidiagonal quadrilateral $ABCD'$

Before we begin to study equidiagonal quadrilaterals, let us define our notations. In a convex quadrilateral $ABCD$, the sides are labeled $a = AB$, $b = BC$, $c = CD$ and $d = DA$, and the diagonals are $p = AC$ and $q = BD$. We use θ for the angle between the diagonals. The line segments connecting the midpoints of opposite sides of a quadrilateral are called the bimedians and are denoted m and n, where m connects the midpoints of the sides a and c.

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2. Characterizations of equidiagonal quadrilaterals

Of the seven characterizations for equidiagonal quadrilaterals that we will prove in this section, three have already appeared in our previous papers [11] and [12]. We include them here anyway for the sake of completeness. One of them is proved in a new way.

It is well known that the midpoints of the sides in any quadrilateral are the vertices of a parallelogram, called Varignon's parallelogram. The diagonals in this parallelogram are the bimedians of the original quadrilateral and the sides in the Varignon parallelogram are half as long as the diagonal in the original quadrilateral that they are parallel to. When studying equidiagonal quadrilaterals, properties of the Varignon parallelogram proves to be useful.

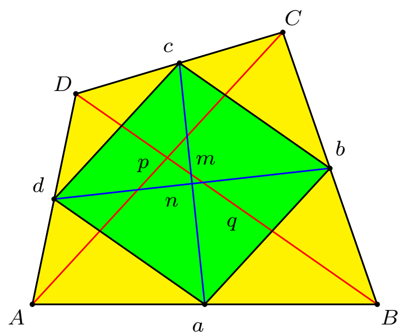

Figure 2. The Varignon parallelogram

Using the parallelogram law in the Varignon parallelogram yields (see Figure 2)

$$m^2 + n^2 = 2\left(\left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2\right)$$

which is equivalent to

$$p^2 + q^2 = 2(m^2 + n^2). \tag{1}$$

This equality is valid in all convex quadrilaterals.

For the product of the diagonals we have a necessary and sufficient condition of equidiagonal quadrilaterals in terms of the bimedians.

Proposition 1. *The product of the diagonals p and q in a convex quadrilateral with bimedians m and n satisfies*

$$pq \leq m^2 + n^2$$

where equality holds if and only if it is an equidiagonal quadrilateral.

Proof. By adding and subtracting $2pq$ to the left hand side of (1), we get

$$2pq \leq (p-q)^2 + 2pq = 2(m^2 + n^2).$$

The inequality follows, with equality if and only if $p = q$. □

The first part in the following theorem was proved by us as Theorem 7 (ii) in [11], but we repeat the short argument here.

Theorem 2. *A convex quadrilateral is equidiagonal if and only if*
(i) *the bimedians are perpendicular, or*
(ii) *the midpoints of its sides are the vertices of a rhombus.*

Proof. (i) It is well known that a quadrilateral has perpendicular diagonals if and only if the sum of the squares of two opposite sides is equal to the sum of the squares of the other two sides (see Theorem 1 in [11]). Hence we get

$$p = q \quad \Leftrightarrow \quad \left(\frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^2 = \left(\frac{q}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \quad \Leftrightarrow \quad m \perp n$$

since opposite sides in a parallelogram are congruent.

(ii) A parallelogram is a rhombus if and only if its diagonals are perpendicular. Since the diagonals in the Varignon parallelogram are the bimedians of the original quadrilateral (see Figure 2), (ii) is equivalent to (i). □

The next characterization is about the area of the quadrilateral. To prove it in a new way compared to what we did in [12, p.19], we need the following area formula for convex quadrilaterals. We cannot find a reference for this formula, but it is similar to one we derived in [10].

Theorem 3. *A convex quadrilateral with diagonals p, q and bimedians m, n has the area*

$$K = \sqrt{m^2n^2 - \left(\frac{p^2 - q^2}{4}\right)^2}.$$

Proof. Rewriting (1), we have in all convex quadrilaterals

$$(m^2 - n^2)^2 + 4m^2n^2 = \left(\frac{p^2 + q^2}{2}\right)^2. \tag{2}$$

Theorem 7 in [10] states that a convex quadrilateral has the area

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (m^2 - n^2)^2}.$$

Inserting (2) yields for the area

$$4K^2 = \frac{4p^2q^2}{4} + 4m^2n^2 - \left(\frac{p^2 + q^2}{2}\right)^2 = 4m^2n^2 - \left(\frac{p^2 - q^2}{2}\right)^2$$

and the formula follows. □

Corollary 4. *The area of a convex quadrilateral is equal to the product of the bimedians if and only if it is an equidiagonal quadrilateral.*

Proof. In Theorem 3, we have that $p = q$ if and only if $K = mn$. □

A direct consequence is another area formula, that also appeared in [12, p.19].

Corollary 5. *A convex quadrilateral with consecutive sides a, b, c, d is equidiagonal if and only if it has the area*

$$K = \tfrac{1}{4}\sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)}$$

where v is the distance between the midpoints of the diagonals.

Proof. The length of the bimedians in a convex quadrilateral are

$$m = \tfrac{1}{2}\sqrt{2(b^2+d^2)-4v^2} \qquad \text{and} \qquad n = \tfrac{1}{2}\sqrt{2(a^2+c^2)-4v^2} \tag{3}$$

according to [10, p.162]. Using these expressions in Corollary 4 directly yields this formula. □

The next characterization is perhaps not so elegant in itself, but it will be used to derive a more symmetric one later on.

Proposition 6. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d is equidiagonal if and only if*

$$ab\cos B + cd\cos D = ad\cos A + bc\cos C.$$

Proof. The quadrilateral is equidiagonal if and only if $2p^2 = 2q^2$, which, according to the law of cosines, is equivalent to (see Figure 2)

$$a^2+b^2-2ab\cos B+c^2+d^2-2cd\cos D = a^2+d^2-2ad\cos A+b^2+c^2-2bc\cos C.$$

Eliminating common terms and factors on both sides, this is equivalent to the equation in the proposition. □

This lemma, which can be thought of as a law of sines for quadrilaterals and is very similar to the previous proposition, will be used in the next proof.

Lemma 7. *In a convex quadrilateral $ABCD$ with consecutive sides a, b, c, d,*

$$ab\sin B + cd\sin D = ad\sin A + bc\sin C.$$

Proof. By dividing the quadrilateral into two triangles using a diagonal, which can be done in two different ways, we have for its area that (see Figure 2)

$$K = \tfrac{1}{2}ab\sin B + \tfrac{1}{2}cd\sin D = \tfrac{1}{2}ad\sin A + \tfrac{1}{2}bc\sin C.$$

The equation in the lemma follows at once by doubling both sides of the second equality. □

Now we come to our main characterization of equidiagonal quadrilaterals.

Theorem 8. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d is equidiagonal if and only if*

$$(a^2-c^2)(b^2-d^2) = 2abcd\big(\cos(A-C)-\cos(B-D)\big).$$

Proof. Squaring both sides of the equation in Lemma 7 yields

$$\begin{aligned} &a^2b^2\sin^2 B + c^2d^2\sin^2 D + 2abcd\sin B\sin D \\ &= a^2d^2\sin^2 A + b^2c^2\sin^2 C + 2abcd\sin A\sin C \end{aligned} \tag{4}$$

which is true in all convex quadrilaterals. Squaring the equation in Proposition 6, we have that a convex quadrilateral is equidiagonal if and only if

$$\begin{aligned} &a^2b^2\cos^2 B + c^2d^2\cos^2 D + 2abcd\cos B\cos D \\ &= a^2d^2\cos^2 A + b^2c^2\cos^2 C + 2abcd\cos A\cos C. \end{aligned} \tag{5}$$

By adding equations (4) and (5) and applying the identity $\sin^2\phi + \cos^2\phi = 1$ four times, we get the following equality that is equivalent to the one in Proposition 6 (due to the property that $x = y$ if and only if $x + z = y + z$ for any z)

$$\begin{aligned} &a^2b^2 + c^2d^2 + 2abcd(\sin B \sin D + \cos B \cos D) \\ &= a^2d^2 + b^2c^2 + 2abcd(\sin A \sin C + \cos A \cos C). \end{aligned}$$

Using the subtraction formula for cosine, this is equivalent to

$$a^2b^2 - a^2d^2 - b^2c^2 + c^2d^2 = 2abcd\cos(A - C) - 2abcd\cos(B - D)$$

which is factored into the equation in the theorem. □

Corollary 9. *Two opposite sides of an equidiagonal quadrilateral are congruent if and only if it is an isosceles trapezoid.*

Proof. Applying the trigonometric formula $\cos\phi - \cos\psi = -2\sin\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}$ and the sum of angles in a quadrilateral, we have that the equation in Theorem 8 is equivalent to

$$(a + c)(a - c)(b + d)(b - d) = -4abcd\sin(A + B)\sin(A + D).$$

Hence $a = c$ or $b = d$ is equivalent to $A + B = \pi$ or $A + D = \pi$, which are well known characterizations of a trapezoid (see [13, p.24]). □

3. A new duality regarding congruent and perpendicular diagonals

Theorem 7 in [11] can be reformulated to say that *a convex quadrilateral is equidiagonal if and only if its Varignon parallelogram is orthodiagonal, and the quadrilateral is orthodiagonal if and only if its Varignon parallelogram is equidiagonal.* Thus it gives a sort of dual connection between a quadrilateral and its Varignon parallelogram. Here we shall prove another duality between a quadrilateral and one quadrilateral associated with it. First let us remind the reader that if squares are erected outwards on the sides of a quadrilateral, then their centers are the vertices of a quadrilateral that is both equidiagonal and orthodiagonal.[1] This result is called van Aubel's theorem. It can be proved using elementary triangle geometry (see the animated proof at [7]) or basic properties of complex numbers as in [2, pp.62–64].

What happens if we exchange the squares for equilateral triangles? Problem 5 on the shortlist for the International Mathematical Olympiad in 1992 asked for a proof that the two line segments connecting opposite centroids of those triangles are perpendicular if the quadrilateral has congruent diagonals [6, p.269]. That problem covered only a quarter of the following theorem, since the converse statement as well as a dual one and its converse are also true. Essentially the same proof of part (i) was given at [15]. We have found no reference to neither the proof nor the statement of part (ii).

[1] An orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals.

Theorem 10. *Suppose equilateral triangles are erected outwards on the sides of a convex quadrilateral $ABCD$. Then the following characterizations hold:*
(i) *$ABCD$ is an equidiagonal quadrilateral if and only if the triangle centroids are the vertices of an orthodiagonal quadrilateral.*
(ii) *$ABCD$ is an orthodiagonal quadrilateral if and only if the triangle centroids are the vertices of an equidiagonal quadrilateral.*

Proof. (i) Let the triangle centroids be G_1, G_2, G_3 and G_4. In an equilateral triangle with side x, the distance from the centroid to a vertex (equal to the circumradius R) is $R = \frac{x}{\sqrt{3}}$. Applying the law of cosines in triangle G_1AG_4 yields (see Figure 3)

$$\begin{aligned}(G_1G_4)^2 &= \left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{d}{\sqrt{3}}\right)^2 - \frac{a}{\sqrt{3}} \cdot \frac{d}{\sqrt{3}} \cos\left(A + \frac{\pi}{3}\right) \\ &= \frac{a^2}{3} + \frac{d^2}{3} - \frac{ad}{3}\left(\frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A\right).\end{aligned}$$

In the same way we have

$$\begin{aligned}(G_2G_3)^2 &= \frac{b^2}{3} + \frac{c^2}{3} - \frac{bc}{3}\left(\frac{1}{2}\cos C - \frac{\sqrt{3}}{2}\sin C\right), \\ (G_1G_2)^2 &= \frac{a^2}{3} + \frac{b^2}{3} - \frac{ab}{3}\left(\frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B\right), \\ (G_3G_4)^2 &= \frac{c^2}{3} + \frac{d^2}{3} - \frac{cd}{3}\left(\frac{1}{2}\cos D - \frac{\sqrt{3}}{2}\sin D\right).\end{aligned}$$

Thus, simplifying and collecting similar terms yields that

$$\begin{aligned}&(G_1G_2)^2 + (G_3G_4)^2 - (G_2G_3)^2 - (G_1G_4)^2 \\ =\ &\tfrac{1}{6}(ad\cos A + bc\cos C - ab\cos B - cd\cos D) \\ +\ &\tfrac{\sqrt{3}}{6}(ab\sin B + cd\sin D - ad\sin A - bc\sin C).\end{aligned}$$

The last parenthesis is equal to zero in all convex quadrilaterals (Lemma 7). Hence we have

$$\begin{aligned}&(G_1G_2)^2 + (G_3G_4)^2 = (G_2G_3)^2 + (G_1G_4)^2 \\ \Leftrightarrow\quad &ab\cos B + cd\cos D = ad\cos A + bc\cos C,\end{aligned}$$

where the first equality is a well known characterization for $G_1G_3 \perp G_2G_4$ (see Theorem 1 in [11]) and the second equality is true if and only if $ABCD$ is equidiagonal according to Proposition 6.

(ii) This statement is trickier to prove with trigonometry, so instead we will use complex numbers. Let the vertices A, B, C and D of a convex quadrilateral be represented by the complex numbers z_1, z_2, z_3 and z_4 respectively. Also, let the centroids G_1, G_2, G_3 and G_4 of the equilateral triangles be represented by the

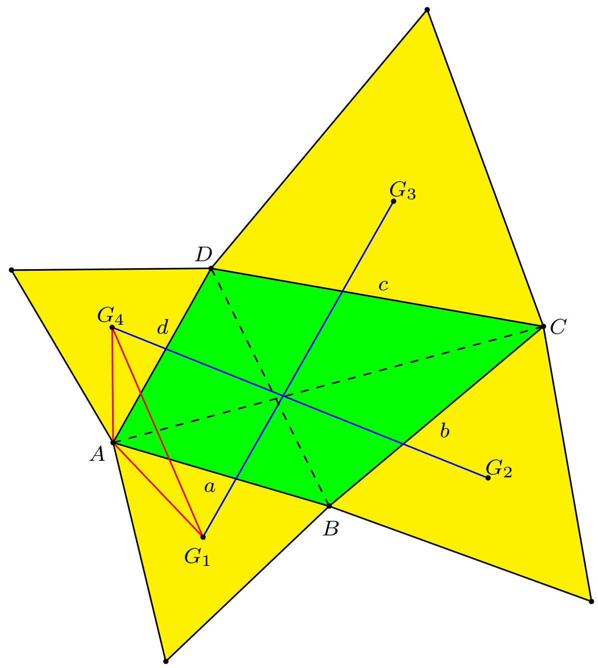

Figure 3. Four equilateral triangles and their centroids

complex numbers g_1, g_2, g_3 and g_4 respectively. The latter are related to the former according to

$$g_1 = \frac{z_1 - z_2}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_2 = \frac{z_2 - z_3}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_3 = \frac{z_3 - z_4}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_4 = \frac{z_4 - z_1}{\sqrt{3}} e^{i\frac{\pi}{6}}.$$

The proof will be in two parts.

($\Rightarrow$) If $ABCD$ is orthodiagonal, then $z_3 - z_1 = i\mathscr{R}(z_4 - z_2)$ for some real number $\mathscr{R} \neq 0$. Using the expressions for the centroids, we get

$$\begin{aligned}\left|\frac{g_3 - g_1}{g_4 - g_2}\right| &= \left|\frac{z_3 - z_4 - (z_1 - z_2)}{z_4 - z_1 - (z_2 - z_3)}\right| = \left|\frac{(z_3 - z_1) - (z_4 - z_2)}{(z_3 - z_1) + (z_4 - z_2)}\right| \\ &= \left|\frac{i\mathscr{R}(z_4 - z_2) - (z_4 - z_2)}{i\mathscr{R}(z_4 - z_2) + (z_4 - z_2)}\right| = \left|\frac{i\mathscr{R} - 1}{i\mathscr{R} + 1}\right| = \frac{\sqrt{1 + \mathscr{R}^2}}{\sqrt{1 + \mathscr{R}^2}} = 1\end{aligned}$$

where the exponential functions and the $\sqrt{3}$ were canceled out in the first equality. This proves that the line segments connecting opposite centroids are congruent, so $G_1G_2G_3G_4$ is an equidiagonal quadrilateral.

($\Leftarrow$) If $G_1G_2G_3G_4$ is equidiagonal, then according to the rewrite in the first part,

$$|(z_3 - z_1) - (z_4 - z_2)| = |(z_3 - z_1) + (z_4 - z_2)|.$$

We shall prove that this implies that $z_3 - z_1$ and $z_4 - z_2$ are perpendicular. Let us define the two new complex numbers w_1 and w_2 according to $w_1 = z_3 - z_1$ and $w_2 = z_4 - z_2$. Thus we are to prove that if $|w_1 - w_2| = |w_1 + w_2|$, then w_1 and w_2

are perpendicular. This is quite obvious from a geometrical perspective considering the vector nature of complex numbers, but we give an algebraic proof anyway. To this end we use the polar form. Thus we have $w_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $w_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. We square the two equal absolute values and rewrite $|w_1 - w_2|^2 = |w_1 + w_2|^2$ to get

$$\begin{aligned}&(r_1\cos\varphi_1 - r_2\cos\varphi_2)^2 + (r_1\sin\varphi_1 - r_2\sin\varphi_2)^2\\ &= (r_1\cos\varphi_1 + r_2\cos\varphi_2)^2 + (r_1\sin\varphi_1 + r_2\sin\varphi_2)^2.\end{aligned}$$

Expanding these expressions and canceling equal terms, this is equivalent to

$$4r_1r_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) = 0 \quad \Leftrightarrow \quad \cos(\varphi_1 - \varphi_2) = 0.$$

The last equation has the valid solutions $\varphi_1 - \varphi_2 = \pm\frac{\pi}{2}$, which proves that the angle between w_1 and w_2 is a right angle. Hence $ABCD$ is orthodiagonal. □

Other generalizations of van Aubel's theorem concerning rectangles, rhombi and parallelograms can be found in [5] and [17].

4. Quadrilaterals that are both equidiagonal and orthodiagonal

Consider Table 1, where three well known properties of the diagonals in seven of the most basic quadrilaterals are shown. The answer "no" refers to the general case for each quadrilateral. One thing is obvious, there is something missing here. No quadrilateral with just the two properties of perpendicular and congruent diagonals is included. This is because no name seems to have been given to this class of quadrilaterals.[2]

Quadrilateral	Bisecting diagonals	Perpendicular diagonals	Congruent diagonals
Trapezoid	No	No	No
Isosceles trapezoid	No	No	Yes
Kite	No	Yes	No
Parallelogram	Yes	No	No
Rhombus	Yes	Yes	No
Rectangle	Yes	No	Yes
Square	Yes	Yes	Yes

Table 1. Diagonal properties in basic quadrilaterals

Before we proceed, we quote in Table 2 in a somewhat expanded form a theorem we proved in [11, p.19]. The four properties on each line in this table are equivalent. The Varignon parallelogram properties follows directly from the fact

[2]In [14, p.50] Gerry Leversha claims that such a quadrilateral is sometimes called a pseudo-square. We can however not find any other reference for that use of the name (neither on the web nor in any geometry books or papers we know of). Instead a Google search indicates that a pseudo-square is a squares with four cut off vertices.

that a parallelogram is a rhombus if and only if its diagonals are perpendicular, and it is a rectangle if and only if its diagonals are congruent [4, p.53].

Original quadrilateral	Diagonal property	Bimedian property	Varignon parallelogram
Equidiagonal	$p = q$	$m \perp n$	Rhombus
Orthodiagonal	$p \perp q$	$m = n$	Rectangle

Table 2. Special cases of the Varignon parallelogram

The bimedians of a convex quadrilateral are the diagonals of its Varignon parallelogram, so the original quadrilateral has congruent and perpendicular diagonals if and only if the Varignon parallelogram has perpendicular and congruent diagonals (see Table 2). For such quadrilaterals, the Varignon parallelogram is a square, and this is a characterization of those quadrilaterals with congruent and perpendicular diagonals since a parallelogram is a square if and only if it is both a rhombus and a rectangle. Thus we have the following two necessary and sufficient conditions.

Theorem 11. *A convex quadrilateral has congruent and perpendicular diagonals if and only if*
(i) *the bimedians are perpendicular and congruent, or*
(ii) *the midpoints of its sides are the vertices of a square.*

So what shall we call these quadrilaterals? They are both equidiagonal and orthodiagonal, but trying to combine the two words yields no good name. The individual words describe the defining properties of these quadrilaterals. With that and Theorem 11 in mind, we propose that a quadrilateral with congruent and perpendicular diagonals is called a *midsquare quadrilateral* (see Figure 4).

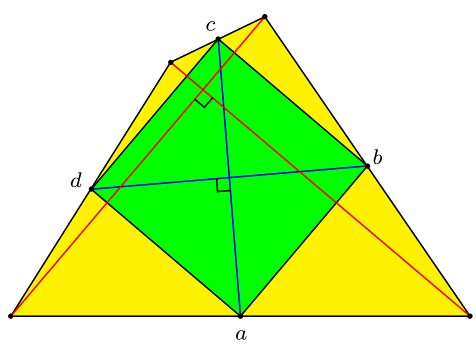

Figure 4. A midsquare quadrilateral and its Varignon square

Three special cases of midsquare quadrilaterals are orthodiagonal isosceles trapezoids, equidiagonal kites and squares.

Proposition 12. *A midsquare quadrilateral is a square if and only if its diagonals bisect each other.*

Proof. If the diagonals of a midsquare quadrilateral bisect each other, it is obvious that it is a square since the diagonals divide it into four congruent right triangles with equal legs.

Conversely it is a well known property that in a square, the diagonals bisect each other. □

After having given a name for this neglected type of quadrilateral, we now consider its area. The first formula in the following proposition has been known at least since 1962 according to [3, p.132].

Proposition 13. *A convex quadrilateral with diagonals p, q and bimedians m, n is a midsquare quadrilateral if and only if its area is given by*

$$K = \tfrac{1}{4}(p^2 + q^2) \qquad \text{or} \qquad K = \tfrac{1}{2}(m^2 + n^2).$$

Proof. Using the identity $(p-q)^2 = p^2+q^2-2pq$, the area of a convex quadrilateral satisfies (see [8])

$$K = \tfrac{1}{2}pq\sin\theta = \tfrac{1}{4}\big(p^2 + q^2 - (p-q)^2\big)\sin\theta \le \tfrac{1}{4}(p^2 + q^2)$$

where equality holds if and only if $p = q$ and $p \perp q$.

The second formula follows at once from the first by using equality (1). □

Since the two diagonals and the two bimedians are individually congruent in a midsquare quadrilateral, its area can be calculated with the four simple formulas

$$K = \tfrac{1}{2}p^2 = \tfrac{1}{2}q^2 = m^2 = n^2. \tag{6}$$

The next proposition gives more area formulas for midsquare quadrilaterals.

Proposition 14. *A convex quadrilateral with consecutive sides a, b, c, d is a midsquare quadrilateral if and only if its area is given by*

$$K = \tfrac{1}{4}\big(2(a^2 + c^2) - 4v^2\big) = \tfrac{1}{4}\big(2(b^2 + d^2) - 4v^2\big)$$

where v is the distance between the midpoints of the diagonals.

Proof. A convex quadrilateral has congruent diagonals if and only if its area is the product of the bimedians according to Corollary 4. Since the diagonals are perpendicular if and only if the bimedians are congruent (Table 2), the two area formulas follows at once using (3). □

Corollary 15. *A convex quadrilateral with consecutive sides a, b, c, d is a square if and only if its area is*

$$K = \tfrac{1}{2}(a^2 + c^2) = \tfrac{1}{2}(b^2 + d^2).$$

Proof. These formulas are a direct consequence of the last proposition since a convex quadrilateral is a square if and only if it is a midsquare quadrilateral with bisecting diagonals ($v = 0$) according to Proposition 12. □

Now we come to an interesting question. Can we calculate the area of a midsquare quadrilateral knowing only its four sides? The answer is yes. The origin for the next theorem is a solved problem we found in the pleasant book [9, pp.179–180]. There Heilbron states that this area is given by

$$K = \frac{1}{4}\left(a^2 + c^2 + \sqrt{2(a^2c^2 + b^2d^2)}\right).$$

He starts his derivation thoroughly, but at the end, when he obtains a quadratic equation, he merely claims that solving it will provide the formula he was supposed to derive. When we started to analyze the solutions to this equation in more detail, we began to smell a rat, and eventually realized that Heilbrons formula is in fact incorrect. We will motivate this after our proof of the correct formula.

Theorem 16. *A midsquare quadrilateral with consecutive sides a, b, c, d has the area*

$$K = \frac{1}{4}\left(a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right).$$

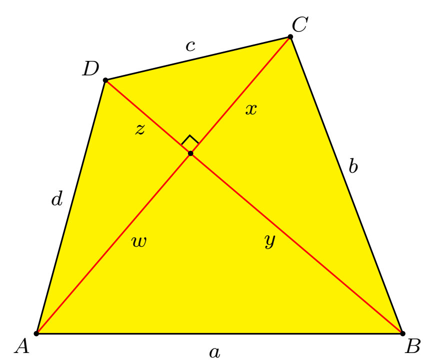

Figure 5. The diagonal parts in a midsquare quadrilateral

Proof. We use notations on the sides and the diagonal parts as in Figure 5, where $w + x = y + z = p$ since the diagonals are congruent. The area is given by $K = \frac{1}{2}p^2$, so we need to express a diagonal p in terms of the sides. Using the Pythagorean theorem, we get

$$a^2 - b^2 = w^2 - x^2 = (w + x)(w - x) = p(2w - p)$$

and similar $b^2 - c^2 = p(2y - p)$. Thus we have

$$a^2 - b^2 + p^2 = 2pw \qquad \text{and} \qquad b^2 - c^2 + p^2 = 2py.$$

Squaring and adding these yields

$$(a^2 - b^2 + p^2)^2 + (b^2 - c^2 + p^2)^2 = 4p^2(w^2 + y^2) = 4p^2a^2$$

where we used the Pythagorean theorem again in the last equality. Expanding and simplifying results in a quadratic equation in p^2:

$$2p^4 - 2(a^2 + c^2)p^2 + (a^2 - b^2)^2 + (b^2 - c^2)^2 = 0.$$

This has the solutions

$$p^2 = \frac{a^2 + c^2 \pm \sqrt{-a^4 - c^4 + 2a^2c^2 - 4b^4 + 4a^2b^2 + 4b^2c^2}}{2}.$$

The radicand can be simplified to

$$-a^4 - c^4 + 2a^2c^2 + 4b^2d^2 = 4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2$$

where we used $a^2 - b^2 + c^2 = d^2$ (see Theorem 1 in [11]). Thus

$$p^2 = \frac{1}{2}\left(a^2 + c^2 \pm \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right). \tag{7}$$

To decide the correct sign we study the special case when the quadrilateral is a square. Using $a = b = c = d$ in (7) yields

$$p^2 = \tfrac{1}{2}(2a^2 \pm 2a^2)$$

where we see that the solution with the negative sign is obviously false. The area formula now follows when inserting (7) into $K = \frac{1}{2}p^2$. □

Note that it is easy to get formulas for the lengths of the diagonals and the bimedians in a midsquare quadrilateral in terms of the sides. We simply have to combine (6) and Theorem 16.

Remark. Let us comment on the formula suggested by Heilbron. It gives the correct area for a square, so we need to do a more thorough investigation. If his formula were correct, it would mean that $2(a^2c^2 + b^2d^2) = (a^2 + c^2)^2$. But then his formula could be simplified to $K = \frac{1}{2}(a^2 + c^2)$. According to Corollary 15, this is a characterization for a square. Hence his formula must be incorrect, since the quadrilateral has perpendicular and congruent diagonals, but need not to be a square. Another way to dispute it is by considering a right kite with $a = d$ and $c = b$. It has the area $K = ac$, but Heilbrons formula gives $K = \frac{1}{4}(a + c)^2$. Equating these expressions yields $(a-c)^2 = 0$ which again imply the quadrilateral must be a square, which it is not.

5. When are certain quadrilaterals equidiagonal?

So far we have several ways of determining when a convex quadrilateral is equidiagonal. An isosceles trapezoid, a rectangle and a square are always equidiagonal, but how can we know when the diagonals are congruent in other basic quadrilaterals, such as a parallelogram or a cyclic quadrilateral?

Theorem 17. *The following characterizations hold:*
(i) *A parallelogram is equidiagonal if and only if it is a rectangle.*
(ii) *A rhombus is equidiagonal if and only if it is a square.*
(iii) *A trapezoid is equidiagonal if and only if it is an isosceles trapezoid.*
(iv) *A cyclic quadrilateral is equidiagonal if and only if it is an isosceles trapezoid.*

Proof. (i) In a parallelogram $ABCD$ with the two different side lengths a and b, the law of cosines yields that

$$p^2 = q^2 \quad \Leftrightarrow \quad a^2 + b^2 - 2ab\cos B = a^2 + b^2 - 2ab\cos A \quad \Leftrightarrow \quad A = B.$$

Two adjacent angles in a parallelogram are equal if and only if it is a rectangle.

(ii) The first part of the proof is the same as in (i) except that $a = b$, which does not effect the outcome. Two adjacent angles in a rhombus are equal if and only if it is a square.

(iii) The lengths of the diagonals in a trapezoid with consecutive sides a, b, c, d are given by (see [13, p.31])

$$p = \sqrt{\frac{ac(a-c) + ad^2 - cb^2}{a-c}} \qquad \text{and} \qquad q = \sqrt{\frac{ac(a-c) + ab^2 - cd^2}{a-c}}$$

where $a \parallel c$ and $a \neq c$. Thus we get

$$p^2 = q^2 \quad \Leftrightarrow \quad ad^2 - cb^2 = ab^2 - cd^2 \quad \Leftrightarrow \quad (a-c)(d^2 - b^2) = 0.$$

Since $a \neq c$, the only valid solution is $b = d$, so we have an isosceles trapezoid.

(iv) In a cyclic quadrilateral we can apply Ptolemy's second theorem, according to which (see [1, p.65])

$$\frac{p}{q} = \frac{ad + bc}{ab + cd}.$$

Hence

$$p = q \quad \Leftrightarrow \quad ab + cd = ad + bc \quad \Leftrightarrow \quad (a-c)(b-d) = 0$$

where the last equality has the two possible solutions $a = c$ and $b = d$. Any cyclic quadrilateral with a pair of opposite congruent sides is an isosceles trapezoid. One way of realizing this is by connecting the vertices to the circumcenter and thus conclude that this cyclic quadrilateral has a line of symmetry (see Figure 6).

Conversely it is well known that an isosceles trapezoid has congruent diagonals. $\square$

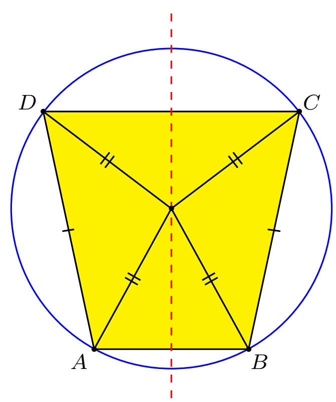

Figure 6. This is an isosceles trapezoid

In the previous section we concluded that an orthodiagonal quadrilateral is also equidiagonal if and only if the midpoints of the sides are the vertices of a square. There don't seem to be any similar easy ways of determining when a kite or a tangential quadrilateral are equidiagonal.

6. The diagonal length in equidiagonal quadrilaterals

We conclude this paper by discussing how the equal length of the diagonals in a general equidiagonal quadrilateral can be calculated given only the four sides, and also how this is related to finding the area of the quadrilateral. Thus this will lead up to a generalization of Theorem 16.

There is a formula relating the four sides and the two diagonals of a convex quadrilateral, sometimes known as Euler's four point relation. It is quite rare to find this relation in geometry books and even rarer to find a proof of it that does not involve determinants, so we start by deriving it here. For this purpose we need the following trigonometric formula.

Lemma 18. *For any two angles α and β we have the identity*

$$\cos^2\alpha + \cos^2\beta + \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta) = 1.$$

Proof. The addition formula for cosines can be rewritten in the form

$$\cos\alpha\cos\beta - \cos(\alpha+\beta) = \sin\alpha\sin\beta.$$

Squaring both sides, we have

$$(\cos\alpha\cos\beta - \cos(\alpha+\beta))^2 = (1-\cos^2\alpha)(1-\cos^2\beta).$$

Now the identity follows after expansion and simplification. □

The following relation has been derived independently by several mathematicians. It cannot be factored, but there are several ways to collect the terms. The version we present with only four terms is definitely one of the most compact, and except for some basic algebra we only use the law of cosines in the short proof.

Theorem 19 (Euler's four point relation)**.** *In all convex quadrilaterals with consecutive sides a, b, c, d and diagonals p, q, it holds that*

$$\begin{aligned}&p^2q^2(a^2+b^2+c^2+d^2-p^2-q^2) - (a^2-b^2+c^2-d^2)(a^2c^2-b^2d^2)\\&-p^2(a^2-d^2)(b^2-c^2) + q^2(a^2-b^2)(c^2-d^2) = 0.\end{aligned}$$

Proof. Let $\alpha = \angle BAC$ and $\beta = \angle DAC$ in quadrilateral $ABCD$. The law of cosines applied in triangles BAC, DAC and ABD yields respectively (see Figure 7)

$$\cos\alpha = \frac{a^2+p^2-b^2}{2ap}, \quad \cos\beta = \frac{d^2+p^2-c^2}{2dp}, \quad \cos(\alpha+\beta) = \frac{a^2+d^2-q^2}{2ad}.$$

Inserting these into the identity in Lemma 18 and multiplying both sides of the equation by the least common multiple $4a^2d^2p^2$, we get after simplification

$$\begin{aligned}&d^2(a^2+p^2-b^2)^2 + a^2(d^2+p^2-c^2)^2 + p^2(a^2+d^2-q^2)^2\\&-(a^2+p^2-b^2)(d^2+p^2-c^2)(a^2+d^2-q^2) = 4a^2d^2p^2.\end{aligned}$$

Now expanding these expressions and collecting similar terms results in Euler's four point relation.[3] □

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[3]The history of this six variable polynomial dates back to the 15th century, and it is closely related to the volume of a tetrahedron. The Italian painter Piero della Francesca was also interested in

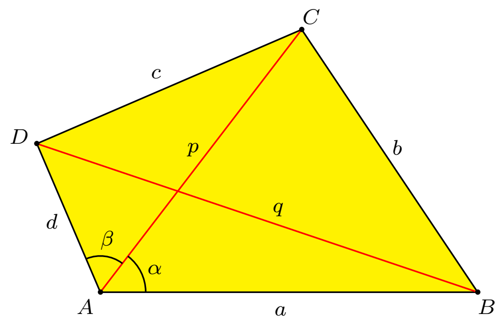

Figure 7. Using the law of cosines in three subtriangles

Returning to the initial goal of calculating the length of the equal diagonals, we set $p = q$ in Theorem 19. This results in the following cubic equation in p^2:

$$2p^6 - (a^2 + b^2 + c^2 + d^2)p^4 + ((a^2 + c^2)(b^2 + d^2) - 2(a^2c^2 + b^2d^2))p^2$$
$$+ (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) = 0.$$

Cubic equations have been solved for five centuries and there are several different solution methods known. However they all have one thing in common as anyone who has used one of them has noticed: the expressions for the roots they produce are very complicated and in most of the times completely useless. In fact, solving a cubic equation with coefficients like the one above with a computer algebra system can produce several pages of output formulas. Should it be necessary in a practical situation, a numerical solution (on a calculator or computer) is almost always preferable.

After having solved the cubic equation numerically, the area of the equidiagonal quadrilateral (with $p = q$) is given by the formula of Staudt (see [16, p.35])

$$K = \tfrac{1}{4}\sqrt{4p^4 - (a^2 - b^2 + c^2 - d^2)^2}.$$

So it may come as a little disappointment that we did not get a nice formula for the diagonals and the area like in the case when the diagonals are also perpendicular. There are however lots of cubic equations arising when solving problems in the geometry of triangles and quadrilaterals, so this is quite a common occurrence. On

geometry and derived a formula for the volume V of a tetrahedron expressed in terms of its six edges. The formula states that the left hand side of the equation in Theorem 19 is equal to $144V^2$. The formula was rediscovered in the 16th century by the Italian mathematician Niccolò Fontana Tartaglia, who was also involved in the first solution of the cubic equation. In the 18th century the famous Swiss mathematician Leonhard Euler solved the same problem. The invention of determinants made it possible for the 19th century British mathematician Arthur Cayley to express the tetrahedron volume in a very compact form using the so called Cayley-Menger determinant. We do not know which one of these gentlemen was the first to conclude that setting the tetrahedron volume equal to zero would result in an interesting identity for quadrilaterals. A trigonometric derivation that did not involve the tetrahedron has surely been known at least since the 19th century when several mathematicians made thorough trigonometric studies of the geometry of quadrilaterals.

the other hand, we can now get a second derivation of Theorem 16. If the diagonals are both congruent and perpendicular, the constant term of the cubic equation vanishes (since $a^2 + c^2 = b^2 + d^2$), so after simplifying the equation and dividing it by the positive number p^2 we get

$$2p^4 - 2(a^2 + c^2)p^2 + (a^2 + c^2)^2 - 2(a^2c^2 + b^2d^2) = 0.$$

This directly yields the solution

$$p^2 = \frac{1}{2}\left(a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right)$$

which we recognize from (7).

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The Miquel Points, Pseudocircumcenter, and Euler-Poncelet Point of a Complete Quadrilateral

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Abstract. We prove over 40 similarities in the configuration of a complete quadrilateral and the Miquel points. Then we introduce a generalized circumcenter and prove a theorem on the Euler-Poncelet point.

1. Introduction

In this paper we will study, using purely synthetical methods, the configuration concerning complete quadrilateral $ABCD$ denoting $P = AC \cap BD$, $Q = AB \cap CD$, and $R = AD \cap BC$. We begin with revealing a large number of similarities within the configuration concerning the associated Miquel points. Then we proceed to introduce a quadrilateral center which generalizes the circumcenter of a cyclic quadrilateral, and finally we will prove a result regarding the Euler-Poncelet Point.

The Euler-Poncelet point X is the common point of the nine-point circles of triangles ABC, BCD, CDA, DAB. Also, it is known to lie on the pedal circles of A with respect to triangle BCD, and the cyclic variants. More on the Euler-Poncelet point can be found in [5].

Our main result is the following:

Theorem 1. *The Euler-Poncelet point X lies on the circumcircle of the triangle PQR.*

It is of particular interest that our result is a strong generalization of the celebrated result by Emelyanov and Emelyanova (see [2]).

Theorem 2 (Emelyanov, Emelyanova)**.** *Let ABC be a triangle with incenter I. Let $AI \cap BC = P$, $BI \cap CA = Q$, $CI \cap AB = R$, then the Feuerbach point F_e lies on the circumcircle of triangle PQR.*

Proof. If we take the complete quadrilateral to be $ABCI$, then the Euler-Poncelet point lies on both the incircle (pedal of I with respect to triangle ABC) and the nine-point circle of ABC, thus it coincides with their point of contact, i.e., the Feuerbach point. Theorem 1 now implies the result. $\square$

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2. Similarities on the Miquel points

In this section we define Miquel points (which were previously studied in [3]) as spiral similarity centers and uncover a surprising amount of similarities within the configuration.

The following two results on spiral similarities are well-known. Their proofs can be found for example in [4].

Proposition 3. *Let A, B, A', B' be points in plane such that no three of them are collinear. Assume that the lines AB and $A'B'$ intersect at P. Then there exists a unique spiral similarity that sends A to A' and B to B'. The center of this spiral similarity is the second intersection of the circles $(AA'P)$ and $(BB'P)$.*

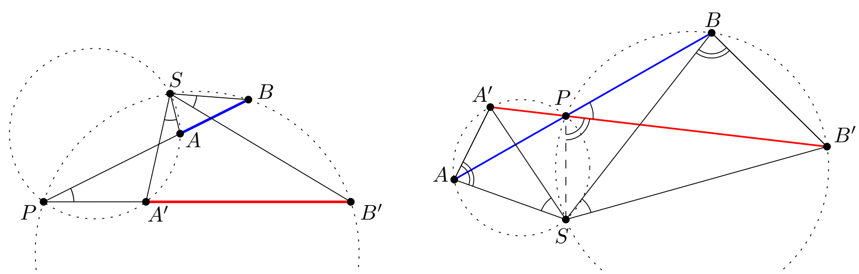

Figure 1.

Proposition 4. *Let $\mathcal{S}(S, k, \varphi)$ be the spiral similarity that maps A to A' and B to B'.*
(a) $\triangle SAB \sim \triangle SA'B'$.
(b) $\triangle SAA' \sim \triangle SBB'$.
(c) *There is a spiral similarity $\mathcal{S}'(S, k', \varphi')$ that maps A to B and A' to B' for suitable choice of k' and φ'.*

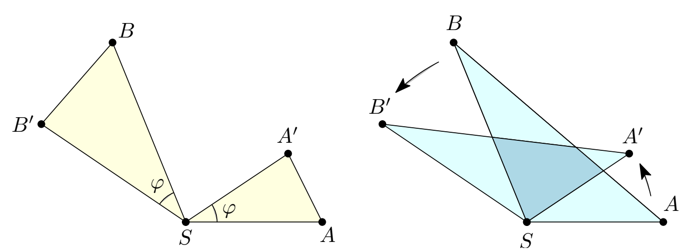

Figure 2.

From these propositions we immediately deduce that there is a unique point which is the center of two spiral similarities.

With this notion we can define three different Miquel points associated to a complete quadrilateral.

Definition. We define the Miquel points M_p, M_q, and M_r as the following spiral similarity centers (segments are considered directed):

Point	Center taking	and (at the same time)
M_p	$AB \mapsto DC$	$AD \mapsto BC$
M_q	$AC \mapsto DB$	$AD \mapsto CB$
M_r	$AC \mapsto BD$	$AB \mapsto CD$

As expected, Proposition 3 gives us many circles passing through the Miquel points. Also let us use the notation M_{XY} for the midpoint of the segment XY.

Proposition 5. *The following sets of points are concyclic:*
(a) $(M_pRAB), (M_pRDC), (M_pQAD), (M_pQBC)$,
(b) $(M_qRAC), (M_qRDB), (M_qPAD), (M_qPCB)$,
(c) $(M_rQAC), (M_rQBD), (M_rPAB), (M_rPCD)$,
(d) $(M_pM_qRM_{AD}M_{BC}), (M_qM_rPM_{AC}M_{DB}), (M_rM_pQM_{AB}M_{CD})$.

Proof. Parts (a), (b), and (c) follow from the definition of the Miquel points and Proposition 3 as each of them is a center of two different spiral similarities.

For part (d) observe that from $\triangle M_pAD \sim \triangle M_pBC$ it follows that

$$\triangle M_pM_{AD}A \sim \triangle M_pM_{BC}B$$

(directly) and hence M_p is the spiral similarity center which takes M_{AD} to A and M_{BC} to B. Proposition 3 now implies that M_p lies on the circumcircle of $\triangle RM_{AD}M_{BC}$. Point M_q lies on the same circle for analogous reasons. The other circles are established in the same way. $\square$

Proposition 6. *The following sets of triangles are directly similar:*
(a) $\triangle M_pAB \sim \triangle M_pDC \sim \triangle M_{BA}M_{BD}M_{AC} \sim \triangle M_{CD}M_{AC}M_{DB}$,
$\triangle M_pAD \sim \triangle M_pBC \sim \triangle M_{DA}M_{DB}M_{AC} \sim \triangle M_{CB}M_{AC}M_{BD}$;
(b) $\triangle M_qAD \sim \triangle M_qCB \sim \triangle M_{DA}M_{DC}M_{AB} \sim \triangle M_{BC}M_{AB}M_{CD}$,
$\triangle M_qAC \sim \triangle M_qDB \sim \triangle M_{CA}M_{CD}M_{AB} \sim \triangle M_{BD}M_{AB}M_{DC}$;
(c) $\triangle M_rAB \sim \triangle M_rCD \sim \triangle M_{BA}M_{BC}M_{AD} \sim \triangle M_{DC}M_{AD}M_{CB}$,
$\triangle M_rAC \sim \triangle M_rBD \sim \triangle M_{CA}M_{CB}M_{AD} \sim \triangle M_{DB}M_{AD}M_{BC}$.

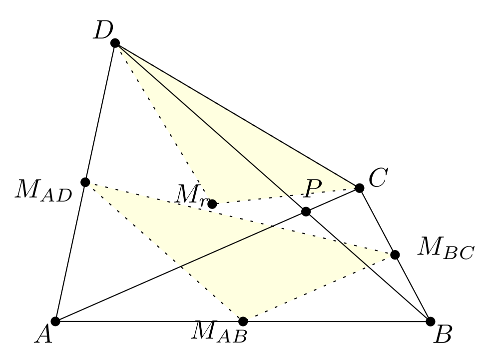

Figure 3.

Proof. Again, from symmetry of the complete quadrilateral it suffices to prove only one part and this time we choose (thinking of a nice diagram) to prove part (c). In the first chain of similarities the first and the third one are immediate from the definition of the Miquel point and the fact that the points M_{BA}, M_{AD}, M_{DC}, M_{CB} form a Varignon parallelogram, respectively.

Note that the lines $M_{AB}M_{BC}$ and $M_{AB}M_{AD}$ are midlines in triangles ABC and ABD, respectively. Angle-chasing (with the use of Proposition 5(c)) now gives

$$\angle(M_{BC}M_{AB}, M_{AD}M_{AB}) = \angle(AC, BD) = \angle(CP, PD) = \angle(CM_r, M_rD)$$

and after the ratio calculation (using $\triangle M_rAC \sim \triangle M_rBD$)

$$\frac{M_{BC}M_{AB}}{M_{AD}M_{AB}} = \frac{AC}{BD} = \frac{CM_r}{DM_r},$$

we have the desired similarity by SAS. The second part is proved likewise. $\square$

Lemma 7. *If $A'B'C'D'$ is the image of $ABCD$ in the inversion with respect to M_p (M_q, M_r, respectively), then quadrilateral $ABCD$ is indirectly similar to $C'D'A'B'$ ($B'A'D'C'$, $D'C'B'A'$, respectively).*

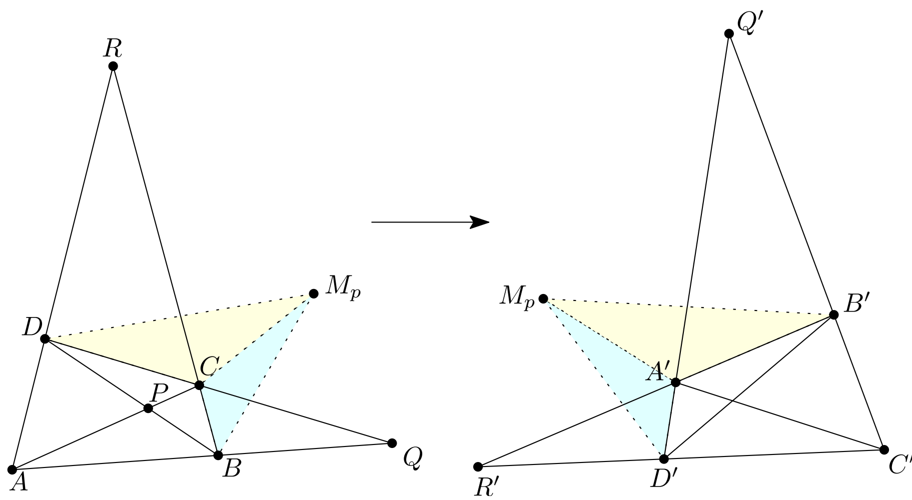

Figure 4.

Proof. From the definition of the Miquel point and inversion, respectively, we have $\triangle M_pBC \sim \triangle M_pAD \sim \triangle M_pD'A'$, where the first similarity is direct and the second one indirect. Similarly, we get $\triangle M_pDC \sim \triangle M_pAB \sim \triangle M_pB'A'$ (again first directly and then indirectly). But this implies indirect similarity of the quadrilaterals M_pDCB and $M_pB'A'D'$, from which we obtain indirect similarity of $\triangle DCB$ and $\triangle B'A'D'$. In the same vein, we find that $\triangle ABD \sim \triangle C'D'B'$ (indirectly) and the desired similarity of quadrilaterals now follows. The proof for M_q and M_r goes along the same lines. $\square$

Proposition 8.
(a) M_p *is the center of spiral similarity which sends* M_rR *to* QM_q.
(b) M_q *is the center of spiral similarity which sends* M_pP *to* RM_r.
(c) M_r *is the center of spiral similarity which sends* M_qQ *to* PM_p.

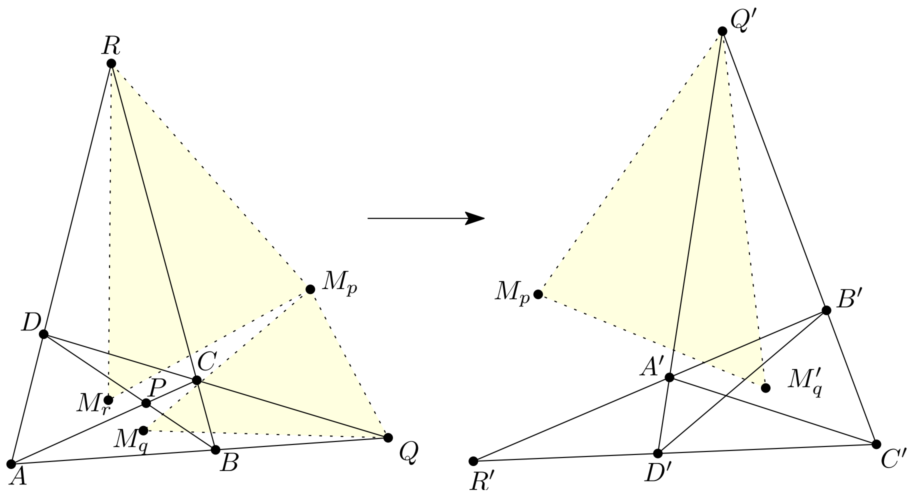

Figure 5.

Proof. From symmetry it is enough to prove part (a). Consider inversion with respect to point M_p and use standard notation for images.

Further, according to Lemma 7 $ABCD$ is indirectly similar to $C'D'A'B'$. In this similarity Q (intersection of AB and CD) corresponds to R' (intersection of $C'D'$ and $A'B'$). Analogously R corresponds to Q'. Also, M_r (second intersection of (QBD) and (QAC)) corresponds to M'_q (second intersection of $(R'D'B')$ and $(R'C'A')$) and M_p (second intersection of (RAB) and (RCD)) corresponds to M_p (second intersection of $(Q'C'D')$ and $(Q'A'B')$).

From these observations we can deduce that $\triangle M_pM_rR \sim \triangle M_pM'_qQ'$ (indirectly). From inversion we also have $\triangle M_pM'_qQ' \sim \triangle M_pM_qQ$ (indirectly). Altogether we have the direct similarity of $\triangle M_pM_rR$ and $\triangle M_pM_qQ$, which implies the result. □

3. The pseudocircumcenter

It is well-known (see e.g. [6]) that in the case of a cyclic quadrilateral $ABCD$ inscribed in a circle centered at O the triangle PQR has M_p, M_q, and M_r as its feet of altitudes and point O as its orthocenter. Also, the Euler-Poncelet point X is symmetric to O with respect to the centroid G of $ABCD$.

We introduce a point O associated to a (not necessarily cyclic) complete quadrilateral which inherits most of these properties.

Theorem 9. *The circles* (PM_qM_r), (QM_rM_p), *and* (RM_pM_q) *meet in one point.*

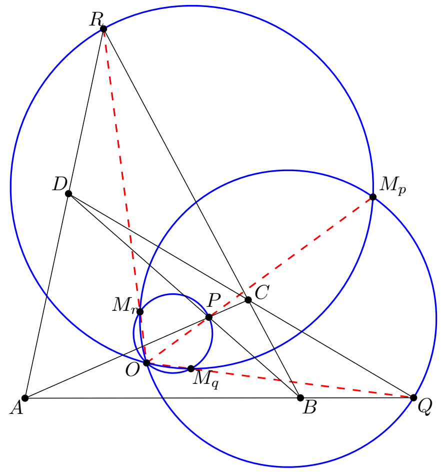

Figure 6.

Proof. Define O as the second intersection of (QM_rM_p) and (RM_pM_q). Then

$$\begin{aligned}\angle(M_qO, OM_r) &= \angle(M_qO, OM_p) + \angle(M_pO, OM_r)\\ &= \angle(M_qR, RM_p) + \angle(M_pQ, QM_r).\end{aligned}$$

Now using the circles from Proposition 5 we obtain

$$\angle(M_qR, RM_p) = \angle(M_qR, RA) + \angle(AR, RM_p) = \angle(M_qC, CA) + \angle(AB, BM_p)$$

and likewise

$$\begin{aligned}\angle(M_pQ, QM_r) = \angle(M_pQ, QC) + \angle(CQ, QM_r) = \angle(M_pB, BC) + \angle(CA, AM_r),\\ \angle(M_rP, PM_q) = \angle(M_rP, PB) + \angle(BP, PM_q) = \angle(M_rA, AB) + \angle(BC, CM_q)\end{aligned}$$

After careful inspection, the three right-hand sides add up to 0, hence

$$\angle(M_qO, OM_r) - \angle(M_qP, PM_r) = 0$$

and the conclusion follows. □

We call this point O the *pseudocircumcenter* of $ABCD$.

Theorem 10. *The lines PM_p, RM_r, QM_q meet in O.*

Proof. It suffices to prove O, R, and M_r are collinear. With the use of Proposition 8, M_q is the center of spiral similarity which takes M_r to R and P to M_p, therefore $\triangle M_qM_rP \sim \triangle M_qRM_p$ (directly) and so $\angle(M_rP, PM_q) = \angle(RM_p, M_pM_q)$. We can now conclude after using the circles (M_qM_rPO) and (M_pM_qRO) as follows:

$$\angle(M_rO, OM_q) = \angle(M_rP, PM_q) = \angle(RM_p, M_pM_q) = \angle(RO, OM_q)$$

which immediately implies collinearity of O, R, M_r. □

Theorem 11.
(a) *The circles* $(M_{AB}M_{AC}M_{AD})$, $(M_{BA}M_{BC}M_{BD})$, $(M_{CA}M_{CB}M_{CD})$, *and* $(M_{DA}M_{DB}M_{DC})$ *all pass through* O.
(b) O *and* X *are symmetric with respect to* G.

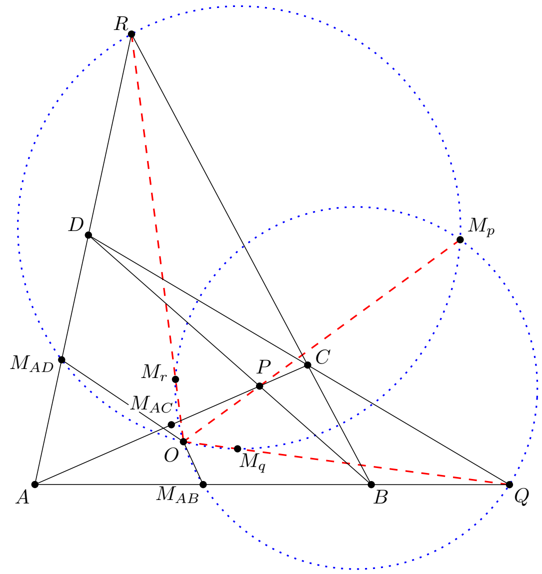

Figure 7.

Proof. From symmetry, it suffices to prove part (a) for one of the mentioned circles, for example $(M_{AB}M_{AC}M_{AD})$. Using midlines in triangles ACD and ABC, we obtain

$$\angle(M_{AD}M_{AC}, M_{AC}M_{AB}) = \angle(CD, CB).$$

Further, point O lies on (RM_pM_{AD}) and (QM_pM_{AB}) (consult Proposition 5(d) and Theorem 9), hence (using also the basic circles from Proposition 5(c))

$$\begin{aligned}\angle(M_{AD}O, OM_{AB}) &= \angle(M_{AD}O, OM_p) + \angle(M_pO, OM_{AB})\\ &= \angle(DR, RM_p) + \angle(M_pQ, QB)\\ &= \angle(DC, CM_p) + \angle(M_pC, CB) = \angle(DC, CB).\end{aligned}$$

Hence O indeed lies on $(M_{AB}M_{AC}M_{AD})$.

For part (b) apply the symmetry with respect to G. It is well-known that M_{AB}, M_{AC}, M_{AD}, M_{BC}, M_{BD}, M_{CD} are sent to M_{CD}, M_{BD}, M_{BC}, M_{AD}, M_{AC}, M_{AB}, respectively. Therefore the circles $(M_{AB}M_{AC}M_{AD})$, $(M_{BA}M_{BC}M_{BD})$, $(M_{CA}M_{CB}M_{CD})$, $(M_{DA}M_{DB}M_{DC})$ are sent to the nine-point circles of the triangles BCD, ACD, ABD, ABC, respectively and these circles have X as their common point. Hence X is the image of O. $\square$

4. Proof of Theorem 1

Lemma 12. *Reflect a triangle ABC about a point P in its plane to triangle $A'B'C'$. Then the circles $(AB'C')$, $(BC'A')$, $(CA'B')$ are concurrent on (ABC).*

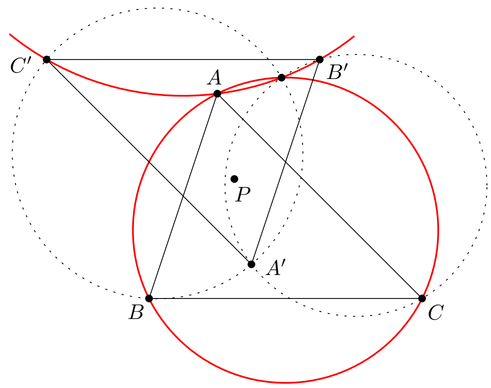

Figure 8.

Proof. Intersect (ABC) and $(AB'C')$ at X. Then angle-chase using the circles and parallel lines

$$\begin{aligned} \angle(BX, XC') &= \angle(BX, XA) + \angle(AX, XC') \\ &= \angle(BC, CA) + \angle(AB', B'C') \\ &= \angle(BC, A'C') + \angle(A'B, BC) \\ &= \angle(BA', A'C'), \end{aligned}$$

which proves that X lies on $(BC'A')$. Analogously, we prove it lies on $(CA'B')$. □

Theorem 1. *Point X lies on (PQR).*

Proof. Let us denote by P', Q', R' the reflections of P, Q, R, respectively, about the centroid G. From Theorem 11, it suffices to prove that O lies on the circle $(P'Q'R')$. We will prove that O lies on the circles $(P'QR)$, $(Q'RP)$, $(R'PQ)$ and then the result will follow from Lemma 12 applied on triangle $P'Q'R'$. In fact, we only need (by symmetry) to prove the circle $(P'QRO)$.

The line $M_{AB}M_{CD}$ is the Newton-Gauss line of $ABCD$ so it passes through the midpoint of PR. As it also passes through G, it is the midline in $\triangle RPP'$. Similarly, we prove that $M_{AD}M_{BC}$ is the midline in $\triangle QPP'$. It follows that

$$\angle(RP', P'Q) = \angle(M_{AB}M_{CD}, M_{AD}M_{BC}).$$

At the same time using Theorems 9 and 10 we obtain

$$\angle(RO, OQ) = \angle(M_rO, OQ) = \angle(M_rM_{CD}, M_{CD}Q) = \angle(M_rM_{CD}, CD).$$

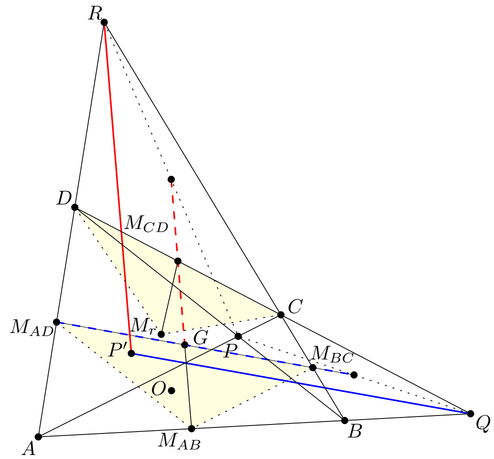

Figure 9.

But in Proposition 6 we proved the direct similarity

$$\triangle M_r CD \sim \triangle M_{BA} M_{BC} M_{AD}.$$

As the angles $\angle(M_r M_{CD}, CD)$ and $\angle(M_{AB} M_{CD}, M_{AD} M_{BC})$ correspond in this similarity (angles by medians), they are equal. It follows that

$$\angle(RP', P'Q) = \angle(RO, OQ),$$

which concludes the entire proof.

□

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A Note on Reflections

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Abstract. We prove some simple results associated with the triangle formed by the reflections of a point in the midpoints of the sides of a given triangle.

Let ABC be a given triangle with midpoints M_a, M_b, M_c of the sides BC, CA, AB respectively. Consider a point P and its reflections X, Y, Z in M_a, M_b, M_c respectively.

Proposition 1. *Triangle XYZ is oppositely congruent to ABC at the complement* (inferior) *of P.*

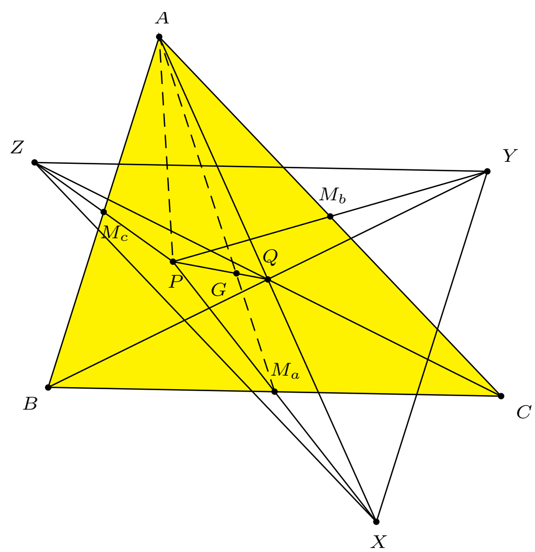

Figure 1

Proof. Let G be the centroid of triangle ABC. Consider triangle APX. It has the segment AM_a as a median, and so has centroid G. If Q is the midpoint of AX, then PQ is another median of triangle APX. Therefore, G divides PQ in the ratio $PG : GQ = 2 : 1$, and Q is the complement of P.

Similarly, the same point Q is the midpoint of BY and CZ. It follows that XYZ is oppositely congruent to ABC at Q. $\square$

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Let X^*, Y^*, Z^* be the reflections of X, Y, Z in the sidelines BC, CA, AB respectively.

Proposition 2. *The circle through X^*, Y^*, Z^* has center O and contains the P and its reflection in O.*

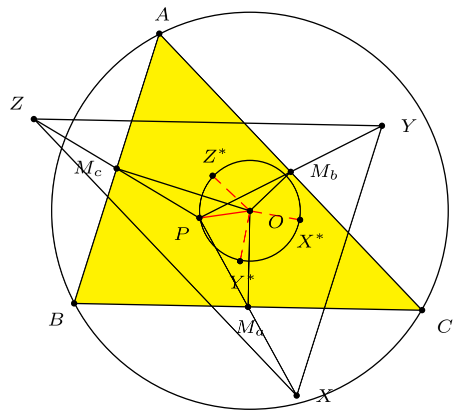

Figure 2

Proof. X^*, Y^*, Z^* are the reflections of P in the perpendicular bisectors of BC, CA, AB respectively. Each of these points is equidistant with P from the circumcenter O of triangle ABC. Therefore the circle through P, center O, also contains the reflection of P in O. □

Theorem 3. *If the Euler lines of PBC, PCA, PAB are concurrent at S, then the Euler lines of AZY, BXZ, CYX are concurrent at the superior* (anticomplement) *of S.*

Proof. Triangle AZY is a translation of triangle PBC.

$$Y - C = (C + A - P) - C = A - P = (A + B - P) - B = Z - B.$$

Clearly the two Euler lines of the two triangles are parallel. Since the centroid of AZY is the superior of the centroid of PBC, every point on the Euler line of AYZ is the superior of a point on the Euler line of PBC.

The same is true for the pairs BXZ, PCA and CYX, PAB. It follows that if the Euler lines of PBC, PCA, PAB are concurrent at S, then those of AZY, BXZ, CYX are concurrent at the superior (anticomplement) of S. □

Remark. It is well known that for $P = I$, the incenter, the Euler lines of IBC, ICA, IAB are concurrent at the Schiffler point $X(21)$ on the Euler line of ABC (see [4]). It follows that the Euler lines of AYZ, BZX, CXY are concurrent at the superior (anticomplement) of $X(21)$ (see Figure 3). This is the triangle center $X(2475)$ of [5], also on the Euler line of ABC.

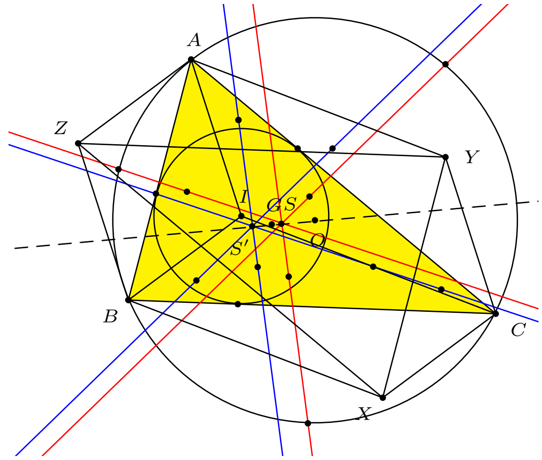

Figure 3

On the other hand, Yiu [8] has noted that, for $P = I$, the Euler lines of XBC, YCA, ZAB are concurrent at the cevian quotient Q/I, where Q is the Spieker center, the inferior (complement) of I.

Lemma 4. *For $P = I$, the incenter, the line AX intersects the Euler line of triangle XBC on the side BC.*

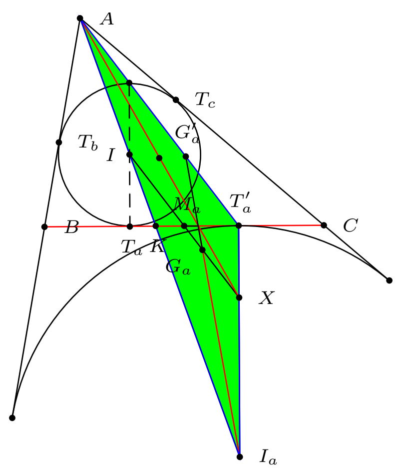

Figure 4

Proof. Let I_a be the center of the excircle tangent to BC at T'_a. Denote by r and r_a the inradius and radius of the A-excircle. We shall make use of the formulas $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where Δ and s are the area and semiperimeter of triangle ABC.

Since IX and BC have a common midpoint M_a, $IBXC$ is a parallelogram. Therefore, BX and CX are perpendicular to CI_a and BI_a respectively. From this we note that X is the orthocenter of triangle I_aBC. Consequently, I_a is the orthocenter of triangle XBC.

Let G_a be the point dividing XM_a in the ratio $XG_a : G_aM_a = 2 : 1$. This is the centroid of triangle XBC, and I_aG_a is the Euler line.

Extend I_aG_a to intersect AT'_a at G'_a. Since the line AT'_a contains the antipode of T_a on the incircle, it is parallel to IX. Therefore, $AG'_a : G'_aT'_a = TG_a : G_aX = 2 : 1$.

Let AI_a intersect BC at K. Consider triangle AT'_aI_a with X on T'_aI_a, K on I_aA, and G'_a on AT'_a. We have

$$\frac{AG'_a}{G'_aT'_a} \cdot \frac{T'_aX}{XI_a} \cdot \frac{I_aK}{KA} = \frac{2}{1} \cdot \frac{r}{r_a - r} \cdot \frac{r_a}{\frac{2\Delta}{a}} = \frac{arr_a}{(r_a - r)\Delta} = \frac{a \cdot \frac{1}{s} \cdot \frac{1}{s-a}}{\frac{1}{s-a} - \frac{1}{s}} = 1.$$

By Ceva's theorem, AX, $I_aG'_a$ and T'_aK are concurrent. This means that AX and the Euler line I_aG_a of triangle XBC intersect on BC. $\square$

Theorem 5. *For $P = I$, the incenter, the Euler lines of triangles XBC, YCA, ZAB are concurrent at the cevian quotient Q/I, where Q is the Spieker center.*

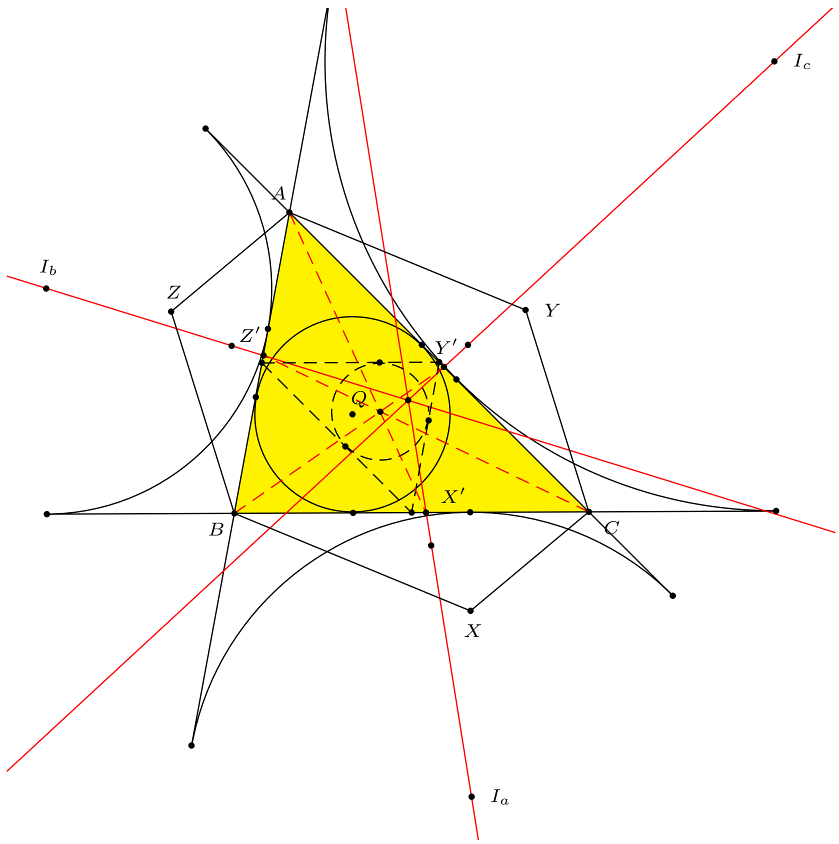

Figure 5

Proof. The line AX contains the Spieker center Q as its midpoint. Denote by $X'Y'Z'$ the cevian triangle of Q. By Lemma 4 above, the Euler line of triangle XBC is I_aX'. Similarly, the Euler lines of YCA and ZAB are the lines I_bY' and I_cZ'. Now, $I_aI_bI_c$ is the anticevian triangle of I, and XYZ is the cevian triangle of S. These lines are concurrent at the cevian quotient Q/I (see [7, §8.3]). □

Remark. For $Q =$ the Spieker center, the cevian quotient Q/I is the triangle center $X(191)$ of [5]. It is the reflection of I in the Schiffler point $X(21)$ (see Remark after Theorem 3 above).

Theorem 6 (Collings)**.** *The circles (AYZ), (BZX), (CXY) are concurrent at a point on the circumcircle of ABC, which is the superior of the center of the rectangular hyperbola through A, B, C, and P.*

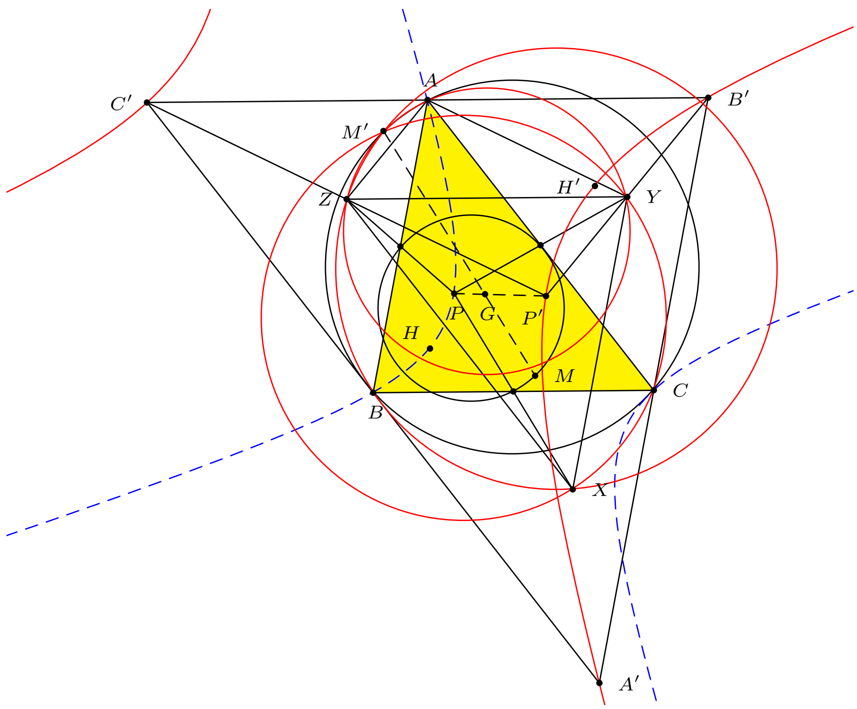

Figure 6

Proof. Collings [1] actually shows that if $A'B'C'$ is the superior (anticomplementary) triangle of ABC, and P' is the superior of P, then the circles (AYZ), (BZX), $CXY)$ and (ABC) intersect at the center M' of the rectangular hyperbola through A', B', C', P' (see Figure 3). Since A', B', C', P' are the superiors of A, B, C, P respectively, M' is the superior of the center M of the rectangular hyperbola through A, B, C, P. This is a point on the nine-point circle of ABC. It follows that M' is a point on the circumcircle. □

For example, if $P = I$, the incenter, the rectangular hyperbola through A, B, C, I (and H) has center the Feuerbach point F. The common point of the circles (AZY), (BXZ), (CYX) is the inferior of F. This is the point $X(100)$.

Peter Moses has informed us [2], among other things, that triangle XYZ is perspective to the mid-arc triangle at $X(100)$ (see Figure 7). The vertices of the mid-arc triangle are the intersections of the angle bisectors with the circumcircle. Since the inferiors (complements) of X, Y, Z are the midpoints X'', Y'', Z'' of IA, IB, IC respectively, it is enough to prove that $X''Y''Z''$ is perspective with the mid-arc triangle of the medial triangle $M_aM_bM_c$.

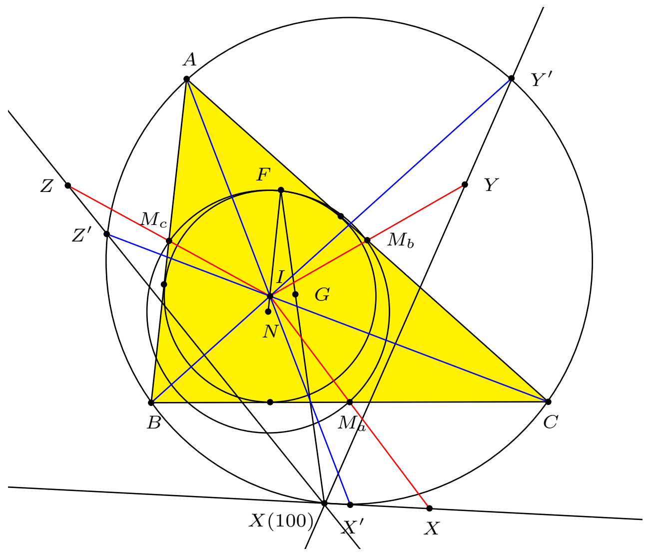

Figure 7

Proposition 7. *Let X'', Y'', Z'' be the midpoints of IA, IB, IC respectively, and M'_a, M'_b, M'_c the midpoints of the the arcs M_bM_c, M_cM_a, M_aM_b of the nine-point circle of triangle ABC not containing M_a, M_b, M_c. The triangles $X''Y''Z''$ and $M'_aM'_bM'_c$ are perspective at the Feuerbach point of triangle ABC.*

Proof. The nine-point circle of triangle ABC is tangent to the incircle at the Feuerbach point F. Let M'_a be the midpoint of the arc M_bM_c of the nine-point circle (not containing M_a).

$$\angle M'_aFM_c = \frac{1}{2}\angle M_bFM_c = \frac{1}{2}\angle M_bM_aM_c = \frac{1}{2}\angle BAC. \tag{1}$$

On the other hand, if X'' is the midpoint of IA, then the circle through X'', M_c and T_c is the nine-point circle of triangle IAB, and it passes through the Feuerbach point F as well (see Remark below), and

$$\angle X''FM_b = \angle X''T_bA = \angle X''AT_b = \frac{1}{2}\angle BAC. \tag{2}$$

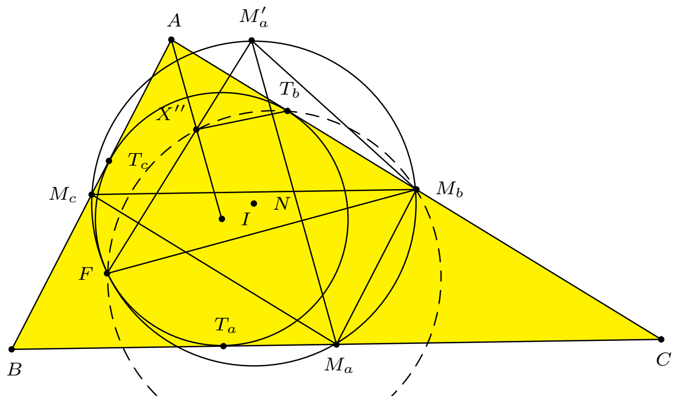

Figure 8

It follows from (1) and (2) that M'_a, X'' and F are collinear.

The same reasoning shows that M'_b, Y'', and F are collinear, so are M'_c, Z'' and F. Therefore, the triangles $X''Y''Z''$ and $M'_aM'_bM'_c$ are perspective at the Feuerbach point. □

Remark. The nine-point circles of IBC, ICA, IAB, and ABC are concurrent at the center of the rectangular hyperbola through A, B, C, I. This is the Feuerbach point F. See Theorem 6 above. A synthetic proof of this fact can be found in [6].

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Antirhombi

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Abstract. First we give the definition and some properties of the non-rhombus quadrilateral that we call antirhombus [1], that is circumscribed around a circle with center the centroid of the quadrilateral. Then we try to cut a triangle ABC with a line to form an antirhombus, we prove that there are three such lines forming with the sides of ABC an hexagon circumscribed around the incircle of ABC and then investigate their interesting configuration.

1. Circumscribed quadrilaterals with the same incenter and centroid

Let $ABCD$ be a quadrilateral circumscribed around a circle with center O and radius r, and x, y, z, w be the distances of the vertices A, B, C, D from the points of tangency T_a, T_b, T_c, T_d (Figure 1).

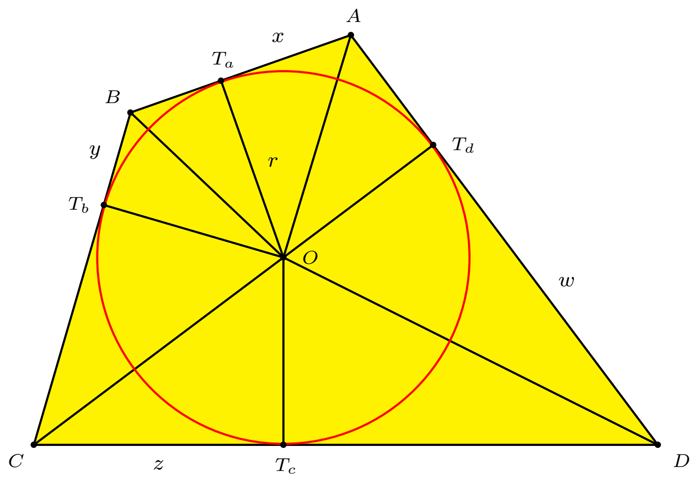

Figure 1

We have

$$x = r\cot\frac{A}{2}, \quad z = r\cot\frac{C}{2}, \qquad OA = \frac{r}{\sin\frac{A}{2}}, \quad OC = \frac{r}{\sin\frac{C}{2}}.$$

Hence,

$$x + z = r\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) = r\cdot\frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}},$$

or

$$\frac{OA\cdot OC}{x+z} = \frac{r}{\sin\frac{A+C}{2}}.$$

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Similarly, we have

$$\frac{OB \cdot OD}{y+w} = \frac{r}{\sin \frac{B+D}{2}}.$$

Since $\sin \frac{A+C}{2} = \sin \frac{B+D}{2}$, we have

$$\frac{OA \cdot OC}{x+z} = \frac{OB \cdot OD}{y+w}. \tag{1}$$

It is obvious that every rhombus is a circumscribed parallelogram and its center is both the incenter and centroid. If the centroid of a circumscribed parallelogram is also its incenter, then this parallelogram is a rhombus. We will investigate this double property for other quadrilaterals.

(1) It is easy to see that if the incenter of a circumscribed trapezium (Figure 2) is also the centroid, i.e., the midpoint of its median , then from the right angled triangles OAB and ODC we get $AB = 2MO = 2ON = CD$. Hence the trapezium is isosceles with $AB = CD = MN$.

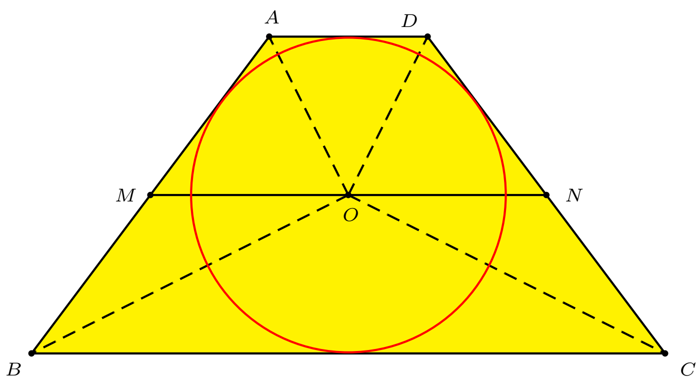

Figure 2

(2) If in a circumscribed trapezium $ABCD$ (with bases BC and AD) we have $OA \cdot OC = OB \cdot OD$, then

$$\begin{aligned} & \frac{r}{\sin \frac{A}{2}} \cdot \frac{r}{\sin \frac{C}{2}} = \frac{r}{\sin \frac{B}{2}} \cdot \frac{r}{\sin \frac{D}{2}} \\ \Longrightarrow \quad & \sin \frac{A}{2} \cos \frac{D}{2} = \cos \frac{A}{2} \sin \frac{D}{2} \\ \Longrightarrow \quad & \tan \frac{A}{2} = \tan \frac{D}{2} \\ \Longrightarrow \quad & A = D. \end{aligned}$$

Hence the trapezium is isosceles and the incenter O is also its centroid.

We will generalize this property by adopting the following definition formulated by George Baloglou [1].

Definition. An antirhombus is a circumscribed quadrilateral $ABCD$ with incenter O that satisfies the condition

$$OA \cdot OC = OB \cdot OD.$$

Theorem 1. *A circumscribed quadrilateral ABCD with incenter O is an antirhombus if and only if $x + z = y + w$.*

Proof. It is obvious from equality (1). □

Theorem 2 ([2, Theorem 13])**.** *A circumscribed quadrilateral with no parallel sides has the same incenter and centroid if and only if it is an antirhombus.*

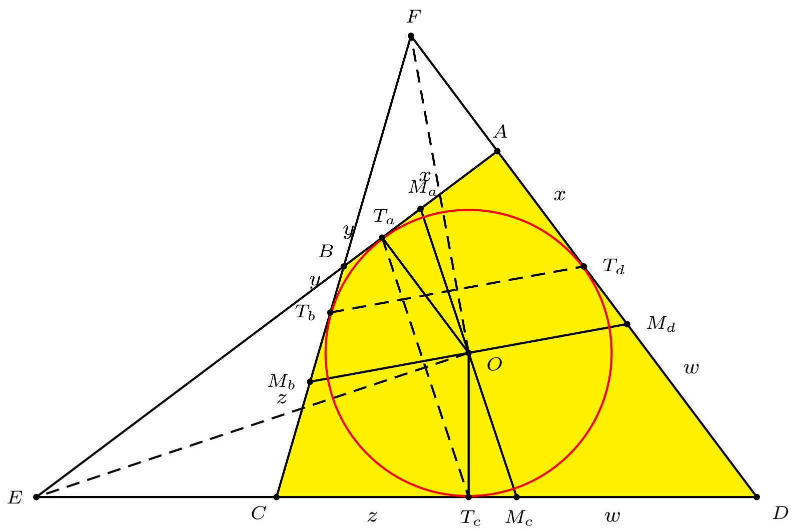

Figure 3

Proof. (1) Let $ABCD$ be a circumscribed quadrilateral with centroid and incenter the point O, and E, F be the intersection points of the opposite sides AB, CD and BC, AD (Figure 3). This point O must be the midpoint of the bimedians M_aM_c, M_bM_d. Let T_a, T_b, T_c, T_d be the tangency points with the incircle. The triangle EM_aM_c is isosceles because EO is a bisector and a median. The triangle ET_aT_c is also isosceles. Therefore, M_aM_c and T_aT_c are parallel, and are both perpendicular to EO. If $x > y$, then $w > z$, and we have $\frac{x-y}{2} = T_aM_a = T_cM_c = \frac{w-z}{2}$. It follows that $x + z = y + w$, and $ABCD$ is an antirhombus. Similarly, M_bM_d and T_bT_d are parallel, and are both perpendicular to FO.

(2) If $ABCD$ is an antirhombus with incenter O, then we have $x+z = y+w$. If $x > y$, then $w > z$ and $T_aM_a = \frac{x-y}{2} = \frac{w-z}{2} = T_cM_c$. The right angled triangles OT_aM_a and OT_cM_c are congruent. Hence, $OM_a = OM_c$. This means that the incenter O lies on the perpendicular bisector $\mathcal{L}_1$ of the bimedian M_aM_c. Similarly O also lies on the perpendicular bisector $\mathcal{L}_2$ of the bimedian M_bM_d. But the only common point of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the centroid of $ABCD$, the common midpoint of the bimedians. Hence the antirhomus has the same incenter and centroid. Again we have $M_aM_c \| T_aT_c$, $M_aM_c \perp EO$ and the same for the other bimedian M_bM_d. □

Corollary 3. *A circumscribed quadrilateral is an antirhombus if and only if a bimedian connecting two opposite sides is perpendicular to the bisector of the angle of these sides and hence parallel to the chord of the corresponding contact points.*

2. Antirhombi from a triangle

Let ABC be a triangle with intouch triangle $A_1B_1C_1$. We begin with the construction of the unique line $\mathscr{L}_a$ which cuts the sidelines of ABC at the points A_a, A_b, A_c and tangent to the incircle of ABC at a point A_2 such that the quadrilateral BCA_bA_c is an antirhombus.

Construction 4. *Let the perpendicular to AI at I intersect AB at J and AC at K. For the quadrilateral BCA_bA_c to be an antirhombus it is sufficient from Theorem 3 that the line JK* (Figure 4) *be a bimedian of the antirhombus. Hence the points A_c and A_b are the symmetrics of B, C in J, K respectively. The line A_bA_c is the line $\mathscr{L}_a$.*

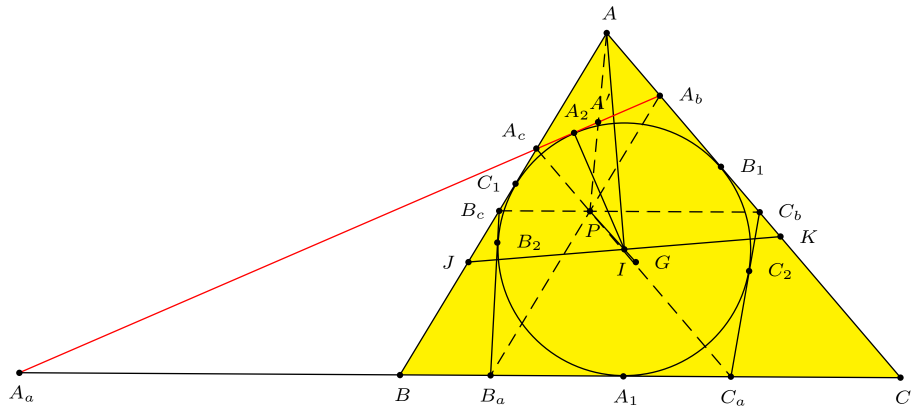

Figure 4

Similarly, we construct the lines $\mathscr{L}_b$ and $\mathscr{L}_c$ intersecting the sidelines at B_a, B_b, B_c, and C_a, C_b, C_c respectively such that CAB_cB_a and ABC_aC_b are antirhomobi.

The following property gives easily the barycentric coordinates of the above points.

Proposition 5. *The lines B_cC_b, C_aA_c, A_bB_a are parallel to the sides of ABC and are concurrent at the image of the centroid G under the homothety* $\mathsf{h}(I, -3)$.

Proof. The parallel from A_b to AC meets the parallel from A_c to AB at a point P. Let A' be the midpoint of A_bA_c (Figure 4). G is the centroid of ABC so we have

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}.$$

Since I is the centroid of the quadrilateral BCA_bA_c,

$$\begin{aligned}
&\overrightarrow{IA_b} + \overrightarrow{IA_c} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&2\overrightarrow{IA'} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&\overrightarrow{IP} + \overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&\overrightarrow{IP} + (\overrightarrow{IG} + \overrightarrow{GA}) + (\overrightarrow{IG} + \overrightarrow{GB}) + (\overrightarrow{IG} + \overrightarrow{GC}) = \overrightarrow{0}.
\end{aligned}$$

Hence $\overrightarrow{IP} = -3\overrightarrow{IG}$, which means that P is the image of G under the homothety $\mathsf{h}(I,-3)$. From $P = 4I - 3G$, we obtain

$$P = (3a - b - c :\ 3b - c - a :\ 3c - a - b) \tag{2}$$

in homogeneous barycentric coordinates. Similarly, the lines B_cC_b, C_aA_c, and A_bB_a are parallel to the sides of ABC and concurrent at P. $\square$

So we have an interesting special case mentioned in [5, §12.1.2] and hence in homogeneous barycentric coordinates we have

$$\begin{aligned} C_b &= (u:0:v+w), \quad B_c = (u:v+w:0);\\ A_c &= (w+u:v:0), \quad C_a = (0:v:w+u);\\ B_a &= (0:u+v:w), \quad A_b = (u+v:0:w). \end{aligned}$$

The equations of the lines are

$$\begin{aligned} \mathscr{L}_a &:= A_bA_c: & -x + \frac{w+u}{v}y + \frac{u+v}{w}z &= 0,\\ \mathscr{L}_b &:= B_cB_a: & \frac{v+w}{u}x - y + \frac{u+v}{w}z &= 0,\\ \mathscr{L}_c &:= C_aC_b: & \frac{v+w}{u}x + \frac{w+u}{v}y - z &= 0, \end{aligned}$$

Since the lines B_cC_b, C_aA_c, A_bB_a are concurrent at P, from the converse of Brianchon's theorem we conclude that there is a conic $\mathscr{C}_1$ inscribed in the hexagon $A_bA_cB_cB_aC_aC_b$. It is known [5] that the coordinates of the lines BC, CA, AB, $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ correspond to points on the dual conic. Since these points are the vertices of ABC and the points $\left(-1 :\ \frac{w+u}{v} :\ \frac{u+v}{w}\right)$, $\left(\frac{v+w}{u} :\ -1 :\ \frac{u+v}{w}\right)$, $\left(\frac{v+w}{u} :\ \frac{w+u}{v} :\ -1\right)$, that all lie on the circumconic

$$\frac{v+w}{x} + \frac{w+u}{y} + \frac{u+v}{z} = 0$$

the dual conic, which is tangent to the 6 lines is the conic with equation

$$\sum_{\text{cyclic}} (v+w)^2x^2 - 2(w+u)(u+v)yz = 0.$$

This conic has center

$$O_1 = (2u + v + w :\ u + 2v + w :\ u + v + 2w).$$

In our case with $P = X_{145}$ given in (2) above, we have

$$\sum_{\text{cyclic}} (s-a)^2x^2 - 2(s-b)(s-c)yz = 0,$$

which is clearly the incircle with center I. Hence, the incenter I of ABC is also the incenter of the quadrilaterals CAB_cB_a and ABC_aC_b.

These quadrilaterals are convex if and only if for the sides $a \leq b \leq c$ of triangle ABC, we have $3a - b - c > 0$.

Proposition 6. *The tangency points A_2, B_2, C_2 of the lines $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ are the vertices of a triangle perspective with ABC.*

Proof. The point A_2 is the pole of $\mathscr{L}_a$ with respect to $\mathscr{C}_1$:

$$\begin{aligned} A_2 = &\begin{pmatrix} -vw & w(w+u) & v(u+v) \end{pmatrix} \begin{pmatrix} 0 & u+v & w+u \\ u+v & 0 & v+w \\ w+u & v+w & 0 \end{pmatrix} \\ = &\begin{pmatrix} (v+w)(w+u)(u+v) & v^2(u+v) & w^2(w+u) \end{pmatrix}, \end{aligned}$$

or

$$A_2 = \left(v+w : \frac{v^2}{w+u} : \frac{w^2}{u+v}\right).$$

Similarly, from the coordinates of the points B_2, C_2 we conclude the triangle $A_2B_2C_2$ is perspective with ABC at the point

$$\left(\frac{u^2}{v+w} : \frac{v^2}{w+u} : \frac{w^2}{u+v}\right).$$

□

In our case with $P = X_{145}$, the perspector is

$$Q = \left(\frac{(3a-b-c)^2}{b+c-a} : \frac{(3b-c-a)^2}{c+a-b} : \frac{(3c-a-b)^2}{a+b-c}\right).$$

This point is not in the current edition of the ENCYCLOPEDIA OF TRIANGLE CENTERS [4]. It has (6-9-13)-search number $0.0267031360104\cdots$. This divides the line IG_e in the ratio

$$IQ : QG_e = 4R + r : -8r.$$

The point of tangency with BC is

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u+v & w+u \\ u+v & 0 & v+w \\ w+u & v+w & 0 \end{pmatrix} = \begin{pmatrix} 0 & u+v & w+u \end{pmatrix}.$$

Proposition 7. *The hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic.*

Proof. Since

$$\begin{aligned} &\frac{BB_a}{B_aC} \cdot \frac{BC_a}{C_aC} \cdot \frac{CC_b}{C_bA} \cdot \frac{CA_b}{A_bA} \cdot \frac{AA_c}{A_cB} \cdot \frac{AB_c}{B_cB} \\ = &\frac{w}{u+v} \cdot \frac{w+u}{v} \cdot \frac{u}{v+w} \cdot \frac{u+v}{w} \cdot \frac{v}{w+u} \cdot \frac{v+w}{u} \\ = &1, \end{aligned}$$

from Carnot's theorem we conclude that the hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic $\mathscr{C}_2$ that has [5, p.141] equation

$$\sum_{\text{cyclic}} vw(v+w)x^2 - u(vw + (w+u)(u+v))yz = 0.$$

The center of this conic is the point

$$O_2 = (u(2vw+u(v+w-u)) : \; v(2wu+v(w+u-v)) : \; w(2uv+w(u+v-w))),$$

which is the midpoint of P and G/P. □

Proposition 8. *The points A_a, B_b, C_c lie on the trilinear polar of X_{5435}.*

Proof. The lines $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ meet the sides BC, CA, AB at the points A_a, B_b, C_c respectively. These are collinear on the Pascal line of the hexagon $A_bA_cB_cB_aC_aC_b$. The line $\mathscr{L}_a$ meets BC at the point $A_a = \left(0 : \; -\frac{v}{w+u} : \; \frac{w}{u+v}\right)$, which is the intersection of BC with the trillinear polar of the point $Q = \left(\frac{u}{v+w} : \; \frac{v}{w+u} : \; \frac{w}{u+v}\right)$. For $P = X_{145}$, this is the point

$$X_{5435} = \left(\frac{3a-b-c}{b+c-a} : \; \frac{3b-c-a}{c+a-b} : \; \frac{3c-a-b}{a+b-c}\right);$$

see [3]. The same holds for B_b and C_c. □

Proposition 9. *The Brianchon points of the quadrilaterals A_bA_cBC, B_cB_aCA, C_aC_bAB are the vertices of a triangle $P_1P_2P_3$ which is perspective with ABC at X_{5435}.*

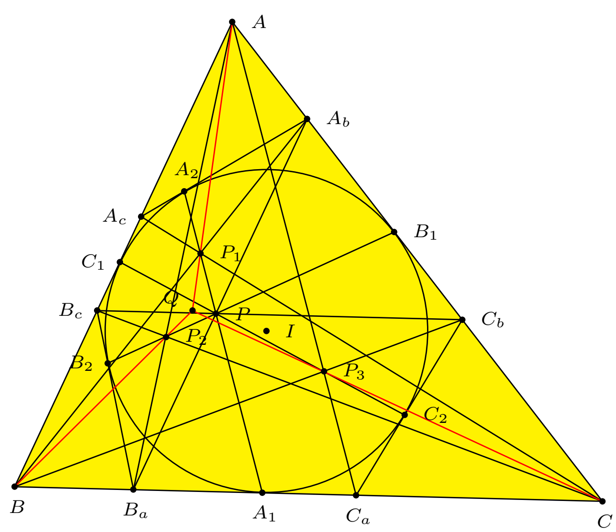

Figure 5

Proof. The Brianchon point P_1 of the circumscribed quadrilateral A_bA_cBC is the common point of the diagonals BA_b, CA_c and the contact chords A_1A_2, B_1C_1 (Figure 5). Hence

$$P_1 = ((w+u)(u+v) : \; v(u+v) : \; w(w+u))$$

and the line AP_1 passes through X_{5435}. Similarly, the lines BP_2 and CP_3 also pass through the same point. □

Proposition 10. *The triangle $P_1P_2P_3$ is perspective with the contacts triangle $A_1B_1C_1$ at P.*

Proof. The points P, A_1, A_2 are collinear because

$$\begin{vmatrix} u & v & w \\ 0 & u+v & w+u \\ v+w & \frac{v^2}{w+u} & \frac{w^2}{u+v} \end{vmatrix} = u(w^2-v^2)+(v+w)(v(w+u)-w(u+v)) = 0.$$

Hence the line A_1P_1 passes through P, and the same holds for the lines B_1P_2 and C_1P_3.

Since the line A_1A_2 is the polar of the point A_a and passes through P, we conclude that the line $\mathscr{L}$ of the points A_a, B_b, C_c (which is the tripolar of X_{5435}) is the polar of P relative to the incircle of ABC. □

Proposition 11. *The lines $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ bound a triangle perspective with ABC at P.*

Proof. It is easy to see that the lines $\mathscr{L}_b$ and $\mathscr{L}_c$ meet at (Figure 6) the point $A_3 = \left(-\frac{u^2}{v+w} : v : w\right)$. Similarly $\mathscr{L}_c$ and $\mathscr{L}_a$ meet at $B_3 = \left(u : -\frac{v^2}{w+u} : w\right)$, and $\mathscr{L}_a$ and $\mathscr{L}_b$ meet at $C_3 = \left(u : v : -\frac{w^2}{u+v}\right)$. From these coordinates it is clear that $A_3B_3C_3$ and ABC are perspective at $P = (u : v : w)$. □

Proposition 12. *The points A, B, C, A_3, B_3, C_3 lie on a conic $\mathscr{C}_3$, and the centers of the conics $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ are collinear.*

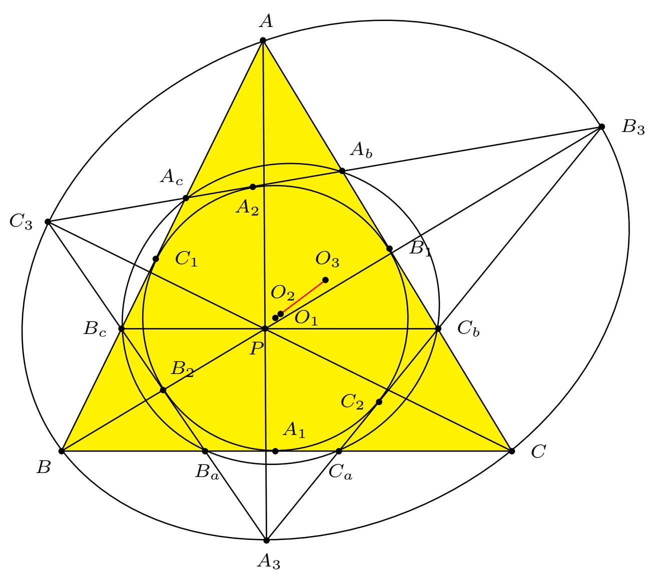

Figure 6

Proof. It is easy to see that the conic $\mathscr{C}_3$ in question is

$$\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z} = 0$$

with center

$$O_3 = (u^2(v^2 + w^2 - u^2) : \ v^2(w^2 + u^2 - v^2) : \ w^2(u^2 + v^2 - w^2)).$$

The centers O_1, O_2, O_3 are all on the line

$$\sum_{\text{cyclic}} (v - w)(u(v + w) - (v^2 + vw + w^2))x = 0.$$

$\square$

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Asymptotic Directions of Pivotal Isocubics

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Abstract. Given the pivotal isocubic $\mathcal{K} = p\mathcal{K}(\Omega, P)$, we seek all the other isocubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ with pole Ω_1 and pivot P_1 which have the same points at infinity, i.e., the same asymptotic directions. We also examine the connection with the Simson lines concurring at a certain given point.

1. Generalities

Recall that the pivotal isocubic $\mathcal{K} = p\mathcal{K}(\Omega, P)$ is the locus of point M such that P, M and the $\Omega-$isoconjugate M^* of M are collinear. With a pole $\Omega(p : q : r)$ and a pivot $P(u : v : w)$, its barycentric equation is

$$\sum_{\text{cyclic}} u\,x(ry^2 - qz^2) = 0 \iff \sum_{\text{cyclic}} p\,yz(wy - vz) = 0 \tag{1}$$

We denote by $\mathcal{F}_{\mathcal{K}}$ the family of all pivotal isocubics having the same points at infinity as $\mathcal{K}$.

Theorem 1 (Pole theorem)**.** *Given $\mathcal{K} = p\mathcal{K}(\Omega, P)$, a pivotal isocubic $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ has the same points at infinity as $\mathcal{K}$ if its pole Ω_1 lies on the cubic $\mathcal{K}_\Omega$ with equation*

$$\sum_{\text{cyclic}} u\,x(y+z-x)(ry-qz) = 0 \tag{2}$$

$$\iff \sum_{\text{cyclic}} pyz(v(x-y+z) - w(x+y-z)) = 0. \tag{3}$$

Theorem 2 (Pivot theorem)**.** *Given $\mathcal{K} = p\mathcal{K}(\Omega, P)$, a pivotal isocubic $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ has the same points at infinity as $\mathcal{K}$ if its pivot P_1 lies on the cubic $\mathcal{K}_P$ with equation*

$$\sum_{\text{cyclic}} u\,(y+z)(ry(x+z) - qz(x+y)) = 0 \tag{4}$$

$$\iff \sum_{\text{cyclic}} p\,(x+y)(x+z)(wy - vz) = 0. \tag{5}$$

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1.1. *Properties of* $\mathcal{K}_\Omega$. $\mathcal{K}_\Omega$ is the pseudo-isocubic $ps\mathcal{K}(\Omega \times cP, G, \Omega)$ where cP is the complement of P and $X_\Omega = \Omega \times cP$ is the barycentric product of Ω and cP. See [3].

$\mathcal{K}_\Omega$ is a circum-cubic passing through Ω and the midpoints M_a, M_b, M_c of BC, CA, AB.

The tangents at A, B, C concur at X_Ω. This point lies on $\mathcal{K}_\Omega$ when G, Ω, P are collinear in which case $\mathcal{K}_\Omega$ is a $p\mathcal{K}$ (see §2).

The equation (2) shows that, for a given Ω, all $\mathcal{K}_\Omega$ form a net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the line BC, the line through the midpoints of AB and AC, the line $A\Omega$, the other two similarly. Generally, this net contains only one circular cubic and only one equilateral cubic.

1.2. *Properties of* $\mathcal{K}_P$. $\mathcal{K}_P$ is the anticomplement of the pseudo-isocubic $\mathcal{K}'_P = ps\mathcal{K}(\Omega \times cP, G, cP)$. See [3].

$\mathcal{K}_P$ is a circum-cubic passing through P and the vertices G_a, G_b, G_c of the antimedial triangle. Moreover, it has the same points at infinity as $\mathcal{K}$ hence, $\mathcal{K}_P$ is a circular cubic (an equilateral cubic) if and only if $\mathcal{K}$ is itself a circular cubic (an equilateral cubic).

The tangents at G_a, G_b, G_c concur at a point which is the anticomplement of X_Ω. This point lies on $\mathcal{K}_P$ when G, Ω, P are collinear in which case $\mathcal{K}_\Omega$ is also a $p\mathcal{K}$ (see §2).

$\mathcal{K}_P$ is itself a pseudo-isocubic if and only if its tangents at A, B, C concur which is realized when G, Ω, P are collinear as above but also when P lies on the circum-conic with center Ω or equivalently when Ω lies on the bicevian conic $\mathcal{C}(G, P)$ with center ccP, the complement of cP. In this latter case, the pseudo-pivot of $\mathcal{K}_P$ lies on the Steiner ellipse and the pseudo-pole lies on $ps\mathcal{K}(ta\Omega, t(G/\Omega), G)$.

The equation (5) shows that, for a given P, all $\mathcal{K}_P$ form another net of circum-cubics which is generated by three decomposed cubics, one of them being the union of the parallels at C, B to AB, AC respectively and the line AP, the other two similarly.

1.3. *A special case.* With $\Omega = P^2$ (barycentric square), $\mathcal{K}_\Omega$ (resp. $\mathcal{K}_P$) is the locus of poles (resp. pivots) of all $p\mathcal{K}$ having asymptotes parallel to the cevian lines of P.

1.4. *Examples.* We show several examples with known cubics for which G, Ω, P are not collinear.

1.4.1. $\mathcal{K}$ *is the McCay cubic.* With $\Omega = K$ and $P = O$, $\mathcal{K}$ is the McCay cubic **K003**. We know that it has three real asymptotes perpendicular to the sidelines of the Morley triangle.

$\mathcal{K}_\Omega$ is **K307** = $ps\mathcal{K}(X_{51}, X_2, X_6)$ with equation :

$$\sum_{\text{cyclic}} a^2 S_A \, x(y+z-x)(c^2y - b^2z) = 0 \tag{6}$$

It passes through K, X_{53}, X_{216}, X_{1249}. The tangents at A, B, C concur at X_{51}, centroid of the orthic triangle. See Figure 1.

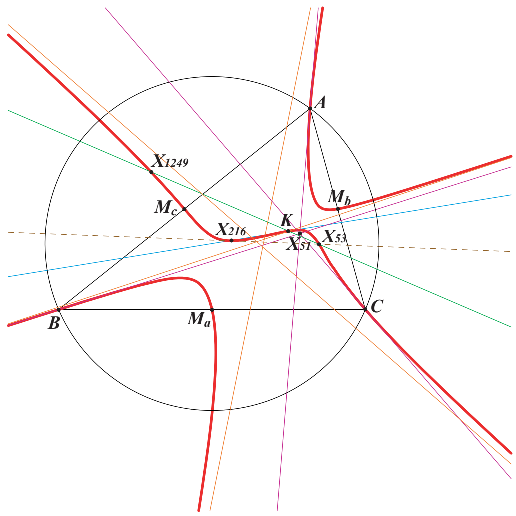

Figure 1. $\mathcal{K}_\Omega$ with $\Omega = K$ and $P = O$

Each point on the curve is the pole of a $p\mathcal{K}$ having asymptotes parallel to those of the McCay cubic. $p\mathcal{K}(X_{53}, X_4)$ = **K049** is the McCay cubic of the orthic triangle, $p\mathcal{K}(X_{216}, X_{20})$ = **K096** contains X_3, X_5, X_{20} and the tangential E_{367} of H in the Darboux cubic.

$\mathcal{K}_P$ is denoted by $\mathcal{K}_O^{++}$ = **K080** : it is a central cubic with center O, having three real asymptotes perpendicular to the sidelines of the Morley triangle and concurring at O. Each point on the curve is the pivot of a $p\mathcal{K}$ having asymptotes parallel to those of the McCay cubic.

Its equation is :

$$\sum_{\text{cyclic}} a^2\,(x+y)(x+z)(c^2 S_C\,y - b^2 S_B\,z) = 0 \qquad (7)$$

$\mathcal{K}_O^{++}$ is a circum-cubic passing through :

– O, H, L, X_{1670}, X_{1671},

– the vertices G_a, G_b, G_c of the antimedial triangle,

– the reflections H_A, H_B, H_C of H in A, B, C,

– the reflections A_O, B_O, C_O of A, B, C in O,

– the points U_a, U_b, U_c which also lie on the Neuberg cubic and on the circle with center L and radius $2R$. These points are the images under $\mathsf{h}(H, 2)$ of the points V_a, V_b, V_c, intersections of the Napoleon cubic with the circumcircle. See Figure 2.

1.4.2. *$\mathcal{K}$ is the orthocubic.* With $\Omega = K$ and $P = H$, $\mathcal{K}$ is the orthocubic **K006**. $\mathcal{K}_\Omega$ is **K260**, a nodal cubic with node K and passes through X_{69}, X_{206}, X_{219},

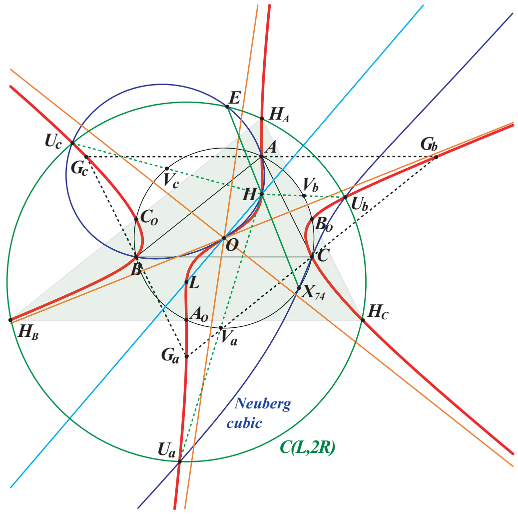

Figure 2. $\mathcal{K}_O^{++}$ and the Neuberg cubic

X_{478}, X_{577}, X_{1249}, X_{2165}. The tangents at A, B, C concur at X_{184}. Its equation is :

$$\sum_{\text{cyclic}} a^4 S_A(y-z)yz - (b^2-c^2)(c^2-a^2)(a^2-b^2)\, xyz = 0 \tag{8}$$

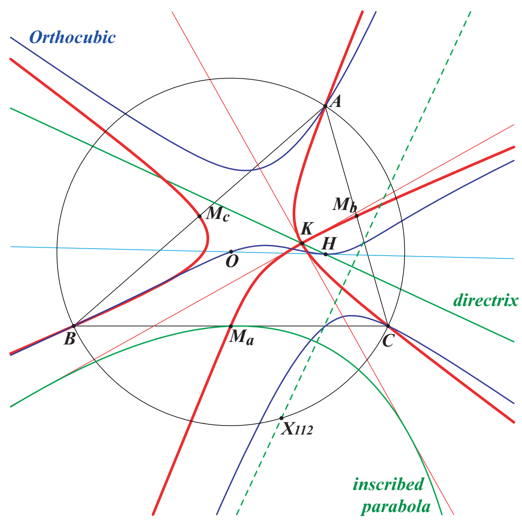

Figure 3. $\mathcal{K}_\Omega$ with $\Omega = K$ and $P = H$

An easy construction of $\mathcal{K}_\Omega$ is the following : the trilinear polar q of any point Q on the Euler line meets the lines KM_a, KM_b, KM_c at Q_a, Q_b, Q_c. The triangles

ABC and $Q_aQ_bQ_c$ are perspective at M and the locus of M is $\mathcal{K}_\Omega$. Furthermore, q envelopes the inscribed parabola with focus X_{112} and directrix the line HK. See Figure 3.

$\mathcal{K}_P$ is **K617**, the anticomplement of **K009**. It is also a nodal cubic with node H and it passes through X_{20}, X_{68}, X_{254}, X_{315}, X_{2996}. The construction seen above for $\mathcal{K}_\Omega$ is easily adapted for $\mathcal{K}_P$: it is enough to replace the Euler line by the line X_2X_{216} and K by H. Similarly we obtain another inscribed parabola with focus X_{107} and directrix the line X_4X_{51}. See Figure 4. The equation of $\mathcal{K}_P$ is :

$$\sum_{\text{cyclic}} a^2\,(x+y)(x+z)(y/S_C - z/S_B) = 0 \tag{9}$$

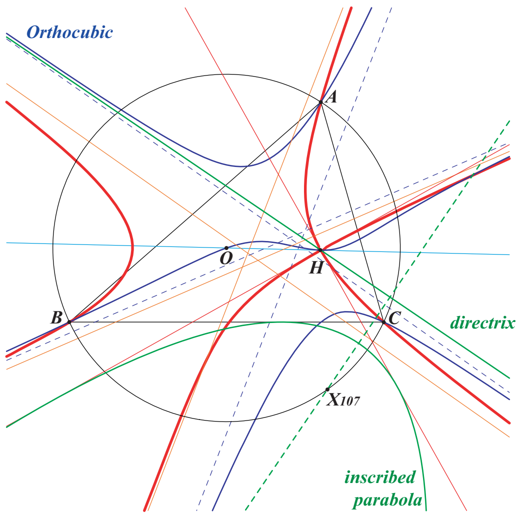

Figure 4. $\mathcal{K}_P$ with $\Omega = K$ and $P = H$

In the two cubics, the tangents at the nodes are parallel to the asymptotes of the Jerabek hyperbola.

From all this, we see that each point Q on the Euler line gives a pole ω on $\mathcal{K}_\Omega$ and a corresponding pivot π on $\mathcal{K}_P$ such that the cubic $p\mathcal{K}(\omega, \pi)$ has its asymptotes parallel to those of the orthocubic. For example, with $Q = G$, $Q = O$ and $Q = H$, we find $p\mathcal{K}(X_{69}, X_{315})$, $p\mathcal{K}(X_{577}, X_{20})$ and $p\mathcal{K}(X_{2165}, X_{68})$ respectively.

1.4.3. *A selection of cubics $\mathcal{K}_\Omega$, $\mathcal{K}_P$ and $\mathcal{K}'_P$.* Table 1 below shows a small selection of these cubics, specially those connected with the usual and well known cubics $\mathcal{K}$ in triangle geometry. When a cubic is not listed in [2], a list of centers is provided.

Some of these examples are detailed below.

Table 1. $\mathcal{K}$ and the related cubics $\mathcal{K}_\Omega$, $\mathcal{K}_P$, $\mathcal{K}'_P$

$\mathcal{K}$	$\mathcal{K}_\Omega$	$\mathcal{K}_P$	$\mathcal{K}'_P$
K001	X_6, X_{1249}, X_{1989}, X_{1990}, X_{3163}	**K449**	**K446**
K002	**K002**	**K007**	**K002**
K003	**K307**	**K080**	**K026**
K004	X_6, X_{393}, X_{1249}	X_4, X_{20}	**K376**
K005	X_6, X_{233}, X_{1249}, X_{2963}	X_4, X_5, X_{20}, X_{2888}	**K569**
K006	**K260**	**K617**	**K009**
K034	**K345**	**K034**	**K345**

2. Pivotal $\mathcal{K}_\Omega$ and $\mathcal{K}_P$

2.1. *Theorem and corollaries.*

Theorem 3. *$\mathcal{K}_\Omega$ and $\mathcal{K}_P$ are pivotal isocubics if and only if*

$$(q-r)u + (r-p)v + (p-q)w = 0$$

i.e. if and only if G, Ω, P are collinear.

Corollary 4. *$\mathcal{K}_\Omega$ has pivot G and pole ω, the barycentric product of Ω and the complement of P. This point is the common tangential of A, B, C.*

This pole lies on the line through Ω, the barycentric square of Ω, the complement of the isotomic conjugate of Ω.

$\mathcal{K}_\Omega$ contains the following points :

- A, B, C, G
- M_a, M_b, M_c
- Ω, ω, cP (complement of P), P^* ($\Omega-$isoconjugate of P)
- the isotomic conjugate of the anticomplement of ω and its complement

Corollary 5. *$\mathcal{K}_P$ is an isotomic pivotal isocubic with pivot the anticomplement of ω. This point is the common tangential of G_a, G_b, G_c.*

This pivot lies on the line through the anticomplement of Ω, the isotomic conjugate of Ω, the anticomplement of the barycentric square of Ω.

$\mathcal{K}_P$ contains the following points :

- A, B, C, G
- G_a, G_b, G_c
- P, the anticomplements of Ω, ω, P^*
- the isotomic conjugate of the anticomplement of ω

Corollary 6. *$\mathcal{K}_\Omega$ is the complement of $\mathcal{K}_P$. The two cubics are tangent at G to the line $G\omega$.*

Corollary 7. *$\mathcal{K}_\Omega$, $\mathcal{K}_P$ have the same points at infinity which are those of $\mathcal{K}$.*

From all this, we notice that $\mathcal{K}_\Omega$ and $\mathcal{K}_P$ have nine common points : A, B, C, G (double), the three points at infinity of $\mathcal{K}$, the isotomic conjugate of the anticomplement of ω.

When we choose Ω or P at G, we obtain :

Corollary 8. (1) *Any isotomic $p\mathcal{K}$ is the locus of pivots of all $p\mathcal{K}$ having the same asymptotic directions as itself.*

(2) *Any $p\mathcal{K}$ with pivot G is the locus of poles of all $p\mathcal{K}$ having the same asymptotic directions as itself.*

See the three examples below.

A line ℓ_G through G meets $\mathcal{K}_\Omega$ at two points Ω_1 and Ω_2 which are $\omega-$isoconjugate and collinear with G.

Denote by P_1, P_2 the anticomplements of Ω_2, Ω_1 respectively (reverse order). These points lie on $\mathcal{K}_P$.

Theorem 9. *The two pivotal isocubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ and $\mathcal{K}_2 = p\mathcal{K}(\Omega_2, P_2)$ have the same points at infinity as $\mathcal{K} = p\mathcal{K}(\Omega, P)$*

Construction : given Ω and P collinear with G, let Γ_ℓ be the inscribed conic in ABC which is tangent at G to ℓ_G. The trilinear pole Q_ℓ of ℓ_G lies on the Steiner circum-ellipse, and the trilinear pole of the tangent at Q_ℓ to the Steiner circum-ellipse is the perspector of Γ_ℓ.

Now draw the two (not necessarily real) tangents to Γ_ℓ passing through ω. These tangents meet ℓ_G at Ω_2, Ω_1. The construction of P_1, P_2 follows easily.

Beware P_1, P_2 are two points on $\mathcal{K}_P$ but are not isotomic conjugates on this cubic.

2.2. *Examples.*

2.2.1. *$\mathcal{K}$ is the Thomson cubic.* When $\mathcal{K}$ is the Thomson cubic **K002** with $\Omega = K$ (Lemoine point) and $P = G$ (centroid), $\mathcal{K}_\Omega$ is the Thomson cubic again and $\mathcal{K}_P$ is the Lucas cubic **K007**. In other words, all the corresponding cubics $\mathcal{K}_1 = p\mathcal{K}(\Omega_1, P_1)$ and $\mathcal{K}_2 = p\mathcal{K}(\Omega_2, P_2)$ have the same points at infinity which are those of the Thomson and Lucas cubics.

The following table shows several examples of corresponding points Ω_1, P_1 and Ω_2, P_2.

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_1	X_8	**K308**	X_1	X_8	**K308**
X_2	X_{69}	**K007**	X_6	X_2	**K002**
X_3	X_{20}	2,3,20,1032,1073,1498	X_4	X_4	**K181**
X_9	X_{329}	2,9,188,282,329,1034,1490	X_{57}	X_7	1,2,7,57,145,174,1488,2089
X_{223}	E_{623}		X_{282}	X_{189}	2,8,9,84,189,282
X_{1073}	X_{253}	2,3,64,69,253,1073,3146	X_{1249}	E_{624}	
E_{630}	?		E_{668}	X_{1034}	
E_{382}	E_{625}		E_{553}	X_{1032}	

Remark. E_{623} is the anticomplement of X_{282}, E_{624} is the anticomplement of X_{1073}, E_{382} is the complement of X_{1032}, E_{630} is the complement of X_{1034}.

2.2.2. *$\mathcal{K}$ is the Grebe cubic.* When $\mathcal{K}$ is the Grebe cubic **K102** with $\Omega = P = K$, we obtain $\mathcal{K}_\Omega = p\mathcal{K}(X_{39}, X_2)$ and $\mathcal{K}_P = p\mathcal{K}(X_2, X_{76})$ = **K141**. We find six cubics with the same points at infinity :

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_3	X_{22}	2,3,22,159	X_{427}	X_4	2,4,76,141,193,427,1843
X_6	X_6	**K102**	X_{141}	X_{69}	2,20,69,141,427
X_{39}	X_2	2,3,6,39,141,427	X_2	X_{76}	**K141**

2.2.3. *$\mathcal{K}$ is an isogonal circular cubic.* The points at infinity of $\mathcal{K}$ need not be real. For example, with $\Omega = K$ and $P = X_{524}$ (infinite point of the line GK), $\mathcal{K}$ is now a circular cubic passing through G, K, X_{111} (Parry point), with singular focus X_{1296} (antipode of the Parry point on the circumcircle). The real asymptote is the parallel to GK at X_{111}.

In this case, $\mathcal{K}_\Omega = p\mathcal{K}(X_{187}, X_2)$ = **K043** (Droussent medial cubic) and $\mathcal{K}_P = p\mathcal{K}(X_2, X_{316})$ = **K008** (Droussent cubic). All the cubics defined in the following table are circular with a real infinite point X_{524}.

Ω_1	P_1	cubic or X_i for $i =$	Ω_2	P_2	cubic or X_i for $i =$
X_3	X_{858}	2,3,66,524,858,895	X_{468}	X_4	**K209**
X_6	X_{524}	1,2,6,111,524,2930	X_{524}	X_{69}	2,20,69,468,524,2373
X_{67}	X_{67}	**K103**	E_{406}	E_{618}	
X_{111}	X_{671}	**K273**	X_{2482}	E_{620}	
X_{187}	X_2	**K043**	X_2	X_{316}	**K008**

Remark. E_{618} is the anticomplement of X_{67}, E_{620} is the anticomplement of X_{111}, E_{406} is the midpoint of X_6, X_{110}.

3. Isogonal and isotomic cubics of the pencil $\mathcal{F}_\mathcal{K}$

We suppose now that $\mathcal{K} = p\mathcal{K}(\Omega, P)$ is neither an isogonal nor an isotomic isocubic, i.e., that Ω is distinct of $K = X_6$ and $G = X_2$.

3.1. *Theorem and consequences.*

Theorem 10. (1) *$\mathcal{F}_\mathcal{K}$ contains one isogonal isocubic if and only if the pivot P of $\mathcal{K}$ lies on the line $\mathcal{L}_g$ passing through H and the $\Omega-$isoconjugate of O.*

(2) *$\mathcal{F}_\mathcal{K}$ contains one isotomic isocubic if and only if the pivot P of $\mathcal{K}$ lies on the line $\mathcal{L}_t$ passing through G and Ω.*

Remark. These two lines are always perfectly defined since Ω is neither K nor G. They coincide if and only if $\Omega = O$: for any $p\mathcal{K}(O, P)$ with P on the Euler line, there is one isogonal isocubic and one isotomic isocubic having asymptotes parallel to those of $p\mathcal{K}(O, P)$.

Corollary 11. *$\mathcal{F}_\mathcal{K}$ contains one isogonal isocubic and one isotomic isocubic if and only if the pivot P of $\mathcal{K}$ is the intersection Q of $\mathcal{L}_g$ and $\mathcal{L}_t$.*

Remark. $\mathcal{L}_g$ and $\mathcal{L}_t$ are parallel if and only if Ω lies on the rectangular hyperbola $\mathcal{H}$ passing through G, O, K, X_{110} (focus of the Kiepert parabola) and also X_{154},

X_{354}, X_{392}, X_{1201}, X_{2574}, X_{2575}. Its asymptotes are parallel to those of the Jerabek hyperbola.

The intersections (other than X_{110}) with the circumcircle are three points on the Thomson cubic which are the vertices of the Thomson triangle. The center of $\mathcal{H}$ is the midpoint of GX_{110}. See Figure 5. The equation of this hyperbola is :

$$\sum_{\text{cyclic}} (b^2 - c^2)(b^2c^2x^2 + a^2S_A\, yz) = 0$$

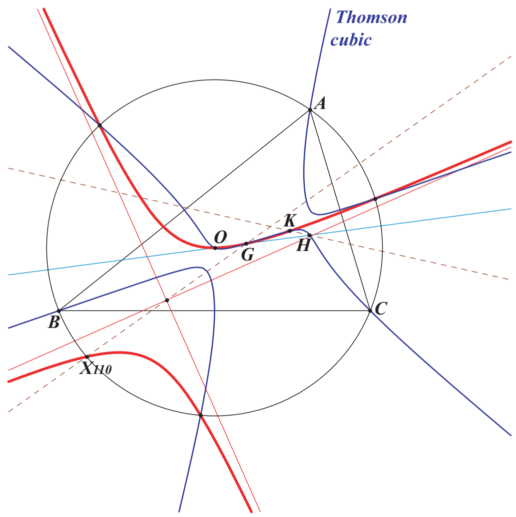

Figure 5. The rectangular hyperbola $\mathcal{H}$

3.2. *Examples.*

- With $\Omega = I$ (incenter), we find $Q = X_8$ (Nagel point). Hence, $p\mathcal{K}(X_1, X_8)$ = **K308** generates a family $\mathcal{F}_{\mathcal{K}}$ of isocubics containing one isogonal isocubic (the Thomson cubic **K002**) and one isotomic isocubic (the Lucas cubic **K007**).
- With $\Omega = O$ and $P = G$, $\mathcal{K}$ = **K168**. The isogonal cubic is $p\mathcal{K}(K, X_{193})$ and the isotomic cubic is $p\mathcal{K}(G, H)$ = **K170**. See Figure 6.

4. Circular $\mathcal{K}_\Omega$ cubics

We have seen that $\mathcal{K}_P$ is circular if and only if $\mathcal{K}$ is itself circular. We have the following theorems for $\mathcal{K}_\Omega$.

Theorem 12. (1) *For any pole Ω distinct of O, there is one and only one circular $\mathcal{K}_\Omega$ which passes through O and Ω.*

(2) *When $\Omega = O$, there are infinitely many circular $\mathcal{K}_\Omega$ forming a pencil of cubics passing through O. In this case, P must lie on the de Longchamps axis.*

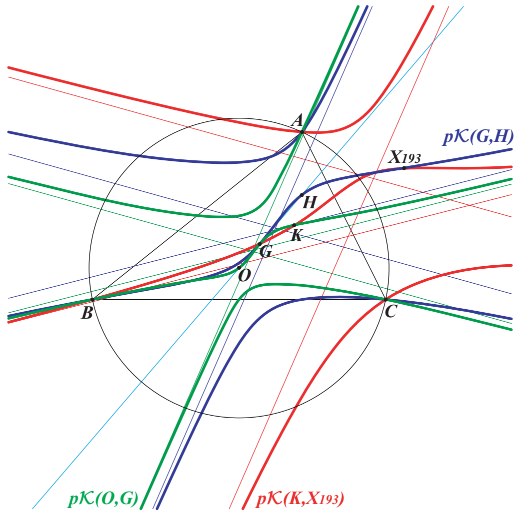

Figure 6. $p\mathcal{K}(O,G)$, $p\mathcal{K}(K,X_{193})$ and $p\mathcal{K}(G,H)$

For example, with $P = X_{858}$, $\mathcal{K}_\Omega$ is the Droussent medial cubic **K043**.

Theorem 13. (1) *For any pivot P distinct of X_{69}, there is one and only one circular $\mathcal{K}_\Omega$.*

(2) *When $P = X_{69}$, there are infinitely many circular $\mathcal{K}_\Omega$ forming a pencil of cubics passing through O and Ω which must lie on the line at infinity. The isogonal conjugate of Ω lies on the cubic (and on the circumcircle).*

The table gives a selection of such cubics.

Ω	centers on the cubic	cubic
X_{30}	X_3, X_4, X_{30}, X_{74}, X_{133}, X_{1511}	**K446**
X_{519}	X_1, X_3, X_{106}, X_{214}, X_{519}, X_{1319}	
X_{524}	X_2, X_3, X_6, X_{67}, X_{111}, X_{187}, X_{468}, X_{524}, X_{1560}, X_{2482}	**K043**
X_{527}	X_3, X_9, X_{57}, X_{527}, X_{1155}, X_{2291}	
X_{532}	X_3, X_{13}, X_{16}, X_{532}, X_{618}, X_{2380}	
X_{533}	X_3, X_{14}, X_{15}, X_{533}, X_{619}, X_{2381}	
X_{758}	X_3, X_{10}, X_{36}, X_{65}, X_{758}, X_{759}	
X_{2393}	X_3, X_{25}, X_{206}, X_{858}, X_{2373}, X_{2393}	

5. $p\mathcal{K}$ with three real asymptotes

Let $\mathcal{K} = p\mathcal{K}(\Omega, P)$ be the pivotal isocubic with pole $\Omega(p : q : r)$ and pivot $P(u : v : w)$. Let $\mathcal{C}_\Omega$ be the circum-conic with perspector Ω and center $O_\Omega = p(q + r - p) : q(r + p - q) : r(p + q - r)$, the $G-$Ceva conjugate of Ω i.e. the perspector of the medial triangle and the anticevian triangle of Ω. Let Γ_Ω be the homothetic of $\mathcal{C}_\Omega$ under $\mathsf{h}(O_\Omega, 3)$.

Theorem 14. *$p\mathcal{K}$ has three real asymptotes if and only if P lies inside* [1] *a tricuspidal quartic $\mathcal{Q}_\Omega$ tritangent to $\mathcal{C}_\Omega$ and having its cusps on Γ_Ω.*

Remark. $\mathcal{Q}_\Omega$ is bitangent to the line at infinity at the two points where $\mathcal{C}_\Omega$ meets the line at infinity.

Corollary 15 (isogonal $p\mathcal{K}$)**.** *From this remark, it is clear that $\mathcal{Q}_\Omega$ is bicircular if and only if $\Omega = K$. In this case, $O_\Omega = O$, $\mathcal{C}_\Omega$ is the circumcircle and $\mathcal{Q}_\Omega$ is a deltoid, the envelope of axes of inscribed parabolas. This result is already mentioned in* [1].

Corollary 16 (isotomic $p\mathcal{K}$)**.** *$\mathcal{Q}_\Omega$ contains A, B, C if and only if $\Omega = G$. In this case, $O_\Omega = G$, $\mathcal{C}_\Omega$ is the Steiner ellipse.*

$\mathcal{Q}_G$ has three cusps lying on the medians of ABC and on Γ_Ω, the homothetic of the Steiner ellipse under $\mathsf{h}(G,3)$. It is tangent at A, B, C to the Steiner ellipse. See Figure 7.

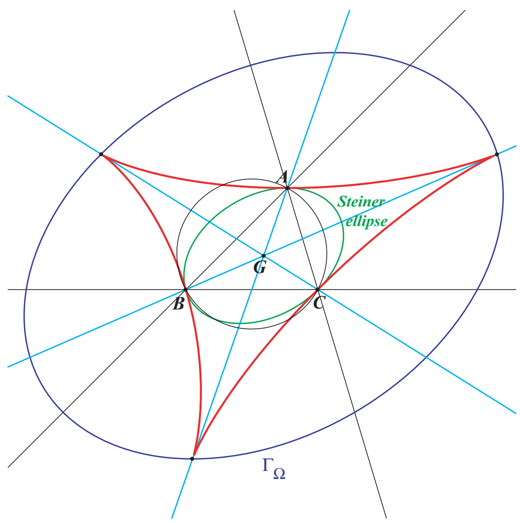

Figure 7. The tricuspidal quartic $\mathcal{Q}_G$

The equation of $\mathcal{Q}_G$ is :

$$32\sum_{\text{cyclic}} x\left(y^3+z^3\right)+61\sum_{\text{cyclic}} y^2z^2+118\sum_{\text{cyclic}} x^2yz=0 \tag{10}$$

Remark. When Ω lies on the inscribed Steiner ellipse, $\mathcal{Q}_\Omega$ decomposes into the line at infinity counted twice and a parabola. In this case, one of the fixed points of the isoconjugation lies at infinity.

[1] P is said to be inside the quartic when it lies in the same region of the plane as O_Ω.

6. Asymptotes and Simson lines

Let $\mathcal{K} = p\mathcal{K}(\Omega, P)$ be the pivotal isocubic with pole $\Omega(p : q : r)$ and pivot $P(u : v : w)$ and let M be a point. It is known that there are three (real or not, distinct or not) Simson lines that pass through M since the envelope of all Simson lines is the well known Steiner deltoid $\mathcal{H}_3$, a tricuspidal bicircular quartic of class 3.

6.1. *Construction of the Simson lines passing through* M. Jean-Pierre Ehrmann has found a simple conic construction of these lines which is as follows. Draw the rectangular circum-hyperbola $\mathcal{H}_M$ passing through M and its image $\mathcal{H}'_M$ under the translation that maps H onto M. $\mathcal{H}'_M$ meets the circumcircle of ABC at four (real or not) points. One of them (always real) is the reflection of H about the center of $\mathcal{H}_M$ and this point also lies on $\mathcal{H}_M$. The three other points (one is always real) are those whose Simson lines pass through M.

Recall that these three points are all real when M lies inside the Steiner deltoid $\mathcal{H}_3$.

6.2. *Concurrent Simson lines and cevian lines.* Three Simson lines concurring at Q are parallel to the cevian lines of a certain point M if and only if M lies on the McCay cubic **K003**. If $M = (\alpha : \beta : \gamma)$, this point Q is given by $Q = (b^2c^2\alpha^2(\beta+\gamma) ::)$. Hence it is the barycentric product $tgM \times ctM$. The mapping $M \mapsto Q$ has numerous properties we shall not consider in this paper. See [5].

For example, if we take $M = X_3 = O$, we obtain a point Q which is X_{5562} in ETC. Its first barycentric coordinate is : $a^2S_A^2\left[a^2(b^2+c^2) - (b^2-c^2)^2\right]$ and its SEARCH number is $1.84961021841713\cdots$. This point is the intersection of many lines such as X_2X_{389}, X_3X_{49}, X_4X_{69}, X_5X_{51}, etc.

Figure 8 shows these Simson lines and the corresponding hyperbolas $\mathcal{H}_O$ (here the Jerabek hyperbola), $\mathcal{H}'_O$ meeting the circumcircle at X_{74} and three points Q_1, Q_2, Q_3 whose Simson lines concur at $Q = X_{5562}$.

6.3. *Theorems.* In this section, we characterize the cubics $\mathcal{K}$ for which the asymptotes are parallel to three Simson lines concurring at a certain point $M(\alpha : \beta : \gamma)$. The equation of these three Simson lines is given by

$$\sum_{\text{cyclic}} a^2\Big(\underbrace{\alpha(y-z) - x(\beta-\gamma)}_{A_1}\Big)\Big(\underbrace{\beta(x+z) - y(\alpha+\gamma)}_{A_2}\Big)\Big(\underbrace{\gamma(x+y) - z(\alpha+\beta)}_{A_3}\Big) = 0,$$

which is then more simply rewritten under the form :

$$\sum_{\text{cyclic}} a^2 A_1A_2A_3 = 0.$$

In this case, the equation of the parallels at M to the asymptotes of $\mathcal{K}$ is :

$$\sum_{\text{cyclic}} p\ (w\,A_2 - v\,A_3)A_2A_3 = 0.$$

When we express that these two equations have the same solutions when $(x : y : z)$ is a point at infinity, we obtain three conditions which are linear with respect

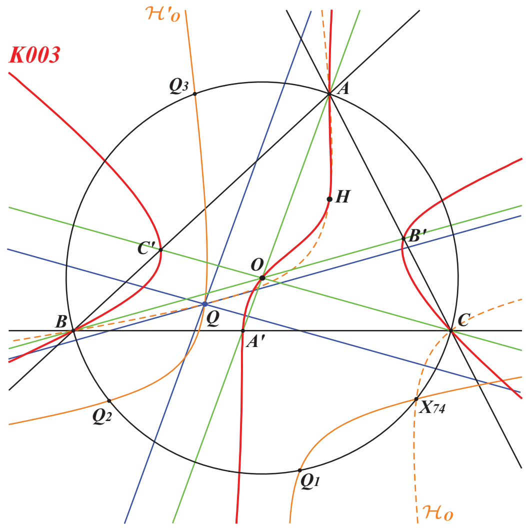

Figure 8. Concurrent Simson lines and cevian lines

to all the variables namely $(p : q : r)$, $(u : v : w)$ and $(\alpha : \beta : \gamma)$. In other words, if two of the three points Ω, P, M are chosen, the coordinates of the third point are given by a system of three linear equations with a corresponding 3×3 matrix generally of rank 3.

Since the system has always the trivial and improper solution $(0 : 0 : 0)$, we will find at least one proper solution if and only if the determinant of each of the three matrices above is zero. This gives three conditions involving two of the three points Ω, P, M. These conditions are

$$(\Omega P) \; : \; \sum_{\text{cyclic}} u\, a^2 S_A (c^2 q - b^2 r) = 0 \iff \sum_{\text{cyclic}} p\, b^2 c^2 (S_B\, v - S_C\, w) = 0, \tag{11}$$

$$\begin{aligned} (\Omega M) \; : \; & \sum_{\text{cyclic}} \alpha\, qr \Big((b^2 - c^2) p - a^2 (q - r) \Big) = 0 \\ \iff & \sum_{\text{cyclic}} a^2 qr \Big(\alpha\,(q - r) + p\,(\beta - \gamma) \Big) = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} (PM) \; : \; & \sum_{\text{cyclic}} \alpha (u + v)(u + w)(S_B\, v - S_C\, w) = 0 \\ \iff & \sum_{\text{cyclic}} a^2 (u + v)(u + w) \Big((\alpha + \beta - \gamma)\, v - (\alpha - \beta + \gamma)\, w \Big) = 0, \end{aligned} \tag{13}$$

Remarks. (1) (ΩP) is linear in Ω and P, (ΩM) and (PM) are linear in M.

(2) (ΩP) identically vanishes when $\Omega = X_6$ or $P = X_4$.

(3) (ΩM) identically vanishes when $\Omega = X_6$ (or when Ω is a midpoint of ABC not giving a proper cubic).

(4) (PM) identically vanishes when $P = X_4$ (or when P is a vertex of the antimedial triangle not giving a proper cubic).

(5) When $P = X_{69}$, the conditions (ΩP) and (PM) show that the points Ω and M must lie on the line $GK = X_2X_6$.

Each time a condition identically vanishes, the system above has one and only one solution since the rank of at least one of the matrices above is 2. This is examined in the next section.

6.4. *Special cases.*

6.4.1. $\Omega = X_6$. When $\Omega = X_6$, the cubic $\mathcal{K}$ is a pivotal isogonal cubic and its asymptotes are parallel to the Simson lines that pass through $M = cP$, the complement of the pivot.

For example, with $P = X_3$ we have $M = cP = X_5$: the asymptotes of the McCay cubic **K003** are the parallels at G to the Simson lines passing through X_5 which are in fact the axes of the Steiner deltoid.

6.4.2. $P = X_4$. When $P = X_4$, the cubic $\mathcal{K}$ is a pivotal isogonal cubic with respect to the orthic triangle. Its asymptotes are parallel to the Simson lines that pass through $M = \Omega \times X_{69}$ (barycentric product). Conversely, if M is given, the pole Ω is that of the isoconjugation that swaps the orthocenter $H = X_4$ and M.

For example, with $M = X_5$ we have $\Omega = X_4 \times X_5 = X_{53}$: the corresponding cubic is the McCay orthic cubic **K049** whose asymptotes are the parallels at X_{51} to the Simson lines passing through X_5 as above.

6.4.3. $\Omega = X_6$ *and* $P = X_4$. This gives the orthocubic **K006** whose asymptotes are parallel to the Simson lines passing through $O = X_3$. In this case, these asymptotes are not concurrent.

6.5. *Interpretation of the three conditions in the general case.*

6.5.1. *The linear conditions.* The conditions (11), (11), (13) above represent actually six equations and four of them are linear with respect to the coordinates of at least one of the three points Ω, P, M. These four equations give the following propositions.

Proposition 17. *For a given pole $\Omega \neq X_6$, the asymptotes of $\mathcal{K}$ are parallel to the Simson lines of a certain point M if and only if its pivot P lies on the line $\mathcal{L}(\Omega, P)$ with equation*

$$\sum_{\text{cyclic}} a^2 S_A(c^2q - b^2r)x = 0.$$

In this case, this point M must lie on the line $\mathcal{L}(\Omega, M)$ with equation

$$\sum_{\text{cyclic}} qr\big((b^2 - c^2)p - a^2(q - r)\big)x = 0.$$

$\mathcal{L}(\Omega, P)$ is the Steiner line of the isogonal conjugate of the infinite point of the trilinear polar of the isogonal conjugate of Ω and therefore passes through X_4.

If Γ_Ω is the circum-conic with perspector Ω, the line $\mathcal{L}(\Omega, M)$ is the conjugated diameter with respect to Γ_Ω of the trilinear polar of the isotomic conjugate of the isogonal conjugate of Ω. Obviously, this line contains the center of Γ_Ω.

Proposition 18. *For a given pivot $P \neq X_4$, the asymptotes of $\mathcal{K}$ are parallel to the Simson lines of a certain point M if and only if its pole Ω lies on the line $\mathcal{L}(P, \Omega)$ with equation*

$$\sum_{\text{cyclic}} b^2c^2(S_B\,v - S_C\,w)x = 0.$$

In this case, this point M must lie on the line $\mathcal{L}(P, M)$ with equation

$$\sum_{\text{cyclic}} (u+v)(u+w)(S_B\,v - S_C\,w)x = 0.$$

$\mathcal{L}(P, \Omega)$ is the trilinear polar of the isogonal conjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ or equivalently the X_4-isoconjugate of P.

$\mathcal{L}(P, M)$ contains cP. It is the trilinear polar of the isoconjugate of the infinite point of the trilinear polar of the barycentric quotient $X_4 \div P$ under the isoconjugation with pole cP.

For example,

- $\mathcal{L}(\Omega = X_2, P)$ is the line through X_4, X_{69}, X_{76}, etc, and $\mathcal{L}(\Omega, P = X_2)$ is the Brocard axis,
- $\mathcal{L}(\Omega = X_2, M)$ and $\mathcal{L}(P = X_2, M)$ coincide into the line X_2, X_6, X_{69}, etc.

6.5.2. *The other conditions.* The remaining two equations are of degree 3 and lead to two cubic curves. They correspond to the choice of a given point M whose isogonal conjugate of isotomic conjugate is denoted gtM, also the barycentric product $M \times X_6$.

Property 1. *For a given point M, there is a cubic $\mathcal{K}$ whose asymptotes are parallel to the Simson lines passing through M if and only if its pole Ω lies on the cubic $\mathcal{K}(M, \Omega)$ which is $ps\mathcal{K}(gtM, X_2, X_6) = ps\mathcal{K}(M \times X_6, X_2, X_6)$ with equation*

$$\sum_{\text{cyclic}} a^2\Big[\alpha\,(y-z) + x\,(\beta-\gamma)\Big]yz = 0.$$

Property 2. *For a given point M, there is a cubic $\mathcal{K}$ whose asymptotes are parallel to the Simson lines passing through M if and only if its pivot P lies on the cubic $\mathcal{K}(M, P)$ which is the anticomplement of $ps\mathcal{K}(gtM, X_2, X_4) = ps\mathcal{K}(M \times X_6, X_2, X_4)$. The equation of $\mathcal{K}(M, P)$ is*

$$\sum_{\text{cyclic}} a^2\Big[(\alpha+\beta-\gamma)\,y - (\alpha-\beta+\gamma)\,z\Big](x+y)(x+z) = 0$$

and that of its complement is very similar to the equation above namely

$$\sum_{\text{cyclic}} a^2 \Big[\alpha\,(y-z) - x\,(\beta-\gamma)\Big]\, yz = 0.$$

Example 1 : When we consider the cubics having their asymptotes parallel to the Simson lines passing through the circumcenter $O = X_3$, we find

- $\mathcal{K}(M = X_3, \Omega)$ = **K260** = $ps\mathcal{K}(X_{184}, X_2, X_6)$ passing through X_6, X_{69}, X_{206}, X_{219}, X_{478}, X_{577}, X_{1249}, X_{2165}.
- $\mathcal{K}(M = X_3, P)$ passing through X_4, X_{20}, X_{68}, X_{254}, X_{315}, X_{2996}. It is the anticomplement of $ps\mathcal{K}(X_{184}, X_2, X_4)$ which contains X_3, X_4, X_{32}, X_{56}, X_{1147}.

Among them, we have the Orthocubic **K006** as already said and also $p\mathcal{K}(X_{69}, X_{315}), p\mathcal{K}(X_{219}, X_{3436}), p\mathcal{K}(X_{577}, X_{20}), p\mathcal{K}(X_{2165}, X_{68}), p\mathcal{K}(X_{32}\times X_{2996}, X_{2996})$.

All these cubics have three asymptotes parallel to the Simson lines passing through X_3 and also to the asymptotes of the Orthocubic **K006**.

The most interesting is probably **K690** = $p\mathcal{K}(X_{2165}, X_{68})$ since it contains X_4, X_{68}, X_{485}, X_{486}, X_{637}, X_{638}. See Figure 9.

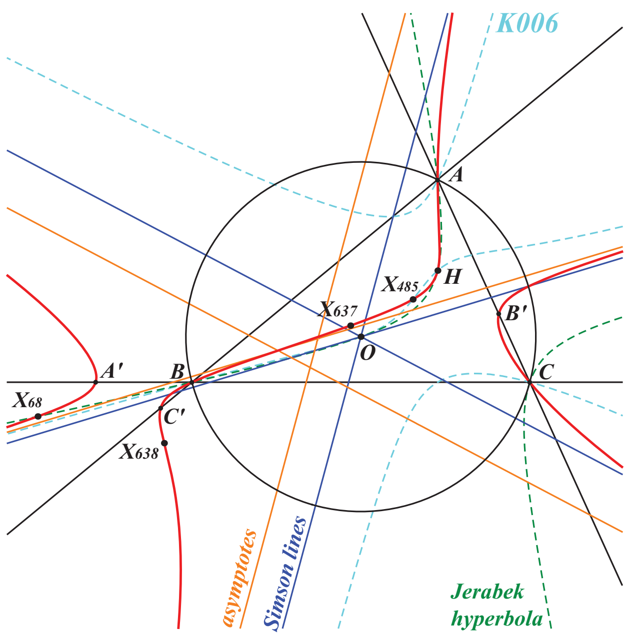

Figure 9. **K690** = $p\mathcal{K}(X_{2165}, X_{68})$

Example 2 : When we consider the cubics having their asymptotes parallel to the Simson lines passing through the incenter $I = X_1$, we find

- $\mathcal{K}(M = X_1, \Omega) = ps\mathcal{K}(X_{31}, X_2, X_6)$ passing through X_6, X_9, X_{19}, X_{478}.

- $\mathcal{K}(M = X_1, P)$ passing through X_4, X_8. It is the anticomplement of $ps\mathcal{K}(X_{31}, X_2, X_1)$ which contains X_1, X_3, X_{56}.

There are two interesting related cubics namely **K691** = $p\mathcal{K}(X_{19}, X_4)$ and **K692** = $p\mathcal{K}(X_6, X_8)$ generating a pencil which contains the decomposed cubic which is the union of the line X_4, X_9 and the circum-conic passing through X_1 and X_4.

Example 3 : The Simson lines passing through the nine point center X_5 form a (decomposed) stelloid and the cubics having their asymptotes parallel to these Simson lines are equilateral cubics.

- $\mathcal{K}(M = X_5, \Omega)$ = **K307** = $ps\mathcal{K}(X_{51}, X_2, X_6)$ passes through X_6, X_{53}, X_{216}, X_{1249}.
- $\mathcal{K}(M = X_5, P)$ passes through X_3, X_4, X_{20}, X_{1670}, X_{1671}. It is the anticomplement of **K026** = $ps\mathcal{K}(X_{51}, X_2, X_3)$ which contains X_3, X_4, X_5.

The most remarkable corresponding cubics are the McCay cubic **K003** = $p\mathcal{K}(X_6, X_3)$, the McCay orthic cubic **K049** = $p\mathcal{K}(X_{53}, X_4)$ and **K096** = $p\mathcal{K}(X_{216}, X_{20})$.

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The Cevian Simson Transformation

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Abstract. We study a transformation whose origin lies in the relation between concurrent Simson lines parallel to cevian lines as seen in [4].

1. Introduction

Let $M = (u : v : w)$ be a point. In [4] we raised the following question: to find a point P such that the three Simson lines passing through P are parallel to the three cevian lines of M.

The answer to this question is that M must lie on the McCay cubic **K003** and, in this case, the corresponding point P is given by

$$P = \left(\frac{u^2(v+w)}{a^2} : \frac{v^2(w+u)}{b^2} : \frac{w^2(u+v)}{c^2}\right),$$

In this case one can find an isogonal pivotal cubic whose asymptotes are also parallel to the cevian lines of M.

We note the strong connection with the cubic **K024** whose equation is

$$\sum_{\text{cyclic}} \frac{x^2(y+z)}{a^2} = 0.$$

When M lies on **K024**, P lies on the line at infinity.

If we denote by gM, tM, cM, aM the isogonal conjugate, the isotomic conjugate, the complement, the anticomplement of M respectively then $P = tgM \times ctM$ where $\times$ is the barycentric product. $\mathcal{L}(M)$ will denote the trilinear polar of M.

In this paper, we extend to the whole plane the mapping CST that sends M onto P which we call the Cevian Simson Transformation.

2. Properties of CST

2.1. *Singular points and consequences.*

Proposition 1. CST *has six singular points which are A, B, C each counted twice.*

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This is obvious from the coordinates of P. It follows that CST transforms any curve $\mathcal{C}$ of degree n into a curve $\mathcal{C}'$ of degree $3n$ which must be reduced according to the number and the nature of the singular points on the original curve.

More precisely, let $G_aG_bG_c$ be the antimedial triangle.

(1) If $\mathcal{C}$ contains only A and is not tangent to G_bG_c, the degree of $\mathcal{C}'$ is $3n-1$,
(2) If $\mathcal{C}$ contains A, B, C and is not tangent at these points to a sideline of $G_aG_bG_c$, the degree of $\mathcal{C}'$ is $3n-3$.
(3) If $\mathcal{C}$ contains A, B, C and has a double contact at these points to a sideline of $G_aG_bG_c$, the degree of $\mathcal{C}'$ is $3n-6$.

In particular,

(4) The transform of a line is generally a cubic which must be tangent to the sidelines of ABC. See §3 below.
(5) The transform of a circum-conic is generally a circum-cubic. See §4 below.
(6) The transform of a circum-cubic tangent at A, B, C to the sidelines of $G_aG_bG_c$ is generally a circum-cubic.

A very special case: the Steiner ellipse is tangent A, B, C to the sidelines of $G_aG_bG_c$ hence its transform is a "curve" of degree 0, namely a point. This point is actually X_{76}, the isotomic conjugate of the Lemoine point $K = X_6$. Note that X_{76} is also CST(X_2).

Consequently, the curve $\mathcal{C}'$ above will have a singular point at X_{76} whose multiplicity is $2n$ lowered according to the singular points on $\mathcal{C}$ as above. The nature of this singular point, i.e., the reality of the nodal tangents, will depend of the nature of the intersections of $\mathcal{C}$ and the Steiner ellipse. If $\mathcal{C}$ contains X_2, the multiplicity must be increased.

This will be developed in the following sections.

2.2. *Fixed points.*

Proposition 2. CST *has one and only one fixed point which is the orthocenter $H = X_4$ of ABC.*

Indeed, M is a fixed point of CST if and only if $P = M \iff ctM = X_6 \iff M = X_4$. It follows that the transform $\mathcal{C}'$ of any curve $\mathcal{C}$ passing through H also passes through H.

2.3. *Some special* CST *images.*

G_a, G_b, G_c are transformed into A, B, C.

The infinite points of the sidelines of ABC are transformed into the traces of the de Longchamps axis $\mathcal{L}(X_{76})$ on these same sidelines.

The infinite points of **K003** are transformed into the cusps of the Steiner deltoid $\mathcal{H}_3$.

The infinite points of an equilateral cubic whose asymptotes are not parallel to those of **K024** are transformed into the cusps of a deltoid inscribed in ABC.

If these asymptotes are parallel to those of **K024**, their infinite points are transformed into the infinite points of the sidelines of ABC.

2.4. *Pre-images of a point.* We already know that X_{76} has infinitely many pre-images which are $G = X_2$ and the points on the Steiner ellipse and also $H = X_4$, being a fixed point, has already at least one pre-image namely itself.

We consider a point P different of X_{76} and not lying on a sideline of ABC or $G_aG_bG_c$. We wish to characterize all the points M such that CST(M) is P.

When expressing that CST$(M) = P$ we obtain three equations representing three nodal circum-cubic curves with nodes at A, B, C. Their isogonal transforms are three conics each passing through one vertex of ABC. These conics have generally three common points hence P has three pre-images M_1, M_2, M_3.

The nature of these points (real or not, distinct or not) depends of the position of P with respect to the sidelines of ABC, the cevian lines of X_{76} and mainly the Ehrmann-MacBeath cubic **K244** which is the locus of the cusps of all the deltoids inscribed in ABC and also the CST image of the line at infinity. For more informations about **K244**, see [1].

More precisely, see Figure 1,

(i) when P lies inside the yellow region (excluding its "edges" mentioned above) there are three real distinct points M_1, M_2, M_3;

(ii) when M lies outside, there is only one real point;

(iii) when P lies on **K244** (but not on the other lines above), there is only one point (counted twice) and this point lies on the line at infinity. For example, when $P = X_{764}$ we obtain X_{513}. This will be detailed in section 3.

The net generated by the three conics above contains the circum-conic which is the isogonal transform of the line passing through X_2 and gtP always defined since $gtP \neq X_2$. This line must contain the points M_1, M_2, M_3.

On the other hand, each cubic which is the union of one conic and the opposite sideline of ABC must contain the isogonal conjugates of the points M_1, M_2, M_3. Hence the three isogonal transforms of these three cubics contain M_1, M_2, M_3. These three latter cubics generate a pencil which contains several simple cubics and, in particular, the $n\mathcal{K}_0(\Omega, \Omega)$ where Ω is the isogonal conjugate of the infinite point of the trilinear polar of tP, a point clearly on the circumcircle of ABC.

This cubic is a member of the class **CL026** and has always three concurring asymptotes and is tritangent at A, B, C to the Steiner ellipse unless it decomposes. See Example 3 below.

The equation of $n\mathcal{K}_0(\Omega, \Omega)$

$$\sum_{\text{cyclic}} (v - w)\, \frac{x^2(y + z)}{a^2} = 0$$

clearly shows that its CST image is the line X_2, P. See §5 for more details.

Furthermore, if the coordinates of $X_2 + \lambda\, gtP$ are inserted into the equation of $n\mathcal{K}_0(\Omega, \Omega)$ then the 3rd degree polynomial in λ has no term in λ^2. Hence the sum of the three values corresponding to the points M_1, M_2, M_3 is zero. It follows that the isobarycenter of M_1, M_2, M_3 is X_2.

In conclusion and generally speaking, we have

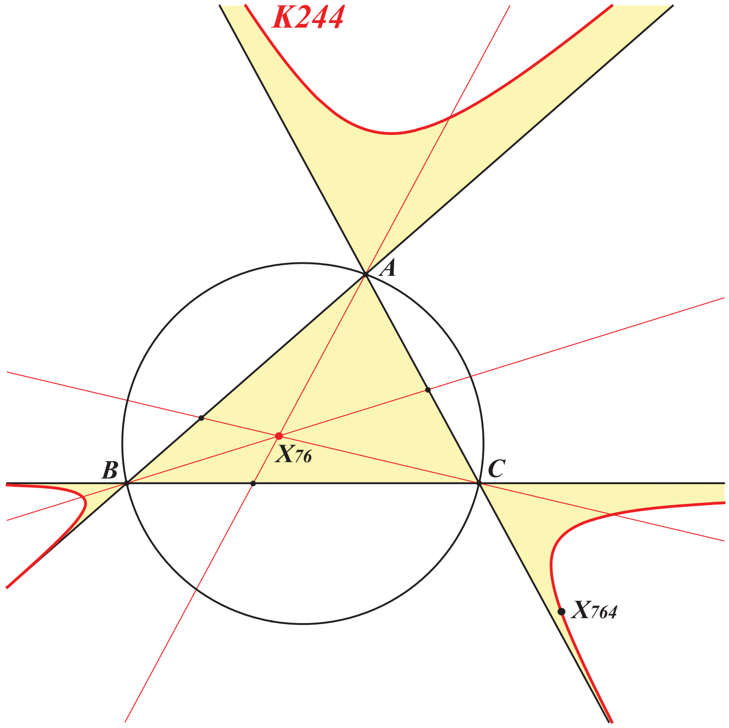

Figure 1. Regions delimited by the cubic **K244**

Proposition 3. (a) *The pre-images* M_1, M_2, M_3 *of a point* $P \neq X_{76}$ *are the intersections of the line joining* X_2 *and* gtP *with the cubic* $n\mathcal{K}_0(\Omega, \Omega)$.
(b) *The* CST *image of this line is the line* X_2, P.
(c) *The centroid* G *of* ABC *is the isobarycenter of* M_1, M_2, M_3.

Example 1. *With* $P = X_4$, *we find the Euler line and* $n\mathcal{K}_0(X_{112}, X_{112})$. *Hence the pre-images of* X_4 *are* X_4, X_{1113}, X_{1114}.

Example 2. *With* $P = X_3$, *we find the line through* X_2, X_{98}, X_{110}, *etc, and* $n\mathcal{K}_0(X_{112}, X_{112})$ *again. One of the pre-images is* X_{110} *and the other are real when* ABC *is acute angled.*

Example 3. *The cubic* $n\mathcal{K}_0(\Omega, \Omega)$ *contains* X_2 *if and only if* P *lies on the line* X_2X_{76}. *In this case, it splits into the Steiner ellipse and the line* X_2X_6.

2.5. CST *images of cevian triangles.* Let $P_aP_bP_c$ be the cevian triangle of $P = (p : q : r)$ and let $Q_aQ_bQ_c$ be its anticomplement.

We have $P_a = (0 : q : r)$, $Q_a = (q + r : -q + r : q - r)$. It is easy to see that $\mathrm{CST}(P_a) = \mathrm{CST}(Q_a) = (0 : c^2q : b^2r) = R_a$. The points R_b, R_c are defined likewise and these three points are the vertices of the cevian triangle of tgP. Hence,

Proposition 4. CST *maps the vertices of the cevian triangle of* P *and the vertices of its anticomplement to the vertices of the same cevian triangle, that of* tgP.

2.6. CST *images of some common triangle centers.* Table 1 gives a selection of some CST images. A (6-9-13)-search number is given for each unlisted point in ETC.

Table 1. CST images of some common triangle centers

M	CST(M)	M	CST(M)	M	CST(M)
X_1	X_{10}	X_2	X_{76}	X_3	X_{5562}
X_4	X_4	X_5	4.342332195522807	X_6	X_{39}
X_7	X_{85}	X_8	X_{341}	X_9	5.493555510910763
X_{10}	5.329221045166122	X_{11}	4.196262646186253	X_{12}	2.698123376290196
X_{13}	0.1427165061182335	X_{14}	5.228738830014126	X_{15}	4.707520749612165
X_{16}	-15.70210201702076	X_{17}	2.708683938139388	X_{18}	12.30617330317703

Peter Moses has kindly provided all the pairs $\{M$,CST$(M)\}$ = $\{X_i, X_j\}$ in the ETC (up to X_{5573}) for these $\{$i , j$\}$. Apart from those listed in Table 1 above and excluding X_2 and all the points on the Steiner ellipse for which CST(M) is X_{76}, he has found

i	66	69	100	101	110	513	651	879	925
j	2353	3926	8	3730	3	764	348	5489	847
i	1113	1114	1379	1380	1576	3952			
j	4	4	3557	3558	3202	1089			

3. CST images of lines

Let $\mathcal{L}$ be the line with equation $px + qy + rz = 0$ and trilinear pole $Q = (qr : rp : pq)$.

3.1. *The general case.* In general, the CST image of $\mathcal{L}$ is a nodal cubic with node X_{76} which is tangent to the sidelines of ABC at the traces A_2, B_2, C_2 of $\mathcal{L}(tgQ)$ and meeting these lines again at the traces A_1, B_1, C_1 of $\mathcal{L}(tgtatQ)$.

Indeed, if $\mathcal{L}$ meets BC at $U = (0 : r : -q)$ and G_bG_c at $U' = (q - r : -p : p)$ then $\mathrm{CST}(U) = A_2 = (0 : c^2r : -b^2q)$ and $\mathrm{CST}(U') = A_1 = (0 : c^2(p+q-r) : -b^2(p - q + r))$.

Note that the CST image of the infinite point of $\mathcal{L}$ is the point

$$\left(\frac{(q-r)^3}{a^2} : \frac{(r-p)^3}{b^2} : \frac{(p-q)^3}{c^2}\right)$$

on the cubic. It is also on **K244** as seen below.

The most remarkable example is obtained when $Q = X_2$. Since $\mathcal{L}$ is the line at infinity and since the two trilinear polars coincide into the de Longchamps axis (the isotomic transform of the circumcircle of ABC), we find the cubic **K244** meeting the sidelines of ABC at three inflexion points on the curve (see Figure 2).

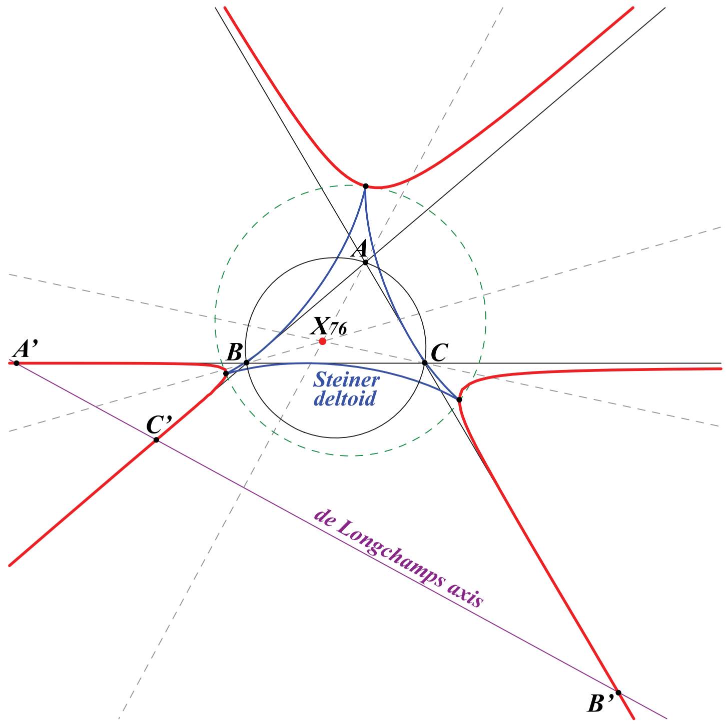

Figure 2. The cubic **K244**

3.2. *Special cases.*

(1) If $\mathcal{L}$ contains X_2 and another point M, the cubic is the line $\mathcal{L}'$ passing through X_{76} and CST(M) counted three times.

More precisely, if $\mathcal{L}$ meets the Kiepert hyperbola again at E then $\mathcal{L}'$ is the line $X_{76}E$.

(2) If $\mathcal{L}$ is tangent to the Steiner ellipse, the cubic is cuspidal (with cusp X_{76}) and the lines $\mathcal{L}(tgQ)$, $\mathcal{L}(tgtatQ)$ envelope the circum-conic and the in-conic with same perspector X_{76} respectively.

4. CST images of circum-conics

Let $\mathcal{C}(Q)$ be the circum-conic with perspector $Q = (p : q : r) \neq X_2$ (to eliminate the Steiner ellipse case) and equation $pyz + qzx + rxy = 0$.

4.1. *The general case.* In general, the CST image of $\mathcal{C}(Q)$ is a nodal circum-cubic with node N_Q passing through X_{76} which turns out to be a $ps\mathcal{K}$ as in [2]. This cubic has the following properties.

(1) Its pseudo-pivot $P_Q = \left(\dfrac{1}{a^2(-p+q+r)} : \cdots : \cdots\right)$ is $tgtaQ$.

(2) Its pseudo-isopivot $P_Q^* = \left(\dfrac{p^2}{a^2} : \cdots : \cdots\right)$ is tgQ^2.

(3) Its node $N_Q = \left(\dfrac{p}{a^2(-p+q+r)} : \cdots : \cdots\right)$ is $P_Q \times Q$.

(4) Its pseudo-pole $\Omega_Q = \left(\dfrac{p^2}{a^4(-p+q+r)} : \cdots : \cdots\right)$ is $P_Q \times P_Q^*$ or $N_Q \times tgQ$, This node is obtained when the intersections of $C(Q)$ with the line through its center and X_2 are transformed under CST.

(5) The isoconjugate X_{76}^* of X_{76} is $Q \times N_Q = \left(\dfrac{p^2}{a^2(-p+q+r)} : \cdots : \cdots\right)$, obviously on the cubic.

The most remarkable example is obtained when $Q = X_6$ since $\mathcal{C}(Q)$ is the circumcircle (O) of ABC. In this case we find the (third) Musselman cubic **K028**, a stelloid which is $ps\mathcal{K}(X_4, X_{264}, X_3)$. See details in [1] and Figure 3.

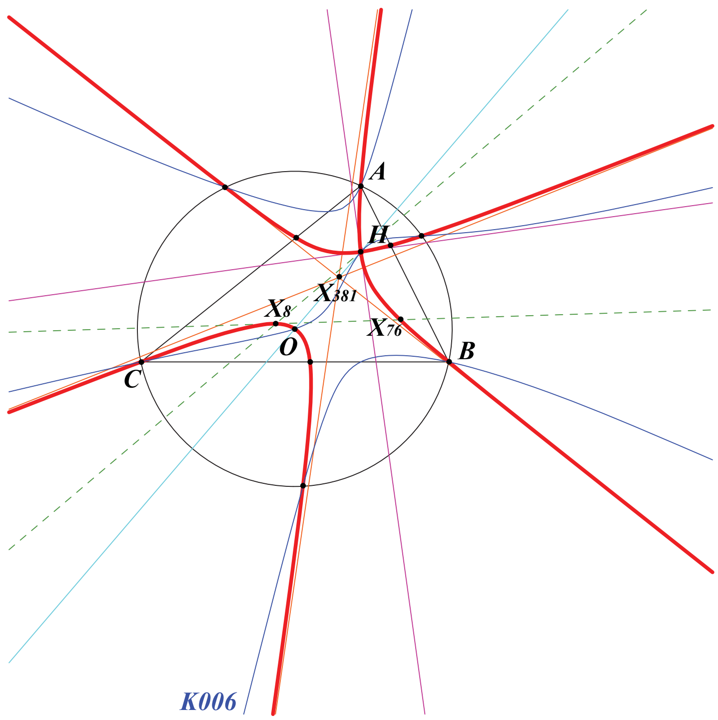

Figure 3. The cubic **K028**

4.2. *Special cases.* The CST image of $\mathcal{C}(Q)$ is a cuspidal circum-cubic if and only if Q lies on two cubics which are the complement of **K196** (the isotomic transform of **K024** with no remarkable center on it) and **K219** (the complement of **K015**) containing X_2, X_{1645}, X_{1646}, X_{1647}, X_{1648}, X_{1649}, X_{1650}. In this latter case, the cusp lies on **K244**.

Figure 4 shows the cubic which is the CST image of $\mathcal{C}(X_{1646})$, a circum-cubic passing through X_{513}, X_{668}, X_{891}, X_{1015}. The cusp is $X_{764} = \text{CST}(X_{513})$. Since X_{891} is a point at infinity, its image also lies on **K244**.

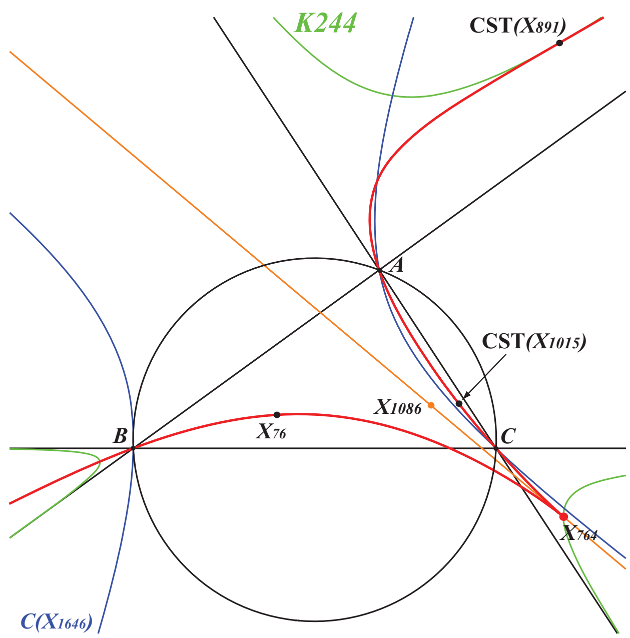

Figure 4. A cuspidal cubic, CST image of $\mathcal{C}(X_{1646})$

4.3. CST *images of some usual circum-conics.* Any $\mathcal{C}(Q)$ which is a rectangular hyperbola must have its perspector Q on the orthic axis, the trilinear polar of X_4. Its CST image $\mathcal{K}(Q)$ is a nodal cubic passing through X_{76} and X_4. Furthermore, its node lies on **K028**, its pseudo-pivot lies on the Steiner ellipse, its pseudo-isopivot lies on the inscribed conic with perspector X_{2052}. The pseudo-pole lies on a complicated quartic.

Table 2 gives a selection of such hyperbolas.

Table 2. CST images of some usual rectangular hyperbolas

Q	$\mathcal{C}(Q)$	$\mathcal{K}(Q)$	N_Q	other centers on $\mathcal{K}(Q)$
X_{523}	Kiepert	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$	X_{76}	
X_{647}	Jerabek	$ps\mathcal{K}(X_{520}, X_{99}, X_3)$	X_3	X_{39}, X_{2353}
X_{650}	Feuerbach	$ps\mathcal{K}(X_{4397}, X_{668}, X_4)$	X_8	X_{10}, X_{85}, X_{341}

Remark. When M lies on the Jerabek hyperbola, the points X_3, M and $\text{CST}(M)$ are collinear. This is also true when M lies on the circumcircle.

More generally, for any point N_Q on **K028**, the points M, CST(M), N_Q are collinear if and only if M lies on two circum-conics γ_1, γ_2.

γ_1 is the isogonal conjugate of the parallel δ_1 at X_3 to the line X_4N_Q. γ_1 is obviously a rectangular hyperbola.

γ_2 is the isogonal conjugate of the perpendicular δ_2 at X_3 to the line X_4N_Q. The perspector of γ_2 lies on the circum-conic passing through X_2, X_6.

Note that δ_2 envelopes the Kiepert parabola and that δ_1, δ_2 meet on the Stammler strophoid **K038**.

The CST images of γ_1, γ_2 are two nodal cubics $ps\mathcal{K}$ with nodes $N_1 = N_Q$, N_2 on the Kiepert hyperbola respectively.

5. CST images of some circum-cubics

5.1. CST *images of the cubics* $n\mathcal{K}_0(P,P)$. If $P = (p : q : r)$, the cubic $n\mathcal{K}_0(P,P)$ has an equation of the form

$$\sum_{\text{cyclic}} \frac{x^2(y+z)}{p} = 0 \iff \sum_{\text{cyclic}} \frac{a^2}{p} \times \frac{x^2(y+z)}{p} = 0,$$

which shows that its CST image is the line $\mathcal{L}(tgP)$.

Recall that $n\mathcal{K}_0(P,P)$ is a member of the class **CL026**. It is a cubic having three asymptotes concurring at X_2.

With $P = X_2, X_6, X_{1989}$ we find the cubics **K016**, **K024**, **K064** whose CST images are the de Longchamps axis, the line at infinity, the perpendicular bisector of OH respectively.

The cubics $n\mathcal{K}_0(X_{112}, X_{112})$, $n\mathcal{K}_0(X_{1576}, X_{1576})$, $n\mathcal{K}_0(X_{32}, X_{32})$ give the Euler line, the Brocard axis, the Lemoine axis.

With $P = gtX_{107}$ we have the cubic whose CST image is the line HK. See Figure 5.

5.2. CST *images of the cubics* $c\mathcal{K}(\#P, P^2) = n\mathcal{K}(P^2, P^2, P)$. If $P = (p : q : r)$, the cubic $c\mathcal{K}(\#P, P^2)$ has an equation of the form

$$\sum_{\text{cyclic}} p^2\, x\, (ry - qz)^2 = 0.$$

It is a nodal cubic with node P. Since it is tangent at A, B, C to the sidelines of the antimedial triangle, its CST image must be a cubic curve with node CST(P). This cubic is tangent to the sidelines of ABC at their intersections with $\mathcal{L}(tgP^2)$ and meets these sidelines again on $\mathcal{L}(tgctP)$.

The most remarkable example is obtained when $P = X_2$ since the CST image of the nodal Tucker cubic **K015** $= c\mathcal{K}(\#X_2, X_2)$ is the cubic **K244**. In this case, the two trilinear polars coincide as already point out above.

We conclude with a summary of interesting CST images.

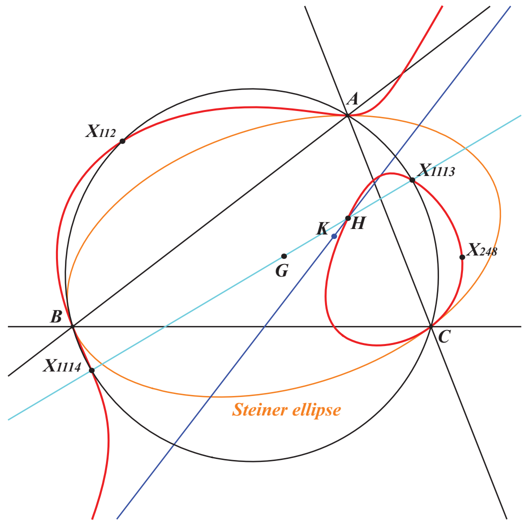

Figure 5. The cubic $n\mathcal{K}_0(P,P)$ with $P = gtX_{107}$

	$\mathcal{C}$	CST($\mathcal{C}$)
Lines	line at infinity	**K244**
	Euler line	line X_4, X_{69}, X_{76}, etc
Conics	Steiner ellipse	X_{76}
	Circumcircle	**K028**
	Kiepert hyperbola	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$
	Jerabek hyperbola	$ps\mathcal{K}(X_{520}, X_{99}, X_3)$
	Feuerbach hyperbola	$ps\mathcal{K}(X_{4397}, X_{668}, X_4)$
Cubics	**K024**	line at infinity
	K015	**K244**
	K242	$ps\mathcal{K}(X_{850} \times X_{76}, X_{670}, X_{76})$
Others	**Q066**	Kiepert hyperbola

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Two Pairs of Archimedean Circles in the Arbelos

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Abstract. We construct four circles congruent to the Archimedean twin circles in the arbelos.

Consider an arbelos formed by semicircles (O_1), (O_2), and (O) of radii a, b, and $a+b$. The famous Archimedean twin circles associated in the arbelos have equal radii $\frac{ab}{a+b}$ (see [2, 3]).

Let CD be the dividing line of the smaller semicircles, and extend their common tangent PQ to intersect (O) at T_a and T_b.

Theorem 1. *Let A' and B' be the orthogonal projections of D on the tangents to (O) at T_a and T_b respectively. The circles with diameters DA' and DB' are congruent to the Archimedean twin circles.*

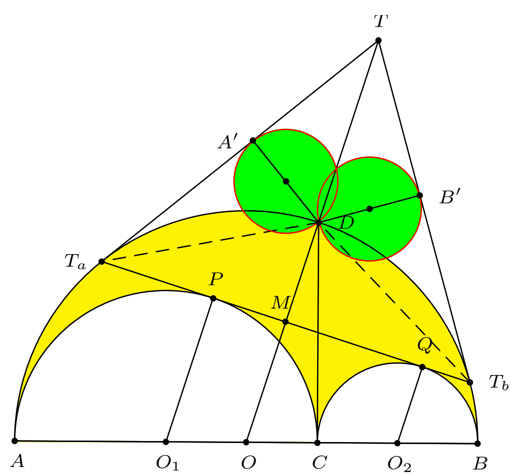

Figure 1

Proof. Let the tangents at T_a and T_b intersect at T. Since OT is the perpendicular bisector of T_aT_b, it intersects the semicircle (O) at the midpoint D of the arc T_aT_b (see [3, §5.2.1]). Since O_1P, OM and O_2Q are parallel, and $O_1P = OO_2 = a$, $O_2Q = O_1O = b$,

$$OM = \frac{a}{a+b}\cdot O_1P + \frac{b}{a+b}\cdot O_2Q = \frac{a^2+b^2}{a+b} \implies DM = OD - OM = \frac{2ab}{a+b}.$$

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Now, $\angle DT_aT = \angle DT_bT_a = \angle DT_aT_b$. Therefore, T_aD bisects angle TT_aT_b. Similarly, T_bD bisects angle TT_bT_a, and D is the incenter of triangle TT_aT_b. It follows that $DA' = DB' = DM$, and the circles with DA' and DB' are congruent to the Archimedean twin circles. □

Remark. The circle with DM as diameter is the Archimedean circle (A_3) in [2] (or (W_4) in [1]).

Theorem 2. *Let A_1A_2 and B_1B_2 be tangents to the smaller semicircles with A_1, B_1 on the line AB and $A_1A_2 = a$, $B_1B_2 = b$. If H and K are the midpoints of the semicircles (O_1) and (O_2) respectively, and $A'' = CH \cap A_1B_2$, $B'' = CK \cap B_1A_2$, then the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles.*

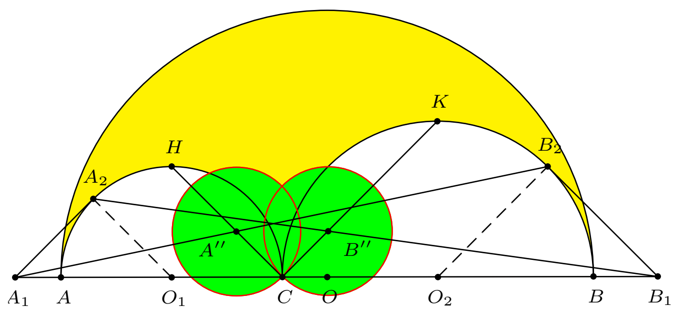

Figure 2

Proof. Clearly, $\angle A''CA_1 = \angle HCO_1 = 45°$. Since $B_1B_2 = O_2B_2 = b$, $\angle B_2B_1O_2 = 45°$, the lines CA'' and B_1B_2 are parallel. Also, $B_1O_2 = \sqrt{2}b$. Similarly, $A_1O_1 = \sqrt{2}a$, and $A_1B_1 = (\sqrt{2}+1)(a+b)$. Therefore,

$$CA'' = B_1B_2 \cdot \frac{A_1C}{A_1B_1} = b \cdot \frac{(\sqrt{2}+1)a}{(\sqrt{2}+1)(a+b)} = \frac{ab}{a+b}.$$

Similarly, $CB'' = \frac{ab}{a+b}$. Therefore, the circles through C with centers A'' and B'' are congruent to the Archimedean twin circles. □

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On Some Triads of Homothetic Triangles

Gotthard Weise

Abstract. To a given reference triangle Δ and three directions $\alpha,\ \beta,\ \gamma$ we construct four triads of homothetic triangles and investigate relations between their homothetic centers, centroids, midway triangles, medial triangles and areas.

1. Two triads of homothetic triangles

Given an arbitrary triangle $\Delta = ABC$ with sidelines $a,\ b,\ c$ and an ordered set $\{\alpha,\ \beta,\ \gamma\}$ of three directions in the plane of Δ. Let

$$\begin{array}{llll} p_1 = \alpha\beta\gamma, & p_2 = \gamma\alpha\beta, & p_3 = \beta\gamma\alpha, & \text{and} \\ p^1 = \gamma\beta\alpha, & p^2 = \beta\alpha\gamma, & p^3 = \alpha\gamma\beta & \end{array}$$

be the even and odd permutations of these directions. Each such permutation $p = \pi_1\pi_2\pi_3$ defines three lines with directions π_1 at A, π_2 at B and π_3 at C, which are the sidelines of two triads T_Δ and T^Δ of homothetic triangles $\Delta_i = U_iV_iW_i$ and $\Delta^i = U^iV^iW^i$ (i = 1, 2, 3) with angles U, V, W. The assignment of the indexed symbols U, V, W to the vertices of these triangles is chosen so that homologous vertices have the same symbol (see Figure 1).

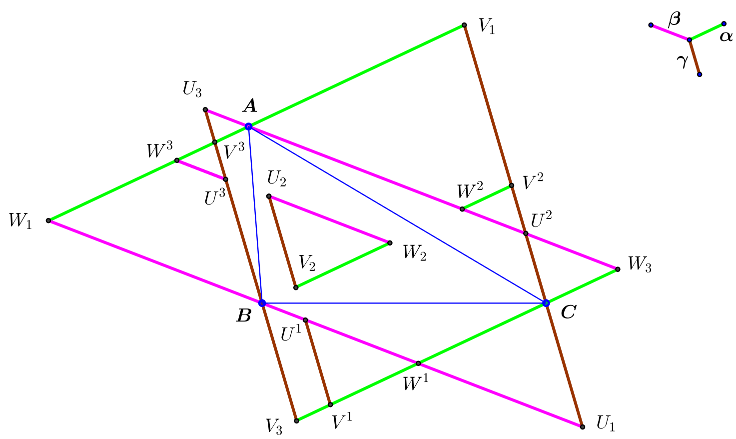

Figure 1

We shall consider some properties of these triads and relationships with other triads of triangles.

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2. Coordinate representations of geometric objects

Geometric objects are described in this paper by homogeneous barycentric coordinates with reference to the triangle Δ. $P = (u : v : w)$ is a point, $l = [p : q : r]$ a line. For a triangle given by its vertices we use the matrix representation (round brackets) with vertice coordinates in the rows. The same triangle can be represented in another matrix form (square brackets), where the rows mean the coordinates of the lines.

We shall regard each direction as a point on the line at infinity with the same name. Then the ordered direction triple $(\alpha,\ \beta, \gamma)$ has the matrix form

$$D = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

with vanishing row sums. By suitable factors in each row it is possible, that not only the row sums of D vanish, but also the column sums, and that all cofactors are equal of unity. In order to verificate analytically the propositions in the sections below, we shall use some other properties of such matrices (see [2]):

$$\beta_1 - \gamma_2 = \gamma_3 - \alpha_1 = \alpha_2 - \beta_3 = \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 =: \lambda_1, \tag{1}$$
$$\gamma_2 - \alpha_3 = \alpha_1 - \beta_2 = \beta_3 - \gamma_1 = \alpha_3\beta_2\gamma_1 - \alpha_1\beta_3\gamma_2 =: \lambda_2, \tag{2}$$
$$\alpha_3 - \beta_1 = \beta_2 - \gamma_3 = \gamma_1 - \alpha_2 = \alpha_2\beta_1\gamma_3 - \alpha_3\beta_2\gamma_1 =: \lambda_3; \tag{3}$$
$$\beta_3 - \gamma_2 = \gamma_1 - \alpha_3 = \alpha_2 - \beta_1 = \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2 =: \mu_1, \tag{4}$$
$$\gamma_2 - \alpha_1 = \alpha_3 - \beta_2 = \beta_1 - \gamma_3 = \alpha_3\beta_1\gamma_2 - \alpha_1\beta_2\gamma_3 =: \mu_2, \tag{5}$$
$$\alpha_1 - \beta_3 = \beta_2 - \gamma_1 = \gamma_3 - \alpha_2 = \alpha_1\beta_2\gamma_3 - \alpha_2\beta_3\gamma_1 =: \mu_3 \tag{6}$$

with

$$\sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} \mu_i = 0 \qquad \text{and} \qquad \sum_{k=1}^{3} \mu_i^2 = 6 + \sum_{k=1}^{3} \lambda_i^2\ ; \tag{7}$$

$$\alpha_2\alpha_3 - \beta_3\gamma_2 = \beta_2\beta_3 - \gamma_3\alpha_2 = \gamma_2\gamma_3 - \alpha_3\beta_2 =: \xi_1, \tag{8}$$
$$\alpha_3\alpha_1 - \beta_1\gamma_3 = \beta_3\beta_1 - \gamma_1\alpha_3 = \gamma_3\gamma_1 - \alpha_1\beta_3 =: \xi_2, \tag{9}$$
$$\alpha_1\alpha_2 - \beta_2\gamma_1 = \beta_1\beta_2 - \gamma_2\alpha_1 = \gamma_1\gamma_2 - \alpha_2\beta_1 =: \xi_3; \tag{10}$$
$$\alpha_2\alpha_3 - \beta_2\gamma_3 = \beta_2\beta_3 - \gamma_2\alpha_3 = \gamma_2\gamma_3 - \alpha_2\beta_3 =: \eta_1, \tag{11}$$
$$\alpha_3\alpha_1 - \beta_3\gamma_1 = \beta_3\beta_1 - \gamma_3\alpha_1 = \gamma_3\gamma_1 - \alpha_3\beta_1 =: \eta_2, \tag{12}$$
$$\alpha_1\alpha_2 - \beta_1\gamma_2 = \beta_1\beta_2 - \gamma_1\alpha_2 = \gamma_1\gamma_2 - \alpha_1\beta_2 =: \eta_3. \tag{13}$$

Furthermore, for each row i of D,

$$\sum_{k=1}^{3} \left(\prod_{j \neq i} d_{jk} \prod_{j \neq k} d_{ij} \right) = 1, \tag{14}$$

and for each column k,

$$\sum_{i=1}^{3}\left(\prod_{j\neq i} d_{jk}\prod_{j\neq k} d_{ij}\right)=1. \tag{15}$$

Here is an example:

$$D=\begin{pmatrix}1 & 2 & -3\\ 1 & 3 & -4\\ -2 & -5 & 7\end{pmatrix}.$$

The 9 lines at the points $A,\ B,\ C$ with the directions $\alpha,\ \beta,\ \gamma$ have following coordinate representations:

	A	B	C
α	$\alpha_A=[0:-\alpha_3:\alpha_2]$	$\alpha_B=[\alpha_3:0:-\alpha_1]$	$\alpha_C=[-\alpha_2:\alpha_1:0]$
β	$\beta_A=[0:-\beta_3:\beta_2]$	$\beta_B=[\beta_3:0:-\beta_1]$	$\beta_C=[-\beta_2:\beta_1:0]$
γ	$\gamma_A=[0:-\gamma_3:\gamma_2]$	$\gamma_B=[\gamma_3:0:-\gamma_1]$	$\gamma_C=[-\gamma_2:\gamma_1:0]$

From this it is easy to determine the matrix forms of the triangles Δ_i and Δ^i in normalized barycentric coordinates (row sums are equal of unity):

$$\Delta_1=\begin{bmatrix}\alpha_A\\ \beta_B\\ \gamma_C\end{bmatrix}\cong\begin{pmatrix}\beta_B\cap\gamma_C\\ \gamma_C\cap\alpha_A\\ \alpha_A\cap\beta_B\end{pmatrix}=\begin{pmatrix}U_1\\ V_1\\ W_1\end{pmatrix}=\begin{pmatrix}\beta_1\gamma_1 & \beta_1\gamma_2 & \beta_3\gamma_1\\ \gamma_1\alpha_2 & \gamma_2\alpha_2 & \gamma_2\alpha_3\\ \alpha_3\beta_1 & \alpha_2\beta_3 & \alpha_3\beta_3\end{pmatrix},$$

$$\Delta_2=\begin{bmatrix}\gamma_A\\ \alpha_B\\ \beta_C\end{bmatrix}\cong\begin{pmatrix}\beta_C\cap\gamma_A\\ \gamma_A\cap\alpha_B\\ \alpha_B\cap\beta_C\end{pmatrix}=\begin{pmatrix}U_2\\ V_2\\ W_2\end{pmatrix}=\begin{pmatrix}\beta_1\gamma_2 & \beta_2\gamma_2 & \beta_2\gamma_3\\ \gamma_3\alpha_1 & \gamma_2\alpha_3 & \gamma_3\alpha_3\\ \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_3\beta_1\end{pmatrix},$$

$$\Delta_3=\begin{bmatrix}\beta_A\\ \gamma_B\\ \alpha_C\end{bmatrix}\cong\begin{pmatrix}\beta_A\cap\gamma_B\\ \gamma_B\cap\alpha_C\\ \alpha_C\cap\beta_A\end{pmatrix}=\begin{pmatrix}U_3\\ V_3\\ W_3\end{pmatrix}=\begin{pmatrix}\beta_3\gamma_1 & \beta_2\gamma_3 & \beta_3\gamma_3\\ \gamma_1\alpha_1 & \gamma_1\alpha_2 & \gamma_3\alpha_1\\ \alpha_1\beta_2 & \alpha_2\beta_2 & \alpha_2\beta_3\end{pmatrix};$$

$$\Delta^1=\begin{bmatrix}\gamma_A\\ \beta_B\\ \alpha_C\end{bmatrix}\cong\begin{pmatrix}\beta_B\cap\gamma_A\\ \gamma_A\cap\alpha_C\\ \alpha_C\cap\beta_B\end{pmatrix}=\begin{pmatrix}U^1\\ V^1\\ W^1\end{pmatrix}=-\begin{pmatrix}\beta_1\gamma_3 & \beta_3\gamma_2 & \beta_3\gamma_3\\ \gamma_2\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_2\\ \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_1\beta_3\end{pmatrix},$$

$$\Delta^2=\begin{bmatrix}\beta_A\\ \alpha_B\\ \gamma_C\end{bmatrix}\cong\begin{pmatrix}\beta_A\cap\gamma_C\\ \gamma_C\cap\alpha_B\\ \alpha_B\cap\beta_A\end{pmatrix}=\begin{pmatrix}U^2\\ V^2\\ W^2\end{pmatrix}=-\begin{pmatrix}\beta_2\gamma_1 & \beta_2\gamma_2 & \beta_3\gamma_2\\ \gamma_1\alpha_1 & \gamma_2\alpha_1 & \gamma_1\alpha_3\\ \alpha_1\beta_3 & \alpha_3\beta_2 & \alpha_3\beta_3\end{pmatrix},$$

$$\Delta^3=\begin{bmatrix}\alpha_A\\ \gamma_B\\ \beta_C\end{bmatrix}\cong\begin{pmatrix}\beta_C\cap\gamma_B\\ \gamma_B\cap\alpha_A\\ \alpha_A\cap\beta_C\end{pmatrix}=\begin{pmatrix}U^3\\ V^3\\ W^3\end{pmatrix}=-\begin{pmatrix}\beta_1\gamma_1 & \beta_2\gamma_1 & \beta_1\gamma_3\\ \gamma_1\alpha_3 & \gamma_3\alpha_2 & \gamma_3\alpha_3\\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_2\end{pmatrix}.$$

In §5 we shall use the concept of the *medial map* m of a point $P=(u:v:w)$, defined by $mP=(v+w:w+u:u+v)$. The *medial image* mP lies on the line PG. The centroid G of Δ divides the segment between P and mP in the ratio 2 : 1. We call a triangle PQR the *medial image* of a triangle UVW, if $P=mU,\ Q=mV$ and $R=mW$.

The isotomic conjugate $P^\bullet$ of a point P has the coordinates $(1/u:\ 1/v:\ 1/w)$.

3. Centroids and homothetic centers of the triangles Δ_i and Δ^i

The centroid of Δ is $G = (1 : 1 : 1)$, G_i are the centroids of the triangles Δ_i. Their j-th coordinates are the sums of the j-th column of the matrix Δ_i. The triangle $\Delta_G = (G_{ij})$ is formed by the centroids G_i. Similarly define G^i and $\Delta^G = (G^{ij})$.

Proposition 1. *The triangles Δ_G and Δ^G have the same centroid G as Δ.*

Proof. The sums of the i-th column sums of $\Delta_1,\ \Delta_2$ and Δ_3 resp. $\Delta^1,\ \Delta^2$ and Δ^3 have the same value 3. □

The centers of homothety P_{ij} of the triangle pairs $(\Delta_i,\ \Delta_j)$ are in detail

$$\begin{aligned}
P_{12} &= (\beta_1(\gamma_1\alpha_2 - \gamma_3\alpha_3) : \gamma_2(\alpha_3\beta_1 - \alpha_2\beta_2) : \alpha_3(\beta_2\gamma_3 - \beta_1\gamma_1)),\\
P_{23} &= (\alpha_1(\beta_3\gamma_1 - \beta_2\gamma_2) : \beta_2(\gamma_2\alpha_3 - \gamma_1\alpha_1) : \gamma_3(\alpha_1\beta_2 - \alpha_3\beta_3)),\\
P_{31} &= (\gamma_1(\alpha_2\beta_3 - \alpha_1\beta_1) : \alpha_2(\beta_1\gamma_2 - \beta_3\gamma_3) : \beta_3(\gamma_3\alpha_1 - \gamma_2\alpha_2)).
\end{aligned}$$

These three points are collinear because each triad of homothetic triangles has collinear homothetic centers. The line g containing them can be written as

$$g = [\delta_2 - \delta_3 : \delta_3 - \delta_1 : \delta_1 - \delta_2], \tag{16}$$

with abbreviations $\delta_i := \alpha_i\beta_i\gamma_i$. Similarly, the homothetic centers P^{ij} of the pairs $(\Delta^i,\ \Delta^j)$ lie on the same line g. □

Proposition 2. *The line g contains the centroid G of Δ.*

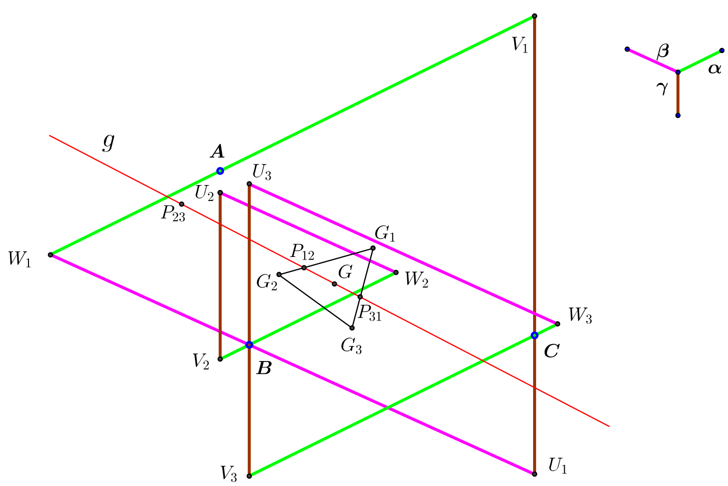

Figure 2

Proof. In accordance with (16) g has a vanishing sum of coordinates. □

Proposition 3. *The homothetic center P_{ij} of the pair $(\Delta_i,\ \Delta_j)$ lies on the side G_iG_j of Δ_G and the homothetic center P^{ij} on the side G^iG^j of Δ^G.*

Proof. Verification. □

4. Areas

Let $|UVW|$ be the area of a triangle UVW and σ the area of the reference triangle Δ. If the vertices of the triangles are given by their normalized barycentric coordinates, then UVW has the area $\sigma \cdot |\det(U,V,W)|$. We shall see below that there is a simple connection between the areas of Δ, Δ_i, Δ^i, Δ_G and Δ^G, independent of $\alpha,\ \beta$ and γ.

Making use of (1) to (6) we find

$$\det(\Delta_i) = \mu_i^2, \qquad \det(\Delta^i) = \lambda_i^2.$$

Let d_{ij} and d^{ij} be the column sums of the matrices Δ_i and Δ^i, respectively, then we have the (normalized) matrices $(G_{ij}) = \frac{1}{3}(d_{ij})$ and $(G^{ij}) = \frac{1}{3}(d^{ij})$, and it is valid

$$\det(\Delta_G) = \frac{1}{27}\det(d_{ij}) = \frac{1}{27}\begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ d_{31} & d_{32} & 3 \end{vmatrix} = \frac{1}{27}\begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ 3 & 3 & 9 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} d_{11}-1 & d_{12}-1 \\ d_{21}-1 & d_{22}-1 \end{vmatrix}.$$

In accordance with

$$\begin{aligned} d_{11}-1 &= \gamma_1\lambda_1 - \beta_1\lambda_2 & \qquad d_{12}-1 &= \alpha_2\lambda_2 - \gamma_2\lambda_3 \\ d_{21}-1 &= \beta_1\lambda_2 - \alpha_1\lambda_3 & \qquad d_{22}-1 &= \gamma_2\lambda_3 - \beta_2\lambda_1 \end{aligned}$$

it follows that

$$\det(\Delta_G) = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) = \frac{1}{6}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

Similarly,

$$\det(\Delta^G) = \frac{1}{3}(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) = \frac{1}{6}(\mu_1^2 + \mu_2^2 + \mu_3^2).$$

From this we obtain

Proposition 4.

$$\sum|\Delta_i| = 6\cdot|\Delta^G| = 6\cdot(|\Delta_G|+|\Delta|), \qquad \sum|\Delta^i| = 6\cdot|\Delta_G| = 6\cdot(|\Delta^G|-|\Delta|),$$

$$|\Delta^G| - |\Delta_G| = |\Delta|, \qquad \sum|\Delta_i| - \sum|\Delta^i| = 6\cdot|\Delta|.$$

5. The triads T_Δ, T^Δ and their midway triangles

Given two labeled triangles $T_1 = X_1Y_1Z_1$ and $T_2 = X_2Y_2Z_2$. Let X_{12} be the midpoint of the line segment X_1X_2, Y_{12} and Z_{12} the midpoints of the segments Y_1Y_2 and Z_1Z_2, respectively. Then the triangle $T_{12} = X_{12}Y_{12}Z_{12}$ is called the *midway triangle of the pair* (T_1, T_2).

The midway triangles of the pairs $(\Delta_i,\ \Delta_j)$ have obvious the normalized representations:

$$\Delta_{12} = \begin{pmatrix} U_{12} \\ V_{12} \\ W_{12} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\beta_1\gamma_3 & -\beta_3\gamma_2 & \beta_3\gamma_1+\beta_2\gamma_3 \\ \gamma_1\alpha_2+\gamma_3\alpha_1 & -\gamma_2\alpha_1 & -\gamma_1\alpha_3 \\ -\alpha_2\beta_1 & \alpha_2\beta_3+\alpha_1\beta_2 & -\alpha_3\beta_2 \end{pmatrix},$$

$$\Delta_{23} = \begin{pmatrix} U_{23} \\ V_{23} \\ W_{23} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \beta_1\gamma_2+\beta_3\gamma_1 & -\beta_2\gamma_1 & -\beta_1\gamma_3 \\ -\gamma_2\alpha_1 & \gamma_2\alpha_3+\gamma_1\alpha_2 & -\gamma_3\alpha_2 \\ -\alpha_1\beta_3 & -\alpha_3\beta_2 & \alpha_3\beta_1+\alpha_2\beta_3 \end{pmatrix},$$

$$\Delta_{31} = \begin{pmatrix} U_{31} \\ V_{31} \\ W_{31} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\beta_2\gamma_1 & \beta_2\gamma_3+\beta_1\gamma_2 & -\beta_3\gamma_2 \\ -\gamma_1\alpha_3 & -\gamma_3\alpha_2 & \gamma_3\alpha_1+\gamma_2\alpha_3 \\ \alpha_1\beta_2+\alpha_3\beta_1 & -\alpha_2\beta_1 & -\alpha_1\beta_3 \end{pmatrix},$$

and we find by calculation

$$|\det(\Delta_{12})| = \frac{1}{4}\mu_3^2, \qquad |\det(\Delta_{23})| = \frac{1}{4}\mu_1^2, \qquad |\det(\Delta_{31})| = \frac{1}{4}\mu_2^2.$$

Proposition 5. *For $i \neq j, k \neq i, j$,*
(a) *the midway triangle Δ_{ij} is the medial image of Δ_k and from this congruent with the medial triangle of Δ_k,*
(b) *G is the homothetic center of the pair $(\Delta_k,\ \Delta_{ij})$.*

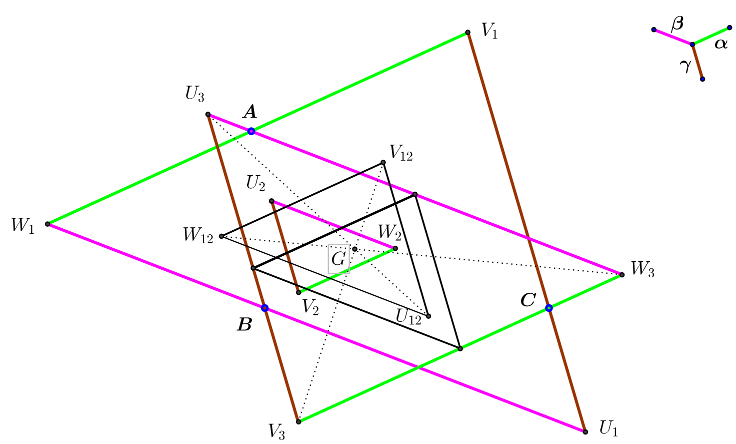

Figure 3

Proof. Verification. □

For the triangles of the triad T^{Δ} the above proposition is true by analogy.

6. Two triads of homothetic inscribed triangles

The above described construction of two triads of homothetic in Δ *circumscribed* triangles Δ_i and Δ^i raises the question whether or not exist triads of homothetic in Δ *inscribed* triangles whose sides have the given directions $\alpha,\ \beta,\ \gamma$. For this purpose we form groups of vertices of Δ^i and Δ_i to new triangles Φ^i and Φ_i, respectively:

$$\Phi^1 = \begin{pmatrix} U^3 \\ V^1 \\ W^2 \end{pmatrix},\ \Phi^2 = \begin{pmatrix} U^2 \\ V^3 \\ W^1 \end{pmatrix},\ \Phi^3 = \begin{pmatrix} U^1 \\ V^2 \\ W^3 \end{pmatrix};$$

$$\Phi_1 = \begin{pmatrix} U_3 \\ V_1 \\ W_2 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} U_2 \\ V_3 \\ W_1 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} U_1 \\ V_2 \\ W_3 \end{pmatrix}.$$

Then we intersect the sidelines of these triangles with certain sidelines of the reference triangle (see Figure 4). The points of intersection A_i, B_i, C_i and A^i, B^i, C^i form triangles Ω_i and Ω^i, respectively, with following normalized barycentric coordinates:

$$\Omega_1 = \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix} := \begin{pmatrix} V^1W^2 \cap a \\ W^2U^3 \cap b \\ U^3V^1 \cap c \end{pmatrix} = -\frac{1}{\mu_1}\begin{pmatrix} 0 & \gamma_2 & -\beta_3 \\ -\gamma_1 & 0 & \alpha_3 \\ \beta_1 & -\alpha_2 & 0 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix} := \begin{pmatrix} U^2V^3 \cap a \\ V^3W^1 \cap b \\ W^1U^2 \cap c \end{pmatrix} = -\frac{1}{\mu_2}\begin{pmatrix} 0 & \beta_2 & -\alpha_3 \\ -\beta_1 & 0 & \gamma_3 \\ \alpha_1 & -\gamma_2 & 0 \end{pmatrix},$$

$$\Omega_3 = \begin{pmatrix} A_3 \\ B_3 \\ C_3 \end{pmatrix} := \begin{pmatrix} W^3U^1 \cap a \\ U^1V^2 \cap b \\ V^2W^3 \cap c \end{pmatrix} = -\frac{1}{\mu_3}\begin{pmatrix} 0 & \alpha_2 & -\gamma_3 \\ -\alpha_1 & 0 & \beta_3 \\ \gamma_1 & -\beta_2 & 0 \end{pmatrix};$$

$$\Omega^1 = \begin{pmatrix} A^1 \\ B^1 \\ C^1 \end{pmatrix} := \begin{pmatrix} V_1U_3 \cap a \\ U_3W_2 \cap b \\ W_2V_1 \cap c \end{pmatrix} = \frac{1}{\lambda_1}\begin{pmatrix} 0 & \alpha_2 & -\beta_3 \\ -\alpha_1 & 0 & \gamma_3 \\ \beta_1 & -\gamma_2 & 0 \end{pmatrix},$$

$$\Omega^2 = \begin{pmatrix} A^2 \\ B^2 \\ C^2 \end{pmatrix} := \begin{pmatrix} U_2W_1 \cap a \\ W_1V_3 \cap b \\ V_3U_2 \cap c \end{pmatrix} = \frac{1}{\lambda_2}\begin{pmatrix} 0 & \gamma_2 & -\alpha_3 \\ -\gamma_1 & 0 & \beta_3 \\ \alpha_1 & -\beta_2 & 0 \end{pmatrix},$$

$$\Omega^3 = \begin{pmatrix} A^3 \\ B^3 \\ C^3 \end{pmatrix} := \begin{pmatrix} W_3V_2 \cap a \\ V_2U_1 \cap b \\ U_1W_3 \cap c \end{pmatrix} = \frac{1}{\lambda_3}\begin{pmatrix} 0 & \beta_2 & -\gamma_3 \\ -\beta_1 & 0 & \alpha_3 \\ \gamma_1 & -\alpha_2 & 0 \end{pmatrix}.$$

Then we have (see Figure 5)

Proposition 6. (a) *The sidelines of the inscribed triangles Ω_i and Ω^i have the directions $\alpha,\ \beta$ and γ.*
(b) *The sides of Δ_i are parallels of the sides a_i, b_i, c_i of Ω_i at A, B, C (in this order). An analogous statement is true for Δ^i.*

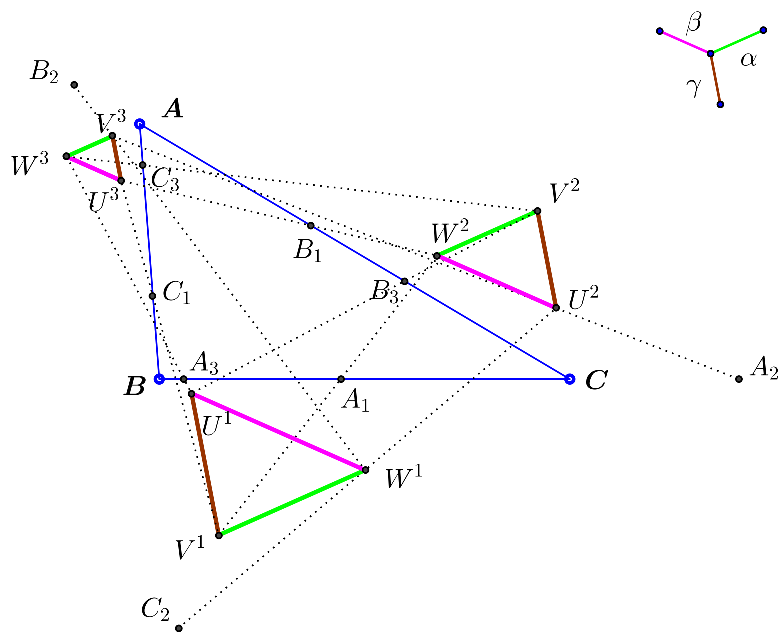

Figure 4

(c) *The points in each of the triples* $\{A_1, B_3, C_2\}$, $\{A_2, B_1, C_3\}$, $\{A_3, B_2, C_1\}$, $\{A^1, B^2, C^3\}$, $\{A^2, B^3, C^1\}$ *and* $\{A^3, B^1, C^2\}$ *are collinear.*
(d) *The product of the areas of* Δ_i *and* Ω_i *and of* Δ^i *and* Ω^i *is independent of* $\alpha,\ \beta,\ \gamma$*:*

$$|\Delta_i| \cdot |\Omega_i| = |\Delta^i| \cdot |\Omega^i| = |\Delta|^2.$$

Proof. Verification.
(a) For instance the line A_1B_1 intersects the infinite line in accordance with (4) at $-\mu_1(\gamma_1 : \gamma_2 : \gamma_3)$, therefor it has the direction γ.

(b) For instance the translation of the line A_1B_1 at C yields the line $[-\gamma_2 : \gamma_1 : 0] = \gamma_C$.

(c) For instance $\det(A_1, B_3, C_2) = \begin{vmatrix} 0 & \gamma_2 & -\beta_3 \\ -\alpha_1 & 0 & \beta_3 \\ \alpha_1 & -\gamma_2 & 0 \end{vmatrix} = 0.$

(d) For instance according to (4)

$$\det(\Omega_1) \cdot \det(\Delta_1) = -\frac{1}{\mu_1^3}(\alpha_3\beta_1\gamma_2 - \alpha_2\beta_3\gamma_1) \cdot \mu_1^2 = 1. \qquad \square$$

For the comparison of homothetic triangles it is useful to give homologous vertices the same position:

$$\Omega_1 = A_1B_1C_1, \quad \Omega_2 = B_2C_2A_2, \quad \Omega_3 = C_3A_3B_3;$$
$$\Omega^1 = C^1B^1A^1, \quad \Omega^2 = B^2A^2C^2, \quad \Omega^3 = A^3C^3B^3.$$

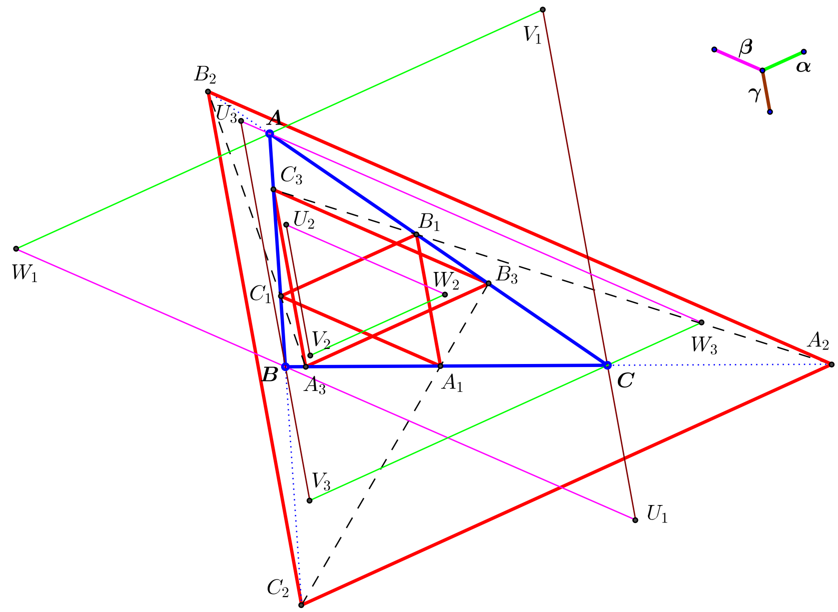

Figure 5

Then for instance the homothetic center Q_{12} of the pair (Ω_1, Ω_2) is the common point of intersection of the lines $A_1B_2,\ B_1C_2$ and C_1A_2. By a simple calculation it follows for the homothetic centers Ω_{ij} and Ω^{ij}

$$\begin{aligned} &Q_{12} = (\beta_1 : \gamma_2 : \alpha_3), \quad Q_{23} = (\alpha_1 : \beta_2 : \gamma_3), \quad Q_{31} = (\gamma_1 : \alpha_2 : \beta_3), \\ &Q^{12} = (\alpha_1 : \gamma_2 : \beta_3), \quad Q^{23} = (\gamma_1 : \beta_2 : \alpha_3), \quad Q^{31} = (\beta_1 : \alpha_2 : \gamma_3). \end{aligned}$$

It is clear, that the homothetic centers $Q_{12},\ Q_{23},\ Q_{31}$ and $Q^{12},\ Q^{23},\ Q^{31}$ are collinear on lines g_Ω and g^Ω, respectively, and these lines have according to (8)-(13) the simple representations

$$g_\Omega = [\xi_1 : \xi_2 : \xi_3], \quad g^\Omega = [\eta_1 : \eta_2 : \eta_3].$$

The centroids S_i of Ω_i and S^i of Ω^i, respectively, are

$$\begin{aligned} S_1 &= (\beta_1 - \gamma_1 : \gamma_2 - \alpha_2 : \alpha_3 - \beta_3), & S^1 &= (\beta_1 - \alpha_1 : \alpha_2 - \gamma_2 : \gamma_3 - \beta_3), \\ S_2 &= (\alpha_1 - \beta_1 : \beta_2 - \gamma_2 : \gamma_3 - \alpha_3), & S^2 &= (\alpha_1 - \gamma_1 : \gamma_2 - \beta_2 : \beta_3 - \alpha_3), \\ S_3 &= (\gamma_1 - \alpha_1 : \alpha_2 - \beta_2 : \beta_3 - \gamma_3), & S^3 &= (\gamma_1 - \beta_1 : \beta_2 - \alpha_2 : \alpha_3 - \gamma_3). \end{aligned}$$

A simple computation proves the following proposition:

Proposition 7. *The three centroids of the Ω_i and the three homothetic centers Q_{ij} lie on the line g_Ω, the three centroids of the Ω^i and the three homothetic centers Q^{ij} on the line g^Ω* (see Figure 6).

Marginal node: The triangles Φ_i and Φ^i have else some peculiarities:

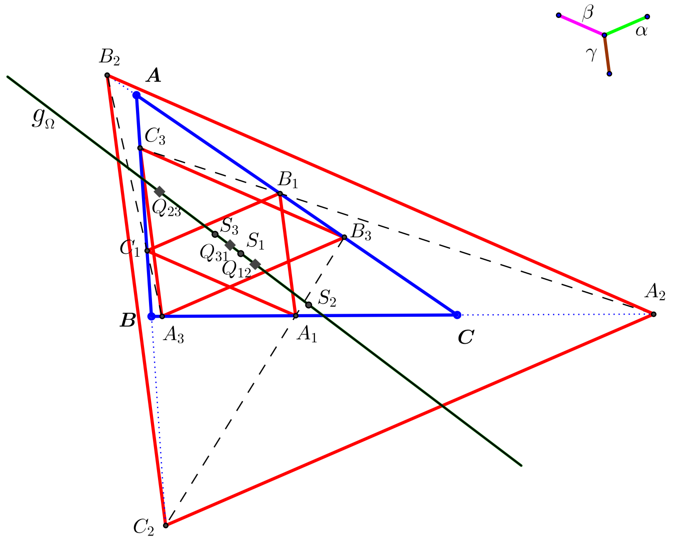

Figure 6

Proposition 8. (a) *The triangles Φ_i and Φ^i have the same area as the reference triangle* (see Figure 7).

(b) *The vertices of Φ_i and Φ^i lie on a circumconic $\mathcal{C}_i$ and $\mathcal{C}^i$ of Δ, respectively. The $\mathcal{C}_i$ are concurrent at $\eta^\bullet$, the $\mathcal{C}^i$ at $\xi^\bullet$* (see Figure 8).

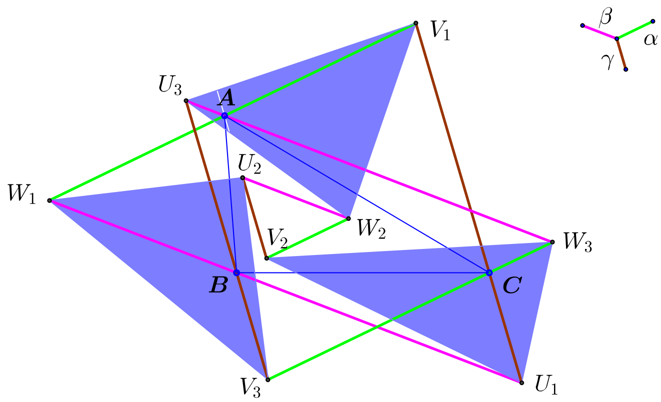

Figure 7

Proof. (a) By use of normalized barycentric coordinates of the vertices of Φ_i and Φ^i it is easy to show, that $|\det(\Phi_i)| = |\det(\Phi^i)| = 1$.

(b) It is simple to verify, that the isotomic conjugates of the vertices of each triangle Φ_i and Φ^i are collinear and that the concerned lines are concurrent at η and ξ, respectively. From this follows the assertion directly. □

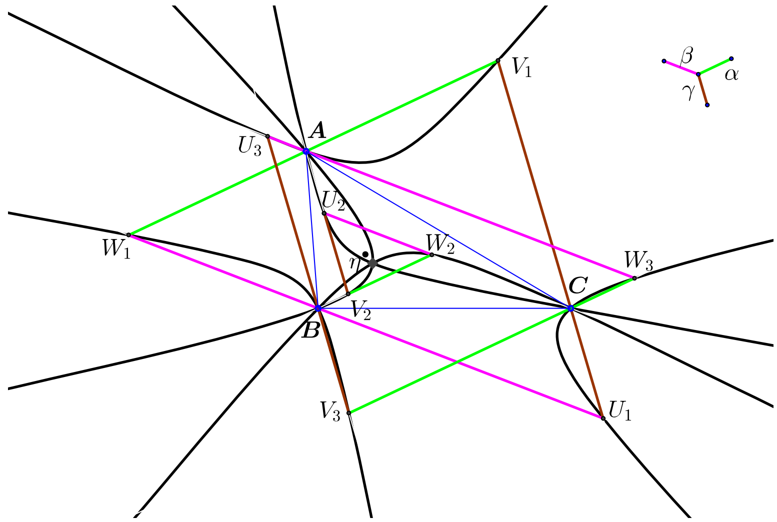

Figure 8

7. A special direction triple ($\alpha,\ \beta,\ \gamma$)

In the end we consider an interesting special case. Given a direction $\alpha = (\alpha_1 : \alpha_2 : \alpha_3)$. We choose β and γ as iterate Brocardians: $\beta = \alpha_{\leftarrow\leftarrow} = (\alpha_3 : \alpha_1 : \alpha_2)$ and $\gamma = \alpha_{\rightarrow\rightarrow} = (\alpha_2 : \alpha_3 : \alpha_1)$ [1]. The isotomic conjugates $\alpha^{\bullet}, \beta^{\bullet}, \gamma^{\bullet}$ are points on the Steiner ellipse, they form a Brocardian triple, that is the line at two of these points is the dual of the third point.

Let $A',\ B',\ C'$ be the reflections of $A,\ B,\ C$ in G and

$$\mathcal{C}_A:\ x^2 - yz = 0,\quad \mathcal{C}_B:\ y^2 - zx = 0,\quad \mathcal{C}_C:\ z^2 - xy = 0$$

three ellipses created by translation of the Steiner ellipse with centers $A',\ B',\ C'$ and passing through the common point G.

In the following we mention without proofs some properties of points and triangles defined above.

(1) The line triples $(\gamma_A,\ \beta_B,\ \alpha_C),\ (\beta_A,\ \alpha_B,\ \gamma_C)$ and $(\alpha_A,\ \gamma_B,\ \beta_C)$ are concurrent at the points

$P_1 = (\alpha_3\alpha_1 : \alpha_2\alpha_3 : \alpha_1\alpha_2),\ P_2 = (\alpha_1\alpha_2 : \alpha_3\alpha_1 : \alpha_2\alpha_3),\ P_3 = (\alpha_2\alpha_3 : \alpha_1\alpha_2 : \alpha_3\alpha_1),$

respectively, that is each of the triangles Δ^i degenerates into a point on the Steiner ellipse.

(2) The triangles Δ_i have the same centroid G, it is at the same time the homothetic center of each pair $(\Delta_i,\ \Delta_j)$.

(3) The points of each triple $\{U_1, U_2, U_3\}, \{V_1, V_2, V_3\}, \{W_1, W_2, W_3\}$ are collinear. The three lines concurrent in G are the duals of $\alpha,\ \beta,\ \gamma$.

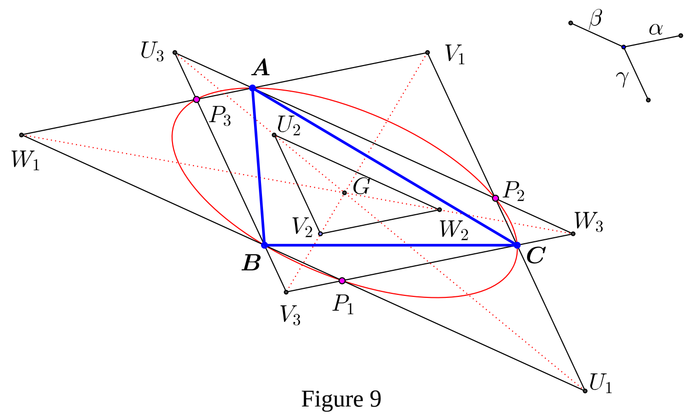

Figure 9

(4) For the vertices of the triangles Δ_i is valid:

$$U_1, V_3, W_2 \in \mathcal{C}_A, \quad U_2, V_1, W_3 \in \mathcal{C}_B, \quad U_3, V_2, W_1 \in \mathcal{C}_C.$$

These triangles are homothetic, congruent and equal in area with Δ.

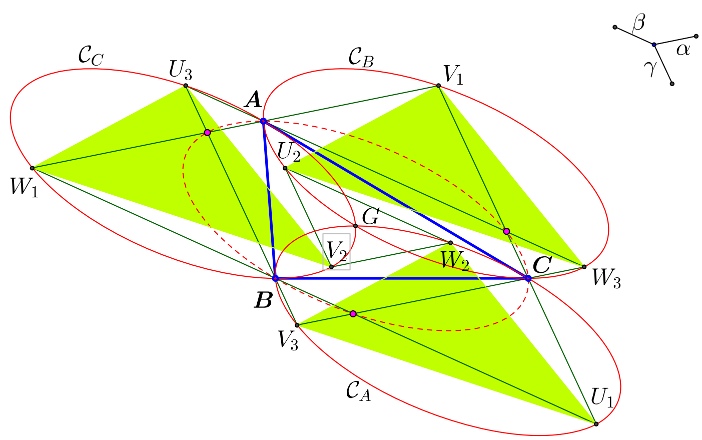

Figure 10

(5) The triangles Φ_k are homothetic and congruent with Δ. The homothetic centers of the pairs $(\Delta,\ \Phi_k)$ are the medial images mP_k of P_k. They are the midpoints of the segments P_iP_j, lie on the Steiner inellipse, have the representations

$$mP_1 = (\alpha_2^2 : \alpha_1^2 : \alpha_3^2),\ mP_2 = (\alpha_3^2 : \alpha_2^2 : \alpha_1^2),\ mP_3 = (\alpha_1^2 : \alpha_3^2 : \alpha_2^2)$$

and form a Brocardian Triple (see Figure 11).

(6) The sum of the areas of Δ_i is sixfold the area of Δ:

$$\sum |\Delta_i| = 6 \cdot |\Delta|.$$

(7) The midway triangle of the pair $(\Delta_i,\ \Delta_j)$ coincides with the medial triangle of Δ_k.

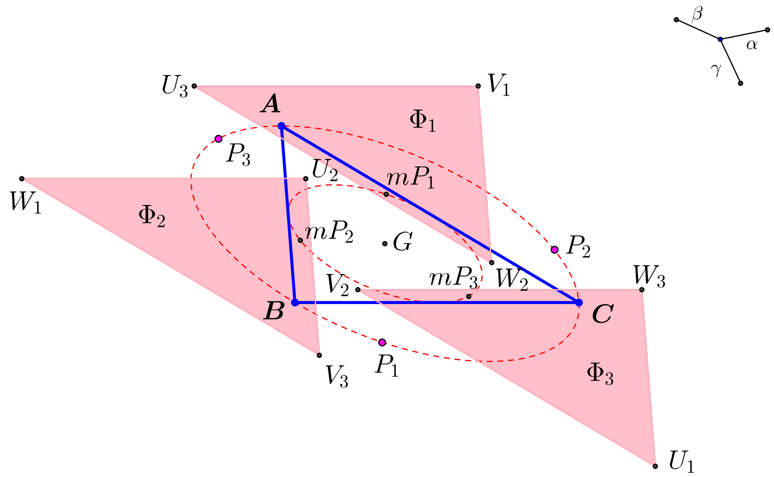

Figure 11

(8) The triangles Ω_i have the representations

$$\Omega_1 = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_2 & 0 & \alpha_3 \\ \alpha_3 & -\alpha_2 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & \alpha_1 & -\alpha_3 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_1 & -\alpha_3 & 0 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\alpha_1 & 0 & \alpha_2 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

and the centroid G. The homothetic centers of the pairs $(\Omega_i,\ \Omega_j)$ coincide with G.

(9) The triangles Ω^i degenerate, their vertices lie on the infinite line.

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[2] G. Weise, Generalization and extension of the Wallace theorem, *Forum Geom.*, 12 (2012) 1–11.

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Symbolic Substitution Has a Geometric Meaning

Manfred Evers

Abstract. By comparing two different metrics in the affine plane, it is shown that symbolic substitution, introduced by Clark Kimberling, has a clear geometric meaning.

1. Introduction

Consider in the real plane $\mathbb{R}^2$ a triangle ABC with vertices

$$A = \left(\frac{a^2 - b^2 + c^2}{2a}, \frac{\sqrt{\sigma}}{2a}\right),\ B = (0,0),\ C = (a,0),$$

where a, b and c are positive real numbers with

$$\sigma := \sigma(a,b,c) = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) > 0.$$

If in $\mathbb{R}^2$ the Euclidean distance d of two points $U = (X, Y)$ and $U' = (X', Y')$ is defined by

$$d(U, U') = \sqrt{(X - X')^2 + (Y - Y')^2},$$

then the reference triangle ABC has side lengths a, b, c. In the following we use Conway's triangle notation:

$$S = \frac{1}{2}\sqrt{\sigma}, \quad S_A = \frac{1}{2}(b^2 + c^2 - a^2), \text{etc.}$$

Thus, we can write $A = \left(\frac{S_B}{a}, \frac{S}{a}\right)$.

Besides Cartesian coordinates we also use barycentric coordinates with respect to the reference triangle. The notation $U = (u, v, w)_{ABC}$ is used if (u, v, w) are the absolute coordinates of U with respect to ABC. If $(u : v : w)$ are homogeneous coordinates of U, then we write $U = (u : v : w)_{ABC}$. The conversion from Cartesian coordinates (X, Y) of a point U to its barycentric coordinates (u, v, w) can be achieved by taking

$$u = \frac{aY}{S}, \quad v = \frac{-((X-a)S + YS_C)}{aS}, \quad w = \frac{XS - YS_B}{aS} \qquad (*)$$

Given two points U and V with absolute barycentric coordinates (u, v, w) and (u', v', w'), the square of the distance between these two points can be calculated

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by the formula

$$d^2(U,U') = S_A(u-u')^2 + S_B(v-v')^2 + S_C(w-w')^2.$$

In addition to the canonical Euclidean metric, we introduce a second metric d_g, which we call *generalized metric* or *g-metric*, for short. This second metric is obtained by giving new lengths to the sides of the reference triangle:

$$d_g(A,B) = c_g, \quad d_g(B,C) = a_g, \quad d_g(C,A) = b_g.$$

We still demand that these lengths a_g, b_g, c_g are positive real numbers [1] but do not claim that

$$\sigma_g := (a_g+b_g+c_g)(-a_g+b_g+c_g)(a_g-b_g+c_g)(a_g+b_g-c_g)$$

is a positive real number.

For different signs of σ_g we get different geometries: $\sigma_g > 0$ delivers an affine version of the Euclidean metric, for $\sigma_g = 0$ the metric is Galilean, and for $\sigma_g < 0$ the metric is a Lorentz-Minkowski metric. With respect to this generalized metric, the square of the distance between two points U and U' with absolute barycentric coordinates (u,v,w) and (u',v',w') is given by

$$d_g^2(U,U') = S_{A,g}(u-u')^2 + S_{B,g}(v-v')^2 + S_{C,g}(w-w')^2 \qquad (**)$$

with $S_{A,g} = \frac{-a_g^2+b_g^2+c_g^2}{2}$ etc. The point with coordinates $(a_g : b_g : c_g)$ will become the generalized incenter I_g. The new circumcenter O_g has coordinates $(a_g^2(-a_g^2 + b_g^2+c_g^2) : \cdots : \cdots)$. More generally, if $x(a,b,c)$ is a barycentric center function of a triangle center X, then $X_g = (x(a_g,b_g,c_g) : \cdots : \cdots)_{ABC}$ is the corresponding g-center. The point G_g still agrees with $G = (1:1:1)_{ABC}$. The square of the new triangle area is $\frac{1}{16}\sigma_g$.

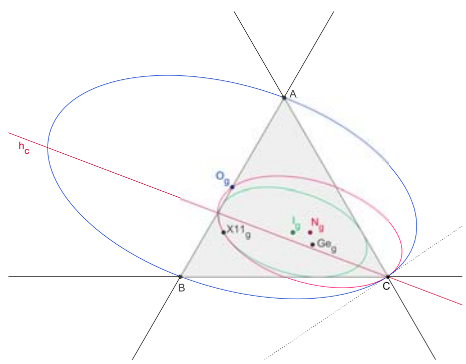

Figure 1. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (3 : 4 : 5)$. The picture shows the circumcircle (blue), the incircle (green) and the nine-point circle (red).

Figures 1 and 2 show the situation for an equilateral triangle ($a = b = c$) on which was imposed a new metric $(a_g, b_g, c_g) = (3,4,5)$ and $(a_g, b_g, c_g) =$

[1] Later on, we dismiss this condition, see 2.4 .

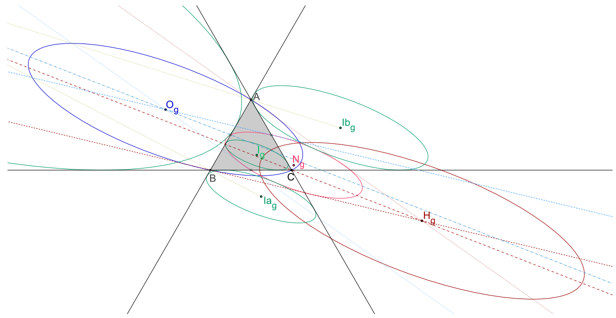

Figure 2. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (6 : 9 : 13)$. Besides the circumcircle, the incircle and the nine-point circle, the picture shows the excircles (green) and the polarcircle (brown).

$(6, 9, 13)$, respectively. Considering these two new metrics, the first triangle is right angled, the second obtuse.

A generalization of the incenter, the circumcenter and the orthocenter of a triangle is given by I. Minevich and P. Morton [12, Section 4]. They introduce the name *generalized* triangle centers. We adopt this terminology.

1.1. *Embedding the affine in a projective plane.* The affine plane can be embedded in a projective plane $\mathcal{P}$ by adding the line at infinity $\mathcal{L}_\infty$. This line consists of all points $(u, v, w)_{ABC}$ with $u + v + w = 0$. Points on $\mathcal{L}_\infty$ are called *infinite* or *improper* while those in the affine plane are called *finite* or *proper*. A line is called *proper* if it differs from $\mathcal{L}_\infty$.

1.2. *g-orthogonality of lines.* If two proper lines are g-orthogonal, we shall say that their infinite points form a pair of g-orthopoints. Two infinite points $U = (u : v : w)_{ABC}$ and $U' = (u' : v' : w')_{ABC}$ are g-orthopoints if and only if $S_{A,g}uu' + S_{B,g}vv' + S_{C,g}ww' = 0$. See [15, p. 55] or [2] for the Euclidean case. The following version of Thales' theorem is valid in all metric affine geometries: Given two lines $\mathcal{L}$ and $\mathcal{L}'$ which meet in a finite point P, these lines are orthogonal precisely when there exist two (not necessarily finite) points Q and R, one on $\mathcal{L}$ and the other on $\mathcal{L}'$ and both different from P, so that the center of the (generalized) circle PQR is a point on the line QR.

In the following, we analyze the three cases $\sigma_g > 0$, $\sigma_g = 0$ and $\sigma_g < 0$.

2. The affine Euclidean case

If $\sigma_g > 0$, then the g-incenter I_g lies inside the medial triangle and the g-symmedian K_g lies inside the Steiner inellipse. The metric given by formula $(**)$ is affine Euclidean. *Proof*: By an affine transformation τ with fixed point I_g, the inconic with center I_g can be mapped onto a circle. Let A', B', C' be the image points of A, B, C under τ. This transformation maps a point $U = (u : v : w)_{ABC}$ to the point $U' = (u : v : w)_{A'B'C'}$. I_g is the incenter of $A'B'C'$; therefore the ratio of the sidelengths of this triangle is $a_g : b_g : c_g$. If x is a barycentric center function, then τ maps the g-center $X_g = (x(a_g, b_g, c_g) : x(b_g, c_g, a_g) : x(c_g, a_g, b_g))_{ABC}$ of

ABC to the triangle center $X = (x(a_g, b_g, c_g) : x(b_g, c_g, a_g) : x(c_g, a_g, b_g))_{A'B'C'}$ of $A'B'C'$. $\square$

2.1. *Description of the generalized circumcircle.* In case of $\sigma_g > 0$, the generalized circles are ellipses. We assume that I_g and I are different points, so that these generalized circles are ellipses with two foci. The focal axes of these ellipses have all the same infinite point. We will determine this point. Furthermore, for the g-circumcircle we calculate the lengths of the two principal axes and the distance between the two foci.

The equation of a g-circle with center $M = (m_a, m_b, m_c)_{ABC}$ and radius ρ is

$$\sum_{\text{cyclic}} S_{A,g}((m_b + m_c)x - m_a(y + z))^2 = \rho^2(m_a + m_b + m_c)^2(x + y + z)^2.$$

We concentrate on the g-circumcircle, which is given by the equation

$$a_g^2 yz + b_g^2 zx + c_g^2 xy = 0.$$

Using $(*)$, we can derive an equation in Cartesian coordinates,

$$\alpha(X - X_0)^2 + 2\beta(X - X_0) + \gamma(Y - Y_0)^2 = \left(\frac{\alpha a_g b_g c_g S}{S_g}\right)^2,$$

with $\alpha = (2a_g S)^2$, $\beta = 2S(a_g^2(b^2 - c^2) - a^2(b_g^2 - c_g^2))$,
$\gamma = a_g^2(b^2 - c^2)^2 - 2a^2(b^2 - c^2)(b_g^2 - c_g^2) - a^4(a_g^2 - 2b_g^2 - 2c_g^2)$, and
(X_0, Y_0) the Cartesian coordinates of the g-circumcenter

$$O_g = (a_g^2(-a_g^2 + b_g^2 + c_g^2) : \cdots : \cdots)_{ABC}.$$

The direction vectors of the two principal axes of the circumellipse are the eigenvectors of the matrix $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. The two eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(\alpha + \gamma \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2}\right) = 2a^2(\theta \pm \phi^2)$$

with

$$\theta = \frac{\alpha + \gamma}{4a^2} = a^2 S_{A,g} + b^2 S_{B,g} + c^2 S_{C,g} = a_g^2 S_A + b_g^2 S_B + c_g^2 S_C,$$
$$\phi = \sqrt[4]{(\theta + 2SS_g)(\theta - 2SS_g)} \quad \text{and } S_g = \frac{1}{2}\sqrt{\sigma_g}.$$

In the affine Euclidean case we have $\lambda_+ > \lambda_- > 0$. A calculation shows that the lengths of the major and minor half axes are

$$r_{max} = \frac{aa_g b_g c_g S}{S_g\sqrt{\lambda_-}} \quad \text{and} \quad r_{min} = \frac{aa_g b_g c_g S}{S_g\sqrt{\lambda_+}}.$$

The distance f of the foci from the center O_g is therefore

$$f = \sqrt{r_{max}^2 - r_{min}^2} = \frac{a_g b_g c_g \phi}{\sigma_g}.$$

The eigenspaces of λ_+ and λ_- are $\mathbb{R}(\lambda_\pm - \gamma, \beta)$, so that the focal line is $(X_0, Y_0) + \mathbb{R}(\lambda_- - \gamma, \beta)$. The infinite point on this line is

$$(a^2\beta : -S(\lambda_- - \gamma) - S_C\beta : S(\lambda_- - \gamma) - S_B\beta)_{ABC}.$$

The Cartesian coordinates of the two foci $F_\pm$ are $(X_0 \pm fv_1, Y_0 \pm fv_2)$, (v_1, v_2) being a unit eigenvector of λ_-. Subsequently, the barycentric coordinates of the two foci can be calculated:

$$F_\pm = (p \pm p', q \pm q', r \pm r')_{ABC},$$

where (p, q, r) are the absolute barycentric coordinates of O_g and

$$p' = \frac{afv_2}{S}, \quad q' = \frac{f(-v_1S - v_2S_C}{aS}, \quad r' = \frac{f(v_1S - v_2S_B)}{aS}.$$

The foci of a circumconic and an inconic can be obtained by a compass-ruler-construction, see Gibert [8]. Dergiades [3] calculated the lengths of the axes of an inellipse with given center and also described a compass-ruler-construction of the foci.

2.2. *Criterion for the g-orthogonality of lines / Theorem of Brianchon and Poncelet.* Let two different lines meet the line at infinity in the points U, V. These lines are g-orthogonal if and only if there exists a hyperbola through the points A, B, C, U, V and the g-orthocenter $H_g = \left(\frac{1}{-a_g^2+b_g^2+c_g^2} : \cdots : \cdots\right)$.

2.3. *The g-angle between two lines.* We want to determine the g-measure $\phi_g \in \left[0, \frac{\pi}{2}\right]$ of the angle $\angle(\mathcal{L}, \mathcal{L}')$ between the proper lines $\mathcal{L} : lx + my + nz = 0$ and $\mathcal{L}' : l'x + m'y + n'z = 0$. This measure is fixed as soon as we know the value of $\sin\phi_g$ or $\cos\phi_g$. To get these two values, let

$$k := l(m' - n') + m(n' - l') + n(l' - m'),$$

and calculate the g-area $|PQR|_g$ of a triangle PQR, where

$$P = \left(\frac{mn' - m'n}{k}, \frac{l'n - ln'}{k}, \frac{lm' - l'm}{k}\right)_{ABC}$$

is the intersection of $\mathcal{L}$ and $\mathcal{L}'$,

$$Q = \left(\frac{mn' - m'n}{k} + m - n, \cdots, \cdots\right)_{ABC}$$

is a second point on $\mathcal{L}$, and

$$R = \left(\frac{mn' - m'n}{k} + m' - n', \cdots, \cdots\right)_{ABC}$$

is a second point on $\mathcal{L}'$. For this,

$$|PQR|_g = |k| \cdot |ABC|_g = \frac{1}{4}|k|\sqrt{\sigma_g} = \frac{1}{2}|k|S_g.$$

Thus, we get

$$\begin{aligned}
\sin\phi_g =& \left|\frac{2k\cdot|ABC|_g}{d_g(P,Q)d_g(P,R)}\right|, \\
\cos\phi_g =& \left|\frac{S_{A,g}(m-n)(m'-n')+S_{B,g}(n-l)(n'-l')+S_{C,g}(l-m)(l'-m')}{d_g(P,Q)d_g(P,R)}\right|, \\
d_g^2(P,Q) =& \ S_{A,g}(m-n)^2+S_{B,g}(n-l)^2+S_{B,g}(l-m)^2, \\
d_g^2(P,R) =& \ S_{A,g}(m'-n')^2+S_{B,g}(n'-l')^2+S_{B,g}(l'-m')^2.
\end{aligned}$$

We interpret $\vec{l}=\begin{pmatrix}m-n\\n-l\\l-m\end{pmatrix}$ and $\vec{l'}=\begin{pmatrix}m'-n'\\n'-l'\\l'-m'\end{pmatrix}$ as direction vectors of $\mathcal{L}$ and $\mathcal{L}'$ respectively, and the expression

$$S_{A,g}(m-n)(m'-n')+S_{B,g}(n-l)(n'-l')+S_{C,g}(l-m)(l'-m')$$

as a dot product $\vec{l}\cdot\vec{l'}=<\vec{l}|\vec{l'}>_g$ of these two vectors, so that we can write

$$\cos(\phi_g)=\frac{|\vec{l}\cdot\vec{l'}|}{\sqrt{\vec{l}\cdot\vec{l}}\sqrt{\vec{l'}\cdot\vec{l'}}}.$$

2.4. *Negative sidelengths* a_g, b_g, c_g. There is no reason to forbid negative values for a_g, b_g, c_g. For example, if we choose $a_g=-a$, $b_g=b$ and $c_g=c$ or $a_g=a$, $b_g=-b$ and $c_g=-c$, we just swap the incenter I with the excenter I_a and the excenter I_b with the excenter I_c. If a weak center, consisting of the main point and its three mates, is not considered as a quadruple but as a set of four points, changing signs leaves this weak center invariant. Strong centers keep their positions, anyway. This situation does not change if we dismiss the condition $\sigma_g>0$.

3. The Galilean case

If $\sigma_g=0$, the point H_g is an infinite point and agrees with O_g and N_g. H_g is the absolute pole of the Galilean plane: two finite points lying on the same line through H_g have Galilean distance 0. A line through the absolute pole is therefore called a null line. In particular, the three generalized triangle altitudes, considered as lines, are null lines. This is in accordance with the area of the triangle: $\frac{\sqrt{\sigma_g}}{4}=0$. The g-symmedian point K_g is a point on the Steiner inellipse. The incenter and the excenters are the barycentric square roots of K_g. One of these roots agrees with H_g, so it is an infinite point. The other three roots lie on the sidelines of the medial triangle of ABC.

3.1. *g-circles in Galilean geometry.* Generalized circles in the Galilean plane are parabolas. Some of them may degenerate. For example, the g-incircle and the g-excircles are double lines (lines with multiplicity 2), and if the incenter is a vertex of the medial triangle, then the g-circumcircle and the g-nine-point circle each consist of two parallel lines. It should be noted that a degenerate g-circle in Galilean geometry has more than one center. All these centers lie on a "centerline" of this g-circle. Michel Bataille [2] calculated the barycentric coordinates of the focus F of

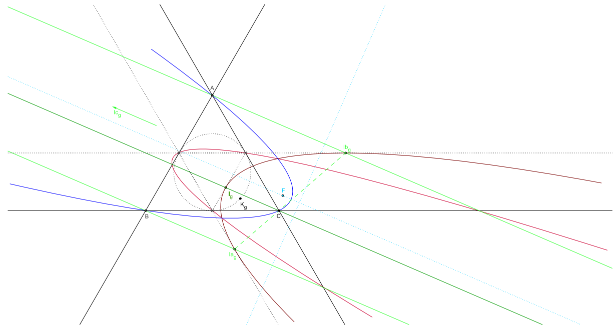

Figure 3. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (2 : 3 : 5)$. The picture shows the g-circumcircle (blue) with focus F, the g-nine-point circle (red) and the g-polar circle (brown). The g-polar circle runs through the g-incenter and the g-excenters. The green lines are the g-altitudes. As double lines they are the g-incircle and two g-excircles.

a circumparabola $\mathcal{C}$ of the triangle ABC: if the infinite point of $\mathcal{C}$ has barycentric coordinates $(u : v : w)$, then the focus is

$$F = \left(\frac{u^2}{vw} + \frac{a^2vw}{a^2vw + b^2wu + c^2uv} : \cdots : \cdots\right)_{ABC}.$$

Figure 3 gives an illustration of a triangle in the Galilean plane.

3.2. *g-Orthogonality of lines in a Galilean plane.* Two lines which meet in a finite point are orthogonal if and only if one of these is a null line.

3.3. *The g-angle between two lines.* It makes sense to define the g-measure of the angle between two non-null-lines $\mathcal{L}$ and $\mathcal{L}'$ to be the g-distance of the poles of these lines with respect to the g-circumcircle of the triangle ABC. *Proof*: We may assume that the finite part of the g-circumcircle of the triangle ABC in $\mathbb{R}^2$ is given by the equation $y = kx^2$, $k \neq 0$, and that $y = mx + n$ and $y = m'x + n'$ are the equations of the lines $\mathcal{L}$ and $\mathcal{L}'$. The poles of these two lines with respect to the g-circumcircle are $P = (\frac{m}{2k},\ -n)$ resp. $P' = (\frac{m'}{2k},\ -n')$. There exists a real number $k' \neq 0$ so that the Galilean distance between any two points (x, y) and (x', y') is $k'|x - x'|$. Therefore, the distance between P and P' is $\frac{k'}{2|k|}|m - m'|$. This is consistent with the usual definition of the measure of the angle between the two lines $\mathcal{L}$ and $\mathcal{L}'$ in Galilean geometry, see for example [7, Chap 4] or [13, Chap 23].$\square$

Remarks. (1) Using barycentric coordinates, the pole of the line $\mathcal{L} : lx+my+nz = 0$ with respect to the circumcircle $\mathcal{C} : a_g^2yz + b_g^2zx + c_g^2xy = 0$ is

$$P = \left(a_g^2(-a_g^2l + b_g^2m + c_g^2n) : \cdots : \cdots\right)_{ABC}.$$

(2) For the reference triangle ABC, the ratio of the measure of an angle and the length of its opposite side is $\frac{1}{2}$. (The measure α of the angle between AB and AC is $\frac{1}{2}a$ *etc*).

(3) For the medial triangle of ABC, the measure of an angle equals the measure of its opposite side.

4. The Lorentz-Minkowski case

If $\sigma_g < 0$ then the metric d_g is a Lorentz-Minkowski metric or an LM metric, for short. In this case, generalized circles are hyperbolas which might eventually degenerate. Given a g-metric, all the g-circles run through the same two points on the line at infinity $\mathcal{L}_\infty$. These two points P_1 and P_2 are the absolute poles of the LM plane. The distance between two different finite points U and V is 0 if and only if (either) P_1 or P_2 is a point on the line UV; in this case UV is called a null line. In LM geometry, not only the distance between two points can be negative but also the square of the distance. Following A. Einstein [6], we will define: Two points U and V are in a *spacelike* position if $d_g^2(U,V) > 0$, and they are in a *timelike* position if $d_g^2(U,V) < 0$. If two points U, V are in a spacelike (respectively timelike) position, then all finite points P, Q on the line UV with $P \neq Q$, are in a spacelike (respectively timelike) position. Therefore, we can call the line UV *spacelike* (respectively *timelike*). Obviously, in LM geometry triangles can have real as well as imaginary sidelengths. Triangles having both, a real and an imaginary sidelength $\neq 0$, do not have weak triangle centers. All strong centers still exist.

4.1. *Description of the generalized circles.* Generalized circles in the LM plane are hyperbolas. The focal axes of two g-circles do not have to be parallel but may be orthogonal, as can be seen in Figure 7. The principal axes and the foci can be calculated the same way as in §2.1 for the Euclidean case, except for the distance between the foci and the center O_g. This is now

$$f = \sqrt{r_{max}^2 + r_{min}^2} = -\frac{a_g b_g c_g \sqrt{|\theta|}}{\sigma_g}.$$

The two absolute poles are

$$P_1 = \ (-a^2(\omega+\beta) : (a^2+S_B)(\omega+\beta)+S\gamma : -S_B(\omega+\beta)-S\gamma)_{ABC},$$
$$P_2 = \ (-a^2(\omega-\beta) : (a^2-S_B)(\omega-\beta)+S\gamma : S_B(\omega-\beta)-S\gamma)_{ABC}.$$

Here we use the notations of §2.1 and $\omega := \sqrt{\beta^2-\alpha\gamma} = a^2\sqrt{-\sigma\sigma_g}$.

4.2. *g-Orthogonality of lines.* A pair (U,V) of points on the line at infinity is a pair of g-orthopoints if and only if the quadruple (U,V,P_1,P_2) is harmonic. It can be easily checked that if a line is spacelike then all its g-orthogonal lines are timelike. Particularly, if a sideline of a triangle is spacelike then the corresponding altitude, considered as a line, is timelike and vice versa. Figure 5 shows a triangle with $d_g(B,C) = 0$, and Figure 6 illustrates the situation for a triangle with $\mathrm{d}_g(A,B) = \mathrm{d}_g(A,C) = 0$. In both cases, the generalized circumcircle as well as the generalized nine point circle degenerate to a pair of lines. In Figure 7 the generalized symmedian K_g is outside the triangle, therefore weak generalized centers do not exist in the real plane.

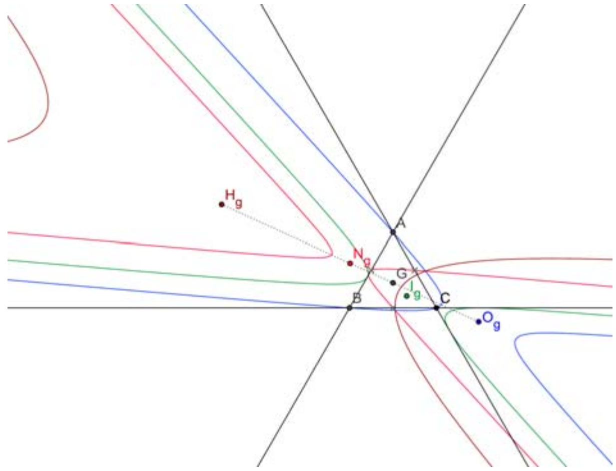

Figure 4. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (3 : 5 : 11)$. The picture shows the g-circumcircle (blue), the g-incircle (green), the g-nine-point circle (red) and the g-polar circle (brown). All g-circles go through the same infinite points.

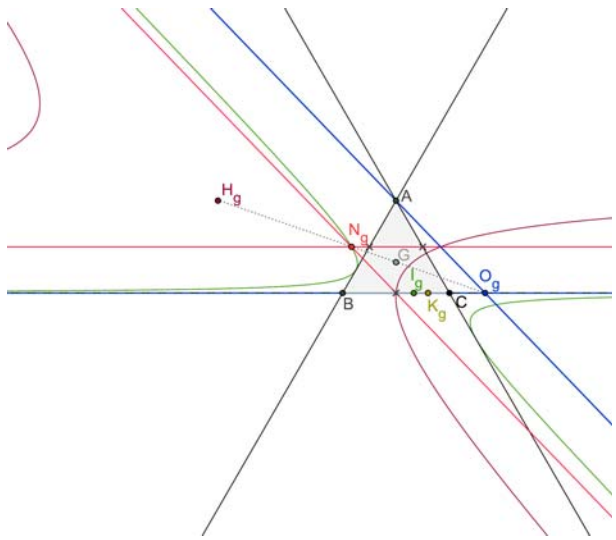

Figure 5. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (0 : 1 : 2)$. The picture shows the following generalized circles: the circumcircle (it consists of two blue lines intersecting in O_g), the incircle (green), the nine-point circle (it consists of two red lines intersecting in N_g) and the polarcircle (brown). I_g, O_g and K_g are points on the sideline BC, N_g is a point on the g-incircle.

4.3. *The g-angle between two lines.* We want to calculate the g-measure ϕ_g of the angle $\angle_g(\mathcal{L}, \mathcal{L}')$ between the proper lines $\mathcal{L} : lx + my + nz = 0$ and $\mathcal{L}' : l'x+m'y+n'z = 0$. To get a real value for ϕ_g, these lines have to be both spacelike or both timelike. The value of $\phi_g \geq 0$ is determined by the value of $\cosh(\phi_g)$. A calculation similar to the one in the Euclidean case gives the following result (we

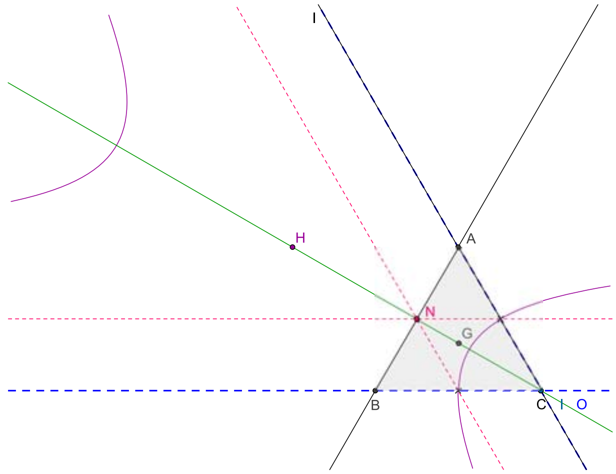

Figure 6. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (0 : 0 : 1)$. The points I_g, O_g and K_g agree with the vertex C.

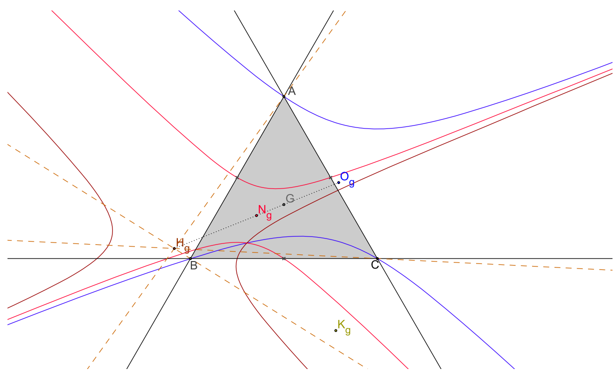

Figure 7. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (2i : 2 : 3)$.

use notations defined in §2.3):

$$\cosh(\phi_g) = \frac{|\vec{l} \cdot \vec{l}'|}{\sqrt{|\vec{l} \cdot \vec{l}|}\sqrt{|\vec{l}' \cdot \vec{l}'|}}.$$

5. Symbolic substitutions

In [10], C. Kimberling gives the following definition of the transfigured plane of a triangle:

> Suppose a, b, c are variables (or indeterminates) over the field of complex numbers and that x, y, z are homogeneous algebraic functions of $(a, b, c) : x = x(a, b, c)$, $y = y(a, b, c), z = z(a, b, c)$, all of the same degree of homogeneity and not all identically zero. Triples (x, y, z) and (x', y', z') are equivalent if $xy' = yx'$ and

> $yz' = zy'$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a 'point'. Let $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$. These three points define the reference triangle ABC. The set of all points is the transfigured plane.

We denote this transfigured plane by $\mathcal{P}$. A model of $\mathcal{P}$ can be obtained by taking a, b, c as the side lengths of the triangle ABC and $(x(a, b, c) : y(a, b, c) : z(a, b, c))$ as barycentric coordinates of a point. A second model is constructed by taking $(x : y : z)$ as trilinear coordinates of a point. If x', y', z' are functions of a, b, c, all of the same degree of homogeneity and not all identically zero, then the substitution $(a, b, c) \mapsto (x'(a, b, c), y'(a, b, c), z'(a, b, c))$ induces a transformation on $\mathcal{P}$. While in the first model, in addition to the vertices A, B, C of the reference triangle, its centroid is a fixed point of the transformations, in the second the vertices and the incenter stay fixed. The advantage of the first model, as compared to the second, is that the transformations leave the line at infinity invariant. Kimberling [9] (see also [4, §2.3]) names these substitutions and the hereby induced transformations *symbolic* because, as he says, they suffer from a geometric meaning:

> The adjective *symbolic* is applied to substitutions $(\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$ in order to distinguish between these and geometric transformations. Consider the substitution $(a, b, c) \to (bc, ca, ab)$; if a, b, c are the sidelengths $2, 4, 5$ of a triangle then bc, ca, ab are not sidelengths of a triangle. Moreover, for general a, b, c, a geometric construction of a point $(x(a, b, c) : y(a, b, c) : z(a, b, c))$ offers no clues for constructing the point $x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab)$.

The first of these two arguments can be dismissed, as in the former sections was shown that every triangle can be given any side lengths a_g, b_g, c_g except for $a_g = b_g = c_g = 0$ by taking a suitable generalized metric d_g. Furthermore, this new metric can be used for the construction of new triangle centers. We give two examples:

(1) The substitution

$$
\begin{aligned}
&(x\ , y\ , z\ ) = (-a, b, c) \mapsto (x', y', z')\\
=&(a^2(b^2+c^2) - (b^2-c^2)^2, b^2(a^2+c^2) - (c^2-a^2)^2, c^2(a^2+b^2) - (a^2-b^2)^2)
\end{aligned}
$$

maps the excenter $I_a = (-a, b, c)_{ABC}$ to the nine-point center N, the incenter I to the anticevian point ${}_aN$ of N, and the symmedian point K to N^2 (the barycentric square of N). If we take ${}_aN$ as the generalized incenter I_g, then N is going to be the generalized excenter $I_{a,g}$ and N^2 the perspector of the g-circumcircle.

(2) Construction of the contact triangle of the MacBeath inconic, starting from its center N. The substitution

$$
\begin{aligned}
&(x\ , y\ , z\ ) = (a, b, c) \mapsto (x', y', z')\\
=&(a^2(b^2+c^2) - (b^2-c^2)^2, b^2(a^2+c^2) - (c^2-a^2)^2, c^2(a^2+b^2) - (a^2-b^2)^2)
\end{aligned}
$$

maps I to N and the Gergonne point G_{e} to G/O, the isotomic conjugate of the circumcenter [2]. I is the center and G_{e} is the perspector of the incircle. Therefore, G_{e} is the isotomic conjugate of the anticomplement of I. Thus, we can construct G/O as isotomic conjugate of the anticomplement of N. The cevian triangle of G/O is the contact triangle of the MacBeath inconic.

5.1. *The geometric meaning of a symbolic substitution.* Let $x(a,b,c)$ be a barycentric center function. If in the affine plane the square of the distance between two points $U=(u,v,w)_{ABC}$ and $U'=(u',v',w')_{ABC}$ is given by

$$d^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-a^2+b^2+c^2)(u-u')^2,$$

the point X that corresponds to this center function has barycentric coordinates $(x(a,b,c) : x(b,c,a) : x(c,a,b))$. The same point is the incenter of the reference triangle when using a distance function d_x with

$$d_x^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-(x(a,b,c))^2+(x(b,c,a))^2+(x(c,a,b))^2)(u-u')^2.$$

Let x', y', z' be functions of a, b, c, all of the same degree of homogeneity and not all identically zero. The point

$$X' = (x(x'(a,b,c), y'(a,b,c), z'(a,b,c)) : \cdots : \cdots)_{ABC}$$

can be interpreted as a point with center function x, when the square of the distance d' of two points $U=(u,v,w)$ and $U'=(u',v',w')$ is given by

$$d'^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-(x'(a,b,c))^2+(y'(a,b,c))^2+(z'(a,b,c))^2)(u-u')^2.$$

6. Generalized versions of Feuerbach's conic theorem

Feuerbach's conic theorem establishes a connection between two different metrics of the affine plane.

6.1. *Feuerbach's conic theorem* (see [5, 1]). Each circumconic of a triangle running through the orthocenter H has its center on the nine point circle.

If we interpret this circumconic as a g-circumcircle of the reference triangle with center O_g, then the g-orthocenter H_g is a point on the Euclidean circumcircle. Thus, we can formulate the following generalized versions (6.1.1) and (6.1.2) of Feuerbach's conic theorem:

6.1.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two circumconics of triangle ABC. We interpret $\mathcal{C}$ and $\mathcal{C}'$ as generalized circumcircles with center O_g and $O_{g'}$. Then H_g is a point on $\mathcal{C}'$ if and only if H'_g is a point on $\mathcal{C}$.

[2] G/O is X_{284} in [9].

6.1.2. A conic runs through all four points of a g-orthocentric system if and only if its center lies on the g-orthic circle, which is the g-circumcircle of the g-orthic triangle.

6.2. *Corollaries.*

6.2.1. Each conic through the vertices of a quadrangle has its center on the nine-point conic of this (complete) quadrangle, see for example [4, Lemma 19.1.2], [12] and [15].

6.2.2. Together with the incenter I, the excenters I_a, I_b, I_c of the triangle ABC form an orthocentric system, the reference triangle ABC being the orthic triangle. Each conic running through these four points has its center on the circumcircle of the triangle ABC. Taking one of these conics and interpreting its center as a g-orthocenter, this conic is the g-polar circle of ABC. The triangle ABC is self polar with respect to it.

6.2.3. We shall call an LM metric *canonical* if H_g is a point on the (common) circumcircle of ABC. In this case the g-circumconic is a rectangular circumhyperbola of ABC, thus running through H. The center O_g of this conic is the complement of H_g and lies on the nine-point circle. The g-nine-point circle is a rectangular circumhyperbola through the vertices of the medial triangle and its orthocenter (= circumcenter of ABC). The perspector of the g-circumconic, the g-symmedian point K_g, is a point on the tripolar line of H. If H is inside the triangle, K_g has to lie outside, and weak g-centers do not exist in the real plane. We now assume that K_g lies inside the triangle ABC. In this case ABC is obtuse, the g-incenter and the g-excenters exist, and these four points lie on the polar circle of ABC.

6.2.4. If $P = (p : q : r)$ and $P' = (p' : q' : r')$ are the centers of two different circumconics $\mathcal{C}$ and $\mathcal{C}'$ of ABC, then the perspectors of these conics have coordinates $(u : v : w) = (p(-p+q+r) : \cdots : \cdots)$ respectively $(u' : v' : w') = (p'(-p'+q'+r') : \cdots : \cdots)$. The fourth (nontrivial) intersection point of $\mathcal{C}$ and $\mathcal{C}'$ has coordinates

$$\left(\frac{1}{vw'-v'w} : \frac{1}{wu'-w'u} : \frac{1}{uv'-u'v}\right).$$

In the case of $P' = O$, the fourth intersection point is called the Collings transform of P, see [9] for a definition and a bibliography.

6.2.5. Let $ABCD$ be a not degenerate quadrangle, the point D having barycentric coordinates $(d : e : f)$ with respect to ABC. Then the centers of circumconics of $ABCD$ lie on the nine-point conic of the complete quadrangle $ABCD$. This conic is also the bicevian conic of D and G with respect to the triangle ABC. The barycentric equation of this conic is

$$efx(-x+y+z) + fdy(x-y+z) + dez(x+y-z) = 0. \qquad (***)$$

Its center is the midpoint of the Varignon parallelogram of the quadrangle $ABCD$ and has coordinates $(2d+e+f : \cdots : \cdots)$. We will denote this midpoint by M_D. If by a homothety χ with center G and scale factor $\frac{1}{4}$, the quadrangle $ABCD$ is mapped onto a quadrangle $A'B'C'D'$, and D' agrees with M_D, as can be proved by a simple calculation. If we now assume that D is a point on a circumconic $\mathcal{C}$ of ABC, the point $D' = M_D$ lies on the image $\mathcal{C}'$ of $\mathcal{C}$ under χ. Suppose that the center of $\mathcal{C}$ has coordinates $(k : l : m)$, then the center of $\mathcal{C}'$ has coordinates $(2k + l + m : \cdots : \cdots)$. *Proof* of $(***)$: Taking $(d : e : f)_{ABC}$ as the g-orthocenter H_g of ABC, the symmedian point K_g has coordinates $(a_g^2 : b_g^2 : c_g^2) = (d(e+f) : e(f+d) : f(d+e))$. We can now use the equation of the g-nine-point circle $\sum_{\text{cyclic}}(S_{A,g}x^2 - a_g^2yz = 0)$ (see [16] for the ordinary version) to get the equation $(***)$. $\square$

7. Additional change of the centroid

In the plane $\mathcal{P}$ of the reference triangle ABC we choose a point $P = pA + qB + rC$ with $p + q + r = 1$ and $pqr \neq 0$. The mapping, which assigns each point Q the barycentric product $P \cdot Q$, is a bijective projective transformation of the plane. It maps the quadrangle $ABCG$ (G being the centroid) to the quadrangle $ABCP$ and the line at infinity to the tripolar line $\mathcal{T}_P$ of P. Making P the *new* centroid G_n of triangle ABC and $\mathcal{T}_P$ the *new* line at infinity, we can define a *new* metric on $\mathcal{P} - \mathcal{T}_P$: The triangle ABC is given side lengths a_g, b_g, c_g, where a_g, b_g, c_g may take any real or purely imaginary values except for $a_g = b_g = c_g = 0$. The square of the *new* distance between two points $U = (u : v : w)_{ABC}$ and $V = (u' : v' : w')_{ABC}$ is now defined by

$$d^2_{\text{new}}(U, U') = \frac{\sum_{\text{cyclic}} S_{A,g}((uv' - u'v)r + (uw' - u'w)q)^2}{(uqr + vrp + wpq)^2(u'qr + v'rp + w'pq)^2}.$$

If X_g is a triangle center of ABC with respect to the metric d_g^2 as defined in Section 1, then $X_g \cdot P$ is the corresponding center of triangle ABC with respect to the *new* metric. Figure 8 gives an illustration by showing the most important triangle centers and triangle circles.

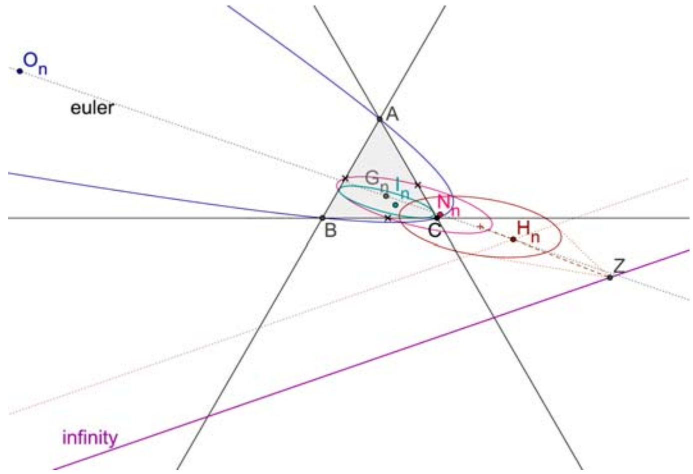

Figure 8. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (6 : 9 : 13)$. Here, the *new* line at infinity is the polar line $\mathcal{T}_P$ of the point $P = G_n$ with barycentric coordinates $(2 : 3 : 4)$. There exists a point Z on the *new* line at infinity with the following properties: The mirror image of Z in a *new* circle is its *new* center, and the polar line of Z with respect to this *new* circle is parallel to $\mathcal{T}_P$.

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Quasi-circumcenters and a Generalization of the Quasi-Euler Line to a Hexagon

Michael de Villiers

Abstract. This short note first proves an elementary property of the quasi-circumcenter of a quadrilateral, and then generalizes the quasi-Euler line of a quadrilateral to a hexagon involving its quasi-circumcenter, its quasi-orthocenter and its lamina centroid.

1. Introduction

The term "quasi-circumcenter" of a quadrilateral seems to have first been introduced by Myakishev in [2], where it is defined as follows: Given a quadrilateral $ABCD$, denote by O_a the circumcenter of triangle BCD, and similarly, O_b, for triangle ACD, O_c for triangle ABD, and O_d for triangle ABC, then the quasi-circumcenter for the quadrilateral is given by $O = O_aO_c \cap O_bO_d$.

From a problem posed in [1] to find the "best" place to build a water reservoir for four villages of more or less equal size, if the four villages are not concyclic, the following theorem was experimentally discovered and proved. It followed from the classroom discussion of a proposed solution by an undergraduate student, Renate Lebleu Davis, at Kennesaw State University during 2006.

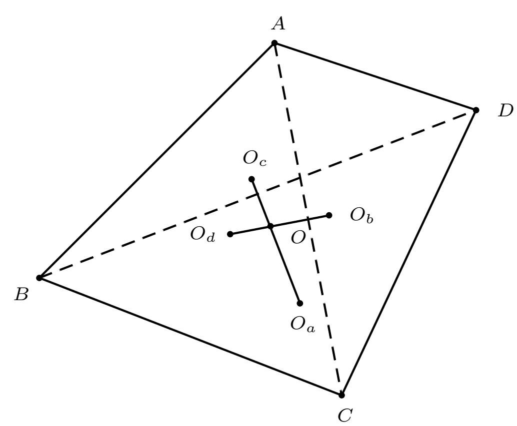

Figure 1

Theorem 1. *For a general quadrilateral $ABCD$, the quasi-circumcenter O is equidistant from A and C, and also from B and D.* (See Figure 1).

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Proof. Since both O_a and O_c lie on the perpendicular bisector of the BD, all points on the line O_aO_c are equidistant from B and D. Similarly, all points on the line O_bO_d are equidistant from A and C. Thus, the intersection O of lines O_aO_c and O_bO_d is equidistant from the two pairs of opposite vertices. □

This result was used in the Kennesaw State Mathematics Competition for High School students in 2007, as well as in the World InterCity Mathematics Competition for Junior High School students in 2009. Of interest too is that an analogous result exists as given below for the "quasi-incenter" of a quadrilateral, defined in the same way as quasi-circumcenter. The proof is left to the reader.

Theorem 2. *Given a general quadrilateral $ABCD$, then the quasi-incenter I is equidistant from AD and BC, as well as equidistant from AB and CD.*

2. The quasi-circumcenter of a hexagon

The point of concurrency given in the Theorem 3 below defines the quasi-circumcenter of a hexagon.

Theorem 3. *If the quasi-circumcenters P, Q, R, S, T, and U, respectively of the quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, and $FABC$ subdividing an arbitrary hexagon $ABCDEF$ are constructed, then the lines connecting opposite vertices of the hexagon formed by these quasi-circumcenters are concurrent.* (See Figure 2).

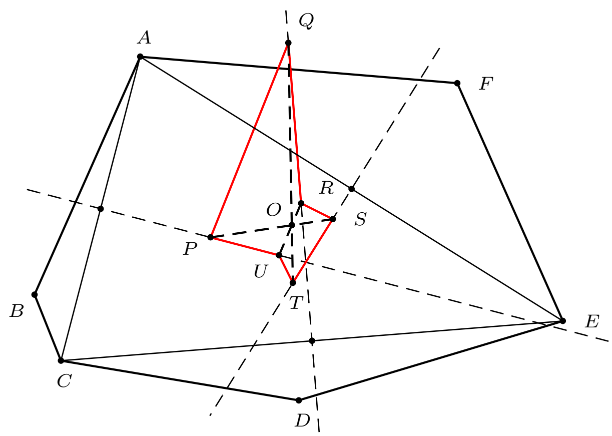

Figure 2

Proof. The result follows directly from the dual of the theorem of Pappus, which can be conveniently formulated as follows: *The diagonals of a plane hexagon whose sides pass alternatively through two fixed points, meet at a point.* With reference to Figure 2, note that alternate sides QR, ST and UP respectively lie on the perpendicular bisectors of sides AE, EC and CA of triangle ACE, and are therefore concurrent. Similarly, the other set of alternate sides are concurrent (with respect to triangle BDF). Hence, according to the dual of Pappus, the lines connecting the opposite vertices (the main diagonals) are concurrent. □

3. The quasi-Euler line of a hexagon

In [2], it is shown how the Euler line for a triangle generalizes to the Ganin - Rideau - Myakishev theorem, e.g. a quasi-Euler line for a general quadrilateral $ABCD$, which involves its lamina centroid G, its quasi-circumcenter O, and its quasi-orthocenter H (which is defined in the same way as the quasi-circumcenter), and $OH : HG = 3 : -2$. Using the result of Theorem 3, this result generalizes to a hexagon as follows.

Theorem 4. *In any hexagon, its lamina centroid G, its quasi-circumcenter O, and its quasi-orthocenter H are collinear, and $OH : HG = 3 : -2$.* (See Figure 3).

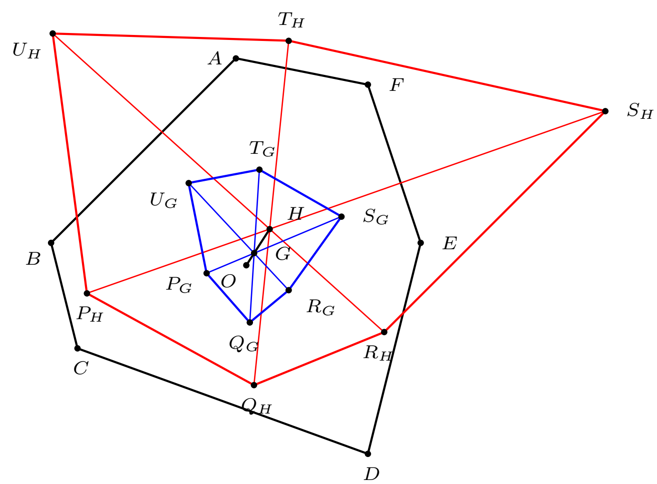

Figure 3

Proof. Subdivide the hexagon $ABCDEF$ into the same six quadrilaterals as in Theorem 3 above, and determine the quasi-circumcenter O, the lamina centroid G, and the quasi-orthocenter H of each quadrilateral, respectively labelling the formed hexagons as

$$\mathcal{P} : PQRSTU, \qquad \mathcal{P}_G : P_GQ_GR_GS_GT_GU_G, \quad \mathcal{P}_H : P_HQ_HR_HS_HT_HU_H.$$

Since the same affine relations hold between O, G and H in each quadrilateral, affine mappings exist that will map $\mathcal{P}$ onto $\mathcal{P}_G$ and $\mathcal{P}_H$. But since the diagonals of $\mathcal{P}$ are concurrent, it follows that the diagonals of $\mathcal{P}_G$ and $\mathcal{P}_H$ would also be concurrent. Respectively label and define those two points of concurrency of $\mathcal{P}_G$ and $\mathcal{P}_H$ as the centroid G and quasi-orthocenter H for the whole hexagon. Finally, since affine transformations preserve collinearity as well as ratios into which segments are divided, we note from the affine mappings between the various hexagons that the result holds. $\square$

We can similarly define the quasi-ninepoint center of a hexagon in terms of the six quasi-ninepoint centers of the subdividing quadrilaterals, from which it follows by the same affine transformations that the quasi-ninepoint center N bisects the segment OH. An interactive Java applet to illustrate and explore Theorem 4 is available for the reader at:

`http://dynamicmathematicslearning.com/quasi-euler-line-hexagon.html`.

The result unfortunately does not generalize further to an octagon as subdividing it in the same way into quadrilaterals or hexagons do not produce octagons that generally have concurrent diagonals.

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A Simple Property of Isosceles Triangles with Applications

Surajit Dutta

Abstract. In this paper we prove a simple property of isosceles triangles and give two applications: construction of third proportional line segments and construction of the inverse point with respect to a circle.

1. Introduction

Here we give an interesting property of isosceles triangles. It is known that if a and b are two given line segments, then their third proportional line segment c can be constructed geometrically; see [2, VI.11]. We can more easily construct the third proportional line segment by using the simple geometric property of the isosceles triangles. Two points A and B are inverse points with respect to the inversion circle with center O and radius r, if $OA \cdot OB = r^2$; see [3]. We can construct the inverse point with respect to a circle by using the same property of the isosceles triangle.

2. The property

Lemma 1. *Let ABC be an isosceles triangle with $AB = BC$. Let D be a point on the ray BC and let h be the ray obtained by reflecting the ray AD in the line AC. Then the ray h cuts the ray BC in a point E which lies outside the segment BC if the point D lies inside this segment* (see Figure 1) *and inside the segment BC if the point D lies outside* (see Figure 2).

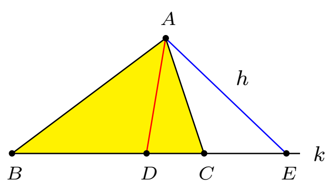

Figure 1

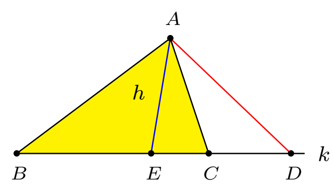

Figure 2

Proof. For Figure 1, let k denote the ray which is the part of the ray BC outside the segment BC. Then,

$$\begin{aligned}\angle(k, CA) + \angle(AC, h) &= \pi - \angle ACB + \angle(AC, h) \\ &= \pi - \angle BAC + \angle(AC, h) \\ &= \pi - \angle BAC + \angle DAC \\ &< \pi.\end{aligned}$$

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The second equality holds since the triangle ABC is isosceles, the third by reflection, and the fourth since D is an interior point of the segment BC. Now according to Euclid's fifth postulate the rays k and h meet in a point E. By the properties of reflection it is obvious that this intersection point E must lie on the ray BC outside the segment BC.

For Figure 2, $\angle(h, AC) = \angle CAD < \angle ACB = \angle BAC$. The first equality holds by reflection, the inequality by the Exterior Angle Theorem, the second equality since the triangle ABC is isosceles. Thus, the ray h runs first within the triangle ABC and meets the side BC in an interior point E. $\square$

Theorem 2. *If ABC is an isosceles triangle and points D, E are given as in* Lemma 1, *then $BC^2 = BD \cdot BE$, i.e., BC is the geometric mean of BD and BE.*

Proof. Firstly, we assume the point D inside the segment BC. The triangles ABD and EBA share the angle at vertex B. Now consider the angle sums of the triangles ABC and ABD.

$$\angle ABC + \angle BCA + \angle CAB = \pi,$$
$$\angle ABD + \angle BDA + \angle DAB = \pi.$$

Note, that $\angle CAB = \angle CAD + \angle DAB$. Thus, comparison of the two angle sums yields $\angle BDA = \angle BCA + \angle CAD$. But $\angle BCA = \angle CAB$ since the triangle ABC is isosceles and $\angle CAD = \angle EAC$ by reflection. Thus,

$$\angle BDA = \angle CAB + \angle EAC = \angle EAB.$$

Therefore the triangles ABD and EBA are inversely similar, and

$$BC : BD = AB : BD = BE : BA = BE : BC.$$

From this, $BC^2 = BD \cdot BE$.

Secondly, if the point D lies outside the segment BC then interchanging the roles of the points D and E in the previous argument yields the same result. $\square$

3. Applications

3.1. *Construction of third proportional line segments.* Let a and b be the lengths of two line segments, and we have to draw a line segment of length c such that the square of b equals the product of a and c.

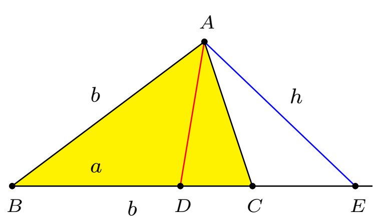

Figure 3

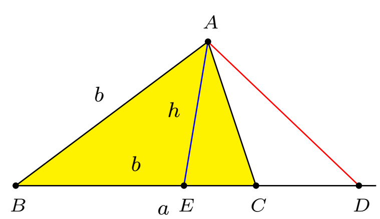

Figure 4

For this, construct an isosceles triangle ABC with $AB = BC = b$ (see Figures 3 and 4). Let D be a point on the ray BC such that $BD = a$ and let h be the line obtained by reflecting the ray AD in the line AC. By Lemma 1 the ray h cuts the ray BC in a point E. Theorem 2 implies $BE = c$.

3.2. *Construction of the inverse point with respect to a circle.* Consider a circle $\mathcal{C}$ with center B and D a point which may lies inside or outside of the circle $\mathcal{C}$. In both cases we can follow the same steps to construct the inverse point of D with respect to the circle $\mathcal{C}$, a case distinction as in the usual treatments, see for example [1, pp.108–109], is not needed.

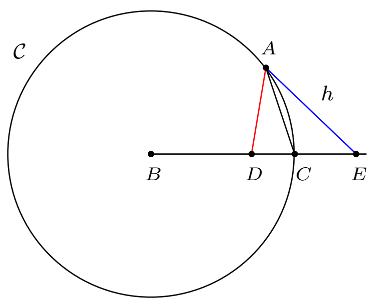

Figure 5

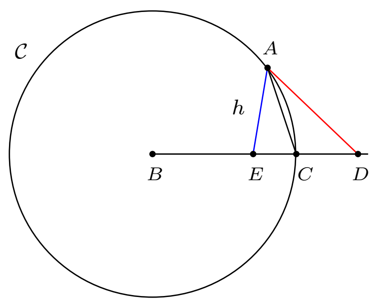

Figure 6

Take the intersection point C of the ray BD with the circle $\mathcal{C}$, see Figures 5 and 6. Connect the point C with an arbitrary point A on the circle $\mathcal{C}$ (different from C) and let h be the ray obtained by reflecting the ray AD in the line AC. The ray h cuts the ray BC in a point E by Lemma 1 which is the inverse point of D with respect to the circle $\mathcal{C}$ in view of Theorem 2.

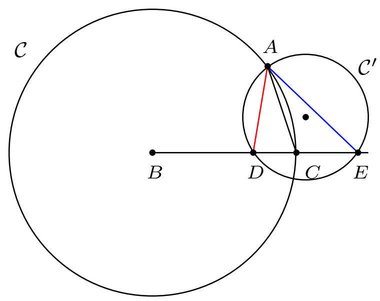

Figure 7

Note that the circumcircle $\mathcal{C}'$ of the triangle ADE is orthogonal to the circle $\mathcal{C}$, since it is invariant under the inversion at the circle $\mathcal{C}$ (see Figure 7).

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A Note on Haga's Theorems in Paper Folding

Hiroshi Okumura

Abstract. Haga's three theorems in the mathematics of square paper folding are unified in a simple way.

1. Introduction

Haga's famous theorems in the mathematics of square paper folding consists of three main parts [1, 2, 3]. Let us assume that $ABCD$ is a piece of square paper with a point E on the side AD. We fold the paper so that the corner C coincides with E and the side BC is carried into $B'E$, which intersects the side AB at a point F (see Figure 1). We call this Haga's fold of the first kind. Haga discovered if E is the midpoint of AD, then F divides AB in the ratio $2 : 1$ internally (first theorem). Also if F is the midpoint of AB, then E divides AD in the ratio $2 : 1$ internally (third theorem; see Figure 2).

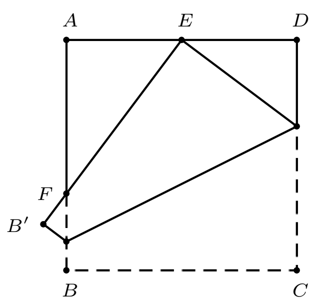

Figure 1

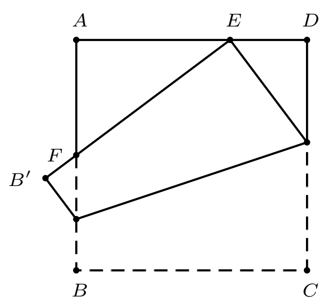

Figure 2

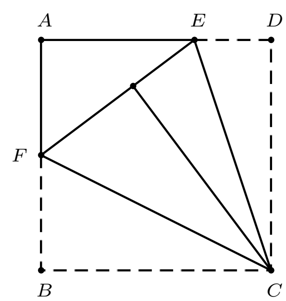

Figure 3

Let F be a point on the side AB such that the reflection of B in the line CF coincides with the reflection of D in the line CE (see Figure 3). This is called Haga's fold of the second kind, with the crease lines CE and CF. He discovered if F is the midpoint of AB, then E divides AD in the ratio $2 : 1$ internally (second theorem). In this note, we show that these three facts are unified in a simple way.

2. Main theorem

It is easy to show AF as a function of DE. Indeed, the following fact is given in [1]: If $AB = 1$, then $AF = \frac{2DE}{1+DE}$ holds for the fold of the first kind. Also he pointed out that his fold of the first kind derived from the fold of the second kind, and vice versa. In fact for the fold of the first kind, the reflection of B in the line CF coincides with the reflection of D in the line CE (see Figure 4). Haga's results are unified as follows.

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Theorem. *The relation* $\frac{AF}{FB} = 2 \cdot \frac{DE}{EA}$ *holds for Haga's folds of the first and second kinds*.

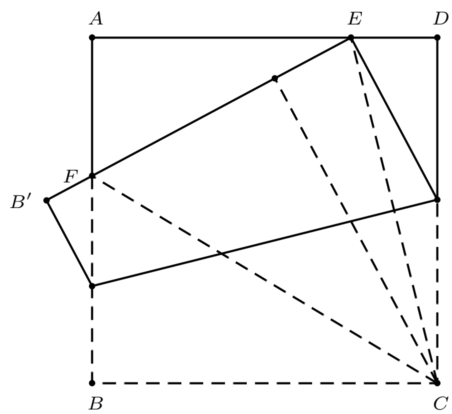

Figure 4

Proof. Let $AB = 1$. The theorem can be proved using the relation $AF = \frac{2DE}{1+DE}$. We give a proof using trigonometry. By the above remark, it is sufficient to prove for the fold of the second kind. Let $\theta = \angle DCE$ and $t = \tan\theta$. Then we get $DE = t$ and $EA = 1 - t$ (see Figure 4). This implies $\frac{DE}{EA} = \frac{t}{1-t}$. While $\angle BCF = \frac{\pi}{4} - \theta$ leads to $FB = \tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-t}{1+t}$ and $AF = 1 - FB = \frac{2t}{1+t}$. Hence we get $\frac{AF}{FB} = \frac{2t}{1-t}$. The theorem is now proved. □

By the theorem $AF : FB = k : 1$ is equivalent to $DE : EA = k : 2$ for a positive real number k. Haga's results are obtained when $k = 1$ and $k = 2$.

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Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon

Nikolaos Dergiades

Abstract. We reformulate and give an elegant proof of a wonderful theorem of Dao Thanh Oai concerning the centers of the circumcircles of the six triangles each bounded by the lines containing three consecutive sides of the hexagon.

In slightly different notations Dao Thanh Oai [3] has posed the problem of proving the following remarkable theorem.

Theorem (Dao). *Let A_i, $i = 1, 2, \dots, 6$, be six points on a circle. Taking subscripts modulo 6, we denote, for $i = 1, 2, \dots, 6$, the intersection of the lines A_iA_{i+1} and $A_{i+2}A_{i+3}$ by B_{i+3}, and the circumcenter of the triangle $A_iA_{i+1}B_{i+2}$ by C_{i+3}. The lines C_1C_4, C_2C_5, C_3C_6 are concurrent.*

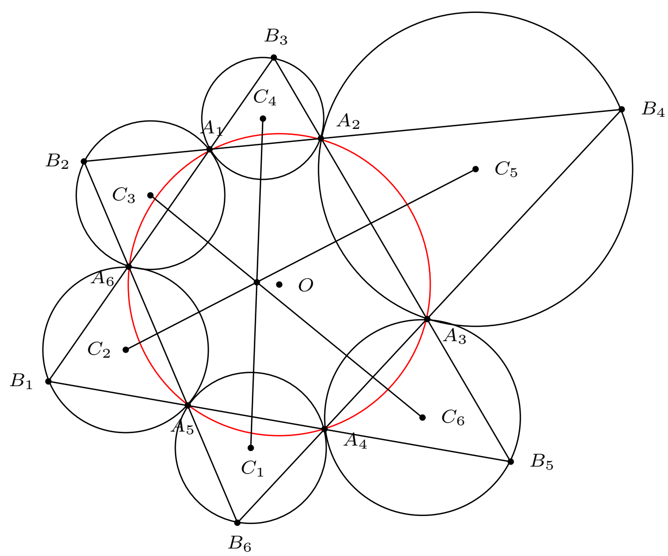

Figure 1

Indeed a proof with tedious computer aided calculations with barycentric coordinates has been given in [4]. In this note we give an elegant proof using complex numbers by considering the given hexagon as inscribed on the unit circle with center 0 in the complex plane.

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Lemma 1. *If A, B, C, D are points on the unit circle with affixes a, b, c, d respectively, and the lines AB, CD intersect at E, then the circumcenter P of triangle ACE has affix $p = \frac{ac(b-d)}{ab-cd}$.*

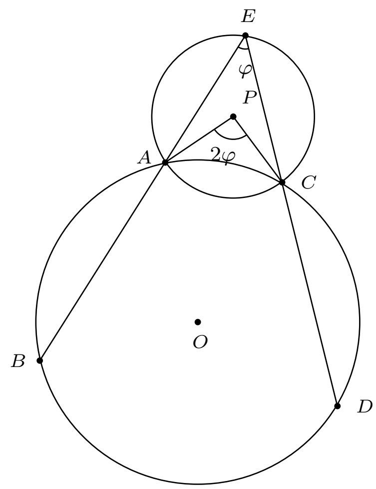

Figure 2

Proof. If the oriented angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $\varphi = \angle AEC$, then $\angle APC = 2\varphi$. If $z = \cos\varphi + i\sin\varphi$, then since for a point A on the unit circle, the conjugate of its affix is $\overline{a} = \frac{1}{a}$, we conclude that

$$(c-p) = (a-p)z^2. \qquad (*)$$

Now,

$$\frac{d-c}{|d-c|} = \frac{b-a}{|b-a|}z \implies \frac{(d-c)^2}{|d-c|^2} = \frac{(b-a)^2}{|b-a|^2}z^2 \implies \frac{d-c}{\overline{d}-\overline{c}} = \frac{b-a}{\overline{b}-\overline{a}}z^2$$
$$\implies \frac{d-c}{\frac{1}{d}-\frac{1}{c}} = \frac{b-a}{\frac{1}{b}-\frac{1}{a}}z^2 \implies cd = abz^2.$$

This reduces to $cd = abz^2$, and from $(*)$, $(c-p)ab = (a-p)cd$. From this, $p = \frac{ac(b-d)}{ab-cd}$. $\square$

To avoid excessive use of subscripts, we reformulate and prove Dao's Theorem in the following form.

Theorem 2. *Let A, B, C, X, Y, Z be arbitrary points on the unit circle with complex affixes a, b, c, x, y, z respectively. The lines ZB, XC, YA and CY, AZ, BX bound the triangles $A'B'C'$ and $A''B''C''$. If A_1, B_1, C_1, A_2, B_2, C_2 are the circumcenters of the circles $(A'YC)$, $(B'ZA)$, $(C'XB)$, $(A''BZ)$, $(B''CX)$, $(C''AY)$, then the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.*

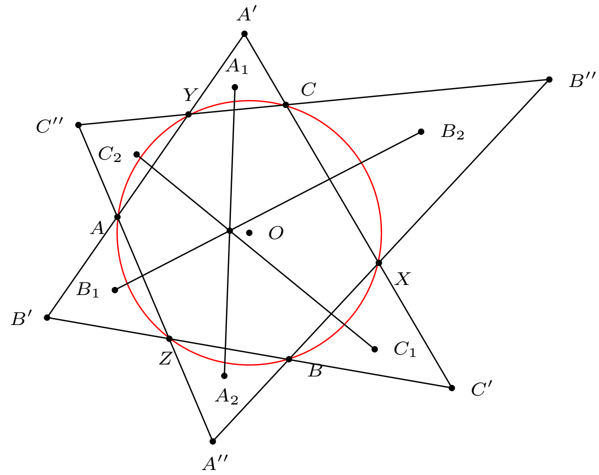

Figure 3

Proof. From Lemma 1, we have the affixes of the circumcenters:

$$a_1 = \frac{cy(x-a)}{cx-ay}, \qquad a_2 = \frac{bz(a-x)}{az-bx},$$
$$b_1 = \frac{az(y-b)}{ay-bz}, \qquad b_2 = \frac{cx(b-y)}{bx-cy},$$
$$c_1 = \frac{bx(z-c)}{bz-cx}, \qquad c_2 = \frac{ay(c-z)}{cy-az}.$$

For every point W on the line A_1A_2, the number $t = \frac{w-a_1}{w-a_2}$ is real. Therefore, $t = \frac{\overline{w}-\overline{a_1}}{\overline{w}-\overline{a_2}}$. This gives the equation of the line A_1A_2 as $\begin{vmatrix} w & \overline{w} & 1 \\ a_1 & \overline{a_1} & 1 \\ a_2 & \overline{a_2} & 1 \end{vmatrix} = 0$ (see [1]). Since a, b, c, x, y, z are unit complex numbers,

$$\overline{a_1} = \frac{\overline{cy}(\overline{x}-\overline{a})}{\overline{cx}-\overline{ay}} = \frac{\frac{1}{c}\cdot\frac{1}{y}\left(\frac{1}{x}-\frac{1}{a}\right)}{\frac{1}{c}\cdot\frac{1}{x}-\frac{1}{a}\cdot\frac{1}{y}} = \frac{x-a}{cx-ay}.$$

Similarly, $\overline{a_2} = \frac{a-x}{az-bx}$. From these we obtain the equation of the line A_1A_2, and likewise those of B_1B_2 and C_1C_2. These are

$$(az+cx-ay-bx)w + (bcxy+abyz-cayz-bczx)\overline{w} + (a-x)(cy-bz) = 0,$$
$$(bx+ay-bz-cy)w + (cayz+bczx-abzx-caxy)\overline{w} + (b-y)(az-cx) = 0,$$
$$(cy+bz-cx-az)w + (abzx+caxy-bcxy-abyz)\overline{w} + (c-z)(bx-ay) = 0.$$

The three lines are concurrent if and only if the determinant

$$\begin{vmatrix} az+cx-ay-bx & bcxy+abyz-cayz-bczx & (a-x)(cy-bz) \\ bx+ay-bz-cy & cayz+bczx-abzx-caxy & (b-y)(az-cx) \\ cy+bz-cx-az & abzx+caxy-bcxy-abyz & (c-z)(bx-ay) \end{vmatrix} = 0.$$

This clearly is true since each column sum is equal to 0. □

From the equations of the lines it is clear that the point of concurrency is O if and only if

$$(a-x)(cy-bz) = (b-y)(az-cx) = (c-z)(bx-ay) = 0.$$

Assume the points A, B, C, X, Y, Z distinct. This condition is satisfied precisely when the unit complex affixes satisfy $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. From this we conclude that XYZ is obtained from ABC by a rotation.

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Two Tangent Circles from Jigsawing Quadrangle

Tran Quang Hung

Abstract. We establish the tangency of two circles associated with the jigsawing quadrangle of Floor van Lamoen.

Floor van Lamoen [1] has given a construction, for an acute angled triangle ABC, a pair of isotomic points QP, Q on BC such that when perpendiculars at P, Q to BC are constructed to intersect AB and AC at P' and Q' respectively, the quadrangle $PP'Q'Q$ satisfies $P'Q' = PP' + QQ'$ (see Figure 1). If the triangles BPP' and CQQ' are rotated about P' and Q' respectively, so that the images of P and Q coincide at a point on $P'Q'$, then the images of B and C coincide at a point A' such that the quadrangle $AP'A'Q'$ is cyclic. van Lamoen also showed that AA' passes through the circumcenter O of triangle ABC. Denote by S the center of the circle through the four points. We show that this circle is tangent at A' to another circle naturally associated with the triangle. Let the tangents at B and C to the circumcircle of ABC intersect at T.

Theorem. *The circles through A, P', A', Q' is tangent at A' to the circle, center T, passing through B and C.*

Proof. (1) We first show that A' also lies on the circle, center T, passing through B and C. Note that triangles $P'BA'$ and $Q'CA'$ are isosceles (see Figure 1). Therefore,

$$\begin{aligned}
\angle BA'C =&\ 360^\circ - \angle BA'P' - \angle P'A'Q' - \angle Q'A'C \\
=&\ 360^\circ - \frac{180^\circ - \angle BP'A'}{2} - (180^\circ - \angle BAC) - \frac{180^\circ - \angle CQ'A'}{2} \\
=&\ \frac{\angle BP'A' + \angle CQ'A'}{2} + \angle BAC \\
=&\ \frac{\angle AQ'A' + \angle CQ'A'}{2} + \angle BAC \\
=&\ 90^\circ + \angle BAC \\
=&\ 180^\circ - \frac{1}{2}\angle BTC.
\end{aligned}$$

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This shows that A' lies on the circle, center T, passing through B and C.

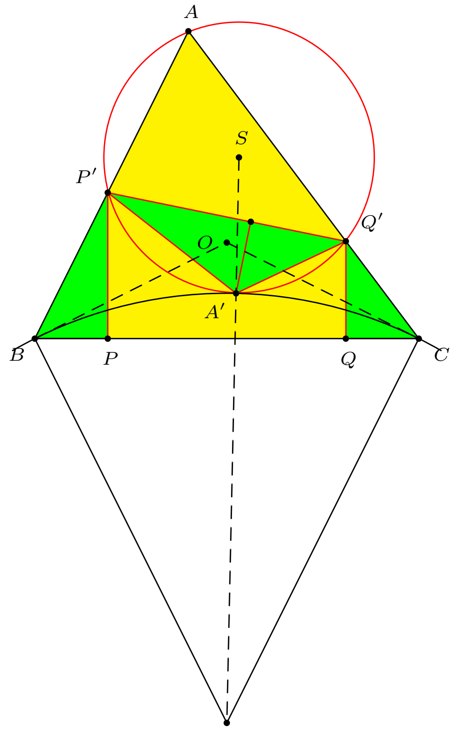

Figure 1.

(2) We show that the points S, A', and T' are collinear.

$$\begin{aligned}
& \angle SA'Q' + \angle Q'A'C + \angle CA'T \\
= \; & (90^\circ - \angle A'P'Q') + \angle Q'CA' + (90^\circ - \angle A'BC) \\
= \; & 180^\circ - \angle BP'P + \angle ACB - (180^\circ - \angle BA'C) \\
= \; & 180^\circ - (90^\circ - \angle ABC) + \angle ACB - (90^\circ - \angle BAC) \\
= \; & \angle ABC + \angle ACB + \angle BAC \\
= \; & 180^\circ.
\end{aligned}$$

Therefore, A' lies on the segment ST.

It follows that the two circles are tangent externally at A'. □

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Two More Pairs of Archimedean Circles in the Arbelos

Tran Quang Hung

Abstract. We construct two more pairs of Archimedean circles in the arbelos. One of them is a pair constructed by Floor van Lamoen in another way.

In addition to the two pairs of Archimedean circles associated with the arbelos constructed by Dao Thanh Oai [1], we construct two more pairs. Given a segment AB with an interior point C, consider the semicircles (O), (O_1), (O_2) with diameters AB, AC, and CB, all on the same side of AB. The perpendicular to AB at C intersects (O) at D. Let a and b be the radii of the semicircles (O_1) and (O_2) respectively. The Archimedean circles have radii $\frac{ab}{a+b}$.

Theorem 1. *Let the perpendiculars to AB at O_1 and O_2 intersect (O) at E and F respectively. If AF intersects (O_1) at H and BE intersects (O_2) at K, then the circles tangent to CD with centers H and K are Archimedean circles.*

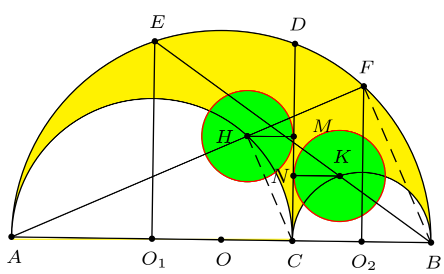

Figure 1

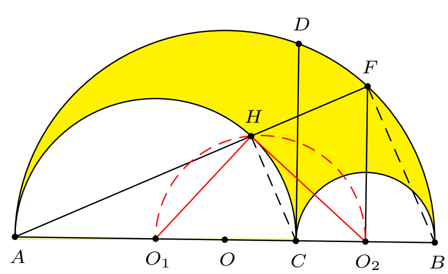

Figure 2

Proof. Let M and N be the orthogonal projections of H and K on CD respectively. Since CH and BF are both perpendicular to AF, the right triangles CHM and FBO_2 are similar (see Figure 1).

$$\frac{HM}{BO_2} = \frac{CH}{FB} = \frac{AC}{AB} \implies HM = BO_2 \cdot \frac{AC}{AB} = b \cdot \frac{2a}{2a+2b} = \frac{ab}{a+b}.$$

Therefore the circle $H(M)$ is Archimedean; similarly for $K(N)$. □

Floor van Lamoen has kindly pointed out that this pair has appeared before in a different construction, as (K_1) and (K_2) in [3] (see also (A_{25a}) and (A_{25b}) in [4]). We show that H and K are intersections of (O_1) and (O_2) with the mid-semicircle with diameter O_1O_2. It is enough to show that $\angle O_1HO_2 = \angle O_1KO_2 = 90°$.

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In Figure 2, O_2 is the midpoint of BC, and BF, CH are parallel. The parallel through O_2 to these lines is the perpendicular bisector of FH. This means that $O_2F = O_2H$, and

$$\begin{aligned}\angle O_1HO_2 &= 180^\circ - \angle O_1HA - \angle O_2HF\\ &= 180^\circ - \angle O_1AH - \angle O_2FH\\ &= \angle AO_2F = 90^\circ.\end{aligned}$$

Similarly, $\angle O_1KO_2 = 90^\circ$.

Theorem 2. *Let P be the intersection of AD with the semicircle with diameter AO_2, and Q that of BD with the semicircle with diameter BO_1. The circles tangent to CD with centers P and Q are Archimedean.*

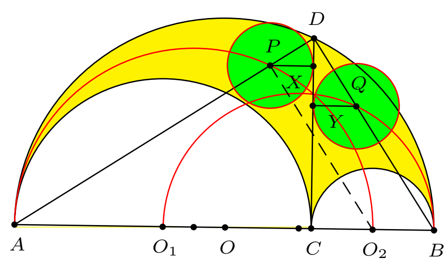

Figure 3

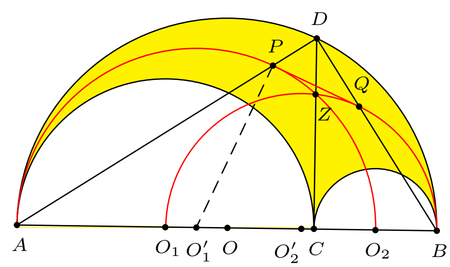

Figure 4

Proof. Let X and Y be the orthogonal projections of P and Q on CD (see Figure 3). Since BD and O_2P are both perpendicular to AD, they are parallel.

$$\frac{PX}{AC} = \frac{DP}{DA} = \frac{BO_2}{BA} \implies PX = AC \cdot \frac{BO_2}{BA} = 2a \cdot \frac{b}{2a+2b} = \frac{ab}{a+b}.$$

Therefore, the circle $P(X)$ is Archimedean; similarly for $Q(Y)$. $\square$

We show that PQ is a common tangent to the semicircles with diameters AO_2 and BO_1 (see [5]). In Figure 4, these two semicircles intersect at a point Z on CD satisfying $CZ^2 = 2a \cdot b = a \cdot 2b$. Now, $DP \cdot DA = DZ(DC + ZC) = DQ \cdot DB$. It follows that $\frac{DP}{DQ} = \frac{DB}{DA}$, so that the right triangles DPQ and DBA are similar. Now, if O_1' is the midpoint of AO_2, then

$$\begin{aligned}\angle O_1'PQ &= 180^\circ - \angle O_1'PA - \angle DPQ\\ &= 180^\circ - \angle BAD - \angle DBA\\ &= \angle ADB = 90^\circ.\end{aligned}$$

Therefore, PQ is tangent to the semicircle on AO_2 at P. Similarly, it is also tangent to the semicircle on BO_1 at Q. It is a common tangent of the two semicircles.

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A Special Point in the Arbelos Leading to a Pair of Archimedean Circles

Floor van Lamoen

Abstract. In 2011 Quang Tuan Bui found a beautiful and simple pair of Archimedean circles, which were published on a website. From this pair we find a special point in the Arbelos leading to a related pair of Archimedean circles.

In 2011 Quang Tuan Bui found a beautiful and elegant pair of Archimedean circles. These were published by Alexander Bogomolny on his website [1]. In 2013 the circle pair was found independently by Hiroshi Okumura [2, 3].

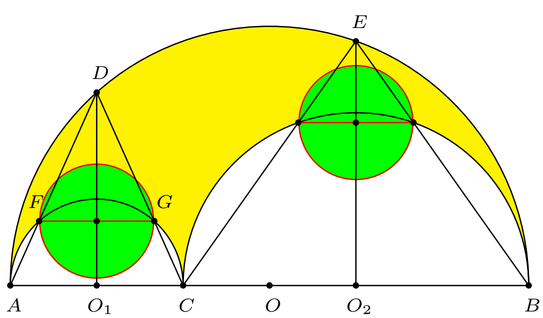

Figure 1.

Consider an arbelos with (O) being the semicircle with diameter AB, while the point C on AB defines the smaller semicircles (O_1) and (O_2) on AC and BC respectively. Let the perpendiculars to AB from O_1 and O_2 meet (O) in D and E respectively. The segments DA and DC meet (O_1) in two points F and G. The segment FG is the diameter of an Archimedean circle (see Figure 1). Likewise an Archimedean circle is found from E.

To prove the correctness of the finding of Bui, we let r, r_1 and r_2 be the radii of (O), (O_1) and (O_2) respectively. Note that $AF : AD = r_1 : r$, so that $AD : FD = r : r_2$. Of course G divides CD in the same ratio. So, by similarity $FG = \frac{r_2}{r} \cdot AC = 2 \cdot \frac{r_1 r_2}{r}$. the Archimedean diameter.

Now one may wonder what the locus of points P is such that PA and PC cut a chord ST off (O_1) congruent to FG. See Figure 2.

For ST to be congruent to FG, it is clear that arcs FS and GT must be congruent. From this the angles DAP and DCP must be congruent, and we conclude that $ACDP$ is cyclic. The locus of P is thus the circumcircle of triangle ACD.

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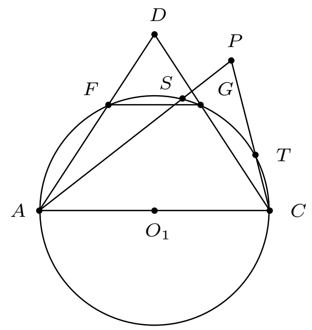

Figure 2.

Similarly, the locus of P for PB and PC to cut congruent to the one cut out by EB and EC is the circumcircle BCE. Now the circumcircles of ACD and BCE intersect, apart from C, in a point L. This point is thus the only point leading to an Archimedean circle on each of the semicircles (O_1) and (O_2). A notable point (see Figure 3).

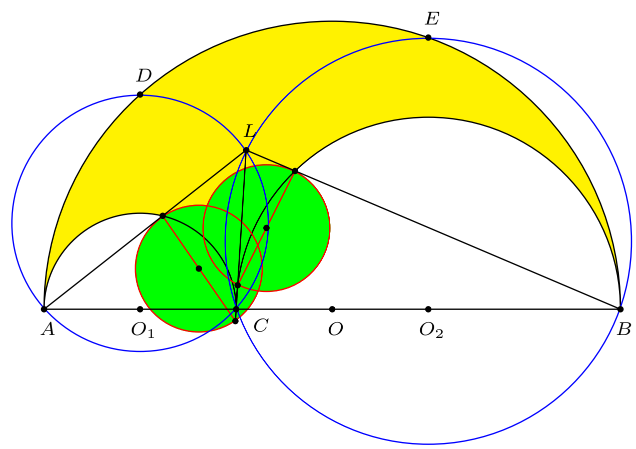

Figure 3.

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Three Constructions of Archimedean Circles in an Arbelos

Paul Yiu

Abstract. We give ruler and compass constructions of three Archimedean circles in an arbelos, each with the endpoints of a diameter on the smaller semicircles. In the first case, the diameter contains the intersection of the defining smaller semicircles of the arbelos. In the second case, these endpoints are the intersections of the smaller semicircles with the lines joining the endpoints of the base of the arbelos to a fixed point on the dividing perpendicular line. In the third case, the diameter containing these endpoints is parallel to the base line of the arbelos.

1. Introduction

We consider three constructions of Archimedean circles in an arbelos. Given a segment AB with an interior point C, the semicircles (O), (O_1), (O_2) with diameters AB, AC, CB on the same side of AB bound the arbelos, with dividing line CD perpendicular to AB. Let a and b be the radii of the semicircles (O_1) and (O_2). Circles with radius $t := \frac{ab}{a+b}$ are called Archimedean. They are congruent to the Archimedean twin circles [1, 2, 3]. We shall make use of the Archimedean circles with centers O_1 and O_2 respectively.

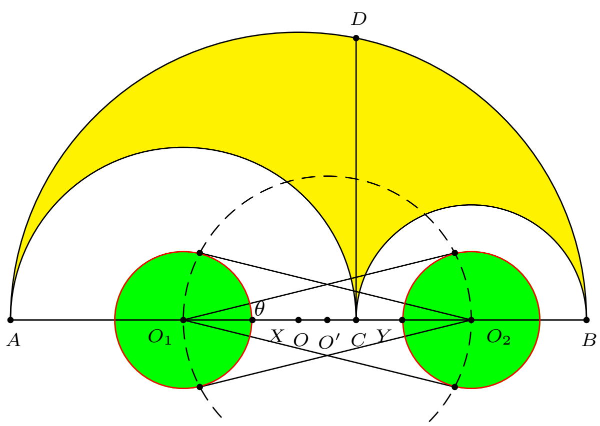

Figure 1.

In particular, we shall encounter below lines making an angle θ with AB defined by

$$\sin\theta = \frac{t}{a+b}. \tag{1}$$

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The author thanks Floor van Lamoen for his excellent suggestion of searching for the Archimedean circle in the last section of this paper.

Such lines are parallel to the tangents from O_1 to the Archimedean circle with center O_2 or vice versa. The points of tangency are the intersections with the circle with diameter O_1O_2 and center O' (see Figure 1). We adopt a Cartesian coordinate system with origin at C, so that the points A and B have coordinates $(-2a, 0)$ and $(2b, 0)$ respectively. The equations of the circles (O_1) and (O_2) are

$$\begin{aligned}(x+a)^2 + y^2 =&\ a^2,\\ (x-b)^2 + y^2 =&\ b^2.\end{aligned}$$

2. Archimedean circles with diameter through C and endpoints on (O_1) and (O_2)

Consider the construction of a line $\mathscr{L}$ through C intersecting the circles (O_1) and (O_2) at A' and B' respectively so that the segment $A'B'$ has length $2t$, and the circle with diameter $A'B'$ is Archimedean. If $\mathscr{L}$ has slope m, then these intersections are $A' = \left(\frac{-2a}{1+m^2}, \frac{-2am}{1+m^2}\right)$ and $B' = \left(\frac{2b}{1+m^2}, \frac{2bm}{1+m^2}\right)$. Since the difference between the x-coordinates is $\frac{2(a+b)}{1+m^2}$, $A'B'^2 = \frac{4(a+b)^2}{1+m^2}$. This is equal to $(2t)^2$ if and only if

$$1 + m^2 = \frac{(a+b)^4}{a^2b^2} = \frac{(a+b)^2}{t^2} = \csc^2\theta$$

for the angle θ defined by (1). It follows that the slope $m = \pm\cot\theta$, and the line $\mathscr{L}$ is perpendicular to a tangent from O_1 to the Archimedean circle at O_2.

Theorem 1. *A line through C intersecting (O_1) and (O_2) at the endpoints of a diameter of an Archimedean circle if and only if it is perpendicular to a tangent from O_1 to the Archimedean circle with center O_2.*

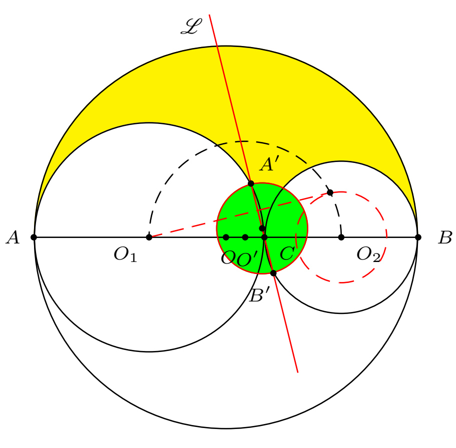

Figure 2(a)

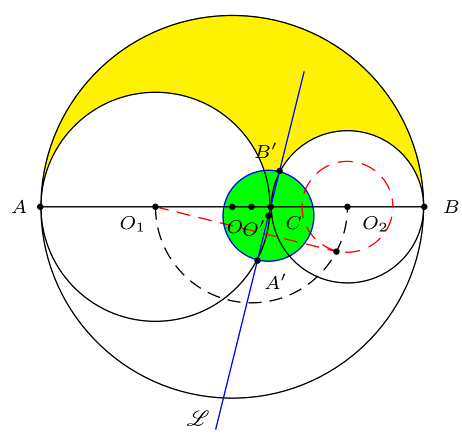

Figure 2(b)

3. Archimedean circle from intersections of QA, QB with Q on CD

We construct a point Q on the line CD such that the intersections of AQ with (O_1) and BQ with (O_2) are the endpoints of a diameter of an Archimedean circle (see Figure 3). Let $Q = (0, q)$. These intersections are $A'' = \left(\frac{-2aq^2}{q^2+4a^2}, \frac{4a^2q}{q^2+4a^2}\right)$ and $B'' = \left(\frac{2bq^2}{q^2+4b^2}, \frac{4b^2q}{q^2+4b^2}\right)$. From these coordinates,

$$A''B''^2 = \frac{4(a+b)^2q^4}{(q^2+4a^2)(q^2+4b^2)}.$$

This is equal to $(2t)^2$ if and only if $(a+b)^4q^4 - a^2b^2(q^2+4a^2)(q^2+4b^2) = 0$. Rewriting this as

$$((a+b)^4 - a^2b^2)q^4 - 4a^2b^2(a^2+b^2)q^2 - 16a^4b^4 = 0, \tag{2}$$

we see that there is a unique positive root.

Theorem 2. *There is a unique point Q on the dividing line CD such that the intersections of QA with (O_1) and QB with (O_2) are the endpoints of a diameter of an Archimedean circle.*

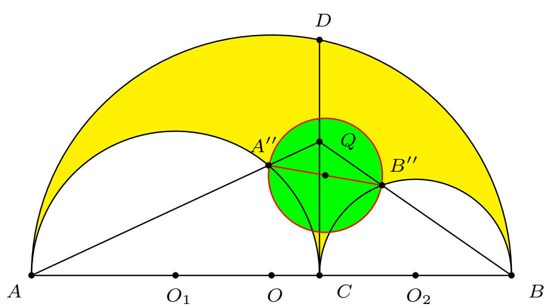

Figure 3.

From (2), we obtain explicitly

$$q^2 = \frac{a^2b^2}{(a+b)^4 - a^2b^2}\left(2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}\right).$$

Now, $\frac{a^2b^2}{(a+b)^4-a^2b^2} = \frac{t^2}{(a+b)^2-t^2} = \tan^2\theta$ for θ defined by (1). It is enough to construct a segment CX on AB with

$$CX^2 = 2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}. \tag{3}$$

Let M be the "highest" point of (O), i.e., the intersection of (O) with the perpendicular to AB at O (see Figure 4). For the construction of X, we make use of the following.

(i) $2(a^2+b^2) = (a+b)^2 + (a-b)^2 = OM^2 + OC^2 = CM^2$,

(ii) $(a-b)^2 + 4(a+b)^2 = 4(O'O^2 + OM^2) = 4 \cdot O'M^2$.

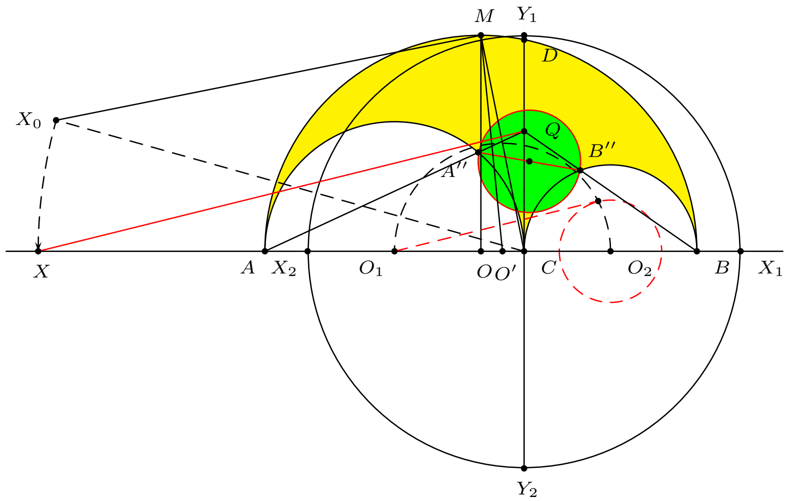

Figure 4.

Construction. (1) On different sides of AB on the perpendicular at C, choose points Y_1 and Y_2 such that $CY_1 = OM$ and $CY_2 = O'M$. Construct the circle with diameter Y_1Y_2, to intersect the line AB in a segment X_1X_2.

(2) On the perpendicular to MC at M, choose a point X_0 such that $MX_0 = X_1X_2$.

(3) Let X be a point on AB such that $CX = CX_0$. The segment CX has length given by (3) above.

(4) Construct a parallel through X to a tangent from O_1 to the Archimedean circle with center O_2, to intersect CD at Q. This is the unique point Q in Theorem 2.

4. Archimedean circle with a diameter parallel to AB and endpoints on (O_1) and (O_2)

We consider the possibility of an Archimedean circle with a diameter parallel to AB having its endpoints one on each of the semicircles (O_1) and (O_2). If the diameter is at a distance d from AB, its endpoints are among the points

$$X_- = (-a - \sqrt{a^2 - d^2}, d), \quad X_+ = (-a + \sqrt{a^2 - d^2}, d);$$
$$Y_- = (b - \sqrt{b^2 - d^2}, d), \quad Y_+ = (b + \sqrt{b^2 - d^2}, d).$$

The differences between the lengths of the various segments and the diameter of an Archimedean circle are

$$\begin{aligned}
X_-Y_+ - 2t &= a + b - 2t + \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2},\\
X_-Y_- - 2t &= a + b - 2t + \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2},\\
X_+Y_+ - 2t &= a + b - 2t - \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2},\\
X_+Y_- - 2t &= a + b - 2t - \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}.
\end{aligned}$$

The condition

$$(X_-Y_+ - 2t)(X_-Y_- - 2t)(X_+Y_+ - 2t)(X_+Y_- - 2t) = 0$$

simplifies into

$$4t(a-t)(b-t)(a+b-t) - (a+b-2t)^2d^2 = 0. \tag{4}$$

This clearly has a unique positive root d.

Theorem 3. *There is a unique Archimedean circle with a diameter parallel to AB, having endpoints one on each of the semicircles (O_1) and (O_2).*

Now, by Heron's formula, $t(a-t)(b-t)(a+b-t)$ is the square of the area of a triangle with sides a, b, and $a+b-2t$. From (4), d is the altitude of the triangle on the side $a+b-2t$. This leads to the following simple construction.

Construction. Let the Archimedean circle with center O_1 intersect O_1C at X and that with center O_2 intersect CO_2 at Y. Construct a point Z (on the same side of the arbelos) such that $XZ = a$ and $YZ = b$. The parallel to AB through Z is the line which intersects (O_1) and (O_2) at two points at a distance $2t$ apart.

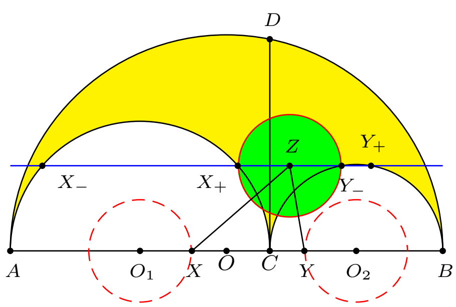

Figure 5a: $a^3 < a^2b + ab^2 + b^3$

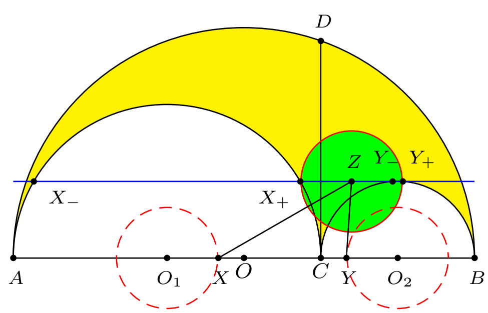

Figure 5b: $a^3 > a^2b + ab^2 + b^3$

The point Z is indeed the center of the Archimedean circle in question.

Assume $a \geq b$ without loss of generality. Note that $O_1X_+ = XZ = a$, and they are parallel since

$$\sin X_+O_1C = \frac{d}{a} = \sin ZXY.$$

This means that O_1XZX_+ is a parallelogram, and $ZX_+ = XO_1 = t$. The circle, center Z, passing through X_+ is Archimedean. The other end of the the diameter is Y_- or Y_+ according as a^3 is less than or greater than $a^2b + ab^2 + b^3$ (see Figures 5a and 5b). This follows from the simple fact

$$\sqrt{b^2 - d^2} = \begin{cases} \frac{-a^3b+a^2b^2+ab^3+b^4}{(a+b)(a^2+b^2)} & \text{if } a^3 < a^2b + ab^2 + b^3, \\ \frac{a^3b-a^2b^2-ab^3-b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 > a^2b + ab^2 + b^3. \end{cases}$$

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A Purely Synthetic Proof of Dao's Theorem on Six Circumcenters Associated with a Cyclic Hexagon

Telv Cohl

Abstract. We present a purely synthetic proof of Dao's theorem on six circumcenters associated with a cyclic hexagon.

Nikolaos Dergiades [4] has given an elegant proof using complex numbers of the following theorem.

Theorem (Dao [2]). *Let six points A, B, C, D, E, F lie on a circle, and $U = AF \cap BC$, $V = AB \cap CD$, $W = BC \cap DE$, $X = CD \cap EF$, $Y = DE \cap FA$, $Z = EF \cap AB$. Denote by O_1, O_2, O_3, O_4, O_5, O_6 the circumceneters of the six triangles ABU, BCV, CDW, DEX , EFY, FAZ. The three lines O_1O_4, O_2O_5, O_3O_6 are concurrent.*

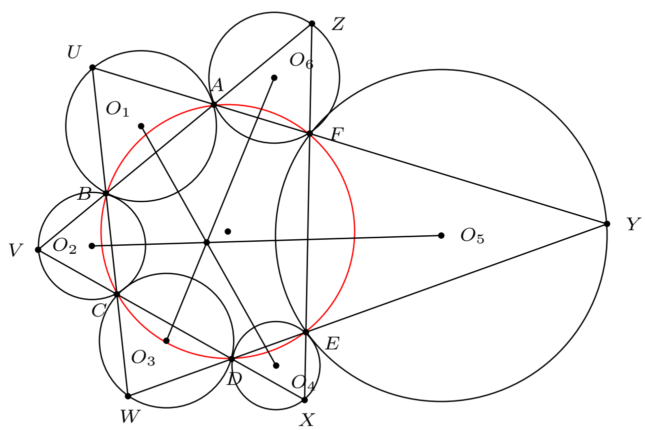

Figure 1

In this note we present a purely synthetic proof.

Lemma 1. *Let A, B, C, A', B', C' be six points* (in cyclic order) *on a circle (O), and $X = AB \cap A'C$, $X' = A'B' \cap AC'$. Let O_1, O'_1 be the circumcenters of (XBC), $(X'B'C')$ respectively. The lines $O_1O'_1$, BB', CC' are concurrent* (see Figure 2).

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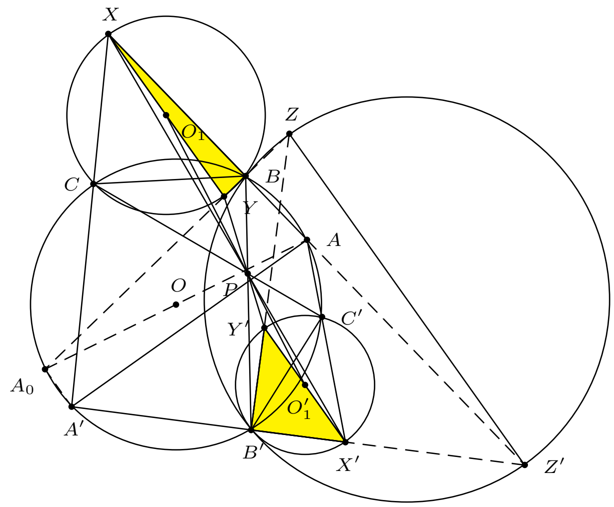

Figure 2

Proof. Since the triangles XBC and $XA'A$ are inversely similar, the diameter XY of (O_1) is an altitude of triangle $XA'A$. Similarly, the diameter $X'Y'$ of (O_1') is an altitude of triangle $XA'A$. Hence, XY and $X'Y'$ are parallel, and XX', YY' intersect at a point P that divides these segments in the ratio of the radii of the circles. Clearly, P also lies on the segment O_1O_1'. The lines $XB, X'B'$ and their perpendiculars $YB, Y'B'$ meet at the points Z', Z respectively. If AA_0 is a diameter of (O), then $A'A_0 \perp A'A$. Since the points Z, B, B', Z' are concyclic, we have $ZZ'||A'A_0$ because they are both antiparallels to BB' relative to A_0Z, $A'Z'$. Hence $XY||X'Y'||ZZ'$, and are perpendicular to $A'A$. By Desargues' theorem, the triangles (XYB) and $(X'Y'B')$ are perspective. Hence, BB' passes though P. Similarly we prove that CC' passes through P. □

We reformulate and prove Dao's theorem in the following form.

Theorem 2. *Divide a circle in six consecutive arcs c_2, a_1, b_2, c_1, a_2, b_1 with the arbitrary points A, B, C, A', B', C'. Let the chords of the arcs a_2, b_2, c_2 bound a triangle $A_1B_1C_1$, and those of the arcs a_1, b_1, c_1 bound a triangle $A_2B_2C_2$. If O_1, O_1', O_2, O_2', O_3, O_3' are the circumcenters of the circles (A_1BC), $(A_2B'C')$, $(B_1C'A)$, (B_2CA'), $(C_1A'B')$, (C_2AB) respectively, then the lines O_1O_1', O_2O_2', O_3O_3' are concurrent* (see Figure 3).

Proof. Let $A_3 = BB' \cap CC'$, $B_3 = CC' \cap AA'$, and $C_3 = AA' \cap BB'$. By Lemma 1 the points A_3, B_3, C_3 lie on the lines O_1O_1', O_2O_2', O_3O_3' respectively. Denote

$$\angle O_1BA_3 = A_b, \qquad \angle O_2C'B_3 = B_c, \qquad \angle O_3A'C_3 = C_a,$$
$$\angle O_1CA_3 = A_c, \qquad \angle O_2AB_3 = B_a, \qquad \angle O_3B'C_3 = C_b.$$

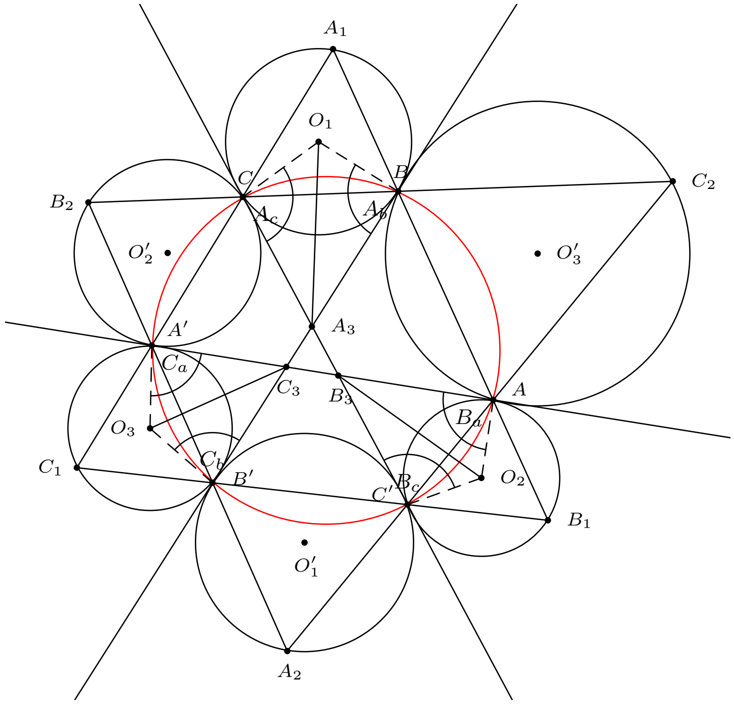

Figure 3

We have $A_b = \angle O_1BC + \angle CBA_3 = 90^o - \angle CA_1B + \angle CBB'$ or

$$A_b = \ 90^\circ - \frac{a_2 + b_1 + c_1 - a_1}{2} + \frac{b_2 + c_1}{2} = 90^\circ + \frac{a_1 + b_2 - a_2 - b_1}{2}.$$

Similarly,

$$B_a = \ 90^\circ - \frac{b_2 + c_1 + a_1 - b_1}{2} + \frac{a_2 + c_1}{2} = 90^\circ - \frac{a_1 + b_2 - a_2 - b_1}{2}.$$

From these, $A_b + B_a = 180^\circ$, and $\sin A_b = \sin B_a$. Similarly, $\sin B_c = \sin C_b$ and $\sin C_a = \sin A_c$.

Consider $O_1A_3O'_1$, $O_2B_3O'_2$, and $O_3C_3O'_3$ as lines through the vertices of triangle $A_3B_3C_3$. Let R_1 be the radius of the circle (O_1). Since

$$\frac{\sin A_b}{\sin C_3A_3O'_1} = \frac{\sin A_b}{\sin BA_3O_1} = \frac{O_1A_3}{R_1} = \frac{\sin A_c}{\sin O_1A_3C} = \frac{\sin A_c}{\sin O'_1A_3B_3},$$

we have $\frac{\sin C_3A_3O'_1}{\sin O'_1A_3B_3} = \frac{\sin A_b}{\sin A_c}$. Similarly, $\frac{\sin A_3B_3O'_2}{\sin O'_2B_3C_3} = \frac{\sin B_c}{\sin B_a}$, and $\frac{\sin B_3C_3O'_3}{\sin O'_3C_3A_3} = \frac{\sin C_a}{\sin C_b}$. Therefore,

$$\frac{\sin C_3A_3O'_1}{\sin O'_1A_3B_3} \cdot \frac{\sin A_3B_3O'_2}{\sin O'_2B_3C_3} \cdot \frac{\sin B_3C_3O'_3}{\sin O'_3C_3A_3} = \frac{\sin A_b}{\sin A_c} \cdot \frac{\sin B_c}{\sin B_a} \cdot \frac{\sin C_a}{\sin C_b} = 1.$$

By the converse of Ceva's theorem, we conclude that the lines $O_1O'_1$, $O_2O'_2$, $O_3O'_3$ are concurrent. □

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The Triangle of Reflections

Jesus Torres

Abstract. This paper presents some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer's Sketchpad, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $\mathbf{T}^\dagger$ of a given triangle $\mathbf{T}$ (obtained by reflecting the vertices in their opposite sides), resulting in interesting triangle centers and simple loci such as circles and conics. These lead to some new triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers. In particular, we show that the Parry reflection point (reflection of circumcenter in the Euler reflection point) is the common point of two triads of circles, one associated with the tangential triangle, and another with the excentral triangle. More interestingly, we show that a certain rectangular hyperbola through the vertices of $\mathbf{T}^\dagger$ appears as the locus of the perspector of a family of triangles perspective with $\mathbf{T}^\dagger$, and in a different context as the locus of the orthology center of $\mathbf{T}^\dagger$ with another family of triangles.

1. Introduction

This paper is a revision of the author's master thesis [14]. We present some results in triangle geometry discovered with the aids of a dynamic software, namely, the Geometer's Sketchpad©, and confirmed with computations using Mathematica 9.0. With the method of barycentric coordinates, we study geometric problems associated with the triangle of reflections $\mathbf{T}^\dagger$ of a given triangle $\mathbf{T}$ (obtained by reflecting the vertices in their opposite sides). We use the notations and basic formulas in triangle geometry as presented in [15]. In particular, coordinates of triangle centers are expressed in the Conway notation, so as to reduce the degrees of polynomials involved. We obtain a number of interesting triangle centers with reasonably simple coordinates, and also new properties of some known, classical centers.

1.1. *Summary*. Let $\mathbf{T}$ be a given triangle. The triangle of reflections $\mathbf{T}^\dagger$ is the one whose vertices are the reflections of the vertices of $\mathbf{T}$ in their opposite sides. This is introduced in CHAPTER 2. Propositions 2.1 and 2.2 explain the significance of the nine-point center of $\mathbf{T}$ in the geometry of $\mathbf{T}^\dagger$. The homogeneous barycentric coordinates of a few classical centers on the Euler line are computed. While the calculations for the centroid and the circumcenter are easy (Proposition 2.4), the

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coordinates of the orthocenter and nine-point center can only be computed with the aids of Mathematica. These two centers will feature in CHAPTER 7.

CHAPTERS 3 and 4 give a number of simple results related to perspectivity and orthology with $\mathbf{T}^\dagger$. In §4.1, we give a simple computational proof of Sondat's theorem (Theorem 4.1) which states that if two nondegenerate triangles are both perspective and orthologic, then the perpector and the two orthology centers are collinear. This applies to $\mathbf{T}^\dagger$ and the orthic triangle of $\mathbf{T}$ (Theorem 4.3). The line containing these centers has a remarkably simple equation. This line also appears as a locus discussed in §7.2. Also, the orthology center $\mathsf{cev}(H)^\perp(\mathbf{T}^\dagger)$ in Theorem 4.3 is a new center which reappears in a number of places in later chapters.

In CHAPTERS 5 and 6 we construct a number of circles associated with $\mathbf{T}^\dagger$. In §5.3 we construct a triad of circles in relation to the tangential triangle of $\mathbf{T}$, and show that they are concurrent at the Parry reflection point (which is the reflection of the circumcenter of $\mathbf{T}$ in its Euler reflection point). Another triad of circles is constructed in §5.4, this time in connection with the excentral triangle of $\mathbf{T}$. This triad of circles are also concurrent at the same Parry reflection point. A new Tucker circle (through the pedals of the vertices of $\mathbf{T}^\dagger$ on the sidelines of $\mathbf{T}$) is constructed in §6.3. The center of this circle bears a very simple relationship with the Parry reflection point and the Hatzipolakis reflection point in §6.1.

In CHAPTER 7 we present two locus problems related to $\mathbf{T}^\dagger$ and resulting in conic loci, which can be easily identified as rectangular hyperbolas. Specifically, we show that the rectangular circum-hyperbola through the vertices of $\mathbf{T}^\dagger$ and the orthocenter of $\mathbf{T}$ arises as the locus of the perspector of a family of triangles perspective with $\mathbf{T}^\dagger$ (Theorem 7.1), and also as the locus of the orthology center of $\mathbf{T}^\dagger$ with another family of triangles (Theorem 7.4(a)). Some of the triangle centers and lines constructed in earlier chapters also feature in the solutions of the loci problems discussed in this chapter.

Appendix A lists a number of triangle centers catalogued in ETC [7] that feature in this paper with properties related to $\mathbf{T}^\dagger$. Appendix B is a summary of new triangle centers appearing in this thesis, listed in order of their search numbers in ETC.

2. The triangle of reflections and the nine-point center

Given a reference $\mathbf{T} := ABC$, consider the reflections of the vertices in the respective opposite sides. In homogeneous barycentric coordinates, these are the points

$$\begin{aligned} A' &= \ (-(S_B+S_C) :\ 2S_C :\ 2S_B), \\ B' &= \ (2S_C :\ -(S_C+S_A) :\ 2S_A), \\ C' &= \ (2S_B :\ 2S_A :\ -(S_A+S_B)). \end{aligned}$$

The triangle $\mathbf{T}^\dagger := A'B'C'$ is called the *triangle of reflections* of $\mathbf{T}$. It is the main object of study of this paper.

2.1. *Perspectivity with* $\mathbf{T}$. Clearly, $\mathbf{T}$ and $\mathbf{T}^\dagger$ are perspective at the orthocenter H.

Proposition 2.1. *The perspectrix of* $\mathbf{T}$ *and* $\mathbf{T}^\dagger$ *is the trilinear polar of the nine-point center* N.

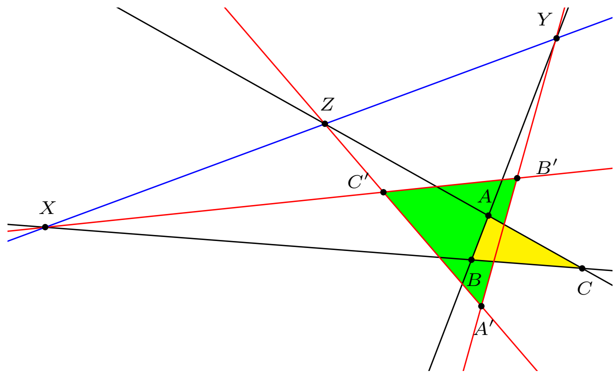

Figure 2.1

Proof. The equation of the line $B'C'$ is

$$(-3S_{AA}+S^2)x+2(S^2+S_{AB})y+2(S^2+S_{CA})z=0.$$

It is clear that $B'C'\cap BC=(0:-(S^2+S_{CA}):S^2+S_{AB})$. The equations of the lines $C'A'$ and $A'B'$, and their intersections with the corresponding sidelines, can be written down easily by cyclic permutations of parameters. These are

$$\begin{aligned} C'A'\cap CA= & \ (S^2+S_{BC}:0:-(S^2+S_{AB})), \\ A'B'\cap AB= & \ (-(S^2+S_{BC}):S^2+S_{CA}:0). \end{aligned}$$

The line containing these points is

$$\frac{x}{S^2+S_{BC}}+\frac{y}{S^2+S_{CA}}+\frac{z}{S^2+S_{AB}}=0,$$

the trilinear polar of the nine-point center N. □

2.2. *Homothety between* $\mathbf{T}^\dagger$ *and the reflection triangle of* N.

Proposition 2.2. *The triangle of reflections* $\mathbf{T}^\dagger$ *is the image of the reflection triangle of the nine-point center* N *under the homothety* $\mathsf{h}(O,2)$.

Proof. If D is the midpoint of BC, it is well known that $2\cdot OD=AH=H_a^\dagger A'$ (see Figure 2.2). If $N_a^\dagger$ is the reflection of N in BC, then

$$NN_a^\dagger=OD+HH_a=\frac{1}{2}(2{\cdot}OD+2HH_a)=\frac{1}{2}(HH_a+H_aH_a^\dagger+H_a^\dagger A')=\frac{1}{2}HA'.$$

Since N is the midpoint of OH, it follows that $N_a^\dagger$ is the midpoint of OA'.

A similar reasoning shows that $N_b^\dagger$ and $N_c^\dagger$ are the midpoints of OB' and OC'. □

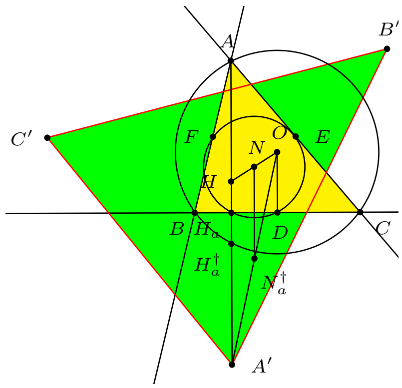

Figure 2.2

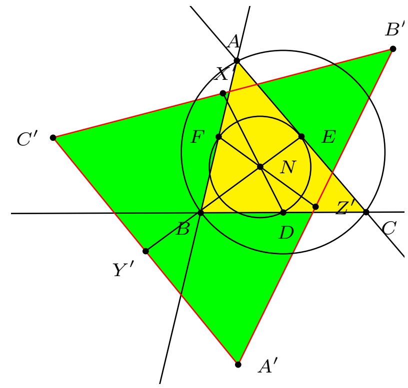

Figure 2.3

Proposition 2.3. *The medial triangles of* $\mathbf{T}$ *and* $\mathbf{T}^\dagger$ *are perspective at* N.

Proof. The midpoint of $B'C'$ is the point

$$\begin{aligned} X' &= \frac{1}{2}(B'+C') \\ &= \frac{1}{2}\left(\frac{(2S_C, -(S_C+S_A), 2S_A)}{S_C+S_A} + \frac{(2S_B, 2S_A, -(S_A+S_B))}{S_A+S_B}\right) \\ &= \frac{(2(S^2+S_{BC}), (S_A-S_B)(S_C+S_A), (S_A-S_C)(S_A+S_B))}{2(S_C+S_A)(S_A+S_B)}. \end{aligned}$$

In homogeneous barycentric coordinates, this is

$$X' = (2(S^2+S_{BC}) : (S_A-S_B)(S_C+S_A) : (S_A-S_C)(S_A+S_B)).$$

The line joining X' to the midpoint of BC has equation

$$\begin{vmatrix} 2(S^2+S_{BC}) & (S_A-S_B)(S_C+S_A) & (S_A-S_C)(S_A+S_B) \\ 0 & 1 & 1 \\ x & y & z \end{vmatrix} = 0,$$

or

$$S_A(S_B-S_C)x + (S^2+S_{BC})(y-z) = 0.$$

This line clearly contains the nine-point center N, since

$$S_A(S_B-S_C)(S^2+S_{BC}) + (S^2+S_{BC})((S^2+S_{CA}) - (S^2+S_{AB})) = 0.$$

Similarly, the lines joining the midpoints of $C'A'$ and $A'B'$ to those of CA and AB also contain N. We conclude that the two medial triangles are perspective at N (see Figure 2.3). $\square$

2.3. *The Euler line of* $\mathbf{T}^\dagger$.

Proposition 2.4. (a) *The centroid, circumcenter, and orthocenter of* $\mathbf{T}^\dagger$ *are the points*

$$\begin{aligned}
G' =&\ (a^2(S_{AA} - S_A(S_B + S_C) - 3S_{BC}) : \cdots : \cdots),\\
O' =&\ (a^2(-3S_A^3(S_B + S_C) + S_{AA}(5S_{BB} + 6S_{BC} + 5S_{CC})\\
&\ + 9S_{ABC}(S_B + S_C) + 4S_{BB}S_{CC})) : \cdots : \cdots),\\
H' =&\ (a^2(2(S_A + S_B + S_C)S^6 + S_{BC}(2(-S_A + 4S_B + 4S_C)S^4\\
&\ + (S_{AA} - 3S^2)((7S_A + 5S_B + 5S_C)S^2 + S_{ABC}))) : \cdots : \cdots),\\
N' =&\ (a^2(3a^4S_A^5 + 2a^2(a^4 + 9S_{BC})S_A^4 - (a^8 + 26a^4S_{BC} - 16S_{BC}^2)S_A^3\\
&\ - 4a^2S_{BC}(4a^4 + 19S_{BC})S_{AA} - S_AS_{BB}S_{CC}(29a^4 + 48S_{BC}) - 14a^2(S_{BC})^3)\\
&\ : \cdots : \cdots).
\end{aligned}$$

(b) *The equation of the Euler line of* $\mathbf{T}^\dagger$*:*

$$\sum_{\text{cyclic}} (S_C + S_A)(S_A + S_B)(S_B - S_C)f(S_A, S_B, S_C)x = 0,$$

where

$$f(S_A, S_B, S_C) = 2(8S_A + S_B + S_C)S^4 - S_AS_B + S_C)((5S_A + 7S_B + 7S_C)S^2 + S_{ABC}).$$

Remarks. (1) The centroid G' is $X(3060)$ in ETC, defined as the external center of similitude of the circumcircle of $\mathbf{T}$ and the nine-point circle of the orthic triangle.

(2) The circumcenter O' is the reflection of O in N^*.

(3) The orthocenter H' has ETC (6-9-13)-search number $31.1514091170\cdots$.

(4) The nine-point center N' has ETC (6-9-13)-search number $5.99676405896\cdots$.

(5) The Euler line of $\mathbf{T}^\dagger$ also contains the triangle center $X(156)$, which is the nine-point center of the tangential triangle.

2.4. *Euler reflection point of* $\mathbf{T}^\dagger$. A famous theorem of Collings [2] and Longuet-Higgins [8] states that the reflections of a line $\mathscr{L}$ in the sidelines of $\mathbf{T}$ are concurrent if and only if $\mathscr{L}$ contains the orthocenter H. If this condition is satisfied, the point of concurrency is a point on the circumcircle.

Applying this to the Euler line of $\mathbf{T}$, we obtain the Euler reflection point

$$E = \left(\frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2}\right).$$

The Euler reflection point of $\mathbf{T}^\dagger$ is a point E' on the circumcircle of $\mathbf{T}^\dagger$ (see Figure 2.4). Its coordinates involve polynomial factors of degree 12 in S_A, S_B, S_C; it has ETC (6-9-13)-search number $-1.94515015138\cdots$.

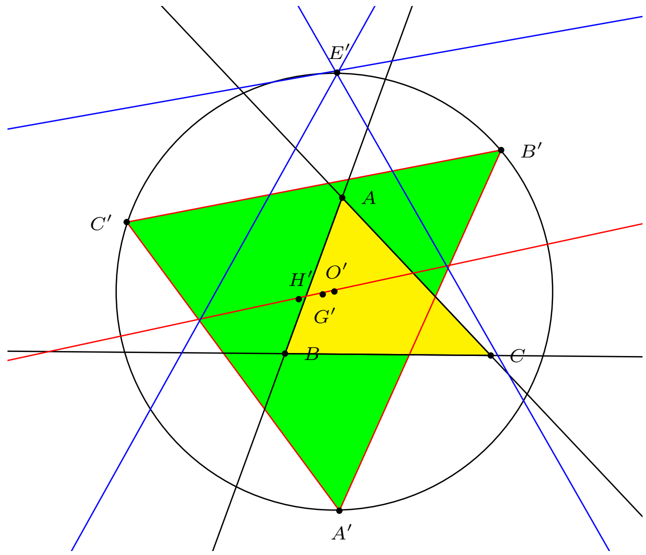

Figure 2.4

3. Perspectivity

3.1. *Triangles perspective with* $\mathbf{T}^\dagger$.

3.1.1. *The excentral triangle.* The excentral triangle of $\mathbf{T}$ has the excenters as vertices. L. Evans [3] has shown that this is perspective with $\mathbf{T}^\dagger$ at the triangle center

$$X(484) = (a(a^3 + a^2(b+c) - a(b^2+bc+c^2) - (b+c)(b-c)^2) : \cdots : \cdots).$$

This triangle center is often called the Evans perspector. It is the inverse of I in the circumcircle of the excentral triangle, and divides OI in the ratio

$$OX(484) : X(484)I = R + 2r : -4r.$$

3.1.2. *The Fermat triangles.* Hatzipolakis and Yiu [5] have shown that the only Kiepert triangles perspective with $\mathbf{T}^\dagger$ are the Fermat triangles, consisting of vertices of equilateral triangles erected on the sides of $\mathbf{T}$. The perspectors are the isodynamic points, $X(16)$ or $X(15)$ according as the vertices of the equilateral triangles are on the same or opposite sides of the vertices of $\mathbf{T}$.

3.2. *Triangles bounded by reflections of the sidelines of* $\mathbf{T}$ *and* $\mathbf{T}^\dagger$ *in each other.* Let a, b, c be the sidelines BC, CA, AB of triangle $\mathbf{T} := ABC$, and a', b', c' those of $\mathbf{T}^\dagger$. The reflections of these lines in a, b, c (and vice versa) give rise to interesting examples of perspective triangles. Let $\mathscr{L}_a$ be the reflection of a in a', and $\mathscr{L}'_a$ that of a' in a. Similarly, define $\mathscr{L}_b$, $\mathscr{L}'_b$, and $\mathscr{L}_c$, $\mathscr{L}'_c$.

Since a, a' intersect at X, the lines $\mathscr{L}_a$, $\mathscr{L}'_a$ intersect BC at the same point. Similarly, the reflections of b and b' in each other intersect b at Y; so do those of c and c' at Z. By Proposition 2.1, X, Y, Z define the trilinear polar of N.

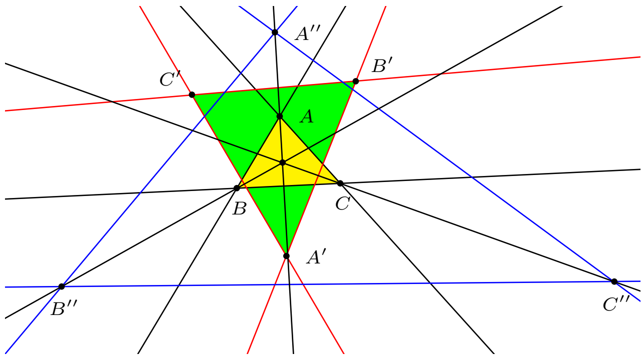

Figure 3.1

Therefore, $\mathbf{T}$, $\mathbf{T}^\dagger$, the triangles $\mathbf{T}^*$ bounded by $\mathscr{L}'_a$, $\mathscr{L}'_b$, $\mathscr{L}'_c$ (see Figure 3.1), and $\mathbf{T}'' = A''B''C''$ bounded by $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ are line-perspective to each other, all sharing the same perspectrix XYZ. They are also vertex-perspective.

The following table gives the $(6-9-13)$-search numbers of the perspectors, with the highest degree of the polynomial factors (in S_A, S_B, S_C) in the coordinates.

	$\mathbf{T}^\dagger$	$\mathbf{T}^*$	$\mathbf{T}''$
$\mathbf{T}$	H	$3.99180618013\cdots(5)$	$8.27975385194\cdots(7)$
$\mathbf{T}^\dagger$		$-9.04876879620\cdots(11)$	$-7.90053389552\cdots(16)$
$\mathbf{T}^*$			$-0.873191727540\cdots(26)$

Here is the perspector of $\mathbf{T}$ and $\mathbf{T}^*$ in homogeneous barycentric coordinates:

$$\left(\frac{a^2}{(3S_A-S_B-S_C)S^4+S_{BC}((6S_A+5S_B+5S_C)S^2+7S_{ABC})}:\cdots:\cdots\right).$$

4. Orthology

4.1. *Orthology and perspectivity.*

Theorem 4.1 (Sondat [12]; see also [13, 9])**.** *If two nondegenerate triangles are both perspective and orthologic, then the perspector and the two orthology centers are collinear.*

Proof. Assume triangles ABC and XYZ are perspective at a point $P=(u:v:w)$ and have perpendiculars from X to BC, Y to CA, and Z to AB concurrent at a point $Q=(u':v':w')$. Now the point X is the intersection of the line AP and the perpendicular from Q to BC. It has coordinates

$$(v((S_B+S_C)w'+S_Bu')-w(S_Cu'+(S_B+S_C)v'):\ v(S_Bv'-S_Cw'):\ w(S_Bv'-S_Cw')).$$

Similarly, Y is the intersection of BP and the perpendicular from Q to CA, and Z is that of CP and the perpendicular from Q to AB. Their coordinates can be written down from those of X by cyclic permutations of parameters. Since the triangles are orthologic, we find the second orthology center. The perpendculars from A to YZ, B to ZX, and C to AB are the lines

$$(S_Au - S_Bv)(u'w - w'u)y - (S_Cw - S_Au)(v'u - u'v)z = 0,$$
$$(S_Bv - S_Cw)(v'u - u'v)z - (S_Au - S_Bv)(w'v - v'w)x = 0,$$
$$(S_Cw - S_Au)(w'v - v'w)x - (S_Bv - S_Cw)(u'w - w'u)y = 0.$$

These lines are concurrent at the point

$$Q' = \left(\frac{S_Bv - S_Cw}{w'v - v'w} : \frac{S_Cw - S_Au}{u'w - w'u} : \frac{S_Au - S_Bv}{v'u - u'v}\right).$$

This clearly lies on the line PQ:

$$(w'v - v'w)x + (u'w - w'u)y + (v'u - u'v)z = 0.$$

Therefore the perspector P and the orthology centers Q and Q' are collinear. $\square$

Here is an illustrative example. Let $\mathbf{T}^{\perp}(P)$ be the pedal triangle of a point P. It is clear that the perpendiculars from the vertices of $\mathbf{T}^{\perp}(P)$ to $\mathbf{T}$ are concurrent at P. Therefore, the perpendiculars from A, B, C to the corresponding sides of the pedal triangle are also concurrent. This is the isogonal conjugate of P.

Since the reflection triangle is homothetic to the pedal triangle. The same result holds, namely,

$$(\mathbf{T}^{\dagger}(P))^{\perp}(\mathbf{T}) = P \quad \text{and} \quad \mathbf{T}^{\perp}(\mathbf{T}^{\dagger}(P)) = P^*.$$

4.2. *Orthology with* $\mathbf{T}$. Clearly the perpendiculars from A', B', C' to the sidelines of $\mathbf{T}$ are concurrent at H. We find the other orthology center.

Proposition 4.2. *The orthology center* $\mathbf{T}^{\perp}(\mathbf{T}^{\dagger})$ *is* N^*.

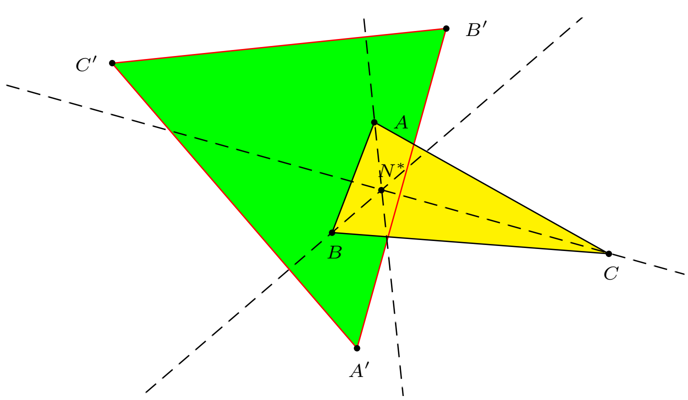

Figure 4.1

Proof. The perpendiculars from A to $B'C'$, B to $C'A'$, and C to $A'B'$ are the lines

$$\begin{array}{rcrcrcl} & & (S_A+S_B)(S^2+S_{CA})y & - & (S_C+S_A)(S^2+S_{AB})z & = & 0,\\ -(S_A+S_B)(S^2+S_{BC})x & & & + & (S_B+S_C)(S^2+S_{AB})z & = & 0,\\ (S_C+S_A)(S^2+S_{BC})x & - & (S_B+S_C)(S^2+S_{CA})y & & & = & 0. \end{array}$$

These are concurrent at

$$\left(\frac{S_B+S_C}{S^2+S_{BC}} : \frac{S_C+S_A}{S^2+S_{CA}} : \frac{S_A+S_B}{S^2+S_{AB}}\right),$$

which is the isogonal conjugate of the nine-point center N. □

4.3. *Orthic triangle.*

Theorem 4.3. *The triangle of reflections* $\mathbf{T}^\dagger$ *is orthologic to the orthic triangle.*

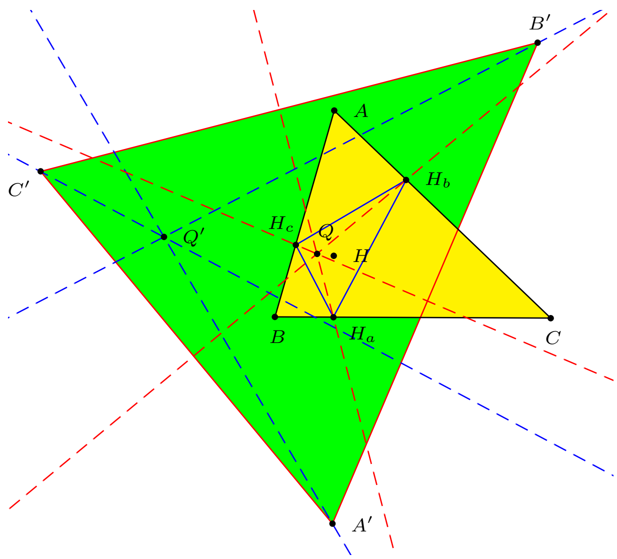

Figure 4.2

Proof. The perpendiculars from H_a to $B'C'$, H_b to $C'A'$, H_c to $A'B'$ are the lines

$$\begin{aligned} 2S^2(S_B-S_C)S_Ax+(S^2(3S_A+S_B+S_C)+S_{ABC})(S_By-S_Cz) &= 0,\\ 2S^2(S_C-S_A)S_By+(S^2(S_A+3S_B+S_C)+S_{ABC})(S_Cz-S_Ax) &= 0,\\ 2S^2(S_A-S_B)S_Cz+(S^2(S_A+S_B+3S_C)+S_{ABC})(S_Ax-S_By) &= 0. \end{aligned}$$

These are concurrent at

$$Q := (S_{BC}(S_B+S_C)(S_{AA}-3S^2)(S^2(3S_A+S_B+S_C)+S_{ABC}) : \cdots : \cdots).$$

The perpendiculars from A' to H_bH_c, B' to H_cH_a, C' to H_aH_b are the lines

$$\begin{aligned}
2S^2(S_B-S_C)(x+y+z) &= (S_B+S_C)(-(S_A+S_B)S_Cy+(S_C+S_A)S_Bz),\\
2S^2(S_C-S_A)(x+y+z) &= (S_C+S_A)(-(S_B+S_C)S_Az+(S_A+S_B)S_Cx),\\
2S^2(S_A-S_B)(x+y+z) &= (S_A+S_B)(-(S_C+S_A)S_Bx+(S_B+S_C)S_Ay).
\end{aligned}$$

These lines are concurrent at

$$Q' = ((S_B+S_C)(3S_A^3(S_B+S_C)-S_{AA}(S_B-S_C)^2-5S_{ABC}(S_B+S_C)-4S_{BC}^2) : \cdots : \cdots).$$

$\square$

Remarks. (1) The orthology center $Q := \mathsf{cev}(H)^{\perp}(\mathbf{T}^{\dagger})$ has ETC (6-9-13)-search number $12.4818250323\cdots$. This also appears in Proposition 6.3 and §7.1.1 below.

(2) The orthology center $Q' := \mathbf{T}^{\dagger\perp}(\mathsf{cev}(H))$ has ETC (6-9-13)-search number $-8.27009636449\cdots$.

(3) Since the two triangles are perspective at H, the line joining these two orthology centers contains H. This is the line

$$\sum_{\text{cyclic}} a^2S_A(S_B-S_C)(3S^2-S_{AA})x = 0.$$

See Theorem 7.4(b) below.

4.4. *Tangential triangle.* Since the tangential triangle is homothetic to the orthic triangle, the results of §4.2 also shows that the triangle of reflections is orthologic to the tangential triangle. Clearly, $(\mathbf{T}^{\dagger})^{\perp}(\mathsf{cev}^{-1}(K)) = (\mathbf{T}^{\dagger})^{\perp}(\mathsf{cev}(H)) = Q'$.

Proposition 4.4. *The orthology center* $\mathsf{cev}^{-1}(K)^{\perp}(\mathbf{T}^{\dagger})$ *is the circumcenter of* $\mathbf{T}^{\dagger}$.

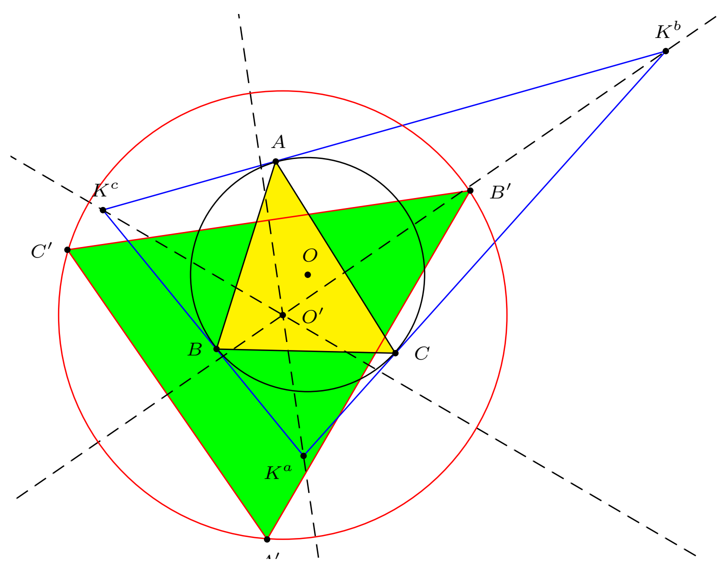

Figure 4.3

Proof. The vertex $K^a = (-(S_B + S_C) : S_C + S_A : S_A + S_B)$ is equidistant from B' and C'. In fact,

$$K^a B'^2 = K^a C'^2 = \frac{S_A(9S^2 + S_{BB} + S_{BC} + S_{CC}) + (S_B + S_C)S_{BC}}{4S_{AA}}.$$

Therefore, K^a lies on the perpendicular bisector of $B'C'$. Similarly, K^b and K^c lie on the perpendicular bisectors of $C'A'$ and $A'B'$ respectively. From this the result follows. □

5. Triads of circles

In this chapter we consider triads of circles related to $\mathbf{T}^\dagger$. The circumcircle of the reflection flanks are considered in §5.1. In §5.2, we construct a triad of coaxial circles associated with pedals and with the line HK as axis. In §5.3,4, we show that a common triangle center, the Parry reflection point $X(399)$, the reflection of the circumcenter in the Euler reflection point of $\mathbf{T}$, occurs as the point of concurrence of two triads of circles, one associated with the tangential triangle (Proposition 5.7), and another with the excentral triangle (§5.4).

We shall make frequent use of the following fundamental theorem.

Theorem 5.1 ([5, Proposition 18])**.** *If the circles XBC, AYC, ABZ have a common point, so do the circles AYZ, XBZ, XYC.*

5.1. *The reflection flanks and their circumcircles.* We shall refer to the triangles $\mathbf{T}^{\dagger a} := AB'C'$, $\mathbf{T}^{\dagger b} := A'BC'$, and $\mathbf{T}^{\dagger c} := A'B'C$ as the reflection flanks.

Proposition 5.2. *The reflection flank $\mathbf{T}^{\dagger a}$ is degenerate if and only if $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.*

Proof. The line $B'C'$ has equation

$$(-3S_{AA} + S^2)x + 2(S^2 + S_{AB})y + 2(S^2 + S_{CA})z = 0.$$

(See §2.1). This contains the vertex A if and only if $3S_{AA} = S^2$, $\cot^2 A = \frac{1}{3}$, i.e., $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. □

The circumcircle of $\mathbf{T}^{\dagger a}$:

$$\begin{aligned}&(S^2 - 3S_{AA})(a^2yz + b^2zx + c^2xy)\\ &- 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0.\end{aligned}$$

Its center is the point

$$O_a = 2S^2(3S^2 - S_{AA})(1, 0, 0) + b^2c^2(S^2 + S_{BC}, S^2 + S_{CA}, S^2 + S_{AB}).$$

Proposition 5.3. (a) *The circumcenters of the reflection flanks form a triangle perspective with ABC at the nine-point center N.*

(b) *The orthocenters of the reflection flanks form a triangle perspective with ABC at N^*.*

Proof. From the coordinates of O_a, we note that the line AO_a contains the nine-point center $N = (S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB})$. Similarly, for the circumcenters O_b and O_c of $\mathbf{T}^{\dagger b}$ and $\mathbf{T}^{\dagger c}$ the lines BO_b and CO_c also contain the nine-point center.

(b) is equivalent to the orthology of $\mathbf{T}$ and $\mathbf{T}^\dagger$. It follows from Proposition 4.2. □

Since the circles $(A'BC)$, $(AB'C)$, (ABC') are concurrent at H, the circumcircles of the reflection flanks are also concurrent.

Proposition 5.4. *The circumcircles of the reflection flanks are concurrent at $X(1157)$, the inverse of N^* in the circumcircle of $\mathbf{T}$.*

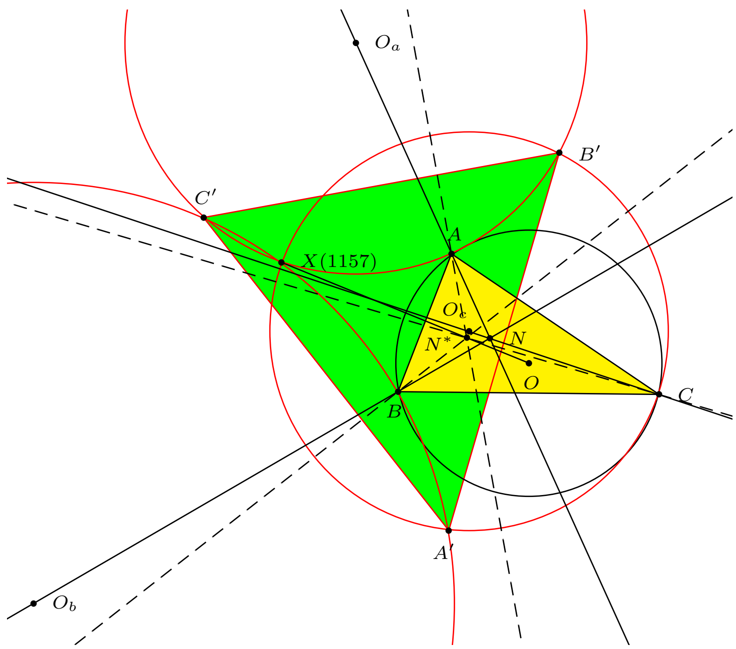

Figure 5.1

Proof. The point of concurrency is necessarily the radical center of the circles. From the equations of the circumcircles of the reflection flanks:

$$(S^2 - 3S_{AA})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(c^2(S^2 + S_{CA})y + b^2(S^2 + S_{AB})z) = 0,$$
$$(S^2 - 3S_{BB})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(a^2(S^2 + S_{AB})z + c^2(S^2 + S_{BC})x) = 0,$$
$$(S^2 - 3S_{CC})(a^2yz + b^2zx + c^2xy) - 2(x + y + z)(b^2(S^2 + S_{BC})x + a^2(S^2 + S_{CA})y) = 0,$$

we obtain the radical center as the point $(x : y : z)$ satisfying

$$\frac{c^2(S^2+S_{CA})y+b^2(S^2+S_{AB})z}{S^2-3S_{AA}} = \frac{a^2(S^2+S_{AB})z+c^2(S^2+S_{BC})x}{S^2-3S_{BB}} = \frac{b^2(S^2+S_{BC})x+a^2(S^2+S_{CA})y}{S^2-3S_{CC}}.$$

Rewriting this as

$$\frac{\frac{(S^2+S_{CA})y}{b^2}+\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})} = \frac{\frac{(S^2+S_{AB})z}{c^2}+\frac{(S^2+S_{BC})x}{a^2}}{b^2(S^2-3S_{BB})} = \frac{\frac{(S^2+S_{BC})x}{a^2}+\frac{(S^2+S_{CA})y}{b^2}}{c^2(S^2-3S_{CC})},$$

we have

$$\frac{\frac{(S^2+S_{CA})y}{b^2}+\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})} = \frac{\frac{(S^2+S_{AB})z}{c^2}+\frac{(S^2+S_{BC})x}{a^2}}{b^2(S^2-3S_{BB})} = \frac{\frac{(S^2+S_{BC})x}{a^2}+\frac{(S^2+S_{CA})y}{b^2}}{c^2(S^2-3S_{CC})}.$$

From these,

$$\begin{aligned}
&\frac{\frac{(S^2+S_{BC})x}{a^2}}{-a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC})}\\
=&\frac{\frac{(S^2+S_{CA})y}{b^2}}{a^2(S^2-3S_{AA})-b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC})}\\
=&\frac{\frac{(S^2+S_{AB})z}{c^2}}{a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})-c^2(S^2-3S_{CC})},
\end{aligned}$$

and

$$x : y : z = \frac{a^2(-a^2(S^2-3S_{AA})+b^2(S^2-3S_{BB})+c^2(S^2-3S_{CC}))}{S^2+S_{BC}} : \cdots : \cdots .$$

This gives the triangle center $X(1157)$ in ETC, the inverse of N^* in the circumcircle of $\mathbf{T}$. □

5.2. *Three coaxial circles.* Let $H_aH_bH_c$ be the orthic triangle, and H'_a, H'_b, H'_c the pedals of A on $B'C'$, B on $C'A'$, and C on $A'B'$ respectively.

Proposition 5.5. *The lines $H_aH'_a$, $H_bH'_b$, $H_cH'_c$ are concurrent at*

$$(a^2S_{BC}((5S_A+S_B+S_C)S^4+S_{ABC}(S^2-2S_{AA})) : \cdots : \cdots).$$

Remark. This has ETC (6-9-13)-search number $3.00505308538\cdots$.

Theorem 5.6. *The three circles $AH_aH'_a$, $BH_bH'_b$, $CH_cH'_c$ are coaxial with radical axis HK.*

Proof. The centers of the circles AXX' are the point

$$\begin{aligned}
O'_a =&\ (S_A(S_B-S_C) : -(S^2+S_{CA}) : S^2+S_{AB}),\\
O'_b =&\ (S^2+S_{BC} : S_B(S_C-S_A) : -(S^2+S_{AB}),\\
O'_c =&\ (-(S^2+S_{BC}) : S^2+S_{CA} : S_C(S_A-S_B)).
\end{aligned}$$

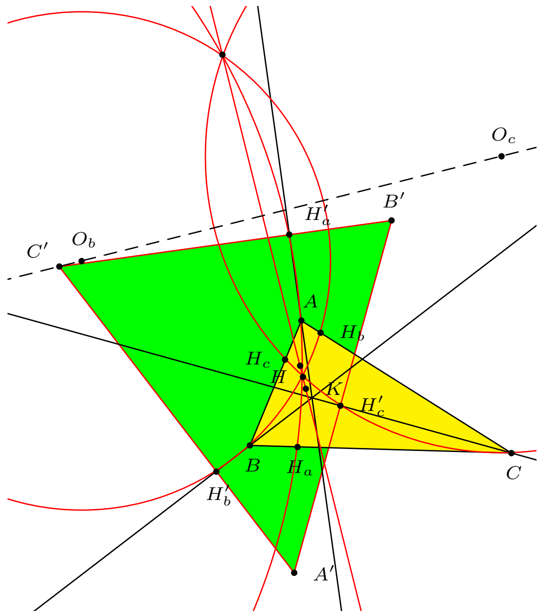

Figure 5.2

These centers lie on the line

$$a^2 S_A x + b^2 S_B y + c^2 S_C z = 0.$$

The circles have equations

$$\begin{aligned}
S_A(S_B - S_C)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_B(S^2+S_{AB})y - S_C(S^2+S_{CA})z) &= 0,\\
S_B(S_C - S_A)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_C(S^2+S_{BC})z - S_A(S^2+S_{AB})x) &= 0,\\
S_C(S_A - S_B)(a^2yz + b^2zx + c^2xy) - (x+y+z)(S_A(S^2+S_{CA})x - S_B(S^2+S_{BC})y) &= 0.
\end{aligned}$$

The radical axis is

$$(S_B - S_C)S_{AA}x + (S_C - S_A)S_{BB}y + (S_A - S_B)S_{CC}z = 0,$$

which clearly contains H and K (see Figure 5.2). □

Remark. The radical axes of the circumcircle with these three circles are concurrent at

$$X(53) = \left(\frac{S^2 + S_{BC}}{S_A} : \frac{S^2 + S_{CA}}{S_B} : \frac{S^2 + S_{AB}}{S_C}\right).$$

5.3. $\mathbf{T}^\dagger$ *and the tangential triangle.* Let $\text{cev}^{-1}(K) := K^aK^bK^c$ be the tangential triangle. The centers of the circles $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ are the points

$$
\begin{aligned}
O''_a &= (S_{ABC} + (3S_A - S_B - S_C)S^2 : \ b^2c^2S_C : \ b^2c^2S_B), \\
O''_b &= (c^2a^2S_C : \ S_{ABC} + (3S_B - S_C - S_A)S^2 : \ c^2a^2S_A), \\
O''_c &= (a^2b^2S_B : \ a^2b^2S_A : \ S_{ABC} + (3S_C - S_A - S_B)S^2).
\end{aligned}
$$

The triangle $O''_aO''_bO''_c$ is perspective with $\mathbf{T}$ at H. Therefore, the two triangles are orthologic. The perpendiculars from O''_a, O''_b, O''_c to BC, CA, AB respectively are concurrent at

$$X(265) = \left(\frac{S_A}{3S_{AA} - S^2} : \frac{S_B}{3S_{BB} - S^2} : \frac{S_C}{3S_{CC} - S^2}\right).$$

Proposition 5.7 ([5, §5.1.2]). *the circles $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ are concurrent at the Parry reflection point $X(399)$.*

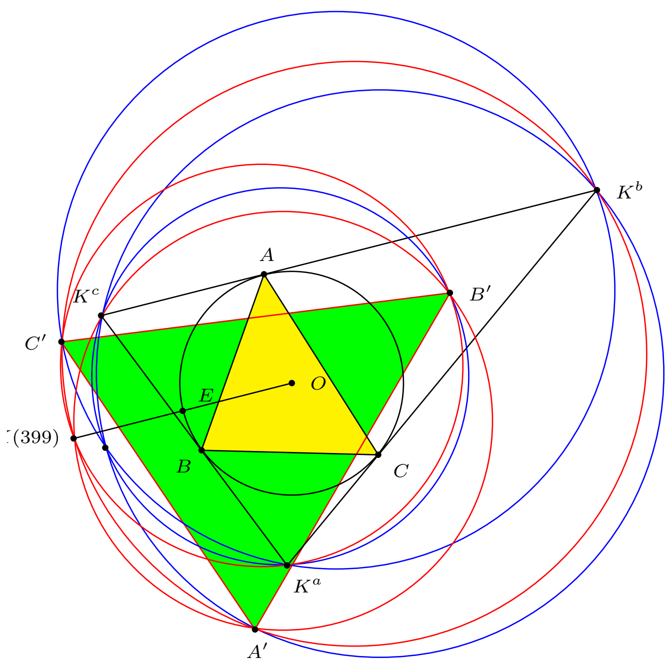

Figure 5.3

Proof. The equations of the circles are

$$
\begin{aligned}
2S_A(a^2yz + b^2zx + c^2xy) + (x+y+z)(b^2c^2x + 2c^2S_Cy + 2b^2S_Bz) &= 0, \\
2S_B(a^2yz + b^2zx + c^2xy) + (x+y+z)(2c^2S_Cx + c^2a^2y + 2a^2S_Az) &= 0, \\
2S_C(a^2yz + b^2zx + c^2xy) + (x+y+z)(2b^2S_Bx + 2a^2S_Ay + a^2b^2z) &= 0.
\end{aligned}
$$

The radical center of the three circles is the point $(x:y:z)$ satisfying

$$\frac{b^2c^2x+2c^2S_Cy+2b^2S_Bz}{S_A}=\frac{2c^2S_Cx+c^2a^2y+2a^2S_Az}{S_B}=\frac{2b^2S_Bx+2a^2S_Ay+a^2b^2z}{S_C}.$$

This is

$$(x:y:z)=(a^2(-8S^4+3b^2c^2(S^2+3S_{BC})):\ \cdots:\ \cdots),$$

the triangle center $X(399)$, the Parry reflection point, which is the reflection of O in the Euler reflection point E (see Figure 5.3). □

The circles $A'K^bK^c$, $B'K^cK^a$, $C'K^aK^b$ are also concurrent (see [11]). The point of concurrency is a triangle center with coordinates

$$(a^2f(S_A,S_B,S_C):b^2f(S_B,S_C,S_A):c^2f(S_C,S_A,S_B)),$$

with ETC (6-9-13)-search number $1.86365616601\cdots$. The polynomial $f(S_A,S_B,S_C)$ has degree 10.

5.4. $\mathbf{T}^\dagger$ *and the excentral triangle.*

Proposition 5.8 ([5, §5.1.3]). *The circles $A'I^bI^c$, $I^aB'I^c$, I^aI^bC' have the Parry reflection point as a common point.*

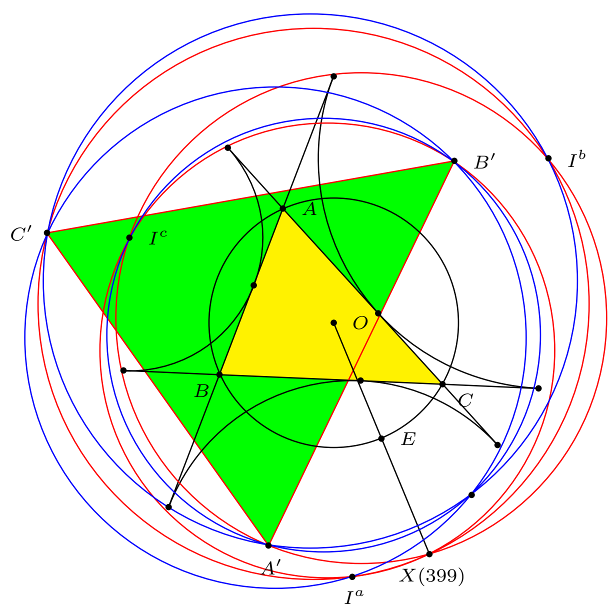

Figure 5.4

The centers of these circles are perspective with the excentral triangle at a point with ETC (6-9-13)-search number $-27.4208873972\cdots$.

On the other hand, the circles $I^aB'C'$, $A'I^bC'$ and $A'B'I^c$ have a common point with ETC (6-9-13)-search number $7.08747856659\cdots$. Their centers are perspective with ABC at the point $X(3336)$ which divides OI in the ratio $OX(3336):X(3336)I = 2R+3r:-4r$.

6. Pedals of vertices of $\mathbf{T}^\dagger$ on the sidelines of $\mathbf{T}$

6.1. *The triad of triangles AB_aC_a, BC_bA_b, CA_cB_c.* Consider the pedals of A', B', C' on the sidelines of $\mathbf{T}$. These are the points

	BC	CA	AB
A'		$B_a = (S_{CC}-S^2:0:2S^2)$,	$C_a = (S_{BB}-S^2:2S^2:0)$;
B'	$A_b = (0:S_{CC}-S^2:2S^2)$		$C_b = (2S^2:S_{AA}-S^2:0)$;
C'	$A_c = (0:2S^2:S_{BB}-S^2)$,	$B_c = (2S^2:0:S_{AA}-S^2)$.	

Proposition 6.1. *The Euler lines of the triangles AB_aC_a, BC_bA_b, CA_cB_c are concurrent at the Hatzipolakis reflection point $X(1986)$.*

Proof. The circumcenter of AB_aC_a is $H_a = (0:S_C:S_B)$. The centroid is the point

$$2S^2(S_B+S_C, S_C+S_A, S_A+S_B) - (3S_{ABC}+S^2(S_A+S_B+S_C))(1,0,0).$$

From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines

$$\begin{aligned}
2S^2\cdot S_A(S_B-S_C)x + (3S_{ABC}-S^2(-S_A+S_B+S_C))(S_By-S_Cz) &= 0,\\
2S^2\cdot S_B(S_C-S_A)y + (3S_{ABC}-S^2(S_A-S_B+S_C))(S_Cz-S_Ax) &= 0,\\
2S^2\cdot S_C(S_A-S_B)z + (3S_{ABC}-S^2(S_A+S_B-S_C))(S_Ax-S_By) &= 0.
\end{aligned}$$

These three lines are concurrent at a point with coordinates given above. It is the triangle center $X(1986)$ (see Figure 6.1). □

Remark. Here is a definition of the Hatzipolakis reflection point $X(1986)$ equivalent to the one given in ETC. Let $H_aH_bH_c$ be the orthic triangle. $X(1986)$ is the common point of the reflections of the circles AB_bH_c in H_bH_c, BH_cH_a in H_cH_a, and CH_aH_b in H_aH_b.

Proposition 6.2. *The circumcircles of the triangles AB_aC_a, BC_bA_b, CA_cB_c have radical center $X(68)$.*

Proof. These are the circles with centers H_a, H_b, H_c, passing through A, B, C respectively.

$$\begin{aligned}
a^2(a^2yz+b^2zx+c^2xy) - (x+y+z)((S_{BB}-S^2)y+(S_{CC}-S^2)z) &= 0,\\
b^2(a^2yz+b^2zx+c^2xy) - (x+y+z)((S_{CC}-S^2)z+(S_{AA}-S^2)x) &= 0,\\
c^2(a^2yz+b^2zx+c^2xy) - (x+y+z)((S_{AA}-S^2)x+(S_{BB}-S^2)y) &= 0.
\end{aligned}$$

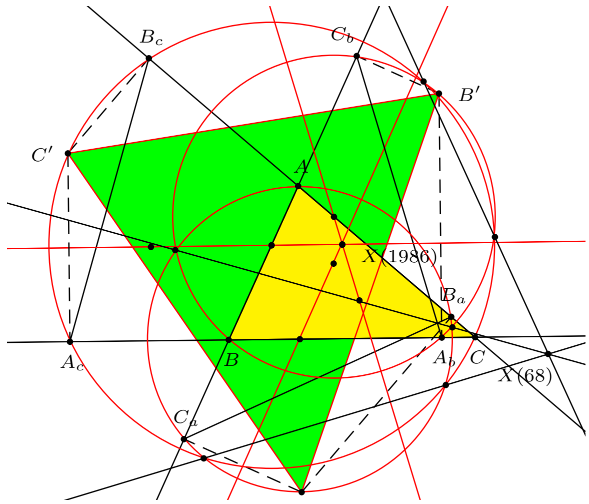

Figure 6.1

The radical center is the point defined by

$$\frac{(S_{BB}-S^2)y+(S_{CC}-S^2)z}{a^2} = \frac{(S_{CC}-S^2)z+(S_{AA}-S^2)x}{b^2} = \frac{(S_{AA}-S^2)x+(S_{BB}-S^2)y}{c^2}.$$

From these,

$$\frac{(S_{AA}-S^2)x}{b^2+c^2-a^2} = \frac{(S_{BB}-S^2)y}{c^2+a^2-b^2} = \frac{(S_{CC}-S^2)z}{a^2+b^2-c^2}$$

and

$$x:y:z = \frac{S_A}{S_{AA}-S^2} : \frac{S_B}{S_{BB}-S^2} : \frac{S_C}{S_{CC}-S^2}.$$

This is the triangle center $X(68)$. □

6.2. *The triad of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$.*

Theorem 6.3. *The Euler lines of the triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$ are concurrent at*

$$Q = (S_{BC}(S_B+S_C)(S_{AA}-3S^2)(S_{ABC}+S^2(3S_A+S_B+S_C)) : \cdots : \cdots).$$

Proof. The circumcenter of $A'B_aC_a$ is $H_a = (0 : S_C : S_B)$, the same as AB_aC_a. The centroid is the point

$$(-(S_{ABC}+S^2(3S_A+S_B+S_C)) : 2(S_C+S_A)(2S^2-S_{AB}) : 2(S_A+S_B)(2S^2-S_{CA})).$$

From these we find the equation of its Euler line; similarly for the other two triangles. The Euler lines of the triangles are the lines

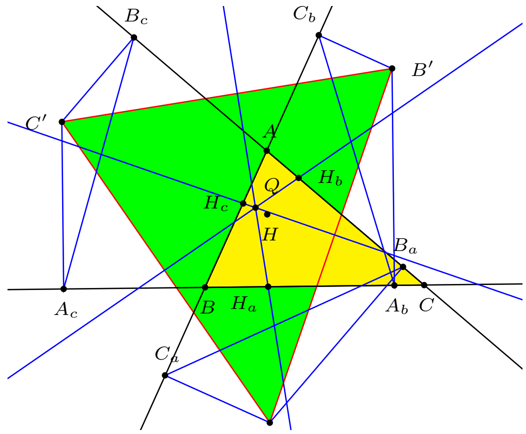

Figure 6.2

$$
\begin{aligned}
2S^2 \cdot S_A(S_B - S_C)x + (S_{ABC} + S^2(3S_A + S_B + S_C))(S_B y - S_C z) &= 0,\\
2S^2 \cdot S_B(S_C - S_A)y + (S_{ABC} + S^2(S_A + 3S_B + S_C))(S_C z - S_A x) &= 0,\\
2S^2 \cdot S_C(S_A - S_B)z + (S_{ABC} + S^2(S_A + S_B + 3S_C))(S_A x - S_B y) &= 0.
\end{aligned}
$$

These three lines are concurrent at a point with coordinates given above. □

Remark. This is the same as the orthology center $\mathsf{cev}(H)^{\perp}(\mathbf{T}^{\dagger})$ in Theorem 4.3.

6.3. *A Taylor-like circle.* It is well known that the six pedals of H_a, H_b, H_c on the sidelines of $\mathbf{T}$ are concyclic. The circle containing them is the Taylor circle (see, for example, [6, §9.6]). The center of the circle, called the Taylor center, is the triangle center

$$X(389) = (S^4 - S_{AA}S_{BC} : \ S^4 - S_{BB}S_{CA} : \ S^4 - S_{CC}S_{AB}).$$

Analogous to the Taylor circle, the pedals of A', B', C' on the sidelines of $\mathbf{T}$ are also concyclic (see [1]). In fact, the circle containing them is a Tucker circle (Figure 6.3), since
(i) the segments B_cC_b, C_aA_c, A_bB_a are parallel to BC, CA, AB respectively, and
(ii) the segments B_aC_a, C_bA_b, A_cB_c are antiparallel to BC, CA, AB respectively, *i.e.*, they are parallel to the corresponding sides of the orthic triangle $H_aH_bH_c$.

The equation of the circle containing these six pedals is

$$
\begin{aligned}
&a^2b^2c^2(a^2yz + b^2zx + c^2xy)\\
&- 2S^2(x+y+z)\left((S_{AA} - S^2)x + (S_{BB} - S^2)y + (S_{CC} - S^2)z\right) = 0.
\end{aligned}
$$

The center of the circle is the point

$$(a^2(S_{AA}-S^2)(S^2+S_{BC}) : b^2(S_{BB}-S^2)(S^2+S_{CA}) : c^2(S_{CC}-S^2)(S^2+S_{AB})).$$

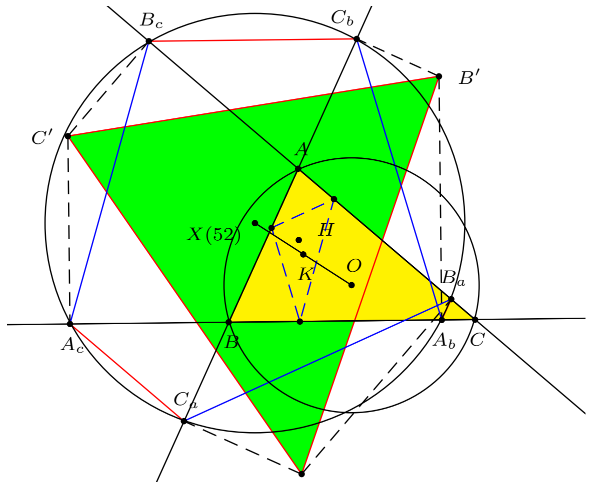

Figure 6.3

This is the triangle center $X(52)$. It is the reflection of O in the center of the Taylor circle. It is also the orthocenter of the orthic triangle (see Figure 6.3).

7. Some locus problems leading to conics

The website [4] CATALOGUE OF TRIANGLE CUBICS of B. Gibert contains a vast number of cubic and higher degree curves arising from locus problems in triangle geometry. In this chapter we consider a few loci related to perspectivity and orthology with $\mathbf{T}^\dagger$ which are conics. To avoid presenting excessively complicated algebraic manipulations, we present two problems in which the conic loci can be easily identified as rectangular hyperbolas. For this we recall a basic fact in triangle geometry: A (circum-)conic passing through the vertices of a triangle is a rectangular hyperbola if and only if it also passes through the orthocenter of the triangle.

7.1. *Reflection of* $\mathbf{T}$ *in a point.* Let P be a point with homogeneous barycentric coordinates $(x : y : z)$. The reflections of $\mathbf{T}$ in P is the triangle $\mathbf{T}_P^\dagger$ with vertices

$$A_P^\dagger = (x - y - z : 2y : 2z),$$
$$B_P^\dagger = (2x : y - z - x : 2z),$$
$$C_P^\dagger = (2x : 2y : z - x - y).$$

7.1.1. *Perspectivity of* $\mathbf{T}^\dagger$ *with reflection of* $\mathbf{T}$ *in a point.*

Theorem 7.1. *The locus of P for which $\mathbf{T}_P^\dagger$ is perspective with $\mathbf{T}^\dagger$ is the rectangular circum-hyperbola of the orthic triangle containing the orthocenter H.*

Proof. The equation of the line $A'A_P^\dagger$ is

$$\begin{vmatrix} -(S_B+S_C) & 2S_C & 2S_B \\ x-y-z & 2y & 2z \\ \mathbb{X} & \mathbb{Y} & \mathbb{Z} \end{vmatrix} = 0$$

or

$$2(S_By - S_Cz)\mathbb{X} - (S_Bx - S_By + S_Cz)\mathbb{Y} + (S_Cx - S_By + S_Cz)\mathbb{Z} = 0.$$

Similarly, we have the equations of the lines $B'B_P^\dagger$ and $C'C_P^\dagger$. These three lines are concurrent if and only if

$$\begin{vmatrix} 2(S_By-S_Cz) & -(S_Bx-S_By+S_Cz) & S_Cx-S_By+S_Cz \\ S_Ax+S_Ay-S_Cz & 2(S_Cz-S_Ax) & -(S_Ax+S_Cy-S_Cz) \\ -(-S_Ax+S_By+S_Az) & -S_Ax+S_By+S_Bz & 2(S_Ax-S_By) \end{vmatrix} = 0.$$

Expanding this determinant, we obtain

$$(x+y+z)\left(\sum_{\text{cyclic}}(S_B-S_C)(S_{AA}x^2+S_{BC}yz)\right) = 0.$$

Since P is a finite point, $x+y+z \neq 0$. Therefore $(x:y:z)$ must satisfy

$$\sum_{\text{cyclic}}(S_B-S_C)(S_{AA}x^2+S_{BC}yz) = 0.$$

The clearly defines a conic. By setting $x = 0$, we obtain

$$(S_By - S_Cz)(S_B(S_C-S_A)y - S_C(S_A-S_B)z) = 0.$$

It is clear that the conic contains $H_a = (0 : S_C : S_B)$. Similarly, it also contains H_b and H_c. Therefore, it is a circumconic of the orthic triangle $\mathsf{cev}(H) = H_aH_bH_c$. We also verify that the conic contains the following two points:
(i) the orthocenter $H = \left(\frac{1}{S_A} : \frac{1}{B} : \frac{1}{S_C}\right)$ (easy),
(ii) the triangle center

$$X(52) = ((S_B+S_C)(S^2-S_{AA})(S^2-S_{AA})(S^2+S_{BC}) : \cdots : \cdots)$$

(with the help of Mathematica). This latter, according to ETC, is the orthocenter $H^\perp$ of the orthic triangle (see Figure 7.1).

It follows that the locus of P is the rectangular circum-hyperbola of the orthic triangle containing H. □

Remarks. (1) For $P = H^\perp = X(52)$, the perspector of $\mathbf{T}^\dagger$ and $\mathbf{T}_P^\dagger$ is the orthology center $Q = (\mathbf{T}^\dagger)^\perp(\mathsf{cev}(H))$ in Proposition 4.3.

(2) Since the conic intersects the sidelines of $\mathbf{T}$ at the traces of H, the second intersections with the sidelines are the traces of another point. This is $X(847)$.

(3) The center of the conic is the point $X(1112)$.

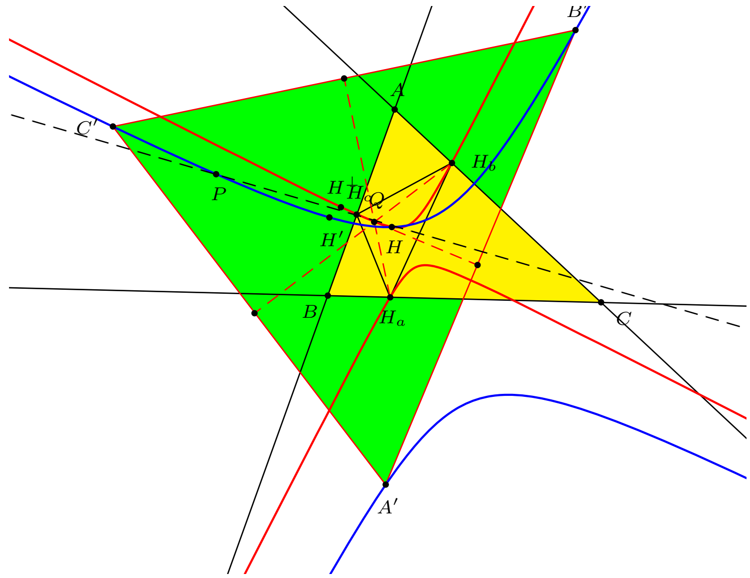

Figure 7.1

7.1.2. *Locus of perspector.*

Theorem 7.2. *For P on the rectangular circum-hyperbola of the orthic triangle containing H, the locus of the perspector of $\mathbf{T}^\dagger$ and $\mathbf{T}^\dagger_P$ is the rectangular circum-hyperbola of $\mathbf{T}^\dagger$ containing the orthocenter H.*

Proof. Rearranging the equation of $A'A^\dagger_P$ in the form

$$(S_B\mathbb{Y} - S_C\mathbb{Z})x - (2\mathbb{X} + \mathbb{Y} + \mathbb{Z})(S_By - S_Cz) = 0,$$

and likewise those of $B'B^\dagger_P$ and $C'C^\dagger_P$, we obtain the condition for concurrency:

$$\begin{vmatrix} S_B\mathbb{Y} - S_C\mathbb{Z} & -S_B(2\mathbb{X} + \mathbb{Y} + \mathbb{Z}) & S_C(2\mathbb{X} + \mathbb{Y} + \mathbb{Z}) \\ S_A(\mathbb{X} + 2\mathbb{Y} + \mathbb{Z}) & S_C\mathbb{Z} - S_A\mathbb{X} & -S_C(\mathbb{X} + 2\mathbb{Y} + \mathbb{Z}) \\ -S_A(\mathbb{X} + \mathbb{Y} + 2\mathbb{Z}) & S_B(\mathbb{X} + \mathbb{Y} + 2\mathbb{Z}) & S_A\mathbb{X} - S_B\mathbb{Y} \end{vmatrix} = 0.$$

Expanding the determinant, we obtain

$$(\mathbb{X} + \mathbb{Y} + \mathbb{Z})\left(\sum_{\text{cyclic}}(S_B - S_C)(2S_{AA}\mathbb{X}^2 - (S_A(S_B + S_C) - 2S_{BC})\mathbb{Y}\mathbb{Z})\right) = 0.$$

Since this perspector cannot be an infinite point, it must lie on the conic

$$\sum_{\text{cyclic}}(S_B - S_C)(2S_{AA}\mathbb{X}^2 - (S_A(S_B + S_C) - 2S_{BC})\mathbb{Y}\mathbb{Z}) = 0. \tag{7.1}$$

Construct the parallels to $A'C'$ and $A'B'$ through H_c and H_b respectively to intersect at a point X. The reflections of B and C in X lie respectively on $A'C'$

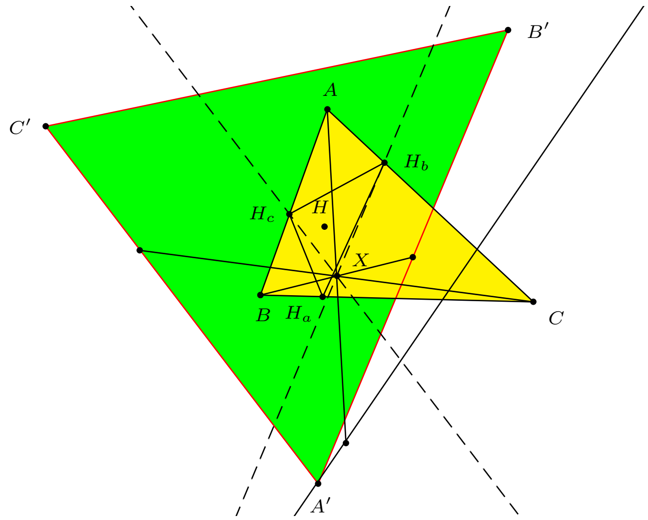

Figure 7.2

and $A'B'$ (Figure 7.2). Therefore, $\mathbf{T}^\dagger_X$ is perspective to $\mathbf{T}^\dagger$ at A'. This shows that the conic (7.1) contains A'. The same reasoning shows that it also contains B' and C'. It is a circumconic of $\mathbf{T}^\dagger$.

Now we claim that this conic contains the following two points:
(i) the orthocenter H (easy, take $P = H$),
(ii) the orthocenter H' of $\mathbf{T}^\dagger$.

In Proposition 4.3, we have constructed the orthology center $Q = (\mathsf{cev}(H))^\dagger(\mathbf{T}^\dagger)$. We claim that the reflection $\mathbf{T}^\dagger_Q$ is perspective with $\mathbf{T}^\dagger$ at the orthocenter of $\mathbf{T}^\dagger$. Note that in triangle $AA'A^\dagger_Q$, Q and H_a are the midpoints of $AA^\dagger_Q$ and AA'. Therefore, $A'A^\dagger_Q$ is parallel to H_aQ. Since H_aQ is perpendicular to $B'C'$, $A'A^\dagger_Q$ is the altitude of $\mathbf{T}^\dagger$ (through A'). Similarly, $B'B^\dagger_Q$ and $C'C^\dagger_Q$ are also altitudes of the same triangle (see Figure 7.3). The three lines are concurrent at H', the orthocenter of $\mathbf{T}^\dagger$, which therefore lies on the conic defined by (7.1).

It follows that (7.1) defines the rectangular circum-hyperbola of $\mathbf{T}^\dagger$ containing H. $\square$

7.2. *Orthology of* $\mathbf{T}^\dagger$ *with reflection triangle of* P. For a given point P, the reflection triangle $\mathbf{T}^\dagger(P)$ has vertices the reflections of P in the sidelines:

$$\begin{aligned}
P^\dagger_a &= (-(S_B+S_C)x : 2S_Cx + (S_B+S_C)y : 2S_Bx + (S_B+S_C)z),\\
P^\dagger_b &= (2S_Cy + (S_C+S_A)x : -(S_C+S_A)y : 2S_Ay + (S_C+S_A)z),\\
P^\dagger_c &= (2S_Bz + (S_A+S_B)x : 2S_Az + (S_A+S_B)y : -(S_A+S_B)z).
\end{aligned}$$

It is well known that the reflection triangle $\mathbf{T}^\dagger(P)$ is degenerate if and only if P lies on the circumcircle. It is clear that the perpendiculars from A, B, C to the

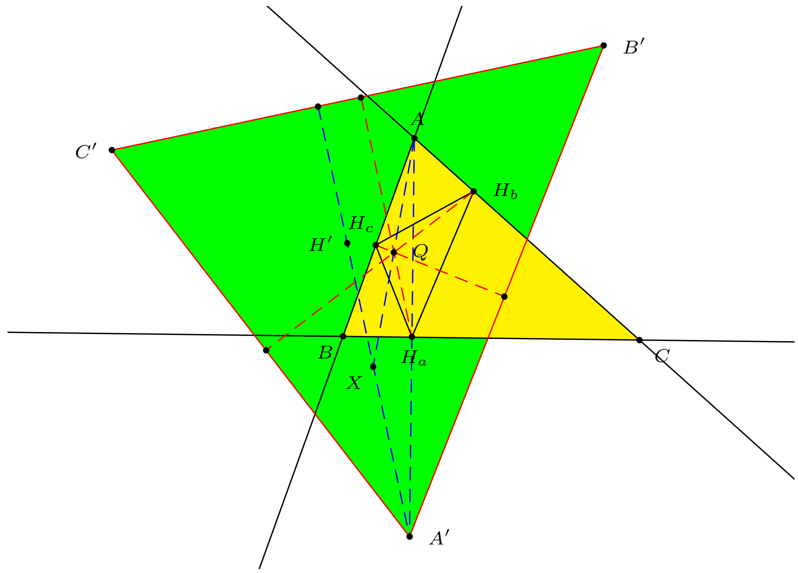

Figure 7.3

line containing $P_a^\dagger$, $P_b^\dagger$, $P_c^\dagger$ are parallel. However, the perpendiculars from these points to the sidelines of $\mathbf{T}$ are concurrent if and only if P is an intersection of the Euler line with the circumcircle. Henceforth, we shall consider P not on the circumcircle, so that its reflection triangle is nondegenerate. We study the locus of P for which $\mathbf{T}^\dagger(P)$ is orthologic to $\mathbf{T}^\dagger$.

7.2.1. *Locus of P whose reflection triangle is orthologic to $\mathbf{T}^\dagger$.*

Theorem 7.3. *The reflection triangle of P is orthologic to $\mathbf{T}^\dagger$ if and only if P lies on the Euler line.*

Proof. Let P be a point outside the circumcircle, and with homogeneous barycentric coordinates $(x : y : z)$. The perpendiculars from the vertices of $\mathbf{T}^\dagger$ to the sidelines of $\mathbf{T}^\dagger(P)$ are the lines

$$\begin{aligned}
2(-c^2S_Cy+b^2S_Bz)\mathbb{X}+(c^2(S_B-S_C)y+2b^2S_Bz)\mathbb{Y}+(-2c^2S_Cy+b^2(S_B-S_C)z)\mathbb{Z}&=0,\\
(-2a^2S_Az+c^2(S_C-S_A)x)\mathbb{X}+2(-a^2S_Az+c^2S_Cx)\mathbb{Y}+(a^2(S_C-S_A)z+2c^2S_Cx)\mathbb{Z}&=0,\\
(b^2(S_A-S_B)x+2a^2S_Ayx)\mathbb{X}+(-2b^2S_Bx+a^2(S_A-S_B)y)\mathbb{Y}+2(-b^2S_Bx+a^2S_Ay)\mathbb{Z}&=0.
\end{aligned}$$

These are concurrent if and only if

$$\begin{vmatrix}
2(-c^2S_Cy+b^2S_Bz) & (c^2(S_B-S_C)y+2b^2S_Bz) & (-2c^2S_Cy+b^2(S_B-S_C)z)\\
(-2a^2S_Az+c^2(S_C-S_A)x) & 2(-a^2S_Az+c^2S_Cx) & (a^2(S_C-S_A)z+2c^2S_Cx)\\
(b^2(S_A-S_B)x+2a^2S_Ayx) & (-2b^2S_Bx+a^2(S_A-S_B)y) & 2(-b^2S_Bx+a^2S_Ay)
\end{vmatrix}=0.$$

Expanding this determinant, we obtain

$$-a^2b^2c^2(a^2yz+b^2zx+c^2xy)(S_A(S_B-S_C)x+S_B(S_C-S_A)y+S_C(S_A-S_B)z)=0.$$

Apart from the circumcircle, this defines the Euler line. $\square$

7.2.2. *Loci of orthology centers.*

Theorem 7.4. *Let P be a point on the Euler line.*

(a) *The locus of the orthology center $\mathbf{T}^{\dagger\perp}(\mathbf{T}^\dagger(P))$ is the rectangular circum-hyperbola of $\mathbf{T}^\dagger$ containing the orthocenter H of $\mathbf{T}$.*

(b) *The locus of the orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger)$ is the line joining the orthocenter H of $\mathbf{T}$ to the nine-point center of $\mathbf{T}^\dagger$.*

Proof. Let $P=(S_{BC}+t, S_{CA}+t, S_{AB}+t)$ be a point on the Euler line.

(a) The perpendiculars from A' to the line $P_b^\dagger P_c^\dagger$ is

$$\begin{aligned}
&2S_A(S_B-S_C)(S^2+t)x\\
&+(S_A(S_A(2S_{BB}+S_{BC}-S_{CC})+S_{BC}(3S_B-S_C))+(S_A(3S_B-S_C)+S_B(S_B+S_C)t)y\\
&+(S_A(S_A(S_{BB}-S_{BC}-2S_{CC})+S_{BC}(S_B-3S_C))+(S_A(S_B-3S_C)-S_C(S_B+S_C))t)z\\
&=0.
\end{aligned}$$

The coefficients are linear in t; similarly for the equations of the perpendiculars from B', C' to $P_c^\dagger P_a^\dagger$ and $P_a^\dagger P_b^\dagger$. The point of concurrency of the three perpendiculars therefore has coordinates given by quadratic functions in t. Therefore, the locus of the orthology center $\mathbf{T}^{\dagger\perp}(\mathbf{T}^\dagger(P))$ is a conic. Note that for $P=O$, this orthology center is H. Also, for $P=N$, this is the orthocenter of $\mathbf{T}^\dagger$. (Proof: Since the reflection triangle of N is homothetic to $\mathbf{T}^\dagger$ (Proposition 2.2), they are orthologic, and $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(N))=H'$, the orthocenter of $\mathbf{T}^\dagger$).

We claim that A', B', C' are also three such orthology centers. Consider the Jerabek hyperbola (the rectangular circum-hyperbola of $\mathbf{T}$ through O and H. This is the isogonal conjugate of the Euler line. It also contains the symmedian point K. If the lines $A'K$, $B'K$, $C'K$ intersect the hyperbola again at X^*, Y^*, Z^* respectively, then their isogonal conjugates X, Y, Z are on the Euler line. If $X_a^\dagger X_b^\dagger X_c^\dagger$ is the reflection triangle of X, then the lines $X_a^\dagger X_b^\dagger$ and $X_a^\dagger X_c^\dagger$ are perpendicular to $A'C'$ and $A'B'$ respectively. This shows that A' is the orthology center $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(X))$, and it lies on the conic locus. The same reasoning shows that $B'=(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(Y))$ and $C'=(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(Z))$ are also on the same conic. This shows that the conic is a circumconic of $\mathbf{T}^\dagger$, containing its orthocenter H' and H.

(b) On the other hand, we compute the equations of the lines from $P_a^\dagger$ to $B'C'$, $P_b^\dagger$ to $C'A'$, and $P_c^\dagger$ to $A'B'$. These three lines are concurrent at a point whose coordinates are linear functions of t. Therefore, the locus of the orthology center is a line. This line has equation

$$\sum_{\text{cyclic}} a^2S_A(S_B-S_C)(3S^2-S_{AA})\mathbb{X}=0.$$

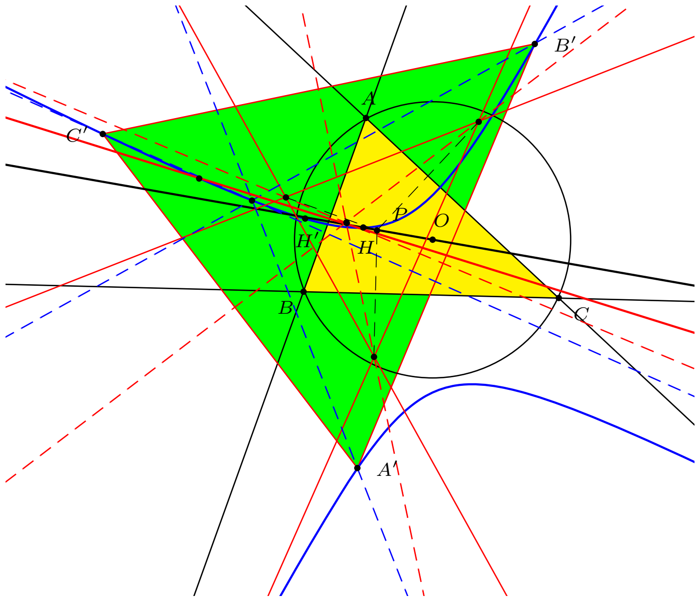

Figure 7.4

It is routine to check that this line contains the orthocenter H. It also contains the nine-point center N' of $\mathbf{T}^\dagger$ (with coordinates given in Proposition 2.4). □

Remarks. (1) This is the same line joining the orthology centers of $\mathbf{T}^\dagger$ and the orthic triangle in Theorem 4.3.

(2) The orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger)$ is

(i) the orthocenter H for $P = X(3520) = (a^2 S_{BC}(5S_{AA}+S^2) : \dots : \dots)$, which can be constructed as the reflection of H in $X(1594)$, which is the intersection of the Euler line with the line joining the Jerabek and Taylor centers,

(ii) the nine-point center of $\mathbf{T}^\dagger$ for

$$P = (-S^4(S_A+S_B+S_C)+S_{BC}((16S_A+7S_B+7S_C)S^2+S_{ABC}) : \dots : \dots)$$

with ETC (6-9-13)-search number $7.47436627857\cdots$.

8. Epilogue

In two appendices we list the triangle centers encountered in this paper. Appendix A lists those catalogued in ETC that feature in this paper with properties related to $\mathbf{T}^\dagger$. Appendix B lists new triangle centers in order of their search numbers in ETC. Here we present two atlases showing the positions of some of the centers in Appendix A in two groups.

Figure 8.1 shows a number of centers related to N and its isogonal conjugate N^*. On the line ON^* there are $X(195)$ (the circumcenter of $\mathbf{T}^\dagger$) and $X(1157)$ (the common point of the circumcircles of the reflection flanks (Proposition 5.4)). The line NN^* intersects the circumcircle at $X(1141)$, which also lies on the rectangular

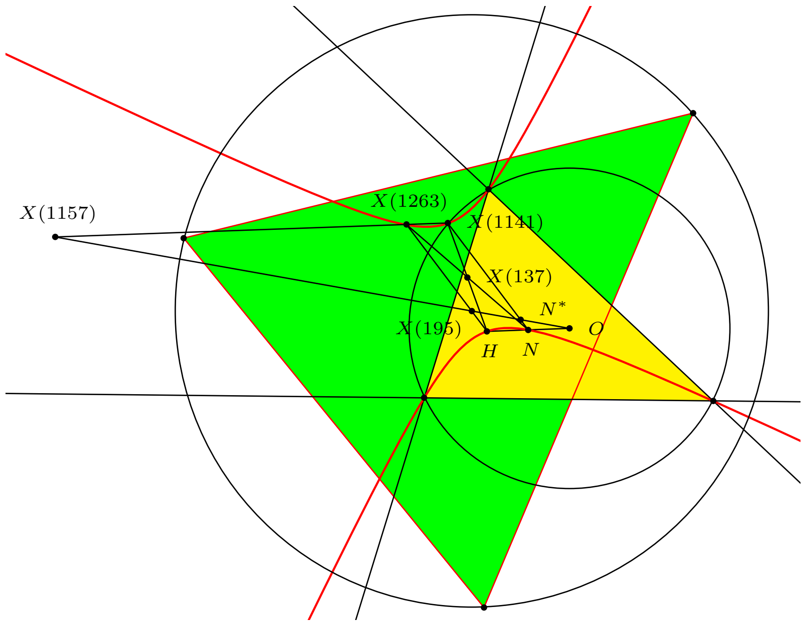

Figure 8.1

circum-hyperbola of $\mathbf{T}$ containing N. The center of the hyperbola is $X(137)$ on the nine-point circle.

The line joining $X(1141)$ to $X(1157)$ is parallel to the Euler line of $\mathbf{T}$. This line intersects the hyperbola at the antipode of N, which is the triangle center $X(1263)$.

Figure 8.2 shows the Euler line and its isogonal conjugate, the Jerabek hyperbola with center $X(125)$. The four triangle centers O, $X(125)$, $X(52)$ (center of the Taylor-like circle in §6.3), and $X(1986)$ (the Hatzipolakis reflection point) form a parallelogram, with center $X(389)$, the center of the Taylor circle. The line joining O to $X(125)$ intersects the hyperbola at $X265)$ (the reflection conjugate of O), which appears several times in this paper.

It is a well known fact that the line the hyperbola intersects the circumcircle at $X(74)$, the antipode of H. The antipode of $X(74)$ on the circumcircle is the Euler reflection point E. The Parry reflection point $X(399)$ is the reflection of O in E.

Figure 8.2 also shows the construction given in §7.2.2 of $X(3520)$ on the Euler line.

Appendix A: Triangle centers in ETC *associated with* $\mathbf{T}^\dagger$.

(1) $X(5) = N$: nine-point center.
- Reflection triangle homothetic to $\mathbf{T}^\dagger$ (Proposition 2.2).
- Perspector of the medial triangles of $\mathbf{T}^\dagger$ and $\mathbf{T}$ (Proposition 2.3).
- Perspector of the circumcenters of $AB'C'$, $A'BC'$, $A'B'C$ (Proposition 5.3(a)).

(2) $X(15)$: isodynamic point.

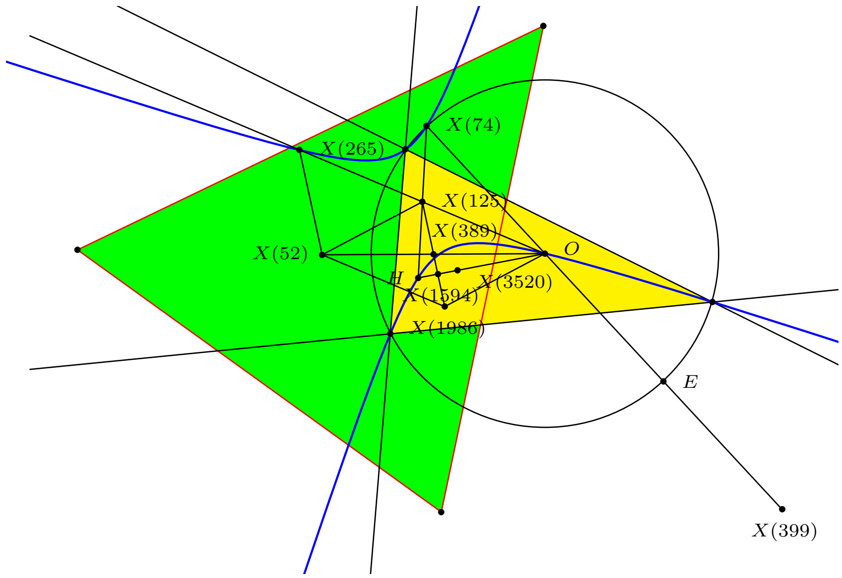

Figure 8.2

- Perspector of negative Fermat triangle with $\mathbf{T}^\dagger$ (§3.1.2).

(3) $X(16)$: isodynamic point.
- Perspector of positive Fermat triangle with $\mathbf{T}^\dagger$ (§3.1.2).

(4) $X(52)$: orthocenter of orthic triangle.
- Center of the circle through the 6 pedals of A' on CA, AB etc. (§6.3).

(5) $X(53)$: symmedian point of orthic triangle.
- Point of concurrency of the radical axes of the circumcircle with the circles $AH_aH'_a$, $BH_bH'_b$, $CH_cH'_c$, where $H_aH_bH_c$ is the orthic triangle, and H'_a, H'_b, H'_c are the pedals of A on $B'C'$ etc (Theorem 5.6).

(6) $X(54) = N^*$.
- orthology center from ABC to $\mathbf{T}^\dagger$ (Proposition 4.2).
- Perspector of the orthocenters of $AB'C'$, $A'BC'$, $A'B'C$ (Proposition 5.3(b)).
- perspector of the orthocenters of the reflection flanks (Proposition 5.3(b)).
- radical center of the nine-point circles of $A'BC$, $AB'C$ and ABC'.
- Perspector of the triangle bounded by the radical axes of the circumcircle with the circles each through one vertex and the reflections of the other two in their opposite sides. (The triangle in question is indeed the anticevian triangle of N^*.)

(7) $X(68)$:
- radical center of the circles AB_aC_aA', BC_bA_bB', CA_cB_cC' (Proposition 6.2).

(8) $X(195)$: reflection of O in N^*.
- The circumcenter of $\mathbf{T}^\dagger$ (§2.3).
- Orthology center from tangential triangle to $\mathbf{T}^\dagger$ (Proposition 4.4).

(9) $X(265)$: reflection conjugate of O.

- Orthology center from circumcenters of $K^aB'C'$, $K^bC'A'$, $K^cA'B'$ to $\mathbf{T}$ (§5.3).
- Point of concurrency of parallels through A to Euler line of $A'BC$ etc.

(10) $X(399)$: Parry reflection point (reflection of O in Euler reflection point):
- Common point of circles $K^aB'C'$, $A'K^bC'$, $A'B'K^c$ (Proposition 5.7).
- Common point of circles $A'I^bI^c$, $B'I^cI^a$, $C'I^aI^b$ (Proposition 5.8).

(11) $X(484)$ Evans perspector
- Perspector of $\mathbf{T}^\dagger$ and the excentral triangle (§3.1.1).

(12) $X(1141)$: intersection of circumcircle with the rectangular circum-hyperbola through N.
- Perspector of the reflection triangle of $X(1157)$.
- Cevian triangle perspective with $\mathbf{T}^\dagger$ at $X(1157)$.

(13) $X(1157)$: inverse of N^* in circumcircle.
- Common point of the circles $AB'C'$, $A'BC'$ and $A'B'C$ (Proposition 5.4).
- Perspector of $\mathbf{T}^\dagger$ with the anticevian triangle of N^*.
- Perspector of $\mathbf{T}^\dagger$ with the cevian triangle of $X(1141)$.

(14) $X(1986)$: Hatzipolakis reflection point.
- Reflection of Jerabek point in Taylor center.
- Common point of the Euler lines of AB_aC_a etc., where B_a and C_a are the pedals of A' on AC and AB respectively (Theorem 6.1).

(15) $X(3060)$
- Centroid $\mathbf{T}^\dagger$ (§2.3).

(16) $X(3336)$
- Perspector of centers of circles $I^aB'C'$, $I^bC'A'$, $I^cA'B'$ (§5.4).

(17) $X(3520)$
- Point on the Euler line with orthology center $(\mathbf{T}^\dagger)^\perp(\mathbf{T}^\dagger(X(3520)) = H$ (§7.2.2, Remark 2(i)).

Appendix B: Triangle centers not in ETC.

(1) $-27.4208873972\cdots$: perspector of excentral triangle with centers of circles $A'I^bI^c$, $B'I^cI^a$, $C'I^aI^b$ (§5.4).
(2) $-9.04876879620\cdots$: perspector of $\mathbf{T}^\dagger$ and triangle bounded by reflections of a' in a etc (§3.2).
(3) $-8.27009636449\cdots$: orthology center from $\mathbf{T}^\dagger$ to orthic triangle (§4.3).
(4) $-7.90053389552\cdots$: perspector of $\mathbf{T}^\dagger$ and triangle bounded by reflections of a in a' etc ((§3.2)).
(5) $-5.94879118842\cdots$: symmedian point of $\mathbf{T}^\dagger$.
(6) $-1.94515015138\cdots$: Euler reflection point of $\mathbf{T}^\dagger$ (§2.4).
(7) $-0.873191727540\cdots$: perspector of the triangles bounded by the reflections of a' in a etc and those of a in a' etc (§3.2).
(8) $1.86365616601\cdots$: common point of circles $A'K^bK^c$, $B'K^cK^a$, $C'K^aK^b$ (§5.4).
(9) $2.99369649092\cdots$: radical center of pedal circles of A', B', C'.
(10) $3.00505308538\cdots$: point of concurrency of H_aH_a', H_bH_b', H_cH_c', where H_a', H_b', H_c' are the pedals of A on $B'C'$, B on $C'A'$, and C on $A'B'$ respectively (Proposition 5.5).
(11) $3.99180618013\cdots$: perspector of $\mathbf{T}$ and triangle bounded by reflections of a' in a etc (§3.2).
(12) $4.22924780831\cdots$: perspector of $\mathbf{T}$ and the reflection triangle of $X(1263)$ (§4.1).
(13) $5.99676405896\cdots$: nine-point center of $\mathbf{T}^\dagger$ (§2.3).

(14) $7.08747856659\cdots$: common point of circles $I^aB'C'$, $A'I^bC'$ and $A'B'I^c$ (§5.4).
(15) $7.47436627857\cdots$: P for which the orthology center $\mathbf{T}^\dagger(P)^\perp(\mathbf{T}^\dagger) = N^\dagger$ (§7.2.2, Remark 2(ii)).
(16) $8.27975385194\cdots$: perspector of $\mathbf{T}$ and triangle bounded by reflections of a in a' etc (§3.2).
(17) $12.48182503\cdots$: $Q :=$ orthology center from orthic triangle to $\mathbf{T}^\dagger$ (§4.3).
- orthology center from reflection triangle of $X(1594)$ to $\mathbf{T}^\dagger$,
- point of concurrency of the Euler lines of triangles $A'B_aC_a$, $B'C_bA_b$, $C'A_cB_c$ (Proposition 6.3).
- reflection of $\mathbf{T}$ in this point is perspective with $\mathbf{T}^\dagger$ (at the orthocenter of $\mathbf{T}^\dagger$) (Proof of Theorem 7.3).

(18) $31.1514091170\cdots$:
- orthocenter of $\mathbf{T}^\dagger$ (§2.3).
- perspector of $\mathbf{T}^\dagger$ and reflection of $\mathbf{T}$ in Q (§7.1.2).

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A Gallery of Conics by Five Elements

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Abstract. This paper is a review on conics defined by five elements, i.e., either lines to which the conic is tangent or points through which the conic passes. The gallery contains all cases combining a number (n) of points and a number $(5-n)$ of lines and also allowing coincidences of some points with some lines. Points and/or lines at infinity are handled as particular cases.

1. Introduction

In the following we review the construction of conics by five elements: points and lines, briefly denoted by $(\alpha P \beta T)$, with $\alpha + \beta = 5$. In these it is required to construct a conic passing through α given points and tangent to β given lines. The six constructions, resulting by giving α, β the values 0 to 5 and considering the data to be in general position, are considered the most important ([18, p. 387]), and are to be found almost on every book about conics. It seems that constructions for which some coincidences are allowed have attracted less attention, though they are related to many interesting theorems of the geometry of conics. Adding to the six main cases those with the projectively different possible coincidences we land to 12 main constructions figuring on the first column of the classifying table below. The six added cases can be considered as limiting cases of the others, in which a point tends to coincide with another or with a line. The twelve main cases are the projectively inequivalent to which every other case can be reduced by means of a projectivity of the plane. There are, though, interesting classical theorems for particular euclidean inequivalent cases worth studying, as, for example, the much studied case of parabolas tangent to four lines (case $(0P5T_1)$ in §7.2). In this review the frame is that of euclidean geometry and consequently a further distinction of *ordinary* from points and lines *at infinity* is taken into account. All of the (50) constructions classified below are known and can be found in the one or the other book on conics (e.g. [7, pp. 136]), but nowhere come to discussion all together, so far I know. In a few cases (e.g. §11.1, §3.2) I have added a proof, which seems to me interesting and have not found elsewhere.

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	1	2	3	4	5	6
2	$(5P0T)$	$(5P_10T)$	$(5P_20T)$			
3	$(4P1T)$	$(4P1T_1)$	$(4P_11T)$	$(4P_21T)$		
4	$(3P2T)$	$(3P2T_1)$	$(3P_12T)$	$(3P_22T)$		
5	$(2P3T)$	$(2P3T_1)$	$(2P_13T)$	$(2P_23T)$		
6	$(1P4T)$	$(1P4T_1)$	$(1P_14T)$			
7	$(0P5T)$	$(0P5T_1)$				
8	$(4P1T)_1$	$(4P_11T_1)$	$(4P_11T)_1$	$(4P_11T)_i$	$(4P_21T)_1$	$(4P_21T)_i$
9	$(3P2T)_1$	$(3P2T_1)_1$	$(3P_12T_1)$	$(3P_12T)_i$	$(3P_22T)_1$	$(3P_22T)_i$
10	$(3P2T)_2$	$(3P_12T_1)_1$	$(3P_12T)_{1i}$	$(3P_22T)_{2i}$		
11	$(2P3T)_1$	$(2P3T_1)_1$	$(2P_13T_1)$	$(2P_13T)_i$	$(2P_23T)_i$	
12	$(2P3T)_2$	$(2P3T_1)_2$	$(2P_13T_1)_1$	$(2P_13T)_{1i}$	$(2P_23T)_{2i}$	
13	$(1P4T)_1$	$(1P4T_1)_1$	$(1P_14T_1)$	$(1P_14T)_i$		

The above table serves as the table of contents of this paper, the row labels are the section numbers and the column labels the subsection numbers. The column with label 1 lists the symbols of the twelve projectively inequivalent cases. Each row of the table comprises the cases, which are projectively equivalent to the one of the first column. The notation used is a slight modification of the one introduced by Chasles ([3, p. 304]). The symbol P_n means that n of the given points are at infinity and T_1 means that one of the tangent lines is the line at infinity, later meaning that the conic, to be constructed, is a parabola. The indices, which adhere to the right parenthesis are optional. When absent, it means that the configuration is in *general position*, i.e. none of the ordinary points is coincident with an ordinary line. When present it means that one or two of the points are correspondingly coincident with one or two tangents. The indices $i, 2i$ mean that one/two ordinary lines are correspondingly coincident with one/two points at infinity. The index $1i$ means that an ordinary point coincides with a line and also a point at infinity coincides with the point at infinity of an ordinary line.

Except for the coincidence suggested by the corresponding symbol, the other data are assumed to be in general positions in the projective sense. For example, the symbol $(3P_12T)_i$ stands for the construction of a hyperbola given two points, an asymptote and a tangent. These four elements are assumed in general position, implying that no further coincidences are present, that the intersection of the asymptote and the tangent are not collinear to the two points, that the line of the two points is not parallel to the tangent or the asymptote, etc. The statements on the number of solutions or non existence in each case, presuppose such a restriction. Throughout the text, *existence* is meant in the *real* plane.

The following notation is also used: t_X denotes the tangent at X, $Y = X(A, B)$ denotes the harmonic conjugate of X with respect to (A, B), $X = (a, b)$ denotes the intersection of lines a, b. Points at infinity are occasionally denoted by $[A]$, the same symbol indicating also the direction determined by that point at infinity. For a line e the symbol $[e]$ denotes its point at infinity. For a point at infinity A the symbol XA denotes the parallel from X to the direction determined by A.

Degenerate conics, consisting of a product of two lines a, b, are represented by $a \cdot b$. Regarding hyperbolas, *asymptotics* are distinguished from *asymptotes*. The first term gives the direction only, the second denotes the precise line.

Regarding the organization of the material, there follow three preliminary sections on the background, which comprise: (a) Involutions (§1.1), (b) Pencils and Families of conics (§1.2), (c) The great theorems (§1.3). Then follow fifty sections handling the inequivalent euclidean constructions. The sections are divided in twelve groups, each group headed by the problem to which all other of the group are projectively equivalent.

1.1. *Involutions.* *Involutions* are homographies of projective lines with the property $f^2 = I \;\Leftrightarrow\; f^{-1} = f$. Using proper coordinates, involutions are described by functions of the form

$$y = \frac{ax+b}{cx-a} \quad \Leftrightarrow \quad x = \frac{ay+b}{cy-a},$$

whose graphs are rectangular hyperbolas symmetric with respect to the diagonal line $y = x$. Such functions are completely determined by prescribing their values

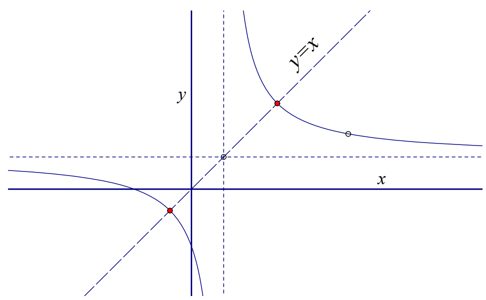

Figure 1. Graph of an involution with existing fixed points

on two elements $(X, f(X))$, $(Y, f(Y))$ and they have either two fixed points or none. Figure 1 shows a case in which there are two fixed points. When the hyperbola has the two branches totally contained in the two sides of the line $y = x$, the corresponding function has no fixed points. An important property of a pair $(X, f(X))$, of related points of an involution, frequently used below, is that it consists of harmonic conjugates with respect to the fixed points of f, when these exist ([7, p. 100], [22, I, p.102], [17, p. 167], [9, p. 35], [2, I, p. 137]).

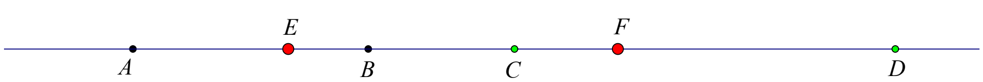

Figure 2. Common harmonics (E, F) of (A, B) and (C, D)

The practical issue of finding the fixed points of involutions is related to the idea of *common harmonics* of two pairs (A, B), (C, D) of collinear points. This is another pair (E, F) of points, which are, as the name suggests, simultaneously harmonic conjugate with respect to (A, B) and with respect to (C, D) (see Figure 2). If such a pair (E, F) exists, then it is easily seen that every circle d passing

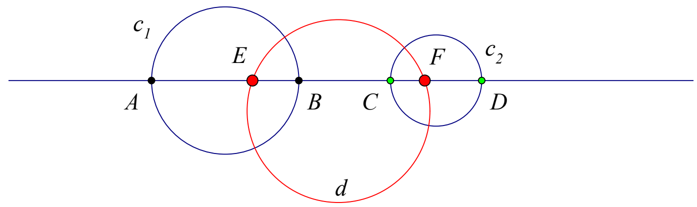

Figure 3. Construction of the common harmonics (E, F) of (A, B) and (C, D)

through E, F is orthogonal to the circles c_1, c_2 with corresponding diameters AB and CD (see Figure 3). Thus, in order to find the common harmonics geometrically, set two circles c_1, c_2 on diameters, respectively, AB and CD and draw a circle d simultaneously orthogonal to c_1 and c_2 (see Figure 4). In case one of the

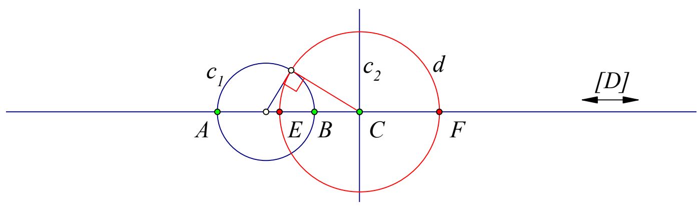

Figure 4. Common harmonics (E, F) of (A, B) and $(C, [D])$

points, D say, is a point at infinity then c_2 is the line orthogonal to AB at C and we can take the circle d to be the one centered at C and orthogonal to c_1, points E, F lying then symmetric with respect to C (see Figure 4).

The common harmonics are precisely the *limiting points* of the *coaxal system* (pencil) of circles determined by c_1 and c_2 ([16, p. 118]). They exist, precisely when the circles c_1, c_2 are non-intersecting.

1.2. *Pencils of conics.* Pencils of conics are *lines* in the five-dimensional projective space of conics. This is reflected in the generation of a pencil as the set $\mathcal{D}$ of linear combinations $c = \alpha \cdot c_1 + \beta \cdot c_2$, where $c_1 = 0$, $c_2 = 0$ are the equations of any two particular members of the pencil. Then $c = 0$ represents the equation of the general member of the pencil for arbitrary real values of α, β with $|\alpha| + |\beta| \neq 0$. The basic pencil, called of *type-I*, is that of all conics passing through four points (see Figure 5). There are five projectively inequivalent pencils, characterized by the fact that all their members intersect by two at the same points, real or imaginary, with the same multiplicities. These are referred to as types I to V pencils ([22, I, p. 128]) and can be considered as limiting cases of the type I pencil. Type II, for instance, results by fixing line $e = AD$ and letting D coincide with A. The resulting pencil

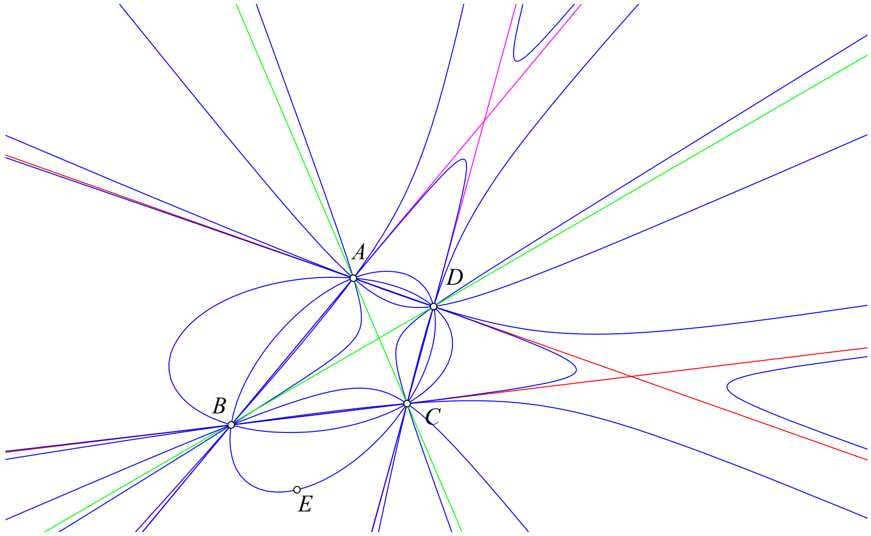

Figure 5. The pencil of conics through A, B, C, D

(seen in §9.1) consists of all conics passing through A, B, C and tangent to e at A. Another type of pencil is obtained from a type I pencil by fixing lines $a = AB$ and $c = CD$ and letting point B converge along a to A and D converge along c to C. The resulting pencil, referred to as type IV pencil (seen in §10.1), consists of all conics tangent to lines a, c correspondingly at their points A and C.

Pencils of conics contain *degenerate members*, at most three ([2, II, p. 124]). In Figure 5 the degenerate members are visible, consisting of the pairs of lines $AB \cdot CD$, $AD \cdot BC$ and $AC \cdot BD$. In the analytic description of pencils $c = \alpha \cdot c_1 + \beta \cdot c_2$, the conics c_1, c_2 can be degenerate members, and this is often convenient and extensively used below.

To every type of pencil corresponds an analogous *range* of conics or *tangential* pencil of conics ([2, II, p. 199], with a notation slightly different from that of Veblen). For example to type I pencils corresponds the range of type I^* of conics, which are tangent to four lines in general positions (see §7.1). Ranges are pencils of conics in the dual projective plane P^* consisting of all lines of the projective plane P. To each pencil of type X corresponds its dual range X^* with properties resulting from those of X by duality.

A particular property of pencils, together with its dual for ranges, is of interest for our subject. For instance, in the case of a pencil $\mathcal{D}$ of conics through points A, B, C, D, it is known ([2, II, p. 197]) that the polars of a fixed point X with respect to all members of the pencil pass through a point Y. This defines a quadratic transformation $Y = f(X)$, which in the coordinates with respect to the projective base $\{E, F, G, A\}$, with $E = (AB, CD)$, $F = (AD, BC)$, $G = (AC, BD)$, is represented by

$$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad z' = \frac{1}{z}.$$

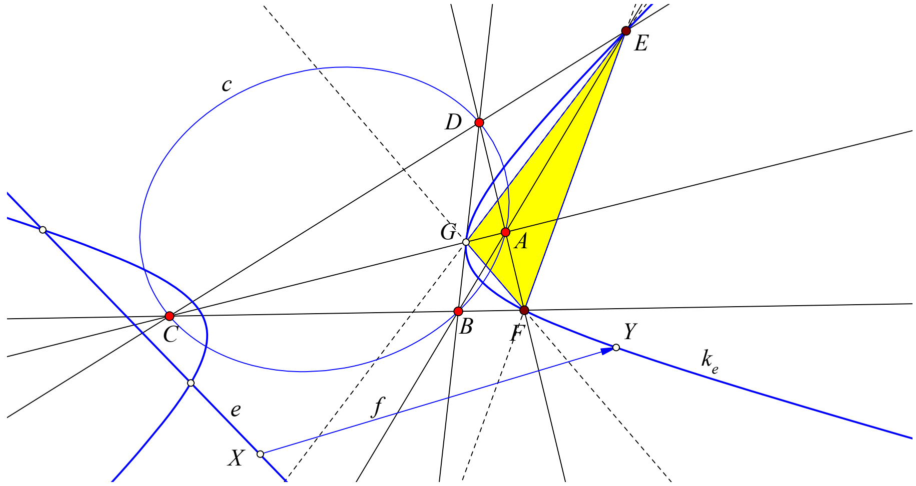

Figure 6. The eleven points conic k_e of A, B, C, D and line e

The image of a line e under this transformation is a conic k_e circumscribing triangle EFG and passing through eight additional points, therefore called an *eleven point conic* ([1, p. 97], [9, p. 66]). Six of the points are the harmonic conjugates $W = V(X, Y)$ of the intersection point $V = (XY, e)$, where X, Y are taken from $\{A, B, C, D\}$. The two remaining points, if real, are the intersection points of k_e with line e and simultaneously the contact points of two members of the pencil $\mathcal{D}$, which are tangent to e (a case handled in §8.1).

The dual to the previous property relates to the range $\mathcal{D}^*$ of conics k tangent to four lines a, b, c, d (see Figure 7). According to this, the poles of a line h with respect to the members of $\mathcal{D}^*$ lie on a line h' and the transformation $h' = F^*(h)$ is a quadratic one of the same nature as the previous one, differing only in that it operates on the dual projective plane P^*. Line h' can be found by a simple criterion, resulting by considering the triangle of diagonals (efg in Figure 7). Lines h, h' intersect each side s of this triangle at points X, Y, which are harmonic conjugate with respect to (U, V), where U, V are the vertices of the quadrilateral lying on s. The images h' under F^* of all lines h passing through a fixed point Q are the tangents of a conic k_Q inscribed in the triangle efg and tangent to eight additional lines, therefore called an *eleven tangents conic*. Six of these lines are the harmonic conjugates $Q(s, s')$ of Q with respect to all pairs (s, s') taken from $\{a, b, c, d\}$. The two remaining tangents, if real, are the tangents through Q of the members of $\mathcal{D}^*$ passing through Q (a case handled in §6.1).

Roughly described, a standard method of constructing a conic by five elements is to find a pencil or range satisfying four of the given conditions, and then use the fifth condition to locate the particular member(s) of the pencil satisfying it. In the case of type I pencils, any fifth point E, different from A, B, C, D, determines exactly one conic of the pencil containing it (a case handled in §2.1).

Pencils of conics, passing through two different points Q, R can be transformed to pencils of circles by a complex projective map, which sends Q, R to the *circular*

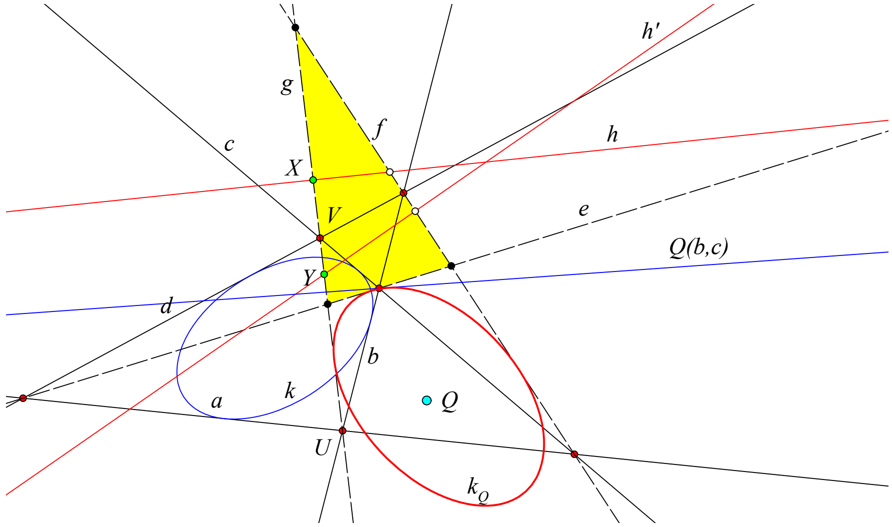

Figure 7. The eleven tangents conic k_Q of a, b, c, d and point Q

points at infinity I, J ([9, p. 71]). By such a map a type I pencil can be transformed to a pencil of intersecting circles and all related conic construction problems reduce to corresponding Apollonius circle construction problems ([6]). This method is concisely expounded in [10].

1.3. *The great theorems.* The basic tool in the context of the present subject is Desargues' theorem, for various types of pencils and ranges, as neatly described in [22, p. 128]. The theorem asserts that a pencil intersects on a fixed line e, through its members, pairs of points (X, Y) *in involution*, later meaning, that there is an involution f on e, such that $Y = f(X) \Leftrightarrow X = f(Y)$. The interesting fact is that f is completely determined by the intersections of e with the *degenerate members* of the pencil, which are products of lines. Line e is assumed to be different from the lines contained in degenerate members of the pencil.

In its dual form, Desargues' theorem, referred to also as Plücker's theorem ([4, p. 25], [2, II, p. 202]) states, that the pairs of tangents (x, y) from a fixed point Q to the members of a range, are *in involution*, later meaning, that there is an involution f^* on the pencil Q^* of lines through Q, such that $y = f^*(x) \Leftrightarrow x = f^*(y)$. This involution is determined again by the *degenerate members* of the range, which are pairs of points. In the case of ranges of conics tangent to four lines, for instance, the degenerate members are pairs of intersection points of the four lines and the involution on Q^* is determined by two pairs of lines joining Q to two such pairs of points ([22, I, p. 129], [9, p. 50], [2, II, p. 197]). Point Q is assumed to be different from the points of degenerate members.

The somewhat difficult to visualize involution on Q^* can be represented by intersecting the rays through Q with a fixed line $e \not\ni Q$. In this way the involution f^* on Q^* defines an involution f on e and the fixed points (rays) of f^* trace on e the fixed points of f. Quite typically for our subject, the requested conics are intimately related to the fixed elements of such involutions ([20, p. 69], [19, p. 365], [2, p. 198, II]).

Pascal's theorem, that the opposite sides of a hexagon, inscribed in a conic, intersect on a line, is used in the present context in order to find additional points on the requested conics. The theorem is used also in its various versions for inscribed pentagons, quadrangles and triangles ([22, I, p. 111], [17, p. 156], [2, II, p. 176]).

Brianchon's theorem, which is dual to Pascal's, asserts that the lines through opposite vertices of a circumscribed to a conic hexagon pass through a fixed point. Again the theorem and its versions for pentagons, quadrangles and triangles is used in order to find additional points on the requested conics.

2. Five points

2.1. *Conic through five points* $(5P0T)$. Construct a conic passing through five points A, B, C, D, E. This is the basic construction, to which, all other constructions may be reduced. Analytically this can be done easily by considering the equations of two line-pairs defined by the five points ([18, p. 232]). Let, for example, the line-pairs $(AB, CD), (AC, BD)$ be given correspondingly by equations $(f = 0, g = 0), (h = 0, j = 0)$. Then the equation

$$k_{\lambda,\mu} = \lambda \cdot (f \cdot g) + \mu \cdot (h \cdot j) = 0,$$

for variable λ and μ, represents the pencil $\mathcal{D}$ of all conics passing through A, B, C, D. The requirement, for such a conic, to pass through E, leads to an equation for λ, μ:

$$k_{\lambda,\mu}(E) = 0,$$

from which λ, μ are determined up to a multiplicative constant, and through this a unique conic is defined as required.

Geometrically, one can use *Pascal's theorem* to produce, from the given five, arbitrary many other points P lying on the conic. Figure 8 illustrates the way this

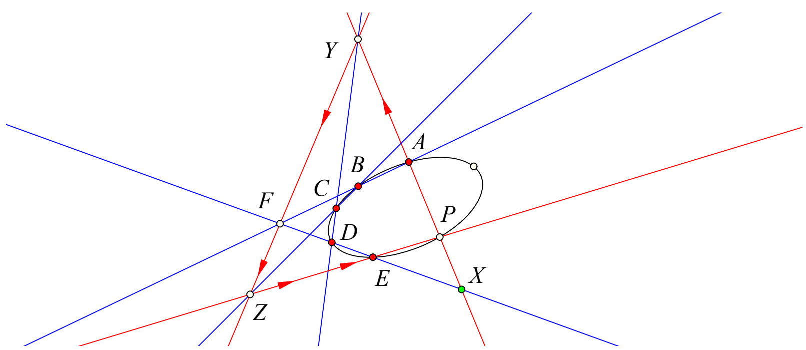

Figure 8. Pascal's theorem producing more points on the conic

is done. Start with an arbitrary point X on line DE and define the intersection point $Y = (CD, AX)$. Join this point to the intersection point $F = (AB, DE)$ and extend the line to find the intersection point $Z = (YF, BC)$. By Pascal's theorem, the intersection point $P = (ZE, AX)$ is on the conic passing through A, B, C, D and E. The books of Russell [17, p. 229] and, for more cases Yiu [24, p. 144], contain many useful constructions, which determine various elements of the conic, such as the intersections with a line, the center, the axes, the foci etc. out of the five given points.

2.2. *Conic through five points, one at infinity* $(5P_1 0T)$. Construct a conic passing through five points $A, B, C, D, [E]$ in general position. The conic is a hyperbola and in some cases a parabola. Additional points can be constructed as in the

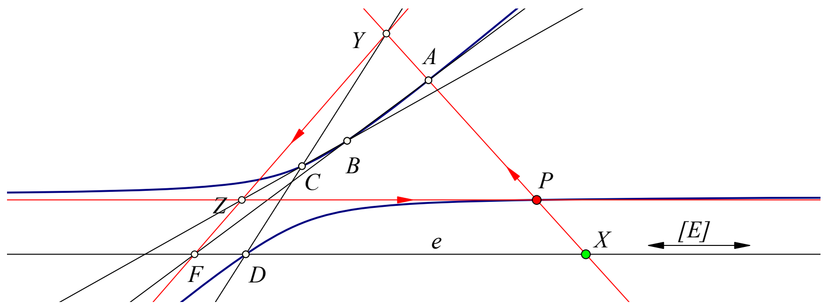

Figure 9. Finding additional points on the conic through $A, B, C, D, [E]$

previous section. Start with a point X on line $e = DE$, find the intersections $Y = (AX, CD)$, and $Z = (FY, BC)$, where $F = (AB, DE)$ (see Figure 9). Point $P = (ZE, XA)$ is on the requested conic, which can be constructed to pass through the five points A, B, C, D, P. There is always a unique solution.

Remark. In general the conic is a hyperbola, and $[E]$ represents the direction of one of its asymptotes. Fixing points A, B, C, D, there are either none or two directions $[E]$, determined by the four points, for which the conic passing through $A, B, C, D, [E]$ is a parabola. This is the case $(4P1T_1)$ of §3.2.

2.3. *Hyperbola from asymptotics and three points* $(5P_2 0T)$. Construct a conic passing through five points $A, B, C, [D], [E]$, thus a hyperbola with asymptotic directions given by $[D], [E]$. The pentagon $ABCDE$ is infinite with DE the line

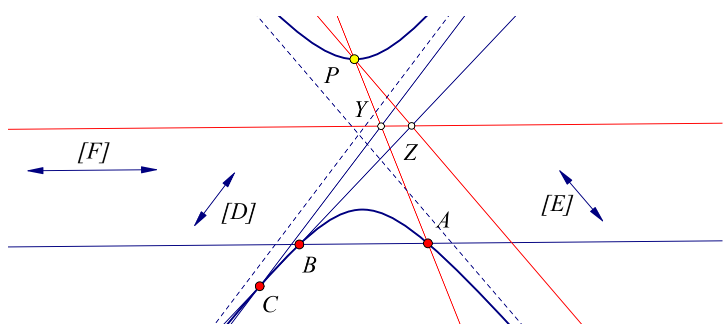

Figure 10. Pascal's theorem with D, E at infinity

at infinity and its intersection F with AB is also at infinity. An arbitrary ray of the pencil A^* of lines through A, and its intersection Y with CD defines the point $Z = (YF, BC)$ and $P = (ZE, AY)$. Last is a point on the requested conic (see Figure 10). There is always one solution. Figure 11 shows a related pencil consist-

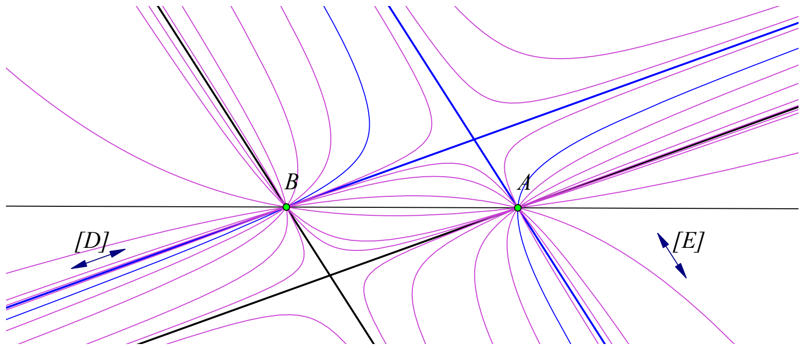

Figure 11. The pencil of conics passing through $A, B, [D], [E]$

ing of all conics passing through $A, B, [D], [E]$, i.e. all hyperbolas with asymptotic directions $[D], [E]$ and passing through the points A, B.

3. Four points and one tangent

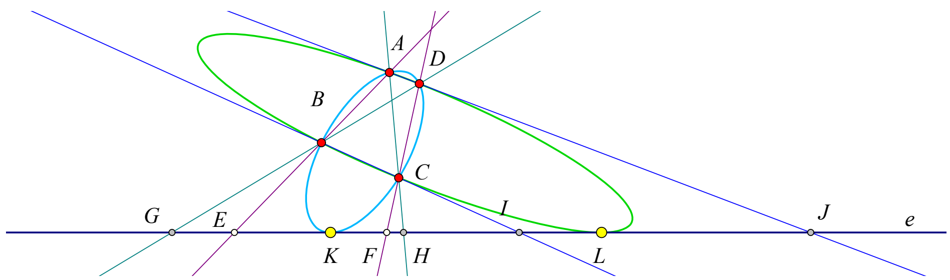

Figure 12. The two conics through A, B, C, D tangent to line e

3.1. *Conic by four points and one tangent* $(4P1T)$. Construct a conic passing through four points A, B, C, D and tangent to one line e. By Desargues' theorem (see §1.3), each member of the pencil $\mathcal{D}$ of conics through A, B, C, D (seen in §1.1) intersects line e to a pair of points in involution. The contact points K, L of the requested conic with e are the fixed points of this involution. Two pairs of points, defining the involution, are the intersections of e with two degenerate members c_1, c_2 of the pencil $\mathcal{D}$, consisting of the pairs of lines $c_1 = AB \cdot CD$ and $c_2 = AD \cdot BC$, intersecting the line respectively in (E, F) and (I, J) (see Figure 12). The fixed points of the involution are the common harmonics (K, L) of the pairs $(E, F), (I, J)$.

An alternative construction for this case relates to the *eleven points conic* of four points and a line (see §1.2), consisting of the poles of line e with respect to all conics of the pencil $\mathcal{D}$. This conic intersects line e precisely at its contact points

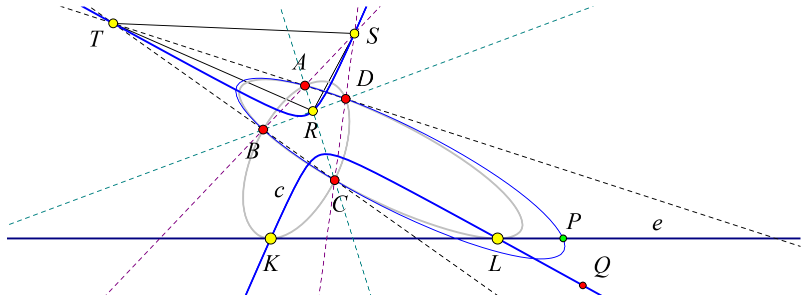

Figure 13. Eleven point conic of $ABCD$ and line e

K, L with the requested conics.

Next two figures display the domains of existence of solutions for variable D, assuming given the positions of A, B, C and line e.

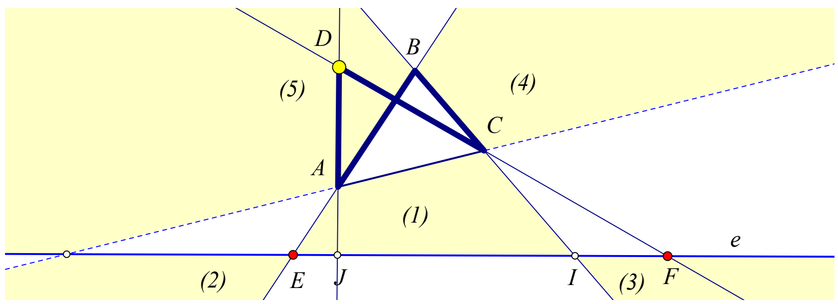

Figure 14. Domains of existence for variable D, e non-intersecting ABC

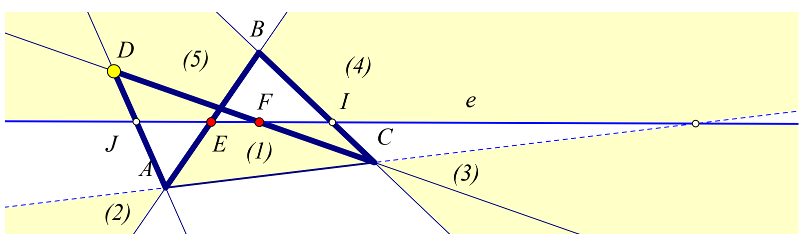

Figure 15. Domains of existence for variable D, e intersecting ABC

3.2. *Parabola through four points* $(4P1T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, passing through four points A, B, C, D. The only difference from the previous case is that line e is now at infinity. The involution on e can be represented on the pencil O^* of lines through the arbitrary but fixed point O ([4, p. 40], [20, p. 69], [17, p. 180]). In fact, draw from O parallels to the lines joining all possible pairs of the four points A, B, C, D. They result in three

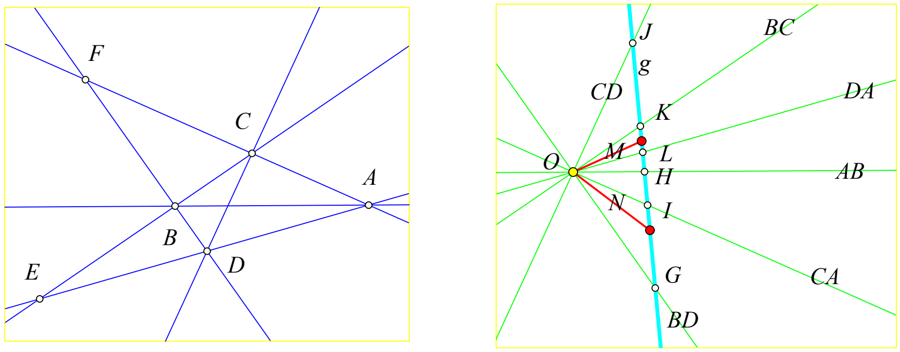

Figure 16. Common harmonic directions

pairs of lines $(BD, CA), (AB, CD), (DA, BC)$, which are *in involution*. This, by intersecting the pencil with an arbitrary line g defines an involution in g (see Figure 16). The corresponding pairs of points $(G, H), (I, J), (K, L)$ of g are in

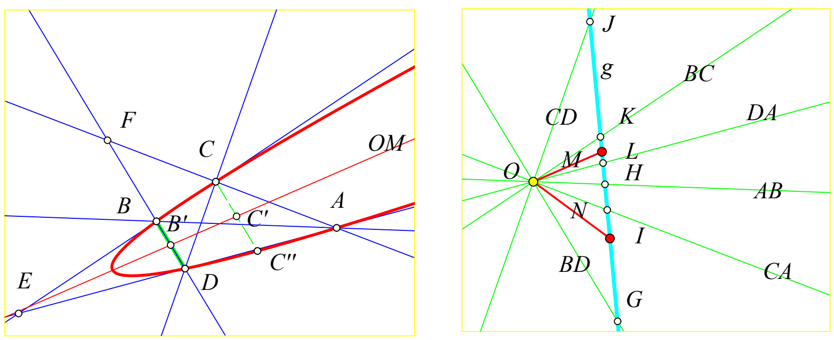

Figure 17. Parabola through A, B, C, D with axis-direction OM

involution. The common harmonics M, N of these pairs, if they exist, define two directions OM and ON, i.e. two points at infinity, which represent the directions of the axis of the requested parabolas, passing through A, B, C, D. Thus, there are either two or none parabola passing through four points A, B, C, D in general position. Figure 17 shows how the construction of one of these two parabolas can be done. Use is made of one of the chords BD of the requested parabola. From the middle B' of BD we draw a parallel h to the direction OM. Then we project C on C' and extend CC' to its double CC''. The projection is by parallels to BD. By a well known property of parabolas, point C'' will also belong to the parabola under construction. Thus, taking C'' as the fifth point we define the parabola as the conic passing through A, B, C, D and C''. An analogous construction can be carried out for the second parabola, whose axis is parallel to the direction ON. The parabolas exist if none of the four given points is contained in the triangle of the other three.

Besides this construction of the two parabolas, which is considered the standard one, there is another approaching the problem from a different point of view. For

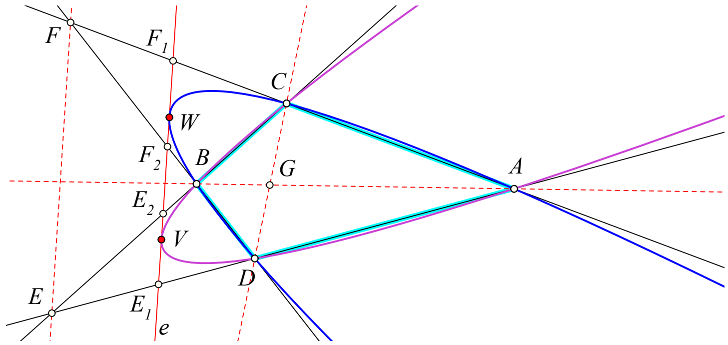

Figure 18. The two parabolas through A, B, C, D

this, consider the line e parallel to the diagonal EF of the quadrangle $ACBD$ at half the distance of $G = (AB, CD)$ from EF (see Figure 18). This line is *the common tangent* to the two requested parabolas. This follows, for example, by considering the conic passing through the eight contact points of the common tangents of two conics ([18, p. 345]) or the properties of the so-called *harmonic locus* of two conics, specialized for two parabolas ([14, II, p. 121]). The contact points V, W of the two parabolas with e are the common harmonics of the point-pairs $(E_1, E_2), (F_1, F_2)$, where $E_1 = (AD, e), E_2 = (BC, e), F_1 = (AC, e), F_2 = (BD, e)$. Once V, W are constructed, the parabolas can be defined to pass through the corresponding fivetuples of points.

Notice that every conic of the pencil $\mathcal{D}$ has a pair of conjugate diameters parallel to the axes of the two parabolas ([19, p. 292]).

3.3. *Conic by 3 points, 1 at infinity, 1 tangent* $(4P_11T)$. Construct a conic passing through four points $A, B, C, [D]$, and tangent to line e. The conic is either a hyperbola with one asymptotic direction $[D]$ or a parabola with axis parallel to $[D]$. Using the method of §3.1 we construct the fixed points D_1, D_2 of the involution,

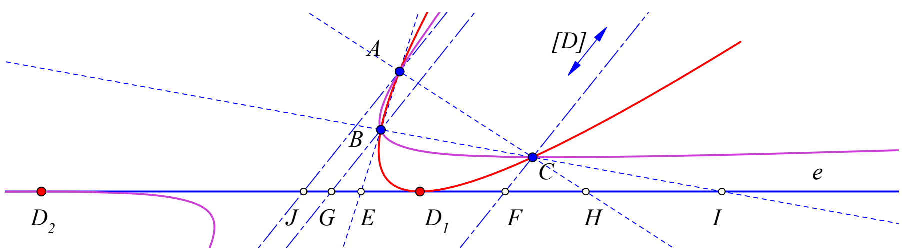

Figure 19. The two conics through $A, B, C, [D]$ tangent to e

defined on line e by its intersections with the line pairs (AB, CD), (AC, BD), (AD, BC). The common harmonics D_1, D_2 of the point-pairs $(E, F), (G, H)$ are the contact points with line e (see Figure 19). Adding one of the D_i to the three points A, B, C we can, as in §8.1, construct a fifth point and pass a conic through the five points. Fixing A, B, C and the position of line e, there are some directions

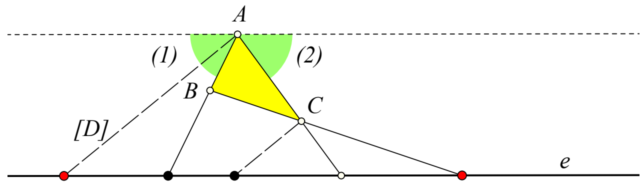

Figure 20. Angular domains of existence

$[D]$ for which no solutions exist. Figure 20 shows a case, in which e does not intersect the interior of ABC and the two angular domains for the direction AD for which there are solutions of the problem.

In general, the conics are two hyperbolas with one asymptotic direction determined by the point at infinity D, or a pair of a hyperbola, as before, and a parabola with axis direction $[D]$. If the line e does not intersect the interior of the triangle ABC, then, for four particular directions $[D]$, there are corresponding parabolas passing through A, B, C, axis direction $[D]$ and tangent to e. The directions $[D]$, for which this happens, can be determined from the triangle ABC and the line e. This is the case $(3P2T_1)$, handled in §4.2. The problem is related to the pencil of conics through $A, B, C, [D]$. This is a specialization of the one in §1.1, resulting from it by sending D at infinity (see Figure 21). In this pencil, all members ex-

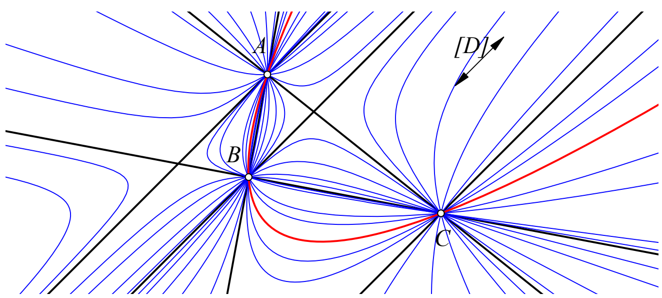

Figure 21. The pencil of conics through $A, B, C, [D]$

cept one are hyperbolas with one asymptotic direction $[D]$. The one exceptional member is the parabola constructed in §8.2.

3.4. *Hyperbola by 2 points, 2 asymptotics, 1 tangent* $(4P_21T)$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e. This is a hyperbola passing through the points A, B, having directions of asymptotes $[C], [D]$ and being tangent to line e. This can be reduced to the case $(5P_20T)$ of §2.3 by locating the contact point S of e with the conic. This is done as in §3.1: Find the intersections $(H, G), (E, F)$ with e of line-pairs $(BD, AC), (BC, AD)$ respectively (see Figure 22). The contact points of the conics with e are the common harmonics S, S' of these two pairs of points.

An alternative solution results by using the *eleven points conic* of $A, B, [C], [D]$ and e as in §3.1, defined as the locus k of poles P of line e with respect to the

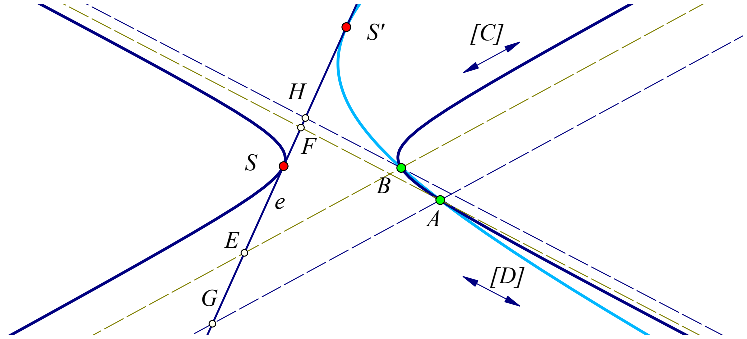

Figure 22. The two hyperbolas through $A, B, [C], [D]$ tangent to e

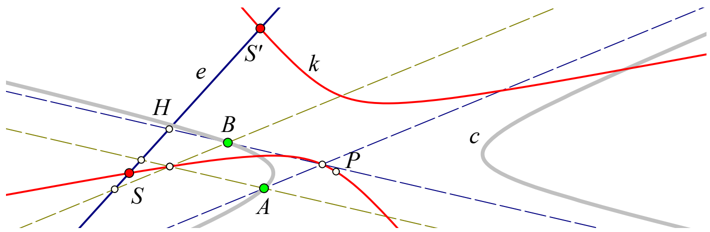

Figure 23. The eleven points conic of $a, b, [C], [D]$ and line e

conics c passing through $A, B, [C], [D]$ (their pencil shown in §2.3). This conic intersects e precisely at the points S, S' (see Figure 23).

Fixing $A, B, [C], [D]$ the lines e for which there are solutions are those defining non-separating segments EF, GH, where $E = (BC, e)$, $F = (AD, e)$, $G =$

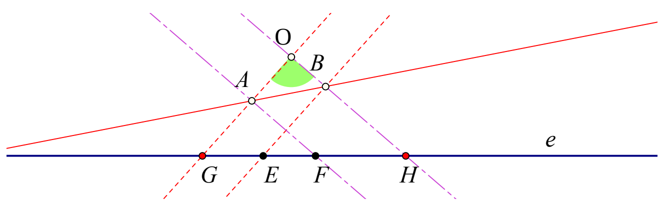

Figure 24. Directions for which exist solutions to $(4P_21T)$

(AC, e), $H = (BD, e)$ (see Figure 24). These are all lines except those, which separate points A, B and their parallels from $O = (AG, BH)$ fall outside the angular domain AOB.

4. Three points and two tangents

4.1. *Conic by three points and two tangents* $(3P2T)$. Construct a conic passing through three points A, B, C and tangent to two lines d, e. The construction can be reduced to that of a conic passing through five points (§2.1) by locating the points of tangency G, H of the two tangents. This can be done by finding the two

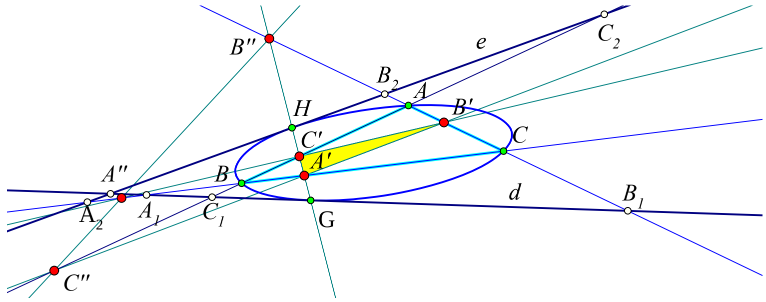

Figure 25. Conic through A, B, C, tangent to d, e

intersection points A', C' of line GH respectively with the known lines BC and AB. The key fact here is, that in all cases of existence of solutions, there is a cevian triangle $A'B'C'$ with respect to ABC, with corresponding tripolar $A''B''C''$, such that the contact points of each one of the requested conics are the intersections of lines e, d with some side of this triangle or its tripolar ([11], [23], [15]). In Figure 25, for example, appears one of the requested conics, tangent to d, e, respectively,

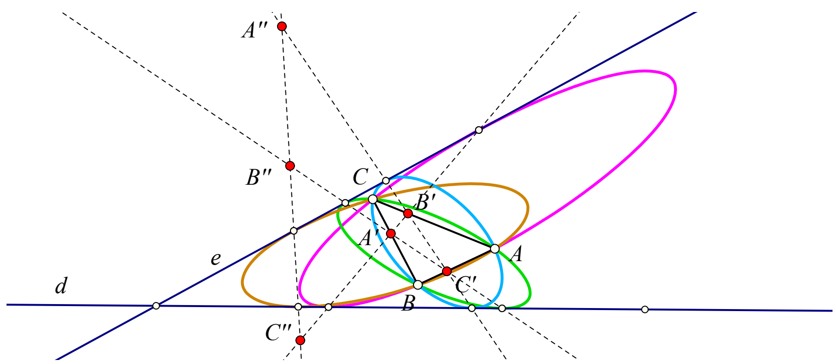

Figure 26. The four conics through A, B, C, tangent to d, e

at the points G, H, which are on the side $A'C'$ of a certain cevian triangle of ABC. The conic is led to pass through the five points A, B, C, G and H. Analogously are constructed three other conics. The determination of the cevian triangle $A'B'C'$ is done again through the construct of common harmonics. For example, points B', B'' are the common harmonics of the point pairs (AC, B_1B_2). These represent the fixed points of an involution, defined, by Desargues' theorem ([17, p. 204]), on line AC, by the intersections with members of the pencil $\mathcal{D}$ of all conics, which are tangent to d, e respectively at G, H (seen in §10.1). Regarding the existence, there are four solutions in the case none of the lines d, e passes through the interior of the triangle ABC (see Figure 26), or both of them intersect the interior of the same couple of sides of this triangle. In all other cases there are no solutions.

An intuitive way to answer, *why four*, offers Figure 27, displaying a cone and a plane ε on which the cone is projected. Plane ε is the one containing the lines e, d

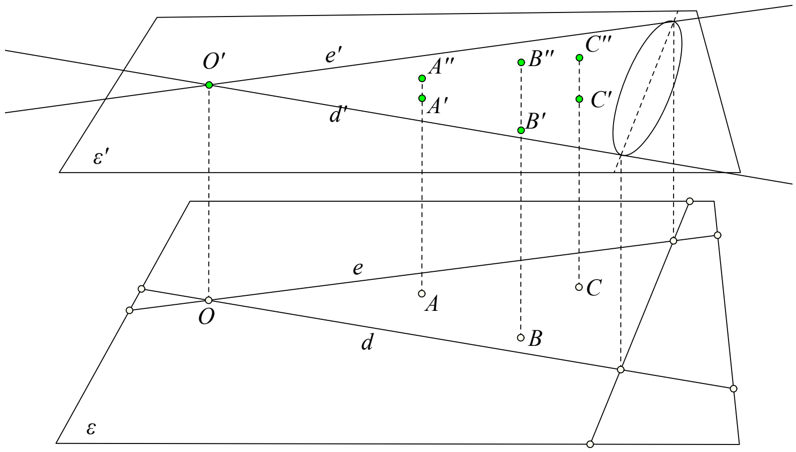

Figure 27. Spacial interpretation of the four solutions

intersecting at O. Plane ε' is parallel to ε from an arbitrary point O' projecting orthogonally to O. The circular cone is constructed so that the parallels d', e' from O' are generators and its axis is contained in ε'. The lines orthogonal to ε at the three points A, B, C intersect the cone respectively at pairs of points (A', A''), (B', B''), (C', C''). The plane through (A', B', C') intersects the cone along a conic, which projects to one of the conics solving the construction problem. The same is true with the triples of points on the conic (A'', B', C'), (A', B'', C'), (A', B', C''). They define respectively, a plane, a conic on the cone, and its projection on ε, representing a solution of the construction problem. The other possible triples of points (e.g such as the triple (A'', B'', C')), because of the reflective symmetry of the cone with respect to the plane ε', deliver conics on the cone, which are reflections of the previous four (e.g. (A'', B'', C') is symmetric to (A', B', C'')), hence by the projection falling onto the same four solutions of the construction problem.

Remark. To handle this, most interesting case and rich in structure of our constructions, one could consider the set of conics passing through three points A, B, C

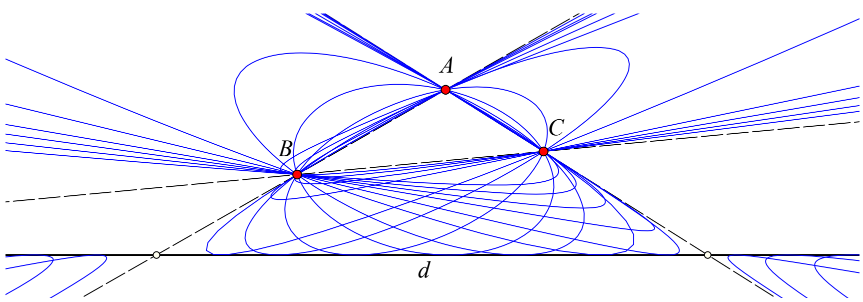

Figure 28. The set of conics through A, B, C tangent to line d

and tangent to one line d, as seen in Figure 28, and attempt to find the members of this set satisfying the fifth condition of tangency with line e. Unfortunately this set of conics is not a pencil, and Desargues' theorem does not apply to it to produce the solutions as usual. The same is true for the set of conics passing

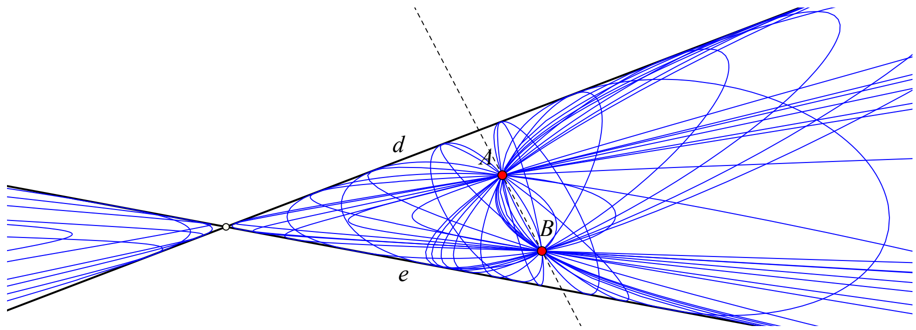

Figure 29. The set of conics through A, B tangent to lines d, e

through two points A, B and tangent to two lines d, e seen in Figure 29. This set of conics admits also a spacial interpretation, as the set of projections of intersections of a cone with all planes passing through points X, Y on the cone, where $X \in \{A', A''\}, Y \in \{B', B''\}$, the points of the two sets projecting respectively on A and B.

4.2. *Parabola by three points and a tangent* $(3P2T_1)$. Construct a conic passing through three points A, B, C and tangent to line d and the line at infinity, thus a parabola. For this, projectively equivalent to the previous, case, the process of de-

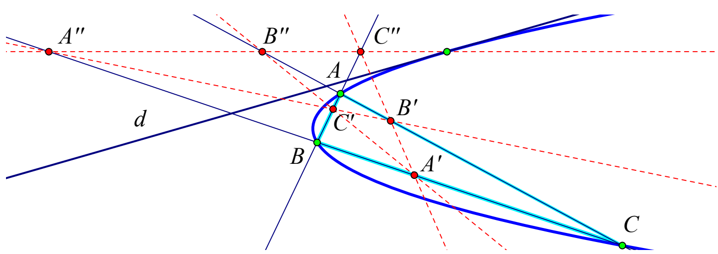

Figure 30. Parabola through A, B, C, tangent to d

termination of the cevian triangle and the tripolar, used there, is even simpler, since line e is now at infinity. The segments $A'A'', B'B'', C'C''$ of common harmonics, determined on each side of ABC, are now bisected by the tangent d and the chords of contact points with lines d, e are parallel to the axes of the parabolas (see Figure 30). Figure 31 displays all four parabolas passing through A, B, C and tangent to line d. There are four solutions if line d does not intersect the interior of triangle ABC and no solution if it does.

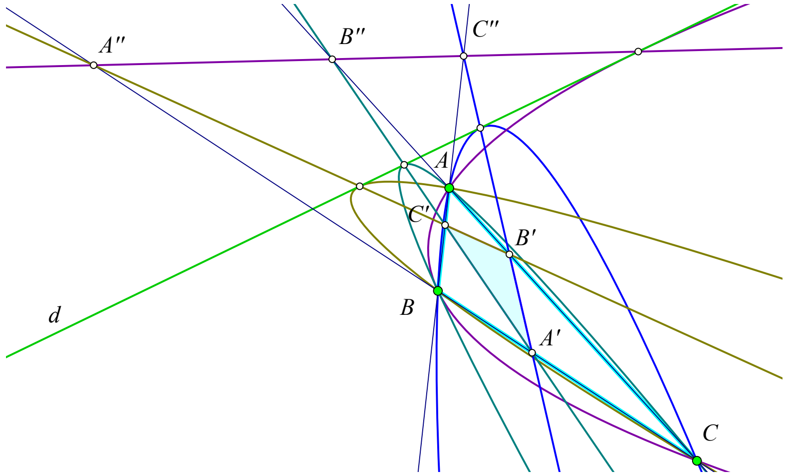

Figure 31. The four parabolas through A, B, C, tangent to d

4.3. *Conic by 2 points, 1 infinity, 2 tangents* $(3P_1 2T)$. Construct a conic passing through points $A, B, [C]$ and tangent to lines d, e. Triangle ABC is infinite with two sides parallel to the direction $[C]$. Points (C', C'') are the common harmonics of (A, B) and of the pair of intersections of AB with lines d, e. Analogously are defined the pairs of points (B', B'') on AC and (A', A'') on BC. $A'B'C'$ is a cevian triangle of ABC and points A'', B'', C'' are on the corresponding tripolar.

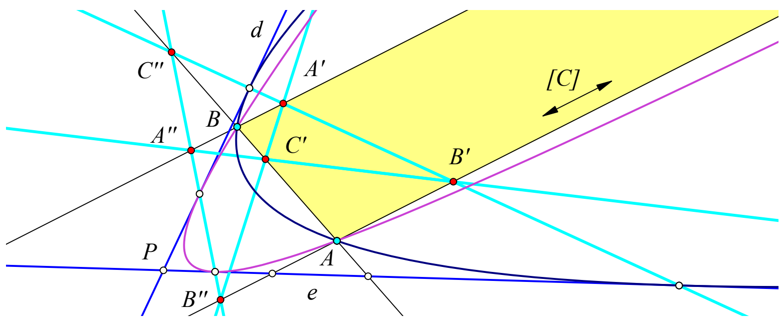

Figure 32. Two (of the four) hyperbolas through $A, B, [C]$ and tangent to lines a, b

Each one of the requested conics passes through A, B, C and is tangent to d, e at their intersection points with one side of the cevian triangle or the tripolar. In Figure 32 only two, out of the four, conics are shown. Additional points on the conics can be found by taking harmonic conjugates with respect to the polar of $P = (d, e)$. The construction of conics can be also completed by using the remark in $(3P2T)_2$ of §10.1. In general the conics are hyperbolas with one asymptotic direction parallel to $[C]$. Fixing A, B and d, e , there are four directions $[C]$ for which the corresponding conic is a parabola. These are determined in $(2P3T_1)$ of §5.2. There are four solutions if the lines d, e either do not intersect the interior of ABC or they intersect the interior of the same pair of sides of this triangle.

4.4. *Hyperbola 1 point 2 asymptotics 2 tangents* $(3P_22T)$. Construct a conic, passing through three points $A, [B], [C]$ and tangent to two lines d, e. This is a hyperbola with asymptotic directions $[B], [C]$. There are again four solutions determined

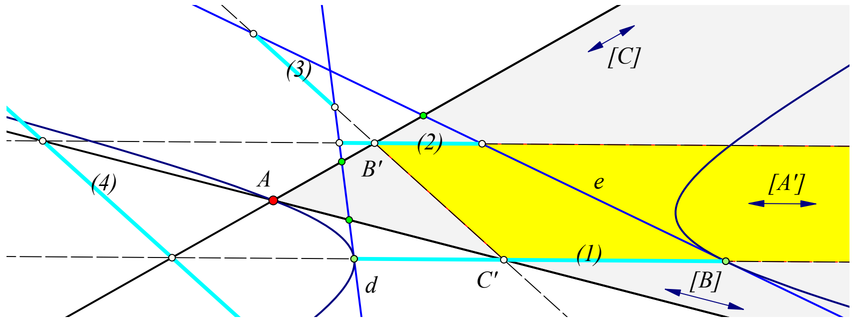

Figure 33. Hyperbola through $A, [B], [C]$ and tangent to lines d, e

by the sides and corresponding tripolar of a cevian triangle $A'B'C'$ of ABC. The triangle ABC has one side-line at infinity. The corresponding cevian triangle is determined as in §4.1, but now one of its vertices is at infinity. The four chords joining contact points of the same conic with lines d, e shown in Figure 33 are denoted by $(1), (2), (3), (4)$. The conic touching d, e at the endpoints of chord (1) is drawn. There are four solutions if the lines d, e do not intersect the interior of ABC or both intersect the interior of the same couple of sides of the triangle. In all other cases there are no solutions.

5. Two points and three tangents

5.1. *Conic by two points and three tangents* $(2P3T)$. Construct a conic passing through two points D, E and tangent to three lines a, b, c. The structure of the solution rests upon the dual theorem of Desargues ([17, p. 215], [9, p. 51], [19, p. 229]) and can be described as follows ([4, p. 58], [11], [23]). The three lines in general positions define a triangle ABC (A opposite to a etc.) and the two given points D, E determine a third point F, with the following properties. A, B, C, F define a *projective base* ([2, p. 95, I]) and in the coordinates with respect to this base the quadratic transformation ([21, p. 127])

$$f : (x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right), \text{ maps } f(D) = E.$$

There are four conics with the prescribed properties (see Figure 34). Each one of them is tangent to the three sides of the triangle ABC and to two additional lines. The pairs of additional lines corresponding to the four conics are (FD, FE), (F_1D, F_1E), (F_2D, F_2E), (F_3D, F_3E), where F_1, F_2, F_3 the harmonic associates ([24, p. 100]) of F. Figure 35 shows the domains for which there are solutions for the $(2P3T)$ problem. The two points D, E must lie, both, in the domain with the same label. Otherwise there are no solutions.

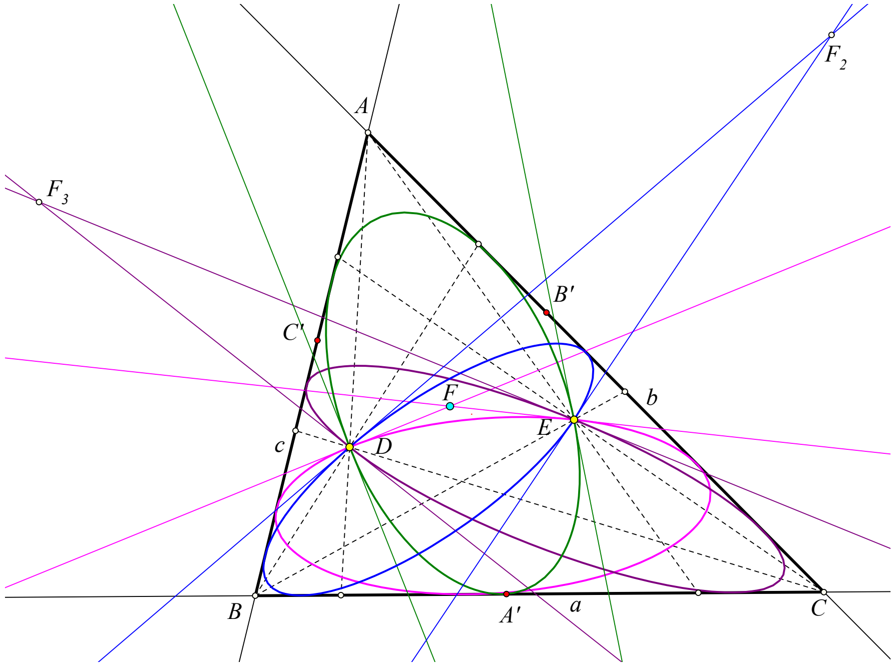

Figure 34. The four conics tangent to a, b, c and passing through D, E

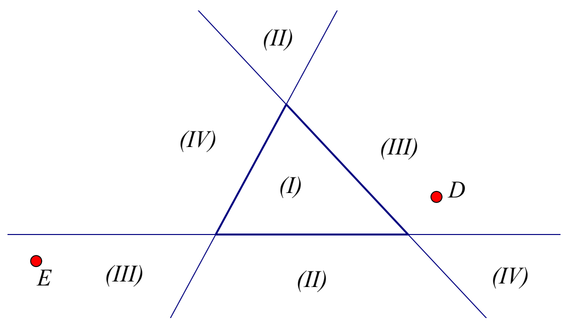

Figure 35. Domains of existence of $2P3T$ conics

This case is the *dual* of the previous one and is reducible to that by taking poles and polars with respect to a fixed conic. For example, taking poles and polars with respect to the conic k with *perspector* F ([24, p. 115]), for each conic c tangent to the sides of ABC and passing through D, E, we obtain a conic c' passing through the vertices of the cevian $A'B'C'$ of F and tangent to d, e, which are the polars of D, E with respect to k (see Figure 36). By this *polarity* ([22, p. 263, I]) the tangents t_D, t_E to c at D, E map to the contact points D_1, E_1 of c' with lines d, e and the intersection point $F = (t_D, t_E)$ maps to line $f = D_1E_1$.

Using a polarity, as before, we could reduce the cases to the half; but it is not the purpose of the present review to produce a least number of pictures.

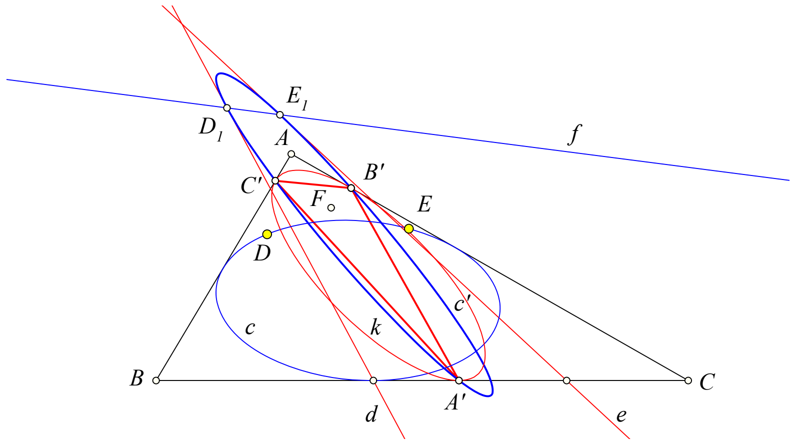

Figure 36. Reduction to the dual by the polarity with respect to k

Remark. In this case, as noticed also in the previous one, some difficulty in handling the construction lies on the fact that the set of conics tangent to a, b, c and passing through D is not a proper pencil of conics (see Figure 37). This set of con-

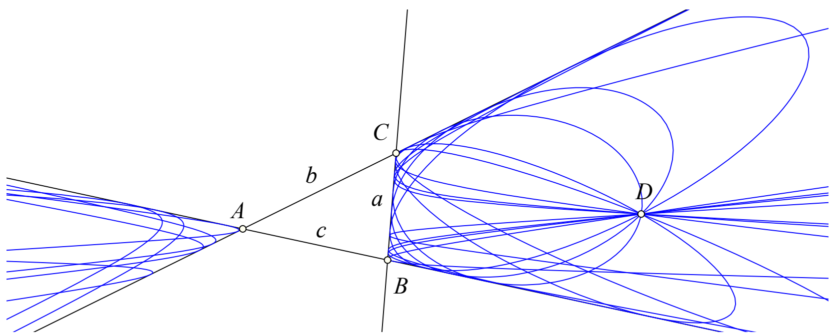

Figure 37. Conics tangent to a, b, c, passing through D

ics admits a spacial interpretation, as in §4.1, representing the conics as projections of intersections of a cone with planes. The cone is constructed as in that section, and the planes intersecting the cone are defined by pairs (X, α) of points and lines. Point $X \in \{D', D''\}$, the two points of the cone being those which project on D. Line α is a tangent to the conic defined on the cone by the plane orthogonal to the plane ε of b, c and intersecting it along line a.

5.2. *Parabola by 2 points, 2 tangents* $(2P3T_1)$. Construct a conic passing through two points A, B and tangent to two lines c, d and the line at infinity, thus, a parabola. Figure 38 shows the process of determination of the point F and its harmonic associates F_1, F_2, F_3 stepping on the previous section. Points B_1, B_2 are the common harmonics of the point-pairs (A', B') and $(G, [d])$, where A', B' are the parallel to c projections of A, B on d and $G = (c, d)$. Analogously are defined on c the common harmonics C_1, C_2 of two similar point-pairs on c. Point F is the

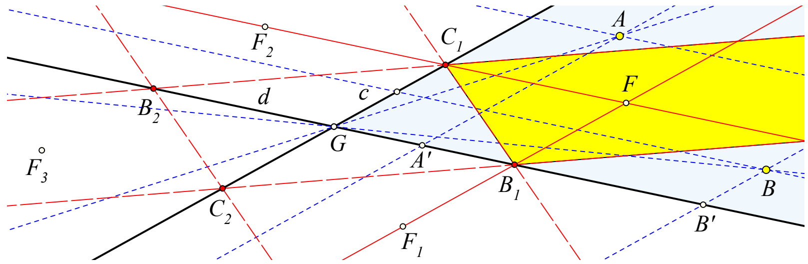

Figure 38. Searching for parabolas through A, B and tangent to c, d

intersection of the parallels from B_1, C_1 respectively to c, d. Figure 39 shows the

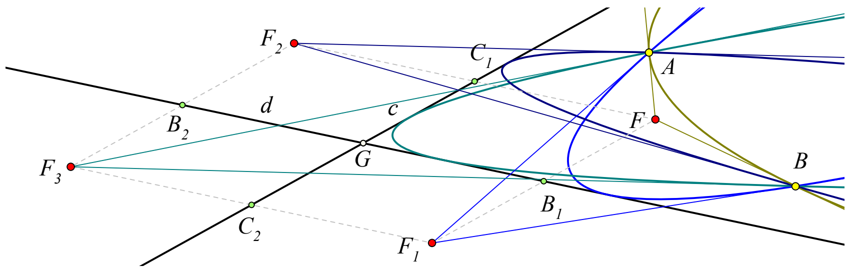

Figure 39. The four parabolas through A, B and tangent to c, d

four parabolas solving the problem. Each of them is tangent to the two given lines c, d and the pair of lines F_iA, F_iB, where F_i are either F or one of its harmonic conjugates (with respect to the triangle with infinite sides $G[c][d]$). There are four solutions if points A, B are in the same or opposite angular domains of lines c, d. Otherwise there are no solutions.

Remark. When line FF_3 passes through the middle M of AB, the two corresponding parabolas, tangent respectively to the line-pairs $(FA, FB), (F_3A, F_3B)$, are homothetic with respect to G and solve problem $(3P_12T_1)$ of §9.3, line FF_3 being then parallel to the axis of the two parabolas and also being harmonic conjugate of AB with respect to lines (c, d).

5.3. *Conic by 1 point, 1 infinity, 3 tangents* $(2P_13T)$. Construct a conic passing through two points $A, [B]$ and tangent to three lines c, d, e. The construction can be done by adapting the one of §5.1. By that method, we first find the cevians of A, B with respect to the triangle $A'B'C'$, whose sides are c, d, e. Then we find the common harmonics A_0, A_1 on line c of the point-pair consisting of the traces of the cevians from A, B and (B', C') . Analogously are defined points B_0, B_1 on d and C_0, C_1 on e (see Figure 40). The six points thus determined define a cevian triangle with perspector F and the corresponding tripolar. Then we define the three harmonic associates F_1, F_2, F_3 of F. Each one of the requested conics is tangent to the three lines c, d, e and also tangent to the two lines F_iA, F_iB, joining some of the points F_i with A and B. There are four solutions if A is on the exterior of

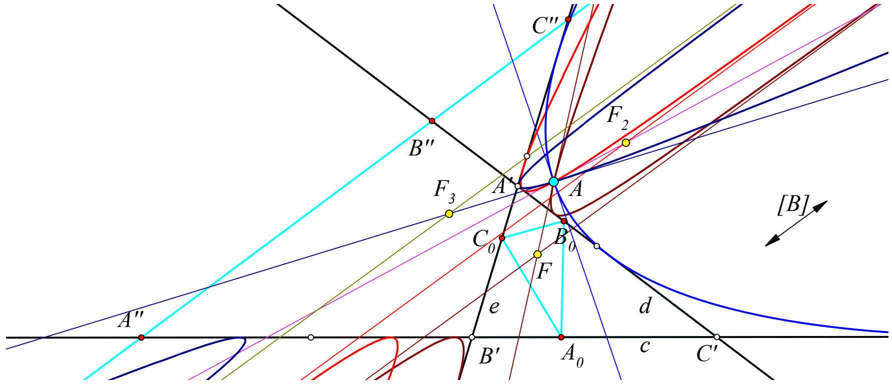

Figure 40. The four conics tangent to c, d, e passing through $A, [B]$

the triangle $A'B'C'$ and in the angular domain containing the parallel to $[B]$ from a vertex. Otherwise there are no solutions. The conics are in general hyperbolas. Fixing the lines c, d, e and point A, there are two directions $[B]$ for which the conics are parabolas. This is handled in $(1P4T_1)$ of §6.2.

5.4. *Hyperbola by 2 asymptotics and 3 tangents* $(2P_23T)$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines a, b, c. This is a hyperbola with asymptotic directions $[A], [B]$ and tangent to three lines. Proceeding as in

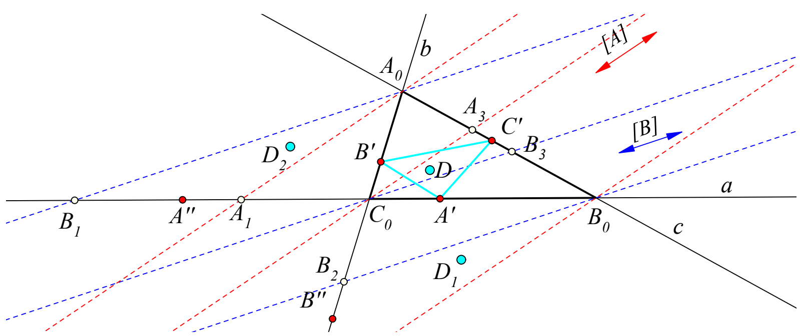

Figure 41. Cevian triangle and perspector in the case $(2P_23T)$

§5.1, we draw parallels to those directions from each vertex of the triangle $A_0B_0C_0$ with side-lines a, b, c. These parallels define on each side two points A_1, B_1 on a, A_2, B_2 on b etc. The common harmonics (A', A'') of point pairs (A_1, B_1) and (C_0, B_0) and the corresponding common harmonics for the other sides, define the cevian triangle $A'B'C'$, its perspector D and the corresponding harmonic associates D_1, D_2, D_3 (see Figure 41). Each of the requested hyperbolas is constructed as a conic tangent to the three lines a, b, c plus the two lines joining D_i to the points at infinity A, B, i.e. the parallels from D_i to $[A]$ and $[B]$ (see Figure 42). Given the directions of lines a, b, c, there are four solutions if drawing parallels to these directions and to $[A], [B]$, later are not separated by the first. Otherwise there are no solutions.

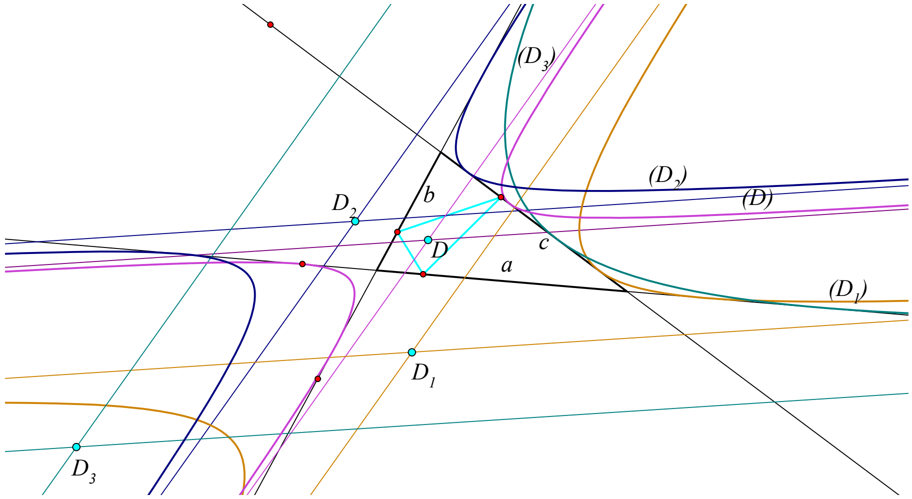

Figure 42. The four hyperbolas tangent to a, b, c with two given asymptotic directions

6. One point and four tangents

6.1. *Conic by 1 point and 4 tangents* $(1P4T)$. Construct a conic tangent to four given lines a, b, c, d and passing through a given point A. One way to the construction is to reduce the problem to its dual $(4P1T)$ of §3.1. In fact, if IJK is

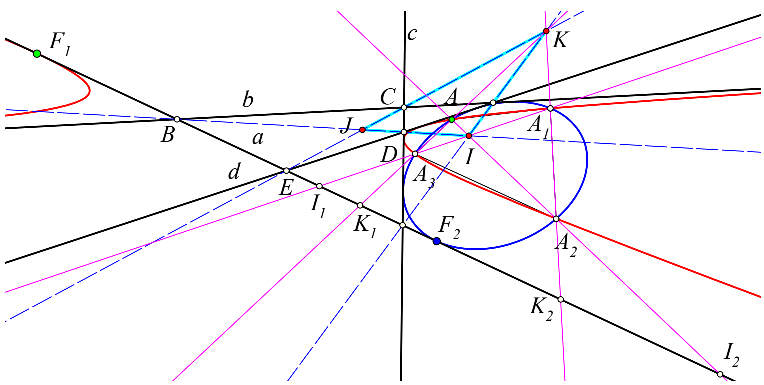

Figure 43. The two conics tangent to a, b, c, d, passing through A

the diagonal triangle of the complete quadrilateral whose sides are a, b, c, d (see Figure 43), then the harmonic associates A_1, A_2, A_3 of A with respect to IJK are also points of the conic. Thus, one can apply the recipe of §3.1 by taking these four points and one of the four given lines.

Another way to define the conics is by using Desargues' theorem in its dual form ([4, p. 57]) in order to locate a fifth tangent to the conic, namely the one passing through A. In fact, according to that theorem, the tangents from an arbitrary point A to the conics of the one-parameter pencil of conics $\mathcal{D}$, which are tangent to four lines a, b, c, d, define an involution on the pencil A^* of all lines passing through that point. The tangents to the members of that pencil, which pass through A are the fixed elements of this involution. By considering the intersection of each line through A with line a, we represent this involution by a corresponding involution of points of a. The fixed elements of the involution in A^* correspond

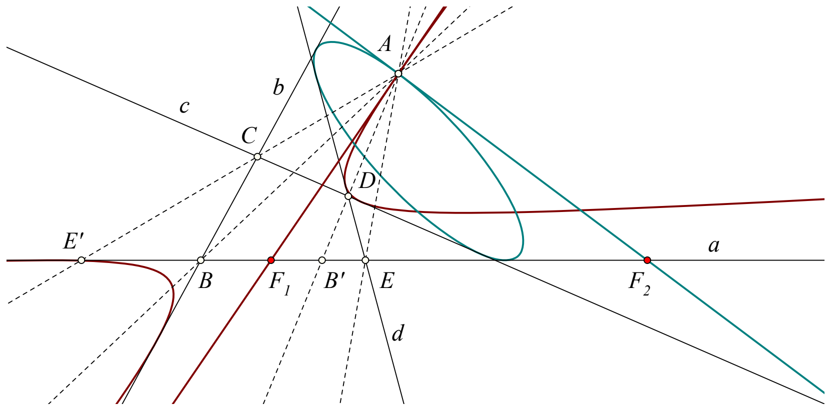

Figure 44. The two conics tangent to a, b, c, d, passing through A

to the fixed points of the corresponding involution on a. It is easy to see that two particular pairs of points in involution on a are the pairs (B, B') and (E, E'), where $B = (b, a), E = (d, a), B' = (AD, a), E' = (AC, a)$. The common harmonics F_1, F_2 of these two pairs define the two requested tangents, which in turn define the two conics (see Figure 44).

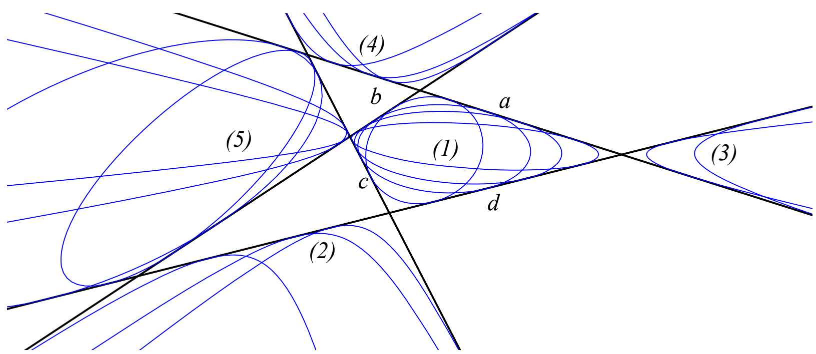

Figure 45. The five domains of existence of solutions

There are two solutions if point A is in one of the five domains $(1) - (5)$ shown in Figure 45. Otherwise there are no solutions. The reason for this is, as is visible in the figure, that the pencil $\mathcal{D}$ of conics tangent to four given lines a, b, c, d does not cover all connected domains defined by the four lines ([2, p. 200, II]). Noticable in the figure is also the fact that for every point in these five domains there are two conics of the pencil passing through the point.

A third method to construct the requested conics is to use the dual of the *eleven points conic* of §3.1, which is the *eleven tangents conic*, defined by four lines a, b, c, d and a point A ([1, p. 97]). This conic is the envelope k of the polars of A with

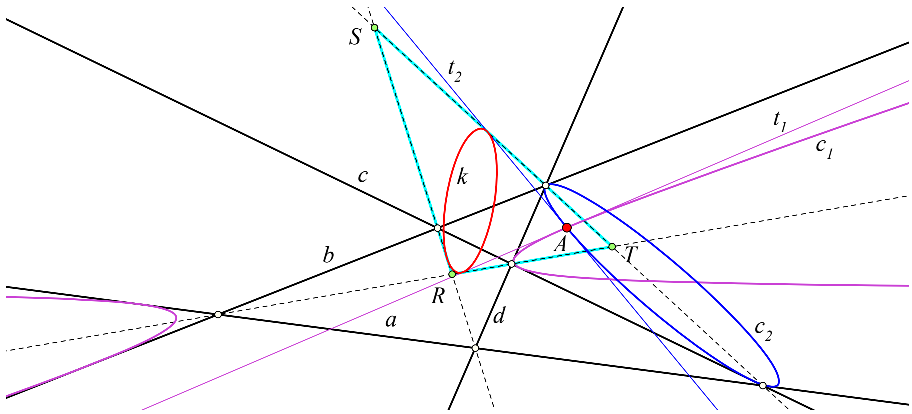

Figure 46. The two conics tangent to a, b, c, d, passing through A

respect to all conics tangent to the four given lines. This conic is tangent to the 3 sides of the diagonal triangle RST of the quadrilateral of the four lines. It is also tangent to the 6 polars of A with respect to all line-pairs of the quadrilateral, and is also tangent to the two tangents t_1, t_2 to the requested conics at A (see Figure 46). Conic k can be constructed by the methods of §7.1 and then t_1, t_2 can be found by drawing the tangents to k from A. The two requested conics can be defined by applying again the methods of the next section and determining the conic tangent to five lines $a, b, c, d, t_i,\ (i = 1, 2)$.

6.2. *Parabola by 1 point, 3 tangents* $(1P4T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, and also tangent to three lines a, b, c and passing through a point E. Any of the methods of the previous section can be modified

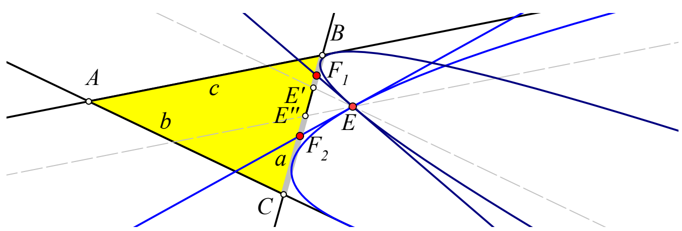

Figure 47. Parabolas tangent to a, b, c, passing through E

to produce the requested parabolas. For example, applying the second method, we draw first parallels to b, c through E intersecting a at E', E'' (see Figure 47). The pairs of lines (EE', EB) and (EE'', EC) through E are related with respect to the involuetion defined by Desargues' theorem. They are tangents from E to degenerate members of the pencil of parabolas tangent to four lines three of which are a, b, c. The common harmonics F_1, F_2 of these pairs define the tangents at E of the requested parabolas passing through E. There are two solutions if point E lies in one angular domain containing the triangle ABC but outside of the triangle. Otherwise there are no solutions.

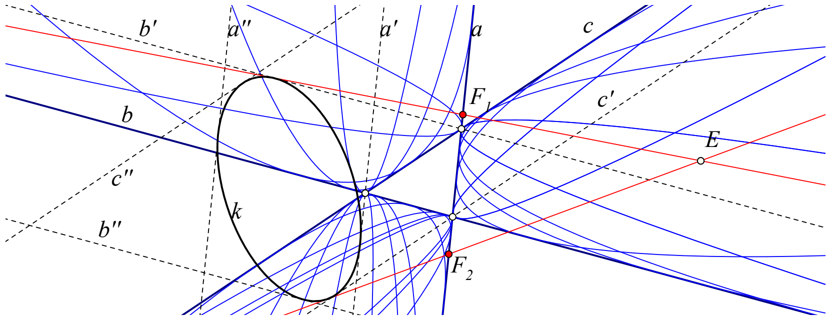

Figure 48. The pencil of parabolas tangent to a, b, c

An alternative solution is obtained by using the *nine tangents conic* of the three lines a, b, c, the line at infinity e and the point E. This is the conic k, defined as envelope of all polars of E with respect to the members of the pencil of parabolas tangent to the three lines a, b, c (see Figure 48). Conic k is tangent to $a', a'', b', b'', c', c''$, where a', a'' are parallel to a, respectively, from point $A = (b, c)$ and the symmetric E_a of E with respect to a, and the other lines are defined analogously. The tangents to the requested parabolas at E are the two tangents from E to k.

6.3. *Conic, 1 infinity, 4 tangents* $(1P_14T)$. Construct a conic tangent to four lines a, b, c, d and passing through a point at infinity $[E]$. Again a solution results by adapting any of the methods of section §6.1. For example, to adapt the sec-

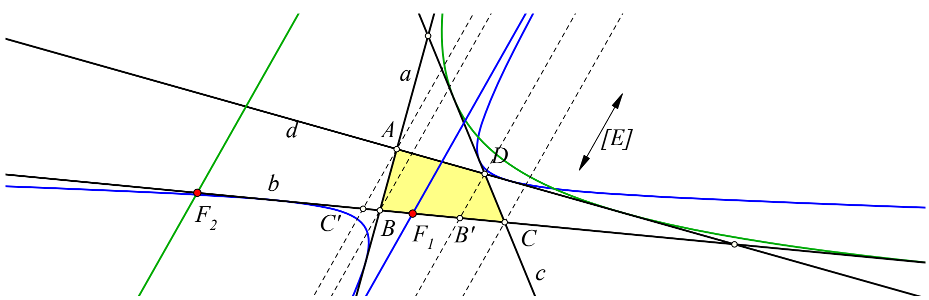

Figure 49. The two conics tangent to a, b, c, d, passing through $[E]$

ond method to the present configuration, define the points B', C' on b, to be the projections of D, A parallel to $[E]$ (see Figure 49). Points F_1, F_2 are the common harmonics of pairs $(B, B'), (C', C)$, and the parallels EF_1, EF_2 define the tangents at E of the requested conics, which are constructed from five tangents. Fixing lines a, b, c, d, there are two solutions when the parallel to $[E]$ from B

falls inside the angle CAB or the parallel from D falls inside the complement of ADC. Otherwise there are no solutions. The above construction assumes that the lines through F_1, F_2 are ordinary and as a consequence the conics are hyperbolas. If the conic is tangent to the line at infinity, thus a parabola, then $[E]$ is uniquely determined from the lines a, b, c, d. This is handled in $(0P5T_1)$ of §7.2.

7. Five tangents

7.1. *Conic by five tangents* $(0P5T)$. Construct a conic tangent to five lines a, b, c, d, e. A first solution is to reduce the construction to its dual of a conic through five points, as in §2.1. For this, use Brianchon's theorem to find the contact points with the sides ([19, p. 225]). In fact, the diagonals BD, CE of the pentagon of the

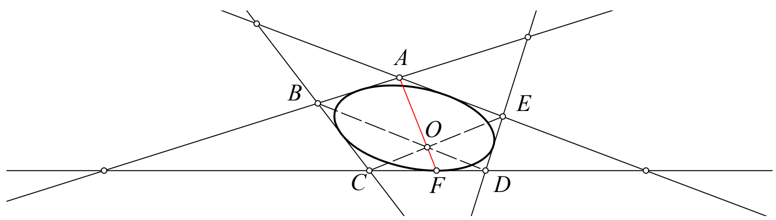

Figure 50. Conic through five tangents, find the contact points

given lines intersect at a point O (see Figure 50), which lies also on the line joining the remaining vertex of the pentagon A to the contact point F of the opposite side. Thus F is constructible from the data. Analogously are found the other contact points with the sides of the pentagon. Using again the theorem of Brianchon in its general form for hexagons circumscribed to a conic, one can construct arbitrary many other tangents to the conic. In fact, take a point F on side AB and define

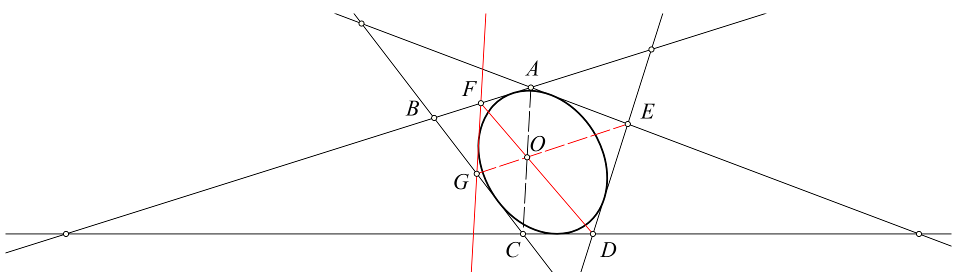

Figure 51. Conic through five tangents, draw arbitrary many tangents FG

the intersection point O of FD and AC (see Figure 51). By Brianchon's theorem the diagonal GE will pass also through O. Hence the position of G can be found by intersecting BC with OE. Thus, moving F on line AB and determining G on BC by the above procedure, we can find arbitrary many tangents FG to the conic. There is always a unique solution.

An image of the pencil of conics tangent to four lines, related to this problem, is contained in §6.1.

7.2. *Parabola by four tangents* $(0P5T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to four given lines a, b, c, d. The case is projectively equivalent to the previous one and the method used there can be adapted to solve the problem. For this, apply Brianchon's theorem to the pentagon $ABCDE$, which

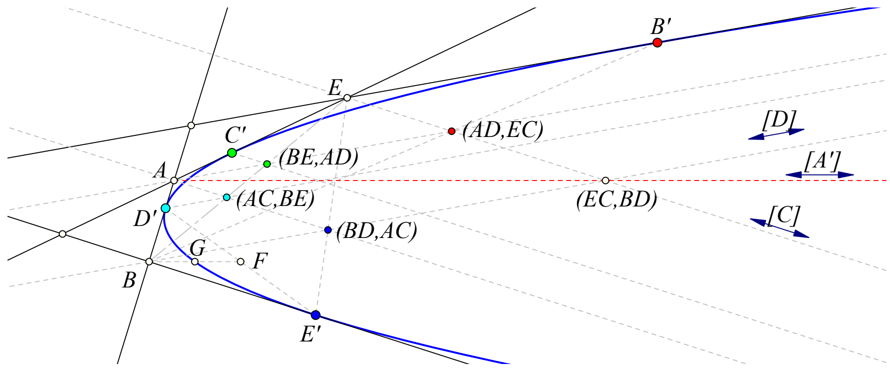

Figure 52. The contact points of the tangent parabola

now has points C, D at infinity (see Figure 52). The contact points of the sides opposite to the vertices are denoted correspondingly A', B', C', D', E'. Point A' is at infinity, and determines the axis of the parabola. By Brianchon's theorem, line AA' passes through the intersection (EC, BD), which is constructible from the data. Analogously are constructible the intersections $(AD, EC$, (BE, AD), (CA, BE), (DB, CA). Join these points correspondingly with the vertices B, C, D, E to find B', C', D', E' through their intersections with the opposite sides of the pentagon. To the four points on the parabola a fifth one G can be defined by taking the middle F of $D'E'$ and the middle G of FB. Thus the parabola can be constructed as a conic passing through the five points B', C', D', E', G ([2, II, p. 212]).

Another way to solve the problem, is through the properties of the created parabola related to the Miquel circles of the quadrilateral of the four given lines ([12, p. 83]). These are the circumcircles of the four triangles formed by the four given lines. A theorem of Miquel asserts that all four circles pass through the same point F (see Figure 53). A theorem of Steiner ([17, p. 161]) completes then the construction, by showing that this point F is the focus of the parabola, while the directrix carries all four orthocenters of the aforementioned triangles ([19, p. 70]). Thus, in order to construct the parabola, it suffices to take the circumcircles and the orthocenters of two such triangles and define F and their orthocenters H_1, H_2 ([12, p.45], [14, p. 100, II]). The parabola then is constructed from its focus and the directrix H_1H_2. There is always a unique solution.

8. Four points one tangent one coincidence

8.1. *Conic by three points and one tangent-at* $(4P1T)_1$. Construct a conic passing through four points A, B, C, D and tangent to a line e at D. In this case it is easy to find a fifth point on the conic and reduce the construction to that of $(5P0T)$ in

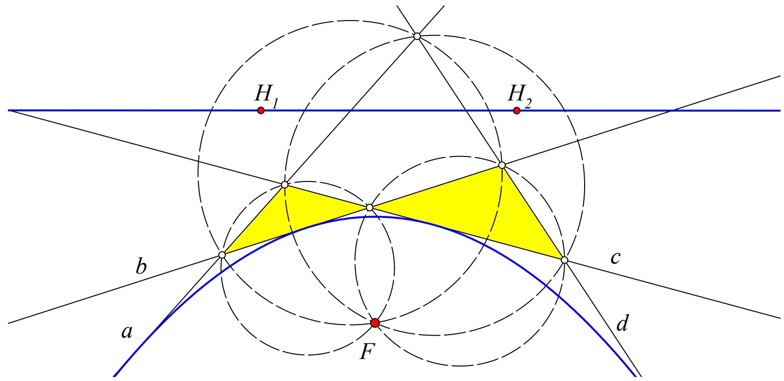

Figure 53. The classical construction of the parabola tangent to four lines

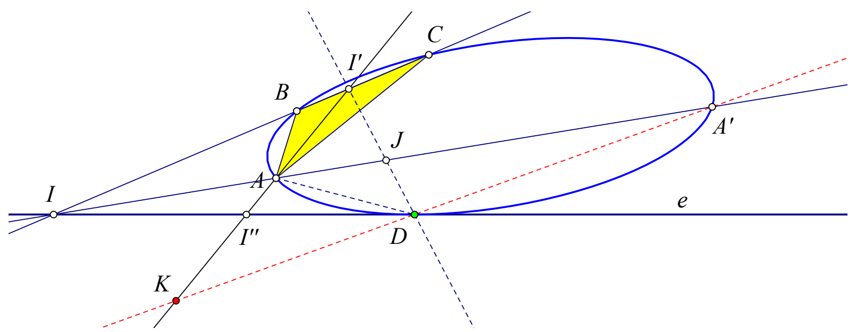

Figure 54. A fifth point A' of the conic through A, B, C tangent to e at D

§2.1. In fact, take $I = (BC, e), I' = I(B, C)$. Then line $I'D$ is the polar of I (see Figure 54). If $J = (I'D, AI)$, then $A' = A(I, J)$ is a point on the conic. There is always a unique solution.

Noticeable in the figure is point $K = (DA', AI')$. It is a fixed point on line AI', since the cross ratio $(A, K, I', I'') = (A, A', J, I) = -1$. Hence points D, A', and through them, the various conics passing through A, B, C and tangent to e, are defined by turning a line about K and considering its intersections with the fixed lines e and IA. Figure 55 shows the structure resulting by making the same construction with respect to the other sides of ABC. In this point E is the tripolar of line e and K_A, K_B, K_C are its harmonic associates with respect to ABC. Each point $D \in e$ defines three other points of the conic passing through A, B, C and tangent to e at D. These points are $A' = (DK_A, K_BK_C), B' = (DK_B, K_CK_A), C' = (DK_C, K_AK_B)$. It is easily seen that points K_A, K_B, K_C are the harmonic associates of D with respect to $A'B'C'$.

8.2. *Parabola by three points and axis-direction* $(4P_11T_1)$. Construct a conic tangent to the line at infinity, hence a parabola, passing through four points $A, B, C, [D]$. The last point, at infinity, defines the direction of the axis of the parabola. The construction is carried out by locating two more points B'', C'' on the parabola and

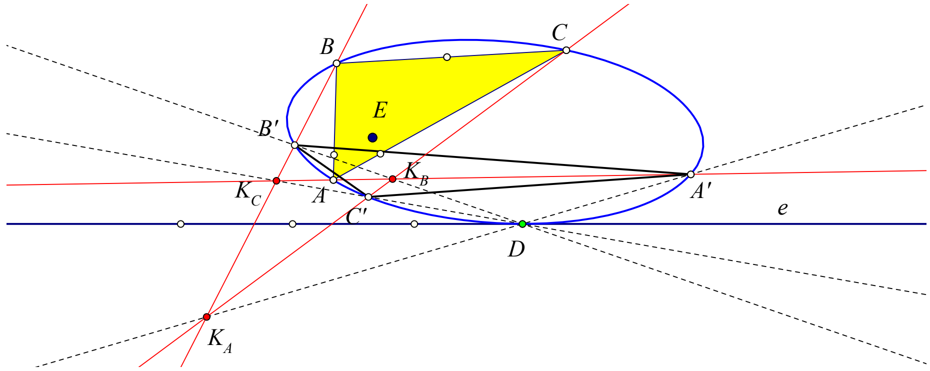

Figure 55. Conic through A, B, C tangent to e at D. Three additional points A', B', C'

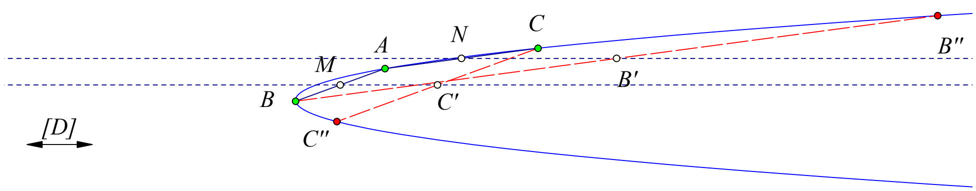

Figure 56. Parabola through A, B, C and given axis-direction $[D]$

passing a conic through A, B, C, B'', C'' (see Figure 56). The construction of the additional points is the same with that of the previous section.

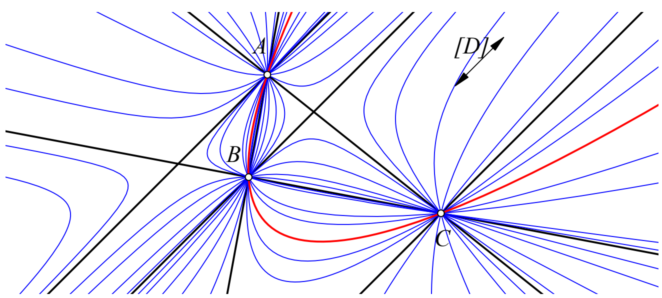

Figure 57. The pencil of conics through $A, B, C, [D]$

The pencil of conics involved is a specialization of the one in §1.1, resulting from it by sending D at infinity (see Figure 57). The construction shows that in this pencil, all members except one are hyperbolas with one asymptotic direction $[D]$. The one exceptional member is the requested parabola.

8.3. *Conic by 2 points, 1 at infinity, 1 tangent-at* $(4P_1 1T)_1$. Construct a conic passing through four points $A, B, C, [D]$ and tangent to a line e at C. Additional points on the conic can be found as in §8.1. In Figure 58 points L, K are two such

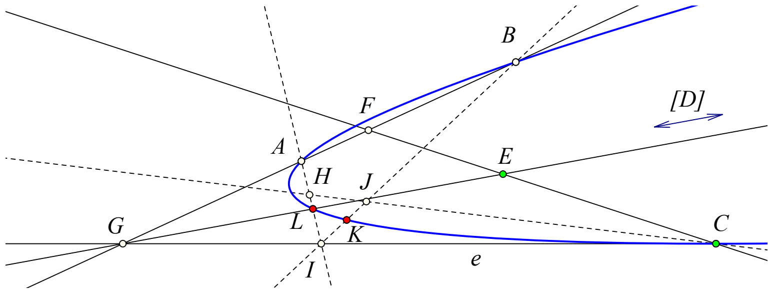

Figure 58. Conic through $A, B, [D]$ tangent to e at C

additional points. They are constructed as conjugates of B and D after constructing the polars CF of $G = (e, AB)$ and CH of $I = (e, AL)$. There is always one solution, which, in general, is a hyperbola. The pencil involved is the one of conics through $A, B, C, [D]$, seen in §8.2. Line e can be considered to turn about point C, defining in each of the obtained locations a corresponding member of that pencil. There is a single line through C, for which the corresponding conic is a parabola with axis $[D]$. In all other cases the conic is a hyperbola with an asymptotic direction $[D]$.

Fixing A, B, C, e, and varying $[D]$ we obtain, in general, hyperbolas. There are, though, two special directions $[D]$, determined in terms of A, B, C, e, for which the resulting conic is a parabola. This is handled in $(3P2T_1)_1$ of §9.2.

8.4. *Hyperbola by 3 points, 1 asymptote* $(4P_1 1T)_i$. Construct a conic passing through four points $A, B, C, [D]$ and tangent to a line e at $[D] \in e$. This is equivalent with the construction of a hyperbola passing through the points A, B, C and

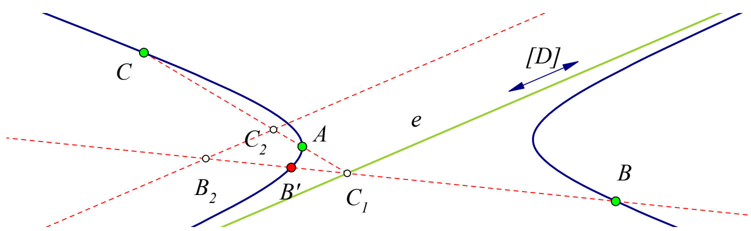

Figure 59. Hyperbola through A, B, C and given asymptote d

having the asymptote line e. Adapting the method of §8.1, we can find two additional points B', C' and construct the requested conic as a $(5PT0)$ conic. For example, to define B', take successively $C_1 = (AC, e), C_2 = C_1(A, C)$. The parallel to e from C_2 is the polar of C_1. Take then the intersection B_2 of that line with BC_1 and $B' = B(C_1, B_2)$, which is a point on the conic. Analogously is defined point C'. There is always one solution. The pencil of conics involved is the one shown in $(4P_1 1T_1)$ of 8.2. A line parallel to $[D]$ determines a unique member of the pencil having this line as an asymptote.

8.5. *Hyperbola by 1 point 2 asymptotics 1 tangent at* $(4P_21T)_1$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e at $B \in e$. The conic is a hyperbola with given asymptotic directions $[C], [D]$, passing through a point A and tangent at a point B to a given line e. In this case, as we did in §8.1,

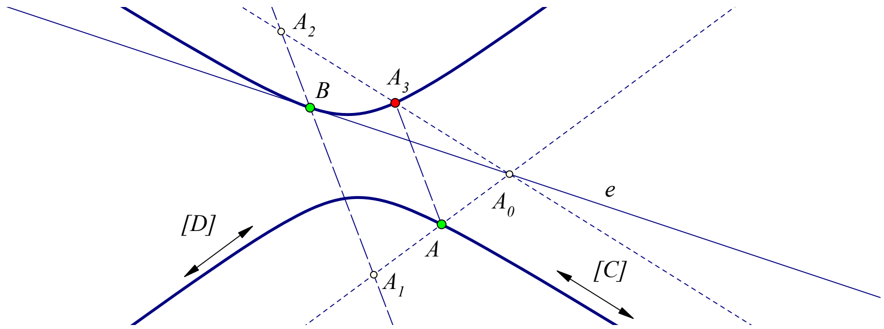

Figure 60. Hyperbola through $A, B, [C], [D]$ tangent to $e \ni B$

we can find additional points on the conic. For this let A_0 be the intersection point with e of the parallel to $[D]$ through A. The symmetric A_1 of A_0 with respect to A defines line BA_1 which is the polar of A_0. The parallel to $[C]$ from A_0 intersects A_1B at A_2 and the middle A_3 of A_0A_2 is on the conic. Repeating the construction with A_3 in place of A and continuing this way, we can construct arbitrary many points on the conic. There is always one solution.

8.6. *Hyperbola by 2 points 1 asymptote 1 asymptotic* $(4P_21T)_i$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e at $[D]$. The requested conic is a hyperbola passing through two points A, B having one asymptotic direction $[C]$ and an asymptote $e \ni [D]$. By a well known property of the hyperbola ([4, p. 42]), the segments AA', BB' intercepted by the asymptotes on the secant AB are equal, hence, knowing AA', we locate B' on AB and given the direction of the other asymptote $[C]$ we determine it completely and find its

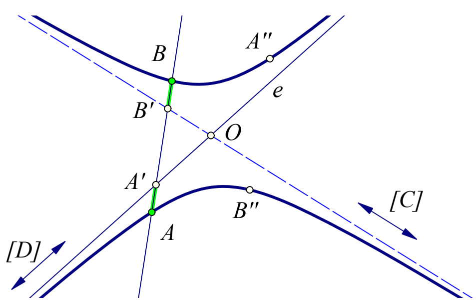

Figure 61. Hyperbola through $A, B, [C], [D]$ and asymptote $e \ni [D]$

intersection point O with the given asymptote e, which is the center of the hyperbola (see Figure 61). Two additional points A'', B'' are immediately constructed, by taking the symmetrics of A, B with respect to O. Arbitrarily many points on the conic can be then constructed by the method of the previous section. There is always a unique solution.

9. Three points two tangents one coincidence

9.1. *Conic by 2 points, 1 tangent, 1 tangent-at* $(3P2T)_1$. Construct a conic passing through three points A, B, C and tangent to two lines $d \ni A, e$. Using again the power of Desargues' theorem, we find first the contact points of the requested

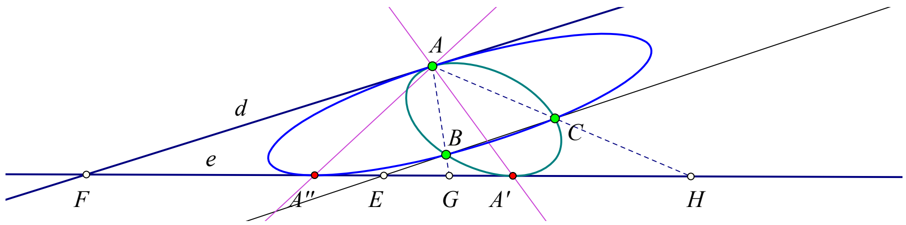

Figure 62. The two conics through A, B, C, tangent to e and also to d at A

conics with line e. These are the common harmonics A', A'' of the point-pairs $(E, F), (G, H)$, where $F = (d, e), E = (e, BC), G = (e, AB), H = (e, AC)$ (see Figure 62). Note that these pairs are defined as intersections of e with two degenerate members of the pencil $\mathcal{D}$ of conics tangent to d at A and passing through B, C. The first pair is the intersection with the degenerate conic of two lines $d \cdot BC$ and the second with the degenerate conic of the two lines $AB \cdot AC$. After locating the contact point, a fifth point on the conic can be obtained by using the polar of F, which is AA' or AA'' and taking the conjugate of B or C. There are no solutions if only one of the lines d, e separates points B, C. Otherwise there are two solutions. Figure 63 displays a pencil $\mathcal{D}$ of conics tangent to line d at a fixed point A and

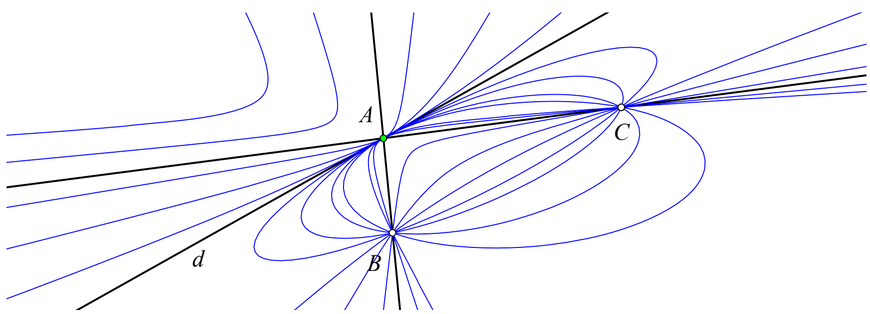

Figure 63. The pencil of conics tangent to d at A, passing through B, C

passing through two points B, C. It is visible there that from every point of the plane passes a unique member of the pencil and that for every line of the plane not separating B, C there are two members tangent to that line ([2, II, p. 193]).

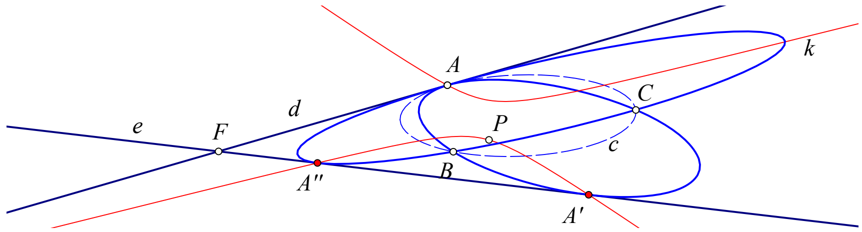

Figure 64. The two conics through A, B, C, tangent to e and also to d at A

For an alternative method, as in §3.1, we consider the poles P of line e with respect to all members c of $\mathcal{D}$. Their locus is the eleven points conic k intersecting the line e at the contact points A', A'' of the requested conics (see Figure 64).

9.2. *Parabola by 2 points, 1 tangent-at* $(3P2T_1)_1$. Construct a conic passing through three points A, B, C, tangent to line d at A and tangent to the line at infinity e, thus a parabola. The involution on e, induced by its intersections with the members of

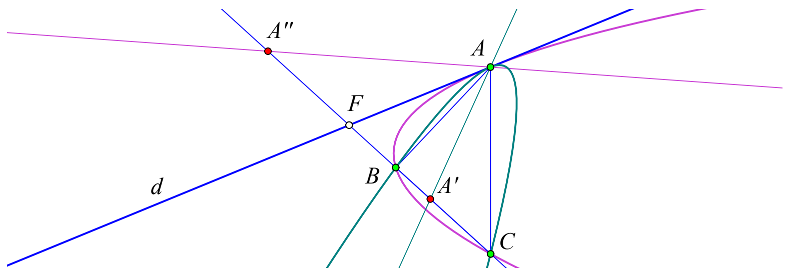

Figure 65. The two parabolas through B, C, tangent to d at A

the pencil $\mathcal{D}$ of conics tangent to d at A and passing through B, C, induces an involution on the pencil A^* of lines through A and, through the intersections of these lines with d, induces also an involution on d. Two, related by this involution point-pairs on d are (B, C) and $(F, [BC])$, where $F = (BC, d)$ and $[BC]$ the point at infinity of BC. The common harmonics A', A'' of these two pairs define, by joining them with A, the directions of the axes of the parabolas (they pass from the contact point at infinity) (see Figure 65). Two additional points on each parabola can be defined by projecting B, C parallel to d on the parallel to the corresponding axis through A and doubling the resulting segments. There are two solutions if the line d does not intersect the interior of segment BC and no solution if it does.
A different way to think about this problem is the following (see Figure 66). All conics c tangent to line d at A and passing through B, C have their centers on a conic k, passing through A and $D = (d, BC)$ and also through the middles P, Q, R of segments AB, AC, BC ([19, p. 299]). This is the *eleven points conic* of $\mathcal{D}$ with respect to the line at infinity. If this conic is a hyperbola, then its points at infinity are the centers of the two requested parabolas. To find the parabolas in this case draw from the middle Q of AC parallels $C'Q, C''Q$ to the asymptotes

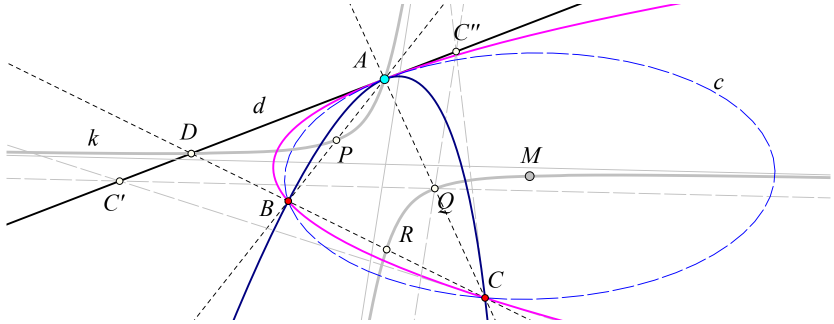

Figure 66. The two parabolas through B, C, tangent to d at A

intersecting d ad C', C'' respectively. One of the requested parabolas is tangent to lines $C'A, C'C$ at A, C correspondingly and passes through B. This is the case $(3P2T)_2$ of §10.1. Analogously can be constructed the other parabola, starting with C'' instead of C'. Figure 67 shows the pencil of all conics passing through

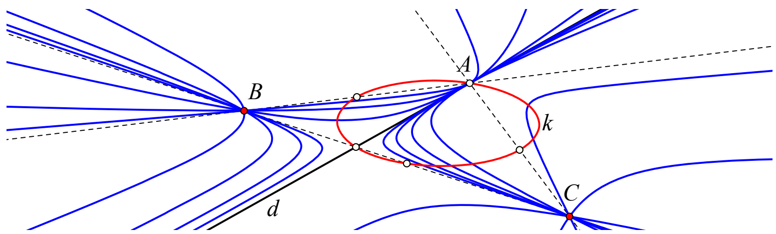

Figure 67. The pencil of conics tangent to d at A and passing through B, C

B, C and tangent to d at A, but having B, C on both sides of d. All conics are hyperbolas and the conic k of their centers is an ellipse. This is the reason of non-existence of solutions in this case.

9.3. *Parabola by 2 points, 1 tangent, axis-direction* $(3P_1 2T_1)$. Construct a conic passing through two points $A, B, [C]$, tangent to line d and also tangent to the line at infinity e, hence a parabola. The point at infinity $[C]$ determines the direction of

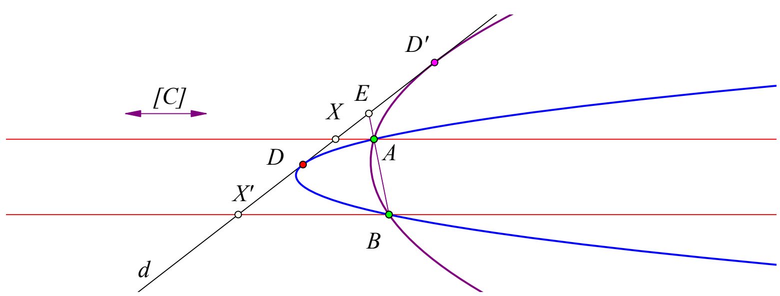

Figure 68. The two parabolas through A, B, tangent to d and axis parallel to $[C]$

the parabola's axis. By Desargues' theorem, the pencil $\mathcal{D}$ of all parabolas passing through $A, B, [C]$, i.e. passing through A, B and having axis direction $[C]$, define through their intersection points X, X' with line d an involution. The fixed points D, D' of this involution are contact points of the requested parabolas. There are two obvious, degenerate, parabolas passing through $A, B, [C]$ defining two pairs of points in involution. One pair consists of the intersections (X, X') with d of the two parallels to $[C]$ from A and B (see Figure 68). The other pair consists of $(E, [d])$, defined by line AB and the line at infinity, where $E = (AB, d)$. The fixed points D, D' of the involution are the common harmonics of these point-pairs. Once the contact points D, D' are known, the requested parabolas are easily constructible by the method of §10.2. If A, B are on the same side of a then there are two solutions, otherwise there is no solution.

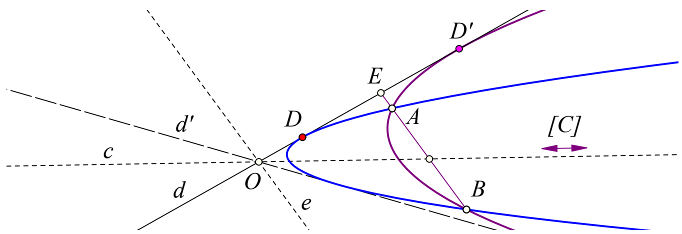

Figure 69. Parabolas through A, B, tangent to d and axis parallel to $[C]$

Another computational solution of the problem results by using the homothety relating the two parabolas. In fact, the intersection point O of d with the parallel c to $[C]$ from the middle of AB is the center of a homothety, mapping one of the parabolas to the other. The other common tangent d' to the two parabolas from O, can be constructed from the given data, since it is the harmonic conjugate of d with respect to the line pair (c, e), where e is the parallel to AB from O. The computations are straightforward and I omit them. See the remark in $(2P3T_1)$ of §5.2, which relates that problem to the present one.

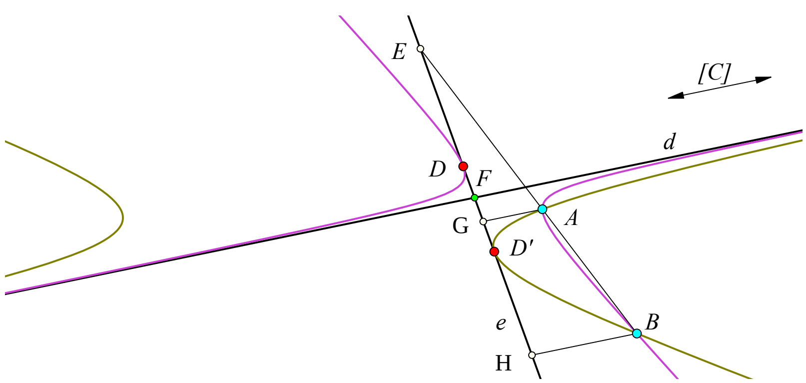

Figure 70. Hyperbola through A, B, asymptote d and tangent to e

9.4. *Hyperbola by 2 points, 1 tangent, 1 asymptote* $(3P_12T)_i$. Construct a conic passing through points $A, B, [C]$ and tangent to line d at $[C]$ and to line e. Thus, d is an asymptote and the conic is a hyperbola. The hyperbolas touch line e at D, D', which are the common harmonics of pairs of points $(E, F), (G, H)$, with $E = (e, AB), F = (d, e), G = (AC, e), H = (BC, e)$ (see Figure 70). Additional points can be found by considering conjugate points with respect to the polar of F. There are two hyperbolas, if A, B are in one and the same angular domain out of the four defined by lines d, e, or they lie on opposite angular domains. Otherwise there are no solutions. Figure 71 shows the pencil $\mathcal{D}$ of hyperbolas through A, B

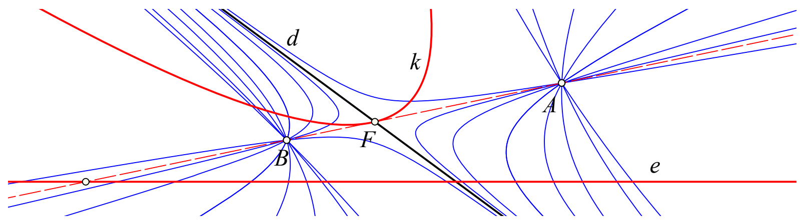

Figure 71. The pencil of hyperbolas through A, B with asymptote d

with one asymptote line d. It shows also a line e, for which there are no tangent members of the pencil. Conic k is the locus of poles of the line d with respect to the members of the pencil (the eleven points conic of $\mathcal{D}$ with respect to e). It is a hyperbola with one asymptote parallel to d, passing through $F = (d, AB)$.

9.5. *Hyperbola by 2 asymptotics, 1 tangent-at* $(3P_22T)_1$. Construct a conic passing through three points $A, [B], [C]$, tangent to $d \ni A$ and tangent also to line e. The conic is a hyperbola with asymptotic directions $[B], [C]$. In analogy to the

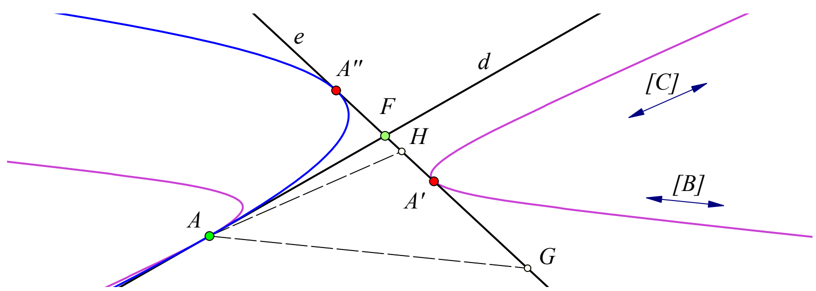

Figure 72. The two hyperbolas through $A, [B], [C]$, tangent to d at A and to line e

method of §9.1, project first A on e parallel to $[B], [C]$ to find respectively points G, H. The contact points A', A'' of the conics with line e are the common harmonics A', A'' of the point-pairs (G, H) and $(F, [e])$, where $F = (d, e)$. Once the two contact points of lines d, e are known, the methods of $(4P_21T)_1$ in §8.5 can be used to complete the construction. There are two solutions if the parallels to $[B], [C]$ from F fall in the same angular domain of lines (d, e). Otherwise there

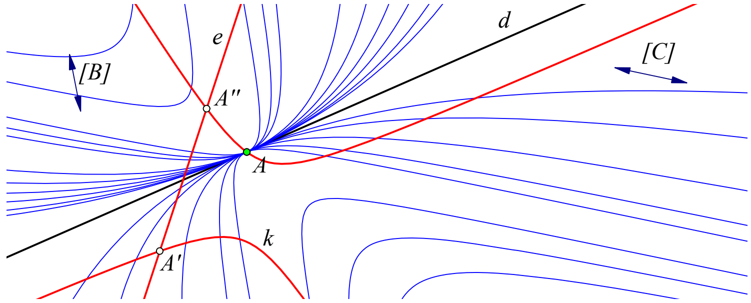

Figure 73. The pencil of hyperbolas with asyptotics $[B], [C]$ tangent to d at A

are no solutions. Figure 73 shows the pencil $\mathcal{D}$ of conics tangent to d at A and asymptotic directions $[B], [C]$. Shown is also the hyperbola k, defined as the locus of poles of a fixed line e with respect to the members of the pencil (the eleven points conic of $\mathcal{D}$ and e). The intersection points A', A'' of k with e are the contact points of requested conics with line e.

9.6. *Conic by 1 asymptote 1 asymptotic 1 point 1 tangent* $(3P_22T)_i$. Construct a conic passing through three points $A, [B], [C]$, tangent to $d \ni [B]$ and tangent to e. This is a hyperbola with an asymptote d, an asymptotic direction $[C]$, passing through point A and tangent to line e. The following construction method is a variation of the one given in §9.1. The pencil of conics $\mathcal{D}$, used in the theorem of Desargues, consists now of all conics tangent to d at $[B]$ and passing through A and $[C]$. This is the pencil of hyperbolas having their centers on line d, one asymptote

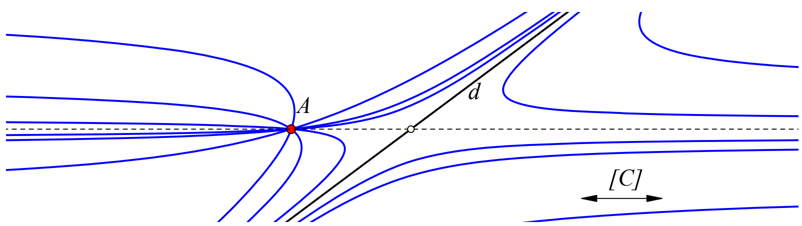

Figure 74. The pencil of hyperbolas through A, asymptote d, asymptotic $[C]$

d, the other parallel to $[C]$ and passing through A (see Figure 74). Two degenerate members of this pencil consist of (a) the product of line AB and the line at infinity, (b) the product of lines $d \cdot AC$. These two members define on e respectively the point-pairs $(G, [e]), (E, F)$, where $G = (e, AB), E = (e, AC)$ and $F = (d, e)$. The contact points D, D' of the requested hyperbolas with line e are the common harmonics of these two point-pairs. They lie on d symmetrically with respect to G (see Figure 75). Once the contact points with line e are known, the methods of $(3P2T)_2$ in §10.1 can be applied to complete the construction of the conics. Fixing the positions of A, d, e, there are two solutions if $[C]$ defines $E = (AC, e)$, such that EF is not separated by G. Otherwise there are no solutions. The points D, D' can be found also as intersections of line e with the conic k which is the locus of

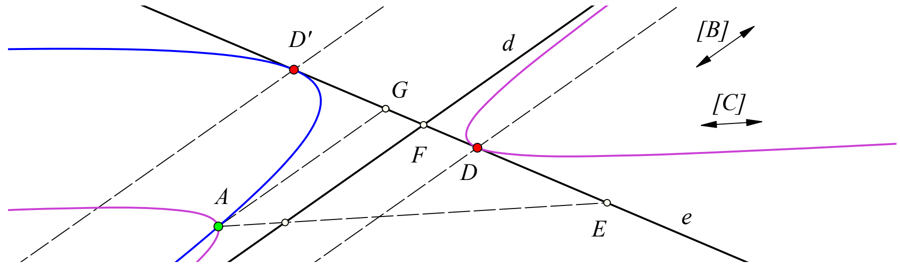

Figure 75. The two hyperbolas through A, asymptote d, asymptotic $[C]$, tangent e

poles of e with respect to the members of the pencil (the eleven points conic). In this case k is a hyperbola with one asymptote parallel to d.

10. Three points two tangents two coincidences

10.1. *Conic by two tangents-at and a point* $(3P2T)_2$. Construct a conic passing through three points A, B, C and tangent to two lines d, e at $A \in d$ and $B \in e$. In this case it is easy to find additional points and pass the conic through five points. Line AB is the polar of $F = (d, e)$ and the conjugate $D = C(F, J)$, where $J = (FC, AB)$ is on the conic. The conjugate $I = J(A, B)$ is the pole of FC and more points can be constructed as shown in Figure 75. The problem has always one solution.

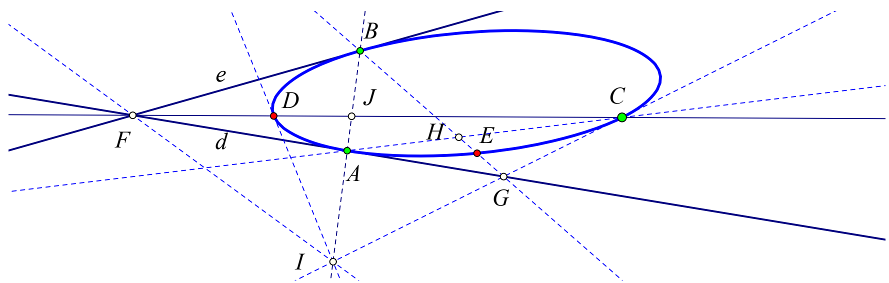

Figure 76. Conic through C, tangent to d, e at A, B

Remark. In this case the simplicity of the analytic solution is worth noticing. Representing lines d, e with two equations respectively $f = 0, g = 0$, and line AB with $h = 0$, the general equation of the conic passing through A, B and tangent there to lines d, e is given by a quadratic equation ([18, p. 234])

$$j = \lambda \cdot (f \cdot g) + \mu \cdot h^2 = 0,$$

where λ and μ are arbitrary constants. The requirement for the conic to pass through C, namely, $j(C) = 0$, determines, the constants λ, μ up to a multiplicative factor, and through these determines a unique conic. The conics resulting for variable λ, μ build the *bitangent* pencil ([2, II, p. 187]), used also in §4.1.

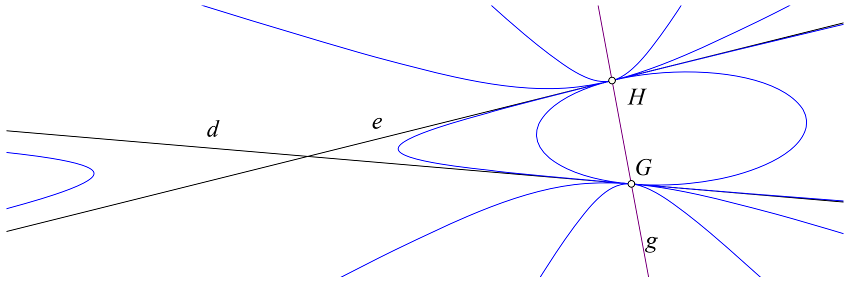

Figure 77. A bitangent pencil of conics $\kappa(d \cdot e) + \lambda(g^2)$

Figure 77 presents such a kind of (type IV) pencil. All conics of the pencil are tangent to the lines d, e at their intersections with line g. The two conics $c_1 = d \cdot e$ and $c_2 = g^2$ are degenerate members of the pencil.

10.2. *Parabola by 1 point, 1 tangent-at, axis-direction* $(3P_12T_1)_1$. Construct a conic passing through three points $A, B, [C]$, tangent to line d at A and also tangent to the line at infinity e. Thus, the conic is a parabola with axis parallel to the direction $[C]$. Project B parallel to d on AC to M and extend BM to the double to find B', which is on the parabola (this map $B \mapsto B'$ is the *affine reflection* with *axis* AC and *conjugate direction* d ([5, p. 203])). Define N to be the symmetric

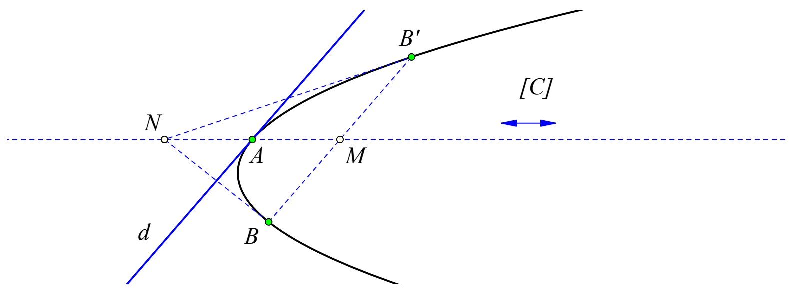

Figure 78. Parabola through A, B, tangent to d and axis parallel to $[C]$

of M with respect to A (see Figure 78). Since BN is tangent at B to the parabola, we can construct arbitrary many points of the parabola by repeating this procedure. There is always one solution.

10.3. *Hyperbola 1 point 1 tangent-at 1 asymptote* $(3P_12T)_{1i}$. Construct a conic passing through points $A, B, [C]$ and tangent to lines $d \ni [C], e \ni B$. This is a hyperbola with asymptote d. A slight variation of the solution in §10.1, leading to the determination of the other asymptote and the center of the hyperbola, is as follows. Let $F = (d, e)$. Then the symmetric F_1 of F with respect to B is on the other asymptote d' of the hyperbola (see Figure 79). Draw also the parallel to d from A intersecting e at D. The symmetric D' of D with respect to A defines the polar BD' of D. The intersection point $E = (BD', d)$ is the pol of line AD. Hence AE is the tangent at A and the symmetric E_1 of E with respect to A defines

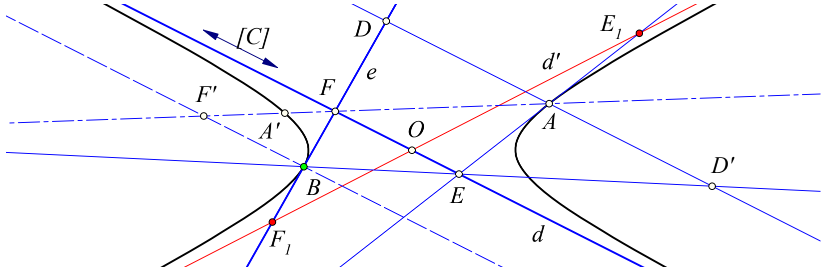

Figure 79. Hyperbola through A, asymptote d and tangent to e at B

a point on the other asymptote d' of the hyperbola. Thus, the other asymptote d' can be constructed to pass from the two points F_1, E_1. The intersection point O of the two asymptotes defines the center of the hyperbola and by the symmetry with respect to O we can find more points on the conic. An additional point on the conic is also A', constructed by first drawing the parallel from B to d. This parallel is the polar of F and if F' is its intersection point with AF, then the harmonic conjugate of A with respect to F, F' is on the conic. There is always one solution.

10.4. *Hyperbola from two asymptotes and a point* $(3P_2 2T)_{2i}$. Construct of a conic passing through three points $A, [B], [C]$ and tangent to two lines $d \ni B, e \ni C$. This is a hyperbola with asymptotes the lines d, e passing through a point A. This

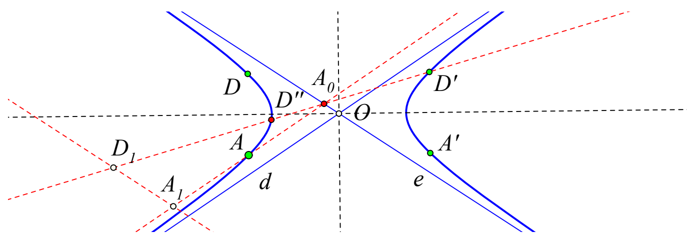

Figure 80. Hyperbola with given asymptotes d, e and passing through A

can be done by determining the successive symmetrics D, D', A' of A with respect to the axes and a fifth additional point D'' easily constructible from the data (see Figure 80). In fact, draw a parallel to the asymptote d intersecting the other asymptote in A_0. The polar of A_0 is the line parallel to e, such that its intersection A_1 with AA_0 is the symmetric of A_0 with respect to A. Consider the intersection D_1 of that polar with line A_0D'. The harmonic conjugate D'' of D' with respect to (A_0, D_1) is on the conic and coincides with the middle of D_1D'. Note the A_0 divides $D'D''$ in ratio $(2:1)$. There is always a unique solution.

11. Two points three tangents one coincidence

11.1. *Conic by 1 point, 1 tangent-at, 2 tangents* $(2P3T)_1$. Construct a conic passing through two points D, E and tangent to three lines $a \ni D, b, c$. The solution can be given by applying a special case of the dual of Desargues' theorem, referred to also as Plücker's theorem ([4, p. 25], [2, p. 202, II]). This case concerns the one-parameter pencil $\mathcal{D}$ of all conics, which are tangent to line a at D and also tangent to two lines b, c (see Figure 84 below in this section). If E is another, arbi-

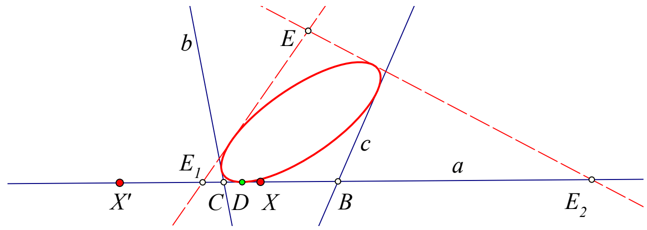

Figure 81. Conics tangent a, b, c, passing through $D \in a, E$

trary, but fixed point, not lying on any of a, b, c, Desargues' theorem asserts, that the pairs of tangents to these conics from E define an involution on the pencil E^* of lines through E. By intersecting the rays of this pencil with a line, such as a, we can represent this involution through one which permutes the points of that line. Thus, the tangents from E to an arbitrary conic of that pencil intersect line a at a pair of points (E_1, E_2), related by this involution (see Figure 81). The requested conics are those, which pass through E and their tangents at E pass through the fixed points X, X' of this involution. In order to construct these points it suffices to find two easily constructible pairs of points in involution on a. One such pair consists of the points $C = (a, b), B = (a, c)$. Another pair is found by drawing

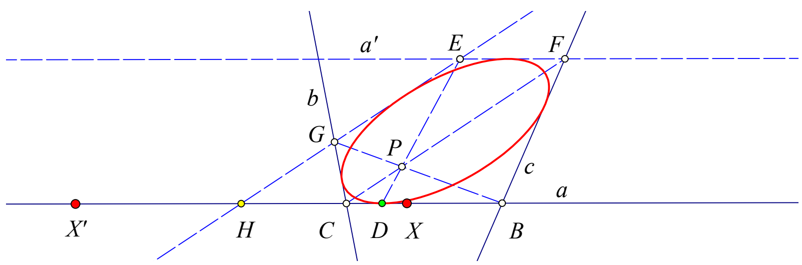

Figure 82. A particular conic tangent to a, b, c, passing through $D \in a$

the parallel a' to a through E, intersecting c at F (see Figure 82). The second tangent from E to the conic inscribed in the quadrilateral with sides a', b, a, c and passing through D, can be found by applying Brianchon's theorem to the pentagon $CBFEG$. This theorem guarantees that lines DE, CF, BG pass through a common point P. Thus, P is constructed by intersecting DE with CF and G is found as the intersection $G = (PB, b)$. In this case the two tangents from E are EG, EF and consequently H corresponds by the involution to the point at infinity of line a. It follows that the fixed points X, X' of the involution are common harmonics of pairs $(H, [a])$ and (B, C). Once the tangents at E are found, each one of the two conics can be constructed by locating one more point on it and applying the recipe of §10.1 using Brianchon's theorem. There are two solutions if points D, E

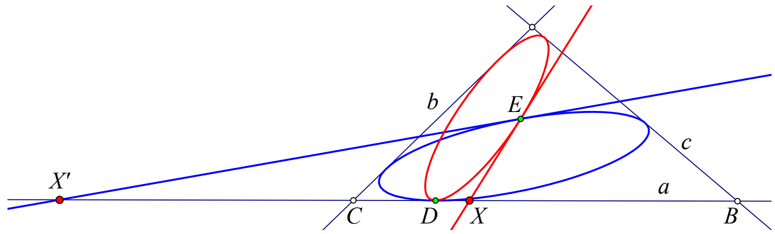

Figure 83. The two conics tangent to a, b, c, passing through $D \in a$, E

are in the same angular domain defined by lines b, c or they are in opposite angular domains. In all other cases there are no solutions. This is visible also in Figure 84 ,

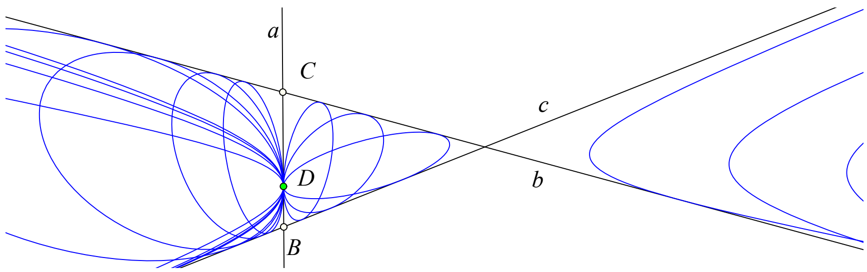

Figure 84. A pencil of conics tangent to a, b, c and passing through $D \in a$

which displays a pencil of conics tangent to a, b, c and passing through $D \in a$. When D is on the exterior of segment BC, then the conics are all located in the angular domain of b, c containing D and its opposite ([2, II, p. 201]).

An alternative solution of the problem is the following. Consider the triangle with sides a, b, c and the conic k passing through its vertices and tangent to EB, EC at B, C respectively, which is a construction of the type $(3P2T)_2$ of

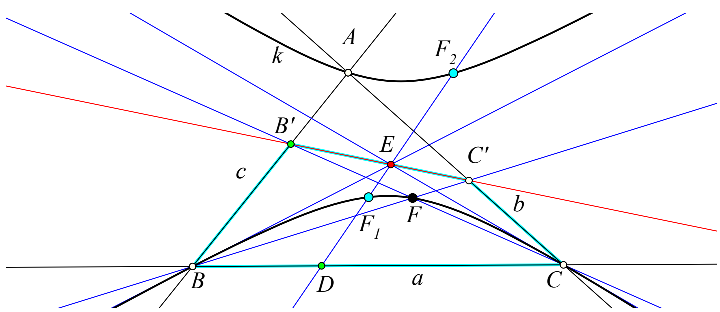

Figure 85. The locus of $F = (B'C, C'B)$ for $B'C'$ passing through E

§10.1 (see Figure 85). It is easy to see, using Maclaurin's theorem ([1, p. 77], [18, p. 230]), that this conic is the geometric locus of points F, which are intersections of diagonals of quadrilaterals $B'BCC'$ with $B'C'$ passing through E. From Brianchon's theorem follows that if $B'C'$ were the tangent at E to our requested conic, then the diagonals $B'C, C'B$ would intersect on line DE. Thus, their intersection point F must coincide with the intersection points F_1, F_2 of line DE with the conic k. Having these two points, the construction of the two tangents at E is immediate and the rest, of the construction of conics, goes as before.

11.2. *Parabola by 1 point, 1 tangent, 1 tangent-at* $(2P3T_1)_1$. Construct a conic passing through two points A, B, tangent to two lines $c \ni A, d$ and tangent also to the line at infinity, thus a parabola. By Desargues' theorem, applied as in §11.1,

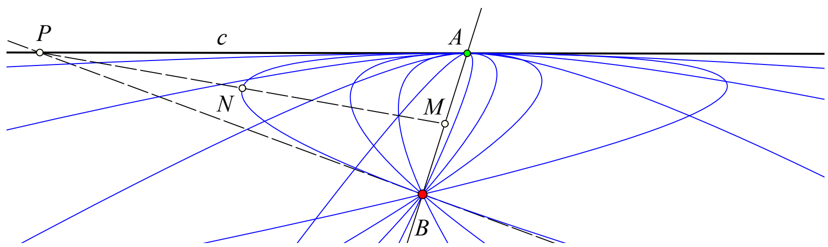

Figure 86. Parabolas through A, B and tangent to c at A

the tangents at B to the members of the pencil $\mathcal{D}$ (see Figure 86) of all conics tangent to c at A, passing through B and also tangent to d and the line at infinity (thus parabolas), define an involution on the pencil B^* of all lines through B. The

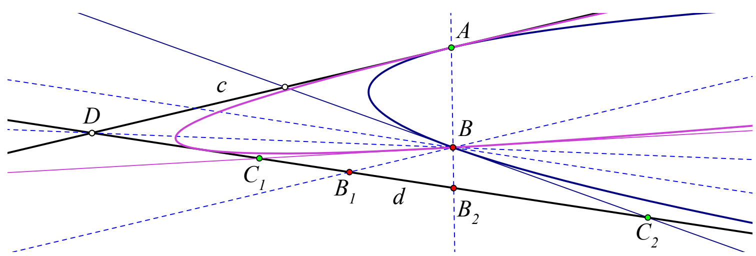

Figure 87. Parabolas through A, B and tangent to $c \ni A, d$

tangents at B to the requested parabolas are the fixed elements of this involution. We can represent this involution through points on the line d, by corresponding to each ray through B its intersection point with d. There are two particular degenerate parabolas of this pencil, coinciding with the lines parallel to c, d through B. The parallel to d defines the pair of corresponding points $(B_2, [d])$, where $B_2 = (AB, d)$. The parallel to c defines the pair of corresponding points (B_1, D), where $B_1 = (B[c], d)$ and $D = (c, d)$. The rays through B representing the fixed elements of the involution in B^* are BC_1, BC_2, where C_1, C_2 are the common harmonics of the pairs $(B_1, D), (B_2, [d])$. Once the tangents through B are located, the parabolas are constructed as in the next section. There are two solutions if points A, B are not separated by line d and no solution if they are.

11.3. *Parabola by 1 point, 2 tangents, axis-direction* $(2P_13T_1)$. Construct a conic passing through two points $A, [B]$, tangent to two lines c, d and also tangent to the line at infinity, thus a parabola. Point $[B]$ determines the direction of the axis of

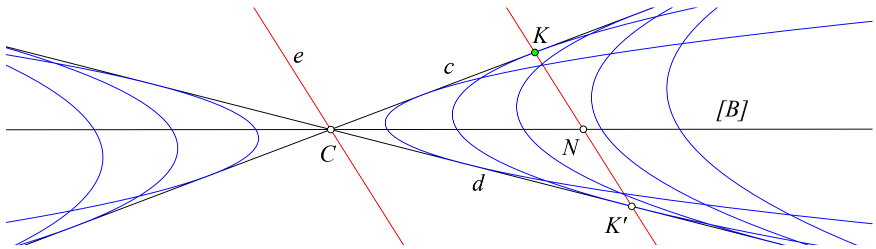

Figure 88. A pencil of parabolas tangent to c, d and axis direction $[B]$

the parabola. The pencil of parabolas tangent to c, d with axis direction $[B]$ can be easily constructed by taking the harmonic conjugate e of CB with respect to c, d. This is namely the direction of chords bisected by CB, where $C = (c, d)$. Having that direction, we can define a second point A' on the requested parabola. This is the result of the affine reflexion on CB parallel to e (see Figure 88). The rest of the construction is thus reducible to that of §9.3, which gives either two solutions or none, if point A is outside of the angular domains determined by c, d, which contain the parallel CB to the given direction $[B]$.

Another, computational, method to locate the two parabolas could be the one using the equation of the parabola with respect to the axes e, CB (see Figure 89).

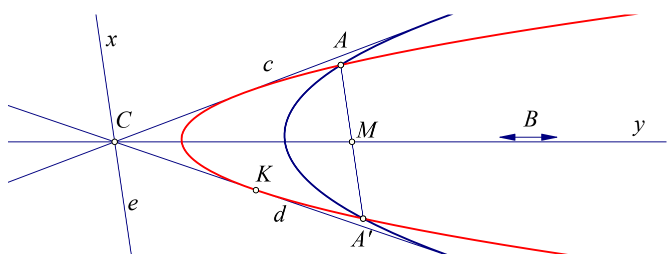

Figure 89. The two parabolas through A, tangent to c, d and axis direction $[B]$

In these axes the equation of the parabola has the form

$$y = \alpha x^2 + \beta,$$

and constants α, $\beta$are easily determined by the data. In fact, the given point A has known coordinates (x_1, y_1) with respect to these axes and the coordinates (x_2, y_2) of the contact point K with c satisfy $y_2 = 2\beta$ and $\frac{y_2}{x_2} = \lambda$, later being a constant determined by the data. It turns out that α, βsatisfy the two equations

$$y_1 = \alpha x_1^2 + \beta, \quad \text{and} \quad \alpha\beta = \frac{\lambda^2}{4},$$

which determine the same solutions under the same conditions as before.

11.4. *Hyperbola by 1 point, 1 asymptote, 2 tangents* $(2P_13T)_i$. Construct a conic passing through two points $A, [B]$, tangent to three lines $c, d, e \ni B$. This is a

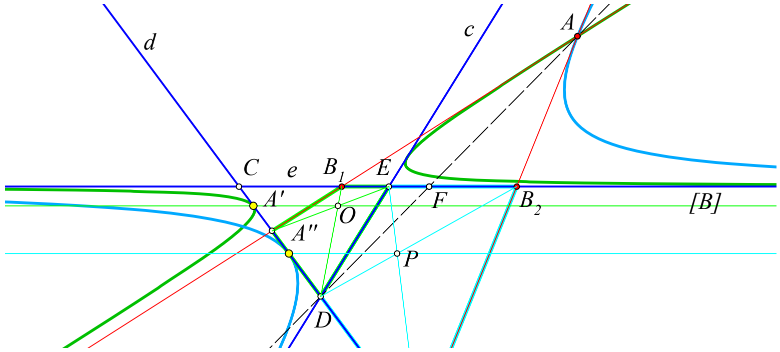

Figure 90. The two hyperbolas through A with asymptote e and tangent to c, d

hyperbola with an asymptote e. By the method of §11.1, the tangents at A of the requested conics are determined by the common harmonics B_1, B_2 of the two pairs of points (C, E) and $(F, [B])$, where $C = (e, d), E = (e, c), D = (c, d), F = (AD, e)$ (see Figure 90). If O is a diagonal point of the quadrilateral formed by the three tangents c, d, e, AB_1, then, by Brianchon's theorem, the intersection point A' of d with the parallel to e through O will be the contact point of d with the conic. The problem reduces then to the construction of the conic tangent at $A' \in d, A \in AB_1$ and passing through A, which is $(3P2T)_2$ of §10.1. Analogous properties hold for the other conic with tangent at A the line AB_2. Fixing the lines c, d, there are two solutions if $A, [B]$ are in the same or opposite angular domains defined by c, d. Otherwise there are no solutions.

11.5. *Hyperbola 1 asymptote 1 asymptotic 2 tangents* $(2P_23T)_i$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines: a at $[A], b, c$. This is a hyperbola with one asymptote a, the other asymptotic direction $[B]$ and two other tangents b, c. The problem is projectively equivalent to $(2P3T)_1$ of §11.1. Here

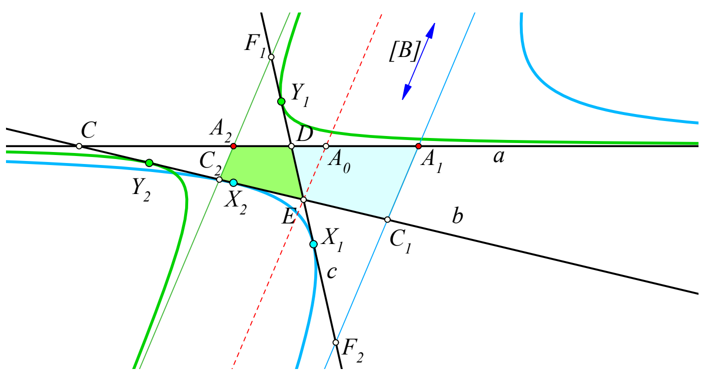

Figure 91. The two hyperbolas with asymptote a, asymptotic $[B]$ and tangents b, c

again the recipe is essentially the one of §11.1, with some simplifications allowing for a faster determination of the conics. In Figure 91, displaying the conics, $A_0 = (a, EB)$ and points A_1, A_2 are the common harmonics of pairs $(C, D), (A_0, [A])$. Parallels to $[B]$ define the two quadrilaterals DA_2C_2E, DA_1C_1E. Each quadrilateral determines a conic tangent to its sides. In this case the contact points of the conics with the sides of the quadrilateral are easily determined. In fact, by the well known property of segments intercepted between asymptotes follows, that the contact points X_1, X_2 are respectively the middles of DF_2, CC_1 and the contact points Y_1, Y_2 are the middles of DF_1, CC_2. There are two solutions if, drawing parallels from a point to b, c and to $[B]$, later does not fall between the two first. Otherwise there are no solutions.

12. Two points three tangents two coincidences

12.1. *Conic by two tangents-at and a tangent* $(2P3T)_2$. Construct a conic tangent to three lines a, b, c, passing through two points A, B with $A \in a$ and $B \in b$. In this case the contact point C of the requested conic with the third line is easily

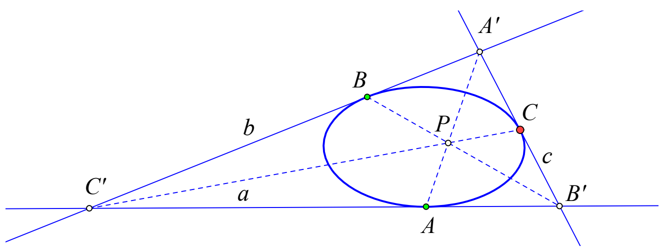

Figure 92. Conic tangent to a, b, c at $A \in a, B \in b$

constructed, since all lines joininig the vertices of the triangle formed by the three lines to the opposite contact point pass through the same point P, the *perspector* of the conic with respect to that triangle $A'B'C'$ (see Figure 92). There is always a unique solution.

12.2. *Parabola by 2 tangents-at* $(2P3T_1)_2$. Construct a conic passing through two points A, B, tangent to two lines $c \ni A, d \ni B$ and also tangent to the line at infinity, thus a parabola. If point $C = (a, b)$, taking the middle D of AB and

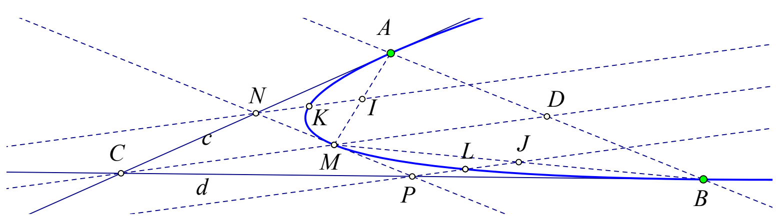

Figure 93. Parabola through A, B and tangent to $a \ni A, b \ni B$

the middle M of CD we construct a new point on the parabola. Analogously are obtained new points K, L from the middles of MA, MB respectively. The parabola is led as a conic through the five points A, B, M, K, L. There is always one solution identified with one first-kind *Artzt parabola* of triangle ABC ([13, p. 518]). Triangle ABC is referred by times as an *Archimedes triangle* ([8, p. 239]).

12.3. *Parabola 1 tangent 1 tangent-at, axis-direction* $(2P_1 3T_1)_1$. Construct a conic passing through two points $A, [B]$, tangent to two lines $c \ni A, d$ and tangent also to the line at infinity, thus, a parabola with axis parallel to $[B]$. One solution is to

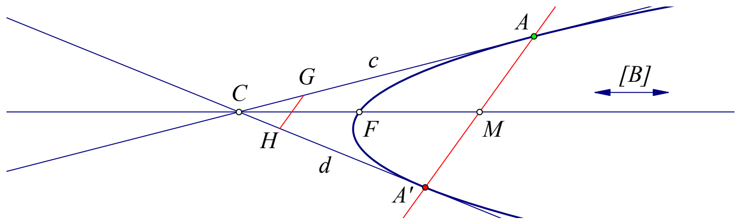

Figure 94. The parabola tangent to d and c at A and axis direction $[B]$

construct, as in the previous section, the direction GH of chords of the parabola, which are bisected by CB. Then, find the point A' on the parabola, such that AA' is parallel to GH and bisected by CB. Point A' is the contact point of the parabola with d and the construction reduces to that of $(2P3T_1)_2$ of §12.2. There is always a unique solution.

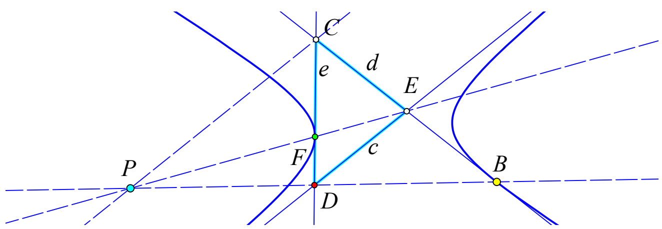

Figure 95. Hyperbola with asymptote c tangent to $d \ni B$ and tangent to e

12.4. *Hyperbola 1 asymptote 1 tangent 1 tangent-at* $(2P_1 3T)_{1i}$. Construct a conic passing through two points $[A], B$ and tangent to three lines $c \ni A, d \ni B, e$. This is a hyperbola with one asymptote c, tangent to d at B and also tangent to e. The triangle CDE with sides the tangents c, d, e is known and the perspector P of the conic, tangent to the sides of this triangle, can be found (see Figure 95). In fact, draw from $C = (d, e)$ parallel to the asymptote c and find its intersection P with BD, where $D = (c, e)$. If $E = (c, d)$, then line PE passes through the contact point F of e with the conic. The case, as the one of §12.1, has always a unique solution.

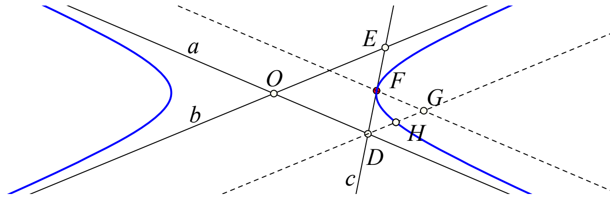

Figure 96. Hyperbola with asymptotes a, b and tangent c

12.5. *Hyperbola 2 asymptotes 1 tangent* $(2P_23T)_{2i}$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines $a \ni A, b \ni B, c$. This is a hyperbola with given asymptotes a, b and a tangent c. This is an easy case, since the contact point of the tangent c is the middle F of DE, where $D = (a, c), E = (b, c)$. The parallel to a from F is the polar of D and the parallel to the other asymptote b from D intersects the first parallel at G. The middle H of DG is a point of the hyperbola. An analogous point can be constructed starting with E. Taking the symmetrics with respect to the center $O = (a, b)$ of the hyperbola we have enough points to define the conic through five points. There is always a unique solution.

13. One point four tangents one coincidence

13.1. *Conic by one tangent-at and three tangents* $(1P4T)_1$. Construct a conic tangent to a given line a at a given point D and also tangent to three other lines b, c, e. The basic underlying structure results from Brianchon's theorem. In fact, consider the intersection point O of the diagonals of the quadrilateral $BCGF$ defined by the

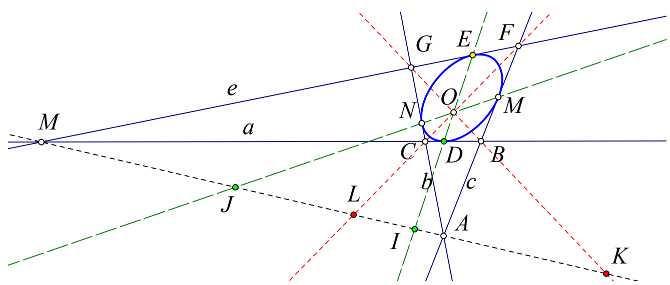

Figure 97. Conic tangent to a, b, c, e, passing through $D \in a$

four lines (see Figure 97). According to Brianchon's theorem, the lines joining opposite contact points DE, MN intersect also at O. Thus, point E is constructible from the given data. Further, if $K = (b, c), M = (a, e), I = (AM, DE)$, the line MN of the other two contact points defines point $J = (MN, AM)$, such that $(KLIJ) = -1$. This allows the determination of J and from this the points M, N, by intersecting line JO with the sides b, c. The problem is thus reducible to $(4P1T)_1$ of §8.1 and has one solution. The pencil involved here is the one of conics tangent to a at D and also tangent to b, c, appearing also in $(2P3T)_1$ of §11.1.

Remark. Besides quadrangle $BCGF$, the complete quadrilateral, defined by lines a, b, c, e, contains also the quadrangles $AGMB$, whose diagonals intersect at L and $CMFA$, whose diagonals intersect at K. It is also easily seen, that the contact points N, E, M are the harmonic associates of D with respect to the diagonal triangle OLK of the complete quadrilateral. Thus the definition of N, E, M from D, does not depend on which one of the three quadrangles (and corresponding intersection of diagonals O, K or L) we select to work with.

13.2. *Parabola by 1 tangent-at, 2 tangents* $(1P4T_1)_1$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to line a at $E \in a$ and tangent to two lines b, c. Here the construction is somewhat simpler than that of the previous case,

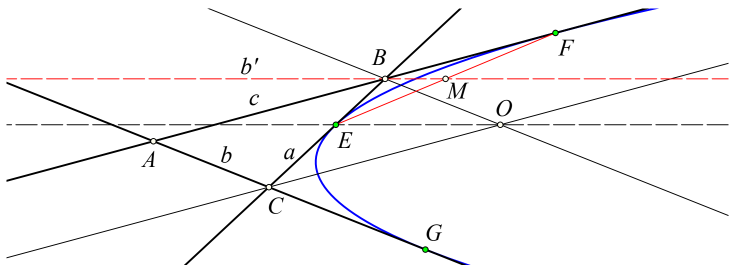

Figure 98. Parabola tangent to a at E and tangent to b, c

because of the nice properties of parabolas. In fact, let O be the intersection of the parallels from B to b and from C to c (see Figure 98). Then the line EO is parallel to the axis of the parabola. Having the direction of the axis, we can construct more points on the parabola using the method of $(2P_13T_1)$ of §11.3. Using this we can find the direction of the chords bisected by the parallel to the axis from B and determine the contact point F with line c. Analogously we can find the contact point G of b, and from these points by similar methods find other arbitrary many points on the parabola. Alternatively, we can use the fact that points $\{F, G, O\}$ are collinear and the line d, carrying them, intersects line BC at the harmonic conjugate $E' = E(B, C)$. There is always a unique solution.

13.3. *Parabola by axis-direction, 3 tangents* $(1P_14T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to three lines a, b, c and passing through $[D]$, i.e. with given axis-direction. This is a case similar to the previous one. Again we construct the intersection point O of the parallels to b, c from the

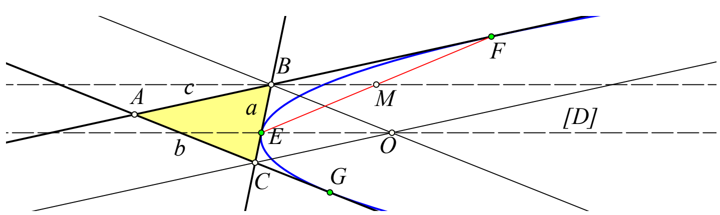

Figure 99. Parabola tangent to a, b, c with given axis-direction

opposite vertices. The line through O, parallel to the given axis-direction, determines now on a the contact point E with the parabola. From there the construction of the other contact points F, G with sides c, b and the completion of the parabola construction is the one described in the previous section. There is always a unique solution.

13.4. *Hyperbola, 1 asymptote, 3 tangents* $(1P_1 4T)_i$. Construct a conic tangent to line e at its point at infinity $[E]$, i.e. a hyperbola with an asymptote e, and tangent to three lines a, b, c. In analogy to §13.1 we can find the contact points of the

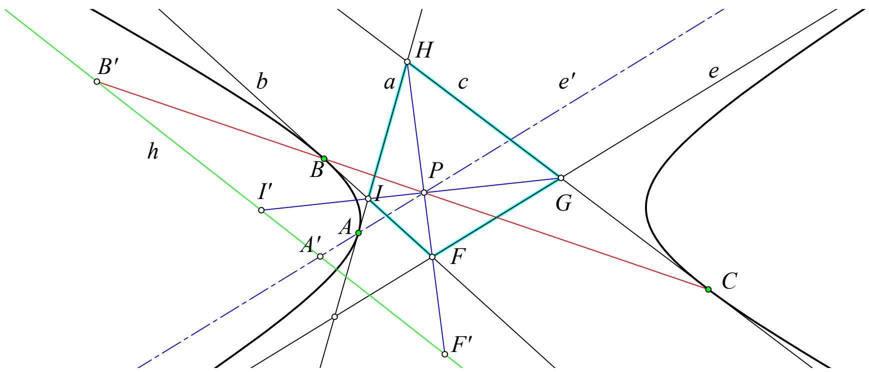

Figure 100. Hyperbola tangent to a, b, c with asymptote e

three tangents with the hyperbola. In fact, consider the intersection point P of the diagonals of quadrilateral $FGHI$, whose all sides are given and are tangents to the hyperbola. By Brianchon's theorem, the line e' parallel to the asymptote e from point P will intersect the side a of the quadrilateral at its contact point A with the hyperbola (see Figure 100). Consider now an arbitrary line h and its intersection points $A' = (e', h), I' = (IG, h), F' = (FH, h)$. The other chord of contact-points BC will intersect line h at the harmonic conjugate B' of A' with respect to (I', F'). Thus, B' is constructible from the given data, and drawing PB' we determine the positions B, C of the contact points on the tangents b, c respectively. Having one asymptote and the contact points on the tangents, we can determine the other asymptote, the center, and, by symmetry to that center, three more points on the conic. The method is described already in $(3P_1 2T)_{1i}$ of §10.3. There is always one solution.

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On Two Triads of Triangles Associated With the Perpendicular Bisectors of the Sides of a Triangle

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Abstract. We discover some properties of the triangle centers related to the two triads of triangles associated with the perpendicular bisectors of the sides of a triangle.

1. Introduction

Given a triangle $\mathbf{T} := ABC$, let the perpendicular bisector of the side BC intersect the sidelines AC and AB at B_c and C_b respectively. Define C_a, A_c, A_b, B_a similarly. In this paper we study the two triads of triangles
(i) $\mathbf{T}_a := AA_bA_c$, $\mathbf{T}_b := B_aBB_c$, $\mathbf{T}_c := C_aC_bC$ (see Figure 1a) and
(ii) $\mathbf{T}'_a := AB_aC_a$, $\mathbf{T}'_b := A_bBC_b$, $\mathbf{T}'_c := A_cB_cC$ (see Figure 1b).

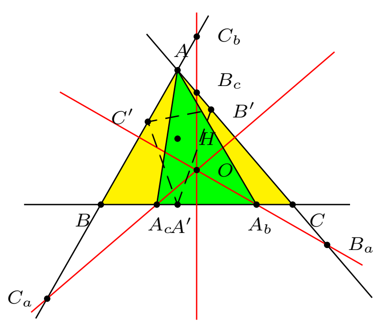

Figure 1a. $\mathbf{T}_a$ and orthic triangle

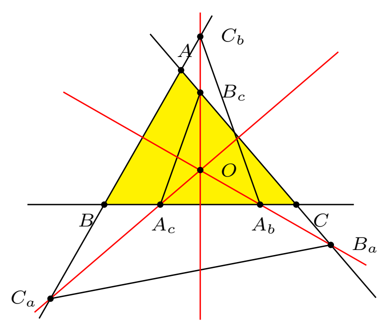

Figure 1b. The triangles $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$

Homogeneous barycentric coordinates are used throughout this work. With the usual notations in triangle geometry, a, b, c for the lengths of the sides BC, CA, AB, and

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2},$$

the points on the perpendicular bisectors are

$$\begin{aligned} A_b &= (0 : -S_A + S_B : S_A + S_B), \quad & A_c &= (0 : S_C + S_A : S_C - S_A);\\ B_c &= (S_B + S_C : 0 : -S_B + S_C), \quad & B_a &= (S_A - S_B : 0 : S_A + S_B);\\ C_a &= (-S_C + S_A : S_C + S_A : 0), \quad & C_b &= (S_B + S_C : S_B - S_C : 0). \end{aligned}$$

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Denote by $\mathbf{T}'$ the orthic triangle of ABC. Its vertices are

$$A' = (0 : S_C : S_B), \qquad B' = (S_C : 0 : S_A), \qquad C' = (S_B : S_A : 0).$$

The sidelengths a', b', c' of $B'C'$, $C'A'$, $A'B'$ of the orthic triangle are given by

$$a'^2 = \frac{a^2 S_{AA}}{b^2c^2}, \qquad b'^2 = \frac{b^2 S_{BB}}{c^2a^2}, \qquad c'^2 = \frac{c^2 S_{CC}}{a^2b^2}.$$

From these,

$$a'^2 : b'^2 : c'^2 = a^4 S_{AA} : \ b^4 S_{BB} : \ c^4 S_{CC}.$$

Corresponding to S_A, S_B, S_C, we have

$$\begin{aligned} S'_A &:= \ \frac{1}{2}(b'^2 + c'^2 - a'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{AA}}{S_A}, \\ S'_B &:= \ \frac{1}{2}(c'^2 + a'^2 - b'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{BB}}{S_B}, \\ S'_C &:= \ \frac{1}{2}(a'^2 + b'^2 - c'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{CC}}{S_C}, \end{aligned}$$

and

$$S'_A : S'_B : S'_C = S_{BC}(S^2 - S_{AA}) : S_{CA}(S^2 - S_{BB}) : S_{AB}(S^2 - S_{CC}).$$

Lemma 1. (a) *The triangles AA_bA_c, B_aBB_c, and C_aC_bC are all similar to $\mathbf{T}'$.*
(b) *The triangles AB_aC_a, A_bBC_b, and A_cB_cC are all similar to $\mathbf{T}$.*

2. A common point of circumcircles

Theorem 2. *The circumcircles of triangles in the two triads $(\mathbf{T}_a, \mathbf{T}_b, \mathbf{T}_c)$ and $(\mathbf{T}'_a, \mathbf{T}'_b, \mathbf{T}'_c)$ all contain the Euler reflection point*

$$E = \left(\frac{a^2}{b^2 - c^2} : \ \frac{b^2}{c^2 - a^2} : \ \frac{c^2}{a^2 - b^2} \right)$$

of the circumcircle of $\mathbf{T}$.

Proof. We compute the coordinates of E with respect to these triangles, and show from these coordinates that E lies on the circumcircle of each. With respect to $\mathbf{T}$,

$$E = ((S_B{+}S_C)(S_C{-}S_A)(S_A{-}S_B) : (S_B{-}S_C)(S_C{+}S_A)(S_A{-}S_B) : (S_B{-}S_C)(S_C{-}S_A)(S_A{+}S_B))$$

with coordinate sum $\sigma = a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 6S_{ABC}$.

The coordinates of E in triangle $\mathbf{T}_a = AA_bA_c$ are

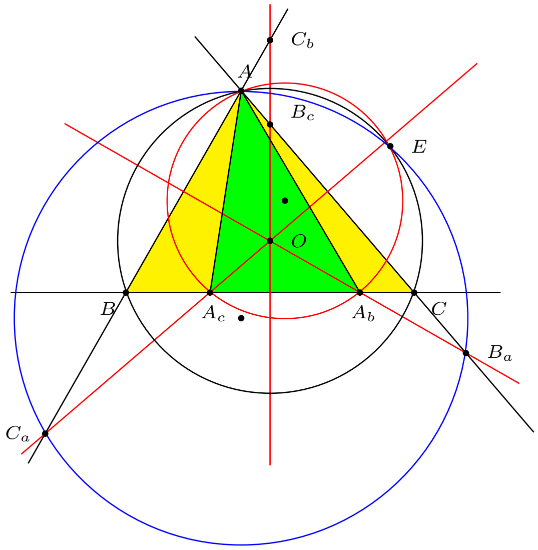

Figure 2

$$
\begin{aligned}
&\text{Area}(EA_bA_c):\ \text{area}(AEA_c):\ \text{area}(AA_bE)\\
=&\frac{1}{4\sigma S_{BC}}\begin{vmatrix}(S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\\ 0 & -(S_A-S_B) & S_A+S_B\\ 0 & S_C+S_A & S_C-S_A\end{vmatrix}\\
&:\ \frac{1}{2\sigma S_C}\begin{vmatrix}1 & 0 & 0\\ (S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\\ 0 & S_C+S_A & S_C-S_A\end{vmatrix}\\
&:\ \frac{1}{2\sigma S_B}\begin{vmatrix}1 & 0 & 0\\ 0 & -(S_A-S_B) & S_A+S_B\\ (S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\end{vmatrix}\\
=&\ \frac{S_A(S_C-S_A)(S_A-S_B)(S_B+S_C)^2}{2\sigma S_{BC}}:\frac{S_B(S_B-S_C)(S_C-S_A)(S_C+S_A)}{\sigma S_C}:\frac{S_C(S_A-S_B)(S_B-S_C)(S_A+S_B)}{\sigma S_B}\\
=&\ \frac{S_A(S_B+S_C)^2}{2(S_B-S_C)}:\frac{S_{BB}(S_C+S_A)}{S_A-S_B}:\frac{S_{CC}(S_A+S_B)}{S_C-S_A}\\
=&\ \frac{S_{AA}(S_B+S_C)^2}{2S_A(S_B-S_C)}:\frac{S_{BB}(S_C+S_A)^2}{(S_C+S_A)(S_A-S_B)}:\frac{S_{CC}(S_A+S_B)^2}{(S_A+S_B)(S_C-S_A)}\\
=&\ \frac{a'^2}{2S_A(S_B-S_C)}:\frac{b'^2}{(S_C+S_A)(S_A-S_B)}:\frac{c'^2}{(S_A+S_B)(S_C-S_A)}
\end{aligned}
$$

This is the isogonal conjugate (in triangle $\mathbf{T}_a = AA_bA_c$) of the point

$$(2S_A(S_B-S_C) : (S_C+S_A)(S_A-S_B) : (S_A+S_B)(S_C-S_A)),$$

which is an infinite point since the coordinate sum

$$2S_A(S_B-S_C)+(S_C+S_A)(S_A-S_B)+(S_A+S_B)(S_C-S_A)=0.$$

This shows that E is on the circumcircle of triangle $\mathbf{T}_a$. Similarly, E is also on the circumcircles of the triangles $\mathbf{T_b}$ and $\mathbf{T}_c$, with coordinates

$$\left(\frac{a'^2}{(S_B+S_C)(S_A-S_B)} : \frac{b'^2}{2S_B(S_C-S_A)} : \frac{c'^2}{(S_A+S_B)(S_B-S_C)}\right)$$

and

$$\left(\frac{a'^2}{(S_B+S_C)(S_C-S_A)} : \frac{b'^2}{(S_C+S_A)(S_B-S_C)} : \frac{c'^2}{2S_C(S_A-S_B)}\right)$$

respectively. A similar calculation shows that E is also on the circumcircles of triangles $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$, with coordinates

$$\left(\frac{a^2}{2S_A(S_B-S_C)} : \frac{b^2}{(S_C+S_A)(S_A-S_B)} : \frac{c^2}{(S_A+S_B)(S_C-S_A)}\right),$$
$$\left(\frac{a^2}{(S_B+S_C)(S_A-S_B)} : \frac{b^2}{2S_B(S_C-S_A)} : \frac{c^2}{(S_A+S_B)(S_B-S_C)}\right),$$
$$\left(\frac{a^2}{(S_B+S_C)(S_C-S_A)} : \frac{b^2}{(S_C+S_A)(S_B-S_C)} : \frac{c^2}{2S_C(S_A-S_B)}\right)$$

respectively in these triangles. □

3. Counterparts of a point in the triad $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$

Let P be a point with homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle $\mathbf{T} = ABC$. The counterparts of P in the triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ are the points A_P, B_P, C_P which have the same coordinates $(x : y : z)$ in these triangles. In homogeneous barycentric coordinates,

$$\begin{aligned}
A_P = {} & (2S_{BC}x : S_C(-S_A+S_B)y + S_B(S_C+S_A)z : S_C(S_A+S_B)y + S_B(S_C-S_A)z),\\
B_P = {} & (S_A(S_B+S_C)z + S_C(S_A-S_B)x : 2S_{CA}y : S_A(-S_B+S_C)z + S_C(S_A+S_B)x),\\
C_P = {} & (S_B(-S_C+S_A)x + S_A(S_B+S_C)y : S_B(S_C+S_A)x + S_A(S_B-S_C)y : 2S_{AB}z)
\end{aligned}$$

Denote by $\mathbf{T}(P)$ the triangle $A_PB_PC_P$. Basic properties of $\mathbf{T}(P)$ can be found in [1].

Theorem 3 (Bui). *The triangle $\mathbf{T}(P)$ is*
(a) *oppositely similar to $\mathbf{T}$,*
(b) *orthologic to $\mathbf{T}$,*
(c) *perspective with $\mathbf{T}$ if and only if P lies on the Jerabek hyperbola. In this case, the perspector traverses the Euler line.*

If $P = (x : y : z)$, the perpendiculars from A to B_PC_P, B to C_PA_P, C to A_PB_P concur at

$$Q = \left(\frac{1}{-b^2c^2S_{BC}x + c^2S_AS_{CC}y + b^2S_AS_{BB}z} : \cdots : \cdots\right),$$

which lies on the circumcircle of $\mathbf{T}$ (see Figure 3).

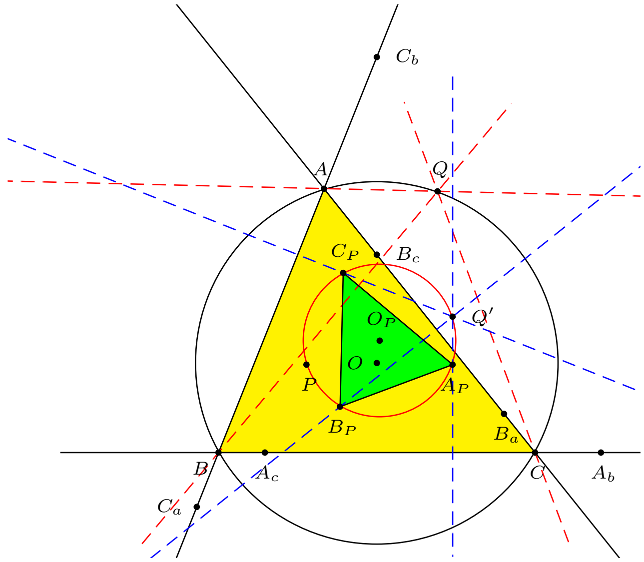

Figure 3.

On the other hand, the perpendiculars from A_P to BC, B_P to CA, C_P to AB concur at

$$\begin{aligned} Q' = \ & (S_{BC}(a^2(S_{AA} - S_{BC}) - S_A(S_{BB} + S_{CC})x + c^2a^2S_AS_{CC}y + a^2b^2S_AS_{BB}z \\ & : \cdots : \cdots). \end{aligned}$$

According to [1], the coordinates of Q' with respect to $\mathbf{T}(P)$ are the same as those of Q with respect to $\mathbf{T}$. It follows that Q' is a point on the circumcircle of $\mathbf{T}(P)$.

Proposition 4. *The triangle $\mathbf{T}(P)$ has orthocenter O.*

Proof. The sum of the coordinates of A_P given at the beginning of the present section is $S_{BC}(x+y+z)$; similarly for B_P and C_P. The orthocenter of $\mathbf{T}(P)$ has coordinates

$$\begin{aligned} & (2S_{BC}x,\ S_C(-S_A + S_B)y + S_B(S_C + S_A)z,\ S_C(S_A + S_B)y + S_B(S_C - S_A)z), \\ + \ & (S_A(S_B + S_C)z + S_C(S_A - S_B)x,\ 2S_{CA}y,\ S_A(-S_B + S_C)z + S_C(S_A + S_B)x), \\ + \ & (S_B(-S_C + S_A)x + S_A(S_B + S_C)y,\ S_B(S_C + S_A)x + S_A(S_B - S_C)y,\ 2S_{AB}z) \\ = \ & (x+y+z)(S_A(S_B + S_C),\ S_B(S_C + S_A),\ S_C(S_A + S_B)). \end{aligned}$$

This is the circumcenter O of $\mathbf{T}$. $\square$

Theorem 5. *The points A_P, B_P, C_P and P are concyclic.*

Proof. The coordinates of P with respect to $\mathbf{T}(P) = A_PB_PC_P$ are

$$
\begin{aligned}
&\text{Area}(PB_PC_P) : \text{ area}(A_PPC_P) : \text{ area}(A_PB_PP)\\
=&\frac{(b^2S_Bx-a^2S_Ay)(a^2S_Az-c^2S_Cx)}{4S_{ABC}S_A(x+y+z)^2} : \frac{(c^2S_Cy-b^2S_Bz)(b^2S_Bx-a^2S_Ay)}{4S_{ABC}S_B(x+y+z)^2}\\
&\quad : \frac{(a^2S_Az-c^2S_Cx)(c^2S_Cy-b^2S_Bz)}{4S_{ABC}S_C(x+y+z)^2}\\
=&\frac{a^2}{a^2S_A(c^2S_Cy-b^2S_Bz)} : \frac{b^2}{b^2S_B(a^2S_Az-c^2S_Cx)} : \frac{c^2}{c^2S_C(b^2S_Bx-a^2S_Ay)}\\
=&\frac{a^2}{\frac{y}{b^2S_B}-\frac{z}{c^2S_C}} : \frac{b^2}{\frac{z}{c^2S_C}-\frac{x}{a^2S_A}} : \frac{c^2}{\frac{x}{a^2S_A}-\frac{y}{b^2S_B}}.
\end{aligned}
$$

This shows that P is the isogonal conjugate (in triangle $\mathbf{T}(P)$) of an infinite point. It is a point on the circumcircle of $\mathbf{T}(P)$. □

Since $\mathbf{T}(P)$ is similar to $\mathbf{T}$, its circumcenter is the point

$$O_P := \frac{a^2S_A\cdot A_P + b^2S_B\cdot B_P + c^2S_C\cdot C_P}{2S^2}$$

in absolute barycentric coordinates. In homogeneous barycentric coordinates with respect to $\mathbf{T}$,

$$
\begin{aligned}
O_P = (&S_{BC}(4S^2\cdot S_A - a^2b^2c^2)x + c^2a^2S_{CCA}y + a^2b^2S_{ABB}z\\
&: \ b^2c^2S_{BCC}x + S_{CA}(4S^2\cdot S_B - a^2b^2c^2)y + a^2b^2S_{AAB}z\\
&: \ b^2c^2S_{BBC}x + c^2a^2S_{CAA}y + S_{AB}(4S^2\cdot S_C - a^2b^2c^2)z).
\end{aligned}
$$

4. Counterparts of a point in the triad $\mathbf{T}_a'$, $\mathbf{T}_b'$, $\mathbf{T}_c'$

For $P = (x : y : z)$ with respect to $\mathbf{T}$, we also consider its counterparts A_P', B_P', C_P' in the triangles $\mathbf{T}_a'$, $\mathbf{T}_b'$, $\mathbf{T}_c'$. These are the points

$$
\begin{aligned}
A_P' &= (2S_Ax - (a^2-b^2)y - (a^2-c^2)z : \ b^2z : \ c^2y),\\
B_P' &= (a^2z : \ 2S_By - (b^2-c^2)z - (b^2-a^2)x : \ c^2x),\\
C_P' &= (a^2y : \ b^2x : \ 2S_Cz - (c^2-a^2)x - (c^2-b^2)y).
\end{aligned}
$$

Denote by $\mathbf{T}'(P)$ the triangle $A_P'B_P'C_P'$.

Proposition 6. *For $P = (x : y : z)$, the triangle $\mathbf{T}'(P)$ is*
(a) *oppositely similar to the orthic triangle $\mathbf{T}'$,*
(b) *perspective with $\mathbf{T}$ at the isogonal conjugate of P in $\mathbf{T}$, namely,*

$$P^* = \left(\frac{a^2}{x} : \ \frac{b^2}{y} : \ \frac{c^2}{z}\right).$$

(c) *orthologic to $\mathbf{T}$ if and only if P lies on the Euler line. In this case,*
(i) *the perpendiculars from A, B, C to $B_P'C_P'$, $C_P'A_P'$, $A_P'B_P'$ are concurrent at*

$$X(265) = \left(\frac{S_A}{S^2-3S_{AA}} : \ \frac{S_B}{S^2-3S_{BB}} : \ \frac{S_C}{S^2-3S_{CC}}\right),$$

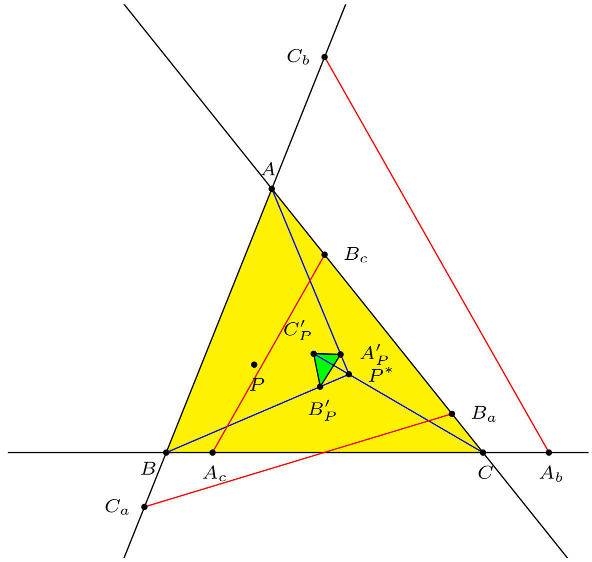

Figure 4.

(ii) *if* $OP : PH = t : 1 - t$, *the perpendiculars from* A'_P, B'_P, C'_P *to* BC, CA, AB *are concurrent at* Q *where* $OQ : QH = 1 - t : t$.

Remark. $X(265)$ is the reflection conjugate of O. Equivalently, it is the reflection of O in the Jerabek center $X(125)$.

Proposition 7. *The points* A'_P, B'_P, C'_P, P *are concyclic if and only if* P *lies on the circumconic which is the isogonal conjugate* (*in* $\mathbf{T}$) *of the perpendicular bisector of the segment* OH.

Proof. With respect to $\mathbf{T}'(P)$, the point P has coordinates

$$\begin{aligned}
& x' : y' : z' \\
= & \text{Area}(PB'_PC'_P) : \text{area}(A'_PPC'_P) : \text{area}(A'_PB'_PP) \\
= & \frac{2a^2S_Ax^2 + a^4yz - (S^2 + 2S_{BC} - S_{CC})zx - (S^2 + 2S_{BC} - S_{BB})xy}{-4S_{BC}(x+y+z)^2} \\
& : \frac{2b^2S_By^2 + b^4zx - (S^2 + 2S_{CA} - S_{AA})xy - (S^2 + 2S_{CA} - S_{CC})yz}{-4S_{CA}(x+y+z)^2} \\
& : \frac{2c^2S_Cz^2 + c^4xy - (S^2 + 2S_{AB} - S_{BB})yz - (S^2 + 2S_{AB} - S_{AA})zx}{-4S_{AB}(x+y+z)^2} \\
= & S_A(2a^2S_Ax^2 + a^4yz - (S^2 + 2S_{BC} - S_{CC})zx - (S^2 + 2S_{BC} - S_{BB})xy) \\
& : S_B(2b^2S_By^2 + b^4zx - (S^2 + 2S_{CA} - S_{AA})xy - (S^2 + 2S_{CA} - S_{CC})yz) \\
& : S_C(2c^2S_Cz^2 + c^4xy - (S^2 + 2S_{AB} - S_{BB})yz - (S^2 + 2S_{AB} - S_{AA})zx)
\end{aligned}$$

The point P lies on the circumcircle of $\mathbf{T}'(P)$ (which is similar to $\mathbf{T}'$) if and only if

$$\begin{aligned}0 =&\ a'^2y'z' + b'^2z'x' + c'^2x'y'\\ =&\ a^4S_{AA}y'z' + b^4S_{BB}z'x' + c^4S_{CC}x'y'\\ =&\ 2S_{ABC}\cdot S^2(x+y+z)^2|HP|^2\left(\sum_{\text{cyclic}} a^4(S^2-3S_{AA})yz\right).\end{aligned}$$

There are two possibilities.

(i) $|HP| = 0 \implies P = H$. In this case, $A'_H = B'_H = C'_H = O$.

(ii) P lies on the circumconic $\sum_{\text{cyclic}} a^4(S^2 - 3S_{AA})yz = 0$, which is the isogonal conjugate (in $\mathbf{T}$) of the perpendicular bisector of the segment OH. $\square$

5. Triangle centers of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ and $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$

Consider the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$. Note that A_O, B_O, C_O are not the circumcenters of the triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ respectively. Indeed, these three points coincide with O: $A_O = B_O = C_O = O$. Instead, the circumcenters of triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ are the points

$$\begin{aligned}O_a =&\ (a^4(S^2-S_{AA}) :\ b^2(2S^2\cdot S_B - S_C(S^2-S_{BB})) :\ c^2(2S^2\cdot S_C - S_B(S^2-S_{CC}))),\\ O_b =&\ (a^2(2S^2\cdot S_A - S_C(S^2-S_{AA})) :\ b^4(S^2-S_{BB}) :\ c^2(2S^2\cdot S_C - S_A(S^2-S_{CC}))),\\ O_c =&\ (a^2(2S^2\cdot S_A - S_B(S^2-S_{AA})) :\ b^2(2S^2\cdot S_B - S_A(S^2-S_{BB})) : c^4(S^2-S_{CC})).\end{aligned}$$

These form the vertices of $\mathbf{T}(P)$ for

$$P = X(1147) = (a^4S_A(S^2-S_{AA}) :\ b^4S_B(S^2-S_{BB}) :\ c^4S_C(S^2-S_{CC})),$$

which, according to the ENCYCLOPEDIA OF TRIANGLE CENTERS [2], is the midpoint of O and $X(155)$, the orthocenter of the tangential triangle (see Figure 5). The three circumcenters are concyclic with $X(1147)$ (see Theorem 5). The center of the circle containing them is $X(156)$, the nine-point center of the tangential triangle

More generally, let P be a triangle center of ABC, with coordinates expressed in terms of a, b, c. The same triangle center P_a of $\mathbf{T}_a = AA_bA_c$ is the point with coordinates in which a, b, c are replaced by a', b', c' respectively (likewise S_A, S_B,

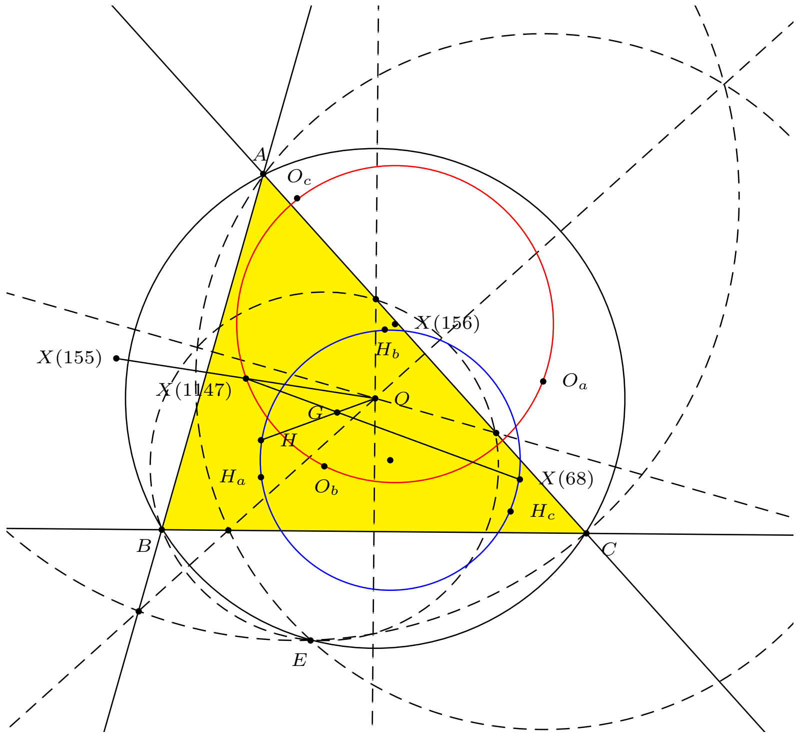

Figure 5.

S_C by S'_A, S'_B, S'_c respectively). For example, for orthocenters,

$$\begin{aligned}
H_a &= \left(\frac{1}{S'_A} : \frac{1}{S'_B} : \frac{1}{S'_C}\right) \\
&= \left(\frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}}\right) \text{ with respect to } \mathbf{T}_a \\
&= \frac{S_A}{S^2 - S_{AA}} \cdot (1,0,0) + \frac{S_B}{S^2 - S_{BB}} \cdot \frac{(0, -S_A + S_B, S_A + S_B)}{2S_B} \\
&\quad + \frac{S_C}{S^2 - S_{CC}} \cdot \frac{(0, S_C + S_A, S_C - S_A)}{2S_C} \text{ in absolute barycentric coordinates} \\
&= ((S^2 - S_{BB})(S^2 - S_{CC}) : a^2 S_C (S^2 - S_{AA}) : a^2 S_B (S^2 - S_{AA}))
\end{aligned}$$

with respect to $\mathbf{T}$.

The orthocenters H_a, H_b, H_c are the vertices of $\mathbf{T}(Q)$ for

$$Q = X(68) = \left(\frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}}\right).$$

The triangle center $X(68)$ is the superior of $X(1147)$. It lies on the circle containing the three orthocenters (see Figure 5 and Theorem 5). The circle $H_aH_bH_c$ also contains the orthocenter H.

The centroids G_a, G_b, G_c are the points

$$\begin{aligned} G_a =&\ (2S_{BC} :\ 2S_{BC} + S_A(S_B - S_C) :\ 2S_{BC} - S_A(S_B - S_C)),\\ G_b =&\ (2S_{CA} - S_B(S_C - S_A) :\ 2S_{CA} :\ 2S_{CA} + S_B(S_C - S_A)),\\ G_c =&\ (2S_{AB} + S_C(S_A - S_B) :\ 2S_{AB} - S_C(S_A - S_B) :\ 2S_{AB}). \end{aligned}$$

In this case, $G_aG_bG_c = \mathbf{T}(G)$.

On the other hand, since the triangles $\mathbf{T}'_a = AB_aC_a$, $\mathbf{T}'_b = A_bBC_b$, $\mathbf{T}'_c = C_aC_bC$ are similar to ABC, we have

$$P'_a = A'_P, \qquad P'_b = B'_P, \qquad P'_c = C'_p.$$

For example, the circumcenters are

$$\begin{aligned} O'_a = A'_O =&\ (3a^2S_{AA} - S_A(S_B - S_C)^2 - a^2S_{BC} :\ b^2c^2S_C :\ b^2c^2S_B),\\ O'_b = B'_O =&\ (c^2a^2S_C :\ 3b^2S_{BB} - S_B(S_C - S_A)^2 - b^2S_{CA} :\ c^2a^2S_A),\\ O'_c = C'_O =&\ (a^2b^2S_B :\ a^2b^2S_A :\ 3c^2S_{CC} - S_C(S_A - S_B)^2 - c^2S_{AB}). \end{aligned}$$

$$\begin{aligned} G'_a =&\ (2b^2 + 2c^2 - 3a^2 :\ b^2 :\ c^2),\\ G'_b =&\ (a^2 :\ 2c^2 + 2a^2 - 3b^2 :\ c^2),\\ G'_c =&\ (a^2 :\ b^2 :\ 2a^2 + 2b^2 - 3c^2). \end{aligned}$$

$H'_a = H'_b = H'_c = O$.

6. Orthology with T and pedal triangles

6.1. *The triangle $O_aO_bO_c$.*

Proposition 8. *The triangle $O_aO_bO_c$ is orthologic to ABC.*
(a) *The perpendiculars from O_a to BC, O_b to CA, and O_c to AB concur at the triangle center* [1]

$$Y_{(1)} := \left(a^2(S_{AA}(S_{BB} + S_{BC} + S_{CC}) - S_{BB}S_{CC}) : \cdots : \cdots\right).$$

(b) *The perpendiculars from A to O_bO_c, B to O_cO_a, and C to O_aO_b concur at the triangle center $X(74)$ on the circumcircle of $\mathbf{T}$.*

Remark. The orthology center $Y_{(1)}$ lies on the circle $O_aO_bO_c$.

Proposition 9. *The triangle $O_aO_bO_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*
(a) *the perpendiculars from O_a, O_b, O_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$16S^2 \cdot S_{ABC} \sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz - (x + y + z)(f_ax + f_by + f_cz) = 0,$$

[1]The triangle center $Y_{(1)}$ does not appear in the current edition of [2]. It has (6-9-13)-search number $-5.64011769173\cdots$.

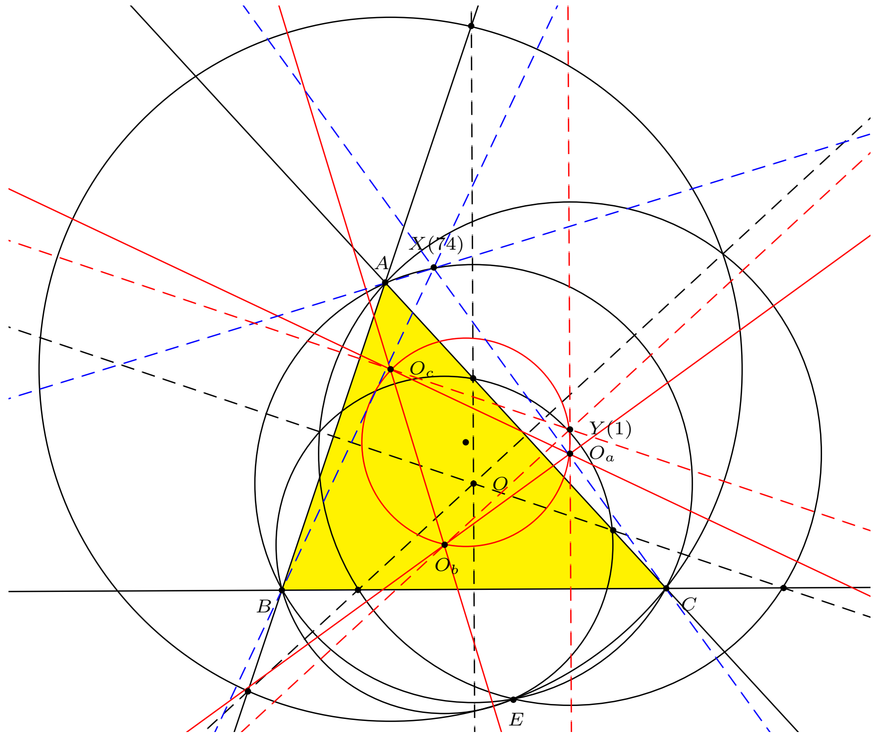

Figure 6

where

$$f_a = b^2c^2(S_B - S_C)(a^4S_A^3 + a^2S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - (8S_A + S_B + S_C)S_{BB}S_{CC}),$$

and f_b, f_c *are defined cyclically,*
(b) *the perpendiculars from the vertices of the pedal triangle of* P *to the corresponding sides of* $O_aO_bO_c$ *are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)}{a^2S_{AA} - S_A(S_B - S_C)^2 - a^2S_{BC}}x = 0.$$

6.2. *The triangle* $G_aG_bG_c$.

Proposition 10. *The triangle* $G_aG_bG_c$ *is orthologic to* ABC.
(a) *The perpendiculars from* G_a *to* BC, G_b *to* CA, *and* G_c *to* AB *concur at the triangle center* [2]

$$Y_{(2)} := (a^2(S_{BB} + S_{BC} + S_{CC})S_{AA} + S_{BB}S_{CC}(2S_A - S_B - S_C) : \cdots : \cdots).$$

[2]The triangle center $Y_{(2)}$ does not appear in the current edition of [2]. It has (6-9-13)-search number $-5.65228493146\cdots$.

(b) *The perpendiculars from A to G_bG_c, B to G_cG_a, and C to G_aG_b concur at the triangle center*

$$X(1294) = \left(\frac{1}{S_{AA}(S_{BB} - S_{BC} + S_{CC}) - S_{BB}S_{CC}} : \cdots : \cdots\right)$$

on the circumcircle.

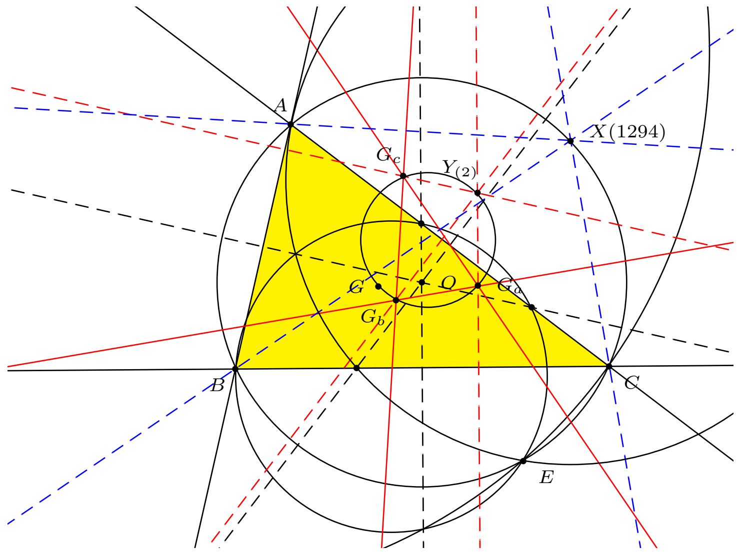

Figure 7

Remark. The circle $G_aG_bG_c$ contains the centroid G and the orthology center $Y_{(2)}$.

Proposition 11. *The triangle $G_aG_bG_c$ is orthologic with the pedal triangle of P if and only if P lies on the line*

$$\sum_{\text{cyclic}} b^2c^2(S_B - S_C)(a^2S_A - S_{BC})x = 0,$$

which passes through O and $X(64) = \left(\frac{a^2}{a^2S_A - S_{BC}} : \frac{b^2}{b^2S_B - S_{CA}} : \frac{c^2}{c^2S_C - S_{AB}}\right)$.

If P lies on this line,

(a) *the perpendiculars from G_a, G_b, G_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$6S_{ABC} \sum_{\text{cyclic}} (S_B - S_C)(a^2S_A - S_{BC})yz$$
$$- (x + y + z)\left(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BB}S_{CC})x\right) = 0.$$

(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $G_aG_bG_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)(a^2S_A - S_{BC})}{a^2S_A - 2S_{BC}}x = 0.$$

6.3. *The triangle $H_aH_bH_c$.*

Proposition 12. *The triangle $H_aH_bH_c$ is orthologic to ABC.*
(a) *The perpendiculars from H_a to BC, H_b to CA, and H_c to AB concur at the orthocenter H of $\mathbf{T}$.*
(b) *The perpendiculars from A to H_bH_c, B to H_cH_a, and C to H_aH_b concur at*

$$X(1300) = \left(\frac{1}{S_A(a^2(S_{AA} - S_{BC}) - S_A(S_B - S_C)^2)} : \cdots : \cdots\right)$$

on the circumcircle of $\mathbf{T}$.

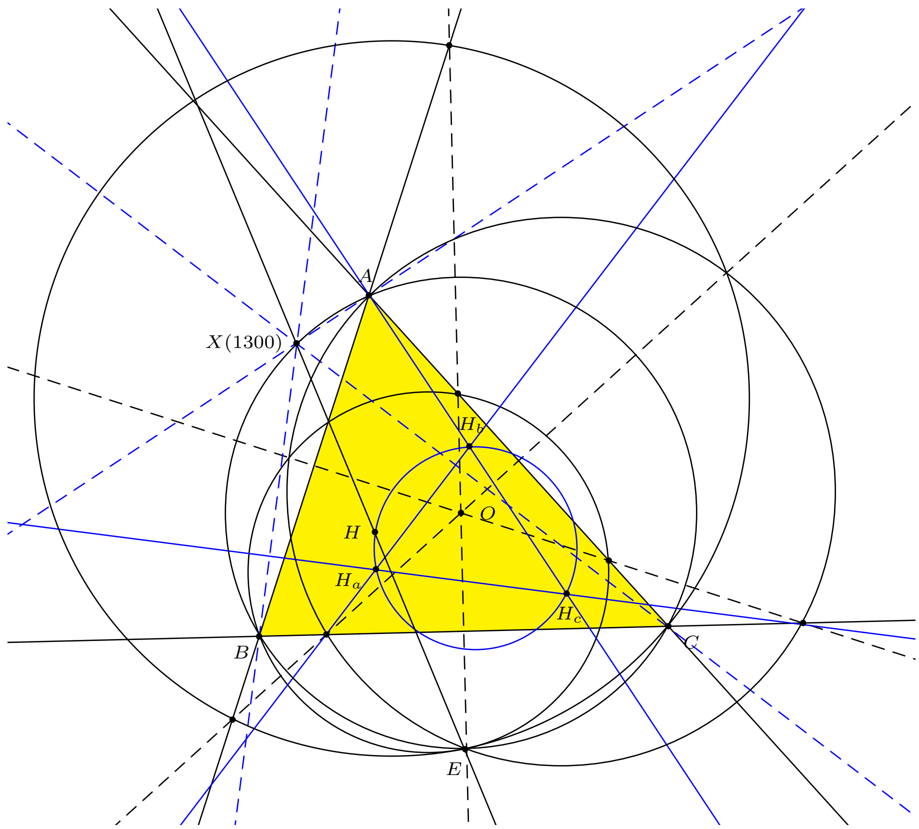

Figure 8

Remark. $X(1300)$ is the second intersection of the circumcircle with the line EH.

Proposition 13. *The triangle $H_aH_bH_c$ is orthologic with the pedal triangle of P if and only if P lies on the line*

$$\frac{S_B - S_C}{a^2 S_A}x + \frac{S_C - S_A}{b^2 S_B}y + \frac{S_A - S_B}{c^2 S_C}z = 0$$

joining the circumcenter O to $X(394) = (a^2 S_{AA} : b^2 S_{BB} : c^2 S_{CC})$.

If P lies on this line,

(a) *the perpendiculars from H_a, H_b, H_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$4S^2 \sum_{\text{cyclic}} S_{BC}(S_B - S_C)yz + (x+y+z)\left(\sum_{\text{cyclic}} a^2 S_A(S_B - S_C)(S^2 - S_{AA})x\right) = 0,$$

with center at the nine-point center N;

(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $H_aH_bH_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{S_A(a^4 S_{AA} - a^2 S_A(S_{BB} + S_{CC}) - S_{BC}(S_B - S_C)^2)}x = 0.$$

6.4. *The triangle $O'_aO'_bO'_c$.*

Proposition 14. *The triangle $O'_aO'_bO'_c$ is orthologic to ABC.*

(a) *The perpendiculars from O'_a to BC, O'_b to CA, and O'_c to AB concur at the orthocenter H of $\mathbf{T}$.*

(b) *The perpendiculars from A to $O'_bO'_c$, B to $O'_cO'_a$, and C to $O'_aO'_b$ concur at the triangle center*

$$X(265) = \left(\frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}}\right)$$

(see Proposition 6).

Proposition 15. *The triangle $O'_aO'_bO'_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*

(a) *the perpendiculars from O'_a, O'_b, O'_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$\begin{aligned} &4S^2 \sum_{\text{cyclic}} a^2 S_A(S_B - S_C)yz \\ &+ a^2b^2c^2(x+y+z)(S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z) = 0, \end{aligned}$$

with center N,

(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $O'_aO'_bO'_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{a^2 S_A - 2S_{BC}}x = 0.$$

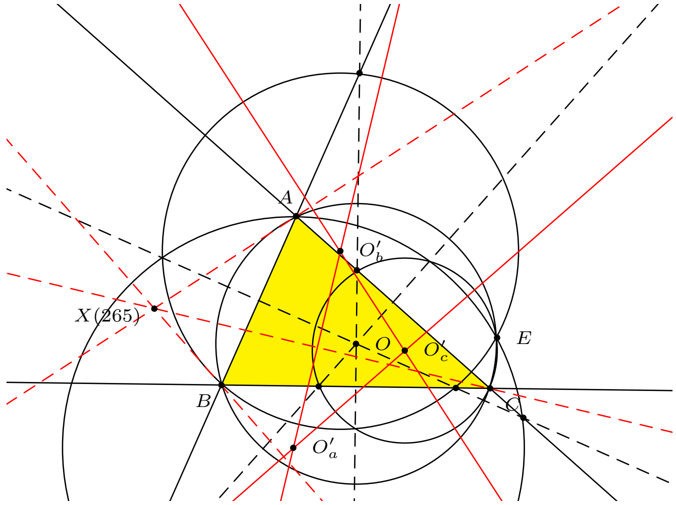

Figure 9

6.5. *The triangle $G'_aG'_bG'_c$.*

Proposition 16. *The triangle $G'_aG'_bG'_c$ is orthologic to ABC.*
(a) *The perpendiculars from G'_a to BC, G'_b to CA, and G'_c to AB concur at the triangle center*

$$X(381) = \left(a^2S_A + 4S_{BC} : \; b^2S_B + 4S_{CA} : \; c^2S_C + 4S_{AB}\right)$$

on the Euler line.
(b) *The perpendiculars from A to $G'_bG'_c$, B to $G'_cG'_a$, and C to $G'_aG'_b$ concur at the triangle center*

$$X(265) = \left(\frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}}\right)$$

(see Proposition 6).

Proposition 17. *The triangle $G'_aG'_bG'_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*
(a) *the perpendiculars from G'_a, G'_b, G'_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$6\sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(S_B - S_C)x\right) = 0$$

with center at the centroid G,
(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $G'_aG'_bG'_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{a^2S_A - 2S_{BC}}x = 0.$$

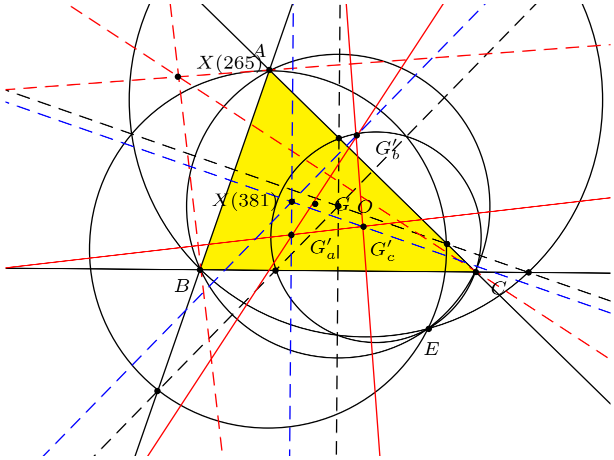

Figure 10

7. The triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$

Proposition 18. *The triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$ are perspective if and only if the line PQ contains the circumcenter O. In this case, the triangles are homothetic at O.*

Proof. Let $P = (x : y : z)$ and $Q = (u : v : w)$. The line containing A_P and A_Q has equation

$$\begin{aligned} a^2S_A(wy - vz)\mathbb{X} + \; &(c^2S_C(vx - uy) - S_B(S_C - S_A)(uz - wx))\mathbb{Y} \\ &+ \; (b^2S_B(uz - wx) + S_C(S_A - S_B)(vx - uy))\mathbb{Z} = 0. \end{aligned}$$

Similarly, we have the equations of the lines B_PB_Q and C_PC_Q. The three lines are concurrent if and only if $f \cdot g = 0$, where

$$\begin{aligned} f :=\; &(c^2S_Cv - b^2S_Bw)x + (a^2S_Aw - c^2S_Cu)y + (b^2S_Bu - a^2S_Av)z, \\ g :=\; &(u + v + w)^2 \left(\sum_{\text{cyclic}} a^4S_{AA}yz \right) \\ &- (x + y + z) \left(\sum_{\text{cyclic}} ((c^2S_Cv + b^2S_Bw)^2 - 4S_{ABC} \cdot S_Avw)x \right). \end{aligned}$$

Note that
(i) the equation $f = 0$ represents the line OQ;
(ii) the equation $g = 0$ represents a conic with center Q. Since the conic also contains Q and intersects the sidelines at imaginary points, its represents only the point Q.

From these we conclude that the triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$ are perspective if and only if O, P, Q are collinear. Each of the lines A_PA_Q, B_PB_Q, C_PC_Q contains O. The perspector is O. □

Theorem 19. *The triangle bounded by the lines A_PA_Q, B_PB_Q, C_PC_Q has circumcenter O.*

Proof. Since O is the orthocenter of the triangle $\mathbf{T}(P)$, the circles OB_PC_P, OC_PA_P, and OA_PB_P have equal radii. Note that O is also the incenter of each of triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$, and these triangles are all similar to $\mathbf{T}'$. Therefore, $\angle OA_PA_Q = \angle OB_PB_Q = \angle OC_PC_Q$. Let $A''B''C''$ be the triangle bounded by the lines A_PA_Q, B_PB_Q and C_PC_Q. Applying the law of sines to triangles $OA''C_P$, $OB''A_P$, $OC''B_P$, we conclude that $OA'' = OB'' = OC''$. □

Proposition 20. *The triangles $\mathbf{T}'(P)$ and $\mathbf{T}'(Q)$ are perspective if and only if Q lies on the line HP. If the condition is satisfied, the triangles are homothetic at O.*

8. The triangles $\mathbf{T}(P)$ and $\mathbf{T}'(P)$

Theorem 21. *The triangles $\mathbf{T}(P)$ and $\mathbf{T}'(P)$ are perspective if and only if P lies on*
(a) *the Jerabek hyperbola or*
(b) *the line* $\sum_{\text{cyclic}} \frac{x}{S_A(S_B-S_C)} = 0$.

Remarks. (1) If P lies on the Jerabek hyperbola, the perspector is the circumcenter O.

(2) The line in (b) contains the Jerabek center $X(125)$ and $X(122)$ (also $X(684)$, $X(1650)$, $X(2972)$). In this case the lines $A_PA'_P$, $B_PB'_P$, $C_PC'_P$ are parallel.

Theorem 22. *Let P be a point with coordinates $(x:y:z)$ in $\mathbf{T}$.*
(a) *The circumcenter of $\mathbf{T}(P)$ has coordinates $(x:y:z)$ in triangle $O'_aO'_bO'_c = \mathbf{T}'(O) = A'_OB'_OC'_O$.*
(b) *The circumcenter of $\mathbf{T}'(P)$ has coordinates $(x:y:z)$ in $O_aO_bO_c$, which is $\mathbf{T}(Q)$ for*

$$Q = X(1147) = (a^4S_A(S^2-S_{AA}) : \ b^4S_B(S^2-S_{BB}) : \ c^4S_C(S^2-S_{CC})).$$

Proof. (a) The point with coordinates $(x:y:z)$ with respect to $O'_aO'_bO'_c$ is

$$\begin{aligned}\overline{O} = (&S_{BC}(4S^2\cdot S_A - a^2b^2c^2)x + c^2a^2S_{CCA}y + a^2b^2S_{ABB}z\\ :\ &b^2c^2S_{BCC}x + S_{CA}(4S^2\cdot S_B - a^2b^2c^2)y + a^2b^2S_{AAB}z\\ :\ &b^2c^2S_{BBC}x + c^2a^2S_{CAA}y + S_{AB}(4S^2\cdot S_C - a^2b^2c^2)z).\end{aligned}$$

Its square distance from A_P is

$$\frac{a^2b^2c^2Q(x,y,z)}{16S^2\cdot(S_{ABC})^2(x+y+z)^2},$$

where

$$\begin{aligned}Q(x,y,z) = {}&b^2c^2S_{BB}S_{CC}x^2 + c^2a^2S_{CC}S_{AA}y^2 + a^2b^2S_{AA}S_{BB}z^2\\ &- 2S_{ABC}(a^2S_{AA}yz + b^2S_{BB}zx + c^2S_{CC}xy).\end{aligned}$$

The symmetry of Q in S_A, S_B, S_C and x, y, z shows that this is the same if A_P is replaced by B_P or C_P. The point $\overline{O}$ therefore is the circumcenter of $A_PB_PC_P$. □

Theorem 23. (a) *Triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the circumcenter O of triangle ABC.*
(b) *The six circumcenters O_a, O_b, O_c, O'_a, O'_b, O'_c are concyclic. The center of the circle containing them is $X(156)$, the nine-point center of the tangential triangle.*

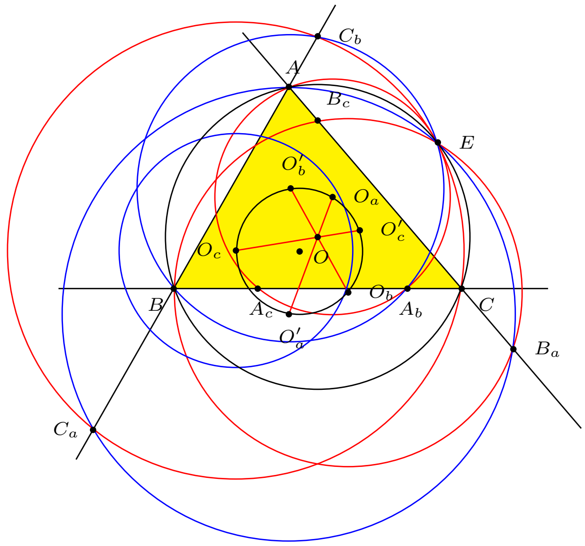

Figure 11

Proof. With respect to $O_aO_bO_c$,

$$O'_a = (-a^2S_{BC} : S_C(a^2S_A + 2S_{BC}) : S_B(a^2S_A + 2S_{BC})) = \left(\frac{-a^2}{a^2S_A + 2S_{BC}} : \frac{b^2}{b^2S_B} : \frac{c^2}{c^2S_C}\right),$$

$$O'_b = (S_C(b^2S_B + 2S_{CA}) : -b^2S_{BC} : S_A(b^2S_B + 2S_{CA})) = \left(\frac{a^2}{a^2S_A} : \frac{-b^2}{b^2S_B + 2S_{CA}} : \frac{c^2}{c^2S_C}\right),$$

$$O'_c = (S_B(c^2S_C + 2S_{AB}) : S_A(c^2S_C + 2S_{AB}) : -c^2S_{BC}) = \left(\frac{a^2}{a^2S_A} : \frac{b^2}{b^2S_B} : \frac{-c^2}{c^2S_C + 2S_{AB}}\right).$$

These expressions show that
(a) triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the orthocenter of $O_aO_bO_c$,
(b) O'_a, O'_b, O'_c are on the the circumcircle of $O_aO_bO_c$. $\square$

The orthocenter of $O_aO_bO_c$ is the circumcenter O. The circumcenter is $X(156)$, which is the nine-point center of the tangential triangle.

Proposition 24. *The triangle $O_aO_bO_c$ is perspective with the inferior triangle of the tangential triangle at a point on its circumcircle.*

Proof. With respect to $O_aO_bO_c$, the midpoint of the A-side of the tangential triangle has coordinates

$$\begin{aligned}&(a^2S_{AA}(2S_{BB} - 7S_{BC} + 2S_{CC}) + S_{ABC}(3S_{BB} - 2S_{BC} + 3S_{CC}) + a^2S_{BB}S_{CC})\\ &: a^2b^2S_A(c^2S_C - 2S_{AB}) : c^2a^2S_A(b^2S_B - 2S_{CA}));\end{aligned}$$

similarly for the midpoints of the other two sides of the tangential triangle. From these coordinates, it is clear that the medial triangle of the tangential triangle is perspective to $O_aO_bO_c$ at the point with coordinates

$$\left(\frac{a^2}{a^2S_A-2S_{BC}}:\ \frac{b^2}{b^2S_B-2S_{CA}}:\ \frac{c^2}{c^2S_C-2S_{AB}}\right).$$

Since triangle $O_aO_bO_c$ is similar to $\mathbf{T}$, this perspector is a point on the circumcircle of $O_aO_bO_c$. It is the isogonal conjugate (with respect to $O_aO_bO_c$) of the infinite point of the Euler line of the triangle. □

Remark. With respect to $\mathbf{T}$, this perspector has coordinates

$$(a^2(S_{AA}(S_{BB}+S_{BC}+S_{CC})-S_{BB}S_{CC}):\cdots:\cdots).$$

See Proposition 8.

Proposition 25. *The triangle $O'_aO'_bO'_c$ is perspective with the tangential triangle at*

$$X(195)=(a^2(-3a^2S_A^3+(5S_{BB}+6S_{BC}+5S_{CC})S_{AA}+9a^2S_{ABC}+4S_{BB}S_{CC}):\cdots:\cdots).$$

Remark. $X(195)$ is the circumcenter of the triangle of reflections (see [3]).

9. A family of circumcircles of $\mathbf{T}(P)$ containing the Euler reflection point

Proposition 26. *Let $P=(x:y:z)$. The circumcircle of $\mathbf{T}(P)=A_PB_PC_P$ contains the Euler reflection point E if and only if P lies on the conic*

$$a^4S_{AA}yz+b^4S_{BB}zx+c^4S_{CC}xy=0.$$

Proof. The coordinates of A_P, B_P, C_P are given at the beginning of §3. Computing the coordinates of E with respect to triangle $A_PB_PC_P$, we have

$$\begin{aligned}
&x':y':z'\\
=&S_{BC}\left(\mu\left(\sum_{\text{cyclic}}a^4S_{AA}yz\right)+\tau(x+y+z)\left(\frac{S_{BC}x}{S_B-S_C}+\frac{2S_{AAC}y}{b^2(S_A-S_B)}+\frac{2S_{AAB}z}{c^2(S_C-S_A)}\right)\right)\\
&:S_{CA}\left(\mu\left(\sum_{\text{cyclic}}a^4S_{AA}yz\right)+\tau(x+y+z)\left(\frac{2S_{BBC}x}{a^2(S_A-S_B)}+\frac{S_{CA}y}{S_C-S_A}+\frac{2S_{ABB}z}{c^2(S_B-S_C)}\right)\right)\\
&:S_{AB}\left(\mu\left(\sum_{\text{cyclic}}a^4S_{AA}yz\right)+\tau(x+y+z)\left(\frac{2S_{BCC}x}{a^2(S_C-S_A)}+\frac{2S_{CCA}y}{b^2(S_B-S_C)}+\frac{S_{AB}z}{S_A-S_B}\right)\right)
\end{aligned}$$

where

$$\begin{aligned}
\mu=&\ a^2S_{BC}+b^2S_{CA}+c^2S_{AB}-6S_{ABC},\\
\tau=&\ (S_{BB}-S_{CC})(S_{CC}-S_{AA})(S_{AA}-S_{BB}).
\end{aligned}$$

This is a point on the circumcircle of triangle $A_PB_PC_P$ (which is similar to ABC) if and only if

$$(S_B+S_C)y'z'+(S_C+S_A)z'x'+(S_A+S_B)x'y'=0.$$

This reduces to

$$\mu^2 \cdot \left(\sum_{\text{cyclic}} a^4 S_{AA} yz\right) \cdot G = 0,$$

where

$$\begin{aligned} G =& \sum_{\text{cyclic}} (b^2c^2 S_{BB}S_{CC}x^2 - 2S_{ABC} \cdot a^2 S_{AA} yz) \\ =& \; S^2 \left(\sum_{\text{cyclic}} S_{BC}^2 x^2 - 2S_{CA}S_{AB} yz\right) + S_{ABC}^2 (x+y+z)^2 \end{aligned}$$

$G = 0$ is the equation of a conic with center O. Since $G(O) = 0$, we conclude that $G = 0$ defines only the point O. Therefore, the circle $A_P B_P C_P$ contains E if and only if P lies on the circumconic $\sum_{\text{cyclic}} a^4 S_{AA} yz$. □

Theorem 27. *Let Q be a triangle center on the circumcircle. The circle $Q_a Q_b Q_c$ contains the Euler reflection point E.*

Proof. Let $Q = \left(\frac{a^2}{(b^2-c^2)(a^2+t)} : \frac{b^2}{(c^2-a^2)(b^2+t)} : \frac{c^2}{(a^2-b^2)(c^2+t)}\right)$ be a triangle center on the circumcircle. For

$$P = \left(\frac{a'^2}{(b'^2 - c'^2)(a'^2+t)} : \frac{b'^2}{(c'^2 - a'^2)(b'^2+t)} : \frac{c'^2}{(a'^2 - b'^2)(c'^2+t)}\right)$$

we have $Q_a = A_P$, $Q_b = B_P$, $Q_c = C_P$. The circle $Q_a Q_b Q_c$ is the same as $A_P B_P C_P$. With respect to triangle $O'_a O'_b O'_c$, the center of $A_P B_P C_P$ has coordinates given above. Therefore the locus of the center of $Q_a Q_b Q_c$ is the circle $O'_a O'_b O'_c$, which is the nine-point circle of the tangential triangle of triangle ABC.

Note that P lies on the conic $\sum_{\text{cyclic}} a^4 S_{AA} yz = 0$. By Proposition 26, the circle $A_P B_P C_P$ contains the Euler reflection point E. □

Theorem 28. *The circumcircle of $\mathbf{T}'(P)$ contains the Euler reflection point if and only if P lies on the circumcircle.*

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Archimedean Circles Related to the Schoch Line

Hiroshi Okumura

Abstract. We give several Archimedean circles of the arbelos related to the Schoch line.

Let us consider an arbelos consisting of three semicircles α, β and γ with diameters AO, BO and AB, respectively, where O is a point on the segment AB. Thomas Schoch has considered the circles α' and β' with centers A and B and passing through the point O. He has found that the circle touching the two circles externally and the semicircle γ internally is Archimedean [1]. The perpendicular to AB from the center of this circle is called the Schoch line (see Figure 1). In this note we give several Archimedean circles touching this line or one of the two circles.

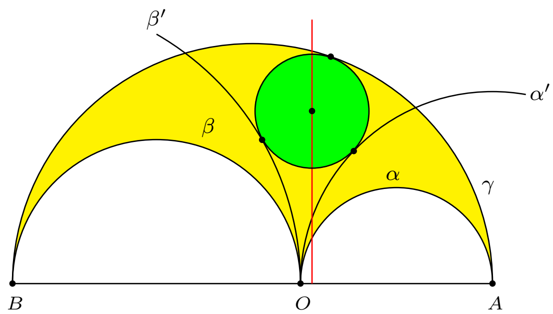

Figure 1

Let a and b be the radii of α and β, respectively. The radius of Archimedean circles is $\frac{ab}{a+b}$, which is denoted by r_A. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that all the semicircles are constructed in the region $y > 0$. Let $s = r_A \cdot \frac{b-a}{b+a}$. The Schoch line is expressed by the equation $x = s$ (see [2]). If $b > a$, then $s > 0$, and the Schoch line intersects the semicircle α.

Let O_α and O_β be the centers of α and β respectively, and μ the circle with $O_\alpha O_\beta$ as diameter (see Figure 2).

Theorem 1. (1) *The two circles each touching the circle* μ *and* α (respectively β), *all externally, and the Schoch line from the side opposite to* B (respectively A) *are Archimedean.*

(2) *If* $b > a$, *then the circle touching the circle* μ *internally,* α *externally, and the Schoch line from the side opposite to* A *is Archimedean.*

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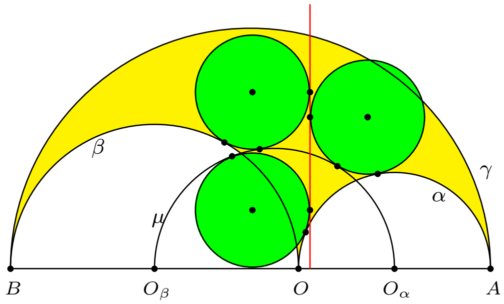

Figure 2

Proof. Let x be the radius of the circle touching the circle μ and α externally, and the Schoch line from side opposite to B in (1). By the Pythagorean theorem,

$$(a+x)^2 - (s+x-a)^2 = \left(\frac{a+b}{2}+x\right)^2 - \left(s+x-\frac{a-b}{2}\right)^2.$$

Solving the equation for x, we get $x = r_A$, i.e., the circle is Archimedean. The rest of the theorem is proved similarly. □

Let $\mathcal{L}_\alpha$ be the perpendicular to AB from the point of intersection of γ and α'. The line $\mathcal{L}_\beta$ is defined similarly. Each of the two lines touches one of the twin circles of Archimedes [1] (see Figure 3). The proof of the following theorem is similar to that of Theorem 1, and is omitted.

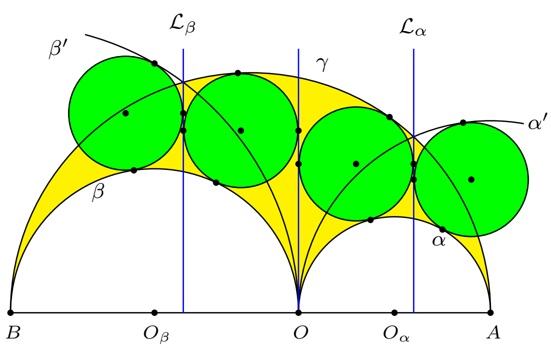

Figure 3

Theorem 2. *Each of the circles touching α externally, α' internally, and $\mathcal{L}_\alpha$ from the side opposite to O* (respectively β, β' and $\mathcal{L}_\beta$) *is Archimedean.*

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Optimal Packings of Two Ellipses in a Square

Thierry Gensane and Pascal Honvault

Abstract. For each real number E in $]0,1]$, we describe the densest packing $\mathcal{P}_E$ of two non-overlapping congruent ellipses of aspect ratio E in a square. We find three different patterns as E belongs to $]0,1/2]$, $[1/2,E_0]$ where $E_0 = \sqrt{(6\sqrt{3}-3)/11}$, and $[E_0,1]$. The technique of unavoidable sets – used by Friedman for proving the optimality of square packings – allows to prove the optimality of each packing $\mathcal{P}_E$.

1. Introduction

We consider the following generalization of the classical disk packing problem in a compact convex domain K: Let $n \in \mathbb{N}$ and $E \in]0,1]$, what is the densest packing of n non-overlapping congruent ellipses of aspect ratio E in K?

In this paper, we describe for each aspect ratio E in $]0,1]$, the densest packing $\mathcal{P}_E$ of two congruent unit ellipses of aspect ratio E in the square $K = [0,1]^2$ and we prove the optimality of these packings. In Figure 1, we display six representative optimal packings of two congruent ellipses. For $E = 1$, the optimal packing $\mathcal{P}_1$ is composed of two disks lying in opposite corners, see [4] for a large list of dense packings of congruent disks in the square. An introductory bibliography on disk packing problems can be found in [1, 3]. When E decreases from 1 to $E_0 = \sqrt{(6\sqrt{3}-3)/11} \approx 0.8198$, the ellipses of optimal packings $\mathcal{P}_E$ flatten by keeping a constant tilted angle equals to $-\pi/4$. For $E \in [1/2, E_0]$, the angle of the two ellipses of $\mathcal{P}_E$ decreases and when $E = 1/2$ the ellipses reach a third side of the square. When E decreases from $1/2$ to 0, the ellipses slide along the sides and move towards the diagonal.

In all the following we consider only *unit ellipses* that is, ellipses whose equation is $x^2 - y^2/E^2 = 1$ when their major and minor axes coincide with the cartesian axes. We can reformulate our problem: What is the side length $s_2(E)$ of the smallest square which contains two non-overlapping unit ellipses of aspect ratio E?

In order to prove the optimality of square packings, Friedman [2] used *sets of unavoidable points*. We adapt his definition to the case of ellipse packings: Let $E \in]0,1]$ and let P be a set of $n-1$ points in the square $K_s = [0,s]^2$ with $s > 0$. We say that P is a *set of unavoidable points* in K_s if any unit ellipse of aspect ratio E in K_s contains an element of P (possibly on its boundary). If P is a set of unavoidable points in K_s, then $s_n(E) \geq s$. For the convenience of the reader, we recall the proof given by Friedman: Shrinking the square K_s by a factor of $1 - \varepsilon/s$ gives a set P' of $n-1$ points in a square $K_{s-\varepsilon}$ so that any unit ellipse in $K_{s-\varepsilon}$ contains an element of P' in its interior. Therefore no more than

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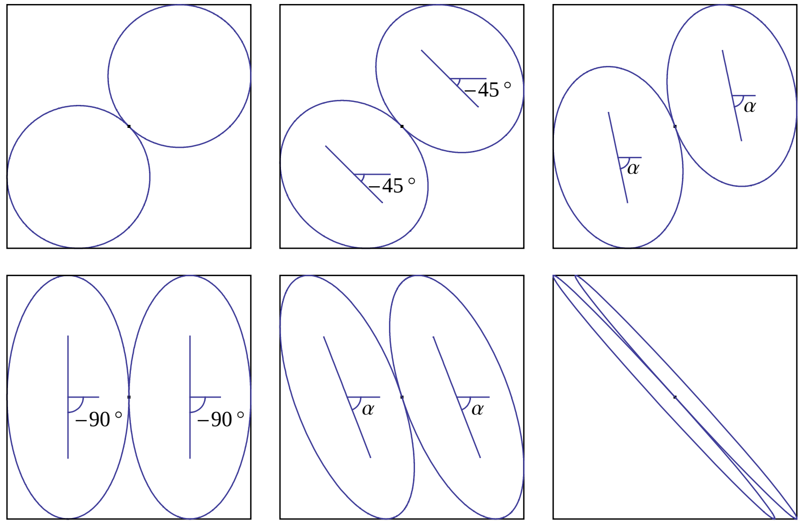


Figure 1. Six optimal packings $\mathcal{P}_E$ of two ellipses for $E = 1, 0.85, 0.69, 0.5, 0.4, 0.05$.

$n-1$ non-overlapping unit ellipses can be packed into a square of side $s-\varepsilon$, and $s_n(E) > s-\varepsilon$. Since this is true for all $\varepsilon > 0$, we must have $s_n(E) \geq s$. The upper bound $s_n(E) \leq s$ is obtained by constructing a packing of n non-overlapping unit ellipses in K_s.

In the case of $n = 2$ ellipses and in order to get $s_2(E) \geq s$, it suffices to show that the center Ω of $K_s = [0, s]^2$ belongs to each unit ellipse $\mathbf{e} \subset K_s$ of aspect ratio E. In fact, we will consider only unit ellipses $\mathbf{e}_\alpha = \mathbf{e}_{(\lambda,\mu),\alpha,E}$ centered at (λ, μ) with $\lambda > 0, \mu > 0$, tilted at an angle $\alpha \in [-\pi/2, -\pi/4]$ and which are tangent to the axes $x = 0$ and $y = 0$:

Fact 1. Let $K_s = [0, s]^2$ be a square of side length s and $\Omega = (s/2, s/2)$ its center. If for all $\alpha \in [-\pi/2, -\pi/4]$, the ellipse $\mathbf{e}_\alpha$ contains the point Ω, then all unit ellipses $\mathbf{e}$ included in K_s contain Ω.

This fact is trivially obtained by contraposition (if a unit ellipse $\mathbf{e} \subset K_s$ does not contain the point Ω, we apply a translation followed by a reflection or a rotation and we get an ellipse $\mathbf{e}_\alpha$ with $\alpha \in [-\pi/2, -\pi/4]$ which does not contain the point Ω). As we want to find the minimal value of s such that each ellipse $\mathbf{e}_\alpha \subset K_s$ contains the center Ω, we consider the intersection points $I = (x_I(\alpha), x_I(\alpha))$ and J of the diagonal $y = x$ and the ellipse $\mathbf{e}_\alpha$, the abscissa of I being larger than the one of J. In Section 2 and 3 we will prove that:

• If $0 < E \leq 1/2$, there exists a unique $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $x_I(\alpha_0) = \mu(\alpha_0) = \mu_0$. The center $\Omega = (\mu_0, \mu_0)$ is an unavoidable point in $K_{2\mu_0}$ and then $s_2(E) \geq 2\mu_0$. The square $K_{2\mu_0}$ is displayed on the right hand side of Figure 2.

• If $1/2 < E < E_0$, the abscissa $x_I(\alpha)$ has a minimum value for a unique $\alpha_0 \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(\alpha_0), x_I(\alpha_0))$ is an unavoidable point

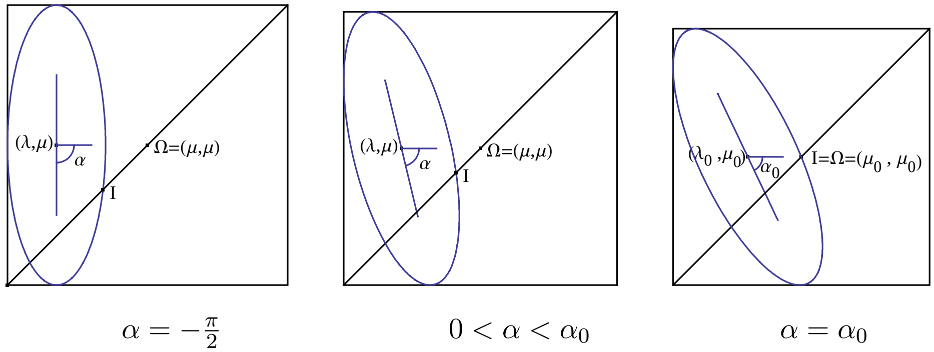


Figure 2. Three ellipses $\mathbf{e}_\alpha \subset K_{2\mu}$ in the case $E \leq \frac{1}{2}$

in $K_{2x_I(\alpha_0)}$ and then $s_2(E) \geq 2x_I(\alpha_0)$. The three squares in Figure 3 represent $K_{2x_I(\alpha_0)}$.

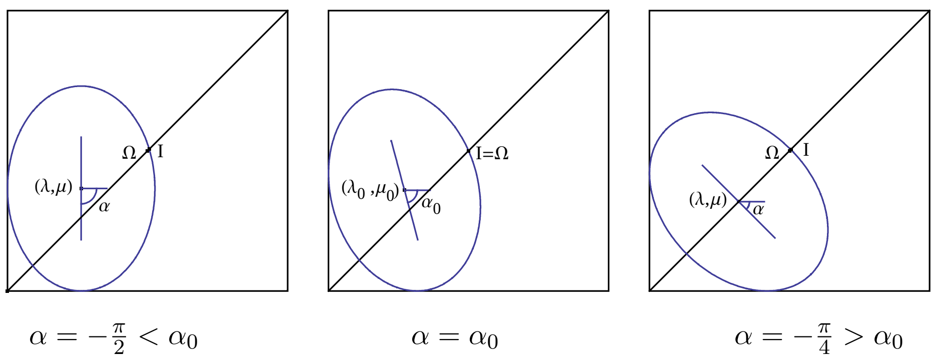


Figure 3. For $\frac{1}{2} < E < E_0$ and if $\alpha \neq \alpha_0$, the center of $K_{2x_I(\alpha_0)}$ belongs to the interior of $\mathbf{e}_\alpha$.

• If $E_0 \leq E \leq 1$, the abscissa $x_I(\alpha)$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(-\pi/4), x_I(-\pi/4))$ is an unavoidable point in $K_{2x_I(-\pi/4)}$ and then $s_2(E) \geq 2x_I(-\pi/4)$, see Figure 4 in Section 3.

We finish the paper by remarking that among all the optimal packings $\mathcal{P}_E$, the densest optimal packings of two congruent ellipses in the square is $\mathcal{P}_{1/2}$.

2. Technical lemmas.

First we precise the coordinates of the center of the ellipse $\mathbf{e}_\alpha$ and the parametrization of $\mathbf{e}_\alpha$ used in Lemma 3.

Lemma 1. *(a) The coordinates of the center of the ellipse* $\mathbf{e}_\alpha$ *are equal to*

$$\lambda = \sqrt{\cos^2\alpha + E^2\sin^2\alpha} \text{ and } \mu = \sqrt{\sin^2\alpha + E^2\cos^2\alpha}. \qquad (1)$$

The function μ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$ and we have $\mu(-\pi/2) = 1$ and $\mu(-\pi/4) = \sqrt{(1+E^2)/2}$.
(b) We have

$$(2\lambda\mu)^2 = 4E^2 + (1-E^2)^2 \sin^2 2\alpha. \tag{2}$$

(c) The ellipse $\mathbf{e}_\alpha$ can be parameterized by

$$\mathbf{e}_\alpha(t) = \begin{pmatrix} 2\lambda\cos^2\left(\frac{t+\varphi}{2}\right) \\ 2\mu\cos^2\left(\frac{t+\psi}{2}\right) \end{pmatrix}, \tag{3}$$

where the angles $\varphi \in [-\frac{\pi}{2}, 0]$ and $\psi \in [-\pi, -\frac{\pi}{2}]$ are the respective arguments of the complex numbers $\cos\alpha + iE\sin\alpha$ and $\sin\alpha - iE\cos\alpha$.

Proof. (a) Let us consider the parametrization of the ellipse $\mathbf{e}_\alpha$

$$\mathbf{e}_\alpha(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos t \\ E\sin t \end{pmatrix}. \tag{4}$$

Let us recall that the *orthoptic curve* of $\mathbf{e}_\alpha$, i.e the locus of all points where the curve's tangents meets at right angles, is the circle centered at $\omega = (\lambda, \mu)$ with radius $\sqrt{1+E^2}$. Since the axes $x = 0$ and $y = 0$ are orthogonal, the origin $(0,0)$ belongs to this circle and we have

$$\lambda^2 + \mu^2 = 1 + E^2. \tag{5}$$

The ellipse $\mathbf{e}_\alpha$ touches tangentially the axe $x = 0$. Therefore for some t we have $x(t) = x'(t) = 0$, which gives

$$\lambda = -\cos\alpha\cos t + E\sin\alpha\sin t, \tag{6}$$
$$0 = -\cos\alpha\sin t - E\sin\alpha\cos t. \tag{7}$$

By adding the squares of equations (6-7), we find $\lambda^2 = \cos^2\alpha + E^2\sin^2\alpha$. The value of μ comes from (5).
(b) We have

$$\begin{aligned}(\lambda\mu)^2 &= (\cos^2\alpha + E^2\sin^2\alpha)(\sin^2\alpha + E^2\cos^2\alpha) \\ &= E^2(\cos^4\alpha + \sin^4\alpha) + (1+E^4)\cos^2\alpha\sin^2\alpha \\ &= E^2(1 - 2\sin^2\alpha\cos^2\alpha) + (1+E^4)\cos^2\alpha\sin^2\alpha,\end{aligned}$$

which gives the result.
(c) By definition of φ, we have $\lambda(\cos\varphi + i\sin\varphi)$=$\cos\alpha + iE\sin\alpha$. We get

$$x(t) = \lambda + \cos\alpha\cos t - \sin\alpha E\sin t = \lambda + \lambda\cos(t+\varphi) = 2\lambda\cos^2\left(\frac{t+\varphi}{2}\right).$$

The expression of $y(t)$ is obtained similarly. □

In the proof of Lemma 3, we change the variable α to the variable T that we now define:

Lemma 2. *Let us consider the angles φ and ψ associated to the ellipse $\mathbf{e}_\alpha$ and defined in Lemma 1. We set $\delta = \frac{\psi-\varphi}{2} \in [-\frac{\pi}{4}, 0[$ and $T = -\cot\delta$. Then the real number T decreases monotonically from 1 at $\alpha = -\pi/2$ to E at $\alpha = -\pi/4$.*

Proof. The definitions of φ and ψ give

$$\begin{aligned}\frac{\mu}{\lambda}e^{i(\psi-\varphi)} &= \frac{\sin(\alpha)-iE\cos(\alpha)}{\cos\alpha+iE\sin\alpha}\\ &= \frac{1}{\lambda^2}\left(\sin\alpha\cos\alpha(1-E^2)-iE\right).\end{aligned}$$

Hence, for $\alpha \in [-\pi/2, -\pi/4]$ we have

$$\cos(\psi-\varphi) = \frac{\sin 2\alpha(1-E^2)}{2\lambda\mu} < 0 \text{ and } \sin(\psi-\varphi) = \frac{-E}{\lambda\mu} < 0. \tag{8}$$

By (8) and the formula $\tan u = (1-\cos 2u)/\sin 2u$, we find

$$\tan\left(\frac{\psi-\varphi}{2}\right) = \frac{1-\frac{\sin 2\alpha(1-E^2)}{2\lambda\mu}}{\frac{-E}{\lambda\mu}} = \frac{2\lambda\mu-\sin 2\alpha(1-E^2)}{-2E}. \tag{9}$$

With Lemma 1 (b) we find

$$\tan\left(\frac{\psi-\varphi}{2}\right) = \frac{1}{2E}\left(-\sqrt{4E^2+(1-E^2)^2\sin^2 2\alpha}+(1-E^2)\sin 2\alpha\right).$$

As $\sin 2\alpha$ decreases monotonically from 0 to -1 on the interval $[-\pi/2, -\pi/4]$, we find that $\tan((\psi-\varphi)/2)$ decreases from -1 to $-1/E$. Thus the real number $T = -1/\tan((\psi-\varphi)/2)$ decreases from 1 to E. □

Let us recall that $I = (x_I, x_I)$ is the intersection point of the diagonal $y = x$ and the ellipse $\mathbf{e}_\alpha$ with a maximal x_I.

Lemma 3. *Let us consider* $E_0 = \sqrt{(6\sqrt{3}-3)/11} \approx 0.8198$.
(a) If $E \in [E_0, 1]$, x_I *is decreasing for* $\alpha \in [-\pi/2, -\pi/4]$.
(b) If $E \in]1/2, E_0[$, *the function* x_I *reaches a unique minimal value for a unique real number* $\alpha_0 \in [-\pi/2, -\pi/4]$.
(c) If $E \in [0, 1/2]$, *the function* x_I *is increasing for* $\alpha \in [-\pi/2, -\pi/4]$.

Proof. First we prove that

$$x_I(\alpha) = \frac{E}{\sqrt{T}\left(\sqrt{ET^2+(1+E^2)T+E}-T\sqrt{2E}\right)}, \tag{10}$$

where $T = -\cot\delta$ has been defined in Lemma 2. We use the parametrization of the ellipse $\mathbf{e}_\alpha$ given by (3) and we look for some $t \in [0, 2\pi]$ such that

$$\sqrt{\lambda}\cos\left(\frac{t+\varphi}{2}\right) = \varepsilon\sqrt{\mu}\cos\left(\frac{t+\psi}{2}\right), \tag{11}$$

where $\varepsilon = \pm 1$. Since the point I occurs for some $t \in [0, \pi]$ (and J for some $t \in [\pi, 2\pi]$), we have $(t+\varphi)/2 \in [-\pi/4, \pi/2]$, $(t+\psi)/2 \in [-\pi/2, \pi/4]$ and we get $\varepsilon = +1$ because the two cosines in (11) are positive. In this equality we expand $\cos((t+\psi)/2) = \cos((t+\varphi)/2+\delta)$ and we find

$$\sqrt{\lambda}\cos\left(\frac{t+\varphi}{2}\right) = \sqrt{\mu}\left(\cos\left(\frac{t+\varphi}{2}\right)\cos\delta - \sin\left(\frac{t+\varphi}{2}\right)\sin\delta\right),$$

which gives

$$\tan\left(\frac{t+\varphi}{2}\right) = \frac{\sqrt{\mu}\cos\delta - \sqrt{\lambda}}{\sqrt{\mu}\sin\delta}$$

and

$$x_I = 2\lambda\cos^2\left(\frac{t+\varphi}{2}\right) = \frac{2\lambda}{1+\tan^2\left(\frac{t+\varphi}{2}\right)} = \frac{2\lambda\mu\sin^2\delta}{\lambda+\mu-2\sqrt{\lambda\mu}\cos\delta}. \tag{12}$$

By (8) and Lemma 1 (b) and since $\sin^2 2\delta = 1-\cos^2(\psi-\varphi)$, we have

$$\sin^2 2\delta = \frac{(2\lambda\mu)^2-(1-E^2)^2\sin^2 2\alpha}{(2\lambda\mu)^2} = \frac{4E^2}{(2\lambda\mu)^2}.$$

The previous equality gives $2\lambda\mu = -2E/\sin 2\delta$ and by (5) we have $\lambda+\mu = (1+E^2+2\lambda\mu)^{1/2}$. Substituting these values in (12) we find

$$\begin{aligned} x_I &= \frac{-E\sin\delta}{\cos\delta\left(\sqrt{1+E^2-\frac{E}{\sin\delta\cos\delta}}-\sqrt{\frac{-2E\cos\delta}{\sin\delta}}\right)} \\ &= \frac{E}{\sqrt{T}\left(\sqrt{(1+E^2)T+\frac{E}{\sin^2\delta}}-\sqrt{T}\sqrt{T}\sqrt{2E}\right)}. \end{aligned}$$

It remains to use $1/\sin^2\delta = 1+T^2$ and we get (10). Let us denote by $f(T)$ the denominator of the right hand side of (10). We find

$$f'(T) = \frac{g(T)-h(T)}{2\sqrt{T}\sqrt{ET^2+(1+E^2)T+E}},$$

where $h(T) = 3\sqrt{2E}\,T\sqrt{ET^2+(1+E^2)T+E}$ and $g(T) = 3ET^2+2(1+E^2)T+E$. Since the functions $g(T)$ and $h(T)$ are positive, the sign of $f'(T)$ is equal to the one of the polynomial $P(T) = g^2(T)-h^2(T)$, that is

$$P(T) = -9E^2T^4-6E(1+E^2)T^3+4(E^4-E^2+1)T^2+4E(1+E^2)T+E^2. \tag{13}$$

We get $P'(T) = -36E^2T^3-18E(1+E^2)T^2+8(E^4-E^2+1)T+4E(1+E^2)$. The discriminant of $P''(T) = -4(27E^2T^2+9E(1+E^2)T-2(E^4-E^2+1))$ is $\Delta = 16\cdot 27E^2(11E^4-2E^2+11) > 0$. We remark that $P''(0) > 0$ and $\lim_{T\to\infty} P''(T) = -\infty$, then $P''(T)$ has a unique positive root T_2. Since $P'(0) > 0$ and $\lim_{T\to\infty} P'(T) = -\infty$, the polynomial $P'(T)$ has a unique root $T_1 > T_2$ and $P'(T) \geq 0$ for all $T \in [0,T_1]$. Finally, $P(0) = E^2 > 0$ implies that the polynomial $P(T)$ has a unique positive root T_0.

Moreover, $P(E) = -E^2(11E^4+6E^2-9)$ vanishes at a unique positive value $E_0 = \sqrt{(-3+6\sqrt{3})/11}$. We remark that $T_0 = E_0$ if and only if $E = E_0$. In the three following cases we conclude with Lemma 2:

(a) If $E \in [E_0,1]$, we have $P(E) \leq 0$. Then $T_0 \leq E \leq 1$ which implies that $P(T) \leq 0$ for all $T \in [E,1]$. So $f(T)$ is decreasing on $[E,1]$ and then x_I is increasing with respect to T.

(b) If $E \in]1/2, E_0[$, we have $P(E) > 0$ and $P(1) = 2(E+1)^2(2E-1)(E-2) < 0$. Then $E < T_0 < 1$ which implies that $f(T)$ is increasing on $[E, T_0]$ and decreasing on $[T_0, 1]$. Thus x_I is decreasing for $T \in [E, T_0]$ and increasing for $T \in [T_0, 1]$.

(c) If $E \in]0, 1/2]$, we have $P(E) \geq 0$ and $P(1) \geq 0$ which implies that $P(T) \geq 0$ on $[E, 1]$. Thus x_I is decreasing for $T \in [E, 1]$. □

3. Calculation of $s_2(E)$

Now we can describe the various optimal packings of two ellipses in the square and the corresponding side lengths $s_2(E)$.

Theorem 4. *If $E \leq 1/2$, then*

$$s_2(E) = \sqrt{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}}. \tag{14}$$

The minimum value $s_2(E)$ is obtained for two parallel ellipses $\mathbf{e}_1 = \mathbf{e}_{\alpha_0}$ and $\mathbf{e}_2$ with

$$\alpha_0 = -\arccos \frac{1}{2}\sqrt{\frac{4 - s_2^2(E)}{1 - E^2}}, \tag{15}$$

and where $\mathbf{e}_2$ is the reflection of $\mathbf{e}_1$ through the center of the square.

Proof. If $\alpha = -\pi/2$, the center $\Omega = (\mu, \mu)$ of $K_{2\mu}$ does not belong to $\mathbf{e}_\alpha$ (except for $E = 1/2$). If $\alpha = -\pi/4$, the center Ω is also the center of the ellipse $\mathbf{e}_\alpha$. We know by Lemma 1 (a) and Lemma 3 (c) that $x_I - \mu$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$. Then there exists a unique angle $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $I = \Omega_0 = (\mu_0, \mu_0)$ with $\mu_0 = \mu(\alpha_0)$. We note that the center Ω_0 belongs to the boundary of the ellipse $\mathbf{e}_{\alpha_0}$, see Figure 2. Let us show that in the square $K_{2\mu_0}$, the center Ω_0 is an unavoidable point. By Fact 1, it suffices to show that any ellipses $\mathbf{e}_\alpha$ included in $K_{2\mu_0}$ contain Ω_0:

• If $\alpha < \alpha_0$, the ellipse $\mathbf{e}_\alpha$ is not contained in $K_{2\mu_0}$ because it intersects the upper side $y = 2\mu_0$ (μ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$).

• If $\alpha > \alpha_0$, the point Ω_0 belongs to the interior of the ellipse $\mathbf{e}_\alpha$ because x_I is increasing for $\alpha \in [\alpha_0, -\pi/4]$.

It remains to calculate α_0 and μ_0. First, we show that

$$\mu_0 - \lambda_0 = \frac{E}{\mu_0}. \tag{16}$$

We have by (4) that $\mathbf{e}_\alpha(t) = (\mu, \mu)$ if and only if

$$\begin{aligned} \mu - \lambda &= \cos\alpha\cos t - E\sin\alpha\sin t, \\ 0 &= \sin\alpha\cos t + E\cos\alpha\sin t. \end{aligned}$$

Substituting $-E\cos\alpha\sin t/\sin\alpha$ for $\cos t$ in the first equality, we find

$$\begin{aligned} -E\sin(t) &= (\mu - \lambda)\sin\alpha, \\ \cos t &= (\mu - \lambda)\cos\alpha, \end{aligned}$$

which implies $E^2 = (\mu - \lambda)^2(\sin^2\alpha + E^2\cos^2\alpha) = (\mu - \lambda)^2\mu^2$, and then (16) since $\mu \geq \lambda$ for $\alpha \in [-\pi/2, -\pi/4]$. By (5) we get for $\alpha = \alpha_0$,

$$1 + E^2 + 2\lambda_0\mu_0 = (\lambda_0 + \mu_0)^2 = (\lambda_0 - \mu_0 + 2\mu_0)^2 = \left(2\mu_0 - \frac{E}{\mu_0}\right)^2.$$

Since (16) implies $\mu_0^2 - E = \lambda_0\mu_0$, we have

$$2(\mu_0^2 - E) = 4\mu_0^2 + \frac{E^2}{\mu_0^2} - 4E - (1 + E^2).$$

This equation in μ_0^2 leads to

$$4\mu_0^2 = (1 + E)^2 + \varepsilon\sqrt{(1 + E)^4 - 8E^2},$$

where $\varepsilon = \pm 1$. The case $\varepsilon = -1$ leads to $4\lambda_0\mu_0 = 4\mu_0^2 - 4E = (1 - E)^2 - \sqrt{(1 + E)^4 - 8E^2} \leq 0$ what is impossible.

Then $s_2(E) \geq 2\mu_0 = \sqrt{(1 + E)^2 + \sqrt{(1 + E)^4 - 8E^2}}$. We can pack in $K_{2\mu_0}$ the reflection of $\mathbf{e}_{\alpha_0}$ through Ω_0 and we get the equality (14). We finally obtain the angle (15) by considering $\mu_0^2 = \sin^2\alpha_0 + E^2\cos^2\alpha_0$ and $s_2(E) = 2\mu_0$. □

Theorem 5. *If $1/2 < E < E_0$, then*

$$s_2(E) = \frac{2E}{\sqrt{T_0}\left(\sqrt{ET_0^2 + (1 + E^2)T_0 + E} - T_0\sqrt{2E}\right)}, \tag{17}$$

where T_0 is the unique positive root of (13). The minimum value $s_2(E)$ is obtained for two parallel ellipses $\mathbf{e}_1 = \mathbf{e}_{\alpha_0}$ and $\mathbf{e}_2$ with

$$\alpha_0 = -\frac{1}{2}\left(\pi + \arcsin\frac{E(T_0^2 - 1)}{T_0(1 - E^2)}\right), \tag{18}$$

and where the ellipse $\mathbf{e}_2$ is the reflection of $\mathbf{e}_1$ through the center of the square.

Proof. We denote by α_0 the unique angle α such that $T_0 = T(\alpha)$ and by I_0 the intersection of $\mathbf{e}_{\alpha_0}$ with $y = x$. Since the continuous function x_I is decreasing for $\alpha \in [-\pi/2, \alpha_0]$ and increasing for $\alpha \in [\alpha_0, -\pi/4]$, the point I_0 belongs to the interior of $\mathbf{e}_\alpha$ if $\alpha \neq \alpha_0$. Then any ellipse $\mathbf{e}_\alpha$ in $K_{2x_{I_0}}$ contains I_0. As Fact 1 gives that I_0 is an unavoidable point, we have $s_2(E) \geq 2x_{I_0}$. We can pack the reflection of the ellipse $\mathbf{e}_{\alpha_0}$ through I_0 and we get $s_2(E) \leq 2x_{I_0}$. The value of x_{I_0} is given by (10).

By (2), (9) and since $-1/T = \tan(\psi - \varphi)/2$, we have

$$(2\lambda\mu)^2 = \left(\frac{2E}{T} + (1 - E^2)\sin 2\alpha\right)^2 = 4E^2 + (1 - E^2)^2\sin^2 2\alpha,$$

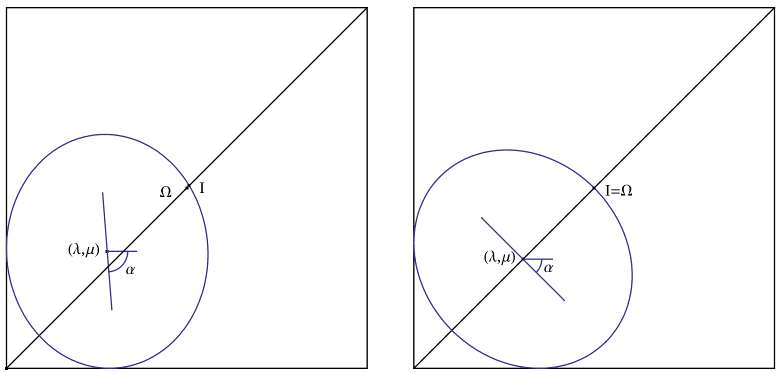
which gives $\sin 2\alpha = E(T^2 - 1)/((1 - E^2)T)$ and (18) for $2\alpha \in [-\pi, -\pi/2]$. □

Theorem 6. *If $E_0 \leq E \leq 1$, then*

$$s_2(E) = \sqrt{2}\left(\sqrt{1 + E^2} + E\right). \tag{19}$$

The minimum value $s_2(E)$ is obtained for two parallel ellipses $\mathbf{e}_1 = \mathbf{e}_\alpha$ and $\mathbf{e}_2$ with $\alpha = -\pi/4$ and where $\mathbf{e}_2$ is the reflection of $\mathbf{e}_1$ through the center of the square.

Proof. We consider again the intersection point I_0 of the ellipse $\mathbf{e}_{-\pi/4}$ and the diagonal $y = x$. Since x_I is decreasing for $\alpha \in [-\pi/2, -\pi/4]$, any ellipse $\mathbf{e}_\alpha$ in $K_{2x_{I_0}}$ contains I_0. As Fact 1 gives that I_0 is an unavoidable point, we get $s_2(E) \geq 2x_{I_0} = 2x_I(-\pi/4) = \sqrt{2}(\sqrt{1+E^2}+E)$. As in the two previous cases, we can pack the reflection of the ellipse $\mathbf{e}_{-\pi/4}$ through I_0 and we get the equality (19). □



$-\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$ $\qquad$ $\alpha = -\frac{\pi}{4}$

Figure 4. For $E_0 \leq E \leq 1$ and $\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$, the center of $K_{2x_I(-\frac{\pi}{4})}$ belongs to the interior of $\mathbf{e}_\alpha$.

It is not surprising that among all the optimal packings $\mathcal{P}_E$, the densest one is $\mathcal{P}_{1/2}$, see Figure 1. We denote by $d(E)$ the density of $\mathcal{P}_E$ and we have for all E in $]0,1]$,

$$d(E) = \frac{2\pi E}{s_2^2(E)}.$$

The formulas (14), (17), (19) for $s_2(E)$ give that $d(E)$ equals to

$$\begin{aligned}
d_1(E) = &\quad \frac{2\pi E}{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}} \text{ if } E \in]0, \frac{1}{2}], \\
d_2(E) = &\quad \frac{\pi}{2E} T_0 \left(\sqrt{ET_0^2 + (1+E^2)T_0 + E} - T_0\sqrt{2E} \right)^2 \text{ if } E \in]\frac{1}{2}, E_0[, \\
d_3(E) = &\quad \frac{\pi E}{\left(\sqrt{1+E^2}+E\right)^2} \text{ if } E \in [E_0, 1].
\end{aligned}$$

The optimality of $\mathcal{P}_E$ for all E on each interval $]0,1/2]$, $]1/2, E_0[$, $[E,1]$ implies the continuity of $d(E)$ on $]0,1]$. It is easy to verify that $d_1(1/2) = d_2(1/2) =$

$\pi/4$ and $d_2(E_0) = d_3(E_0) = \pi E_0/\left(\sqrt{1+E_0^2}+E_0\right)^2$. We show that $d(E)$ is increasing on $]0, 1/2]$ and decreasing on $[1/2, 1]$. For $E \in]0, \frac{1}{2}]$, we get

$$d_1'(E) = \frac{2\pi(1-E^2)}{\sqrt{(E+1)^4 - 8E^2}\left(1+2E+E^2+\sqrt{(E+1)^4-8E^2}\right)} > 0$$

and for $E \in [E_0, 1]$,

$$d_3'(E) = \frac{\pi\left(\sqrt{1+E^2}-2E\right)}{\sqrt{1+E^2}\left(\sqrt{1+E^2}+E\right)^2} < 0.$$

In the case of Theorem 5, we have $s_2(E) = 2E/f(T_0)$ and $d_2(E) = (\pi/(2E))f^2(T_0)$. Since $T_0 = T_0(E)$ is a single root of (13), $T_0(E)$ is differentiable at $E \in]1/2, E_0[$ and we get

$$d_2'(E) = \frac{\pi}{2E^2}\left(2Ef(T_0)f'(T_0)\frac{dT_0}{dE} - f^2(T_0)\right).$$

As $f'(T_0) = 0$, we obtain $d_2'(E) < 0$.

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The Diagonal Point Triangle Revisited

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Abstract. We derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral in terms of the four tangent lengths, and prove a characterization for a tangential trapezoid.

1. Introduction

Consider a convex quadrilateral with no pair of opposite parallel sides. Let the two diagonals intersect at E and the extensions of opposite sides intersect at F and G. Then the triangle EFG is called the *diagonal point triangle* or sometimes just the diagonal triangle (see Figure 1).

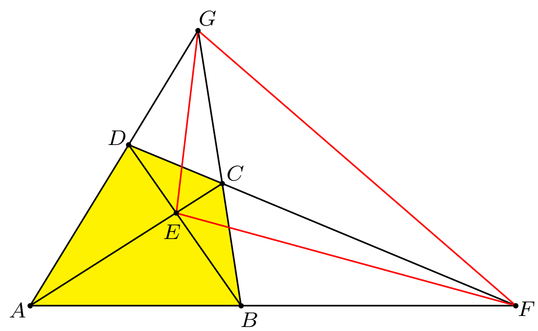

Figure 1. The diagonal point triangle EFG

The significance of the diagonal point triangle is most evident in projective geometry, where it is studied in connection with the complete quadrilateral. It is for instance a well known property that the diagonal point triangle associated with a cyclic quadrilateral is self-conjugate.

In [5] we derived a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the four sides. In this note we shall derive a formula for this triangle area in connection with a tangential quadrilateral (a quadrilateral with an incircle), but here it will be in terms of the tangent lengths instead. The tangent lengths e, f, g, h in a tangential quadrilateral are defined to be the distances from the vertices to the points where the incircle is tangent to the sides (see Figure 2).

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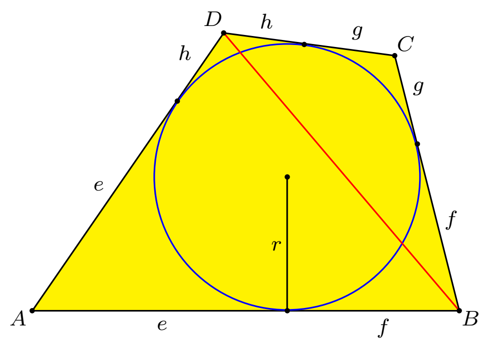

Figure 2. A tangential quadrilateral with its tangent lengths and a diagonal

2. More on the area of the diagonal point triangle

We will use Richard Guy's version of Hugh ApSimon's formula to derive a formula for the area of the diagonal point triangle belonging to a tangential quadrilateral. According to it (see [2]), the diagonal point triangle belonging to a convex quadrilateral $ABCD$ has the area

$$T = \frac{2T_1T_2T_3T_4}{K(T_1T_2 - T_3T_4)} \tag{1}$$

where T_1, T_2, T_3, T_4 are the areas of triangles ABC, ACD, ABD, BCD respectively, and K is the area of the quadrilateral.

Theorem 1. *If e, f, g, h are the tangent lengths in a tangential quadrilateral with no pair of opposite parallel sides, then the associated diagonal point triangle has the area*

$$T = \frac{2efghK}{|ef - gh||eh - fg|}$$

where

$$K = \sqrt{(e+f+g+h)(efg+fgh+ghe+hef)}$$

is the area of the quadrilateral.

Proof. In a tangential quadrilateral, triangle ABD has the area (see Figure 2)

$$T_3 = \frac{1}{2}(e+f)(e+h)\sin A = (e+f)(e+h)\sin\frac{A}{2}\cos\frac{A}{2}.$$

According to Theorem 8 in [4], we have that

$$\sin^2\frac{A}{2} = \frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)}.$$

Using the trigonometric Pythagorean theorem yields

$$\cos^2\frac{A}{2} = 1 - \sin^2\frac{A}{2} = \frac{(e+f)(e+g)(e+h) - (efg+fgh+ghe+hef)}{(e+f)(e+g)(e+h)}$$
$$= \frac{e^2(e+f+g+h)}{(e+f)(e+g)(e+h)}.$$

Thus we get the subtriangle area

$$T_3 = \frac{(e+f)(e+h)e\sqrt{(efg+fgh+ghe+hef)(e+f+g+h)}}{(e+f)(e+g)(e+h)} = \frac{eK}{e+g}.$$

The last equality is due to formula (2) in [4] which gives the area of a tangential quadrilateral in terms of the tangent lengths. By symmetry we also have

$$T_1 = \frac{fK}{f+h}, \qquad T_2 = \frac{hK}{f+h}, \qquad T_4 = \frac{gK}{e+g}.$$

Combining the last four formulas gives

$$T_1T_2 - T_3T_4 = \frac{fhK^2}{(f+h)^2} - \frac{egK^2}{(e+g)^2} = K^2\left(\frac{(e+g)^2fh - eg(f+h)^2}{(e+g)^2(f+h)^2}\right).$$

Expanding the numerator, canceling the two double products and factoring it yields

$$(e+g)^2fh - eg(f+h)^2 = e^2fh + fg^2h - ef^2g - egh^2 = (ef-gh)(eh-fg).$$

Now by inserting the expressions for the triangle areas T_1, T_2, T_3, T_4 into (1), we get the area of the diagonal point triangle belonging to a tangential quadrilateral. Hence this is

$$T = \frac{2efghK^4}{K(e+g)^2(f+h)^2} \cdot \frac{(e+g)^2(f+h)^2}{K^2(ef-gh)(eh-fg)}$$

and the formula in the theorem follows by simplification and adding an absolute value to the denominator to cover all cases. $\square$

Except in projective geometry, where the notion of area is irrelevant, we have only found one source where the diagonal point triangle associated with a tangential quadrilateral is treated. This is in the old extensive paper [1] on quadrilateral geometry by Dostor. He derives a formula for this triangle area,[1] but that formula is wrong. It states incorrectly (using other notations) that the area is given by

$$T = \frac{4efghK}{(e^2-g^2)(f^2-h^2)}$$

where e, f, g, h are the tangent lengths and K is the area of the quadrilateral. In [5] we concluded that Dostor's formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral is also wrong, and then derived the correct formula.

An interesting observation is that the correct formula in [5] for a cyclic quadrilateral has the exact same *form* as Dostor's incorrect formula for a tangential

[1]Formula CCXVII on page 308 in [1]. We used e, f, g, h in the citation of his formula to easily be able to compare it to Theorem 1 in this note.

quadrilateral (except for a factor of 2). But there is one big difference. The letters used in Dostor's formula stands for the tangent lengths in a tangential quadrilateral, whereas in Theorem 1 in [5], we used a, b, c, d which stands for the side lengths in a cyclic quadrilateral.

3. A characterization of tangential trapezoids

If two opposite sides in the quadrilateral are parallel, then one of the points F or G becomes a point at infinity. Then the area of the diagonal point triangle is infinite. This is equivalent to having a denominator in Theorem 1 that is zero, so we get a necessary and sufficient condition for parallel opposite sides this way. Hence opposite sides are parallel if and only if $ef = gh$ or $eh = fg$.

Now we give a second proof of these characterizations of a *tangential trapezoid* (a trapezoid with an incircle; see Figure 3) where it is easier to determine which pair of opposite sides that are parallel in each case.

Theorem 2. *The opposite sides AB and CD in a tangential quadrilateral $ABCD$ with tangent lengths e, f, g, h are parallel if and only if*

$$eh = fg.$$

The opposite sides AD and BC are parallel if and only if

$$ef = gh.$$

Proof. According to Theorem 3 in [6], the opposite sides AB and CD in a convex quadrilateral are parallel if and only if

$$\tan\frac{A}{2}\tan\frac{D}{2} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

Since $\tan\frac{A}{2} = \frac{r}{e}$, $\tan\frac{B}{2} = \frac{r}{f}$, $\tan\frac{C}{2} = \frac{r}{g}$ and $\tan\frac{D}{2} = \frac{r}{h}$ in a tangential quadrilateral with inradius r (see Figure 2), we have that AB and CD are parallel if and only if

$$\frac{r}{e}\cdot\frac{r}{h} = \frac{r}{f}\cdot\frac{r}{g} \qquad \Leftrightarrow \qquad eh = fg.$$

The second condition is proved in the same way using the angle characterization

$$\tan\frac{A}{2}\tan\frac{B}{2} = \tan\frac{C}{2}\tan\frac{D}{2}$$

for when AD and BC are parallel in a convex quadrilateral $ABCD$. □

In [4, p.129] we concluded that the inradius in a tangential trapezoid with tangent lengths e, f, g, h is given by

$$r = \sqrt[4]{efgh}.$$

Combining this with Theorem 2 yields that the formula for the inradius in a tangential trapezoid $ABCD$ with bases AB and CD can be simplified to

$$r = \sqrt{eh} = \sqrt{fg}.$$

These formulas can also be derived without the use of trigonometry. We give two other short proofs. Let the incircle be tangent to AB and CD at W and Y respectively, and I be the incenter (see Figure 3). Then triangles AWI and IYD are similar (AAA), so $\frac{r}{h} = \frac{e}{r}$. Whence $r^2 = eh$ and the second formula follows in a similar way. Another derivation starts by noting that the angle AID is a right angle when $AB \parallel CD$. Using the Pythagorean theorem in the three triangles AWI, DYI and AID yields $AI^2 = e^2 + r^2$, $DI^2 = h^2 + r^2$ and $AI^2 + DI^2 = (e+h)^2$. Combining these, we have $r^2 + e^2 + r^2 + h^2 = (e+h)^2$, and thus $r^2 = eh$.

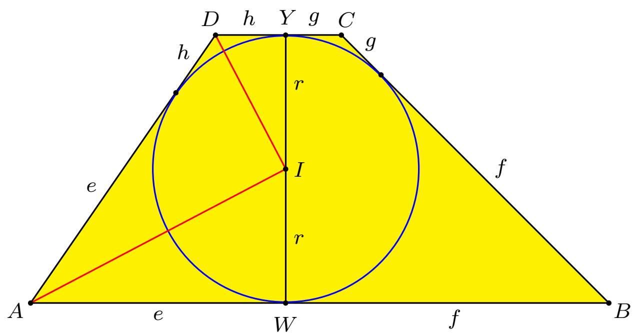

Figure 3. A tangential trapezoid

When the bases of the tangential trapezoid instead are AD and BC, the corresponding formulas are

$$r = \sqrt{ef} = \sqrt{gh}.$$

They can be derived in the same way by any of the three methods used above.

As a final remark, we note that the related equality $eg = fh$ gives another necessary and sufficient condition in tangential quadrilaterals. Two different proofs (both using other notations) were given in [7] and [3, p.104] that this is a characterization for when a tangential quadrilateral is also cyclic.

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A Simple Construction of an Inconic

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Abstract. We give a simple construction of an inscribed inconic with given perspector and its traces by constructing two extra points on the conic, without explicitly constructing the center of the conic.

Let ABC be a given triangle. Consider a point P with its cevian triangle XYZ. It is well known that there is an inscribed conic tangent to the sidelines at X, Y, Z. The center Q of the conic is the inferior of the isotomic conjugate of P (see [1, p.128]). The reflections of X, Y, Z in Q are three extra points on the conic. We give below a construction of two extra points Y' and Z' on the conic, without first locating the center Q.

Proposition 1. *Given a triangle ABC and a point P with cevian triangle XYZ, let X', Y', Z' be the second intersections of cevians AX, BY , CZ and the inconic with perspector P. The cross ratios (AX, PX'), (BY, PY'), and (CZ, PZ') are all equal to* $4 : 1$ (see Figure 1a).

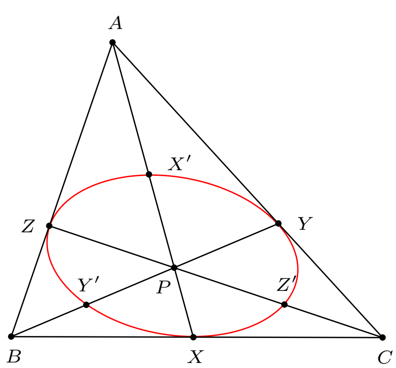

Figure 1a

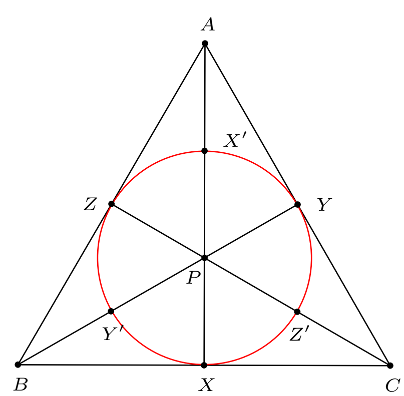

Figure 1b

Proof. It is enough to prove this for the incircle of the equilateral triangle, regarded as the Steiner inellipse with perspector at the centroid (see Figure 1b). Then a projective transformation that maps our triangle ABC and P into the equilateral triangle with its centroid preserves these cross ratios. Now, for the equilateral triangle, clearly,

$$(AX, PX') = \frac{AP}{PX} : \frac{AX'}{X'X} = \frac{2}{1} : \frac{1}{2} = 4 : 1.$$

$\square$

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We simply use the usual method to construct a point with a given cross ratio. Let D be the point that divides the segment AP in the ratio $3 : 1$. We construct the intersection points $M = DZ \cap CA$ and $N = DY \cap AB$, draw parallels to AX through N and M that intersect BY and CZ at Y' and Z' respectively (see Figure 2). If J is the infinite point of the line AP, then

$$(BY, PY') = (AD, PJ) = \frac{AP}{PD} : \frac{AJ}{JD} = -4 : -1 = 4 : 1.$$

Therefore, Y' lies on the inconic with perspector P.

The same reasoning shows that Z' lies on the same conic.

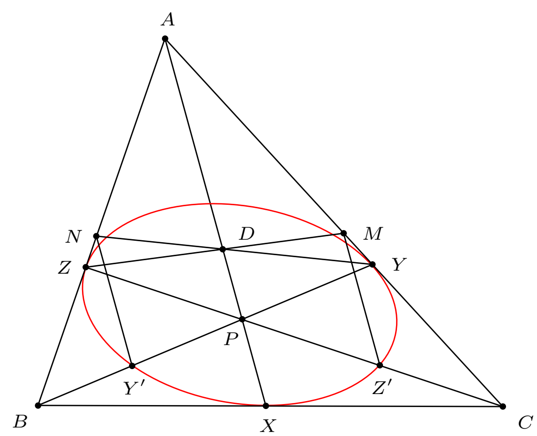

Figure 2

Reference

[1] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
`http://math.fau.edu/Yiu/Geometry.html`

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On a Circle Containing the Incenters of Tangential Quadrilaterals

Albrecht Hess

Abstract. When we fix one side and draw different tangential quadrilaterals having the same side lengths but different angles we observe that their incenters lie on a circle. Based on a known formula expressing the incircle radius of a tangential quadrilateral by its tangent lengths, some older results will be presented in a new light and the equation of the before mentioned circle will appear. This circle encodes information about tangential and bicentric quadrilaterals that leads to an apparently new characterization of tangential quadrilaterals. Curiously enough, no trigonometric formulae are needed.

1. Introduction

Figure 1 shows a tangential quadrilateral $ABCD$, its incircle with incentre I and radius r. Let W, X, Y, Z, be the tangency points and denote the tangent lengths AW etc. by e, f, g and h. While the tangent lengths determine the sides of a tangential quadrilateral $AB = a = e + f$, $BC = b = f + g$, $CD = c = g + h$, $DA = d = h + e$, the tangent lengths cannot be derived unambiguously from the sides - in contrast to the triangle, where the tangent lengths are $e = s-a$, $f = s-b$, $g = s - c$ with semiperimeter s.

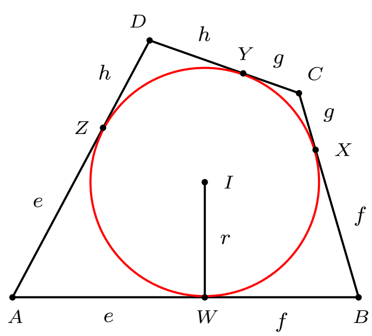

Figure 1

The reason is that the condition $a + c = b + d$ for the sides in a tangential quadrilateral produces one degree of freedom in the solutions of the equations for the tangent lengths. This dichotomy between ambiguous and unambiguous tangent lengths in quadrilaterals and triangles continues to hold in polygons of an even and odd number of vertices.

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2. Inradius and distances of incenter from vertices of the quadrilateral

Given the tangent lengths e, f, g, h, the radius r of the inscribed circle of the tangential quadrilateral is determined according to the formula

$$r^2 = \frac{fgh + egh + efh + efg}{e + f + g + h} \tag{1}$$

cf. [8], [9, Lemma 2], [13, (1)], [16]. In [8] this equation is derived from

$$\mathrm{Im}((r + ei)(r + fi)(r + gi)(r + hi)) = \sigma_1 r^3 - \sigma_3 r = 0, \tag{2}$$

wherein σ_k are the k-th degree elementary symmetric functions of e, f, g, h. Formula (2) says that the four right-angled triangles with the legs r and e, r and f, r and g, and r and h can be put together to form an angle of $180°$. From one pair of each of these triangles one can form a tangential quadrilateral as shown in Figure 2.

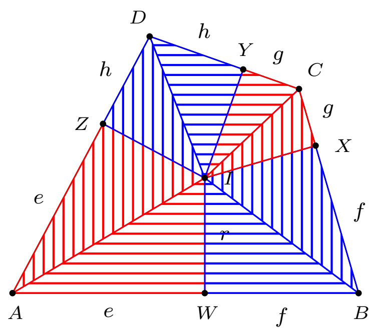

Figure 2

The generalization of formula (2) for arbitrary polygons can be resolved unambiguously for $r > 0$ only in the case of the triangle and the tangential quadrilateral and leads for the triangle to the formula

$$r^2 = \frac{efg}{e + f + g} = \frac{(s - a)(s - b)(s - c)}{s},$$

which expresses the radius r of its incircle by the tangent lengths or the side lengths. This also gives a short proof of Heron's triangle area formula [18] in the spirit of "proofs without words". From (1) we obtain

$$AI^2 = r^2 + e^2 = \frac{fgh + egh + efh + efg}{e + f + g + h} + e^2 = \frac{(e + f)(e + g)(e + h)}{e + f + g + h}, \tag{3}$$

compare with [13, (5)]. Either multiply out the right hand side, or observe that $\sigma_1(r^2 + e^2) = \sigma_3 + e^2\sigma_1$ is a monic third degree polynomial in e vanishing for $e = -f$, $e = -g$ and $e = -h$. Similar formulae hold for the other distances between the vertices and the incenter I.

For later use, we note

$$\frac{AI \cdot CI}{BI \cdot DI} = \frac{e + g}{f + h}, \tag{4}$$

which is an immediate consequence of (3). Similar formulae are in [7, Theorem 8].

3. Construction of the bicentric quadrilateral from its side lengths

With given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, that fulfill the condition $a+c = b+d$, we can construct in general different incongruent tangential quadrilaterals. Among them, the tangential quadrilateral with the greatest area, and therefore with the greatest radius of its incircle, is the bicentric quadrilateral with sides a, b, c, d, (cf. [3, p.135, (5)], [4, p. 238, (35)], [20]). The conditions for its constructability $a < b + c + d$ etc. as a cyclic quadrilateral from its side lengths are obviously satisfied and the construction can be carried out by use of the circle of Apollonius as in [11, pp. 82-83], or by joining two similar cyclic quadrilaterals such that they form a trapezoid with the same angles, but in a different order (see Figure 3). It seems that Bretschneider must have had this gluing method of similar geometric figures in mind when he produced this impressive series of formulas for triangles and quadrilaterals in [3] and [4]. Recently, Varverakis used this trick in his article [20] to have a short and visual proof of the maximum area property for cyclic quadrilaterals.

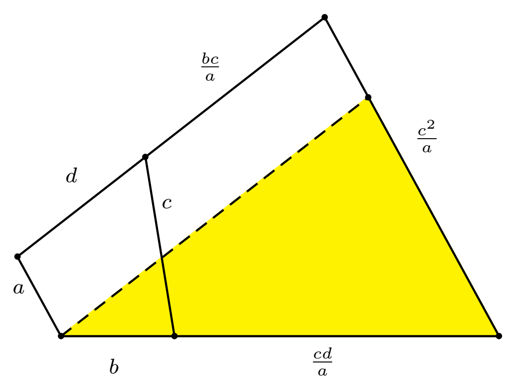

Figure 3

The triangle inequalities for the shaded triangle in Figure 3, i.e. the constructability conditions for the trapezoid and hence for the bicentric quadrilateral are satisfied. Consider, for example, this one

$$\left(d + \frac{bc}{a}\right) + \left(\frac{c^2}{a} - a\right) > \left(b + \frac{dc}{a}\right).$$

It is equivalent to $(c - a)(a + b + c - d) > 0$ and is satisfied if the dilated quadrilateral is glued to the larger of the sides a and c. In the same way the other triangle inequalities are proved. For $c = a$, there is no need to paste two similar quadrilaterals together since cyclic quadrilaterals with a pair of equal opposite sides are trapezoids.

The area K of a bicentric quadrilateral is $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{abcd}$ by Brahmagupta's formula (cf. [3, p.135, (5)], [4, p. 238, (35)], [5, Th.

3.22], [6, pp.62-63], [10], [11, Th. 109], [15]), and the radius of its incircle

$$r_{\max} = \frac{K}{s} = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{b+d} \tag{5}$$

is the largest among the incircle radii of all tangential quadrilaterals with sides a, b, c, d.

4. Construction of a tangential quadrilateral from given tangent lengths and conclusions

If we want to construct a tangential quadrilateral with given tangent lengths e, f, g, h, we could determine its inradius r with some intricate segment multiplications and divisions based on formula (1) and then assemble the right-angled triangles with the legs r and e, r and f, r and g, and r and h as in Figure 2. Another way that takes us further and leads to the main formula (7), starts from the bicentric quadrilateral with side lengths $a = e + f$, $b = f + g$, $c = g + h$, $d = h + e$. We get the same bicentric quadrilateral with sides a, b, c, d from different tangent lengths, the degree of freedom being the choice of one of them, e.g. e. Then the other tangent lengths are

$$\begin{aligned} BW = \ & BX = f = a - e, \\ CX = \ & CY = g = b - a + e, \\ DY = \ & DZ = b = d - e; \end{aligned} \tag{6}$$

see Figure 4.

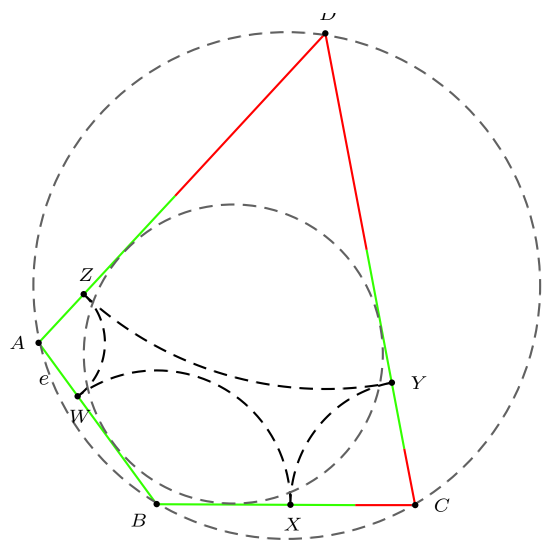

Figure 4

The right hand sides are positive, if a is the smallest side and $e < a$. The green zones indicate where the endpoints of the segments of lengths e, f, g, h are situated, the red zones are excluded. According to (1), the incircle radius for a tangential quadrilateral with the tangent lengths from (6) is

$$\begin{aligned} r^2 &= \frac{(f+g)eh+(h+e)fg}{(f+g)+(h+e)} \\ &= \frac{be(d-c)+d(a-e)(b-a+e)}{b+d} \\ &= -\left(e-\frac{ad}{b+d}\right)^2+\frac{abcd}{(b+d)^2} \\ &= -\left(e-\frac{ad}{s}\right)^2+r_{\max}^2. \end{aligned} \tag{7}$$

This formula has the following consequences.

Theorem 1. *A tangential quadrilateral $ABCD$ with sides a, b, c, d, and semiperimeter $s = a + c = b + d$ is cyclic if and only if its tangent lengths are*

$$e = \frac{ad}{s}, \quad f = \frac{ab}{s}, \quad g = \frac{bc}{s}, \quad h = \frac{cd}{s}. \tag{8}$$

With these formulae, it is easy to deduce the characterization of bicentrics in Problem 10804 in the MONTHLY [19] or in Hajja's article [9, Lemma 1].

Corollary 2. *A tangential quadrilateral is bicentric if and only if $eg = fh$.*

One direction is obvious from (8). If, on the other hand, $eg = fh$, then $eb = e(f+g) = f(h+e) = fd$, and together with $e + f = a$, we find $e = \frac{ad}{b+d}$, $f = \frac{ab}{b+d}$, i.e. (8).

From $\frac{AC}{BD} = \frac{ad+bc}{ab+cd}$, a companion formula of Ptolemy's theorem, valid for all cyclic quadrilaterals (cf. [1, p.130], [2, Lemma 2], [11, p. 85]) we get immediately from (8) the second criterion of [9].

Corollary 3. *A tangential quadrilateral $ABCD$ is bicentric only if*

$$\frac{AC}{BD} = \frac{e+g}{f+h}. \tag{9}$$

To prove the converse of Corollary 3, we invert the points A, B, C, D, with respect to the incircle and insert (4) into the formulae

$$AC = A'C' \cdot \frac{AI \cdot CI}{r^2}, \qquad BD = B'D' \cdot \frac{BI \cdot DI}{r^2}$$

for the distances of the images A', B', C', D', and get

$$\frac{AC}{BD} = \frac{e+g}{f+h} \cdot \frac{A'C'}{B'D'}. \tag{10}$$

Formula (10) says that under condition (9) the diagonals of the parallelogram $A'B'C'D'$ are of equal lengths, so that $A'B'C'D'$ is a rectangle, hence cyclic, and the pre-images A, B, C, D, are cyclic too.

Writing (7) as $r^2 + \left(e - \frac{ad}{s}\right)^2 = r_{\max}^2$, we obtain the following Theorem 4, or more visually, Theorem 4′ as announced in the abstract.

Theorem 4. *For different choices of the tangent lengths e, f, g, h, having the same sums $a = e + f$, $b = f + g$, $c = g + h$, $d = h + e$, we make the side AB of the bicentric quadrilateral $ABCD$ with sides a, b, c, d to the x-axis of a coordinate system with origin A. The points with coordinates (e, r) move on a circle around the point W of tangency of the incircle with AB, which goes through the incenter I of $ABCD$.*

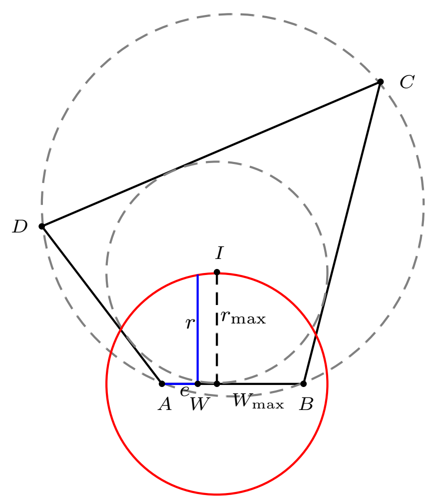

Figure 5

Theorem 4′. *When we fix one side and draw different tangential quadrilaterals having the same side lengths by changing the angles, the incenters will move on a circle.*

This theorem allows us to construct the inradius $r = \sqrt{\frac{fge+egh+efh+efg}{e+f+g+h}}$ of a tangential quadrilateral with the tangent lengths e, f, g, h from the bicentric quadrilateral $ABCD$ with the sides $e+f$, $f+g$, $g+h$, $h+e$. Having once determined r, we can easily construct the tangential quadrilateral with given tangent lengths. To perform this construction of r draw the perpendicular at the point W with $AW = e$ on the side AB of the bicentric quadrilateral $ABCD$. It intersects the above mentioned circle of the incenters around $W_{\max}$ at a point whose distance to AB is r. This construction can be carried out without any restriction if one starts with a tangency point within the green ranges of Figure 4. If, for the sake of simplicity, AB is the shortest side of the quadrilateral, the inequalities

$$AW_{\max} = \frac{ad}{s} \leq r_{\max} = \frac{\sqrt{abcd}}{s},$$
$$BW_{\max} = \frac{ab}{s} \leq r_{\max} = \frac{\sqrt{abcd}}{s}$$

guarantee that the segment AB lies within the circle with the center $W_{\max}$ and the radius $r_{\max}$, see Figure 5.

Another way to understand the formula (7) is the following characterization of tangential quadrilaterals, complementing the investigations of [14] and [17].

Theorem 5. *The points K, L, M, N are situated on the sides a, b, c, d, of a quadrilateral $ABCD$ such that they divide the respective sides in the ratio of the adjacent sides:*

$$\frac{AK}{KB} = \frac{d}{b}, \quad \frac{BL}{LC} = \frac{a}{c}, \quad \frac{CM}{MD} = \frac{b}{d}, \quad \frac{DN}{NA} = \frac{c}{a}. \tag{11}$$

Then $ABCD$ is a tangential quadrilateral if and only if $KLMN$ is cyclic.

Proof. For a tangential quadrilateral we infer from (7) that the points K, L, M, N, lie on a circle around I with radius $r_{\max} = \frac{\sqrt{abcd}}{a+c}$.

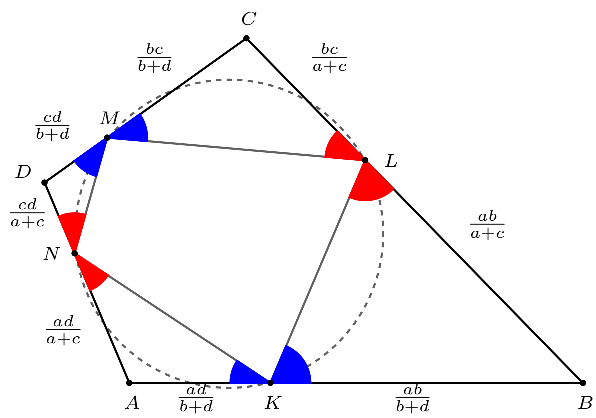

Figure 6.

In order to prove the converse, let $KLMN$ be a cyclic quadrilateral and suppose that $a + c > b + d$. Then $AK = \frac{ad}{b+c} > \frac{ad}{a+c} = AN$ implies that $\angle AKN < \angle KNA$. Similar inequalities hold for the other angles between the sides of $KLMN$ and the sides of $ABCD$. Therefore the sum of the red angles at points N and L, i.e. $\angle KNA + \angle DNM + \angle MLC + \angle BLK$ is greater than the sum of the corresponding blue angles at points K and M. From this we get a contradiction because for a cyclic quadrilateral $KLMN$ both sums must be $180°$. Hence we conclude that the quadrilateral $ABCD$ is tangential. $\square$

It is noteworthy that the radii $r_{\max} = \frac{K}{s} = \frac{\sqrt{abcd}}{a+c}$ of the circles through points K, L, M and N, for incongruent tangential quadrilaterals with equal side lengths but different angles do not depend on the latter. This permits another construction of the bicentric quadrilateral with side lengths a, b, c, d, $a + c = b + d$. This construction is as follows: Draw any tangential quadrilateral $ABCD$ with the side lengths a, b, c, d. Divide its sides in the ratio of the adjacent sides to get the points K, L, M and N (cf. (11)). Measure the distance of the incenter I from one of the points K, L, M and N and construct a circle with this distance as radius, tangent to AB in K. Then complete the construction by drawing tangents from A and B

to this circle whereon the segments $AD = d$ and $BC = b$ are given the prescribed lengths.

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Congruent Contiguous Excircles

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Abstract. In this paper we present some interesting lines in a triangle and we give some of their properties.

1. The e-property

For a given triangle ABC, and a point X on the side BC, consider the contiguous subtriangles ABX and AXC, with their excircles on the sides BX and XC. We shall say that X has the e-property if these two excircles are congruent (see Figure 1).

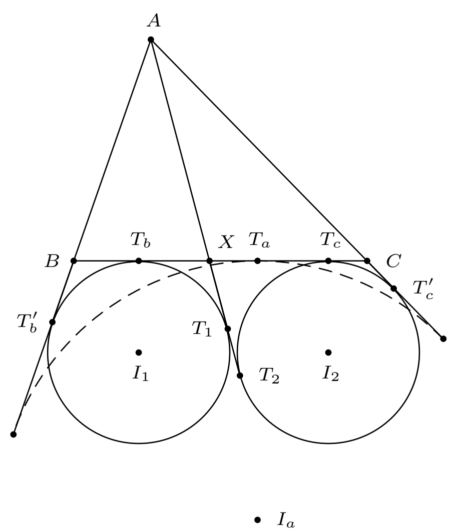

Figure 1

Denote by a, b, c the lengths of the sides BC, CA, AB. The radii of the incircle and excircle on BC are $r = \frac{\Delta}{s}$ and $r_a = \frac{\Delta}{s-a}$, where s and Δ are the semiperimeter and area of triangle ABC. It is known that $BT_a = s - c$ and $T_aC = s - b$.

In the following, we shall denote by s_T and $\Delta(T)$ the semiperimeter and area of a triangle T.

Let I_1, I_2, and I_a be the centers of the excircles of triangles ABX, AXC, and ABC on the sides BX, XC, and BC respectively, with points of tangency indicated in Figure 1. We denote by ρ_a the common exradius of the congruent contiguous excircles of ABX and ACX.

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Theorem 1. *Let X be the point on BC with the e-property.*

(a) $AX = \sqrt{s(s-a)}$.

(b) $\rho_a = \frac{\Delta}{(\sqrt{s}+\sqrt{s-a})\sqrt{s-a}}$.

(c) $BX = \frac{a\left(c+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}$, *and* $CX = \frac{a\left(b+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}$.

Proof. (a) In triangles ABX and AXC we have

$$\begin{aligned}\rho_a &= \frac{\Delta(ABX)}{s_{ABX}-BX} = \frac{\Delta(AXC)}{s_{AXC}-XC}\\ &= \frac{\Delta(ABX)+\Delta(AXC)}{s_{ABX}-BX+s_{AXC}-XC}\\ &= \frac{\Delta}{AX+s-a}.\end{aligned} \tag{1}$$

Since I_1I_2 is parallel to BC,

$$\begin{aligned}\frac{r_a-\rho_a}{r_a} = \frac{I_1I_2}{a} &= \frac{T_bX+XT_c}{a}\\ &= \frac{(s_{ABX}-AX)+(s_{AXC}-AX)}{a} = \frac{s-AX}{a}.\end{aligned}$$

From this, $\frac{r_a-\rho_a}{r_a} = \frac{s-AX}{a}$, and

$$AX-(s-a) = \frac{a\rho_a}{r_a}. \tag{2}$$

From (1) and (2), we have $AX^2-(s-a)^2 = \frac{a\Delta}{r_a} = a(s-a)$. Therefore, $AX^2 = (s-a)^2+a(s-a) = s(s-a)$. This proves (a).

(b) follows from (1) and (a).

(c) Let $BX = x$. We have

$$\rho_a = \frac{\Delta(ABX)}{s_{ABX}-BX} = \frac{\Delta(AXC)}{s_{AXC}-XC} \implies \frac{x}{c+AX-x} = \frac{a-x}{b+AX-(a-x)}.$$

This reduces to $x(b+AX-a+x) = (a-x)(c+AX-x)$, and $x = \frac{a(c+AX)}{b+c+2\cdot AX}$. Making use of (a), we obtain the expression for BX given in (b); similarly for CX. □

Analogous to the congruent contiguous excircles, the problem of congruent contiguous incircles has been studied by P. Yiu ([3] and [4]). It was shown [3, §9.1.5] that for a point X on BC, the incircles of triangles ABX and AXC are congruent if and only if their excircles on the sides BX and XC are congruent, i.e., X has the e-property.

Theorem 2. *For the points of tangency of the excircles indicated in* Figure 1, *we have*

(a) $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$,

(b) $\frac{AT_b'}{AT_c'} = \frac{BX}{CX} = \frac{c+\sqrt{s(s-a)}}{b+\sqrt{s(s-a)}}$.

Proof. (a) From the excircle of triangle ABX on the side BX, we have, making use of Theorem 1 (a) and (c),

$$\begin{aligned}BT_b = \ & s_{ABX} - c = \frac{AX + BX - c}{2} \\ = \ & \frac{\sqrt{s(s-a)} + \frac{a\left(c+\sqrt{s(s-a)}\right)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} - c}{2} \\ = \ & \frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \end{aligned} \tag{3}$$

after simplification. Similarly,

$$CT_c = \frac{(s-b)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}}. \tag{4}$$

From (3) and (4), $\frac{BT_b}{CT_c} = \frac{s-c}{s-b}$. This proves (a).

(b) Note that

$$AT_b' = \ AB+BT_b' = AB+BT_b = c+\frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} = \frac{\sqrt{s}\left(c+\sqrt{s(s-a)}\right)}{\sqrt{s}+\sqrt{s-a}}.$$

Similarly, $AT_c' = \frac{\sqrt{s}\left(b+\sqrt{s(s-a)}\right)}{\sqrt{s}+\sqrt{s(s-a)}}$. From these, (b) follows. □

Theorem 3. *For the points of tangency of the excircles indicated in* Figure 1, *we have*

(a) $T_bT_c = \ I_1I_2 = \frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}$,

(b) $T_1T_2 = \ \frac{|b-c|\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}$.

Proof. (a) Since the excircles are congruent, $I_1I_2T_cT_b$ is a rectangle. Using (3) and (4), we obtain

$$\begin{aligned}T_bT_c = \ & BC - BT_b - CT_c \\ = \ & a - \frac{(s-c)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} - \frac{(s-b)\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \\ = \ & \frac{a\left(\sqrt{s}+\sqrt{s-a}\right) - a\sqrt{s-a}}{\sqrt{s}+\sqrt{s-a}} \\ = \ & \frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}.\end{aligned}$$

(b) If the excircles of triangles ABX and AXC touch the line AX at T_1 and T_2 respectively, then, making use of (a) above and Theorem 1(b), we have

$$\begin{aligned} T_1T_2^2 &= I_1I_2^2 - (\rho_a + \rho_a)^2 \\ &= \left(\frac{a\sqrt{s}}{\sqrt{s}+\sqrt{s-a}}\right)^2 - \frac{4\Delta^2}{\left(\sqrt{s}+\sqrt{s-a}\right)^2(s-a)} \\ &= \frac{a^2s - 4s(s-b)(s-c)}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} \\ &= \frac{s(b-c)^2}{\left(\sqrt{s}+\sqrt{s-a}\right)^2}. \end{aligned}$$

□

Theorem 4. *If X has the e-property, the line AX bisects I_1I_2.*

Proof. Let the bisectors AI_1 and AI_2 of angles BAX and XAC intersect BC at D and E respectively. In triangle ABX with bisector AD, we have, by Theorem 1(a) and (c),

$$DX = \frac{BX \cdot AX}{AB + AX} = \frac{a\sqrt{s(s-a)}}{\left(s+\sqrt{s-a}\right)^2}.$$

Similarly, $EX = \frac{a\sqrt{s(s-a)}}{\left(\sqrt{s}+\sqrt{s-a}\right)^2} = DX$. This shows that X is the midpoint of DE. Since I_1I_2 is parallel to DE, the line AX bisects I_1I_2. □

2. Some identities and inequalities involving points with the e-property

Analogous to the point X on BC with the e-property, there are also points Y on CA, and Z on AB with the e-property, i.e, the contiguous triangles BCY, BYA have congruent excircles on CY, YA, and the pair CAZ, CZB also with congruent excircles on AZ, ZB (see Figure 2). In general, the cevians AX, BY, and CZ are not concurrent. If triangle ABC is isosceles, they do, because of obvious symmetry.

Proposition 5. *The area of triangle XYZ is*

$$\frac{\begin{array}{c}(a+\sqrt{s(s-c)})(b+\sqrt{s(s-a)})(c+\sqrt{s(s-b)}) \\ +(a+\sqrt{s(s-b)})(b+\sqrt{s(s-c)})(c+\sqrt{s(s-a)})\end{array}}{(\sqrt{s}+\sqrt{s-a})^2(\sqrt{s}+\sqrt{s-b})^2(\sqrt{s}+\sqrt{s-c})^2} \cdot \Delta.$$

We establish some identities and inequalities involving the cevian lines through through the points with the e-property. Denote by R the circumradius of triangle ABC.

Theorem 6. (a) $AX \cdot BY \cdot CZ = s\Delta$.

(b) $AX^2 + BY^2 + CZ^2 = s^2$.

(c) $\frac{1}{AX^2} + \frac{1}{BY^2} + \frac{1}{CZ^2} = \frac{4R+r}{rs^2}$.

(d) $\frac{a}{AX^2} + \frac{b}{BY^2} + \frac{c}{CZ^2} = \frac{2(2R-r)}{\Delta}$.

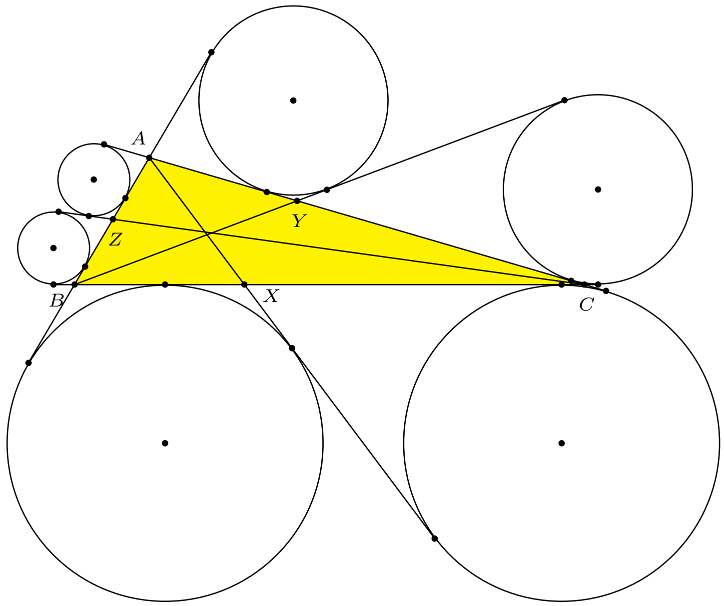

Figure 2

(e) $AX^4 + BY^4 + CZ^4 = s^2(s^2 - 2r^2 - 8Rr)$.
(f) $AX^6 + BY^6 + CZ^6 = s^4(s^2 - 12Rr)$.

Proof. These follow from Theorem 1(a), (c), and the basic relations

$$\begin{aligned} ab + bc + ca &= s^2 + r^2 + 4Rr, \\ a^2 + b^2 + c^2 &= 2(s^2 - r^2 - 4Rr), \\ a^3 + b^3 + c^3 &= 2s(s^2 - 3r^2 - 6Rr); \end{aligned}$$

see [1]. □

Theorem 7. *Let m_a and w_a be the lengths of the median on BC and angle bisector of angle A. If X is the point on BC with the e-property, then $w_a \le AX \le m_a$.*

Proof. For the bisector of angle A,

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} = \frac{2\sqrt{bc}}{b+c}\cdot AX \le AX,$$

since $2\sqrt{bc} \le b + c$.

On the other hand, for the median on BC, the inequality $AX \le m_a$ is equivalent to $s(s-a) \le \frac{2(b^2+c^2)-a^2}{4}$. After some rearrangement, this reduces to $(b+c)^2 \le 2(b^2+c^2)$, which is clearly valid. □

Theorem 8. (a) $AX \cdot BY + BY \cdot CZ + CZ \cdot AX \le s^2$.
(b) $3\sqrt[3]{s\Delta} \le AX + BY + CZ \le \sqrt{3}s$.

Proof. These follow from applying the well known inequalities $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$ and $xy+yz+zx \leq x^2+y^2+z^2$, applied to $x = AX$, $y = BY$, and $z = CZ$, and making use of Theorem 6 (a), (b). □

Remarks. (1) By using Schwarz's Inequality, we have

$$(AX^2+BY^2+CZ^2)\left(\frac{1}{AX^2}+\frac{1}{BY^2}+\frac{1}{CZ^2}\right) \geq 9.$$

By using the formulae in Theorem 6(b), (c), we have $s^2 \cdot \frac{4R+r}{rs^2} \geq 9$. From this the well known Euler's inequality $R \geq 2r$ follows.

(2) Again, it is easy to establish

$$AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 = (4R+r)rs^2.$$

From the inequality

$$AX^2 \cdot BY^2 + BY^2 \cdot CZ^2 + CZ^2 \cdot AX^2 \leq AX^4 + BY^4 + CZ^4,$$

and the formula in Theorem 6(e), we have $(4R+r)rs^2 \leq s^2(s^2-2r^2-8Rr)$. This leads to another known inequality

$$3(4R+r)r \leq s^2;$$

see [1, 2].

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Some Circles Associated with the Feuerbach Points

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Abstract. Consider a triangle with its nine-point circle tangent to the incircle and excircles at the Feuerbach point. We show that the four circles each through the circumcenter, nine-point, and Feuerbach point contain the nine-point center of the intouch triangle or the corresponding extouch triangle. Furthermore, the lines joining these Feuerbach points to the corresponding nine-point centers are concurrent on the nine-point circle of the given triangle.

1. Four coaxial circles through the Feuerbach points

The starting point of this note is the famous Feuerbach theorem, that for a given triangle, the nine-point circle is tangent internally to the incircle and externally to each of the excircles. Given triangle ABC, with incenter I, excenters I_a, I_b, I_c, and nine-point N, the points of tangency of the nine-point circle with these circles are the Feuerbach points $F_{\rm e}$, F_a, F_b, F_c on the lines NI, NI_a, NI_b, NI_c with ratios of division

$$NF_{\rm e} : F_{\rm e}I = \frac{R}{2} : -r,$$

$$NF_a : F_aI_a = \frac{R}{2} : r_a, \quad NF_b : F_bI_b = \frac{R}{2} : r_b, \quad NF_c : F_cI_c = \frac{R}{2} : r_c,$$

where R, r, r_a, r_b, r_c are the circumradius, inradius, and exradii.

Proposition 1. *Let O be the circumcenter of triangle ABC.*
(a) $OI^2 = R(R - 2r)$.
(b) $OI_a^2 = R(R + 2r_a)$.
(c) $II_a^2 = 4R(r_a - r)$.
(d) *The excentral triangle $I_aI_bI_c$ has circumcenter I' at the reflection of I in O, and circumradius $2R$.*

For (a-c), see [1, Theorems 152-154]. For (d), see [4, §4.6.1].

Consider the circle through O, N, and $F_{\rm e}$. Since I is an interior point of the segment $NF_{\rm e}$, it is an interior point of the circle. Our first result relates the endpoint of the chord through O and I with the intouch triangle, whose vertices are the points of tangency of the incircle with the sidelines.

Theorem 2. *The line OI intersects the circle $ONF_{\rm e}$ again at the nine-point center of the intouch triangle.*

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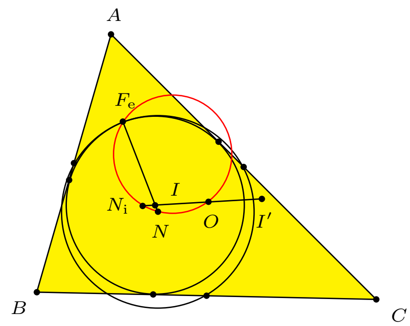

Figure 1

Proof. Let N_i be the second intersection of the line OI with the circle ONF_e. By the intersecting chords theorem, $OI \cdot IN_i = NI \cdot IF_e$. Therefore,

$$\frac{IN_i}{OI} = \frac{NI \cdot IF_e}{OI^2} = \frac{\left(\frac{R}{2} - r\right) r}{R(R-2r)} = \frac{r}{2R}.$$

Note that the intouch triangle is homothetic to the excentral triangle. Since the excentral triangle has circumcenter I' and nine-point center O, its Euler line is the line OI. Since the intouch triangle has circumcenter I, its Euler line is also the line OI. From

$$\frac{IN_i}{I'O} = \frac{IN_i}{OI} = \frac{r}{2R},$$

the homothetic ratio of the intouch and excentral triangles, we conclude that N_i is nine-point center of the intouch triangle. □

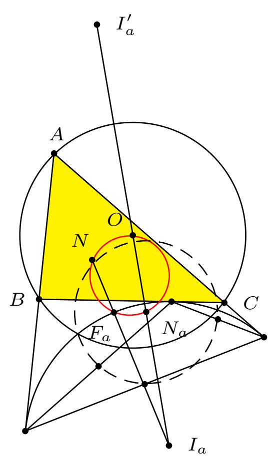

Figure 2.

Analogous results hold if we replace the Feuerbach point F_e by the other Feuerbach points, say, F_a. if the circle through O, N, F_a intersects the line OI_a at N_a, then $\frac{I_aN_a}{I_aO} = \frac{r_a}{2R}$. Now, triangle II_bI_c is homothetic to the A-extouch triangle formed by the points of tangency of the A-excircle with the sidelines of triangle ABC, with homothetic ratio $-\frac{r}{2R}$, since the circumradius of II_bI_c is also $2R$. In fact, the circumcenter of II_bI_c is the reflection of I_a in O. It follows that $\frac{I_aN_a}{I'_aO} = -\frac{I_aN_a}{OI_a} = -\frac{r}{2R}$, and N_a is the nine-point center of the A-extouch triangle (see Figure 2).

Theorem 3. *The points O, N, F_a and N_a are concyclic; so are O, N, F_b, N_b, and O, N, F_c, N_c.*

2. Concurrency of four lines on the nine-point circle

In preparation for the proof of our next main result (Theorem 7 below), we establish an interesting relation between the centers O, N, I, I_a given in Proposition 5. The reformulation as Proposition 6 in terms of directed angles ([2, §§16-19]) makes the proof of Theorem 7 independent of the relative position of O and N with respect to the bisector of angle A.

Lemma 4. *Let N be the nine-point center of triangle ABC.*
(a) *If $A > 60°$, then N and O lie on the same side of the bisector of angle A.*
(b) *If $A = 60°$, then N lies on the bisector of angle A.*
(c) *If $A < 60°$, then N and O lie on opposite sides of the bisector of angle A.*

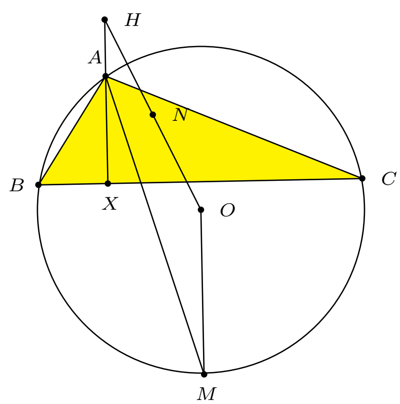

Figure 3A

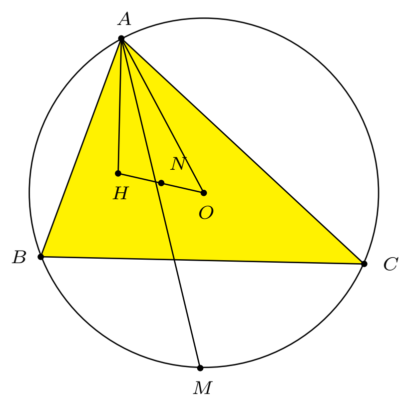

Figure 3B

Proof. First consider the case when A is an obtuse angle (see Figure 3A). Construct the perpendicular from O to BC, to intersect the circumcircle at M on the opposite side of A. The line AM is clearly the bisector of angle A. If X is the orthogonal projection of A on BC, then the orthocenter H and X are on opposite sides of A. It follows that O and H, and their midpoint N, all are on the same side of the bisector AM.

The same conclusion holds if $A = 90°$, since the orthocenter H coincides with A.

Now we assume A an acute angle (see Figure 3B). It is known that $AH = 2R\cos A$, and that the bisector of angle A also bisects the angle OAH. It divides

OH in the ratio $R : 2R\cos A = 1 : 2\cos A$. Therefore, O and N are on the same side of the bisector if and only if $2\cos A < 1$, i.e., $A > 60°$.

If $A = 60°$, then $AH = AO$, and N lies on the bisector. This completes the proof of the theorem. □

Proposition 5. (a) *The angle IOI_a is acute, right, or obtuse according as A is less than, equal to, or greater than* $60°$.

(b) $\angle INI_a = \begin{cases} 2\angle IOI_a, & \text{if } A \leq 60°, \\ 360° - 2\angle IOI_a, & \text{if } A > 60°. \end{cases}$.

Proof. (a) Applying the law of cosines to triangle IOI_a, and using the expressions for the lengths given in Proposition 2, we have

$$\begin{aligned}\cos IOI_a = & \frac{OI^2 + OI_a^2 - II_a^2}{2 \cdot OI \cdot OI_a} \\ = & \frac{R(R-2r) + R(R+2r_a) - 4R(r_a - r)}{2 \cdot OI \cdot OI_a} \\ = & \frac{R(R - r_a + r)}{OI \cdot OI_a}. \end{aligned} \tag{1}$$

The angle IOI_a is acute, right, or obtuse according as $r_a - r$ is less than, equal to or greater than R. Since $\frac{r_a - r}{R} = 4\sin^2\frac{A}{2}$, these are the cases according as A is less than, equal to, greater than $60°$.

(b) From (1), we have

$$\begin{aligned}\cos 2 \cdot IOI_a = & 2\cos^2 IOI_a - 1 \\ = & \frac{(OI^2 + OI_a^2 - II_a^2)^2 - 2 \cdot OI^2 \cdot OI_a^2}{2 \cdot OI^2 \cdot OI_a^2} \\ = & \frac{(2R(R - r_a + r))^2 - 2R(R-2r) \cdot R(R+2r_a)}{2R(R-2r) \cdot R(R+2r_a)} \\ = & \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R-2r)(R+2r_a)}. \end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned}\cos INI_a = & \frac{NI^2 + NI_a^2 - II_a^2}{2NI \cdot NI_a} \\ = & \frac{\left(\frac{R}{2} - r\right)^2 + \left(\frac{R}{2} + r_a\right)^2 - 4R(r_a - a)}{2\left(\frac{R}{2} - r\right)\left(\frac{R}{2} + r_a\right)} \\ = & \frac{R^2 - 6R(r_a - r) + 2(r_a^2 + r^2)}{(R-2r)(R+2r_a)}. \end{aligned} \tag{3}$$

Comparison of (2) and (3) shows that $\cos INI_a = \cos 2 \cdot IOI_a$. Therefore, $\angle INI_a = 2\angle IOI_a$ or $360° - 2\angle IOI_a$. Taking (a) into account, we have (b). □

Remark. When $A = 60°$, $\angle IOI_a = 90°$, and $\angle INI_a = 180°$ (see Lemma 4(b)).

In terms of directed angles, we reformulate Proposition 5 below.

Proposition 6. $(NI,\ NI_a) = -2(OI,\ OI_a)$.

Theorem 7. *The four lines $F_{\mathrm{e}}N_{\mathrm{i}}$, F_aN_a, F_bN_b, F_cN_c are concurrent at a point on the nine-point circle.*

Proof. It is enough to prove this for the lines $F_{\mathrm{e}}N_{\mathrm{i}}$ and F_aN_a (see Figure 4). Let P be the intersection of the line $F_{\mathrm{e}}N_{\mathrm{i}}$ with the nine-point circle. We show that it also lies on the line F_aN_a. For this, it is enough to verify $(F_aN,\ F_aP) = (F_aN,\ F_aN_a)$.

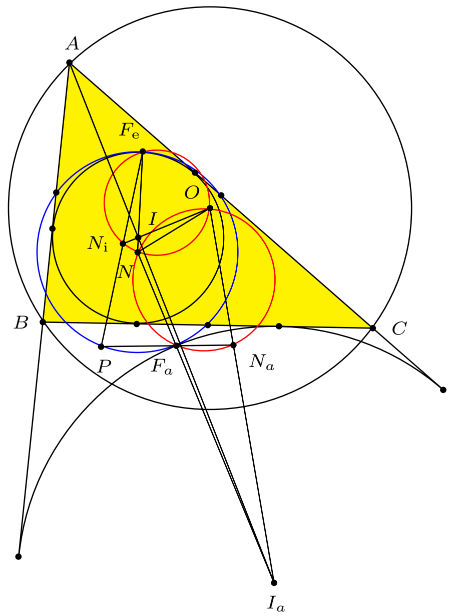

Figure 4.

$$\begin{aligned}
(F_aN,\ F_aP) &= (PF_a,\ PN) && \text{triangle } NF_aP \text{ isosceles}\\
&= (PF_a,\ PF_{\mathrm{e}}) + (PF_{\mathrm{e}},\ PN) &&\\
&= \frac{1}{2}(NF_a,\ NF_{\mathrm{e}}) + (PF_{\mathrm{e}},\ PN) && N = \text{center of circle } PF_{\mathrm{e}}F_a\\
&= \frac{1}{2}(NI_a,\ NI) + (PF_{\mathrm{e}},\ PN) &&\\
&= (OI,\ OI_a) + (PF_{\mathrm{e}},\ PN) && \text{Proposition 6}\\
&= (OI,\ OI_a) + (NF_{\mathrm{e}},\ N_{\mathrm{i}}F_{\mathrm{e}}) && \text{triangle } NPF_{\mathrm{e}} \text{ isosceles}\\
&= (OI,\ OI_a) + (NO,\ N_{\mathrm{i}}O) && O,\ N,\ F_{\mathrm{e}},\ N_{\mathrm{i}} \text{ concyclic}\\
&= (OI,\ ON_a) + (ON,\ OI) &&\\
&= (ON,\ ON_a) &&\\
&= (F_aN,\ F_aN_a) && O,\ N,\ F_a,\ N_a \text{ concyclic.}
\end{aligned}$$

The same reasoning shows that P also lies on the lines F_bN_b and F_cN_c. □

We summarize the main results in this note in Figure 5 below, and conclude with an identification of the triangle centers $N_{\rm i}$ and P. According to the ENCYCLOPEDIA OF TRIANGLE CENTERS [3], the intouch triangle has orthocenter $X(65)$. Its nine-point center $N_{\rm i}$, being the midpoint of $IX(65)$, is $X(942)$. This point lies on the line through $X(11) = F_{\rm e}$ and $X(113)$, which is on the nine-point circle, Therefore P is $X(113)$, which is also the midpoint of $HX(110)$.

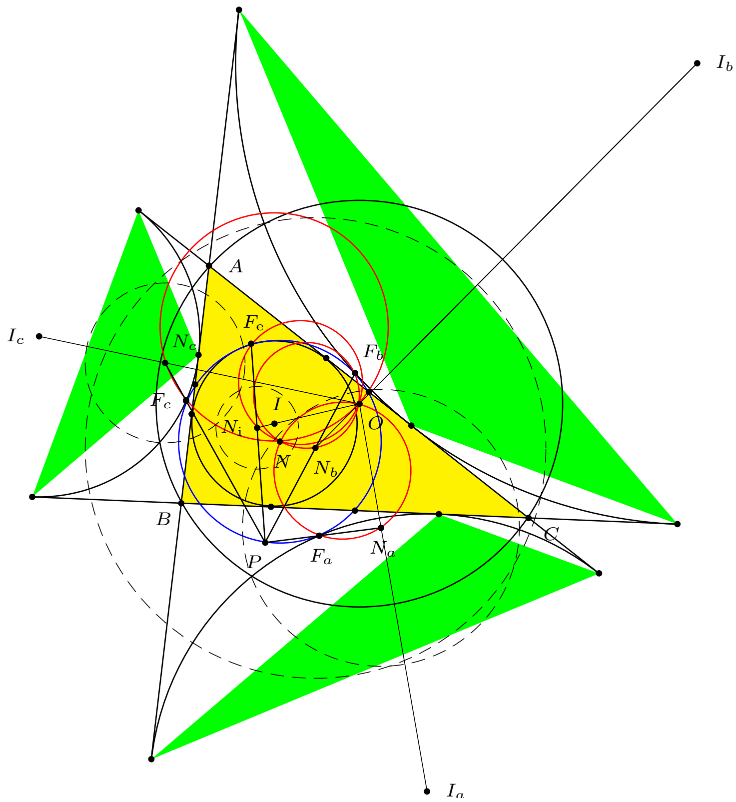

Figure 5.

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Generalized Archimedean Arbelos Twins

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Abstract. We generalize the well known Archimedean arbelos twins by extending the notion of arbelos, and we construct an infinite number of Archimedean circles.

1. Archimedean arbelos

On a segment AB we take an arbitrary point P and with diameters AP, PB, AB we construct the semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, where $R = R_1 + R_2$. If we cut from the large semicircle the small ones then the resulting figure is called from antiquity (Archimedes) arbelos (the shoemaker's knife). The perpendicular at P to AB meets the large semicircle at Q and divides the arbelos in two mixtilinear triangles with equal incircles.

Theorem 1 (Archimedean arbelos twins)**.** *The two circles $K_1(r_1)$ and $K_2(r_2)$ that are inside the arbelos and are tangent to the arbelos and the line PQ have equal radii $r_1 = r_2 = \frac{R_1R_2}{R_1+R_2}$* (see Figure 1)*. Equivalently,*

$$\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$

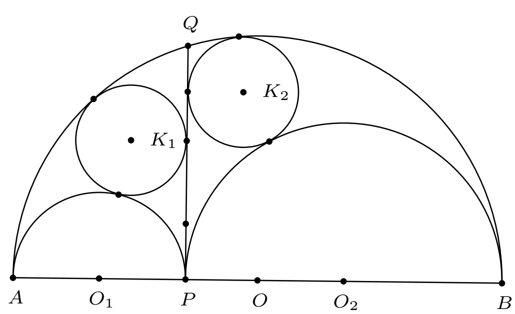

Figure 1.

A circle in the arbelos with radius r_1 is called an Archimedean circle ([6, p.61]).

We can find infinitely many such twin incircles where the above case is a limit special case. We generalize the notion of the arbelos as a triangle whose sides are in general circular arcs. We will investigate some special cases of these arbeloi.

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2. Generalized arbelos

2.1. *Soddy type arbelos with 3 vertices tangency points of the arcs.* Let U_X, U_Y, U_Z (counter clockwise direction) be the vertices of the arbelos whose sides are arcs of circles tangent to each other at the vertices, with centers X, Y, Z and radii r_X, r_Y, r_Z. If the rotation from U_Y to U_Z on the arc with radius r_X is clockwise relative to U_X, then r_X is positive, otherwise it is negative. Similarly we characterize the radii r_Y, r_Z relative to the movement on the appropriate arcs from U_Z to U_X, and from U_X to U_Y. Hence, we can have 8 different cases of arbeloi with the same vertices and the same radii in absolute value. If we change the sign of the radii of the arbelos, then we get the complementary arbelos. Denote by Δ the area of the triangle XYZ and r the radius of a circle tangent to the arcs-sides of the arbelos.

Theorem 2. *The radius r of the circle that is tangent to the sides of the above arbelos is given by*

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} + \frac{2\Delta}{r_X r_Y r_Z},$$

or

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} - \frac{2\Delta}{r_X r_Y r_Z}.$$

One of these corresponds to the complementary arbelos.

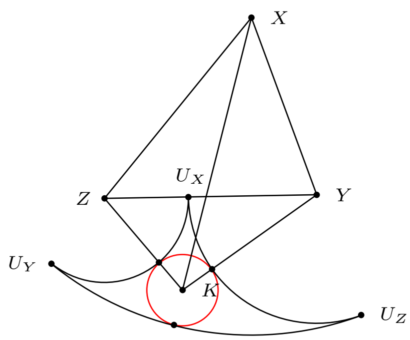

Figure 2a

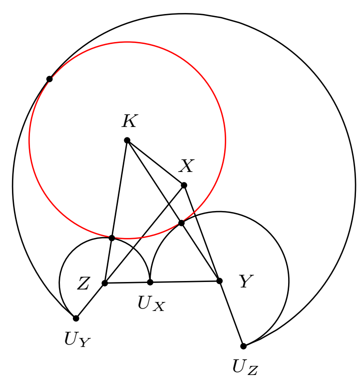

Figure 2b

Proof. If $K(r)$ is the circle tangent to the sides of the arbelos, then in Figure 2a, the radius r_X is negative, the sides of triangle XYZ are $a = |r_Y + r_Z|$, $b = |r_Z + r_X|$, $c = |r_X + r_Y|$, and the tripolar coordinates of K relative to XYZ and the area Δ of triangle XYZ are

$$KX = \lambda = |r_X + r|, \quad KY = \mu = |r_Y + r|, \quad KZ = \nu = |r_Z + r|,$$
$$\Delta = \sqrt{r_X r_Y r_Z (r_X + r_Y + r_Z)}.$$

If we substitute these values into the equality that the tripolar coordinates satisfy ([3]):

$$(\mu^2+\nu^2-a^2)^2\lambda^2+(\nu^2+\lambda^2-b^2)^2\mu^2+(\lambda^2+\mu^2-c^2)^2\nu^2$$
$$-(\mu^2+\nu^2-a^2)(\nu^2+\lambda^2-b^2)(\lambda^2+\mu^2-c^2)=\lambda^2\mu^2\nu^2,$$

we get

$$\left(\frac{1}{r}-\frac{1}{r_X}-\frac{1}{r_Y}-\frac{1}{r_Z}\right)^2=\frac{4\Delta^2}{r_X^2r_Y^2r_Z^2}.$$

Therefore,

$$\frac{1}{r}=\frac{1}{r_X}+\frac{1}{r_Y}+\frac{1}{r_Z}\pm\frac{2\Delta}{r_Xr_Yr_Z}.$$

□

Remarks. (1) The Arcimedean arbelos is of Soddy type with collinear vertices where $r_X=R_1$, $r_Y=R_2$, $r_Z=-R=-(R_1+R_2)$, $\Delta=0$. In this case, there is a double solution. Hence, the inradius of the Archimedean arbelos is given by

$$\frac{1}{r}=\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_1+R_2}.$$

(2) If the radii r_X, r_Y, r_Z are positive, then the radii r refer to the inner and outer Soddy circles ([2]).

In the sequel, we shall adopt the following notation: For a line ℓ, Π_ℓ denotes the *orthogonal projection map* onto ℓ.

Lemma 3. *Let $K(r)$ be a circle tangent externally or internally to the circles $O_1(R_1)$ and $O_2(R_2)$, where R_1, R_2 may assume any real values. For a point F on the radical axis of the circles,*

$$r=\frac{\overrightarrow{O_1O_2}\cdot\Pi_{O_1O_2}(\overrightarrow{FK})}{R_1-R_2}. \tag{1}$$

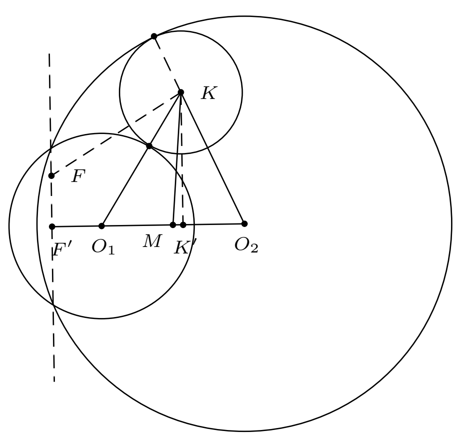

Figure 3.

Proof. Let M be the midpoint of O_1O_2 (Figure 3). We have

$$R_2^2 - R_1^2 = FO_2^2 - FO_1^2 = (\overrightarrow{FO_2} - \overrightarrow{FO_1}) \cdot (\overrightarrow{FO_2} + \overrightarrow{FO_1}) = 2\overrightarrow{O_1O_2} \cdot \overrightarrow{FM},$$
$$|R_1 + r|^2 - |R_2 + r|^2 = KO_1^2 - KO_2^2 = (\overrightarrow{KO_1} - \overrightarrow{KO_2}) \cdot (\overrightarrow{KO_1} + \overrightarrow{KO_2}) = 2\overrightarrow{O_1O_2} \cdot \overrightarrow{MK}.$$

By addition, we get

$$2r(R_1 - R_2) = 2\overrightarrow{O_1O_2} \cdot (\overrightarrow{FM} + \overrightarrow{MK}) = 2\overrightarrow{O_1O_2} \cdot \Pi_{O_1O_2}(\overrightarrow{FK}),$$

and the result follows. □

Remark. For the Archimedean twins (1) gives

$$r_1 = \frac{(2R_1 - r_1)R_2}{R_1 + R_2 + R_1} \implies \frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Theorem 4. *Let $U_XU_YU_Z$ be an arbelos with collinear centers X, Y, Z on a line $\mathcal{L}$, and $K(r)$ be the incircle of the arbelos.*

$$r = \left| \frac{\Pi_{\mathcal{L}}(\overrightarrow{U_YU_Z})}{\frac{r_X - r_Y}{XY} - \frac{r_X - r_Z}{XZ}} \right|.$$

Proof. Since U_Z, U_Y are points on the radical axes of the circle pairs $X(r_X)$, $Y(r_Y)$, and $X(r_X)$, $Z(r_Z)$ respectively, by Lemma 3 we have

$$r = \frac{\overrightarrow{XY} \cdot \Pi_{\mathcal{L}}(\overrightarrow{U_ZK})}{r_X - r_Y} = \frac{\overrightarrow{XZ} \cdot \Pi_{\mathcal{L}}(\overrightarrow{U_YK})}{r_X - r_Z}.$$

Since X, Y, Z are all on the line $\mathcal{L}$, $\Pi_{\mathcal{L}}(\overrightarrow{U_ZK}) = \frac{r(r_X - r_Y)}{XY}\overrightarrow{\jmath}$ and $\Pi_{\mathcal{L}}(\overrightarrow{U_YK}) = \frac{r(r_X - r_Z)}{XZ}\overrightarrow{\jmath}$, where $\overrightarrow{\jmath}$ is a unit vector on $\mathcal{L}$. Hence,

$$\begin{aligned} \Pi_{\mathcal{L}}(\overrightarrow{U_YU_Z}) &= \Pi_{\mathcal{L}}(\overrightarrow{U_YK}) - \Pi_{\mathcal{L}}(\overrightarrow{U_ZK}) \\ &= r\left(\frac{r_X - r_Z}{XZ} - \frac{r_X - r_Y}{XY}\right)\overrightarrow{\jmath} \end{aligned}$$

From this the result follows. □

2.2. *Arbelos of type A.* On a line $\mathcal{L}$ we take the consecutive points U_Z', U_Y, U_Z, U_Y', and construct on the same side of the line the semicircles $(U_Z'U_Z)$, (U_YU_Y'), and (U_YU_Z). Let U_X be the intersection of the first two semicircles (see Figure 4). The arbelos $U_XU_YU_Z$ is of type A. It has arc U_YU_Z positive, and U_ZU_X, U_XU_Y both negative. The diameter U_YU_Z is the base of the arbelos.

2.2.1. *The incircle.* Let $K(r)$ be the incircle of this arbelos and A, B, C the points of tangency. If S is the external center of similitude of the semicircles $(U_Z'U_Z)$ and (U_YU_Y'), then the line BC passes through S, and the inversion with pole S and power $d^2 = SU_Y \cdot SU_Z = SU_X^2 = SB \cdot SC$ swaps the semicircles $(U_Z'U_Z)$ and (U_YU_Y'), and leaves the circle $K(r)$ and the semicircle (U_YU_Z) invariant. Hence, $SA^2 = SB \cdot SC$, and the SA is tangent at A to both $K(r)$ and (U_YU_Z). If the line SA meets the perpendiculars to $\mathcal{L}$ at U_Y, U_Z at D and E respectively, then D is the radical center of $K(r)$, (U_YU_Z), (U_YU_Y'), and E is the radical center of $K(r)$, (U_YU_Z), $(U_Z'U_Z)$. Hence, $DC = DA$ and $EB = EA$.

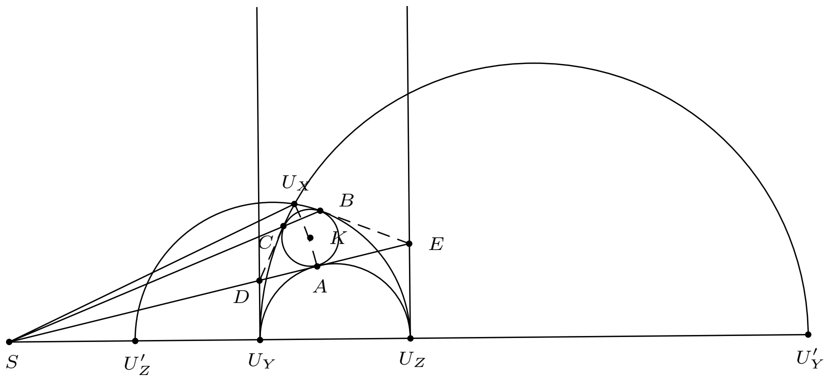

Figure 4.

Construction of the incircle. We construct the external center of similitude S of the semicircles $(U'_Z U_Z)$ and $(U_Y U'_Y)$, and take a point A on the semicircle $(U_Y U_Z)$ such that $SA = SU_X$. Let the line SA meet the perpendiculars to $\mathcal{L}$ through U_Y, U_Z at D, E respectively. We take on $(U'_Z U_Z)$ the point B such that $EB = EA$, and on $(U_Y U'_Y)$ the point C such that $DC = DA$. The circumcircle of ABC is the incircle of the arbelos.

The radius of the incircle. If we take $U'_Z U_Y = 2y$, $U_Y U_Z = 2R$, $U_Z U'_Y = 2z$, then $r_X = R$, $r_Y = -R - y$, $r_Z = -R - z$, $XY = -y$, and $XZ = z$. From Theorem 4, we have

$$r = \left| \frac{2R}{\frac{2R+y}{-y} - \frac{2R+z}{z}} \right| = \frac{Ryz}{R(y+z)+yz},$$

or

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} + \frac{1}{z}. \tag{2}$$

2.3. *Arbelos of type B.* On a line $\mathcal{L}$ we take the consecutive points U'_Y, U'_Z, U_Y, U_Z, and construct on the same side of the line the semicircles $(U'_Z U_Z)$, $(U'_Y U_Y)$, and $(U_Y U_Z)$. Let U_X be the intersection of the first two semicircles (see Figure 5). The arbelos $U_X U_Y U_Z$ is of type B. It has arc $U_Y U_Z$ positive and the arcs $U_Z U_X$, $U_X U_Y$ of different signs. The base of the arbelos is $U_Y U_Z$.

Construction of the incircle. The construction is the same as in type A, but now S is the internal point of similitude of the semicircles $(U'_Z U_Z)$ and $(U'_Y U_Y)$.

The radius of the incircle. We have the same formula as (2), but now since U'_Y is not on the right hand side of U_Z, the distance z must be negative, and so we have

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} - \frac{1}{z}. \tag{3}$$

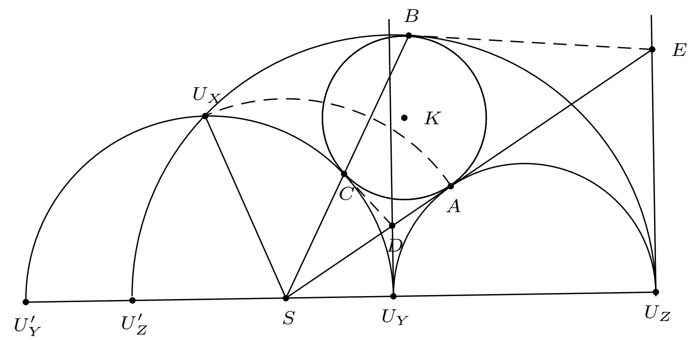

Figure 5.

Remark. For the incircle $K(r)$ of the Archimedean arbelos, since $U_X = A$, $U_Y = P$, $U_Z = B$ with base $2R_2$, $y = R_1$, $z = -R_1 - R_2$, (3) gives

$$\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.$$

3. Generalized Archimedean arbelos twins

We shall construct in the Archimedean arbelos a generalized pair of inscribed equal circles.

Theorem 5. *In the Archimedean arbelos* (Figure 6) *where $AP = 2R_1$, $PB = 2R_2$, $AB = 2R_1 + 2R_2$, we extend AB (to left and right) with equal segments $AA' = 2x = BB'$. The semicircle $(A'P)$ divides the arbelos in two arbeloi with incircles $K_1(r_1)$, $K_3(r_3)$, and the semicircle (PB') divides the arbelos in two arbeloi with incircles $K_2(r_2)$, $K_4(r_4)$. We have a couple of twin circles: $r_1 = r_2$ and $r_3 = r_4$, with*

$$\frac{1}{r_1} = \frac{1}{x} + \frac{1}{R_1} + \frac{1}{R_2} \quad \textit{and} \quad \frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.$$

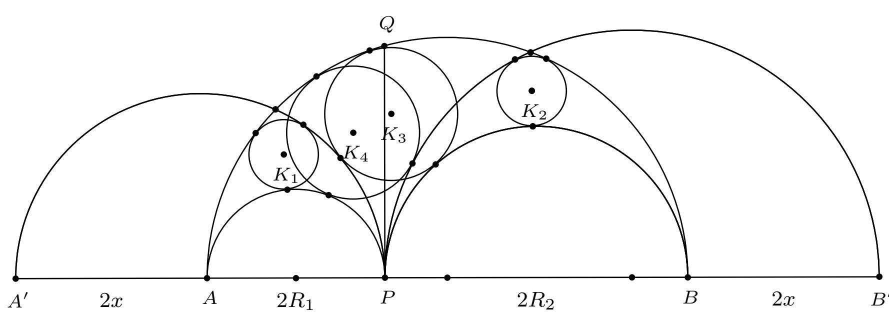

Figure 6.

Proof. The circle $K_1(r_1)$ is the incircle of an arbelos of type A with base AP. Hence, $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} + \frac{1}{R_2}$.

The circle $K_2(r_2)$ is the incircle of an arbelos of type A with base PB. Hence, $\frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{x} + \frac{1}{R_1}$.

The circle $K_3(r_3)$ is the incircle of an arbelos of type B with base PB. Hence, $\frac{1}{r_3} = \frac{1}{R_2} + \frac{1}{R_1} - \frac{1}{R_1+R_2+x}$.

The circle $K_4(r_4)$ is the incircle of an arbelos of type B with base AP. Hence, $\frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1+R_2+x}$.

Therefore, $r_1 = r_2$ and $r_3 = r_4$. □

Remark. If $x \to \infty$, then the semicircles $(A'P)$ and (PB') tend to the perpendicular semiline PQ, and the four incircles tend to the Archimedean twin circles.

4. Bisectors of the Archimedean Arbelos

We have seen that in the Archimedean arbelos, the semiline PQ divides the arbelos in two arbeloi with equal incircles. This semiline as a degenerate semicircle is like a bisector from P of the arbelos that produces the twins (Archimedean circles) with radius r_P such that

$$\frac{1}{r_P} = \frac{1}{R_1} + \frac{1}{R_2}.$$

We find the other two bisectors of the arbelos from the vertices A and B (Figure 7).

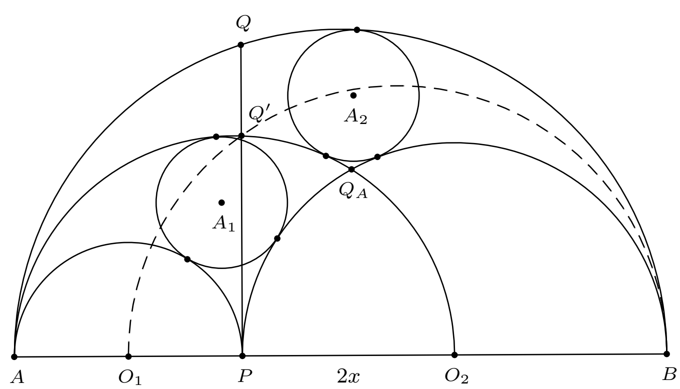

Figure 7.

Let the semicircle (AO_2) bisector of the arbelos meet the semicircle (PB) at Q_A, and $A_1(r_1)$, $A_2(r_2)$ be the incircles of the arbeloi Q_AAP, Q_AAB.

The arbelos Q_AAP with base AP is of type B, so we have $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} - \frac{1}{R_1+R_2}$.

The arbelos Q_AAB with base AB is the complement of an arbelos of type A; so we have $\frac{1}{r_2} = -\left(\frac{1}{R_1+R_2} - \frac{1}{R_1} - \frac{1}{R_2-x}\right)$. In order to have $r_1 = r_2 = r_A$, we set $2x = R_2$. Hence the point O_2 is the midpoint of PB, and the bisector semicircle (AO_2)meets PQ at the point Q' such that $PQ' = \sqrt{AP \cdot PO_2} = \sqrt{2R_1R_2}$, and $\frac{1}{r_A} = \frac{1}{R_1} + \frac{2}{R_2} - \frac{1}{R_1+R_2}$.

Similarly, if O_1 is the midpoint of AP, then the semicircle (O_1B) is the bisector of the arbelos from B that passes also from Q', and $\frac{1}{r_B} = \frac{2}{R_1} + \frac{1}{R_2} - \frac{1}{R_1+R_2}$.

Hence, *the three bisectors of the Archimedean Arbelos are concurrent at Q'.*

5. Infinitely many Archimedean circles

There are many exciting constructions of Archimedean circles or families of these; see [4, 5].

Here we construct a more natural family of Archimedean circles that contains the original Archimedean twins. In the Arcimedean arbelos with diameters AP, PB, AB, and semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, we take at the left of A the point X, and at right of B the point Y such that $AX = BY = 2x$. We rotate clockwise P around A by an angle $\frac{\pi}{2}$ at A_1, and counterclockwise P around B by an angle $\frac{\pi}{2}$ at B_1. The line XA_1 meets the line PQ at D_1. The line YB_1 meets the line PQ at C_1. We take at the left of A the point C such that $CA = PC_1 = 2d_1$, and at right of B the point D such that $BD = PD_1 = 2d_2$. The semicircles (CP), (AY) meet at M, and the semicircles (PD), (XB) meet at N. We show that the incircles $K_1(r_1)$, $K_2(r_2)$ of the arbeloi MAP, NPB are both Archimedean circles.

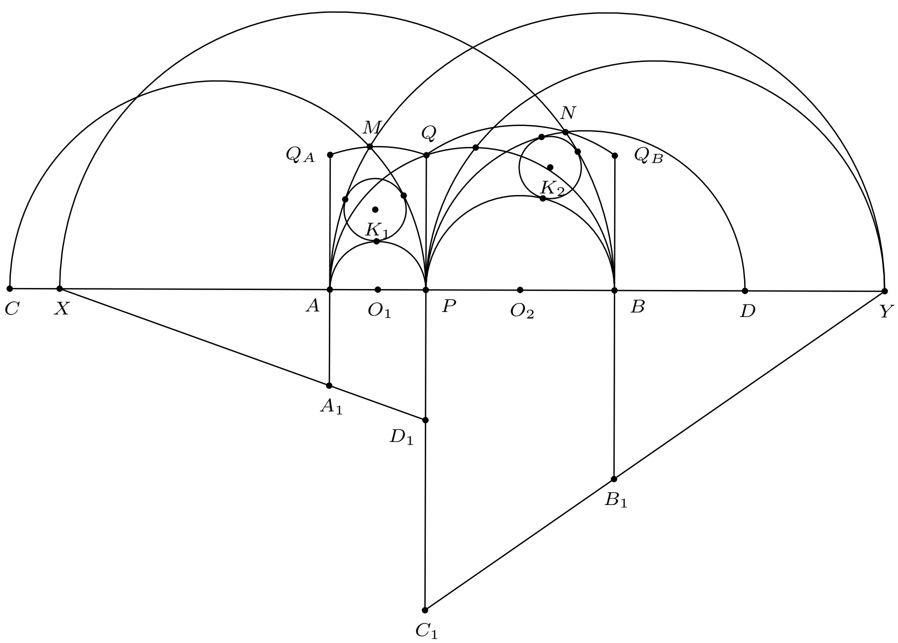

Figure 8.

Proof. Since in triangle XPD_1, PA_1 is a bisector and AA_1 is parallel to PD_1 (Figure 8), we know that $\frac{1}{PD_1} + \frac{1}{XP} = \frac{1}{AA_1}$. Hence,

$$\frac{1}{d_2} + \frac{1}{x + R_1} = \frac{1}{R_1}. \tag{4}$$

Similarly, from triangle PYC_1, we have

$$\frac{1}{d_1} + \frac{1}{x + R_2} = \frac{1}{R_2}. \tag{5}$$

The arbelos MAP is of type A; so we have $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{d_1} + \frac{1}{x+R_2} = \frac{1}{R_1} + \frac{1}{R_2}$ from (5).

The arbelos NPB is also of type A; so we have $\frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{d_2} + \frac{1}{x+R_1} = \frac{1}{R_2} + \frac{1}{R_1}$ from (4).

Hence $r_1 = r_2$ is the radius of the Archimedean circle. □

For $x = 0$ the circles $K_1(r_1)$, $K_2(r_2)$ coincide with the original Archimedean twin circles since the semicircles (AY), (XB) coincide with the semicircle (AB), and the semicircles (CP), (PD) coincide with the line PQ. If M_1, M_2 are the midpoints of CP, AY, then applying Stewart's theorem to triangle MM_1M_2, we have

$$O_1M^2 \cdot M_1M_2 = M_1M^2 \cdot O_1M_2 + M_2M^2 \cdot O_1M_1 - O_1M_1 \cdot O_1M_2 \cdot M_1M_2,$$

or

$$(d_1+R_2+x)O_1M^2 = (d_1+R_1)^2(R_2+x)+(R_1+R_2+x)^2d_1-d_1(R_2+x)(d_1+R_2+x).$$

Substituting $d_1 = \frac{R_2(x+R_2)}{x}$, we get

$$O_1M^2 = R_1^2 + 4R_1R_2 = O_1P^2 + PQ^2 = O_1Q^2.$$

Hence, the locus of M is the circular arc $O_1(Q)$ from the point Q to Q_A on the perpendicular Q_AA to AB. Similarly, we can prove that the locus of N is the circular arc $O_2(Q)$ from the point Q to Q_B on the perpendicular Q_BB to AB.

6. A special generalization of arbelos with arcs of angle 2ϕ

If we substitute the semicircles in Archimedean arbelos with arcs of angle 2ϕ, i.e., $AP = 2R_1 \sin\phi$, $PB = 2R_2 \sin\phi$, $AB = 2(R_1 + R_2)\sin\phi$ (Figure 9) [1], the points A, O_1, O are collinear; so are the points O, O_2, B. The tangent to the arc (AP) at P meets the arc (AB) at Q_A, and the tangent to the arc (PB) at P meets the arc (AB) at Q_B. Let $K_1(r_1)$, $K_2(r_2)$ be the incircles of the arbeloi Q_BAP and Q_APB. If $2\phi = \pi$, then we have the classical arbelos, and Q_A, Q_B coincide with the point Q. We prove that $r_1 = r_2$.

Proof. The point A is on the radical axis of $O_1(R_1)$, $O_-(R_1 + R_2)$). From (1), we have

$$r_1 = \frac{\overrightarrow{O_1O} \cdot \Pi_{O_1O}(\overrightarrow{AK_1})}{r_{O_1} - r_O} = \frac{R_2 \cdot \Pi_{O_1O}(AK_1)}{R_1 + R_1 + R_2}.$$

Also, $r_1 = \Pi_{O_1O}(K_1P)$. Hence,

$$2R_1 \sin^2\phi = \Pi_{O_1O}(AK_1) + \Pi_{O_1O}(K_1P) = r_1 \cdot \frac{2R_1 + R_2}{R_2} + r_1 = r_1 \cdot \frac{2(R_1 + R_2)}{R_2},$$

or $r_1 = \frac{R_1R_2\sin^2\phi}{R_1+R_2}$. Similarly,

$$r_2 = \frac{\overrightarrow{OO_2} \cdot \Pi_{OO_2}(\overrightarrow{BK_2})}{r_O - r_{O_2}} = \frac{R_1 \cdot \Pi_{OO_2}(BK_2)}{-R_1 - R_2 - R_2},$$

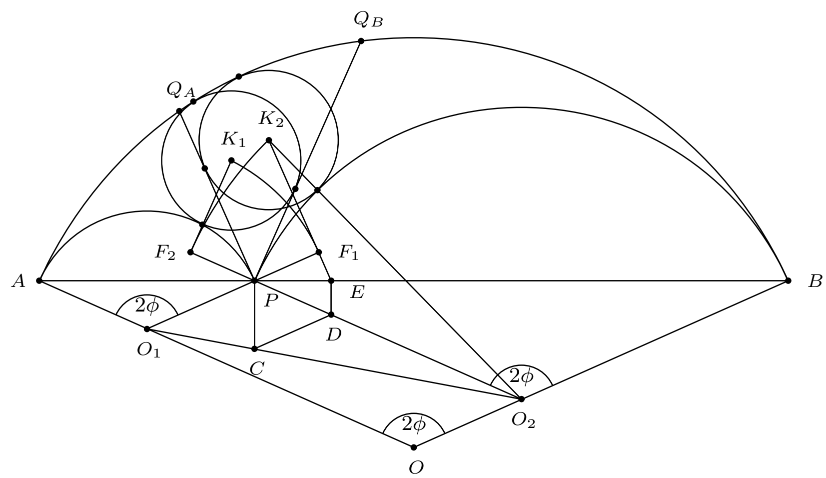

Figure 9.

and $r_2 = \Pi_{OO_2}(PK_2)$. Hence,

$$2R_2 \sin^2\phi = \Pi_{OO_2}(PK_2) + \Pi_{OO_2}(K_2B) = r_2 + r_2 \cdot \frac{R_1 + 2R_2}{R_1} = r_2 \cdot \frac{2(R_1+R_2)}{R_1},$$

or $r_2 = \frac{R_1 R_2 \sin^2\phi}{R_1+R_2}$.

If $2\phi = \pi$, then we have the radius of the Arcimedean circle. □

Construction of the twins. The perpendicular from P to AB meets O_1O_2 at the point C, and the parallel from C to PO_1 meets PO_2 at D. Since PC is a bisector in triangle PO_1O_2, we have $\frac{1}{CD} = \frac{1}{R_1} + \frac{1}{R_2}$. Hence we need the construction of $r_1 = CD \cdot \sin^2\phi$. The perpendicular from D to AB meets AB at E and the perpendicular from E to PO_1 meets this line at the point F_1. The symmetric of F_1 in PC is the point F_2 on the line PO_2. The perpendicular at F_2 to PF_2 meets the circle $O_1(F_1)$ at the point K_1 and the line EF_1 meets the circle $O_2(F_2)$ at the point K_2. These points are the centers of the twin incircles and the construction of these circles is obvious.

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