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Table of Contents

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A Note on Conic Sections and Tangent Circles

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Abstract. This article presents a result on circles tangent to a given conic section and to each other. The result is proved using a set of parameterizations that cover all possible scenarios.

1. Introduction

The objective of this article is to establish the following.

Theorem 1. *Suppose S is a conic section (of eccentricity $\geq \frac{1}{\sqrt{2}}$ if it is an ellipse) There exists a set S', which is either a conic section or a union of two conic sections, with the following property. For two circles P and Q each tangent to S at two points, and to each other externally (at a point not on S if S is a hyperbola), the centers of the two circles R_1 and R_2 that are also tangent externally to P and Q and tangent to S lie on S'.*

For other problems involving tangent circles, see [1, 2, 3, 4].

The proof of Theorem 1 is based on explicit parameterizations of P, Q and R_1, given S. (It is not necessary to check R_2 separately, because of symmetry.) In each case, the proposed set S' is clearly either a conic section or a union of two conic sections, as claimed, but the following conditions also need to be verified:

(a) The proposed point of tangency between any circle (among P, Q and R_1) and S lies on S.
(b) The proposed center of R_1 lies on the proposed curve (i.e., S', or one of its components).
(c) The distance between the proposed center of any circle and the corresponding proposed point of tangency with S is equal to the proposed radius.
(d) The line segment from the proposed center of any circle to the corresponding proposed point of tangency with S is normal to S.
(e) The distance between the proposed centers of two mutually tangent circles is equal to the sum of their proposed radii.

Verifying conditions (c) and (e) is generally done by setting Δx and Δy to be the differences between the proposed x- and y-coordinates respectively for the points in question, and Δz to be the proposed distance (i.e., the radius or the sum of the

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radii), and checking that

$$\Delta x^2 + \Delta y^2 = \Delta z^2 \tag{1}$$

is satisfied.

The notation with Δx and Δy is also used when verifying condition (d).

2. Ellipse

2.1. *Parameterization.* For the ellipse S:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \frac{a}{b} \geq \sqrt{2},$$

S' is the ellipse

$$\left(\frac{x}{a'}\right)^2 + \left(\frac{y}{b'}\right)^2 = 1$$

where

$$a' = \frac{1}{2}\left(a + \sqrt{a^2 - b^2}\right) \text{ and } b' = b - \frac{a}{2b}\left(a - \sqrt{a^2 - b^2}\right).$$

With parameters α and u satisfying

$$\sin\alpha = \frac{b}{a}, \qquad 0 < \alpha \leq \frac{\pi}{4},$$
$$2\alpha \leq u \leq \pi - 2\alpha,$$

we define three circles P, Q, R_1 with centers and radii given below, and verify that their points of tangency with S are as in the rightmost column (see Figure 1).

	Center	Radius	Point of tangency with S
P	$\left(\sqrt{a^2-b^2}\cos(u-\alpha), 0\right)$	$b\sin(u-\alpha)$	$\left(a\frac{\cos(u-\alpha)}{\cos\alpha}, \pm b\sqrt{1-\left(\frac{\cos(u-\alpha)}{\cos\alpha}\right)^2}\right)$
Q	$\left(\sqrt{a^2-b^2}\cos(u+\alpha), 0\right)$	$b\sin(u+\alpha)$	$\left(a\frac{\cos(u+\alpha)}{\cos\alpha}, \pm b\sqrt{1-\left(\frac{\cos(u+\alpha)}{\cos\alpha}\right)^2}\right)$
R_1	$\left(a'\frac{\cos u}{\cos\alpha}, b'\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}\right)$	$(b-b')\sin u$	$\left(a\frac{\cos u}{\cos\alpha}, b\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}\right)$

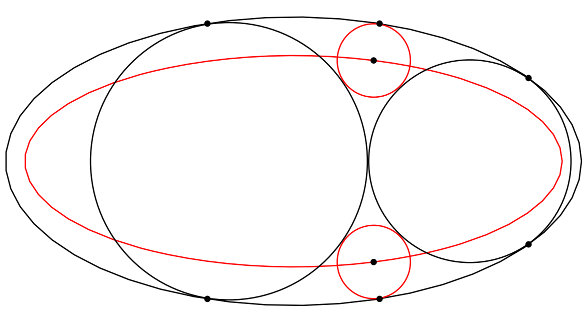

Figure 1. Example of circles tangent to an ellipse

2.2. *Verification.* (a) and (b) are trivial.

(c) For P or Q, we have

$$
\begin{aligned}
&\Delta x^2 + \Delta y^2 \\
= &\left(a^2 - b^2 - \frac{2a\sqrt{a^2-b^2}}{\cos\alpha} + \frac{a^2}{\cos^2\alpha}\right)\cos^2(u\pm\alpha) + b^2\left(1 - \frac{\cos^2(u\pm\alpha)}{\cos^2\alpha}\right) \\
= &\, b^2\left(1-\cos^2(u\pm\alpha)\right) + \cos^2(u\pm\alpha)\left(a - \frac{\sqrt{a^2-b^2}}{\cos\alpha}\right)^2 \\
= &\, b^2\sin^2(u\pm\alpha) = \Delta z^2.
\end{aligned}
$$

For R_1, we have

$$
\begin{aligned}
&\Delta x^2 + \Delta y^2 \\
= &\,\frac{1}{4}\left(a-\sqrt{a^2-b^2}\right)^2\frac{\cos^2 u}{\cos^2\alpha} + \frac{a^2}{4b^2}\left(a-\sqrt{a^2-b^2}\right)^2\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right) \\
= &\,\frac{1}{4}\left(a-\sqrt{a^2-b^2}\right)^2\left(\frac{a^2}{b^2} - \frac{a^2-b^2}{b^2}\frac{\cos^2 u}{\cos^2\alpha}\right) \\
= &\,\frac{a^2}{4b^2}\left(a-\sqrt{a^2-b^2}\right)^2\left(1-\cos^2 u\right) \\
= &\,\left(b-b'\right)^2\sin^2 u = \Delta z^2.
\end{aligned}
$$

(d) The slope of the tangent of S at the point (x,y) is $-\frac{b^2x}{a^2y}$, as can be shown with basic calculus. For P or Q, taking the point of tangency with positive y-coordinate, we have

$$
\frac{\Delta y}{\Delta x} = \frac{b\sqrt{1-\left(\frac{\cos(u\pm\alpha)}{\cos\alpha}\right)^2}}{\left(\frac{a}{\cos\alpha}-\sqrt{a^2-b^2}\right)\cos(u\pm\alpha)} = \frac{b\sqrt{1-\left(\frac{\cos(u\pm\alpha)}{\cos\alpha}\right)^2}}{\left(\frac{a}{\cos\alpha}-\frac{a^2-b^2}{a\cos\alpha}\right)\cos(u\pm\alpha)} = \frac{a^2y}{b^2x}.
$$

For R_1, we have

$$
\frac{\Delta y}{\Delta x} = \frac{b-b'}{a-a'}\frac{\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}}{\frac{\cos u}{\cos\alpha}} = \frac{a\sqrt{1-\left(\frac{\cos u}{\cos\alpha}\right)^2}}{b\left(\frac{\cos u}{\cos\alpha}\right)} = \frac{a^2y}{b^2x}.
$$

(e) The distance between the proposed centers of P and Q is

$$
\begin{aligned}
&\sqrt{a^2-b^2}\left(\cos(u-\alpha)-\cos(u+\alpha)\right) = \sqrt{a^2-b^2}\left(2\sin u\sin\alpha\right) \\
= &\, b\left(2\sin u\cos\alpha\right) = b\left(\sin(u-\alpha)+\sin(u+\alpha)\right),
\end{aligned}
$$

which equals the sum of the proposed radii.

If we instead consider either P or Q together with R_1, the expressions for Δx, Δy and Δz are as follows:

$$\begin{aligned}
\Delta x^2 = & \left(\sqrt{a^2-b^2}\cos\left(u\pm\alpha\right)-\frac{1}{2}\left(a+\sqrt{a^2-b^2}\right)\frac{\cos u}{\cos\alpha}\right)^2 \\
= & \left(\sqrt{a^2-b^2}\left(\cos u\cos\alpha\mp\sin u\sin\alpha\right)-\frac{1}{2}\left(a+\sqrt{a^2-b^2}\right)\frac{\cos u}{\cos\alpha}\right)^2 \\
= & \left(a^2-b^2\right)\left(\cos^2 u\cos^2\alpha+\sin^2 u\sin^2\alpha\mp 2\sin u\cos u\sin\alpha\cos\alpha\right) \\
& +\frac{1}{4}\left(2a^2-b^2+2a\sqrt{a^2-b^2}\frac{\cos^2 u}{\cos^2\alpha}\right) \\
& -\left(a^2-b^2+a\sqrt{a^2-b^2}\right)\left(\cos^2 u\mp\frac{\sin u\cos u\sin\alpha}{\cos\alpha}\right), \\
\Delta y^2 = & \left(b-\frac{a^2}{2b}+\frac{a\sqrt{a^2-b^2}}{2b}\right)^2\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right) \\
= & \left(b^2+\frac{2a^4-a^2b^2}{4b^2}-a^2+a\sqrt{a^2-b^2}-\frac{a^3\sqrt{a^2-b^2}}{2b^2}\right)\left(1-\frac{\cos^2 u}{\cos^2\alpha}\right), \\
\Delta z^2 = & \left(b\sin\left(u\pm\alpha\right)+\frac{a}{2b}\left(a-\sqrt{a^2-b^2}\right)\sin u\right)^2 \\
= & \left(b\left(\sin u\cos\alpha\pm\cos u\sin\alpha\right)+\frac{a}{2b}\left(a-\sqrt{a^2-b^2}\right)\sin u\right)^2 \\
= & \ b^2\sin^2 u\cos^2\alpha+b^2\cos^2 u\sin^2\alpha\pm 2b^2\sin u\cos u\sin\alpha\cos\alpha \\
& +\frac{\left(2a^4-a^2b^2\right)\sin^2 u}{4b^2}-\frac{a^3\sqrt{a^2-b^2}\sin^2 u}{2b^2} \\
& +a\left(a-\sqrt{a^2-b^2}\right)\left(\sin^2 u\cos\alpha\pm\sin u\cos u\sin\alpha\right).
\end{aligned}$$

The reader can verify that when we insert these expressions into (1) the terms with $\pm$ or $\mp$ cancel out, while collecting the remaining terms yields

$$\Xi\sin^2 u+\Xi\cos^2 u=\Xi,$$

where

$$\Xi=\frac{2a^4-5a^2b^2+4b^4+2a\left(-a^2+2b^2\right)\sqrt{a^2-b^2}}{4b^2},$$

which of course also implies cancellation.

3. Parabola

3.1. *Parameterization.* For the parabola S:

$$y=cx^2,$$

S' is the parabola

$$y=c\left(\frac{4x}{3}\right)^2+\frac{1}{8c}.$$

With parameter $u \geq \frac{1}{c}$, we define the circles P, Q, R_1 in the table below, and verify that the points of tangency with S are as given in the rightmost column (see Figure 2).

	Center	Radius	Point of tangency with S
P	$(0, cu^2 - u + \frac{1}{2c})$	$u - \frac{1}{2c}$	$(\pm\sqrt{u^2 - \frac{u}{c}}, cu^2 - u)$
Q	$(0, cu^2 + u + \frac{1}{2c})$	$u + \frac{1}{2c}$	$(\pm\sqrt{u^2 + \frac{u}{c}}, cu^2 + u)$
R_1	$\left(\frac{3}{4}\sqrt{u^2 - \frac{1}{4c^2}}, cu^2 - \frac{1}{8c}\right)$	$\frac{u}{4}$	$\left(\sqrt{u^2 - \frac{1}{4c^2}}, cu^2 - \frac{1}{4c}\right)$

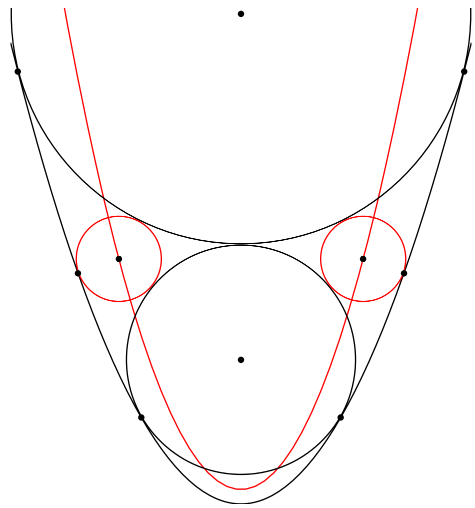

Figure 2. Example of circles tangent to a parabola

3.2. *Verification.* (a) Trivial, since all proposed points of tangency have the form $(\pm\sqrt{m}, cm)$.

(b) With $x = \frac{3}{4}\sqrt{u^2 - \frac{1}{4c^2}}$, we have

$$y = c\left(\frac{4x}{3}\right)^2 + \frac{1}{8c} = c\left(u^2 - \frac{1}{4c^2}\right) + \frac{1}{8c} = cu^2 - \frac{1}{8c}$$

as required.

(c) For P or Q, we have

$$\Delta x^2 + \Delta y^2 = \left(u^2 \pm \frac{u}{c}\right) + \left(\frac{1}{2c}\right)^2 = \left(u \pm \frac{1}{2c}\right)^2 = \Delta z^2.$$

For R_1, we have

$$\Delta x^2 + \Delta y^2 = \frac{1}{16}\left(u^2 - \frac{1}{4c^2}\right) + \left(\frac{1}{8c}\right)^2 = \left(\frac{u}{4}\right)^2 = \Delta z^2.$$

(d) The slope of the tangent of S at the point $(\sqrt{m}, cm)$ is $2c\sqrt{m}$, and so the slope of the line segment from the center of the corresponding circle to the point of tangency must be $-\frac{1}{2c\sqrt{m}}$. For P and Q and taking the point of tangency with positive x-coordinate, we have trivially

$$\frac{\Delta y}{\Delta x} = \frac{-\frac{1}{2c}}{\sqrt{m}} = -\frac{1}{2c\sqrt{m}},$$

and for R_1 we have almost equally trivially

$$\frac{\Delta y}{\Delta x} = \frac{-\frac{1}{8c}}{\frac{1}{4}\sqrt{m}} = -\frac{1}{2c\sqrt{m}}.$$

(e) The distance between the proposed centers of P and Q is $2u$, which is equal to the sum of their radii. If we instead consider P or Q along with R_1, we have

$$\begin{aligned}\Delta x^2 + \Delta y^2 = \; & \frac{9}{16}\left(u^2 - \frac{1}{4c^2}\right) + \left(u \pm \frac{5}{8c}\right)^2 \\ = \; & \frac{25}{16}u^2 \pm \frac{5u}{4c} + \frac{1}{4c^2} = \left(\frac{5}{4}u \pm \frac{1}{2c}\right)^2 = \Delta z^2.\end{aligned}$$

4. Hyperbola

4.1. *Parameterization.* For the hyperbola S:

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1.$$

S' is the union of two hyperbolas S_1 and S_2. The **first** component of S' is the hyperbola S_1

$$\left(\frac{x}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = 1,$$

where

$$a' = \frac{1}{2}\left(a + \sqrt{a^2+b^2}\right), \qquad b' = b + \frac{a}{2b}\left(a - \sqrt{a^2+b^2}\right).$$

With parameter $u \geq \frac{2b}{a}$, we define the circles P, Q, R_1

	Center	Radius
P	$\left(\frac{a^2+b^2}{a^2}\left(\sqrt{(a^2+b^2)\,u^2+a^2} - bu\right), 0\right)$	$\frac{b}{a^2}\left((a^2+b^2)\,u - b\sqrt{(a^2+b^2)\,u^2+a^2}\right)$
Q	$\left(\frac{a^2+b^2}{a^2}\left(\sqrt{(a^2+b^2)\,u^2+a^2} + bu\right), 0\right)$	$\frac{b}{a^2}\left((a^2+b^2)\,u + b\sqrt{(a^2+b^2)\,u^2+a^2}\right)$
R_1	$\left(a'\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b'\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$	$\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right)$

and verify that the points of tangency with S are as follows (see Figure 3).

	Point of tangency with S
P	$\left(\sqrt{(a^2+b^2)\,u^2+a^2} - bu, \pm\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2+a^2} - bu\right)^2 - a^2}\right)$
Q	$\left(\sqrt{(a^2+b^2)\,u^2+a^2} + bu, \pm\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2+a^2} + bu\right)^2 - a^2}\right)$
R_1	$\left(a\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$

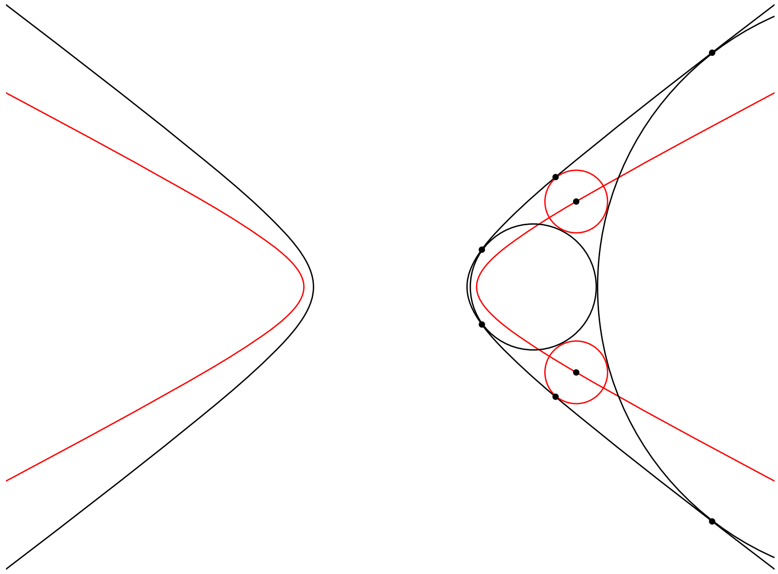

Figure 3. Example of circles tangent to a hyperbola with center of R_1 on S_1

The **second** component of S' is the hyperbola S_2

$$\left(\frac{x}{a''}\right)^2 - \left(\frac{y}{b''}\right)^2 = 1,$$

where

$$a'' = a + \frac{b}{2a}\left(b - \sqrt{a^2+b^2}\right), \qquad b'' = \frac{1}{2}\left(b + \sqrt{a^2+b^2}\right).$$

With parameter $u \geq \frac{b}{\sqrt{a^2+b^2}}$, we define the circles P, Q, R_1

	Center	Radius
P	$\left(0, \frac{a^2+b^2}{b^2}\left(\sqrt{(a^2+b^2)\,u^2-b^2}-au\right)\right)$	$\frac{a}{b^2}\left(\left(a^2+b^2\right)u - a\sqrt{(a^2+b^2)\,u^2-b^2}\right)$
Q	$\left(0, \frac{a^2+b^2}{b^2}\left(\sqrt{(a^2+b^2)\,u^2-b^2}+au\right)\right)$	$\frac{a}{b^2}\left(\left(a^2+b^2\right)u + a\sqrt{(a^2+b^2)\,u^2-b^2}\right)$
R_1	$\left(a''\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b''\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$	$\frac{u}{2a}\left(a^2+b^2-b\sqrt{a^2+b^2}\right)$

and verify that the points of tangency with S are as follows (see Figure 4).

	Point of tangency with S
P	$\left(\pm\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2-b^2}-au\right)^2+b^2}, \sqrt{(a^2+b^2)\,u^2-b^2}-au\right)$
Q	$\left(\pm\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2-b^2}+au\right)^2+b^2}, \sqrt{(a^2+b^2)\,u^2-b^2}+au\right)$
R_1	$\left(a\sqrt{u^2+\frac{a^2}{a^2+b^2}}, b\sqrt{u^2-\frac{b^2}{a^2+b^2}}\right)$

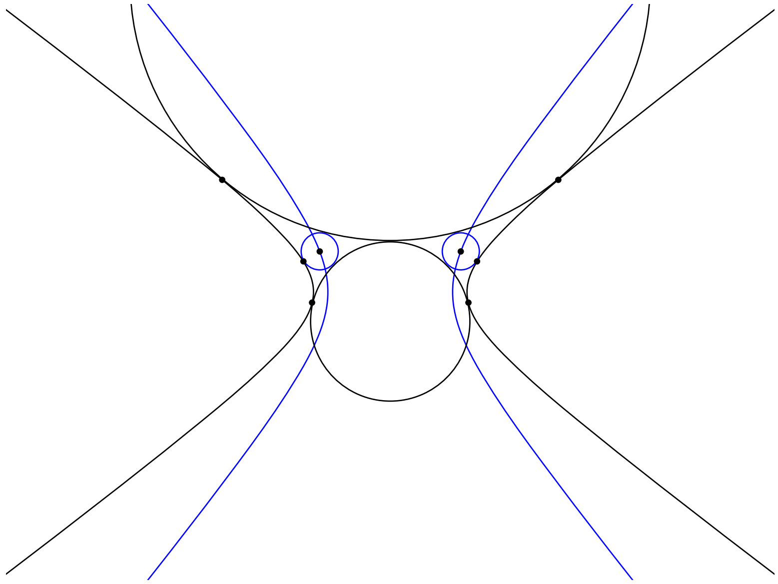

Figure 4. Example of circles tangent to a hyperbola with center of R_1 on S_2

4.2. *Verification.* (a) For the first component of S', both P and Q have proposed points of tangency with S of the form $\left(m, \pm\frac{b}{a}\sqrt{m^2-a^2}\right)$.

For the second component, they have proposed points of tangency of the form $\left(\pm\frac{a}{b}\sqrt{m^2+b^2}, m\right)$.

In both cases, it is clear that $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$ is satisfied. For R_1, we get in both cases

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1.$$

(b) In a similar fashion, we have

$$\left(\frac{x}{a'}\right)^2 - \left(\frac{y}{b'}\right)^2 = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1,$$
$$\left(\frac{x}{a''}\right)^2 - \left(\frac{y}{b''}\right)^2 = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1.$$

(c) For P and Q with the **first** component S_1, starting with the proposed center and points of tangency with S for P or Q, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\;&\frac{b^4}{a^4}\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2+\frac{b^2}{a^2}\left(\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2-a^2\right)\\
=\;&\frac{b^2}{a^4}\left(\left(a^2+b^2\right)\left(\left(a^2+b^2\right)u^2+a^2+b^2u^2\pm 2bu\sqrt{(a^2+b^2)\,u^2+a^2}\right)-a^4\right)\\
=\;&\frac{b^2}{a^4}\left(\left(a^2+b^2\right)u\pm b\sqrt{(a^2+b^2)\,u^2+a^2}\right)^2=\;\Delta z^2.
\end{aligned}$$

For R_1, we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\;&(a-a')^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+(b-b')^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\;&\frac{1}{4}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+\frac{a^2}{4b^2}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\;&\frac{1}{4}\left(-a+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2u^2}{b^2}\right)=\;\left(\frac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)\frac{u}{b}\sqrt{a^2+b^2}\right)^2\\
=\;&\left(\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right)\right)^2=\;\Delta z^2.
\end{aligned}$$

For P and Q with the **second** component S_2, we have similarly

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\;&\frac{a^2}{b^2}\left(\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2+b^2\right)+\frac{a^4}{b^4}\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2\\
=\;&\frac{a^2}{b^4}\left(\left(a^2+b^2\right)\left(\left(a^2+b^2\right)u^2-b^2+a^2u^2\pm 2au\sqrt{(a^2+b^2)\,u^2-b^2}\right)+b^4\right)\\
=\;&\frac{a^2}{b^4}\left(\left(a^2+b^2\right)u\pm a\sqrt{(a^2+b^2)\,u^2-b^2}\right)^2=\;\Delta z^2,
\end{aligned}$$

and for R_1 we have

$$\begin{aligned}
&\Delta x^2+\Delta y^2\\
=\;&(a-a'')^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+(b-b'')^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\;&\frac{b^2}{4a^2}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{a^2}{a^2+b^2}\right)+\frac{1}{4}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2-\frac{b^2}{a^2+b^2}\right)\\
=\;&\frac{1}{4}\left(-b+\sqrt{a^2+b^2}\right)^2\left(u^2+\frac{b^2u^2}{a^2}\right)\\
=\;&\left(\frac{1}{2}\left(-b+\sqrt{a^2+b^2}\right)\frac{u}{a}\sqrt{a^2+b^2}\right)^2\\
=\;&\left(\frac{u}{2a}\left(a^2+b^2-b\sqrt{a^2+b^2}\right)\right)^2=\;\Delta z^2.
\end{aligned}$$

(d) The slope of the tangent of S at the point (x,y) is $\frac{b^2x}{a^2y}$, and so the slope of the line segment from the center of the corresponding circle to its point of tangency (x,y) with S must be $-\frac{a^2y}{b^2x}$. For P or Q for the verification for the first component of S', taking the point of tangency with positive y-coordinate, we have

$$\frac{\Delta y}{\Delta x}=\frac{\frac{b}{a}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)^2-a^2}}{\left(-\frac{b^2}{a^2}\right)\left(\sqrt{(a^2+b^2)\,u^2+a^2}\pm bu\right)}=-\frac{a^2y}{b^2x}.$$

For R_1, we have

$$\frac{\Delta y}{\Delta x}=\frac{(b-b')\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{(a-a')\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{b\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a^2y}{b^2x}.$$

For the verification for the second component of S', taking the point of tangency with positive x-coordinate for P or Q, we have similarly

$$\frac{\Delta y}{\Delta x}=\frac{\left(-\frac{a^2}{b^2}\right)\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)}{\frac{a}{b}\sqrt{\left(\sqrt{(a^2+b^2)\,u^2-b^2}\pm au\right)^2+b^2}}=-\frac{a^2y}{b^2x},$$

and for R_1, we have

$$\frac{\Delta y}{\Delta x}=\frac{(b-b'')\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{(a-a'')\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a\sqrt{u^2-\frac{b^2}{a^2+b^2}}}{b\sqrt{u^2+\frac{a^2}{a^2+b^2}}}=-\frac{a^2y}{b^2x}.$$

(e) Verifying the condition for P and Q is rather trivial for both components of S'. For P or Q along with R_1, with the first component, we have

$$\begin{aligned}
&\Delta x^2\\
=\ &\left(\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a^2}-\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)\right)\sqrt{u^2+\frac{a^2}{a^2+b^2}}\pm\frac{a^2+b^2}{a^2}bu\right)^2\\
=\ &\left(\frac{\left(a^2+b^2\right)^3}{a^4}+\frac{2a^2+b^2}{4}-\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a}-\frac{\left(a^2+b^2\right)^2}{a^2}+\frac{a}{2}\sqrt{a^2+b^2}\right)\left(u^2+\frac{a^2}{a^2+b^2}\right)\\
&+\frac{\left(a^2+b^2\right)^2b^2u^2}{a^4}\pm 2\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a^2}-\frac{a}{2}-\frac{\sqrt{a^2+b^2}}{2}\right)\sqrt{(a^2+b^2)\,u^2+a^2}\,\frac{bu\sqrt{a^2+b^2}}{a^2},
\end{aligned}$$

$$
\begin{aligned}
\Delta y^2 =& \left(b^2 + a\left(a - \sqrt{a^2+b^2}\right) + \frac{a^2}{4b^2}\left(2a^2+b^2-2a\sqrt{a^2+b^2}\right)\right)\left(u^2 - \frac{b^2}{a^2+b^2}\right),\\
\Delta z^2 =& \left(\frac{u}{2b}\left(a^2+b^2-a\sqrt{a^2+b^2}\right) + \frac{b}{a^2}\left(\left(a^2+b^2\right)u \pm b\sqrt{\left(a^2+b^2\right)u^2+a^2}\right)\right)^2\\
=& \frac{\left(a^2+b^2\right)^2u^2}{4b^2} + \frac{a^2u^2\left(a^2+b^2\right)}{4b^2} + \frac{b^2u^2\left(a^2+b^2\right)^2}{a^4} + \frac{b^4\left(\left(a^2+b^2\right)u^2+a^2\right)}{a^4}\\
&- \frac{au^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{2b^2} + \frac{u^2\left(a^2+b^2\right)^2}{a^2} - \frac{u^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{a}\\
&\pm \frac{bu}{a^2}\sqrt{a^2+b^2}\left(\sqrt{a^2+b^2} - a + \frac{2b^2}{a^2}\sqrt{a^2+b^2}\right),
\end{aligned}
$$

and the reader can verify that (1) is satisfied.

Likewise, for the second component of S', we have

$$
\begin{aligned}
\Delta x^2 =& \left(a + \frac{b^2}{2a} - \frac{b\sqrt{a^2+b^2}}{2a}\right)^2\left(u^2 + \frac{a^2}{a^2+b^2}\right)\\
=& \left(a^2 + \frac{a^2b^2+2b^4}{4a^2} + b^2 - b\sqrt{a^2+b^2} - \frac{b^3\sqrt{a^2+b^2}}{2a^2}\right)\left(u^2+\frac{a^2}{a^2+b^2}\right),\\
\Delta y^2 =& \left(\left(\frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{b^2} - \frac{b}{2} - \frac{\sqrt{a^2+b^2}}{2}\right)\sqrt{u^2 - \frac{b^2}{a^2+b^2}} \pm \frac{au\left(a^2+b^2\right)}{b^2}\right)^2\\
=& \left(\frac{\left(a^2+b^2\right)^3}{b^4} + \frac{a^2+2b^4}{4} - \frac{\left(a^2+b^2\right)^{\frac{3}{2}}}{b} - \frac{\left(a^2+b^2\right)^2}{b^2} + \frac{b\sqrt{a^2+b^2}}{2}\right)\\
&\times\left(u^2 - \frac{b^2}{a^2+b^2}\right) + \frac{a^2u^2\left(a^2+b^2\right)^2}{b^4}\\
&\pm 2\left(\frac{au\left(a^2+b^2\right)^{\frac{5}{2}}}{b^4} - \frac{au\left(a^2+b^2\right)}{2b} - \frac{au\left(a^2+b^2\right)^{\frac{3}{2}}}{2b^2}\right)\sqrt{u^2 - \frac{b^2}{a^2+b^2}},\\
\Delta z^2 =& \left(\frac{au\left(a^2+b^2\right)}{b^2} + \frac{\left(a^2+b^2\right)u}{2a} - \frac{bu\sqrt{a^2+b^2}}{2a} \pm \frac{a^2\sqrt{\left(a^2+b^2\right)u^2-b^2}}{b^2}\right)^2\\
=& \frac{\left(a^6+2a^4b^2+a^2b^4\right)u^2}{b^4} + \frac{\left(a^4+3a^2b^2+2b^4\right)u^2}{4a^2} + \frac{\left(a^2+b^2\right)^2u^2}{b^2}\\
&- \frac{\left(a^2+b^2\right)u^2\sqrt{a^2+b^2}}{b} - \frac{bu^2\left(a^2+b^2\right)\sqrt{a^2+b^2}}{2a^2} + \frac{a^4\left(\left(a^2+b^2\right)u^2-b^2\right)}{b^4}\\
&\pm 2\frac{a^2u\sqrt{\left(a^2+b^2\right)u^2-b^2}}{b^2}\left(\frac{a^3+ab^2}{b^2} + \frac{a^2+b^2}{2a} - \frac{b\sqrt{a^2+b^2}}{2a}\right),
\end{aligned}
$$

for which (1) again is satisfied.

5. Conclusion

The parameterizations cover all possible scenarios, and Theorem 1 is thus proved. Strictly speaking, the set of points where R_1 can have its center is in general not the entire conic section, but a subset as implied by the conditions that we have included for the parameters.

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A Toroidal Approach to The Archimedean Quadrature

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Abstract. An "Archimedean" quadrature is attempted "borrowing" π from the 3-dimensional space of a horn torus.

Archimedes' essay entitled *Measurement of the Circle* [1] starts with the following theorem: "*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle*." Indeed, letting this triangle be the one given by $AB\Gamma$ in Figure 1, with the right angle at vertex B, and letting R designate the radius of the circle centered at A, with $AB = R$, if $B\Gamma = 2\pi R$, the area of triangle $AB\Gamma$ will be:

$$R(2\pi R)/2 = \pi R^2,$$

which is half the area of the rectangle $AB\Gamma\Lambda$. Given next sides AB and $B\Gamma$ of the rectangle, the side x of the square with area equal to πR^2 could be constructed geometrically based on the right angle altitude theorem: x is the altitude of the right triangle with hypotenuse equal to $AB + B\Gamma$ so that $x^2 = (AB)(B\Gamma)$.

Nevertheless, the number π is an irrational number which is not geometrically constructible. Therefore, to square the circle based on Archimedes theorem, the line segment $B\Gamma$ has to be identified with the length $2\pi R$ *a priori*, which is not in the spirit of the problem of quadrature. Confined to the Euclidean plane, π has to be constructed too, and this is what Archimedes tried through a spiral approach. In this article, a squaring of the circle is attempted "borrowing" π from the 3-dimensional space. It has been established ever since Clifford's work [2] on tori that the surface of a horn torus is representable in the plane as a square, and since this surface is equal to $4\pi^2R^2$, the edge of the corresponding square should be equal to $2\pi R$ given, of course, that R is the radius of the defining circle and of the tube. So:

Let there be a horn torus $\mathbb{T}$ of radius R in the 3-dimensional Euclidean space.

Problem: Given a square which produces the torus $\mathbb{T}$ when both pairs of opposite edges are glued, find the square that squares the circle of radius R.

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Analysis: Consider again Figure 1. Suppose that $B\Gamma\Delta E$ is the given square and hence, that its edge is equal to $2\pi R$ since the area of the square should be equal to the area of the surface of $\mathbb{T}$ which is $4\pi^2 R^2$. The congruent right-angled triangles $AB\Gamma$ and HBE may then be drawn based on Archimedes theorem, with $AB = HB = R$ and $A\Gamma = HE = E\sqrt{1+4\pi^2}$, given that $B\Gamma = BE = 2\pi R$ and that R is known. Consequently, HA is equal to $R\sqrt{2}$ and parallel to diagonal ΓE since $\angle HAB = \angle AE\Gamma = \pi/4$. It follows that $A\Gamma EH$ is an isosceles trapezoid and so is the trapezoid $\Phi\Gamma EZ$ formed when $\Phi Z \parallel \Gamma E$ is drawn through the intersection point B of the diagonals of $A\Gamma EH$. Both are tangential trapezoids because

$$\angle HAE + \angle EA\Gamma + \angle H\Gamma A + \angle H\Gamma E = \angle AH\Gamma + \angle \Gamma HE + \angle AEH + \angle AE\Gamma = \pi$$

and $\Phi Z \parallel \Gamma E$: $\angle HAE = \angle H\Gamma E = \angle AH\Gamma = \angle AE\Gamma = \pi/4$ and $\angle EA\Gamma + \angle H\Gamma A = \angle \Gamma HE + \angle AEH = \pi/2$. So drawing $B\Theta \perp \Gamma E$ and the perpendicular bisector of $B\Theta$ cutting it at point K, this point is the center of the circle inscribed in trapezoid $\Phi\Gamma EZ$, and $A\Gamma$ and HE are tangent to this circle.

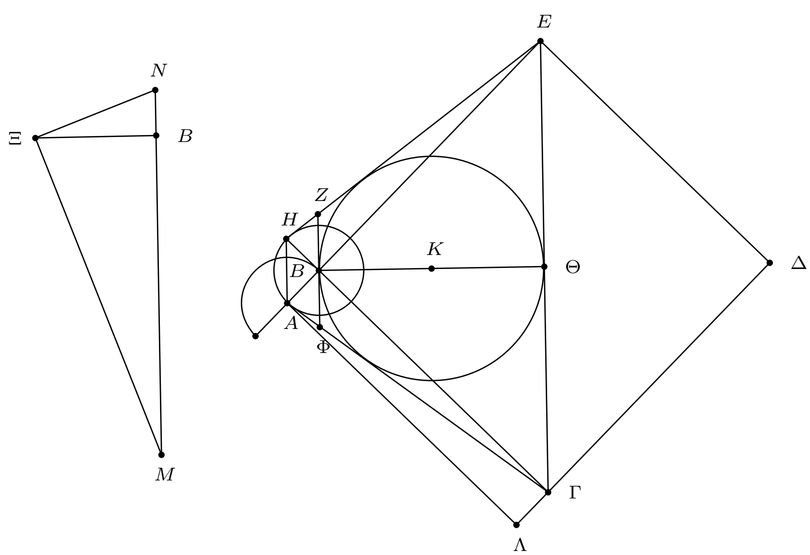

Figure 1. $BN = AB, BM = B\Lambda$

Construction: Given a square $B\Gamma\Delta E$, draw the diagonal ΓE, from vertex B draw perpendicular to ΓE cutting it at point Θ, construct the perpendicular bisector of $B\Theta$ crossing it at point K, draw circle $(K, K\Theta = KB)$, from vertex Γ draw tangent at this circle meeting the extension of edge BE at point A so as AB can be defined, draw a right triangle $MN\Xi$ with hypotenuse MN equal to $AB + B\Gamma$ and with the vertex Ξ so as ΞB can be the altitude of the triangle: ΞB is the side of the square that squares circle (A, AB).

Proof: Relating $B\Gamma\Delta E$ with some torus $\mathbb{T}$, the Construction reproduces Figure 1 and the subsequent relations apart in so far as the identification of AB with R is concerned. The Analysis assumes that R is given, but R is what we are looking for given a square of side $2\pi R$. We proceed by *reductio ad absurdum*. Suppose that $AB \neq R$; but then, the hypotenuse $A\Gamma$ would be tangent to a circle other than circle $(K, K\Theta = KB)$, which is the one inscribed in $\Phi\Gamma EZ$. So, $AB = R$. The construction next of right triangle $MN\Xi$ is made in connection with the right angle altitude theorem.

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The Golden Ratio and Regular Polygons

Manfred Pietsch

Abstract. We give a simple construction of a chord of the circumcircle of an isosceles triangle whose intersections with the legs of the triangles divide symmetrically the chord in the golden ratio.

Proposition 1. *Let ABC be an isosceles triangle with $AB = AC$, and D, E, F points on BC, CA, AB respectively, such that $BDEF$ is a parallelogram with $BD = DF = FE$. If EF is extended to intersect the circumcircle of ABC at G and H (so that F is between E and G), then F divides EG in the golden ratio.*

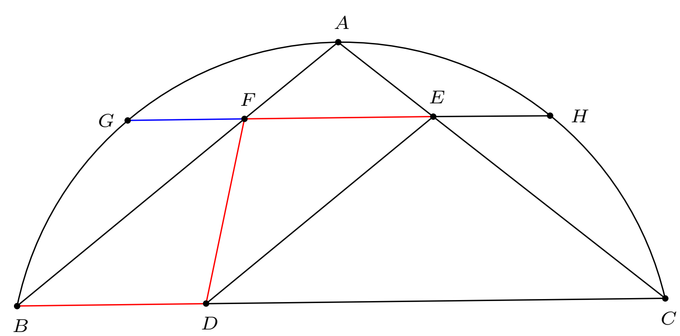

Figure 1

Proof. As EF is parallel to BD, the corresponding angles DBF and EFA are equal. Therefore the isosceles triangles DFB and AFE must be similar. Let $x = EF(= DF = BD)$, $y = FG(= EH)$, $a = AF$, and $b = FB$. The similarity of the triangles DFB and AFE implies $BD : BF = FA : FE$. Thus, $x : b = a : x$, and

$$x^2 = ab. \tag{1}$$

Using the intersecting chords theorem we get $GF \cdot FH = AF \cdot FB$, or

$$y(x + y) = ab. \tag{2}$$

From (1) and (2), $x^2 = (x + y)y$, or

$$\frac{x}{y} = \frac{x + y}{x}.$$

This means that F divides EG in the golden ratio, □

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The construction of the parallelogram can be effected by a central dilation. If the perpendicular bisector of AB intersects BC at D', and the parallelogram $ABD'E'$ is completed, then $BD' = D'A = AE'$. Let BE' intersect AC at E, and the parallel through E to AB and BC intersect BC at D and AB at F respectively, then $BDEF$ is a parallelogram satisfying $BD = DF = FE$ (see Figure 2).

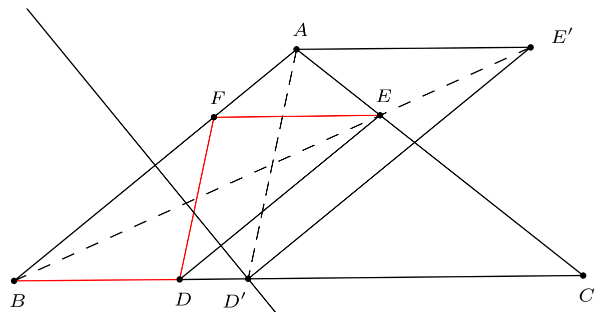

Figure 2

Previous golden ratio constructions based on an equilateral triangle (Odom [1]) or on a square (Tran [2]) are special cases of the present method. Our construction actually gives a simple alternative to the construction in [3], which applies to general regular polygons. In Figure 3, ABC is an isosceles triangle with $AB = AC$, and D, D' are points on BC, E on AB, E' on AC such that $BDE'E$ and $CD'EE'$ are parallelograms with $BD = DE = EE'$ and $CD' = D'E' = E'E$. If B' and C' are the reflections of D in AB and D' in AC respectively, then BB' and CC' are tangents to the circumcircle of ABC at B and C, and E, E' are the trisection points of the segment $B'C'$ (see [3, Proposition 1 and Figure 4]).

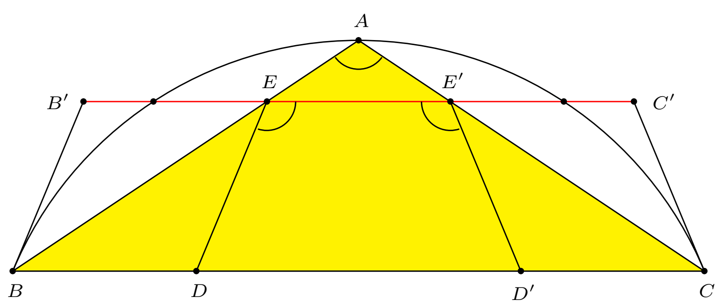

Figure 3

In Figure 3, suppose that AB and AC are consecutive sides of a regular n-gon. Since $\angle DEE' = \angle EE'D' = \angle BAC$, D, E, E', D' are consecutive vertices of a similar regular n-gon (see Figures 4A-C and 5A).

A most interesting case occurs for the pentagons. Figure 5A shows that the smaller regular pentagon has one vertex at the center of the circle circumscribing

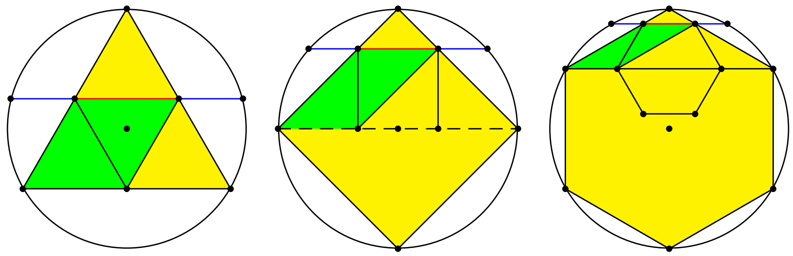

Figure 4A Figure 4B Figure 4C

the given regular pentagon. The areas of the two regular pentagons are in the ratio $1 : 5$. Figure 5B shows a regular pentagon and its reflection about a diameter of its circumcircle parallel to a side. On each side there are two intersection points, dividing it into a solid segment in middle with two dashed segments. In this configuration, each linear segment consisting of a solid segment and a dashed segment is divided in the golden ratio at the junction point.

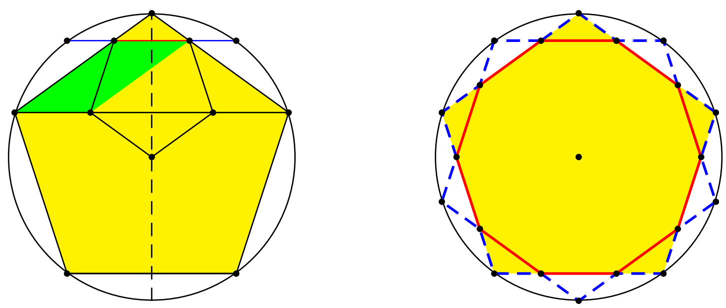

Figure 5A Figure 5B

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Circle Chains Inscribed in Symmetrical Lunes and Integer Sequences

Giovanni Lucca

Abstract. We derive the conditions for inscribing, inside a symmetrical lune, a chain of mutually tangent circles having the property that the ratio between the radii of the largest circle and any circle in the chain is an integer.

1. Introduction

Given two congruent, non-concentric circles, the area inside one but outside the other is called a *symmetric lune*. The aim of this paper is to show some connections between infinite chains of mutually tangent circles that can be inscribed inside a symmetrical lune and certain integer sequences. A generic example of chain is shown in Figure 1.

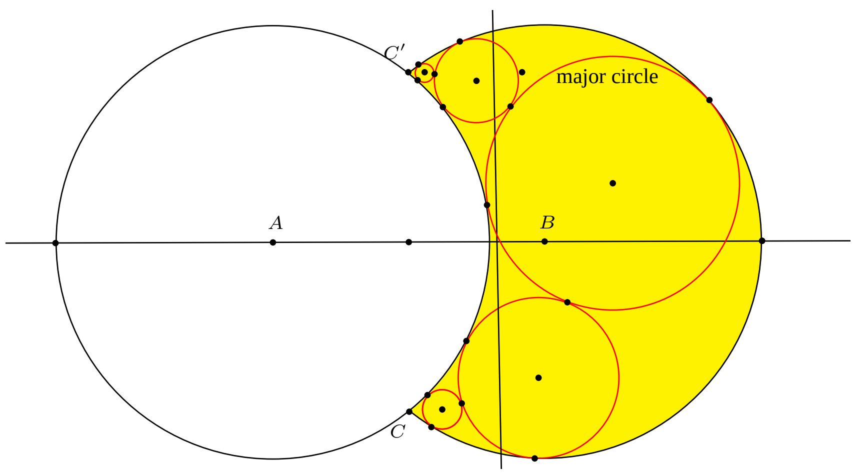

Figure 1. Example of a circle chain inscribed inside a symmetrical lune

As far as we know, only one paper [3] can be found in literature dealing with the problem of an infinite chain of mutually tangent circles inscribed inside a lune. By using the inversion technique [5], the author derived, expressions for center coordinates and radius of the circles belonging to the chain. The results are general and they are valid also for a generic lune not necessarily symmetrical. For circle chains in symmetric lenses, see [2].

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2. Some definitions and useful expressions

For the following, it is convenient to define the *major circle* in the chain (see Figure 1) as the one having the largest radius and to label it by index 0; thus, we can subdivide a generic chain into two sub-chains: an *up chain* starting from the major circle and converging to point C' and a *down chain* starting from the major circle and converging to point C. The characteristics of the circles chain are strictly related to the ratios d/R and y_0/R where

- R is the radius of the two intersecting circles forming the lune,
- $2d$ (with $d < R$) is the distance between the centers of the two intersecting circles (length of segment AB in Figure 2),
- y_0 is the ordinate of the center relevant to the major circle in the chain (point C in Figure 2).

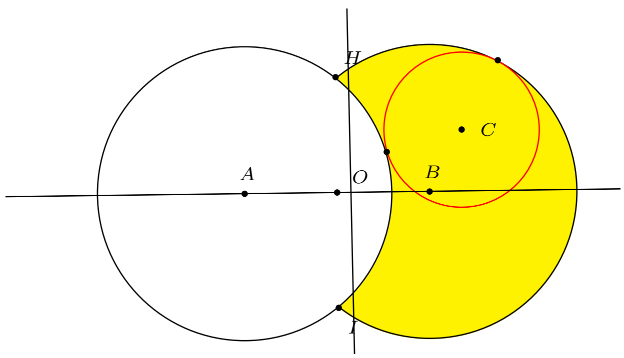

Figure 2. A symmetrical lune with major circle of the chain

In this paper, we want to investigate if some conditions exist so that the ratios τ_k $(k = 1, 2, \dots)$ between the radius of the major circle and the one of the generic k-th circle is an integer number for both the up and down sub-chains. In other words, which are the conditions (provided they exist) in order that the radius of any generic circle of the chain is a sub-multiple of the major circle radius?

In [3], it is shown that the radius r_0 of the major circle depends on y_0 by means of the relation:

$$r_0(y_0) = d\sqrt{\frac{y_0^2}{d^2 - R^2} + 1}. \tag{1}$$

Note that, for the major circle ordinate, the following relation must hold:

$$-d\sqrt{1 - \left(\frac{d}{R}\right)^2} \leq y_0 \leq d\sqrt{1 - \left(\frac{d}{R}\right)^2}. \tag{2}$$

In correspondence of two particular values for y_0, we have two symmetrical dispositions for the up and down chains

- if $y_0 = 0$, we have that $r_0 = d$ and the major circle is bisected by the x-axis (central symmetry).
- if $y_0 = \pm d\sqrt{1-\left(\frac{d}{R}\right)^2}$, we have that $r_0 = d\sqrt{1-\left(\frac{d}{R}\right)^2}$ and two equal major circles, (one for the up chain and one for the down chain), both tangent to x-axis, exist (bi-central symmetry).

It is worthwhile to add the formula for the abscissa x0 of the major circle:

$$x_0 = \frac{R}{d} r_0. \tag{3}$$

On the basis of the contents of [3], and by applying the inversion technique [5], one can derive the expressions for the radius and center coordinates of the k-th circle for both the up and down chain. We here summarize the main formulas and results.

By choosing the inversion circle with center in $(X_{\text{c\_inv}},\ Y_{\text{c\_inv}})$ and radius ρ respectively given by

$$X_{\text{c\_inv}} = 0, \quad Y_{\text{c\_inv}} = -\sqrt{R^2-d^2}, \quad \rho = 2\sqrt{R^2-d^2}. \tag{4}$$

We have the following:

- The two intersecting circles forming the lunes are transformed into two straight lines having equations:

$$y = \pm\frac{d}{\sqrt{R^2-d^2}}x + \sqrt{R^2-d^2}. \tag{5}$$

- The major circle is transformed into another circle having radius R_0 given by:

$$R_0 = \left(\frac{4(R^2-d^2)}{R^2\left(\frac{y_0^2}{d^2-R^2}+1\right) + (y_0+\sqrt{R^2-d^2})^2 - d^2\left(\frac{y_0^2}{d^2-R^2}+1\right)}\right) r_0. \tag{6}$$

- All the inverted circles of the chain retain the property of being tangent to the neighbour ones and to the two straight lines given by (5); the coordinates of their center and radius are

$$\begin{aligned} x'_{\text{c}k} &= \frac{\omega^k}{d} R R_0, \\ y'_{\text{c}k} &= \sqrt{R^2-d^2}, \\ r'_k &= \omega^k R_0 \end{aligned} \tag{7}$$

for $k = 0, \pm 1, \pm 2, \ldots$.

- As far as center coordinates and radius of the k-th circle of the chain are concerned, one has, for $k = 0, \pm 1, \pm 2, \ldots$,

$$\begin{aligned} x_{\mathrm{c}k} &= s_k \left(\frac{\omega^k}{d} R R_0 \right), \\ y_{\mathrm{c}k} &= \sqrt{R^2 - d^2}(-1 + 2 s_k), \\ r_k &= |s_k| \omega^k R_0, \end{aligned} \tag{8}$$

where

$$\omega = \frac{R-d}{R+d}, \tag{9}$$

$$s_k = \left[\frac{\omega^{2k} 4 (R^2 - d^2) \left(\frac{y_0^2}{d^2 - R^2} + 1 \right)}{R^2 \left(\frac{y_0^2}{d^2 - R^2} + 1 \right) + (y_0 + \sqrt{R^2 - d^2})^2 - d^2 \left(\frac{y_0^2}{d^2 - R^2} + 1 \right)} \right]^{-1}. \tag{10}$$

In Figure 3, an example of a circle chain inside a symmetrical lune together its inverse image is shown.

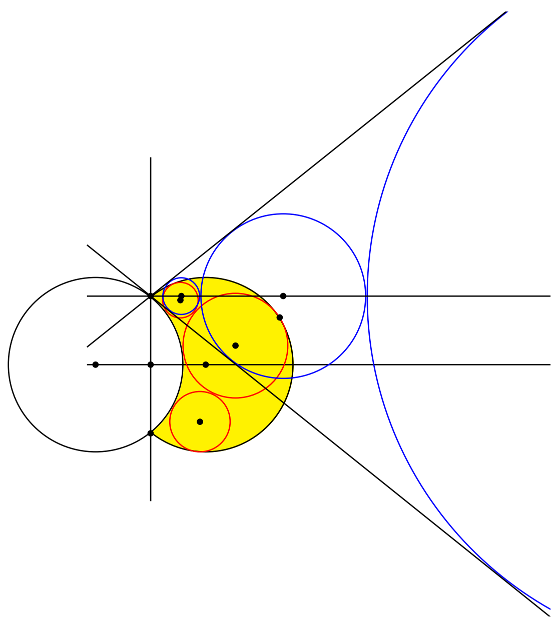

Figure 3. Circle chain inside a lune and its inverse images

Thus, by the aid of the previous formulas (6), (8), (10) and by means of little algebra, one can write:

$$r_k = \begin{cases} r_o \left[\omega^k \frac{\sqrt{R^2 - d^2} + y_0}{2\sqrt{R^2 - d^2}} + \omega^{-k} \frac{\sqrt{R^2 - d^2} - y_0}{2\sqrt{R^2 - d^2}} \right]^{-1}, & k = 0, 1, 2, \ldots, \\ r_o \left[\omega^k \frac{\sqrt{R^2 - d^2} - y_0}{2\sqrt{R^2 - d^2}} + \omega^{-k} \frac{\sqrt{R^2 - d^2} + y_0}{2\sqrt{R^2 - d^2}} \right]^{-1}, & k = 0, -1, -2, \ldots. \end{cases} \tag{11}$$

If k is positive, we have the up chain while, if k is negative, we have the down chain. From (11), we can define the sequence $\{\tau_k\}$ of the ratios between the major circle radius and the k-th circle radius i.e.:

$$\tau_k = \begin{cases} \omega^k \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}}, & k = 0, 1, 2, \dots, \\ \omega^k \frac{\sqrt{R^2-d^2}-y_0}{2\sqrt{R^2-d^2}} + \omega^{-k} \frac{\sqrt{R^2-d^2}+y_0}{2\sqrt{R^2-d^2}}, & k = 0, -1, -2, \dots. \end{cases} \tag{12}$$

By looking at (12), one can notice that the sequence $\{\tau_k\}$ can be expressed by means of a Binet-like formula of the type:

$$\tau_k = \begin{cases} \alpha\omega^k + \beta\omega^{-k}, & k = 0, 1, 2, \dots, \\ \beta\omega^k + \alpha\omega^{-k}, & k = 0, -1, -2, \dots. \end{cases} \tag{13}$$

where

$$\alpha = \frac{\sqrt{R^2 - d^2} + y_0}{2\sqrt{R^2 - d^2}}, \qquad \beta = \frac{\sqrt{R^2 - d^2} - y_0}{2\sqrt{R^2 - d^2}}. \tag{14}$$

It is important, for the following, to point out that J. Kocik showed in [1], how Binet-like formulas can be expressed by means of second order recursive relations that, in the context of the present work, are of the type:

$$\tau_{k+2} = \left(\omega + \frac{1}{\omega}\right) \tau_{k-1} - \tau_k. \tag{15}$$

3. Conditions for $\{\tau_k\}$ to be an integer sequence

In the general case, the sequence $\{\tau_k\}$ is composed by real numbers; here we want to find under which conditions $\{\tau_k\}$ is entirely composed by integer numbers. In other words which are the values for the pair $(d/R, y_0/R)$ so that $\{\tau_k\}$ is an integer sequence?

From the general point of view, it is possible to impose, by means of (12), that the ratios τ_1 and τ_{-1} are equal to two given real numbers μ and λ ($\mu > 1$ and $\lambda > 1$) respectively.

So, by making, for simplicity, the following variable substitutions:

$$\begin{array}{ll} X = \frac{d}{R}, & 0 < X < 1, \\ Y = \frac{y_0}{R}, & -\frac{1-X^2}{2} \leq Y \leq \frac{1-X^2}{2}. \end{array} \tag{16}$$

One can write, from (12) and for $k = 1$ and $k = -1$, the following system of equations:

$$\begin{cases} \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}} + \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}} = \mu, \\ \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}} + \left(\frac{1-X}{1+X}\right) \frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}} = \lambda. \end{cases} \tag{17a}$$

One can verify that the only solution of system (17a) satisfying the constraints in (16) is:

$$(X,Y)=\left(\sqrt{\frac{\lambda+\mu-2}{\lambda+\mu+2}},\ \frac{2(\mu-\lambda)}{(\lambda+\mu+2)\sqrt{\lambda+\mu-2}}\right). \tag{18a}$$

In analogous way, one can impose from (12) that $\tau_1=\lambda$ and $\tau_{-1}=\mu$ which yields the following system:

$$\begin{cases}\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}}+\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}}=\lambda,\\ \left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}-Y}{2\sqrt{1-X^2}}+\left(\frac{1-X}{1+X}\right)\frac{\sqrt{1-X^2}+Y}{2\sqrt{1-X^2}}=\mu.\end{cases} \tag{17b}$$

having the only solution:

$$(X,\ Y)=\left(\sqrt{\frac{\lambda+\mu-2}{\lambda+\mu+2}},\ -\frac{2(\mu-\lambda)}{(\lambda+\mu+2)\sqrt{\lambda+\mu-2}}\right). \tag{18b}$$

Hence, solutions (18a)-(18b) are the conditions relevant to the half width d of the segment AB (shown in Figure 2) and to the ordinate y_0 of the major circle in order to have the ratios $\tau_1=\mu$ and $\tau_{-1}=\lambda$ or $\tau_1=\lambda$ and $\tau_{-1}=\mu$ respectively.

In particular, we are interested to the case where $\mu=n-1$ and $\lambda=m-1$ being m and n a pair of integers with $m\geq 2$ and $n\geq 2$. Thus, we state the following property:

Consider a symmetrical lune characterized by a given ratio d/R. The condition for inscribing inside it a circle chain generating an integer sequence $\{\tau_k\}$ is that

$$\frac{d}{R}=\sqrt{\frac{m+n-4}{m+n}}, \tag{19}$$

$$\frac{y_0}{R}=\frac{2(n-m)}{(m+n)\sqrt{m+n-4}}. \tag{20}$$

where (m,n) is a pair of generic integers with $m\geq 2$, $n\geq 2$ and $(m,n)\neq(2,2)$.

Note that if $m>n$, one has $y_0<0$.

In order to demonstrate this property, by taking into account (9) and (19), one has:

$$\omega+\frac{1}{\omega}=m+n-2 \tag{21}$$

so that $\omega+\frac{1}{\omega}$ is an integer.

Furthermore, as one can easily verify:

$$\tau_{\mathrm{u}0}=\tau_{\mathrm{d}0}=1. \tag{22}$$

As far as $\tau_{\mathrm{u}1}$ and $\tau_{\mathrm{d}1}$ are concerned, we recall that

$$\tau_{\mathrm{u}1}=n-1,\qquad \tau_{\mathrm{d}1}=m-1. \tag{23}$$

Now, let us focus our attention on the up chain; being τ_{u0}, τ_{u1}, and $\omega + \frac{1}{\omega}$ integer numbers, we have, from (15), that τ_{u2} is an integer too; being (15) a recursive relation, it follows that τ_{uk} is an integer for any value of k.

The same reasoning can be applied to the down chain too. Therefore, the sequence $\{\tau_k\}$ is entirely composed by integer numbers.

To conclude, given a symmetrical lune of ratio d/R, if a pair (m, n) of integers exists so that relation (19) is satisfied and by choosing the ordinate y0 so that (20) too is satisfied, then it is possible to inscribe inside the lune itself a circles chain generating two integer sequences $\{\tau_{uk}\}$ and $\{\tau_{dk}\}$.

Conversely, relations (19) and (20) can be used to create an inscribed chain starting from an arbitrary pair of integers (m, n) provided that $(m, n) \neq (2, 2)$.

4. Examples of chains generating integer sequences catalogued in OEIS

By means of suitable choices for the values (d/R) and (y_0/R) given by formulas (19) and (20) respectively, one can obtain circles chains associated to integer sequences. Some interesting particular cases are represented by the symmetrical chains. As mentioned in Section 2, depending on the ordinate y_0 of the major circle center, one can have two different kinds of symmetrical chains that, consequently, generate identical sequences $\{\tau_{uk}\}$ and $\{\tau_{dk}\}$ i.e.:

- the case with $m = n$ ($m \geq 3$, $n \geq 3$); central symmetry
- the case with $m = 2$ and $n \geq 3$ or $m \geq 3$ and $n = 2$; bi-central symmetry

A certain number of these sequences can be found in OEIS ([4], the *On Line Encyclopedia of Integer Sequences*); in Tables I and II some of them are listed:

Table I: sequences listed in OEIS and related to chains having central symmetry

Values of m and n	Classification in OEIS
$m = n = 3$	$A001075$
$m = n = 4$	$A001541$
$m = n = 5$	$A001091$
$m = n = 6$	$A001079$
$m = n = 7$	$A023038$
$m = n = 8$	$A011943$
$m = n = 9$	$A001081$
$m = n = 10$	$A023039$
$m = n = 11$	$A001085$
$m = n = 12$	$A077422$
$m = n = 13$	$A077424$
$m = n = 14$	$A097308$
$m = n = 15$	$A097310$
$m = n = 16$	$A068203$
$m = n = 18$	$A056771$
$m = n = 20$	$A078986$
$m = n = 24$	$A174748$
$m = n = 25$	$A114051$
$m = n = 26$	$A174751$
$m = n = 27$	$A114052$

Table II: sequences listed in OEIS and related to chains having bi-central symmetry

Values of m and n	Classification in OEIS
$m = 2, n = 3$ or $n = 2, m = 3$	A122367
$m = 2, n = 4$ or $n = 2, m = 4$	A079935
$m = 2, n = 5$ or $n = 2, m = 5$	A004253
$m = 2, n = 6$ or $n = 2, m = 6$	A001653
$m = 2, n = 7$ or $n = 2, m = 7$	A049685
$m = 2, n = 8$ or $n = 2, m = 8$	A070997
$m = 2, n = 9$ or $n = 2, m = 9$	A070998
$m = 2, n = 10$ or $n = 2, m = 10$	A138288
$m = 2, n = 11$ or $n = 2, m = 11$	A078922
$m = 2, n = 12$ or $n = 2, m = 12$	A077417
$m = 2, n = 13$ or $n = 2, m = 13$	A085260
$m = 2, n = 14$ or $n = 2, m = 14$	A001570
$m = 2, n = 15$ or $n = 2, m = 15$	A160682
$m = 2, n = 16$ or $n = 2, m = 16$	A157456
$m = 2, n = 17$ or $n = 2, m = 17$	A161595
$m = 2, n = 18$ or $n = 2, m = 18$	A007805
$m = 2, n = 20$ or $n = 2, m = 20$	A075839
$m = 2, n = 22$ or $n = 2, m = 22$	A157014
$m = 2, n = 24$ or $n = 2, m = 24$	A159664
$m = 2, n = 26$ or $n = 2, m = 26$	A153111
$m = 2, n = 27$ or $n = 2, m = 27$	A097835
$m = 2, n = 28$ or $n = 2, m = 28$	A159668
$m = 2, n = 30$ or $n = 2, m = 30$	A157877
$m = 2, n = 31$ or $n = 2, m = 31$	A111216
$m = 2, n = 32$ or $n = 2, m = 32$	A159674
$m = 2, n = 34$ or $n = 2, m = 34$	A077420
$m = 2, n = 36$ or $n = 2, m = 36$	A238379
$m = 2, n = 38$ or $n = 2, m = 38$	A097315
$m = 2, n = 40$ or $n = 2, m = 40$	A269028

In Table III, some other examples of integer sequences not associated to any symmetry between up and circles chains are shown; one can notice that by interchanging the values of m and n, up and down chains are interchanged too.

Table III: sequences listed in OEIS and related to chains having no symmetry

Values of m and n	Classification of the sequence According to OEIS	
$m = 3, n = 4$	A002320 up	A002310 down
$m = 4, n = 3$	A002310 up	A002320 down
$m = 3, n = 5$	A038723 up	A038725 down
$m = 5, n = 3$	A038725 up	A038723 down
$m = 3, n = 6$	A033889 up	A172968 up
$m = 6, n = 3$	A172968 down	A033889 down
$m = 4, n = 6$	A105426 up	A144479 up
$m = 6, n = 4$	A144479 down	A105426 down

Let us see two examples of circles chains generated by sequences listed in OEIS.

Example 1. Circle chain with central symmetry derived from sequence $A001075$.

The first terms of $A001075$ are: $\{1, 2, 7, 26, 97, 362, \dots\}$.
From the second term, we have that $\tau_{-1} = \tau_1 = 2$; by remembering (23), one has $m = n = 3$ and finally from (20) and (19), one obtains:

$$\frac{y_0}{R} = 0, \qquad \frac{d}{R} = \frac{1}{\sqrt{3}}.$$

Example 2. Circle chain with central symmetry derived from sequence $A122367$.

The first terms of $A122367$ are: $\{1, 2, 5, 13, 34, 89, \dots\}$. Due to the fact that we are considering a chain with bi-symmetry, we have $r_0 = |y_0|$. From (1) one can write

$$y_0 = d\sqrt{\frac{y_0^2}{d^2 - R^2} + 1}.$$

By considering the up chain (so that $y_0 > 0$) and by substituting (20) and (19) in the previous formula, we obtain the following relation between m and n:

$$n - m = \sqrt{(m + n - 4)(mn - m - n)}.$$

Moreover, from the second term of the sequence, we have that $\tau_1 = 2$; by remembering (23), one has that $n = 3$. By substituting $n = 3$ in the above equation, one gets $m = 2$. Finally, from (20) and (19), one has:

$$\frac{y_0}{R} = \frac{2}{5}, \quad \frac{r}{R} = \frac{1}{\sqrt{5}}.$$

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Counting the Number of Isosceles Triangles in Rectangular Regular Grids

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Abstract. In general graph theory, the only relationship between vertices are expressed via the edges. When the vertices are embedded in an Euclidean space, the geometric relationships between vertices and edges can be interesting objects of study. We look at the number of isosceles triangles where the vertices are points on a regular grid and show that they satisfy a recurrence relation when the grid is large enough. We also derive recurrence relations for the number of acute, obtuse and right isosceles triangles.

1. Introduction

In general graph theory, the relationship between vertices are expressed via the edges of the graph. In geometric graph theory, the vertices and edges have geometric attributes that are important as well. For instance, a random geometric graph is constructed by embedding the vertices randomly in some metric space and connecting two vertices by an edge if and only if they are close enough to each other. When the vertices lie in an Euclidean space, the edges of vertices can form geometric objects such as polygons. In [4], the occurrence of polygons is studied. In [1] the number of nontrivial triangles is studied. In this note, we consider this problem when the vertices are arranged on a regular grid. The study of the abundance (or sparsity) of such subgraphs or network motifs [3] is important in the characterization of complex networks.

Consider an n by k rectangular regular grid G with $n, k \geq 2$. A physical manifestation of this pattern, called geoboard, is useful in teaching elementary geometric concepts [2]. Let 3 distinct points be chosen on the grid such that they form the vertices of a triangle with nonzero area (i.e. the points are not collinear). In OEIS sequences A271910, A271911, A271912, A271913, A271915 [5], the number of such triangles that are isosceles are listed for various k and n. Neil Sloane made the conjecture that for a fixed $n \geq 2$, the number of isosceles triangles in an n by k grid, denoted as $a_n(k)$, satisfies the recurrence relation

$$a_n(k) = 2a_n(k-1) - 2a_n(k-3) + a_n(k-4)$$

for $k > K(n)$ for some number $K(n)$. The purpose of this note is to show that this conjecture is true and give an explicit form of $K(n)$. In particular, we show that

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$K(n) = (n-1)^2 + 3$ if n is even and $K(n) = (n-1)^2 + 2$ if n is odd and that this is the best possible value for $K(n)$.

2. Counting isosceles triangles

We first start with some simple results:

Lemma 1. *If x, y, u, w are integers such that $0 < x, u \leq n$ and $y > \frac{n^2}{2}$, then $x^2 + y^2 = u^2 + w^2$ implies that $x = u$ and $y = w$.*

Proof. If $y \neq w$, then $|y^2 - w^2| \geq 2y - 1 > n^2 - 1$. On the other hand, $|y^2 - w^2| = |u^2 - x^2| < n^2$, a contradiction. $\square$

Lemma 2. *If x, y, u, w are integers such that $0 \leq x, u \leq n$ and $y > \frac{n^2+1}{2}$, then $x^2 + y^2 = u^2 + w^2$ implies that $x = u$ and $y = w$.*

Proof. If $y \neq w$, then $|y^2 - w^2| \geq 2y - 1 > n^2$. On the other hand, $|y^2 - w^2| = |u^2 - x^2| \leq n^2$, a contradiction. $\square$

Lemma 3.

$$2 \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2m) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor.$$

Proof. If n is odd, then

$$2 \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2m) = 2 \sum_{m=1}^{\frac{n-1}{2}} (n - 2m) = n(n-1) - 2\frac{n-1}{2}\frac{n+1}{2} = \frac{(n-1)^2}{2}.$$

If n is even, then

$$\begin{aligned} 2 \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2m) &= 2 \sum_{m=1}^{\frac{n-2}{2}} (n - 2m) \\ &= n(n-2) - 2\frac{n-2}{2}\frac{n}{2} = \frac{(n-1)^2 - 1}{2} = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor. \end{aligned}$$

$\square$

Our main result is the following:

Theorem 4. *Let $a_n(k)$ be the number of isosceles triangles of nonzero area formed by 3 distinct points in an n by k grid. Then*

$$a_n(k) = 2a_n(k-1) - 2a_n(k-3) + a_n(k-4)$$

for $k > (n-1)^2 + 3$.

Proof. For $k > 2$, let the n by k array be decomposed into 3 parts consisting of the first column, the middle part (of size n by $k-2$) and the last column denoted as p_1, p_2 and p_3 respectively.

Let $b_n(k)$ be the number of isosceles triangles in the n by k array with vertices in the last column p_3 and let $c_n(k)$ be the number of isosceles triangles in the n

by k array with vertices in the first and last columns p_1 and p_3. It is clear that $b_n(k) = a_n(k) - a_n(k-1)$. Furthermore, $b_n(k) = b_n(k-1) + c_n(k)$. Let us partition the isosceles triangles corresponding to $c_n(k)$ into 2 groups, $A(k)$ and $B(k)$. $A(k)$ are triangles with all 3 vertices in p_1 or p_3 and $B(k)$ are triangles with a vertex in each of p_1, p_2 and p_3.

Thus triangles in $A(k)$ must have 2 vertices in p_1 or 2 vertices in p_3. Since $k > n$, the edges not in p_1 (resp. p_3) must be longer than the edge in p_1 (resp. p_3). Therefore all triangles in $A(k)$ must be of the form where the 2 vertices in p_1 (resp. p_3) are an even number of rows apart (i.e., there are an odd number of rows between the 2 vertices) and the third vertex is in p_3 (resp. p_1) in the middle row between them. Since $k > (n-1)^2 + 1$, these triangles are all acute (we'll revisit this later). Let us count how many such triangles there are. There are $n - 2m$ pairs of vertices which are $2m$ rows apart, for $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. Thus the total number of triangles in $A(k)$ is

$$2 \sum_{m=1}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2m) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor .$$

by Lemma 3.

Next we consider the isosceles triangles in $B(k)$. Let e_1 be the edge between the vertex in p_1 and the vertex in p_2 and e_2 be the edge between the vertex in p_2 and the vertex in p_3 and e_3 be the edge between the vertices in p_1 and p_3. There are 2 cases.

In case 1, the length of e_1 is equal to the length of e_2 and is expressed as $x^2 + y^2 = u^2 + w^2$ with $0 \leq x, u \leq n-1$ and $y + w = k-1$. Without loss of generality, we pick y to be the larger of y and w, i.e., $y \geq \frac{k}{2}$. This is illustrated in Fig. 1.

Since $k > (n-1)^2 + 1$, Lemma 2 implies that $x = u$ and $y = w$ and this is only possible if k is odd and the vertex in p_1 and in p_3 must be at the same row and p_2 in a different row. Note that such triangles are right triangles for $n = 2$ and obtuse for $n > 2$.

In case 2, the length of e_3 is equal to the length of either e_1 or e_2. Lemma 2 shows that in this case $y = k - 1$ which is not possible. This implies that $c_n(k) = P(n,2) + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$ if k is odd and $c_n(k) = \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$ if k is even. Thus we have

$$\begin{aligned} a_n(k) &= a_n(k-1) + b_n(k) = a_n(k-1) + b_n(k-1) + c_n(k) \\ &= a_n(k-1) + a_n(k-1) - a_n(k-2) + c_n(k) \\ &= 2a_n(k-1) - a_n(k-2) + c_n(k). \end{aligned}$$

Since $k - 2 > n$ and $k - 2 > (n-1)^2 + 1$, we can apply the same analysis to find that $c_n(k-2) = c_n(k)$. Since $a_n(k-2) = 2a_n(k-3) - a_n(k-4) + c_n(k-2)$, we get

$$\begin{aligned} a_n(k) &= 2a_n(k-1) - 2a_n(k-3) + a_n(k-4) - c_n(k-2) + c_n(k) \\ &= 2a_n(k-1) - 2a_n(k-3) + a_n(k-4). \end{aligned}$$

□

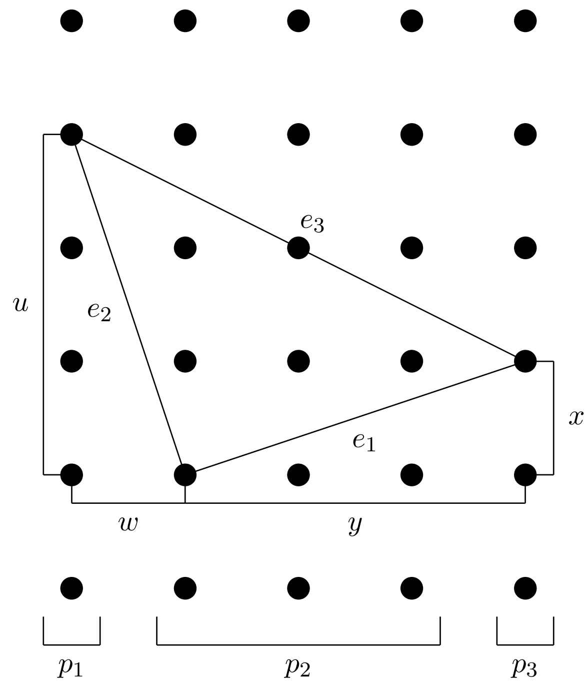

Figure 1. Illustrating an isosceles triangle in $B(k)$ and the definitions of u, x, w and y. For this illustrative example, we did not require that $k > (n-1)^2 + 1$.

It is clear that $a_n(k) = a_k(n)$. The above argument also shows that:

Corollary 5. *Suppose that $k > (n-1)^2 + 1$. Then* $a_n(k) = 2a_n(k-1) - a_n(k-2) + n(n-1) + \lfloor \frac{(n-1)^2}{2} \rfloor$ *if k is odd and* $a_n(k) = 2a_n(k-1) - a_n(k-2) + \lfloor \frac{(n-1)^2}{2} \rfloor$ *if k is even.*

3. Obtuse, acute and right isosceles

As noted above, the triangles in $A(k)$ are acute and the triangles in B(k) are right for n = 2 and obtuse for $n > 2$. This implies that Theorem 4 is also true when restricted to the set of acute isosceles triangles and restricted obtuse or right isosceles triangles if $n > 2$. As for Corollary 5, we have the following similar results for acute isosceles triangles:

Corollary 6. *Let $a_n^a(k)$ be the number of acute isosceles triangles of nonzero area formed by 3 distinct points in an n by k grid. Then*

$$a_n^a(k) = 2a_n^a(k-1) - a_n^a(k-2) + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor$$

for $k > (n-1)^2 + 1$.

Similarly, the same arguments can be applied to obtuse isosceles triangles:

Corollary 7. *Let $a_n^o(k)$ be the number of obtuse isosceles triangles of nonzero area formed by 3 distinct points in an n by k grid. Suppose $k > \max(3, (n-1)^2+1)$. Then*
$a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2) + n(n-1)$ *if k is odd and*
$a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2)$ *if k is even.*

And right isosceles triangles:

Corollary 8. *Let $a_n^r(k)$ be the number of right isosceles triangles of nonzero area formed by 3 distinct points in an n by k grid. Then*

$$a_n^r(k) = 2a_n^r(k-1) - a_n^r(k-2)$$

for $k > \max(3, (n-1)^2+1)$.

4. Pythagorean triples and a small improvement

In Theorem 4 we have given an explicit form for the bound $K(n)$ in the conjecture described in Section 1. We next show that for odd n, this bound can be improved by 1. Let us consider the case $k = (n-1)^2+1$. Consider the triangles in $B(k)$. Again case 2, where the length of e_3 is equal to the length of either e_1 or e_2, is impossible. For case 1, the length of e_1 is equal to the length of e_2 and is expressed as $x^2+y^2 = u^2+w^2$ with $0 \leq x, u \leq n-1$ and $y \geq \frac{k}{2}$ and $y \geq w$. If $x > 0$, this implies that $u > 0$ and we can apply Lemma 1 to show that the only isosceles triangles in $B(k)$ are as in Theorem 4. Suppose $x = 0$. We can eliminate $y = w$ since this results in $u = 0$ and a collinear set of vertices. For $n = 2$, $k = 2$, it is clear that $w = 0$ and $y = w+1 = 1$. For $n > 2$, since $u \leq n-1$, this again means that $y = w+1$ as otherwise $y^2 - w^2 \geq 4y - 4 \geq 2k - 4 > (n-1)^2$. This implies that k must necessarily be even such that a, b, c are integers forming a Pythagorean triple satisfying $a^2 = b^2 + c^2$ where $a = \frac{k}{2}$, $b = a - 1$ and $c = \sqrt{a^2 - b^2} = \sqrt{k-1}$. If n is odd, then $k = (n-1)^2+1$ is odd, and the case of $x = 0$ cannot occur. Thus Theorem 4 can be improved to:

Theorem 9. *Suppose n is odd. Then $a_n(k) = 2a_n(k-1) - 2a_n(k-3) + a_n(k-4)$ for $k > (n-1)^2+2$.*

We can rewrite this in an inhomogeneous form of lower degree:

Corollary 10. *Suppose that n is odd and $k > (n-1)^2$. Then*
$a_n(k) = 2a_n(k-1) - a_n(k-2) + n(n-1) + \lfloor \frac{(n-1)^2}{2} \rfloor$ *if k is odd and*
$a_n(k) = 2a_n(k-1) - a_n(k-2) + \lfloor \frac{(n-1)^2}{2} \rfloor$ *if k is even.*

Again, Theorem 9 is still valid when restricted to acute isosceles triangles. When $n = 3$ and $k = (n-1)^2+1 = 5$, two of the triangles in $B(k)$ are right isosceles and for $n > 3$ and $k = (n-1)^2+1$, all triangles in $B(k)$ are obtuse triangles. Thus we have

Corollary 11. *Suppose n is odd. Then*

$$a_n^a(k) = 2a_n^a(k-1) - 2a_n^a(k-3) + a_n^a(k-4) \text{ for } k > (n-1)^2 + 2.$$

Furthermore, $a_n^o(k) = 2a_n^o(k-1) - 2a_n^o(k-3) + a_n^o(k-4)$ *and* $a_n^r(k) = 2a_n^r(k-1) - 2a_n^r(k-3) + a_n^r(k-4)$ *for* $k > \max(7, (n-1)^2 + 2)$.

Similarly we have for obtuse isosceles triangles:

Corollary 12. *Suppose n is odd and $k > \max(5, (n-1)^2)$. Then* $a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2) + n(n-1)$ *for k odd and* $a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2)$ *if k is even.*

When n is even and $k = (n-1)^2 + 1$, the above analysis shows that there are 4 extra isosceles triangles in $B(k)$ due to the case $x = 0$. These triangles are right triangles for $n = 2$ and obtuse for $n > 2$. This is illustrated in Fig. 2 for the case $k = 10$, $n = 4$. This means that when restricted to acute isosceles triangles (or right isosceles triangles with $n > 2$), the condition that n is odd is not necessary in Theorem 9. In addition Corollaries 6 and 8 can be improved to:

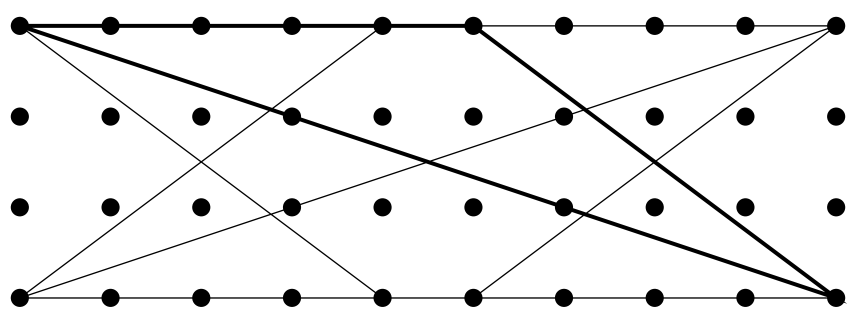

Figure 2. 4 obtuse isosceles triangles in $B(k)$ corresponding to $x = 0$ for the case $k = 10$, $n = 4$. One of the triangles is highlighted and the other 3 are obtained via symmetries.

Corollary 13. *Suppose $k > (n-1)^2$. Then* $a_n^a(k) = 2a_n^a(k-1) - a_n^a(k-2) + \lfloor \frac{(n-1)^2}{2} \rfloor$.

This can be rewritten in homogeneous form as:

Corollary 14. *Suppose $k > (n-1)^2 + 1$. Then* $a_n^a(k) = 3a_n^a(k-1) - 3a_n^a(k-2) + a_n^a(k-3)$.

Proof. From Corollary 13, we have $a_n^a(k) = 2a_n^a(k-1) - a_n^a(k-2) + \lfloor \frac{(n-1)^2}{2} \rfloor$ and $a_n^a(k-1) = 2a_n^a(k-2) - a_n^a(k-3) + \lfloor \frac{(n-1)^2}{2} \rfloor$. Subtracting these two equations and combining terms we reach the conclusion. $\square$

A similar homogeneous linear recurrence for right isosceles triangles is:

Corollary 15. *Suppose* $n > 2$ *and* $k > \max(5, (n-1)^2)$. *Then* $a_n^r(k) = 2a_n^r(k-1) - a_n^r(k-2)$.

Similarly, we have the following results which complements Corollaries 5 and 7.

Corollary 16. *Suppose that* n *is even and* $k = (n-1)^2 + 1$. *Then* $a_n(k) = 2a_n(k-1) - a_n(k-2) + \lfloor \frac{(n-1)^2}{2} \rfloor + 4$.

Again, this can be rewritten as:

Corollary 17. *Suppose that* $n > 2$ *is even and* $k = (n-1)^2 + 1$. *Then* $a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2) + 4$.

Because of the recurrence relation in Theorem 4, the generating functions for $a_n(k)$ for $k \geq 1$ will be of the form $\frac{p_n(x)}{(x-1)^3(x+1)}$. The denominator for the generating function of $a_n^a(k)$ can be reduced to $(x-1)^3$ corresponding to the recurrence relation in Corollary 14. Similarly, the denominator for the generating function of $a_n^r(k)$ can be reduced to $(x-1)^2$ corresponding to the recurrence relation in Corollary 8.

5. Optimal value for $K(n)$

Corollary 16 shows that $(n-1)^2 + 3$ is the best value for $K(n)$ when n is even. Next we show that $(n-1)^2 + 2$ is the best value for $K(n)$ when n is odd.

Consider the case where $n > 2$ is odd and $k = (n-1)^2$. Then k is even and by setting $y = \frac{k}{2}$, $w = \frac{k}{2} - 1$, $x = 1$, $u = n - 1$, we get $x^2 + y^2 = u^2 + w^2$ where $x \neq u$ and $y \neq w$ and $y + w = k - 1$. This corresponds to 4 additional isosceles triangles in $B(k)$. These triangles are right triangles for $n = 3$ (Fig. 3) and obtuse for $n > 3$.

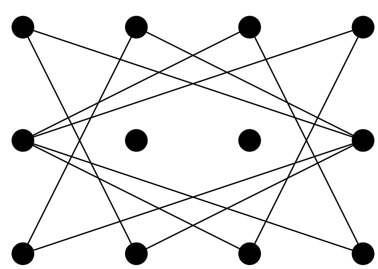

Figure 3. 4 right isosceles triangles in $B(k)$ corresponding to $x = 1$ for the case $k = 4, n = 3$.

Theorem 18. *Suppose that* n *is odd and* $k = (n-1)^2$. *Then*

$$a_n(k) = 2a_n(k-1) - a_n(k-2) + \left\lfloor \frac{(n-1)^2}{2} \right\rfloor + 4.$$

Corollary 19. *Suppose that $n > 3$ is odd and $k = (n-1)^2$. Then*

$$a_n^o(k) = 2a_n^o(k-1) - a_n^o(k-2) + 4.$$

This means that $K(n) = (n-1)^2 + 3$ for n is even and $K(n) = (n-1)^2 + 2$ for n is odd are the best possible values for $K(n)$ in the conjecture in Section 1 as expressed in Theorems 4 and 9. They are also optimal when restricted to the set of obtuse isosceles and $n > 3$. In addition,

$$a_n(k) = 2a_n(k-1) - 2a_n(k-3) + a_n(k-4) - 4$$

if $k = (n-1)^2 + 3$ and n is even or if $k = (n-1)^2 + 2$ and n is odd. This is due to the fact that for these values of n and k, $c_n(k-2) = c_n(k) + 4$.

6. Subsets of points on a regular grid containing the vertices of isosceles triangles

So far we looked at a regular grid of points and counted the number of isosceles triangles with these points as vertices. A related problem is the minimal number of points on the grid such that an isosceles triangle can always be found.

Consider again an n by k regular grid. Let $S(n,k) = r$ be defined as the smallest number r such that any r points in the grid contain the vertices of an isosceles triangle of nonzero area. This is equivalent to finding the constellation with the largest number of points $T(n,k)$ such that no three points form an isosceles triangle of nonzero area as $S(n,k) = T(n,k) + 1$. The values of $T(n,k)$ for small n and k are listed in OEIS A271914 where it was conjectured that $T(n,k) \leq n + k - 1$. The next Lemma shows that this conjecture is true for $k = 1, 2$.

Lemma 20. *$T(n,k)$ satisfies the following properties:*
(1) $T(n,k) = T(k,n)$. If $m \leq n$, then $T(m,k) \leq T(n,k)$.
(2) $T(n,k) \geq \max(n,k)$.
(3) $T(n,1) = n$.
(4) (subadditivity) $T(n,k+m) \leq T(n,k) + T(n,m)$.
(5) $T(n,2) = n$ for $n > 3$.
(6) Suppose $n > 4$. Then $T(n,3) \geq n+1$ if n is odd and $T(n,3) \geq n+2$ if n is even.

Proof. Property (1) is trivial. Properties (2) and (3) are clear, by considering points all in one row (or column) only. To show property (4), suppose $T(n,k)+T(n,m)+1$ points are chosen in an n by $k+m$ grid, and consider a decomposition into an n by k grid and an n by m grid. Either the n by k grid has more than $T(n,k)$ points or the n by m grid has more than $T(n,m)$ points. In either case, there is an isosceles triangle. Property (4) implies that $T(n+2,2) \leq T(n,2) + T(2,2) = T(n,2) + 2$ and property (2) implies $T(n,2) \geq n$ for $n \geq 2$. Thus property (5) follows from properties (2) and (4). Suppose $n > 4$. For n odd, the points $\{(1,i) : 2 \leq i \leq n\}$ along with the points $(2,1)$ and $(3,1)$ shows that $T(n,3) \geq n+1$ (Fig. 4). If n is even, the points $\{(1,i) : 2 \leq i \leq n-1\}$ along with the points $(2,1)$, $(3,1)$, $(2,n)$ and $(3,n)$ shows that $T(n,3) \geq n+2$ (Fig. 5). $\square$

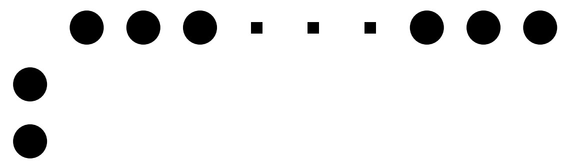

Figure 4. A constellation of $n+1$ points for which there are no 3 points forming an isosceles triangle for the case when $n > 4$ is odd.

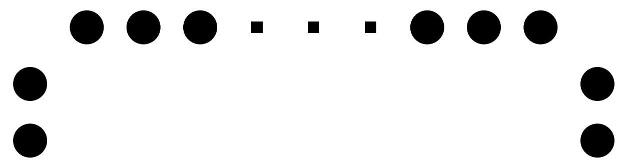

Figure 5. A constellation of $n+2$ points for which there is are 3 points forming an isosceles triangle for the case when $n > 4$ is even.

Note that for n odd, the constellation in Fig. 5 for $n-1$ which has $n-1+2 = n+1$ points will also show that $T(n,3) \geq n+1$. We conjecture the following:

Conjecture 1. If n is even, then $T(n,k) \leq n+k-2$ for $k \geq 2n$.

Conjecture 2. For $n > 4$, $T(n,3) = n+1$ if n is odd and $T(n,3) = n+2$ if n is even.

Note that by Lemma 20 Conjecture 1 is true for $n = 2$.

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Chains of Tangent Circles Inscribed in a Triangle

Giovanni Lucca

Abstract. Starting from the incircle of a generic triangle, we construct three infinite chains of circles having the property that the generic i-th circle of the chain is tangent to the $(i-1)$-th and $(i+1)$-th ones and to two sides of the triangle. Furthermore, we look for the conditions which guarantee that the ratio between the inradius and the radius of every circle in the three chains is an integer.

Figure 1 shows a generic triangle with three circle chains, each, beginning with the incircle, consisting of circles tangent to two sides of the triangle and to its two neighbours. We study the possibilities that the inradius is an integer multiple of the radius of each circle in this configuration.

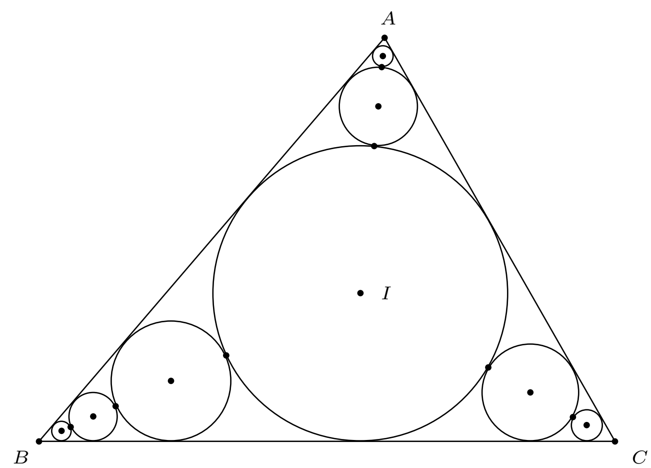

Figure 1. Three circle chains inside a triangle originating from the incircle

We begin with the construction of a circle chain beginning with a circle $\mathscr{C}_0$ (with center A_0) tangent to two lines ℓ and ℓ' intersecting at O (Figure 2). Construct the segment OA_0.

(1) Let OA_0 intersect the circle $\mathscr{C}_0$ at P_1.
(2) Construct the perpendicular to OA_0 at P_1 to intersect ℓ at Q_1.
(3) Construct the bisector of angle OQ_1P_1 to intersect OA_0 at A_1.
(4) Construct the circle $\mathscr{C}_1$ with center A_1 passing through P_1. This is tangent to $\mathscr{C}_0$ and both lines ℓ and ℓ'.

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(5) Repeat (1)-(4) with $\mathscr{C}_1$ replacing $\mathscr{C}_0$, resulting in P_2, A_2, and the circle $\mathscr{C}_2$ tangent to $\mathscr{C}_1$ and both lines ℓ and ℓ'.

(6) Continuing to form a circle chain $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \dots$.

Lemma 1. *Let ℓ and ℓ' be two lines meeting at O at an angle θ. The radii of the circles in a chain tangent to both lines form a geometric progression of common ratio $\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$.*

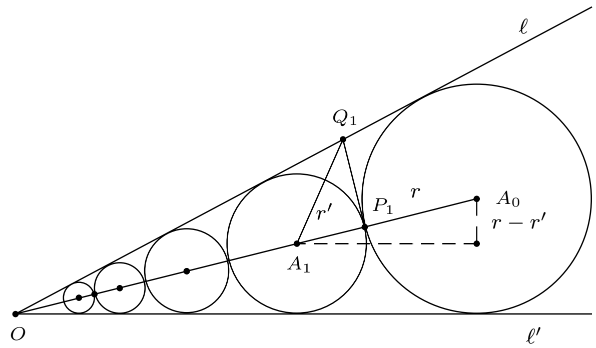

Figure 2

Proof. Consider in Figure 2 two neighboring circles with radii $r > r'$. Clearly,

$$\sin\frac{\theta}{2} = \frac{r-r'}{r+r'} \implies \frac{r}{r'} = \frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$$

□

Now consider a triangle ABC with three circle constructed in its three angles, each beginning with the incircle. By Lemma 1, the inradius is an integer multiple of the radius of each circle in the three chains if and only if

$$\frac{1+\sin\frac{A}{2}}{1-\sin\frac{A}{2}} = k, \qquad \frac{1+\sin\frac{B}{2}}{1-\sin\frac{B}{2}} = m, \qquad \frac{1+\sin\frac{C}{2}}{1-\sin\frac{C}{2}} = n,$$

for integers k, m, $n > 1$. From these,

$$\sin\frac{A}{2} = \frac{k-1}{k+1}, \qquad \sin\frac{B}{2} = \frac{m-1}{m+1}, \qquad \sin\frac{C}{2} = \frac{n-1}{n+1}.$$

Since $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$, we must have

$$\begin{aligned} \frac{k-1}{k+1} = &\ \sin\frac{A}{2} = \cos\left(\frac{B}{2} + \frac{C}{2}\right) \\ = &\ \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2} \\ = &\ \frac{2\sqrt{m}}{m+1}\cdot\frac{2\sqrt{n}}{n+1} - \frac{m-1}{m+1}\cdot\frac{n-1}{n+1} \\ = &\ \frac{4\sqrt{mn} - (m-1)(n-1)}{(m+1)(n+1)}. \end{aligned} \tag{1}$$

It follows that $\sqrt{mn}$ must be rational, and

$$mn \text{ is the square of an integer.} \tag{2}$$

Now, the condition $\frac{B}{2} + \frac{C}{2} < \frac{\pi}{2}$ poses another restriction:

$$\arcsin \frac{m-1}{m+1} + \arcsin \frac{n-1}{n+1} < \frac{\pi}{2}. \tag{3}$$

The only integers $m \leq n$ satisfying (2) and (3)

$$(m, n) = (3, 3),\ (2, 2),\ (2, 8),$$

corresponding to $k = 3$, 8, 2 respectively. This results in only two triangles with

$$(k, m, n) = (3, 3, 3),\ (8, 2, 2).$$

The case $(k, m, n) = (3, 3, 3)$ is clearly realized by equilateral triangles. For $(k, m, n) = (8, 2, 2)$, $B = C = 2 \arcsin \frac{1}{3} = \arccos \frac{7}{9}$ and $A = \pi - 4 \arcsin \frac{1}{3}$. Since the triangle is isosceles, $BC : CA : AB = 14 : 9 : 9$ (see Figure 3).

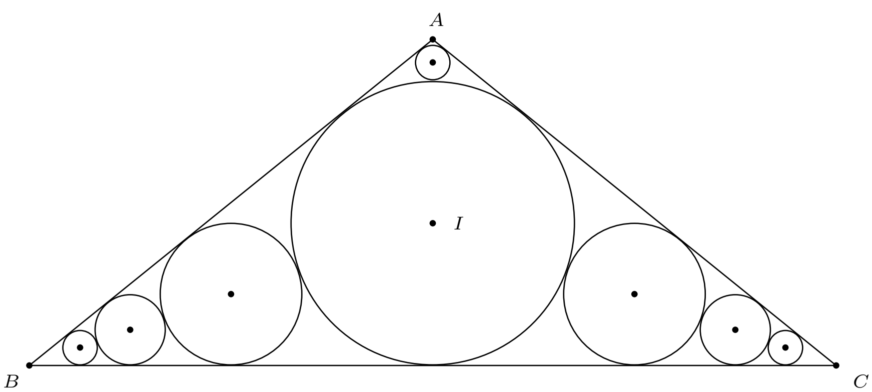

Figure 3. Triangle with circle chains with common ratios $(k, m, n) = (8, 2, 2)$

It is interesting to note a degenerate case. For $(m, n) = (4, 9)$, we have $k = 1$ from (1), and (3) is an equality: $\arcsin \frac{3}{5} + \arcsin \frac{8}{10} = \frac{\pi}{2}$. In this case, $\frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ and $A = 0$. This yields a degenerate triangle with two parallel lines making an angle $2 \arcsin \frac{3}{5} = \arcsin \frac{24}{25}$ with the base (see Figure 4).

We summarize the results in the following theorem.

Theorem 2. *There are three classes of triangles ABC in which the radii of the circle chains in the angles A, B, C beginning with the incircle are geometric progressions with* integer *common ratios k, m, n respectively:*

(k, m, n)	A	B	C
$(1, 4, 9)$	0	$2 \arcsin \frac{3}{5}$	$\pi - 2 \arcsin \frac{3}{5}$
$(3, 3, 3)$	$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{3}$
$(8, 2, 2)$	$\pi - 4 \arcsin \frac{1}{3}$	$2 \arcsin \frac{1}{3}$	$2 \arcsin \frac{1}{3}$

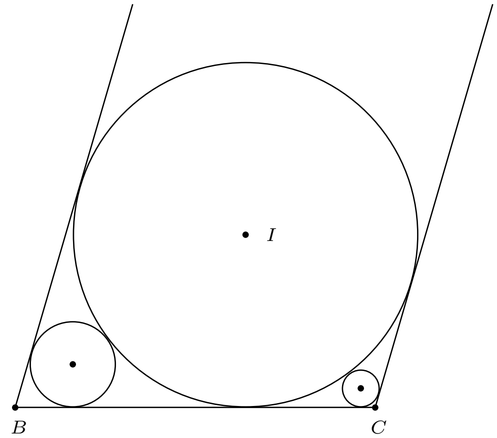

Figure 4. Degenerate triangle with circle chains with common ratios $(k, m, n) = (1, 4, 9)$

In [1, 2], we relate circle chains inscribed in symmetric lenses and lunes with certain integer sequences. The integer sequences encountered in the present note of the ratios of radii of successive circles in the circle chains are classified in OEIS [3] as

- $A000012$ for $\{1^k\}$,
- $A000079$ for $\{2^k\}$,
- $A000302$ for $\{4^k\}$,
- $A001018$ for $\{8^k\}$,
- $A001019$ for $\{9^k\}$.

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Do Dogs Know The Bifurcation Locus?

Li Zhou

Abstract. A dog runs at speed r and swims at speed s, with $r > s$. For a fixed point A in a lake with a straight shoreline, where are all the points B in the lake such that the direct swimming path from A to B takes the dog the same time as the fastest indirect swimming- running-swimming path? We give a simple geometric solution to this bifurcation-locus problem.

In [1], the authors discuss a challenging situation for the remarkable dog Elvis: As in Figure 1, Elvis is initially at a point A in the lake and a ball is thrown to a point B in the lake. What path should Elvis choose in order to minimize his time to reach the ball? This is challenging because Elvis runs at speed r and swims at speed s, with $r > s$, so a direct swimming (S) path AB may be slower than an indirect swimming-running-swimming (SRS) path $AXYB$.

Figure 1. Elvis' dilemma

In [2], the author gives a simple ruler-compass determination of the optimal path (S or the fastest SRS) for any two given points A and B.

We now ask a more interesting question. For a fixed point A, a point B is called a bifurcation point of A if the S-path from A to B takes the same time as the fastest SRS-path from A to B. What is the locus of bifurcation points of A? This question has a nice answer and a simple geometric proof.

As in Figure 2, let B be a bifurcation point of A. The fastest SRS-path from A to B is $AXY\ B$ where AX and YB form the angle $\theta = \arccos \frac{s}{r}$ with the shoreline (see [1] or [2]). Let A' be the reflection of A across the shoreline. Draw the line d through A' and perpendicular to $A'X$.

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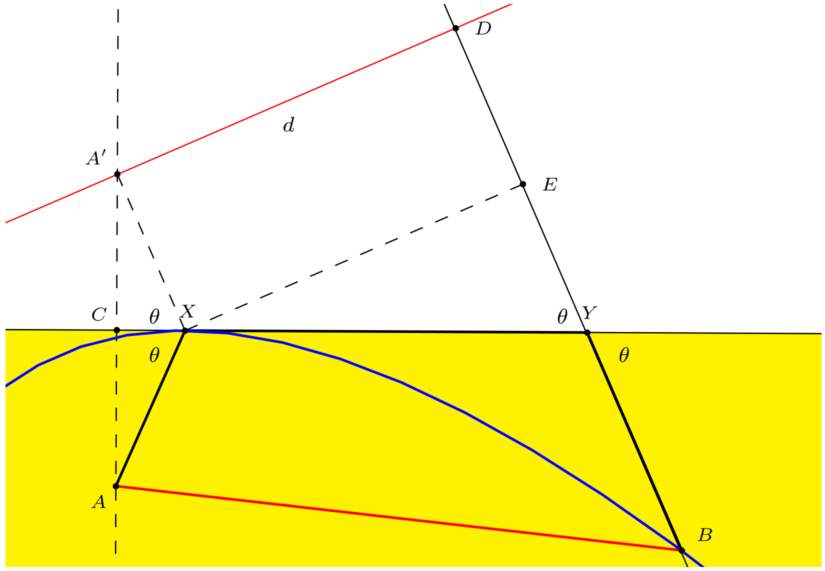

Figure 2. Bifurcation locus of A

Theorem 1. *The bifurcation locus of A is part of the parabola with focus A and directrix d.*

Proof. Note that $BY \perp d$ with foot D. Locate E on BD such that $XE \parallel d$. Then $AX = A'X = DE$, and the dog swims the distance EY in the same time as he runs the distance XY. Thus, the time to swim the distance BD is the same as the time for the SRS-path $AXYB$, thus also the same as the time to swim the distance AB. Hence, $BD = BA$, completing the proof.

Of course, the locus is the part of the parabola starting at X and moving away from A. $\square$

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Another Simple Construction of the Golden Section with Equilateral Triangles

Tran Quang Hung

Abstract. A simple construction of the golden section with equilateral triangles.

Three congruent equilateral triangles $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ are positioned in such a way that B_1 and B_3 are respectively midpoints of the segments A_2C_2 and A_2B_2, while the vertices A_1, A_2, A_3 are on a line perpendicular to both the segments A_1C_1 and A_2C_2.

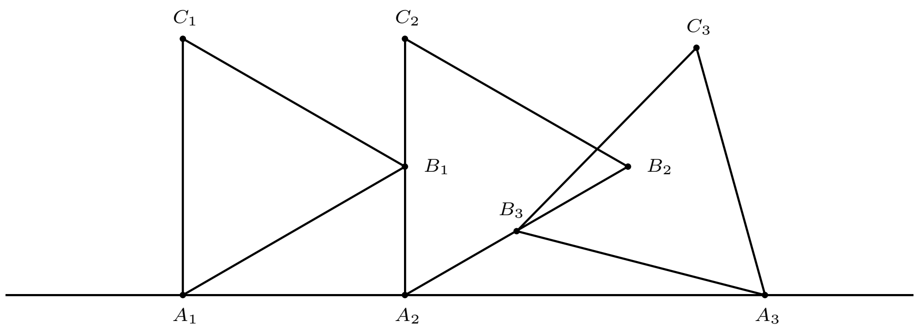

Figure 1

In this arrangement, A_2 divides A_1A_3 in the golden ratio.

Proof. Let a be a side of the equilateral triangles. Because $\angle B_1A_1A_2 = 30^\circ$,

$$A_1A_2 = a\cos 30^\circ = \frac{\sqrt{3}}{2}a. \tag{1}$$

Again, since $\angle B_3A_2A_3 = 30^\circ$, by the law of cosines,

$$\begin{aligned} A_3B_3^2 &= A_2B_3^2 + A_2A_3^2 - 2A_2B_3 \cdot A_2A_3\cos 30^\circ, \\ a^2 &= \frac{a^2}{4} + A_2A_3^2 - 2\cdot\frac{a}{2}\cdot A_2A_3\cdot\cos 30^\circ \\ \frac{3}{4}a^2 &= A_2A_3^2 - A_2A_3\cdot a\cos 30^\circ. \end{aligned}$$

From (1), $A_1A_2^2 = A_2A_3^2 - A_1A_2 \cdot A_2A_3$. Hence,

$$\frac{A_2A_3}{A_1A_2} = \frac{\sqrt{5}+1}{2}.$$

□

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Reference.

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Steiner-Lehmus Type Results Related to The Gergonne Point of a Triangle

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Abstract. A *Gergonne cevian* (abbreviated GC) is a line segment joining the vertex of a triangle with the point of tangency of the triangle's incircle with the opposite side. In this paper, we determine monotonicity results for various segments determined by the intersection of Gergonne cevians and angle bisectors (both internal and external) within a triangle. We first consider the problem, in response to a prior question, of comparing certain segment lengths determined by the intersection of a fixed GC with the external bisectors of the other two angles of a triangle. We then consider the analogous problem wherein one fixes an angle bisector and considers the segment lengths determined by its intersection with the GC's emanating from the other two vertices. Finally, we prove some results for a related question comparing segments determined by the intersection of the GC from $\angle B$ with an angle bisector from $\angle C$ within a triangle ABC to those determined by the intersection of a bisector from $\angle B$ with the GC from $\angle C$.

1. Introduction

Segments within an isosceles triangle that are symmetric with respect to the line of symmetry of the triangle are always congruent. Conversely, one might ask when congruence of some particular pair of symmetric segments within a triangle implies congruence of the corresponding sides. For example, it is readily shown that congruence of altitudes or medians to two sides of a triangle implies congruence of the sides. The comparable problem involving angle bisectors is more difficult and the fact that the same result holds for angle bisectors is known as the Steiner-Lehmus theorem. Many proofs have been given of this result and we refer the reader to [2, 3, 4, 7] and also to the references contained within [10, 11]. See [5] for stronger versions of the theorem and [1] for a version involving extensions of the angle bisector. Results such as these arise frequently as particular cases of more general monotonic behavior. For example, a median, altitude or angle bisector to a shorter side is always longer, and conversely. On the other hand, the congruence of a pair of segments that would be congruent within an isosceles triangle by virtue of their symmetry need not imply that the triangle within which they lie is isosceles. See, e.g., [6] for an example.

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Here, we consider variants of the Steiner-Lehmus result involving the Gergonne point. A *Gergonne cevian* (GC) joins a vertex of a triangle with the point of tangency of the opposite side with the triangle's incircle. The Gergonne cevians of a triangle are concurrent, and their point of concurrency is known as the *Gergonne point* of the triangle (see, e.g., [3, p. 13]). In [9], a Gergonne analogue of the Steiner-Lehmus theorem was proven. That is, if BD and CE are GC's within $\triangle ABC$, then $BD = CE$ implies $AB = AC$. It is also shown in [9] that if the internal angle bisectors of $\angle B$ and $\angle C$ within $\triangle ABC$ intersect the Gergonne cevian AD at E and F, respectively, then $BE = CF$ implies $AB = AC$. The comparable question involving external instead of internal angle bisectors is raised at the conclusion of [9], and here we provide an affirmative answer to this question by establishing a more general monotonicity criterion.

This paper is organized as follows. In the next section, after addressing the problem from [9] described above, we consider the analogous problem wherein the roles of the GC's and angle bisectors are reversed. That is, given an angle bisector, we compare the distances, when traveled along a GC, from the other two vertices to the bisector. We address this problem in the case of both an internal and an external bisector. In the third section, we consider a related question which extends work from [6] wherein we compare segment lengths determined by the intersection of the GC from $\angle B$ with the bisector (either internal or external) from $\angle C$ within $\triangle ABC$ to those determined by the intersection of the comparable bisector from $\angle B$ with the GC from $\angle C$. We prove a more general monotonicity property from which the specific Steiner-Lehmus type result follows in each case. Our arguments primarily make use of establishing certain trigonometric and/or algebraic inequalities.

We will use at times the following results. The first may be obtained by slightly generalizing the argument presented for [9, Theorem 2].

Lemma 1. *Let BD and CE be Gergonne cevians within $\triangle ABC$. If $AB < AC$, then $BD < CE$.*

The following formula (see, e.g., [8, p. 46]) for the sine of a half-angle in terms of the side lengths of a triangle will also be useful.

Lemma 2. *If s denotes the semiperimeter of a triangle ABC, then $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$.*

In the proofs below, we let $a = BC$, $b = AC$ and $c = AB$, with $s = \frac{a+b+c}{2}$.

2. An answer to a previous question and related result

In Figure 1 below, the point of tangency of the incircle of $\triangle ABC$ with side BC is denoted by D. The exterior angle bisectors of $\angle B$ and $\angle C$ meet AD (extended) at points E and F, respectively.

Our first result shows that $BE = CF$ in Figure 1 implies $AB = AC$, answering a question raised in [9].

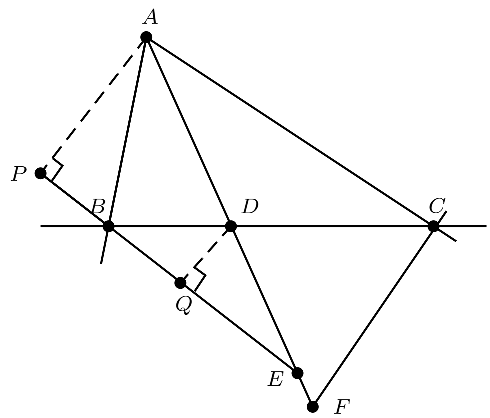

Figure 1. Intersection of Gergonne cevian with exterior angle bisectors of the other two angles.

Theorem 3. *If $AB < AC$ in Figure 1, then $BE < CF$. In particular, if $BE = CF$, then $AB = AC$.*

Proof. Let P and Q be the feet, respectively, of points A and D on the line $\overleftrightarrow{BE}$. Since $\angle ABP = \angle DBQ = \frac{\pi}{2} - \frac{\angle B}{2}$, we have $AP = c\cos\frac{B}{2}$, $DQ = (s-b)\cos\frac{B}{2}$, $BP = c\sin\frac{B}{2}$ and $PQ = BP + BQ = c\sin\frac{B}{2} + (s-b)\sin\frac{B}{2}$. Let $x = EQ$. Since $\triangle DQE \sim \triangle APE$, we have $\frac{QE}{PE} = \frac{DQ}{AP}$, which implies

$$\frac{x}{x + (s-b+c)\sin\frac{B}{2}} = \frac{s-b}{c}$$

and thus $x = \frac{(s-b+c)(s-b)\sin\frac{B}{2}}{b+c-s}$. We then have

$$BE = EQ + BQ = \frac{s-b+c}{b+c-s}(s-b)\sin\frac{B}{2} + (s-b)\sin\frac{B}{2} = \frac{2c(s-b)\sin\frac{B}{2}}{b+c-s}.$$

By symmetry, we have $CF = \frac{2b(s-c)\sin\frac{C}{2}}{c+b-s}$. Thus, $BE < CF$ if and only if $c(s-b)\sin\frac{B}{2} < b(s-c)\sin\frac{C}{2}$. By Lemma 2, we have $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$, and thus the preceding inequality is equivalent to $\sqrt{c(s-b)} < \sqrt{b(s-c)}$, i.e., $c(a-b+c) < b(a+b-c)$. The last inequality holds since $a(c-b) < 0 < b^2 - c^2$, which completes the proof. □

We now consider the analogous problem in which the roles of the angle bisectors and GC's are reversed. In Figure 2 below, BD and CE are GC's in $\triangle ABC$. The exterior and interior bisectors of $\angle A$ meet BD (extended) at the points F and J, respectively, and CE at G and I, respectively.

The first part of the following theorem is an analogue of [9, Theorem 3].

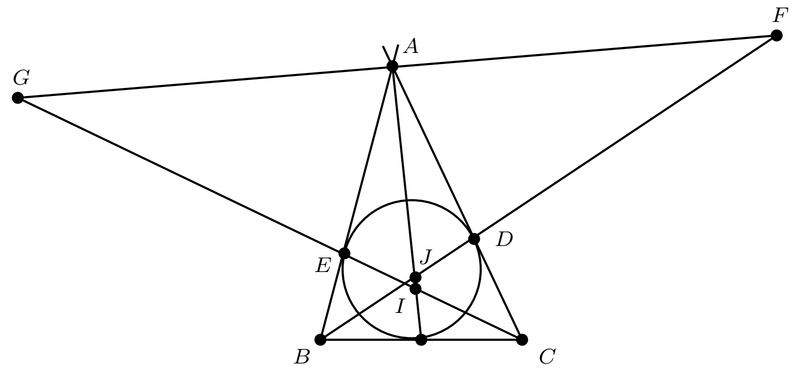

Figure 2. Intersection of interior and exterior angle bisector with the GC's to the two adjacent sides.

Theorem 4. *If $AB < AC$ in Figure 2, then $BJ < CI$ and if $BC \leq AB < AC$, then $CG < BF$.*

Proof. By the angle bisector theorem applied to $\triangle ABD$, we have

$$BJ = \frac{BJ}{BJ + JD} BD = \frac{1}{1 + \frac{AD}{AB}} BD = \frac{c}{c + s - a} BD,$$

and similarly $CI = \frac{b}{b+s-a} CE$. Since $BD < CE$, by Lemma 1, and since $c < b$ and $s > a$ implies $\frac{c}{c+s-a} < \frac{b}{b+s-a}$, it follows that $BJ < CI$.

To prove the second statement, we first find an expression for BF. By the law of cosines in $\triangle ABF$,

$$\begin{aligned} BF^2 &= AF^2 + AB^2 - 2AF \cdot AB \cos(\angle BAF) \\ &= AF^2 + c^2 - 2cAF \cos\left(\frac{A}{2} + \frac{\pi}{2}\right) \\ &= AF^2 + 2cAF \sin\frac{A}{2} + c^2. \end{aligned}$$

By the proof of Theorem 3 above, we have $AF = \frac{2c(s-a)\sin\frac{A}{2}}{s-b}$. Thus,

$$\begin{aligned} BF^2 &= \frac{4c^2(s-a)^2 \sin^2\frac{A}{2}}{(s-b)^2} + \frac{4c^2(s-a)\sin^2\frac{A}{2}}{s-b} + c^2 \\ &= \frac{4c^2(s-a)[(s-a)+(s-b)]\sin^2\frac{A}{2}}{(s-b)^2} + c^2 \\ &= \frac{4c^3(s-a)\sin^2\frac{A}{2}}{(s-b)^2} + c^2 = \frac{4c^2(s-a)(s-c)}{b(s-b)} + c^2, \end{aligned}$$

where we have applied Lemma 2 in the last equality.

Noting the comparable expression for CG^2, it follows that $BF > CG$ if and only if

$$\frac{4c^2(s-a)(s-c) + bc^2(s-b)}{b(s-b)} > \frac{4b^2(s-a)(s-b) + b^2c(s-c)}{c(s-c)},$$

which we rewrite as

$$4(s-a)(c^3(s-c)^2 - b^3(s-b)^2) > bc(b^2-c^2)(s-b)(s-c). \tag{1}$$

Dividing both sides of (1) by $b-c$ gives

$$4(s-a)f(b,c) > bc(b+c)(s-b)(s-c), \tag{2}$$

where

$$f(b,c) = -(b^2+bc+c^2)s^2 + 2s(b^2+c^2)(b+c) - (b^4+b^3c+b^2c^2+bc^3+c^4).$$

Simplifying the expression for $f(b,c)$ using $s = \frac{a+b+c}{2}$, inequality (2) is equivalent to

$$bc(b^2+bc+c^2)+a(b^2+c^2)(b+c)-\frac{1}{4}(b^2+bc+c^2)(a+b+c)^2 > \frac{bc(b+c)(s-b)(s-c)}{4(s-a)}. \tag{3}$$

Dividing both sides by b^2+bc+c^2, and noting $bc \le \frac{1}{2}(b^2+c^2)$, to show (3), it suffices to show

$$bc + \frac{2}{3}a(b+c) - \frac{1}{4}(a^2+2a(b+c)+(b+c)^2) > \frac{(b+c)(s-b)(s-c)}{12(s-a)}.$$

Let $b+c = 2\ell$ where $\ell > a$, with $b = \ell + y$ and $c = \ell - y$ for some $0 < y < \frac{a}{2}$. Rewriting the last inequality in terms of y and ℓ, we have

$$-\frac{a^2}{4} - y^2 > \frac{\ell\left(\frac{a^2}{4} - y^2\right)}{3(2\ell-a)} - \frac{a\ell}{3}. \tag{4}$$

To prove (4), we consider the cases $a < \ell < \frac{3a}{2}$ or $\ell \ge \frac{3a}{2}$. In the first case, since $c = \ell - y \ge a$, by assumption, it suffices to verify (4) when $y = \ell - a$. That is, we must show

$$-\frac{a^2}{4} + \left(\frac{\ell}{3(2\ell-a)} - 1\right)(\ell-a)^2 > \frac{a^2\ell}{12(2\ell-a)} - \frac{a\ell}{3}, \qquad a < \ell < \frac{3a}{2}. \tag{5}$$

Clearing fractions in (5), rearranging, and dividing by $\ell - a$, we see that (5) is equivalent to $4(3a-5\ell)(\ell-a) > a(3a-8\ell)$ for $a < \ell < \frac{3a}{2}$, which is readily verified. If $\ell \ge \frac{3a}{2}$, then since $0 < y < \frac{a}{2}$, to show (4) in this case, it suffices to show that it holds (possibly with equality) when $y = \frac{a}{2}$, which is seen to be the case for all such ℓ. This implies inequality (4), and thus (3), which completes the proof. $\square$

Remark: The second part of the prior theorem may not hold if b or c less than a is allowed. To show this, let $a = 1$ and $\epsilon := b - c$ where $0 < \epsilon < 1$. If ϵ is fixed, then both sides of (2) are seen to be functions of $s > 1$. Let $\epsilon = .04$. If $b = .54$ and $c = .50$, then $s = 1.02$ and a direct calculation shows that the right side of (2) exceeds the left. If $b = 1.54$ and $c = 1.5$, then $s = 2.02$ and the left side of (2) exceeds the right in this case. By continuity, there exists some $s' \in (1.02, 2.02)$ such that there is equality in (2). Then pick b and c such that $b - c = .04$ and $b + c = 2s' - 1$ which yields a triangle ABC having $BF = CG$ with $AB \ne AC$. Similar examples can be found for other small ϵ. On the other hand, if $AB, AC \geq BC$, then $BF = CG$ implies $AB = AC$, by the previous theorem.

3. Other monotonicity results

In this section, we establish some comparable monotonicity results in which we consider pairs of segment lengths determined by the intersection of angle bisectors with Gergonne cevians. We first consider the case of exterior angle bisectors. In Figure 3 below, BD and CE are GC's in $\triangle ABC$. The exterior bisectors of $\angle C$ and $\angle B$ intersect BD and CE (extended) at F and G, respectively.

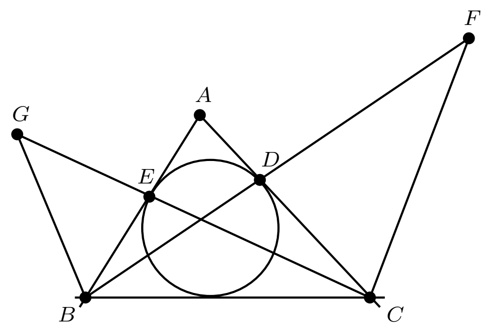

Figure 3. Intersection of GC's with the exterior angle bisectors of the respective angles.

Our next result compares the distances from F and G to the points B and C.

Theorem 5. *If $AB < AC$ in Figure 3, then (i) $BG < CF$ and (ii) $CG < BF$.*

Proof. To show (i), first note that $BG < CF$ if and only if $\frac{(s-b)\sin\frac{B}{2}}{s-c} < \frac{(s-c)\sin\frac{C}{2}}{s-b}$, by the formulas found in the proof of Theorem 3 above. The last inequality, together with the relations $s - b = r\cot\frac{B}{2}$ and $s - c = r\cot\frac{C}{2}$ where r is the inradius of $\triangle ABC$, implies $\cot^2\frac{B}{2}\sin\frac{B}{2} < \cot^2\frac{C}{2}\sin\frac{C}{2}$, i.e., $\cot\frac{B}{2}\cos\frac{B}{2} < \cot\frac{C}{2}\cos\frac{C}{2}$. That $BG < CF$ follows from $\angle B > \angle C$ and the fact that cosine and cotangent are decreasing on $(0, \frac{\pi}{2})$.

For (ii), note first note that by the law of cosines in $\triangle BCF$ and the formula for CF, we have that BF^2 equals

$$\begin{aligned}
&CF^2 + BC^2 - 2CF\cdot BC\cos\left(\frac{C}{2}+\frac{\pi}{2}\right)\\
&= \frac{4a^2(s-c)^2\sin^2\frac{C}{2}}{(s-b)^2} + \frac{4a^2(s-c)\sin^2\frac{C}{2}}{s-b} + a^2\\
&= \frac{4a^2(s-c)[(s-c)+(s-b)]\sin^2\frac{C}{2}}{(s-b)^2} + a^2\\
&= \frac{4a^3(s-c)\sin^2\frac{C}{2}}{(s-b)^2} + a^2.
\end{aligned}$$

Thus $CG < BF$ if and only if

$$\frac{(s-b)\sin^2\frac{B}{2}}{(s-c)^2} < \frac{(s-c)\sin^2\frac{C}{2}}{(s-b)^2},$$

which implies $\cot^3\frac{B}{2}\sin^2\frac{B}{2} < \cot^3\frac{C}{2}\sin^2\frac{C}{2}$. The last inequality is true since $\cot\frac{B}{2}\cos^2\frac{B}{2} < \cot\frac{C}{2}\cos^2\frac{C}{2}$ as $\pi > \angle B > \angle C > 0$, which completes the proof. $\square$

Before proving our next result, we will need the following trigonometric inequality.

Lemma 6. *If $\angle B > \angle C$ in $\triangle ABC$, then*

$$\frac{\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2} + \sin B + \sin(B+C)}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2} + \sin C + \sin(B+C)} < \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}. \tag{6}$$

Proof. To show the right-hand inequality in (6), note first that

$$\sin\frac{C}{2}(\sin B+\sin(B+C)) = 2\sin\frac{C}{2}\sin\left(B+\frac{C}{2}\right)\cos\frac{C}{2} = \sin C\sin\left(B+\frac{C}{2}\right)$$

and $\sin\frac{B}{2}(\sin C + \sin(B+C)) = \sin B\sin\left(\frac{B}{2}+C\right)$. Also, we have

$$\begin{aligned}\sin\frac{C}{2}\tan\frac{C}{2} - \sin\frac{B}{2}\tan\frac{B}{2} &= \frac{1-\cos^2\frac{C}{2}}{\cos\frac{C}{2}} - \frac{1-\cos^2\frac{B}{2}}{\cos\frac{B}{2}} \\ &= \frac{\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(1+\cos\frac{B}{2}\cos\frac{C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}.\end{aligned}$$

Subtracting, to prove the right inequality in (6), we must show

$$\begin{aligned}&\sin C\sin\left(B+\frac{C}{2}\right) - \sin B\sin\left(\frac{B}{2}+C\right) \\ &\quad + \frac{2\left(\cos\frac{B}{2}-\cos\frac{C}{2}\right)\left(1+\cos\frac{B}{2}\cos\frac{C}{2}\right)\sin^2\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}} < 0.\end{aligned} \tag{7}$$

Since $\angle B > \angle C$ and cosine is decreasing, the third term on the left-hand side of (7) is negative. Thus, to show (7), it is enough to show $\sin C\sin\left(B+\frac{C}{2}\right) < \sin B\sin\left(\frac{B}{2}+C\right)$.

Applying the product-to-difference cosine formula to both sides of this last inequality, and rearranging, gives

$$\cos\left(\frac{C}{2}-B\right) - \cos\left(\frac{B}{2}-C\right) < \cos\left(\frac{3C}{2}+B\right) - \cos\left(\frac{3B}{2}+C\right).$$

By the difference-to-product sine formula, this preceding inequality may be rewritten as

$$-\sin\left(\frac{3(B-C)}{4}\right)\sin\left(\frac{B+C}{4}\right) < \sin\left(\frac{5(B+C)}{4}\right)\sin\left(\frac{B-C}{4}\right). \tag{8}$$

To show (8), first note that since $\angle B > \angle C$, the left side of (8) is negative whereas the right side is non-negative if $B+C \leq \frac{4\pi}{5}$. On the other hand, if $B+C > \frac{4\pi}{5}$, then $-\sin\left(\frac{5(B+C)}{4}\right) = \sin\left(\frac{5(B+C)}{4} - \pi\right) < \sin\left(\frac{B+C}{4}\right)$ since $\angle B + \angle C < \pi$ and sine is increasing in the first quadrant. Furthermore, $\sin\left(\frac{B-C}{4}\right) < \sin\left(\frac{3(B-C)}{4}\right)$ since $0 < \frac{B-C}{4} < \frac{3(B-C)}{4} < \pi - \left(\frac{B-C}{4}\right)$, whence (8) follows in this case by multiplication. This implies inequality (7), which completes the proof of the right inequality in (6).

To prove the left inequality in (6), first note that if $\frac{a}{b} > \frac{c}{d}$ with $\frac{c}{d} < 1$, then $\frac{a+x}{b+x} > \frac{c}{d}$ if all variables represent positive quantities. Thus it suffices to show

$$\frac{\cos^2 \frac{B}{2}}{\cos^2 \frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2} + \sin B}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2} + \sin C}. \tag{9}$$

By the sine and cosine double-angle formulas, inequality (9) may be rewritten as

$$\begin{aligned}(\sin B - \sin C)\sin^2\left(\frac{B+C}{2}\right) &< \sin B\cos^2\frac{C}{2} - \sin C\cos^2\frac{B}{2} \\ &= \sin B\left(\frac{1+\cos C}{2}\right) - \sin C\left(\frac{1+\cos B}{2}\right) \\ &= \frac{1}{2}(\sin B - \sin C) + \frac{1}{2}\sin(B-C).\end{aligned}$$

Since $\cos 2x = 1 - 2\sin^2 x$, the last inequality is equivalent to

$$-(\sin B - \sin C)\cos(B+C) < \sin(B-C),$$

i.e.,

$$-2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)\cos(B+C) < 2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B-C}{2}\right). \tag{10}$$

Since $\angle B > \angle C$, inequality (10) is seen to hold as $0 < \cos\left(\frac{B+C}{2}\right) < \cos\left(\frac{B-C}{2}\right)$ with $-\cos(B+C) < 1$, which completes the proof. □

We will also need the following well-known collinearity result known as the Theorem of Menelaus (see, e.g., [3, p. 66]).

Lemma 7. *If ABC is a triangle and D is on an extension of AB, E is on side BC, and F is on side AC, then the three points D, E, and F are collinear if and only if*

$$\left(\frac{AD}{DB}\right)\left(\frac{BE}{EC}\right)\left(\frac{CF}{FA}\right) = -1. \tag{11}$$

Here, the measure of a segment in one direction is considered to be the opposite of its measure in the other. In practice, when using the "only if" direction of this result, one often takes the absolute value of both sides of (11) and no longer considers segments as signed.

We now consider the comparable version of the prior problem wherein exterior are replaced by interior angle bisectors. In Figure 4 below, the interior angle bisectors BH and CJ intersect Gergonne cevians CE and BD at Q and P, respectively.

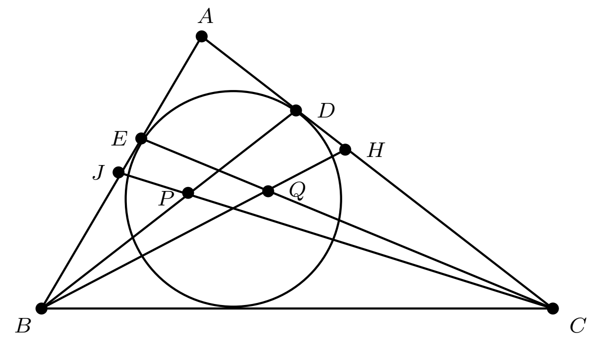

Figure 4. Intersection of GC's with the interior angle bisectors of the respective angles.

In our next result, we compare the internal segments of BH determined by Q with those of CJ determined by P.

Theorem 8. *If $AB < AC$ in Figure 4, then $BQ < CP$. If $\angle A \geq \frac{\pi}{3}$ in $\triangle ABC$, then $AB < AC$ implies $JP < HQ$.*

Proof. By Lemma 7 applied to $\triangle AJC$ with transversal $\overleftrightarrow{BPD}$, we have $\frac{AB}{BJ} \cdot \frac{JP}{PC} \cdot \frac{CD}{DA} = 1$. Note that CJ bisecting $\angle C$ implies $\frac{AB}{BJ} = \frac{AJ+BJ}{BJ} = \frac{b}{a} + 1$. Also, $CD = s - c = r \cot \frac{C}{2}$ and $DA = s - a = r \cot \frac{A}{2} = r \tan\left(\frac{B+C}{2}\right)$. It follows that

$$\frac{JP}{PC} = \frac{BJ}{AB} \cdot \frac{DA}{CD} = \frac{\tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right)}{\frac{b}{a} + 1} = \frac{\tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B+C)}{\sin B + \sin(B+C)},$$

where we have used the fact $\frac{b}{a} = \frac{\sin B}{\sin(B+C)}$ in the last equality. By a formula for the length of the angle bisector, we have $CJ = \frac{2ab \cos \frac{C}{2}}{a+b} = \frac{2a \sin B \cos \frac{C}{2}}{\sin B + \sin(B+C)}$ and thus

$$\begin{aligned} CP &= \left(\frac{CP}{CP+JP}\right) CJ \\ &= \frac{\sin B + \sin(B+C)}{\sin B + \sin(B+C) + \tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B+C)} \cdot \frac{2a \sin B \cos \frac{C}{2}}{\sin B + \sin(B+C)} \\ &= \frac{a \sin B \sin C}{\sin \frac{C}{2} \left(\sin B + \sin(B+C) + \tan \frac{C}{2} \tan\left(\frac{B+C}{2}\right) \sin(B+C)\right)}, \end{aligned}$$

where we have used the identity $\sin 2x = 2 \sin x \cos x$ in the last equality. Noting the comparable expression for BQ, the inequality $BQ < CP$ follows from the right inequality in Lemma 6.

For the second statement, first note that by the formula for the ratio $\frac{JP}{PC}$ found above, we have

$$\begin{aligned} JP &= \left(\frac{JP}{CP+JP}\right) CJ \\ &= \left(\frac{\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)}{\sin B+\sin(B+C)+\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)}\right)\left(\frac{2a\sin B\cos\frac{C}{2}}{\sin B+\sin(B+C)}\right) \\ &= \frac{a\sin B\sin C\tan\left(\frac{B+C}{2}\right)\sin(B+C)}{\cos\frac{C}{2}(\sin B+\sin(B+C))\left(\sin B+\sin(B+C)+\tan\frac{C}{2}\tan\left(\frac{B+C}{2}\right)\sin(B+C)\right)}, \end{aligned}$$

where we have used $\sin C = 2\tan\frac{C}{2}\cos^2\frac{C}{2}$. Noting the comparable expression for HQ, it follows that $JP < HQ$ if and only if

$$\begin{aligned} &\frac{\cos\frac{B}{2}}{(\sin B+\sin(B+C))\left(\sin B+\sin(B+C)+2\tan\frac{C}{2}\sin^2\left(\frac{B+C}{2}\right)\right)} \\ &< \frac{\cos\frac{C}{2}}{(\sin C+\sin(B+C))\left(\sin C+\sin(B+C)+2\tan\frac{B}{2}\sin^2\left(\frac{B+C}{2}\right)\right)}. \end{aligned} \tag{12}$$

Since $\sin x + \sin(x+y) = 2\sin\left(x+\frac{y}{2}\right)\cos\frac{y}{2}$, inequality (12) may be rewritten as

$$\frac{\sin\left(\frac{B}{2}+C\right)\cos^2\frac{B}{2}}{\sin\left(B+\frac{C}{2}\right)\cos^2\frac{C}{2}} < \frac{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{C}{2}+\sin B+\sin(B+C)}{2\sin^2\left(\frac{B+C}{2}\right)\tan\frac{B}{2}+\sin C+\sin(B+C)}. \tag{13}$$

Since $\angle B > \angle C$ and $\angle A \geq \frac{\pi}{3}$, we have $0 < \frac{B}{2}+C < B+\frac{C}{2} \leq \pi - \left(\frac{B}{2}+C\right)$ and therefore $0 < \sin\left(\frac{B}{2}+C\right) \leq \sin\left(B+\frac{C}{2}\right)$. Thus inequality (13) follows from the left inequality in Lemma 6, which completes the proof. □

Remark: The second part of the prior theorem may not hold if $\angle B + \angle C > \frac{2\pi}{3}$. For example, if $\angle B = \frac{\pi}{2}$ and $\angle C = \frac{22\pi}{45}$, then the left-hand side of (13) is seen to be strictly larger than the right. If $\angle B = \frac{\pi}{2}$ and $\angle C \leq \frac{\pi}{6}$, then we know from the previous theorem that the right side of (13) is larger. Thus, by continuity, there exists a triangle ABC in which $JP = HQ$ with $AB \neq AC$. On the other hand, if $\angle A \geq \frac{\pi}{3}$, then $JP = HQ$ implies $AB = AC$.

Our final two results compare the lengths of the internal segments within the GC's that are determined by the points P and Q in Figure 4 above. At this point, we will assume without loss of generality that $a = BC = 1$ and thus $s = \frac{b+c+1}{2}$.

Theorem 9. *If $AB < AC$ in Figure 4, then $BP < CQ$.*

Proof. We first find an expression for BP. By the angle bisector theorem, we have

$$BP = \frac{BP}{BP+DP}BD = \frac{BC}{BC+DC}BD = \frac{1}{1+s-c}BD,$$

and thus by the law of cosines in $\triangle BCD$ and $\triangle ABC$,

$$\begin{aligned} BP^2 &= \frac{1}{(1+s-c)^2}((s-c)^2+1-2(s-c)\cos C) \\ &= \frac{1}{b(1+s-c)^2}(b(s-c)^2+b-(s-c)(1+b^2-c^2)). \end{aligned}$$

By the comparable expression for CQ^2, it follows that $BP < CQ$ if and only if

$$\begin{aligned} &c(1+s-b)^2(b(s-c)^2+b-(s-c)(1+b^2-c^2)) \\ &< b(1+s-c)^2(c(s-b)^2+c-(s-b)(1+c^2-b^2)). \end{aligned} \tag{14}$$

To show (14), we first rewrite it as

$$\begin{aligned} &bc((s-c)^2-(s-b)^2+2(b-c)(s-b)(s-c)) \\ &+ bc(2(c-b)+(s-b)^2-(s-c)^2) \\ &+ b(1+s-c)^2(s-b)-c(1+s-b)^2(s-c) \\ &+ (c^2-b^2)(c(1+s-b)^2(s-c)+b(1+s-c)^2(s-b)) < 0. \end{aligned} \tag{15}$$

Note that $(s-c)^2-(s-b)^2 = (2s-b-c)(b-c) = b-c$ and

$b(1+s-c)^2(s-b)-c(1+s-b)^2(s-c) = (c-b)(b+c-s)+(b-c)(s-b)(s-c)(s+2)$.

Dividing both sides of (15) by $b-c>0$, we then have

$$\begin{aligned} &bc(1+2(s-b)(s-c))-3bc+s-b-c+(s+2)(s-b)(s-c) \\ &- (b+c)(c(1+s-b)^2(s-c)+b(1+s-c)^2(s-b)) \\ &= 2bc((s-b)(s-c)-1)+1-s+(s+2)(s-b)(s-c) \\ &- (b+c)(c(1+s-b)^2(s-c)+b(1+s-c)^2(s-b)) < 0. \end{aligned} \tag{16}$$

To show (16), since $s-b, s-c<1$ and $s>1$, it is enough to show

$$(s+2)(s-b)(s-c)-(b+c)(c(1+s-b)^2(s-c)+b(1+s-c)^2(s-b)) < 0. \tag{17}$$

Observe that

$$\begin{aligned} &c(1+s-b)^2(s-c)+b(1+s-c)^2(s-b) \\ &> c(s-c)+b(s-b)+2(b+c)(s-b)(s-c) \\ &> c(s-c)(s-b)+b(s-b)(s-c)+2(b+c)(s-b)(s-c) \\ &= 3(b+c)(s-b)(s-c). \end{aligned}$$

For (17), it then suffices to show

$$(s+2)(s-b)(s-c)-3(b+c)^2(s-b)(s-c) < 0,$$

i.e.,

$$(s+2-3(2s-1)^2)(s-b)(s-c) = (s-1)(1-12s)(s-b)(s-c) < 0,$$

which is true since $s>1$. This establishes inequality (14) and completes the proof. $\square$

A similar result applies when comparing the segments DP and EQ in the case of an acute triangle.

Theorem 10. *If $\triangle ABC$ is acute and $AB < AC$ in Figure 4, then $EQ < DP$.*

Proof. First note that $DP^2 > EQ^2$ if and only if

$$\begin{aligned}&\frac{(s-c)^2}{b(1+s-c)^2}(b(s-c)^2+b-(s-c)(1+b^2-c^2))\\ >&\frac{(s-b)^2}{c(1+s-b)^2}(c(s-b)^2+c-(s-b)(1+c^2-b^2)),\end{aligned}$$

which may be rewritten as

$$\begin{aligned}&bc[(s-c)^4-(s-b)^4+2(s-b)(s-c)((s-c)^3-(s-b)^3)\\ &+(s-b)^2(s-c)^2((s-c)^2-(s-b)^2)]\\ &+bc[(s-c)^2-(s-b)^2+2(s-b)(s-c)((s-c)-(s-b))]\\ &+b(1+s-c)^2(s-b)^3-c(1+s-b)^2(s-c)^3\\ &+(c^2-b^2)(c(1+s-b)^2(s-c)^3+b(1+s-c)^2(s-b)^3)\\ &>0. \qquad (18)\end{aligned}$$

One may verify

$$b(s-b)^3-c(s-c)^3=s^3(b-c)+3s^2(c^2-b^2)+3s(b^3-c^3)+c^4-b^4,$$

$$b(s-b)^3(s-c)-c(s-b)(s-c)^3=(s-b)(s-c)(s^2(b-c)+2s(c^2-b^2)+b^3-c^3),$$

and

$$b(s-b)^3(s-c)^2-c(s-b)^2(s-c)^3=(s-b)^2(s-c)^2(s(b-c)+c^2-b^2).$$

Dividing both sides of (18) by $b-c$, and noting $b+c=2s-1$, we then have

$$\begin{aligned}&bc(2+2(s-b)(s-c)-(s-b)^2(s-c)^2)+s^3-3s^2(2s-1)+3s((2s-1)^2-bc)\\ &-(2s-1)((2s-1)^2-bc)+2(s-b)(s-c)(s^2-2s(2s-1)+(2s-1)^2-bc)\\ &+(s-b)^2(s-c)^2(1-s)-(b+c)(c(1+s-b)^2(s-c)^3+b(1-s-c)^2(s-b)^3)\\ &>0,\end{aligned}$$

i.e.,

$$\begin{aligned}&bc(s-(s-b)^2(s-c)^2)-s^3+(3+2(s-b)(s-c))s^2-(3+4(s-b)(s-c)\\ &+(s-b)^2(s-c)^2)s+(1+(s-b)(s-c))2-(b+c)(c(1+s-b)^2(s-c)^3\\ &+b(1+s-c)^2(s-b)^3)>0. \qquad (19)\end{aligned}$$

Let $b=\frac{2s-1}{2}+x$ and $c=\frac{2s-1}{2}-x$ for some $0<x<\frac{1}{2}$. Then $bc=s^2-s+\frac{1}{4}-x^2$, with $s-b=\frac{1}{2}-x$ and $s-c=\frac{1}{2}+x$. Substituting this into (19), and simplifying, gives

$$
\begin{aligned}
&(2+2\gamma-\gamma^2)s^2-3(1+\gamma)s-3\gamma^3+(1+\gamma)^2\\
&-(2s-1)\left[\left(s-\frac{1}{2}-x\right)\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3+\left(s-\frac{1}{2}+x\right)\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3\right]\\
&>0,
\end{aligned}
$$

where $\gamma=\frac{1}{4}-x^2$.

Since $\triangle ABC$ is acute, we have $b^2+c^2>1$ and $c^2+1>b^2$. Translating these conditions in terms of x and s, to complete the proof, we must show $f(x,s)>0$, for x and s satisfying

$$
\frac{1}{2}+\sqrt{\frac{1}{2}-x^2}<s<\frac{1}{2}+\frac{1}{4x},\quad 0<x<\frac{1}{2},
$$

where

$$
\begin{aligned}
&f(x,s)\\
&=\left(\frac{39}{16}-\frac{3}{2}x^2-x^4\right)s^2-\left(\frac{15}{4}-3x^2\right)s-\left(\frac{1}{4}-x^2\right)^3+\left(\frac{5}{4}-x^2\right)^2\\
&-(2s-1)\left[\left(s-\frac{1}{2}-x\right)\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3+\left(s-\frac{1}{2}+x\right)\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3\right].
\end{aligned}
$$

In order to do so, we seek to minimize $f(x,s)$ over its domain C. We consider restrictions on C as follows. Given $0<\delta<\frac{1}{2}$ (δ to be determined), let $D=D_\delta$ denote the closure of the set of points (x,s) in C such that $\delta<x<\frac{1}{2}$. By *Maple*, there exist no points (x,s) in $\mathbb{R}^2$ such that $\frac{df}{dx}(x,s)=\frac{df}{ds}(x,s)=0$ with $0<x<\frac{1}{2}$. This implies that the minimum value of f on the compact set D must occur on its boundary. Let $\alpha(x)=\frac{1}{2}+\frac{1}{4x}$ and $\beta(x)=\frac{1}{2}+\sqrt{\frac{1}{2}-x^2}$. First observe that

$$
f(x,\alpha(x))=\frac{(2x+1)(2x-1)^3(16x^4+32x^3+32x^2-21)}{256x^2}.
$$

Since the factor $16x^4+32x^3+32x^2-21$ is negative for $0<x<\frac{1}{2}$, it is seen that $f(x,\alpha(x))>0$ for $\delta<x<\frac{1}{2}$. By *Maple*, the derivative of the function $g(x)=f(x,\beta(x))$ has no zeros on the interval $(0,\frac{1}{2})$ with $g(0)\approx .009$ and $g\left(\frac{1}{2}\right)=0$, which implies $g(x)>0$ on the interval.

We now consider the values of f on the third piece of the boundary of D, namely, along the vertical line $x=\delta$. Note first that

$$
f(0,s)=\frac{21}{16}s^2-\frac{21}{8}s+\frac{81}{64}>0,\qquad s\geq\frac{1+\sqrt{2}}{2}.
$$

If $h(s)=f(x,s)$ for a given x, then the position of the vertex of the graph of the parabolic curve $h(s)$ is seen to be a continuous function of the parameter x. Note that the coefficient of s^2 in $f(x,s)$ is given by

$$\frac{39}{16}-\frac{3}{2}x^2-x^4-2\left(\frac{3}{2}-x\right)^2\left(\frac{1}{2}+x\right)^3-2\left(\frac{3}{2}+x\right)^2\left(\frac{1}{2}-x\right)^3,$$

which can be shown to be positive for $0<x<\frac{1}{2}$, and thus the vertex of $h(s)$ always corresponds to a minimum. By continuity, we may then select $\delta^*>0$ such that for all $0<\delta<\delta^*$, we have $f(\delta,s)>0$ for all s such that $\beta(\delta)\leq s\leq\alpha(\delta)$. If $0<\delta<\delta^*$, it follows from the preceding that $f(x,s)\geq 0$ on D_δ, with equality occurring only at the point $(x,s)=\left(\frac{1}{2},1\right)$. In particular, we have $f(x,s)>0$ for all interior points of D_δ. Upon allowing δ to approach zero, it follows that $f(x,s)$ is positive for all points in C, which completes the proof. □

Remark: The preceding result need not hold if $\triangle ABC$ is obtuse. For example, $f(x,s)<0$, i.e., $DP<EQ$, if $x=.1$ and $s=1.1$, which corresponds to $\triangle ABC$ having side lengths $AB=.5$, $AC=.7$ and $BC=1$. Note that if $\triangle ABC$ is acute, then Theorem 10 implies 9, though not conversely, by Lemma 1.

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Putting the Icosahedron into the Octahedron

Paris Pamfilos

Abstract. We compute the dimensions of a regular tetragonal pyramid, which allows a cut by a plane along a regular pentagon. In addition, we relate this construction to a simple construction of the icosahedron and make a conjecture on the impossibility to generalize such sections of regular pyramids.

1. Pentagonal sections

The present discussion could be entitled *Organizing calculations with Menelaos*, and originates from a problem from the book of Sharygin [2, p. 147], in which it is required to construct a pentagonal section of a regular pyramid with quadrangular

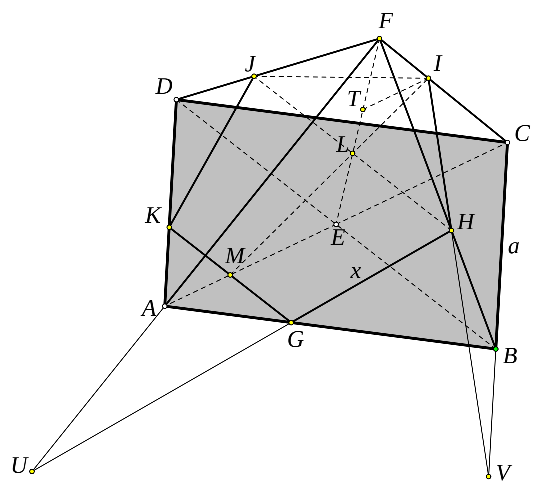

Figure 1. Pentagonal section of quadrangular pyramid

basis. The basis of the pyramid is a square of side-length a and the pyramid is assumed to be regular, i.e. its apex F is located on the orthogonal at the center E of the square at a distance $h = |EF|$ from it (See Figure 1). The exercise asks for the determination of the height h if we know that there is a section $GHIJK$ of the pyramid by a plane which is a regular pentagon. The section is tacitly assumed to be symmetric with respect to the diagonal symmetry plane AFC of the pyramid and defined by a plane through the three points K, G, I. The first two, K and G, lying symmetric with respect to the diagonal AC of the square. The third point I, lying on the edge CF. The, otherwise, excellent book, uses in this case some calculations, that I could not put in a systematic schedule somehow. So, I decided

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to do them in my way, trying to deduce them from some organizing principle. Here are the results of this effort, which, among other things, show that the exercise has some interesting extensions, but also prove that the corresponding calculations, made in the book, were not correct.

The organizing principle for doing the calculations is the theorem of Menelaos ([1, p.153]). We apply this theorem two times. The first on the triangle FBC, using the secant line VHI. This leads to the determination of the side x of the pentagon in terms of the side a of the square basis. Then we apply the theorem on the triangle ABF, using the secant UGH. This leads to the determination of the edge's length, $d = |AF|$, in terms of a and x and completes the proof of the following theorem.

Theorem 1. *A regular pyramid on a square basis of side a can be intersected by a plane along a regular pentagon lying symmetric with respect to one of its planes of symmetry if and only if it has equilateral faces i.e. the pyramid is a half regular octahedron.*

Before to delve into the details, let us fix the notation used in the figure shown. Points M and L are respectively the middles of KG and JH. Point L is located on the symmetry axis EF of the pyramid. Point T is the orthogonal projection of I on this axis. Finally, points U and V are respectively the intersections of line-pairs (AF, GH) and (BC, HI).

2. Menelaos once

The first application of the theorem of Menelaos to FBC with secant VHI leads to the equation

$$\frac{VB}{VC} \cdot \frac{IC}{IF} \cdot \frac{HF}{HB} = 1. \tag{1}$$

The three quotients entering the equation can be calculated easily:

$$\frac{VB}{VC} = \frac{ME}{MC} = \frac{ME}{ME + EC} = \frac{1}{1 + \frac{EC}{ME}} = \frac{1}{1 + \frac{a/\sqrt{2}}{a/\sqrt{2} - x/2}} = \frac{\sqrt{2}a - x}{2\sqrt{2}a - x}, \tag{2}$$

$$\frac{IC}{IF} = \frac{TE}{TF} = \frac{FE - FT}{TF} = \frac{FE}{TF} + 1 = 1 - \frac{EC}{TI} = 1 - \frac{EC}{ME}\frac{ME}{TI}. \tag{3}$$

In the last expression we can replace

$$\frac{EC}{ME} = \frac{a/\sqrt{2}}{a/\sqrt{2} - x/2}, \quad \frac{ME}{TI} = \frac{ML}{LI} = \phi, \tag{4}$$

where $\phi = \frac{\sqrt{5}+1}{2}$ the golden section. Introducing this into equation (3) we obtain

$$\frac{IC}{IF} = \frac{\sqrt{2}a(1 - \phi) - x}{\sqrt{2}a - x}. \tag{5}$$

Finally the third quotient entering (1) evaluates to

$$\frac{HF}{HB} = \frac{HF}{FB - FH} = \frac{1}{\frac{FB}{HF} + 1} = \frac{1}{1 - \frac{\sqrt{2}a}{\phi x}} = \frac{\phi x}{\phi x - \sqrt{2}a}. \tag{6}$$

Using equations (2), (5) and (6), equation (1) transforms to

$$\frac{\sqrt{2}a - x}{2\sqrt{2}a - x} \cdot \frac{\sqrt{2}a(1-\phi) - x}{\sqrt{2}a - x} \cdot \frac{\phi x}{\phi x - \sqrt{2}a} = 1, \tag{7}$$

which, by taking into account the equation satisfied by $\phi: \quad \phi^2 - \phi - 1 = 0$, and simplifying reduces to

$$x = \frac{\sqrt{2}a}{\phi + 1}. \tag{8}$$

3. Menelaos twice

Before to start with the second application of Menelaos theorem, let us compute the side $y = |UA|$ of the triangle UAG lying on the plane of the face ABF. The side UG has the length ϕx of the diagonal JH of the regular pentagon. Also $|AG| = x/\sqrt{2}$ and the $\cos(\omega)$, where $\omega = \widehat{UAG}$, is

$$\cos(\omega) = -\cos(\pi - \omega) = -\frac{a}{2d}.$$

Thus, the cosine theorem applied to side UG of triangle UAG leads to equation

$$\begin{aligned}(\phi x)^2 &= y^2 + \frac{x^2}{2} + 2y\frac{x}{\sqrt{2}}\frac{a}{2d} \\ \Leftrightarrow y^2 &+ \frac{ax}{\sqrt{2}d}y + (x^2/2 - (\phi x)^2) = 0. \end{aligned} \tag{9}$$

Now, coming to the Menelaos equation for triangle ABF and the secant UGH we have:

$$\frac{UA}{UF} \cdot \frac{HF}{HB} \cdot \frac{GB}{GA} = 1. \tag{10}$$

Using equation (6) for the middle factor and computing the last one we get for their product

$$\frac{GB}{GA} = \frac{AB - AG}{GA} = \frac{AB}{GA} + 1 = 1 - \frac{a}{(x/\sqrt{2})} = \frac{x - \sqrt{2}a}{x}, \tag{11}$$

$$\frac{HF}{HB} \cdot \frac{GB}{GA} = \frac{\phi x}{\phi x - \sqrt{2}a} \cdot \frac{x - \sqrt{2}a}{x} = \frac{\phi(x - \sqrt{2}a)}{\phi x - \sqrt{2}a}. \tag{12}$$

On the other side, the first factor is

$$\frac{UA}{UF} = \frac{UA}{UA + AF} = \frac{1}{1 + \frac{AF}{UA}} = \frac{1}{1 + \frac{d}{y}}. \tag{13}$$

Combining equations (10), (12), (13) and using (8) for x we come at

$$1 + \frac{d}{y} = \frac{\phi(x - \sqrt{2}a)}{\phi x - \sqrt{2}a} = \frac{\phi(\frac{\sqrt{2}a}{\phi+1} - \sqrt{2}a)}{\phi\frac{\sqrt{2}a}{\phi+1} - \sqrt{2}a} = \phi^2 = \phi + 1,$$

which implies

$$d = y \cdot \phi. \tag{14}$$

This, taking into account (8) and (9), implies, after a short calculation

$$d^2 = a^2 \qquad \Leftrightarrow \qquad d = a. \tag{15}$$

This implies that the tetragonal pyramid has equilateral faces, i.e. it is a half-octahedron and its height is

$$h = \sqrt{d^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}. \tag{16}$$

This completes the proof of the first half of the assertion of the theorem, the other half being obvious, since the steps of the proof can be reversed, if we assume that the pyramid is a half regular octahedron.

4. Relation to Icosahedron

Using (8) in formula (5) and calculating similarly the other ratios we see that the golden ratio appears in all side divisions:

$$\frac{CI}{IF} = \frac{BG}{GA} = \frac{FJ}{JD} = \phi. \tag{17}$$

Drawing also the line orthogonal to the plane of the pentagon at its center O and taking its intersection N with the pyramid's edge AF, we see easily that $\frac{AN}{NF} = \phi$ and that the pentagonal pyramid $NGHIJK$ is the pentagonal gasket of the pla-

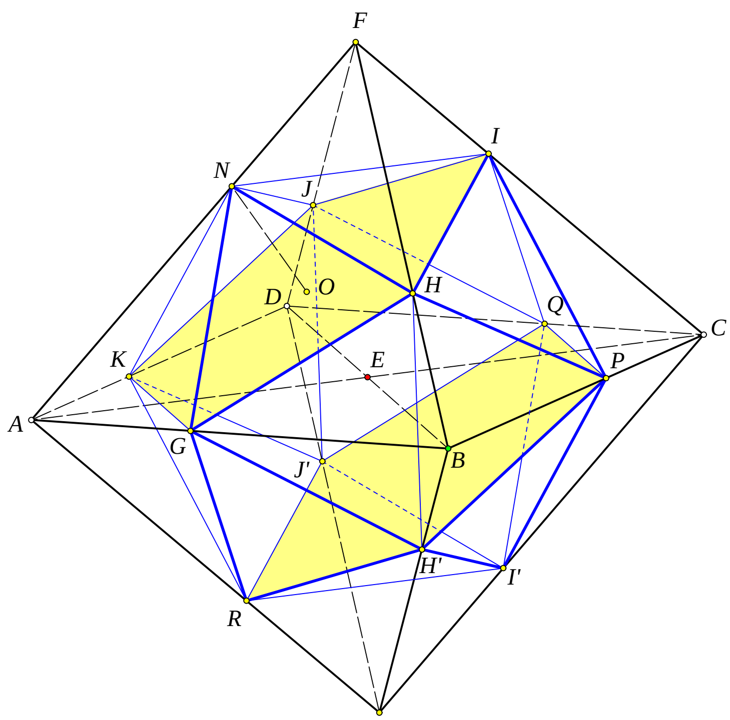

Figure 2. Icosahedron inscribed in the octahedron

tonic regular icosahedron (See Figure 2). The symmetric pentagon $KGH'I'J'$ with respect to the plane of the basic square produces analogously another pentagonal gasket, and the two gaskets can be easily completed to an icosahedron. This can be done by taking the symmetrics of these two pentagons with respect to the symmetry plane BFD of the pyramid. The vertices of the four, thus defined, pentagons determine the vertices of an icosahedron. This shows that the initial exercise relates to the inclusion of the icosahedron into the octahedron, defined by the square $ABCD$. The described procedure gives also a very simple method to construct the icosahedron. It suffices to construct the octahedron, which is simple, and divide its edges in the golden ratio using a certain rule suggested by the figure. The twelve resulting division points are the vertices of an icosahedron. Eight out

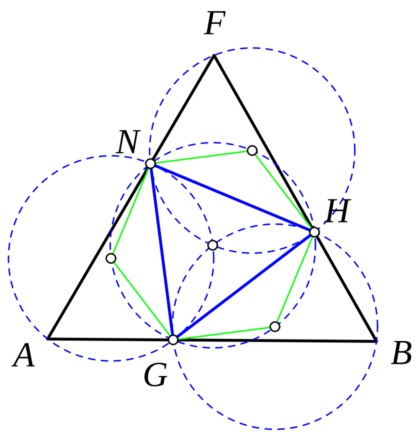

Figure 3. Face in face

of the twenty faces of the icosahedron are inscribed in corresponding faces of the octahedron (See Figure 3), the other twelve sitting entirely inside the octahedron.

5. Generalization

The original problem could be generalized to arbitrary regular pyramids. For triangular basis the answer is easy to supply. The only regular triangular pyramid admiting a square section is the regular tetrahedron, the section being the square of the middles of the segments joining the endpoints of two opposite lying edges (See Figure 4). This, assuming that one side HE of the square is parallel to the edge BC, we have that also the opposite side of the square FG is parallel to BC and since $\frac{AE}{AB} = \frac{HE}{BC} = \frac{FG}{BC} = \frac{DF}{DB}$, we have that EF and AD must be parallel. Since analogously $\frac{BE}{BA} = \frac{AE}{AB}$ we deduce that E and analogously F are the middles of the respective edges AB and BD. Thus, $x = |AE| = |EH| = |EF| = |EB| = |BF|$. This implies that the faces of the pyramid are equilateral triangles and shows the claim.

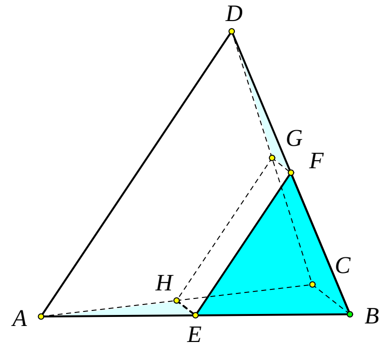

Figure 4. Square section of regular tetrahedron

At this point one is tempted to seek regular $(n+1)-$ sided polygon sections for regular pyramids with basis having $n > 4$ sides. The method used in the preceding paragraphs can be transferred in this more general question and used in order to determine the special (isosceles) triangles, which play the role of the faces of a pyramid allowing a regular polygonal section with $n+1$ sides. Next section, however shows that there are not hexagonal regular sections in a regular pentagonal pyramid and suggests the conjecture that there are no similarly constructible sections for $n > 4$.

6. Impossibility of hexagonal regular sections

In this section, working a bit more general than the case requires, we assume that the basis of the regular pyramid is a regular (odd) n-sided polygon and ask for the particular one which allows a plane-cut along an $(n+1)$-sided regular polygon. In the following discussion we think and calculate for general (odd) n, though the figure we refer to corresponds to $n = 5$ and ultimately it is for this figure that we

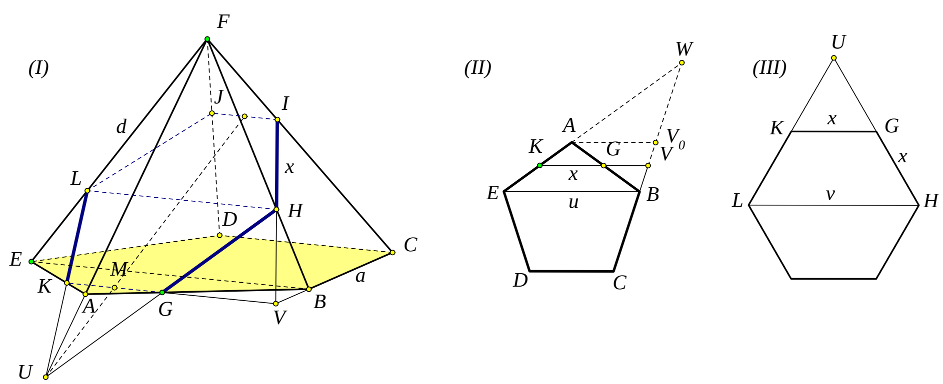

Figure 5. Pyramids with odd sided regular polygon basis

apply the results of the calculation. Our aim is to prove the following theorem (See Figure 5).

Theorem 2. *There is no regular pentagonal pyramid $FABCDE$ admitting a section symmetric with respect to one symmetry plane of the pyramid, which, in addition is a regular hexagon.*

For the proof we apply Menelaos theorem on the face BFC with secant VHI leading to equation:

$$\frac{VC}{VB} \cdot \frac{IF}{IC} \cdot \frac{HB}{HF} = 1.$$

The first ratio entering the equation is again easily computed

$$\frac{VC}{VB} = \frac{VB + BC}{VB} = 1 + \frac{BC}{VB}. \tag{18}$$

By assumption the basis is a regular n-sided polygon, and, as suggested by the figure 5-(II), the length of VB can be computed in terms of the side $x = |KG|$ of the $(n+1)-$gonal section and $a = |AB|$ the side of the basis. In fact, introducing the half-angle ω of the n-sided basis and using the shortcut $s = \sin(\omega)$, we have

$$\begin{aligned}
\omega &= \frac{n-2}{2n}\pi, \qquad |AG| = \frac{x}{2s}, \\
|GB| &= a - |AG| = \frac{2as - x}{2s}, \\
|VB| &= \frac{|GB|}{|AB|}|BV_0| = \frac{2as - x}{2as}|BV_0|.
\end{aligned} \tag{19}$$

Referring to the figure 5-(II), the segment $u = |EB| = 2a\sin(\omega)$ is the smallest diagonal of the $n-$gon and BV_0 satisfies

$$\begin{aligned}
&\frac{BV_0}{BW} = \frac{EA}{EW} = \frac{a}{a + \frac{a/2}{\cos(\pi - 2\omega)}} = \frac{2\cos(2\omega)}{2\cos(2\omega) - 1} \\
\Rightarrow\ &|BV_0| = \frac{EA}{EW}|BW| = \frac{2\cos(2\omega)}{2\cos(2\omega) - 1} \cdot \frac{a/2}{\cos(\pi - 2\omega)}, \\
&|BV_0| = \frac{a}{1 - 2\cos(2\omega)} = \frac{a}{4s^2 - 1}.
\end{aligned}$$

which, by (18) and (19) gives

$$\begin{aligned}
|VB| &= \frac{2as - x}{2as} \cdot \frac{a}{4s^2 - 1} = \frac{2as - x}{2s(4s^2 - 1)} \\
\Rightarrow \frac{VC}{VB} &= 1 + \frac{a}{|VB|} = \frac{8as^3 - x}{2as - x}.
\end{aligned} \tag{20}$$

The second ratio entering the Menelaos theorem is

$$\frac{IF}{IC} = \frac{IF}{FC - FI} = \frac{1}{\frac{FC}{IF} + 1} = \frac{1}{1 - \frac{a}{x}} = \frac{x}{x - a}. \tag{21}$$

Finally the third ratio is

$$\frac{HB}{HF} = \frac{FB - FH}{HF} = \frac{FB}{HF} + 1 = 1 - \frac{EB}{LH} = 1 - \frac{u}{v}. \tag{22}$$

The length $v = |LH|$ is the smallest diagonal of the $(n+1)-$ sided cut of the pyramid and can be expressed in terms of x:

$$|KU| = \frac{x}{2\cos(\psi)}, \quad \frac{v}{x} = \frac{x + |KU|}{x} = \frac{2\cos(\psi) + 1}{2\cos(\psi)} \;\Rightarrow\; v = x\frac{2c+1}{2c},$$

where we have set $\psi = \widehat{GKU}$ for the complement of the angle of the $(n+1)-$ sided regular polygon and used the shortcut $c = \cos(\psi)$, the relevant elements being displayed in figure 5-(III). Replacing this in (22) we get

$$\frac{HB}{HF} = \frac{x(2c+1) - 4asc}{x(2c+1)}. \tag{23}$$

By substitution into the Menelaos condition we obtain the equation

$$\frac{8as^3 - x}{2as - x} \cdot \frac{x}{x-a} \cdot \frac{x(2c+1) - 4asc}{x(2c+1)} = 1.$$

$$\Leftrightarrow \; \frac{x}{2s} = a\frac{16cs^3 - 2c - 1}{16cs^3 + 8s^3 - 2s - 2c - 1}. \tag{24}$$

If there were a hexagonal section, then the quantity $\frac{x}{2s} = |AG|$, should have the value on the right of the last equation. This, in the case $n = 5$, for which $s = \phi/2, c = 1/2$, evaluates to

$$a\frac{\phi^2 - 2}{2\phi^2 - \phi - 2} = a\frac{2\phi - 1}{3\phi} \approx 0.46065 \cdot a. \tag{25}$$

But this value leads to a contradiction visible in figure 6. In fact, the left figure

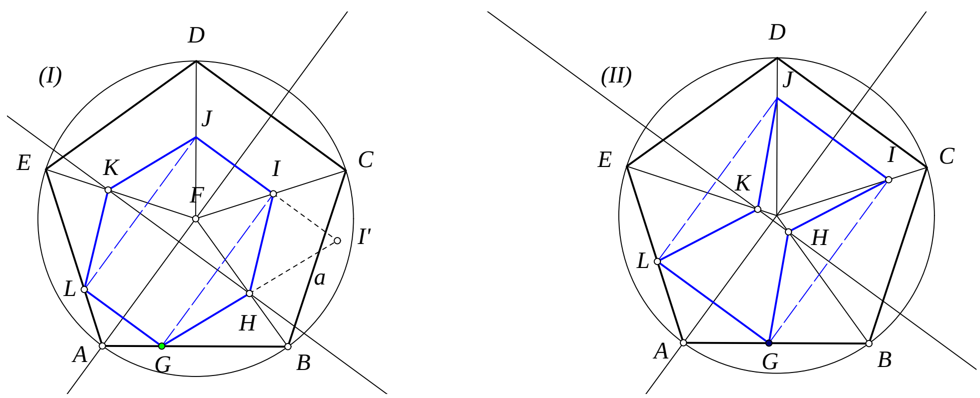

Figure 6. Impossible configuration

6-(I) shows how should look the vertical projection of the regular hexagon on the plane of the basis of the pyramid, if such an hexagon was existing. The right part (II) of the figure, however, shows how the projection should look, for the value $|AG| \approx 0.46065 \cdot a$ according to equation (25). In fact, opposite sides LG, IJ of the projected hexagon are equal segments and $LGIJ$ is a rectangle. The other two vertices project at points K, H, which coincide, respectively, with the intersections of the medial line of GI with the radii FE, FB of the circumcircle of the pentagon.

Thus, the shape of the projected hexagon is completely determined by the length of $|AG|$. However for the value obtained above, it is easy to show that the location of the intersection points K, H of the medial line of GI with the radii FE, FB results to a non-convex hexagon (See Figure 6-II), which is impossible.

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Pascal's Hexagram and the Geometry of the Ricochet Configuration

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Abstract. This paper is a study of a geometric arrangement called the 'ricochet configuration' (or R-configuration), which arises in the context of Pascal's theorem. We give a synthetic proof of the fact that a specific pair of Pascal lines is coincident for a sextuple in R-configuration. Furthermore we calculate the symmetry group of a generic R-configuration, and consequently the degree of the subvariety $\mathcal{R} \subseteq \mathbb{P}^6$ of all such configurations. We also find a set of equivariant defining equations for $\mathcal{R}$, and show that it is an intersection of two invariant hypersurfaces.

1. Introduction

The 'ricochet configuration' is a specific arrangement of six points on a conic which arises in the context of Pascal's theorem. It was discovered by the author in [1]. We recall some of this background below for ease of reading.

1.1. Let $\mathcal{K}$ denote a nonsingular conic in the complex projective plane. Consider six distinct points A, B, C, D, E, F on $\mathcal{K}$, arranged into an array $\left[\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right]$. Then Pascal's theorem says that the three cross-hair intersection points

$$AE \cap BF, \quad BD \cap CE, \quad AD \cap CF$$

(corresponding to the three minors of the array) are collinear.

The line containing them is called the Pascal line, or just the Pascal, of the array; we will denote it by $\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}$. It is easy to see that the Pascal remains unchanged if we permute the rows or the columns of the array; for instance

$$\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}, \quad \left\{\begin{array}{ccc} F & E & D \\ A & B & C \end{array}\right\}, \quad \left\{\begin{array}{ccc} E & D & F \\ B & C & A \end{array}\right\}$$

all denote the same line.

Any essentially different arrangement of the same sextuple of points, say

$$\left\{\begin{array}{ccc} E & A & C \\ B & F & D \end{array}\right\},$$

corresponds *a priori* to a different line. Hence we have a total of $\frac{6!}{2!3!} = 60$ notionally distinct Pascals. It is a theorem due to Pedoe [7], that these 60 lines are pairwise distinct for a *general* choice of the initial sextuple. In other words, there must be something geometrically special about the sextuple (always assumed to consist of distinct points) if some of its Pascals are to coincide.

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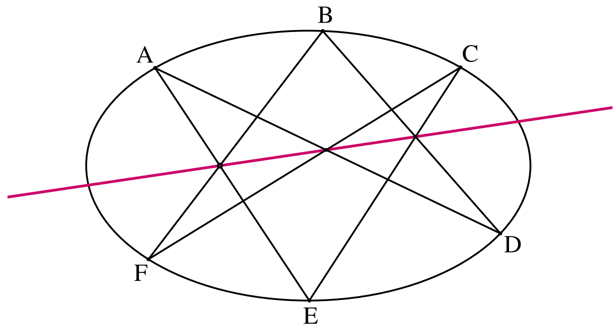

Figure 1. Pascal's theorem

The main theorem on [1, p. 12] characterises all such special situations. It says that if some of the Pascals coincide, then the sextuple must either be in *involutive configuration*, or in *ricochet configuration*. We will describe both of these below. The first is very classical (cf. [8, §260]); whereas the second is probably not. To the best of my knowledge, it had not previously appeared in the literature before it was discovered in the process of proving this theorem.

1.2. *The involutive configuration.* The sextuple $\Gamma = \{A, \ldots, F\}$ is said be in involutive configuration, (or in involution for short), if there exists a point Q in the plane with three lines L, L', L'' through it such that

$$\Gamma = (L \cup L' \cup L'') \cap \mathcal{K}.$$

With points labelled as in the diagram, it turns out that the Pascals

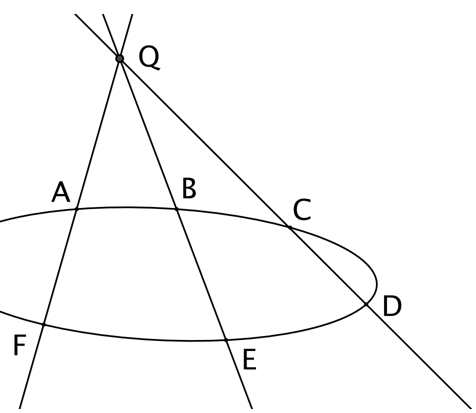

Figure 2. The involutive configuration

$$\left\{ \begin{matrix} A & B & C \\ F & E & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} F & B & C \\ A & E & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} A & E & C \\ F & B & D \end{matrix} \right\}, \quad \left\{ \begin{matrix} A & B & D \\ F & E & C \end{matrix} \right\},$$

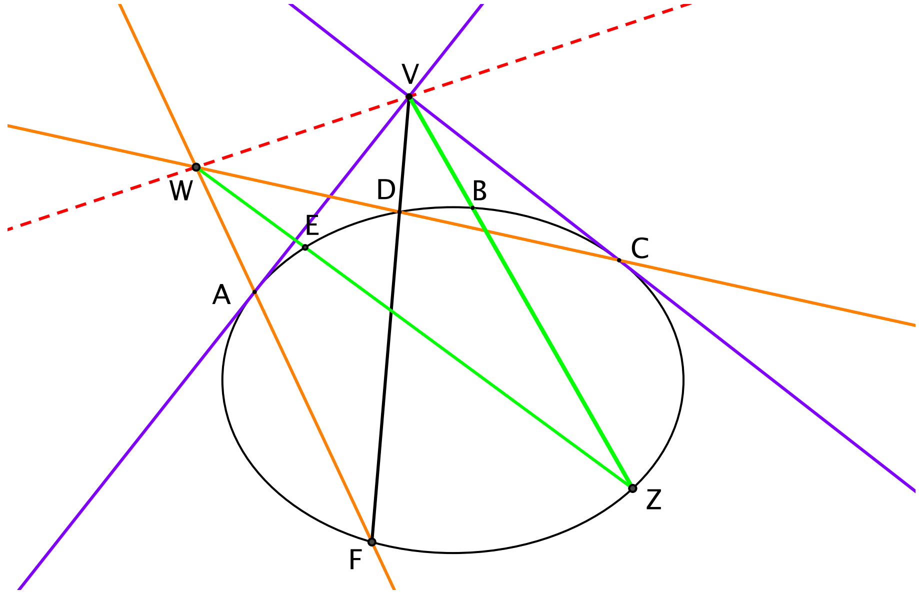

Figure 3. The ricochet configuration

all coincide (see [1, p. 9]). The pattern is straightforward: fix any column in the first array and switch its entries to get another array. The common Pascal is the polar of Q with respect to the conic.

1.3. *The ricochet configuration.* The construction in this case is rather more elaborate. Start with arbitrary distinct points A, B, C, D on the conic. We will define two more points E and F to complete the sextuple (see Diagram 3).

- Draw tangents to the conic at A and C. Let V denote their intersection point.
- Extend VD so that it intersects the conic again at F.
- Let W be the intersection point of AF and CD.
- Now mark off Z on the conic such that V, B, Z are collinear, and finally E such that W, Z, E are collinear.

One may think of B as a billiard ball which is struck by V so that it bounces off the conic at Z, and gets redirected to W; hence the name 'ricochet'. For such a sextuple, the Pascals

$$\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}, \qquad \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\} \tag{1}$$

coincide (see [1, p. 10]). The common Pascal is the line VW, something which is not altogether obvious from the diagram. It is *prima facie* a little odd that the Pascal only depends on A, C, D and not on B. All of this will be clarified in §3.

As mentioned above, the main result of [1] can be paraphrased as saying that any sextuple for which some of the Pascals coincide must fit into either Diagram 2 or Diagram 3, up to a relabelling of points.

1.4. *A summary of results.* This paper is a study of the algebro-geometric properties of the ricochet configuration (henceforth called the R-configuration).

(1) The fact that the two Pascals in (1) coincide was proved by an inelegant brute-force calculation in [1]. We will give a geometric proof in §3.
(2) In §4, we determine the group of symmetries of a generic R-configuration; it turns out to be the 8-element dihedral group. If one thinks of a sextuple as an element of

$$\mathrm{Sym}^6 \mathcal{K} \simeq \mathrm{Sym}^6 \mathbb{P}^1 \simeq \mathbb{P}^6,$$

then all sextuples in ricochet configuration form a 4-dimensional subvariety $\mathcal{R} \subseteq \mathbb{P}^6$. The symmetry group will be used to prove that $\mathcal{R}$ has degree 60.
(3) The special linear group $SL(2, \mathbb{C})$ acts on the projective plane by linear automorphisms in such a way that $\mathcal{K}$ is stabilized (more on this in §2.1 below). Since the R-configuration is constructed synthetically, the subvariety $\mathcal{R}$ is stabilized by the induced group action on $\mathrm{Sym}^6 \mathcal{K}$. Hence $\mathcal{R}$ must be defined by $SL(2)$-invariant homogeneous equations; or in classical language, by the vanishing of certain covariants of binary sextic forms. We will find such equations explicitly in §5. It turns out that $\mathcal{R}$ is defined by the vanishing of two invariants, one each in degrees 6 and 10. This means that $\mathcal{R}$ is a complete intersection; that is to say, it is defined by the smallest possible number of equations for its dimension.

All the necessary background in projective geometry may be found in [2, 9, 10]. We will use [11] as the standard reference for algebraic geometry, but nothing beyond the most basic notions will be needed.

2. Preliminaries

Our entire set-up agrees with the one used in [1, Ch. 3]. We will recall only some of it below, and refer the reader to the earlier paper for details. Section 3 is in any event mostly geometric, and apart from §2.2 on involutions, it does not need any of the algebraic preliminaries given here.

2.1. For $m \geq 0$, let S_m denote the vector space of homogeneous polynomials of degree m in the variables $\mathbf{x} = \{x_1, x_2\}$. In classical language, elements of S_m are the binary m-ics. Given $A \in S_m$ and $B \in S_n$, their r-th transvectant will be denoted by $(A, B)_r$. It is a binary form of degree $m + n - 2r$.

We will use $\mathbb{P}^2 = \mathbb{P}S_2$ as our working projective plane; thus a nonzero quadratic form $Q = a_0\, x_1^2 + a_1\, x_1\, x_2 + a_2\, x_2^2$ represents a point $[Q] \in \mathbb{P}^2$. Consider the Veronese imbedding

$$\mathbb{P}S_1 \xrightarrow{\phi} \mathbb{P}S_2, \quad [u] \longrightarrow [u^2].$$

The image of ϕ will be our conic $\mathcal{K}$. The point $[Q]$ lies on $\mathcal{K}$, iff Q is the square of a linear form. Thus $\mathcal{K}$ is defined by the equation $a_1^2 = 4\, a_0\, a_2$. Henceforth we will write Q for $[Q]$ etc., if no confusion is likely. We will sometimes use affine

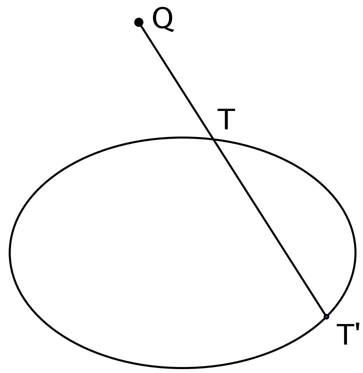

Figure 4. $T \longrightarrow \sigma_Q(T) = T'$

coordinates on $\mathcal{K} \simeq \mathbb{C} \cup \{\infty\}$, so that $\alpha \in \mathbb{C}$ corresponds to $\phi(x_1 - \alpha\, x_2)$, and ∞ to $\phi(x_2)$.

The advantage of such a set-up is that the action of the special linear group is naturally built into it. A matrix $M = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \in SL(2, \mathbb{C})$ gives an automorphism of S_m defined by a linear change of variables $f(x_1, x_2) \to f(\alpha\, x_1 + \beta\, x_2, \gamma\, x_1 + \delta\, x_2)$; this in turn induces an automorphism of the projective space $\mathbb{P}S_m \simeq \mathbb{P}^m$. The operation of transvection commutes with a linear change of variables; in particular all the notions involving points and lines in $\mathbb{P}^2$, as well as polarities with respect to $\mathcal{K}$ are expressible in the language of transvectants.

2.2. *Involutions.* Every point $Q \in \mathbb{P}^2 \setminus \mathcal{K}$ defines an involution (i.e., a degree 2 automorphism) σ_Q on $\mathcal{K}$. It takes a point T to the other intersection of QT with $\mathcal{K}$ (see Diagram 4). In particular, $\sigma_Q(T) = T$ exactly when QT is tangent to $\mathcal{K}$.

Now let Y be any point in $\mathbb{P}^2$, and let $\mathrm{y}_1, \mathrm{y}_2$ be the intersection points of its polar with respect to $\mathcal{K}$. (That is to say, $\mathrm{y}_i Y$ are tangent to the conic.) We define $Z = \sigma_Q(Y)$ to be the pole of the line joining $\mathrm{z}_1 = \sigma_Q(\mathrm{y}_1), \mathrm{z}_2 = \sigma_Q(\mathrm{y}_2)$. Thus σ_Q extends to an involution of the entire plane (see Diagram 5). The points Y, Q and $\sigma_Q(Y)$ are collinear. This has the consequence that if ℓ is a line passing through Q, then $\sigma_Q(\ell) = \ell$ as a set.

2.3. *Algebraic form of the R-configuration.* We will express the notion of an R-configuration in the language of §2.1. Consider the set of letters $\textsc{ltr} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}\}$. Define a hexad to be an *injective* map $\textsc{ltr} \xrightarrow{h} \mathcal{K}$, and write

$$A = h(\mathbb{A}), \quad B = h(\mathbb{B}), \quad \dots \quad F = h(\mathbb{F})$$

for the corresponding distinct points on the conic. Then $\Gamma = \text{image}(h) = \{A, \dots, F\}$ is the associated sextuple. A hexad h will be called an *alignment* if the two rows of the table

$$\begin{array}{c|c|c|c|c|c} A & B & C & D & E & F \\ \hline 0 & t & \infty & 1 & \frac{t-1}{t+1} & -1 \end{array} \tag{2}$$

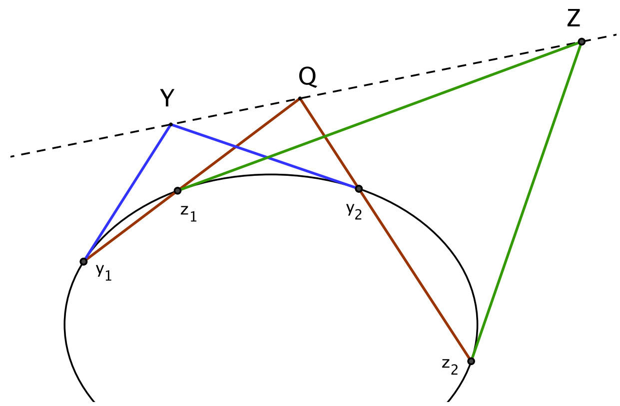

Figure 5. $Y \longrightarrow \sigma_Q(Y) = Z$

are projectively isomorphic for some complex number t. In other words, there should exist an automorphism of $\mathcal{K}$ (or what is the same, a fractional linear transformation of $\mathbb{P}^1$) which takes $A, B, \dots, F$ respectively to $0, t, \dots, -1$. Since the points are required to be distinct, we must have $t \neq 0, 1, \sqrt{-1}$. For later reference, let $\Sigma(t)$ denote the sextuple corresponding to the second row of (2).

We will say that a sextuple Γ is in R-configuration (or, it is an R-sextuple), if it admits at least one alignment. To see that this definition agrees with the geometric construction, choose coordinates on $\mathcal{K}$ such that A, C, D respectively correspond to $0, \infty, 1$. This can always be done by the fundamental theorem of projective geometry. Using binary forms,

$$A = x_1^2, \quad C = x_2^2, \quad D = (x_1 - x_2)^2.$$

Following the geometric construction in §1.3, we get $V = x_1\, x_2, F = (x_1 + x_2)^2$, and hence[1]

$$W = (x_1(x_1 + x_2), x_2\,(x_1 - x_2))_1 = \Box\,(x_1^2 - 2\,x_1\,x_2 - x_2^2).$$

Now let $B = (x_1 - t\,x_2)^2$ for some t. Then $Z = \sigma_V(B) = \Box\,(x_1 + t\,x_2)^2$, and finally

$$E = \sigma_W(Z) = \Box\,(x_1 - \frac{t-1}{t+1}\,x_2)^2.$$

This agrees exactly with (2). The ricochet $B \rightsquigarrow Z \rightsquigarrow E$ corresponds to $t \to -t \to \frac{t-1}{t+1}$. Notice that A, C, D, F is a harmonic quadruple, i.e., the cross-ratio $\langle A, C, D, F\rangle = -1$. Thus one can think of the R-configuration as a 'fixed' harmonic quadruple, joined by a moving pair of points B and E. It will be convenient to introduce the partition

$$\text{LTR} = \underbrace{\{\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{F}\}}_{\text{H-LTR}} \cup \{\mathbb{B}, \mathbb{E}\}, \tag{3}$$

[1]As in [1], we will use $\Box$ to indicate a nonzero multiplicative scalar whose precise value is irrelevant.

where H-LTR is to thought of as the 'harmonic' subset of letters.

The fractional linear transformation

$$\varphi(t) = \frac{t-1}{t+1}, \tag{4}$$

will appear many times below. Its inverse is given by $\varphi^{-1}(t) = \frac{1+t}{1-t}$.

2.4. *Example.* The table

$-\frac{1}{3}$	1	2	$\frac{1}{4}$	$\frac{1}{18}$	$-\frac{3}{2}$
0	4	∞	1	$\frac{3}{5}$	-1

is so arranged that the second row is $\Sigma(4)$, and $s \to \frac{3s+1}{2-s}$ transforms the first row into the second. Hence the first row (and of course, also the second) is in R-configuration.

2.5. We identify the projective space $\mathbb{P}^6$ with $\mathbb{P}S_6$. A nonzero binary sextic form $\mathcal{H}$ will factor as $\prod_{i=1}^{6}(\alpha_i\, x_1 - \beta_i\, x_2)$, and as such corresponds to the sextuple of points $\{\beta_i/\alpha_i : 1 \leqslant i \leqslant 6\}$ on $\mathbb{P}^1 \simeq \mathcal{K}$. The points are distinct if $\mathcal{H}$ has no repeated linear factors. Hence the set of sextuples of distinct points on $\mathcal{K}$ can be identified with the complement of the discriminant hypersurface in $\mathbb{P}^6$.

Let $\mathcal{R} \subseteq \mathbb{P}^6$ denote the Zariski closure of the set of all R-configurations; in other words, it is the Zariski closure of the union of $SL(2)$-orbits of the sextic forms

$$G_t = x_1\, x_2\, (x_1 - x_2)\, (x_1 + x_2)\, (x_1 - t\, x_2)\, (x_1 - \varphi(t)\, x_2),$$

over all complex numbers $t \neq 0, 1, \sqrt{-1}$. Since an R-configuration is built from an arbitrary choice of A, B, C, D on the conic, $\mathcal{R}$ is an irreducible 4-dimensional rational projective variety.

3. The double ricochet

We are aiming for the following theorem:

Theorem 1. If $\Gamma = \{A, \dots, F\}$ is a sextuple in R-configuration, then either of the Pascals

$$\left\{\begin{array}{ccc} A & B & C \\ F & E & D \end{array}\right\}, \quad \left\{\begin{array}{ccc} A & E & C \\ D & B & F \end{array}\right\}$$

coincides with the line VW.

The first task is to construct a specific automorphism ψ of the conic, which is expressible as a product of two involutions in two distinct ways.

3.1. Diagram 6 is a minor variant of Diagram 3. The points A, C, D, F, V, W are exactly as before, but B and E are not yet in the picture. The lines AD, CF intersect in the newly shown point U.

Lemma 2. The points U, V, W are collinear.

Proof. The involution σ_V preserves the points A, C, and interchanges D, F. Hence it takes $U = AD \cap CF$ to $W = AF \cap CD$. Thus U, V, W are collinear. $\square$

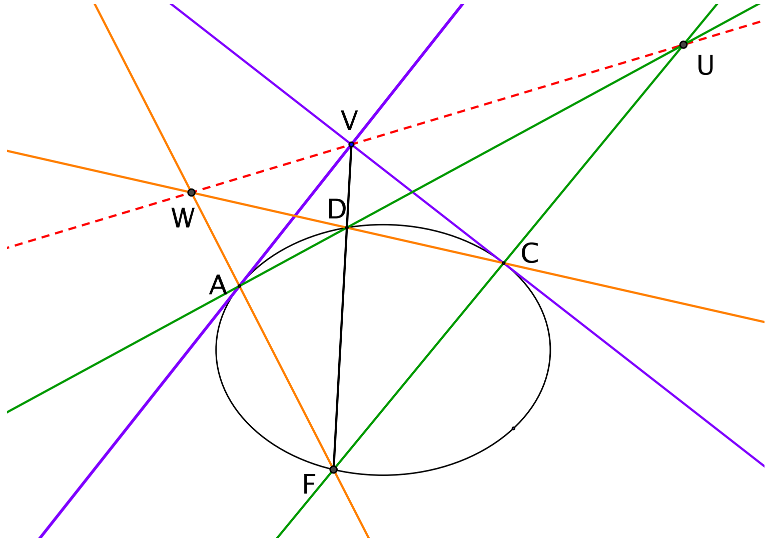

Figure 6. The harmonic quadruple $\{A, C, D, F\}$

Lemma 3. The two automorphisms $\sigma_W \circ \sigma_V$ and $\sigma_V \circ \sigma_U$ of $\mathcal{K}$ are equal.

Proof. Since $\mathcal{K} \simeq \mathbb{P}^1$, by the fundamental theorem of projective geometry it will suffice to show that the two agree on three distinct points. Now $\sigma_W \circ \sigma_V(A) = \sigma_W(A) = F$, and $\sigma_V \circ \sigma_U(A) = \sigma_V(D) = F$. Similarly, it is easy to check that both automorphisms send C to D, and D to A. $\square$

Now let $\psi : \mathcal{K} \longrightarrow \mathcal{K}$ denote this common automorphism, and let $\mathbb{L}$ denote the line UVW. For arbitrary points B and E on the conic, consider the Pascal $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$. By definition, it must pass through the point $U = AD \cap CF$. As B and E move on the conic, the Pascal will pivot around U. We should like to know under what conditions it will equal $\mathbb{L}$. This is answered by the next proposition.

Proposition 4. We have $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\} = \mathbb{L}$, exactly when $\psi(B) = E$.

Diagram 7 shows the action of $\psi = \sigma_W \circ \sigma_V = \sigma_V \circ \sigma_U$. One can pass from B to E either by a ricochet at Z or at Z'. Let us assume the proposition for now, and deduce theorem 1. If $\psi(B) = E$, then $\psi(Z') = Z$. Applying the proposition with Z' in place of B, we have

$$\left\{ \begin{array}{ccc} A & Z' & C \\ F & Z & D \end{array} \right\} = \mathbb{L}.$$

Now apply σ_V to this equation. Since $\mathbb{L}$ passes through V, we have $\sigma_V(\mathbb{L}) = \mathbb{L}$ by §2.2. But then

$$\left\{ \begin{array}{ccc} \sigma_V(A) & \sigma_V(Z') & \sigma_V(C) \\ \sigma_V(F) & \sigma_V(Z) & \sigma_V(D) \end{array} \right\} = \left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\} = \mathbb{L},$$

which is what we wanted.

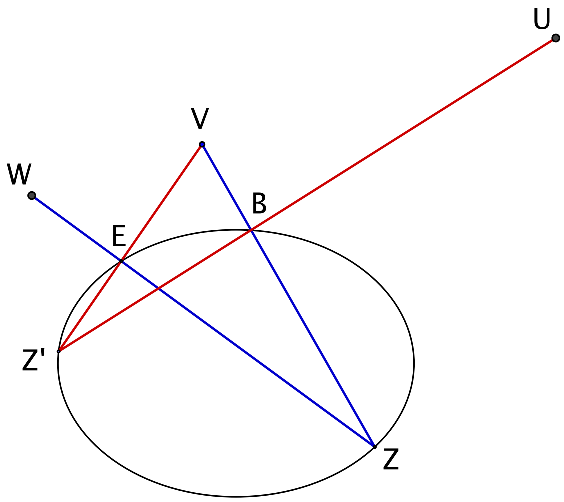

Figure 7. $\sigma_W \circ \sigma_V(B) = E = \sigma_V \circ \sigma_U(B)$

3.2. It remains to prove the proposition. Let A, C, D, F be as in Diagram 6. Given an arbitrary point B on the conic, we will define another point $\omega(B)$ such that

$$\left\{ \begin{array}{ccc} A & B & C \\ F & \omega(B) & D \end{array} \right\} = \mathbb{L}. \tag{5}$$

Afterwards we will prove that ω and ψ are the same morphism. One can define $\omega(B)$ in either of the following two ways; the identity in (5) is then simply the definition of the Pascal.

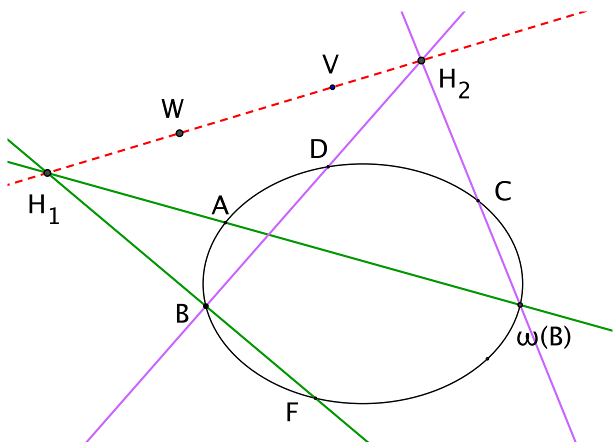

Figure 8. $B \longrightarrow \omega(B)$

(1) Intersect BF with $\mathbb{L}$ to get a point H_1, and define $\omega(B)$ to be the other intersection of AH_1 with $\mathcal{K}$. Then the Pascal in (5) is $H_1U = \mathbb{L}$.

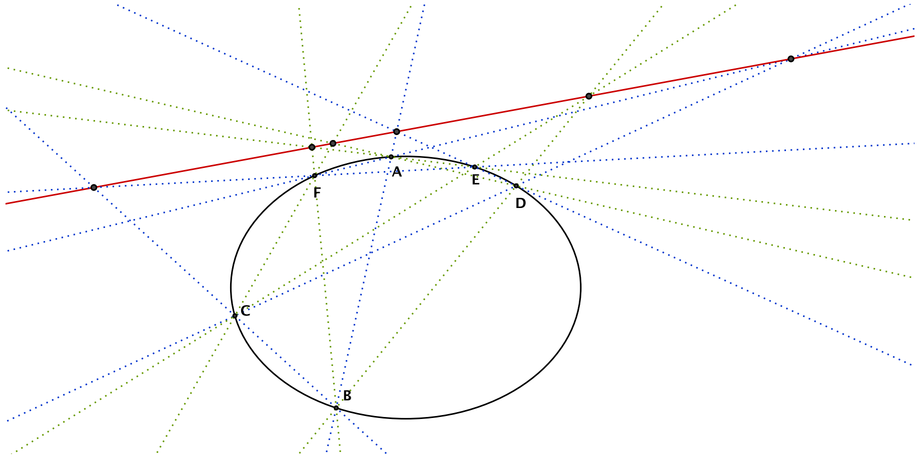

Figure 9. The double Pascal of an R-configuration

(2) Intersect BD with $\mathbb{L}$ to get a point H_2, and define $\omega(B)$ to be the other intersection of CH_2 with $\mathcal{K}$. Once again, the Pascal in (5) $H_2U = \mathbb{L}$.

This defines a morphism $\omega : \mathcal{K} \longrightarrow \mathcal{K}$, which is bijective since the construction can be reversed to define ω^{-1}. One point should be clarified. Throughout this paper, we have considered sextuples of distinct points only. However, ω is defined for *all* positions of B on $\mathcal{K}$, even those which coincide with other points. For instance, if B coincides with D, then we interpret BD as the tangent at D.

Now observe that

- If $B = A$, then $H_1 = W$ and $\omega(B) = F$.
- If $B = C$, then $H_2 = W$ and $\omega(B) = D$.
- If $B = D$, then $H_1 = V$ and $\omega(B) = A$ since V lies on the tangent at A.

Thus A, C, D are respectively mapped to F, D, A by ψ as well as ω, hence they must be the same morphism. This proves the proposition. $\square$

Theorem 1 is now completely proved. The equality of the two automorphisms seems difficult to prove directly, since their definitions are rather disparate. But the fundamental theorem of projective geometry allows us to conclude the argument by comparing their values only at three chosen points.

Diagram 9 on page 82 shows the R-configuration with its double Pascal. All the joins belonging to $\left\{ \begin{array}{ccc} A & B & C \\ F & E & D \end{array} \right\}$ are shown in green, and those belonging to $\left\{ \begin{array}{ccc} A & E & C \\ D & B & F \end{array} \right\}$ are in blue. Although a certain amount of clutter is unavoidable in a diagram of this kind, it is hard to miss the red Pascal line containing the six intersection points.

In [1, p. 17], Gröbner basis computations are used to prove that the converse of this theorem is also true, i.e., assuming that the two Pascals coincide forces Γ to be in R-configuration. It would be interesting to have a purely geometric proof of this fact, but I do not see one.

4. The shuffle group and the degree of the ricochet locus

In this section we will determine the group of combinatorial symmetries of a generic R-sextuple. This calculation will be of use in finding the degree of the variety $\mathcal{R}$. As in [1], let $\mathfrak{G}(X)$ denote the group of bijections $X \longrightarrow X$ on a set X.

4.1. Let $\Gamma = \Sigma(t)$ as in (2). Fix the alignment $h : \textsc{ltr} \longrightarrow \Sigma(t)$ such that

$$\mathbb{A} \to 0, \quad \mathbb{B} \to t, \quad \mathbb{C} \to \infty, \quad \mathbb{D} \to 1, \quad \mathbb{E} \to \frac{t-1}{t+1}, \quad \mathbb{F} \to -1.$$

Consider the subgroup $H(t) \subseteq \mathfrak{G}(\textsc{ltr})$ of elements z such that $h \circ z$ is also an alignment. In other words, $H(t)$ measures in how many ways the same sextuple can be seen to be in R-configuration. We may call it the shuffle group corresponding to t.

Lemma 5. The elements

$$u = (\mathbb{A}\,\mathbb{D}\,\mathbb{C}\,\mathbb{F}), \quad v = (\mathbb{A}\,\mathbb{D})(\mathbb{B}\,\mathbb{E})(\mathbb{C}\,\mathbb{F}) \tag{6}$$

are in $H(t)$.

By our convention, the 4-cycle u takes $\mathbb{A}$ to $\mathbb{D}$ etc.

Proof. The proof for u is captured by the following table:

1	t	-1	∞	$\frac{t-1}{t+1}$	0
0	$\frac{t-1}{t+1}$	∞	1	$-\frac{1}{t}$	-1

The hexad $h \circ u$ is given by $\mathbb{A} \to \mathbb{D} \to 1, \mathbb{B} \to \mathbb{B} \to t$ etc, all of which is described by the first row. The fractional linear transformation $s \to \frac{s-1}{s+1}$ converts it into the second row, which is $\Sigma(\frac{t-1}{t+1})$. Hence $h \circ u$ is also an alignment, i.e., $u \in H(t)$.

Similarly, the hexad $h \circ v$ is the first row of the table:

1	$\frac{t-1}{t+1}$	-1	0	t	∞
0	$\frac{1}{t}$	∞	1	$\frac{1-t}{1+t}$	-1

The transformation $s \to \frac{1-s}{1+s}$ converts it into the second row, which is $\Sigma(\frac{1}{t})$. Hence $v \in H(t)$. $\square$

These two elements satisfy the relations $u^4 = v^2 = (u\,v)^2 = e$, hence the subgroup of $\mathfrak{G}(\textsc{ltr})$ generated by them is the dihedral group with 8 elements.

4.2. We already know that $\{A, C, D, F\}$ is a harmonic quadruple inside $\Sigma(t)$. But some other quadruple, say $\{B, D, C, E\}$, will be harmonic exactly when the cross-ratio

$$\langle B, D, C, E \rangle = \frac{2}{t^2+1},$$

is $-1, \frac{1}{2}$ or 2. This can happen only for finitely many values of t, and of course likewise for all such cases. Hence, $\{A, C, D, F\}$ is the unique harmonic quadruple inside $\Sigma(t)$, for all but finitely many values of t (that is to say, for a 'generic' t).

Proposition 6. For generic t, the group $H(t)$ is generated by u and v.

Proof. By what has been said, every element in $H(t)$ must preserve the subset $\text{H-LTR} \subseteq \text{LTR}$. This gives a morphism $f : H(t) \longrightarrow \mathfrak{G}(\text{H-LTR})$.

Let $G \subseteq \mathfrak{G}(\text{H-LTR})$ denote the group of permutations δ such that

$$\langle h \circ \delta(\mathbb{A}), h \circ \delta(\mathbb{C}), h \circ \delta(\mathbb{D}), h \circ \delta(\mathbb{F})\rangle = -1.$$

Now G is the 8-element dihedral group generated by $u = (\mathbb{A}\,\mathbb{D}\,\mathbb{C}\,\mathbb{F})$ and $v' = (\mathbb{A}\,\mathbb{D})(\mathbb{C}\,\mathbb{F})$; this is a standard fact about the symmetries of the cross-ratio and in particular those of a harmonic quadruple (see [13, Ch. IV]). We know, *a priori*, that the image of f is contained in G. Now f surjects onto G, since $f(v) = v'$. It is easy to check that $(\mathbb{B}\,\mathbb{E}) \notin H(t)$, and hence f is also injective. It follows that f is an isomorphism onto G, and thus $H(t)$ is generated by u and v. $\square$

4.3. The group $H(t)$ may be larger for special values of t. If $t = \sqrt{-3}$, then $\{B, D, C, E\}$ is also a harmonic quadruple, which allows more possibilities for elements in $H(t)$. A routine computation shows that $H(\sqrt{-3})$ is the 16-element group generated by u and v, together with the additional element $(\mathbb{A}\,\mathbb{B})(\mathbb{C}\,\mathbb{D})(\mathbb{E}\,\mathbb{F})$. There are several such special values of t, but we do not attempt to classify them.

In general, an element of $H(t)$ does not extend to an automorphism of the entire conic. For instance, let T denote the intersection of the lines AD, BE. If v were to extend to an automorphism of $\mathcal{K}$, it would have to coincide with the involution σ_T, since both have identical actions on the four points A, B, D, E. However this is a contradiction, since the line CF will not pass through T for generic t.

4.4. *The degree of* $\mathcal{R}$. Let us write $H = H(t)$ in the generic case, since the group is independent of t. We will use it to determine the degree of $\mathcal{R}$ as a projective subvariety in $\mathbb{P}^6$. If $z \in \mathcal{K}$ is an arbitrary point, then $\{\Gamma \in \mathbb{P}^6 : z \in \Gamma\}$ is a hyperplane in $\mathbb{P}^6$. Since the degree of $\mathcal{R}$ is the number of points in its intersection with four general hyperplanes, we are reduced to the following question: Given a set of four general points $\Omega = \{z_1, \dots, z_4\} \subseteq \mathbb{P}^1 \simeq \mathcal{K}$, find the number of R-sextuples Γ which contain Ω.

Thus the degree of $\mathcal{R}$ can be understood in the following intuitive way. The R-configuration has four degrees of freedom; that is to say, four general points on the conic will fit into only finitely many R-configurations. We want to know how many.[2] The following two examples should capture the gist of the matter.

Let $\Omega = \{2, 3, 5, 7\}$, and assign them respectively to positions $\mathbb{A}, \mathbb{C}, \mathbb{D}, \mathbb{E}$. This means that, in the table below

A	B	C	D	E	F
2	b	3	5	7	f
0	t	∞	1	$\varphi(t)$	-1

[2]Since there are only finitely many values of t for which $H(t)$ is larger than H, the subclass of such R-configurations has only three degrees of freedom. Hence a general set of four points on the conic is not extendable to any such configuration. This fact will play a role in the degree calculation below.

we want to find all pairs (b, f) such that the second row is projectively isomorphic to the third row for some t. The transformation $\mu(s) = \frac{9s-4}{3s-2}$ takes $0, \infty, 1$ respectively to $2, 3, 5$. Hence $f = \mu(-1) = \frac{13}{5}$. Then $\varphi(t) = \mu^{-1}(7) = \frac{5}{6}$, and hence $t = \varphi^{-1}(\frac{5}{6}) = 11$. Finally $b = \mu(11) = \frac{95}{31}$. Thus we have a unique pair (b, f) which extends the given initialisation of Ω to an R-sextuple.

Now assign the same numbers respectively to $\mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{E}$. This leads to the table:

A	B	C	D	E	F
2	3	c	5	7	f
0	t	∞	1	$\varphi(t)$	-1

As before, we are searching for all pairs (c, f) such that the second and the third rows are projectively isomorphic for some t. Now $\nu(s) = \frac{(c-5)\,(s-2)}{3\,(c-s)}$ takes $2, c, 5$ respectively to $0, \infty, 1$. Hence $t = \nu(3) = \frac{(c-5)}{3\,(c-3)}$, which leads to $\varphi(t) = \frac{t-1}{t+1} = \frac{c-2}{7-2c}$. But $\varphi(t)$ is also equal to $\nu(7) = \frac{5c-25}{3c-21}$. Equating the two leads to the quadratic equation $13\,c^2 - 112\,c + 217 = 0$, and hence two values of c. Since $f = \nu^{-1}(-1) = \frac{c+10}{8-c}$ is completely determined by c, we get two pairs (c, f).

The crucial difference between the two examples is that three of the elements in $\{A, C, D, F\}$ are specified in the first, and only two in the second. The idea behind the degree computation is to keep a tally of all such possibilities.

4.5. We define an *initialisation* to be a bijection $\beta : \mathcal{F} \to \Omega$, for some 4-element subset $\mathcal{F} \subseteq$ LTR. An extension of β is an alignment $\beta' :$ LTR $\to \Gamma$ such that $\beta'|_{\mathcal{F}} = \beta$. Here Γ is necessarily an R-configuration containing Ω. Define the type of β to be the cardinality of the set H-LTR $\cap\, \mathcal{F}$.

For instance, the first example corresponds to the initialisation

$$\mathbb{A} \to 2, \quad \mathbb{C} \to 3, \quad \mathbb{D} \to 5, \quad \mathbb{E} \to 7, \tag{7}$$

which is of type 3. It admits a unique extension

$$\mathbb{B} \to \frac{95}{31}, \quad \mathbb{F} \to \frac{13}{5}.$$

Proposition 7. An initialisation has respectively $2, 1$ or 0 extensions according to whether its type is $2, 3$ or 4.

Proof. Assume that the type is 4. But since a general quadruple Ω is not harmonic, it cannot occupy the positions $\{A, C, D, F\}$ in an R-configuration. Hence there cannot be any extensions.

The first example in §4.4 illustrates type 3, and the second illustrates type 2. The proofs in the general case are exactly on the same lines, hence we leave them to the reader. The only issue which perhaps requires comment is the following: if the type is 2, then we get a quadratic equation for one of the unknown letters. Since the z_i are general, the equation has two distinct roots rather than a repeated root. Either of the roots determines the other unknown uniquely. $\square$

The geometry of the type 3 case is utterly straightforward. If the values of three letters from H-LTR are specified, so is the fourth by harmonicity. This determines all points in Diagram 6, and then specifying either B or E also specifies the other.

As to a type 2 case, assume for instance that $\{A, B, C, E\}$ are specified. Then so are V, Z and hence the line ZE is specified (on which W must lie). Since V, D, F are collinear, knowing D is tantamount to knowing F. Now, for a variable point D on the conic, the function $D \longrightarrow AD \cap CF$ traces a conic in the plane. It intersects ZE in two points, which are the two acceptable positions of W. But D, F are determined once W is determined, hence we get two extensions. The other type 2 cases are similar.

4.6. Fix a general quadruple $\Omega = \{z_1, \dots, z_4\} \subseteq \mathcal{K}$. It takes an elementary counting argument to see that there are

- 144 initialisations of type 2,
- 192 initialisations of type 3, and
- 24 initialisations of type 4.

For instance, to form a type 2 initialisation, choose two letters from H-LTR in 6 ways. Those, combined with $\{\mathbb{B}, \mathbb{E}\}$, can be distributed in 24 ways over the z_i. Hence there are $24 \times 6 = 144$ such initialisations.

Now observe that the group H will act on the sets of initialisations and extensions. For instance, the element v in (6) will change the initialisation in (7) to

$$\mathbb{D} \to 2, \quad \mathbb{F} \to 3, \quad \mathbb{A} \to 5, \quad \mathbb{B} \to 7,$$

and its extension to

$$\mathbb{E} \to \frac{95}{31}, \quad \mathbb{C} \to \frac{13}{5}.$$

Of course, both initialisations lead to the same R-sextuple, namely $\left\{2, 3, 5, 7, \frac{95}{31}, \frac{13}{5}\right\}$.

Since elements of H preserve the harmonic subset H-LTR, they do not affect the type of an initialisation. If two initialisations β_1, β_2 are in the same H-orbit, then the R-configurations obtained by extending them will be the same. Conversely, suppose that β_1, β_2 are two initialisations with respective extensions β_1', β_2' such that $\Gamma = \text{image}(\beta_1') = \text{image}(\beta_2')$. But since Ω is general, the shuffle group of Γ is exactly H, and no larger. Hence β_1, β_2 must be in the same H-orbit.

Now we can count the number of possible R-configurations which contain a given Ω. There are $\frac{144}{8} \times 2 = 36$ configurations coming from all initialisations of type 2, and $\frac{192}{8} \times 1 = 24$ from those of type 3. There are none coming from initialisations of type 4, which gives a total of $24 + 36 = 60$. This proves that

Theorem 8. The degree of $\mathcal{R}$ is 60. $\square$

Said differently, if we start with a randomly chosen quadruple of points on the conic, then there are 60 ways to add two more points so that the resulting sextuple is in R-configuration.

Now we will look for equations which define the variety $\mathcal{R}$. Since the dimension of $\mathcal{R}$ is two less than that of its ambient space $\mathbb{P}^6$, no fewer than two equations would suffice. The simplest situation would be that of an ideal-theoretic complete intersection; i.e., $\mathcal{R}$ would be defined by two equations of degrees m and n such that $m\,n = 60$. As we will see, this is not too good to be true.

5. Equations for the ricochet locus

We begin with a short introduction to classical invariant theory, which should motivate some of the calculations to follow. The most readable classical references on this subject are [3, 5, 8]. Modern accounts may be found in [6, Appendix B] and [12, Ch. 4].

5.1. *Invariants and Covariants.* The invariant theory of binary quartics is as good an illustration as any. Consider a degree 4 homogeneous polynomial

$$\Phi = a_0\, x_1^4 + a_1\, x_1^3\, x_2 + a_2\, x_1^2\, x_2^2 + a_3\, x_1\, x_2^3 + a_4\, x_2^4, \qquad (a_i \in \mathbb{C})$$

in the variables $\mathbf{x} = \{x_1, x_2\}$. Its Hessian, which we denote by $\mathrm{He}(\Phi)$, is defined to be the self-transvectant $(\Phi, \Phi)_2 = \frac{1}{72} \begin{vmatrix} \Phi_{x_1 x_1} & \Phi_{x_1 x_2} \\ \Phi_{x_2 x_1} & \Phi_{x_2 x_2} \end{vmatrix}$. It has the expression

$$\mathrm{He}(\Phi) = \left(\frac{1}{3}\, a_0\, a_2 - \frac{1}{8}\, a_1^2\right)\, x_1^4 + \left(a_0\, a_3 - \frac{1}{6}\, a_1\, a_2\right)\, x_1^3\, x_2 + \cdots + \left(\frac{1}{3}\, a_2\, a_4 - \frac{1}{8}\, a_3^2\right)\, x_2^4.$$

This is an example of a *covariant*, since its construction is compatible with a linear change of variables in the following sense. A 2×2 matrix with determinant 1, for instance $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$, corresponds to the change of variables:

$$x_1 \longrightarrow 2\, x_1 + 5\, x_2, \qquad x_2 \longrightarrow 3\, x_1 + 8\, x_2. \tag{8}$$

Now consider the following two processes:

- Use the substitution (8) in Φ to form another polynomial Φ', and take its Hessian $\mathrm{He}(\Phi')$.
- Use the substitution (8) in $\mathrm{He}(\Phi)$ to form $[\mathrm{He}(\Phi)]'$.

The outcomes are identical, i.e., $\mathrm{He}(\Phi') = [\mathrm{He}(\Phi)]'$. (This is a consequence of the general fact that the formation of transvectants commutes with a linear change of variables. Of course, it can also be checked by a brute-force calculation.) Since the Hessian is of degree 2 in the a_i, and degree 4 in the $\mathbf{x}$, it is called a covariant of degree-order $(2, 4)$. A covariant of order 0, i.e., one which contains no $\mathbf{x}$-terms, is called an invariant. For instance,

$$(\Phi, (\Phi, \Phi)_2)_4 = a_0\, a_2\, a_4 - \frac{3}{8}\, a_1^2\, a_4 - \frac{3}{8}\, a_0\, a_3^2 + \frac{1}{8}\, a_1\, a_2\, a_3 - \frac{1}{36}\, a_2^3,$$

is an invariant of degree 3. It is a foundational theorem in the subject that every covariant is expressible as a compound transvectant; that is to say, it can be written as a linear combination of terms of the form

$$(\ldots (\Phi, (\Phi, \Phi)_{r_1}))_{r_2}, \ldots)_{r_k}.$$

Any invariant of binary quartics is a polynomial in the two fundamental invariants $(\Phi, \Phi)_4$ and $(\Phi, (\Phi, \Phi)_2)_4$. A similar statement is true of covariants, but the corresponding list is longer.

5.2. The expression $\mathrm{He}(\Phi)$ is identically zero, if and only if Φ is the fourth power of a linear form. This illustrates the principle that any (topologically closed) property of a polynomial which is unaffected by a linear change of variables is equivalent to the vanishing of a finite number of covariants.[3]

As another illustration, if $\alpha_i\, x_1 + \beta_i\, x_2, (i = 1, \dots, 4)$ are the linear factors of Φ, then $(\Phi, (\Phi, \Phi)_2)_4$ is identically zero exactly when the four points $[\alpha_i, \beta_i] \in \mathbb{P}^1$ are harmonic, i.e., their cross-ratio in some order is -1.

5.3. All of this carries over to polynomials of arbitrary degree d, but the size and complexity of the minimal set of covariants (the so-called 'fundamental system') grow rapidly with d. Our immediate interest lies in the case $d = 6$, when the fundamental system has a total of five invariants, namely one each in degrees $2, 4, 6, 10, 15$. We will denote them by I_2, I_4 etc. Explicit transvectant expressions for the I_r are given in [5, p. 156], but we will not reproduce them here.

Now, to return to the subject of R-configurations, we are looking for covariants which vanish on the binary sextic

$$G_t = \underbrace{x_1\, x_2\, (x_1 - x_2)\, (x_1 + x_2)}_{\Theta}\, \underbrace{(x_1 - t\, x_2)\, (x_1 - \varphi(t)\, x_2)}_{\Delta_t}, \tag{9}$$

irrespective of the value of t. There is no general procedure which is assured to solve such a problem. However, let us take two plausible decisions at the outset:

(1) It will be easier to look for invariants, rather than arbitrary covariants.
(2) The decomposition $G_t = \Theta\, \Delta_t$ is likely to be helpful, especially since $(\Theta, (\Theta, \Theta)_2)_4 = 0$ by the harmonicity of Θ.

These decisions will eventually be vindicated by the fact that they lead to a complete solution. Had this not happened, one would have to start anew and try another strategy. There is no prior guarantee of success.

5.4. Each invariant of G_t is expressible[4] as a polynomial in

(1) the individual invariants of Θ and Δ_t, together with
(2) joint invariants of Θ and Δ_t.

According to the list given in [5, p. 168], the individual invariants are

$$\begin{aligned} &\theta_{20} = (\Theta, \Theta)_4, \ \theta_{30} = (\Theta, (\Theta, \Theta)_2)_4 \quad \text{for } \Theta; \\ &\delta_{02} = (\Delta_t, \Delta_t)_2, \quad \text{for } \Delta_t; \end{aligned}$$

and the joint ones are

$$\beta_{12} = (\Theta, \Delta_t^2)_4, \quad \beta_{22} = (H, \Delta_t^2)_4, \quad \beta_{33} = (T, \Delta_t^3)_6,$$

[3] This can be made precise as follows: The space of $\mathbb{P}^m$ of binary m-ics has coordinate ring $S = \mathbb{C}[a_0, \dots, a_m]$. The action of $SL(2)$ endows S with the structure of a graded representation. The locus of polynomials which satisfy a certain invariant property is an $SL(2)$-stable subvariety $X \subseteq \mathbb{P}^m$, whose ideal $I_X \subseteq S$ is a subrepresentation. Since S is a noetherian ring, we can choose a finite number of covariants whose coefficients generate this ideal.

[4] This would technically be true of any sextic form, but there is nothing to be gained by chopping up an arbitrary sextic into a quartic and a quadratic. This is worth doing here precisely because Θ and Δ_t are simpler than in the general case.

where $H = (\Theta, \Theta)_2$ and $T = (\Theta, H)_1$. The notation is such that if ζ stands for any of the letters θ, δ or β, then ζ_{ij} is of degree i in Θ and j in Δ_t.

In our case we have $\theta_{20} = \frac{1}{2}$, and $\theta_{30} = 0$. The remaining invariants are also easy to calculate; they are as follows:

$$\delta_{02} = -\tfrac{1}{2}\,\tfrac{(t^2+1)^2}{(t+1)^2}, \quad \beta_{12} = \tfrac{1}{2}\,\tfrac{(t^2+2t-1)\,(t^2-2t-1)}{(t+1)^2}, \quad \beta_{22} = \tfrac{\delta_{02}}{3}, \quad \beta_{33} = -\tfrac{1}{4}\,\tfrac{t(t-1)(t^2+1)}{(t+1)^2}. \tag{10}$$

This implies that $\beta_{33}^2 = \frac{1}{32}(\delta_{02}\beta_{12}^2 - \delta_{02}^3)$, and hence β_{22}, β_{33}^2 are, in effect, redundant. All of this simplifies things considerably.

5.5. The actual derivation of the formulae for I_r needs the symbolic calculus, as explained in [4] or [5]. Such calculations are tedious and often unpleasant to read through, hence we will sketch the derivation of I_2 as an example, and leave the rest as exercises for the patient reader.

We will follow the recipe of [4, §3.2.5]. Write $\Theta = a_{\mathbf{x}}^4 = b_{\mathbf{x}}^4$, and $\Delta_t = p_{\mathbf{x}}^2 = q_{\mathbf{x}}^2$, where a, b, p, q are symbolic (or *umbral*) letters. Then $I_2 = (a_{\mathbf{x}}^4\, p_{\mathbf{x}}^2, b_{\mathbf{x}}^4\, q_{\mathbf{x}}^2)_6$ is a sum of $6! = 720$ terms, which are of three kinds:

- 48 terms of the form $(a\,b)^4\,(p\,q)^2$,
- 288 terms of the form $(a\,b)^2(a\,q)^2\,(p\,b)^2$, and
- 384 terms of the form $(p\,q)\,(a\,q)\,(p\,b)(a\,b)^3$.

We have identities

$$\begin{aligned} (a\,b)^4\,(p\,q)^2 &= \theta_{20}\,\delta_{02}, \\ (a\,b)^2(a\,q)^2\,(p\,b)^2 &= \frac{1}{3}\,\theta_{20}\,\delta_{02} + \beta_{22}, \\ (p\,q)\,(a\,q)\,(p\,b)(a\,b)^3 &= \frac{1}{2}\,\theta_{20}\,\delta_{02}. \end{aligned}$$

The first is immediate from the definition. The second and the third follow by a straightforward expansion after using the Plücker syzygy $(a\,q)\,(p\,b) = (a\,p)\,(q\,b) + (a\,b)\,(p\,q)$. And then,

$$I_2 = \frac{1}{720}\,[(48 + 288/3 + 384/2)\,\theta_{20}\,\delta_{02} + 288\,\beta_{22}] = \frac{7}{15}\,\theta_{20}\,\delta_{02} + \frac{2}{5}\,\beta_{22},$$

which is the required formula. As it stands, it is applicable to any binary sextic written as a product of a quartic and a quadratic. But now we can use the simplifications in (10) to get

$$I_2 = \frac{11}{30}\,\delta_{02}. \tag{11}$$

5.6. With rather more work of the same kind, one deduces the following formulae[5] for the remaining invariants:

[5] The expression for I_{15} is very intricate. We can afford to omit it here, because it won't be needed in this calculation.

$$
\begin{aligned}
I_4 &= \frac{2}{1125}\,\beta_{12}^2 + \frac{124}{5625}\,\delta_{02}^2, \\
I_6 &= \frac{91}{253125}\,\delta_{02}\,\beta_{12}^2 + \frac{98}{1265625}\,\delta_{02}^3, \\
I_{10} &= \frac{416}{284765625}\,\delta_{02}\,\beta_{12}^4 + \frac{1141}{284765625}\,\delta_{02}^3\,\beta_{12}^2 - \frac{1372}{7119140625}\,\delta_{02}^5.
\end{aligned}
$$

Notice that each I_r is expressible as a polynomial in only two 'variables' $x = \delta_{02}$ and $y = \beta_{12}$. It follows that $I_2^3, I_2\,I_4, I_6$ are linear combinations of the two-element set $\{x^3, x\,y^2\}$, and hence must be linearly dependent. The actual dependency relation is easily found by solving a set of linear equations; it turns out to be

$$\underbrace{4032\,I_2^3 - 25025\,I_2\,I_4 + 45375\,I_6}_{\mathbf{U}_6} = 0. \tag{12}$$

Thus we have found a degree 6 invariant vanishing on G_t. The readers may wish to convince themselves that a parallel argument gives nothing in degrees 2 or 4.

5.7. We can use the same line of argument to find another such invariant in degree 10. (As before, there is nothing new to be found in degree 8.) The space of degree 10 invariants for binary sextics is spanned by the six elements

$$I_2^5, \quad I_2^3\,I_4, \quad I_2\,I_4^2, \quad I_2^2\,I_6, \quad I_4\,I_6, \quad I_{10}. \tag{13}$$

It is contained in the span of the three-element set $\{x\,y^4, x^3\,y^2, x^5\}$, and hence there must be three linearly independent invariants of degree 10 vanishing on G_t. The identities $\mathbf{U}_6\,I_2^2 = \mathbf{U}_6\,I_4 = 0$ account for two of these, which leaves room for a new invariant which is not a multiple of $\mathbf{U}_6$. Once again, a routine calculation in linear algebra shows that one can take it to be

$$\underbrace{358278336\,I_2^2\,I_6 - 2772533775\,I_4\,I_6 + 6933745\,I_2\,I_4^2 + 1207483200\,I_{10}}_{\mathbf{U}_{10}} = 0.$$

We have arrived at the following statement:

Proposition 9. For an arbitrary t, we have $\mathbf{U}_6(G_t) = \mathbf{U}_{10}(G_t) = 0$. $\square$

Let $\mathcal{Y} \subseteq \mathbb{P}^6$ tentatively denote the 4-dimensional variety defined by the equations $\mathbf{U}_6 = \mathbf{U}_{10} = 0$. By Bézout's theorem, $\mathcal{Y}$ has degree $6 \times 10 = 60$. Now $\mathcal{R} \subseteq \mathcal{Y}$ by the proposition, and since they have the same degrees, we must have $\mathcal{R} = \mathcal{Y}$. We have proved the following:

Theorem 10. Let Φ be a binary sextic representing a set of six distinct points $\Gamma \subseteq \mathcal{K}$. Then Γ is in R-configuration, if and only if $\mathbf{U}_6(\Phi) = \mathbf{U}_{10}(\Phi) = 0$. $\square$

We have $I_{\mathcal{R}} = (\mathbf{U}_6, \mathbf{U}_{10})$, i.e., $\mathcal{R}$ is an ideal-theoretic complete intersection. This implies that any covariant which vanishes on $\mathcal{R}$ is expressible in the form $f\,\mathbf{U}_6 + f'\,\mathbf{U}_{10}$ for some f, f'. Hence there is no such essentially new covariant waiting to be found.

Thus we have completely succeeded in finding invariant-theoretic necessary and sufficient conditions which characterise the R-configuration. A large share of our success is due to the fact that the product of degrees of the two invariants turned out to be exactly the degree of $\mathcal{R}$. In this we have been fortunate, to the extent that such a term has any meaning in mathematics.

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A Simple Synthetic Proof of Lemoine's Theorem

Tran Quang Hung

Abstract. Using similar triangles and cyclic quadrilaterals, we shall give a simple synthetic proof of the Lemoine's theorem that the symmedian point of a triangle is the unique point which is the centroid of its own pedal triangle.

This article is to give a new proof of Lemoine's theorem on the symmedian point of a triangle. The symmedian point K of a triangle ABC is the isogonal conjugate of its centroid G.

Lemoine's Theorem. *Given a triangle ABC, a point P is the centroid of its own pedal triangle with reference to ABC if and only if it is the symmedian point of triangle ABC.*

Proof. ($\Leftarrow$) Let P be the symmedian point K of ABC, and M the midpoint of BC, N the reflection of G in M. Then $BGCN$ is a parallelogram, and that the quadrilaterals $KEAF$, $KDBF$, and $KDCE$ are cyclic.

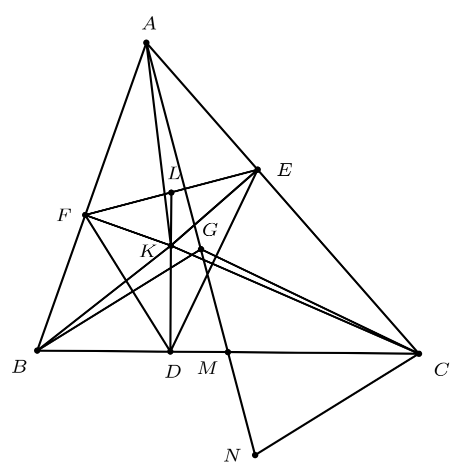

Figure 1

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The author thanks Paul Yiu for his help in the preparation of this note.

Chasing angles, we have

$$\begin{aligned}\angle CNG = \ & \angle BGN = \ \angle GAB + \angle GBA \\ = \ & \angle KAC + \angle KBC = \ \angle KFE + \angle KFD \\ = \ & \angle DFE, \end{aligned} \tag{1}$$

and

$$\begin{aligned}\angle NCG = \ & \angle NCB + \angle BCG = \ \angle GBC + \angle BCG \\ = \ & \angle KBA + \angle KCA = \ \angle KDF + \angle KDE \\ = \ & \angle FDE. \end{aligned} \tag{2}$$

From (1) and (2), it follows that triangles CGN and DEF are similar. Let the line DK intersect EF at L. Then

$$\angle MCN = \angle GBC = \angle KBA = \angle KDF = \angle FDL.$$

This means triangles CMN and DLF are similar, and there is a *similarity* transforming C, G, N, M to D, E, F, L respectively. Since M is the midpoint of GN, L is the midpoint of EF. This means that the line DK bisects EF. Similarly, the lines EK and FK bisect FD and DE respectively. Hence, K is the centroid of triangle DEF.

($\Rightarrow$) Suppose P is the centroid of its own pedal triangle triangle DEF. Let Q be the isogonal conjugate of P with respect to triangle ABC. Extend AQ to N such that CN is parallel to QB. Note that the quadrilaterals $PEAF$, $PFBD$, and $PDCE$ are cyclic.

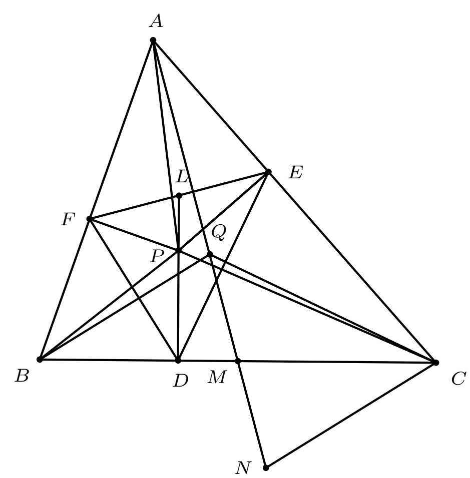

Figure 2

By angle chasing,

$$\begin{aligned}\angle CNQ &= \angle BQN = \angle QAB + \angle QBA\\ &= \angle PAC + \angle PBC = \angle PFE + \angle PFD\\ &= \angle DFE, \end{aligned} \tag{3}$$

and

$$\begin{aligned}\angle NCQ &= \angle NCB + \angle BCQ = \angle QBC + \angle QCB\\ &= \angle PBA + \angle PCA = \angle PDF + \angle PDE\\ &= \angle FDE. \end{aligned} \tag{4}$$

From (3) and (4), we deduce that the triangles CQN and DEF are similar. Let the line DP intersect EF at L. Because P is the centroid of triangle DEF, L is the midpoint of EF. Let the line CB intersect QN at M. Then

$$\angle MCN = \angle QBC = \angle PBA = \angle PDF = \angle LDF.$$

This means the triangles CMN and DLF are similar. There is a similarity transforming D, E, F, L to C, Q, N, M respectively. Since L is the midpoint of EF, M is the midpoint of QN. Since CN is parallel to BQ, the triangles BMQ and CMN are congruent. This implies that M is the midpoint of BC, and the line AQ bisects BC. A similar proof shows that the line BQ bisects the segment AC. Hence Q is the centroid of triangle ABC, and P, being the isogonal conjugate of Q, is the symmedian point of ABC. □

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An Improved Inequality for the Perimeter of a Quadrilateral

Elliott A. Weinstein and John D. Klemm

Abstract. We demonstrate that a new inequality for the perimeter of a quadrilateral is strict even in the nonconvex case.

In [2] the following problem was proposed:

> Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that $|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|$. (Here $|XY|$ denotes the distance from X to Y.)

In words, the perimeter of a convex quadrilateral is at least equal to the sum of the diagonals plus twice the length of the line segment connecting their midpoints. The solution [3] reveals that the statement is true even if the word *convex* is removed. In [5] it was stated without detailed proof that for a convex quadrilateral, the inequality is strict. We show here that the inequality is strict for any nondegenerate quadrilateral (i.e., for which no three vertices are collinear).

Theorem 1. *For any nondegenerate quadrilateral $ABCD$, let E be the midpoint of AC, and let F be the midpoint of BD. Then*

$$|AB| + |BC| + |CD| + |DA| > |AC| + |BD| + 2|EF|. \tag{1}$$

Proof. The solution (see [3]) to the original problem even with *convex* removed follows immediately by letting A, B, C, $D \in \mathbb{C}$ and $x = A - B$, $y = B - C$, $z = C - D$ in Hlawka's inequality

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|.$$

(Note: $|A - B|$ is equivalent to $|AB|$, and $|C - D + A - B| = 2\left|\frac{A+C}{2} - \frac{B+D}{2}\right| = 2|EF|$.) We now prove strictness. In Proof 1 of [1], multiplying both sides of Hlawka's inequality by $|x| + |y| + |z| + |x + y + z|$ leads to the equivalent form

$$\begin{aligned}
&(|x| + |y| - |x + y|)(|z| - |x + y| + |x + y + z|)\\
&+ (|y| + |z| - |y + z|)(|x| - |y + z| + |x + y + z|)\\
&+ (|z| + |x| - |z + x|)(|y| - |z + x| + |x + y + z|) \geq 0,
\end{aligned}$$

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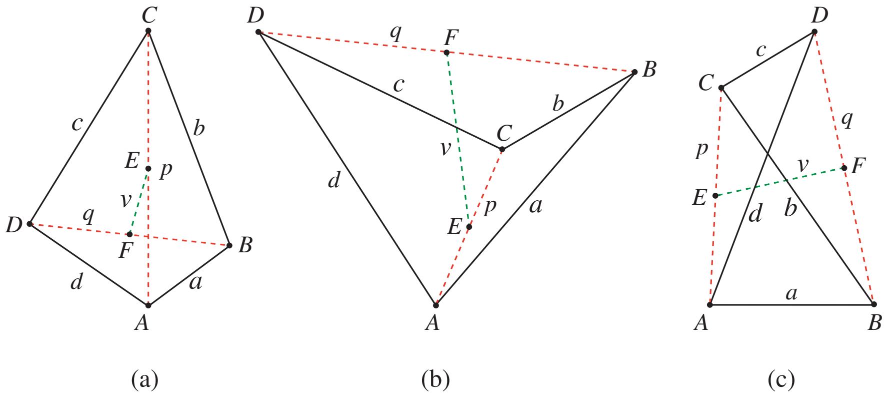

Figure 1

and since each of the six factors is nonnegative by the triangle inequality, Hlawka's inequality is proved. Inserting the given values,

$$\begin{aligned}&(|A-B|+|B-C|-|A-C|)(|C-D|-|A-C|+|A-D|)\\&+(|B-C|+|C-D|-|B-D|)(|A-B|-|B-D|+|A-D|)\\&+(|C-D|+|A-B|-|C-D+A-B|)\\&\quad\cdot(|B-C|-|C-D+A-B|+|A-D|)\geq 0.\end{aligned}$$

In order for equality to be attained, at least one factor in each of the three pairs of factors above must be equal to 0. Inspection reveals that for the first two pairs of factors, a factor is 0 only if three vertices of the quadrilateral are collinear. Since we are excluding degenerate cases, not just one but in fact the first two products are strictly positive. This proves the strictness in (1). □

There is a geometric interpretation, which may be easier to visualize using more convenient notation. Let $|AB|=a$, $|BC|=b$, $|CD|=c$, $|DA|=d$, $|AC|=p$, $|BD|=q$, $|EF|=v$. Then Theorem 1 can be written as

Theorem 1* *For any nondegenerate quadrilateral with consecutive sides a, b, c, d, diagonals p, q, and v the length of the line segment connecting the midpoints of the diagonals,*

$$a+b+c+d>p+q+2v. \tag{2}$$

The equivalent form of Hlawka's inequality is now

$$(a+b-p)(c-p+d) \tag{3}$$

$$+(b+c-q)(a-q+d) \tag{4}$$

$$+(c+a-2v)(b-2v+d)\geq 0. \tag{5}$$

Whether the quadrilateral is convex as in (a), concave as in (b), or crossing (i.e., nonsimple) as in (c) of Figure 1, the three terms in each factor of (3) and (4) are sides of a triangle and so each of these factors is strictly positive by the triangle inequality.

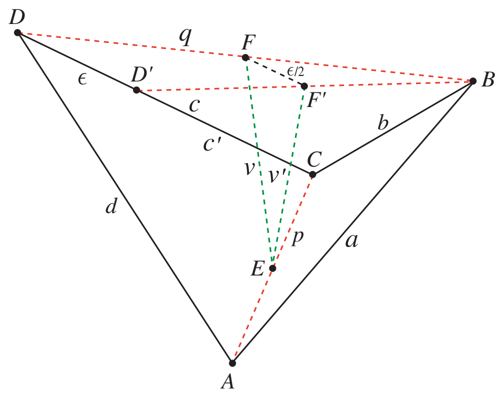

Figure 2

We now show that with but one exception, the product in (5) is strictly positive as well.

Theorem 2. *For any nondegenerate quadrilateral with consecutive sides a, b, c, d, let v be the length of the line segment connecting the midpoints of the diagonals. Then $a + c > 2v$ and $b + d > 2v$, except for a crossing quadrilateral with a pair of parallel sides, in which case the inequality involving the two parallel sides becomes an equality.*

Proof. We know from the proof of Theorem 1 that $a + c \geq 2v$ and $b + d \geq 2v$, so all that remains is to ascertain when equality holds. For $v = 0$ these inequalities obviously are strict, so assume $v > 0$ and $a + c = 2v$. Reverting to the earlier notation used for the third product pair in the proof of Theorem 1, $|C - D| + |A - B| = |C - D + A - B|$ implies that $C - D = \lambda(A - B)$, where $\lambda > 0$, which implies that the quadrilateral is crossing and $AB \parallel CD$, as claimed. Conversely, for any crossing quadrilateral with parallel sides AB and CD, since EF is the midsegment of trapezoid $ABDC$, $|AB| + |CD| = 2|EF|$, that is, $a + c = 2v$. Since only one pair of sides can be crossing, the other inequality remains strict. Relabeling the vertices gives $b + d = 2v$ as the other possibility.

This result also can be proved geometrically, as follows. For a concave quadrilateral $ABCD$ as in Figure 2, choose any point D' on CD not an endpoint, let $c' = |CD'|$ and $\epsilon = |DD'| = c - c'$, and let F' be the midpoint of BD'. Because FF' is the midline of triangle BDD' parallel to DD', $|FF'| = \frac{\epsilon}{2}$. Let $|EF'| = v'$. Then from the triangle inequality applied to $EF'F$, $v' + \frac{\epsilon}{2} > v$ and so $2v' + \epsilon > 2v$. We know from the proof of Theorem 1 that for quadrilateral $ABCD'$, $a + c' \geq 2v'$. Then $a + c = a + c' + \epsilon \geq 2v' + \epsilon > 2v$. An identical argument shows the same for $b + d$. This demonstrates that equality is impossible. Note that this proof fails for either a convex or a crossing quadrilateral since, unlike in the concave case, E, F, and F' may be collinear, but we can restore the proof for these quadrilaterals by observing that E, F, and F' are collinear if and only if the quadrilateral $ABCD$ or $ABDC$ (each for suitably labeled vertices) is a trapezoid (use the fact that $FF' \parallel DD'$). In the first case, the strict inequality still holds, since the sum of the lengths of the parallel sides equals twice the length of the midsegment (which properly contains EF). In the second case, $ABCD$ is a crossing quadrilateral with a pair of parallel sides, and then EF is the midsegment of $ABDC$, hence the exception. Provided neither of these is the case, we can proceed as in the above proof. For an alternative geometric construction in the convex case (and for which the trapezoid presents no real difficulty), Figure 1 in [5] shows all at once that the three terms in every factor of (3), (4), and (5) are sides of a triangle, and so both factors in (5) are strictly positive as well. $\square$

The inequality (2), being true for all (nondegenerate) quadrilaterals, may have application to four-bar linkages in mechanics. See [4] for an introduction and references.

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Equilateral Jacobi Triangles

Glenn T. Vickers

Abstract. Given any triangle and any three angles (α, β, γ), the Jacobi construction produces a triangle (here chosen to be equilateral) in perspective with the first. For scalene triangles there are a finite number of values of β, γ for which two values of α give equilateral triangles. The construction of Morley has this property but so do certain other configurations.

1. Introduction.

The 2000 year gap between the publication of Euclid (c. 300 BC) and the geometric discoveries of Napoleon (c. 1820), Jacobi (1825) and Morley (1899) is as amazing as the beauty of these later results. An investigation into the relationship between theses three theorems is presented which not only re-discovers them but reveals other configurations with similar properties (if less elegance).

The starting point is the geometrical theorem of Jacobi. With ABC being any triangle, construct the points P, Q, R so that

$$\angle RAB = \angle QAC = \alpha, \angle PBC = \angle RBA = \beta \text{ and } \angle QCA = \angle PCB = \gamma.$$

These points form a *Jacobi triangle* for ABC and Jacobi's theorem states that the lines $AP,\ BQ$ and CR are concurrent (at the Jacobi point K), see Figure 1. Proofs of this result are readily available, e.g. [1] and [2]. Let Δ be the area and a, b, c be the lengths of the sides of ABC. With

$$X = 2\Delta(\cot\alpha + \cot A),\ Y = 2\Delta(\cot\beta + \cot B),\ Z = 2\Delta(\cot\gamma + \cot C),$$

the coordinates of the key points are (using areal coordinates based upon ABC)

$$P(-a^2, Z, Y),\ Q(Z, -b^2, X),\ R(Y, X, -c^2),\ K(1/X, 1/Y, 1/Z).$$

The first question to be addressed is 'when is PQR an equilateral triangle?' The theorem associated with the name of Napoleon asserts that this occurs when

$$\alpha = \beta = \gamma = \pm\pi/6 \tag{1}$$

and Morley's theorem tells us that it also occurs when

$$\alpha = -A/3,\ \beta = -B/3,\ \gamma = -C/3. \tag{2}$$

Indeed there are a total of 18 configurations predicted by this last theorem since various internal and external trisections of the angles may be combined. Many proofs of Morley's theorem are available (some quite short e.g. [3] and [4]) but the contrast between the simplicity of its statement and the intricacies of its proof is remarkable.

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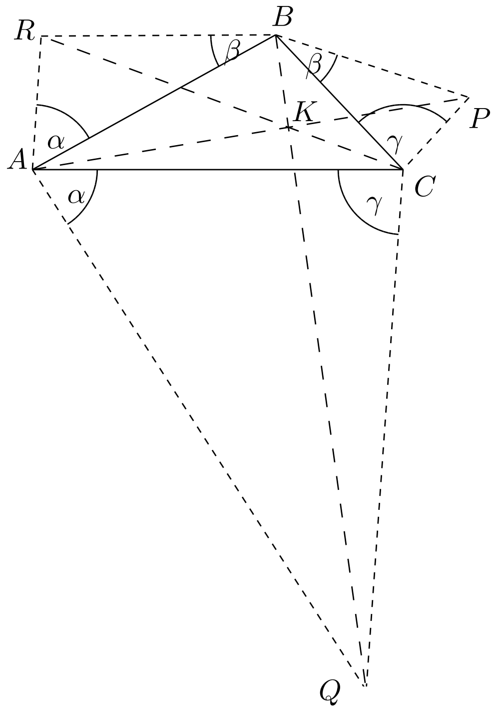


Figure 1. The points P, Q, R are constructed on a base triangle ABC with pairs of angles equal as shown. Jacobi's theorem states that AP, BQ, CR are concurrent at K.

2. The Solution Curves.

With ABC a given triangle, the conditions for PQR being equilateral are simply

$$QR = RP = PQ$$

and these impose two conditions upon the three variables α, β, γ (or X, Y, Z). Thus a 1-parameter family of solutions is to be expected.

The case when ABC is itself equilateral is, not surprisingly, rather special and a consideration of this case is in Appendix A. Another case which lends itself to easy analytic treatment is when $A = B$ and $\alpha = \beta$. This is considered in Appendix B.

Figures 2, 3 and 4 show the curves in the spaces $(\beta, \gamma),\ (\gamma, \alpha),\ (\alpha, \beta)$ respectively, when these angles are constrained by the requirement that PQR is equilateral. Clearly adding a multiple of π to any of α, β, γ does not affect the configuration. In the figures, each of these angles lies in $(0, \pi)$ but some of the formulae given will produce values outside this interval, e.g. equations (1) and (2). All of the figures (except those in the Appendices) refer to the arbitrarily chosen triangle with

$$A = 0.8,\ B = 0.5,\ C = \pi - 1.3.$$

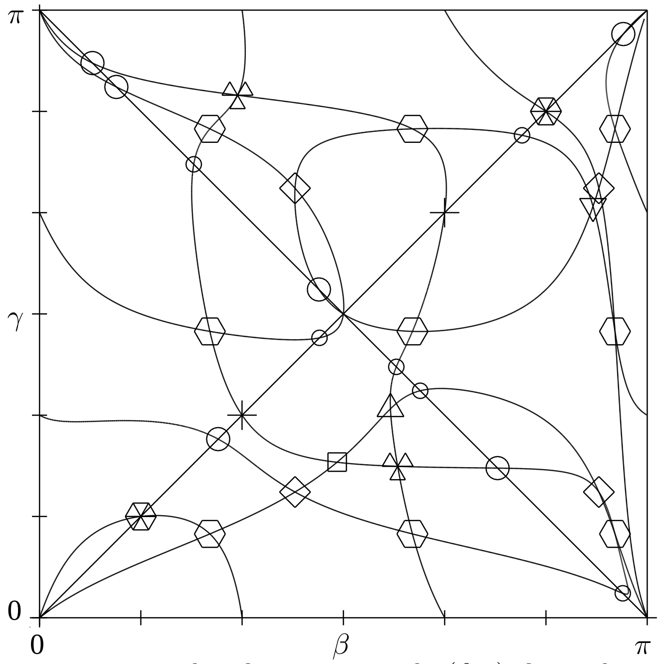
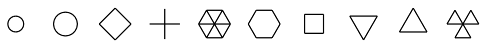


Figure 2. The solution curves in the (β, γ) plane. The intersection points are coded as shown below.



Class: 1 2 3 4 5 6 7 8a 8b 9

Most of the formulae presented will refer to solutions in the (β, γ)-plane, i.e. Figure 2. A cyclic permutation will provide the complete family of solutions. It will be noticed that both diagonal lines occur on each of the figures. Now when $\beta + \gamma = \pi$, the point P is at infinity and if either Q or R is also at infinity then there is a degenerate solution with PQR having infinite size. Such situations occur when

$$-\alpha = \beta = \gamma \text{ or } \alpha = -\beta = \gamma \text{ or } \alpha = \beta = -\gamma.$$

The complexity of the solution curves is remarkable, not only do they follow tortuous paths but they intersect themselves. Except for the appendices, only the points of self-intersection will be considered from here on. At such a point there may be no, one or two proper solutions, i.e. equilateral triangles which have non-zero, finite size.

3. Intersection Points with No Proper Solutions.

3.1. *Triangles of Infinite Size.*
Class 0: $\alpha = \beta = \gamma = \pi/2$**.** Here each of P, Q, R is at infinity.

Class 1: $-\alpha = \beta = \gamma = A/2 \pm \pi/3$**.**
In Figure 2 there are two such points on $\beta = \gamma$ and four on $\beta = -\gamma$ (or $\beta + \gamma = \pi$). These values are found as limiting values as a solution curve approaches one of the

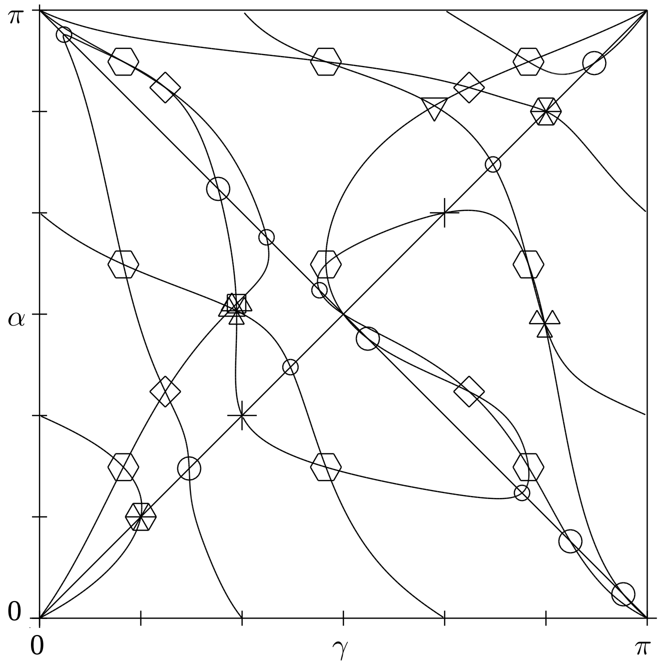


Figure 3. The solution curves in the (γ, α) plane

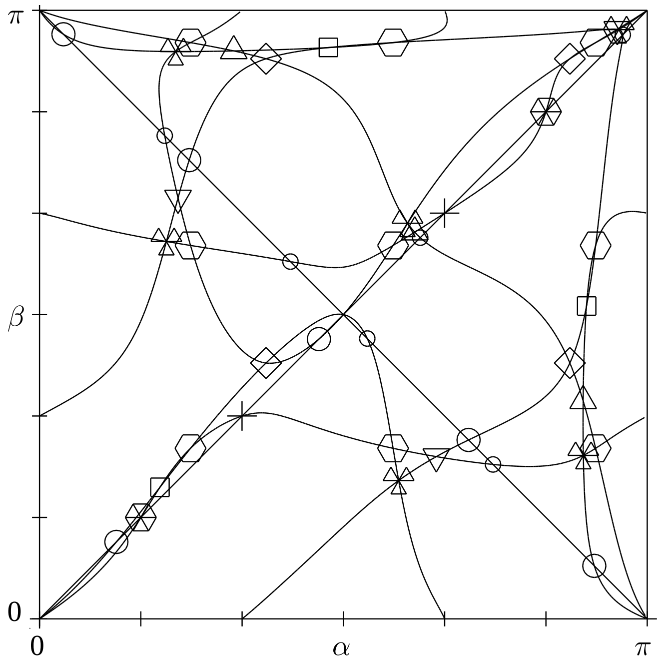


Figure 4. The solution curves in the (α, β) plane.

diagonals. When such a curve crosses $\beta = \gamma$, the value of α is $-\beta$ but when it crosses $\beta = -\gamma$ the value of α is β at two of these points and $-\beta$ at the other two.

Class 2: $-\alpha = \beta = \gamma = A/2 \pm \pi/6$**.**
This gives a set of six points in a similar manner to the previous class.

3.2. *Triangles of Zero Size.*
Class 3: Incentre and Ex-centres.
When

$$\begin{aligned} &\alpha = -A/2,\ \beta = -B/2, \gamma = -C/2 \\ \text{or}\quad &\alpha = -A/2,\ \beta = (\pi - B)/2, \gamma = (\pi - C)/2 \end{aligned}$$

the lines AP, BQ, CR become the interior or exterior bisectors of the angles of ABC. There are four such solutions and the triangles PQR are just points (the incentre and the three ex-centres). The limiting value of α as a solution approaches one of these four points in Figure 2 is $-A/2$ or $(\pi - A)/2$. All four solutions are intersection points in each of the three solution spaces.

4. Intersection Points with One Proper Solution.

Class 4: $\alpha = 0,\ \beta = \gamma = \pm\pi/3$.
The triangle PQR is an equilateral triangle with one side being a side of ABC. The constructed triangle may be interior or exterior to ABC. Note that the values $\alpha = 0, \beta = \gamma = \pi/3$ give an intersection point in Figure 2 but not Figures 3 or 4. This is in contrast to Classes 1 and 2 where each of the six points is an intersection point in each of the three Figures.

Class 5: Napoleon Triangles.
The equilateral triangle associated with the name Napoleon is usually defined to be that formed by the centroids of the three exterior (or interior) equilateral triangles referred to in Class 4. Clearly this is equivalent to

$$\alpha = \beta = \gamma = \pm\pi/6$$

which give the same two points in each Figure.

5. Intersection Points with Two Proper Solutions.

This is by far the most interesting situation. A representative Figure is given for each of the four remaining classes. Each corresponds to a solution in Figure 2. Thus for a pair of values for β, γ there are two values of α which produce equilateral triangles; labelled PQR and $PQ'R'$. Of necessity, B, R, R' and C, Q, Q' are each collinear sets of points. Only in Figure 7 are these lines drawn.

Class 6: Morley Triangles.
With

$$\alpha_i = (i\pi - A)/3,\ \beta_i = (i\pi - B)/3,\ \gamma_i = (i\pi - C)/3\ (i = -1, 0, 1)$$

the values $(\alpha, \beta, \gamma) = (\alpha_i, \beta_j, \gamma_k)$ will give an equilateral triangle provided that $i + j + k \not\equiv 1 (\text{mod } 3)$. Hence there are 18 Morley triangles, each of the 9 Morley points

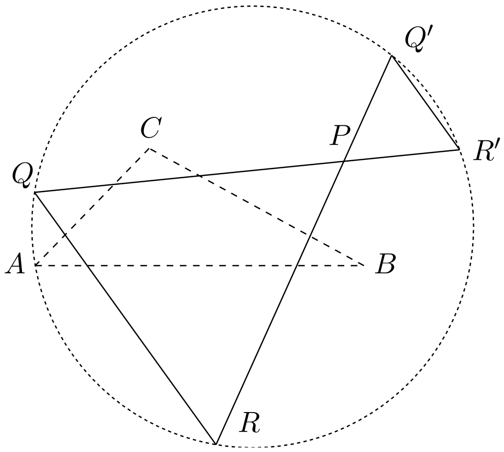


Figure 5. A typical Morley configuration with
$\beta = (\pi - B)/3$ and $\gamma = (\pi - C)/3$.
The values of α are $-A/3$ and $(\pi - A)/3$.

in Figures 2-4 giving rise to two solutions. Figure 5 shows a typical configuration with the two equilateral triangles for one of the Morley points in Figure 2. The two branches of solutions through a Morley point have, as their limiting values, the two values given above. The sets of points P, Q, R' and P, Q', R are always collinear but their ordering along the line is not always the same. It is to be noted that the points $AQQ'R'R$ lie on a circle.

Class 7:

$$\alpha = \pm\pi/6,\ \beta + \gamma = B + C,\ \frac{\sin(B + 2\beta)}{\sin\beta} = \frac{\sin(C + 2\gamma)}{\sin\gamma}. \qquad (3)$$

This is perhaps the most remarkable of the non-classical cases. A typical configuration is shown in Figure 6 and it resembles a Morley solution in that the points P, Q, R' and P, Q', R each form collinear sets and the points $AQQ'R'R$ lie on a circle (see Figures 5 and 6). But the values of α coincide with those associated with Napoleon. So we have what might be described as a Napoleon-Morley hybrid. Further evidence that this Class is related to Morley is provided by the observation that, for Morley, each of

$$\frac{\sin(A + 2\alpha)}{\sin\alpha},\ \frac{\sin(B + 2\beta)}{\sin\beta},\ \frac{\sin(C + 2\gamma)}{\sin\gamma}$$

is ± 1, which implies that at least two are equal and this is true here.

If (β, γ) is a Class 7 intersection point in Figure 2, the values of α being α_1 and α_2, then neither (α_1, β) nor (α_2, β) is an intersection point in Figure 4. This contrasts with the behaviour of Morley points.

The proof of equations (3) consists of verifying that the above conditions do indeed imply that PQR is equilateral. It involves various trigonometric identities

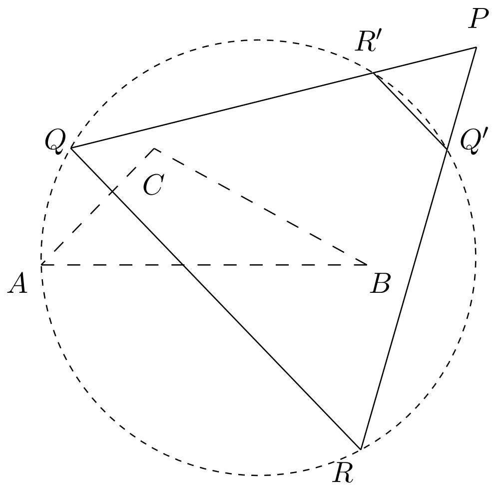


Figure 6. A Class 7 solution showing that $AQQ'RR'$ lie on a circle.

of little interest and is omitted. However, we do mention that

$$\angle QRR' = A = \angle RQQ' = \angle CPB.$$

Classes 8a and 8b:

$$\beta + \gamma = A + \pi \pm \pi/3, \ \frac{\sin(B + 2\beta)}{\sin\beta} = \frac{\sin(C + 2\gamma)}{\sin\gamma} \tag{4}$$

and

$$\sin\alpha\left[\frac{\sin B}{\sin(\alpha+\gamma)} + \frac{\sin C}{\sin(\alpha+\beta)}\right] \\ \pm\frac{\sin A}{\sqrt{3}\sin(\beta+\gamma)}[\cos(C + 2\gamma - \beta) + 2\cos A\cos(C+\gamma)] = 0.$$

Again the existence of such a solution is more interesting than the details of its verification and no proof is included.

If D is defined as the point at which the line $RR'B$ meets $QQ'C$ then the three lines QQ', RR', PD meet at angles of $\pi/3$, see Figure 7.

Class 9:

$$\left(\frac{\cot\alpha + \cot\beta}{\cot\alpha + \cot\gamma}\right)^2 = \frac{\sin C[2\sin A\cot\beta\sin^2\gamma\cos(B+2\beta) - \sin C\sin^2(\beta+\gamma)]}{\sin B[2\sin A\cot\gamma\sin^2\beta\cos(C+2\gamma) - \sin B\sin^2(\gamma+\beta)]}. \tag{5}$$

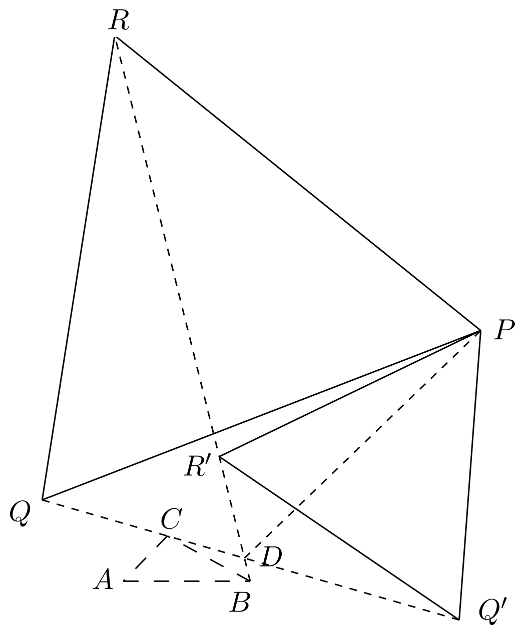


Figure 7. A Class 8 solution showing that QQ', RR', PD meet at angles of $\pi/3$.

and

$$\begin{aligned}
&\frac{\sin C \sin\beta}{\sin B \sin\gamma}\left(3 + \frac{\tan\beta}{\tan\gamma}\right)\cos(A - 2\gamma) \\
&= -2\cos(\beta - \gamma) \\
&= \frac{\sin B \sin\gamma}{\sin C \sin\beta}\left(3 + \frac{\tan\gamma}{\tan\beta}\right)\cos(A - 2\beta).
\end{aligned} \tag{6}$$

Clearly the trigonometric complexity of this solution is of a different order to the other classes and indicates a new approach. A brief description of the technique used is included in Appendix C. The last two equations may be solved for β and γ and then α found from the first.

The configurations do have the interesting property that the lines $Q'R, QR'$ and AP are concurrent and meet at angles of $\pi/3$, see Figure 8. Also a result from the algebraic approach of Appendix C is that $CQ'/CQ = -BR'/BR$.

6. Concluding Remarks.

The nine Morley points in each of Figures 2, 3 and 4 refer to a total of only 18 equilateral triangles. But the number of Class 7 points is 1, 1, 3 in Figures 2, 3, 4 respectively and each of these gives two equilateral triangles without duplication. Indeed, the total number of equilateral triangles for classes 7-9 is 46. But this number may be dependent upon the base triangle ABC.

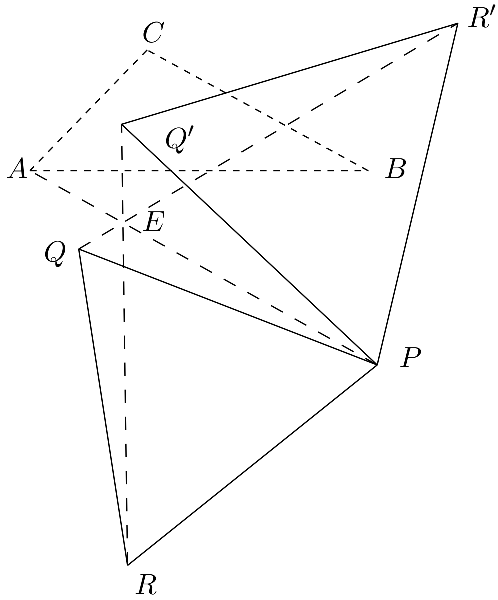


Figure 8. A Class 9 solution showing that $Q'R, QR'$ and AP are concurrent and meet at angles of $\pi/3$.

Not all of the solutions to equations (3), (4) and (5) give valid configurations. For example, the r.h.s. of equation (5) may be negative. However, it is conjectured that (for scalene triangles) all equilateral Jacobi triangles are covered by the above Classes.

The form of the solution presented in Class 9 is quite possibly unduly complicated, a different approach may well produce a simpler answer.

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Appendix A. ABC an Equilateral Triangle.

When ABC and PQR are both equilateral triangles it is claimed that:

(1) When α, β, γ are all distinct then
- K lies on the circumcircle of ABC,
- each of α, β, γ lies in the interval $(\pi/6,\ \pi - \tan^{-1}(\sqrt{3}/5)\ )$,
- P lies on the hyperbola $3x^2 - y^2 - z^2 - yz = 0$.

(2) When $\alpha = \beta \neq \gamma$
- K lies on the line $x = y$,
- the locus of P is the rectangular hyperbola $x^2 - z^2 - yz + zx = 0$.

(3) When $\alpha = \beta = \gamma$ PQR is always equilateral, K is at the circumcentre of ABC and P, Q, R lie on the lines $y = z,\ z = x,\ x = y$ respectively.

1.1. *Proofs.* When $A = B = C = \pi/3$, the coordinates of P, Q, R may be written as

$$P(-2, 1+w, 1+v),\ Q(1+w, -2, 1+u),\ R(1+v, 1+u, -2)$$

where

$$u = \sqrt{3}\cot\alpha,\ v = \sqrt{3}\cot\beta,\ w = \sqrt{3}\cot\gamma.$$

The distances QR, RP, PQ can be expressed as rational functions of u, v, w and the elimination of w (Maple or similar is recommended) from the equations $QR = RP$ and $QR = PQ$ yields either two of u, v, w are equal or

$$u^2 + uv + v^2 + 3u + 3v - 9 = 0. \tag{7}$$

(1) When α, β, γ are distinct, u, v, w are also distinct and (7) implies

$$u^3 + 3u^2 - 9u = v^3 + 3v^2 - 9v.$$

Hence u, v, w are the roots of the cubic equation

$$s^3 + 3s^2 - 9s + q = 0$$

for some value of q. Now this equation has three real, distinct roots provided that $q \in (-27, 5)$. Hence the values $\cot\alpha, \cot\beta, \cot\gamma$ satisfy

$$t^3 + \sqrt{3}t^2 - 3t + p = 0$$

for some $p \in (-3\sqrt{3}, 5\sqrt{3}/9)$ and it follows that each of α, β, γ lies in the interval $(\pi/6,\ \pi - \tan^{-1}(\sqrt{3}/5)\ )$. Furthermore

$$X + Y + Z = a^2(3 + u + v + w)/2 = 0$$

and so K lies on $yz + zx + xy = 0$ which is the circumcircle of ABC.

The coordinates of P are

$$x = -2,\ y = 1 + w,\ z = 1 + v$$

where

$$u + v + w = -3 \text{ and } vw + wu + uv = -9.$$

These imply

$$3x^2 - y^2 - z^2 - yz = 0 \tag{8}$$

which is an hyperbola with centre $(-1, 2, 2)$ on the circumcircle of ABC. Its asymptotes are perpendicular to AB and AC.

(2) The substitution $A = \pi/3$ in (10) gives $\gamma = \alpha$ (which is forbidden here) or

$$\tan\gamma = -\frac{\tan\alpha(\tan\alpha + \sqrt{3})}{5\tan\alpha + \sqrt{3}}.$$

The coordinates of P are

$$x = -2,\ y = 1 + \sqrt{3}\cot\gamma,\ z = 1 + \sqrt{3}\cot\alpha$$

which imply (after eliminating α and γ) that P lies on the rectangular hyperbola

$$x^2 - z^2 - yz + zx = 0. \tag{9}$$

(3) The case $\alpha = \beta = \gamma$ is trivial and is included for completeness.

Appendix B. $A = B$ **and** $\alpha = \beta$**.**

Here we briefly consider the case when ABC is isosceles and the corresponding Jacobi angles are equal. The symmetry of the situation greatly simplifies the trigonometry, for example K has to lie on the median through C. Referring to Figure 9 and using the sine rule, it will be seen that

$$\frac{a}{\sin(\alpha+\gamma)} = \frac{QC}{\sin\alpha} \text{ and } \frac{CR}{\sin(A+\alpha)} = \frac{a}{\sin(\pi/2-\alpha)} = \frac{a}{\cos\alpha}.$$

Thus

$$\frac{\sin(\alpha+\gamma)\sin(A+\alpha)}{\sin 2\alpha} = \frac{CR}{2.QC}.$$

But

$$\frac{CR}{\sin(\pi/3 + A - \gamma)} = \frac{QC}{\sin(\pi/6)}$$

and so

$$\sin(\alpha+\gamma)\sin(A+\alpha) = \sin(2\alpha)\sin(\pi/3 + A - \gamma).$$

There is also a configuration given by replacing $\pi/3$ by $-\pi/3$ and so we have

$$\tan\gamma = \frac{\tan\alpha(\pm\sqrt{3} - \tan\alpha)}{2\tan\alpha \mp \sqrt{3}\tan\alpha\tan A + \tan A}. \tag{10}$$

If the two triangles are denoted by PQR and $P'Q'R'$ then R and R' coincide. Not only are the points P, B, P' collinear but also P, R, Q' (see Figure 9).

For any value of $\alpha = \beta$, there are two values for γ which make PQR equilateral. For example, when $\alpha = \beta = \pi/6$, (10) gives not only the Napoleon value $\gamma = \pi/6$ but also

$$\cot\gamma = -(\sqrt{3} + 3\tan A)/2.$$

There is also a solution with $\alpha = \beta = A = B$ and $\gamma = \pm\pi/6$.

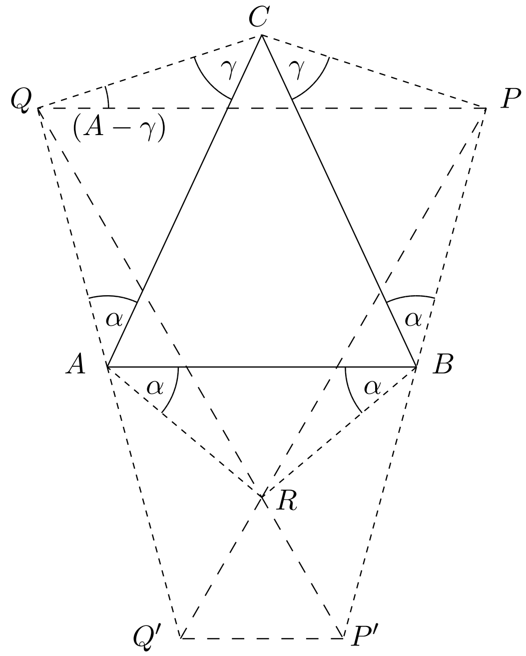

Figure 9. ABC is an isosceles triangle, PQR and $P'Q'R$ are the two equilateral Jacobi triangles associated with ABC when $\alpha = \beta$.

Appendix C. An Outline of the Proof of Class 9.

We use areal coordinates based upon PQR. Let QQ' meet RR' at $D(x_0, y_0, z_0)$. The coordinates of Q' and R' will have the form

$$Q'(x_0, q, z_0) \text{ and } R'(x_0, y_0, r). \tag{11}$$

The requirement that $PQ' = PR'$ gives

$$\frac{q^2 + qz_0 + z_0^2}{(x_0 + q + z_0)^2} = \frac{r^2 + ry_0 + y_0^2}{(x_0 + y_0 + r)^2}$$

and $PQ' = Q'R'$ gives another algebraic condition. With the aid of Maple, it is found that $PQ' = PR' = Q'R'$ when

$$q = \frac{x_0(z_0 - y_0)}{(y_0 - x_0)} \text{ and } r = \frac{x_0(z_0 - y_0)}{(x_0 - z_0)}.$$

These are not the only possibilities. Indeed, here lies an explanation for the different types of intersection points.

Let the circumcircles of PQR and $PQ'R'$ intersect at E (and P). Then each of $\angle Q'ER', \angle R'EP$ and $\angle REP$ is $\pi/3$. Thus the lines QR' and $Q'R$ intersect at E. It is straightforward to verify that E also lies on AP, see Figure 8.

Relative to ABC the coordinates of Q' and R' may be written as

$$Q'(Z, -b^2, X') \text{ and } R'(Y, X', -c^2)$$

and, since the coordinates of P, Q, R are known relative to ABC, these values may be transformed to those in (11). This gives the consistency condition

$$\begin{aligned}
&[a^2(Y+Z+b^2+c^2)-2(Y^2+YZ+Z^2)]X^2\\
&+[2b^2Y^2-4YZ(Y+Z)+2c^2Z^2+2a^2(b^2Y+YZ+c^2Z)-4a^2b^2c^2]X\\
&+a^2b^2c^2(b^2+c^2-3Y-3Z)+2a^2(b^2Y^2+c^2Z^2)\\
&-(b^2-c^2+Y-Z)(b^2Y^2-c^2Z^2)+2YZ(b^2c^2-YZ)=0
\end{aligned}$$

which (reverting to α, β, γ) gives equation (5). This equation also gives

$$\frac{\cot\alpha+\cot\beta}{\cot\alpha'+\cot\beta}=-\frac{\cot\alpha+\cot\gamma}{\cot\alpha'+\cot\gamma}\Rightarrow\ \frac{CQ'}{CQ}=-\frac{BR'}{BR}.$$

Now let $\theta=(X+Y-c^2)/(X+Z-b^2)$ and use this to eliminate X from the basic conditions $PQ=QR=RP$ (expressed in terms of coordinates based upon ABC). The results are a cubic and a quartic in θ. But we know that if θ is a solution then so is $-\theta$ which permits the separation of these equations into even and odd powers of θ. These give (eventually) the equations (6).

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Convexity and Non-Differentiable Singularities in Mortici's Paper on Fermat-Torricelli Points

Martin Celli

Abstract. We give two proofs of Mortici's result on Fermat-Torricelli points for a class of polygons.

In [1], the following generalization of a classical result on the Fermat-Torricelli point of a triangle is proved:

Theorem. *Let T, $A_1, \dots, A_n$ be $n+1$ points in the plane such that*

$$\angle A_1TA_2 = \angle A_2TA_3 = \cdots = \angle A_nTA_1 = \frac{2\pi}{n}.$$

Let $F(M) = A_1M + \cdots + A_nM$. Then for every point M, we have $F(T) \leq F(M)$.

The proof given in [1] involves the complex n-roots of unity. The aim of this note is to present two other proofs. The main argument of each one has already been used in some known proof for the classical case $n = 3$ ([2, 3]). Whereas the first proof presented here makes clear the role of the convexity of the function $F(M)$, the second proof is based on the behavior of $F(M)$ near the vertices A_k.

The first proof. Following [2], we have, for every M, k, $M_0 \neq A_k$,

$$A_kM_0 + \frac{\overrightarrow{A_kM_0}}{A_kM_0} \cdot \overrightarrow{M_0M} = \frac{\overrightarrow{A_kM_0}}{A_kM_0} \cdot (\overrightarrow{A_kM_0} + \overrightarrow{M_0M}) = \frac{\overrightarrow{A_kM_0}}{A_kM_0} \cdot \overrightarrow{A_kM} \leq A_kM$$

by the CBS-inequality. Summing over k, we obtain:

$$F(M_0) + \left(\frac{\overrightarrow{A_1M_0}}{A_1M_0} + \cdots + \frac{\overrightarrow{A_nM_0}}{A_nM_0} \right) \cdot \overrightarrow{M_0M} \leq F(M). \tag{1}$$

We just have to see that, for $M_0 = T$:

$$\frac{\overrightarrow{A_1M_0}}{A_1M_0} + \cdots + \frac{\overrightarrow{A_nM_0}}{A_nM_0} = \overrightarrow{0}.$$

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The inequality (1) means that the graph of $F(M)$ lies above its tangent planes, which is a characterization of convex functions. As a matter of fact, the factor $\frac{\overrightarrow{A_1M_0}}{A_1M_0} + \cdots + \frac{\overrightarrow{A_nM_0}}{A_nM_0}$ is nothing but the gradient of F at the point M_0, since

$$\overrightarrow{\nabla}(A_kM) = \overrightarrow{\nabla}\left(\sqrt{A_kM^2}\right) = \frac{1}{2\sqrt{A_kM^2}}\overrightarrow{\nabla}\left(A_kM^2\right) = \frac{\overrightarrow{A_kM}}{A_kM}.$$

Therefore, we can say that the first proof is based on the fact that every critical point of a convex function corresponds to a minimum.

The second proof. As $F(M)$ is a positive, continuous function, it achieves its minimum value at a point M_0 (and possibly others). Let us assume that $M_0 = A_k$ for some k, for instance, $M_0 = A_1$. As the function $A_2M + \cdots + A_nM$ is differentiable at A_1, we have, for $M \approx A_1$,

$$F(M) = F(A_1) + A_1M + \overrightarrow{U} \cdot \overrightarrow{A_1M} + A_1M\epsilon(M),$$

where

$$\overrightarrow{U} = \overrightarrow{\nabla}_{A_1}(A_2M + \cdot + A_nM) = \frac{\overrightarrow{A_2A_1}}{A_2A_1} + \cdots + \frac{\overrightarrow{A_nA_1}}{A_nA_1}$$

and $\epsilon(M) \longrightarrow 0$ when $M \longrightarrow A_1$. This is equivalent to

$$\frac{F(M) - F(A_1)}{A_1M} = 1 + ||\overrightarrow{U}|| \cos(\overrightarrow{U}, \overrightarrow{A_1M}) + \epsilon(M).$$

For $M \approx A_1$ on the half-line $(A_1, -\overrightarrow{U})$ (for all M in the case $\overrightarrow{U} = \overrightarrow{0}$), we obtain

$$\frac{F(M) - F(A_1)}{A_1M} \approx 1 - ||\overrightarrow{U}||.$$

These computations can also be found in [3], for instance. In Mortici's particular case, we also have

$$||\overrightarrow{U}|| \geq \overrightarrow{U} \cdot \frac{\overrightarrow{TA_1}}{TA_1} = \cos(TA_1A_2) + \cos(TA_1A_3) + \cdots + \cos(TA_1A_n).$$

As the geometric angles of every triangle TA_1A_k add up to π, we have

$$0 \leq TA_1A_k \leq \pi - A_1TA_k \leq \pi,$$

so that $\cos(TA_1A_k) \geq -\cos(A_1TA_k)$. Thus,

$$\begin{aligned} ||\overrightarrow{U}|| \geq\ & -(\cos(A_1TA_2) + \cos(A_1TA_3) + \cdots + \cos(A_1TA_n)) \\ =\ & -\left(\cos\frac{2\pi}{n} + \cos\frac{2 \cdot 2\pi}{n} + \cdots + \cos\frac{(n-1)\cdot 2\pi}{n}\right) \\ =\ & 1, \end{aligned}$$

with equality for $TA_kA_1 = 0$ for all k, which is impossible for $n \geq 3$. So $||\overrightarrow{U}|| > 1$ and, for some M,

$$\frac{F(M) - F(A_1)}{A_1M} \approx 1 - ||\overrightarrow{U}|| < 0,\ F(M) < F(A_1).$$

The function $F(M)$ cannot achieve its minimum value at A_1 or at any A_k, and M_0 cannot be one of the A_k. So M_0 has to be a critical point of $F(M)$. Now, in order to have $M_0 = T$, we just need to prove that T is the only critical point of $F(M)$. As a matter of fact, for every $M \neq T,\ A_1, \ldots,\ A_n$, we have

$$\begin{aligned}
&\left|\left|\vec{\vec{\nabla}}(A_1M + \cdots + A_nM)\right|\right| \\
= &\left|\left|\frac{\overrightarrow{A_1M}}{A_1M} + \cdots + \frac{\overrightarrow{A_nM}}{A_nM}\right|\right| \\
\geq &\left(\frac{\overrightarrow{A_1M}}{A_1M} + \cdots + \frac{\overrightarrow{A_nM}}{A_nM}\right) \cdot \frac{\overrightarrow{TM}}{TM} \\
= &\cos(TMA_1) + \cdots + \cos(TMA_n) \\
\geq &-\left(\cos(MTA_1) + \cdots + \cos(MTA_n)\right) \text{ as in the proof of the previous inequality} \\
= &-\left(\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cdots + \cos\left(\theta + \frac{(n-1)\cdot 2\pi}{n}\right)\right) \\
= &\,0,
\end{aligned}$$

where $\theta = (\overrightarrow{TM},\ \overrightarrow{TA_1})[2\pi]$, with equality for: $MA_kT = 0$ for all k, which is impossible for $n \geq 3$. Therefore, $||\vec{\vec{\nabla}}(A_1M + \cdots + A_nM)|| > 0$.

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Another Purely Synthetic Proof of Lemoine's Theorem

Nguyen Tien Dung

Abstract. In this article, we give another purely synthetic proof the Lemoine's theorem that the symmedian point of a triangle is the unique point which is the centroid of its own pedal triangle.

Tran Quang Hung [6] proposed a new proof using similar triangles and cyclic quadrilaterals of Lemoine's theorem on the symmedian point of a triangle.

Lemoine's Theorem ([6]). *Given a triangle ABC, a point P is the centroid of its own pedal triangle with reference to ABC if and only if P is the symmedian point of the triangle ABC.*

In this article, we shall give another synthetic proof of the theorem, also using similar triangles and cyclic quadrilaterals.

Lemma 1. *Let $ABCD$ be a cyclic quadrilateral. The diagonal AC is a symmedian of triangle ABD if and only if CA is a symmedian of triangle CBD.*

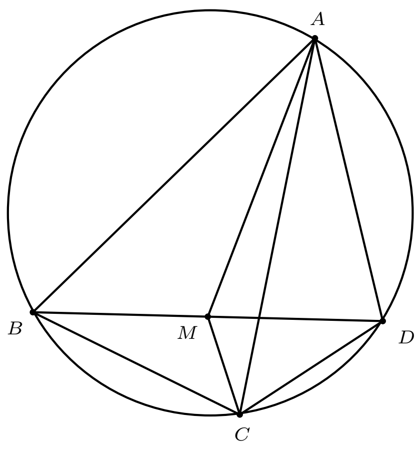

Figure 1

Proof. We only need to prove that if AC is a symmedian of the triangle ABD, then CA is a symmedian of triangle CBD. Let M be the midpoint of the segment BC. As $\angle BAM = \angle CAD$ and $\angle ABM = \angle ACD$, the triangles ABM and

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ACD are similar. It follows that $\dfrac{AB}{AC} = \dfrac{BM}{CD}$, and $AB \cdot CD = AC \cdot BM$. Since $AB \cdot CD + BC \cdot DA = AC \cdot BD$ by Ptolemy's theorem, we have $BC \cdot DA = AC \cdot DM = AC \cdot BM$. Notice that $\dfrac{AC}{BC} = \dfrac{AD}{BM}$ and $\angle CAD = \angle CBM$. The triangles ACD and BCM are similar. Hence, $\angle ACD = \angle BCM$, and CA is a symmedian of triangle CBD. □

Proof of Lemoine's Theorem. Denote by D, E and F the orthogonal projections of P onto BC, CA and AB respectively. The line AP intersects the circumcircle of triangle ABC again at Q. Let M be the midpoint of BC and let L be the reflection of F through P.

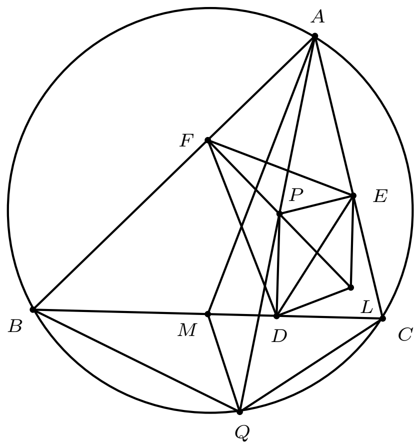

Figure 2

We can see that the quadrilaterals $AEPF$, $BFPD$ and $CDPE$ are cyclic. As $\angle PFE = \angle PAE = \angle QBC$ and $\angle PEF = \angle PAF = \angle BCQ$, the triangles PFE and QBC are similar. It follows that $FE \cdot BQ = FP \cdot BC = FL \cdot BM$. Hence, triangles EFL and MBQ are also similar.

(a) If P is the symmedian point of triangle ABC, AQ is a symmedian of the triangle. According to Lemma 1, QA is the symmedian of the triangle QBC. As the triangles EFL and MBQ are similar, $\angle ELF = \angle MQB = \angle AQC = \angle ABC = \angle DPL$; so $DP \parallel EL$. Similarly, $EP \parallel DL$. The quadrilateral $DPEL$ is a parallelogram. It follows that PL bisects DE. Therefore, FP is a median of the triangle DEF. Similarly, EP is also a median of the same triangle. Hence, P is the centroid of the triangle DEF.

(b) If P is the centroid of triangle DEF, the quadrilateral $DPEL$ is a parallelogram. Since triangles EFL and MBQ are similar, $\angle BQM = \angle FLE = \angle DPL = \angle ABC = \angle AQC$. It follows that QA is a symmedian of triangle QBC. By Lemma 1, AP is a symmedian of triangle ABC. Similarly, BP is also a symmedian of the same triangle ABC. It follows that P is the symmedian point of triangle ABC.

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Side Lengths of Morley Triangles and Tetrahedra

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Abstract. The famous Morley theorem says that the adjacent angle trisectors of a triangle form an equilateral triangle. We recall some known proofs and provide several new. In hyperbolic geometry, we compute the side lengths of the associated Morley triangle and show that the limit is the Euclidean (flat) equilateral case. The perspectivity properties hold also in general. We introduce new invariants such as the Morley polynomial and Morley group. Finally, we consider the space analogue and compute the side lengths of the Morley tetrahedron.

1. Introduction

The Morley trisector theorem, known also as *the Morley miracle*, says that the adjacent angle trisectors of a triangle meet at the vertices of an equilateral triangle. Frank Morley (1860–1937) - algebraic geometer - obtained this wonderful result in 1899 and to this day it continues to attract interest. There are many proofs of this theorem scattered in papers, books and web sites.

H. Coxeter, J. Conway and A. Connes are only some of the well known names who contributed with their own (rather conceptual) proofs of the Morley miracle (see [3], [11], [9]). In fact, Conway's proof was first essentially anticipated by Coxeter and attributed back to R. Bricard.

Actually, since any Euclidean triangle is affine - regular (affine image of a regular triangle), no wonder that by starting from a regular (Morley) triangle we get by an affine transformation a triangle similar to the given triangle. Connes' proof gives a precise algebraic control of the affine transformation involved (and not only over the field of complex numbers).

The Morley miracle was only a very special case of a general theory developed by Morley on Clifford chains. Originally he proved in an algebraic manner that the centers of inscribed cardioids in a triangle are on 9 lines, from which 3 by 3 are parallel in three directions under the angle of $\pi/3$.

In 1933 F. Morley published (together with his son F. Morley) the book Inversive Geometry ([37]). Yet another "Morley's miracle" was his congruence $4^{p-1} \equiv$

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$\pm\binom{p-1}{(p-1)/2}(mod\, p^3)$ for any prime $p > 3$ published in Annals of Math. 1894/5 (see [1]).

An interesting account on the history of the Morley trisector theorem and its proofs was given in 1978 paper in the AMER. MATH. MONTHLY [39] with 150 bibliographic units. Another account presenting more than 30 proofs and about 200 bibliographic units on Morley's theorem is M. Sc. Thesis [32] written in Croatian in 2003 under mentorship of Professor Vladimir Volenec.

Many geometry textbooks or survey papers or problem books or blogs in geometry mention Morley's miracle: Berger, Coxeter, Bollobás, Prasolov, Barnes, Honsberger, Connes et al, Hahn, Gardner, Shklarsky et al, Bogomolny, Tao, Gowers to name just a few well known authors (see throughout the literature [1]-[62], including contents on blogs and web sites).

In this article we consider the classical and the Morley triangle of a hyperbolic triangle. We compute its side lengths and show that in the limit it agrees with the Euclidean (flat) lengths. We consider perspectivity properties, the Morley polynomial and the Morley group. We also consider the space analogue and compute the edge lengths of the Morley tetrahedron.

2. Morley theorem in plane

Let us first recall three short standard proofs and provide a new one. They are based on the sine rule and the triple formula

$$\begin{aligned}\sin(3x) &= 3\sin(x) - 4\sin^3(x)\\ &= 4\sin(x)\sin(x^+)\sin(x^{++})\\ &= 4\sin(x)\sin(x^+)\sin(x^-),\end{aligned}$$

where $x^{\pm} = \frac{\pi}{3} \pm x$. Let the triangle $\triangle ABC$ has the angles $A=3\alpha, B=3\beta, C=3\gamma$. So, $\alpha + \beta + \gamma = \frac{\pi}{3}$. Let R be its circumradius.

Theorem 1 (Morley's theorem (1899))**.** *The three points of intersection of the adjacent trisectors of the angles of a triangle form an equilateral triangle.*

Proof. 1 ([19], 1949)
Let P, Q and R (not to be confused with circumradius R) be vertices of the Morley triangle (see Figure 1). Then $a/\sin(A) = 2R$ etc. We just use the sine rule twice: for the triangles $\triangle ABR$ and $\triangle CPQ$. From the first triangle and the triple formula we easily get $AR = 8R\sin(\beta)\sin(\gamma)\sin(\gamma^+)$. It follows $CQ/\sin(\beta^+) = CP/\sin(\alpha^+) = 8R\sin(\alpha)\sin(\beta)$. Consider the triangle with one side CQ and angles $\alpha^+, \beta^+, \gamma$. By the sine rule it follows that this triangle is congruent to $\triangle CPQ$. Hence $PQ = 8R\sin(\alpha)\sin(\beta)\sin(\gamma)$. By symmetry it follows $PQ = QR = RP$. (Or, one can use the cosine rule for $\triangle CPQ$ with sides CQ and CP and the angle γ to obtain PQ.) $\square$

Proof. 2 ("Chasing the angles")
This is a little variation of Proof 1. Again by the sine rule for $\triangle ACQ$ we get

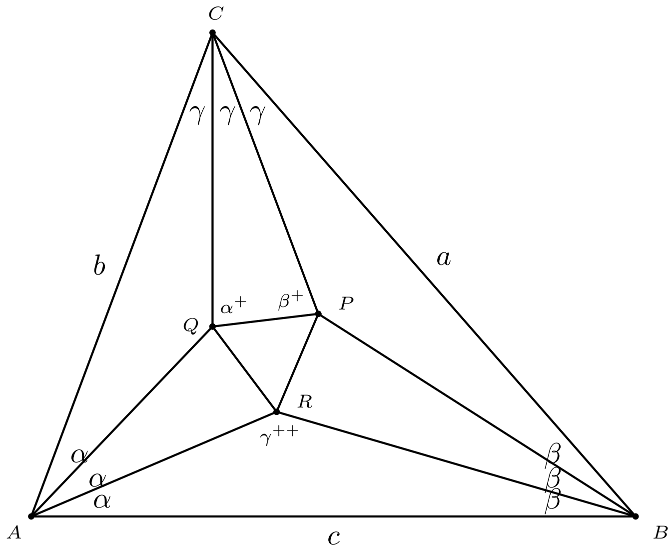

Figure 1

$CQ/AC = \sin(\alpha)/\sin(\beta^{++})$; hence $CQ = 2R\sin(3\beta)\sin(\alpha)/\sin(\beta^{++})$. Similarly, $CP = 2R\sin(3\alpha)\sin(\beta)/\sin(\alpha)$. By the triple formula we obtain $CQ/CP = \sin(\beta^{+})/\sin(\alpha^{+})$. Hence, β^{+} is the angle against CQ and α^{+} against CP in $\triangle CPQ$. And a similar distribution of angles holds around any of the vertices P, Q, R. Therefore, $\angle PQR = 2\pi - (\alpha^{+} + \beta^{++} + \gamma^{+}) = \pi/3$, and so all angles of $\triangle PQR$ are $\pi/3$. □

The triangle $\triangle PQR$ is called the (basic) *Morley triangle* of the original triangle $\triangle ABC$. There are altogether 27 Morley triangles associated not only to trisectors of A, etc., but also of $A + 2\pi$, $A + 4\pi$, $\ldots$ Out of 27, in general only 18 are equilateral triangles. For more details, constructions and figures see [22]. Let us emphasize once more that the side length PQ of the basic Morley triangle $\triangle PQR$ of the triangle $\triangle ABC$ is given in terms of the circumradius R by

$$PQ = 8R\sin\left(\frac{A}{3}\right)\sin\left(\frac{B}{3}\right)\sin\left(\frac{C}{3}\right) .$$

The symmetry of this expression implies that $\triangle PQR$ is regular. Just as a numerical example, if $\triangle ABC$ is the right triangle (the angle C is right), with legs $AC = BC = 1$, then the basic Morley triangle has side length $\sqrt{2} - \sqrt{1.5} \approx 0.189$ and area $(3.5\sqrt{3} - 6)/4 \approx 0.01554$ (about 3% of the area of ABC).

A well known proof of Coxeter (see [11]) goes basically as follows. Suppose for the moment that the triangle $\triangle PQR$ is equilateral. Let the trisectors (or trisectrices) AQ and BP meet at W, i.e. let $W = AQ \cap BP$, and similarly $U = BR \cap CQ$, $V = AR \cap CP$. Then the triangles $\triangle URQ$, $\triangle VRP$ and $\triangle WPQ$ are isosceles and it is easy to find their angles. To prove the Morley theorem, we start from any equilateral triangle $\triangle PQR$, construct the points U, V, W with appropriate angles

α^-, β^-, γ^- (recall $x^\pm = \frac{\pi}{3} \pm x$) and then the triangle $\triangle ABC$ as $A = VR \cap WQ$ etc.

The obtained triangle is similar to the given triangle (with the given angles A, B and C).

By reversing the Coxeter proof we now give a (new) direct proof that the triangles $\triangle URQ$, $\triangle VRP$ and $\triangle WPQ$ are isosceles. This in turn implies immediately Morley's theorem by the distribution of angles $\angle PQW = \gamma^-$, $\angle CQW = \beta^-$ etc. It is enough to prove $QW = PW$.

By the sine law for $\triangle AQC$ we have $AQ = b\sin(\gamma)/\sin(\beta^{++}) = b\sin(\gamma)/\sin(\beta^-)$ and by the sine law for $\triangle ABW$ we have $AW = c\sin(2\beta)/\sin((2\gamma)^+)$. Now $QW = AW - AQ$. Hence

$$QW = c\frac{\sin(2\beta)}{\sin((2\gamma)^+)} - b\frac{\sin(\gamma)}{\sin(\beta^-)}.$$

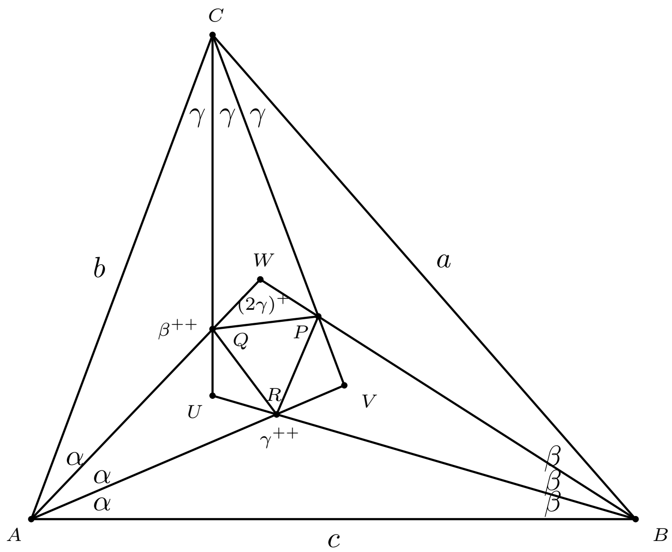

Figure 2

In the same way

$$PW = c\frac{\sin(2\alpha)}{\sin((2\gamma)^+)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

We claim $QW = PW$, and this is equivalent to

$$c\frac{\sin(2\beta)}{\sin((2\gamma)^+)} - b\frac{\sin(\gamma)}{\sin(\beta^-)} = c\frac{\sin(2\alpha)}{\sin((2\gamma)^+)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}$$

$$\iff c\frac{\sin(2\beta) - \sin(2\alpha)}{\sin((2\gamma)^+)} = b\frac{\sin(\gamma)}{\sin(\beta^-)} - a\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

By the sine rule $b/c = \sin(3\beta)/\sin(3\gamma), a/c = \sin(3\alpha)/\sin(3\gamma)$, this is equivalent to

$$\frac{\sin(3\gamma)}{\sin((2\gamma)^+)}(\sin(2\beta) - \sin(2\alpha)) = \sin(3\beta)\frac{\sin(\gamma)}{\sin(\beta^-)} - \sin(3\alpha)\frac{\sin(\gamma)}{\sin(\alpha^-)}.$$

By the triple formula this is equivalent to the trigonometric identity which simplifies to the following simple identity

$$\sin(2\beta) - \sin(2\alpha) = (\sin(\beta)\sin(\beta^+) - \sin(\alpha)\sin(\alpha^+))\frac{\sin((2\gamma)^+)}{\sin(\gamma^+)\sin(\gamma^-)}$$

($\Leftrightarrow \sin(\beta - \alpha) = (\sin(\beta)\sin(\beta^+) - \sin(\alpha)\sin(\alpha^+))/\sin(\gamma^+)$). This identity is easy to check (recall, $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \frac{\pi}{3}$). In a way a similar proof was given in [7].

Note that the hexagon $URVPWQ$ is equal to the intersection of the three middle thirds bounded by trisectors.

There are still new proofs of the classical Morley's theorem, see e.g., S. A. Kuruklis, Trisectors like Bisectors with equilaterals instead of Points, CUBO A Mathematical Journal, Vol. 16 No01, 71-110, June 2014., I. Gorjian, O. A. S. Karamzadeh and M. Namdari, Morley's Theorem is no longer mysterious, Math. Intelligencer, Viewpoints (online 25 Nov. 2015) and M. Smyth, Morley's Theorem: A walk in the park, Math. Intelligencer, Letter (online 25 Nov. 2015).

3. Hyperbolic Morley triangle

Now consider a hyperbolic triangle $\triangle ABC$. We stick to standard notation, but for the sake of brevity we introduce the following extra notation (see Figure 3.).

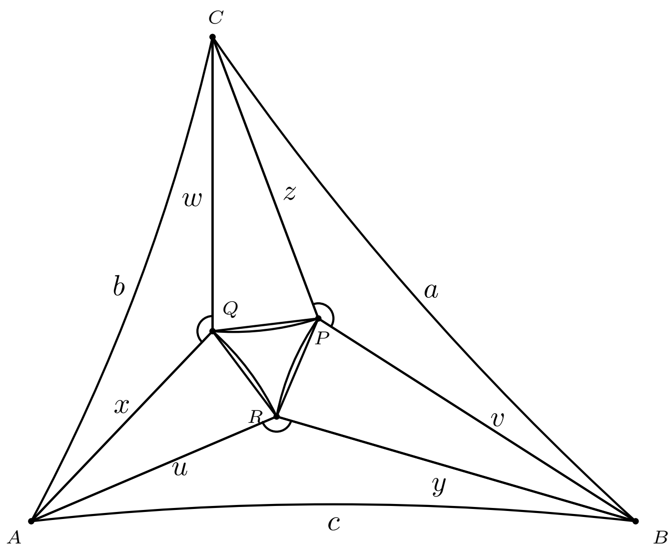

Figure 3

$$AQ = x, AR = u, \quad BR = y, BP = v, \quad CP = z, CQ = w.$$

We shall consider the hyperbolic plane of constant negative curvature $-1/k^2$, but will be working with constant curvature -1. Hence, instead of $\sinh \frac{x}{k}$ etc., we shall write $\sinh x$ etc., except when we consider the limiting process when $k \longrightarrow \infty$. For the notational brevity, we shall use the shorter (and former standard) notations: $\sinh \leftrightarrow \text{sh}, \cosh \leftrightarrow \text{ch}, \tanh \leftrightarrow \text{th}, \coth \leftrightarrow \text{cth}$.
We first need a lemma.

Lemma 2 (A-S-A formula)**.** *For given side c and adjacent angles A and B, the side length a in the hyperbolic triangle $\triangle ABC$ is given by*

$$\text{th}(a) = \frac{\text{sh}(c)\sin(A)}{\text{ch}(c)\sin(A)\cos(B) + \cos(A)\sin(B)}. \tag{1}$$

Proof. Start with the cosine rule $\text{ch}(a) = \text{ch}(b)\text{ch}(c) - \text{sh}(b)\text{sh}(c)\cos(A)$ and $\text{ch}(b) = \text{ch}(a)\text{ch}(c) - \text{sh}(a)\text{sh}(c)\cos(B)$, and the sine rule $\text{sh}(b) = \text{sh}(a)\frac{\sin(B)}{\sin(A)}$. Substitute the last two values into the first equation, use $\text{ch}^2(c) = 1 + \text{sh}^2(c)$, divide the obtained equation by $\text{sh}(c)\text{sh}(a)$ and get $\text{cth}(a)\text{sh}(c) = \text{ch}(c)\cos(B) + \cot(A)\sin(B)$. This implies the formula (1). □

Equivalently, formula (1) can be written as

$$\text{th}(a) = \frac{\text{sh}(c)\sin(A)}{\text{ch}^2\left(\frac{c}{2}\right)\sin(A+B) + \text{sh}^2\left(\frac{c}{2}\right)\sin(A-B)} \tag{2}$$

or (by hyperbolic Napier analogy) as

$$\text{th}(a) = \frac{2\sin(A)}{\text{cth}\left(\frac{c}{2}\right)\sin(A+B) + \text{th}\left(\frac{c}{2}\right)\sin(A-B)}. \tag{3}$$

Theorem 3 (sides of a hyperbolic Morley triangle)**.** *The side lengths of hyperbolic Morley's triangle $\triangle PQR$ of the given hyperbolic triangle $\triangle ABC$ are given by*

$$\text{ch}^2(QR) = \frac{(1 - \text{th}(x)\text{th}(u)\cos(\alpha))^2}{(1 - \text{th}^2(x))(1 - \text{th}^2(u))}, \tag{4}$$

where, by (2) *or* (3), *$x = AQ$ and $u = AR$ are given by*

$$\begin{aligned} \text{th}(x) &= \frac{2\text{th}\left(\frac{b}{2}\right)\sin(\gamma)}{\sin(\alpha+\gamma) + \text{th}^2\left(\frac{b}{2}\right)\sin(\alpha-\gamma)}, \\ \text{th}(u) &= \frac{2\text{th}\left(\frac{c}{2}\right)\sin(\beta)}{\sin(\alpha+\beta) + \text{th}^2\left(\frac{c}{2}\right)\sin(\alpha-\beta)} \end{aligned} \tag{5}$$

and similarly for the side lengths PQ and RP.

Proof. It is now a straightforward computation from the hyperbolic cosine rule for $\text{ch}(QR)$ in the triangle $\triangle AQR$ and using $\text{ch}(x) = \frac{1}{\sqrt{1-\text{th}^2(x)}}$, $\text{sh}(x) = \frac{\text{th}(x)}{\sqrt{1-\text{th}^2(x)}}$ and similarly for $\text{ch}(u)$ and $\text{sh}(u)$. Then use Lemma 2. □

By using the standard (dual) formula

$$\text{th}^2\left(\frac{b}{2}\right) = \frac{\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)},$$

in terms of angles only, we have

$$\text{th}(x) = \frac{2N\sin(\gamma)}{\sin(\alpha+\gamma)\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)+\sin(\alpha-\gamma)\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}, \tag{6}$$

where $3\alpha+3\beta+3\gamma=\pi-\delta$ and

$$N^2 = \sin\left(\frac{\delta}{2}\right)\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right).$$

Let us write now (2) properly with curvature k. We have

$$\begin{aligned}\text{th}\left(\frac{x}{k}\right) &= \frac{\text{sh}\left(\frac{b}{k}\right)\sin(\gamma)}{\text{ch}^2\left(\frac{b}{2k}\right)\sin(\alpha+\gamma)+\text{sh}^2\left(\frac{b}{2k}\right)\sin(\alpha-\gamma)}, \text{ and}\\ \text{th}\left(\frac{u}{k}\right) &= \frac{\text{sh}\left(\frac{c}{k}\right)\sin(\beta)}{\text{ch}^2\left(\frac{c}{2k}\right)\sin(\alpha+\beta)+\text{sh}^2\left(\frac{c}{2k}\right)\sin(\alpha-\beta)}\end{aligned} \tag{7}$$

Taking the limit in (7) as $k\to\infty$ (or $\delta\to 0$ in (6)), we see that $\left(\text{th}\left(\frac{x}{k}\right)\right)k$ tends to

$$x = \frac{b\sin(\gamma)}{\sin(\alpha+\gamma)} = \frac{2R\sin(3\beta)\sin(\gamma)}{\sin(\beta^{++})} = 8R\sin(\beta)\sin(\beta^+)\sin(\gamma),$$

and similarly for u. (Here R is the circumradius of the limiting Euclidean triangle.)

This agrees with the Euclidean expression for $x = AQ$ given in Proof 1 of the Morley theorem.

Next, from (4) we get

$$\text{sh}^2(QR) = \frac{\text{th}^2(x)+\text{th}^2(u)-2\text{th}(x)\text{th}(u)\cos(\alpha)-\text{th}^2(x)\text{th}^2(u)\sin^2(\alpha)}{\left(1-\text{th}^2(x)\right)\left(1-\text{th}^2(u)\right)} \tag{8}$$

By writing (8) also “properly” and by taking $k\to\infty$ (or $\delta\to 0$), by using (7) we easily obtain that $\text{sh}^2\left(\frac{QR}{k}\right)k^2$ tends to

$$\begin{aligned}QR^2 &= \frac{\begin{pmatrix}(8R\sin\beta\sin\beta^+\sin\gamma)^2+(8R\sin\gamma\sin\gamma^+\sin\beta)^2\\ -2(8R\sin\beta\sin\beta^+\sin\gamma)(8R\sin\gamma\sin\gamma^+\sin\beta)\cos\alpha-0\end{pmatrix}}{(1-0)(1-0)}\\ &= (8R\sin(\beta)\sin(\gamma))^2\left[\sin^2(\beta^+)+\sin^2(\gamma^+)-2\sin(\beta^+)\sin(\gamma^+)\cos(\alpha)\right]\\ &= (8R\sin(\alpha)\sin(\beta)\sin(\gamma))^2.\end{aligned}$$

And this agrees with the expression for QR in the Proof 1 of the Euclidean Morley theorem.

Let us note that if the hyperbolic triangle $\triangle ABC$ has circumcircle of hyperbolic radius R, then (6) can be written in terms of R and angles as follows

$$\mathrm{th}(x) = 2\mathrm{th}(R)\frac{\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin(3\beta)}\,\frac{\sin\left(\frac{\pi}{3}-\beta\right)}{\sin\left(\frac{\pi-\delta}{3}-\beta\right)}\,\frac{4\sin(\beta)\sin(\beta^{+})\sin(\gamma)}{1+\frac{\sin\left(\frac{\delta}{2}\right)\sin\left(3\beta+\frac{\delta}{2}\right)}{\sin\left(3\alpha+\frac{\delta}{2}\right)\sin\left(3\gamma+\frac{\delta}{2}\right)}\frac{\sin(\alpha-\gamma)}{\sin(\alpha+\gamma)}} \tag{9}$$

If we let δ to tend to 0, then from (9) it follows that $\mathrm{th}(x/k)k$ tends to

$$x = 2R\cdot 1\cdot 1\cdot\frac{4\sin(\beta)\sin(\beta^{+})\sin(\gamma)}{1+0} = 8R\sin(\beta)\sin(\beta^{+})\sin(\gamma).$$

Again this is in accordance with the expression for $x = AQ$ in the Proof 1 of the Morley theorem in the Euclidean case.

So, there is no evident symmetry in the hyperbolic Morley case, but perhaps some interesting inequalities hold instead (e.g. the equality–cosine law in the flat case and the corresponding inequalities otherwise, see [52]). However, we shall not consider it here.

4. Perspectivity of Morley configurations

In the list of triangle centers (see [31] and web sites [4] and [60]) there is one called "the second Morley point" M. Looking at Figure 1., then $AP\cap BQ\cap CR = M$. We shall prove this by using the natural barycentric coordinates. Recall that the barycentric coordinates of a point P in the plane of a triangle $\triangle ABC$ are given by the proportion $P\ \ldots\ t_1 : t_2 : t_3 = \mathrm{area}(PBC) : \mathrm{area}(PAC) : \mathrm{area}(PAB)$ of oriented areas. The equation of a line joining two points with barycentric coordinates (r_1, r_2, r_3) and (s_1, s_2, s_3) is given by

$$\begin{vmatrix} t_1 & t_2 & t_3 \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0\,.$$

Note that the area S of a triangle and the height h_c from the vertex C (in standard notations) in terms of c, A and B are given by:

$$\begin{aligned} S = {} & \frac{c^2}{2}\frac{\sin(A)\sin(B)}{\sin(A+B)}, & h_c = {} & c\,\frac{\sin(A)\sin(B)}{\sin(A+B)} \\ = {} & \frac{c^2}{2(\cot(A)+\cot(B))} & = {} & \frac{c}{\cot(A)+\cot(B)} \end{aligned}$$

Theorem 4 (P. Yff, 1967)**.** *The Morley equilateral triangle $\triangle PQR$ is perspective to the original triangle $\triangle ABC$ and the center of the perspective is called the second Morley triangle center. It has barycentric coordinates* $\sin(3\alpha)/\cos(\alpha)$: $\sin(3\beta)/\cos(\beta) : \sin(3\gamma)/\cos(\gamma)$.

Proof. By using the above formulas for area and height we easily get the barycentric coordinates of the point $P\ \ldots\ a : 2b\cos(\gamma) : 2c\cos(\beta)$. Similarly we obtain $Q\ \ldots\ 2a\cos(\gamma) : b : 2c\cos(\alpha)$ and $R\ \ldots\ 2a\cos(\beta) : 2b\cos(\alpha) : c$. The barycentric coordinates of the vertices of the original triangle are given by $A\ \ldots\ 1 : 0 : 0$, $B\ \ldots\ 0 : 1 : 0$ and $C\ \ldots\ 0 : 0 : 1$. By using the above determinant it is easy to check that the intersecting point $AP \cap BQ$ also lies on CR. Hence all three lines AP, BQ, CR intersect at a point. It is now easy to check its barycentric coordinates. □

A similar proof but using homogenous trilinear coordinates (the distances from a point to the sides of the triangle) is given in the paper [30], for more general angles rA, and $(1-r)A$, etc. for positive real number $r \neq 1$, and limits when $r \to 0$.

Let us briefly present yet another proof about the first and second Morley points given in [62] based on perspectivity, thus holding in any geometry. (Keep in mind Figure 2.)

Let x_A and y_A be two lines through the vertex A of the triangle ABC and similarly x_B, y_B and x_C, y_C. Let x_A be adjacent to b, x_B to c, and x_C to a. Denote the intersection points

$$\begin{aligned} y_B \cap x_C = P, \quad y_C \cap x_A = Q, \quad y_A \cap x_B = R, \\ x_B \cap y_C = U, \quad x_C \cap y_A = V, \quad x_A \cap y_B = W, \end{aligned}$$

and the lines

$$AP = z_A, BQ = z_B, CR = z_C, \ AU = u_A, BV = u_B, CW = u_C.$$

If l_A, l_B and l_C are any lines through vertices A, B and C, respectively, of the triangle $\triangle ABC$ with side lines a, b and c, denote

$$(a, b, c; l_A, l_B, l_C) := \frac{\sin(\angle(a, l_C))\sin(\angle(b, l_A))\sin(\angle(c, l_B))}{\sin(\angle(l_C, b))\sin(\angle(l_A, c))\sin(\angle(l_B, a))}.$$

Since we keep the first part a, b, c always in this order, we may write it simply as (l_A, l_B, l_C), where l_A can be x_A, y_A, z_A or u_A etc.

By Ceva's theorem for points P, Q, R, respectively, we have

$$(z_A, y_B, x_C) = (x_A, z_B, y_C) = (y_A, x_B, z_C) = 1.$$

Hence, their product is also equal to 1. Ceva's theorem for points U, V and W, respectively, yields that the corresponding product also equals to 1.

Now suppose that the triplets of lines x_A, x_B, x_C and y_A, y_B, y_C are reciprocal, i.e. $(x_A, x_B, x_C)(y_A, y_B, y_C) = 1$. Then from

$$(x_A, x_B, x_C)(y_A, y_B, y_C)(z_A, y_B, x_C)(x_A, z_B, y_C)(y_A, x_B, z_C) = 1,$$

after an "orgy" of cancellations (as Coxeter expressed it once), we conclude that $(z_A, z_B, z_C) = 1$; and similarly for (u_A, u_B, u_C).

By the converse of Ceva's theorem, the lines z_A, z_B and z_C are concurrent or parallel. Since no two of these lines can be parallel, all three are concurrent. The same holds for the lines u_A, u_B, and u_C.

Let $z_A \cap z_B \cap z_C = M$, and $u_A \cap u_B \cap u_C = N$.

Observe that hexagons $ARBPCQ$ and $AVCUBW$ are Brianchon hexagons.

Conversely, if the lines z_A, z_B, z_C meet at M and u_A, u_B, u_C meet at N, then the triples of lines x_A, x_B, x_C and y_A, y_B, y_C are reciprocal lines through the vertices A, B, C of the triangle $\triangle ABC$.

This is a consequence of the fact that if $(z_A, z_B, z_C) = 1$ (or $(u_A, u_B, u_C) = 1$), then from $(z_A, y_B, x_C)(x_A, z_B, y_C)(y_A, x_B, z_C) = 1$ we get (x_A, x_B, x_C) $(y_A, y_B, y_C) = 1$ (and similarly with (u_A, u_B, u_C)).

So, the reciprocity of triples x_A, x_B, x_C and y_A, y_B, y_C implies that the lines z_A, z_B, z_C are concurrent, thus the hexagon $ARBPCQ$ has the Brianchon property. Rearranging the sides of this hexagon in the form $x_C y_B x_A y_C x_B y_A$ we see that the hexagon $PWQURV$ also has the Brianchon property.

Hence we have proved (as was proved in 1938. by M. Zacharias, [62]):

Theorem 5. *If two triplets of lines x_A, x_B, x_C and y_A, y_B, y_C through the vertices of the triangle $\triangle ABC$ are reciprocal, i.e. $(x_A, x_B, x_C)(y_A, y_B, y_C) = 1$, then the lines z_A, z_B, z_C meet at M, the lines u_A, u_B, u_C meet at N and the lines PU, QV and RW meet at one point O. The converse also holds.*

Note that trisectors are reciprocal triplets of lines through the triangle vertices. The point O is called the *first* and the point M the *second Morley point* of the triangle ABC.

As a consequence, there is also a short proof of the Morley theorem as follows.

Proof of the Euclidean Morley theorem (from perspectivity). We stick with the same notation as before (angles $A = 3\alpha, B = 3\beta, C = 3\gamma$ etc.); let P, Q, R be the adjacent trisector intersections of the interior angles of the triangle ABC. Let $AQ \cap BP = W, BR \cap CQ = U$ and $AR \cap CP = V$. Then by Theorem 5 the lines PU, QV and RW meet at the point O. Since AR and BR are angle bisectors of the triangle $\triangle ABW$, it follows that R is the incenter of $\triangle ABW$ and hence RW the angle bisector of $\angle AWB$. Similarly, Q and P are incenters of $\triangle AVC$ and $\triangle BUC$, respectively. Since Q is the incenter of $\triangle AVC$, the angle $\angle CQV = \frac{\pi}{2} + \alpha$, and since P is the incenter of $\triangle CUB$, the angle $\angle QUP = \frac{\pi}{2} - (\beta + \gamma)$. It follows from these two angle–values that $\angle QOU = \frac{\pi}{3}$. Similarly, $\angle ROV = \frac{\pi}{3}$, and so the lines PU, QV and RW meet mutually at the angle $\frac{\pi}{3}$. By the triangle congruences $\triangle OQW \cong \triangle OPW$ and $\triangle OQU \cong \triangle ORU$ it follows the lengths equality $OP = OQ = OR$ and hence the triangle $\triangle PQR$ is equilateral. $\square$

From the above discussion we have that in the trisector case the triangles $\triangle ABC$ and $\triangle PQR$ are perspective from the point M (the second Morley point), and the triangles $\triangle ABC$ and $\triangle UVW$ are perspective from the point N. By Theorem 5, the triangles $\triangle PQR$ and $\triangle UVW$ are also perspective from the point O (the first Morley point).

It is now easy to prove by using Desargues theorem that the points O, M and N are collinear, and also that 12 points $A, B, C, P, Q, R, U, V, W, M, N, O$ and 16 lines $x_A, x_B, x_C, y_A, y_B, y_C, z_A, z_B, z_C, u_A, u_B, u_C, PU, QV, RW$ and MN form a configuration $(12_4; 16_3)$ of 12 points with 4 lines out of 16 through each point and with 3 out of 12 points on each of 16 lines.

Namely, for triangles $\triangle APV$ and $\triangle BQU$ we have that the points $AP \cap BQ = M$, $PV \cap QU = C$, $AV \cap BU = R$ are collinear (on the line z_C). By Desargues' theorem the lines AB, PQ, UV are concurrent and denote the intersection point by S which may be proper or improper. Since the triangles $\triangle APU$ and $\triangle BQV$

are perspective wrt to S, by Desargues' theorem it follows that $AP \cap BQ = M$, $PU \cap QV = O$, $AU \cap BV = N$ are collinear.

Now since the Ceva theorem holds in hyperbolic geometry without any change (see e.g. [17]), it follows by the mutatis mutandis argument that the above discussion on perspectivity holds as well. So with the same proof we can summarize the perspectivity for the hyperbolic triangle in the following theorem.

Theorem 6. *Let $\triangle ABC$ be a hyperbolic triangle, P, Q, R the intersection points of the adjacent trisectors of the interior angles of $\triangle ABC$, respectively* ($P = x_C \cap y_B, Q = x_A \cap y_C, R = x_B \cap y_A$ *where $x_A, \ldots, y_C$ are trisectors*). *Then the corresponding Morley triangle $\triangle PQR$ is perspective to the original triangle $\triangle ABC$ with the perspective center $M = z_A \cap z_B \cap z_C$, where $z_A = AP, z_B = BQ, z_C = CR$. Denote further $x_B \cap y_C = U$, $x_C \cap y_A = V, x_A \cap y_B = W$. The lines $AU = u_A, BV = u_B, CW = u_C$ are concurrent: $u_A \cap u_B \cap u_C = N$. Also, the lines PU, QV and RW are concurrent at the point O. The converse holds as well. The points O* (*the first Morley hyperbolic point*), M (*the second Morley hyperbolic point*) *and N* (*the third Morley hyperbolic point*) *are collinear. So, we have again in the hyperbolic plane, starting with trisectors of a given triangle, a configuration* $(12_4; 16_3)$.

By the same argument as in the last Proof of the Euclidean Morley theorem we still can claim that in the hyperbolic case R is the incenter of $\triangle ABW$ (and also Q the incenter of $\triangle ACV$ and P the incenter of $\triangle BCU$).

5. Morley polynomial and group

Recall that in the Euclidean Morley theorem, the side length of the Morley triangle (see Proof 1 of Theorem 1) is given by $PQ = 8R\sin(A/3)\sin(B/3)\sin(C/3)$, or written as a $\mathbb{Z}_3$–symmetric product $PQ = 8R\prod\sin(A/3) = \frac{abc}{2S}\cdot 4\prod\sin(A/3)$, where $S = \operatorname{area}(\triangle ABC)$ (recall, not any hyperbolic triangle has circumcircle!).

Let us express this side length PQ in terms of the elements of the original triangle $\triangle ABC$.

Let $X = PQ$. Consider the elementary symmetric functions in squared side lengths a^2, b^2, c^2 of $\triangle ABC$:

$$E_1 = a^2 + b^2 + c^2, \quad E_2 = a^2b^2 + b^2c^2 + c^2a^2, \quad e_3 = \sqrt{E_3} = abc.$$

Define the polynomial

$$\mathcal{M}(X) = \prod\left(X - 8R\sin\left(\frac{A + 2k\pi}{3}\right)\sin\left(\frac{B + 2l\pi}{3}\right)\sin\left(\frac{C + 2m\pi}{3}\right)\right),$$

where the product is taken over all integers $0 \le k, l, m \le 2$, such that $k+l+m \equiv 0 \pmod 3$ and $k+l+m \equiv 2 \pmod 3$, or equivalently $A/3+B/3+C/3 = \pm\pi/3 \pmod{2\pi}$.

Namely, as was proved in [19], only in these 18 cases (out of 27) we obtain equilateral Morley triangles corresponding to trisectors of $A + 2k\pi$, $0 \le k \le 2$, etc. (see [19], [22], [6]).

We call $\mathcal{M}(X)$ the *Morley polynomial*. To compute it explicitly in terms of a, b, c and R, we use the triple formula, sine and cosine rules, Heron's formula $(4S)^2 = 4E_2 - E_1^2$ and $4RS = e_3$.

By using computer algebra system (Maple) we obtain the Morley polynomial $\mathcal{M}(X)$ which has as roots the (signed) edges of all 18 Morley triangles. The result of a tricky computation (using complex numbers) is given by the following theorem.

Theorem 7 (Morley polynomial)**.** *The Morley polynomial $\mathcal{M}(X)$ whose roots are side lengths of all 18 equilateral Morley triangles of a given (a, b, c) triangle is given as the product of two irreducible polynomials $\mathcal{M}_1(X)$ and $\mathcal{M}_2(X)$ each of degree 9*

$$\mathcal{M}(X) = \mathcal{M}_1(X)\mathcal{M}_2(X),$$

where

$$\begin{aligned}
&\mathcal{M}_1(X) \\
&= X^9 + 81R^2X^7 - \frac{27}{2}E_1R^2X^5 + 3e_3X^6 - \frac{81}{2}e_3R^2X^4 - \frac{27}{2}E_1e_3R^2X^2 \\
&\quad + \frac{1}{4}\left(-27E_1^2R^2 - 15e_3^2\right)X^3 + e_3^3 \\
&\quad + \sqrt{3}\left(9RX^8 + 81R^3X^6 + \frac{9}{2}e_3RX^5 - \frac{81}{2}E_1R^3X^4 - 81e_3R^3X^3 - \frac{9}{2}e_3^2RX^2\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}_2(X) \\
&= X^9 + 81R^2X^7 - \frac{27}{2}E_1R^2X^5 + 3e_3X^6 - \frac{81}{2}e_3R^2X^4 - \frac{27}{2}E_1e_3R^2X^2 \\
&\quad + \frac{1}{4}\left(-27E_1^2R^2 - 15e_3^2\right)X^3 + e_3^3 - \\
&\quad - \sqrt{3}\left(9RX^8 + 81R^3X^6 + \frac{9}{2}e_3RX^5 - \frac{81}{2}E_1R^3X^4 - 81e_3R^3X^3 - \frac{9}{2}e_3^2RX^2\right).
\end{aligned}$$

If we normalize by setting $Y = X/(3R\sqrt{3})$ together with $g = E_1/(162R^2)$ and $h = 2S\sqrt{3}/(243R^2)$, we obtain by multiplying $\mathcal{M}_1$ and $\mathcal{M}_2$ the polynomial $\mathcal{N}(Y) \in \mathbb{Z}[g, h][Y]$ (up to a scalar multiple):

$$\begin{aligned}
\mathcal{N}(Y) = {} & Y^{18} - 3Y^{16} + 12hY^{15} + 3(1-2g)Y^{14} + 12hY^{13} \\
& - (1 - 6h^2 + 18g^2)Y^{12} - 12h(1+4g)Y^{11} - 3(15g^2 - 2g + 33h^2)Y^{10} \\
& + 4h(1 - 27g^2 - 41h^2)Y^9 + 3(18g^3 - 3g^2 + 6gh^2 + 11h^2)Y^8 \\
& + 6h(15g^2 - 2g + 29h^2)Y^7 + (81g^4 + 270g^2h^2 + 321h^4 - 4h^2)Y^6 \\
& + 12h(9g^3 + 11gh^2 - 2h^2)Y^5 + 12h^2(3g^2 - 7h^2)Y^4 \\
& - 48h^3(3g^2 + 5h^2)Y^3 - 96gh^4Y^2 + 64h^6.
\end{aligned}$$

This agrees and is in accordance with the main result in [6], formula (14) and even refines it in the form of the product of two "$\sqrt{3}$ – conjugate" polynomials $\mathcal{M}_1$ and $\mathcal{M}_2$, each of degree 9.

This has many interesting consequences including those in [6], but we shall not go into further details here.

In [14] it was carefully examined all 27 Morley points in the complex plane. They form 18 equilateral and 9 (in general) nonequilateral triangles. They are all described by the action of a noncommutative group of order 27. This group H acts on the vertices of the basic Morley triangle $\triangle PQR$ ("the heart of $\triangle ABC$") to obtain all 27 Morley points. Call this group H the *Morley group*.

Theorem 8. *The Morley group H is isomorphic to the (discrete) Heisenberg unitriangular group $UT(3,3)$ of order 27.*

Proof. The Morley group is nonabelian of order 27 in which every element except identity is of order 3. This follows from the geometric reasons as it was checked in [14]. Recall, the exponent of a group is the lowest common multiple of the orders of all group elements. By the well known theorem from the finite group theory, for any odd prime p, the unitriangular group of degree 3 over the field of $p(=3)$ elements is the only nonabelian group of order p^3 with exponent p. This extra special group H of exponent $p = 3$ has the presentation given by $< a, b | a^3, b^3, [a,b] = [a,b]^a = [a,b]^b >$, as the Heisenberg group is known. So, $H \cong UT(3,3)$. $\square$

The Heisenberg group $H = H(\mathbb{Z}/3\mathbb{Z}) = UT(3,3)$ modulo 3 as an upper triangular group is generated by two elements

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and relations $z = xyx^{-1}y^{-1}$, $xz = zx$, $yz = zy$, $x^3 = y^3 = z^3 = 1$, where

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Any element $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{Z}/3\mathbb{Z})$ $(a, b, c \in \mathbb{Z}/3\mathbb{Z})$ is obtained by the generators.

In medical words, the basic Morley triangle ("the heart") is by the mechanism of Heisenberg group H transferred to the whole body of all 27 Morley triangles (in Euclidean case 18 equilateral).

6. The Morley tetrahedron

Let $ABCD$ be a (Euclidean) tetrahedron. Let us perform an analogous construction in space of the Morley plane scenario. Recall, to a triangle ABC, we associated the Morley triangle PQR, where $P = (B)_C \cap (C)_B$, $Q = (C)_A \cap (A)_C$, and $R = (A)_B \cap (B)_A$. Here $(X)_Y$ denotes the trisector line of the angle X which is closer to Y than the other trisector of X. We computed the side (or edge) lengths of the Morley triangle in terms of the given triangle (it turned out that these expressions are symmetric in A, B, C and hence equal).

Denote by $(XY)_Z$ the trisecting plane of the dihedral angle (XY) of the tetrahedron that is closer to the vertex Z than the other trisector of (XY). Let

$$\begin{array}{ll} P = (BC)_D \cap (CD)_B \cap (DB)_C, & Q = (AC)_D \cap (CD)_A \cap (DA)_C, \\ R = (AB)_D \cap (BD)_A \cap (DA)_B, & S = (AB)_C \cap (BC)_A \cap (CA)_B. \end{array}$$

We call $PQRS$ the *Morley tetrahedron* of $ABCD$.

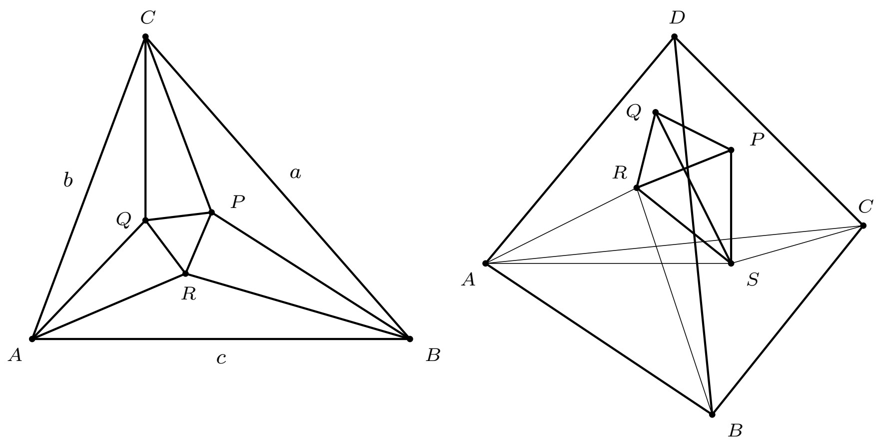

Figure 4

Remark. Let $\Delta = A_0A_1 \ldots A_n \subset \mathbb{R}^n$ be an n–simplex. Consider $(n-2)$–faces of Δ and dihedral angles determined by two adjacent facets (codimension 1 faces) of Δ. By trisecting each dihedral angle in three equal parts by two $(n-1)$–planes (trisectors), let M_i be the intersection of the trisectors adjacent to the face against A_i. The simplex $M_0M_1 \ldots M_n$ is the *Morley simplex* of Δ.

Unfortunately, in dimensions ≥ 3, the Morley simplex is not regular in general as is the case in dimension $n = 2$. It is not even equifacetal. This can be seen by the simple example of the tetrahedron whose vertices are the origin and unit vertices on the coordinate axes (computation is a bit technical but quite elementary).

Analogous to the triangles UQR, VPR and WPQ in Figure 2, in 3–space we also have four pyramids $ZPQR, WPRS, VPQS$ and $UQRS$ whose bases are faces of the Morley tetrahedron $PQRS$ and apexes U, V, W, Z. Unlike the planar isosceles triangles ($WP = WQ$ etc.), here some analogs also fail. Note that the convex polytope with vertices U, V, W, Z and P, Q, R, S is the intersection of six wedges formed by trisecting planes and has 8 faces.

Let us mention only that perspectivity analogous to the plane case as we proved in Section 2 fails in dimension 3.

However, what we shall do for tetrahedron is to express edge lengths of the Morley tetrahedron in terms of the initial tetrahedron as we did in the plane in both geometries.

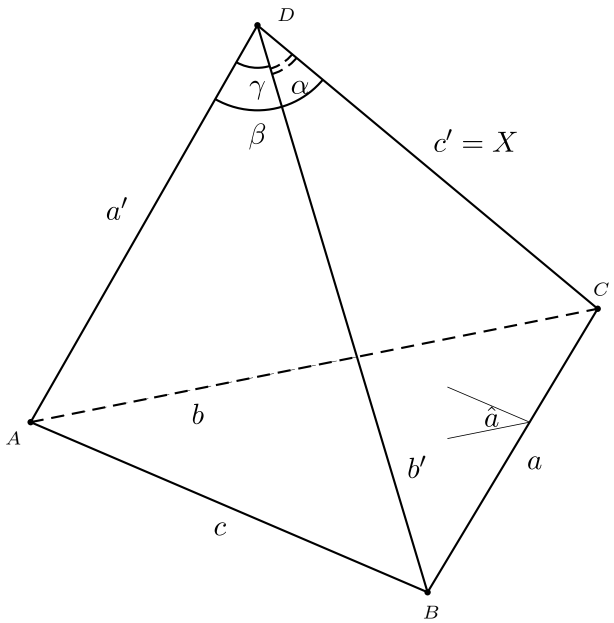

Figure 5

To abbreviate notations, let a, b, c, a', b', c' be the edge lengths of the tetrahedron $T = ABCD$ as in Figure 5, $AB = c$ etc.

Denote also by $\widehat{a}, \widehat{b}, \dots, \widehat{c'}$ the dihedral angles of T, i.e. $\widehat{a} = \widehat{BC}$ etc. Denote by $\widehat{ab}$ the plane angle between edges a and b, i.e. $\widehat{ab} = \angle BCA$ etc. Suppose we are given the base ABC of T and the dihedral angles at the base $\widehat{a}, \widehat{b}, \widehat{c}$. We want to compute the length of the lateral edge $c' = X$.

Lemma 9 (Lateral edge in terms of base and dihedral angles of its edges - "cot rule")**.**

$$X^2 = \left(\frac{ab}{\sum a\cot(\widehat{a})}\right)^2 \left(\cot^2(\widehat{a}) + \cot^2(\widehat{b}) + 2\cot(\widehat{a})\cot(\widehat{b})\cos(\widehat{ab}) + \sin^2(\widehat{ab})\right).$$

Here $\sum a\cot(\widehat{a}) = a\cot(\widehat{a}) + b\cot(\widehat{b}) + c\cot(\widehat{c})$.

Proof. Drop the height h_a from D to BC etc. and let h be the height of D to ABC. Let D_a, D_b and D' be feet to BC, CA and ABC, respectively. Then $h_a\sin(\widehat{a}) = h$. Now, $X^2 = CD'^2 + h^2$. Note that triangles $\triangle D'D_aC$ and $\triangle D'D_bC$ are right triangles, hence CD' is a diameter of the circle through D', D_a, C and D_b. Then by e.g. Ptolemy's theorem, and by the sine law $CD'\sin(\widehat{ab}) = D_aD_b$. By the cosine law we have $D_aD_b^2 = D_aD'^2 + D_bD'^2 - 2D_aD' \cdot D_bD'\cos(\pi - \widehat{ab})$. Next, $D_aD' = h\cot(\widehat{a}), D_bD' = h\cot(\widehat{b})$. We compute h from $ab\sin(\widehat{ab}) = 2area(ABC) = \sum a\, h_a \cos(\widehat{a}) = h\sum a\cot(\widehat{a})$. Now lemma follows easily. □

This lemma can also be thought of as angle–side–angle, or A–S–A rule for a tetrahedron.

Lemma 10 (Dihedral angles in terms of plane angles or edge lengths)**.** *With notations from Figure 5, we have in terms of plane angles at D:*

$$\cos(\widehat{c'}) = \frac{\cos(\gamma) - \cos(\alpha)\cos(\beta)}{\sin(\alpha)\sin(\beta)} = \frac{\cos(\widehat{a'b'}) - \cos(\widehat{b'c'})\cos(\widehat{a'c'})}{\sin(\widehat{b'c'})\sin(\widehat{a'c'})}.$$

Proof. This is the spherical cosine law on the unit sphere around D. Then use the ordinary cosine law to express it by edge lengths. □

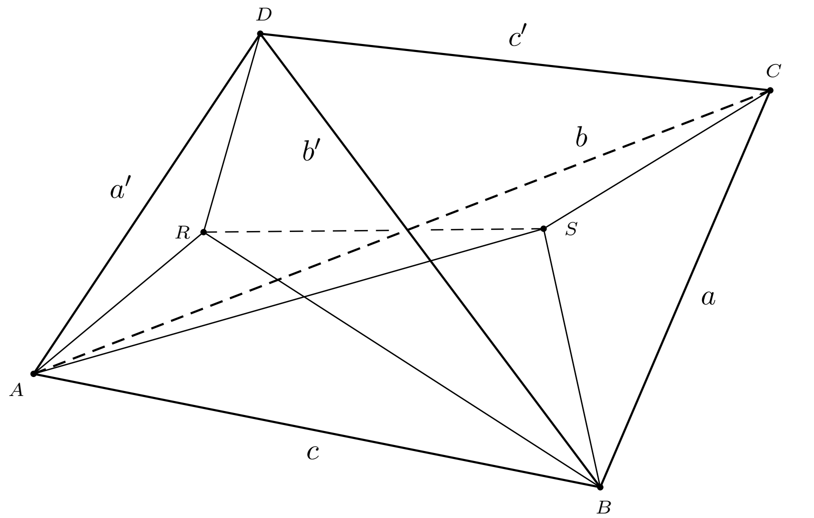

Figure 6

Now by using the above Lemmas 9 and 10 we can compute the edge length, say, RS in terms of $ABCD$ (see Figure 6.). First we compute RA and RB as lateral edges with base triangle (with thirds dihedrals) ABD and SA and SB as lateral edges with base triangle ABC. Then in the tetrahedron $ABRS$ we are given five edge lengths RA, RB, SA, SB and $AB = c$ and the dihedral angle at AB equal to $\widehat{AB}/3 = \widehat{c}/3$, written as $\widehat{c}_3$, which is opposite to RS in the tetrahedron $ABRS$.

Lemma 11 (S–A–S tetrahedron formula - a tetrahedron cosine law)**.** *Let a, b, c, a', b' and dihedral angle $\widehat{c}$ against the edge c' are given. Verbally, given are five edges* *and the dihedral angle against the missing edge. Then the missing edge-length is given by*

$$\begin{aligned} c'^2 =&\ a'^2 + b^2 - 2a'b\left(\cos(\widehat{bc})\cos(\widehat{a'c}) + \sin(\widehat{bc})\sin(\widehat{a'c})\cos(\widehat{c})\right), \\ \cos(\widehat{bc}) =&\ (b^2 + c^2 - a^2)/(2bc), \\ \cos(\widehat{a'c}) =&\ (a'^2 + c^2 - b'^2)/(2a'c). \end{aligned}$$

Proof. It follows from the cosine law for the triangle ACD and the previous Lemma 10. $\square$

By using Lemmas 9 and 11 we shall now express edge lengths of the Morley tetrahedron in terms of the original tetrahedron $ABCD$. Recall that we have introduced the abbreviated notation. So, for example, we write $\widehat{a}_3$ instead of $\widehat{a}/3$ etc.

Theorem 12. *The edge length RS of the Morley tetrahedron associated to the tetrahedron $ABCD$ is given by*

$$RS^2 = SA^2 + RA^2 - 2RA\cdot SA(\cos(\widetilde{\alpha}')\cos(\widetilde{\alpha}) + \sin(\widetilde{\alpha}')\sin(\widetilde{\alpha})\cos(\widehat{c}_3)),$$

where $\widetilde{\alpha}$ is the angle BAS and $\widetilde{\alpha}'$ is the angle BAR, given by

$$\begin{aligned} &\cos\widetilde{\alpha} \\ =&\ \left(c^2 + SA^2 - SB^2\right)/(2cSA) \\ =&\ \frac{c^2/(2cSA)}{\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2}\left[\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2\right. \\ &+ b^2\left(\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc}) + \sin^2(\widehat{bc})\right) \\ &\left. - a^2\left(\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac}) + \sin^2(\widehat{ac})\right)\right] \\ =&\ \frac{c}{2SA\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2}\left[\left(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)\right)^2\right. \\ &+ b^2\left(\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc})\right) \\ &\left. - a^2\left(\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac})\right)\right], \end{aligned}$$

and

$$\cos\widetilde{\alpha}' = \left(c^2 + RA^2 - RB^2\right)/(2cRA)$$

where

$$RA^2 = \left(\frac{a'c}{a'\cot(\widehat{a'}_3) + b'\cot(\widehat{b'}_3) + c\cot(\widehat{c}_3)}\right)^2 \times (\cot^2(\widehat{a'}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a'}_3)\cot(\widehat{c}_3)\cos(\widehat{a'c}) + \sin^2(\widehat{a'c})),$$

$$RB^2 = \left(\frac{b'c}{b'\cot(\widehat{b'}_3) + a'\cot(\widehat{a'}_3) + c\cot(\widehat{c}_3)}\right)^2 \times (\cot^2(\widehat{b'}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b'}_3)\cot(\widehat{c}_3)\cos(\widehat{b'c}) + \sin^2(\widehat{b'c})),$$

$$SA^2 = \left(\frac{bc}{b\cot(\widehat{b}_3) + a\cot(\widehat{a}_3) + c\cot(\widehat{c}_3)}\right)^2 \times (\cot^2(\widehat{b}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{b}_3)\cot(\widehat{c}_3)\cos(\widehat{bc}) + \sin^2(\widehat{bc})),$$

$$SB^2 = \left(\frac{ac}{a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)}\right)^2 \times (\cot^2(\widehat{a}_3) + \cot^2(\widehat{c}_3) + 2\cot(\widehat{a}_3)\cot(\widehat{c}_3)\cos(\widehat{ac}) + \sin^2(\widehat{ac})).$$

Alternatively, let h_S, h_R be the distances from S to AB and R to AB, respectively. Then

$$h_S = ab\sin(\widehat{ab})/(a\cot(\widehat{a}_3) + b\cot(\widehat{b}_3) + c\cot(\widehat{c}_3)),$$

$$h_R = a'b'\sin(\widehat{a'b'})/(a'\cot(\widehat{a'}_3) + b'\cot(\widehat{b'}_3) + c\cot(\widehat{c}_3)),$$

$$RS^2 = SA^2 + RA^2 - 2h_Rh_S\cos(\widehat{c}_3) - 2\sqrt{SA^2 - h_S^2}\sqrt{RA^2 - h_R^2}.$$

We conclude this paper with two problems.

Problem 1. What kind of symmetries (of Regge type or other) can be seen from Theorem 12 (and from possible hyperbolic analogue)?

Problem 2. Can we simulate reverse Coxeter proof in space? Start with (any?) tetrahedron $PQRS$ and reconstruct $ABCD$ such that $PQRS$ is the Morley tetrahedron of $ABCD$ with given dihedrals $\alpha_{i,j}$ (satisfying $\det(\cos(\alpha_{i,j})) = 0$), i.e. $ABCD$ is similar to a given tetrahedron.

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Gergonne Meets Sangaku

Paris Pamfilos

Abstract. In this article we discuss the relation of some hyperbolas, naturally associated with the Gergonne point of a triangle, with the construction of two equal circles known from a Sangaku problem at the temple of Chiba.

1. Hyperbolas related to the Gergonne point

The Gergonne point G of a triangle ABC is the common point of the three cevians $\{AA', BB', CC'\}$ ([8, p. 30]), joining the contact points of the incircle with the opposite vertices (see Figure 1). If we select one of these cevians, AA'

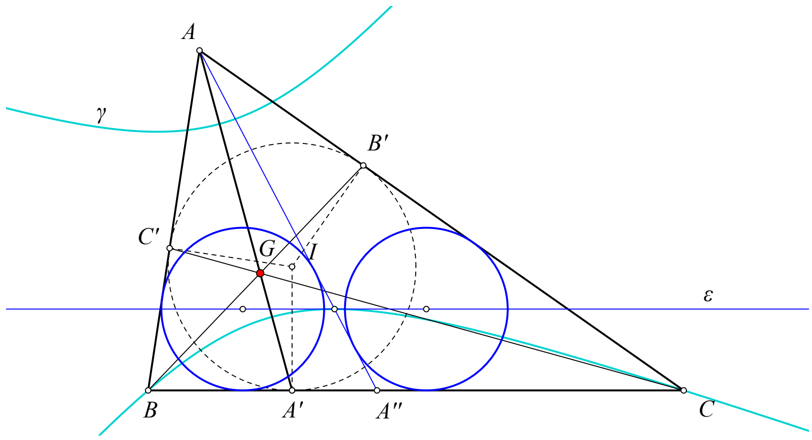

Figure 1. Hyperbola related to the "Gergonne cevian" AA'

say, then there is a unique hyperbola γ with focal points at $\{A, A'\}$, passing through points $\{B, C\}$. It turns out that the two equal "sangaku circles" ([7]), relative to the side BC, have their centers on a tangent ε to this hyperbola, which is parallel to BC.

The present article is devoted to the discussion of this shape, which, among other things, illustrates an easy way to construct the two equal circles. In fact, as it is seen below, the equation of the hyperbola with respect to its axes, of which AA' is per definition its transverse axis, is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where} \quad 2a = |AB'| = |AC'| \quad \text{and} \quad c^2 = a^2 + b^2 = |AA'|^2/4. \tag{1}$$

Once the hyperbola has been constructed and the parallel ε has been found, the centers of the two circles coincide with the intersections of ε with the bisectors of

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the angles $\{\widehat{B}, \widehat{C}\}$. In dealing with the details, we start with the investigation of general triangles on "focal chords" of hyperbolas. Having enough information and properties of this kind of triangles, we combine them with the properties of the sangaku circles, to obtain the reasoning behind this shape.

Before proceeding further, I must make a note on the naming conventions I am used to. I follow mainly the French naming convention, using the word "principal" or "major", for the circle having diametral points the two vertices of the hyperbola, which in English literature is called "auxiliary". The latter name I reserve instead for the two circles centered at the focal points with radius $2a$, which in French are often called "director circles" ([6, p. 367]) and can be confused with the "director" or "orthoptic circle" ([2, p. 229 II]), which is the locus of points viewing an ellipse or certain hyperbolas under a right angle.

2. Triangles on focal chords

A focal chord of a hyperbola, i.e. a chord PQ, passing through one focal point F, defines, using the other focus, a triangle PQF'. Consider now such a triangle

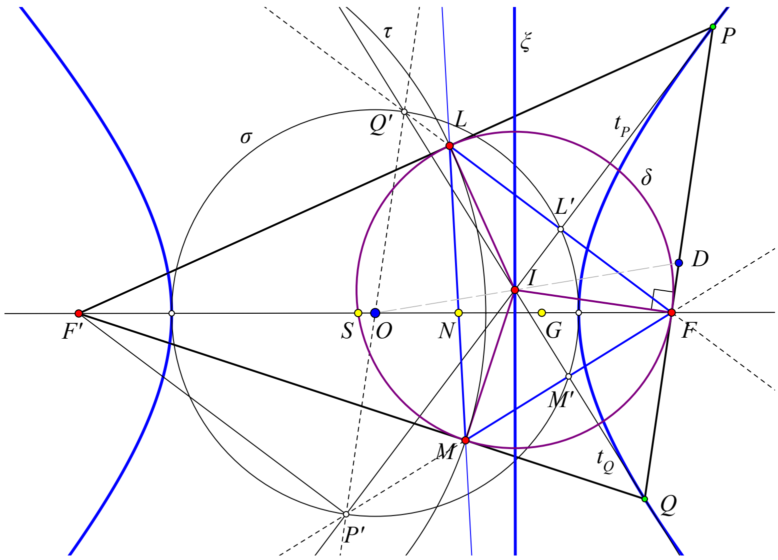

Figure 2. Triangle PQF' on the focal chord PFQ

with respect to the hyperbola with center O, given by equation (1). Let $\{t_P, t_Q\}$ denote the tangents at points $\{P, Q\}$ and $\{\sigma(O, a), \tau(F', 2a)\}$ be respectively the major and the auxiliary circle relative to F' (see Figure 2). Let further ξ be the directrix relative to the focus F. The following lemma formulates the main properties of these triangles. Its proof can be found in almost any book on conics or analytic geometry ([4, p. 76], [2, p. 226, II], [1, p. 8]).

Lemma 1. *Under the previous conventions, the following are valid properties.*

(1) *The tangents $\{t_P, t_Q\}$ are respectively bisectors of the angles $\{\widehat{P}, \widehat{Q}\}$.*

(2) *The projections* $\{L', M'\}$ *of* F *on these tangents are points of the major circle* $\sigma(O, a)$*.*
(3) *The reflected points* $\{L, M\}$ *of* F *on these tangents are points of the auxiliary circle* $\tau(F', 2a)$*.*
(4) *The orthogonal to* PQ *at* F *intersects the directrix* ξ *at a point* I*, which is the incenter of triangle* PQF'*.*
(5) *The radii* $\{IL, IM\}$ *of the incircle* δ *of* PQF' *are tangent to the auxiliary circle* τ*.*

3. The Gergonne point

The following theorem supplies some less known properties of these triangles PQF' (see Figure 2), that seem to me interesting for their own sake, the last one being of use also in our particular problem. In this $E = (AB, CD)$ denotes the intersection of lines $\{AB, CD\}$, $E(ABCD)$ denotes the pencil of four lines or "rays" $\{EA, EB, EC, ED\}$ through E and $D = C(A, B)$ denotes the harmonic conjugate of C relative to $\{A, B\}$.

Theorem 2. *Continuing with the previous conventions, the following are valid properties.*
(1) *Lines* $\{LM, \xi, PQ\}$ *are concurrent at a point* N' *(not shown in Figure 2).*
(2) *The pole* N *of the directrix* ξ*, relative to the auxiliary circle* τ*, coincides with the intersection* $N = (LM, FF')$*. Hence* LM *passes through the fixed* N*, depending only on the hyperbola and not on the particular direction of the chord* PQ *through* F*.*
(3) *The incircle* δ *intersects the transverse axis a second time at a point* S*, which is fixed and independent of the direction of* PQ*.*
(4) *Similarly, the Gergonne point* G *of the triangle* PQF' *is fixed on the transverse axis* FF' *and independent of the direction of* PQ *through* F*.*
(5) *The center* O *of the hyperbola, the incenter* I *of triangle* PQF' *and the middle* D *of side* PQ *are collinear.*

Proof. Nr-1. can be proved by defining initially N' to be the intersection $N' = (\xi, PQ)$ and considering the two pencils $\{F'(N'FQP), F(N'F'ML)\}$, which have the ray FF' in common. Using the characteristic property of the polar ξ, it is easy to see, that both pencils are harmonic. Hence they define the same cross ratio on every line intersecting them and not passing through their centers $\{F, F'\}$. Applying then the well known property of such pencils with a common ray ([5, p. 90]), we see that the three other pairs of corresponding rays intersect at collinear points $N' = (F'N', FN'), M = (F'Q, FM')$ and $L = (F'P, FL')$.

Nr-2. can be proved by first observing that the incenter I is the pole of LM relative to the circle τ and lies also on line ξ. Then, by the reciprocity of pole-polar ([5, p. 166]), the pole of ξ relative to τ must also be contained in LM. Since this pole is also on the orthogonal FF' to line ξ, it must coincide with $N = (LM, FF')$.

Nr-3. follows from *nr-2.* by comparing powers relative to the circles $\{\sigma, \delta\}$. In fact, the power of N relative to δ : $|NS||NF| = |NL||NM| = p$ is also the power of N relative to τ, shown above to be fixed. Hence p is constant, which implies the claim.

Nr-4. follows from the coincidence of the Gergonne point of PQF' with the symmedian point of the triangle FLM, which has PQF' as its tangential triangle ([8, p. 56]). But it is well known that the symmedian point G of FLM is the harmonic conjugate $G = F'(N, F)$ of F' relative to $\{N, F\}$ ([3, p. 160]).

Nr-5. follows by inspecting the quadrilateral $PQP'Q'$. Here $\{P', Q'\}$ denote the second intersections of the principal circle σ respectively with lines $\{FL', FM'\}$. By Lemma 1, follows that the tangents $\{t_P, t_Q\}$ pass respectively through $\{P', Q'\}$ and that $P'Q'$ is a diameter of the circle σ, seen from $\{L', M'\}$ under a right angle. This means that $\{Q'M', P'L'\}$ are two altitudes of the triangle $FP'Q'$ meeting at I. Hence FI extended is also an altitude of this triangle, and, consequently, orthogonal to $P'Q'$. Since FI is also orthogonal to PQ the two lines $\{PQ, P'Q'\}$ are parallel. Hence, $PQP'Q'$ is a trapezium, its diagonals meeting at I, which is collinear with the middles $\{O, D\}$ of its parallel sides $\{P'Q', PQ\}$. □

The next lemma simply stresses the fact that every triangle can be considered to be based on a focal chord of an appropriate hyperbola.

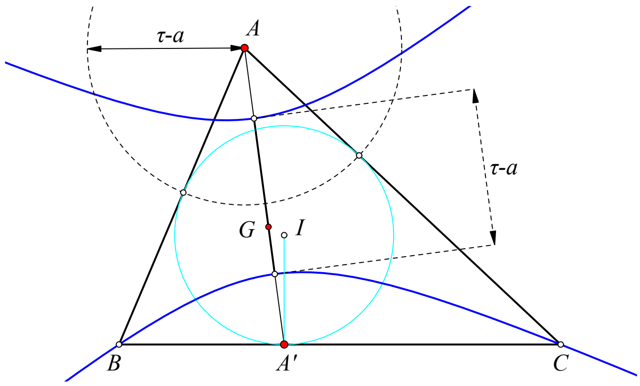

Figure 3. The hyperbola associated to the "Gergonne cevian" AA'

Lemma 3. *For every triangle ABC and each one of the three "Gergonne cevians", i.e. cevians like AA', passing through the Gergonne point G, there is precisely one hyperbola with focal points at $\{A, A'\}$ and passing through the other vertices $\{B, C\}$.*

Proof. By Figure (see Figure 3). Since the hyperbola is uniquely defined by its focal points $\{A, A'\}$ and the property

$$||BA| - |BA'|| = ||CA| - |CA'|| = \tau - a, \tag{2}$$

where τ denotes now the semi-perimeter of the triangle and $a = |BC|$. □

4. The critical tangent

Continuing with the notation established in the preceding sections, we examine now the tangent ε to the hyperbola, which is parallel to the side BC of the triangle.

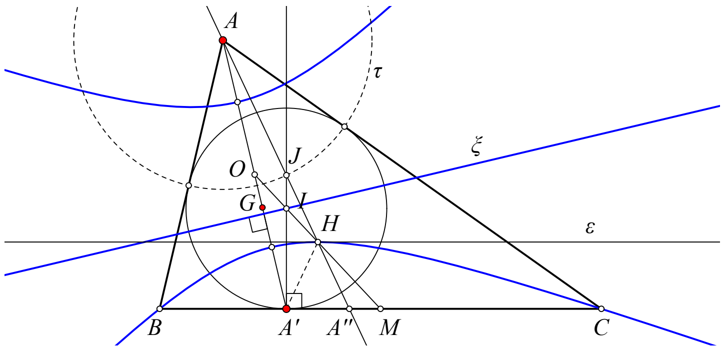

Figure 4. The tangent ε parallel to BC

Lemma 4. *Let M be the middle of the side BC and H be the intersection of the hyperbola with line OM. Then the tangent ε of the hyperbola at H is parallel to BC, points $\{O, I, H, M\}$ are collinear and points $\{A, J, H\}$ are also collinear, J being the symmetric of A' relative to ε.*

Proof. The parallelism of ε to BC is obvious, since line OM, joining the center O of the hyperbola with the middle of the chord BC defines the conjugate direction to that of BC. By Theorem 2, I is on OM, hence the first claimed collinearity. By lemma-1, the symmetric J relative to ε is on τ and points $\{A, J, H\}$ are collinear. $\square$

The connection with the sangaku circles results now from the fact, that the intersection point $A'' = (AJ, BC)$ defines the two "subtriangles" $\{ABA'', AA''C\}$, the incircles of which are the two equal sangaku circles of the triangle ABC relative to the side BC ([7]). Thus, it remains to show that the centers of these circles are on the line ε. But this is a general property of the hyperbola, as seen by the following lemma.

Lemma 5. *With the notation adopted so far, the incenter of the triangle ABA'' is on the line ε.*

Proof. Consider the intersection R of the bisector t_B of the angle $\widehat{B}$ with line ε. It suffices to show that the other bisector, of angle $\widehat{BA''A}$, passes through R. To see this, examine the triangle $A'HR$. Because R is the intersection point of two tangents $\{t_B, \varepsilon\}$ of the hyperbola at points $\{B, H\}$, the angle $\widehat{BA'H}$ from the focus A' to the points of tangency is bisected by $A'R$ ([1, p. 12]). Since $\{BC, \varepsilon\}$ are parallel, we have also $\widehat{RA'B} = \widehat{A'RH}$. Thus, $A'HR$ is isosceles and $|HR| = |HA'| = |HA''|$. Last equality being valid because $JA'A''$ is a right

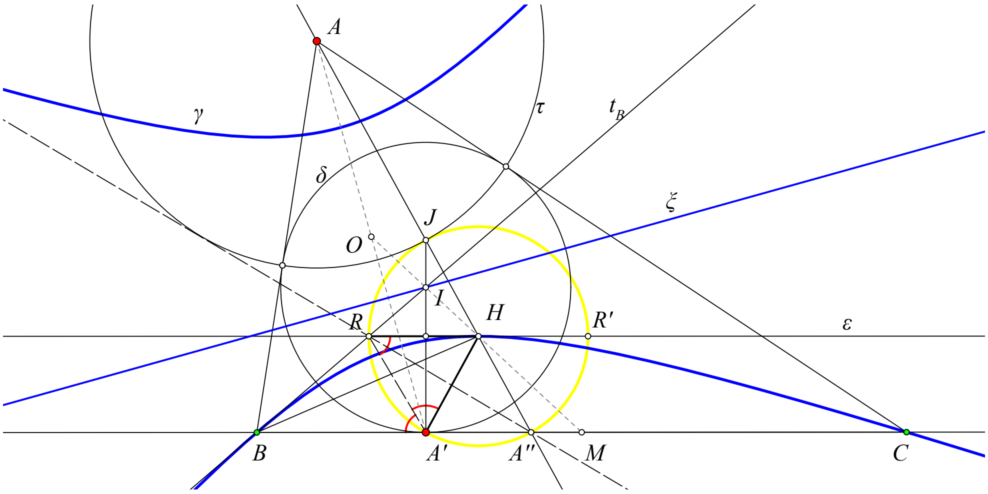

Figure 5. Incenter of triangle ABA''

triangle and H is the middle of its hypotenuse. This shows that triangle RHA'' is also isosceles and $\widehat{HA''R} = \widehat{HRA''} = \widehat{RA''A'}$, thereby proving that $A''R$ is a bisector of the angle $\widehat{BA''A}$. □

From the arguments, used in the proof given above, results also an alternative way to find the centers of the two sangaku circles, expressed by the following corollary.

Corollary 6. *The centers of the two equal sangaku circles relative to the side BC of the triangle are the diametral points $\{R, R'\}$ of a diameter parallel to BC of the circle with diameter JA''.*

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New Interpolation Inequalities to Euler's $R \geq 2r$

Dorin Andrica and Dan Ştefan Marinescu

Abstract. The purpose of this paper is to obtain some interpolation inequalities to the well-known Euler's inequality $R \geq 2r$ in terms of new geometric elements given by the radii R_A, R_B, R_C of the tangent circles at the vertices to the circumcircle of a triangle and to the opposite sides. The main results are given in Theorems 4-8.

1. Introduction

At the first 2015 Romanian IMO Team Selection Test the first author of this paper has proposed the following problem: Let R_A be the radius of the tangent circle at A to the circumcircle of triangle ABC and to the side BC. Similarly, define the radii R_B and R_C. The following inequality holds

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r},$$

where r is the inradius of triangle ABC.

In this short paper we discuss some proofs to the above inequality and we complete it to the left hand-side in order to get a new interpolation for the well-known Euler's inequality $R \geq 2r$, where R is the circumradius of triangle ABC. Also, we give other interpolation inequalities to the Euler's inequality in terms of the radii R_A, R_B, R_C. For other interpolation and improvements inequalities to the Euler's inequality we refer to the excellent monograph [2].

2. Some auxiliary results

As usual, we denote by a, b, c the lengths of the sides opposite to the vertices A, B, C, respectively, and by $K[ABC]$ the area of triangle ABC. We need the following helpful results.

Lemma 1. *In triangle ABC denote by h_a, h_b, h_c the lengths of the altitudes from the vertices A, B, C, respectively. The relation*

$$\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}$$

holds.

Proof. Just use the formula for the area of a triangle. □

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Lemma 2. *If R_A is the radius of the interior (exterior) tangent circle at A to the circum- circle of triangle ABC and to the side BC, then*

$$\frac{r}{R_A} = \frac{a}{s} \cos^2 \frac{B-C}{2},$$

where s denotes the semiperimeter of triangle ABC.

Proof 1. If $B = C$, then clearly we have $R_A = \frac{h}{2}$. Using the relations $K[ABC] = sr = a\frac{h_a}{2}$, the conclusion follows.

When $B \neq C$, let us suppose that $B > C$. Consider T the intersection point of the common tangent line at A to the two circles with the line BC (see Figure 1). In triangle TAB, we have $\hat{T} = B - C$ and from the Law of Sines we obtain

$$\frac{c}{\sin(B-C)} = \frac{TA}{\sin B} \implies TA = \frac{bc}{2Rsin(B-C)}.$$

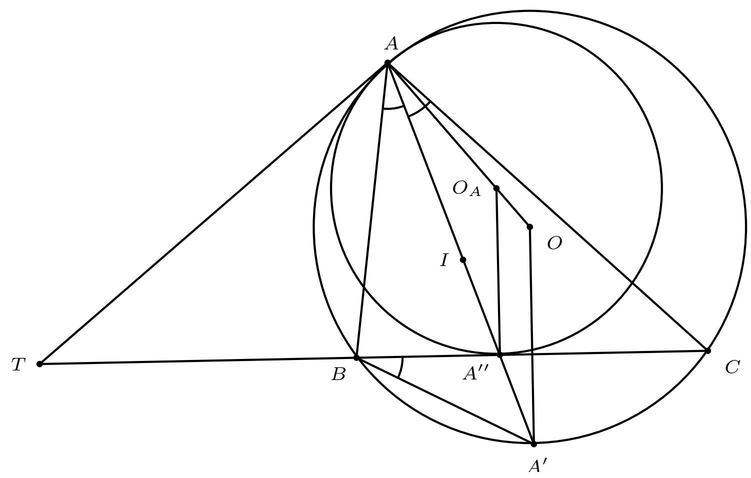

Figure 1

Because $\tan\frac{B-C}{2} = \frac{R_A}{TA}$, it follows that

$$R_A = \frac{bc}{2Rsin(B-C)} \cdot \frac{\sin\frac{B-C}{2}}{\cos\frac{B-C}{2}} = \frac{bc}{2R\cos^2\frac{B-c}{2}}.$$

Therefore,

$$\frac{r}{R_A} = \frac{r}{bc} \cot 4R \cos^2 \frac{B-C}{2} = \frac{ar}{4RK} \cos^2 \frac{B-c}{2} = \frac{a}{s} \cos^2 \frac{B-C}{2},$$

where $K = K[ABC]$, and the proof is complete. □

Proof 2. Let γ_A be the circle tangent at A to the circumcircle of triangle ABC and tangent at T to the line BC. Assume that $R = 1$, and consider the inversion of pole A and unit power. In what follows, X' will denote the image of the point $X \neq A$ by this inversion.

Under this inversion, the line BC is transformed into a circle $AB'C'$ centered at some point Ω. The circle ABC is transformed into the line $B'C'$, and γ_A is transformed into a line ℓ through T' and parallel to $B'C'$.

Let D be the orthogonal projection of A on the line BC. Then $AD = \frac{1}{AD} = \frac{1}{h_a}$, where h_a is the length of the altitude from the vertex A in the triangle ABC, and $\Omega T' = \Omega A = \frac{1}{2h_a}$.

Next, let A_1 be the antipode of A in circle γ_A, so A_1' is the orthogonal projection of A on line ℓ, and $AA_1' = \frac{1}{AA_1'} = \frac{1}{2R_A}$.

Finally, let O denote the circumcenter of the triangle ABC and notice the angles OAD, $\Omega AA_1'$ are both congruent to the absolute value of the difference of the internal angles of triangle ABC at B and C, to obtain

$$\cos(B-C) = \frac{AA_1' - \Omega T'}{\Omega A} = \frac{\frac{1}{2R_A} - \frac{1}{2h_a}}{\frac{1}{2R_A}} = \frac{h_a}{R_A} - 1 = \frac{2K}{aR_A} - 1,$$

where $K = K[ABC]$ and the desired formula follows after standard transformations. $\square$

Lemma 3. *In every triangle ABC the following inequality holds*

$$\cos^2 \frac{B-C}{2} \geq \frac{2r}{R}.$$

We have equality if and only if $2a = b + c$.

Proof 1. We have

$$\begin{aligned}
\cos\frac{B-C}{2} &= \cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2} \\
&= \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2} \\
&= \cos\frac{B+C}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2} \\
&= \sin\frac{A}{2} + 2\sin\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

Therefore,

$$\cos\frac{B-C}{2} \geq 2\sqrt{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 2\sqrt{2\cdot\frac{r}{4R}} = \sqrt{\frac{2r}{R}},$$

and the conclusion follows. The equality holds if and only if $\sin\frac{A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2}$, that is $a^2 = 4(s-a)^2$, hence $2a = b + c$. $\square$

Proof 2. Let I be the incenter of triangle ABC, and consider A' the intersection point of the ray AI with the circumcircle of triangle ABC. We have

$$A'A^2 = (A'I + AI)^2 \geq 4A'I \cdot AI = 8Rr,$$

where the last equality is obtained from the power of I with respect to the circumcircle of triangle ABC. Clearly, the equality holds if and only if $A'I = AI$. But

$$AA' = 2R\sin\left(B + \frac{A}{2}\right) = 2R\cos\frac{B-C}{2},$$

hence the desired inequality follows. As we already mentioned, the equality holds if and only if $A'I = AI$, that is $AA' = 2IA' = 2BA'$, so

$$\cos\frac{B-C}{2} = 2\sin\frac{A}{2}.$$

We obtain

$$\sin\frac{A}{2} = 2\sin\frac{B}{2}\sin\frac{C}{2},$$

therefore $2a = b + c$. □

3. The main results

The first interpolation result is directly connected to the original problem mentioned in the introduction and it is contained in the following theorem.

Theorem 4. *With the above notations the following inequalities hold*

$$\frac{4}{R} \leq \frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \leq \frac{2}{r}. \tag{1}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. From Lemma 2 we have $\frac{r}{R_A} \leq \frac{a}{s}$, with equality if and only if $B = C$. Similarly, $\frac{r}{R_B} \leq \frac{b}{s}$ with equality when $C = A$, and $\frac{r}{R_C} \leq \frac{c}{s}$ with equality when $A = B$. Summing up these inequalities it follows the right hand-side inequality, with equality if and only if $A = B = C$, that is the triangle is equilateral.

From Lemma 3 and Lemma 2 we have $\frac{r}{R_A} \geq \frac{a}{s} \cdot \frac{2r}{R}$, with equality if and only if $2a = b + c$, and two analogous inequalities for the radii R_B and R_C. Summing up these inequalities we obtain

$$\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C} \geq \frac{2}{R} \cdot \frac{a+b+c}{s} = \frac{4}{R},$$

and we are done. □

Remark. (1) It is possible to give a direct geometric argument for the right hand-side inequality in (1). Consider OA to be the center of the tangent circle at A to the circumcircle of triangle ABC and to the side BC, and A'' the tangency point of this circle with the line BC (see Figure 1). Using the triangle inequality in triangle AO_AA'' we have $h_a \leq AA''leAO_A + O_AA'' = 2R_A$, hence we obtain $\frac{1}{2R_A} \leq \frac{1}{h_a}$, and other two similar inequalities for R_B and R_C. Summing up these inequalities the conclusion follows from Lemma 1.

Theorem 5. *With the above notations the following inequalities hold*

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}} \leq 1. \tag{2}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. Multiplying the inequalities obtained from Lemma 2, we obtain

$$\frac{r^3}{R_AR_BR_C} \leq \frac{abc}{s^3},$$

hence

$$\frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}} \leq 1.$$

On the other hand, multiplying the inequalities obtained from Lemma 2 and using Lemma 3, it follows that

$$\frac{abc}{s^3} \cdot \frac{8r^3}{R^3} \leq \frac{r^3}{R_AR_BR_C}.$$

That is

$$\frac{2r}{R} \leq \frac{K[ABC]}{\sqrt[3]{abcR_AR_BR_C}},$$

and we complete the left hand-side of (2). Clearly, the equality holds if and only if the triangle ABC is equilateral. $\square$

From the relation $\frac{r}{R_A} = \frac{a}{s} \cdot \cos^2 \frac{B-C}{2}$ proved in Lemma 2, we obtain

$$R_A = \frac{K}{a\cos^2\frac{B-C}{2}}.$$

In the second proof of Lemma 3 we have shown that $AA' = 2R\sin\left(B + \frac{A}{2}\right) = 2R\cos\frac{B-C}{2}$, hence $\cos\frac{B-C}{2} = \frac{AA'}{2R}$. It is clear that the point A'' is the feet of the bisector of the angle A of triangle ABC. Denote by ℓ_a the length of bisector of angle A of triangle ABC, i.e. the length of the segment $[AA'']$. Triangles $AA''O_A$ and $AA'O$ are similar, therefore we obtain

$$\frac{R_A}{R} = \frac{\ell_a}{AA'} = \frac{\ell_a^2}{\ell_a \cdot AA'}.$$

From the Law of Sines in triangle ACA'', it follows that

$$\frac{\ell_a}{\sin C} = \frac{b}{\sin\left(C + \frac{A}{2}\right)}.$$

But, clearly we have

$$\sin\left(C + \frac{A}{2}\right) = \sin\left(B + \frac{A}{2}\right) = \cos\frac{B-C}{2},$$

hence $\ell_a \cdot AA' = 2Rb\sin C$. We obtain

$$R_A = \frac{\ell_a^2}{2b\sin C} = \frac{\ell_a^2}{2h_a} = \frac{a \cdot \ell_a^2}{4K}, \tag{3}$$

where $K = K[ABC]$ is the area of triangle ABC.

Theorem 6. *With the above notations the following inequalities hold*

$$\frac{9}{2}r \le R_A + R_B + R_C \le \frac{9}{4}R. \tag{4}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. The left hand-side inequality can be proved using the inequality

$$R_A = \frac{K}{a\cos^2\frac{B-C}{2}} = \frac{sr}{a\cos^2\frac{B-C}{2}} \ge \frac{sr}{a},$$

where the equality holds if and only if $B = C$, and other two similar inequalities for R_B and R_C. We obtain

$$R_A + R_B + R_C \ge rs\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{r}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge \frac{9}{2}r,$$

with equality if and only if $a = b = c$.

From (3) and from the well-known formula $\ell_a^2 = \frac{4bc}{(b+c)^2}s(s-a)$, the right hand-side inequality is equivalent to

$$\sum_{\text{cyclic}} \frac{4abc}{4K}\cdot\frac{s(s-a)}{(b+c)^2} \le \frac{9}{4}R,$$

hence X

$$\sum_{\text{cyclic}} \frac{4RK}{K}\cdot\frac{s(s-a)}{(b+c)^2} \le \frac{9}{4}R,$$

that is

$$\sum_{\text{cyclic}} \frac{s(s-a)}{(b+c)^2} \le \frac{9}{16}. \tag{5}$$

The inequality (5) is equivalent to

$$\sum_{\text{cyclic}} \frac{1-\frac{a}{s}}{\left(\frac{b}{s}+\frac{c}{s}\right)^2} \le \frac{9}{16}.$$

Let $\frac{a}{s} = 2x$, $\frac{b}{s} = 2y$, $\frac{c}{s} = 2z$, where $x, y, z > 0$ and $x+y+z = 1$. The inequality (5) os equivalent to

$$\sum_{\text{cyclic}} \frac{1-2x}{(y+z)^2} \le \frac{9}{4},$$

for every $x, y, z > 0$ with $x+y+z = 1$. Hence, it is reduced to

$$\sum_{\text{cyclic}} \frac{1-2x}{(1-x)^2} \le \frac{9}{4},$$

for every $x, y, z > 0$ with $x+y+z = 1$.

The function $f : (0,1) \to \mathbb{R}$ defined by $f(t) = \frac{1-2t}{(1-t)^2}$ has second derivative

$$f''(t) = \frac{-4t-2}{(1-t)^4} < 0.$$

That is, it is concave on the interval $(0,1)$. From Jensen's inequality it follows that

$$f(x)+f(y)+f(z) \leq 3f\left(\frac{x+y+z}{3}\right) = 3f\left(\frac{1}{3}\right) = \frac{9}{4}, \tag{6}$$

and the result is completely proved. $\square$

The function f in the proof of Theorem 6 satisfies $f'' < -2$ on the interval $(0,1)$. Therefore, using the result of [1], the function $g: (0,1) \to \mathbb{R}$ defined by $g(t) = f(t) + t^2$ is concave on $(0,1)$. Applying the Jensen's inequality for g, we get the following inequality for f:

$$f(x)+f(y)+f(z) \leq 3f\left(\frac{x+y+z}{3}\right) - \frac{1}{3}\left((x-y)^2+(y-z)^2+(z-x)^2\right). \tag{7}$$

Considering $x = \frac{a}{2s}$, $y = \frac{b}{2s}$, $z = \frac{c}{2s}$, the inequality (7) is equivalent to

$$\sum_{\text{cyclic}} \frac{4s(s-a)}{(b+c)^2} \leq \frac{9}{4} - \frac{1}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right),$$

that is

$$\sum_{\text{cyclic}} \frac{4RK}{K} \cdot \frac{4s(s-a)}{(b+c)^2} \leq \frac{9}{4}R - \frac{R}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right),$$

and we obtain the following refinement of right-hand side inequality in Theorem 6:

Theorem 7. *With the above notations the following inequality holds*

$$R_A + R_B + R_C \leq \frac{9}{4}R - \frac{R}{12s^2}\left((a-b)^2+(b-c)^2+(c-a)^2\right), \tag{8}$$

with equality if and only if the triangle ABC is equilateral.

Remark. (2) The radius R_A can be expressed in terms of the exradius r_a of the triangle ABC as follows:

$$R_A = \frac{\ell_a^2}{2h_a} = \frac{4bc}{(b+c)^2} \cdot \frac{s(s-a)}{4K/a} = \frac{abcs}{r_a(b+c)^2},$$

and similar formulas for R_B and R_C. We obtain the following formula connecting all the radii R_A, R_B, R_C, R, r:

$$\frac{1}{\sqrt{R_A r_a}} + \frac{1}{\sqrt{R_B r_b}} + \frac{1}{\sqrt{R_C r_c}} = \frac{2}{\sqrt{Rr}}. \tag{9}$$

Using the Cauchy-Schwarz inequality and formula (9) we can write

$$\begin{aligned}
\frac{4}{Rr} &= \left(\frac{2}{\sqrt{Rr}}\right)^2 = \left(\frac{1}{\sqrt{R_A r_a}} + \frac{1}{\sqrt{R_B r_b}} + \frac{1}{\sqrt{R_C r_c}}\right)^2 \\
&\leq \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right) \\
&= \left(\frac{1}{R_A} + \frac{1}{R_B} + \frac{1}{R_C}\right)\frac{1}{r},
\end{aligned}$$

where we have used the well-known formula $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$. Therefore, we have obtained a new proof to the left-hand side inequality in Theorem 4.

The last result contains two weighted interpolation results.

Theorem 8. *With the above notations the following inequalities hold:*

$$6r \le \frac{a}{a+b+c}R_A+\frac{b}{a+b+c}R_B+\frac{c}{a+b+c}R_C \le 3R; \tag{10}$$

$$\frac{s}{2R} \le \frac{R_A}{a}+\frac{R_B}{b}+\frac{R_C}{c} \le \frac{s}{4r}. \tag{11}$$

We have equality if and only if the triangle ABC is equilateral.

Proof. From Lemma 2 we have a $aR_A=\frac{rs}{a\cos^2\frac{B-C}{2}}$, and using the inequality in Lemma 3, it follows that $rs \le aR_A \le \frac{sR}{2}$, and two similar inequalities for R_B and R_C. Summing up these inequalities we get (10).

For the right-hand side inequality in (11), from $\frac{R_A}{a}=\frac{\ell_a}{4s}$, using the inequality $\ell_a^2 \le s(s-a)$, we obtain

$$\frac{R_A}{a}+\frac{R_B}{b}+\frac{R_C}{c} \le \sum_{\text{cyclic}} \frac{s(s-a)}{4s}=\frac{s}{4r}.$$

For the left-hand side inequality in (11), we use $R_A=\frac{rs}{a\cos^2\frac{B-C}{2}} \ge \frac{rs}{a}$, and we obtain $\frac{R_A}{a} \ge \frac{rs}{a^2}$, and two similar inequalities for R_B and R_C. Then

$$\begin{aligned}\frac{R_A}{a}+\frac{R_B}{b}+\frac{R_C}{c} &\ge rs\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\\ &\ge rs\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\\ &= rs\cdot\frac{2s}{abc}=rs\cdot\frac{2s}{4Rrs}=\frac{s}{2R},\end{aligned}$$

and we are done. □

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On the Tucker Circles

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Abstract. Parametrizing Tucker circles by the lengths of their antiparallel sides, we find conditions for which Tucker circles are congruent, orthogonal, or tangential. In particular, we show that the Gallatly circle, which is the common pedal circle of the Brocard points, is the smallest Tucker circle, not orthogonal to any Tucker circle, and congruent Tucker circles are symmetric with respect to the line joining the Brocard points. Some orthology results are also obtained.

1. The Tucker hexagon $\mathscr{T}(t)$

Given triangle ABC, let B_a and C_a be points on the sidelines AC and AB such that triangle AB_aC_a is oppositely similar to ABC. The line B_aC_a is antiparallel to BC, meaning that B_aC_a is parallel to the side H_bH_c of the orthic triangle $H_aH_bH_c$ of triangle ABC (see Figure 1). Thus, we have through
B_a the antiparallel to BC to intersect AB at C_a. Continue to construct through
C_a the parallel to CA to intersect BC at A_c, then through
A_c the antiparallel to AB to intersect CA at B_c, then through
B_c the parallel to BC to intersect AB at C_b, then through
C_b the antiparallel to CA to intersect BC at A_b, then through
A_b the parallel to AB to intersect the line CA.
This last intersection is the same as the point B_a, thus completing a hexagon $B_aC_aA_cB_cC_bA_b$ whose sides are alternately antiparallel and parallel to the sides of triangle ABC. This is called a Tucker hexagon.

Let a, b, c be the lengths of the sides BC, CA, AB of triangle ABC, and R its circumradius. Suppose $B_aC_a = t$, positive or negative according as B_a and C_a are on the half-lines AC and AB or their complementary half-lines. Then $AC_a = \frac{bt}{a}$, $AB_a = \frac{ct}{a}$. It follows that $BA_b = \frac{ct}{b}$, $BC_b = \frac{at}{b}$, $CA_c = \frac{bt}{c}$, $CB_c = \frac{at}{c}$. Triangles A_bBC_b and A_cB_cC are also oppositely similar to ABC. Also, $B_aC_a = C_bA_b = A_cB_c = t$. The three antiparallel sides of $\mathscr{T}(t)$ have equal lengths t. With reference to triangle ABC, the vertices of the Tucker hexagon $\mathscr{T}(t)$ have homogeneous barycentric coordinates

$B_a = (ab - ct : 0 : ct)$	$C_a = (ca - bt : bt : 0)$
$C_b = (at : bc - at : 0)$	$A_b = (0 : ab - ct : ct)$
$A_c = (0 : bt : ca - bt)$	$B_c = (at : 0 : bc - at)$

(1)

We shall also make use of the *absolute* barycentric coordinates of finite points by normalizing their homogeneous coordinates, i.e., by dividing by their coordinate sum.

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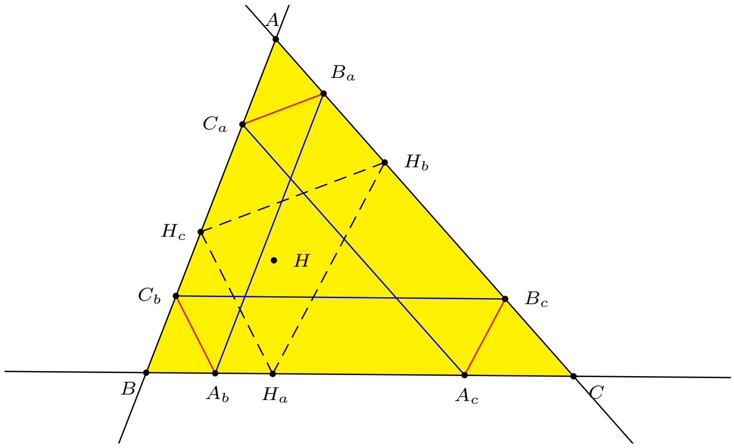

Figure 1. A Tucker hexagon

It is convenient to make use of the elementary symmetric functions of a^2, b^2, c^2:

$$\lambda := a^2 + b^2 + c^2, \quad \mu := b^2c^2 + c^2a^2 + a^2b^2, \quad \nu := a^2b^2c^2. \tag{2}$$

We shall also denote by S *twice* the area of triangle ABC.

In absolute barycentric coordinates, the circumcenter and the symmedian point of triangle ABC are the points

$$O = \frac{1}{4S^2}(a^2(\lambda - 2a^2),\ b^2(\lambda - 2b^2),\ c^2(\lambda - 2c^2)), \tag{3}$$

$$K = \frac{1}{\lambda}(a^2,\ b^2,\ c^2). \tag{4}$$

Lemma 1. (a) $4\mu - \lambda^2 = 4S^2$.
(b) $\lambda^2 - 3\mu \geq 0$.

Proof. (a)

$$\begin{aligned} 4\mu - \lambda^2 &= 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c) \\ &= 4S^2. \end{aligned}$$

(b)

$$\begin{aligned} \lambda^2 - 3\mu &= a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2 \\ &= \frac{1}{2}\left((b^2-c^2)^2 + (c^2-a^2)^2 + (a^2-b^2)^2\right). \end{aligned}$$

$\square$

Proposition 2. *The midpoints L_a, L_b, L_c of the antiparallel sides of $\mathscr{T}(t)$ are on the symmedians AK, BK, CK respectively, and divide AK, BK, CK in the same ratio*

$$AL_a : L_aK = BL_b : L_bK = CL_c : L_cK = \lambda t : 2\sqrt{\nu} - \lambda t \tag{5}$$

Proof. The midpoint of the antiparallel side B_aC_a is

$$\begin{aligned} L_a &= \frac{1}{2}(B_a + C_a) \\ &= \frac{1}{2abc}(2abc - (b^2+c^2)t,\ b^2t,\ c^2t) \\ &= \frac{1}{2\sqrt{\nu}}\left((2\sqrt{\nu} - \lambda t,\ 0,\ 0) + (a^2t, b^2t, c^2t)\right) \\ &= \frac{1}{2\sqrt{\nu}}(2\sqrt{\nu} - \lambda t)A + \lambda tK). \qquad (6) \end{aligned}$$

This shows that L_a is a point on the symmedian AK, and it divides AK in the ratio $AL_a : L_aK = \lambda t : 2\sqrt{\nu} - \lambda t$ (see Figure 2).

The same is true for L_b and L_c. □

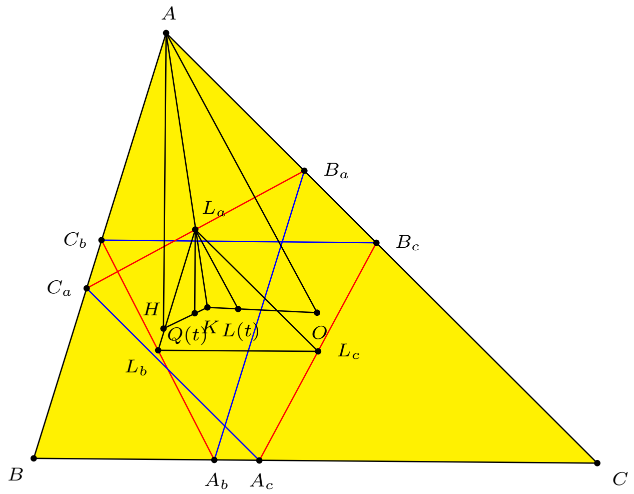

Figure 2

Proposition 3. (a) *The triangles ABC and $L_aL_bL_c$ are homothetic at the symmedian point K.*

(b) *They are also orthologic.*

(i) *The perpendiculars from A to L_bL_c, B to L_cL_a, and C to L_aL_b are concurrent at the orthocenter H.*

(ii) *The perpendiculars from L_a to BC, L_b to CA, and L_c to AB are concurrent at the point Q dividing HK in the ratio $HQ(t) : Q(t)K = \lambda t : 2\sqrt{\nu} - \lambda t$.*

Proof. (a) follows from (5).

(b) The orthology follows from the homothety.

(i) is clear.

(ii) The perpendicular from L_a to BC, being parallel to AH, intersects HK at a point $Q(t)$ such that $HQ(t) : Q(t)K = AL_a : L_aK = \lambda t : 2\sqrt{\nu} - \lambda t$ (see Figure 2). By Proposition 2, the perpendiculars from L_b to CA and L_c to AB intersect HK at the same point $Q(t)$. □

Proposition 4. (a) *The perpendicular bisectors of the antiparallel sides B_aC_a, C_bA_b, A_cB_c of the Tucker hexagon $\mathscr{T}(t)$ are concurrent at the point $L(t)$ dividing OK in the ratio*

$$OL(t) : L(t)K = \lambda t : 2\sqrt{\nu} - \lambda t.$$

(b) *The point $L(t)$ is at a distance $\frac{2\sqrt{\nu}-\lambda t}{4S}$ from each of the antiparallels.*

Proof. (a) From (6) it follows that

$$2\sqrt{\nu}L_a + (\lambda t - 2\sqrt{\nu})A = \lambda t K,$$

and

$$2\sqrt{\nu}L_a + (\lambda t - 2\sqrt{\nu})(A - O) = (2\sqrt{\nu} - \lambda t)O + \lambda t K.$$

This means that the parallel through L_a to OA intersects OK at a point $L(t)$ dividing OK in the ratio

$$OL(t) : L(t)K = \lambda t : 2\sqrt{\nu} - \lambda t; \tag{7}$$

see Figure 2. Since the coefficients are all symmetric functions of a^2, b^2, c^2, the analogues of (7) hold when L_a, A are replaced by L_b, B, and L_c, C respectively. This means that the parallels through L_a, L_b, L_c to OA, OB, OC are concurrent at the same point $L(t)$ (see Figure 3).

(b) The antiparallel side B_aC_a, being parallel to the side H_bH_c of the orthic triangle, is perpendicular to the circumradius OA. Equation (7) shows that $L(t)$ is at a distance

$$\left(1 - \frac{\lambda t}{2\sqrt{\nu}}\right) \cdot R = \frac{2\sqrt{\nu} - \lambda t}{2\sqrt{\nu}} \cdot \frac{\sqrt{\nu}}{2S} = \frac{2\sqrt{\nu} - \lambda t}{4S}$$

from B_aC_a. This is the same for the antiparallels C_bA_b and A_cB_c. $\square$

Remark. In homogeneous barycentric coordinates,

$$\begin{aligned} L(t) = \; & (a^2(abc(b^2 + c^2 - a^2) + t(a^2(b^2 + c^2) - (b^4 + c^4)) \\ & : \; b^2(abc(c^2 + a^2 - b^2) + t(b^2(c^2 + a^2) - (c^4 + a^4)) \\ & : \; c^2(abc(a^2 + b^2 - c^2) + t(c^2(a^2 + b^2) - (a^4 + b^4))). \end{aligned} \tag{8}$$

Corollary 5 (Construction of Tucker hexagon)**.** *Let $H_aH_bH_c$ be the orthic triangle of ABC, and L a point on the Brocard axis. If the parallels through L to the circumradii OA, OB, OC intersect the symmedians AK, BK, CK at L_a, L_b, L_c respectively, then the parallels through L_a to H_bH_c, L_b to H_cH_a, and L_c to H_aH_b intersect the sidelines of triangle ABC at the vertices of a Tucker hexagon* (see Figure 3).

2. The Tucker circle $\mathscr{C}(t)$

Proposition 6. *The vertices of the Tucker hexagon $\mathscr{T}(t)$ are concyclic. The circle containing them has center $L(t)$ and radius $\mathcal{R}(t)$ given by*

$$\mathcal{R}(t)^2 = \frac{\mu t^2 - \lambda\sqrt{\nu}t + \nu}{4S^2}. \tag{9}$$

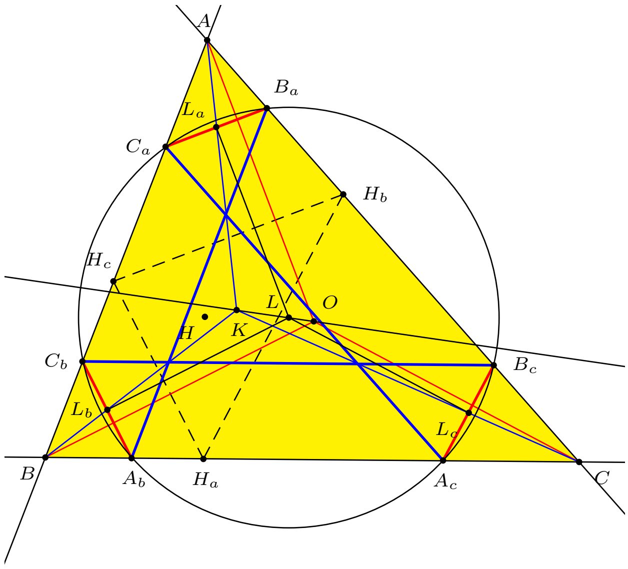

Figure 3. The Tucker circle $\mathscr{C}(t)$

Proof. Since the antiparallels B_aC_a, C_bA_b, A_cB_c have equal lengths t and are perpendicular to the circumradii OA, OB, OC respectively (see Figure 2), by Proposition 4(b), each of the six vertices of the Tucker hexagon $\mathscr{T}(t)$ is at a distance $\mathcal{R}(t)$ from $L(t)$ given by

$$\begin{aligned}\mathcal{R}(t)^2 &= \left(\frac{\lambda t - 2\sqrt{\nu}}{4S}\right)^2 + \left(\frac{t}{2}\right)^2 = \frac{(\lambda t - 2\sqrt{\nu})^2 + 4S^2t^2}{16S^2} \\ &= \frac{(\lambda^2 + 4S^2)t^2 - 4\lambda\sqrt{\nu}t + 4\nu}{16S^2} = \frac{\mu t^2 - \lambda\sqrt{\nu}t + \nu}{4S^2}.\end{aligned}$$

□

We call the circumcircle of the Tucker hexagon $\mathscr{T}(t)$ the Tucker circle $\mathscr{C}(t)$ (see Figure 3).

Remark. If $t = \frac{\tau\sqrt{\nu}}{\lambda}$, then the vertices of the Tucker hexagon $\mathscr{T}(t)$ are

$B_a = (\lambda - \tau c^2 : 0 : \tau c^2)$	$C_a = (\lambda - \tau b^2 : \tau b^2 : 0)$
$C_b = (\tau a^2 : \lambda - \tau a^2 : 0)$	$A_b = (0 : \lambda - \tau c^2 : \tau c^2)$
$A_c = (0 : \tau b^2 : \lambda - \tau b^2)$	$B_c = (\tau a^2 : 0 : \lambda - \tau a^2)$

and the radius of the Tucker circle $\mathscr{C}(t)$ is given by

$$\mathcal{R}(t)^2 = \frac{1}{4}\left((\tau - 2)^2R^2 + \left(\frac{\tau\sqrt{\nu}}{\lambda}\right)^2\right). \tag{10}$$

3. Special Tucker circles

Tucker circle	Parameter	Center	Radius
First Lemoine circle	$t_1 = \frac{\sqrt{\nu}}{\lambda}$	$L_1 = X(182)$	$\frac{1}{2}\sqrt{R^2 + t_1^2}$
Second Lemoine circle	$t_2 = \frac{2\sqrt{\nu}}{\lambda}$	$L_2 = K$	$\frac{\sqrt{\nu}}{\lambda} = t_1$
Third Lemoine circle	$t_3 = \frac{3\sqrt{\nu}}{\lambda}$	$L_3 = 2L_2 - L_1$	$\frac{1}{2}\sqrt{R^2 + t_3^2}$
Bui's circle	$t_{3/2} = \frac{3\sqrt{\nu}}{2\lambda}$	$X(575) = \frac{3}{2}L_2 - L_1$	$\frac{1}{2}\sqrt{\frac{1}{4}R^2 + t_{3/2}^2}$
Apollonius circle	$t = -s$	$X(970)$	$\frac{s^2+r^2}{4r}$
Taylor circle	$t = \frac{S}{2R}$	$X(389)$	Proposition 7
Torres circle	$t = \frac{S}{R}$	$X(52)$	§3.4
Gallatly circle	$t = \frac{\lambda\sqrt{\nu}}{2\mu}$	$X(39)$	$\frac{\sqrt{\nu}}{2\sqrt{\mu}}$
First van Lamoen circle	$t = \frac{2\sqrt{\nu}}{\lambda+2\sqrt{3}S}$	$X(15)$	$\frac{2\sqrt{\nu}}{\lambda+2\sqrt{3}S}$
Second van Lamoen circle	$t = \frac{2\sqrt{\nu}}{\lambda-2\sqrt{3}S}$	$X(16)$	$\frac{2\sqrt{\nu}}{\lambda-2\sqrt{3}S}$
First Kenmotu circle	$t = \frac{2\sqrt{\nu}}{\lambda+2S}$	$X(371)$	$\frac{\sqrt{2}\sqrt{\nu}}{\lambda+2S}$
Second Kenmotu circle	$t = \frac{2\sqrt{\nu}}{\lambda-2S}$	$X(372)$	$\frac{\sqrt{2}\sqrt{\nu}}{\lambda-2S}$

Table 1. Tucker circles

3.1. *The Lemoine circles.* The famous Lemoine circles are among the Tucker circles, with very simple parameters. In fact, for $n = 1, 2, 3$, the n-th Lemoine circle is the Tucker circle with parameter $t_n = \frac{n\sqrt{\nu}}{\lambda}$. Figure 4 shows the n-th Lemoine circles for $n = 1, 2, 3$, along with the circumcircle, which may be regarded as a Lemoine circle for $n = 0$.

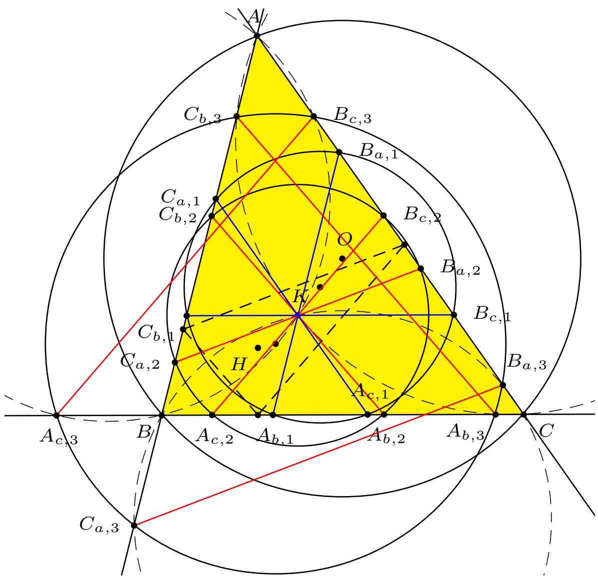

Figure 4. The Lemoine circles for $n = 0, 1, 2, 3$

The vertices of the corresponding Lemoine hexagons are constructed as follows.

(1) $B_{c,1}$, $C_{b,1}$ are the intercepts with the parallel to BC through the symmedian point K.

(2) $B_{a,2}$, $C_{a,2}$ are the intercepts with the antiparallel to BC through the symmedian point K.

(3) $B_{a,3}$, $C_{a,3}$ are the second intersections of the circle (KBC) with AC and AB.

3.2. *Bui's circle.* Q. T. Bui [1] has introduced a Tucker circle by considering the three circles each passing through the symmedian point K and tangent to the circumcircle at a vertex. Thus, the circle through K tangent to the circumcircle at A intersects AC and AB again at B_c and C_b respectively (see Figure 5); similarly for the other two circles leading to C_a, A_c, and A_b, B_a.

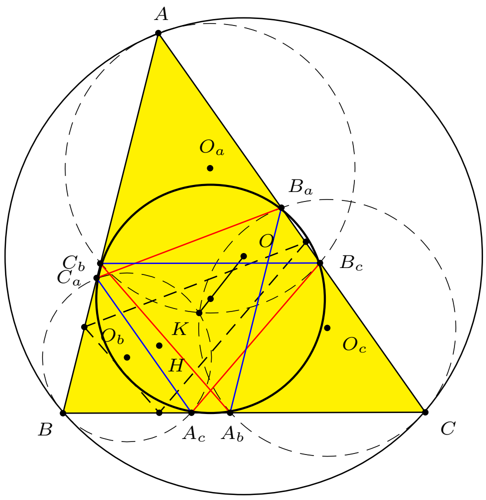

Figure 5. Bui's circle

$B_a = (2a^2 + 2b^2 - c^2 : 0 : 3c^2)$	$C_a = (2c^2 + 2a^2 - b^2 : 3b^2 : 0)$
$C_b = (3a^2 : 2b^2 + 2c^2 - a^2 : 0)$	$A_b = (0 : 2a^2 + 2b^2 - c^2 : 3c^2)$
$A_c = (0 : 3b^2 : 2c^2 + 2a^2 - b^2)$	$B_c = (3a^2, 0, 2b^2 + 2c^2 - a^2)$

These six points lie on a Tucker circle with parameter $\frac{3\sqrt{\nu}}{2\lambda}$, radius $\dfrac{\sqrt{9\mu - 2\lambda^2}}{2\lambda}R$, and center $X(575)$ dividing OK in the ratio $3:1$. We call this Bui's circle.

3.3. *The Taylor circle.* For the Taylor hexagon, the intersection of two antiparallel sides is the midpoint of the third side of the orthic triangle, i.e., C_bA_b and A_cB_c intersect at the midpoint M_a of H_bH_c; similarly for the other two pairs (see Figure 6).

We establish a simple formula for the radius of the Taylor circle.

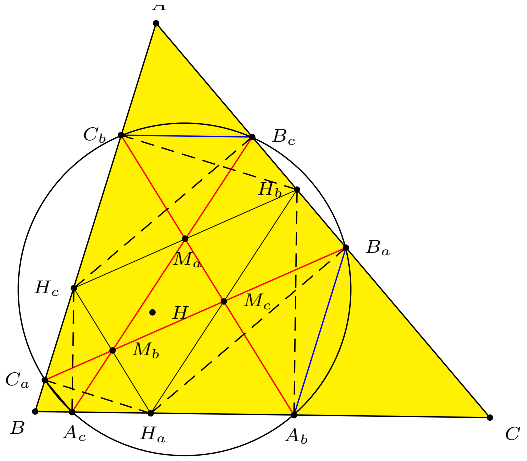

Figure 6. The Taylor circle

Proposition 7. *The radius of the Taylor circle is*

$$R_T = R\sqrt{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C}.$$

Proof. The parameter of the Taylor hexagon being $t = \frac{\sqrt{\nu}}{4R^2}$, by Proposition 6, the radius R_T of the Taylor circle is given by

$$\begin{aligned} R_T^2 = & \frac{\mu\left(\frac{\sqrt{\nu}}{4R^2}\right)^2 - \lambda\sqrt{\nu}\left(\frac{\sqrt{\nu}}{4R^2}\right) + \nu}{4S^2} = \nu \cdot \frac{\mu - 4R^2\lambda + 16R^4}{16R^4 \cdot 4S^2} \\ = & \; 4R^2S^2 \cdot \frac{\mu - 4R^2\lambda + 16R^4}{16R^4 \cdot 4S^2} = \frac{\mu - 4R^2\lambda + 16R^4}{16R^2} \\ = & \; \frac{b^2c^2 + c^2a^2 + a^2b^2 - 4(a^2+b^2+c^2)R^2 + 16R^4}{16R^2}. \end{aligned}$$

With $a = 2R\sin A$, $b = 2R\sin B$, and $c = 2R\sin C$, this becomes

$$\begin{aligned} R_T^2 = & \; R^2(\sin^2 B \sin^2 C + \sin^2 C \sin^2 A + \sin^2 A \sin^2 B \\ & - (\sin^2 A + \sin^2 B + \sin^2 C) + 1) \\ = & \; R^2(\sin^2 A \sin^2 B \sin^2 C + (1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C)) \\ = & \; R^2(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C). \end{aligned}$$

□

3.4. *Torres' Tucker circle.* Let A', B', C' be the reflections of A, B, C in their own opposite sides. These are the points

$$\begin{aligned} A' = & \; (-a^2 : a^2 + b^2 - c^2 : c^2 + a^2 - b^2), \\ B' = & \; (a^2 + b^2 - c^2 : -b^2 : b^2 + c^2 - a^2), \\ C' = & \; (c^2 + a^2 - b^2 : b^2 + c^2 - a^2 : -c^2). \end{aligned}$$

The pedals of A', B', C' on the sidelines of triangle ABC are the points

Pedal	of	on	coordinates
B_a	A'	AC	$(a^2b^2 - 2S^2 : 0 : 2S^2)$
C_a	A'	AB	$(c^2a^2 - 2S^2 : 2S^2 : 0)$
C_b	B'	AB	$(2S^2 : b^2c^2 - 2S^2 : 0)$
A_b	B'	BC	$(0 : a^2b^2 - 2S^2 : 2S^2)$
A_c	C'	BC	$(0 : 2S^2 : c^2a^2 - 2S^2)$
B_c	C'	AC	$(2S^2 : 0 : b^2c^2 - 2S^2)$

J. Torres [9] has shown that these are the vertices of a Tucker hexagon, and the center of the Tucker circle is $X(52)$, the orthocenter of the orthic triangle. This is the Tucker circle $\mathscr{C}\left(\frac{S}{R}\right)$ (see Figure 7).

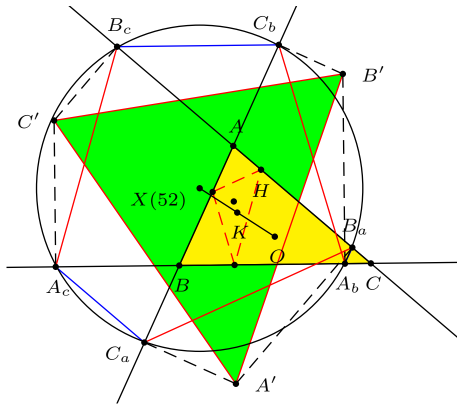

Figure 7. Torres' Tucker circle

3.5. *The Gallatly circle.* From formula (9) for the radius of the Tucker circle $\mathscr{C}(t)$, we note that the minimum of $\mathcal{R}(t)$ occurs when $t = \frac{\lambda\sqrt{\nu}}{2\mu}$. From Table 1, this is the parameter of the Gallatly circle, with center $X(39)$, the midpoint of the Brocard points. It follows that the Gallatly circle is the *smallest* Tucker circle. It is the common pedal circle of the Brocard points (see Figure 8).

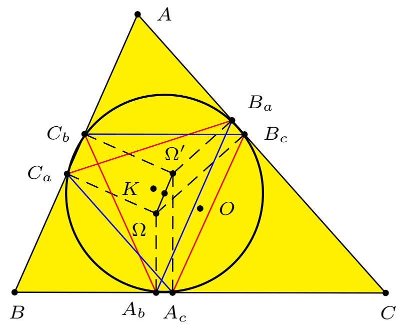

Figure 8. The Gallatly circle

3.6. *van Lamoen's and Kenmotu's circles.* Van Lamoen [7] has explained the construction of a Tucker hexagon given its center on the Brocard axis. Dergiades modifies this construction by using the rotations of the sidelines of triangle ABC about the center. Let the center be the isogonal conjugate of the Kiepert perspector $K(\theta)$. The rotations of the lines BC, CA, AB about $K(\theta)^*$ by an angle 2θ intersect the lines CA, AB, BC at the points B_c, C_a, A_b respectively. From these points the parallel to BC, CA, AB intersect AB, BC, CA at C_b, A_c, B_a. Then B_aC_a, C_bA_b, A_cB_c are the antiparallels and B_cC_b, C_aA_c, A_bB_a the parallels of the Tucker hexagon with center $K(\theta)^*$.

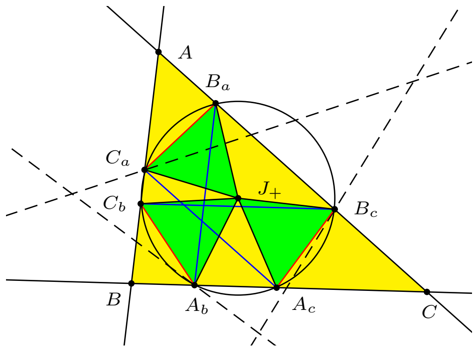

Figure 9A. Tucker circle with center J_+

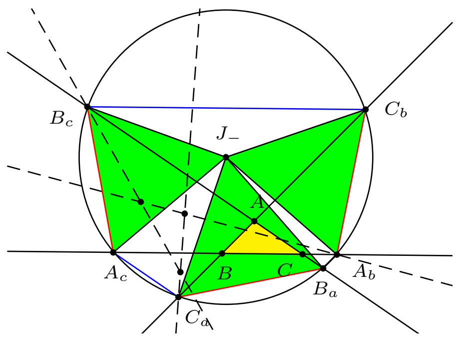

Figure 9B. Tucker circle with center J_-

With $\theta = \varepsilon \cdot \frac{\pi}{6}$, $\varepsilon = \pm 1$, we obtain the two Tucker hexagons each centered at an isodynamic point J_ε containing three congruent equilateral triangles (see Figures 9A, B).

3.7. *Tucker circles through the vertices.* For $t = \frac{bc}{a}$, we obtain the A-Tucker circle passing through the vertex A. The vertices of the A-Tucker hexagon $\mathscr{T}_a$ are

$B_a^a = (a^2 - c^2 : 0 : c^2)$	$C_a^a = (a^2 - b^2 : b^2 : 0)$
$C_b^a = (1 : 0 : 0)$	$A_b^a = (0 : a^2 - c^2 : c^2)$
$A_c^a = (0 : b^2 : a^2 - b^2)$	$B_c^a = (1 : 0 : 0)$

The segments AA_b^a and AA_c^a are the antiparallel segments. They have equal lengths $\frac{bc}{a}$. Therefore the center of the A-Tucker circle $\mathscr{C}_a$ lies on the A-altitude of triangle ABC (and the Brocard axis OK); see Figure 10. It is the point

$$L^a = (2a^2(2S^2 - b^2c^2) : \ b^2c^2(a^2 + b^2 - c^2) : \ b^2c^2(c^2 + a^2 - b^2)).$$

Likewise, there are the B- and C-Tucker circles passing through B and C, with centers on the B- and C-altitudes respectively.

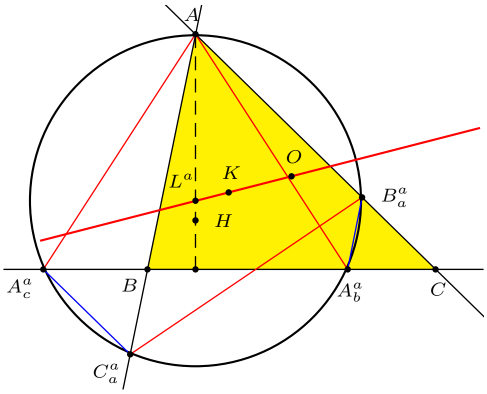

Figure 10. The A-Tucker circle

4. Congruent Tucker circles

Let $\mathscr{C}(t)$ and $\mathscr{C}(t')$ be distinct Tucker circles which are congruent. Writing $t = \frac{\tau\sqrt{\nu}}{\lambda}$ and $t' = \frac{\tau'\sqrt{\nu}}{\lambda}$ for $\tau \neq \tau'$, we have, by (10) in the Remark following Proposition 6,

$$(\tau + \tau' - 4)R^2 + \frac{(\tau + \tau')\nu}{\lambda^2} = 0.$$

From this,

$$\tau + \tau' = \frac{4R^2}{R^2 + \frac{\nu}{\lambda^2}} = \frac{4R^2}{R^2 + \frac{4R^2S^2}{\lambda^2}} = \frac{4\lambda^2}{\lambda^2 + 4S^2} = \frac{4\lambda^2}{4\mu} = \frac{\lambda^2}{\mu}.$$

Equivalently, $t + t' = \dfrac{\lambda\sqrt{\nu}}{\mu}$.

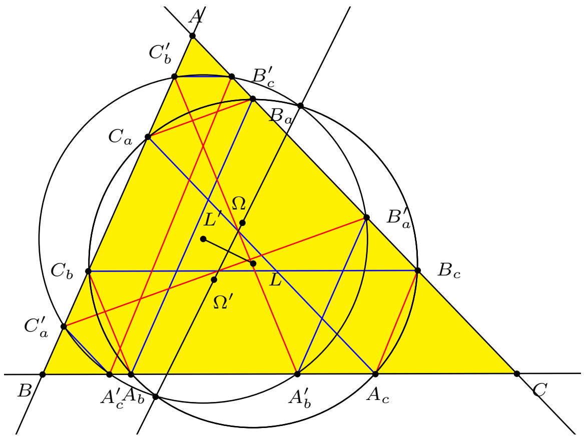

Figure 11. Congruent Tucker circles

Proposition 8. *The Tucker circles $\mathscr{C}(t)$ and $\mathscr{C}(t')$ are congruent if and only if*

$$t + t' = \frac{\lambda\sqrt{\nu}}{\mu}.$$

Corollary 9. *Two Tucker circles are congruent if and only if they are symmetric with respect to the line joining the Brocard points* (see Figure 11).

5. Orthogonal and tangential Tucker circles

Proposition 10. *The distance $L(t,t')$ between the centers of the Tucker circles $\mathscr{C}(t)$ and $\mathscr{C}(t')$ is given by*

$$L(t,t')^2 = \frac{1}{4S^2}(t'-t)^2(\lambda^2 - 3\mu).$$

Proof. The length of the segment OK is given by

$$OK^2 = \frac{1 - 4\sin^2\omega}{\cos^2\omega} \cdot R^2$$

where ω is the Brocard angle satisfying $\sin^2\omega = \frac{S^2}{\mu}$; see [5, Theorems 435 and 450]. Therefore,

$$OK^2 = \frac{\mu - 4S^2}{\mu - S^2} \cdot \frac{\nu}{4S^2} = \frac{(\mu - 4S^2)\nu}{4S^2(\mu - S^2)} = \frac{(\mu - (4\mu - \lambda^2))\nu}{S^2 \cdot \lambda^2} = \frac{(\lambda^2 - 3\mu)\nu}{S^2 \cdot \lambda^2}.$$

By Proposition 4(a), $L(t)$ and $L(t')$ divide OK in the ratios $\frac{\lambda t}{2\sqrt{\nu}} : 1 - \frac{\lambda t}{2\sqrt{\nu}}$ and $\frac{\lambda t'}{2\sqrt{\nu}} : 1 - \frac{\lambda t'}{2\sqrt{\nu}}$ respectively, the distance $L(t,t') = \frac{\lambda}{2\sqrt{\nu}}(t'-t) \cdot OK$. It follows that

$$L(t,t')^2 = \frac{\lambda^2}{4\nu}(t'-t)^2 \cdot \frac{(\lambda^2 - 3\mu)\nu}{S^2 \cdot \lambda^2} = \frac{1}{4S^2}(t'-t)^2(\lambda^2 - 3\mu).$$

□

5.1. *Orthogonal Tucker circles.*

Proposition 11. *There are $k = 0, 1, 2$ Tucker circles orthogonal to a given Tucker circle $\mathscr{C}(t)$ according as*

$$F := (2\lambda^2 - 7\mu)(2\mu t - \lambda\sqrt{\nu})^2 - 2(4\mu - \lambda^2)^2\nu$$

is negative, zero, or positive.

Proof. The Tucker circle $\mathscr{C}(t')$ is orthogonal to $\mathscr{C}(t)$ if and only if

$$\mathcal{R}(t')^2 + \mathcal{R}(t)^2 = L(t,t')^2.$$

This is equivalent to

$$(\mu t^2 - \lambda\sqrt{\nu}t + \nu) + (\mu t'^2 - \lambda\sqrt{\nu}t' + \nu) = (\lambda^2 - 3\mu)(t'-t)^2.$$

Written as a quadratic equation t':

$$(4\mu - \lambda^2)t'^2 + (2(\lambda^2 - 3\mu)t - \lambda\sqrt{\nu})t' + ((4\mu - \lambda^2)t^2 - \lambda\sqrt{\nu}t + 2\nu) = 0,$$

this has discriminant given by

$$\begin{aligned}
&(2(\lambda^2-3\mu)t-\lambda\sqrt{\nu})^2-4(4\mu-\lambda^2)((4\mu-\lambda^2)t^2-\lambda\sqrt{\nu}t+2\nu)\\
=&4\mu(2\lambda^2-7\mu)t^2-4\lambda\sqrt{\nu}(2\lambda^2-7\mu)t+(9\lambda^2-32\mu)\nu\\
=&4(2\lambda^2-7\mu)(\mu t^2-\lambda\sqrt{\nu}t+\nu)-(4\mu-\lambda^2)\nu\\
=&\frac{(2\lambda^2-7\mu)(2\mu t-\lambda\sqrt{\nu})^2-2(4\mu-\lambda^2)^2\nu}{\mu}.
\end{aligned}$$

From this the result follows. □

Corollary 12. *There is no Tucker circle orthogonal to the Gallatly circle.*

Proof. For the Gallatly circle with $t=\frac{\lambda\sqrt{\nu}}{2\mu}$, the discriminant in Proposition 11 is $F=-2(4\mu-\lambda^2)^2\nu<0$. □

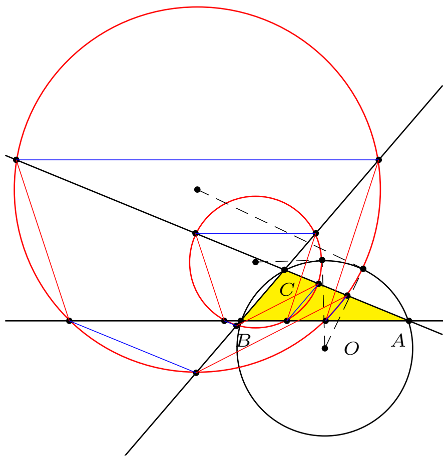

Figure 12. Two Tucker circles orthogonal to the cirucmcircle of the $(2,4,5)$ triangle

Remark. For the circumcircle, $t'=0$. A Tucker circle of parameter t is orthogonal to the circumcircle if and only if

$$(4\mu-\lambda^2)t^2-\lambda\sqrt{\nu}t+2\nu=0.$$

This has discriminant $\lambda^2\nu-8(4\mu-\lambda^2)\nu=(9\lambda^2-32\mu)\nu$. Apart from ν, this is

$$d(a,b,c):=9(a^4+b^4+c^4)-14(b^2c^2+c^2a^2+a^2b^2).$$

Since $d(2,3,4)<0$, there is no Tucker circle orthogonal to the cirumcircle of the $(2,3,4)$-triangle. On the other hand, $d(2,4,5)>0$. There are two Tucker circles orthogonal to the circumcircle of the $(2,4,5)$-triangle; see Figure 12.

5.2. *Tangential Tucker circles.*

Proposition 13. *If triangle ABC is non-equilateral, there are always two Tucker circles tangent to a given Tucker circle $\mathscr{C}(t)$.*

Proof. The Tucker circle $\mathscr{C}(t')$ is tangent to $\mathscr{C}(t)$ if and only if

$$(\mathcal{R}(t)+\mathcal{R}(t')-L(t,t'))(\mathcal{R}(t)-\mathcal{R}(t')-L(t,t'))(-\mathcal{R}(t)+\mathcal{R}(t')-L(t,t'))=0.$$

Multiplying by $\mathcal{R}(t)+\mathcal{R}(t')+L(t,t')>0$, and simplifying, we have

$$2\mathcal{R}(t)^2\mathcal{R}(t')^2+2(\mathcal{R}(t)^2+\mathcal{R}(t')^2)L(t,t')^2-\mathcal{R}(t)^4-\mathcal{R}(t')^4-L(t,t')^4=0.$$

This is $-\frac{(4\mu-\lambda^2)(t-t')^2}{16S^4}$ times

$$(4\mu-\lambda^2)t'^2+2((\lambda^2-2\mu)t-\lambda\sqrt{\nu})t'+((4\mu-\lambda^2)t^2-2\lambda\sqrt{\nu}t+3\nu).$$

This latter quadratic in t' has leading coefficient $4\mu-\lambda^2\neq 0$ and discriminant

$$\begin{aligned}
& 4((\lambda^2-2\mu)t-\lambda\sqrt{\nu})^2-4(4\mu-\lambda^2)((4\mu-\lambda^2)t^2-2\lambda\sqrt{\nu}t+3\nu)\\
=\ & 16(\lambda^2-3\mu)(\mu t^2-\lambda\sqrt{\nu}t+\nu)\\
=\ & 16(\lambda^2-3\mu)\cdot 4S^2\mathcal{R}(t)^2\\
>\ & 0
\end{aligned}$$

since $\lambda^2-3\mu>0$ (by Lemma 1(b)) and $\mathcal{R}(t)>0$ for every t. Therefore, there are always two distinct Tucker circles tangent to a given $\mathscr{C}(t)$. □

For the circumcircle of ABC corresponding to $t=0$, the two tangent Tucker circles have parameters

$$t'=\frac{\lambda\pm 2\sqrt{\lambda^2-3\mu}}{4\mu-\lambda^2}\sqrt{\nu};$$

see Figure 13.

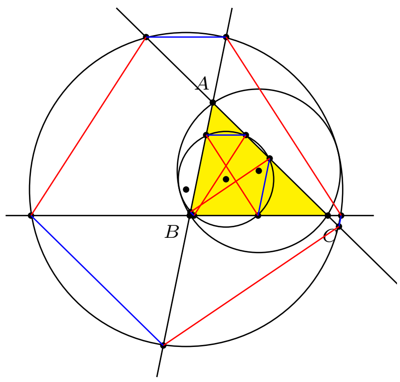

Figure 13. Tucker circles tangent to the circumcircle

6. The envelope of the Tucker circles

Proposition 14. *The barycentric equation of the Tucker circle* $\mathscr{C}(t)$ *is*

$$(a^2yz+b^2zx+c^2xy)-(x+y+z)\left(\frac{bc}{a}x+\frac{ca}{b}y+\frac{ab}{c}z\right)t+(x+y+z)^2t^2=0. \tag{11}$$

Proof. From the homogeneous barycentric coordinates of the vertices of $\mathscr{T}(t)$ given in (1), we determine the equation of the Tucker circle in the form

$$a^2yz+b^2zx+c^2xy-(x+y+z)(px+qy+rz)=0$$

where p, q, r are constants. In fact, p, q, r are respectively the powers of A, B, C relative to the Tucker circle. This means

$$p=AB_a\cdot AB_c=\left(b\cdot\frac{ct}{ab}\right)\left(b\cdot\frac{bc-at}{bc}\right)=\frac{t(bc-at)}{a},$$

and similarly $q=\frac{t(ca-bt)}{b}$ and $r=\frac{t(ab-ct)}{c}$. Therefore, the equation of the Tucker circle is

$$a^2yz+b^2zx+c^2xy-(x+y+z)t\left(\left(\frac{bc}{a}-t\right)x+\left(\frac{ca}{b}-t\right)y+\left(\frac{ab}{c}-t\right)z\right)=0.$$

This can be easily rearranged to give equation (11) above. □

Corollary 15. *The radical axis of two distinct Tucker circles is parallel to the Lemoine axis.*

Proof. Since the centers of Tucker circles are on the Brocard axis OK, the radical axis of any two distinct Tucker circle is a line perpendicular to OK, and is parallel to the Lemoine axis. □

Theorem 16. *The envelope of the Tucker circles is the Brocard ellipse.*

Proof. The equation of the envelope is $\Delta=0$, where Δ is the discriminant of the quadratic in t given in (11):

$$\begin{aligned}\Delta=&\left(\frac{bc}{a}x+\frac{ca}{b}y+\frac{ab}{c}z\right)^2-4(a^2yz+b^2zx+c^2xy)\\ =&\frac{b^4c^4x^2+c^4a^4y^2+a^4b^4z^2-2a^4b^2c^2yz-2a^2b^4c^2zx-2a^2b^2c^4xy}{a^2b^2c^2}.\end{aligned}$$

The equation $\Delta=0$ represents the Brocard inellipse with foci at the two Brocard points, tangent to the sides of triangle ABC at the traces of K, and has center at the midpoint of the Brocard points (see Figure 14). □

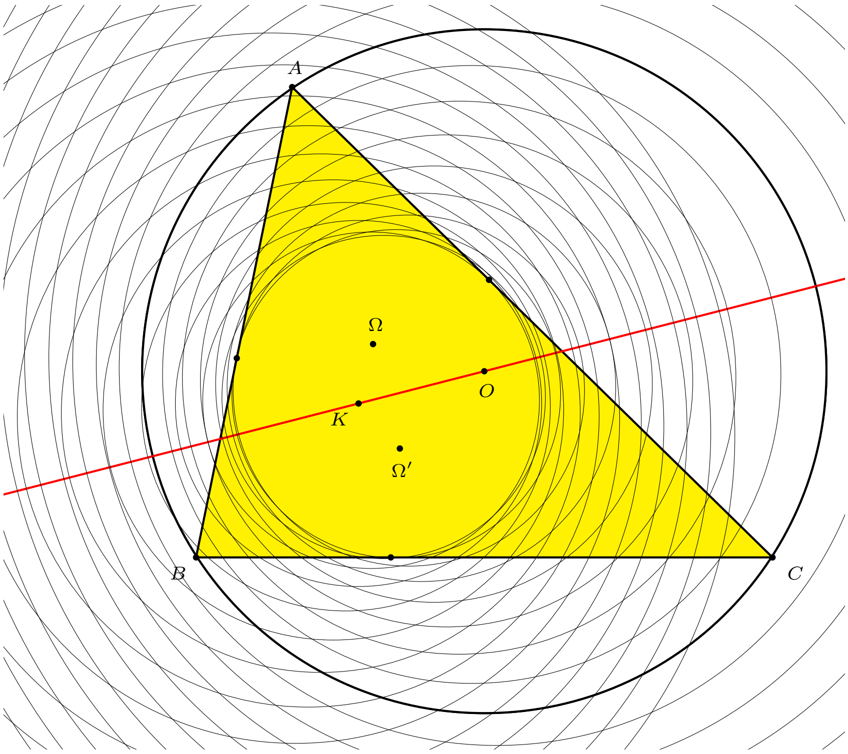

Figure 14. The envelope of Tucker circles

7. Orthology

Consider the two triangles formed by
(i) the midpoints of the segments A_bA_c, B_cB_a, C_aC_b along the sidelines of triangle ABC:

$$\begin{aligned} A' &= (0 : abc + (b^2 - c^2)t : abc - (b^2 - c^2)t), \\ B' &= (abc - (c^2 - a^2)t : 0 : abc + (c^2 - a^2)t), \\ C' &= (abc + (a^2 - b^2)t : abc - (a^2 - b^2)t : 0), \end{aligned}$$

and
(ii) the midpoints of the parallel sides B_cC_b, C_aA_c, A_bB_a of the Tucker hexagon $\mathscr{T}(t)$:

$$\begin{aligned} A'' &= (2at : bc - at : bc - at), \\ B'' &= (ca - bt : 2bt : ca - bt), \\ C'' &= (ab - ct : ab - ct : 2ct). \end{aligned}$$

Since the perpendiculars from these midpoints to the sidelines of triangle ABC are concurrent at $L(t)$, the center of the Tucker circle, each of the triangles $A'B'C'$

and $A''B''C''$ is orthologic to triangle ABC at $L(t)$. We determine the other two orthology centers.

Proposition 17. *The perpendiculars from A to $B'C'$, B to $C'A'$, and C to $A'B'$ are concurrent at the isogonal conjugate of $L(t)$.*

Proof. The quadrilateral $AC'L(t)B'$ is cyclic with $AL(t)$ as a diameter. Hence the perpendicular from A to $B'C'$ passes through the isogonal conjugate of $L(T)$ because it is the A-altitude of triangle $AB'C'$. Hence this perpendicular is isogonal to the A-diameter of $AB'C'$ that is the line $AL(t)$. Similarly the perpendiculars from B, C to $C'A'$, $A'B'$ pass through the isogonal conjugate of $L(t)$. □

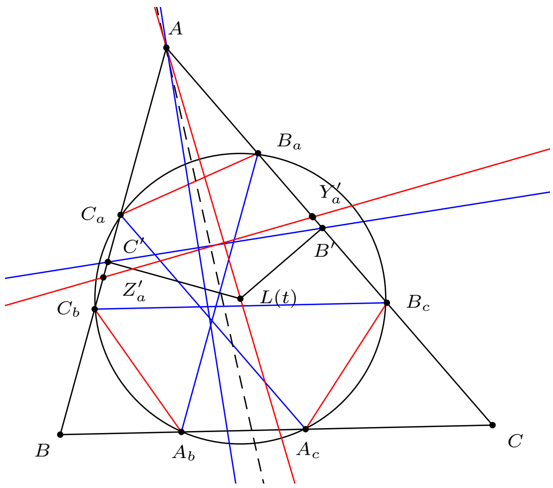

Figure 15. $B'C'$ and its reflection in the A-bisector

It follows from (8) that the isogonal conjugate of $L(t)$ is the point

$$Q'(t) = \left(\frac{1}{abc(b^2+c^2-a^2)+t(a^2(b^2+c^2)-(b^4+c^4))} : \cdots : \cdots \right).$$

Proposition 18. *The perpendiculars from A to $B''C''$, B to $C''A''$, and C to $A''B''$ are concurrent at the isogonal conjugate of the harmonic conjugate of $L(t)$ in OK* (see Figure 16).

Proof. The line $B''C''$ has barycentric equation

$$\begin{aligned}&(-a^2bc+a(b^2+c^2)t+3bct^2)x\\ &+(ca-bt)(ab-3ct)y+(ab-ct)(ca-3bt)z=0,\end{aligned}$$

and the perpendicular from A to this line is

$$(b(ca-bt)(c^2+a^2-b^2)-2c^2a^2t)y-(c(ab-ct)(a^2+b^2-c^2)-2a^2b^2t)z=0;$$

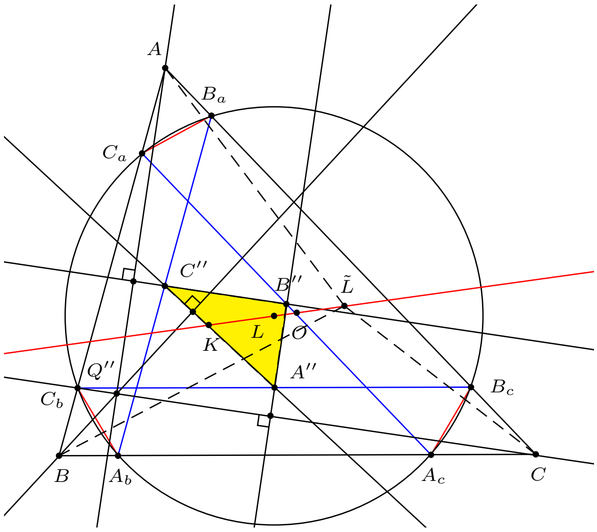

Figure 16. Orthology of midpoint triangle of parallel sides of Tucker hexagon

similarly for the other two perpendiculars. The three perpendiculars are concurrent at

$$Q''(t) = \left(\frac{1}{abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)} : \cdots : \cdots\right).$$

This is the isogonal conjugate of the point

$$(a^2(abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)) : \cdots : \cdots).$$

Now,

$$\begin{aligned}
&(a^2(abc(b^2+c^2-a^2) - t(a^2(b^2+c^2-a^2)+2b^2c^2)), \ldots, \ldots)\\
=\ &\frac{1}{2}\left((2abc-(a^2+b^2+c^2)t)(a^2(b^2+c^2-a^2), \cdots, \cdots) - 4S^2t(a^2,b^2,c^2)\right)\\
=\ &2S^2\left((2\sqrt{\nu}-\lambda t)O - \lambda t K\right).
\end{aligned}$$

These are the homogeneous barycentric coordinates of the point on the Brocard axis dividing OK in the ratio $-\lambda t : 2\sqrt{\nu} - \lambda t = -\tau : 2-\tau$. This is the harmonic conjugate of $L(t)$ in OK; see Figure 16, where the harmonic conjugate of $L = L(t)$ is indicated by $\tilde{L}$. Therefore, the orthology center $Q''(t)$ is the isogonal conjugate of the harmonic conjugate of $L(t)$ in OK. □

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Projective Geometry in Relation to the Excircles

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Abstract. In this article, we explore a small collection of results in relation to the polars of a triangle's vertices with respect to its excircles.

1. Introduction

In triangle ABC, let ℓ_a be the polar of A with respect to the excircle opposite A. Similarly define ℓ_b and ℓ_c. Let the triangle defined by ℓ_a, ℓ_b and ℓ_c be $A'B'C'$.

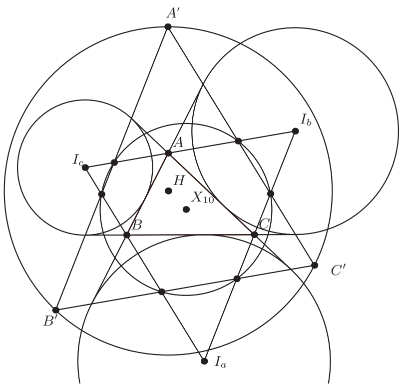

Figure 1

We will explore a small number of interesting results associated with the triangle $A'B'C'$ motivated by the following three statements:

- The circumcenter of $A'B'C'$ is the orthocenter of ABC (Theorem 3).
- The six intersection points of the triangles $A'B'C'$ and $I_aI_bI_c$ lie on a circle centered at the Spieker center (Theorem 7).
- This circle is the radical circle of the excircles (Theorem 12).

2. Common Triangle Centers

Lemma 1. $AA' \perp BC$.

Proof. We know that $X_cZ_c \perp I_cB$. Since both I_cB and $A'C'$ intersect BC at an angle of $\frac{\pi}{2} - \frac{\beta}{2}$, we know that $I_cB \parallel A'C'$ and so $X_cZ_c \perp A'C'$. Analogously, $X_bY_b \perp A'B'$. Therefore $S = X_cZ_c \cap X_bY_b$ is the orthocenter of $A'X_cX_b$. Hence it suffices to show that A, A', and S are collinear (see Figure 2).

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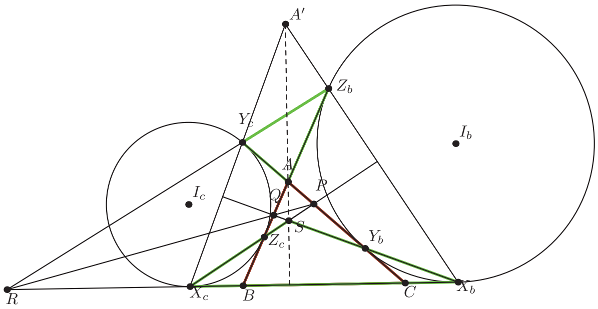

Figure 2

Consider the points $P = X_cZ_c \cap AC$, $Q = X_bY_b \cap AB$ and, $R = PQ \cap BC$. By Menelaus' theorem (in undirected form) applied on line X_cZ_cP and triangle ABC,

$$\frac{X_cC}{X_cB}\frac{Z_cB}{Z_cA}\frac{PA}{PC} = 1 \implies \frac{s}{s-a}\frac{s-a}{s-c}\frac{PA}{PC} = 1 \implies \frac{PA}{PC} = \frac{s-c}{s}.$$

Analogously, using and line X_bY_bQ and triangle ABC, we get that $\frac{QA}{QB} = \frac{s-b}{s}$. Using Menelaus again, but with line PQR and triangle ABC, we get that

$$\frac{RC}{RB}\frac{QB}{QA}\frac{PA}{PC} = 1 \implies \frac{RC}{RB} = \frac{s-c}{s-b}.$$

However, this means that

$$\frac{RC}{RB}\frac{Z_bB}{Z_bA}\frac{Y_cA}{Y_cC} = \frac{s-c}{s-b}\frac{s}{s-c}\frac{s-b}{s} = 1.$$

Thus, by Menelaus' theorem again with triangle ABC, we have that R, Z_b, Y_c are collinear.

Therefore, the triangles Y_cAZ_b and X_cSX_b are perspective about line PQR. By Desargues' theorem, this means that they are perspective about a point. Hence, X_cY_c, Z_bX_b, and AS must concur, namely at A'. Thus, A, A', S must be collinear. □

Lemma 2. $A'X_a \parallel AI_a$.

Proof. Using the law of sines on triangle $A'Y_cA$ (see Figure 3), we have that

$$\frac{AA'}{\sin\left(\frac{\pi}{2}+\frac{\gamma}{2}\right)} = \frac{s-b}{\sin\frac{\gamma}{2}} \implies AA' = \frac{s-b}{\tan\frac{\gamma}{2}}.$$

Because triangle I_aX_aC is right angled, we know that $\tan\frac{\gamma}{2} = \frac{s-b}{I_aX_a}$. It follows that $AA' = I_aX_a$.

By Lemma 1 , both AA' and I_aX_a are perpendicular to BC, $AA' \parallel I_aX_a$. Hence, we can conclude that $A'X_aI_aA$ is a parallelogram. Lemma 2 follows. □

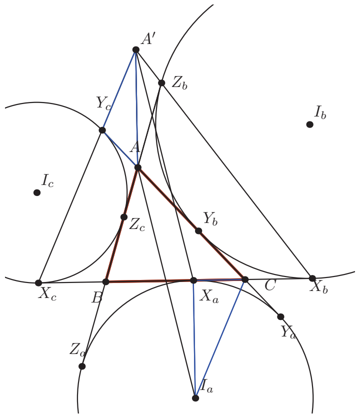

Figure 3

Theorem 3. *The circumcenter of $A'B'C'$ is the orthocenter of ABC.*

Proof. Since $\angle BX_bA' = \frac{\pi}{2} - \frac{\beta}{2}$, and $AA' \perp BC$ (see Figure 3), it follows that $\angle X_bA'A = \frac{\beta}{2}$. Since $\angle(X_cX_b, I_aA) = \frac{\alpha}{2} + \beta$ and $\angle(X_cX_b, X_cA') = \frac{\pi}{2} - \frac{\gamma}{2}$, we have that $\angle(I_aA, X_cA') = \frac{\alpha}{2} + \beta - (\frac{\pi}{2} - \frac{\gamma}{2}) = \frac{\beta}{2}$.

However, due to Lemma 2 , $A'X_a \parallel AI_a$ and so, this implies that $\angle(A'X_a, A'X_b) = \angle(I_aA, A'X_b) = \frac{\beta}{2}$. Hence, $A'A$ and $A'X_a$ are are reflections about the angle bisector of $\angle B'A'C'$.

It also follows that $\angle B'A'X_a = \angle X_bA'A = \frac{\beta}{2}$. However, checking the angles in quadrilateral $BX_cB'Z_a$,

$$\begin{aligned}
\angle A'B'C' &= 2\pi - (\angle X_cBZ_a + \angle BX_cB' + \angle BZ_aB) \\
&= 2\pi - \left(\beta + \frac{\pi}{2} + \frac{\gamma}{2} + \frac{\pi}{2} + \frac{\alpha}{2}\right) \\
&= \frac{\pi}{2} - \frac{\beta}{2}.
\end{aligned}$$

It follows that $A'X_a \perp B'C'$.

This means that $A'X_a$, $B'Y_b$ and $C'Z_c$ concur at the orthocenter of $A'B'C'$. Hence, AA', BB', CC' must concur at the circumcenter of $A'B'C'$ (the isogonal conjugate of the orthocenter). By Lemma 1 , these lines also concur at the orthocenter of ABC. Hence, the orthocenter of ABC is the circumcenter of $A'B'C'$. □

3. Six Concyclic Points

Let the intersections of BI_b and CI_c with ℓ_a be A_1 and A_2. Similarly define B_1, B_2, C_1, and C_2 so that the points A_1, A_2, B_1, B_2, C_1, and C_2 have the same clockwise/anticlockwise orientation as A, B, and C.

Lemma 4. *The points B, C, A_1, and A_2 lie on the circle with diameter BC.*

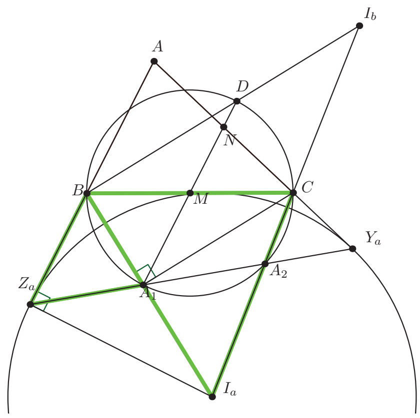

Figure 4

Proof. The triangles BZ_aA_1 and BI_aC are clearly similar by a spiral similarity centered at B. This implies that the triangles BZ_aI_a and BA_1C are also similar by a spiral similarity centered at B. Hence, BA_1C has a right angle at A_1.

Similarly, using the spiral similarity between triangles CY_aA_2 and CI_aB, we find that CA_2B has a right angle at A_2. Hence as,

$$\angle BA_1C = \angle CA_2B = \frac{\pi}{2},$$

we know that B, C, A_1, and A_2 lie on a circle with diameter BC. □

Lemma 5. *The line A_1B_2 bisects BC and AC (and so $A_1B_2 \parallel AB$).*

Proof. Let M and N and P be the midpoints of BC, AC, and AB respectively. Note that by Lemma 4 ,

$$\angle A_1MB = \pi - 2 \cdot \angle CBA_1 = \pi - 2(\frac{\pi}{2} - \frac{\beta}{2}) = \beta = \angle ABM.$$

Hence, $A_1M \parallel AB$. Similarly, $B_2N \parallel AB$. However, we also know that $MN \parallel AB$, and so A_1, M, N, and B_2 all lie on a line parallel to AB. □

Corollary 6. *A_1B_2 and BI_b intersect on the circle with diameter BC.*

Proof. Let D be the intersection of A_1B_2 and BI_b. By Lemma 5 ,

$$\angle MDB = \angle DBA = \angle DBM$$

Hence, MDB is isosceles. This means that $MB = MD$, and so the result follows. □

Theorem 7. *The six intersection points of the triangles $A'B'C'$ and $I_aI_bI_c$ lie on a circle centered at the Spieker center.*

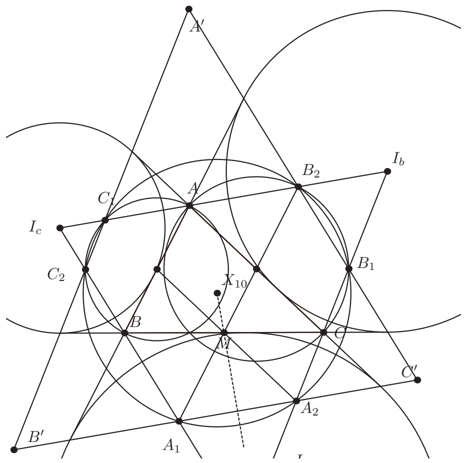

Figure 5

Proof. Consider the circles with diameters AB and AC. Since these circles intersect at A and the projection of A onto BC, their radical axis is the altitude from A to BC.

By Lemma 1 , this implies that A' is on the radical axis of these two circles. By Lemma 4 , it follows that $A'B_1 \cdot A'B_2 = A'C_1 \cdot A'C_2$. Hence, $B_1B_2C_1C_2$ is cyclic.

By Lemma 4 , ABC_1C_2 is cyclic, so combining this with Lemma 5 , we have,

$$\angle B_2C_1C_2 = \angle AC_1C_2 = \pi - \angle ABC_2 = \pi - \angle B_2A_1C_2.$$

Hence $B_2C_1C_2A_1$ is cyclic. Since we showed that $B_1B_2C_1C_2$ is cyclic, which also means that $A_1A_2B_1B_2$ is cyclic, we can conclude that $A_1A_2B_1B_2C_1C_2$ is cyclic as desired.

The center of the circle is the intersection of the perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2. By Lemma 4 , the triangle MA_1A_2 is isosceles, so the perpendicular bisector of A_1A_2 is the angle bisector of $\angle A_1MA_2$.

By Lemma 5 , A_1, M, and N are collinear as are A_2, M, and P. Hence the perpendicular bisector of A_1A_2 is actually the angle bisector of $\angle NMP$. Analogously, it follows that perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2 concur at the incenter of triangle MNP. □

4. Radical circle

Lemma 8. *A_2 and B_1 are inverses about the A-excircle of ABC.*

Proof. Let U be the inverse of A with respect to the A-excircle of ABC. Because Z_aY_a is the polar of A with respect to the A-excircle, we have that $\angle I_aUA_2 = \frac{\pi}{2}$.

Hence, if V is the inverse of A_2 with respect to the A-excircle, then it must hold that $\angle I_aVA = \angle I_aUA_2 = \frac{\pi}{2}$ and that V lies on I_aA_2. Since these conditions uniquely determine some point, by Lemma 4 (A, C, B_1, B_2 lie on a circle with diameter AC), we have that $V = B_1$. $\square$

Corollary 9. *Z_a, X_a, and B_1 lie on a line perpendicular to $A'C'$.*

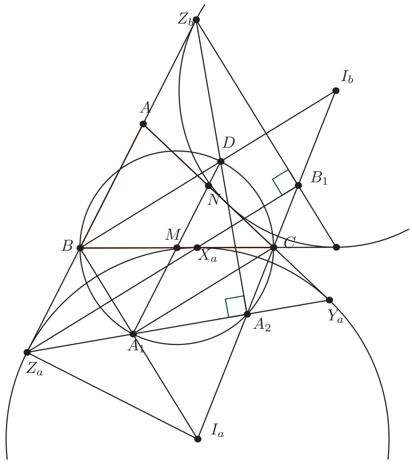

Figure 6

Proof. By Lemmas 3 and 5, B lies on the polar of B_1 with respect to the A-excircle. Hence, B_1 lies on the polar of B with respect to the A-excircle, which is the line Z_aX_a. Perpendicularity follows because $A'C' \parallel I_aI_c$. $\square$

Corollary 10. *Z_a, Z_b, A_2, and B_1 lie on a circle centered at the midpoint of AB.*

Proof. Because of Corollary 9 ,

$$\angle Z_aA_2Z_b = \angle Z_aB_1Z_b = \frac{\pi}{2}.$$

Hence the four points lie on a circle centered at the midpoint of Z_aZ_b. Since $AZ_b = BZ_a = s - c$, Z_aA_b and AB have the same midpoint. The result follows. $\square$

Corollary 11. *A_1B_2, BI_b, A_2Z_b concur on the circle with diameter BC.*

Proof. By Lemma 6 , the concurrency is sufficient to prove. Let D be the intersection point of A_1B_2 and BI_b. Clearly $\angle DA_2A_1 = \frac{\pi}{2}$ because BDA_1A_2 is cyclic. Hence, the result follows because of Corollary 9 . $\square$

Theorem 12. *A_1, A_2, B_1, B_2, C_1, and C_2 lie on the radical circle of the excircles of ABC.*

Proof. By Lemma 8 , A_2 and B_1 are inverses about the A-excircle of ABC, as are A_1 and C_2. Hence the given circle, which is the circumcircle of $A_1A_2B_1C_2$, is invariant with respect to inversion about the A-excircle. Hence, the circle is orthogonal to the A-excircle. Similarly, the circle is orthogonal to the excircles opposite B and C. The result follows. $\square$

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Triangle Constructions Based on Angular Coordinates

Thomas D. Maienschein and Michael Q. Rieck

Abstract. Two very different, yet related, triangle constructions are examined, based on a given reference triangle and on a triple of signed angles. These produce triangles that are in perspective with the reference triangle and with each other, using the same center of perspective. The first construction is rather well-known, and produces a Kiepert-Morley-Hofstadter-Kimberling triangle. A new circumconic is associated with this construction. The second construction generalizes work of D. M. Bailey and J. Van Yzeren. A number of known central triangles are obtainable using one or both of these constructions.

1. Introduction

This article is concerned with two very different triangle constructions based on a given reference triangle. Each of these is also based an a triple of signed angles (ψ_1, ψ_2, ψ_3). These two constructions produce triangles that are in perspective with the reference triangle and with each other, using the same point of perspective. If it happens that $\psi_1 + \psi_2 + \psi_3 \equiv 0$ (mod π), then the point of perspective will just be the point whose angular coordinates are (ψ_1, ψ_2, ψ_3). The first construction is rather well-known, and produces the Kiepert-Morley-Hofstadter-Kimberling (KMHK) triangle, with ψ_1, ψ_2 and ψ_3 serving as the swing angles. The second construction generalizes work of D. M. Bailey [1] and J. Van Yzeren [7]. It focuses attention on a certain triple of circles, where each circle passes through two of the reference triangle vertices.

Section 2 carefully introduces the notions of "directed angles" and "angular coordinates," in the sense in which we will be using these phrases. Section 3 details the construction of a Kiepert-Morley-Hofstadter-Kimberling triangle. Most of this material is admittedly already presented adequately in Chapter 6 of [4]. However, there is a result at the end of Section 3 here that appears to be new. Section 4 details our extension of [1] and [7], and this results in the construction of another triangle, as mentioned earlier.

In Section 5, straightforward methods are presented for testing the trilinear coordinates of a given triangle to determine whether or not it can be obtained by means of one of the two constructions. In Section 6, the results of thus testing the examples of central triangles in [4] are presented. Many of these central triangles passed

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one or both of these tests. Some of these central triangle were known already to be thus obtainable, but some of the results appear to be new.

2. Directed angles and angular coordinates

We will require the following definition. Let A, B, and P be points in the plane. Define the *directed angle* $\measuredangle APB$ to be the angle through which the line $\overleftrightarrow{AP}$ can be rotated about P to coincide with the line $\overleftrightarrow{BP}$. The angle is signed, with positive values indicating counterclockwise rotation, and is only well-defined modulo π. Any equation involving directed angles should be considered modulo π. We will fix a triangle ΔABC with circumcenter O and circumradius R and with A, B, and C not collinear. The interior angles at A, B, and C will be denoted by θ_1, θ_2, and θ_3, respectively.

Having fixed the triangle ΔABC, define the *angular coordinates* of a point P to be the triple (ϕ_1, ϕ_2, ϕ_3) of directed angles where

$$\phi_1 = \measuredangle BPC, \;\; \phi_2 = \measuredangle CPA, \;\; \phi_3 = \measuredangle APB. \tag{1}$$

Remark. This agrees with Yzeren's definition in [7]. Some sources (e.g. [2, Chapter II], [6]) define angular coordinates only for points inside ΔABC in terms of absolute angles. Clearly $\phi_1 + \phi_2 + \phi_3 = 0$ (mod π).

Observe that the inscribed angle theorem can be written in terms of directed angles as follows:

Lemma 1. *Let A, B, P, and Q be points in the plane. Then A, B, P, and Q are concyclic if and only if $\measuredangle APB = \measuredangle AQB$ if and only if $\measuredangle PAQ = \measuredangle PBQ$.*

Proof. This follows from the traditional inscribed angle theorem along with the following consideration: If P and Q are on opposite sides of a chord AB of a circle, then $\angle APB = \pi - \angle AQB$. But the directed angles $\measuredangle APB$ and $\measuredangle AQB$ must have opposite orientation in this case, so $\measuredangle APB = \pi + \measuredangle AQB = \measuredangle AQB$. □

The following lemma is a direct consequence of Lemma 1.6 and Corollary 2.8 of [5], so we omit the proof. It also follows from a result in [2, Chapter II], but only for the case that P is inside ΔABC. The condition that P is not on the circumcircle or sidelines is equivalent to the condition that $\phi_i \neq 0, \theta_i$ for each i.

Lemma 2. *Suppose P is not on the circumcircle or sidelines of ΔABC, and that P has angular coordinates (ϕ_1, ϕ_2, ϕ_3). Then P has homogeneous trilinear coordinates*

$$\left[\frac{\sin(\phi_1)}{\sin(\theta_1 - \phi_1)} : \frac{\sin(\phi_2)}{\sin(\theta_2 - \phi_2)} : \frac{\sin(\phi_3)}{\sin(\theta_3 - \phi_3)}\right].$$

3. The first triangle construction

Let us begin by reexamining the construction presented in [4]. This is a generalization of the construction in [3] that is used to define Hofstadter points. Using a reference triangle ΔABC, with directed interior angles $\theta_1, \theta_2, \theta_3$, and given a triple of directed angles (ψ_1, ψ_2, ψ_3), another triangle $\Delta A'B'C'$ is produced that is in perspective to ΔABC. We refer to this resulting triangle as the Kiepert-Morley-Hofstadter-Kimberling triangle. This triangle and the following theorem are illustrated in Figure 1. (The figure also contains some red circles and their intersections that should be ignored for the moment.)

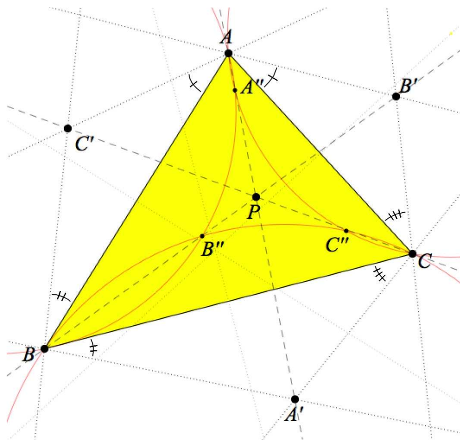

Figure 1. The two constructions

Theorem 3. *Let* (ψ_1, ψ_2, ψ_3) *be any triple of directed angles such that* $\psi_i \neq 0, \theta_i$. *Let* A', B', *and* C' *be the points satisfying*

$$\measuredangle BAC' = \measuredangle B'AC = \psi_1,$$
$$\measuredangle CBA' = \measuredangle C'BA = \psi_2,$$
$$\measuredangle ACB' = \measuredangle A'CB = \psi_3.$$

Then

(i) $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, *and* $\overleftrightarrow{CC'}$ *are concurrent, meeting in a point* P*;*

(ii) *the homogeneous trilinear coordinates of the point* A' *are*

$$\left[\frac{\sin\psi_2 \sin\psi_3}{\sin(\theta_2-\psi_2)\sin(\theta_3-\psi_3)} : \frac{\sin\psi_2}{\sin(\theta_2-\psi_2)} : \frac{\sin\psi_3}{\sin(\theta_3-\psi_3)} \right] \tag{2}$$

and similarly for B' *and* C'*; and*

(iii) P *has homogeneous trilinear coordinates*

$$\left[\frac{\sin\psi_1}{\sin(\theta_1-\psi_1)} : \frac{\sin\psi_2}{\sin(\theta_2-\psi_2)} : \frac{\sin\psi_3}{\sin(\theta_3-\psi_3)} \right]$$

Proof. We here follow the same reasoning as in [3]. First, note that a given line through A, B, or C includes all points with some fixed ratio of trilinear coordinates $[\ell_2 : \ell_3]$, $[\ell_1 : \ell_3]$, or $[\ell_1 : \ell_2]$, respectively. Points on $\overleftrightarrow{CA'}$ satisfy

$$[\ell_1 : \ell_2] = [\,\sin\psi_3 : \sin(\theta_3-\psi_3)\,]$$

and points on $\overleftrightarrow{BA'}$ satisfy

$$[\ell_1 : \ell_3] = [\,\sin\psi_2 : \sin(\theta_2 - \psi_2)\,].$$

Hence A' has the homogeneous trilinear coordinates claimed in (ii). Moreover, A' satisfies

$$[\ell_2 : \ell_3] = [\,\sin\psi_2\sin(\theta_3 - \psi_3) : \sin\psi_3\sin(\theta_2 - \psi_2)\,]. \tag{3}$$

The other points on $\overleftrightarrow{AA'}$ must also have this ratio of trilinear coordinates. Analogous reasoning shows that $\overleftrightarrow{BB'}$ is given by

$$[\ell_1 : \ell_3] = [\,\sin\psi_1\sin(\theta_3 - \psi_3) : \sin\psi_3\sin(\theta_1 - \psi_1)\,] \tag{4}$$

and $\overleftrightarrow{CC'}$ is given by

$$[\ell_1 : \ell_2] = [\,\sin\psi_1\sin(\theta_2 - \psi_2) : \sin\psi_2\sin(\theta_1 - \psi_1)\,]. \tag{5}$$

The point P with the homogeneous trilinear coordinates given in (iii) satisfies each of (3), (4), and (5), so it must be the common intersection of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$, thus establishing (i). □

Remark. In the case that $\psi_2 = -\psi_3$, the lines which would intersect to form A' are parallel. In this case the expression (3) gives the line through A parallel to both of these, and the proof continues with this line in place of $\overleftrightarrow{AA'}$. The same principle holds for B' and C'.

In the case that $\psi = r\theta$ and $r \neq 0, 1$, this construction yields the Hofstadter r-point, as defined in [3]. If $\psi_1 = \psi_2 = \psi_3 = -\pi/3$, then P is the first isogonic center. If $\psi_1 = \psi_2 = \psi_3 = \pi/3$, then P is the second isogonic center. By Theorem 3, it follows that the angular coordinates of the first and second isogonic centers are $(-\pi/3, -\pi/3, -\pi/3)$ and $(\pi/3, \pi/3, \pi/3)$, respectively. If $\psi = \theta/2$, then P is the incenter I. It does not follow that the angular coordinates of I are $\psi = \theta/2$, because in this case $\psi_1 + \psi_2 + \psi_3 \neq 0$. Indeed, it is straightforward to deduce that the angular coordinates of I are in fact $\psi = (\theta + \pi)/2$ (and therefore repeating the construction using these angles still produces the incenter I). The following result, illustrated in Figure 2 and Figure 3, appears to be new.

Theorem 4. *Let $\Delta A'B'C'$ be a KMHK triangle (with respect to ΔABC). The orthogonal projections D, E, F of A', B', C' onto the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$ are the vertices of a cevian triangle (with respect to ΔABC). Letting Q denote its center of perspective, if P is held fixed, but A', B', C' are allowed to vary, then the point Q traces out a circumconic of ΔABC.*

Proof. Using formulas from [4], if $[\,\ell : m : n\,]$ are the trilinear parameters for the line $\overleftrightarrow{A'D}$, then $\ell = m\cos\theta_3 + n\cos\theta_2$ (perpendicular lines), and $\ell\sin\psi_2\sin\psi_3 + m\sin\psi_2\sin(\theta_3 - \psi_3) + n\sin\psi_3\sin(\theta_2 - \psi_2) = 0$ (line contains A'). So,

$$m\sin\psi_2\cos\psi_3\sin\theta_3 + n\sin\psi_3\cos\psi_2\sin\theta_2 = 0,$$

and we may thus take $m = \sin\psi_3\cos\psi_2\sin\theta_2$ and $n = -\sin\psi_2\cos\psi_3\sin\theta_3$.

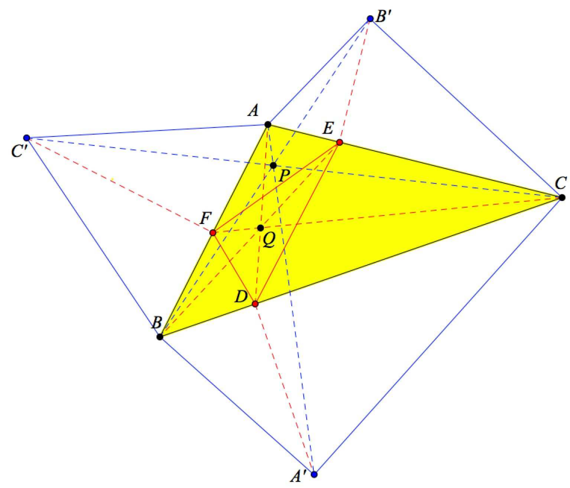

Figure 2. A cevian triangle from a KMHK triangle

Letting $[\,0 : \mu : \nu\,]$ be the trilinear coordinates for D, we may take $\mu = \sin\theta_3 \cot\psi_3$ and $\nu = \sin\theta_2 \cot\psi_2$. The line $\overleftrightarrow{AD}$ has trilinear parameters $[\,0 : \nu : -\mu\,]$. Similarly for the lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$. These three lines intersect at a point Q whose trilinear coordinates are $[\,\csc\theta_1 \tan\psi_1 : \csc\theta_2 \tan\psi_2 : \csc\theta_3 \tan\psi_3\,]$.

If we let (ϕ_1, ϕ_2, ϕ_3) be the angular coordinates of P, then its trilinear coordinates are

$$[\,\sin\phi_1/\sin(\theta_1 - \phi_1) : \sin\phi_2/\sin(\theta_2 - \phi_2) : \sin\phi_2/\sin(\theta_2 - \phi_2)\,] =$$
$$[\,\sin\psi_1/\sin(\theta_1 - \psi_1) : \sin\psi_2/\sin(\theta_2 - \psi_2) : \sin\psi_2/\sin(\theta_2 - \psi_2)\,]\,.$$

Therefore, there is a parameter λ such that, for $i = 1, 2, 3$,

$$\frac{\sin\phi_i}{\sin(\theta_i - \phi_i)} = \lambda \cdot \frac{\sin\psi_i}{\sin(\theta_i - \psi_i)}, \text{ and so } \cot\psi_i = \lambda\cot\phi_i + (1-\lambda)\cot\theta_i.$$

The i-th trilinear coordinate of Q thus becomes $1/[\lambda\sin\theta_i\cot\phi_i + (1-\lambda)\cos\theta_i]$, and the isogonal conjugate Q^{-1} of Q has i-th trilinear coordinate $\lambda\sin\theta_i\cot\phi_i + (1-\lambda)\cos\theta_i$. Varying λ, we see that Q^{-1} traces out a line, and therefore Q traces out a circumconic. $\square$

Figure 3 illustrates this circumconic, which is a circumhyperbola here. With P fixed, as A', B', C' are allowed to vary, the point Q moves along the green curve, which is the circumhyperbola.

4. The second triangle construction

Our second triangle construction generalizes a construction studied in [1], [5], [6] and [7]. These studies all essentially concern an arbitrary point P, and the three circles through P that also pass through two of the reference triangle vertices. We

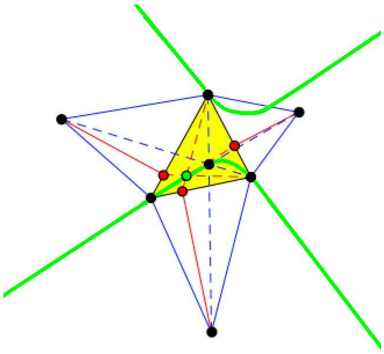

Figure 3. A circumconic associated with the KMHK construction

will instead begin with the reference triangle ΔABC, and with a triple of directed angles (ψ_1, ψ_2, ψ_3), as we did in the first construction.

Starting with the triple of directed angles, construct three circles as follows: Let P be any point for which $\measuredangle BPC = \psi_1$ and let $\mathcal{C}_X$ denote the circle BPC. By Lemma 1, this construction is well-defined. Using C and A (*resp.* A and B) in place of B and C, we obtain a circle $\mathcal{C}_Y$ (*resp.* $\mathcal{C}_Z$). Finally, let X, Y, and Z denote the centers of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$, respectively.

If the three circles $\mathcal{C}_X$, $\mathcal{C}_Y$, and $\mathcal{C}_Z$ have a common point of intersection, then by definition that point has angular coordinates (ψ_1, ψ_2, ψ_3) and so $\psi_1 + \psi_2 + \psi_3 \equiv 0$ (mod π). We do not assume, however, that our original triple of directed angles satisfies this equation, and so the three circles do not generally have a common point of intersection.

Let A'' (*resp.* B'', *resp.* C'') be the point of intersection of $\mathcal{C}_Y$ and $\mathcal{C}_Z$ (*resp.* $\mathcal{C}_Z$ and $\mathcal{C}_X$, *resp.* $\mathcal{C}_X$ and $\mathcal{C}_Y$), other than A (*resp.* B, *resp.* C). Figure 1 shows these circles and their intersections, and also illustrates the theorem to be presented concerning these.

Theorem 5. *For a triangle ΔABC, and for a triple of directed angles (ψ_1, ψ_2, ψ_3) such that $\psi_i \neq 0, \theta_i$, let A'', B'', C'' be the circle intersection points considered above. Let A', B', C' be the points in Theorem 3. Then*

(i) *A, A' and A'' are collinear, as are B, B' and B'', as are C, C' and C'';*

(ii) *the homogeneous trilinear coordinates of the point A'' are*

$$\left[\frac{\sin(\psi_2 + \psi_3)}{\sin(\psi_2 + \psi_3 - \theta_2 - \theta_3)} : \frac{\sin(\psi_2)}{\sin(\theta_2 - \psi_2)} : \frac{\sin(\psi_3)}{\sin(\theta_3 - \psi_3)} \right] \tag{6}$$

and similarly for B'' and C''; and

(iii) *if $\psi_1 + \psi_2 + \psi_3 \equiv 0$ (mod π), then $A'' = B'' = C'' = P$ (with P as in Theorem 3), and (ψ_1, ψ_2, ψ_3) are the angular coordinates of P.*

Proof. Observe that the point A'' has angular coordinates $(\psi_1'', \psi_2, \psi_3)$ for some value ψ_1'': the last two angular coordinates are known by Lemma 1 and the construction of A''. It follows that $\psi_1'' = -\psi_2 - \psi_3$. Part (ii) is then established by converting angular to trilinear coordinates (Lemma 2) and replacing θ_1 with $\pi - \theta_2 - \theta_3$.

Part (iii) follows immediately from (ii).

Since the ratio $[\ell_2 : \ell_3]$ is shared by points A' and A'' (as in Figure 2 and Figure 6), it must be the case that A' and A'' are on the same line through A. This is part (i). □

Following a simple lemma, a characterization is now presented of triangles that can be obtained via the second construction, using the same center of perspective.

Lemma 6. *Let E, G, H and F be concyclic points, occurring in this cyclic order. Let X and Y be points such that E, G and X are collinear, F, H and Y are collinear, and the lines $\overleftrightarrow{GH}$ and $\overleftrightarrow{XY}$ are parallel. Then, E, X, Y and F are concyclic points. Conversely, if U and V are points such that E, U, V and F are concyclic, E, G and U are collinear, and F, H and V are collinear, then $\overleftrightarrow{GH}$ and $\overleftrightarrow{UV}$ are parallel.*

Proof. $\measuredangle YFE = \measuredangle HFE = -\measuredangle EGH = \measuredangle HGX = -\measuredangle GXY = -\measuredangle EXY$. Therefore, E, X, Y and F are concyclic. (Euclid's theorem on cyclic quadrilaterals is used in both directions.) Also, $\measuredangle GUV = \measuredangle EUV = -\measuredangle VFE = -\measuredangle HFE = \measuredangle EGH = -\measuredangle HGU$. So $\overleftrightarrow{GH}$ and $\overleftrightarrow{UV}$ are parallel. □

Theorem 7. *Suppose that $\Delta A''B''C''$ can be obtained from the reference triangle ΔABC, using the second construction. Let P denote the center of perspective. Suppose that ΔXYZ is another triangle, homothetic to $\Delta A''B''C''$, with P as the homothetic center. Then ΔXYZ can also be obtained from ΔABC by means of the second construction. Conversely, all triangles obtainable via the second construction, and having P as the center of perspective, are related to $\Delta A''B''C''$ in this manner.*

Proof. A, B, A'', B'' are concyclic. A, A'', X are collinear, as are B, B'', Y, with the two lines intersecting at P. The lines $\overleftrightarrow{A''B''}$ and $\overleftrightarrow{XY}$ are parallel. So by the lemma, A, B, X, Y are concyclic. Similarly, B, C, Y, Z are concyclic, and C, A, Z, X are concyclic. This reasoning can be reversed to establish that all triangles obtainable via the second construction, and having P as the center of perspective, are related to $\Delta A''B''C''$ in this manner. □

Remark. Let O denote the circumcenter of the reference triangle ΔABC. In [5], it is demonstrated that the triangle whose vertices are the centers of $\mathcal{C}_X$, $\mathcal{C}_Y$ and $\mathcal{C}_Z$, is in an orthological relation with the reference triangle, with P as the orthology center of the latter with respect to the former, and with O as the orthology center of the former with respect to the latter. Conversely, given a triangle T with this orthological relation to the reference triangle (still using P and O as orthology centers), the vertices of the reference triangle can be reflected about the corresponding sides of T to obtain the vertices of a triangle that is also obtainable via the construction discussed in this section.

5. Triangles obtainable via the two constructions

The question of whether a given triangle can or cannot be obtained from the reference triangle by means of one of the two constructions discussed above shall now be considered. Here we will suppose that we are presented with the homogeneous trilinear coordinates of the vertices of some triangle. As is customary, we will assume this is presented in the form of a 3×3 matrix with each row providing the trilinear coordinates of a vertex:

$$L = \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix} .$$

We wish to know if this is the trilinear coordinates matrix of a KMHK triangle. We know that any KMHK triangle has the following as its trilinear coordinates matrix:

$$M = \begin{pmatrix} \rho_2\rho_3 & \rho_2 & \rho_3 \\ \rho_1 & \rho_3\rho_1 & \rho_3 \\ \rho_1 & \rho_2 & \rho_1\rho_2 \end{pmatrix} ,$$

where $\rho_i = \sin\psi_i / \sin(\theta_i - \psi_i)$, with θ_i and ψ_i as before. It is required therefore that $\ell_{12}\ell_{23}\ell_{31} = \ell_{21}\ell_{32}\ell_{31}$. If this is so then the rows of L can easily be rescaled (each row being multiplied by a scalar) to cause $\ell_{21} = \ell_{31}$, $\ell_{12} = \ell_{32}$ and $\ell_{13} = \ell_{23}$. Assume that this has been done already. If L is indeed the trilinear coordinate matrix of a KMHK triangle, then it must equal λM for some scalar λ. But this means that $\ell_{12}\ell_{13}/\ell_{11} = \ell_{21}\ell_{23}/\ell_{22} = \ell_{31}\ell_{32}/\ell_{33}$. Conversely, if this condition concerning the entries of L is satisfied, then it is straightforward to see that L is indeed the trilinear coordinates matrix of a KMHK triangle.

We turn now to the question of whether or not L is the trilinear coordinates matrix of some triangle that can be obtained using the second construction, the one based on intersecting circles. The approach taken to answering this question differs substantially from the approach used for the first construction. However, it is again clear that $\ell_{12}\ell_{23}\ell_{31} = \ell_{21}\ell_{32}\ell_{13}$ is still a necessary condition, so we will assume that this is the case. Let O be the circumcenter of the reference triangle. Let A'', B'' and C'' be the points having the first, second and third rows of L as their trilinear coordinates. Essentially following the notation used in [5], let

$$c_{ij} = R \cdot \frac{\ell_{i2}\ell_{i3}\sin\theta_1 + \ell_{i3}\ell_{i1}\sin\theta_2 + \ell_{i2}\ell_{i3}\sin\theta_3}{\ell_{i1}\sin\theta_1 + \ell_{i2}\sin\theta_2 + \ell_{i3}\sin\theta_3} \cdot \frac{1}{\ell_{ij}} ,$$

where R is the circumradius of the reference triangle. In [5] it is demonstrated that $|c_{11}|$ is the distance between O and the center of the circle containing A'', B, C. Similarly, $|c_{12}|$ is the distance between O and the center of the circle containing A'', C, A, and $|c_{13}|$ is the distance between O and the center of the circle containing

A'', A, B. Likewise for $|c_{21}|$, $|c_{22}|$ and $|c_{23}|$ ($|c_{31}|$, $|c_{32}|$ and $|c_{33}|$) with B'' (C'') taking the place of A''.

Now, the triangle $A''B''C''$ is obtained from the triangle ABC using the second construction if and only if B, C, B'' and C'' are concyclic, and C, A, C'' and A'' are concyclic, and A, B, A'' and B'' are concyclic. This is so if and only if $|c_{21}| = |c_{31}|$, $|c_{32}| = |c_{12}|$ and $|c_{13}| = |c_{23}|$. If we divide both sides of these three equations by R, we obtain three equations that can easily be used to test whether or not L is the trilinear coordinates matrix of a triangle that can be obtained using the second construction.

6. Relationship with central triangles

The center of perspective P used in the two constructions will henceforth be assumed to be a triangle center for the reference triangle. The trilinear coordinates of the vertices produced by the two constructions, as presented in Sections 3 and 4, make it clear that the constructed triangle is a central triangle of type 1, as defined in Chapter 2 of [4]. Recall that this means that the matrix has the form

$$\begin{pmatrix} f(a,b,c) & g(b,c,a) & g(c,a,b) \\ g(a,b,c) & f(b,c,a) & g(c,a,b) \\ g(a,b,c) & g(b,c,a) & f(c,a,b) \end{pmatrix}.$$

for triangle center functions f and g, where a, b, c are the triangle side lengths. More explicitly, f and g must be homogeneous and must be invariant under a swapping of their second and third arguments.

We now ask, which of the central triangles presented in Chapter 6 of [4] can be obtained using one of the two constructions? Many of the central triangles there are presented using a trilinear coordinates matrix that manifests the triangle to be of type 1. These triangles can be tested immediately using the tests given in the previous section. Most of the other triangles are presented using a trilinear coordinate matrix whose rows can be rescaled so as to produce a matrix of the above form. This then shows that the triangle is actually of type 1, and provides a matrix that can be used in the tests in the previous section. The matrices in Chapter 6 that can be adjusted in this way all have the form

$$\begin{pmatrix} * & \gamma\alpha'\beta'' & \beta\alpha'\gamma'' \\ \gamma\beta'\alpha'' & * & \alpha\beta'\gamma'' \\ \beta\gamma'\alpha'' & \alpha\gamma'\beta'' & * \end{pmatrix}.$$

where $\alpha = \alpha(a,b,c)$, $\alpha' = \alpha'(a,b,c)$ and $\alpha'' = \alpha''(a,b,c)$ are triangle center functions, and $\beta = \alpha(b,c,a), \beta' = \alpha'(b,c,a), \beta'' = \alpha''(b,c,a), \gamma = \alpha(c,a,b), \gamma' = \alpha'(c,a,b), \gamma'' = \alpha''(c,a,b)$. To bring this matrix into the desired form, just divide the first row by $\alpha'\beta\gamma$, divide the second row by $\alpha\beta'\gamma$, and divide the third row by $\alpha\beta\gamma'$. This yields the following matrix:

$$\begin{pmatrix} * & \beta''/\beta & \gamma''/\gamma \\ \alpha''/\alpha & * & \gamma''/\gamma \\ \alpha''/\alpha & \beta''/\beta & * \end{pmatrix}.$$

MATHEMATICA was used to conduct the tests on the triangles in Chapter 6 of [4]. Cevian triangles are trivially KHMK triangles, or at least limiting cases of such as the swing angles go to zero in some fixed proportion. Similarly, circumcervian triangles are trivially examples of the second construction since their vertices and those of the reference triangle are concyclic. The triangles in the table on page 198 of [4] are, as stated there, KHMK triangles. Apart from these, our testing also determined that the excentral triangle (6.7 of [4]), the hexyl triangle (6.36 of [4]), the half-altitude triangle (6.38 of [4]), and the BCI triangle (6.39 of [4]) are KHMK triangles. For the excentral triangle, the claim is known and it is straightforward to check that $\psi_i = \frac{1}{2}(\pi - \theta_i)$ $(i = 1, 2, 3)$. For the half-altitude triangle, the claim is also known and it is straightforward to check that $\tan\psi_i = \frac{1}{2}\tan\theta_i$ $(i = 1, 2, 3)$. The facts concerning the other two triangles are less immediate.

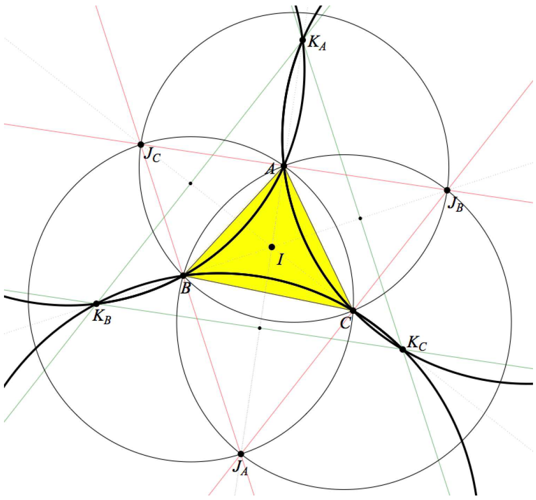

Figure 4. Circles for the excenters and their reflections about the incenter

Skipping the reference triangle itself, and the circumcevian examples, the only other triangles listed in Chapter 6 of [4] that can be obtained using our second construction are as follows: the orthic triangle (6.4 of [4]), the excentral triangle (6.7 of [4]), the reflections of the circumcenter about the reference triangle vertices (6.13 of [4]), and the reflections of the excenters about the incenter (6.42 of [4]). This fact concerning the orthic triangle and the excentral triangle are well-known, and indeed they are related since the excentral (orthic) triangle of the orthic (excentral) triangle is the reference triangle. Figure 4 exhibits the situation for the excentral triangle and for the reflections of the excenters about the incenter. Here I is the incenter, J_A, J_B, J_C are the excenters, and K_A, K_B, K_C are their reflections about I.

We may deduce that the triangle obtained by reflecting the feet of the altitudes about the vertices of the reference triangle can also be obtained via the second construction. This is so since this triangle has the same inverse relationship to the triangle obtained by reflecting the excenters about the incenter that the orthic triangle has to the excentral triangle. The triangle whose vertices are the reflected excenters of the reference triangle, has the reference triangle's incenter I as its orthocenter. The feet of its altitudes can be seen in Figure 4 as small dots. The reflection of these about I are just the vertices of the reference triangle.

Similarly, but rather trivially, the triangle obtained by reflecting the circumcenter about the reference triangle vertices has the same inverse relation to the triangle obtained from the reference triangle by taking as vertices the midpoints on the segments connecting the reference triangle's circumcenter to its vertices. Therefore, the latter is also an example of a central triangle obtainable by means of the second construction. Alternatively, Theorem 7 can be used to establish this and similar claims.

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About a Strengthened Version of the Erdős-Mordell Inequality

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Abstract. In this paper, we use barycentric coordinates to prove the strengthened version of the Erdős-Mordell inequality, proposed by Dao, Ngyuen and Pham in [3].

One of the most beautiful results in geometry is represented by the Erdős-Mordell ([4]) inequality that for any point P inside a triangle ABC,

$$PA + PB + PC \geq 2d(P, AB) + 2d(P, BC) + 2d(P, CA),$$

where $d(P, AB)$ denotes the distance from the point P to the line AB. There are a number of references on this result; see, for example, [1, 5]. Recently, Dao, Nguyen and Pham [3] improved the Erdős-Mordell inequality by replacing the lengths PA, PB, PC by the distances from P to the tangents to the circumcircle at A, B, C respectively.

The aim of this paper is to prove a further strengthened version of the theorem of Dao-Nguyen-Pham. We use barycentric coordinates to obtain new inequalities (Corollaries 4, 5), and the inequality of Dao-Nguyen-Pham in Corollary 6. Finally, we complete with an interesting application (Corollary 7).

In this paper, $X \in [Y, Z]$ means that X, Y, Z are collinear, and X is an interior or a boundary point of the segment YZ.

We start with the following lemma.

Lemma 1. *Let A, B, C be points on a line ℓ and $B \in [A, C]$ and $k := \frac{AB}{AC}$ be the ratio of directed lengths. Then*

$$d(B,\ \ell) = (1 - k)d(A,\ \ell) + kd(C,\ \ell).$$

Proof. Denote by U, V, W the orthogonal projections of the points A, B, C respectively onto the line ℓ. Let $T \in [C, W]$ such that $AT \perp CW$ and $AT \cap BV = \{S\}$ (see Figure 1).

Then $AUVS$ and $SVWT$ are rectangles and $AU = SV = TW$. On the other side, $\Delta ASB \sim \Delta ATC$. Then $\frac{BS}{CT} = \frac{AB}{AC} = k$; so $BS = k \cdot CT$. Furthermore,

$$\begin{aligned}(1-k)d(A,\ \ell) + kd(C,\ \ell) &= (1-k)AU + kCW \\ &= (1-k)SV + kSV + kCT \\ &= SV + BS = BV = d(B,\ \ell).\end{aligned}$$

□

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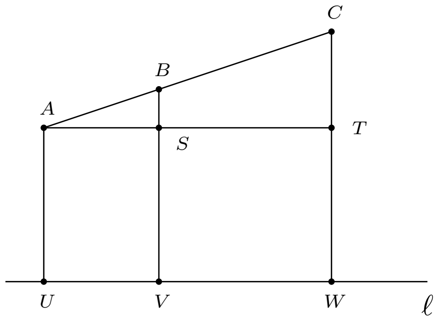

Figure 1

We recall that for any point P inside or the sides of triangle ABC, there are $x,\ y,\ z \in [0,\ 1]$ with $x + y + z = 1$ such that

$$x\overrightarrow{PA} + y\overrightarrow{PB} + z\overrightarrow{PC} = 0.$$

These numbers are unique and are called the *barycentric coordinates of P with reference to* triangle ABC. Moreover, we have

$$x = \frac{[PBC]}{[ABC]}, \quad y = \frac{[PCA]}{[ABC]}, \quad z = \frac{[PAB]}{[ABC]},$$

where $[XYZ]$ denotes the (oriented) area of triangle XYZ.

Lemma 2. *Let ABC be a triangle with vertices on the same side of a line ℓ, and P a point inside or on the sides of the triangle. If $x,\ y,\ z$ are the barycentric coordinates of P with reference to ABC, then*

$$d(P,\ \ell) = xd(A,\ \ell) + yd(B,\ \ell) + zd(C,\ \ell).$$

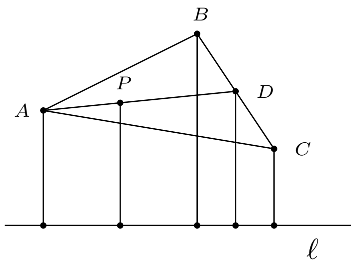

Figure 2

Proof. Let $AP \cap BC = \{D\}$ so that $x = \frac{[PBC]}{[ABC]} = \frac{PD}{AD}$. From Lemma 1,

$$d(P,\ \ell) = (1 - x)d(D,\ \ell) + xd(A,\ \ell). \tag{1}$$

On the other hand, $y = \frac{[PCA]}{[ABC]}$ and $z = \frac{[PAB]}{[ABC]}$, so that $\dfrac{y}{z} = \dfrac{[PCA]}{[PAB]} = \dfrac{CD}{BD}$, and $\dfrac{CD}{CB} = \dfrac{y}{y+z}$. From Lemma 1,

$$d(D,\ \ell) = \left(1 - \frac{y}{y+z}\right) d(C,\ \ell) + \frac{y}{y+z} d(B,\ \ell).$$

Since $x + y + z = 1$, this is equivalent to

$$(1-x)d(D,\ \ell) = (y+z)d(D,\ \ell) = zd(C,\ \ell) + yd(B,\ \ell).$$

Together with (1), this gives

$$d(P,\ \ell) = xd(A,\ \ell) + yd(B,\ \ell) + yd(B,\ \ell) + zd(C,\ \ell).$$

□

Consider triangle ABC with $A' \in [B, C]$, $B' \in [A, C]$, and $C' \in [A, B]$. Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, and P be a point in the plane of the triangle. We investigate the inequality:

$$\begin{aligned} &\alpha^2 d(P,\ BC) + \beta^2 d(P,\ AC) + \gamma^2 d(P,\ AB) \\ &\geq 2\beta\gamma d(P,\ B'C') + 2\alpha\gamma d(P,\ A'C') + 2\alpha\beta d(P,\ A'B'). \end{aligned} \tag{2}$$

Proposition 3. *The following assertions are equivalent:*

(a) *For any point P inside or on the sides of triangle $A'B'C'$, the inequality* (2) *holds.*

(b) *For any point $P \in \{A',\ B',\ C'\}$, the inequality* (2) *holds, i.e., for α, β, $\gamma \in \mathbb{R}$,*

$$\begin{aligned} \beta^2 d(A',\ AC) + \gamma^2 d(A',\ AB) &\geq 2\beta\gamma d(A',\ B'C'), \\ \alpha^2 d(B',\ BC) + \gamma^2 d(B',\ AB) &\geq 2\alpha\gamma d(B',\ A'C'), \\ \alpha^2 d(C',\ BC) + \beta^2 d(C',\ AC) &\geq 2\alpha\beta d(C',\ A'B'). \end{aligned}$$

Proof. (a) ⇒ (b): clear.

(b) ⇒ (a). Let x, y, z be the barycentric coordinates of the point P with reference to triangle $A'B'C'$. By Lemma 2, we have

$$\begin{aligned} d(P,\ BC) &= xd(A',\ BC) + yd(B',\ BC) + zd(C',\ BC) \\ &= yd(B',\ BC) + zd(C',\ BC), \end{aligned}$$

and analogous results for the lines CA, AB replacing BC. Then

$$\begin{aligned} &\alpha^2 d(P,\ BC) + \beta^2 d(P,\ AC) + \gamma^2 d(P,\ AB) \\ &= \alpha^2(yd(B',\ BC) + zd(C',\ BC)) + \beta^2(xd(A',\ AC) + zd(C',\ AC)) \\ &\qquad + \gamma^2(xd(A',\ AB) + yd(B',\ AB)) \\ &= x(\beta^2 d(A',\ AC) + \gamma^2 d(A',\ AB)) + y(\alpha^2 d(B',\ BC) + \gamma^2 d(B',\ AB)) \\ &\qquad + z(\alpha^2 d(C',\ BC) + \beta^2 d(C',\ AC)) \\ &\geq x \cdot 2\beta\gamma d(A',\ B'C') + y \cdot 2\alpha\gamma d(B',\ A'C') + z \cdot 2\alpha\beta d(C',\ A'B'). \end{aligned} \tag{3}$$

Since

$d(P,\ B'C') = xd(A',\ B'C') + yd(B',\ B'C') + zd(C',\ B'C') = xd(A',\ B'C'),$

and similarly $d(P,\ C'A') = yd(B',\ A'C')$, $d(P,\ A'B') = zd(C',\ A'B')$, the last term of (3) is equal to

$$2\beta\gamma d(P,\ B'C') + 2\alpha\gamma d(P,\ A'C') + 2\alpha\beta d(P,\ A'B').$$

This completes the proof of (b) $\Rightarrow$ (a). $\square$

Corollary 4. *Let the incircle of triangle ABC touch the sides BC, CA, AB at A', B', C' respectively. The inequality* (2) *holds for any point P inside or on the sides of triangle $A'B'C'$.*

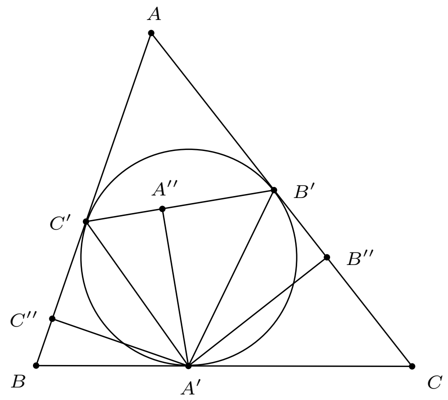

Figure 3

Proof. By using Proposition 3, it enough to prove the inequality (3) only for $P \in \{A',\ B',\ C'\}$. We suppose $P = A'$. Denote by A'', B'', C'' the orthogonal projections of the point A' onto the lines $B'C'$, AC, AB respectively. Let r be the radius of the incircle of the triangle ABC. Then

$$A'C'' = A'C' \sin C''C'A' = A'C' \sin A'B'C' = 2r \sin^2 A'B'C'.$$

Similarly, $A'B'' = 2r \sin^2 A'C'B'$. Now we have

$$\begin{aligned} 2\beta\gamma A'A'' &= \beta\gamma A'C' \sin A'C'B' + \beta\gamma A'B' \sin A'B'C' \\ &= 2\beta\gamma r \sin A'B'C' \sin A'C'B' + 2\beta\gamma r \sin A'B'C' \sin A'C'B' \\ &= 4\beta\gamma r \sin A'B'C' \sin A'C'B' \\ &\leq 2\gamma^2 r \sin^2 A'B'C' + 2\beta^2 r \sin^2 A'C'B' \\ &= \gamma^2 A'C'' + \beta^2 A'B''. \end{aligned}$$

Also,

$$\gamma^2 d(A',\ AB) + \beta^2 d(A',\ AC) \geq 2\beta\gamma d(A',\ B'C'),$$

and the proof is complete. $\square$

Corollary 5. *Let the incircle of triangle ABC touch the sides BC, CA, AB at A', B', C' respectively. For any point P inside or on the sides of triangle $A'B'C'$,*

$$d(P,\ BC)+d(P,\ AC)+d(P,\ AB) \geq 2d(P,\ B'C')+2d(P,\ A'C')+2d(P,\ A'B').$$

Proof. We apply Corollary 4 for $\alpha = \beta = \gamma = 1$. □

Now, the inequality of Dao-Nguyen-Pham ([3]) is an easy consequence of the previous results.

Corollary 6 (Dao-Nguyen-Pham [3])**.** *Let ABC be a triangle inscribed in a circle (O), and P be a point inside the triangle, with orthogonal projections D, E, F onto BC, CA, AB respectively, and H, K, L onto the tangents to (O) at A, B, C respectively. Then*

$$PH + PK + PL \geq 2(PD + PE + PF).$$

Proof. The conclusion follows by using Corollary 5 for the triangle determined by all three tangents, and the fact that the circle (O) is the incircle of this triangle. □

In fact, Corollary 4 and a similar reasoning lead us to the weighted version of the previous inequality(see [3, Theorem 4]). Now, we conclude our paper with the following application, motivated by a recent problem posed in American Mathematical Monthly ([2]).

Corollary 7. *Let ABC be a triangle inscribed into a circle (O), and P be a point inside the triangle, with orthogonal projections D, E, F onto the tangents to (O) at A, B, C respectively. Then*

$$\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} \geq \frac{1}{R},$$

where R is the circumradius of triangle ABC.

Proof. The circumcircle (O) is the incircle of the triangle bounded by the three tangents at the vertices. Applying Corollary 4 with $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$, we have

$$\begin{aligned}
\frac{PD}{a^2} + \frac{PE}{b^2} + \frac{PF}{c^2} &\geq \frac{2d(P,\ BC)}{bc} + \frac{2d(P,\ AC)}{ac} + \frac{2d(P,\ AB)}{ab} \\
&= \frac{2}{abc}\left(a \cdot d(P,\ BC) + b \cdot d(P,\ AC) + c \cdot d(P,\ AB)\right) \\
&= \frac{2}{abc}(2[PBC] + 2[PCA] + 2[PAB]) \\
&= \frac{4[ABC]}{abc} \\
&= \frac{1}{R},
\end{aligned}$$

and the proof is complete. □

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On the Orthogonality of a Median and a Symmedian

Navneel Singhal

Abstract. We give a synthetic proof of F. J. García Capitán's theorem on the lemniscate as the locus of a vertex of a triangle, given the other two vertices, such that the corresponding median and symmedian are orthogonal.

1. Introduction

In his paper [1], F. J. García Capitán proved the following theorem using Cartesian coordinates.

Theorem. *Let B and C be fixed points in the plane. The locus of a point A such that the A-median and the A-symmedian of triangle ABC are orthogonal is the lemniscate of Bernoulli with endpoints at B and C.*

We give a synthetic proof of this theorem, beginning with a series of lemmas.

Lemma 1. *Let $ABCD$ be a cyclic quadrilateral. The points $AB \cap CD$, $BC \cap DA$, $AC \cap BD$ form the vertices of a self polar triangle with respect to the circumcircle of $ABCD$.*

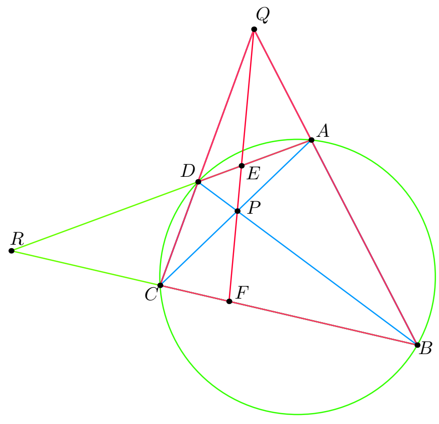

Figure 1

Proof. Let AD meet BC at R, AB meet CD at Q, and AC meet BD at P. Let QP intersect BC, AD at F, E, respectively. We know from triangles DAQ and BCQ that $(R, E; A, D)$ and $(R, F; B, C)$ are harmonic. So it follows that EF is the polar of R. Hence PQ is the polar of R. Similarly PR is the polar of Q and RQ is the polar of P. So PQR is a self polar triangle with respect to the circumcircle of $ABCD$. □

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Lemma 2. *Let ω be the circle with BC as diameter and M as center. Denote by A_1 the inverse of A in ω.*

(a) *The circles ω_1, ω_2 through $\{A, B\}, \{A, C\}$ and tangent to BC pass through A_1.*

(b) *Let H be the orthocenter of ABC. The circle with diameter AH and the circumcircle of BHC meet on A_1.*

(c) *If the A-symmedian cuts Ω, the circumcircle of ABC, at A_2, then A_1, A_2 are the reflections of each other in BC.*

Proof. (a) Since ω_1, ω_2 are tangent to BC, and their radical axis bisects BC, we know that M is on their radical axis. Also the inversion in ω preserves the circles and so A_1 is the other intersection point of ω_1, ω_2.

(b) By (a), $\angle BA_1C = 180^\circ - \angle A_1BC - \angle A_1CB = 180^\circ - \angle BAC$, so A_1 is on the circumcircle of BHC. Let the orthic triangle of ABC be DEF, where D is on BC etc. The inversion at A, and of radius $\sqrt{AH \cdot AD}$ maps BC to the circle with AH as diameter and thus A_1 to the intersection of tangents to that circle at E, F, which is the midpoint of BC. So A_1 is on the circle with AH as diameter as well by inverting back.

(c) Since the circumcircles of BHC and ABC are reflections of each other across BC, so the reflection of A_1 in BC is on Ω. Since we also have $\angle A_2CB = \angle A_2AB = \angle MAC = \angle A_1CB$, so the reflection of A_1 in BC is A_2. □

Remark. A_1 is the vertex of the D-triangle of ABC corresponding to A (see, for example, [2]), and has a lot of interesting properties, which we will not pursue in this paper.

Lemma 3. (a) *All conics through an orthocentric quadruple of points are equilateral* (rectangular) *hyperbolas, and all equilateral hyperbolas through the vertices of a triangle pass through its orthocenter.*

(b) *The locus of the centers of equilateral hyperbolas through the vertices of a triangle is the nine-point circle of the triangle.*

2. Proof of the Main Theorem

2.1. *A on lemniscate $\Rightarrow$ orthogonality of A-median and A-symmedian.*

Let the equilateral hyperbola with BC as transverse axis and passing through B, C be $\mathcal{H}$. It is the inverse image of the lemniscate with B, C as its endpoints in ω. We are going to show that if A_1 is on $\mathcal{H}$, $AA_2 \perp AM$.

Since A_2 is the reflection of A_1 in BC, A_2 is also on $\mathcal{H}$. As the perpendicular from A_1 to BC meets $\mathcal{H}$ at A_2, $\{A_1, A_2, B, C\}$ is an orthocentric quadruple by Lemma 3(a). So $A_1B \cap A_2C, A_1C \cap A_2B$ are on ω.

In view of Lemma 1, A_1, A_2 are conjugate points with respect to ω. Now as A is the inverse of A_1 in ω, the line through A and perpendicular to AM is the polar of A_1 with respect to ω, which passes through A_2. Therefore, the A-median is perpendicular to the A-symmedian.

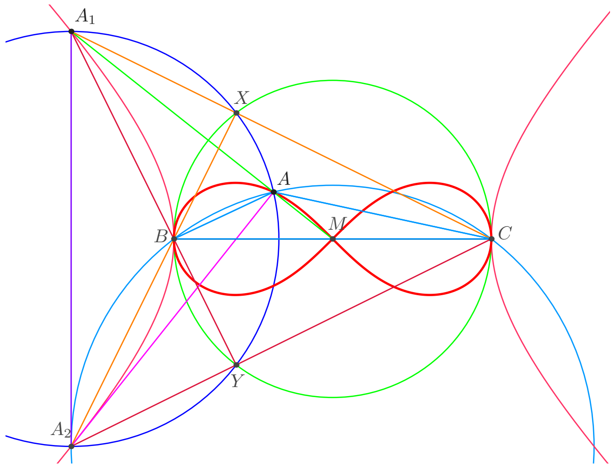

Figure 2

2.2. *Orthogonality of A-median and A-symmedian $\Rightarrow$ A on lemniscate.*

Now we suppose that $AM \perp AA_2$. Since A is the inverse of A_1 in ω, AA_2 is the polar of A_1 with respect to ω. Let $A_2B \cap A_1C = X$. Then

$$\begin{aligned}
\angle BXC &= 180^\circ - \angle XBC - \angle XCB \\
&= \angle A_2BC - \angle A_1CB \\
&= \angle A_1BC - \angle A_1CB \\
&= \angle BAM - \angle CAM \\
&= 90^\circ + \angle BAA_2 - \angle MAC \\
&= 90^\circ
\end{aligned}$$

wherein we have used Lemma 2 repeatedly.

So $X \in \omega$. Similarly, $A_1B \cap A_2C \in \omega$.

Thus $\{A_1, A_2, B, C\}$ is an orthocentric quadruple. Invoking Lemma 3(b), as M is on the nine point circle of A_1BC, let $\mathcal{H}$ be the hyperbola through A_1, A_2, B, C and with center M.

Now that $MA_1 = MA_2$, A_1, A_2 are on a circle with M as center, so the perpendicular bisector of A_1A_2 is an axis of $\mathcal{H}$. Thus BC is the transverse axis of $\mathcal{H}$ (as B, C are on $\mathcal{H}$), and so A_1 is on the fixed hyperbola $\mathcal{H}$, thereby establishing the fact that A is on the lemniscate with B, C as endpoints by inversion in ω.

This completes the proof of the Main Theorem.

3. Some interesting properties

Property 1. *$A_1A_2 \cap BC$ is the inverse of the trace of the symmedian point on BC in ω.*

Proof. The pole of AA_2 with respect to Ω is A_1 and that of BC is the point at ∞ in the direction perpendicular to BC, so the claim follows. $\square$

Property 2. *The line A_1A_2 is tangent to Ω.*

Proof. Follows from the previous property and the fact that the pole of AA_2 with respect to Ω is the intersection of the tangents to Ω at A, A_2 and BC. □

Property 3. *Irrespective of the condition of orthogonality of the median and the symmedian, the points A, X, Y, A_1, A_2 are on the A-Apollonius circle.*

Proof. Since the quadrilateral $BCAA_2$ is harmonic, we know that A_2 is on the A-Apollonius circle. Now the A-Apollonius circle is symmetric with respect to BC, so A_1 is on it as well. $\angle YA_2A = \angle ACB = \angle AA_1C$, so Y and analogously X are on the A-Apollonius circle. □

Property 4. *Irrespective of the condition of orthogonality of the median and the symmedian, if AC and AB meet the A-Apollonius circle at A_b, A_c respectively, then the arcs A_1A_c and A_2A_b are congruent.*

Proof. Simple angle chasing. □

Property 5. *Irrespective of the condition of orthogonality of the median and the symmedian, the tangent to Ω at A_2, the A-Apollonius circle and the line through A and parallel to BC are concurrent.*

Proof. Using cross ratios,

$$\begin{aligned} -1 &= (B, C; A, A_2) \\ &\stackrel{A_2}{=} (X, Y; A, A_2A_2 \cap \odot(AXY)) \end{aligned}$$

By projecting this through A_1, we have our conclusion. □

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On the Elementary Single-Fold Operations of Origami: Reflections and Incidence Constraints on the Plane

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Abstract. This article reviews the so-called "axioms" of origami (paper folding), which are elementary single-fold operations to achieve incidences between points and lines in a sheet of paper. The geometry of reflections is applied, and exhaustive analysis of all possible incidences reveals a set of eight elementary operations. The set includes the previously known seven "axioms", plus the operation of folding along a given line. This operation has been ignored in past studies because it does not create a new line. However, completeness of the set and its regular application in practical origami dictate its inclusion. Formal definitions and conditions of existence of solutions are given for all the operations.

1. Introduction

Three decades ago, Justin [18] introduced a set of 7 elementary single-fold operations which have become known as the "axioms" of origami, the Japanese art of paper folding [1, 2, 11]. Each operation is defined by one or more alignments (incidences) between points and lines on a sheet of paper, that must be achieved with a single fold. The number of solutions of each operation must be finite; however, depending on the relative position of the given points and lines, some of the operations may have none, one or multiple solutions. It has been shown that the set of operations constitutes a more powerful geometrical tool than the combination of straight edge and compass [1]. For example, the operations allow for the trisection of arbitrary angles [14], solving the problem of duplicating the cube [24], constructing heptagons [10] and solving cubic and fourth order equations [1, 9], all of which may be not done by straight edge and compass alone.

Justin's work [18] seems to have been overlooked at its time, and the same fold operations have been rediscovered later and expressed under various forms by Huzita [15], Hatori (according to [2]) and other enthusiasts of origami mathematics [1, 3, 9, 20, 23]. The operations are popularly known today as Huzita's axioms, Huzita-Justin's axioms or Huzita-Hatori's axioms. [1] Let us note that the designation as "axioms" is not correct since some of the operations may be derived from others, and further, some of them may not be possible depending on the configuration of

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[1]In his work, Huzita [15] listed six of Justin's fold operations. The seventh was rediscovered by Hatori, in 2001 (according to [2]).

given points and lines. It has been claimed that the set is complete, in the sense that it contemplates all possible alignments between points and lines with a finite number of solutions, and excluding redundant alignments [2].

More recently, Kasem et al. [19] showed several inconsistencies in the operations owing to the rather imprecise form in which they had been stated. The inconsistencies included impossibility of some folds, infinite solutions and superfluous conditions. Seeking a more rigorous treatment, Ghourabi et al. [11] expressed the operations in formal algebraic terms and analyzed their number of solutions and conditions of existence. Such a formalization is a necessary step for adapting folding techniques to industrial applications. In fact, recent years have seen a surge of applications of origami to science and technology, e.g., in aerospace and automotive technology [4], materials science [27], biology [22], civil engineering [8], robotics [7] and acoustics [13]. Further, a number of computational systems of origami simulation have been developed [16]. The present article follows the call for normalization and analyzes the elementary operations by applying the geometry of reflections. Folding a sheet of paper along a straight line superposes the paper on one side of the fold line to the other side. As a result, all points and lines on each side of fold line are reflected across it onto the other side [23]. In fact, the geometry of paper folding may be reproduced by using a semi-reflective mirror called "Mira" [5]. A full geometrical characterization of Mira constructions has been given in terms of "primitive actions" [6], which are equivalent to the fold operations studied here.

First, the analysis will determine all possible incidences between given points and lines on a plane that may be achieved by a reflection. Next, it will derive all possible fold operations that may be defined so as to satisfy combinations those incidences. In this way, a total of eight elementary operations will be obtained, i.e., one more than the previous set, where the new operation is to fold along a given line. This operation has been commonly disregarded by previous studies on the argument that it does not create a new line; however, completeness of the set demands its inclusion. Further, it has applications in actual origami folding, as will be discussed later (Section 5).

2. Reflections on a plane

The medium on which all folds are performed is assumed to be an infinite Euclidean plane [9]. Points are denoted by capital letters (P, Q etc.), lines by small letters (m, n etc.), except the fold line which is denoted by the special symbol χ, and $P \in m$ means that point P is on line m.

A reflection is defined as follows [23]:

Definition 1. *Given a line χ, the reflection $\mathcal{F}$ in χ is the mapping on the set of points in the plane such that for point P,*

$$\mathcal{F}(P) = \begin{cases} P & \text{if } P \in \chi, \\ P' & \text{if } P \notin \chi \text{ and } \chi \text{ is the perpendicular bisector of segment } PP'. \end{cases}$$

(see Figure 1).

It is easy to see that $\mathcal{F}(P) = P'$ if and only if $\mathcal{F}(P') = P$.

Reflection of a line m in χ is obtained by reflecting every point in m. Therefore, $\mathcal{F}(m) = \{\mathcal{F}(P)|P \in m\}$. Let $m' = \mathcal{F}(m)$ and consider the following cases:

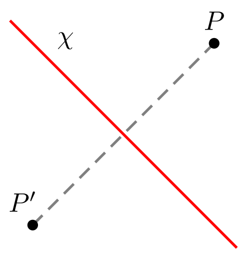

Figure 1

1. If m and χ are parallel ($m \parallel \chi$), then $m \parallel m'$ (Figure 2, left).

2. If m and χ are not parallel ($m \nparallel \chi$) then line χ is a bisector of the angle between m and m' (Figure 2, center).

3. If $m = \chi$, then every point in m is its own reflection, and therefore $m' = m$.

4. If m and χ are perpendicular ($m \perp \chi$), then the reflection of every point of m is also on m; therefore, $m = m'$ (Figure 2, right). Note also that χ divides m into two halves, and each half is reflected onto the other. Thus, for every point P on m and not on the intersection with χ, $\mathcal{F}(P) \neq P$.

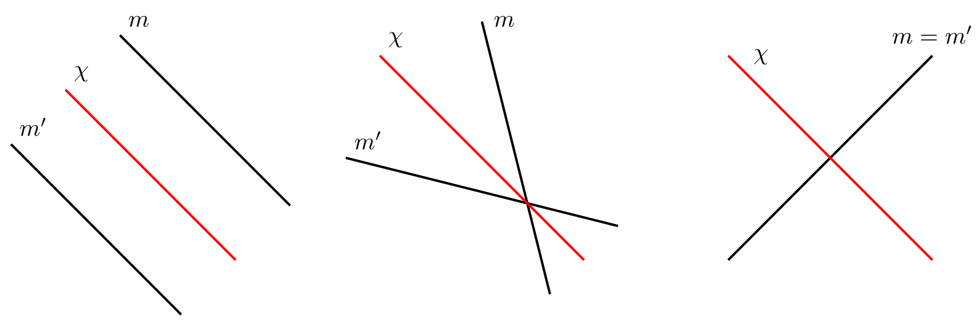

Figure 2

3. Incidence constraints

Elementary single-fold operations are defined in terms of incidence constraints between pairs of objects (points or lines) that must be satisfied with a fold [2, 11, 18]. Each constraint involves an object α and the image $\mathcal{F}(\beta)$ of an object β (including the case $\alpha = \beta$) by reflection in the fold line. The symmetry of the reflection mapping implies that all incidence relations are also symmetric.

A total of six different incidences are possible on a plane, an they are defined and analyzed in the next subsections (see also Table 1). In order to facilitate the posterior definitions of the fold operations, incidences involving distinct objects (i.e., $\alpha \neq \beta$) are distinguished from those involving the same object (i.e., $\alpha = \beta$).

3.1. *Incidences involving distinct objects.*

Incidence I1. $\mathcal{F}(P) = Q$, with $P \neq Q$.

In this incidence, the reflection of a given point P coincides with another given point Q. According to Definition 1, its solution is the unique fold line χ which is the perpendicular bisector of segment PQ.

Incidence I2. $\mathcal{F}(m) = n$, with $m \neq n$.

In this incidence, the reflection of a given line m coincides with another given line n.

Two cases are possible:

(1) When $m \nparallel n$, there are two possible fold lines that satisfy the incidence, which are the bisectors to the angles defined by m and n (Figure 3).

(2) When $m \parallel n$, there is only one solution, which is a fold line parallel and equidistant to both m and n (Figure 2, left, with $m' = n$).

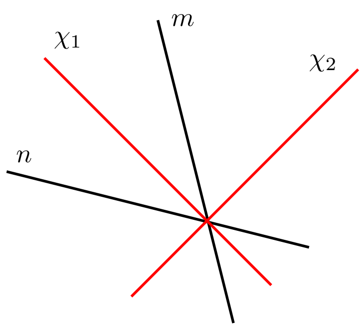

Figure 3

Incidence I3. $\mathcal{F}(P) \in m$, with $P \notin m$.

In this incidence, the reflection of a given point P is on a given line m, and the case in which P is already on m is excluded. It has been shown that the fold lines that satisfy the incidence are tangents to a parabola with focus P and directrix m [1, 23].

It is useful to derive equations of the fold lines and the associated parabola, for later application to the analysis of fold operations. Without loss of generality, choose a Cartesian system of coordinates x, y so that P is located at $(0, 1)$ and line m is $y = -1$ (Figure 4). Also, let $P' = \mathcal{F}(P)$ be located at $(t, -1)$, where t is a free parameter.

The slope of segment PP' is $-2/t$. The fold line χ is perpendicular to PP' and therefore has a slope of $t/2$. Further, χ passes through the midpoint of PP', which is located at $(t/2, 0)$. Thus, χ has an equation

$$y = \frac{t}{2}\left(x - \frac{t}{2}\right). \tag{1}$$

Next, consider point T located at the intersection of χ with a vertical line through P'. Its coordinates may be obtained by evaluating Equation (1) at $x = t$, which produces $(t, t2/4)$. Those coordinates describe parametrically a parabola with equation

$$y = \frac{x^2}{4}, \tag{2}$$

which is denoted by Ψ. This is precisely the equation of a parabola with focus at $(0, 1)$ and directrix $y = -1$. [2] Further, note that the slope of a tangent to Ψ at point T is $y'(t) = t/2$, which is the same slope of χ. Therefore, χ is a line tangent to Ψ at point T.

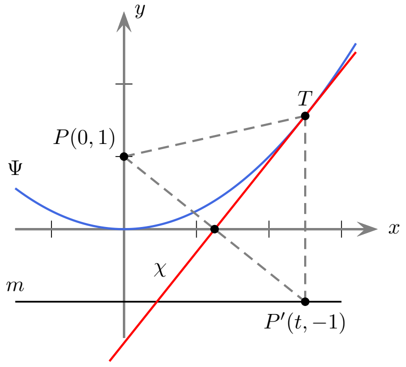

Figure 4

Since t in Equation (1) is a free parameter, then the solution to this incidence is a family of fold lines with one parameter (Figure 5).

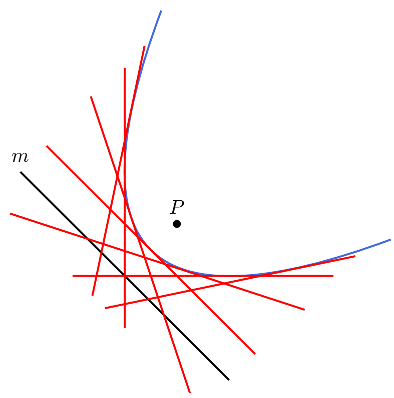

Figure 5

[2]The general equation of a parabola with vertical axis and vertex at $(0, 0)$ is $y = x^2/(4a)$, where a is the distance from the vertex to the directrix $y = -a$ or the focus $(0, a)$ [26].

The case $P \in m$ is excluded because under such condition any fold line passing through P or perpendicular to m satisfies the incidence. Those two possibilities are considered in incidences I4 and I5, respectively.

3.2. *Incidences involving an object and its reflected image.*

Incidence I4. $\mathcal{F}(P) = P$.

In this incidence, the reflection of a given point P coincides with itself, and it is satisfied by any fold line χ passing through P . An arbitrary direction for line χ may be defined by its angle θ with, e.g., the x-axis in a Cartesian coordinate system. Therefore, the solution to the incidence is a family of fold lines with one parameter (Figure 6).

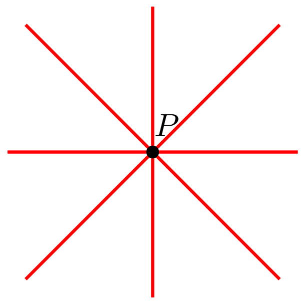

Figure 6

Incidence I5. $\mathcal{F}(m) = m$, and $\exists P \in m, \mathcal{F}(P) \neq P$.

Both this and the next incidence consider the reflection of a line m to itself. As discussed in Section 2, there are two ways in which such a reflection may be achieved. In the current case, one half of m, defined from an arbitrary point $R \in m$, is reflected upon the opposite half.

The position of point R may be specified by its distance s from a particular point $P_0 \in m$. Therefore, the solution to the incidence is a family of fold lines with one free parameter (Figure 7).

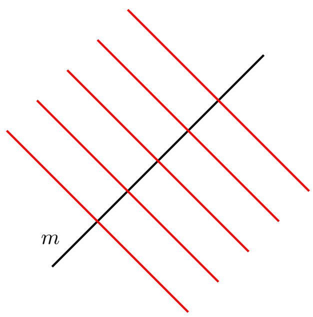

Figure 7

Incidence I6. $\mathcal{F}(m) = m$, and $\exists P \in m, \mathcal{F}(P) = P$.
This is the second case of reflection of line m to itself. In this case, each point $P \in m$ is reflected to itself, and therefore the incidence is satisfied by the unique fold line $\chi = m$.

4. Elementary fold operations

4.1. *Definition.* A straight line on a plane is an object with two degrees of freedom. [3] When an incidence constraint is set for χ, satisfying the constraint consumes a number of degrees of freedom, and that number is called the codimension of the constraint. Incidences I1, I2 and I6 have either a unique or a finite number of solutions; therefore, those incidences have codimension 2. On the other hand, each of incidences I3, I4 and I5 have a family of solutions with one free parameter and therefore they have codimension 1 (Table 1).

Table 1: Incidence constraints.

Incidence	Definition[a]	Codimension
I1	$\mathcal{F}(P) = Q$, with $P \neq Q$	2
I2	$\mathcal{F}(m) = n$, with $m \neq n$	2
I3	$\mathcal{F}(P) \in m$, with $P \notin m$	1
I4	$\mathcal{F}(P) = P$	1
I5	$\mathcal{F}(m) = m$, and $\exists P \in m, \mathcal{F}(P) \neq P$	1
I6	$\mathcal{F}(m) = m$, and $\exists P \in m, \mathcal{F}(P) = P$	2

[a] P and Q are points; m and n are lines.

An elementary single-fold operation is defined as a minimal set of alignments between points and lines that is satisfied with a single fold and has a finite number of solutions [2]. Equivalently, it is the resolution of a set of incidence constraints which have a total codimension of 2.

Each of the incidences I1, I2 and I6 already define an elementary operation. The other three incidences must be applied in pairs (including pairing incidences of the same type), and there are a total of 6 possible pairs. However, incidence I5 can be not be used twice. If it is, then for given lines m and n the fold line χ has to satisfy both $\mathcal{F}(m) = m$ and $\mathcal{F}(n) = n$. Therefore, χ must be perpendicular to both m and n, and two cases are possible:

(1) If $m \nparallel n$, then a perpendicular to both lines does not exist (in the Euclidean plane).
(2) If $m \parallel n$, then any perpendicular to m or n is a valid fold line.

Thus, the pair of constraints has none or infinite solutions, and so it does not define a valid elementary fold operation.

[3] A fold line χ may be defined by an equation of the form $ax + by + c = 0$, where a, b and c are constants, and (a, b) is a normal vector to χ. A vector in arbitrary direction may be defined by letting $a = \cos\theta$, $b = \sin\theta$, with $0 \leq \theta < 2\pi$. Therefore, two parameters must be set in order to define any fold line, namely, θ and c.

A total of eight elementary fold operations may be then defined, and they are analyzed in the next subsections.

4.2. *Elementary operations defined by codimension* 2 *incidences.*
The following three operations are defined by incidences I1, I2 and I6, respectively.

Operation O1. Given points P and Q, with $P \neq Q$, construct a fold line so that $\mathcal{F}(P) = Q$.

Operation O2. Given lines m and n, with $m \neq n$, construct a fold line so that $\mathcal{F}(m) = n$.

Operation O3. Given a line m, construct a fold line so that $\mathcal{F}(m) = m$, and $\exists P \in m, \mathcal{F}(P) = P$.

5. Elementary operations defined by pairs of codimension 1 incidences

Operation O4. Given points P and Q, with $P \neq Q$, construct a fold line so that $\mathcal{F}(P) = P$ and $\mathcal{F}(Q) = Q$.

This operation is defined by application of incidence I4 to two distinct points. It has a unique solution, which is a fold line χ passing through points P and Q (Figure 8).

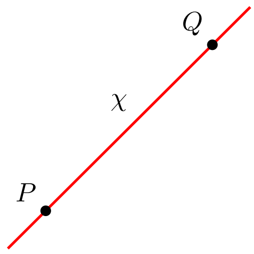

Figure 8

Operation O5. Given a point P and a line m, construct a fold line such that $\mathcal{F}(P) = P$ and $\mathcal{F}(m) = m$, and $\exists Q \in m$, $\mathcal{F}(Q) \neq Q$.

This operation is defined by application of incidences I4 and I5. It has a unique solution, which is a fold line χ perpendicular to m and passing through P (Figure 9). Note that the case $P \in m$ is allowed, which has the same unique solution.

Operation O6. Given points P, Q, and a line m, with $P \notin m$, construct a fold line such that $\mathcal{F}(P) \in m$ and $\mathcal{F}(Q) = Q$.

This operation is defined by application of incidences I3 and I4. Its solution is a fold line that is tangent to a parabola with focus P and directrix m, and passes through point Q.

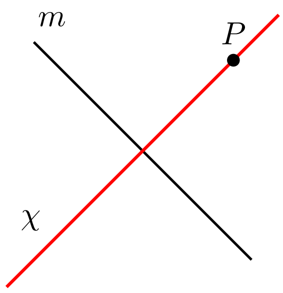

Figure 9

Assume the same parabola of Figure 4, given by Equation (2), and a point Q at the position (x_q, y_q). Replacing the coordinates of Q in Equation (1) produces the quadratic equation

$$t^2 - 2x_q t + 4y_q = 0. \tag{3}$$

The discriminant of Equation (3) is $\Delta = 4x_q^2 - 16y_q$, and $\Delta = 0$ yields $y_q = x_q^2/4$, which implies $Q \in \Psi$. Since Ψ is the locus of points that are equidistant from P and m, we may conclude that the fold operation has a unique solution when Q is equidistant to P and m, two solutions when Q is closer to m (i.e., $y_q < x_q^2/4$), and no solution when Q is closer to P (i.e., $y_q > x_q^2/4$).

Figure 10 shows an example for the case of two solutions.

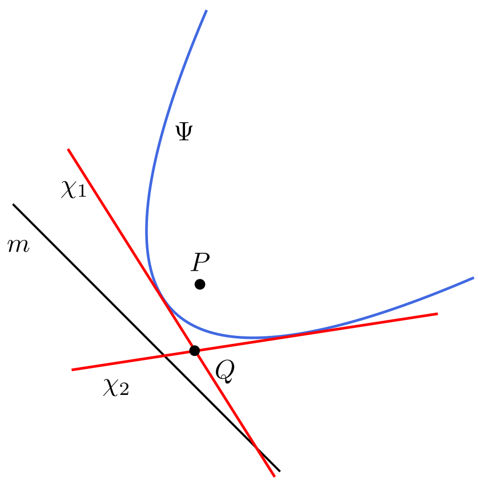

Figure 10

Operation O7. Given points P, Q, and lines m, n, with $P \notin m$, $Q \notin n$, and $P \neq Q$ or $m \neq n$, construct a fold line so that $\mathcal{F}(P) \in m$ and $\mathcal{F}(Q) \in n$.

This operation derives from the application of incidence I3 to two distinct point-line pairs. Its solution is a fold line that is tangent to both a parabola Ψ with focus P and directrix m, and a parabola Θ with focus Q and directrix n.

Again, assume the same parabola Ψ of Figure 4, given by Equation (2). Assume also that Q is located at (x_q, y_q), and its reflection $Q' = \mathcal{F}(Q)$ is at (x'_q, y'_q). Then, the segment $\overline{QQ'}$ has slope $\frac{(y_q - y'_q)}{(x_q - x'_q)}$. The fold line χ, given by Equation (1), has slope $t/2$ and is perpendicular to $\overline{QQ'}$ (because it reflects Q onto Q'). Therefore,

$$\frac{t}{2} = -\frac{x_q - x'_q}{y_q - y'_q}. \tag{4}$$

Further, χ passes through the midpoint of $\overline{QQ'}$, which is located at $((x_q + x'_q)/2,\ (y_q + y'_q)/2)$. Replacing these coordinates into Equation (1) produces

$$2(y_q + y'_q) = t(x_q + x'_q - t). \tag{5}$$

Finally, eliminating t from Equations (4) and (5) produces

$$(y_q + y'_q)(y_q - y'_q)^2 = -(x_q^2 - {x'_q}^2)(y_q - y'_q) - 2(x_q - x'_q)^2. \tag{6}$$

For a given line n, the coordinates of Q' satisfy an equation of the form

$$a x'_q + b y'_q + c = 0, \tag{7}$$

where a, b and c are constants.

Equations (6) and (7) may be solved for x'_q and y'_q. Substituting this solution into Equation (4) yields t, which defines the fold line χ in Equation (1). Two cases may be considered:

(1) If $m \parallel n$, then Q' is on a horizontal line and so $y'_q = -c/b$. In this case, Equation (6) is quadratic in x'_q and may have zero to two solutions.
(2) If $m \nparallel n$, solving Equation (7) for x'_q or y'_q and replacing in Equation (6) produces a cubic equation with one to three solutions. An example for the latter case is shown in Figure 11.

Let us investigate further the conditions to ensure the existence of solutions. As noted above, the operation may not have a solution only in the case of $m \parallel n$. Rearranging Equation (6) produces

$$(2 - y_q + y'_q)(x_q - x'_q)^2 + 2x_q(y_q - y'_q)(x_q - x'_q) + (y_q + y'_q)(y_q - y'_q)^2 = 0, \tag{8}$$

which is a quadratic equation in $(x_q - x'_q)$. The discriminant is

$$\Delta = 4x_q^2(y_q - y'_q)^2 - 4(2 - y_q + y'_q)(y_q + y'_q)(y_q - y'_q)^2, \tag{9}$$

and letting $\Delta \geq 0$ produces

$$x_q^2 + (y_q - 1)^2 \geq (y'_q + 1)^2. \tag{10}$$

The left side of Equation (10) is the squared distance between P and Q, and the right side is the squared distance between m and n. This result does not seem reported in the literature, and may be stated as a theorem:

Theorem 1. *Given points P, Q, and lines m, n, with $P \notin m$, $Q \notin n$, and $P \neq Q$ or $m \neq n$, a fold line that places P on m and Q on n exists if and only if the distance between P and Q is larger than or equal to the distance between m and n.*

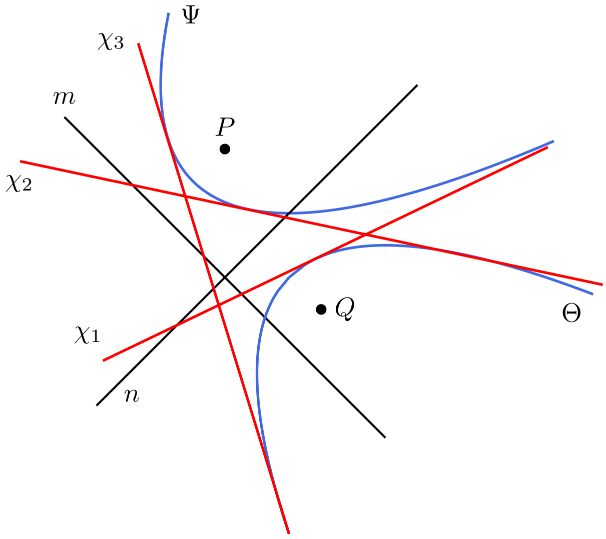

Figure 11

Operation O8. Given point P and lines m and n, with $P \notin m$, construct a fold line so that $\mathcal{F}(P) \in m$ and $\mathcal{F}(n) = n$ and $\exists Q \in n, \mathcal{F}(Q) \neq Q$.

This operation is defined by application of incidences I3 and I5. Its solution is a fold line that is tangent to a parabola with focus P and directrix m, and is perpendicular to line n.

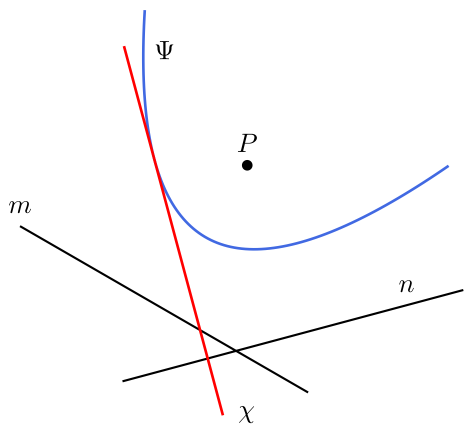

Figure 12

As in the previous operation, assume the same parabola of Figure 4 given by Equation (2), and a line n with equation $ax + by + c = 0$. The fold line χ has slope $t/2$ and is perpendicular to n. Two cases may be considered:

(1) If $m \nparallel n$, then $a \neq 0$. Therefore,

$$\frac{t}{2} = -\frac{b}{a} \tag{11}$$

which has a unique solution for t (Figure 12). Knowing t, Equation (11) defines the fold line χ.

(2) If $m \parallel n$, then n is a horizontal line and cannot be perpendicular to any tangent to parabola Ψ. In this case, the operation does not have a solution.

5.1. *Summary.* Table 2 lists the incidence constraints that define each operation and their number of solutions, Table 3 lists the conditions for the existence of solutions, and Table 4 restates the operations as folding actions of a medium $\mathcal{O}$.

6. Discussion

The complete set of elementary single-fold operations contains eight operations, listed in Table 4. Operations O1, O2 and O4 to O8 constitute Justin's original set [18], and operation O3 is the new addition proposed here. O3 does not create a new line and has been ignored in previous studies on origami constructions [2, 11]. [4] However, it is a valid elementary single-fold operation and completeness of the set demands its inclusion.

Table 2: Incidence constraints and number of solutions of the elementary single-fold operations.

Operation	Incidence constraints	Solutions
O1	$\mathcal{F}(P) = Q$, with $P \neq Q$	1
O2	$\mathcal{F}(m) = n$, with $m \neq n$	$1, 2$
O3	$\mathcal{F}(m) = m$,and $\exists P \in m,\ \mathcal{F}(P) = P$	1
O4	$\mathcal{F}(P) = P$ and $\mathcal{F}(Q) = Q$, with $P \neq Q$	1
O5	$\mathcal{F}(P) = P$ and $\mathcal{F}(m) = m$, and $\exists Q \in m$, $\mathcal{F}(Q) \neq Q$	1
O6	$\mathcal{F}(P) \in m$, with $P \notin m$, and $\mathcal{F}(Q) = Q$	$0 - 2$
O7	$\mathcal{F}(P) \in m$, with $P \notin m$, $\mathcal{F}(Q) \in n$, with $Q \notin n$, and $P \neq Q$ or $m \neq n$	$0 - 3$
O8	$\mathcal{F}(P) \in m$, with $P \notin m$, and $\mathcal{F}(n) = n$, and $\exists Q \in n$, $\mathcal{F}(Q) \neq Q$	$0, 1$

Table 3: Conditions for the existence of solutions of the elementary single-fold operations.

Operation	Conditions
O1 to O5	none
O6	distance between P and Q larger than distance between Q and m
O7	distance between P and Q larger than distance between m and n
O8	$m \nparallel n$

[4] In his formulation, Justin [18] allowed for solutions where the fold line itself coincides with an existent line. However, such action was not considered as a fold operation on its own.

Table 4: Elementary single-fold operations restated as folding actions.

No.	Action[a]
O1	Given two distinct points P and Q, fold $\mathcal{O}$ to place P onto Q.
O2	Given two distinct lines m and n, fold $\mathcal{O}$ to align m and n.
O3	Fold along a given a line m.
O4	Given two distinct points P and Q, fold $\mathcal{O}$ along a line passing through P and Q.
O5	Given a line m and a point P, fold $\mathcal{O}$ along a line passing through P to reflect half of m onto its other half.
O6	Given a line m, a point P not on m and a point Q, fold $\mathcal{O}$ along a line passing through Q to place P onto m.
O7	Given two lines m and n, a point P not on m and a point Q not on n, where m and n are distinct or P and Q are distinct, fold $\mathcal{O}$ to place P onto m, and Q onto n.
O8	Given two lines m and n, and a point P not on m, fold $\mathcal{O}$ to place P onto m, and to reflect half of n onto its other half.

[a] $\mathcal{O}$ denotes the medium in which folds are performed; e.g., a sheet of paper, fabric, plastic, metal or any other foldable material.

There is also a more practical reason for not ignoring O3. Folding a sheet of paper along a line superposes the paper on both sides of the fold line, in two layers. In origami mathematics, it is assumed that all lines and points marked on one layer are also defined on the layers above and below, as if the paper were "transparent" [9, 23]. However, it is not so in actual paper folding: a given origami work may require to fold, e.g., the top layer along a line marked on the layer below it. Such is the case when folding parallel lines for building tessellation grids [12, see instruction 5 for a triangle grid in p. 8]. Figure 13 shows a simple example. Assume a sheet of paper (not necessarily square) in which two parallel lines m and n are marked, and assume that we want to create a third equidistant parallel line o. Both steps (1) and (2) require to fold along given lines (n and m, respectively). Naturally, it would be also possible to use the points of intersections of m and n with the borders of the paper as references, instead of the lines themselves. Thus, the instruction for step (2) could be: fold the top layer along a line passing through points P and Q on the bottom layer, where P and Q are the intersections of m with the upper and lower edges of the paper. However, in actual practice it is much simpler and convenient to perform the fold by aligning it with line m. Further, it might be the case that the borders of the paper are not well defined (or have not been defined) or that it is considered as a theoretical infinite plane.

O3 is also a common instruction also for building figurative models [21, see steps 8 and 16 of "Baby" in page 88, and step 6 of "Songbird 2" in page 340]. Thus, the operation should be in the repertoire of, e.g., computational systems for origami simulation and design [17, 25].

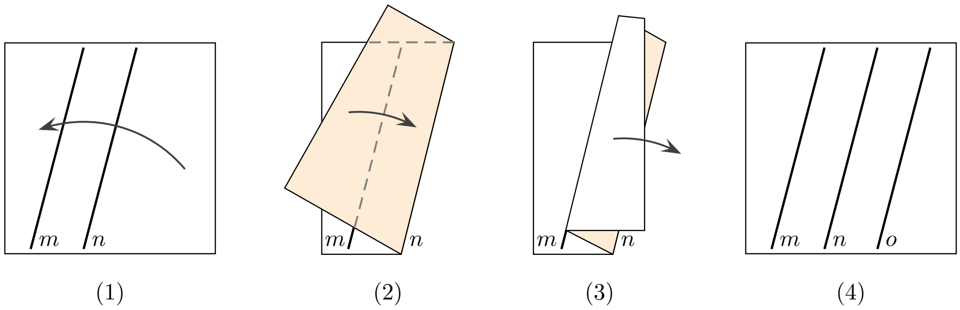

Figure 13. Given parallel lines m and n, create a third equidistant parallel line at the left of n. (1) Fold along line n. (2) Fold the top layer along line m in the bottom layer. (3) Unfold. (4) Final result.

7. Conclusion

Analysis of reflections of points and lines on a plane subject to incidence constraints has determined a complete set of eight elementary single-fold operations. Precise definitions and conditions of existence of solutions of all operations are given in Tables 2 to 4, which may be useful to scientific and technological applications of origami.

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Properties of the Tangents to a Circle that Forms Pascal Points on the Sides of a Convex Quadrilateral

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Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. The theory deals with the properties of the Pascal points on the sides of a convex quadrilateral, the properties of "circles that form Pascal points", and the special properties of "the circle coordinated with the Pascal points formed by it".

In the present paper we shall continue developing the theory and prove six new theorems that describe the properties of the tangents to the circle that forms Pascal points.

1. Introduction: General concepts and theorems of the theory of a convex quadrilateral and a circle that forms Pascal points

In order to understand the new theorems, we include in the introduction a short review of the main concepts and the fundamental theorem of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides. In addition, we present two general theorems that we shall employ in proving the new theorems.

The theory investigates the situation in which $ABCD$ is a convex quadrilateral and ω is a circle that satisfies the following two requirements:

(I) It passes through both point E, which is the point of intersection of the diagonals, and point F, which is the point of intersection of the continuations of sides BC and AD.

(II) It intersects sides BC and AC at their inner points M and N, respectively (see Figure 1).

In this case, the fundamental theorem of the theory holds (see [2], [3]):

The Fundamental Theorem.
Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the continuations of these sides, and that passes through the point of intersection of the diagonals.
In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the continuation of a diagonal.
Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.

Or, by notation (see Figure 2):

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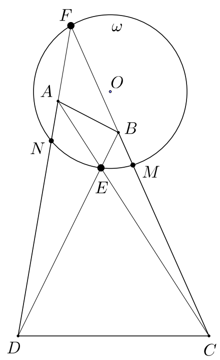

Figure 1.

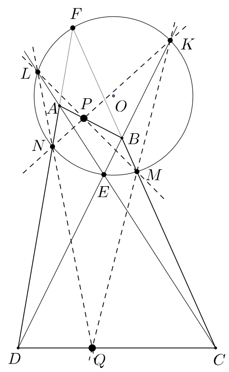

Figure 2.

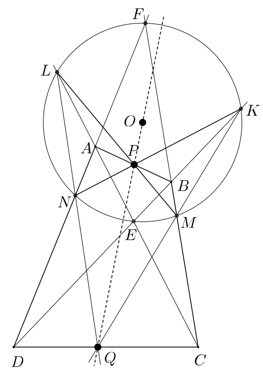

Figure 3.

Given: convex quadrilateral $ABCD$, in which $E = AC \cap BD$, $F = BC \cap AD$. Circle ω that satisfies $E, F \in \omega$; $M = \omega \cap [BC]$; $N = \omega \cap [AD]$; $K = \omega \cap BD$; $L = \omega \cap AC$.

Prove: $KN \cap LM = P \in [AB]$; $KM \cap LN = Q \in [CD]$.

We prove the fundamental theorem using the general Pascal Theorem (see [2]).

Definitions:

Since the proof of the properties of the points of intersection P and Q is based on Pascal's Theorem, we shall call

(I) these points *"Pascal points" on sides AB and CD of the quadrilateral*.

(II) the circle that passes through the points of intersection E and F and through two opposite sides *"a circle that forms Pascal points on the sides of the quadrilateral"*.

Of all the circles that form Pascal points, there is one particular special circle whose center is located on the same straight line together with the Pascal points that are formed by it.

(III) A circle whose center is collinear with the "Pascal points" formed by it will be called: *"the circle coordinated with the Pascal points formed by it"*.

For example, in Figure 3 the center of circle ω (point O) is collinear with the Pascal points P and Q formed using the circle. Therefore, circle ω is coordinated with the Pascal points formed by it.

We also use the following general theorems of the theory (see proofs in [2]):

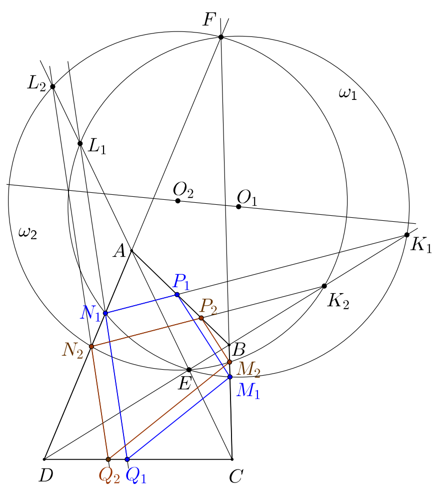

Figure 4.

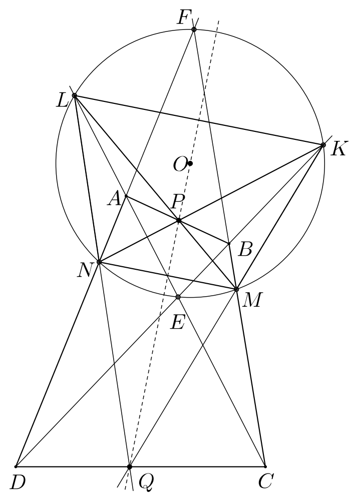

Figure 5.

General Theorem A.
Let $ABCD$ be a convex quadrilateral. Also, let ω_1 and ω_2 be circles defined as follows:
ω_1 is a circle that intersects sides BC and AD at points M_1 and N_1, respectively, and intersects the continuations of the diagonals at points K_1 and L_1, respectively, and circle ω_1 forms Pascal points P_1 and Q_1 on sides AB and CD, respectively (see Figure 4)*;*
ω_2 is a circle that intersects sides BC and AD at points M_2 and N_2, respectively, and intersects the continuations of the diagonals at points K_2 and L_2, respectively, and circle ω_2 forms Pascal points P_2 and Q_2 on sides AB and CD, respectively.
Then, the corresponding sides of quadrilaterals $P_1M_1Q_1N_1$ and $P_2M_2Q_2N_2$ are parallel to each other.

General Theorem B.
Let $ABCD$ be a convex quadrilateral, and let ω be a circle coordinated with the Pascal points P and Q formed by it, where ω intersects a pair of opposite sides of the quadrilateral at points M and N, and also intersects the continuations of the diagonals at points K and L (see Figure 5)*.*
Then there holds:

(a) *$KL||MN$;*
(b) *quadrilateral $PMQN$ is a kite;*
(c) *in a system in which circle ω is the unit circle, the complex coordinates of points K, L, M, and N satisfy the equality $mn = kl$;*
(d) *inversion relative to circle ω transforms points P and Q one into the other.*

2. New properties of the tangents to a circle that forms Pascal points

Theorem 1. *Let $ABCD$ be a quadrilateral* (*convex*) *in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is an arbitrary circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and also intersects the continuations of diagonals BD and AC at points K and L, respectively;
P and Q are the Pascal points formed by ω;
R is the point of intersection of the tangents to the circle at points K and L;
T is the point of intersection of the tangents to the circle at points M and N.
Then:

(a) *points R and T belong to "Pascal point line" PQ* (*see Figure 6*).
(b) *points Q, T, P, and R form a harmonic quadruple, i.e., there holds* $\dfrac{PT}{TQ} = \dfrac{PR}{RQ}$.

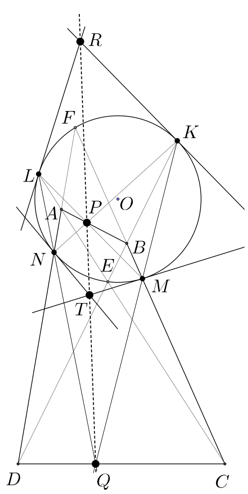

Figure 6.

Proof. (a) In proving the theorem we shall make use of the following properties of a pole and its polar with respect to the given circle. (Note: The definition and the properties of a pole and its polar appear, for example, in [1, Chapter 6, Paragraph 1] or [4, Sections 204, 205, 211]):

(i) *For a given pole, X, that lies outside the circle, polar x is a straight line that passes the points of tangency of the two tangents to the circle that issue from point X* (see Figure 7a).
(ii) *In a quadrilateral inscribed in a circle in which the continuations of the opposite sides intersect at points X and Y, and the diagonals intersect at point Z* (see Figure 7b), *there holds that straight line ZY is the polar of point X, and the straight line ZX is the polar of point Y.*

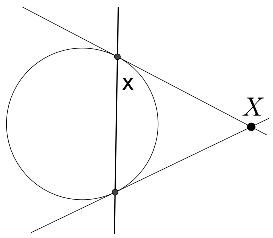

Figure 7a.

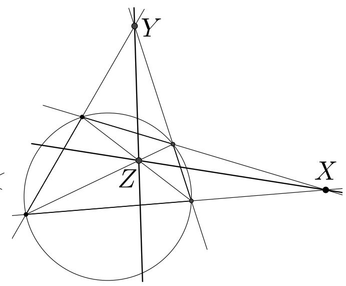

Figure 7b.

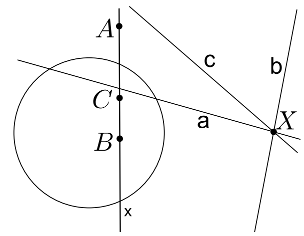

Figure 7c.

(iii) *If the straight lines* a, b, c, … *pass through the same point* X, *then their poles,* A, B, C, … (*relative to a given circle*) *belong to the same straight line,* x, *which is the polar of pole X* (see Figure 7c).

Let us carry out the following additional constructions (see Figure 8):

We connect points K, L, N, and M by segments, to form quadrilateral $KLNM$. We continue sides KL and MN to intersect at point S.

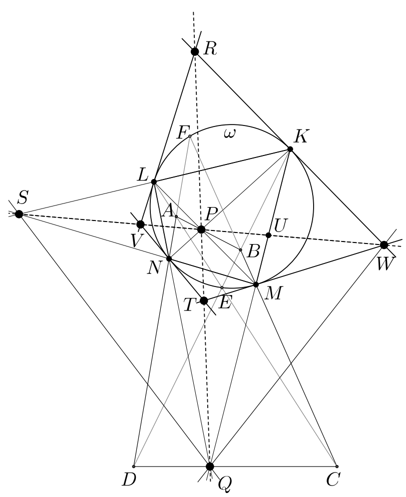

Figure 8.

From property (i), straight line KL is the polar of point R, and straight line MN is the polar of point T with respect to circle ω.

From property (ii), straight line PQ is the polar of point S, and straight line PS is the polar of point Q with respect to circle ω.

Therefore, from property (iii), straight line QS is the polar of point P with respect to circle ω.

We thus obtained that points R, T, P, and Q are poles whose polars (straight lines

KL, MN, QS, and PS, respectively) are straight lines that pass through the same point S.
From here, it also follows that these four points belong to the same straight line (line PQ, which is the polar of S).
(b) Let us prove that the four points R, P, T, and Q form a harmonic quadruple.
We denote by V the point of intersection of the tangents to the circle at points L and N, and by W the point of intersection of the tangents to the circle at points K and M.
Similar to section (a), it can be proven that points S, V, P, and W belong to the same straight line (line PS). We shall make use of the following two well-known properties of a harmonic quadruple of points:

(1) *If point X lies outside circle ω and straight line x is its polar with respect to this circle, than for any straight line that passes through point X and intersects circle ω at points A and B, and polar x at point Y* (see Figure 9a), *there holds that the four points X, A, Y, and B form a harmonic quadruple.*
(2) *A central projection* (see Figure 9b) *preserves the double ratio of the four points that lie on the same straight line.*

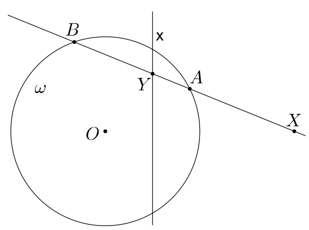

Figure 9a.

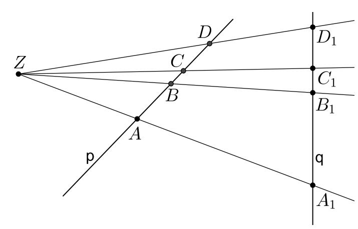

Figure 9b.

In Section (a) we saw that the straight line PS is the polar of point Q with respect to circle ω.

The straight line QK passes through pole Q, intersects circle ω at points M and K, and intersects polar PS at point U (see Figure 8). From property (1), points Q and U divide chord KM of circle ω by a harmonic division. In other words, points Q, M, U, and K constitute a harmonic quadruple on straight line QK.

In the central projection (projective transformation) from point W, the points of straight line QK are transformed to the points of straight line QP, and in particular, points Q, M, U, and K are transformed to points Q, T, P and R, respectively. In accordance with property (2), the double ratio of points Q, M, U, and K on straight line QK equals the double ratio of points Q, T, P and R on straight line QP, and therefore points R, P, T and Q must also constitute a harmonic quadruple. $\square$

Theorem 2. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, the continuations of sides BC and AD intersect at point F, and the*

continuations of sides AB and CD intersect at point G;
ω is an arbitrary circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
ω_1 is an arbitrary circle that passes through points E and G, and intersects sides AB and CD at points M_1 and N_1, respectively, and intersects the continuations of diagonals BD and AC at points K_1 and L_1, respectively (*see Figure 10*).
Then:

(a) *The angle between the tangents to circle ω at points K and L* (*the angle between the tangents to circle ω_1 at points K_1 and L_1*) *does not depend on the choice of circle, but depends only on the angle between the diagonals of the quadrilateral $ABCD$.*
(b) *The angle between the tangents to circle ω at points M and N* (*the angle between the tangents to circle ω_1 at points M_1 and N_1*) *does not depend on the choice of the circle, but depends only on the angle between the continuations of sides BC and AD* (*the angle between the continuations of sides AB and CD*).

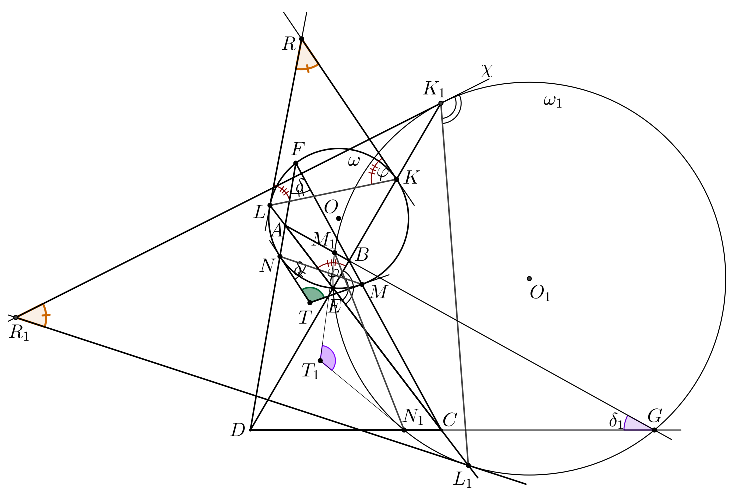

Figure 10.

Proof. We denote by R the point of intersection of the tangents to circle ω at points K and L, and by T the point of intersection of the tangents to this circle at points M and N.

We denote by φ the size of angle $\measuredangle KEL$. In circle ω, the inscribed angle $\measuredangle KEL$ equals each of the angles $\measuredangle KLR$ and $\measuredangle LKR$, which are the angles between chord KL and the tangents at the points L and K, respectively (see Figure 10), in other words: $\measuredangle LKR = \measuredangle KLR = \measuredangle KEL = \varphi$.

Therefore, in the triangle KLR there holds: $\measuredangle KRL = 180^\circ - 2\varphi$.

Thus, the size of angle $\measuredangle KRL$ depends only on angle φ, where φ is the angle between the diagonals of quadrilateral $ABCD$, and therefore φ does not depend on the choice of circle ω. Therefore, angle $\measuredangle KRL$ also does not depend on the choice of circle ω.

Angle $\measuredangle MFN$ is also an inscribed angle in circle ω. We denote this angle by δ, and by a similar way can show that the angles in circle ω also satisfy the following equality: $\measuredangle NMT = \measuredangle MNT = \measuredangle MFN = \delta$.

Hence, in the triangle MNT there holds: $\measuredangle MTN = 180^\circ - 2\delta$.

For the purpose of the proof, we shall assume that angle φ is acute. Then arc $\widehat{KL}$ of circle ω is smaller than 180°. In this case, the center, O, of ω lies between chords KL and MN, and therefore, point R and points O and T lie on different sides of line KL (as shown in Figure 10).

Angles $\measuredangle K_1EL_1$ and $\measuredangle KEL$ are adjacent, therefore $\measuredangle K_1EL_1 = 180^\circ - \varphi$. In other words, $\measuredangle K_1EL_1$ is an obtuse angle, and therefore arc $\widehat{K_1GL_1}$ of circle ω_1 is greater than 180°. In this case, chords K_1L_1 and M_1N_1 of circle ω_1 lie on the same side relative to the center, O_1, of circle ω_1. Therefore, point O_1 and points R_1 and T_1 are located on different sides relative to line K_1L_1.

Let us now find angle $\measuredangle K_1R_1L_1$ between the tangents to circle ω_1 at points K_1 and L_1.

For angle $\measuredangle L_1K_1X$ between the tangent to circle ω_1 at point K_1 and chord K_1L_1 there holds: $\measuredangle L_1K_1X = \measuredangle K_1EL_1 = 180^\circ - \varphi$.

Hence it follows that $\measuredangle R_1K_1L_1 = 180^\circ - \measuredangle L_1K_1X = 180^\circ - (180^\circ - \varphi) = \varphi$, and therefore in the isosceles triangle $K_1L_1R_1$ there holds:

$\measuredangle K_1R_1L_1 = 180^\circ - 2\varphi$.

In a similar manner, it is easy to prove that if the angle between straight lines AB and CD equals δ_1, then angle $\measuredangle M_1T_1N_1$ between the tangents to circle ω_1 at points M_1 and N_1 is $180^\circ - 2\delta_1$. $\square$

Note: In the case that angle φ is obtuse, it is easy to prove that the reciprocal relation of points R, O, T, and line KL, and the reciprocal relation of points R_1, O_1, T_1, and line K_1L_1 will change accordingly.
In other words: point O and points T and R will be located on different sides relative to line KL, and point R_1 and points O_1, T_1 will be located on different sides relative to line K_1L_1.
Therefore, Theorem 2 also holds in this case.

Conclusions from Theorem 2:

(1) *The angle between the tangents to circle ω at points K and L equals the angle between the tangents to circle ω_1 at points K_1 and L_1.*
(2) *The angle between the tangents to circle ω at points M and N and the angle between the tangents to circle ω_1 at points M_1 and N_1 are usually not equal.*
(3) *In a quadrilateral in which the diagonals are perpendicular, the tangents to circle ω at points K and L (and the tangents to circle ω_1 at points K_1 and L_1) are parallel.*

Note: One can arrive at Conclusion (3) by either of two methods:

(I) In such a quadrilateral $\measuredangle KEL$ is an inscribed right angle. It therefore rests on diameter KL of circle ω, and therefore the tangents to a circle at the ends of a diameter are parallel to each other.

(II) If $\alpha = 90°$, then from the formula we obtained in proving Theorem 2, the angle between the tangents to circle ω at points K and L is $180° - 2 \cdot 90° = 0°$. Therefore the tangents are parallel.

Theorem 3. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
In addition, circle ω is coordinated with the Pascal points P and Q formed by it (i.e., the center, O, of circle ω belongs to line PQ);
The four tangents to circle ω at points K, L, M, and N intersect pairwise at points V, W, X, and Y. In other words, the tangents at points K and M intersect at point V, the tangents at points L and N intersect at point W, the tangents at points K and N intersect at point X, and the tangents at points L and M intersect at point Y (see Figure 11).
Then: Straight line PQ is a mid-perpendicular to segment VW (point P is the middle of segment VW) and PQ is also a mid-perpendicular to segment XY (point Q is the middle of segment XY).

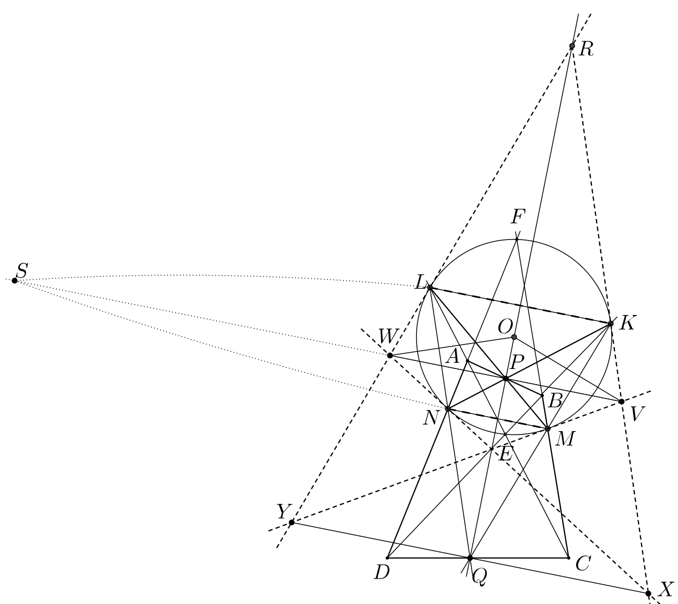

Figure 11.

Proof. We denote by S the point of intersection of lines KL and MN. In the proof of Theorem 1, we saw that points V, W, P, and S lie on the same straight line and form a harmonic quadruple. From Section (a) of General Theorem B, it follows that in the case that circle ω is coordinated with the Pascal points formed by it, straight lines KL and MN are parallel to each other, and therefore their point of intersection, S, is a point at infinity.
Hence it follows that:

(i) Point P is the middle of segment VW (see, for example, [4, Section 199];
(ii) Straight line VW is parallel to lines KL and MN (because line VW also passes through point S);
(iii) Quadrilateral $PMQN$ is a kite, and therefore $PQ \perp MN$ (see Section (b) in General Theorem B).

From these three properties, it follows that straight line PQ is a mid-perpendicular to segment VW, bisecting segment VW at point P.

We will now prove that line PQ is also a mid-perpendicular to segment XY, bisecting XY at point Q.
From Section (d) in General Theorem B, an inversion transformation with respect to circle ω transforms points P and Q one into the other. Therefore:

(i) Points O, P, and Q lie on the same straight line (line OP);
(ii) The polar of point P with respect to circle ω is the straight line that passes through point Q, and is perpendicular to line OP.

Straight line LM is the polar of point Y with respect to circle ω, and point P belongs to LM. Therefore, from the principal property of a pole and its polar (see [1, Chapter 6, Paragraph 1]), the polar of point P (with respect to circle ω) passes through point Y.
Similarly, because straight line KN is the polar of point X (with respect to circle ω), and because $P \in KN$, it follows that the polar of point P passes through point X.
We thus obtained that the polar of point P (with respect to circle ω) passes through the three points, Q, Y, and X, and also is perpendicular to straight line OP. Therefore, straight line XY passes through point Q and is parallel to straight line VW (because $VW \perp OP$ and $XY \perp OP$).
We denote by R the point of intersection of the tangents at points K and L (see Figure 11). From Theorem 1, point R belongs to the straight line PQ.
We consider triangle RXY, for which there holds:

(i) Segment VW, whose ends V and W lie on two sides of the triangle, is parallel to the third side, XY;
(ii) Line RP bisects segment VW (at point P).

Therefore, line RP also bisects segment XY (at point Q).
In summary, straight line RP (which is also line PQ) is a mid-perpendicular to segment XY. □

Theorem 4. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*

ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
R is the point of intersection of the tangents to circle ω at points K and L;
P and Q are the Pascal points formed by circle ω;
We denote: $\measuredangle MPN = \alpha$, $\measuredangle MQN = \beta$, $\measuredangle KRL = \gamma$, $\measuredangle AFB = \delta$.
Then:

(a) *When the center, O, of circle ω lies between chords KL and MN* (see Figure 12), *there holds:*
(i) $\alpha + \beta + \gamma = 180°$ *and* (ii) $\beta + \delta + \dfrac{\gamma}{2} = 90°$*;*

(b) *When chord KL is the diameter of ω, there holds:*
(i) $\alpha + \beta = 180°$ *and* (ii) $\beta + \delta = 90°$*;*

(c) *When chords KL and MN lie on the same side relative to the center O,* (see Figure 13b), *there holds:*
(i) $\alpha + \beta - \gamma = 180°$ *and* (ii) $\beta + \delta - \dfrac{\gamma}{2} = 90°$*;*

(d) *In each of the three cases listed above there holds:* $\delta = \dfrac{\alpha - \beta}{2}$.

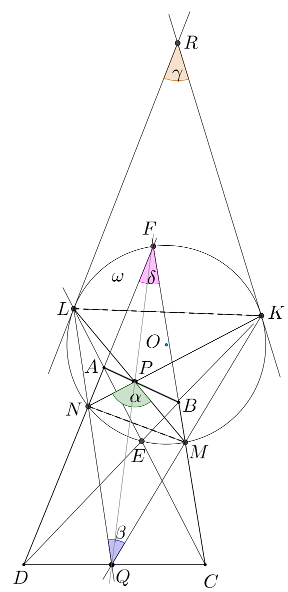

Figure 12.

Proof. In accordance with Theorem 2, in any circle that forms Pascal points and passes through a pair of opposite sides, the angle between the tangents to the circle at the points of intersection of the circle with the continuations of the diagonals (the tangents at the points K and L in Figure 12) is a fixed value that does not depend on the selection of the circle.

From General Theorem A, any circle that forms Pascal points P and Q on sides AB and CD and that intersects the two other sides at points M and N defines a quadrilateral, $PMQN$, with fixed angles that do not depend on the choice of circle.

Therefore, angles α, β and γ do not depend on the choice of circle ω. To prove the theorem, we shall choose ω to be the circle coordinated with the Pascal points formed by it.

(a)(i) The center, O, of ω lies between chords KL and MN (see Figure 13a), therefore angle $\measuredangle KNL$ (the inscribed angle resting on arc $\widehat{KL}$) equals angle $\measuredangle LKR$ (the angle between tangent KR and chord KL). Angle $\measuredangle LKR$ is the base angle in the isosceles triangle $\triangle RKL$. Therefore there holds: $\measuredangle LKR = 90 - \dfrac{\gamma}{2}$, and therefore also $\measuredangle KNL = 90 - \dfrac{\gamma}{2}$.

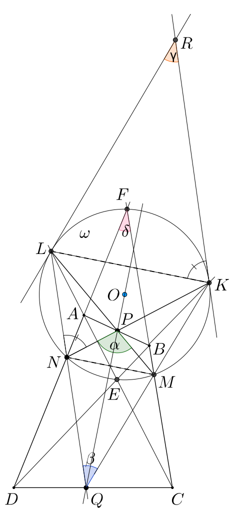

Figure 13a.

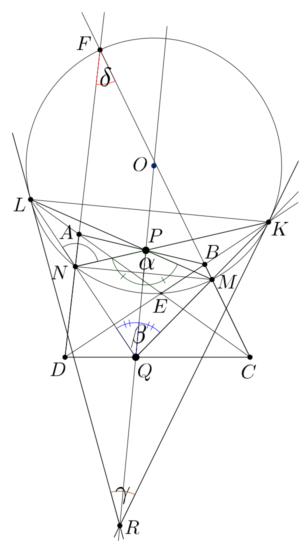

Figure 13b.

In addition, $\measuredangle KNL$ is an exterior angle of triangle $\triangle PQN$. The angles of this triangle satisfy: $\measuredangle NPQ = \dfrac{\alpha}{2}$ and $\measuredangle NQP = \dfrac{\beta}{2}$ (because $PMQN$ is a kite and segment PQ is its main diagonal). Therefore $\measuredangle KNL = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$. Hence it follows that $\dfrac{\alpha}{2} + \dfrac{\beta}{2} = 90^\circ - \dfrac{\gamma}{2}$ or $\alpha + \beta + \gamma = 180^\circ$.

(ii) $\measuredangle KQL$ is an exterior angle of circle ω (see Figure 13a), and therefore there holds:

$$\begin{aligned}\measuredangle KQL &= \frac{1}{2}\left(\widehat{KL}-\widehat{MN}\right)\\ &= \frac{1}{2}\widehat{KL}-\frac{1}{2}\widehat{MN}\\ &= \measuredangle LNK-\measuredangle MFN\\ &= 90^\circ-\frac{\gamma}{2}-\delta.\end{aligned}$$

It also follows that $\beta = 90^\circ - \frac{\gamma}{2} - \delta$ or $\beta + \delta + \frac{\gamma}{2} = 90^\circ$.

(b)(i) Chord KL is a diameter of ω, therefore the tangents to the circle at points K and L are parallel to each other, and therefore angle γ between the tangents equals 0 (R is a point at infinity). $\measuredangle KNL$ is an exterior angle that rests on the diameter, and therefore, it equals 90°.

In addition, as we have seen in Section (a), we have $\measuredangle KNL = \frac{\alpha}{2} + \frac{\beta}{2}$.

Therefore $\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ$, or $\alpha + \beta = 180^\circ$.

(ii) For angle $\measuredangle KQL = \beta$ there holds:

$$\measuredangle KQL = \frac{1}{2}\left(\widehat{KL}-\widehat{MN}\right) = \frac{1}{2}\cdot 180^\circ - \frac{1}{2}\widehat{MN} = 90^\circ - \measuredangle MFN = 90^\circ - \delta,$$

and from here we have $\beta + \delta = 90^\circ$.

(c)(i) Chords KL and MN lie on the same side relative to center O (as described in Figure 13b). For angle $\measuredangle KNL$ there holds:

$$\measuredangle KNL = \frac{1}{2}\widehat{KFL} = \frac{1}{2}\left(360^\circ - \widehat{KEL}\right) = 180^\circ - \measuredangle KLR.$$

Angle $\measuredangle KLR$ is the angle between the tangent to the circle (at point L) and chord LK. In addition, it is also the base angle in isosceles triangle $\triangle KLR$. Therefore $\measuredangle KLR = 90^\circ - \frac{\gamma}{2}$, and hence $\measuredangle KNL = 90^\circ + \frac{\gamma}{2}$.

In Section (a) we proved that $\measuredangle KNL = \frac{\alpha}{2} + \frac{\beta}{2}$, therefore $\frac{\alpha}{2} + \frac{\beta}{2} = 90^\circ + \frac{\gamma}{2}$ or $\alpha + \beta - \gamma = 180^\circ$.

(ii) For angle $\measuredangle KQL = \beta$ there holds (see Figure 13b):

$$\measuredangle KQL = \frac{1}{2}\left(\widehat{KFL}-\widehat{MEN}\right) = \measuredangle KNL - \measuredangle MNF.$$

Therefore, after substituting the appropriate expressions for the angles, we obtain $\beta = 90^\circ + \frac{\gamma}{2} - \delta$, and hence $\beta + \delta = 90^\circ + \frac{\gamma}{2}$.

(d) We show that the equality holds for each of the three locations of the center, O, of circle ω relative to chords KL and MN.

If the center, O, of ω is between chords KL and MN, there holds:

$2\beta + 2\delta = 180^\circ - \gamma \Rightarrow 2\beta + 2\delta = \alpha + \beta$, and therefore $\delta = \frac{\alpha - \beta}{2}$.

If chord KL is a diameter of ω, there holds $\beta + \delta = 90^\circ$ and also $\dfrac{\alpha + \beta}{2} = 90^\circ$, therefore $\beta + \delta = \dfrac{\alpha + \beta}{2} \Rightarrow \delta = \dfrac{\alpha - \beta}{2}$.
If chords KL and MN are on the same side relative to the center, O, there holds: $2\beta + 2\delta = 180^\circ + \gamma \Rightarrow 2\beta + 2\delta = \alpha + \beta$, and therefore $\delta = \dfrac{\alpha - \beta}{2}$. $\square$

Theorem 5. *Let $ABCD$ be a quadrilateral in which the diagonals intersect at point E, and the continuations of sides BC and AD intersect at point F;*
ω is a circle that passes through points E and F, and intersects sides BC and AD at points M and N, respectively, and intersects the continuations of diagonals BD and AC at points K and L, respectively;
In addition, circle ω is coordinated with the Pascal points P and Q formed by it (*i.e., the center, O, of circle ω belongs to line PQ*)*;*
I and J are the points of intersection of straight line PQ and circle ω;
R is the point of intersection of the tangents to circle ω at points K and L (see Figure 14).
Then:

(a) *Point I is the center of the circle inscribed in kite $PMQN$, and point J is the center of the circle that is tangent to the continuations of the sides of kite $PMQN$.*
(b) *Point J is the center of the incircle of triangle RKL, and point I is the center of the excircle in triangle RKL which is tangent to side KL.*

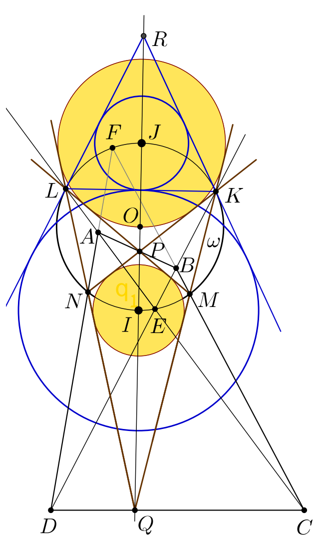

Figure 14.

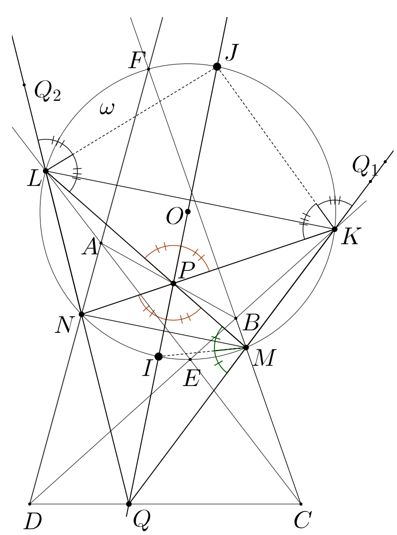

Figure 15.

Proof. (a) Given: circle ω whose center, O, is collinear with the Pascal points P and Q formed by it. Therefore, from Section (d) of General Theorem B, inversion relative to circle ω transforms points P and Q one into the other. Therefore, these points together with the points of intersection I and J form a harmonic quadruple. Since points I and J divide segment PQ harmonically (I – internal division, J – external division), circle ω is a circle of Apollonius of segment PQ (see, for example, [1, Chapter 5, paragraph 4]).
Point M belongs to the circle of Apollonius ω, and therefore, for M, it holds that segment MI bisects angle $\measuredangle PMQ$ in triangle PMQ (see Figure 15).
Quadrilateral $PMQN$ is a kite. The main diagonal of the kite (segment PQ) bisects the two angles $\measuredangle MPN$ and $\measuredangle MQN$. It follows that point I is the point of intersection of three angle bisectors in quadrilateral $PMQN$. Therefore, point I is equidistant from all four sides of the quadrilateral. It thus follows that point I is the center of the circle inscribed in quadrilateral $PMQN$.
We now consider segment KJ. Since point K belongs to the circle of Apollonius ω (whose diameter is IJ, and $\measuredangle IKJ = 90°$), it follows that segment KJ bisects the exterior angle of triangle $\triangle PKQ$ (the angle $\measuredangle PKQ_1$).
Similarly, we prove that LJ bisects angle $\measuredangle PLQ_2$.
In addition, ray PJ bisects angle $\measuredangle KPL$ (because PQ bisects angle $\measuredangle MPN$, which is vertically opposite to angle $\measuredangle KPL$). It follows that point J is located at equal distances from four rays: PK, PL, KQ_1 and LQ_2, which are the continuations of the sides of kite $PMQN$ (See Figure 15).
Therefore, J is the center of the circle tangent to the continuations of the sides of the quadrilateral $PMQN$.

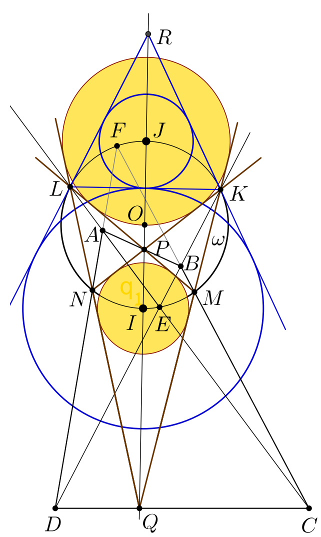

Figure 16.

(b) Let us prove that point J is the center of the incircle of triangle RKL. Point J belongs to ray RO, which bisects angle $\measuredangle KRL$ between the tangents to circle ω

that issue from point R (See Figure 16).
In addition, point J is the middle of arc $\widehat{KL}$ (because RO is a mid-perpendicular to chord KL of circle ω).
Therefore, $\measuredangle LKJ = \measuredangle JKR$ (because $\measuredangle LKJ = \frac{1}{2}\widehat{KJ}$ and $\measuredangle JKR = \frac{1}{2}\widehat{JK}$), and therefore KJ bisects angle $\measuredangle LKR$.
It follows that point J is the point of intersection of the two angle bisectors in triangle $\triangle RKL$. Therefore, point J is the center of the incircle in this triangle.
Angle $\measuredangle JKI$ is an inscribed angle resting on diameter IJ of the circle ω, therefore $\measuredangle JKI = 90°$. In addition, angles $\measuredangle LKR_1$ and $\measuredangle LKR$ are adjacent angles. Hence, segment KI bisects $\measuredangle LKR_1$, where $\measuredangle LKR_1$ is exterior to triangle $\triangle LKR$ (see Figure 16).
Similarly, we prove that segment LI bisects angle $\measuredangle KLR_2$, which is also an exterior angle to triangle $\triangle LKR$.
We obtained that point I is the point of intersection of two angle bisectors. These angles are exterior angles in triangle $\triangle RKL$, therefore point I is located at equal distances from segment KL (side of triangle $\triangle RKL$), and from rays KR_1 and LR_2 (the continuations of the two other sides of the triangle).
Therefore, I is the center of the excircle of triangle $\triangle RKL$. □

Theorem 6. *The data of this Theorem is the same as the data of Theorem 5.*
We also denote by:
Σ_1 *the circle inscribed in kite* $PMQN$,
Σ_2 *the circle tangent to the continuations of the sides of kite* $PMQN$,
Σ_3 *the incircle of triangle* RKL,
Σ_4 *the excircle of triangle* RKL (see Figure 17)*;*
r_i *the radius of circle* Σ_i $(i \in \{1, 2, 3, 4\})$, *and*
$\measuredangle MPN = \alpha$, $\measuredangle MQN = \beta$, $\measuredangle KRL = \gamma$.
Then:

(a) *The following relations hold for the radii of circles* Σ_i*:*
 (i) $r_1 + r_2 < r_3 + r_4$;
 (ii) $\dfrac{r_1}{r_2} = \dfrac{\sin\dfrac{\alpha}{2} - \sin\dfrac{\beta}{2}}{\sin\dfrac{\alpha}{2} + \sin\dfrac{\beta}{2}}$.

(b) *The following relations hold for the areas of circles* Σ_i*:*
 (i) *If* $\alpha + \beta < 180°$, *then* $S_{\Sigma_1} + S_{\Sigma_2} < S_{\Sigma_3} + S_{\Sigma_4}$.
 (ii) *If* $\alpha + \beta = 180°$, *then* $S_{\Sigma_1} + S_{\Sigma_2} = S_{\Sigma_3} + S_{\Sigma_4}$.
 (iii) *If* $\alpha + \beta > 180°$, *then* $S_{\Sigma_1} + S_{\Sigma_2} > S_{\Sigma_3} + S_{\Sigma_4}$.

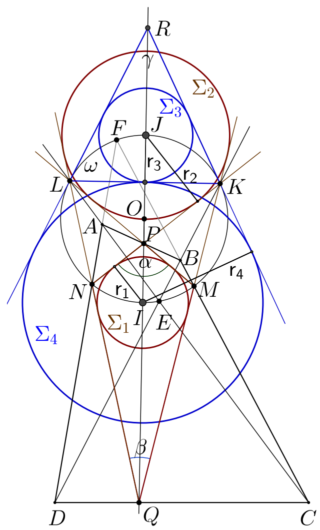

Figure 17.

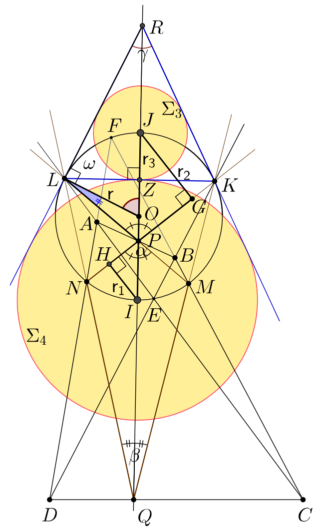

Figure 18.

Proof. (a) We denote by H the point of tangency of circle Σ_2 on side PN of kite $PMQN$ (see Figure 18). Therefore $IH = r_1$. In right triangle $\triangle IPH$ there holds: $\sin\dfrac{\alpha}{2} = \dfrac{IH}{IP}$, and therefore $IP = \dfrac{r_1}{\sin\dfrac{\alpha}{2}}$.

We denote by G the point of tangency of circle Σ_1 on side PN of kite $PMQN$. Therefore $JG = r_2$. In the right triangle $\triangle PJG$ there holds: $\sin\measuredangle GPJ = \dfrac{JG}{JP}$, and therefore $JP = \dfrac{r_2}{\sin\dfrac{\alpha}{2}}$.

For segment IJ we obtain: $IJ = IP + PJ = \dfrac{r_1}{\sin\dfrac{\alpha}{2}} + \dfrac{r_2}{\sin\dfrac{\alpha}{2}} = \dfrac{r_1 + r_2}{\sin\dfrac{\alpha}{2}}$.

We denote by r the length of the radius of circle ω, and therefore there holds: $OI = OJ = OL = r$ and also $IJ = 2r$, and hence $r_1 + r_2 = 2r\sin\dfrac{\alpha}{2}$.

Point Z is the point of tangency of circles Σ_3 and Σ_4 on side KL of triangle $\triangle RKL$ (see Figure 18), and therefore $JZ = r_3$ and $IZ = r_4$. Therefore, for segment IJ there also holds: $IJ = JZ + IZ = r_3 + r_4$, and hence: $r_3 + r_4 = 2r$.

It follows that the radii of the four circles Σ_i satisfy the equality:

$$r_1 + r_2 = (r_3 + r_4)\sin\frac{\alpha}{2}.$$

Angle $\dfrac{\alpha}{2}$ is acute, therefore $\sin\dfrac{\alpha}{2} < 1$, and the following inequality holds:

$$r_1 + r_2 < r_3 + r_4.$$

We express the angles of triangle POL using α and β.
In $\triangle POL$ there holds: $\measuredangle OPL = \dfrac{\alpha}{2}$, side OL is the radius of circle ω, and also side OL is perpendicular to the tangent to the circle at point L. I In other words, $\measuredangle OLP = 90°$. Therefore, in the right triangle $\triangle RLO$ there holds:
$\measuredangle ROL = 90° - \dfrac{\gamma}{2}$, and hence: $\measuredangle POL = 90° + \dfrac{\gamma}{2}$.
We will now show that in all cases where chord KL is not a diameter (and therefore point R is not a point at infinity) there holds $\measuredangle PLO = \dfrac{\beta}{2}$.
If the center, O, is between chords KL and MN, then angle $\measuredangle ROL$ is an exterior angle of triangle $\triangle POL$, and angle $\measuredangle ROL$ is not adjacent to angle $\measuredangle PLO$ (see Figure 18).
We use the formula $\dfrac{\alpha}{2} + \dfrac{\beta}{2} + \dfrac{\gamma}{2} = 90°$ from Section (a) of Theorem 4.
There holds:

$$\begin{aligned}\measuredangle PLO = \measuredangle ROL - \measuredangle OPL = & \left(90° - \frac{\gamma}{2}\right) - \frac{\alpha}{2}\\ = & \ 90° - \left(\frac{\alpha}{2} + \frac{\gamma}{2}\right) = \frac{\beta}{2}.\end{aligned}$$

If chords KL and MN are located on the same side relative to the center, O, then angle $\measuredangle ROL$ is an exterior angle of triangle $\triangle POL$ (see Figure 19), and there holds: $\measuredangle ROL = 90° - \dfrac{\gamma}{2}$.
We use the equality $\measuredangle OPL = \dfrac{\alpha}{2}$ and the formula $\dfrac{\alpha}{2} + \dfrac{\beta}{2} - \dfrac{\gamma}{2} = 90°$ from Section (c) of Theorem 4 and obtain:

$$\measuredangle PLO = 180° - \frac{\alpha}{2} - \left(90° - \frac{\gamma}{2}\right) = 90° - \frac{\alpha}{2} + \frac{\gamma}{2} = \frac{\beta}{2}.$$

From the Law of Sines in the triangle $\triangle POL$ it follows that:
$\dfrac{OP}{\sin \measuredangle PLO} = \dfrac{OL}{\sin \measuredangle LPO}$ and hence:

$$OP = \frac{r \cdot \sin \dfrac{\beta}{2}}{\sin \dfrac{\alpha}{2}}.$$

For segment IP there holds: $IP = IO - OP$ (see Figures 18 and 19).
We substitute the obtained expressions for the segments that appear in the last equality, to obtain:

$$\frac{r_1}{\sin \dfrac{\alpha}{2}} = r - \frac{r \cdot \sin \dfrac{\beta}{2}}{\sin \dfrac{\alpha}{2}},$$

and hence: $r_1 = r\left(\sin \dfrac{\alpha}{2} - \sin \dfrac{\beta}{2}\right)$.
For segment PJ there holds: $PJ = PO + OJ$.

We substitute the corresponding expressions in this equality, to obtain:

$$\frac{r_2}{\sin\dfrac{\alpha}{2}} = \frac{r\cdot\sin\dfrac{\beta}{2}}{\sin\dfrac{\alpha}{2}} + r,$$

and hence: $r_2 = r\left(\sin\dfrac{\alpha}{2} + \sin\dfrac{\beta}{2}\right)$.

Therefore for the ratio $\dfrac{r_1}{r_2}$, we obtain: $\dfrac{r_1}{r_2} = \dfrac{\sin\dfrac{\alpha}{2} - \sin\dfrac{\beta}{2}}{\sin\dfrac{\alpha}{2} + \sin\dfrac{\beta}{2}}$.

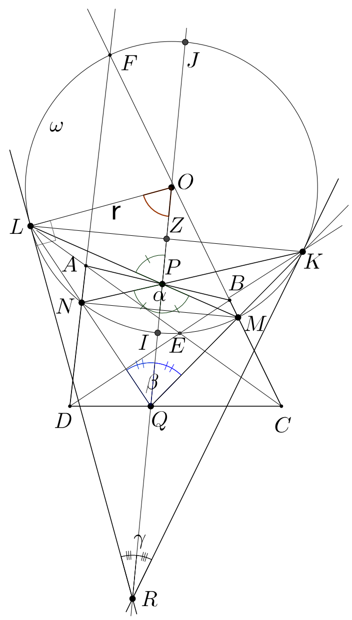

Figure 19.

(b) We consider separately the sum of the areas of circles $\varSigma_1$ and $\varSigma_2$, and the sum of the areas of circles $\varSigma_3$ and $\varSigma_4$:

$$\begin{aligned} S_{\varSigma_1} + S_{\varSigma_2} &= \pi\left(r_1^2 + r_2^2\right) \\ &= \pi\left(r^2\left(\sin\frac{\alpha}{2} - \sin\frac{\beta}{2}\right)^2 + r^2\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2}\right)^2\right) \\ &= 2\pi r^2\left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\right); \end{aligned}$$

$$\begin{aligned} S_{\Sigma_3}+S_{\Sigma_4} &= \pi\left(r_3^2+r_4^2\right) \\ &= \pi\left((r_3+r_4)^2-2r_3r_4\right) \\ &= \pi\left((2r)^2-2r_3r_4\right). \end{aligned}$$

Let us express the product of radii r_3r_4 using α, β and r.
For diameter IJ in circle ω there holds: $IJ = IZ + ZJ = r_4 + r_3$.
Segment ZL is perpendicular to diameter IJ (at point Z), and the other end of segment ZL (point L) belongs to circle ω (see Figures 18 and 19). Therefore the length of segment ZL is the geometric mean of the lengths of segments IZ and ZJ, in other words $IZ \cdot ZJ = ZL^2$ or $r_3 \cdot r_4 = ZL^2$.
In right triangle OLZ there holds: $\sin\measuredangle LOZ = \dfrac{ZL}{OL}$ and hence:

$$ZL = r\sin\left(90^\circ-\frac{\gamma}{2}\right) = r\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right).$$

Therefore, for the sum of circle areas Σ_3 and Σ_4 there holds:

$$\begin{aligned} S_{\Sigma_3}+S_{\Sigma_4} &= \pi\left(4r^2-2r_3r_4\right) \\ &= \pi\left(4r^2-2ZL^2\right) \\ &= \pi\left(4r^2-2r^2\sin^2\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\right). \end{aligned}$$

Let us consider the difference of the area sums:

$$\begin{aligned} &(S_{\Sigma_1}+S_{\Sigma_2})-(S_{\Sigma_3}+S_{\Sigma_4}) \\ &= \pi r^2\left(2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\beta}{2}\right)-2\pi r^2\left(2-\sin^2\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\right) \\ &= \pi r^2\left(2-(\cos\alpha+\cos\beta)\right)-2\pi r^2\left(1+\cos^2\left(\frac{\alpha+\beta}{2}\right)\right) \\ &= \pi r^2\left(2-2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2-2\cos^2\left(\frac{\alpha+\beta}{2}\right)\right) \\ &= -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right). \end{aligned}$$

In other words we have:

$$(S_{\Sigma_1}+S_{\Sigma_2})-(S_{\Sigma_3}+S_{\Sigma_4}) = -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right).$$

Let us investigate the sign of expression

$$\mathrm{A} = -2\pi r^2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right)$$

as a function of the sum of angles $\alpha+\beta$:

(i) If $\alpha+\beta<180^\circ$, which is the case where the center, O, lies between chords KL and MN, then angle $\dfrac{\alpha+\beta}{2}$ is acute and therefore $\cos\dfrac{\alpha+\beta}{2} > 0$, and

$\left(\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}\right)>0$. Therefore in this case, A <0, which gives $S_{\Sigma_1}+S_{\Sigma_2}<S_{\Sigma_3}+S_{\Sigma_4}$.

(ii) If $\alpha+\beta=180^\circ$, which is the case where chord KL is a diameter of circle ω, then angle $\dfrac{\alpha+\beta}{2}$ equals 90° and therefore $\cos\dfrac{\alpha+\beta}{2}=0$.
In this case, A $=0$, and $S_{\Sigma_1}+S_{\Sigma_2}=S_{\Sigma_3}+S_{\Sigma_4}$.

(iii) If $\alpha+\beta>180^\circ$, which is the case where chords KL and MN are on the same side relative to the center, O, then angle $\dfrac{\alpha+\beta}{2}$ is obtuse and therefore $\cos\dfrac{\alpha+\beta}{2}<0$.

Let us investigate the sign of B $=\left(\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}\right)$.

There holds: $\cos\dfrac{\alpha-\beta}{2}+\cos\dfrac{\alpha+\beta}{2}=2\cos\dfrac{\alpha}{2}\cos\dfrac{-\beta}{2}=2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}$.

Since angles $\dfrac{\alpha}{2}$ and $\dfrac{\beta}{2}$ are acute, it holds that $\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}>0$, and hence B >0.
Therefore, in this case, A >0, which gives $S_{\Sigma_1}+S_{\Sigma_2}>S_{\Sigma_3}+S_{\Sigma_4}$. $\square$

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The Two Incenters of an Arbitrary Convex Quadrilateral

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Abstract. For an arbitrary convex quadrilateral $ABCD$ with area $\mathcal{A}$ and perimeter p, we define two points I_1, I_2 on its Newton line that serve as incenters. These points are the centers of two circles with radii r_1, r_2 that are tangent to opposite sides of $ABCD$. We then prove that $\mathcal{A} = pr/2$, where r is the harmonic mean of r_1 and r_2. We also investigate the special cases with $I_1 \equiv I_2$ and/or $r_1 = r_2$.

1. Introduction

We have recently shown [1] that many of the classical two-dimensional figures of Euclidean geometry satisfy the relation

$$\mathcal{A} = pr/2\ , \tag{1}$$

where $\mathcal{A}$ is the area, p is the perimeter, and r is the inradius. For figures without an incircle (parallelograms, rectangles, trapezoids), the radius r is the harmonic mean of the radii r_1 and r_2 of two internally tangent circles to opposite sides, that is

$$r = 2r_1r_2/(r_1 + r_2)\ . \tag{2}$$

Here we prove the same results for convex quadrilaterals with and without an incircle. These results were anticipated, but unexpectedly, the two tangent circles in the case without an incircle are not concentric, unlike in all the figures studied in previous work [1]. This is a surprising result because it implies that the arbitrary convex quadrilateral does not exhibit even this minor symmetry (a common incenter) in its properties, yet it satisfies equations (1) and (2) by permitting two different incenters I_1 and I_2 on its Newton line for the radii r_1 and r_2, respectively. This unusual property of the convex quadrilateral prompted us to also investigate all the special cases with $I_1 \equiv I_2$ and/or $r_1 = r_2$.

2. Arbitrary convex quadrilateral

Consider an arbitrary convex quadrilateral $ABCD$ with Newton line MN [3], where M and N are the midpoints of the diagonals AC and BD (Figure 1). Let the lengths of the sides of $ABCD$ be $AB = a, BC = b, CD = c$, and $DA = d$. We extend sides AB and DC to a common point E. Similarly, we extend sides AD and BC to a common point F. We bisect $\angle E$ and $\angle F$. The angle bisectors EI_1 and FI_2 intersect the Newton line MN at I_1 and I_2, respectively.

Definition. We define I_1 as the incenter of $ABCD$ that is equidistant from sides AB and CD at a distance of r_1. We also define I_2 as the incenter of $ABCD$ that is equidistant from sides BC and DA at a distance of r_2.

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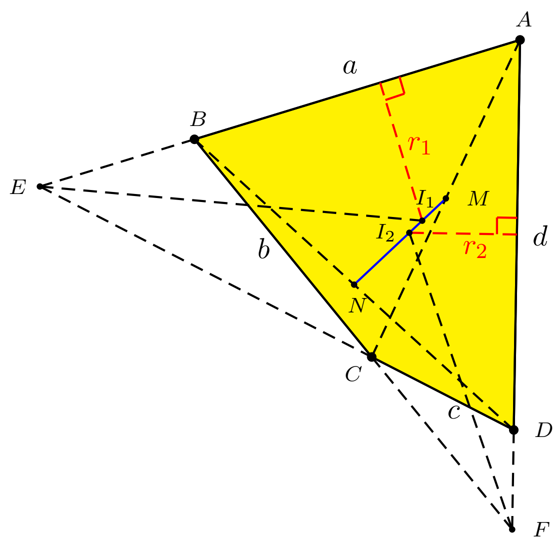

Figure 1. Convex quadrilateral $ABCD$ with two interior incenters I_1 and I_2 on its Newton line MN.

Remark 1. Points I_1, I_2 are usually interior to $ABCD$, but one of them can also be outside of $ABCD$ (as in Figure 2).

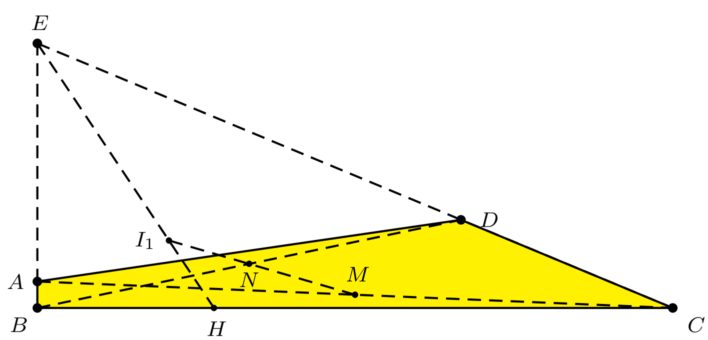

Figure 2. Incenter I_1 lies outside of this quadrilateral $ABCD$.

Lemma 1 (Based on Léon Anne's Theorem [2, # 555])**.** *Let $ABCD$ be a quadrilateral with M, N the midpoints of its diagonals AC, BD, respectively. A point O satisfies the equality of areas*

$$(OAB) + (OCD) = (OBC) + (ODA)\,, \tag{3}$$

if and only if O lies on the Newton line MN.

Proof. Using the cross products of the vectors of the sides of $ABCD$, equation (3) implies that

$$\begin{aligned} & (OAB) - (OBC) + (OCD) - (ODA) = 0 \\ \Longleftrightarrow\ & \overrightarrow{OA} \times \overrightarrow{OB} + \overrightarrow{OC} \times \overrightarrow{OB} + \overrightarrow{OC} \times \overrightarrow{OD} + \overrightarrow{OA} \times \overrightarrow{OD} = \overrightarrow{0} \\ \Longleftrightarrow\ & (\overrightarrow{OA} + \overrightarrow{OC}) \times \overrightarrow{OB} + (\overrightarrow{OC} + \overrightarrow{OA}) \times \overrightarrow{OD} = \overrightarrow{0} \\ \Longleftrightarrow\ & (\overrightarrow{OA} + \overrightarrow{OC}) \times (\overrightarrow{OB} + \overrightarrow{OD}) = \overrightarrow{0} \\ \Longleftrightarrow\ & 2\overrightarrow{OM} \times 2\overrightarrow{ON} = \overrightarrow{0}, \end{aligned} \tag{4}$$

therefore point O lies on the line MN (see also [5]). □

Since for signed areas it holds that $(OAB) + (OBC) + (OCD) + (ODA) = (ABCD)$, we readily prove the following theorem:

Theorem 2 (Arbitrary Convex Quadrilateral)**.** *The area of $ABCD$ is given by equation* (1)*, where the radius r is given by equation* (2) *and the two internally tangent circles to opposite sides $\odot I_1$ and $\odot I_2$ are centered on two different points on the Newton line MN.*

Proof. Since I_1 lies on the Newton line, we find for the area $\mathcal{A}$ of $ABCD$ that

$$\mathcal{A}/2 = (I_1AB) + (I_1CD) = (a + c)r_1/2\,, \tag{5}$$

or

$$a + c = \mathcal{A}/r_1\,. \tag{6}$$

Similarly, we find for the incenter I_2 that

$$b + d = \mathcal{A}/r_2\,. \tag{7}$$

Adding the last two equations and using the definition of perimeter $p = a+b+c+d$, we find that

$$p = \mathcal{A}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \mathcal{A}\,\frac{r_1 + r_2}{r_1 r_2}\,, \tag{8}$$

which is equation (1) with r given by equation (2). □

3. Tangential quadrilateral

In the special case of a tangential quadrilateral, the two incenters I_1 and I_2 coincide with point I and, obviously, $r_1 = r_2$. Using Lemma 1, we prove the following theorem:

Theorem 3 (Based on Newton's Theorem [2, # 556])**.** *If a quadrilateral $ABCD$ is tangential with incenter I and inradius r, then I lies on the Newton line MN* (*as in Figure 3*)*, $I_1 \equiv I_2 \equiv I$, $r_1 = r_2 = r$, and the area of the figure is given by $\mathcal{A} = pr/2$.*

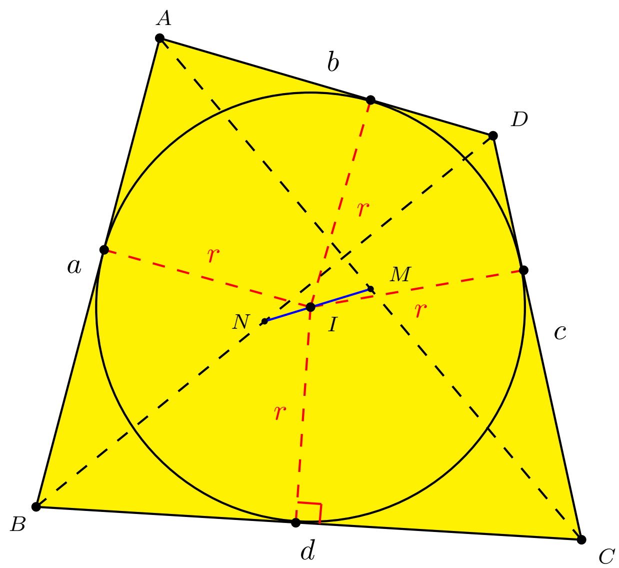

Figure 3. Tangential quadrilateral $ABCD$ with a single incenter I on its Newton line MN.

Proof. If $ABCD$ has an incircle $\odot I$ of radius r, then by the tangency of its sides

$$AB + CD = BC + DA\,. \tag{9}$$

Multiplying by $r/2$ across equation (9), we find that

$$(IAB) + (ICD) = (IBC) + (IDA)\,, \tag{10}$$

where again parentheses denote the areas of the corresponding triangles. Therefore, by Lemma 1, the incenter I lies on the Newton line MN of $ABCD$ and the area of the figure is

$$\mathcal{A} = (a + c)r/2 + (b + d)r/2 = pr/2\,. \tag{11}$$

Since I is the point of intersection of the bisectors of $\angle E$ and $\angle F$ (seen in Figure 1), it follows that $I_1 \equiv I_2 \equiv I$, $r_1 = r_2 = r$. □

Remark 2. The proof of the converse of Theorem 3 is trivial: If $I_1 \equiv I_2 \equiv I$ and $r_1 = r_2 = r$, then equation (11) implies equation (10) which in turn implies equation (9).

Theorem 4. *In quadrilateral $ABCD$, if the incircle $I_1(r_1)$ is tangent to a third side, then $ABCD$ is tangential.*

Proof. Let the incircle be tangent to side BC in addition to sides AB and CD (as in Figure 4). If it were not tangent to side AD as well, then we would draw another segment AD' tangent to the incircle and hence the incenter I_1 that lies on the Newton line MN would also lie on the "Newton line" MN' of the tangential quadrilateral $ABCD'$. Since AB, CD are not parallel, the two Newton lines do

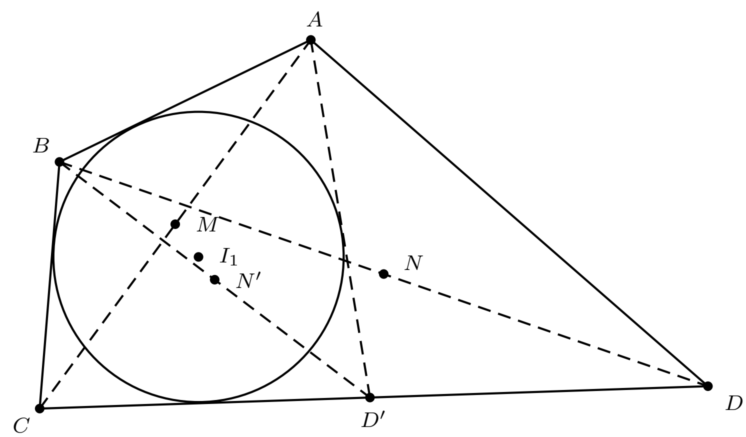

Figure 4. Quadrilateral $ABCD$ with an incircle I_1 tangent to three of its sides.

not coincide, hence $I_1 \equiv M$. In a similar fashion, if we draw a tangent line to the incircle from vertex D, we find that $I_1 \equiv N$. But M, N cannot coincide because $ABCD$ is not a parallelogram, thus the two equivalences of I_1 are impossible, in which case the incircle $I_1(r_1)$ must necessarily be tangent to the fourth side AD, making $ABCD$ a tangential quadrilateral. The same holds true for the incircle $I_2(r_2)$ when it is tangent to three sides of $ABCD$. □

4. Cyclic quadrilateral

In the special case of a cyclic quadrilateral, the two incenters I_1 and I_2 coincide, but $r_1 \neq r_2$. The following theorem has been proven in the distant past albeit in a different way:

Theorem 5 (Based on Theorem [2, # 387])**.** *In a cyclic quadrilateral $ABCD$, the incenters I_1, I_2 coincide with point I on the segment MN of the Newton line* (*Figure 5*)*.*

Proof. Since I_1, I_2 are located on the Newton line of $ABCD$, it is sufficient to show that the intersection I of the two angle bisectors from $\angle E$ and $\angle F$ (Figure 5) lies on the Newton line MN of $ABCD$. We find that $\angle ECF = \angle EIF + \phi + \omega$ and $\angle EAF = \angle EIF - \phi - \omega$ from which we get $\angle EIF = (\angle ECF + \angle EAF)/2 = 90^o$, so $\triangle EC_1A_1$ is isosceles and I is the midpoint of C_1A_1.

Now let the diagonals be $AC = e$ and $BD = f$ and define $x = f/(e+f)$ and $y = e/(e+f)$, such that $x + y = 1$. From the similar triangles $\triangle FAB \sim \triangle FCD$, $\triangle FAC \sim \triangle FBD$, and the angle bisector theorem, we find the proportions

$$\frac{AA_1}{A_1B} = \frac{FA}{FB} = \frac{e}{f} = \frac{FC}{FD} = \frac{CC_1}{C_1D} = \frac{y}{x}\,, \tag{12}$$

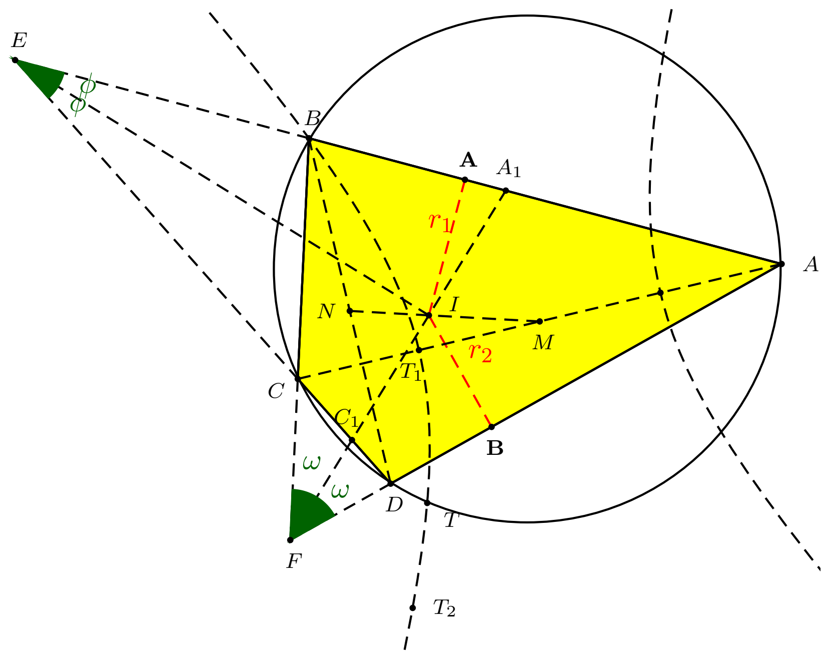

Figure 5. Cyclic quadrilateral with $I_1 \equiv I_2 \equiv I$.

which imply that $A_1 = xA + yB$, $C_1 = xC + yD$, and finally for the midpoint I of C_1A_1 that

$$I = (A_1 + C_1)/2 = x(A + C)/2 + y(B + D)/2 = xM + yN\,, \qquad (13)$$

which shows that I lies on segment MN of the Newton line and divides it in a ratio of $MI : IN = y : x = e : f$, just as A_1, C_1 divide AB, CD, respectively. □

5. Bicentric quadrilateral

In the special case of a cyclic or tangential quadrilateral, we derive the conditions under which it is also tangential or cyclic, respectively, thus it is bicentric with a single incenter I and inradius r. We prove the following two theorems:

Theorem 6 (Cyclic Quadrilateral is Bicentric)**.** *Consider $\triangle ABC$ inscribed in $\odot O$ (Figure 6). Any point D chosen on minor $\widehat{CA}$ defines a cyclic quadrilateral $ABCD$ with side lengths $AB = a, BC = b, CD = c$, and $DA = d$. Of these quadrilaterals, there exists only one that is also tangential (therefore it is bicentric) to a single incircle $\odot I$. Its vertex D lies at the point of intersection of minor $\widehat{CA}$ with the branch of the hyperbola with foci A and C that passes through vertex B. An analogous property holds when point D is chosen to lie on minor $\widehat{AB}$ or on minor $\widehat{BC}$.*

Proof. Consider a hyperbola with foci A and C such that one of its branches passes through vertex B and intersects minor $\widehat{CA}$ at point D (Figure 6). Then by the

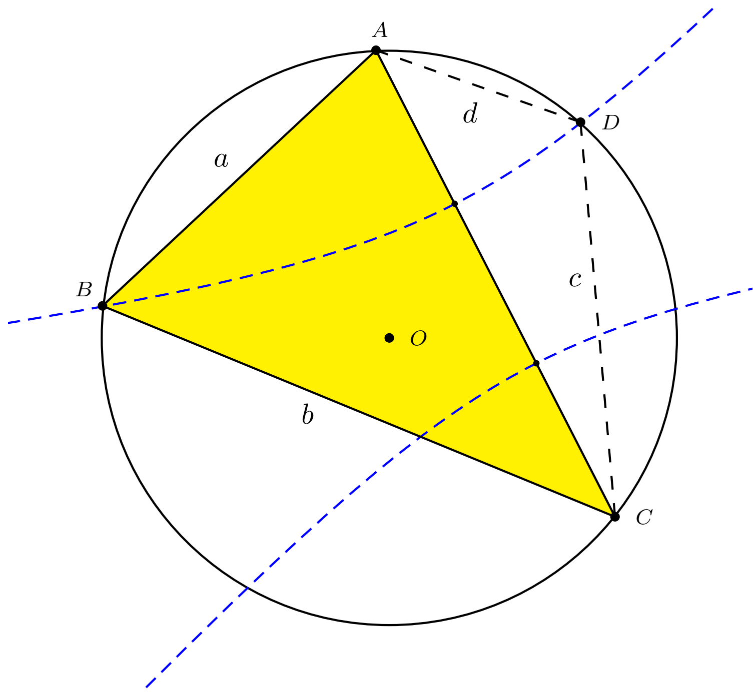

Figure 6. Triangle ABC inscribed in $\odot O$ and the bicentric quadrilateral $ABCD$ constructed by locating vertex D on minor $\overset{\frown}{CA}$ as stated in Theorem 6. The hyperbola with foci A and C with one branch through vertex B is shown by dashed lines.

geometric definition (the locus) of the hyperbola, we can write that

$$b - a = c - d\,, \tag{14}$$

where $a < b$ and $d < c$, as in Figure 6. Similarly, in the case with $a > b$ and $d > c$, we can write that

$$a - b = d - c\,. \tag{15}$$

Both equations are equivalent to

$$a + c = b + d\,, \tag{16}$$

which implies that $ABCD$ is tangential (thus also bicentric). □

Theorem 7 (Tangential Quadrilateral is Bicentric)**.** *Consider a tangential quadrilateral $ABCD$ with incenter I* (*as in Figure 3*)*. Then $ABCD$ is cyclic* (*thus also bicentric*) *only if vertex D lies on the hyperbola with foci A and C that passes through vertex B* (*Figure 6*)*. An analogous property holds when any other vertex is chosen instead of D.*

Proof. Consider the tangential quadrilateral $ABCD$ with incircle $\odot I$ shown in Figure 3. By the tangency of its sides, equation (9) is valid and it can be written in the form of equation (16). We re-arrange terms in equation (16) to obtain:

$$b - a = c - d\,. \tag{17}$$

This equation defines a hyperbola with foci A and C one branch of which passes through vertices B and D (as shown in Figure 6 for the case $a < b, d < c$). If D also lies on the circumcircle of $\triangle ABC$, then $ABCD$ is cyclic (thus also bicentric). $\square$

6. A Euclidean construction of vertex D

Given $\triangle ABC$ inscribed in $\odot O(R)$ (as in Figure 7), we construct on the lower $\overset{\frown}{CA}$ of $\odot O(R)$ the point D, without using the hyperbola mentioned above, such that the quadrilateral $ABCD$ is convex and tangential with $AB + CD = BC + DA$.

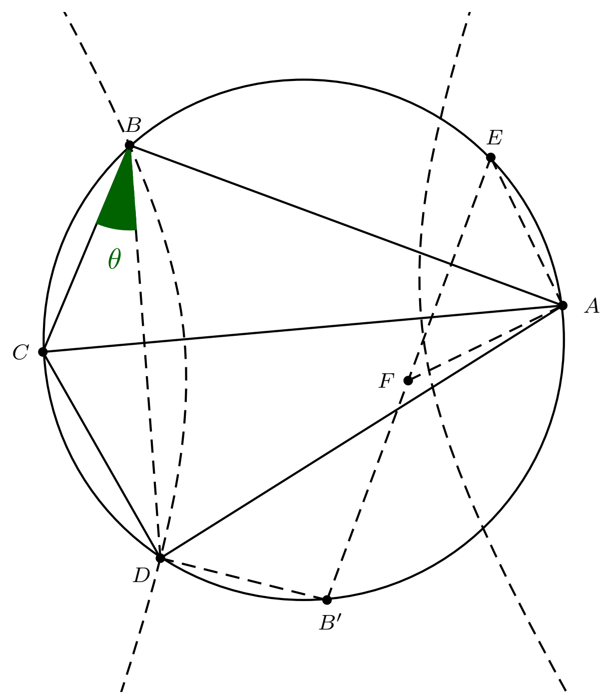

Figure 7. A Euclidean construction of D on the circumcircle of $\triangle ABC$ without using the hyperbola.

Construction. Let $AB > BC$, as in Figure 7. From the midpoint B' of the lower $\overset{\frown}{CA}$, we draw a perpendicular to AB that meets the circle at E, and a perpendicular to EA at A that meets EB' at F. On the minor $\overset{\frown}{CB'}$, we locate the required point D such that $DB' = FA$.

Proof. Since $\overset{\frown}{BCB'} + \overset{\frown}{AE} = 180^o$, then $2A + B + \overset{\frown}{AE} = A + B + C$ or $\overset{\frown}{AE} = C - A$. Using this result, we find that

$$\begin{aligned} DB' &= FA \\ &= AE \tan \tfrac{B}{2} \\ &= 2R \sin \tfrac{C-A}{2} \tan \tfrac{B}{2}\,. \end{aligned} \tag{18}$$

Next we define $\angle CBD = \theta$ and we find that

$$\begin{aligned} & AB + CD = BC + DA \\ \iff\ & 2R\sin C + 2R\sin\theta = 2R\sin A + 2R\sin(B-\theta) \\ \iff\ & 2R\sin\tfrac{C-A}{2}\cos\tfrac{C+A}{2} = 2R\sin(\tfrac{B}{2}-\theta)\cos\tfrac{B}{2} \\ \iff\ & 2R\sin(\tfrac{B}{2}-\theta) = 2R\sin\tfrac{C-A}{2}\tan\tfrac{B}{2} \\ \iff\ & DB' = 2R\sin\tfrac{C-A}{2}\tan\tfrac{B}{2}\,, \end{aligned} \tag{19}$$

as was also found in equation (18). □

7. A quadrilateral with $I_1 \equiv I_2$

Theorem 8. *If $I_1 \equiv I_2$, then the quadrilateral is tangential, or cyclic, or bicentric.*

Proof. Using barycentric coordinates in the basic $\triangle EDA$, let $E = (1:0:0)$, $D = (0:1:0)$, $A = (0:0:1)$, $DA = a$, $AE = b$, and $ED = c$. Also let $I_1 \equiv I_2 \equiv I_o$ on the bisector of $\angle DEA$ with barycentric coordinates $I_o = (k : b : c)$. Finally, let I and I_e be the incenter and E-excenter of $\triangle EDA$, respectively (Figure 8).

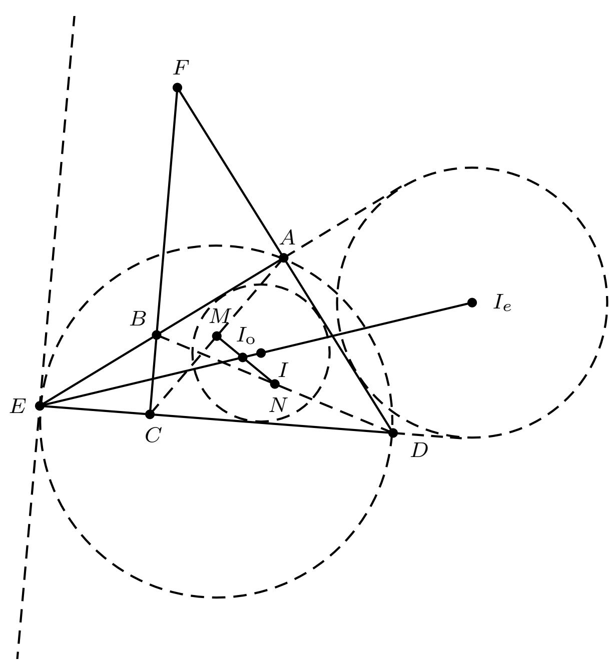

Figure 8. Quadrilateral $ABCD$ with $I_1 \equiv I_2 \equiv I_o$.

Since $F = (0:m:1)$ and $B = (1:0:n)$, we obtain the equation $mnx + y - mz = 0$ for the line BC with point at infinity $P_\infty = (1+m : -m - mn : -1+mn)$, $C = (1 : -mn : 0)$, as well as the midpoints of the diagonals $M = (1 : -mn : 1-mn)$ and $N = (1 : 1+n : n)$.

Since the points M, N, and I_o are collinear, the determinant $det(M, N, I_o)$ must be zero from which we find that

$$n = \frac{b+c-k}{(b-c+k)+(b-c-k)m}\,. \tag{20}$$

If a point $(x_o : y_o : z_o)$ is equidistant from the lines $a_i x + b_i y + c_i z = 0$ ($i = 1, 2$), the following equality holds [4]:

$$S_2(a_1x_o + b_1y_o + c_1z_o)^2 = S_1(a_2x_o + b_2y_o + c_2z_o)^2\,, \tag{21}$$

where

$$S_i = (b^2+c^2-a^2)(b_i-c_i)^2+(c^2+a^2-b^2)(c_i-a_i)^2+(a^2+b^2-c^2)(a_i-b_i)^2\,, \tag{22}$$

and $i = 1, 2$. Point I_o is equidistant from the lines BC (given above) and DA ($x = 0$). Applying equation (21) to I_o and using equation (20) to eliminate n, we find that

$$(1+m)(k^2-a^2)(b-cm)\,[b(b-c+k)-(b-c-k)cm] = 0\,. \tag{23}$$

The solutions of this equation can be classified as follows:

(a) $m = -1$ is rejected because then $F = (0 : -1 : 1)$ becomes a point at infinity.

(b) $k = \pm a$ implies that I_o coincides with I or I_e, thus $\odot I_1(r_1)$ is tangent to a third side and by Theorem 4 this quadrilateral is tangential.

(c) $b = cm$ and $[b(b-c+k)-(b-c-k)cm] = 0$ both imply that $P_\infty = (b^2-c^2 : -b^2 : c^2)$ which means that BC is parallel to the tangent to the circumcircle of $\triangle EDA$ at point E, thus this quadrilateral is cyclic.

Solutions (b) and (c) of equation (23) are the only ones that are valid. When one solution from (b) and one solution from (c) are simultaneously valid, then the quadrilateral is bicentric. This completes the proof. □

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A Group Theoretic Interpretation of Poncelet's Theorem Part 2: The Complex Case

Albrecht Hess

Abstract. Poncelet's theorem about polygons that are inscribed in a conic and at the same time circumscribe another one has a greater companion, in which different conics touch the sides of the polygon, while all conics belong to a fixed pencil. In the first part of these investigations on Poncelet's theorem [11] the case of a pencil of circles in the real plane was analyzed. It was shown that the question of whether 'circuminscribed' polygons exist for every starting point can be decided by considering the action of a group. As a continuation of this work we now examine the case of a pencil of conics in the complex plane. It will be shown that in this case the answer to the same question about the existence of a continuous family of 'circuminscribed' polygons depends on the addition of an elliptic curve.

1. Introduction

In [12, p. 285], Jacobi derives a formula relating the 'circuminscribed' polygons in Poncelet's theorem to the repeated addition of a parameter t in the argument of an elliptic function. Further ahead on p. 291 he shows that in the general case of tangents to circles of a pencil one has to add in the argument of the same elliptic function parameters $t, t', t'', \ldots$ depending on the elements of this pencil. In [3], there is given a summary of Jacobi's approach. We present here a proof of Poncelet's theorem in the general case based on the relation of an elliptic curve to Poncelet configurations. These configurations have been studied with other means elsewhere (see e.g. [1], [2], [3], [4], [7], [8], [9], [10], [12], [14], [15], [16], [17] and literature cited there.). As already mentioned in the first part: There is nothing novel about the ingredients but perhaps this particular view could shed a new light on them.

In sections 2 and 3, we collect those results with proofs from [2], [14] and [15], that are important for the understanding of the rest of the article. Lemma 3 takes therein a central place. Its proof shows that the 6 points, where conics of a pencil touch the sides of a triangle inscribed in a conic of this pencil, are the intersecting points of a complete quadrilateral.

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In section 4 and 5 we show how the construction of points where the secants of a conic touch other conics of a pencil can be related to the addition on an elliptic curve. This will be explained in Theorem 8. The main result, however, is Theorem 15. By virtue of the previously defined addition on an elliptic curve this theorem provides a criterion to decide, whether a given set of conics from a pencil forms a Poncelet configuration, which means that starting from each point on a fixed conic we can construct closed 'circuminscribed' polygons whose sides touch the other conics. This result connects the analytic addition of arguments of an elliptic function in Jacobi's article [12] with the geometric addition on an elliptic curve.

2. Preliminaries: Pencils of Quadrics

A *quadric* k in the projective space $\mathbb{P}^n$ (all spaces are over the field of complex numbers) is the zero set of a homogeneous quadratic polynomial $k(X) = \sum_{ij} k_{ij}x_ix_j$, whose coefficients are the entries of a symmetric matrix $(k_{ij}) \in \mathbb{C}^N,\ N = n(n+1)/2$, the zero matrix excluded. A quadric k is called *nondegenerate* if $\det(k_{ij}) \neq 0$. In this case, k is a smooth analytic hypersurface of $\mathbb{P}^n$. Since the multiples of (k_{ij}) are the only symmetric matrices that define the same quadric k, we see that the variety $Q\mathbb{P}^n$ of all quadrics in $\mathbb{P}^n$ is itself a projective space $\mathbb{P}^{N-1}$.

A line $\mathcal{K}$ (one dimensional subspace) in $Q\mathbb{P}^n$ is called a pencil of quadrics. If k_0 and k_1 are two distinct quadrics from $\mathcal{K}$, then

$$\mathcal{K} = \{k_\lambda : \lambda \in \mathbb{P}^1\},\ k_\lambda(X) = \lambda_0 k_0(X) + \lambda_1 k_1(X).$$

A pencil is in general position if it can be defined by two nondegenerate quadrics that intersect transversally. As in this article only the cases $n = 1$ and $n = 2$ are relevant, a discussion of both cases will clarify these definitions.

2.1. *Quadrics in* $\mathbb{P}^1$. There is nothing exciting about zero-dimensional quadrics $k(X) = k_{00}x^2 + 2k_{01}xy + k_{11}y^2 = 0$ in $\mathbb{P}^1$. They correspond to the subsets of $\mathbb{P}^1$, that consist of two points, or one double point in the case of a degenerate quadric, for which $\det(k_{ij}) = k_{00}k_{11} - k_{01}^2 = 0$.

A bit more interesting are pencils of such quadrics that are in general position. Suppose that such a pencil is a line through the nondegenerate quadrics $k_0(X)$ and $k_1(X)$. If we transform the zeros of $k_0(X)$ by a birational map of $\mathbb{P}^1$ into $[0,1]$ and $[1,0]$, the equation of k_0 becomes $k_0(X) = xy = 0$. If the distinct zeros of k_1 are $[u,1]$ and $[v,1]$, its equation is $k_1(X) = (x-uy)(x-vy) = 0$. The fact that k_0 and k_1 intersect transversally means in this case, that $uv \neq 0$. Then we get (see [2, §14.2.8]).

Theorem 1 (**Desargues' Involution Theorem in** $\mathbb{P}^1$). *For a pencil $\mathcal{K}$ of quadrics in $\mathbb{P}^1$, that is in general position, there exists a birational involution ϕ of $\mathbb{P}^1$, i.e. $\phi^2 = \mathrm{id}$, such that for any $k \in \mathcal{K}$, $k = \{M, N\}$, one has $\phi(M) = N$ (and obviously $\phi(N) = M$).*
Furthermore, in such a pencil there are exactly two degenerate quadrics corresponding to the two fixed points F, F' of the involution ϕ. These fixed points form

harmonic ranges

$$(F, F', M, N) = -1 \tag{1}$$

with each of the nondegenerate quadrics $k = \{M, N\}$.

Proof. From Vieta's formula applied to the zeros $M = [M_0, M_1]$ and $N = [N_0, N_1]$ of $\lambda_0 k_0(X) + \lambda_1 k_1(X) = \lambda_0 xy + \lambda_1(x - uy)(x - vy) = 0$ we get $M_0 N_0 = uvM_1 N_1$. Hence,

$$\phi \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} = \begin{pmatrix} 0 & uv \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix}$$

is the involution with $\phi(M) = N$ we have been looking for.
The equation (1) follows from $(F, F', N, M) = (\phi(F), \phi(F'), \phi(M), \phi(N)) = (F, F', M, N)$ and the properties of the cross-ratio, to be found in any textbook on projective geometry. □

2.2. *Quadrics in* $\mathbb{P}^2$. A *conic* is the zero set of $k(X) = k_{00}x^2 + k_{11}y^2 + k_{22}z^2 + 2k_{01}xy + 2k_{02}xz + 2k_{12}yz$. This is a one-dimensional quadric in $\mathbb{P}^2$ which degenerates, $\det(k_{ij}) = 0$, iff $k(X)$ factorizes into a product of two lines. This can be seen by diagonalizing the matrix (k_{ij}). The cases of $\operatorname{rank}(k_{ij}) = 1, 2, 3$, that are projectively equivalent to $0 = k(X) = x^2$; $x^2 + y^2$; $x^2 + y^2 + z^2$; respectively, correspond to a double line, two crossing lines and a nondegenerate conic.

Let $\mathcal{K}$ be a pencil of conics in general position in the projective plane $\mathbb{P}^2$. This means that all conics of $\mathcal{K}$ have four base points B_1, B_2, B_3, B_4 in common, no three of them collinear. The base points can be moved by a projective transformation of $\mathbb{P}^2$ to $[\pm 1, \pm 1, 1]$, (see Figure 1). The three degenerate conics of the pencil are $k_x : y^2 - z^2 = 0, k_y : z^2 - x^2 = 0, k_z : x^2 - y^2 = 0$. Any conic k_λ of the pencil is the zero set of an equation

$$k_\lambda : \lambda_0(y^2 - z^2) + \lambda_1(z^2 - x^2) = 0,\ \lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1. \tag{2}$$

To a line g, not passing through any base point, Desargues' involution Theorem 1 applies by restriction of the equations of the conics k_λ to g. We get (see [2], [5], [6])

Theorem 2 (**Desargues' Involution Theorem in** $\mathbb{P}^2$). *Given a pencil* $\mathcal{K}$ *of conics in* $\mathbb{P}^2$ *and a line* g*, not containing any base point of the pencil. Then*

- i *The pairs of intersections of* g *with the conics of* $\mathcal{K}$ *generate an involution* ϕ *on* g.
- ii *The intersections of* g *with two conics of the pencil determine the involution* ϕ *uniquely.*
- iii *The fixed points* F_1, F_2 *of* ϕ *are the points, where conics from the pencil touch* g.
- iv *The intersections* L_1, L_2 *of* g *with any conic* k_λ *from* $\mathcal{K}$ *form together with the fixed points* F_1, F_2 *a harmonic range*

$$(L_1, L_2, F_1, F_2) = -1. \tag{3}$$

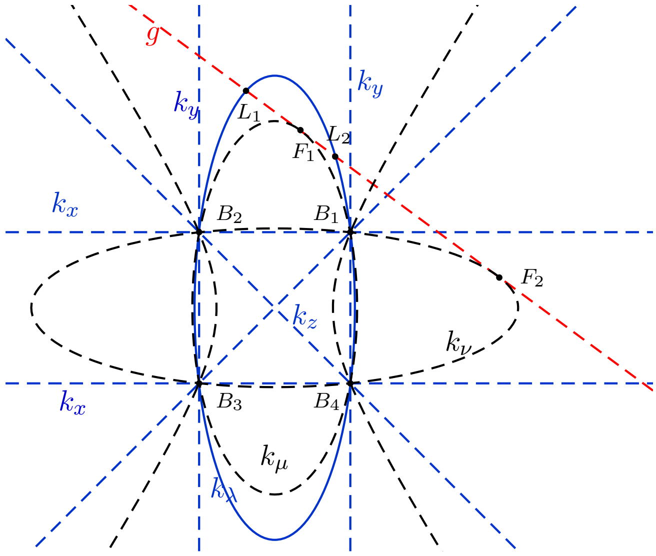

Figure 1

Remark. While in Theorem 1 it is important that the pencil is in general position - if not, all quadrics would have a common point - the essential condition of Theorem 2 is that the line g avoids the base points. The theorem remains true if some base points coincide.

In Figure 2 two generators of pencils of types I-V are represented from left to right. These different types reflect coincidences between some of the base points (cf. [2, §16.5]). Sure, these are just one-dimensional sketches of surfaces of real dimension two in a space of real dimension four.

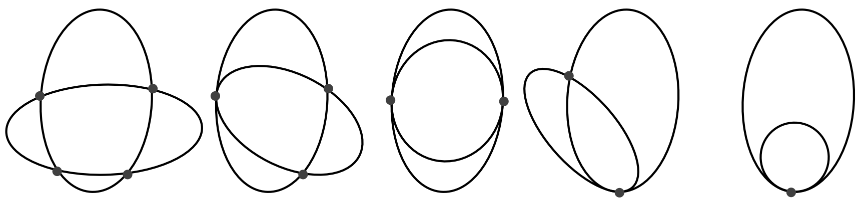

Figure 2

3. The Interplay of Involutions

According to Desargues' Theorem 2, a pencil $\mathcal{K}$ of conics in $\mathbb{P}^2$ induces a biratonal involution on every line not meeting any base point of the pencil. How these involutions on different lines fit together will be explained in the following lemmas[1]. For the first two see Figure 3.

Lemma 3 (Collinearity of Fixed Points). *On a conic k_0 from a pencil $\mathcal{K}$ in general position[2], take three points P_0, Q_0, P, distinct from the base points B_i. The intersections with $\mathcal{K}$ induce involutions on the sides of the triangle P_0Q_0P. Let $T_0 \in P_0Q_0$ and $R \in P_0P$ be fixed points of these involutions. Then $S = T_0R \cap PQ_0$ is a fixed point on PQ_0.*

Proof. The conic k_1 touches P_0Q_0 at T_0 and the conic k_λ touches P_0P at R. The line $g = T_0R$ intersects k_1 again at T. Let $\mathcal{L}$ be the pencil containing k_1 and the degenerate conic $P_0Q_0 \cup PT$. Since P_0Q_0 touches k_1, there are two possibilities: Either PT intersects k_1 at two points T, T' and $\mathcal{L}$ is a pencil of type II, or PT is tangent to k_1 at T and $\mathcal{L}$ is a pencil of type III, see Figure 2. We show that the latter occurs and that the double line g is in $\mathcal{L}$.

Step 1: The restrictions of $\mathcal{K}$ and $\mathcal{L}$ induce the same involution on P_0P.

Indeed, one common zero-dimensional quadric is $\{P_0, P\}$ and the other is the intersection of P_0P and the conic k_1 from $\mathcal{K} \cap \mathcal{L}$. Hence, besides the conic $k_\lambda \in \mathcal{K}$, tangent to P_0P at R by assumption, there must be a conic $l \in \mathcal{L}$, tangent to P_0P at this point R.

Step 2: The pencil $\mathcal{L}$ contains the double line g.

The line g intersects the conic l at three points T_0, T and R. Obviously, T is distinct from T_0. Indeed, otherwise g would have a double intersection with k_1 at T_0 and would hence coincide with P_0Q_0, which is absurd. If $T = R$, since there is only one conic from the pencil $\mathcal{K}$ through this point, then $k_1 = k_\lambda$ and $\mathcal{L}$ is of type III containing the double line g.

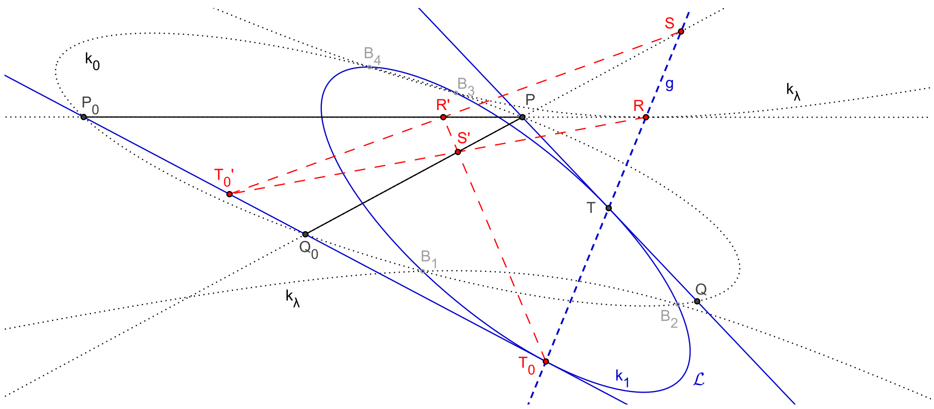

Figure 3

If $T \neq R$, then the line g is a part of l, both having three intersections, so l is degenerate. Like all conics of $\mathcal{L}$, l has a double intersection with P_0Q_0 at T_0. By

[1] The origin of the ideas behind the statements of this section is [15]. A more recent presentation is in [2].

[2] The letter $\mathcal{K}$ from now on will always denote a pencil in general position.

definition, l is tangent to P_0P at R, hence intersects it twice at R. Therefore, l is the double line g. In either way, $\mathcal{L}$ is of type III. The double line g connects the tangency points of P_0Q_0 and PT with k_1.

Step 3: The intersection S of g and PQ_0 is a fixed point on PQ_0.

Applying the argument from step 1 again, we see that the involutions induced on the line PQ_0 by the intersections with $\mathcal{L}$ and $\mathcal{K}$ are the same. Hence, the intersection S of the double line $g = T_0R$ from $\mathcal{L}$ with PQ_0 must be a fixed point of both pencils. So we conclude that the fixed points T_0, R and S are collinear. □

Corollary 4. *For fixed points T_0, R, S of the involutions on the sides of the triangle P_0Q_0P we have the following dichotomy: Either these points are collinear* (*Menelaus configuration*) *or their respective connexions with the vertices P, Q_0, P_0 are concurrent* (*Ceva configuration*)*.*

Proof. From $(Q_0, P, S, S') = -1$, cf. Figure 3 and (3), it follows that

$$\frac{P_0T_0}{T_0Q_0}\frac{Q_0S}{SP}\frac{PR}{RP_0} = -\frac{P_0T_0}{T_0Q_0}\frac{Q_0S'}{S'P}\frac{PR}{RP_0}.$$

One triplet T_0, R, S or T_0, R, S' is collinear (Menelaus configuration) according to Lemma 3. The other triplet forms a Ceva configuration by the theorems of Menelaus and Ceva. □

The second intersection Q of PT with k_0 produces a quadrangle P_0Q_0PQ inscribed in the conic k_0. Both P_0Q_0 and PQ touch k_1. We call *$\mathcal{K}-$parallelograms* quadrangles inscribed in k_0 with a pair of opposite sides touching the same conic of the pencil $\mathcal{K}$. This naming will become transparent after Definition 1 below. In $\mathcal{K}-$parallelograms not only one but each pair of opposite sides is tangent to one conic of the pencil. This is the content of Lemma 5, see Figure 4.

Lemma 5. *The $\mathcal{K}-$parallelogram P_0Q_0PQ is inscribed in the conic k_0 with P_0Q_0 and PQ tangent to k_1 at T_0 and T, respectively. Then PQ_0 and QP_0 touch a conic $k_\mu \in \mathcal{K}$ and the tangency points S and U are on the line $g = T_0T$. The same is true for the other pair of opposite sides PP_0 and QQ_0.*

Proof. The same reasoning as in Lemma 3 shows that the involutions induced by the pencil $\mathcal{L}$, containing $P_0Q_0 \cup PQ, k_1$ and the double line g, and the pencil $\mathcal{K}$ on each of the lines PQ_0 and QP_0 must coincide. Hence, the intersections S and U with the double line $g \in \mathcal{L}$ are tangency points of the lines PQ_0 and QP_0 with conics $k_\mu, k_\nu \in \mathcal{K}$, possibly different. Let us show that $k_\mu = k_\nu$ by introducing the pencil $\mathcal{M}$ of all conics through P_0, Q_0, P, Q. The same involutions are induced by $\mathcal{M}$ and $\mathcal{K}$ on g. In fact, $P_0Q_0 \cup PQ \in \mathcal{M}$ as well as $k_1 \in \mathcal{K}$, intersect g at $\{T_0, T\}$ and k_0 is in both pencils. So there must be one conic in $\mathcal{K}$ going through the points $\{S, U\}$, where g intersects $PQ_0 \cup QP_0 \in \mathcal{M}$. This conic is $k_\mu = k_\nu$. □

Lemma 6 (**Completing to a $\mathcal{K}-$parallelogram**)**.** *If T_0 and R are fixed points of the involutions induced on P_0Q_0 and P_0P by the pencil $\mathcal{K}$, then there exists one and only one $\mathcal{K}-$parallelogram P_0Q_0PQ for which T_0 and R are amongst the points where pairs of opposite sides touch a conic of the pencil.*

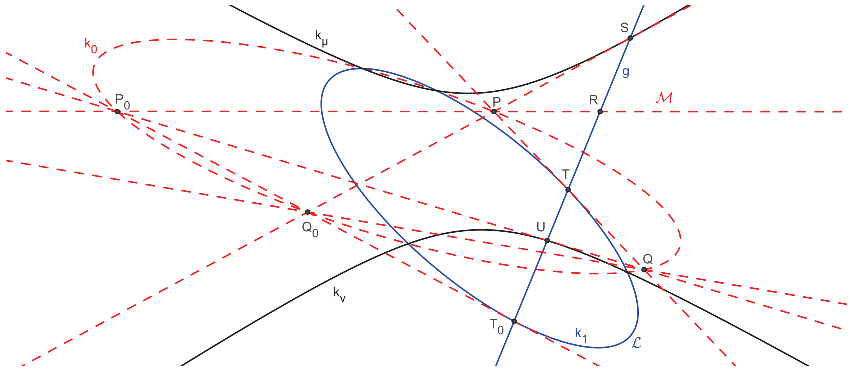

Figure 4

Proof. Since the line $g = T_0R$ contains all tangency points where pairs of opposite sides of $\mathcal{K}-$parallelograms completing P_0Q_0P touch the same conic of the pencil according to Lemma 5, the construction of T and Q carried out in the proof to Lemma 3 yields a necessary condition to construct the $\mathcal{K}-$parallelogram. Lemma 5 shows that the completion of P_0Q_0P by Q does indeed give the desired result. □

In the next section, when we define a parallelism of Poncelet structures we need this parallelism to be transitive. This transitivity is related to the content of the next lemma.

Lemma 7 (**Transitivity**). *Let P_0, P_1, P_2 be distinct points on a conic k_0 so that P_iT_i are tangent at T_i to the same conic k_1. Then the intersections $R_{i+2} = T_iT_{i+1} \cap P_iP_{i+1}$ $(mod\ 3)$, being fixed points of the involutions induced by the pencil $\mathcal{K}$ through k_0 and k_1 on P_iP_{i+1} by Lemma 5, form a Ceva configuration on the sides of the triangle $P_0P_1P_2$, (cf. Corollary 4).*

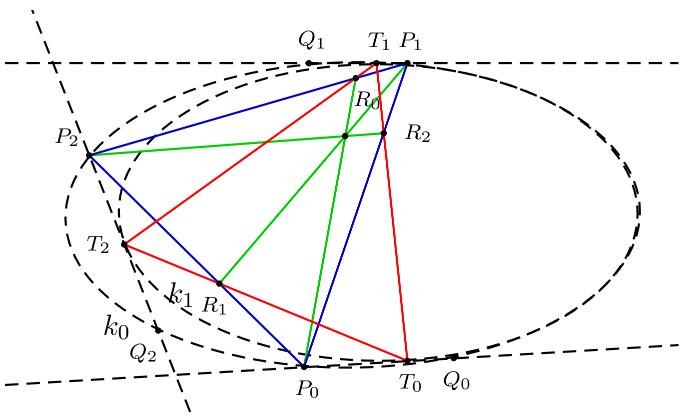

Figure 5

Proof. If we set $Q_i = P_iT_i \cap k_0$, then $P_iQ_iP_{i+1}Q_{i+1}$ are $\mathcal{K}-$parallelograms to which Lemma 5 applies and shows that R_{i+2} are fixed points of the involutions on the respective sides of $P_0P_1P_2$, see Figure 5. Apart from cases of coinciding tangency points if $P_2 \to Q_0$ and $Q_2 \to P_0$, see Figure 6, that we kindly ask the reader to clarify by himself, the points T_i are distinct. Tangents at three distinct points to a conic are never concurrent, hence the triangles $P_0P_1P_2$ and $T_0T_1T_2$ lack a center of perspectivity. According to Desargues' two triangle theorem ([5, §2.3]) they lack an axis of perspectivity too. This excludes the possibility for the intersections R_i of corresponding sides of these triangles to be collinear, so they must form a Ceva configuration according to Corollary 4. □

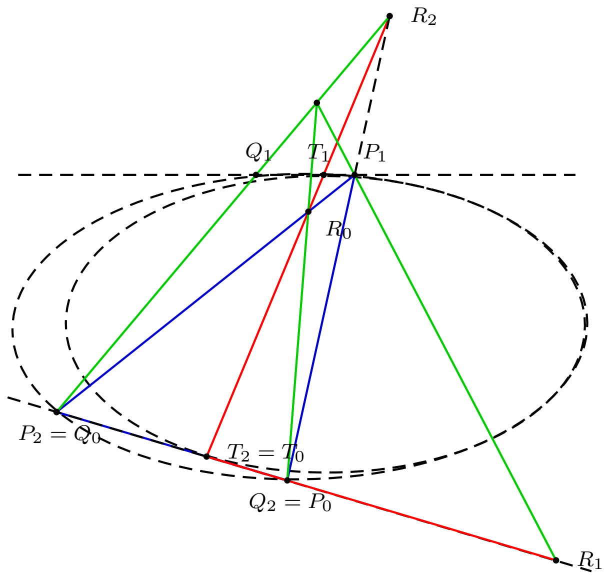

Figure 6

4. The Cubic

From now on let $\mathcal{K}$ be the pencil of conics through the base points $B_i = [\pm 1, \pm 1, 1]$, and let $k_\lambda : \lambda_0(y^2 - z^2) + \lambda_1(z^2 - x^2) = 0$ be one of the nondegenerate conics from $\mathcal{K}$ and $O = [u, v, w]$ a point of k_λ, which is not a base point.

If tangents are drawn from O to all conics of $\mathcal{K}$, their contact points $T = [x, y, z]$ are on a cubic E_O. To obtain its equation

$$E_O(x, y, z) = ux(y^2 - z^2) + vy(z^2 - x^2) + wz(x^2 - y^2) = 0 \qquad (4)$$

we have to eliminate the parameters μ_i from the conic $\mu_0(y^2 - z^2) + \mu_1(z^2 - x^2) = 0$ and the equation of the polar line $\mu_0(vy - wz) + \mu_1(wz - ux) = 0$ of the pole O (for polars cf. [2, §14.5]). This cubic already appeared in Lebesgue's article, section II, although without coordinates, see [15, p.121]. It is curious that

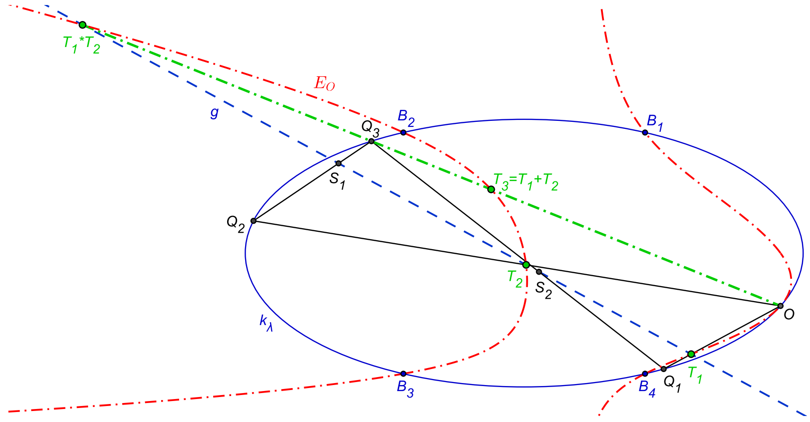

Figure 7

its properties as an elliptic curve with an addition closely related to the properties of Poncelet configurations were not exploited there.

The Weierstrass form (see [13, p.56]) of E_O is, thanks to Maple,

$$\begin{aligned} y^2 &= x^3 + Ax + B, \\ A &= -\frac{1}{6}(\Delta_{uv}^2 + \Delta_{vw}^2 + \Delta_{wu}^2), \\ B &= \frac{1}{27}(\Delta_{uv} - \Delta_{vw})(\Delta_{vw} - \Delta_{wu})(\Delta_{wu} - \Delta_{uv}), \end{aligned} \tag{5}$$

with the shortcuts $\Delta_{uv} = u^2 - v^2, \Delta_{vw} = v^2 - w^2, \Delta_{wu} = w^2 - u^2$. From (5) we obtain, using the relation $\Delta_{uv} + \Delta_{vw} + \Delta_{wu} = 0$, the discriminant Δ of the cubic E_O as

$$\Delta = -16(4A^3 + 27B^2) = 16\Delta_{uv}^2\Delta_{vw}^2\Delta_{wu}^2.$$

Consequently, the discriminant Δ vanishes on k_λ only at the base points. Since O is not a base point, the curve E_O is nonsingular, see [13, p.58], hence an elliptic curve[3].

The next theorem is fundamental. It establishes the link between Poncelet configurations and the addition on the elliptic curve E_O with origin O.

Theorem 8. *Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1 and OQ_2, respectively. Completing OQ_1Q_2 to a $\mathcal{K}-$parallelogram yields $OQ_1Q_2Q_3$ as in Lemma 6. The line $g = T_1T_2$ intersects Q_2Q_3 at S_1 where this secant touches the same conic k_μ as OQ_1 at T_1. The same is true for the intersection $S_2 = g \cap Q_1Q_3$ and a conic k_ν touching OQ_2 and Q_1Q_3 at T_2 and S_2. The addition on E_O produces $T_3 = T_1 + T_2$ on OQ_3. This point T_3 forms Ceva configurations together with T_1 and S_2 on the sides of OQ_1Q_3, and together with T_2 and S_1 on the sides of OQ_2Q_3, see Figure 7.*

[3] Who neither trusts the program Maple and is not willing either to check the Weierstrass form by hand, may apply Proposition 11 with the not yet defined addition on E_O substituted by Lemma 6 to convince himself that E_O is an elliptic curve.

Proof. The line $g = T_1T_2$ intersects all sides of the quadrangle $OQ_1Q_2Q_3$ at tangency points with conics from $\mathcal{K}$. Its intersection $T_1 * T_2$ with the line OQ_3 is the third intersection of T_1T_2 with E_O. The sum $T_3 = T_1 + T_2$ is the third intersection of the line through O and $T_1 * T_2$ with E_O. From Corollary 4 the remaining assertions of the theorem follow easily. □

The addition on the elliptic curve E_O can be made more visual if we identify tangency points on elliptic curves with different origins, see Figure 8.

Definition 1. Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1 and OQ_2. We say that T_1 and a point $S_1 \in E_{O_2}$ are equivalent $(S_1 \sim T_1)$ or that the lines OT_1 and Q_2S_1 are $\mathcal{K} - parallel$ along OT_2Q_2 if T_1, T_2, S_1 are collinear and OT_1 and Q_2S_1 touch the same conic.

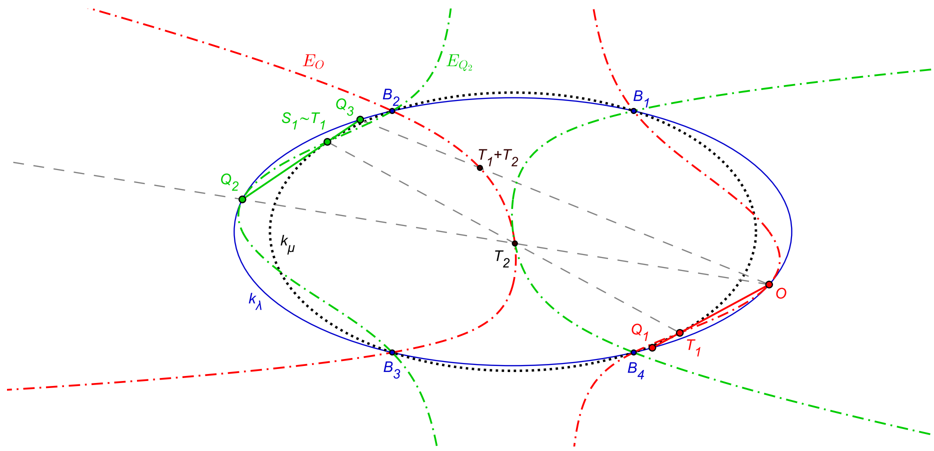

Figure 8

With this identification, writing the tangency points that form a Ceva configuration between the start points and end points, we have

$$OT_2Q_2 + Q_2T_1Q_3 = O(T_1 + T_2)Q_3 \tag{6}$$

as if this were an addition of vectors. From Theorem 8 it follows with a look at Figure 7 that T_2 and S_2 are equivalent too. More precisely, they are equivalent along OT_1Q_1, and

$$OT_1Q_1 + Q_1T_2Q_3 = O(T_1 + T_2)Q_3. \tag{7}$$

Corollary 9. *Let $T_1, T_2 \in E_O$ be tangency points on the secants OQ_1, OQ_2 and its sum $T_3 = T_1 + T_2$ the corresponding tangency point on the secant OQ_3. The tangency point S_1 on Q_2Q_3 is equivalent to T_1 if and only if S_1, T_2 and T_3 form a Ceva configuration on the sides of OQ_2Q_3.*

Proof. Look at $T_1 + T_2$ and $T_1 * T_2$ in the proof of Theorem 8 and apply Corollary 4. □

This definition of $\mathcal{K}-$parallelism explains also the name of $\mathcal{K}-$parallelograms: They are quadrangles with $\mathcal{K}-$parallel opposite sides. The next proposition says that for secants these two concepts are equivalent: touching the same conic and being $\mathcal{K}-$parallel.

Proposition 10. *If secants OQ_1 and Q_2Q_3 of k_λ touch the same conic of the pencil $\mathcal{K}$ at T_1 and S_1 then there exists a unique point $T_2 \in E_O$ so that T_1 and S_1 are equivalent and the secants are $\mathcal{K}-$parallel along OT_2Q_2.*

Proof. From Lemma 5 it follows that T_1S_1 intersects OQ_2 at a point T_2, that is one of the fixed points of the involution induced by $\mathcal{K}$ on OQ_2 and hence belongs to E_O. Then T_1 and S_1 are equivalent along OT_2Q_2 by definition. The uniqueness of T_2 is obvious. □

Griffiths and Harris showed ([9, p.38]) by abstract considerations that the incidence variety

$$E = \{(X, \xi) \in k_\lambda \times k_\mu^* : X \in \xi\} \tag{8}$$

of all pairs of secants ξ with starting point $X \in k_\lambda$, which are tangent to a fixed conic k_μ (hence elements of k_μ^*) is an elliptic curve. Here we have a concrete bijection from E to E_O:

Proposition 11. *Let $T_1 \in E_O$ be a tangency point so that OT_1 is tangent to the conic k_μ. Further let $p : k_\mu^* \to k_\mu$ be the pole map restricted to k_μ^*. Then a bijection from E onto E_O is*

$$(X, \xi) \mapsto T = T_1 p(\xi) \cap OX. \tag{9}$$

Proof. Let $q : E_O \to k_\lambda$ associate to each tangency point $T \in E_O$ the second intersection of OT with k_λ. Then $T \mapsto (q(T), q(T)q(T + T_1))$ is the inverse of (9), see Figure 9.

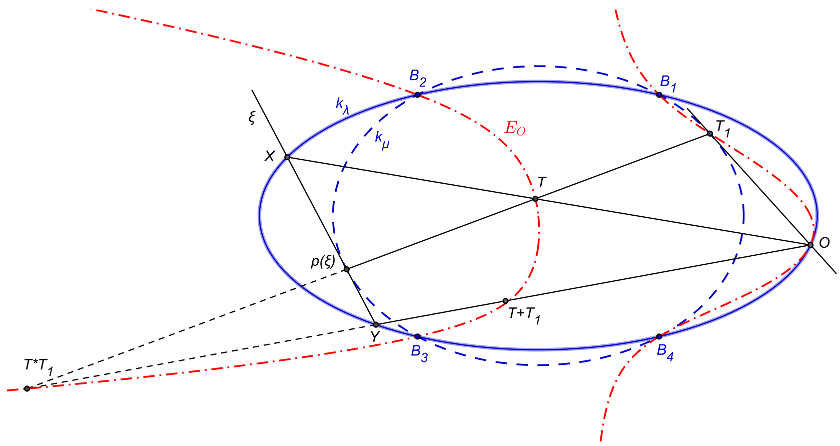

Figure 9

□

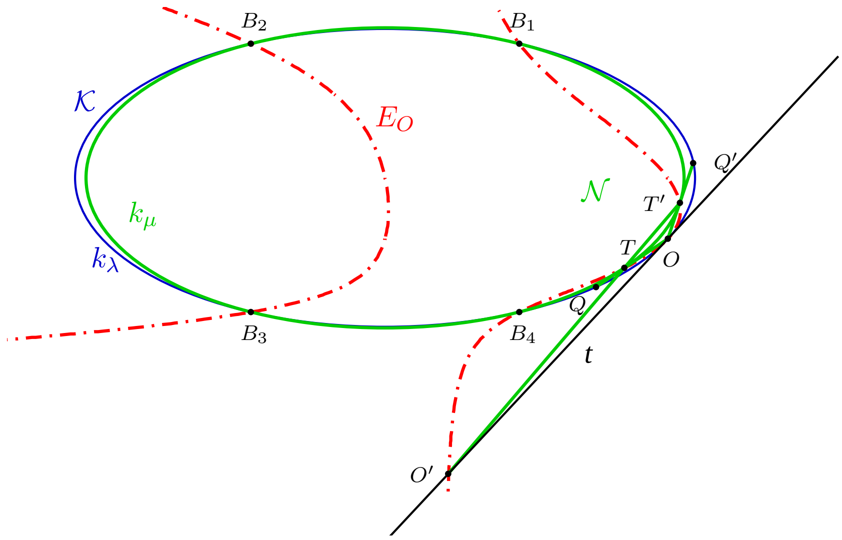

Figure 10

Just two remarks: First, although equivalence mappings between the elliptic curve E_O and the cubics E_{B_i} cannot exist since the equations $\pm x(y^2 - z^2) \pm y(z^2 - x^2) + z(x^2 - y^2) = 0$ of the latter factorize into the product of three lines through B_i, one has the mappings

$$T \mapsto q(B_i)q(B_i + T) = B_i q(B_i + T) \tag{10}$$

from E_O onto the line bundle through B_i which extend continuously the $\mathcal{K}-$ parallelism to the secants through the base points. Below it will be shown that OT and $O(-T)$ touch the same conic, hence $B_i q(B_i + T)$ and $B_i q(B_i - T)$ too. Then $B_i q(B_i + T) = B_i q(B_i - T)$, since B_i belongs to all conics and through a point on a conic only one tangent can be drawn. Therefore, the mapping (10) is a double cover of the $\mathbb{P}^1$-isomorphic line bundle through B_i, branched over the four lines $B_i B_j$.

Second, in Proposition 11 it was shown that all elliptic curves E_{O_i} are isomorphic to the elliptic curve E from (8). Before proving that the equivalences of tangency points along secants via a tangency point are isomorphisms of the elliptic curves E_{O_i} we keep the promise and confirm that OT and $O(-T)$ touch the same conic.

Proposition 12. *If OT is tangent to k_μ at $T \in E_O$ then $(-T)$ is the tangency point of the other tangent from O to k_μ.*

Proof. Let OQ and OQ' with tangency points T and T' be the tangents from O to a conic $k_\mu \in \mathcal{K}$, see Figure 10. The main observation is that k_λ and E_O share the tangent line t at O. To see this, we can consider a secant OP with a tangency point S close to O and let P and S approach O or calculate the tangent to E_O in O from the equation (4) and compare it with the tangent at this point to the conic k_λ.

Let $\mathcal{N}$ be the pencil of type III containing k_μ, the degenerate conic $OT \cup OT'$ and therefore also the double line TT'. Both pencils $\mathcal{K}$ and $\mathcal{N}$ induce by intersections the same one-dimensional pencil on t because both produce the double point O and the intersections with k_μ. Consequently, the intersection O' of the double line TT' with t must also be a tangency point with a conic from $\mathcal{K}$, hence an element of E_O. Since t already has a double intersection with E_O in O, the point $O' = O * O$ is the third intersection of t with the cubic and $T + T' = O$. $\square$

Corollary 13. *For $T \in E_O$ let $q(T) = Q$ be the second intersection of OT with k_λ. The other preimage of Q under q is the third intersection $O * T$ of the line OT with E_O, given by*

$$O * T = O * O - T. \tag{11}$$

Proposition 14. *The equivalence $\epsilon_{OT_2Q_2}$ of tangency points from E_O via T_2 with those from E_{Q_2} according to Definition 1. is an isomorphism of these elliptic curves*[4]*. The composition of equivalences is transitive* (*cf. also (6) and (7) and contemplate Figure 8*).

$$\epsilon_{Q_2T_1Q_3} \circ \epsilon_{OT_2Q_2} = \epsilon_{O(T_1+T_2)Q_3}, \tag{12}$$

Proof. All elliptic curves E_{O_i} go through seven points: four base points $B_i = [\pm 1, \pm 1, 1]$ and the respective singular points $X = [1,0,0], Y = [0,1,0], Z = [0,0,1]$ of the singular conics $k_x : y^2 - z^2 = 0, k_y : z^2 - x^2 = 0, k_z : x^2 - y^2 = 0$ of $\mathcal{K}$. The remaining two from the nine intersections of E_O and E_{Q_2} are obviously the tangency points T_2 and T_2' on the line OQ_2. If we take any point $T \in E_O$, the line TT_2 will intersect E_{Q_2} in three points: T_2, S and S'. To choose the correct image of T under $\epsilon_{OT_2Q_2}$ among S and S', i.e. the $\mathcal{K}-$equivalent one, we take the second intersection of TT_2 with the conic through T. This intersection is uniquely defined and can be found by eliminating the parameters μ_i from $\mu_0 k_x(T) + \mu_1 k_y(T) = 0$ and intersecting $k_\mu(X) = k_y(T)k_x(X) - k_x(T)k_y(X) = 0$ with the line TT_2, see Lemma 6 and Figure 8. Another way to see that $S = \epsilon_{OT_2Q_2}(T)$ is obtained from T by a regular mapping is to write it as

$$\epsilon_{OT_2Q_2}(T) = q(T_2)q(T + T_2) \cap TT_2,$$

with q defined in the proof to Proposition 11.

These mappings $\epsilon_{OT_2Q_2}$ are non-constant isogenies (biholomorphic mappings, respecting the origins) of elliptic curves, hence isomorphisms of the underlying group structure, see [13, chap VI.4], and [18, chap III.4]. This can be seen by lifting the mapping to the universal cover $\mathbb{C}$ of the elliptic curves, where its derivative is an everywhere bounded analytic function, hence constant. So, the covering mapping, being linear and respecting the origins, is a multiplication with a complex number and as such an isomorphism of the group structure. This remains true if the covering mapping descends from the cover to the elliptic curves.

---

[4] The mere fact that all $E_O, O \in k_\lambda \setminus \{B_i\}$, are isomorphic, without indicating explicit isomorphisms, can also be deduced from their $j-$invariants, expressing them with the Weierstrass form (5) only by the parameter $\lambda = [\lambda_0, \lambda_1]$ of k_λ. In order to do this, we only have to eliminate the coordinates $[u, v, w]$ using equation (2). For $j-$invariant see [18, Proposition 1.4]

The equation (12) is a consequence of the transitivity explained in Lemma 7 and of Corollary 9. □

Note that there is another isomorphism from E_O to E_{Q_2}, i.e. $\epsilon_{OT_2'Q_2}$ which goes via the second intersection T_2' of E_O and E_{Q_2}. By Proposition 12, each of them is the negative of the other

$$\epsilon_{OT_2'Q_2} = -\epsilon_{OT_2Q_2}. \tag{13}$$

5. Poncelet Configurations

As before, on a nondegenerate conic k_λ from the pencil $\mathcal{K}$ a point O, distinct from the base points of $\mathcal{K}$, is fixed. Let E_O be the elliptic curve defined in (4).

Definition 2. We call the conics $k_0, \ldots, k_{n-1} \in \mathcal{K}$ a *Poncelet configuration* if for every $Q_0 \in k_\lambda$ there is a closed polygonal chain $(Q_0, \ldots, Q_{n-1})$ inscribed in k_λ so that its secants $Q_iQ_{i+1}, i = 0, 1, \ldots \pmod n$ touch all conics $k_0, \ldots, k_{n-1}$ in some order.

Theorem 15. *In the same settings as above, let $k_0, \ldots, k_{n-1} \in \mathcal{K}$ be given together with points $\Delta T_i \in E_O \cap k_i$, where tangents from O touch k_i. These conics form a Poncelet configuration if and only if some of the sums $\Sigma = \pm\Delta T_0 \pm \cdots \pm \Delta T_{n-1}$ vanish.*

Proof. Suppose that any of the above sums vanishes. Since ΔT_i and $-\Delta T_i$ are the points where tangents from O touch k_i (Proposition 12), let us assume, by switching signs if necessary, that $\Sigma = \Delta T_0 + \cdots + \Delta T_{n-1} = 0$. For $Q_0 \in k_\lambda$ take a $T_0 \in E_O$ with $q(T_0) = Q_0$, for the mapping q see Corollary 13, and define inductively $T_{i+1} = T_i + \Delta T_i$ and $Q_{i+1} = q(T_{i+1})$. By Proposition 14, both secants $q(T_i)q(T_{i+1})$ and $O\Delta T_i$ are $\mathcal{K}-$parallel and therefore touch the same conic k_i. This argument is valid if among the end points of the secants there is no base point. By the continuous extension (10), the tangency to the same conic persists if some of the end points of the secants are base points. The polygonal chain $(Q_0, \ldots, Q_n)$ is closed since $T_n = T_1 + \Sigma = T_1$.

Now suppose that none of the sums is zero. We will show that $\Sigma = \Delta T_0 + \cdots + \Delta T_{n-1} \neq 0$ implies that there are only four $T_0 \in E_O$ for which a polygonal chain, starting in $Q_0 = q(T_0)$ and going through consecutive points Q_i via tangency points S_i that are equivalent to ΔT_i, will close after n steps at Q_0. The choice of the other tangent from Q_i to k_i results in a sign change of ΔT_i (Proposition 12). Since none of these sign changes yields the sum zero of the $\pm\Delta T_i$, due to the assumption made above, it follows that there are at most $4 \cdot 2^n$ starting points, all possible sign changes taken into account and all permutations of the k_i neglected for having no effect on the end point, for which the polygonal chain is closed. But for a Poncelet configuration there should be a closed chain for every starting point.

Drawing from $Q_0 = q(T_0)$ the tangent to k_0 through the point $\epsilon_{OT_0Q_0}(\Delta T_0) \in E_{Q_0}$ of tangency we arrive by (6) at the point $Q_1 = q(T_1), T_1 = T_0 + \Delta T_0$. These formulas for Q_1 and T_1 remain true even if T_0 is one of the base points and there is no isomorphism from E_O onto E_{Q_0}, see (10). Eventually, we arrive at $Q_{n-1} = q(T_{n-1}), T_{n-1} = T_0 + \Delta T_0 + \cdots + \Delta T_{n-2}$. For the closing of the polygon by a tangent from Q_{n-1} to k_{n-1} through the tangency point $S_{n-1} = \epsilon_{OT_{n-1}Q_{n-1}}(\Delta T_{n-1}) \in E_{Q_{n-1}}$ it is necessary that

$$q(T_{n-1} + \Delta T_{n-1}) = q(T_0 + \Sigma) = q(T_0).$$

Since $T_0 + \Sigma \neq T_0$, the line $T_0(T_0 + \Sigma)$ contains O, see Figure 11. Hence, $O = T_0 * (T_0 + \Sigma)$ and

$$T_0 + (T_0 + \Sigma) = O * O. \tag{14}$$

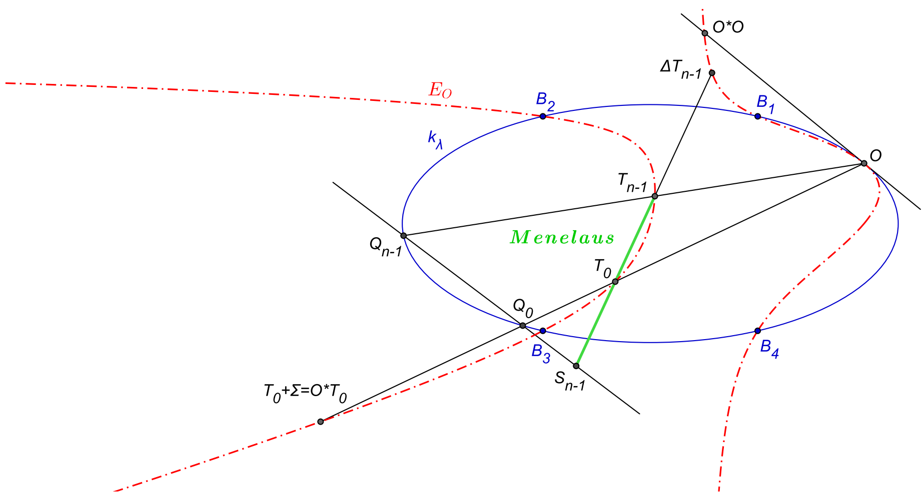

Figure 11

But (14) has just four solutions: $T_0 = \frac{O*O-\Sigma}{2} + U$, where $U \in E_O$ is a point of order two. It follows from $U = -U$ and Proposition 12. that there exists a conic k_μ from $\mathcal{K}$ such that OU is the unique tangent from O to k_μ. If k_μ is nondegenerate, then $k_\mu = k_\lambda$ and $U = O$. If k_μ is one of the degenerate conics k_x, k_y, k_z of $\mathcal{K}$ formed by two lines, then U is their respective intersection $X[1,0,0], Y[0,1,0]$ or $Z[0,0,1]$.

The equation $2U = O$ for the points U of order two can be solved as well by the following observation. Its four solutions are those points U for which the secant line $O'U$ through U and the point $O' = O * O = [1/u, 1/v, 1/w]$ is tangent to E_O at U. It can easily be verified that this occurs at the previously determined points $U = O, X[1,0,0], Y[0,1,0], Z[0,0,1]$.

This proves the claim about the existence of at most four starting points $Q_0 = q(T_0)$ for which we get a closed polygonal chain along tangents to k_i, that are parallel to ΔT_i. □

The next proposition states a numerical criterion to check out if a polygon $Q_0, \ldots, Q_{n-1}$ produces via given tangency points points $S_i \in Q_iQ_{i+1}$ the conics of a Poncelet configuration.

Proposition 16. *In the same settings as above, let $Q_0, \dots, Q_{n-1} \in k_\lambda \setminus \{B_i\}$ be given (the base points are excluded since (15) makes no sense for these points) and on each secant Q_iQ_{i+1} a fixed point S_i of the involution induced by $\mathcal{K}$ on it. Then*

$$\frac{Q_0S_0}{S_0Q_1}\frac{Q_1S_1}{S_1Q_2}\cdot \ldots \cdot \frac{Q_{n-1}S_{n-1}}{S_{n-1}Q_0} = \pm 1. \tag{15}$$

If the result is 1, the conics k_i tangent to Q_iQ_{i+1} at S_i form a Poncelet configuration.

If the result is -1, take an odd number of points S_i on Q_iQ_{i+1} and substitute them by the other tangent point S_i' to change the sign in (15). With an appropriate change of the correponding conics we obtain again a Poncelet configuration.

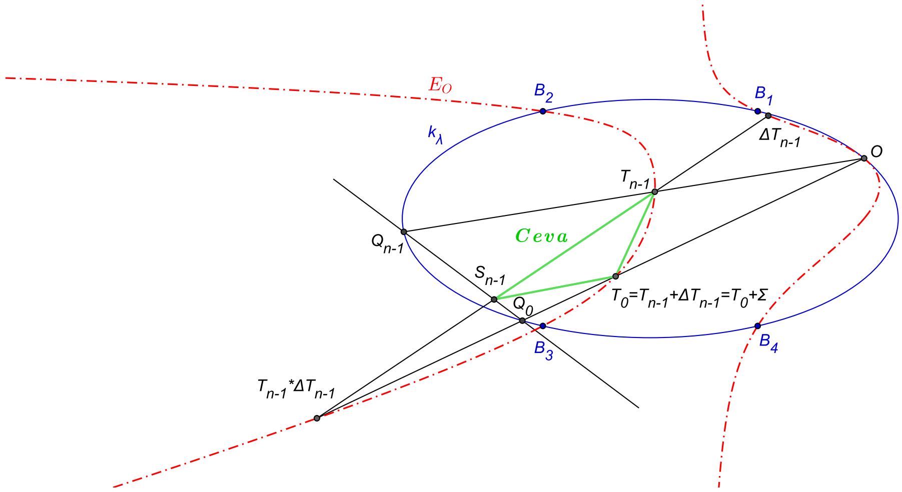

Figure 12

Proof. Go through the construction of T_i with $q(T_i) = Q_i$ as before and take always the next T_{i+1} that forms a Ceva configuration together with T_i and S_i on the sides of OQ_iQ_{i+1}. By Corollary 9. we have $T_{i+1} = T_i + \Delta T_i$ and $q(T_{i+1}) = Q_{i+1}$ for $\Delta T_i = \epsilon_{Q_iT_iO}(S_i)$. You can close up the polygon and get $\sum \Delta T_i = 0$ if and only if the last triangle OQ_0Q_{n-1} with T_0, T_{n-1} and S_{n-1} on its sides forms a Ceva configuration, see Figure 12. In this case (15) is obtained by multiplying the Ceva relations for all triangles OQ_iQ_{i+1}. If the last triangle is a Menelaus configuration, (15) gives -1 from multiplying this Menelaus relation with the other Ceva relations. The substitution of a tangency point on Q_iQ_{i+1} by the other one changes the sign in (15) because of (3). $\square$

In order to conclude this article, let us ask if there could be defined an action of the group E_O on polygons inscribed in k_λ, because something similar was done in the real case, see [11]. It can be seen very quickly that this is impossible in the complex case. There is no reasonable group action of E_O on k_λ. The above mentioned points $O, X[1,0,0]$, $Y[0,1,0]$, $Z[0,0,1]$ of order two are the only elements S of E_O for which the translation $\tau_S : T \mapsto S+T$ of E_O descends via $q : E_O \to k_\lambda$ to a well defined mapping τ_S on the conic. From Corollary 13 we have that $q(O * O - T) = q(T)$. If we want τ_S to descend from E_O to a mapping on k_λ, then we must have $q(S + O * O - T) = q(S+T)$ for all T. This is possible only if $S + O * O - T = O * O - (S+T)$, i.e. if $S = -S$ is an element of order two. Only the four translations $\tau_O, \tau_X, \tau_Y, \tau_Z$ can be transferred to the conic k_λ. They give the Frégier involutions through the Frégier points X, Y, Z (see [2, §16.3]), or in the concrete example of $\mathcal{K}$ the reflections across the $x-$, $y-$ and $z-$ axis that form together with the identity a very small group, the Klein four-group.

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On The Relations Between Truncated Cuboctahedron, Truncated Icosidodecahedron and Metrics

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Abstract. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, we give two new metrics to be their spheres an Archimedean solids truncated cuboctahedron and truncated icosidodecahedron.

1. Introduction

A polyhedron is a three-dimensional figure made up of polygons. When discussing polyhedra one will use the terms faces, edges and vertices. Each polygonal part of the polyhedron is called a face. A line segment along which two faces come together is called an edge. A point where several edges and faces come together is called a vertex. That is, a polyhedron is a solid in three dimensions with flat faces, straight edges and vertices.

A regular polyhedron is a polyhedron with congruent faces and identical vertices. There are only five regular convex polyhedra which are called Platonic solids. A convex polyhedron is said to be semiregular if its faces have a similar configuration of nonintersecting regular plane convex polygons of two or more different types about each vertex. These solids are commonly called the Archimedean solids. Archimedes discovered the semiregular convex solids. However, several centuries passed before their rediscovery by the renaissance mathematicians. Finally, Kepler completed the work in 1620 by introducing prisms and antiprisms as well as four regular nonconvex polyhedra, now known as the Kepler–Poinsot polyhedra. Construction of the dual solids of the Archimedean solids was completed in 1865 by Catalan nearly two centuries after Kepler (See [12]). The duals are known as the Catalan solids. The Catalan solids are all convex. They are face-transitive

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when all its faces are the same but not vertex-transitive. Unlike Platonic solids and Archimedean solids, the face of Catalan solids are not regular polygons.

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [13].

Some mathematicians have studied and improved metric geometry in $\mathbb{R}^3$. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [1, 2, 4, 5, 7, 8, 9, 10, 11] the authors give some metrics of which the spheres of the 3-dimensional analytical space equipped with these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space equipped with these metrics are truncated cuboctahedron and truncated icosidodecahedron. Also we give some properties about these metrics.

2. Truncated cuboctahedron and its metric

The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result. For more detailed knowledge, see [3] and [6].

It has been stated in [14], an Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other.

The Archimedean solids are the only 13 polyhedra that are convex, have identical vertices, and their faces are regular polygons (although not equal as in the Platonic solids).

Five Archimedean solids are derived from the Platonic solids by truncating (cutting off the corners) a percentage less than $1/2$.

Two special Archimedean solids can be obtained by full truncating (percentage $1/2$) either of two dual Platonic solids: the Cuboctahedron, which comes from trucating either a Cube, or its dual an Octahedron. And the Icosidodecahedron, which comes from truncating either an Icosahedron, or its dual a Dodecahedron. Hence their "double name".

The next two solids, the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) apparently seem to be derived from truncating the two preceding ones. However, it is apparent from the above discussion on the percentage of truncation that one cannot truncate a solid with unequally shaped faces and end up with regular polygons as faces. Therefore, these two solids need be constructed with another technique. Actually, they can be built from the original platonic solids by a process called expansion. It consists on separating apart the faces of the original polyhedron with spherical symmetry, up to a point where they can be linked through new faces which are regular polygons. The name of the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and of the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) again seem to indicate that they can be derived from truncating the Cuboctahedron and the Icosidodecahedron. But, as reasoned above, this is not possible.

Finally, there are two special solids which have two chiral (specular symmetric) variations: the Snub Cube and the Snub Dodecahedron. These solids can be constructed as an alternation of another Archimedean solid. This process consists on deleting alternated vertices and creating new triangles at the deleted vertices.

One of the Archimedean solids is the truncated cuboctahedron. It has 12 square faces, 8 regular hexagonal faces, 6 regular octagonal faces, 48 vertices and 72 edges. Since each of its faces has point symmetry (equivalently, 180 rotational symmetry), the truncated cuboctahedron is a zonohedron ([15]).

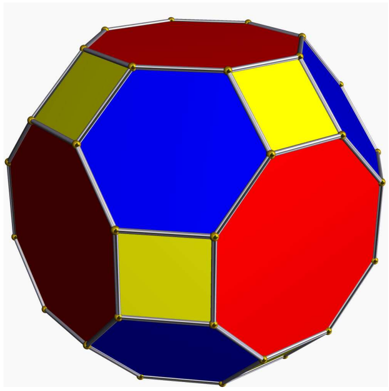
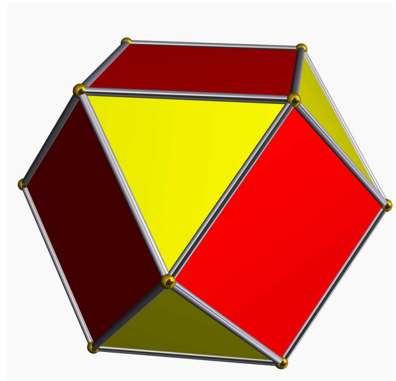
 

Figure 1(a) truncated cuboctahedron   Figure 1(b) Cuboctahedron

We describe the metric that unit sphere is truncated cuboctahedron as following:

Definition 1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{TC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated cuboctahedron distance between P_1 and P_2 is defined by

$$d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l} X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}X_{12},\ \frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},\ X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\},\\ Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}Y_{12},\ \frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12},\ Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\},\\ Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}Z_{12},\ \frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12},\ Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\end{array}\right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to truncated cuboctahedron distance, there are three different paths from P_1 to P_2. These paths are

(i) a line segment which is parallel to a coordinate axis.

(ii) union of three line segments which one is parallel to a coordinate axis and other line segments makes $\arctan(\frac{195}{28})$ angle with another coordinate axes.

(iii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{3}{4})$ angle with another coordinate axis.

Thus truncated cuboctahedron distance between P_1 and P_2 is for (i) Euclidean lengths of line segment, for (ii) $\frac{3-\sqrt{2}}{3}$ times the sum of Euclidean lengths of mentioned three line segments, and for (iii) $\frac{10-\sqrt{2}}{14}$ times the sum of Euclidean lengths of mentioned two line segments.

Figure 2 illustrates truncated cuboctahedron way from P_1 to P_2 if maximum value is $|y_1 - y_2|$, $\frac{3-\sqrt{2}}{3}\left(|y_1 - y_2| + \frac{1}{14}\left(|x_1 - x_2| + |z_1 - z_2|\right)\right)$, $\frac{10-\sqrt{2}}{14}\left(|y_1 - y_2| + \frac{1}{2}|z_1 - z_2|\right)$, or $\frac{10-\sqrt{2}}{14}\left(|y_1 - y_2| + \frac{1}{2}|x_1 - x_2|\right)$.

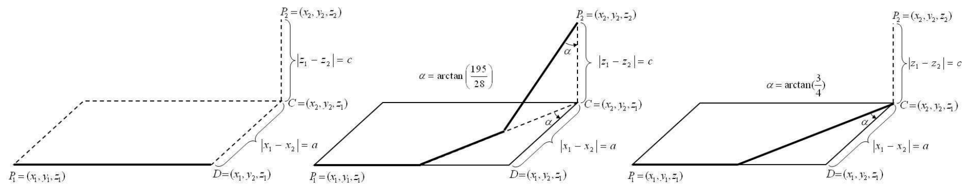


Figure 2: TC way from P_1 to P_2

Lemma 1. *Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|, |y_1-y_2|, |z_1-z_2|$, respectively. Then*

$$d_{TC}(P_1,P_2)\geq \frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12}, \frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}, X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right),$$

$$d_{TC}(P_1,P_2)\geq \frac{3-\sqrt{2}}{3}\left(Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{12}, \frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}, Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\}\right),$$

$$d_{TC}(P_1,P_2)\geq \frac{3-\sqrt{2}}{3}\left(Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{12}, \frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12}, Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\right).$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 2. *The distance function d_{TC} is a metric. Also according to d_{TC}, the unit sphere is a truncated cuboctahedron in $\mathbb{R}^3$.*

Proof. Let $d_{TC}:\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ be the truncated cuboctahedron distance function and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ are distinct three points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|, |y_1-y_2|, |z_1-z_2|$, respectively. To show that d_{TC} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3\in\mathbb{R}^3$.

(M1) $d_{TC}(P_1,P_2)\geq 0$ and $d_{TC}(P_1,P_2)=0$ iff $P_1=P_2$

(M2) $d_{TC}(P_1,P_2)=d_{TC}(P_2,P_1)$

(M3) $d_{TC}(P_1,P_3)\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{TC}(P_1,P_2)\geq 0$. If $d_{TC}(P_1,P_2)=0$ then there are possible three cases. These cases are

(1) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12}, \frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}, X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right)$

(2) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(Y_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{12}, \frac{3\sqrt{2}+2}{14}(X_{12}+Z_{12}),\\ Y_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}, Y_{12}+\frac{3\sqrt{2}+3}{2}X_{12}\end{array}\right\}\right)$

(3) $d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(Z_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{12}, \frac{3\sqrt{2}+2}{14}(X_{12}+Y_{12}),\\ Z_{12}+\frac{3\sqrt{2}+3}{2}X_{12}, Z_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}\end{array}\right\}\right)$.

Case I: If

$$d_{TC}(P_1,P_2)=\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12}, \frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12}, X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right),$$

then

$$\frac{3-\sqrt{2}}{3}\left(X_{12}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}\right)=0$$

$$\Leftrightarrow X_{12}=0 \text{ and } \frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{12},\frac{3\sqrt{2}+2}{14}(Y_{12}+Z_{12}),\\ X_{12}+\frac{3\sqrt{2}+3}{2}Y_{12},X_{12}+\frac{3\sqrt{2}+3}{2}Z_{12}\end{array}\right\}=0$$

$$\Leftrightarrow x_1=x_2,\, y_1=y_2,\, z_1=z_2$$

$$\Leftrightarrow (x_1,y_1,z_1)=(x_2,y_2,z_2)$$

$$\Leftrightarrow P_1=P_2$$

The other cases can be shown by similar way in Case I. Thus we get $d_{TC}(P_1,P_2)=0$ iff $P_1=P_2$.

Since $|x_1-x_2|=|x_2-x_1|$, $|y_1-y_2|$=$|y_2-y_1|$ and $|z_1-z_2|=|z_2-z_1|$, obviously $d_{TC}(P_1,P_2)=d_{TC}(P_2,P_1)$. That is, d_{TC} is symmetric.

$X_{13},Y_{13},Z_{13},X_{23},Y_{23},Z_{23}$ denote $|x_1-x_3|\,,|y_1-y_3|\,,|z_1-z_3|\,,|x_2-x_3|\,,$ $|y_2-y_3|\,,|z_2-z_3|$, respectively.

$$\begin{aligned}
&d_{TC}(P_1,P_3)\\
&=\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}
X_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}X_{13},\frac{3\sqrt{2}+2}{14}(Y_{13}+Z_{13}),\\ X_{13}+\frac{3\sqrt{2}+3}{2}Y_{13},X_{13}+\frac{3\sqrt{2}+3}{2}Z_{13}\end{array}\right\},\\
Y_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Y_{13},\frac{3\sqrt{2}+2}{14}(X_{13}+Z_{13}),\\ Y_{13}+\frac{3\sqrt{2}+3}{2}Z_{13},Y_{13}+\frac{3\sqrt{2}+3}{2}X_{13}\end{array}\right\},\\
Z_{13}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}Z_{13},\frac{3\sqrt{2}+2}{14}(X_{13}+Y_{13}),\\ Z_{13}+\frac{3\sqrt{2}+3}{2}X_{13},Z_{13}+\frac{3\sqrt{2}+3}{2}Y_{13}\end{array}\right\}
\end{array}\right\}\\
&\leq\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}
X_{12}+X_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}\left(X_{12}+X_{23}\right),\\ \frac{3\sqrt{2}+2}{14}(Y_{12}+Y_{23}+Z_{12}+Z_{23}),\\ X_{12}+X_{23}+\frac{3\sqrt{2}+3}{2}\left(Y_{12}+Y_{23}\right),\\ X_{12}+X_{23}+\frac{3\sqrt{2}+3}{2}\left(Z_{12}+Z_{23}\right)\end{array}\right\},\\
Y_{12}+Y_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}\left(Y_{12}+Y_{23}\right),\\ \frac{3\sqrt{2}+2}{14}(X_{12}+X_{23}+Z_{12}+Z_{23}),\\ Y_{12}+Y_{23}+\frac{3\sqrt{2}+3}{2}\left(Z_{12}+Z_{23}\right),\\ Y_{12}+Y_{23}+\frac{3\sqrt{2}+3}{2}\left(X_{12}+X_{23}\right)\end{array}\right\},\\
Z_{12}+Z_{23}+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{c}\frac{22+12\sqrt{2}}{7}\left(Z_{12}+Z_{23}\right),\\ \frac{3\sqrt{2}+2}{14}(X_{12}+X_{23}+Y_{12}+Y_{23}),\\ Z_{12}+Z_{23}+\frac{3\sqrt{2}+3}{2}\left(X_{12}+X_{23}\right),\\ Z_{12}+Z_{23}+\frac{3\sqrt{2}+3}{2}\left(Y_{12}+Y_{23}\right)\end{array}\right\}
\end{array}\right\}\\
&=I.
\end{aligned}$$

Therefore one can easily find that $I\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$ from Lemma 1. So $d_{TC}(P_1,P_3)\leq d_{TC}(P_1,P_2)+d_{TC}(P_2,P_3)$. Consequently, truncated cuboctahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that truncated cuboctahedron distance is 1 from $O = (0,0,0)$ is $S_{TC} =$

$$\left\{ (x,y,z) \colon \frac{3-\sqrt{2}}{3} \max \left\{ \begin{array}{l} |x| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|x|, \frac{3\sqrt{2}+2}{14}(|y|+|z|), \\ |x| + \frac{3\sqrt{2}+3}{2}|y|, |x| + \frac{3\sqrt{2}+3}{2}|z| \end{array} \right\}, \\ |y| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|y|, \frac{3\sqrt{2}+2}{14}(|x|+|z|), \\ |y| + \frac{3\sqrt{2}+3}{2}|z|, |y| + \frac{3\sqrt{2}+3}{2}|x| \end{array} \right\}, \\ |z| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|z|, \frac{3\sqrt{2}+2}{14}(|x|+|y|), \\ |z| + \frac{3\sqrt{2}+3}{2}|x|, |z| + \frac{3\sqrt{2}+3}{2}|y| \end{array} \right\} \end{array} \right\} = 1 \right\}.$$

Thus the graph of S_{TC} is as in Figure 3:

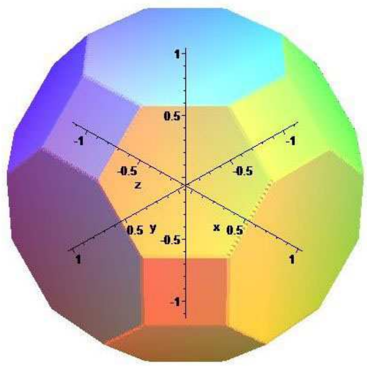


Figure 3 The unit sphere in terms of d_{TC}: Truncated cuboctahedron

□

Corollary 3. *The equation of the truncated cuboctahedron with center* (x_0, y_0, z_0) *and radius* r *is*

$$\frac{3-\sqrt{2}}{3} \max \left\{ \begin{array}{l} |x-x_0| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|x-x_0|, \frac{3\sqrt{2}+2}{14}(|y-y_0|+|z-z_0|), \\ |x-x_0| + \frac{3\sqrt{2}+3}{2}|y-y_0|, |x-x_0| + \frac{3\sqrt{2}+3}{2}|z-z_0| \end{array} \right\}, \\ |y-y_0| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|y-y_0|, \frac{3\sqrt{2}+2}{14}(|x-x_0|+|z-z_0|), \\ |y-y_0| + \frac{3\sqrt{2}+3}{2}|z-z_0|, |y-y_0| + \frac{3\sqrt{2}+3}{2}|x-x_0| \end{array} \right\}, \\ |z-z_0| + \frac{3\sqrt{2}-2}{14} \max \left\{ \begin{array}{l} \frac{22+12\sqrt{2}}{7}|z-z_0|, \frac{3\sqrt{2}+2}{14}(|x-x_0|+|y-y_0|), \\ |z-z_0| + \frac{3\sqrt{2}+3}{2}|x-x_0|, |z-z_0| + \frac{3\sqrt{2}+3}{2}|y-y_0| \end{array} \right\} \end{array} \right\} = r$$

which is a polyhedron which has 26 *faces and* 48 *vertices. Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all permutations of the three axis components and all possible* $+/-$ *sign components of the points* $(\frac{\sqrt{2}+3}{7}r, \frac{2\sqrt{2}-1}{7}r, r)$.

Lemma 4. *Let* l *be the line through the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p, q, r), *then*

$$d_{TC}(P_1, P_2) = \mu(P_1 P_2) d_E(P_1, P_2)$$

where

$$\mu(P_1P_2)=\frac{\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}|p|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|p|,\frac{3\sqrt{2}+2}{14}(|q|+|r|),\\|p|+\frac{3\sqrt{2}+3}{2}|q|,|p|+\frac{3\sqrt{2}+3}{2}|r|\end{array}\right\},\\|q|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|q|,\frac{3\sqrt{2}+2}{14}(|p|+|r|),\\|q|+\frac{3\sqrt{2}+3}{2}|r|,|q|+\frac{3\sqrt{2}+3}{2}|p|\end{array}\right\},\\|r|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|r|,\frac{3\sqrt{2}+2}{14}(|p|+|q|),\\|r|+\frac{3\sqrt{2}+3}{2}|p|,|r|+\frac{3\sqrt{2}+3}{2}|q|\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $r\in\mathbb{R}$. Thus,$d_{TC}(P_1,P_2)$ is equal to

$$|\lambda|\left(\frac{3-\sqrt{2}}{3}\max\left\{\begin{array}{l}|p|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|p|,\frac{3\sqrt{2}+2}{14}(|q|+|r|),\\|p|+\frac{3\sqrt{2}+3}{2}|q|,|p|+\frac{3\sqrt{2}+3}{2}|r|\end{array}\right\},\\|q|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|q|,\frac{3\sqrt{2}+2}{14}(|p|+|r|),\\|q|+\frac{3\sqrt{2}+3}{2}|r|,|q|+\frac{3\sqrt{2}+3}{2}|p|\end{array}\right\},\\|r|+\frac{3\sqrt{2}-2}{14}\max\left\{\begin{array}{l}\frac{22+12\sqrt{2}}{7}|r|,\frac{3\sqrt{2}+2}{14}(|p|+|q|),\\|r|+\frac{3\sqrt{2}+3}{2}|p|,|r|+\frac{3\sqrt{2}+3}{2}|q|\end{array}\right\}\end{array}\right\}\right)$$

and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result. □

The above lemma says that d_{TC}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 5. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1,X)=d_E(P_2,X)$ if and only if $d_{TC}(P_1,X)=d_{TC}(P_2,X)$.*

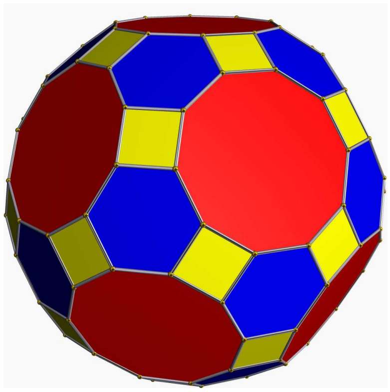
Corollary 6. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TC}(X,P_1)\;/\;d_{TC}(X,P_2)=d_E(X,P_1)\;/\;d_E(X,P_2)\;.$$

That is, the ratios of the Euclidean and $d_{TC}-$distances along a line are the same.

3. Truncated icosadodecahedron and its metric

The truncated icosidodecahedron is an Archimedean solid, one of thirteen convex isogonal nonprismatic solids constructed by two or more types of regular polygon faces. It has 30 square faces, 20 regular hexagonal faces, 12 regular decagonal faces, 120 vertices and 180 edges more than any other convex nonprismatic uniform polyhedron. Since each of its faces has point symmetry (equivalently, 180 rotational symmetry), the truncated icosidodecahedron is a zonohedron ([16]; see Figure 4(a)).



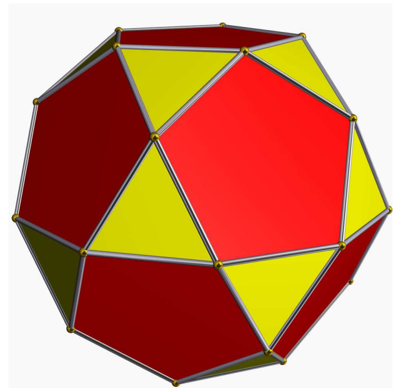


Figure 4(a)Truncated icosidodecahedron Figure 4(b) Icosidodecahedron

We describe the metric that unit sphere is truncated icosidodecahedron as following:

Definition 2. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{TI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated icosidodecahedron distance between P_1 and P_2 is defined by
$d_{TI}(P_1, P_2) =$

$$\max \left\{ \begin{array}{l} \frac{4-\sqrt{5}}{3} X_{12} + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19} X_{12}, \frac{15\sqrt{5}+17}{19} \left(Y_{12}+Z_{12}\right), \frac{\sqrt{5}+29}{19} \left(X_{12}+Z_{12}\right), \\ X_{12} + \frac{21\sqrt{5}+39}{38} Z_{12} + \frac{9\sqrt{5}+33}{38} Y_{12}, \frac{18\sqrt{5}+104}{95} X_{12} + \frac{48\sqrt{5}+138}{95} Y_{12} \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3} Y_{12} + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19} Y_{12}, \frac{15\sqrt{5}+17}{19} \left(X_{12}+Z_{12}\right), \frac{\sqrt{5}+29}{19} \left(X_{12}+Y_{12}\right), \\ Y_{12} + \frac{21\sqrt{5}+39}{38} X_{12} + \frac{9\sqrt{5}+33}{38} Z_{12}, \frac{18\sqrt{5}+104}{95} Y_{12} + \frac{48\sqrt{5}+138}{95} Z_{12} \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3} Z_{12} + \frac{7\sqrt{5}-13}{12} \max \left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19} Z_{12}, \frac{15\sqrt{5}+17}{19} \left(X_{12}+Y_{12}\right), \frac{\sqrt{5}+29}{19} \left(Y_{12}+Z_{12}\right), \\ Z_{12} + \frac{21\sqrt{5}+39}{38} Y_{12} + \frac{9\sqrt{5}+33}{38} X_{12}, \frac{18\sqrt{5}+104}{95} Z_{12} + \frac{48\sqrt{5}+138}{95} X_{12} \end{array} \right\} \end{array} \right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to truncated icosidodecahedron distance, there are five different paths from P_1 to P_2. These paths are

(i) a line segment which is parallel to a coordinate axis,

(ii) union of three line segments each of which is parallel to a coordinate axis,

(iii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{\sqrt{5}}{2})$ angle with another coordinate axis,

(iv) union of three line segments which two of them are parallel to a coordinate axis and other line segment makes $\arctan(\frac{937-215\sqrt{5}}{1824})$ angle with other coordinate axis,

(v) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{1}{2})$ angle with another coordinate axis.

Thus truncated cuboctahedron distance between P_1 and P_2 is for (i) Euclidean length of line segment, for (ii) $\frac{4-\sqrt{5}}{3}$ times the sum of Euclidean lengths of mentioned three line segments, for (iii) $\frac{3\sqrt{5}-1}{6}$ times the sum of Euclidean lengths of two line segments, for (iv) $\frac{\sqrt{5}+1}{4}$ times the sum of Euclidean lengths of three line

segments, and for (v) $\frac{\sqrt{5}+7}{10}$ times the sum of Euclidean lengths of two line segments. Figure 5 shows that the path between P_1 and P_2 in case of the maximum is
$|y_1 - y_2|$,
$\frac{4-\sqrt{5}}{3}\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right)$,
$\frac{3\sqrt{5}-1}{6}\left(|y_1 - y_2| + \frac{3-\sqrt{5}}{2}|x_1 - x_2|\right)$,
$\frac{\sqrt{5}+1}{4}\left(|y_1 - y_2| + |z_1 - z_2| + \frac{12\left(\sqrt{5}-1\right)}{19}|x_1 - x_2|\right)$, or
$\frac{\sqrt{5}+7}{10}\left(|y_1 - y_2| + \frac{\sqrt{5}-1}{2}|z_1 - z_2|\right)$.

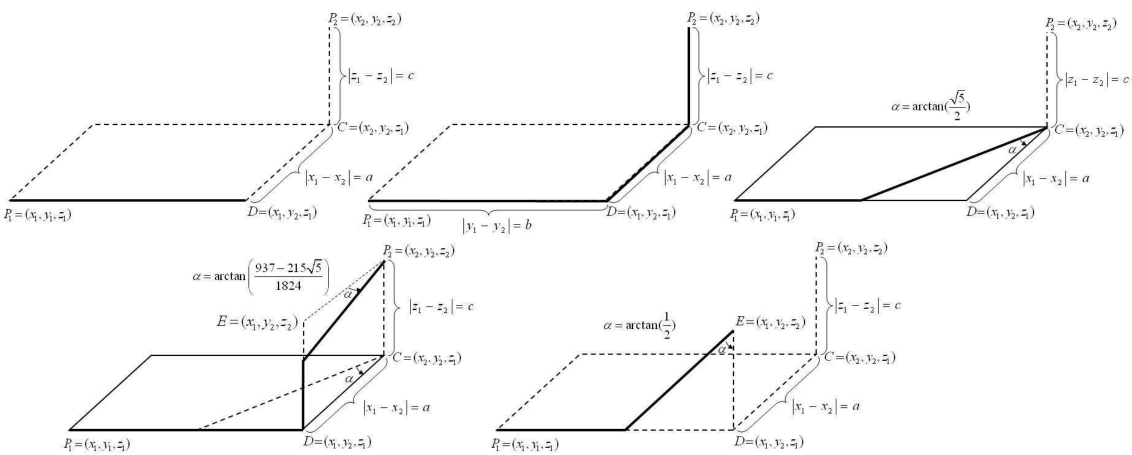


Figure 5: TI way from P_1 to P_2

Lemma 7. *Let* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *be distinct two points in* $\mathbb{R}^3$*. Then*

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}X_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}X_{12}, \frac{15\sqrt{5}+17}{19}\left(Y_{12}+Z_{12}\right), \frac{\sqrt{5}+29}{19}\left(X_{12}+Z_{12}\right), \\ X_{12} + \frac{21\sqrt{5}+39}{38}Z_{12} + \frac{9\sqrt{5}+33}{38}Y_{12}, \frac{18\sqrt{5}+104}{95}X_{12} + \frac{48\sqrt{5}+138}{95}Y_{12} \end{array}\right\}$$

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}Y_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Y_{12}, \frac{15\sqrt{5}+17}{19}\left(X_{12}+Z_{12}\right), \frac{\sqrt{5}+29}{19}\left(X_{12}+Y_{12}\right), \\ Y_{12} + \frac{21\sqrt{5}+39}{38}X_{12} + \frac{9\sqrt{5}+33}{38}Z_{12}, \frac{18\sqrt{5}+104}{95}Y_{12} + \frac{48\sqrt{5}+138}{95}Z_{12} \end{array}\right\}$$

$$d_{TI}(P_1,P_2) \geq \frac{4-\sqrt{5}}{3}Z_{12} + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{6\sqrt{5}+22}{19}Z_{12}, \frac{15\sqrt{5}+17}{19}\left(X_{12}+Y_{12}\right), \frac{\sqrt{5}+29}{19}\left(Y_{12}+Z_{12}\right), \\ Z_{12} + \frac{21\sqrt{5}+39}{38}Y_{12} + \frac{9\sqrt{5}+33}{38}X_{12}, \frac{18\sqrt{5}+104}{95}Z_{12} + \frac{48\sqrt{5}+138}{95}X_{12} \end{array}\right\}.$$

where $X_{12}=|x_1 - x_2|$, $Y_{12}=|y_1 - y_2|$, $Z_{12}=|z_1 - z_2|$.

Proof. Proof is trivial by the definition of maximum function. □

Theorem 8. *The distance function* d_{TI} *is a metric. Also according to* d_{TI}*, unit sphere is a truncated icosidodecahedron in* $\mathbb{R}^3$.

Proof. One can easily show that the truncated icosidodecahedron distance function satisfies the metric axioms by similar way in Theorem 2.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated icosidodecahedron distance is 1 from $O = (0, 0, 0)$ is $S_{TI} =$

$$\left\{ (x,y,z) : \max \left\{ \begin{array}{l} \frac{4-\sqrt{5}}{3}|x| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|y|+|z|), \frac{\sqrt{5}+29}{19}(|x|+|z|), \\ \frac{6\sqrt{5}+22}{19}|x|, |x| + \frac{21\sqrt{5}+39}{38}|z| + \frac{9\sqrt{5}+33}{38}|y|, \\ \frac{18\sqrt{5}+104}{95}|x| + \frac{48\sqrt{5}+138}{95}|y| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|y| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|x|+|z|), \frac{\sqrt{5}+29}{19}(|x|+|y|), \\ \frac{6\sqrt{5}+22}{19}|y|, |y| + \frac{21\sqrt{5}+39}{38}|x| + \frac{9\sqrt{5}+33}{38}|z|, \\ \frac{18\sqrt{5}+104}{95}|y| + \frac{48\sqrt{5}+138}{95}|z| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|z| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{15\sqrt{5}+17}{19}(|x|+|y|), \frac{\sqrt{5}+29}{19}(|y|+|z|), \\ \frac{6\sqrt{5}+22}{19}|z|, |z| + \frac{21\sqrt{5}+39}{38}|y| + \frac{9\sqrt{5}+33}{38}|x|, \\ \frac{18\sqrt{5}+104}{95}|z| + \frac{48\sqrt{5}+138}{95}|x| \end{array} \right\} \end{array} \right\} = 1 \right\}.$$

Thus the graph of S_{TI} is as in Figure 6:

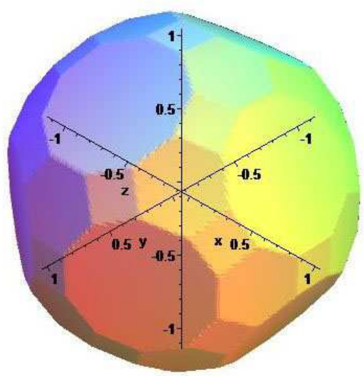


Figure 6 The unit sphere in terms of d_{TI}: Truncated icosidodecahedron

□

Corollary 9. *The equation of the truncated icosidodecahedron with center* (x_0, y_0, z_0) *and radius* r *is*

$$\max \left\{ \begin{array}{l} \frac{4-\sqrt{5}}{3}|x-x_0| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|x-x_0|, \frac{15\sqrt{5}+17}{19}(|y-y_0|+|z-z_0|), \\ \frac{\sqrt{5}+29}{19}(|x-x_0|+|z-z_0|), \\ |x-x_0| + \frac{21\sqrt{5}+39}{38}|z-z_0| + \frac{9\sqrt{5}+33}{38}|y-y_0|, \\ \frac{18\sqrt{5}+104}{95}|x-x_0| + \frac{48\sqrt{5}+138}{95}|y-y_0| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|y-y_0| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|y-y_0|, \frac{15\sqrt{5}+17}{19}(|x-x_0|+|z-z_0|), \\ \frac{\sqrt{5}+29}{19}(|x-x_0|+|y-y_0|), \\ |y-y_0| + \frac{21\sqrt{5}+39}{38}|x-x_0| + \frac{9\sqrt{5}+33}{38}|z-z_0|, \\ \frac{18\sqrt{5}+104}{95}|y-y_0| + \frac{48\sqrt{5}+138}{95}|z-z_0| \end{array} \right\}, \\ \frac{4-\sqrt{5}}{3}|z-z_0| + \frac{7\sqrt{5}-13}{12}\max\left\{ \begin{array}{l} \frac{6\sqrt{5}+22}{19}|z-z_0|, \frac{15\sqrt{5}+17}{19}(|x-x_0|+|y-y_0|), \\ \frac{\sqrt{5}+29}{19}(|y-y_0|+|z-z_0|), \\ |z-z_0| + \frac{21\sqrt{5}+39}{38}|y-y_0| + \frac{9\sqrt{5}+33}{38}|x-x_0|, \\ \frac{18\sqrt{5}+104}{95}|z-z_0| + \frac{48\sqrt{5}+138}{95}|x-x_0| \end{array} \right\} \end{array} \right\} = r$$

which is a polyhedron which has 62 *faces and* 120 *vertices. Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all possible* $+/-$ *sign components of the points*

$$\begin{array}{lll}
\left(\frac{2\sqrt{5}-3}{11}r, \frac{2\sqrt{5}-3}{11}r, r\right), & \left(r, \frac{2\sqrt{5}-3}{11}r, \frac{2\sqrt{5}-3}{11}r\right), & \left(\frac{2\sqrt{5}-3}{11}r, r, \frac{2\sqrt{5}-3}{11}r\right), \\
\left(\frac{4\sqrt{5}-6}{11}r, \frac{3\sqrt{5}+1}{22}r, \frac{5\sqrt{5}+9}{22}r\right), & \left(\frac{5\sqrt{5}+9}{22}r, \frac{4\sqrt{5}-6}{11}r, \frac{3\sqrt{5}+1}{22}r\right), & \left(\frac{3\sqrt{5}+1}{22}r, \frac{5\sqrt{5}+9}{22}r, \frac{4\sqrt{5}-6}{11}r\right), \\
\left(\frac{2\sqrt{5}-3}{11}r, \frac{\sqrt{5}+4}{11}r, \frac{5\sqrt{5}-2}{11}r\right), & \left(\frac{5\sqrt{5}-2}{11}r, \frac{2\sqrt{5}-3}{11}r, \frac{\sqrt{5}+4}{11}r\right), & \left(\frac{\sqrt{5}+4}{11}r, \frac{5\sqrt{5}-2}{11}r, \frac{2\sqrt{5}-3}{11}r\right), \\
\left(\frac{7\sqrt{5}-5}{22}r, \frac{7-\sqrt{5}}{11}r, \frac{\sqrt{5}+15}{22}r\right), & \left(\frac{\sqrt{5}+15}{22}r, \frac{7\sqrt{5}-5}{22}r, \frac{7-\sqrt{5}}{11}r\right), & \left(\frac{7-\sqrt{5}}{11}r, \frac{\sqrt{5}+15}{22}r, \frac{7\sqrt{5}-5}{22}r\right), \\
\left(\frac{3\sqrt{5}+1}{22}r, \frac{21-3\sqrt{5}}{22}r, \frac{3\sqrt{5}+1}{11}r\right), & \left(\frac{3\sqrt{5}+1}{11}r, \frac{3\sqrt{5}+1}{22}r, \frac{21-3\sqrt{5}}{22}r\right), & \left(\frac{21-3\sqrt{5}}{22}r, \frac{3\sqrt{5}+1}{11}r, \frac{3\sqrt{5}+1}{22}r\right).
\end{array}$$

Lemma 10. *Let* l *be the line through the points* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p, q, r)*, then*

$$d_{TI}(P_1, P_2) = \mu(P_1P_2) d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\left\{\begin{array}{l}
\frac{4-\sqrt{5}}{3}|p| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|q|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|r|), \\ \frac{6\sqrt{5}+22}{19}|p|, |p| + \frac{21\sqrt{5}+39}{38}|r| + \frac{9\sqrt{5}+33}{38}|q|, \\ \frac{18\sqrt{5}+104}{95}|p| + \frac{48\sqrt{5}+138}{95}|q| \end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|q| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|p|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|q|), \\ \frac{6\sqrt{5}+22}{19}|q|, |q| + \frac{21\sqrt{5}+39}{38}|p| + \frac{9\sqrt{5}+33}{38}|r|, \\ \frac{18\sqrt{5}+104}{95}|q| + \frac{48\sqrt{5}+138}{95}|r| \end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|r| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|p|+|q|), \frac{\sqrt{5}+29}{19}(|q|+|r|), \\ \frac{6\sqrt{5}+22}{19}|r|, |r| + \frac{21\sqrt{5}+39}{38}|q| + \frac{9\sqrt{5}+33}{38}|p|, \\ \frac{18\sqrt{5}+104}{95}|r| + \frac{48\sqrt{5}+138}{95}|p| \end{array}\right\}
\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $r \in \mathbb{R}$. Thus,

$$d_{TI}(P_1, P_2) = |\lambda| \max\left\{\begin{array}{l}
\frac{4-\sqrt{5}}{3}|p| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|q|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|r|), \\ \frac{6\sqrt{5}+22}{19}|p|, |p| + \frac{21\sqrt{5}+39}{38}|r| + \frac{9\sqrt{5}+33}{38}|q|, \\ \frac{18\sqrt{5}+104}{95}|p| + \frac{48\sqrt{5}+138}{95}|q| \end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|q| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|p|+|r|), \frac{\sqrt{5}+29}{19}(|p|+|q|), \\ \frac{6\sqrt{5}+22}{19}|q|, |q| + \frac{21\sqrt{5}+39}{38}|p| + \frac{9\sqrt{5}+33}{38}|r|, \\ \frac{18\sqrt{5}+104}{95}|q| + \frac{48\sqrt{5}+138}{95}|r| \end{array}\right\}, \\
\frac{4-\sqrt{5}}{3}|r| + \frac{7\sqrt{5}-13}{12}\max\left\{\begin{array}{l} \frac{15\sqrt{5}+17}{19}(|p|+|q|), \frac{\sqrt{5}+29}{19}(|q|+|r|), \\ \frac{6\sqrt{5}+22}{19}|r|, |r| + \frac{21\sqrt{5}+39}{38}|q| + \frac{9\sqrt{5}+33}{38}|p|, \\ \frac{18\sqrt{5}+104}{95}|r| + \frac{48\sqrt{5}+138}{95}|p| \end{array}\right\}
\end{array}\right\}$$

and $d_E(A, B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The above lemma says that d_{TI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 11. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TI}(P_1, X) = d_{TI}(P_2, X)$.*

Corollary 12. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TI}(X, P_1) \;/\; d_{TI}(X, P_2) = d_E(X, P_1) \;/\; d_E(X, P_2) \; .$$

That is, the ratios of the Euclidean and $d_{TI}-$distances along a line are the same.

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Another Construction of the Golden Ratio in an Isosceles Triangle

Tran Quang Hung

Abstract. Given an isosceles triangle, consider the circle with its top vertex as center and passing through the other two vertices. Using a symmedian, we construct a chord of the circle parallel to the base of the triangle to intersect the equal sides so that for each segment formed by three contiguous points is divided in the golden ratio.

Given an isosceles triangle ABC with $AB = AC$, construct
(i) the circle (ω) with center A and passing through B, C,
(ii) the symmedian BE,
(iii) the circle (K) passing through C, E, and tangent to AB at F,
(iv) the parallel line from F to BC to intersect CA at G, and the circle (ω) at M and N (see Figure 1).

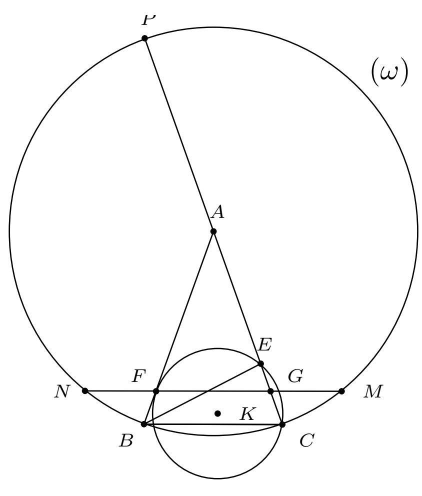

Figure 1.

Proposition. G divides FM in the golden ratio.

Proof. We show that

$$GM \cdot FM = FG^2. \tag{1}$$

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Extend CA to intersect the circle (ω) at P, so that CP is a diameter of the circle. By symmetry and the intersecting chords theorem,

$$\begin{aligned} FM \cdot GM &= NG \cdot GM = PG \cdot GC \\ &= (PA + AG)(AC - AG) \\ &= (AC + AG)(AC - AG) \\ &= AC^2 - AG^2 \\ &= AC^2 - AF^2 \\ &= AC^2 - AE \cdot AC. \end{aligned}$$

Since BE is a symmedian of triangle ABC,

$$\frac{AE}{AC} = \frac{AB^2}{AB^2 + BC^2}$$

(see [1, Theorem 561]). It follows that

$$\begin{aligned} FM \cdot GM &= AC^2 \left(1 - \frac{AE}{AC}\right) = AC^2 \left(1 - \frac{AB^2}{AB^2 + BC^2}\right) \\ &= \frac{AC^2 \cdot BC^2}{AB^2 + BC^2}. \end{aligned} \tag{2}$$

On the other hand, $\dfrac{AF}{AB} = \dfrac{FG}{BC}$ by Thales' theorem. From this,

$$\begin{aligned} FG^2 &= BC^2 \cdot \frac{AF^2}{AB^2} = BC^2 \cdot \frac{AE \cdot AC}{AB^2} = BC^2 \cdot \frac{AE \cdot AC}{AC^2} \\ &= BC^2 \cdot \frac{AE}{AC} = BC^2 \cdot \frac{AB^2}{AB^2 + BC^2} \\ &= \frac{AC^2 \cdot BC^2}{AB^2 + BC^2}. \end{aligned} \tag{3}$$

Comparing (2) and (3), we obtain (1). This proves that G divides FM in the golden ratio. $\square$

By symmetry, F also divides GN in the golden ratio.

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On the Feuerbach Triangle

Dasari Naga Vijay Krishna

Abstract. We study the relations among the Feuerbach points of a triangle and the feet of the angle bisectors. From these points we construct 6 points, pairwise on the three sides of the triangle, which lie on a conic. In addition, we also establish some collinearity and perspectivity results.

1. Perspectivity of Feuerbach and incentral triangles

In this note we prove some interesting properties of the Feuerbach points of a triangle. Recall that by the famous Feuerbach theorem, the nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles. The points of tangency are the Feuerbach points. If a triangle ABC has side lengths $BC = a$, $CA = b$, $AB = c$, its incenter and the excenters are the points

$$I = (a : b : c), \quad I_a = (-a : b : c),\ I_b = (a : -b : c),\ I_c = (a : b : -c)$$

in homogeneous barycentric coordinates with reference to ABC. On the other hand, the nine-point center is the point

$$N = (a^2(b^2{+}c^2){-}(b^2{-}c^2)^2 :\ b^2(C^2{+}a^2){-}(c^2{-}a^2)^2 :\ c^2(a^2{+}b^2){-}(a^2{-}b^2)^2).$$

From the formulas for the circumradius R and the inradius r

$$R = \frac{abc}{4\Delta} \quad \text{and} \quad r = \frac{2\Delta}{a+b+c}$$

in terms of a, b, c, and the area Δ of the triangle, we obtain the coordinates of the Feuerbach points.

Proposition 1. *The nine-point circle is tangent to the incircle at*

$$F_{\mathrm{e}} = ((b-c)^2(b+c-a) :\ (c-a)^2(c+a-b) :\ (a-b)^2(a+b-c)),$$

and to the A-, B-, C-excircles respectively at

$$\begin{aligned}
F_a &= \ (-(b-c)^2(a+b+c) :\ (c+a)^2(a+b-c) :\ (a+b)^2(c+a-b)),\\
F_b &= \ ((b+c)^2(a+b-c) :\ -(c-a)^2(a+b+c) :\ (a+b)^2(b+c-a)),\\
F_c &= \ ((b+c)^2(c+a-b) :\ (c+a)^2(b+c-a) :\ -(a-b)^2(a+b+c)).
\end{aligned}$$

We call $F_aF_bF_c$ the *Feuerbach triangle*.

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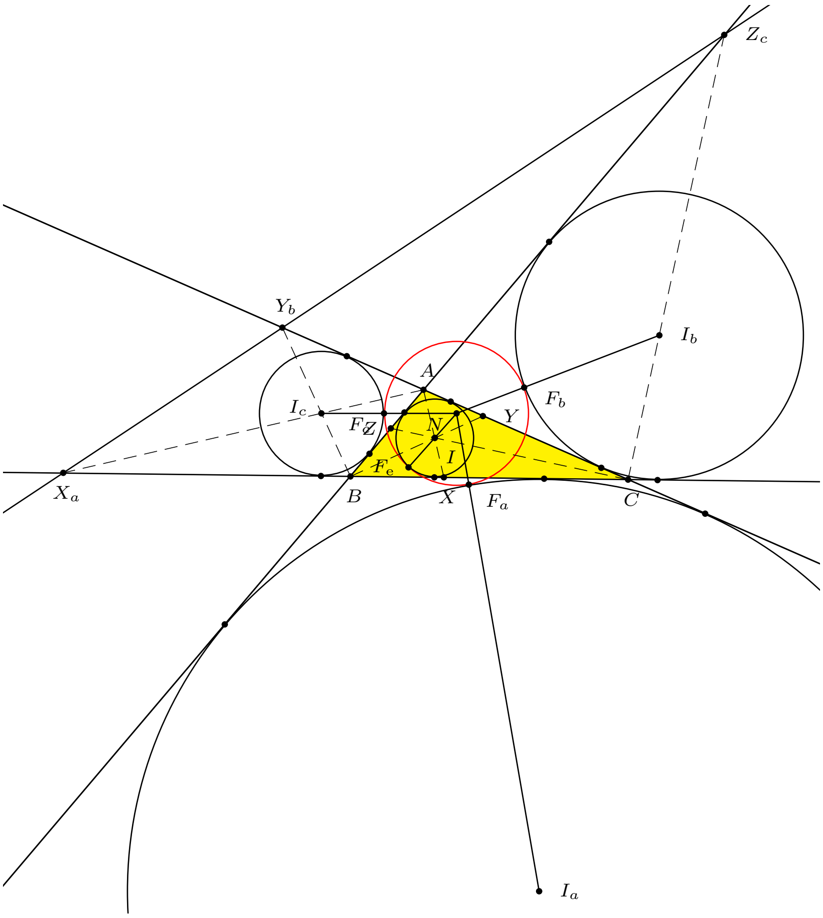

Figure 1

We also consider the intersections of the angle bisectors with the sides. Let the internal and external bisectors of angle A intersect the line BC at X and X_a respectively. Similarly define Y, Y_b, Z, Z_c as the intersections of the internal and external bisectors of angles B and C with their opposite sides (see Figure 1). In homogeneous barycentric coordinates,

$X = (0 : b : c)$	$X_a = (0 : b : -c)$
$Y = (a : 0 : c)$	$Y_b = (-a : 0 : c)$
$Z = (a : b : c)$	$Z_c = (a : -b : 0)$

We call XYZ the *incentral triangle*.

Line	Equation
YZ	$-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$
ZX	$\frac{x}{a}-\frac{y}{b}+\frac{z}{c}=0$
XY	$\frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$
$F_{\mathrm{e}}F_a$	$(b^2-bc+c^2-a^2)x+c(b-c)y-b(b-c)z=0$
$F_{\mathrm{e}}F_b$	$-c(c-a)x+(c^2-ca+a^2-b^2)y+a(c-a)z=0$
$F_{\mathrm{e}}F_c$	$b(a-b)x-a(a-b)y+(a^2-ab+b^2-c^2)z=0$
F_bF_c	$-(b^2+bc+c^2-a^2)x+c(b+c)y+b(b+c)z=0$
F_cF_a	$c(c+a)x-(c^2+ca+a^2-b^2)y+a(c+a)z-0$
F_aF_b	$b(a+b)x+a(a+b)y-(a^2+ab+b^2-c^2)z=0$

Table 1. Equations of lines.

From the equations of the lines in Table 1, it is clear that

The line	YZ	ZX	XY	$F_{\mathrm{e}}F_a$	$F_{\mathrm{e}}F_b$	$F_{\mathrm{e}}F_c$	F_bF_c	F_cF_a	F_aF_b
contains	X_a	Y_b	Z_c	X	Y	Z	X_a	Y_b	Z_c

Table 2: Incidence of points and lines.

Note that X_a, Y_b, Z_c are collinear on $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$, the trilinear polar of $I=(a:b:c)$.

Proposition 2. *The triangles $F_aF_bF_c$ and XYZ are perspective at F_{e} and has perspectrix the trilinear polar of $I=(a:b:c)$.*

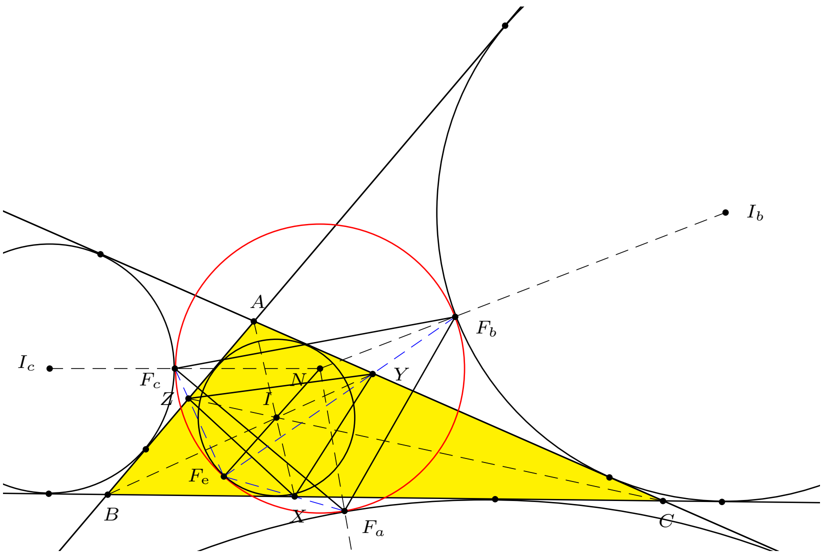

Figure 2

Proof. From Table 2, the lines F_aX, F_bY, F_cZ are concurrent at the Feuerbach point F_e. This means that the triangles $F_aF_bF_c$ and XYZ are perspective at F_e (see Figure 2).

By Desargues' theorem, the two triangles $F_aF_bF_c$ and XYZ are also line perspective. This means that the three points

$$F_bF_c \cap YZ, \qquad F_aF_c \cap XZ, \qquad F_aF_b \cap XY$$

are collinear. From Table 2, these are respectively the points X_a, Y_b, Z_c, they are collinear on the trilinear polar of I. This is the perspectrix of the triangles. □

Proposition 3. *The following pairs of triangles are perspective.*

	Triangle	*Triangle*	*Perspector*	*Perspectrix*
(i)	$F_eF_cF_b$	XY_bZ_c	F_a	YZ
(ii)	$F_cF_eF_a$	X_aYZ_c	F_b	ZX
(iii)	$F_bF_aF_e$	X_aY_bZ	F_c	XY

Proof. We shall (i) only.
From Table 2, it is clear that the lines F_eX, F_cY_b, F_bZ_c concur at F_a. Also,

$$F_cF_b \cap Y_bZ_c = X_a, \quad F_bF_e \cap Z_cX = Y, \quad F_eF_c \cap XY_b = Z.$$

This shows that the line YZ is the perspectrix of the triangles $F_eF_cF_b$ and XY_bZ_c. □

2. Similarity of the Feuerbach and incentral triangles

Proposition 4. *Triangles $F_aF_bF_c$ and XYZ are similar.*

Proof. We show that

$$\frac{F_bF_c}{YZ} = \frac{F_cF_a}{ZX} = \frac{F_aF_b}{XY}. \tag{1}$$

For the feet Y, Z of the bisectors of angles B, C, we have, by applying the law of cosines to triangle AYZ,

$$\begin{aligned}
YZ^2 =&\ AY^2 + AZ^2 - 2\cdot AY\cdot AZ\cos A \\
=&\ \left(\frac{bc}{c+a}\right)^2 + \left(\frac{cb}{b+a}\right)^2 - 2\cdot\frac{bc}{c+a}\cdot\frac{cb}{b+a}\cdot\frac{b^2+c^2-a^2}{2bc} \\
=&\ \frac{bc}{(c+a)^2(a+b)^2}\big(bc((a+b)^2+(c+a)^2)-(c+a)(a+b)(b^2+c^2-a^2)\big) \\
=&\ \frac{bc}{(c+a)^2(a+b)^2}\big(bc(2a(a+b+c)+(b^2+c^2))-(c+a)(a+b)(b^2+c^2) \\
&\quad +a^2(c+a)(a+b))\big) \\
=&\ \frac{bc}{(c+a)^2(a+b)^2}\big(2abc(a+b+c)-a(a+b+c)(b^2+c^2) \\
&\quad +a^3(a+b+c)+a^2bc)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{abc}{(c+a)^2(a+b)^2}\left((a+b+c)\left(2bc-(b^2+c^2)+a^2\right)+abc\right)\\
&= \frac{abc}{(c+a)^2(a+b)^2}\left((a+b+c)(a-b+c)(a+b-c)+abc\right)\\
&= \frac{4\Delta R}{(c+a)^2(a+b)^2}\left(8\Delta r_a+4\Delta R\right)\\
&= \frac{16\Delta^2}{(c+a)^2(a+b)^2}\cdot R(R+2r_a)\\
&= \frac{16\Delta^2\cdot OI_a^2}{(c+a)^2(a+b)^2}.
\end{aligned}
$$

Therefore, $YZ = \frac{4\Delta\cdot OI_a}{(c+a)(a+b)R} = \frac{abc\cdot OI_a}{(c+a)(a+b)R}$. From [7, Theorem 3], $F_bF_c = \frac{(b+c)R^2}{OI_b\cdot OI_c}$. It follows that

$$\frac{F_bF_c}{YZ} = \frac{(b+c)R^2}{OI_b\cdot OI_c}\cdot\frac{(c+a)(a+b)R}{abc\cdot OI_a} = \frac{(b+c)(c+a)(a+b)R^3}{abc\cdot OI_a\cdot OI_b\cdot OI_c}.$$

Since this ratio is symmetric in a, b, c, it is also equal to $\frac{F_cF_a}{ZX}$ and $\frac{F_aF_b}{XY}$. This proves (1), and we conclude that triangles $F_aF_bF_c$ and XYZ are similar. $\square$

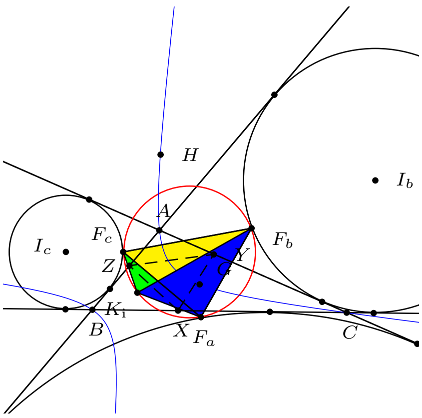

Figure 3A

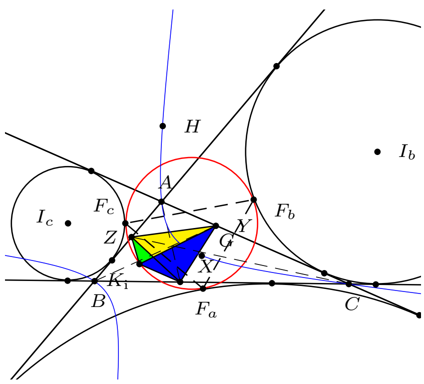

Figure 3B

Remark. In fact, the similarity of triangles $F_aF_bF_c$ and XYZ is direct. This means that there is a center of similarity P such that

$$\Delta PF_bF_c : \Delta PF_cF_a : \Delta PF_aF_b = \Delta PYZ : \Delta PZX : \Delta PXY.$$

In this case, the center of similarity is the Kiepert center

$$K_i = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$$

(which is the center of the Kiepert circum-hyperbola through the orthocenter H and the centroid G of triangle ABC) is a center of similarity (see Figures 3A and 3B). For notational convenience, let

$$\Sigma := a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2, \tag{2}$$

$$F(u, v, w) := uvw + (u + v + w)(w + u - v)(u + v - w). \tag{3}$$

Note that the coordinate sum of $K_{\rm i}$ is 2Σ, and F is symmetric in v and w. Now,

$$\Delta K_{\rm i}YZ = \frac{(a-b)(a-c)(b+c)F(a,b,c)}{2(c+a)(a+b)\Sigma},$$

$$\Delta K_{\rm i}F_bF_c = \frac{abc(a-b)(a-c)(b+c)^3(c+a)(a+b)}{2F(b,c,a)F(c,a,b)\Sigma}.$$

From this,

$$\frac{\Delta K_{\rm i}F_bF_c}{\Delta K_{\rm i}YZ} = \frac{abc(b+c)^2(c+a)^2(a+b)^2}{F(a,b,c)F(b,c,a)F(c,a,b)}$$

is symmetric in a, b, c. This means that

$$\Delta K_{\rm i}F_bF_c : \Delta K_{\rm i}F_cF_a : \Delta K_{\rm i}F_aF_b = \Delta K_{\rm i}YZ : \Delta K_{\rm i}ZX : \Delta K_{\rm i}XY,$$

and the triangles $F_aF_bF_c$ and XYZ are directly similarly with $K_{\rm i}$ as a center of similarity.

3. The Feuerbach conic

We consider the points at which the sidelines of the Feuerbach triangle intersect the sidelines of triangle ABC:

	$X_b = BC \cap F_cF_a$	$X_c = BC \cap F_aF_b$
$Y_a = CA \cap F_bF_c$		$Y_c = CA \cap F_aF_b$
$Z_a = AB \cap F_bF_c$	$Z_b = AB \cap F_cF_a$	

Proposition 5. *The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a conic.*

Proof. Since the four points F_b, Y_a, Z_a, F_c are collinear, and the line F_bF_c passes through X_a, so does the line Y_aZ_a. Similarly, the lines Z_bX_b passes through Y_b and X_cY_c through Z_c. Furthermore, the points X_a, Y_b, Z_c are collinear, being on the trilinear polar of the incenter I. It follows from Pascal's theorem that the six points X_b, X_c, Y_c, Y_a, Z_a, Z_b are on a conic (see Figure 4). □

We call the conic through these six points the *Feuerbach conic* of triangle ABC. Proposition 5 is true when the Feuerbach triangle is replaced by any triangle perspective with ABC.

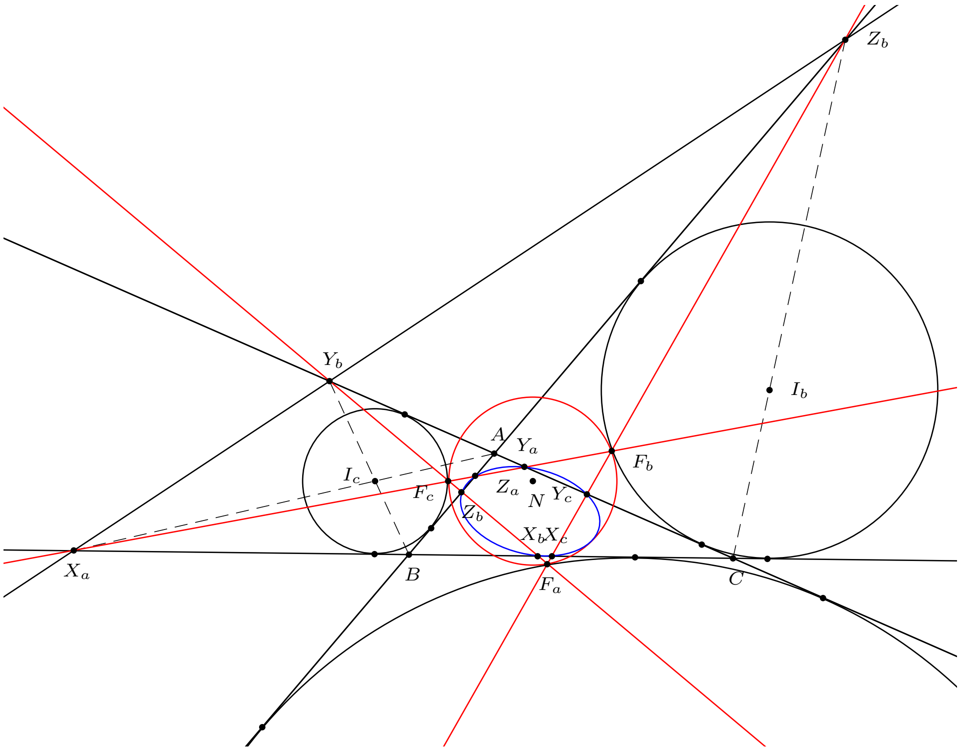

Figure 4

From the equations of the lines given in Table 1, we determine the coordinates of the points in Proposition 5:

$$\begin{aligned} X_b &= (0:\ a(c+a):\ c^2+a^2-b^2+ca); \\ X_c &= (0:\ a^2+b^2-c^2+ab:\ a(a+b)), \\ Y_c &= (a^2+b^2-c^2+ab:\ 0:\ b(a+b)); \\ Y_a &= (b(b+c):\ 0:\ b^2+c^2-a^2+bc), \\ Z_a &= (c(b+c):\ b^2+c^2-a^2+bc:\ 0), \\ Z_b &= (c^2+a^2-b^2+ca:\ c(c+a):\ 0). \end{aligned}$$

Proposition 6. *The barycentric equation of the Feuerbach conic is*

$$\sum_{\text{cyclic}} \frac{b^2+bc+c^2-a^2}{b+c}x^2 + \frac{(a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b)}{a(c+a)(a+b)}yz = 0. \tag{4}$$

Proof. With $x = 0$, equation (4) becomes

$$\begin{aligned} 0 = {} & \frac{c^2+ca+a^2-b^2}{c+a}y^2 + \frac{a^2+ab+b^2-c^2}{a+b}z^2 \\ & + \frac{(a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b)}{a(c+a)(a+b)}yz \\ = {} & \frac{\begin{pmatrix} a(a+b)(c^2+ca+a^2-b^2)y^2 + a(c+a)(a^2+ab+b^2-c^2)z^2 \\ +((a+b+c)(b-c)^2(b+c)-2a^2(c+a)(a+b))yz \end{pmatrix}}{a(c+a)(a+b)} \end{aligned}$$

The numerator factors as

$$((c^2+ca+a^2-b^2)y - a(c+a)z)(a(a+b)y - (a^2+ab+b^2-c^2)z)$$

since the coefficient of yz in this product is equal to

$$\begin{aligned} & -a^2(c+a)(a+b) - (c^2+ca+a^2-b^2)(a^2+ab+b^2-c^2) \\ = {} & a^2(c+a)(a+b) - (c^2+ca+a^2-b^2)(a^2+ab+b^2-c^2) - 2a^2(c+a)(a+b) \\ = {} & a^2(c+a)(a+b) - (a(c+a)-(b^2-c^2))(a(a+b)+(b^2-c^2)) \\ & \quad - 2a^2(c+a)(a+b) \\ = {} & a(a+b)(b^2-c^2) - a(c+a)(b^2-c^2) + (b^2-c^2)^2 - 2a^2(c+a)(a+b) \\ = {} & (a(a+b)-a(c+a)+(b^2-c^2))(b^2-c^2) - 2a^2(c+a)(a+b) \\ = {} & (a(b-c)+(b^2-c^2))(b^2-c^2) - 2a^2(c+a)(a+b) \\ = {} & (a+b+c)(b-c)^2(b+c) - 2a^2(c+a)(a+b). \end{aligned}$$

This means that the conic defined by (4) intersects the line BC at the points

$$(0 : a(c+a) : c^2+ca+a^2-b^2) \quad \text{and} \quad (0 : a^2+ab+b^2-c^2 : a(a+b)).$$

These are the points X_b and X_c.

Similarly the conic intersects CA at Y_c, Y_a, and AB at Z_a, Z_b. It is therefore the Feuerbach conic. □

Remark. The coordinates of the center of a conic with known barycentric equation can be computed using the formula in [11, §10.7.2]. For the Feuerbach conic, the center has homogeneous barycentric coordinates

$$(bc(b+c)^2 g(a,b,c) :\ ca(c+a)^2 g(b,c,a) :\ ab(a+b)^2 g(c,a,b))$$

for a polynomial $g(u,v,w)$ of degree 10 symmetric in v and w. It has ETC-(6,9,13) search number $1.93698582914\ldots$.

4. Some collinearity and perspectivity results

Proposition 7. *The points*

$$V_a := BY_c \cap CZ_b, \qquad V_b := CZ_a \cap AX_c, \qquad V_c := AX_b \cap BY_a$$

are collinear and the triangles ABC and $V_aV_bV_c$ are perspective.

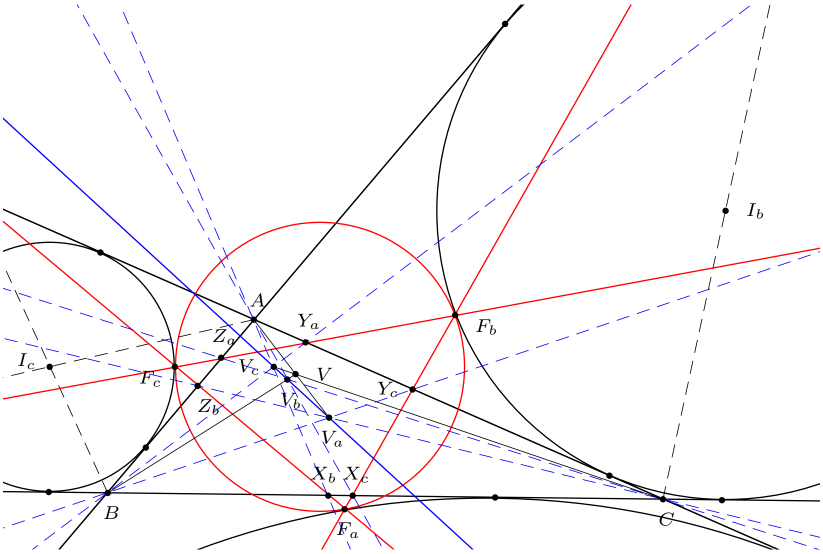

Figure 5

Proof. The lines BY_c and CZ_b have barycentric equations

$$\begin{aligned} -b(a+b)x \qquad\qquad & +(a^2+ab+b^2-c^2)z &= 0, \\ -c(c+a)x \quad +(c^2+ca+a^2-b^2)y & &= 0. \end{aligned}$$

They intersect at the point

$$\begin{aligned} V_a &= \left(1 : \frac{c(c+a)}{c^2+ca+a^2-b^2} : \frac{b(a+b)}{a^2+ab+b^2-c^2}\right) \\ &= \left(\frac{1}{bc} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} V_b &= CZ_a \cap AX_c \\ &= \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{1}{ca} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right), \\ V_c &= AX_b \cap BY_a \\ &= \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{1}{ab}\right). \end{aligned}$$

From these coordinates, it is clear that triangles $V_aV_bV_c$ is perspective with ABC at

$$V = \left(\frac{b+c}{a(b^2+bc+c^2-a^2)} : \frac{c+a}{b(c^2+ca+a^2-b^2)} : \frac{a+b}{c(a^2+ab+b^2-c^2)}\right). \tag{5}$$

The three points V_a, V_b, V_c are collinear. The line containing them has barycentric equation

$$\sum_{\text{cyclic}} (b-c)(b^2+bc+c^2-a^2)x = 0, \tag{6}$$

This line is the perspectrix of the triangles ABC and $V_aV_bV_c$. $\square$

Remarks. (1) The perspector V given in (5) is the triangle center $X(6757)$ of [5]. It lies on the perpendicular to the Euler line at the nine-point center:

$$\sum_{\text{cyclic}} a^2(b^2+bc+c^2-a^2)(b^2-bc+c^2-a^2)x = 0.$$

(2) The perspectrix (the line $V_aV_bV_c$) contains, among others, the triangle centers

- $X(79) = \left(\frac{1}{b^2+bc+c^2-a^2} : \cdots : \cdots\right)$, which is the perspector of ABC and the reflection triangle of the incenter,
- $X(2160) = \left(\frac{a}{b^2+bc+c^2-a^2} : \cdots : \cdots\right)$, which is the perspector of ABC and the triangle bounded by the radical axes of the circumcircle with the circles tangent to two sides of the reference triangle and center on the third side (see Figure 6).

Proof. These circles have barycentric equations

$$\begin{aligned}
&4(b+c)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((a+b+c)^2(b+c-a)^2x+(c^2+a^2-b^2)^2y+(a^2+b^2-c^2)^2z)=0,\\
&4(c+a)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((b^2+c^2-a^2)^2x+(a+b+c)^2(c+a-b)^2y+(a^2+b^2-c^2)^2z)=0,\\
&4(a+b)^2(a^2yz+b^2zx+c^2xy)-(x+y+z)\times\\
&\quad((b^2+c^2-a^2)^2x+(c^2+a^2-b^2)^2y+(a+b+c)^2(a+b-c)^2z)=0.
\end{aligned}$$

Their radical axes with the circumcircle are the lines

$$\frac{(a+b+c)^2(b+c-a)^2x+(c^2+a^2-b^2)^2y+(a^2+b^2-c^2)^2z}{(b+c)^2}=0,$$

$$\frac{(b^2+c^2-a^2)^2x+(a+b+c)^2(c+a-b)^2y+(a^2+b^2-c^2)^2z}{(c+a)^2}=0,$$

$$\frac{(b^2+c^2-a^2)^2x+(c^2+a^2-b^2)^2y+(a+b+c)^2(a+b-c)^2z}{(a+b)^2}=0.$$

These lines bound a triangle with vertices

$$\begin{aligned}
A' &= \left(f(a,b,c) : \frac{b}{c^2+ca+a^2-b^2} : \frac{c}{a^2+ab+b^2-c^2}\right),\\
B' &= \left(\frac{a}{b^2+bc+c^2-a^2} : f(b,c,a) : \frac{c}{a^2+ab+b^2-c^2}\right),\\
C' &= \left(\frac{a}{b^2+bc+c^2-a^2} : \frac{b}{c^2+ca+a^2-b^2} : f(c,a,b)\right),
\end{aligned}$$

where

$$f(u,v,w) := \frac{F(u,v,w)((u+v+w)(u^3-(v+w)(v-w)^2)+2u^2vw)}{(v^2+w^2-u^2)^2(w^2+wu+u^2-v^2)(u^2+uv+v^2-w^2)},$$

and F is defined in (3). From the coordinates of A', B', C', it is clear that ABC and $A'B'C'$ are perspective at

$$\left(\frac{a}{b^2+bc+c^2-a^2} : \frac{b}{c^2+ca+a^2-b^2} : \frac{c}{a^2+ab+b^2-a^2}\right).$$

□

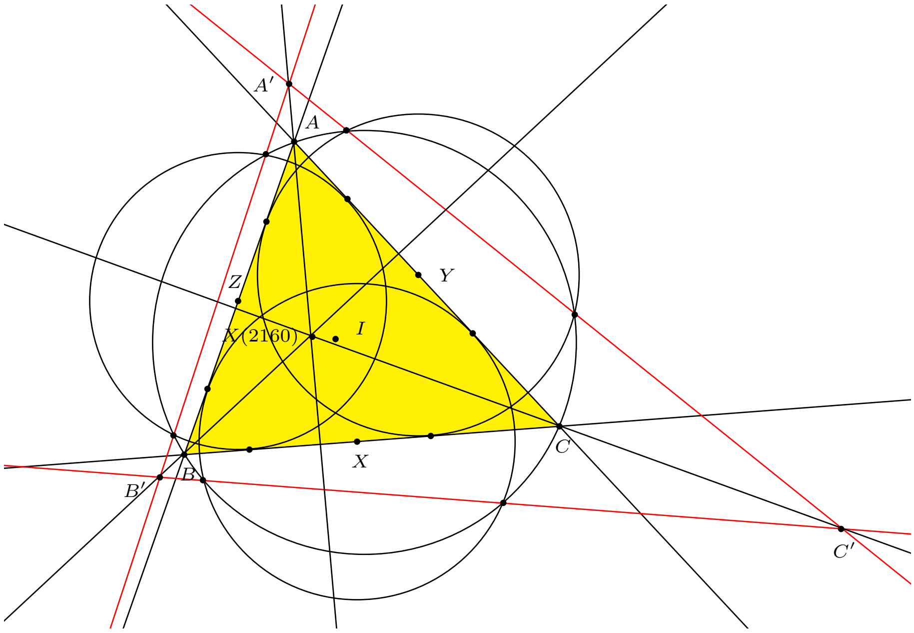

Figure 6

(3) On the other hand, the points

$$BY_a \cap CZ_a, \qquad CZ_b \cap AX_b, \qquad \text{and} \qquad AX_c \cap BY_c$$

are on the bisectors of angles A, B, C respectively, as is easily verified.

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Orthocenters of Simplices on Spheres

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Abstract. We consider orthocenters of simplices of the unit sphere of the n-dimensional Euclidean space. For $n = 3$, orthocenters always exist for all simplices, but for $n \geqq 4$, they do not necessarily exist. Moreover, unlike the case of the Euclidean space, it is possible that there exist infinite numbers of orthocenters. In this paper, we give characterizations of the existence and the uniqueness of orthocenters.

1. Introduction.

For a simplex of the unit sphere $\mathbb{S}^{n-1}$ of the Euclidean space $\mathbb{R}^n$, if great-circles which are passing through vertices and perpendicular to the opposite faces are concurrent, their common point is called an orthocenter. For $n \geqq 4$, orthocenters do not necessarily exist. In particular, the existence of orthocenters is equivalent to

$$(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)(\mathbf{p}_j^* \cdot \mathbf{p}_\ell^*) = (\mathbf{p}_i^* \cdot \mathbf{p}_\ell^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*)$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$ (Theorem 1). Moreover, if there exist orthocenters, the uniqueness of them is equivalent that there exist at least two pairs of vertices which are not perpendicular, respectively (Theorem 2) (remark that "uniqueness" means the existence of just one pair of orthocenters antipodal each other, because the antipodal of an orthocenter is another orthocenter). Notice that, for a simplex of the Euclidean space, orthocenters exist only at most one point, and the existence of the orthocenter is equivalent to

$$(\mathbf{q}_i - \mathbf{q}_k) \cdot (\mathbf{q}_j - \mathbf{q}_\ell) = 0$$

for arbitrary pairwise distinct vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$, $\mathbf{q}_\ell$ (see, e.g., §1 of [1], (1.1) of [3]).

2. Preliminaries.

The following notations are valid throughout this paper. For a spherical simplex S on $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$, i.e., for

$$S = \{\mathbf{x} \in \mathbb{S}^{n-1} : \mathbf{p}_i \cdot \mathbf{x} \geqq 0,\ {}^\forall i = 0, \ldots, n-1\},$$

where $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1} \in \mathbb{S}^{n-1}$ are linearly independent, let

$$S^* = \{\mathbf{y} \in \mathbb{S}^{n-1} : \mathbf{x} \cdot \mathbf{y} \geqq 0,\ {}^\forall \mathbf{x} \in S\},$$

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and, for $j = 0,\ldots, n-1$, let $\mathbf{p}_j^* \in \mathbb{S}^{n-1}$ be the unique vector such that

$$\mathbf{p}_i \cdot \mathbf{p}_j^* = 0 \ \text{ for } \ {}^\forall i = 0, \ldots, \widehat{j}, \ldots, n-1 \ \text{ and } \ \mathbf{p}_j \cdot \mathbf{p}_j^* > 0,$$

where the circumflex indicates that the term below it has been omitted. Then we have

$$\begin{aligned} S &= \left(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\right) \cap \mathbb{S}^{n-1} \\ &= \{\mathbf{x} \in \mathbb{S}^{n-1} : \mathbf{x} \cdot \mathbf{y} \geqq 0, \ {}^\forall \mathbf{y} \in S^*\}, \\ S^* &= \left(\mathbb{R}^+ \cdot \mathbf{p}_0 + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}\right) \cap \mathbb{S}^{n-1} \\ &= \{\mathbf{y} \in \mathbb{S}^{n-1} : \mathbf{p}_j^* \cdot \mathbf{y} \geqq 0, \ {}^\forall j = 0, \ldots, n-1\}, \end{aligned}$$

where $\mathbb{R}^+$ is the set of non-negative real numbers, i.e., $\mathbf{p}_0^*,\ldots, \mathbf{p}_{n-1}^*$ and $\mathbf{p}_0,\ldots, \mathbf{p}_{n-1}$ are vertices of S and S^*, respectively (see [4] and notice $S^* = -S^\circ$ and $\mathbf{p}_j^* = -\mathbf{p}_j^\circ$). The symbols Δ and Δ^* means

$$\Delta\,(k_0 \cdots k_m) = \det \begin{pmatrix} \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_m} \\ \vdots & & \vdots \\ \mathbf{p}_{k_m} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_m} \cdot \mathbf{p}_{k_m} \end{pmatrix},$$

$$\Delta^*\,(k_0 \cdots k_m) = \det \begin{pmatrix} \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_m}^* \\ \vdots & & \vdots \\ \mathbf{p}_{k_m}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_m}^* \cdot \mathbf{p}_{k_m}^* \end{pmatrix},$$

for $m = 0,\ldots, n-1$, and pairwise distinct indices $k_0,\ldots, k_m = 0,\ldots, n-1$, and

$$\Delta \begin{pmatrix} k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \end{pmatrix} = \det \begin{pmatrix} \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0} \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_{k_0} \cdot \mathbf{p}_j \\ \vdots & & \vdots & \vdots \\ \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_{k_{m-1}} \cdot \mathbf{p}_j \\ \mathbf{p}_i \cdot \mathbf{p}_{k_0} & \cdots & \mathbf{p}_i \cdot \mathbf{p}_{k_{m-1}} & \mathbf{p}_i \cdot \mathbf{p}_j \end{pmatrix},$$

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \end{pmatrix} = \det \begin{pmatrix} \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_{k_0}^* \cdot \mathbf{p}_j^* \\ \vdots & & \vdots & \vdots \\ \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_{k_{m-1}}^* \cdot \mathbf{p}_j^* \\ \mathbf{p}_i^* \cdot \mathbf{p}_{k_0}^* & \cdots & \mathbf{p}_i^* \cdot \mathbf{p}_{k_{m-1}}^* & \mathbf{p}_i^* \cdot \mathbf{p}_j^* \end{pmatrix},$$

for $m = 0,\ldots, n-2$, and pairwise distinct indices $k_0,\ldots, k_{m-1}, i, j = 0,\ldots, n-1$. Especially, we set

$$\Delta \begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i \cdot \mathbf{p}_j, \quad \text{and} \quad \Delta^* \begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i^* \cdot \mathbf{p}_j^*,$$

for distinct indices $i, j = 0,\ldots, n-1$.

Remark 1. From Lemma 5.1 and the second equation of Lemma 3.4 of [4], for an index $k = 0, \ldots, n-1$, we have

$$\mathbf{p}_k^* = (-1)^{n-1-k}\varepsilon \frac{\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1} \rangle\!\rangle}{\sqrt{\Delta\left(0 \cdots \widehat{k} \cdots n-1\right)}}, \tag{1}$$

$$\mathbf{p}_k = (-1)^{n-1-k}\varepsilon \frac{\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^* \rangle\!\rangle}{\sqrt{\Delta^*\left(0 \cdots \widehat{k} \cdots n-1\right)}}, \tag{1*}$$

where $\langle\!\langle \mathbf{q}_0, \ldots, \mathbf{q}_{n-2} \rangle\!\rangle$ is the unique vector in $\mathbb{R}^n$ such that

$$\langle\!\langle \mathbf{q}_0, \ldots, \mathbf{q}_{n-2} \rangle\!\rangle \cdot \mathbf{q}_{n-1} = \det(\mathbf{q}_0, \ldots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1}), \quad \text{for } {}^\forall \mathbf{q}_{n-1} \in \mathbb{R}^n,$$

$(\mathbf{q}_0, \ldots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1})$ is the matrix that has $\mathbf{q}_0, \ldots, \mathbf{q}_{n-2}, \mathbf{q}_{n-1}$ as column vectors with Cartesian coordinate system, and

$$\varepsilon = \mathrm{sgn}(\det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1})) = \mathrm{sgn}(\det(\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*)) \in \{1, -1\}.$$

Orthocenters are defined as follows:

Definition 1. The point $\mathbf{h}$ on $\mathbb{S}^{n-1}$ is called an *orthocenter* if, for each index $i = 0, \ldots, n-1$, there exists a great circular arc C_i of $\mathbb{S}^{n-1}$ such that C_i is passing through $\mathbf{p}_i$, $\mathbf{p}_i^*$, and $\mathbf{h}$.

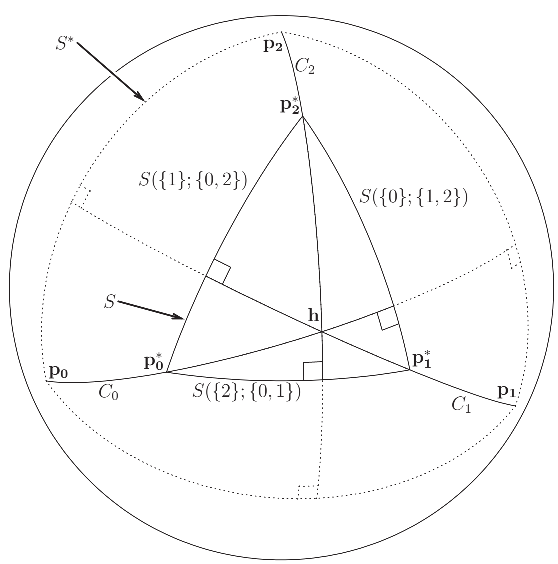

Figure 1.

Remark 2. C_i is passing through $\mathbf{p}_i$, so, C_i is perpendicular to the opposite face of the vertex $\mathbf{p}_i^*$:

$$\begin{aligned} & S(\{i\};\{0,\dots,\widehat{i},\dots,n-1\}) \\ &= \{\mathbf{x} \in S : \mathbf{p}_i \,.\, \mathbf{x} = 0\} \\ &= \left(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_i^*} + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\right) \cap \mathbb{S}^{n-1}. \end{aligned}$$

Remark 3. If $\mathbf{h}$ is an orthocenter of S, then the antipode $-\mathbf{h}$ is another orthocenter.

Remark 4. If $\mathbf{h}$ is an orthocenter of S, it is also an orthocenter of S^*.

The following two theorems are main purposes of this paper.

Theorem 1. *The followings are equivalent:*

(a) *there exists an orthocenter* $\mathbf{h}$*;*

(b) *the equation* $\Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta\begin{pmatrix} i \\ \ell \end{pmatrix}\Delta\begin{pmatrix} j \\ k \end{pmatrix}$ *holds for arbitrary pairwise distinct indices* i, j, k, $\ell = 0,\dots, n-1$*;*

(b)* *the equation* $\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\Delta^*\begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta^*\begin{pmatrix} i \\ \ell \end{pmatrix}\Delta^*\begin{pmatrix} j \\ k \end{pmatrix}$ *holds for arbitrary pairwise distinct indices* i, j, k, $\ell = 0,\dots, n-1$.

Theorem 2. *The followings are equivalent:*

(c) *there exist just two orthocenters and they are antipodal each other,* $\pm\mathbf{h}$*;*

(d) (b) *holds and there exist at least two pairs of distinct indices* i, k *and* j, ℓ *such that neither* $\Delta\begin{pmatrix} i \\ k \end{pmatrix}$ *nor* $\Delta\begin{pmatrix} j \\ \ell \end{pmatrix}$ *is equal to* 0*;*

(d)* (b)* *holds and there exist at least two pairs of distinct indices* i, k *and* j, ℓ *such that neither* $\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}$ *nor* $\Delta^*\begin{pmatrix} j \\ \ell \end{pmatrix}$ *is equal to* 0.

Remark 5. Assume (d). Then, we have that either there exist pairwise distinct indices i, j, k, ℓ such that

$$\Delta\begin{pmatrix} i \\ k \end{pmatrix} \neq 0 \neq \Delta\begin{pmatrix} j \\ \ell \end{pmatrix}, \tag{2}$$

or there exist pairwise distinct indices i, k, ℓ such that

$$\Delta\begin{pmatrix} i \\ k \end{pmatrix} \neq 0 \neq \Delta\begin{pmatrix} i \\ \ell \end{pmatrix}. \tag{3}$$

If (2) holds, we have

$$0 \neq \Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta\begin{pmatrix} i \\ \ell \end{pmatrix}\Delta\begin{pmatrix} j \\ k \end{pmatrix},$$

so (3) holds, i.e., (d) implies (3) for some pairwise distinct indices i, k, ℓ. Similarly, (d)* implies

$$\Delta^*\begin{pmatrix} i \\ k \end{pmatrix} \neq 0 \neq \Delta^*\begin{pmatrix} i \\ \ell \end{pmatrix}, \tag{3*}$$

for some pairwise distinct indices i, k, ℓ.

Remark 6. Assume (b) and the negation of (d). Then, if there exists no pair of distinct indices i, j with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, the set of orthocenters is the whole unit sphere $\mathbb{S}^{n-1}$. Otherwise, i.e., if there exists the unique pair of distinct indices i, j with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, the set of orthocenters is the unit circle passing through $\mathbf{p}_i$ and $\mathbf{p}_j$ (see the proof of Lemma 5).

Remark 7. From Theorem 1, orthocenters always exist for $n = 3$.

Later, theorems above are proved completely and the pair of orthocenters are represented explicitly if three conditions of Theorem 2 hold. Now we can prove (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (b)* immediately by the following two lemmas:

Lemma 3. *The condition* (a) *implies*

$(\mathbf{h} \cdot \mathbf{p}_i)(\mathbf{p}_j \cdot \mathbf{p}_k) = (\mathbf{h} \cdot \mathbf{p}_j)(\mathbf{p}_i \cdot \mathbf{p}_k)$, (*resp.* $(\mathbf{h} \cdot \mathbf{p}_i^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*) = (\mathbf{h} \cdot \mathbf{p}_j^*)(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)$)

for arbitrary pairwise distinct indices i, j, $k = 0, \ldots, n-1$.

Proof. If $\mathbf{p}_k = \mathbf{p}_k^*$, we have $\mathbf{p}_i \cdot \mathbf{p}_k = \mathbf{p}_j \cdot \mathbf{p}_k = 0$. So the both sides of the conclusion are equal to 0. Otherwise, i.e., if $\mathbf{p}_k \neq \mathbf{p}_k^*$, there exists a great circular arc C_k passing through $\mathbf{p}_k$, $\mathbf{p}_k^*$, and $\mathbf{h}$, so, we can represent $\mathbf{h} = A\mathbf{p}_k + A^*\mathbf{p}_k^*$. Hence, we have

$$(\mathbf{h} \cdot \mathbf{p}_i)(\mathbf{p}_j \cdot \mathbf{p}_k) = A(\mathbf{p}_k \cdot \mathbf{p}_i)(\mathbf{p}_j \cdot \mathbf{p}_k) = (\mathbf{p}_k \cdot \mathbf{p}_i)(\mathbf{p}_j \cdot \mathbf{h}).$$

□

Lemma 4. *If there exists a point* $\mathbf{n} \in \mathbb{S}^{n-1}$ *which satisfies*

$(\mathbf{n} \cdot \mathbf{p}_i)(\mathbf{p}_j \cdot \mathbf{p}_k) = (\mathbf{n} \cdot \mathbf{p}_j)(\mathbf{p}_i \cdot \mathbf{p}_k)$ (*resp.* $(\mathbf{n} \cdot \mathbf{p}_i^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*) = (\mathbf{n} \cdot \mathbf{p}_j^*)(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)$)

for arbitrary pairwise distinct indices i, j, k, *then the condition* (b) (*resp.* (b)*) *holds.*

Proof. Let i, j, k, and ℓ be pairwise distinct indices. Then, fix an index h with $\mathbf{n} \cdot \mathbf{p}_h \neq 0$. If $h = i$, we have

$$(\mathbf{p}_i \cdot \mathbf{p}_k)(\mathbf{p}_j \cdot \mathbf{p}_\ell) = \frac{\mathbf{n} \cdot \mathbf{p}_j}{\mathbf{n} \cdot \mathbf{p}_i}(\mathbf{p}_i \cdot \mathbf{p}_k)(\mathbf{p}_i \cdot \mathbf{p}_\ell) = (\mathbf{p}_j \cdot \mathbf{p}_k)(\mathbf{p}_i \cdot \mathbf{p}_\ell).$$

Otherwise, i.e., if $h \neq i$, we have

$$\begin{aligned}(\mathbf{p}_i \cdot \mathbf{p}_k)(\mathbf{p}_j \cdot \mathbf{p}_\ell) &= \frac{\mathbf{n} \cdot \mathbf{p}_k}{\mathbf{n} \cdot \mathbf{p}_h}(\mathbf{p}_i \cdot \mathbf{p}_h)(\mathbf{p}_j \cdot \mathbf{p}_\ell) \\ &= \frac{\mathbf{n} \cdot \mathbf{p}_\ell}{\mathbf{n} \cdot \mathbf{p}_h}(\mathbf{p}_i \cdot \mathbf{p}_h)(\mathbf{p}_j \cdot \mathbf{p}_k) \\ &= (\mathbf{p}_i \cdot \mathbf{p}_\ell)(\mathbf{p}_j \cdot \mathbf{p}_k).\end{aligned}$$

□

The following lemma is useful to prove the equivalence (d) $\Leftrightarrow$ (d)*.

Lemma 5. *We have the following equivalences:*

(α) *there exists no pair of distinct indices i, j with $\Delta\binom{i}{j} \neq 0$ if and only if there exists no pair of distinct indices i', j' with $\Delta^*\binom{i'}{j'} \neq 0$,*

(β) *there exists the unique pair of distinct indices i, j with $\Delta\binom{i}{j} \neq 0$ if and only if there exists the unique pair of distinct indices i', j' with $\Delta^*\binom{i'}{j'} \neq 0$,*

(γ) *there exist at least two pairs of distinct indices i, k and j, ℓ with $\Delta\binom{i}{k} \neq 0 \neq \Delta\binom{j}{\ell}$ if and only if there exist at least two pairs of distinct indices i', k' and j', ℓ' with $\Delta^*\binom{i'}{k'} \neq 0 \neq \Delta^*\binom{j'}{\ell'}$.*

Notice that, if the conditions of (β) hold, the pair i, j is equal to the pair i', j'.

Proof. (α): If there exists no pair i, j with $\Delta\binom{i}{j} \neq 0$, then $\mathbf{p}_k = \mathbf{p}_k^*$ holds for an arbitrary index $k = 0,\ldots, n-1$ (so, for each $\mathbf{x} \in \mathbb{S}^{n-1}$, there exists a great circular arc C_k passing through $\mathbf{p}_k$, $\mathbf{p}_k^*$, and $\mathbf{x}$. See Remark 6). Hence there exists no pair i', j' with $\Delta^*\binom{i'}{j'} \neq 0$. The proof of the converse is similar. (β): Assume that there exists the unique pair i, j with $\Delta\binom{i}{j} \neq 0$. Then, $\mathbf{p}_k = \mathbf{p}_k^*$ holds for each index $k \neq i, j$ (see Remark 6 again). Hence we have $\Delta^*\binom{i''}{j''} = 0$ for each pair of distinct indices i'', j'' except the pair i, j (we have $\Delta^*\binom{i}{j} \neq 0$ because, if not, we have $\Delta\binom{i}{j} = 0$ from the equivalence (α)). The proof of the converse is similar. (γ): It is from (α) and (β). □

3. Equivalence of (b) and (b)*, and Equivalence of (d) and (d)*

In this section, we prove the equivalences (b) $\Leftrightarrow$ (b)* and (d) $\Leftrightarrow$ (d)*. To prove them, we need the following two lemmas. Notice that equations except (8) do not require the assumption (b).

Lemma 6. *Let $\{k_0, \ldots, k_{n-1}\}$ be a permutation of the set of all indices $\{0, \ldots, n-1\}$. Then we have*

$$\Delta^*(k_0 \cdots k_m) = \frac{\Delta(k_{m+1} \cdots k_{n-1})}{\Delta(0 \cdots n-1)} \prod_{r=0}^{m} (\mathbf{p}_{k_r} \cdot \mathbf{p}_{k_r}^*)^2, \tag{4}$$

for $m = 0,\dots, n-1$, *and*

$$\Delta^*\left(k_0\cdots k_{m-1}\,\begin{matrix} i\\ j\end{matrix}\right) = \frac{-\Delta\left(k_m\cdots\widehat{i}\cdots\widehat{j}\cdots k_{n-1}\,\begin{matrix} i\\ j\end{matrix}\right)}{\Delta\left(0\cdots n-1\right)}(\mathbf{p}_i\cdot\mathbf{p}_i^*)(\mathbf{p}_j\cdot\mathbf{p}_j^*)\prod_{r=0}^{m-1}(\mathbf{p}_{k_r}\cdot\mathbf{p}_{k_r}^*)^2, \tag{5}$$

for $m = 0,\dots, n-2$ *and distinct indices* i, $j = k_m,\dots, k_{n-1}$ (*We can use the notation* $k_m\cdots\widehat{k_s}\cdots\widehat{k_t}\cdots k_{n-1}$ *whether* $s < t$ *or not. If* $s > t$, *it means* $k_m\cdots\widehat{k_t}\cdots\widehat{k_s}\cdots k_{n-1}$), *and*

$$\mathbf{p}_k\cdot\mathbf{p}_k^* = \frac{\sqrt{\Delta\left(0\cdots n-1\right)}}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}}, \tag{6}$$

for an index $k = 0,\dots, n-1$.

Proof. We have (6) by

$$\begin{aligned}\mathbf{p}_k^*\cdot\mathbf{p}_k &= \frac{(-1)^{n-1-k}\varepsilon\det(\mathbf{p}_0,\dots,\widehat{\mathbf{p}_k},\dots,\mathbf{p}_{n-1},\mathbf{p}_k)}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}}\\ &= \frac{\varepsilon\det(\mathbf{p}_0,\dots,\mathbf{p}_{n-1})}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}}\\ &= \frac{|\det(\mathbf{p}_0,\dots,\mathbf{p}_{n-1})|}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}}\\ &= \frac{\sqrt{\Delta\left(0\cdots n-1\right)}}{\sqrt{\Delta\left(0\cdots\widehat{k}\cdots n-1\right)}},\end{aligned}$$

where the first equality is from (1). The equations (4) and (5) are from (6) and Lemma 5.3 of [4]:

$$\Delta^*\left(k_0\cdots k_m\right) = \frac{\left(\Delta\left(0\cdots n-1\right)\right)^m}{\prod_{r=0}^{m-1}\Delta\left(0\cdots\widehat{k_r}\cdots n-1\right)}\cdot\frac{\Delta\left(k_{m+1}\cdots k_{n-1}\right)}{\Delta\left(0\cdots\widehat{k_m}\cdots n-1\right)},$$

$$\begin{aligned}\Delta^*\left(k_0\cdots k_{m-1}\,\begin{matrix} i\\ j\end{matrix}\right) = {}& \frac{\left(\Delta\left(0\cdots n-1\right)\right)^m}{\prod_{r=0}^{m-1}\Delta\left(0\cdots\widehat{k_r}\cdots n-1\right)}\\ &\cdot\frac{-\Delta\left(k_m\cdots\widehat{i}\cdots\widehat{j}\cdots k_{n-1}\,\begin{matrix} i\\ j\end{matrix}\right)}{\sqrt{\Delta\left(0\cdots\widehat{i}\cdots n-1\right)}\sqrt{\Delta\left(0\cdots\widehat{j}\cdots n-1\right)}}.\end{aligned}$$

$\square$

Remark 8. We also have similar equations:

$$\Delta\left(k_0\cdots k_m\right)=\frac{\Delta^*\left(k_{m+1}\cdots k_{n-1}\right)}{\Delta^*\left(0\cdots n-1\right)}\prod_{r=0}^{m}(\mathbf{p}_{k_r}\cdot\mathbf{p}^*_{k_r})^2, \tag{4*}$$

$$\Delta\left(k_0\cdots k_{m-1}\,{i \atop j}\right)=\frac{-\,\Delta^*\left(k_m\cdots\widehat{i}\cdots\widehat{j}\cdots k_{n-1}\,{i \atop j}\right)}{\Delta^*\left(0\cdots n-1\right)} \cdot(\mathbf{p}_i\cdot\mathbf{p}^*_i)(\mathbf{p}_j\cdot\mathbf{p}^*_j)\prod_{r=0}^{m-1}(\mathbf{p}_{k_r}\cdot\mathbf{p}^*_{k_r})^2, \tag{5*}$$

$$\mathbf{p}_k\cdot\mathbf{p}^*_k=\frac{\sqrt{\Delta^*\left(0\cdots n-1\right)}}{\sqrt{\Delta^*\left(0\cdots\widehat{k}\cdots n-1\right)}}. \tag{6*}$$

Lemma 7. *We have*

$$\Delta\left(k_0\cdots k_m\right)=\Delta\left(k_0\cdots k_{m-1}\right)-\sum_{\ell=0}^{m-1}\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\Delta\left(k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,{k_m \atop k_\ell}\right), \tag{7}$$

for $m=0,\ldots,n-1$ *and pairwise distinct indices* $k_0,\ldots,k_m$. *Moreover, the condition* (b) *implies*

$$\Delta\left(k_0\cdots k_{m-1}\,{i \atop j}\right)=\Delta\left(k_0\cdots k_{m-2}\,{i \atop j}\right)-\Delta\begin{pmatrix}i\\ k_{m-1}\end{pmatrix}\Delta\left(k_0\cdots k_{m-2}\,{k_{m-1} \atop j}\right), \tag{8}$$

for $m=0,\ldots,n-2$ *and pairwise distinct indices* $k_0,\ldots,k_{m-1},i,j$.

Proof. We have

$$\begin{aligned}\Delta\left(k_0\cdots k_m\right)&=\sum_{\ell=0}^{m-1}(-1)^{m-\ell}\cdot\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\cdot\det M_\ell+1\cdot\Delta\left(k_m\right)\cdot\Delta\left(k_0\cdots k_{m-1}\right)\\ &=-\sum_{\ell=0}^{m-1}\Delta\begin{pmatrix}k_\ell\\ k_m\end{pmatrix}\Delta\left(k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,{k_m \atop k_\ell}\right)+\Delta\left(k_0\cdots k_{m-1}\right),\end{aligned}$$

where

$$M_\ell=\begin{pmatrix}\mathbf{p}_{k_0}\cdot\mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0}\cdot\mathbf{p}_{k_{m-1}}\\ \vdots & & \vdots\\ \widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_0}} & \cdots & \widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_{m-1}}}\\ \vdots & & \vdots\\ \mathbf{p}_{k_m}\cdot\mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_m}\cdot\mathbf{p}_{k_{m-1}}\end{pmatrix}.$$

Moreover, if the condition (b) holds, we also have

$$
\begin{aligned}
&\Delta\left(k_0\cdots k_{m-1}\,\begin{matrix} i\\ j\end{matrix}\right)\\
&=\sum_{\ell=0}^{m-2}(-1)^{m-1-\ell}\cdot\Delta\begin{pmatrix} k_\ell\\ k_{m-1}\end{pmatrix}\cdot\det M'_\ell\\
&\qquad+1\cdot\Delta\left(k_{m-1}\right)\cdot\Delta\left(k_0\cdots k_{m-2}\,\begin{matrix} i\\ j\end{matrix}\right)\\
&\qquad+(-1)\cdot\Delta\begin{pmatrix} i\\ k_{m-1}\end{pmatrix}\cdot\Delta\left(k_0\cdots k_{m-2}\,\begin{matrix} k_{m-1}\\ j\end{matrix}\right)\\
&=\sum_{\ell=0}^{m-2}0+\Delta\left(k_0\cdots k_{m-2}\,\begin{matrix} i\\ j\end{matrix}\right)-\Delta\begin{pmatrix} i\\ k_{m-1}\end{pmatrix}\Delta\left(k_0\cdots k_{m-2}\,\begin{matrix} k_{m-1}\\ j\end{matrix}\right),
\end{aligned}
$$

where the last equality is from the parallelism of the lower 2 rows of

$$
M'_\ell=\begin{pmatrix}
\mathbf{p}_{k_0}\cdot\mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_0}\cdot\mathbf{p}_{k_{m-2}} & \mathbf{p}_{k_0}\cdot\mathbf{p}_j\\
\vdots & & \vdots & \vdots\\
\widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_0}} & \cdots & \widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_{k_{m-2}}} & \widehat{\mathbf{p}_{k_\ell}\cdot\mathbf{p}_j}\\
\vdots & & \vdots & \vdots\\
\mathbf{p}_{k_{m-1}}\cdot\mathbf{p}_{k_0} & \cdots & \mathbf{p}_{k_{m-1}}\cdot\mathbf{p}_{k_{m-2}} & \mathbf{p}_{k_{m-1}}\cdot\mathbf{p}_j\\
\mathbf{p}_i\cdot\mathbf{p}_{k_0} & \cdots & \mathbf{p}_i\cdot\mathbf{p}_{k_{m-2}} & \mathbf{p}_i\cdot\mathbf{p}_j
\end{pmatrix}.
$$

$\square$

The equation (8) enables us to prove the following two lemmas and two corollaries by induction (the equation (7) is used in Appendix).

Lemma 8. *The condition* (b) *implies*

$$
\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i\\ j\end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i'\\ j'\end{matrix}\right)=\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i'\\ j\end{matrix}\right)\Delta\left(k'_0\cdots k'_{m-1}\,\begin{matrix} i\\ j'\end{matrix}\right),
$$

for ℓ, $m=0,\ldots,n-2$ *and indices* $k_0,\ldots,k_{\ell-1}$, $k'_0,\ldots,k'_{m-1}$, i, j, i', j' *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

Proof. The conclusion above is called the type (ℓ,m). Without loss of generality, we can assume $\ell\geqq m$. The type $(0,0)$ is obvious, because, if $i\neq i'$ and $j\neq j'$ then it is (b), otherwise it is the identity. If $\ell>0$, the type $(\ell,0)$ is shown by:

$$
\begin{aligned}
&\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i\\ j\end{matrix}\right)\Delta\begin{pmatrix} i'\\ j'\end{pmatrix}-\Delta\left(k_0\cdots k_{\ell-1}\,\begin{matrix} i'\\ j\end{matrix}\right)\Delta\begin{pmatrix} i\\ j'\end{pmatrix}\\
&=\left(\Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} i\\ j\end{matrix}\right)-\Delta\begin{pmatrix} i\\ k_{\ell-1}\end{pmatrix}\Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} k_{\ell-1}\\ j\end{matrix}\right)\right)\Delta\begin{pmatrix} i'\\ j'\end{pmatrix}\\
&\quad-\left(\Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} i'\\ j\end{matrix}\right)-\Delta\begin{pmatrix} i'\\ k_{\ell-1}\end{pmatrix}\Delta\left(k_0\cdots k_{\ell-2}\,\begin{matrix} k_{\ell-1}\\ j\end{matrix}\right)\right)\Delta\begin{pmatrix} i\\ j'\end{pmatrix}
\end{aligned}
$$

$$= \Delta \begin{pmatrix} k_0 \cdots k_{\ell-2} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} i' \\ j' \end{pmatrix} - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-2} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} i \\ j' \end{pmatrix}$$
$$= 0,$$

where the last equality is from the type $(\ell-1, 0)$. If $\ell \geqq m > 0$, the type (ℓ, m) is shown by:

$$\begin{aligned}
&\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i \\ j' \end{matrix} \end{pmatrix} \\
&= \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \left(\Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} i' \\ k'_{m-1} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} k'_{m-1} \\ j' \end{matrix} \end{pmatrix} \right) \\
&\qquad - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \left(\Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} i \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} i \\ k'_{m-1} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} k'_{m-1} \\ j' \end{matrix} \end{pmatrix} \right) \\
&= \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} i \\ j' \end{matrix} \end{pmatrix} \right) \\
&\qquad - \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} i' \\ k'_{m-1} \end{pmatrix} \right. \\
&\qquad\qquad \left. - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} i \\ k'_{m-1} \end{pmatrix} \right) \Delta \begin{pmatrix} k'_0 \cdots k'_{m-2} \, \begin{matrix} k'_{m-1} \\ j' \end{matrix} \end{pmatrix} \\
&= 0,
\end{aligned}$$

where the last equality is from the types $(\ell, m-1)$ and $(\ell, 0)$. □

Lemma 9. *The condition* (b) *implies*

$$\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} k \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} = \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} k \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix},$$

for ℓ, $m = 0, \ldots, n-3$ *and indices* $k_0, \ldots, k_{\ell-1}$, $k'_0, \ldots, k'_{m-1}$, i, j, i', j', k *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

Proof. We have

$$\begin{aligned}
&\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} k \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} k \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} \\
&= \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} i \\ k \end{pmatrix} \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} k \\ j \end{matrix} \end{pmatrix} \right) \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} \\
&\qquad - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \left(\Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} - \Delta \begin{pmatrix} i' \\ k \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} k \\ j' \end{matrix} \end{pmatrix} \right) \\
&= -\Delta \begin{pmatrix} i \\ k \end{pmatrix} \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} k \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix} \right. \\
&\qquad\qquad \left. - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, \begin{matrix} i' \\ j \end{matrix} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, \begin{matrix} k \\ j' \end{matrix} \end{pmatrix} \right) \\
&= 0,
\end{aligned}$$

where the second and last equalities are the types $(\ell, 0)$ and (ℓ, m) of the previous lemma, respectively. □

Corollary 10. *The condition* (b) *also implies*

$$\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k \begin{matrix} i \\ j'\end{matrix}\end{pmatrix},$$

for ℓ, $m = 0,\ldots, n-3$ *and indices* $k_0,\ldots, k_{\ell-1}$, $k'_0,\ldots, k'_{m-1}$, i, j, i', j', k *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

Proof. In particular, if $k \neq i'$, it is obvious from two previous lemmas. Otherwise, we have

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i' \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i' \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}\\
&= \left(\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i \\ i'\end{pmatrix}\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\right)\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\\
&\quad -\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\left(\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i \\ i'\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\right)\\
&=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}\\
&=0.
\end{aligned}$$

□

Corollary 11. *The condition* (b) *also implies*

$$\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k' \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k' \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k \begin{matrix} i \\ j'\end{matrix}\end{pmatrix},$$

for ℓ, $m = 0,\ldots, n-3$ *and indices* $k_0,\ldots, k_{\ell-1}$, $k'_0,\ldots, k'_{m-1}$, i, j, i', j', k, k' *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

Proof. If $k = k'$ then the conclusion is from Lemma 8, so assume $k \neq k'$. If $k \neq i'$, then we have

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k' \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\\
&=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k'k \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\\
&=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}k' \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}k \begin{matrix} i \\ j'\end{matrix}\end{pmatrix},
\end{aligned}$$

where the first and last equalities are from Lemma 9 and Corollary 10, respectively. If $k' \neq i$, the proof is similar. Otherwise, i.e, if $k = i'$ and $k' = i$, then

$$\begin{aligned}
&\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i' \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i' \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}\\
&=\Delta\begin{pmatrix}k_0\cdots k_{\ell-1}i' \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\left(\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i' \\ i\end{pmatrix}\Delta\begin{pmatrix}k'_0\cdots k'_{m-1} \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}\right)\\
&\quad -\left(\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i' \\ j\end{matrix}\end{pmatrix}-\Delta\begin{pmatrix}i' \\ i\end{pmatrix}\Delta\begin{pmatrix}k_0\cdots k_{\ell-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\right)\Delta\begin{pmatrix}k'_0\cdots k'_{m-1}i' \begin{matrix} i \\ j'\end{matrix}\end{pmatrix}
\end{aligned}$$

$$= \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} i' \, {i \atop j} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i' \atop j'} \end{pmatrix} - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i' \atop j} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} i' \, {i \atop j'} \end{pmatrix} \right)$$
$$- \Delta \begin{pmatrix} i' \\ i \end{pmatrix} \left(\Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} i' \, {i \atop j} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i \atop j'} \end{pmatrix} \right.$$
$$\left. - \Delta \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i \atop j} \end{pmatrix} \Delta \begin{pmatrix} k'_0 \cdots k'_{m-1} i' \, {i \atop j'} \end{pmatrix} \right)$$
$$= 0,$$

where the last equality is from Lemma 9 and Corollary 10. □

Now, we prove the purposes of this section.

Lemma 12. *The condition* (b) *is equivalent to the condition* (b)*.

Proof. Assume that the condition (b) holds. Then, for pairwise distinct indices i, i', j, j', we have

$$\begin{aligned} \Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \Delta^* \begin{pmatrix} i' \\ j' \end{pmatrix} &= \frac{-\Delta \begin{pmatrix} 0 \cdots \widehat{i} \cdots \widehat{j} \cdots n-1 \, {i \atop j} \end{pmatrix}}{\Delta\,(0 \cdots n-1)} (\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*) \\ &\quad \cdot \frac{-\Delta \begin{pmatrix} 0 \cdots \widehat{i'} \cdots \widehat{j'} \cdots n-1 \, {i' \atop j'} \end{pmatrix}}{\Delta\,(0 \cdots n-1)} (\mathbf{p}_{i'} \cdot \mathbf{p}_{i'}^*)(\mathbf{p}_{j'} \cdot \mathbf{p}_{j'}^*) \\ &= \frac{-\Delta \begin{pmatrix} 0 \cdots \widehat{i'} \cdots \widehat{j} \cdots n-1 \, {i' \atop j} \end{pmatrix}}{\Delta\,(0 \cdots n-1)} (\mathbf{p}_{i'} \cdot \mathbf{p}_{i'}^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*) \\ &\quad \cdot \frac{-\Delta \begin{pmatrix} 0 \cdots \widehat{i} \cdots \widehat{j'} \cdots n-1 \, {i \atop j'} \end{pmatrix}}{\Delta\,(0 \cdots n-1)} (\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_{j'} \cdot \mathbf{p}_{j'}^*) \\ &= \Delta^* \begin{pmatrix} i' \\ j \end{pmatrix} \Delta^* \begin{pmatrix} i \\ j' \end{pmatrix}, \end{aligned}$$

where the first and last equalities are from (5) and the second equality is from Corollary 11. The proof of the converse is similar. □

Remark 9. From the previous lemma, the condition (b) implies similar equations:

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i' \atop j'} \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i' \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i \atop j'} \end{pmatrix},$$
$$\Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} k \, {i \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i' \atop j'} \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} k \, {i' \atop j'} \end{pmatrix},$$
$$\Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} k \, {i \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} \, {i' \atop j'} \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} \, {i' \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} k \, {i \atop j'} \end{pmatrix},$$
$$\Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} k \, {i \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} k' \, {i' \atop j'} \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{\ell-1} k' \, {i' \atop j} \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m-1} k \, {i \atop j'} \end{pmatrix}.$$

Corollary 13. *The condition* (d) *is equivalent to the condition* (d)*.

Proof. It is an natural consequence of Lemma 12 and the equivalence (γ) of Lemma 5. $\square$

4. μ_k, ν_k, μ_k^*, and ν_k^*

To represent orthocenters, we use four types of values: μ_k, ν_k, μ_k^*, and ν_k^*. To define them and to check their fundamental relations, we need two lemmas.

Lemma 14. *The condition* (b) *implies*

$$\begin{aligned}&\Delta\begin{pmatrix}k_0\cdots k_m \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta^*\begin{pmatrix}k_0\cdots k_m \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\\ &=\Delta\begin{pmatrix}k_0\cdots k_{m-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta^*\begin{pmatrix}k_0\cdots k_{m-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}(\mathbf{p}_{k_m}\cdot\mathbf{p}^*_{k_m})^2,\end{aligned}$$

for $m=0,\ldots,n-3$ *and indices* $k_0,\ldots,k_m$, i, j, i', j' *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *are distinct.*

Proof. This is a natural consequence of (5) and

$$\begin{aligned}&\Delta\begin{pmatrix}k_0\cdots k_m \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k_{m+1}\cdots\widehat{i'}\cdots\widehat{j'}\cdots k_{n-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}\\ &=\Delta\begin{pmatrix}k_0\cdots k_{m-1} \begin{matrix} i \\ j\end{matrix}\end{pmatrix}\Delta\begin{pmatrix}k_m\cdots\widehat{i'}\cdots\widehat{j'}\cdots k_{n-1} \begin{matrix} i' \\ j'\end{matrix}\end{pmatrix},\end{aligned}$$

which is from Lemma 9, where $\{k_0,\ldots,k_m,k_{m+1},\ldots,k_{n-1}\}$ is a permutation of the set of all indices $\{0,\ldots,n-1\}$. $\square$

Lemma 15. *The condition* (b) (*which is equivalent to* (b)*) *implies*

$$\begin{aligned}\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\binom{i'}{j'}&=\Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\binom{i}{j},\\ \Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\begin{pmatrix}k\begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}&=\Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\begin{pmatrix}k\begin{matrix} i \\ j\end{matrix}\end{pmatrix},\\ \Delta^*\binom{i}{k}\Delta^*\binom{k}{j}\Delta^*\binom{i'}{j'}&=\Delta^*\binom{i'}{k}\Delta^*\binom{k}{j'}\Delta^*\binom{i}{j},\\ \Delta^*\binom{i}{k}\Delta^*\binom{k}{j}\Delta^*\begin{pmatrix}k\begin{matrix} i' \\ j'\end{matrix}\end{pmatrix}&=\Delta^*\binom{i'}{k}\Delta^*\binom{k}{j'}\Delta^*\begin{pmatrix}k\begin{matrix} i \\ j\end{matrix}\end{pmatrix},\end{aligned}$$

for indices i, j, i', j', k *such that two indices of them appearing in an identical* $\Delta(\cdots)$ *or in an* $\Delta^*(\cdots)$ *are distinct.*

Proof. For the first equation, if $\{i,j\}\cap\{i',j'\}\neq\emptyset$ then it is obvious, for example, if $i=i'$ then $i\neq j'$, so we have

$$\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\binom{i}{j'}=\Delta\binom{i}{k}\Delta\binom{k}{j'}\Delta\binom{i}{j},$$

from (b). Otherwise, i.e., if $\{i, j\} \cap \{i', j'\} = \emptyset$, we have

$$\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\binom{i'}{j'} = \Delta\binom{i'}{k}\Delta\binom{k}{j}\Delta\binom{i}{j'} = \Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\binom{i}{j},$$

where the both equalities are from (b). For the second equation, we have

$$\begin{aligned}\Delta\binom{i}{k}\Delta\binom{k}{j}\Delta\left(k\,{}^{i'}_{j'}\right) &= \Delta\binom{i}{k}\Delta\binom{k}{j}\left(\Delta\binom{i'}{j'} - \Delta\binom{i'}{k}\Delta\binom{k}{j'}\right)\\ &= \Delta\binom{i'}{k}\Delta\binom{k}{j'}\left(\Delta\binom{i}{j} - \Delta\binom{i}{k}\Delta\binom{k}{j}\right)\\ &= \Delta\binom{i'}{k}\Delta\binom{k}{j'}\Delta\left(k\,{}^{i}_{j}\right),\end{aligned}$$

where the second equality is from the first equation. Similarly, the third and last equations are from (b)*, so they are from (b). □

From the previous lemma, the following values are determined only by the index k.

Definition 2. On the assumption (b), for an index $k = 0, \ldots, n-1$, if there exists a pair of distinct indices i, $j \neq k$ with $\Delta\binom{i}{j} \neq 0$ (resp. $\Delta\left(k\,{}^{i}_{j}\right) \neq 0$, $\Delta^*\binom{i}{j} \neq 0$, $\Delta^*\left(k\,{}^{i}_{j}\right) \neq 0$), we define

$$\mu_k = \frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\binom{i}{j}} \qquad \Bigg(\text{resp. } \nu_k = -\frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\left(k\,{}^{i}_{j}\right)},$$

$$\mu_k^* = \frac{\Delta^*\binom{i}{k}\Delta^*\binom{k}{j}}{\Delta^*\binom{i}{j}}, \qquad \nu_k^* = -\frac{\Delta^*\binom{i}{k}\Delta^*\binom{k}{j}}{\Delta^*\left(k\,{}^{i}_{j}\right)}\Bigg),$$

which depends only the index k.

Remark 10. For distinct indices k and ℓ, if there exist μ_k and μ_ℓ, then, for some pairs of distinct indices i, j and i', j' with $\ell \neq i \neq k \neq j$ and $i' \neq \ell \neq j' \neq k$, we have

$$\mu_k\mu_\ell = \frac{\Delta\binom{i}{k}\Delta\binom{k}{j}}{\Delta\binom{i}{j}}\frac{\Delta\binom{i'}{\ell}\Delta\binom{\ell}{j'}}{\Delta\binom{i'}{j'}}$$

$$= \frac{\Delta\begin{pmatrix} i \\ \ell \end{pmatrix}\Delta\begin{pmatrix} k \\ j \end{pmatrix}}{\Delta\begin{pmatrix} i \\ j \end{pmatrix}} \frac{\Delta\begin{pmatrix} i' \\ \ell \end{pmatrix}\Delta\begin{pmatrix} k \\ j' \end{pmatrix}}{\Delta\begin{pmatrix} i' \\ j' \end{pmatrix}} = \left(\Delta\begin{pmatrix} k \\ \ell \end{pmatrix}\right)^2.$$

If there exist μ_k and ν_k, then for some pairs of distinct indices i, $j \neq k$ and i', $j' \neq k$, we have

$$(1-\mu_k)(1-\nu_k) = \left(1 - \frac{\Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} k \\ j \end{pmatrix}}{\Delta\begin{pmatrix} i \\ j \end{pmatrix}}\right)\left(1 + \frac{\Delta\begin{pmatrix} i' \\ k \end{pmatrix}\Delta\begin{pmatrix} k \\ j' \end{pmatrix}}{\Delta\begin{pmatrix} k & i' \\ & j' \end{pmatrix}}\right)$$

$$= \frac{\Delta\begin{pmatrix} k & i \\ & j \end{pmatrix}}{\Delta\begin{pmatrix} i \\ j \end{pmatrix}} \frac{\Delta\begin{pmatrix} i' \\ j' \end{pmatrix}}{\Delta\begin{pmatrix} k & i' \\ & j' \end{pmatrix}} = 1,$$

where the last equality is from Lemma 9 (We also have

$$(1-\mu_k^*)(1-\nu_k^*) = 1$$

if μ_k^* and ν_k^* exist. It is obvious that the existence of μ_k (resp. ν_k, μ_k^*, and ν_k^*) and $\mu_k \neq 1$ (resp. $\nu_k \neq 1$, $\mu_k^* \neq 1$, and $\nu_k^* \neq 1$) implies the existence of ν_k (resp. μ_k, ν_k^*, and μ_k^*)).

The values μ_k^* (resp. μ_k) and ν_k (resp. ν_k^*) exist simultaneously and satisfy an equation.

Lemma 16. *Assume that the condition* (b) *holds. Then, for an index* $k = 0, \ldots, n-1$, *there exists* μ_k^* (*resp.* μ_k) *if and only if there exists* ν_k (*resp.* ν_k^*). *Moreover, if there exists* ν_k (*resp.* ν_k^*), *we have*

$$1-\mu_k^* = (1-\nu_k)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2 \qquad (\textit{resp. } 1-\mu_k = (1-\nu_k^*)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2).$$

Proof. Suppose that there does not exist μ_k^*, i.e., for an arbitrary pair of distinct indices i, $j \neq k$,

$$\mathbf{p}_i^* \cdot \mathbf{p}_j^* = \Delta^*\begin{pmatrix} i \\ j \end{pmatrix} = 0 \tag{9}$$

holds. Then, $\{\mathbf{p}_0^*, \ldots, \mathbf{p}_{k-1}^*, \mathbf{p}_k, \mathbf{p}_{k+1}^*, \ldots, \mathbf{p}_{n-1}^*\}$ is an orthonormal basis. Hence, we have

$$\mathbf{p}_k^* = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\mathbf{p}_k + \sum_{\ell=0}^{\substack{\ell\neq k\\ n-1}} \Delta^*\begin{pmatrix} \ell \\ k \end{pmatrix}\mathbf{p}_\ell^*, \tag{10}$$

$$\mathbf{p}_\ell = (\mathbf{p}_\ell \cdot \mathbf{p}_\ell^*)\mathbf{p}_\ell^* + \Delta\begin{pmatrix} \ell \\ k \end{pmatrix}\mathbf{p}_k \qquad \text{for an index } {}^\forall\ell \neq k. \tag{11}$$

So, we have

$$\Delta\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i \cdot \mathbf{p}_j = \Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} j \\ k \end{pmatrix}. \tag{12}$$

Therefore, we also have $\Delta\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix} = 0$, which means that there does not exist ν_k. Similarly, we can prove the nonentity of μ_k implies the nonentity of ν_k^*. Conversely, suppose that there exists μ_k, i.e., there exists a pair of distinct indices i, $j \neq k$ with $\Delta\begin{pmatrix} i \\ j \end{pmatrix} \neq 0$. If there also exists μ_k^*, i.e., if there exists a pair of distinct indices i', $j' \neq k$ with $\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix} \neq 0$, we have $\Delta^*\begin{pmatrix} k\,{}^{i'}_{j'}\end{pmatrix} \neq 0$ from

$$\Delta\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix} \Delta^*\begin{pmatrix} k\,{}^{i'}_{j'}\end{pmatrix} = \Delta\begin{pmatrix} i \\ j \end{pmatrix} \Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix} (\mathbf{p}_k \cdot \mathbf{p}_k^*)^2,$$

which is from Lemma 14. Hence there exists ν_k^*. Otherwise, i.e., if there does not exist μ_k^*, we have

$$\begin{aligned}
\Delta\begin{pmatrix} i \\ j \end{pmatrix} &= \Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} j \\ k \end{pmatrix} = \left(-\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\frac{\mathbf{p}_i \cdot \mathbf{p}_i^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\right)\left(-\Delta^*\begin{pmatrix} j \\ k \end{pmatrix}\frac{\mathbf{p}_j \cdot \mathbf{p}_j^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\right) \\
&= \frac{(\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*)}{(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2}\left(\Delta^*\begin{pmatrix} i \\ k \end{pmatrix}\Delta^*\begin{pmatrix} j \\ k \end{pmatrix} - \Delta^*\begin{pmatrix} i \\ j \end{pmatrix}\right) \\
&= -\frac{(\mathbf{p}_i \cdot \mathbf{p}_i^*)(\mathbf{p}_j \cdot \mathbf{p}_j^*)}{(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2}\Delta^*\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix},
\end{aligned}$$

where the first and third equality is from (12) and (9), respectively, and the second equality is from

$$0 = \mathbf{p}_i \cdot \mathbf{p}_k^* = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\Delta\begin{pmatrix} i \\ k \end{pmatrix} + (\mathbf{p}_i \cdot \mathbf{p}_i^*)\Delta^*\begin{pmatrix} i \\ k \end{pmatrix},$$
$$0 = \mathbf{p}_j \cdot \mathbf{p}_k^* = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\Delta\begin{pmatrix} j \\ k \end{pmatrix} + (\mathbf{p}_j \cdot \mathbf{p}_j^*)\Delta^*\begin{pmatrix} j \\ k \end{pmatrix},$$

whose last equalities are from (10) or (11). So we have $\Delta^*\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix} \neq 0$, hence there exists ν_k^*. Similarly, we can prove that the existence of μ_k^* implies the existence of ν_k. Moreover, if there exists ν_k, there also exists μ_k^*, so, there exist pairs of distinct indices i, $j \neq k$ and i', $j' \neq k$ with $\Delta\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix} \neq 0 \neq \Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix}$. Hence, we have

$$1 - \mu_k^* = \frac{\Delta^*\begin{pmatrix} k\,{}^{i'}_{j'}\end{pmatrix}}{\Delta^*\begin{pmatrix} i' \\ j' \end{pmatrix}} = \frac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}}{\Delta\begin{pmatrix} k\,{}^{i}_{j}\end{pmatrix}}(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2 = (1 - \nu_k)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2,$$

where the second equality is from Lemma 14. Similarly, the existence of ν_k^* implies the existence of μ_k and

$$1 - \mu_k = (1 - \nu_k^*)(\mathbf{p}_k \cdot \mathbf{p}_k^*)^2.$$

$\square$

5. Main results

We define the vector $\tilde{\mathbf{h}}_k$ (resp. $\tilde{\mathbf{h}}^*_k$) whose normalization $\mathbf{h}_k$ (resp. $\mathbf{h}^*_k$) is an orthocenter (see Lemma 20).

Definition 3. On the assumption (b), for an index $k = 0,\ldots, n-1$, if there exists ν_k (resp. ν^*_k), then we define

$$\tilde{\mathbf{h}}_k = \mathbf{p}^*_k - (1-\nu_k)(\mathbf{p}_k \cdot \mathbf{p}^*_k)\mathbf{p}_k = \mathbf{p}^*_k - \frac{1-\mu^*_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\mathbf{p}_k$$
$$\Big(\text{resp. } \tilde{\mathbf{h}}^*_k = \mathbf{p}_k - (1-\nu^*_k)(\mathbf{p}_k \cdot \mathbf{p}^*_k)\mathbf{p}^*_k = \mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\mathbf{p}^*_k\Big),$$

which is in $\mathbb{R}^n$. Moreover, if $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (resp. $\tilde{\mathbf{h}}^*_k \neq \mathbf{0}$), then we define

$$\mathbf{h}_k = \frac{\tilde{\mathbf{h}}_k}{|\tilde{\mathbf{h}}_k|} \qquad \Big(\text{resp. } \mathbf{h}^*_k = \frac{\tilde{\mathbf{h}}^*_k}{|\tilde{\mathbf{h}}^*_k|}\Big),$$

which is in $\mathbb{S}^{n-1}$.

The equations of the following lemma are relations of $\tilde{\mathbf{h}}_0,\ldots, \tilde{\mathbf{h}}_{n-1}, \tilde{\mathbf{h}}^*_0,\ldots, \tilde{\mathbf{h}}^*_{n-1}$, which means that existing ones of normalizations of them, $\mathbf{h}_0,\ldots, \mathbf{h}_{n-1}, \mathbf{h}^*_0,\ldots, \mathbf{h}^*_{n-1}$, are consistent, with the exception of the sign.

Lemma 17. *Assume that the condition* (b) *holds. Then, for pairwise distinct indices $k, \ell, i = 0,\ldots, n-1$, if there exist ν^*_k (resp. ν_k) and ν^*_ℓ (resp. ν_ℓ), we have*

$$\Delta\binom{i}{k}\tilde{\mathbf{h}}^*_\ell = \Delta\binom{i}{\ell}\tilde{\mathbf{h}}^*_k \qquad \Big(\textit{resp. } \Delta^*\binom{i}{k}\tilde{\mathbf{h}}_\ell = \Delta^*\binom{i}{\ell}\tilde{\mathbf{h}}_k\Big),$$

if there exist ν_k (resp. ν^*_k) and ν^*_ℓ (resp. ν_ℓ), we have

$$\nu_k\tilde{\mathbf{h}}^*_\ell = \Delta\binom{k}{\ell}\frac{\tilde{\mathbf{h}}_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k} \qquad \Big(\textit{resp. } \nu^*_k\tilde{\mathbf{h}}_\ell = \Delta^*\binom{k}{\ell}\frac{\tilde{\mathbf{h}}^*_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\Big),$$

if there exist ν_k and ν^*_k, we have

$$\nu_k\tilde{\mathbf{h}}^*_k = \mu_k\frac{\tilde{\mathbf{h}}_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k} \qquad \Big(\textit{resp. } \nu^*_k\tilde{\mathbf{h}}_k = \mu^*_k\frac{\tilde{\mathbf{h}}^*_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\Big).$$

Proof. We have the following equations by comparing the inner product of $\mathbf{p}_0,\ldots, \mathbf{p}_{n-1}$ and both sides:

$$\Delta\binom{i}{k}\left(\mathbf{p}_\ell - \frac{1-\mu_\ell}{\mathbf{p}_\ell \cdot \mathbf{p}^*_\ell}\mathbf{p}^*_\ell\right) = \Delta\binom{i}{\ell}\left(\mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\mathbf{p}^*_k\right),$$
$$\nu_k\left(\mathbf{p}_\ell - \frac{1-\mu_\ell}{\mathbf{p}_\ell \cdot \mathbf{p}^*_\ell}\mathbf{p}^*_\ell\right) = \Delta\binom{k}{\ell}\left(\frac{\mathbf{p}^*_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k} - (1-\nu_k)\mathbf{p}_k\right),$$
$$\nu_k\left(\mathbf{p}_k - \frac{1-\mu_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k}\mathbf{p}^*_k\right) = \mu_k\left(\frac{\mathbf{p}^*_k}{\mathbf{p}_k \cdot \mathbf{p}^*_k} - (1-\nu_k)\mathbf{p}_k\right).$$

□

The following three lemmas show the implication (d) $\Rightarrow$ (c).

Lemma 18. *Assume* (d). *Then there exists* ν_k *(resp.* ν_k^**), and*

$$\tilde{\mathbf{h}}_k \neq \mathbf{0} \qquad (\textit{resp.}\ \tilde{\mathbf{h}}_k^* \neq \mathbf{0}),$$

for some index $k = 0,\ldots, n-1$.

Proof. The condition (d) is equivalent to the condition (d)*, so, we can assume (3)* for some pairwise distinct indices i, k, ℓ. From the former and latter inequalities of (3)*, we have $\mathbf{p}_k \neq \mathbf{p}_k^*$ and the existence of μ_k^* and $\tilde{\mathbf{h}}_k$, respectively. The linear independence of $\mathbf{p}_k$ and $\mathbf{p}_k^*$ implies $\tilde{\mathbf{h}}_k \neq \mathbf{0}$. □

Lemma 19. *Assume* (b) *and the existence of* ν_k (*resp.* ν_k^*) *and* $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (*resp.* $\tilde{\mathbf{h}}_k^* \neq \mathbf{0}$) *for some index* $k = 0,\ldots, n-1$. *Then, orthocenters exist only at most two points and they are antipodal each other if there exist two orthocenters.*

Proof. From the existence of μ_k^*, there exist pair of distinct indices i, $j \neq k$ such that $\Delta^* \binom{i}{j} \neq 0$. We have $\mathbf{p}_i \neq \mathbf{p}_i^*$ (resp. $\mathbf{p}_j \neq \mathbf{p}_j^*$), so, there exists the unique great circular C_i (resp. C_j) passing through $\mathbf{p}_i$ and $\mathbf{p}_i^*$ (resp. $\mathbf{p}_j$ and $\mathbf{p}_j^*$). If there exist orthocenters, they are in the intersection of C_i and C_j, so, it is enough to show $C_i \neq C_j$. So, assume $C_i = C_j$. Then, $\{\mathbf{p}_i, \mathbf{p}_j\}$ is a basis of the 2-dim Euclidean space L spanned by C_i, so $\mathbf{p}_k^*$ is perpendicular to L. $\mathbf{p}_i^*$ and $\mathbf{p}_j^*$ are in L, so we have

$$\Delta^* \binom{i}{k} = 0 = \Delta^* \binom{j}{k},$$

which implies $\mu_k^* = 0$. From

$$\mathbf{0} \neq \tilde{\mathbf{h}}_k = \mathbf{p}_k^* - \frac{1-\mu_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*}\mathbf{p}_k = \mathbf{p}_k^* - \frac{\mathbf{p}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*},$$

we also have $\mathbf{p}_k \neq \mathbf{p}_k^*$. Hence there exists an index $\ell \neq k$ such that $\Delta^* \binom{\ell}{k} \neq 0$. Inequalities $i \neq \ell \neq j$ implies

$$0 \neq \Delta^* \binom{i}{j} \Delta^* \binom{\ell}{k} = \Delta^* \binom{i}{k} \Delta^* \binom{\ell}{j} = 0 \cdot \Delta^* \binom{\ell}{j},$$

which is a contradiction. □

Lemma 20. *Assume* (b) *and the existence of* ν_k (*resp.* ν_k^*) *and* $\tilde{\mathbf{h}}_k \neq \mathbf{0}$ (*resp.* $\tilde{\mathbf{h}}_k^* \neq \mathbf{0}$(*for some index* $k = 0,\ldots, n-1$. *Then*

$$\pm\,\mathbf{h}_k \qquad (\textit{resp.}\ \ \pm\,\mathbf{h}_k^*)$$

are an pair of orthocenters.

Proof. It is enough to show that either $\mathbf{p}_i = \mathbf{p}_i^*$ holds or $\tilde{\mathbf{h}}_k$ is a linear combination of $\mathbf{p}_i$ and $\mathbf{p}_i^*$ for each index $i \neq k$. The proof is divided into 4 cases. First, suppose

$\Delta^* \binom{i}{j} \neq 0$ for some index $j \neq i$, k and the existence of μ_i^*. Then we have

$$\tilde{\mathbf{h}}_k = \frac{\Delta^* \binom{k}{j}}{\Delta^* \binom{i}{j}} \tilde{\mathbf{h}}_i = \frac{\Delta^* \binom{k}{j}}{\Delta^* \binom{i}{j}} \left(\mathbf{p}_i^* - \frac{1-\mu_i^*}{\mathbf{p}_i \cdot \mathbf{p}_i^*} \mathbf{p}_i \right).$$

Secondly, suppose $\Delta^* \binom{i}{j} \neq 0$ for some index $j \neq i$, k and the nonentity of μ_i^*. Then the set $\{\mathbf{p}_0^*, \ldots, \mathbf{p}_{i-1}^*, \mathbf{p}_i, \mathbf{p}_{i+1}^*, \ldots, \mathbf{p}_{n-1}^*\}$ is an orthonormal basis, so, we have

$$\mu_k^* = \frac{\Delta^* \binom{i}{k} \Delta^* \binom{k}{j}}{\Delta^* \binom{i}{j}} = \frac{\Delta^* \binom{i}{k} \cdot 0}{\Delta^* \binom{i}{j}},$$

$$\mathbf{p}_k = (\mathbf{p}_k \cdot \mathbf{p}_k^*)\mathbf{p}_k^* + \Delta \binom{k}{i} \mathbf{p}_i,$$

hence, we also have

$$\tilde{\mathbf{h}}_k = \mathbf{p}_k^* - \frac{1-\mu_k^*}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \mathbf{p}_k = \mathbf{p}_k^* - \frac{\mathbf{p}_k}{\mathbf{p}_k \cdot \mathbf{p}_k^*} = -\frac{\Delta \binom{k}{i}}{\mathbf{p}_k \cdot \mathbf{p}_k^*} \mathbf{p}_i.$$

Thirdly, suppose $\Delta^* \binom{i}{j'} = 0$ for an arbitrary index $j' \neq i$, k and $\Delta^* \binom{k}{j} \neq 0$ for some index $j \neq i$, k. Then there exists μ_i^* and

$$\tilde{\mathbf{h}}_i = \frac{\Delta^* \binom{i}{j}}{\Delta^* \binom{k}{j}} \tilde{\mathbf{h}}_k = \frac{0}{\Delta^* \binom{k}{j}} \tilde{\mathbf{h}}_k,$$

so, $\mathbf{p}_i = \mathbf{p}_i^*$. At last, suppose $\Delta^* \binom{i}{j'} = 0 = \Delta^* \binom{k}{j'}$ for an arbitrary index $j' \neq i$, k. Then vectors $\mathbf{p}_k$, $\mathbf{p}_k^*$, $\mathbf{p}_i$, $\mathbf{p}_i^*$ are in a 2-dim Euclidean space, so, either $\mathbf{p}_i = \mathbf{p}_i^*$ holds, or $\tilde{\mathbf{h}}_k$, a linear combination of $\mathbf{p}_k$ and $\mathbf{p}_k^*$, is a linear combination of $\mathbf{p}_i$ and $\mathbf{p}_i^*$. $\square$

Now, we can prove the conclusions of this paper.

Proof of Theorems 1 and 2. The equivalences (b) $\Leftrightarrow$ (b)* and (d) $\Leftrightarrow$ (d)* are already shown by Lemma 12 and Corollary 13, respectively. And the implications (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are shown by Lemmas 3 and 4, and Lemmas 18, 19, and 20, respectively. So we only have to show the converses (b) $\Rightarrow$ (a) and

$\neg$(d) $\Rightarrow$ $\neg$(c), where $\neg$ means the negation. To show them, because of the implications (c) $\Rightarrow$ (a) and $\neg$(b) $\Rightarrow$ $\neg$(a), it is enough to show (b) $\wedge$ $\neg$(d) $\Rightarrow$ (a) $\wedge$ $\neg$(c), which is already mentioned in Remark 6. $\square$

6. Appendix.

For an orthocentric simplex, by repeating to replace a vertex with one of its orthocenters, new vertex does not appear except original vertices, their antipodals, and orthocenters. To prove it, we consider a generic case, that is, we assume that neither $\Delta\begin{pmatrix} i \\ j \end{pmatrix}$, $\Delta\begin{pmatrix} k & i \\ & j \end{pmatrix}$, $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}$, nor $\Delta^*\begin{pmatrix} k & i \\ & j \end{pmatrix}$ is equal to 0 for pairwise distinct i, j, $k = 0,\ldots, n-1$, for an orthocentric simplex S. Then, for $k = 0,\ldots, n-1$, all of μ_k, ν_k, μ_k^*, ν_k^* exist, neither of them is equal to 0, and neither of them is equal to 1. The simplex replacing the last vertex with the orthocenter $\mathbf{h}_{n-1}$ is denoted by $\mathcal{M}(S)$, that is, vertices of $S' = \mathcal{M}(S)$ are

$$\mathbf{p}_k'^* = \mathbf{p}_k^*, \qquad \mathbf{p}_{n-1}'^* = \mathbf{h}_{n-1}, \tag{13}$$

and vertices of S'^* are

$$\mathbf{p}_k' = \frac{\mathbf{p}_k - \dfrac{\Delta\begin{pmatrix} k \\ n-1 \end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}}{\sqrt{-\mu_k\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)}}, \qquad \mathbf{p}_{n-1}' = \operatorname{sgn}\nu_{n-1}\cdot\mathbf{p}_{n-1}, \tag{14}$$

for $k = 0,\ldots, n-2$. Notice that S' is also orthocentric (see Lemma 3). Then, we have

$$\nu_k' = \frac{1}{1 - \dfrac{1}{\mu_k'}} = \frac{1}{1 - \dfrac{\Delta'\begin{pmatrix} i \\ j \end{pmatrix}}{\Delta'\begin{pmatrix} i \\ k \end{pmatrix}\Delta'\begin{pmatrix} k \\ j \end{pmatrix}}} = \frac{1}{1 - \dfrac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}\nu_{n-1}\mu_k\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)}{\Delta\begin{pmatrix} i \\ k \end{pmatrix}\Delta\begin{pmatrix} k \\ j \end{pmatrix}}}$$
$$= \frac{1}{1 - \nu_{n-1}\left(\dfrac{1}{\nu_k} + \dfrac{1}{\nu_{n-1}}\right)} = -\frac{\nu_k}{\nu_{n-1}}, \tag{15}$$

for $k = 0,\ldots, n-2$, where the third equality is from

$$\Delta'\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}_i'\cdot\mathbf{p}_j' = \frac{\Delta\begin{pmatrix} i \\ j \end{pmatrix} - \dfrac{2\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta\begin{pmatrix} j \\ n-1 \end{pmatrix}}{\mu_{n-1}} + \dfrac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta\begin{pmatrix} j \\ n-1 \end{pmatrix}}{\mu_{n-1}^2}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j} + \dfrac{1}{\nu_{n-1}}\right)}}$$

$$= \frac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}(1 - 2 + \dfrac{1}{\mu_{n-1}})}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j} + \dfrac{1}{\nu_{n-1}}\right)}}$$

$$= \frac{-\,\Delta\begin{pmatrix} i \\ j \end{pmatrix}/\nu_{n-1}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}\sqrt{-\mu_j\left(\dfrac{1}{\nu_j} + \dfrac{1}{\nu_{n-1}}\right)}}, \tag{16}$$

for an arbitrary pair of distinct i, $j = 0,\ldots, n-2$. We also have

$$\nu'_{n-1} = \frac{1}{1 - \dfrac{1}{\mu'_{n-1}}} = \frac{1}{1 - \dfrac{\Delta'\begin{pmatrix} i \\ j \end{pmatrix}}{\Delta'\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta'\begin{pmatrix} n-1 \\ j \end{pmatrix}}} = \frac{1}{1 + \dfrac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}\nu_{n-1}}{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}\Delta\begin{pmatrix} n-1 \\ j \end{pmatrix}}}$$

$$= \frac{1}{1 + \dfrac{\nu_{n-1}}{\mu_{n-1}}} = \frac{1}{\nu_{n-1}}, \tag{17}$$

where the third equality is from (16) and

$$\Delta'\begin{pmatrix} i \\ n-1 \end{pmatrix} = \operatorname{sgn}\nu_{n-1}\cdot\frac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix} - \dfrac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}}{\mu_{n-1}}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}} = \operatorname{sgn}\nu_{n-1}\cdot\frac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}/\nu_{n-1}}{\sqrt{-\mu_i\left(\dfrac{1}{\nu_i} + \dfrac{1}{\nu_{n-1}}\right)}},$$

for arbitrary $i = 0,\ldots, n-2$. For $k = 0,\ldots, n-2$, it is obvious that

$$\nu'^*_k = \nu^*_k, \tag{18}$$

because

$$\Delta'^*\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{p}'^*_i \cdot \mathbf{p}'^*_j = \mathbf{p}^*_i \cdot \mathbf{p}^*_j = \Delta^*\begin{pmatrix} i \\ j \end{pmatrix},$$

for an arbitrary pair of distinct i, $j = 0,\ldots, n-2$. To calculate ν'^*_{n-1}, we need preparation. From (8), for $m = 0,\ldots, n-2$ and distinct i, $j = m,\ldots, n-1$, we can prove

$$\Delta\begin{pmatrix} 0\cdots m-1 \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} = \frac{\Delta\begin{pmatrix} i \\ j \end{pmatrix}}{(1-\nu_0)\cdots(1-\nu_{m-1})} \tag{19}$$

$$\left(\text{resp.}\quad \Delta^*\begin{pmatrix} 0\cdots m-1 \, \begin{matrix} i \\ j \end{matrix} \end{pmatrix} = \frac{\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}}{(1-\nu^*_0)\cdots(1-\nu^*_{m-1})}\right), \tag*{(19)*}$$

by induction of m. From (7) and (19), for $m = 0,\ldots, n-1$, we can also prove

$$\Delta\,(0\cdots m) = \frac{1-\nu_0-\cdots-\nu_m}{(1-\nu_0)\cdots(1-\nu_m)} \tag{20}$$

$$\left(\text{resp.}\quad \Delta^*\,(0\cdots m) = \frac{1-\nu_0^*-\cdots-\nu_m^*}{(1-\nu_0^*)\cdots(1-\nu_m^*)}\right), \tag{20*}$$

by induction. Hence, we have

$$(\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*)^2 = \frac{\Delta^*\,(0\cdots n-1)}{\Delta^*\,(0\cdots n-2)} = \frac{1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*}{(1-\nu_0^*-\cdots-\nu_{n-2}^*)(1-\nu_{n-1}^*)}, \tag{21}$$

and

$$\begin{aligned}|\tilde{\mathbf{h}}_{n-1}|^2 &= \left|\mathbf{p}_{n-1}^* - \frac{\mathbf{p}_{n-1}}{(1-\nu_{n-1}^*)\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*}\right|^2 \\ &= 1 - \frac{2}{1-\nu_{n-1}^*} + \frac{1}{(1-\nu_{n-1}^*)^2(\mathbf{p}_{n-1}\cdot\mathbf{p}_{n-1}^*)^2} \\ &= \frac{\nu_{n-1}^*(\nu_0^*+\cdots+\nu_{n-2}^*+\nu_{n-1}^*)}{(1-\nu_{n-1}^*)(1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*)}. \end{aligned} \tag{22}$$

Now, we can calculate $\nu_{n-1}'^*$:

$$\begin{aligned}\nu_{n-1}'^* &= \frac{1}{1-\dfrac{1}{\mu_{n-1}'^*}} = \frac{1}{1-\dfrac{\Delta'^*\binom{i}{j}}{\Delta'^*\binom{i}{n-1}\Delta'^*\binom{j}{n-1}}} = \frac{1}{1-\dfrac{\Delta^*\binom{i}{j}|\tilde{\mathbf{h}}_{n-1}|^2}{\Delta^*\binom{i}{n-1}\Delta^*\binom{j}{n-1}}} \\ &= \frac{1}{1-\dfrac{|\tilde{\mathbf{h}}_{n-1}|^2}{\mu_{n-1}^*}} = 1-\nu_0^*-\cdots-\nu_{n-2}^*-\nu_{n-1}^*, \end{aligned} \tag{23}$$

where the third equality is from $\Delta'^*\binom{i}{j} = \Delta^*\binom{i}{j}$ and

$$\Delta'^*\binom{i}{n-1} = \mathbf{p}_i'^*\cdot\mathbf{p}_{n-1}'^* = \frac{\mathbf{p}_i^*\cdot\tilde{\mathbf{h}}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} = \frac{\mathbf{p}_i^*\cdot\mathbf{p}_{n-1}^*}{|\tilde{\mathbf{h}}_{n-1}|} = \frac{\Delta^*\binom{i}{n-1}}{|\tilde{\mathbf{h}}_{n-1}|},$$

for an arbitrary pair of distinct $i, j = 0,\ldots, n-2$. To get the purpose of this section, it is enough to consider two following cases. In the case where $S' = \mathcal{M}(S)$ and $S'' = \mathcal{M}(S')$ holds, S'' is consistent with S. In particular, we have

$$\mathbf{p}_k''^* = \mathbf{p}_k'^* = \mathbf{p}_k^*,$$

for $k = 0,\ldots, n-2$, and

$$\mathbf{p}_{n-1}''^* = \mathbf{h}_{n-1}' = \mathbf{p}_{n-1}^*,$$

where the last equality is from

$$\begin{aligned}
\tilde{\mathbf{h}}'_{n-1} &= \mathbf{p}'^*_{n-1} - (1-\nu'_{n-1})(\mathbf{p}'_{n-1}\cdot\mathbf{p}'^*_{n-1})\mathbf{p}'_{n-1} \\
&= \frac{\tilde{\mathbf{h}}_{n-1} - \left(1 - \dfrac{1}{\nu_{n-1}}\right)(\mathbf{p}_{n-1}\cdot\tilde{\mathbf{h}}_{n-1})\mathbf{p}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} \\
&= \frac{\left(\mathbf{p}^*_{n-1} - (1-\nu_{n-1})\,(\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1})\mathbf{p}_{n-1}\right) - \left(1 - \dfrac{1}{\nu_{n-1}}\right)\nu_{n-1}(\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1})\mathbf{p}_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|} \\
&= \frac{\mathbf{p}^*_{n-1}}{|\tilde{\mathbf{h}}_{n-1}|}.
\end{aligned}$$

On the other hand, in the case where $S' = \mathcal{M}(S)$ holds, S'' is the simplex replacing last two vertices of S' each other, i.e.,

$$\mathbf{p}''^*_k = \mathbf{p}'^*_k \quad \text{for } k = 0,\dots,n-3, \qquad \mathbf{p}''^*_{n-2} = \mathbf{p}'^*_{n-1}, \qquad \mathbf{p}''^*_{n-1} = \mathbf{p}'^*_{n-2},$$

and $S''' = \mathcal{M}(S'')$ holds, S''' is the simplex with vertices

$$\mathbf{p}'''^*_k = \mathbf{p}''^*_k = \mathbf{p}'^*_k = \mathbf{p}^*_k,$$

for $k = 0,\dots, n-3$,

$$\mathbf{p}'''^*_{n-2} = \mathbf{p}''^*_{n-2} = \mathbf{p}'^*_{n-1} = \mathbf{h}_{n-1},$$

and

$$\mathbf{p}'''^*_{n-1} = \mathbf{h}''_{n-1} = \operatorname{sgn}\left(\frac{-\Delta\dbinom{n-2}{n-1}}{(1-\nu^*_{n-2})\nu_{n-1}}\right)\mathbf{p}^*_{n-1},$$

where the last equality is from

$$\begin{aligned}
\tilde{\mathbf{h}}''_{n-1} &= \mathbf{p}''^*_{n-1} - \frac{\mathbf{p}''_{n-1}}{(1-\nu''^*_{n-1})\mathbf{p}''_{n-1}\cdot\mathbf{p}''^*_{n-1}} \\
&= \mathbf{p}'^*_{n-2} - \frac{\mathbf{p}'_{n-2}}{(1-\nu'^*_{n-2})\mathbf{p}'_{n-2}\cdot\mathbf{p}'^*_{n-2}} \\
&= \mathbf{p}^*_{n-2} - \frac{\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\dbinom{n-2}{n-1}\mathbf{p}_{n-1}}{(1-\nu^*_{n-2})\left(\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\dbinom{n-2}{n-1}\mathbf{p}_{n-1}\right)\cdot\mathbf{p}^*_{n-2}} \\
&= \mathbf{p}^*_{n-2} - \frac{\mathbf{p}_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\dbinom{n-2}{n-1}\mathbf{p}_{n-1}}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}} \\
&= \; - \frac{\tilde{\mathbf{h}}^*_{n-2} + \dfrac{1-\nu_{n-1}}{\nu_{n-1}}\Delta\dbinom{n-2}{n-1}\mathbf{p}_{n-1}}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}}
\end{aligned}$$

$$= -\frac{\dfrac{\Delta\dbinom{n-2}{n-1}}{\nu_{n-1}}\left(\dfrac{\tilde{\mathbf{h}}_{n-1}}{\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1}}+(1-\nu_{n-1})\mathbf{p}_{n-1}\right)}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}}$$

$$= -\frac{\dfrac{\Delta\dbinom{n-2}{n-1}}{\nu_{n-1}}\cdot\dfrac{\mathbf{p}^*_{n-1}}{\mathbf{p}_{n-1}\cdot\mathbf{p}^*_{n-1}}}{(1-\nu^*_{n-2})\mathbf{p}_{n-2}\cdot\mathbf{p}^*_{n-2}}.$$

From two cases above, if we denote the orthocenters of S by $\pm\,\mathbf{p}^*_n$, then orthocenters of the simplex with vertices $\mathbf{p}^*_0,\ldots,\widehat{\mathbf{p}^*_k},\ldots,\mathbf{p}^*_n$ are $\pm\,\mathbf{p}^*_k$, for $k=0,\ldots,n-1$ (notice that neither μ_i, ν_i, μ^*_i, nor ν^*_i changes if a vertex is replaced with its antipodal for arbitrary $i=0,\ldots,n-1$). Moreover, for $k=0,\ldots,n$, from (18), ν^*_k does not change if S is replaced with the simplex with vertices $\mathbf{p}^*_0,\ldots,\widehat{\mathbf{p}^*_i},\ldots,\mathbf{p}^*_n$ for some $i=0,\ldots,\widehat{k},\ldots,n-1$. If we denote ν'^*_{n-1} for $S'=\mathcal{M}(S)$ by ν^*_n, we have

$$\nu^*_0+\cdots+\nu^*_{n-1}+\nu^*_n=1, \tag{24}$$

from (23) (see the equations (2) of [1] and (4) of [2]).

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On Centers and Central Lines of Triangles in the Elliptic Plane

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Abstract. We determine barycentric coordinates of triangle centers in the elliptic plane. The main focus is put on centers that lie on lines whose euclidean limit (triangle excess $\to 0$) is the Euler line or the Brocard line. We also investigate curves which can serve in elliptic geometry as substitutes for the euclidean nine-point-circle, the first Lemoine circle or the apollonian circles.

Introduction

In the first section we give a short introduction to metric geometry in the projective plane. We assume the reader is familiar with this subject, but we recall some fundamental definitions and theorems, in order to introduce the terminology and to fix notations. The second section provides appropriate tools (definitions, theorems, rules) for calculating the barycentric coordinates (in section 3) of a series of centers lying on four central lines of a triangle in the elliptic plane.
The content of this work is linked to results presented by Wildberger [27], Wildberger and Alkhaldi [28], Ungar [25], Horváth [8], Vigara [26], Russell [18].

1. Metric geometry in the projective plane

1.1. *The projective plane, its points and its lines.*

Let V be the three dimensional vector space $\mathbb{R}^3$, equipped with the canonical dot product $\boldsymbol{p}\cdot\boldsymbol{q} = (p_0, p_1, p_2)\cdot(q_0, q_1, q_2) = p_0q_0 + p_1q_1 + p_2q_2$ and the induced norm $\|.\|$, and let $\mathcal{P}$ denote the projective plane $(V - \{\mathbf{0}\})/\mathbb{R}^\times$. The image of a non-zero vector $\boldsymbol{p} = (p_0, p_1, p_2) \in V$ under the canonical projection $\Pi : V \to \mathcal{P}$ will be denoted by $(p_0 : p_1 : p_2)$ and will be regarded as a point in this plane.

Given two different points P and Q in this projective plane, there exists exactly one line that is incident with these two points. It is called the *join* $P \vee Q$ of P and Q. If $\boldsymbol{p} = (p_0, p_1, p_3)$, $\boldsymbol{q} = (q_0, q_1, q_2)$ are two non zero vectors with $\Pi(\boldsymbol{p}) = P$ and $\Pi(\boldsymbol{q}) = Q$, then the line $P \vee Q$ through P and Q is the set of points $\Pi(s\boldsymbol{p} + t\boldsymbol{q})$ with $s, t \in \mathbb{R}$. One can find linear forms $l \in V^* - \{\mathbf{0^*}\}$ with $ker(l) = span(\boldsymbol{p}, \boldsymbol{q})$. A suitable l is, for example, $l = *(\boldsymbol{p}\times\boldsymbol{q}) = (\boldsymbol{p}\times\boldsymbol{q})^*$, where $\times$ stands for the canonical cross product on $V = \mathbb{R}^3$ and $*$ for the isomorphism $V \to V^*, *(\boldsymbol{r}) = (.)\cdot\boldsymbol{r}$. The linear form l is uniquely determined up to a nonzero

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real factor, so there is a 1:1-correspondence between the lines in the projective plane and the elements of $\mathcal{P}^* = (V^* - \{\mathbf{0^*}\})/\mathbb{R}^\times$. We identify the line $l = P \vee Q$ with the element $(p_1q_2 - p_2q_1 : p_2q_0 - p_0q_2 : p_0q_1 - p_1q_0)^* \in \mathcal{P}^*$.

In the projective plane, two different lines $k = (k_0 : k_1 : k_2)^*, l = (l_0 : l_1 : l_2)^*$ always meet in one point $k \wedge l = \Pi((k_0, k_1, k_2) \times (l_0, l_1, l_2))$, the so called *meet* of these lines.

1.2. *Visualizing points and lines.*

Using an orthogonal coordinate system, we know how to visualize (in a canonical way) a point with cartesian coordinates (p_1, p_2) in the affine plane $\mathbb{A}^2$. A point $P = (1 : p_1 : p_2) \in \mathcal{P}$ will be visualized as the point (p_1, p_2) in the affine plane, and we will call $P^\vee := (1, p_1, p_2) \in \mathbb{R}^3$ the visualizing vector of P. But in $\mathcal{P}$ there exist points $(0 : p_1 : p_2)$ which can not be visualized in the affine plane. These points are considered to be points on the "line at infinity". For these points we define $P^\vee := (0, 1, p_2/p_1)$, if $p_1 \neq 0$, and $P^\vee := (0, 0, 1)$, otherwise. In this way, we ensure that the triple $P^\vee$ is strictly positive with respect to the lexicographic order.

A line appears as a "stright line" in the coordinate system.

1.3. *Collineations and correlations.*

A *collineation* on $\mathcal{P}$ is a bijective mapping $\mathcal{P} \to \mathcal{P}$ that maps lines to lines. These collineations form the group of automorphisms of $\mathcal{P}$.
Collineations preserve the cross ratio $(P, Q; R, S)$ of four points on a line.

A *correlation* on the projective plane is either a point-to-line transformation that maps collinear points to concurrent lines, or it is a line-to-point transformation that maps concurrent lines to collinear points.
Correlations, as collineationes, preserve the cross ratio.

1.4. *Metrical structures on $\mathcal{P}$.*

1.4.1. *The absolute conic.* One of the correlations is the *polarity* with respect to the *absolute conic* $\mathcal{C}$. This correlation assigns each point $P = (p_0 : p_1 : p_2)$ its polar line $P^\delta = (p_0 : \sigma p_1 : \varepsilon\sigma p_2)^*$ and assigns each line $l = (l_0 : l_1 : l_2)^*$ the corresponding pole $l^\delta = (\varepsilon\sigma l_0 : \varepsilon l_1 : l_2)$. Here, $\sigma \in \mathbb{R}^\times$, $\varepsilon \in \{-1, 1\}$, and the absolute conic consists of all points P with $P \in P^\delta$. P^δ and l^δ are called the dual of P and l, respectively.

Besides the norm $\|.\|$ we introduce a seminorm $\|.\|_{\sigma,\varepsilon}$ on V; this is defined by: $\|\boldsymbol{p}\|_{\sigma,\varepsilon} = \sqrt{|p_0^2 + \sigma p_1^2 + \varepsilon\sigma p_2^2|}$ for $\boldsymbol{p} = (p_0, p_1, p_2)$. It can be easily checked that $P = \Pi(\boldsymbol{p}) \in \mathcal{C}$ exactly when $\|\boldsymbol{p}\|_{\sigma,\varepsilon} = 0$. Points on $\mathcal{C}$ are called *isotropic points*.

The dual of an isotropic point is an *isotropic line*. As isotropic points, also isotropic lines form a conic, the dual conic $\mathcal{C}^\delta$ of $\mathcal{C}$.

1.4.2. *Cayley-Klein geometries.* Laguerre and Cayley were presumably the first to recognize that conic sections can be used to define the angle between lines and the distance between points, cf. [1]. An important role within this connection plays the cross ratio of points and of lines. We do not go into the relationship between conics and measures; there are many books and articles about Cayley-Klein-geometries (for example [13, 15, 17, 22, 23]) treating this subject. Particularly extensive investigations on cross-ratios offers Vigara [26].

Later, systematic studies by Felix Klein [13] led to a classification of metric geometries on $\mathcal{P}$. He realized that not only a geometry determines its automorphisms, but one can make use of automorphisms to define a geometry [12]. The automorphisms on $\mathcal{P}$ are the collineations. By studying the subgroup of collineations that keep the absolute conic fixed (as a whole, not pointwise), he was able to find different metric geometries on $\mathcal{P}$.

If $\varepsilon = 1$ and $\sigma > 0$, there are no real points on the absolute conic, so there are no isotropic points and no isotropic lines. The resulting geometry was called *elliptic* by Klein. It is closely related to spherical geometry. From the geometry on a sphere we get an elliptic geometry by identifying antipodal points. Already Riemann had used spherical geometry to get a new metric geometry with constant positive curvature, cf. [14, ch. 38].

Klein [13] also showed that in the elliptic case the euclidean geometry can be received as a limit for $\sigma \to 0$. (For $\sigma \to \infty$ one gets the polar-euclidean geometry.)

1.5. *Metrical structures in the elliptic plane.*

In the following, we consider just the elliptic case. Thus, we assume $\varepsilon = 1$ and $\sigma > 0$. Nearly all our results can be transferred to other Cayley-Klein geometries and even to a "mixed case" where the points lie in different connected components of $\mathcal{P} - \mathcal{C}$, cf. [10, 18, 25, 27, 28]. Nevertheless, it is less complicated to derive results in the elliptic case, because: First, there are no isotropic points and lines. Secondly, if we additionally put σ to 1 - and this is what we are going to do - , then the norm $\|.\|_{\sigma,\varepsilon}$ agrees with the standard norm $\|.\|$ and this simplifies many formulas. For example, we have $(p_0{:}p_1{:}p_2)^\delta = (p_0{:}p_1{:}p_2)^*$.

1.5.1. *Barycentric coordinates of points.* For $P \in \mathcal{P}$, define the vector P° by $P^\circ := P^\vee/\|P^\vee\|$. Given n points $P_1, \cdots, P_n \in \mathcal{P}$, we say that a point P is a (*linear*) *combination* of $P_1, \cdots, P_n$, if there are real numbers $t_1, \cdots, t_n$ such that $P = \Pi(t_1P_1^\circ + \cdots + t_nP_n^\circ)$, and we write $P = t_1P_1 + \cdots + t_nP_n$.
The points $P_1, \cdots, P_n$ form a *dependent system* if one of the n points is a combination of the others. Otherwise, $P_1, \cdots, P_n$ are *independent*. A single point is always independent, so are two different points. Three points are independent exactly when they are not collinear. And more than three points in $\mathcal{P}$ always form a dependent system.

If $\Delta = ABC$ is a triple of three non-collinear points A, B, C, then every point $P \in \mathcal{P}$ can be written as a combination of these. If $P = s_1A + s_2B + s_3C$ and $P = t_1A + t_2B + t_3C$ are two such combinations, then there is always a real number $c \neq 0$ such that $c(s_1, s_2, s_3) = (t_1, t_2, t_3)$. Thus, the point P is determined by Δ and the homogenous triple $(s_1 : s_2 : s_2)$. We write $P = [s_1 : s_2 : s_2]_\Delta$ and call this the *representation of* P *by barycentric coordinates* with respect to Δ. The terminology is not uniform here; the coordinates are also named *gyrobarycentric* (Ungar [25]), *circumlinear* (Wildberger, Alkhaldi [28]), *triangular* (Horváth [10]).

Barycentric coordinates of a point P can be calculated the following way: Because $A^\circ, B^\circ, C^\circ$ form a basis of $\mathbb{R}^3$, there is a unique way of representing P°

by a linear combination $P^\circ = s_1A^\circ + s_2B^\circ + s_3C^\circ$ of the base vectors; and the coordinates are

$$s_1 = \frac{P^\circ \cdot (B^\circ \times C^\circ)}{A^\circ \cdot (B^\circ \times C^\circ)},\ s_2 = \frac{P^\circ \cdot (C^\circ \times A^\circ)}{B^\circ \cdot (C^\circ \times A^\circ)},\ s_3 = \frac{P^\circ \cdot (A^\circ \times B^\circ)}{C^\circ \cdot (A^\circ \times B^\circ)}.$$

Since $A^\circ{\cdot}(B^\circ \times C^\circ) = B^\circ{\cdot}(C^\circ \times A^\circ) = C^\circ{\cdot}(A^\circ \times B^\circ)$, we get

$$s_1 : s_2 : s_3 = P^\circ{\cdot}(B^\circ \times C^\circ) : P^\circ{\cdot}(C^\circ \times A^\circ) : P^\circ{\cdot}(A^\circ \times B^\circ).$$

1.5.2. *Orthogonality.* We define orthogonality via polarity: A line k is *orthogonal* (or *perpendicular*) to a line l exactly when the dual k^δ of k is a point on l. It can easily be shown that, if k is orthogonal to l, then l is orthogonal to k.
The orthogonality between points is also defined: Two points are orthogonal precisely when their dual lines are. We can make use of the dot product to check if two points are orthogonal: Two points P and Q are orthogonal precisely when $P^\circ{\cdot}\, Q^\circ = 0$. Obviously, the set of points orthogonal to a point P is its polar line P^δ.

1.5.3. *The distance between points and the length of line segments in elliptic geometry.* Lines in elliptic geometry are without boundary. They are all the same length, usually set to π; this equals one half of the length of the great circle on a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. In this case, the distance between two points P and Q is $d(P,Q) = \varphi$ with $\cos(\varphi) = |P^\circ{\cdot}\, Q^\circ|$. $P^\circ{\cdot}\, Q^\circ$ always takes values in the interval $]-1;1]$, so the distance between two points P and Q lies in the interval $[0, \frac{\pi}{2}]$, and $d(P,Q) = \frac{\pi}{2}$ implies that $P^\circ{\cdot}\, Q^\circ = 0$ and P and Q are orthogonal.

Two different points P and Q determine the line $P \vee Q$. The set $P \vee Q - \{P,Q\}$ consists of two connected components, the closure of these are called the *line segments* of P, Q. One of these two segments contains all points $\Pi(sP^\circ + (1-s)Q^\circ)$ with $s(1-s){\geq}0$, while the other contains all points $\Pi(sP^\circ + (1-s)Q^\circ)$ with $s(1-s){\leq}0$. The first segment will be denoted by $[P,Q]_+$, the second by $[P,Q]_-$.

We show that $P + Q$ is the midpoint of $[P,Q]_+$ by proving the equation $(P + Q)^\circ{\cdot}P^\circ = (P+Q)^\circ{\cdot}Q^\circ$:

$$(P+Q)^\circ{\cdot}P^\circ = \frac{(P^\circ+Q^\circ){\cdot}P^\circ}{\sqrt{(P^\circ+Q^\circ){\cdot}(P^\circ+Q^\circ)}} = \frac{1+P^\circ{\cdot}Q^\circ}{\sqrt{2(1+P^\circ{\cdot}Q^\circ)}} = \sqrt{\frac{1+P^\circ{\cdot}Q^\circ}{2}} = (P+Q)^\circ{\cdot}Q^\circ.$$

In the same way it can be verified that $P - Q$ is the midpoint of $[P,Q]_-$. Since $(P^\circ + Q^\circ){\cdot}(P^\circ - Q^\circ) = 0$, the two points $P + Q$ and $P - Q$ are orthogonal.
We now can calculate the measures (lengths) of the segments $[P,Q]_+$ and $[P,Q]_-$:

$$\mu([P,Q]_+) = \arccos(P^\circ \cdot Q^\circ) \in [0,\pi[\text{ and } \mu([P,Q]_-) = \pi - \mu([P,Q]_+).$$

For further calculations the following formula will be useful:

$$\sin(\mu([P,Q]_+)) = \sin(\mu([P,Q]_-)) = \sin(d(P,Q)) = \|P^\circ \times Q^\circ\|.$$

Proof of this formula: $\sin(\mu([P,Q]_+)) = \sin(\mu([P,Q]_-)) = \sin(d(P,Q))$, because $\sin(\pi - x) = \sin(x)$ for $x \in [0,\pi[$. The correctness of the last equation can be proved by verifying the equation $\|P^\circ{\times}\, Q^\circ\|^2 = 1 - (P^\circ{\cdot}\, Q^\circ)^2$. $\square$

1.5.4. *Angles.* The (angle) distance between two lines k and l we get by dualizing the distance of two points: $d(k,l) = d(k^\delta, l^\delta)$. We even use the same symbol d for the distance between lines as between points and do not introduce a new sign.
By dualizing line segments, we get angles as subsets of the pencil of lines through a point which is the vertex of this angle:
Given three different points Q, R and S, we define the angles

$$\angle_+ QSR := \{S \vee P |\, P \in [Q,R]_+\} \text{ and } \angle_- QSR := \{S \vee P |\, P \in [Q,R]_-\}.$$

Using the same symbol μ for the measure of angles as for line segments, we have

$$\mu(\angle_+ QSR) = \arcsin \|(S^\circ \times Q^\circ)^\circ \times (S^\circ \times R^\circ)^\circ\| \text{ and } \mu(\angle_- QSR) = \pi - \mu(\angle_+ QSR).$$

1.5.5. *Perpendicular line through a point/perpendicular point on a line.* Consider a point $P = (p_0 : p_1 : p_2)$ and a line $l = (l_0 : l_1 : l_2)^*$. We assume that $P \neq l^\delta$. The perpendicular from P to l is

$$perp(l,P) := P \vee l^\delta = (p_1 l_2 - p_2 l_1 : p_2 l_0 - p_0 l_2 : p_0 l_1 - p_1 l_0)^*.$$

The line $perp(l,P)$ intersects l at the point

$$\begin{aligned} Q &= l \wedge perp(l,P) \\ &= (l_0(l_1p_1 + l_2p_2) - p_0(l_1^2 + l_2^2) : l_1(l_0p_0 + l_2p_2) - p_1(l_0^2 + l_2^2) : l_2(l_0p_0 + l_1p_1) - p_2(l_0^2 + l_1^2)). \end{aligned}$$

This point Q is called the *orthogonal projection* of P on l or the *pedal* of P on l.

Given two different points P and Q, the perpendicular bisector of $[P,Q]_+$ is the line $(P+Q) \vee (P \times Q)^\delta$ and the perpendicular bisector of $[P,Q]_-$ is the line $(P-Q) \vee (P \times Q)^\delta$. A point on either of these perpendicular bisectors has the same distance from the endpoints P and Q of the segment.

There is exactly one point Q on the line l with $d(Q,P) = \pi/2$; this is

$$Q = l \wedge P^\delta = (p_1 l_2 - p_2 l_1 : p_2 l_0 - p_0 l_2 : p_0 l_1 - p_1 l_0).$$

Proof. Most of the results can be obtained by straight forward computation. Here we just show that a point on the perpendicular bisector of $[P,Q]_-$ is equidistant from P and Q:
If R is a point on this perpendicular bisector, then there exist real numbers s and t such that $R^\circ = s(P^\circ - Q^\circ)^\circ + t(P^\circ \times Q^\circ)^\circ$. Then,

$$\begin{aligned} |R^\circ \cdot P^\circ| = |s(P^\circ - Q^\circ)^\circ \cdot P^\circ| &= |s(1 - Q^\circ \cdot P^\circ)|/\|P^\circ - Q^\circ\| \\ &= |s(1 - P^\circ \cdot Q^\circ)|/\|P^\circ - Q^\circ\| = |R^\circ \cdot Q^\circ|. \end{aligned}$$

□

1.5.6. *Parallel line through a point.* Given a line $l = (l_0 : l_1 : l_2)^*$ and a point $P = (p_0 : p_1 : p_2) \neq l^\delta$, the *parallel to l through P*, $par(l,P)$, is the line $perp(perp(l,P),P)$ (cf. [27]):

$$\begin{aligned} &par(l,P) = \\ &(p_0(l_1p_1 + l_2p_2) - l_0(p_1^2 + p_2^2) : p_1(l_0p_0 + l_2p_2) - l_1(p_0^2 + p_2^2) : p_2(l_0p_0 + l_1p_1) - l_2(p_0^2 + p_1^2))^*. \end{aligned}$$

1.5.7. *Reflections.* The mirror image of a point $P = (p_0 : p_1 : p_2)$ in a point $S = (s_0 : s_1 : s_2)$ is the point $Q = (q_0 : q_1 : q_2)$ with

$$\begin{aligned} q_0 &= p_0(s_0^2 - s_1^2 - s_2^2) + 2s_0(p_1 s_1 + p_2 s_2), \\ q_1 &= p_1(-s_0^2 + s_1^2 - s_2^2) + 2s_1(p_0 s_0 + p_2 s_2), \\ q_2 &= p_2(-s_0^2 - s_1^2 + s_2^2) + 2s_2(p_0 s_0 + p_1 s_1). \end{aligned}$$

Proof. By using a computer algebra system (CAS) it can be confirmed that

$$\frac{(p_0, p_1, p_2)}{\|(p_0, p_1, p_2)\|} + \frac{(q_0, q_1, q_2)}{\|(q_0, q_1, q_2)\|} = \frac{2(p_0 s_0 + p_1 s_1 + p_2 s_2)}{\|(p_0, p_1, p_2)\|\|(s_0, s_1, s_2)\|}(s_0, s_1, s_2).$$

From this equation results that S is a midpoint of either $[P, Q]_+$ or $[P, Q]_-$. □

Remark. A reflexion in a point S can also be interpreted as

- a rotation about S through an angle of $\frac{\pi}{2}$,
- a reflexion in the line S^δ.

1.5.8. *Circles.* For two points $M = (m_0 : m_1, m_2)$ and $P = (p_0 : p_1 : p_2)$, the circle $\mathcal{C}(M, P)$ with center M through the point P consists of all points $X = (x_0 : x_1 : x_2)$ with $X^\circ \cdot M^\circ = P^\circ \cdot M^\circ$. Thus, the coordinates of X must satisfy the quadratic equation

$$(x_0^2 + x_1^2 + x_3^2)(m_0 p_0 + m_1 p_1 + m_2 p_2)^2 - (m_0 x_0 + m_1 x_1 + m_2 x_2)^2 (p_0^2 + p_1^2 + p_3^2) = 0.$$

2. The use of barycentric coordinates

2.1. *Triangles, their sites and inner angles.*

We now fix a reference triple ABC of non-collinear points in $\mathcal{P}$. The set $\mathcal{P} - (A \vee B \cup B \vee C \cup C \vee A)$ consists of four connected components. Their closures are called *triangles*. Thus, there are four triangles $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ that share the same vertices A, B, C. Inside each triangle, there is exactly one of the four points $G_0 := A + B + C$, $G_1 := -A + B + C$, $G_2 := A - B + C, G_3 := A + B - C$, and we enumerate the triangles such that $G_i \in \Delta_i$ for $i = 0, 1, 2, 3$. In this case, the point G_i is the *centroid* of the triangle Δ_i (for $i = 0, 1, 2, 3$).

Besides the vertices, the four triangles Δ_i have all the same *sidelines* $B \vee C$, $C \vee A$, $A \vee B$, but they do not have the same *sides*. For example, the sides of Δ_0 are $[B, C]_+, [C, A]_+, [A, B]_+$, while Δ_1 has the sides $[B, C]_+, [C, A]_-, [A, B]_-$. The lengths of the sides of the triangle Δ_0 we denote by $a_0 := \mu([B, C]_+), b_0 := \mu([C, A]_+), c_0 := \mu([A, B]_+)$. The side lengths of the other triangles Δ_i are named accordingly; for example, $a_1 := \mu([B, C]_+), b_1 := \mu([C, A]_-), c_1 := \mu([A, B]_-)$.

We introduced angles as a subset of the pencil of lines through a point. The (inner) angles of Δ_0 are $\angle_+ BAC, \angle_+ CBA, \angle_+ ACB$, the angles of Δ_1 are $\angle_+ BAC$, $\angle_- CBA$, $\angle_- ACB$, etc. The measures of these angles are $\alpha_0 = \mu(\angle_+ BAC)$, $\beta_0 = \mu(\angle_+ CBA)$, $\gamma_0 = \mu(\angle_+ ACB)$, $\alpha_1 = \mu(\angle_+ BAC)$, $\beta_1 = \mu(\angle_- CBA) = \pi - \mu(\angle_+ CBA)$, etc.

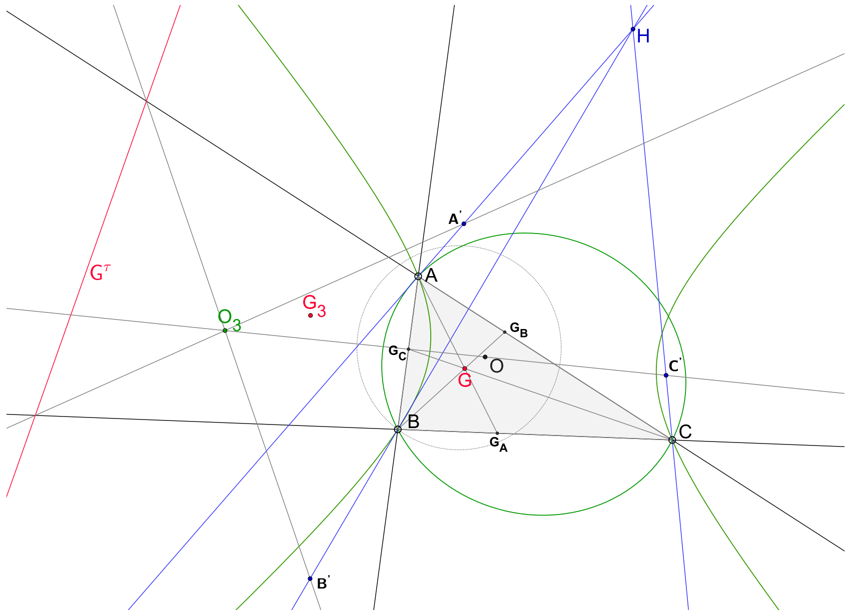

Figure 1. The picture shows the triangle Δ_0, the dual triple $A'B'C'$ of ABC and the orthocenter H, furthermore the centroids G and G_3 of Δ_0 and Δ_3, the tripolar line of G, as well as the circumcircles of Δ_0 and Δ_3 together with their centers O and O_3. Since the absolute conic $\mathcal{C}$ has no real points, the dotted circle $\tilde{\mathcal{C}} := \{(x_0 : x_1 : x_2) |\, x_0^2 = x_1^2 + x_2^2\}$ serves as a substitute for constructions. For example, the pole A' of the line $B \vee C$ with respect to $\mathcal{C}$ can be obtained as follows: Construct the pole of $B \vee C$ with respect to $\tilde{\mathcal{C}}$, then its mirror image in the center $(1 : 0 : 0)$ of $\tilde{\mathcal{C}}$ is A'.
The figures were created with the software program GeoGebra [29].

In the following we concentrate mainly on the triangle Δ_0. After having calculated the coordinates of triangle centers for this triangle, the results can be easily transferred to the triangles $\Delta_i, i > 0$. To simplify the notation, we write $a, b, c, \alpha, \beta, \gamma$ instead of $a_0, b_0, c_0, \alpha_0, \beta_0, \gamma_0$.
To shorten formulas, we will use abbreviations:
For $x \in \mathbb{R}$ define $\mathsf{c}_x := \cos x$ and $\mathsf{s}_x := \sin x$.
The semiperimeter of the triangle Δ_0 is $s := (a + b + c)/2$.
We put

$$\mathsf{S}_A := \mathsf{c}_a - \mathsf{c}_b\mathsf{c}_c = \mathsf{s}_s\mathsf{s}_{s-a} - \mathsf{s}_{s-b}\mathsf{s}_{s-c},$$
$$\mathsf{S}_B := \mathsf{c}_b - \mathsf{c}_c\mathsf{c}_a, \quad \mathsf{S}_C := \mathsf{c}_c - \mathsf{c}_a\mathsf{c}_b,$$

and for the barycentric coordinates of a point P with respect to the reference triple $\Delta = ABC$ we use the short form $[p_1 : p_2 : p_3]$ instead of $[p_1 : p_2 : p_3]_\Delta$.

2.2. *Rules of elliptic trigonometry.*

In the following, we will make use of the following rules of elliptic trigonometry:

Cosine rules:

$$\cos\alpha = \frac{\cos a - \cos b\,\cos c}{\sin b\,\sin c}\,,\ \cos a = \frac{\cos\alpha + \cos\beta\,\cos\gamma}{\sin\beta\sin\gamma} = 1 + \frac{2\sin\epsilon\,\sin\epsilon\text{-}\alpha}{\sin\beta\,\sin\gamma},$$

where $2\epsilon = \alpha+\beta+\gamma-\pi$ is the *excess* of Δ_0.

Sine rule:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} = \frac{|\mathsf{S}|}{\sin a\,\sin b\,\sin c},$$

with

$$\mathsf{S} = \mathsf{S}(\Delta) = \det\begin{pmatrix} A^\circ \\ B^\circ \\ C^\circ \end{pmatrix}$$

and

$$\begin{aligned} |\mathsf{S}| &= \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\,\cos b\,\cos c} \\ &= 2\sqrt{\sin s\,\sin(s-a)\,\sin(s-b)\,\sin(s-c)}. \end{aligned}$$

Proof.

$$\begin{aligned} \cos\alpha &= \frac{(A^\circ\times B^\circ)\cdot(A^\circ\times C^\circ)}{\|A^\circ\times B^\circ\|\|A^\circ\times C^\circ\|} = \frac{(A^\circ\cdot A^\circ)(B^\circ\cdot C^\circ) - (A^\circ\cdot C^\circ)(B^\circ\cdot A^\circ)}{\mathsf{s}_c\mathsf{s}_b} \\ &= \frac{\mathsf{c}_a - \mathsf{c}_b\mathsf{c}_c}{\mathsf{s}_b\mathsf{s}_c}. \end{aligned}$$

Before giving a proof for the second cosine rule, we prove the sine rule.

$$\frac{\sin\alpha}{\sin a} = \frac{1}{\mathsf{s}_a}\frac{\|(A^\circ\times C^\circ)\times(A^\circ\times B^\circ)\|}{\|A^\circ\times B^\circ\|\|A^\circ\times C^\circ\|} = \frac{|\mathsf{S}(\Delta)|}{\mathsf{s}_a\,\mathsf{s}_b\,\mathsf{s}_c}$$

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \big(\frac{\mathsf{c}_a - \mathsf{c}_b\mathsf{c}_c}{\mathsf{s}_b\mathsf{s}_c}\big)^2 = \frac{1 - \mathsf{c}_a^2 - \mathsf{c}_b^2 - \mathsf{c}_c^2 + 2\mathsf{c}_a\mathsf{c}_b\mathsf{c}_c}{\mathsf{s}_b^2\mathsf{s}_c^2}$$

$$\begin{aligned} \sin^2\alpha &= (1+\cos\alpha)(1-\cos\alpha) = (1 + \frac{\mathsf{c}_a - \mathsf{c}_b\mathsf{c}_c}{\mathsf{s}_b\mathsf{s}_c})(1 - \frac{\mathsf{c}_a - \mathsf{c}_b\mathsf{c}_c}{\mathsf{s}_b\mathsf{s}_c}) \\ &= \frac{\mathsf{c}_a - \mathsf{c}_{b+c}}{\mathsf{s}_b\mathsf{s}_c}\,\frac{\mathsf{c}_{b-c} - \mathsf{c}_a}{\mathsf{s}_b\mathsf{s}_c} = \frac{4\mathsf{s}_s\mathsf{s}_{s-a}\mathsf{s}_{s-b}\mathsf{s}_{s-c}}{\mathsf{s}_b^2\mathsf{s}_c^2}. \end{aligned}$$

Proof of the second cosine rule:

$$\begin{aligned} \frac{\cos\alpha + \cos\beta\,\cos\gamma}{\sin\beta\sin\gamma} &= \frac{\frac{\cos a - \cos b\,\cos c}{\sin b\,\sin c} + \frac{\cos b - \cos c\,\cos a}{\sin c\,\sin a}\,\frac{\cos c - \cos a\,\cos b}{\sin a\,\sin b}}{\sin\beta\,\sin\gamma} \\ &= \frac{\mathsf{c}_a(1 - \mathsf{c}_a^2 - \mathsf{c}_b^2 - \mathsf{c}_c^2 + 2\mathsf{c}_a\mathsf{c}_b\mathsf{c}_c)}{\mathsf{s}_\beta\,\mathsf{s}_\gamma\mathsf{s}_a{}^2\mathsf{s}_b\mathsf{s}_c} = \mathsf{c}_a \end{aligned}$$

$$\begin{aligned} \cos a &= 1 + \frac{\cos\alpha + \cos\beta\cos\gamma - \sin\beta\sin\gamma}{\sin\beta\sin\gamma} = 1 + \frac{\cos\alpha + \cos\beta{+}\gamma}{\sin\beta\sin\gamma} \\ &= 1 + \frac{2\,\cos\frac{1}{2}(\alpha{+}\beta{+}\gamma)\,\cos\frac{1}{2}(\beta{+}\gamma{-}\alpha)}{\sin\beta\sin\gamma} = 1 + \frac{2\sin\epsilon\,\sin\epsilon\text{-}\alpha}{\sin\beta\sin\gamma}. \end{aligned}$$

$\square$

A detailed collection of spherical trigonometry formulas, including their proofs, can be found in [4, 24].

2.3. *Calculations with barycentric coordinates.*
By using barycentric coordinates, many concepts can be transferred directly from metric affine (for example euclidean) geometry to elliptic geometry.

2.3.1. *The distance between two points in barycentric coordinates.* We introduce the matrix

$$\mathfrak{T} = (\mathfrak{t}_{ij})_{i,j=1,2,3} := \begin{pmatrix} A^\circ \cdot A^\circ & A^\circ \cdot B^\circ & A^\circ \cdot C^\circ \\ B^\circ \cdot A^\circ & B^\circ \cdot B^\circ & B^\circ \cdot C^\circ \\ C^\circ \cdot A^\circ & C^\circ \cdot B^\circ & C^\circ \cdot C^\circ \end{pmatrix} = \begin{pmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{pmatrix},$$

which we call the *characteristic matrix* of Δ.
Besides the dot product $\cdot$ for 3-vectors we introduce another scalar product $*$:

$$\begin{aligned}&(p_1, p_2, p_3) * (q_1, q_2, q_3)\\ &= (p_1, p_2, p_3)\,\mathfrak{T} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\\ &= p_1q_1 + p_2q_2 + p_3q_3 + (p_2q_3{+}p_3q_2)\cos a + (p_3q_1{+}p_1q_3)\cos b + (p_1q_2{+}p_2q_2)\cos c\end{aligned}$$

and use the abbreviations

$$(p_1, p_2, p_3)^2 := (p_1, p_2, p_3) \cdot (p_1, p_2, p_3), (p_1, p_2, p_3)^{*2} := (p_1, p_2, p_3) * (p_1, p_2, p_3),$$
$$\|(p_1, p_2, p_3)\|_* := \sqrt{(p_1, p_2, p_3)^{*2}}.$$

With the help of these products and the resulting norms, the distance between two points $P = [p_1 : p_2 : p_3]$ and $Q = [q_1 : q_2 : q_3]$ can be calculated as follows:

$$\begin{aligned} d(P, Q) &= \arccos\, \frac{|(p_1A^\circ + p_2B^\circ + p_3C^\circ) \cdot (q_1A^\circ + q_2B^\circ + q_3C^\circ)|}{\|p_1A^\circ + p_2B^\circ + p_3C^\circ\|\ \|q_1A^\circ + q_2B^\circ + q_3C^\circ\|} \\ &= \arccos\, \frac{|(p_1, p_2, p_3) * (q_1, q_2, q_3)|}{\|(p_1, p_2, p_3)\|_*\, \|(q_1, q_2, q_3)\|_*}. \end{aligned}$$

2.3.2. *Circles.* For two points $M = [m_1 : m_2 : m_3]$ and $P = [p_1 : p_2 : p_3]$, the circle $\mathcal{C}(M, P)$ with center M through the point P consists of all points $X = [x_1 : x_2 : x_3]$ with

$$((m_1, m_2, m_3){*}(x_1, x_2, x_3))^2(p_1, p_2, p_3)^{*2} = ((m_1, m_2, m_3){*}(p_1, p_2, p_3))^2(x_1, x_2, x_3)^{*2}.$$

2.3.3. *Lines.* Given two different points $P = [p_1 : p_2 : p_3]$ and $Q = [q_1 : q_2 : q_3]$, a third point $X = [x_1 : x_2 : x_3]$ is a point on $P \vee Q$ exactly when

$$((p_1, p_2, p_3) \times (q_1, q_2, q_3)) \cdot (x_1, x_2, x_3) = 0.$$

If $R \vee S$ is a line through $R = [r_1 : r_2 : r_3]$ and $S = [s_1 : s_2 : s_3]$, different from $P \vee Q$, then both lines meet at a point $T = [t_1 : t_2 : t_3]$ with

$$(t_1, t_2, t_3) = ((p_1, p_2, p_3) \times ((q_1, q_2, q_3)) \times ((r_1, r_2, r_3){\times}(s_1, s_2, s_3)).$$

2.3.4. *The midpoint of a segment.* Let $P = [p_1 : p_2 : p_3]$ and $Q = [q_1 : q_2 : q_3]$ are two different points. We can assume that the triples $\boldsymbol{p} = (p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$ are both strictly positive with respect to the lexicographic order. Then the midpoints of the two segments $[P, Q]_\pm$ are

$$[\frac{p_1}{\|\boldsymbol{p}\|_*} \pm \frac{q_1}{\|\boldsymbol{q}\|_*} : \frac{p_2}{\|\boldsymbol{p}\|_*} \pm \frac{q_2}{\|\boldsymbol{q}\|_*} : \frac{p_3}{\|\boldsymbol{p}\|_*} \pm \frac{q_3}{\|\boldsymbol{q}\|_*}].$$

2.3.5. *The dual Δ' of Δ.* We put $A' := (B \vee C)^\delta, B' := (C \vee A)^\delta, C' := (A \vee B)^\delta$. The triple $\Delta' = A'B'C'$ is called the dual of Δ. The points A', B', C' can be written in terms of barycentric coordinates as follows:

$$\begin{aligned} A' &= [1 - \mathsf{c}_a^2 : \mathsf{c}_a\mathsf{c}_b - \mathsf{c}_c : \mathsf{c}_c\mathsf{c}_a - \mathsf{c}_b] = [\mathsf{s}_a^2 : -\mathsf{S}_C : -\mathsf{S}_B] \\ B' &= [\mathsf{c}_a\mathsf{c}_b - \mathsf{c}_c : 1 - \mathsf{c}_b^2 : \mathsf{c}_b\mathsf{c}_c - \mathsf{c}_a] = [-\mathsf{S}_C : \mathsf{s}_b^2 : -\mathsf{S}_A] \\ C' &= [\mathsf{c}_c\mathsf{c}_a - \mathsf{c}_b : \mathsf{c}_b\mathsf{c}_c - \mathsf{c}_a : 1 - \mathsf{c}_c^2] = [-\mathsf{S}_B : -\mathsf{S}_A : \mathsf{s}_c^2] \end{aligned}$$

Proof. Up to a real factor $1/\mathsf{S}$, the characteristic matrix $\mathfrak{T}$ is the matrix that transforms $\begin{pmatrix} B^\circ \times C^\circ \\ C^\circ \times A^\circ \\ A^\circ \times B^\circ \end{pmatrix}$ onto $\begin{pmatrix} A^\circ \\ B^\circ \\ C^\circ \end{pmatrix}$:

$$\begin{pmatrix} A^\circ \\ B^\circ \\ C^\circ \end{pmatrix} = \frac{1}{\mathsf{S}(\Delta)} \mathfrak{T} \begin{pmatrix} B^\circ \times C^\circ \\ C^\circ \times A^\circ \\ A^\circ \times B^\circ \end{pmatrix}.$$

The matrix $\mathfrak{T}^{-1}$ of the inverse transformation is

$$\begin{aligned} \mathfrak{T}^{-1} &= \frac{1}{\mathsf{S}} \begin{pmatrix} \mathfrak{t}_{22}\mathfrak{t}_{33} - \mathfrak{t}_{23}^2 & \mathfrak{t}_{23}\mathfrak{t}_{31} - \mathfrak{t}_{12}\mathfrak{t}_{33} & \mathfrak{t}_{12}\mathfrak{t}_{23} - \mathfrak{t}_{31}\mathfrak{t}_{22} \\ \mathfrak{t}_{23}\mathfrak{t}_{31} - \mathfrak{t}_{12}\mathfrak{t}_{33} & \mathfrak{t}_{33}\mathfrak{t}_{11} - \mathfrak{t}_{31}^2 & \mathfrak{t}_{31}\mathfrak{t}_{12} - \mathfrak{t}_{13}\mathfrak{t}_{11} \\ \mathfrak{t}_{12}\mathfrak{t}_{23} - \mathfrak{t}_{31}\mathfrak{t}_{22} & \mathfrak{t}_{31}\mathfrak{t}_{12} - \mathfrak{t}_{13}\mathfrak{t}_{11} & \mathfrak{t}_{11}\mathfrak{t}_{22} - \mathfrak{t}_{12}^2 \end{pmatrix} \\ &= \frac{1}{\mathsf{S}} \begin{pmatrix} 1 - \mathsf{c}_a^2 & \mathsf{c}_a\mathsf{c}_b - \mathsf{c}_c & \mathsf{c}_c\mathsf{c}_a - \mathsf{c}_b \\ \mathsf{c}_a\mathsf{c}_b - \mathsf{c}_c & 1 - \mathsf{c}_b^2 & \mathsf{c}_b\mathsf{c}_c - \mathsf{c}_a \\ \mathsf{c}_c\mathsf{c}_a - \mathsf{c}_b & \mathsf{c}_b\mathsf{c}_c - \mathsf{c}_a & 1 - \mathsf{c}_c^2 \end{pmatrix}, \end{aligned}$$

which proves the statement. □

2.3.6. *The dual of a point and the dual of a line.* The dual line P^δ of a point $P = [p_1 : p_2 : p_3]$ has the equation (in barycentric coordinates)

$$(p_1, p_2, p_3)\mathfrak{T} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

If ℓ is a line with equation $\ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 = 0$, then its dual is the point $R = [r_1 : r_2 : r_3]$ with $(r_1, r_2, r_3) = (\ell_1, \ell_2, \ell_3)\mathfrak{T}^{-1}$.

2.3.7. *The angle bisectors of two lines.* Let $k : k_1x_1 + k_1x_1 + k_1x_1 = 0$ and $\ell : \ell_1x_1 + \ell_1x_1 + \ell_1x_1 = 0$ be two different lines, then their two angle bisectors are $m : m_1x_1 + m_1x_1 + m_1x_1 = 0$ and $n : n_1x_1 + n_1x_1 + n_1x_1 = 0$, with

$$\begin{aligned}
&(m_1, m_2, m_3) = \sqrt{(\boldsymbol{\ell}\mathfrak{T}^{-1})\cdot\boldsymbol{\ell}}\;\boldsymbol{k} + \sqrt{(\boldsymbol{k}\mathfrak{T}^{-1})\cdot\boldsymbol{k}}\;\boldsymbol{\ell},\\
&(n_1, n_2, n_3) = \sqrt{(\boldsymbol{\ell}\mathfrak{T}^{-1})\cdot\boldsymbol{\ell}}\;\boldsymbol{k} - \sqrt{(\boldsymbol{k}\mathfrak{T}^{-1})\cdot\boldsymbol{k}}\;\boldsymbol{\ell},\\
&\boldsymbol{k} = (k_1, k_2, k_3), \boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3).
\end{aligned}$$

2.3.8. *Chasles' Theorem.* We define:

- Three points P, Q, R are *in a general position* if they are independent and $P \neq (Q \vee R)^\delta, Q \neq (R \vee P)^\delta, R \neq (P \vee Q)^\delta$.
- A triple PQR of three points P, Q, R is a *perspective triple* with perspector S if the triples ABC and PQR are perspective and S is the perspective center, in other words: S is the meet of the lines $A \vee P, B \vee Q$ and $R \vee R$.

If the points A, B, C are in a general position, then the dual $\Delta' = A'B'C'$ of Δ is a perspective triple.

Proof. Using 1.2.3, it is easy to verify that the point

$$\begin{aligned}
H :=&[\frac{1}{\mathsf{c}_b\mathsf{c}_c - \mathsf{c}_a} : \frac{1}{\mathsf{c}_b\mathsf{c}_a - \mathsf{c}_b} : \frac{1}{\mathsf{c}_a\mathsf{c}_b - \mathsf{c}_c}]\\
=&[\frac{1}{\mathsf{S}_A} : \frac{1}{\mathsf{S}_B} : \frac{1}{\mathsf{S}_C}]
\end{aligned}$$

is a common point of the lines $A \vee A', B \vee B', C \vee C'$ and therefore it is the center of perspective. H is the *orthocenter* of Δ. □

2.3.9. *Pedals and antipedals of a point.* The *pedals* of a point $P = [p_1 : p_2 : p_3]$ on the sidelines of Δ are notated $A_{[P]}, B_{[P]}, C_{[P]}$. We calculate the barycentric coordinates $[q_1 : q_2 : q_3]$ of $A_{[P]}$:

$$\begin{aligned}
(q_1, q_2, q_3) &= ((p_1, p_2, p_3) \times (\mathsf{s}_a^2, -\mathsf{S}_C, -\mathsf{S}_B)) \times ((0, 1, 0) \times (0, 0, 1))\\
&= (0, p_1\mathsf{S}_C + p_2\mathsf{s}_a^2, p_1\mathsf{S}_B + p_3\mathsf{s}_a^2).
\end{aligned}$$

Similarly the coordinates of the other two pedals can be calculated:

$$\begin{aligned}
B_{[P]} &= [p_2\mathsf{S}_C + p_1{\mathsf{s}_b}^2 : 0 : p_2\mathsf{S}_A + p_3{\mathsf{s}_b}^2]\\
C_{[P]} &= [p_3\mathsf{S}_B + p_1{\mathsf{s}_c}^2 : p_3\mathsf{S}_A + p_2{\mathsf{s}_c}^2 : 0].
\end{aligned}$$

Define the *antipedal points* $A^{[P]}, B^{[P]}, C^{[P]}$ of P by

$A^{[P]} := perp(B \vee P, B) \wedge perp(C \vee P, C)$ and $B^{[P]}, C^{[P]}$ cyclically.

Straightforward calculation gives

$$A^{[P]} = [-1 : \frac{p_2\mathsf{S}_C + p_1\mathsf{s}_b^2}{p_1\mathsf{S}_C + p_2\mathsf{s}_a^2} : \frac{p_3\mathsf{S}_B + p_1\mathsf{s}_c^2}{p_1\mathsf{S}_B + p_3\mathsf{s}_a^2}].$$

A special case: For $P = H$ we get

$$A^{[H]} = [-\mathsf{c}_a : \mathsf{c}_b : \mathsf{c}_c],\ B^{[H]} = [\mathsf{c}_a : -\mathsf{c}_b : \mathsf{c}_c],\ A^{[H]} = [\mathsf{c}_a : \mathsf{c}_b : -\mathsf{c}_c].$$

2.3.10. *Cevian and anticevian triangles.* If $P = [p_1 : p_2 : p_3]$ is a point different from A, B, C, then the lines $P \vee A$, $P \vee B$, $P \vee C$ are called the *cevians* of P. The cevians meet the sidelines a, b, c in $A_P := [0 : p_2 : p_3], B_P := [p_1 : 0 : p_3], C_P := [p_1 : p_2 : 0]$, respectively. These points are called the *traces* of P. The points $A^P := [-p_1 : p_2 : p_3], B^P := [p_1 : -p_2 : p_3], C^P := [p_1 : p_2 : -p_3]$ are called *harmonic associates* of P.

We now assume that P is not a point on a sideline of ABC and define: The *cevian triangle* of P with respect to Δ_0 is the triangle with vertices A_P, B_P, C_P which contains the point $[|p_1| : |p_1| : |p_1|]$, the cevian triangle of P with respect to Δ_1 is the triangle with vertices A_P, B_P, C_P which contains the point $[-|p_1| : |p_1| : |p_1|]$, and so on. Furthermore, we define: The *anticevian triangle* of P with respect to Δ_0 is the triangle with vertices A^P, B^P, C^P that has all points A, B, C on its sides. The anticevian triangle of P with respect to Δ_1 has the same vertices, but only the point A is on one of its sides, while the points B and C are not.

A special case:
The traces of G_i are the midpoints of the sides of Δ_i, the cevians of G_i are (therefore) called the *medians* of Δ_i. G_i itself is called the *centroid* of Δ_i, and the cevian triangle of G_i with respect to Δ_i is called the *medial triangle* of Δ_i.
Δ_0 is, in general, not the medial triangle of the anticevian triangle of G_0. (The same applies to the other triangles Δ_i.) The *antimedial triangle* of Δ_0 is the anticevian triangle with respect to Δ_0 of the point $G^+ := [\cos a : \cos b : \cos c]$. In the last subsection it was shown that the vertices of this triangle also form the antipedal triple of H. We now prove that the anticevian triangle with respect to Δ_0 of the point G^+ has G^+ as its centroid, cf. Wildberger [27, 28].

Proof. We know already that the points A, B, C lie on the sides of the anticevian triangle of G^+ with respect to Δ_0. Now we show that A is equidistant from B^{G^+} and C^{G^+} by proving the equation $A = B^{G^+} + C^{G^+}$: Define the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ by $\boldsymbol{p} := (\mathtt{c}_a, -\mathtt{c}_b, \mathtt{c}_c)$ and $\boldsymbol{q} := (\mathtt{c}_a, \mathtt{c}_b, -\mathtt{c}_c)$. Since $\boldsymbol{p}^{*2} = \boldsymbol{q}^{*2}$, we have $\boldsymbol{p}/\|\boldsymbol{p}\|_* + \boldsymbol{q}/\|\boldsymbol{q}\|_* = (2\mathtt{c}_c, 0, 0)/\|\boldsymbol{p}\|_*$.
In the same way it is shown that the points B and C are the midpoints of the corresponding sides of the anticevian triangle of G^+ with respect to Δ_0. □

Remark. Wildberger's name for the antimedial triangle is *double triangle* [27, 28].

2.3.11. *Tripolar and tripole.* Given a point $P = [p_1 : p_2 : p_3]$, then the point $[0 : -p_2 : p_3]$ is the harmonic conjugate of A_P with respect to $\{B, C\}$. Correspondingly, the harmonic conjugates of the traces of P on the other sidelines are $[-p_1 : 0 : p_3]$ and $[p_1 : -p_2 : 0]$. These three harmonic conjugates are collinear; the equation of the line l is

$$p_2p_3x_1 + p_3p_1x_2 + p_1p_2x_3 = 0.$$

This line is called the *tripolar line* or the *tripolar* of P and we denote it by P^τ. P is the *tripole* of l and we write $P = l^\tau$.

We calculate the coordinates of the dual point of the tripolar of P and get

$$P^{\tau\delta} = [p_1(p_2\mathsf{S}_B + p_3\mathsf{S}_C) - p_2p_3\mathsf{s}_a^2 : p_2(p_3\mathsf{S}_C + p_1\mathsf{S}_A) - p_3p_1\mathsf{s}_b^2 \\ : p_3(p_1\mathsf{S}_A + p_2\mathsf{S}_B) - p_1p_2\mathsf{s}_c^2].$$

Two Examples:

The line H^τ is called *orthic line* and has the equation

$$\mathsf{S}_A x_1 + \mathsf{S}_B x_2 + \mathsf{S}_C x_3 = 0,$$

its dual is the point

$$H^{\tau\delta} = [2\mathsf{S}_B\mathsf{S}_C - \mathsf{S}_A\mathsf{s}_a^2 : 2\mathsf{S}_C\mathsf{S}_A - \mathsf{S}_B\mathsf{s}_b^2 : 2\mathsf{S}_A\mathsf{S}_B - \mathsf{S}_C\mathsf{s}_c^2].$$

Wildberger [24, 25] names this point *orthostar*, we adopt this terminology.

The tripolar of G has the equation $x_1 + x_2 + x_3 = 0$, the point $G^{\tau\delta}$ has coordinates

$$[(1-\mathsf{c}_a)(1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c) : (1-\mathsf{c}_b)(1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c) : (1-\mathsf{c}_c)(1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c)]$$
$$=[\mathsf{s}_{a/2}^2(\mathsf{s}_{a/2}^2-\mathsf{s}_{b/2}^2-\mathsf{s}_{c/2}^2) : \mathsf{s}_{b/2}^2(-\mathsf{s}_{a/2}^2+\mathsf{s}_{b/2}^2-\mathsf{s}_{c/2}^2) : \mathsf{s}_{c/2}^2(-\mathsf{s}_{a/2}^2-\mathsf{s}_{b/2}^2+\mathsf{s}_{c/2}^2)]$$

In the next subsection we identify this point as the *circumcenter* of the triangle Δ_0.

2.3.12. *The four classical triangle centers of* Δ_0. We already calculated the coordinates of two classical triangle centers of Δ_0, the centroid $G = G_0$ and the orthocenter[1] H :

$$G = [1 : 1 : 1]$$
$$H = [\frac{1}{\cos a - \cos b \cos c} : \frac{1}{\cos b - \cos c \cos a} : \frac{1}{\cos c - \cos a \cos b}]$$

The other two classical centers are the centers $O = O_0$ of the circumcircle and the center $I = I_0$ of the incircle.

We show that the point $G^{\tau\delta}$ is the circumcenter O:
It can be easily checked that the pedals of $G^{\tau\delta}$ are the traces of G, so $G^{\tau\delta}$ is a point on all three perpendicular bisectors $A_G \vee A'$, $B_G \vee B'$, $B_G \vee B'$ of the triangle sides and has, therefore, the same distance from the vertices A, B and C.
The radius of the circumcircle is

$$\mathsf{R} = d(O, A)$$
$$= \arccos\sqrt{\left|\frac{\mathsf{c}_a^2 + \mathsf{c}_b^2 + \mathsf{c}_c^2 - 2\mathsf{c}_a\mathsf{c}_b\mathsf{c}_c - 1}{\mathsf{c}_a^2 + \mathsf{c}_b^2 + \mathsf{c}_c^2 - 2\mathsf{c}_b\mathsf{c}_c - 2\mathsf{c}_c\mathsf{c}_a - 2\mathsf{c}_a\mathsf{c}_b + 2\mathsf{c}_a + 2\mathsf{c}_b + 2\mathsf{c}_c - 3}\right|}$$
$$= \arccos\sqrt{\left|\frac{\mathsf{s}_a^2+\mathsf{s}_b^2+\mathsf{s}_c^2 - (\mathsf{s}_s^2+\mathsf{s}_{s-a}^2+\mathsf{s}_{s-b}^2+\mathsf{s}_{s-c}^2)}{\mathsf{s}_a^2+\mathsf{s}_b^2+\mathsf{s}_c^2 + 2(\mathsf{s}_{s-b}\mathsf{s}_{s-c}+\mathsf{s}_{s-c}\mathsf{s}_{s-a}+\mathsf{s}_{s-a}\mathsf{s}_{s-b}-\mathsf{s}_s(\mathsf{s}_{s-a}+\mathsf{s}_{s-b}+\mathsf{s}_{s-c}))}\right|}\,.$$

[1]Here we have to assume that the vertices are in a general position.

The equation of the circumcircle: A point $X = [x_1 : x_2 : x_3]$ is a point on the circumcircle of Δ_0 precisely when its coordinates satisfy the equation

$$(1 - \cos a)x_2x_3 + (1 - \cos b)x_3x_1 + (1 - \cos c)x_1x_2 = 0,$$

which is equivalent to

$$\sin^2\tfrac{a}{2}\; x_2x_3 + \sin^2\tfrac{b}{2}\; x_3x_1 + \sin^2\tfrac{c}{2}\; x_1x_2 = 0.$$

The incenter $I = [\sin a : \sin b : \sin c]$ of Δ_0 is the meet of the three bisectors of the (inner) angles of Δ_0.

Proof. We show that $I = [\sin a : \sin b : \sin c]$ is a point on the bisector of α by showing that the dual point W_A of this bisector is orthogonal to I. Using equations

$$I^\circ = (|B^\circ \times C^\circ|A^\circ + |C^\circ \times A^\circ|B^\circ + |A^\circ \times B^\circ|C^\circ)^\circ$$

and $$(W_A)^\circ = \Big(\frac{A^\circ \times B^\circ}{\|(A^\circ \times B^\circ)\|} + \frac{A^\circ \times C^\circ}{\|(A^\circ \times C^\circ)\|}\Big)^\circ,$$

it can be easily checked that $I^\circ \cdot (W_A)^\circ = 0$. In a similar way it can be proved that I is also a point on the other two angle bisectors. $\square$

The pedals of I are

$$\begin{aligned}
A_{[I]} &= [0 : \cos a\, \cos b - \sin a\, \sin b - \cos c : \cos c\, \cos a - \sin c\, \sin a - \cos b]\\
B_{[I]} &= [\cos a\, \cos b - \sin a\, \sin b - \cos c : 0 : \cos b\, \cos c - \sin b\, \sin c - \cos a]\\
C_{[I]} &= [\cos c\, \cos a - \sin c\, \sin c - \cos b : \cos b\, \cos c - \sin b\, \sin c - \cos a : 0]
\end{aligned}$$

These three pedal points are also the traces of the *Gergonne point*

$$\begin{aligned}
Ge &= [\frac{1}{\cos(b{+}c) - \cos a} : \frac{1}{\cos(c{+}a) - \cos b} : \frac{1}{\cos(a{+}b) - \cos c}]\\
&= [\frac{1}{\sin(s{-}a)} : \frac{1}{\sin(s{-}b)} : \frac{1}{\sin(s{-}c)}].
\end{aligned}$$

The cevian triangle of Ge is called the *tangent triangle* of Δ_0.

The radius of the inner circle is

$$\mathsf{r} = \arccos\ \frac{2}{\kappa}\sin s$$

with $\kappa = \|(\mathsf{s}_a, \mathsf{s}_b, \mathsf{s}_c)\|_* = \sqrt{\mathsf{s}_a^2 + \mathsf{s}_b^2 + \mathsf{s}_a^2 + 2(\mathsf{c}_a\mathsf{s}_b\mathsf{s}_c + \mathsf{c}_b\mathsf{s}_c\mathsf{s}_a + \mathsf{c}_c\mathsf{s}_a\mathsf{s}_b)}$.

Proof.

$$\cos d(I, A_{[I]}) = \sqrt{\frac{((\mathsf{s}_a, \mathsf{s}_b, \mathsf{s}_c) * (0, \mathsf{c}_a\mathsf{c}_b - \mathsf{s}_a\mathsf{s}_b - \mathsf{c}_c, \mathsf{c}_c\mathsf{c}_a - \mathsf{s}_c\mathsf{s}_a - \mathsf{c}_b))^2}{(\mathsf{s}_a, \mathsf{s}_b, \mathsf{s}_c)^{*2}(0, \mathsf{c}_a\mathsf{c}_b - \mathsf{s}_a\mathsf{s}_b - \mathsf{c}_c, \mathsf{c}_c\mathsf{c}_a - \mathsf{s}_c\mathsf{s}_a - \mathsf{c}_b)^{*2}}}$$

$$= \sqrt{\frac{(2\,\mathsf{s}_a(1 - (\mathsf{c}_a(\mathsf{c}_b\mathsf{c}_c - \mathsf{s}_a\mathsf{s}_b) - \mathsf{s}_a(\mathsf{c}_b\mathsf{s}_c + \mathsf{c}_c\mathsf{s}_b))))^2}{\kappa^2(\mathsf{s}_a^2(1 - (\mathsf{c}_a(\mathsf{c}_b\mathsf{c}_c - \mathsf{s}_a\mathsf{s}_b) - \mathsf{s}_a(\mathsf{c}_b\mathsf{s}_c + \mathsf{c}_c\mathsf{s}_b))))}} = \frac{2}{\kappa}\sin s$$

In the same way it can be shown that $\cos d(I, B_{[I]}) = \cos d(I, C_{[I]}) = (2\sin s)/\kappa$. Thus, the point I is equidistant to the sides of the triangle and $\cos \mathsf{r} = (2\sin s)/\kappa$. □

The equation of the incircle: When $p_1 := \dfrac{1}{\sin(s-a)}, p_2 := \dfrac{1}{\sin(s-b)}, p_3 := \dfrac{1}{\sin(s-c)}$, the equation of the incircle is

$$\frac{x_1^2}{p_1^2} + \frac{x_2^2}{p_2^2} + \frac{x_3^2}{p_3^2} - \frac{2x_2x_3}{p_2p_3} - \frac{2x_3x_1}{p_3p_1} - \frac{2x_1x_2}{p_1p_2} = 0$$

Proof. This equation is correct because it describes a conic which touches the triangle sides at the traces of $Ge = [p_1 : p_2 : p_3]$. □

Remark. The incenter of Δ_0 is not only the circumcenter of the tangent triangle but also the circumcenter of the *dual triangle* Δ'_0. The radius of the circumcircle of the dual triangle Δ'_0 is $d(I, A') = \arccos |\mathsf{S}|/\kappa$.

2.3.13. *Triangle centers of* $\Delta_1, \Delta_2, \Delta_3$. The incenter I of Δ_0 can be written $[f(a,b,c) : f(b,c,a) : f(c,a,b)]$ with $f(a,b,c) = \sin a$. The orthocenter H can also be written $[f(a,b,c) : f(b,c,a) : f(c,a,b)]$, but with a different center function f; a suitable center function for H is $f(a,b,c) = 1/(\cos b \cos c - \cos a)$. The first component $f(a,b,c)$ obviously determines the triangle center, the other two one gets by cyclic permutation.

Knowing $f(a,b,c)$, we can also write down the corresponding triangle centers for the triangles $\Delta_i, i = 1,2,3$:

If $[f(a,b,c) : f(b,c,a) : f(c,a,b)]$ is representation of a triangle center $Z = Z_0$ of Δ_0 by a barycentric coordinates, then

$$\begin{aligned} Z_1 &= [-f(a_1,b_1,c_1) : f(b_1,c_1,a_1) : f(c_1,a_1,b_1)] \\ &= [-f(a,\pi{-}b,\pi{-}c) : f(\pi{-}b,\pi{-}c,a) : f(\pi{-}c,a,\pi{-}b)] \end{aligned}$$

is the corresponding triangle center of Δ_1 and

$$Z_2 = [f(a_2,b_2,c_2) : -f(b_2,c_2,a_2) : f(c_2,a_2,b_2)],$$
$$Z_3 = [f(a_3,b_3,c_3) : f(b_3,c_3,a_3) : -f(c_3,a_3,b_3)]$$

are the triangle centers of Δ_2 and Δ_3, respectively.

In the last subsection we presented the radii of the circumcircle and the incircle of Δ_0 as functions of the side lengths and the semiperimeter: $\mathsf{R} = \mathsf{R}(a,b,c,s), \mathsf{r} = \mathsf{r}(a,b,c,s)$. The radii R_i and r_i of the corresponding circles of Δ_i are: $\mathsf{R}_i = \mathsf{R}(a_i,b_i,c_i,s_i), \mathsf{r}_i = \mathsf{r}(a_i,b_i,c_i,s_i)$, with $s_i = (a_i + b_i + c_i)/2$.

Remark. The triangle centers G_i and I_i, $i = 1, 2, 3$, are harmonic associates of G and I, respectively. The orthocenter H is an absolute triangle center: $H = H_1 = H_2 = H_3$.

2.3.14. *The staudtian and the area of a triangle.* The *staudtian*[2] is a function n which assigns each triple of points a real number; the staudtian of Δ is

$$n(\Delta) = \frac{1}{2}|\mathsf{S}(\Delta)| = \frac{1}{2}|\det \begin{pmatrix} A^\circ \\ B^\circ \\ C^\circ \end{pmatrix}|.$$

The staudtian has some characteristics of an area. There are equations such as:

$$n(\Delta) = \frac{1}{2} \sin a \, \sin b \, \sin \gamma = \frac{1}{2} \, \sin a \, \sin h_a, \text{ with } h_a = d(A, A_H),$$

and for a point $P \in \Delta_0$ the equation:

$$P = [n(BPC) : n(CPA) : n(APB)].$$

But n lacks the property of additivity. For $P \in \Delta_0$ the inequality holds:

$$n(BPC) + n(CPA) + n(APB) > n(ABC),$$

and the value of $n(BPC)+n(CPA)+n(APB)$ takes its maximum for the incenter I. (For a proof of the equivalent statement in the hyperbolic plane, see Horváth [10].)

The proper triangle area is given by the excess 2ϵ; for the triangle Δ_0 this is

$$2\epsilon(\Delta_0) = \alpha + \beta + \gamma - \pi.$$

Adding up all the areas of the triangles $\Delta_i, i = 0, 1, 2, 3$, we get 2π for the area of the whole elliptic plane.

2.3.15. *Isoconjugation.* Let $P = [p_1 : p_2 : p_3]$ be a point not on a sideline of Δ_0 and let $Q = [q_1 : q_2 : q_3]$ be a point different from the vertices of Δ_0, then the point $R = [p_1 q_2 q_3 : p_2 q_3 q_1 : p_3 q_1 q_2]$ is called the *isoconjugate* of Q with respect to the pole P or shorter the P-isoconjugate of Q.
Obviously, if R is the P-isoconjugate of Q, then Q is the P-isoconjugate of R.

Some examples:

G-isoconjugation is also called *isotomic conjugation*. The fixed points of G-isoconjugation are the centroids G, G_1, G_2, G_3.
The isotomic conjugate of the Gergonne point Ge is the *Nagel point* Na. Its traces are the touch points of the excircles of Δ_0 (= incircles of the triangles $\Delta_i, i = 1, 2, 3$) with the sides of Δ_0.

Isogonal conjugation leaves the incenters $I_i, i = 0, 1, 2, 3$, fixed. It is the P-isoconjugation for $P = [\sin^2 a : \sin^2 b : \sin^2 c]$. This point is called *symmedian* and is usually notated by the letter K.

[2]The name is taken in honor of the geometer von Staudt [1798-1867], who was the first to use this function in spherical geometry, see [4, 10].

The circumcenters O_1, O_2, O_3 form a perspective triple. The perspector is the isogonal conjugate of O, and we denote this point by H^-.

Define the point $\tilde{K}$ by

$$\tilde{K} := [1-\cos a, 1-\cos b, 1-\cos c,] = [\sin^2 a/2, \sin^2 b/2, \sin^2 c/2].$$

Horváth uses the name *Lemoine point* for it and we also shall use this name.
The points on the circumcircle: $(1-\cos a)x_2x_3+(1-\cos b)x_3x_1+(1-\cos c)x_1x_2 = 0$ are the $\tilde{K}$-isoconjugates of the points on the tripolar of G: $x_1 + x_2 + x_3 = 0$, a line which is also the dual of O.

2.4. *Conics.*

2.4.1. *Different types of conics.* Let $\mathfrak{M} = (\mathfrak{m}_{ij})_{i,j=1,2,3}$ be a symmetric matrix, then the quadratic equation

$$\mathfrak{m}_{11}x_1^2 + \mathfrak{m}_{22}x_2^2 + \mathfrak{m}_{33}x_3^2 + 2\mathfrak{m}_{23}x_2x_3 + 2\mathfrak{m}_{31}x_3x_1 + 2\mathfrak{m}_{12}x_1x_2 = 0$$

is the equation of a conic which we denote by $\mathcal{C}(\mathfrak{M})$. Given a symmetric matrix $\mathfrak{M}$ and a nonzero real number t, then the conics $\mathcal{C}(\mathfrak{M})$ and $\mathcal{C}(t\mathfrak{M})$ are the same. We can reverse this implication if we restrict ourselves to real conics. Here, we define: A *real conic* in $\mathcal{P}$ is

- either the union of two different real lines,
- or a circle (with radius $r \in [0, \frac{\pi}{2}]$) ,
- or a proper ellipse, that is an irreducible ($det(\mathfrak{M}) \neq 0$) conic with infinitely many real points and which is not a circle.

Remark.

- A double line can be regarded as a circle with radius $r = \frac{\pi}{2}$.
- There is no difference between ellipses, hyperbolae and parabolae in elliptic geometry, cf. [8].

The polar of a point $P = [p_1 : p_2 : p_3]$ with respect to the conic $\mathcal{C}(\mathfrak{M})$ is the line with the equation

$$(x_1, x_2, x_3)\,\mathfrak{M}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0.$$

The pole of the line $\ell : \ell_1x_1 + \ell_2x_2 + \ell_3x_3 = 0$ is the point $P = [p_1 : p_2 : p_3]$ with $(p_1, p_2, p_3) = (\ell_1, \ell_2, \ell_3)\mathfrak{M}^{\#}$, where

$$\mathfrak{M}^{\#} = \begin{pmatrix} \mathfrak{m}_{22}\mathfrak{m}_{33} - \mathfrak{m}_{23}^2 & \mathfrak{m}_{23}\mathfrak{m}_{31} - \mathfrak{m}_{12}\mathfrak{m}_{33} & \mathfrak{m}_{12}\mathfrak{m}_{23} - \mathfrak{m}_{31}\mathfrak{m}_{22} \\ \mathfrak{m}_{23}\mathfrak{m}_{31} - \mathfrak{m}_{12}\mathfrak{m}_{33} & \mathfrak{m}_{33}\mathfrak{m}_{11} - \mathfrak{m}_{31}^2 & \mathfrak{m}_{31}\mathfrak{m}_{12} - \mathfrak{m}_{13}\mathfrak{m}_{11} \\ \mathfrak{m}_{12}\mathfrak{m}_{23} - \mathfrak{m}_{31}\mathfrak{m}_{22} & \mathfrak{m}_{31}\mathfrak{m}_{12} - \mathfrak{m}_{13}\mathfrak{m}_{11} & \mathfrak{m}_{11}\mathfrak{m}_{22} - \mathfrak{m}_{12}^2 \end{pmatrix}$$

is the adjoint of $\mathfrak{M}$.

2.4.2. *The perspector of a conic.* If $\mathfrak{M}$ is a diagonal matrix, then the polar lines of A, B, C are the lines $B \vee C$, $C \vee A$, $A \vee B$, respectively. If $\mathfrak{M}$ is not diagonal, then the poles of $B \vee C$, $C \vee A$, $A \vee B$ with respect to the conic form a perspective triple with perspector

$$[\frac{1}{\mathfrak{m}_{11}\mathfrak{m}_{23} - \mathfrak{m}_{31}\mathfrak{m}_{12}} : \frac{1}{\mathfrak{m}_{22}\mathfrak{m}_{31} - \mathfrak{m}_{12}\mathfrak{m}_{23}} : \frac{1}{\mathfrak{m}_{33}\mathfrak{m}_{12} - \mathfrak{m}_{23}\mathfrak{m}_{31}}].$$

An example: A matrix of the absolute conic is $\mathfrak{T}$ and the perspector is H. Since there are no real points on this conic, it is not a real conic.

2.4.3. *Symmetry points and symmetry axes of real conics.* A point $P = [p_1 : p_2 : p_3]$ is a *symmetry point* of a conic if for every point $Q = [q_1 : q_2 : q_3]$ on this conic the mirror image of Q in P is also a point on this conic. A line l is a *symmetry axis* if its dual l^δ is a symmetry point. The meet of two different symmetry axes is a symmetry point and the join of two different symmetry points is a symmetry axis.

Three examples:

If the conic is the union of two different lines which meet at a point P, then the point P and the duals of the two angle bisectors are the symmetry points. The point P is regarded as the center of the conic.

The symmetry points of a circle with center P and radius $r \in [0, \pi/2]$ are the center P and all points on P^δ.

A proper ellipse has three symmetry points, one lies inside the ellipse and is regarded as its center, the other two lie outside.

How can we find the symmetry points of a real conic $\mathcal{C}(\mathfrak{M})$ in case of a circle with radius $r \in]0, \pi/2[$ or a proper ellipse? In this case, a point $P = [p_1 : p_2 : p_3]$ is a symmetry point precisely when its dual P^δ is identical with the polar of P with respect to the conic, this is when the vector (p_1, p_2, p_3) is an eigenvector of the matrix $\mathfrak{T}^\# \mathfrak{M}$.

Three examples:

A circumconic with perspector $P = [p_1 : p_2 : p_3]$ is described by the equation

$$p_1 x_2 x_3 + p_2 x_3 x_1 + p_3 x_1 x_2 = 0.$$

A comparison of this equation with the equation of the circumcircle shows that the Lemoine point $\tilde{K}$ is the perspector of the circumcircle. The symmetry points are, besides the circumcenter O, the points on the line $G^\tau : x_1 + x_2 + x_3 = 0$.

We want to determine the symmetry points of the circumconic in case of $P = [1 + 2\mathsf{c}_a : 1 + 2\mathsf{c}_b : 1 + 2\mathsf{c}_c]$. If the triangle Δ_0 is not equilateral, then the circumconic is a proper ellipse and the matrix $\mathfrak{T}^\# \mathfrak{M}$ has three different eigenvalues. One is $2(\mathsf{c}_a^2 + \mathsf{c}_b^2 + \mathsf{c}_c^2 - 2\mathsf{c}_a \mathsf{c}_b \mathsf{c}_c - 1)$, belonging to the eigenvector $(1, 1, 1)$. The other two eigenvalues and their corresponding eigenvectors can also be explicitly calculated. Here, the characteristic polynomial of $\mathfrak{T}^\# \mathfrak{M}$ splits into a linear and a quadratic rational factor. But for proper ellipses this is an exception. In general, formulas for the symmetry points of a circumconic with a given perspector are rather complicated.

On the other hand, knowing the center $M = [m_1 : m_2 : m_3]$ of a circumconic, its perspector P can be calculated quite easily:

$$\begin{aligned} P = [m_1(2m_2 m_3 \mathsf{c}_a - m_1^2 + m_2^2 + m_3^2) &: m_2(2m_3 m_1 \mathsf{c}_b + m_1^2 - m_2^2 + m_3^2) \\ &: m_3(2m_1 m_2 \mathsf{c}_c + m_1^2 + m_2^2 - m_3^2)]. \end{aligned}$$

The equation of the inconic with perspector $P = [p_1 : p_2 : p_3]$ is

$$\frac{x_1^2}{p_1^2} + \frac{x_2^2}{p_2^2} + \frac{x_3^2}{p_3^2} + 2\frac{x_2x_3}{p_2p_3} + 2\frac{x_3x_1}{p_3p_1} + 2\frac{x_1x_2}{p_1p_2} = 0$$

If the perspector P is the Gergonne point Ge, then the inconic is the incircle. Its symmetry points are the incenter I and the points on the line

$$\begin{aligned} &\mathsf{s}_{s-a}(\mathsf{s}_a\mathsf{s}_b\mathsf{s}_c - \mathsf{s}_a\mathsf{S}_A + \mathsf{s}_b\mathsf{S}_B + \mathsf{s}_c\mathsf{S}_C)x_1 \\ +&\mathsf{s}_{s-b}(\mathsf{s}_a\mathsf{s}_b\mathsf{s}_c + \mathsf{s}_a\mathsf{S}_A - \mathsf{s}_b\mathsf{S}_B + \mathsf{s}_c\mathsf{S}_C)x_2 \\ +&\mathsf{s}_{s-c}(\mathsf{s}_a\mathsf{s}_b\mathsf{s}_c + \mathsf{s}_a\mathsf{S}_A + \mathsf{s}_b\mathsf{S}_B - \mathsf{s}_c\mathsf{S}_C)x_3 = 0. \end{aligned}$$

2.4.4. *Bicevian conics.* Let $P = [p_1 : p_2 : p_3]$ and $Q = [q_1 : q_2 : q_3]$ be two different points, not on any of the lines of Δ_0. Then

$$(x_1, x_2, x_3)\begin{pmatrix} \frac{2}{p_1q_1} & \frac{1}{p_1q_2} + \frac{1}{p_2q_1} & \frac{1}{p_3q_1} + \frac{1}{p_1q_3} \\ \frac{1}{p_1q_2} + \frac{1}{p_2q_1} & \frac{2}{p_2q_2} & \frac{1}{p_2q_3} + \frac{1}{p_3q_2} \\ \frac{1}{p_3q_1} + \frac{1}{p_1q_3} & \frac{1}{p_2q_3} + \frac{1}{p_3q_2} & \frac{2}{p_3q_3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

is the equation of a conic, which passes through the traces of P and Q. The perspector of this conic is the point $R = [r_1 : r_2 : r_3]$ with

$$(r_1, r_2, r_3) = (p_1q_1(p_2q_3 - p_3q_2), p_2q_2(p_3q_1 - p_1q_3), p_3q_3(p_1q_2 - p_2q_1)).$$

What are the conditions for Q to be a circumcevian conjugate of P or, in other words, what are the conditions for the bicevian conic to be a circle? Here, we should keep in mind that the point triple $A_PB_PC_P$ has four circumcircles and, therefore, there are four circumcevian conjugates of P. First, we calculate the barycentric coordinates of the circumcenter M_0 of the triangle $(A_PB_PC_P)_0$: The dual M_0^δ of M_0 is the line $(A_P - B_P) \vee (A_P - C_P)$, which is described by the equation

$$p_2p_3(-t_1 + t_2 + t_3)x_1 + p_2p_1(t_1 - t_2 + t_3)x_2 + p_1p_2(t_1 + t_2 - t_3)x_3 = 0,$$

with $t_1 := \|(0, p_2, p_3)\|_* = \sqrt{p_2^2 + 2p_2p_3\mathsf{c}_a + p_3^2}$,
$t_2 := \|(p_1, 0, p_3)\|_* = \sqrt{p_3^2 + 2p_3p_1\mathsf{c}_b + p_1^2}$
and $t_3 := \|(p_1, p_2, 0)\|_* = \sqrt{p_1^2 + 2p_1p_2\mathsf{c}_c + p_2^2}$.
We put $(s_1, s_2, s_3) := ((-t_1 + t_2 + t_3)/p_1, (t_1 - t_2 + t_3)/p_2, (t_1 + t_2 - t_3)/p_3)$ and calculate the coordinates $[m_{01} : m_{02} : m_{03}]$ of M_0 and the coordinates $[q_{01} : q_{02} : q_{03}]$ of the circumcevian conjugate Q_0 of P:

$$\begin{aligned} (m_{01}, m_{02}, m_{03}) &= (s_1, s_2, s_3)\mathfrak{T}^\# \\ &= ((-t_1 + t_2 + t_3)/p_1, (t_1 - t_2 + t_3)/p_2, (t_1 + t_2 - t_3)/p_3)\mathfrak{T}^\# \\ (q_{01}, q_{02}, q_{03}) &= (\frac{1}{(s_1^2 - 4)p_1}, \frac{1}{(s_2^2 - 4)p_2}, \frac{1}{(s_3^2 - 4)p_3}) \\ &= (\frac{p_1}{(-t_1+t_2+t_3)^2 - 4p_1^2}, \frac{p_2}{(t_1-t_2+t_3)^2 - 4p_2^2}, \frac{p_3}{(t_1-t_2+t_3)^2 - 4p_3^2}). \end{aligned}$$

For the radius r of the cevian-circle we calculate

$$r = \arccos \frac{2\mathsf{S}^2}{\|(m_{01}, m_{02}, m_{03})\|_*}.$$

The other circumcenters M_i and circumcevian conjugates Q_i , $i = 1, 2, 3$, can be determined in the same way. We give the results for $i = 1$:

$$(m_{11}, m_{12}, m_{13}) = ((t_1{+}t_2{+}t_3)/p_1, (-t_1{-}t_2{+}t_3)/p_2, (-t_1{+}t_2{-}t_3)/p_3)\mathfrak{T}^{\#},$$

$$(q_{11}, q_{12}, q_{13}) = \Big(\frac{p_1}{(t_1{+}t_2{+}t_3)^2 - 4p_1^2}, \frac{p_2}{(-t_1{-}t_2{+}t_3)^2 - 4p_2^2}, \frac{p_3}{(-t_1{+}t_2{-}t_3)^2 - 4p_3^2}\Big).$$

3. Four central lines

3.1. *The orthoaxis $G^+ \vee H$.*

3.1.1. *Triangle centers on the orthoaxis.* Wildberger [27] introduces the name *orthoaxis* for a line incident with several triangle centers: the orthocenter H, the centroid G^+ of the antimedial triangle, the orthostar $H^{\tau\delta}$ and three more triangle centers:

- One is the point $O^+ := [\mathsf{S}_A \mathsf{s}_a^2 : \mathsf{S}_B \mathsf{s}_b^2 : \mathsf{S}_C \mathsf{s}_c^2]$.
  In [27, 28] it is called *basecenter* and introduced as the meet of the lines $A \vee (B_H \vee C_H)^\delta$, $B \vee (C_H \vee A_H)^\delta$, $C \vee (A_H \vee B_H)^\delta$. In [26] the point O^+ is called *pseudo-circumcenter*, and it is shown that it is the meet of the perpendicular pseudo-bisectors $A_{G^+} \vee A'$, $B_{G^+} \vee B'$, $C_{G^+} \vee C'$.
- The second point is
  $L := [\mathsf{c}_a(-\mathsf{c}_a^2{+}\mathsf{c}_b^2{+}\mathsf{c}_c^2{+}1){-}2\mathsf{c}_b\mathsf{c}_c : \mathsf{c}_b(\mathsf{c}_a^2{-}\mathsf{c}_b^2{+}\mathsf{c}_c^2{+}1){-}2\mathsf{c}_a\mathsf{c}_c$
  $: \mathsf{c}_c(\mathsf{c}_a^2{+}\mathsf{c}_b^2{-}\mathsf{c}_c^2{+}1){-}2\mathsf{c}_a\mathsf{c}_b]$.
  It is the common point of the lines $A^{G^+} \vee A'$, $B^{G^+} \vee B'$, $C^{G^+} \vee C'$, and it is called *double dual point* in [27, 28].
- The third point is the intersection of the orthoaxis with the orthic axis H^τ and has coordinates
  $[\mathsf{S}_B\mathsf{S}_C(-2\mathsf{c}_a^2{+}\mathsf{c}_b^2{+}\mathsf{c}_c^2) : \mathsf{S}_C\mathsf{S}_A(\mathsf{c}_a^2{-}2\mathsf{c}_b^2{+}\mathsf{c}_c^2) : \mathsf{S}_A\mathsf{S}_B(\mathsf{c}_a^2{+}\mathsf{c}_b^2{-}2\mathsf{c}_c^2)]$.

3.1.2. *The bicevian conic through the traces of H and G^+.* We prove a conjecture of Vigara [26]:
The orthoaxis $o := H \vee G^+$ is a symmetry axis of the bicevian conic which passes through the traces of H and G^+.

Proof. The orthoaxis is described by the equation:

$$\mathsf{S}_A(\mathsf{c}_b^2 - \mathsf{c}_c^2)x_1 + \mathsf{S}_B(\mathsf{c}_c^2 - \mathsf{c}_a^2)x_2 + \mathsf{S}_C(\mathsf{c}_a^2 - \mathsf{c}_b^2)x_3 = 0,$$

so its dual is the point $P := o^\delta = [\mathsf{c}_a(\mathsf{c}_b^2 - \mathsf{c}_c^2), \mathsf{c}_b(\mathsf{c}_c^2 - \mathsf{c}_a^2), \mathsf{c}_c(\mathsf{c}_a^2 - \mathsf{c}_b^2)]$.
This point P is also the perspector of the conic through the traces of H and G^+; the equation of the conic is

$$(x_1, x_2, x_3)\,\mathfrak{M}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

with

$$\mathfrak{M} = \begin{pmatrix} 2\,\mathsf{S}_A\mathsf{c}_b\mathsf{c}_c & -\mathsf{c}_c(\mathsf{S}_A\mathsf{c}_a+\mathsf{S}_B\mathsf{c}_b) & -\mathsf{c}_b(\mathsf{S}_C\mathsf{c}_c+\mathsf{S}_A\mathsf{c}_a) \\ -\mathsf{c}_c(\mathsf{S}_A\mathsf{c}_a+\mathsf{S}_B\mathsf{c}_b) & 2\,\mathsf{S}_B\mathsf{c}_c\mathsf{c}_a & -\mathsf{c}_a(\mathsf{S}_B\mathsf{c}_b+\mathsf{S}_C\mathsf{c}_c) \\ -\mathsf{c}_b(\mathsf{S}_C\mathsf{c}_c+\mathsf{S}_A\mathsf{c}_a) & -\mathsf{c}_a(\mathsf{S}_B\mathsf{c}_b+\mathsf{S}_C\mathsf{c}_c) & 2\,\mathsf{S}_C\mathsf{c}_a\mathsf{c}_b \end{pmatrix}.$$

The point P is not only the perspector of the conic but also a symmetry point: The polar of P with respect to this conic is calculated by

$$(x_1,x_2,x_3)\,\mathfrak{C}\begin{pmatrix}\mathsf{c}_a(\mathsf{c}_b^2-\mathsf{c}_c^2)\\ \mathsf{c}_b(\mathsf{c}_c^2-\mathsf{c}_a^2)\\ \mathsf{c}_c(\mathsf{c}_a^2-\mathsf{c}_b^2)\end{pmatrix}=0.$$

and this, again, is an equation of the orthoaxis.

Let Q_1 and Q_2 be the intersections of the orthoaxis with the conic, and define M_1 and M_2 by $M_1 = Q_1 + Q_2$ and $M_2 = Q_1 - Q_2$. Then PM_1M_2 is a polar triple: $d(P,M_1) = d(P,M_2) = d(M_1,M_2) = \pi/2$. And the dual of PM_1M_2 with respect to $\mathcal{C}(\mathfrak{M})$ is M_2PM_1. The polar line of each of these points is a symmetry axis of $\mathcal{C}(\mathfrak{M})$. □

Vigara [26] assumes that either M_1 or M_2 equals the point $O^+ + H$. But this is not the case; a simple calculation shows that, in general, $d(P, O^+ \pm H) \neq \pi/2$.

3.2. *The line $G \vee O$.*

3.2.1. *Triangle centers on the line $G \vee O$.* The line $G \vee O$ is the orthoaxis of the medial triangle and has the equation

$$(1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c)(\mathsf{c}_b-\mathsf{c}_c)x_1 + (1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c)(\mathsf{c}_c-\mathsf{c}_a)x_2 \\ + (1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c)(\mathsf{c}_a-\mathsf{c}_b)x_3 = 0.$$

Besides G and O it contains the following triangle centers:

- The isogonal conjugate of O with barycentric coordinates

  $$[\frac{1+\mathsf{c}_a}{1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c} : \frac{1+\mathsf{c}_b}{1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c} : \frac{1+\mathsf{c}_c}{1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c}].$$

  This point is also the common point of the lines $perp(B_G \vee C_G, A)$, $perp(C_G \vee A_G, B)$ and $perp(A_G \vee B_G, C)$; we denote it by H^-.

- The $\tilde{K}$-conjugate of O; its coordinates are

  $$[\frac{1}{1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c} : \frac{1}{1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c} : \frac{1}{1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c}].$$

- The point L was already introduced in the last subsection; it is the intersection of the line $G \vee O$ with the orthoaxis. But it is also a point on the line $I \vee Ge$ and a point on all the lines $G_i \vee O_i, i = 0,1,2,3$. We call this point *de Longchamps point*. The name reason for choosing the name is: This point L is the radical center of the three power circles $\mathcal{C}(B_G + C_G, A), \mathcal{C}(C_G + A_G, B), \mathcal{C}(A_G + B_G, C)$.

Proof of the last statement: We will outline the proof for $perp(B_G \vee C_G, A)$ being the radical line of the first two power circles. In a similar way it can be shown that $perp(C_G \vee A_G, B), perp(A_G \vee B_G, C)$ are the other two radical lines.
We start with the dual of the line through the centers of the first two circles; this is the point

$$P = [(1-\mathsf{c}_a)(1+\mathsf{c}_a+\mathsf{c}_b+\mathsf{c}_c) : -(1+\mathsf{c}_b)(1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c) : -(1+\mathsf{c}_c)(1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c)].$$

Then for every real number t, the vector

$$\begin{aligned} \boldsymbol{p}_t := ((1-\mathsf{c}_a)(1+\mathsf{c}_a+\mathsf{c}_b+\mathsf{c}_c) + t(\mathsf{c}_a(1-\mathsf{c}_a^2+\mathsf{c}_b^2+\mathsf{c}_c^2)-2\mathsf{c}_b\mathsf{c}_c), \\ -(1+\mathsf{c}_b)(1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c) + t(\mathsf{c}_b(1+\mathsf{c}_a^2-\mathsf{c}_b^2+\mathsf{c}_c^2)-2\mathsf{c}_a\mathsf{c}_c), \\ -(1+\mathsf{c}_c)(1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c) + t(\mathsf{c}_c(1+\mathsf{c}_a^2+\mathsf{c}_b^2-\mathsf{c}_c^2)-2\mathsf{c}_a\mathsf{c}_b)). \end{aligned}$$

represents the barycentric coordinates of a point on $P \vee L$. We substitute the components of $\boldsymbol{p}_t$ for x_1, x_2, x_3 in the equations of the two power circles

$$\begin{aligned} &(x_1(\mathsf{c}_b+\mathsf{c}_c)+(\mathsf{c}_a+1)(x_2+x_3))^2 \\ &= (2\mathsf{c}_a x_2x_3+2\mathsf{c}_b x_1x_3+2\mathsf{c}_c x_1x_2+x_1^2+x_2^2+x_3^2)(\mathsf{c}_b+\mathsf{c}_c)^2, \\ &(x_2(\mathsf{c}_a+\mathsf{c}_c)+(\mathsf{c}_b+1)(x_1+x_3))^2 \\ &= (2\mathsf{c}_a x_2x_3+2\mathsf{c}_b x_1x_3+2\mathsf{c}_c x_1x_2+x_1^2+x_2^2+x_3^2)(\mathsf{c}_a+\mathsf{c}_c)^2, \end{aligned}$$

solve for t and get the same solutions for both circles. □

The points O, G, H^-, L form a harmonic range.

Proof. We introduce the vectors
$\boldsymbol{o} = ((1-\mathsf{c}_a)(1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c), (1-\mathsf{c}_b)(1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c), (1-\mathsf{c}_c)(1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c)),$
$\boldsymbol{g} = (1, 1, 1),$
$\boldsymbol{h} = ((1+\mathsf{c}_a)/(1+\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c), (1+\mathsf{c}_b)/(1-\mathsf{c}_a+\mathsf{c}_b-\mathsf{c}_c), (1+\mathsf{c}_c)/(1-\mathsf{c}_a-\mathsf{c}_b+\mathsf{c}_c)),$
$\boldsymbol{l} = (\mathsf{c}_a(-\mathsf{c}_a^2+\mathsf{c}_b^2+\mathsf{c}_c^2+1)-2\mathsf{c}_b\mathsf{c}_c, \mathsf{c}_b(\mathsf{c}_a^2-\mathsf{c}_b^2+\mathsf{c}_c^2+1)-2\mathsf{c}_a\mathsf{c}_c,$
$\mathsf{c}_c(\mathsf{c}_a^2+\mathsf{c}_b^2-\mathsf{c}_c^2+1)-2\mathsf{c}_a\mathsf{c}_b),$
the real numbers
$r = 1 - \mathsf{c}_a^2 - \mathsf{c}_b^2 - \mathsf{c}_c^2 + 2\mathsf{c}_a\mathsf{c}_b\mathsf{c}_c,$
$s = (1 + \mathsf{c}_a - \mathsf{c}_b - \mathsf{c}_c)(1 - \mathsf{c}_a + \mathsf{c}_b - \mathsf{c}_c)(1 - \mathsf{c}_a - \mathsf{c}_b + \mathsf{c}_c),$
$t = (1 + \mathsf{c}_a + \mathsf{c}_b + \mathsf{c}_c),$
and get the equations $s\boldsymbol{o} + t\boldsymbol{h} = 2r\boldsymbol{g}$ and $s\boldsymbol{o} - t\boldsymbol{h} = 2\boldsymbol{l}$. □

3.2.2. *A cubic curve as a substitute for the Euler circle.* For a point $P = [p_1 : p_2 : p_3] \in \mathcal{P}$ we calculate the pedals of P on the sidelines of the medial triangle:

$$\tilde{A}_{[P]} := (P \vee (B_{[G]} \vee C_{[G]})^\delta) \wedge (B_{[G]} \vee C_{[G]}), \tilde{B}_{[P]}, \tilde{C}_{[P]}.$$

The points P for which $\tilde{A}_{[P]}, \tilde{B}_{[P]}, \tilde{C}_{[P]}$ are collinear lie on a cubic that passes through the traces of the points H^- and $G_i, i = 0, 1, 2, 3$. We call this cubic *Euler-Feuerbach cubic*. In metric affine geometries this cubic splits into the nine-point-circle (Euler circle) and the line $G^\tau : x_1 + x_2 + x_3 = 0$.

Proof. Instead of proving the statement for the triangle Δ_0, we present the proof for the antimedial triangle of Δ_0. In this case, the formulae obtained are substantially shorter.
For the coordinates of the pedals $A_{[P]}, B_{[P]}, C_{[P]}$ of a point P on the sidelines of Δ, see 2.3.9. If these three pedals are collinear on a line, this line is called a *Simson line* of P. The locus of points P having a Simson line is the cubic with the equation

$$\mathsf{c}_a x_1(x_2^2\mathsf{s}_c^2 + x_3^2\mathsf{s}_b^2) + \mathsf{c}_b x_2(x_3^2\mathsf{s}_a^2 + x_1^2\mathsf{s}_c^2) + \mathsf{c}_c x_3(x_1^2\mathsf{s}_b^2 + x_2^2\mathsf{s}_a^2) \\ - 2x_1x_2x_3(1 - \mathsf{c}_a\mathsf{c}_b\mathsf{c}_c) = 0.$$

The centroid of the antimedial triangle is $G^+ = [\mathsf{c}_a : \mathsf{c}_b : \mathsf{c}_c]$. The traces of G^+ on the sidelines of the antimedial triangle are A, B, C. The tripolar of G^+ meets the sidelines of the antimedial triangle in $[0 : \mathsf{c}_b : -\mathsf{c}_c], [-\mathsf{c}_a : 0 : \mathsf{c}_c]$ and $[\mathsf{c}_a : -\mathsf{c}_b : 0]$. The traces of the point L on the sidelines of the antimedial triangle are

$$[\mathsf{c}_a(\mathsf{c}_b^2 + \mathsf{c}_c^2) - 2\mathsf{c}_b^2\mathsf{c}_b^2 : \mathsf{c}_b(\mathsf{c}_b^2 - \mathsf{c}_c^2) : \mathsf{c}_c(\mathsf{c}_b^2 - \mathsf{c}_c^2)],$$
$$[\mathsf{c}_a(\mathsf{c}_c^2 - \mathsf{c}_a^2) : \mathsf{c}_b(\mathsf{c}_c^2 + \mathsf{c}_a^2) - 2\mathsf{c}_b^2\mathsf{c}_b^2 : \mathsf{c}_c(\mathsf{c}_c^2 - \mathsf{c}_a^2)],$$
$$[\mathsf{c}_a(\mathsf{c}_a^2 - \mathsf{c}_b^2) : \mathsf{c}_b(\mathsf{c}_a^2 - \mathsf{c}_b^2) : \mathsf{c}_c(\mathsf{c}_a^2 + \mathsf{c}_b^2) - 2\mathsf{c}_a^2\mathsf{c}_b^2].$$

It can now be checked that the coordinates of all the traces satisfy the cubic equation. □

Remark. This circumcubic of ABC is the non pivotal isocubic $n\mathcal{K}(K, G^+, t)$ with pole K (symmedian), root G^+ and parameter $t = -2(1 - \mathsf{c}_a\mathsf{c}_b\mathsf{c}_c) \neq 0$ (terminology adopted from [5, 7]). It is the locus of dual points of the Simson lines of Δ, and it also passes through

- the vertices A', B', C' of the dual triangle,
- and through the points
  $[-\mathsf{S}_B\mathsf{S}_C : \mathsf{c}_b^2\mathsf{S}_B : \mathsf{c}_c^2\mathsf{S}_C], [\mathsf{c}_a^2\mathsf{S}_A : -\mathsf{S}_C\mathsf{S}_A : \mathsf{c}_c^2\mathsf{S}_C],$
  $[\mathsf{c}_a^2\mathsf{S}_A : \mathsf{c}_b^2\mathsf{S}_B : -\mathsf{S}_A\mathsf{S}_B].$

Closely connected with it is the Simson cubic

$$\mathsf{S}_A x_1(x_2^2 + x_3^2) + \mathsf{S}_B x_2(x_3^2 + x_1^2) + \mathsf{S}_C x_3(x_1^2 + x_2^2) - 2x_1x_2x_3(1 - \mathsf{c}_a\mathsf{c}_b\mathsf{c}_c) = 0,$$

which is the locus of tripoles of the Simson lines.

The Euler-Feuerbach cubic belongs to a set of cubics which can be constructed as follows: Consider the pencil of circumconics of Δ which pass through a given point $P = [p_1 : p_2 : p_3]$ different from A, B, C. The symmetry points of all these conics lie on a cubic which passes through the traces of P and the traces of $G_i, i = 0, 1, 2, 3$. In metric affin geometries this cubic splits into the bicevian conic of P and G and the tripolar line of G.

3.2.3. *Triplex points on $G \vee O$.* In euclidean geometry, triplex points were introduced by K. Mütz [16]; further studies on triplex and related points have been carried out by E. Schmidt [20]. By joining the meet of the perpendicular bisector of $A \vee B$ and the side line $A \vee C$ with the vertex B and joining the meet of the

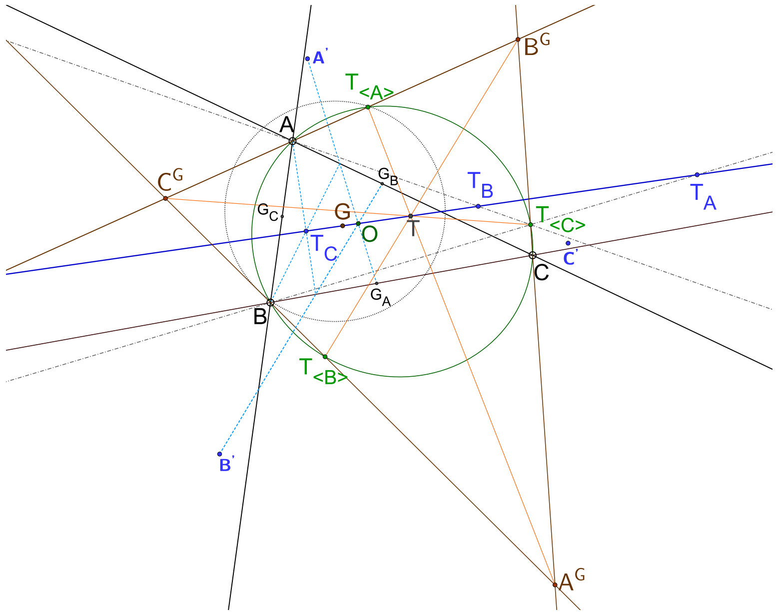

Figure 2. The triplex points on the line $G \vee O$ and the points $T_{<A>}, T_{<B>}, T_{<C>}$ on the circumcircle.

perpendicular bisector of $A \vee C$ and the side line $A \vee B$ with the vertex C, we get two lines that meet at a point, the *triplex point* T_A:

$$T_A = (((C_G \vee C') \wedge (A \vee C)) \vee B) \wedge (((B_G \vee B') \wedge (A \vee B)) \vee C)$$
$$= [1 : \frac{1 - c_b}{c_a - c_c} : \frac{1 - c_c}{c_a - c_b}].$$

The triplex points T_B, T_C are defined accordingly. It can be easily checked that T_A, T_B, T_C are points on the line $G \vee O$. (See [16, 20] for the euclidean version.)

The points $T_{<A>} := (B \vee T_C) \wedge (C \vee T_B), T_{<B>} := (C \vee T_A) \wedge (A \vee T_C), T_{<C>} := (A \vee T_B) \wedge (B \vee T_A)$ lie on the circumcircle; their coordinates are:

$$T_{<A>} = [1 - \mathsf{c}_a : \mathsf{c}_b - \mathsf{c}_c : \mathsf{c}_c - \mathsf{c}_b],$$
$$T_{<B>} = [\mathsf{c}_a - \mathsf{c}_c : 1 - \mathsf{c}_b : \mathsf{c}_c - \mathsf{c}_a],$$
$$T_{<C>} = [\mathsf{c}_a - \mathsf{c}_b : \mathsf{c}_b - \mathsf{c}_a : 1 - \mathsf{c}_c].$$

Furthermore, these points lie on the lines $A \vee A^G, B \vee B^G, C \vee C^G$, respectively, and the lines $A^G \vee T_{<A>}, B^G \vee T_{<B>}, C^G \vee T_{<C>}$ meet at point

$$T := [-3\mathsf{c}_a^2 + \mathsf{c}_b^2 + \mathsf{c}_c^2 - 2\mathsf{c}_b\mathsf{c}_c + 2\mathsf{c}_a\mathsf{c}_c + 2\mathsf{c}_a\mathsf{c}_b + 2\mathsf{c}_a - 2\mathsf{c}_b - 2\mathsf{c}_c + 1 : \cdots : \cdots]$$
$$= [-3\mathsf{s}_{a/2}^4 + \mathsf{s}_{b/2}^4 + \mathsf{s}_{c/2}^4 + 2\mathsf{s}_{a/2}^2\mathsf{s}_{b/2}^2 + 2\mathsf{s}_{a/2}^2\mathsf{s}_{c/2}^2 - 2\mathsf{s}_{b/2}^2\mathsf{s}_{c/2}^2 : \cdots : \cdots]$$

on the line $G \vee O$.

3.3. *The line* $O \vee K$.

3.3.1. *Triangle centers on the line* $O \vee K$. The points $(A \vee T_A) \wedge (B \vee C), (B \vee T_B) \wedge (C \vee A), (C \vee T_C) \wedge (A \vee B)$ are collinear, they all lie on the line $O \vee K$. The line $O \vee K$ is described by the equation

$$\frac{\mathsf{c}_b - \mathsf{c}_c}{1 - \mathsf{c}_a} x_1 + \frac{\mathsf{c}_c - \mathsf{c}_a}{1 - \mathsf{c}_b} x_2 + \frac{\mathsf{c}_a - \mathsf{c}_b}{1 - \mathsf{c}_c} x_3 = 0.$$

It has a tripole on the circumcircle and, besides the points already mentioned, it contains the Lemoine point $\tilde{K}$.

The *Lemoine axis* $\tilde{K}^\tau$ is perpendicular to $O \vee K$, so its dual point also lies on $O \vee K$. This axis $\tilde{K}^\tau$ is also the polar of the Lemoine point with respect to the circumcircle and intersects the sidelines of Δ_0 at the points $L_A = [0 : 1{-}\mathsf{c}_b : \mathsf{c}_c{-}1]$, $L_B = [\mathsf{c}_a{-}1 : 0 : 1{-}\mathsf{c}_c]$, $L_C = [1{-}\mathsf{c}_a : \mathsf{c}_b{-}1 : 0]$.

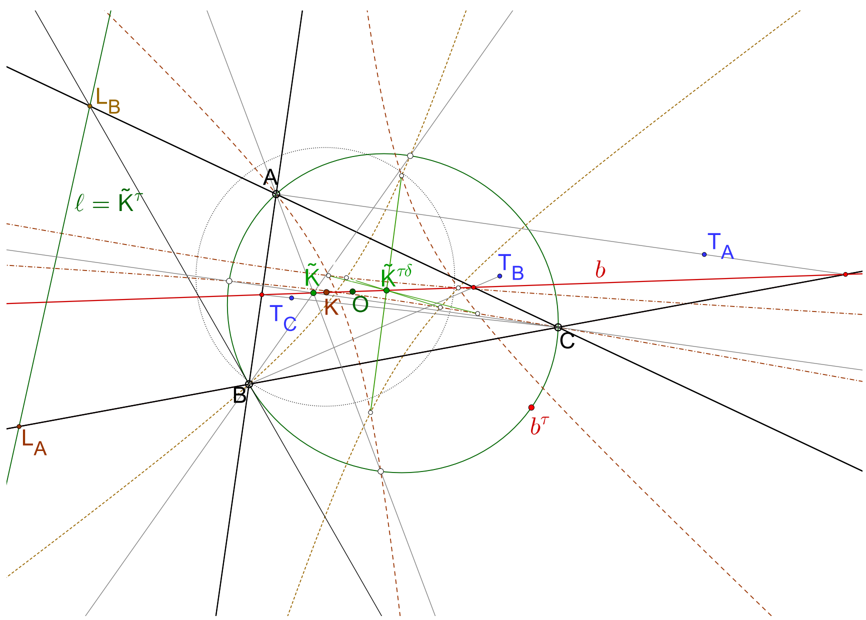

Figure 3. The line $b = O \vee K$, the Lemoine axis $\ell = \tilde{K}^\tau$, the circumcircle and the apollonian circles.

3.3.2. *The Apollonian circles.* The line $O \vee K$ is the common radical axis of the circles $\mathcal{C}(L_A, A)$, $\mathcal{C}(L_B, B)$, $\mathcal{C}(L_C, C)$, which we will call *apollonian circles* of Δ_0.

Proof. It can be easily checked that the polar line of a perspector P of a circumconic with respect to this conic is the line P^τ.

The two points $[(1{-}\mathsf{c}_a)(1{+}\mathsf{c}_a{-}t_\pm) : (1{-}\mathsf{c}_b)(1{+}\mathsf{c}_b{-}t_\pm) : (1{-}\mathsf{c}_b)(1{+}\mathsf{c}_b{-}t_\pm)]$, with $t_\pm = \frac{\sqrt{3}}{6}\left(\sqrt{1 + \mathsf{c}_a + \mathsf{c}_b + \mathsf{c}_c} \pm \sqrt{1 - \mathsf{c}_a^2 - \mathsf{c}_b^2 - \mathsf{c}_c^2 + 2\mathsf{c}_a\mathsf{c}_b\mathsf{c}_c}\right)$, lie on $O \vee K$ and on each of the apollonian circles. So we call these two points *isodynamic points*. $\square$

3.3.3. *The Lemoine conic.* Define the point $P_1 := par(B \vee C, \tilde{K}) \wedge (B \vee C)$ and P_2, P_3 accordingly. Further, define $P_{23} := par(B \vee A, \tilde{K}) \wedge (A \vee C)$ and the points $P_{32}, P_{12}, P_{21}, P_{31}, P_{13}$ accordingly. (Here we consider A, B, C as the first, second and third point of the triangle Δ_0, respectively.)
The points P_1, P_2, P_3 lie on the line $\tilde{K}^\delta$ with the equation

$$(1-c_a+c_b+c_c-2c_bc_c)x_1+(1+c_a-c_b+c_c-2c_cc_a)x_2+(1+c_a+c_b-c_c-2c_ac_b)x_3 = 0.$$

The six points $P_{23}, P_{32}, P_{31}, P_{13}, P_{12}, P_{21}$ lie on a conic with the equation

$$\begin{aligned}\sum_{cyclic}\Big(&\big(\nu_1+\nu_2+\nu_3-2\nu_2\nu_3\big)\big(\nu_1(\nu_2+\nu_3-4\nu_2\nu_3)+(\nu_2-\nu_3)^2\big)\nu_2\nu_3x_1^2\\ &-\Big((\nu_1^4+\nu_1^3(3(\nu_2+\nu_3)-8\nu_2\nu_3))+\nu_1^2(3(\nu_2^2+\nu_3^2)+8\nu_2\nu_3-14\nu_2\nu_3(\nu_1+\nu_3)+20\nu_2^2\nu_3^2)\\ &-\nu_1(\nu_2+\nu_3)(6\nu_2\nu_3(1+\nu_2+\nu_3)-(\nu_2^2+\nu_3^2))+2\nu_2\nu_3(\nu_2+\nu_3)^2\Big)\nu_1x_2x_3\Big) \;=\; 0,\end{aligned}$$

where we put $\nu_1 := 1-c_a = 2s_{a/2}^{\,2}\,,\ \nu_2 := 1-c_b\,,\ \nu_3 := 1-c_c$.
We call this conic *Lemoine conic*.
It can be proved by calculation:

- The line $K \vee O$ is a symmetry line of the Lemoine conic.
- If the line $\tilde{K}^\delta$ has common points with the circumcircle, then these points are also points on the Lemoine conic.
- The pole of $\tilde{K}^\delta$ with respect to the circumcircle is a point on $K \vee O$.
- The pole of $\tilde{K}^\delta$ with respect to the Lemoine conic is a point on $K \vee O$.

3.4. *The Akopyan line $O \vee H^{\tau\delta}$.*

3.4.1. *Triangle centers on the line $O \vee H^{\tau\delta}$.* There are several triangles centers, introduced by Akopyan [2], lying on the join of the circumcenter and the orthostar; therefore, we propose to name this line *Akopyan line*. (Akopyan himself uses the name *Euler line* for it, and Vigara uses the name *Akopyan Euler line*.) Its equation is

$$\begin{aligned}(c_b-c_c)(1+2\mathsf{c}_a-\mathsf{c}_b-\mathsf{c}_c-\mathsf{c}_b\mathsf{c}_c)x_1 + (\mathsf{c}_c-\mathsf{c}_a)(1-\mathsf{c}_a+2\mathsf{c}_b-\mathsf{c}_c-\mathsf{c}_c\mathsf{c}_a)x_2&\\ + (\mathsf{c}_a-\mathsf{c}_b)(1-\mathsf{c}_a-\mathsf{c}_b+2\mathsf{c}_c-\mathsf{c}_a\mathsf{c}_b)x_3 &= 0.\end{aligned}$$

As a first point on this line, apart from O and $H^{\tau\delta}$, we introduce the point G^*, whose cevians bisect the triangle area in equal parts. The existence of such a point was already shown by J. Steiner [21]. (See also [3].) The coordinates of G^* are

$$\begin{aligned}&[\frac{\sqrt{1+\mathsf{c}_a}}{\sqrt{2}\sqrt{1+\mathsf{c}_a}+\sqrt{1+\mathsf{c}_b}\sqrt{1+\mathsf{c}_c}} : \frac{\sqrt{1+\mathsf{c}_b}}{\sqrt{2}\sqrt{1+\mathsf{c}_b}+\sqrt{1+\mathsf{c}_c}\sqrt{1+\mathsf{c}_a}}\\ &\qquad : \frac{\sqrt{1+\mathsf{c}_c}}{\sqrt{2}\sqrt{1+\mathsf{c}_c}+\sqrt{1+\mathsf{c}_a}\sqrt{1+\mathsf{c}_b}}]\\ &= [\frac{\mathsf{c}_{a/2}}{\mathsf{c}_{a/2}+\mathsf{c}_{b/2}\mathsf{c}_{c/2}} : \frac{\mathsf{c}_{b/2}}{\mathsf{c}_{b/2}+\mathsf{c}_{c/2}\mathsf{c}_{a/2}} : \frac{\mathsf{c}_{c/2}}{\mathsf{c}_{c/2}+\mathsf{c}_{a/2}\mathsf{c}_{b/2}}].\end{aligned}$$

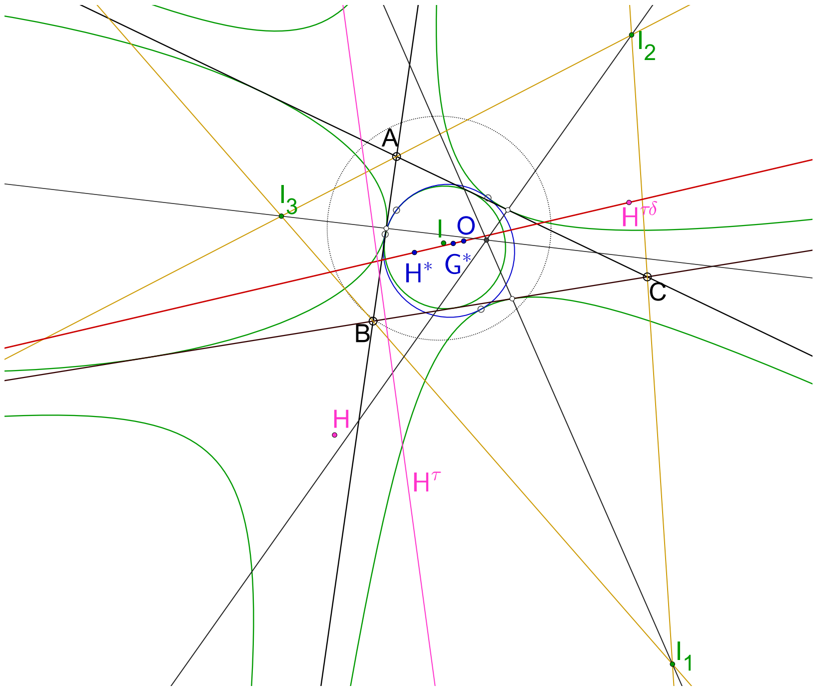

Figure 4. The Akopyan line and the Hart circle together with the incircle and the excircles of Δ_0.

The calculation is carried out according to the construction of G^* described below.

Akopyan [2] shows that the cyclocevian of G^* lies on the line $O \vee G^*$ and has properties that justify to call it a *pseudo-orthocenter*. We, therefore, denote it by H^*. He also shows that the center of the common cevian circle of G^* and H^* - we will call this point N^* - is also a point on $O \vee G^*$. Thus, the cevian circle of G^* can be seen as a good substitute in elliptic geometry for the euclidean nine point circle, even more so, since this circle, as also proved by Akopyan, touches the incircles of Δ_i for $i = 0, 1, 2, 3$. The common cevian circle of G^* and H^* we like to name *Hart circle* of Δ_0, because A. S. Hart [9] calculated 1861 the equation of the circle which touches incircle and the excircles of a spherical triangle, and the name *Hart's circle* is used by G. Salmon in [19]. Salmon showed that its center N^* lies on the lines $G \vee H$ and $O^+ \vee H^-$ and he calculated the (trilinear) coordinates of N^*. The barycentric representation of this point is $N^* = [(\mathsf{c}_a+1)(\mathsf{c}_a(\mathsf{c}_b\,\mathsf{c}_c-1) + 1 - \mathsf{c}_b^2 + \mathsf{c}_b\,\mathsf{c}_c - \mathsf{c}_c^2) : \cdots : \cdots]$.

Remark. The Akopyan line is a line which contains a point together with its circumcevian conjugate and the center of their common cevian circle. Such a line is called *cevian axis* [6]. The orthoaxis and the line $G \vee O$ are, in general, not cevian axes of the triangle Δ_0. But the line $G \vee O$ is a cevian axis of the anticevian triangle $(A^G B^G C^G)_0$ of G, see Figure 2.

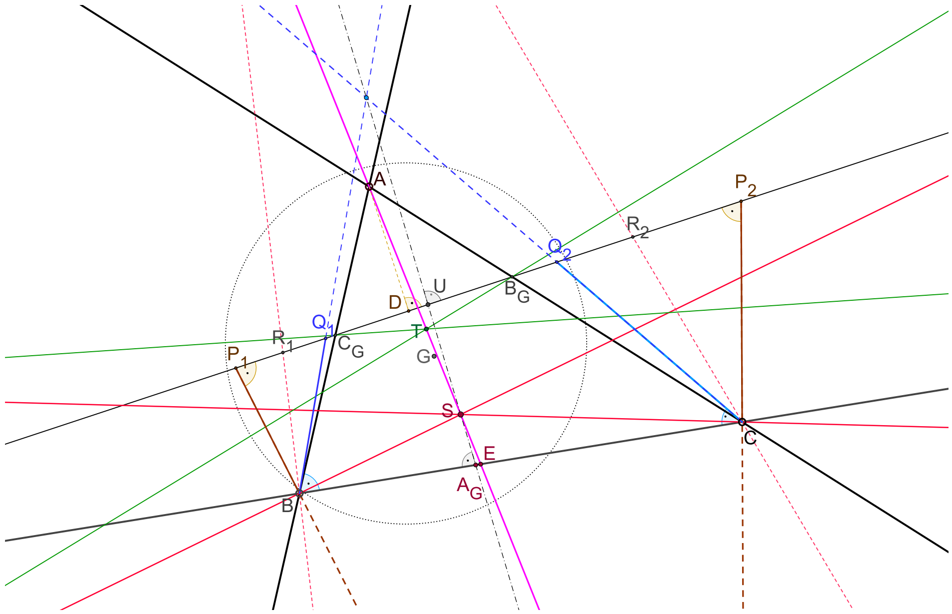

Figure 5. Construction of the bisector $A \vee S$ through the vertex A.

Explanations to Figure 5: By projecting the points A, B, C onto the sideline $B_G \vee C_G$ of the medial triangle of Δ_0, we get the points D, P_1 and P_2, respectively. The area 2ϵ of the triangle Δ_0 is the same as the area of the quadrangle $(AP_1P_2B)_0$ because triangle $(C_GAD)_0$ is congruent to triangle $(C_GBP_1)_0$ and triangle $(B_GAD)_0$ congruent to triangle $(B_GBP_2)_0$.
The lines $perp(B \vee C, B)$ and $perp(B \vee C, C)$ meet $B_G \vee C_G$ at Q_1 and Q_2, respectively. It follows that $2\epsilon = \mu(\angle_+ P_1BQ_1) + \mu(\angle_+ P_2CQ_2)$.
Let W be the meet of the lines $B \vee C$ and $B_G \vee C_G$. Confirm by calculation that the mirror image of P_1BQ_1 in W (or in its dual line $W^\delta = A_G \vee U$) is P_2BQ_2. It follows that $\mu(\angle_+ P_1BQ_1) = \mu(\angle_+ P_2CQ_2) = \epsilon$.
The lines $B \vee R_1$ and $C \vee R_2$ are the internal bisectors, the lines $B \vee S$ and $C \vee S$ the external bisectors of $\angle_+ P_1BQ_1$ resp. $\angle_+ P_2CQ_2$. Define $E := (A \vee S) \wedge (B \vee C)$ and let T be the midpoint of $[A, E]_+$. We can now confirm by calculating that $T \vee C_G = perp(B \vee R_1, C_G)$ and conclude that the area of the triangle $(ABE)_0$ is ϵ.

List of triangle centers:

triangle center P	$P =$ $[f(\alpha,\beta,\gamma){:}f(\beta,\gamma,\alpha){:}f(\gamma,\alpha,\beta)]$, $f(\alpha,\beta,\gamma) =$	euclidean limit point (We adopt the notation from [11].)
G	1	X_2
G^+	$1 + \frac{2\ \sin\epsilon\ \sin\epsilon\text{-}\alpha}{\sin\beta\sin\gamma}$	
G^*	$\sin\alpha\,/\,\sin\frac{1}{2}\epsilon\text{ - }\alpha$	
I	$\sin\alpha$	X_1
O	$\sin\alpha\ \ \cos\epsilon\text{-}\alpha$	X_3
O^+	$\sin 2\alpha$	
H	$\tan\alpha$	X_4
H^-	$\sin\alpha\,/\,\cos\epsilon\text{-}\alpha$	
N^*	$\sin\alpha\ \ \cos\beta{-}\gamma$	X_5
K	$\sin^2\alpha$	X_6
$\tilde{K}$	$\sin\alpha\ \sin\epsilon\text{-}\alpha$	
Ge	$\tan\frac{1}{2}\alpha$	X_7
Na	$\cot\frac{1}{2}\alpha$	X_8
L	$3{\xi_\alpha}^2{-}2\xi_\alpha(\xi_\beta{+}\xi_\gamma){-}(\xi_\beta{-}\xi_\gamma)^2{+}\phi_\alpha({\xi_\alpha}^2{-}{\xi_\beta}^2{-}{\xi_\gamma}^2)$, with $\xi_\alpha = \mathsf{s}_\alpha\mathsf{s}_{\epsilon\text{-}\alpha},\ldots$ and $\phi_\alpha = \frac{2\sin\epsilon\ \sin\epsilon\text{-}\alpha}{\sin\beta\sin\gamma}$	X_{20}
T (cf. 3.2.3.)	$-3{\mathsf{s}_{\alpha/2}}^4 + 2{\mathsf{s}_{\alpha/2}}^2({\mathsf{s}_{\beta/2}}^2{+}{\mathsf{s}_{\gamma/2}}^2) + ({\mathsf{s}_{\beta/2}}^2{-}{\mathsf{s}_{\gamma/2}}^2)^2$	
$H^{\tau\delta}$	$\sin\alpha\,(\cos\alpha - 2\cos\beta\ \cos\gamma)$	X_{30}
$(O\vee K)^\tau$	$\xi_\alpha/(\xi_\beta{-}\xi_\gamma)$ $\xi_\alpha = \mathsf{s}_\alpha\mathsf{s}_{\epsilon\text{-}\alpha}$, $\xi_\beta = \mathsf{s}_\beta\mathsf{s}_{\epsilon\text{-}\beta}$, $\xi_\gamma = \mathsf{s}_\gamma\mathsf{s}_{\epsilon\text{-}\gamma}$	X_{110}
$\tilde{K}^{\tau\delta}$	$\xi_\alpha(\xi_\beta{+}\xi_\gamma){-}{\xi_\beta}^2{-}{\xi_\gamma}^2 + 2\,\mathsf{s}_\epsilon\ \mathsf{s}_{\epsilon\text{-}\alpha}\ \mathsf{s}_{\epsilon\text{-}\beta}\ \mathsf{s}_{\epsilon\text{-}\gamma}$, $\xi_\alpha = \mathsf{s}_\alpha\mathsf{s}_{\epsilon\text{-}\alpha}$, $\xi_\beta = \mathsf{s}_\beta\mathsf{s}_{\epsilon\text{-}\beta}$, $\xi_\gamma = \mathsf{s}_\gamma\mathsf{s}_{\epsilon\text{-}\gamma}$	infinity point on the Brocard axis

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Three Synthetic Proofs of the Butterfly Theorem

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Abstract. We give three synthetic proofs of the butterfly theorem, using Thales' theorem, the notions of isogonal conjugates, and spiral similarity respectively.

1. Introduction

Theorem (The Butterfly Theorem)**.** *Through the midpoint M of a chord PQ of a circle, two other chords AB and CD are drawn such that A and D are on opposite sides of PQ . If AD and BC intersect PQ at X and Y respectively, then M is also the midpoint of XY.*

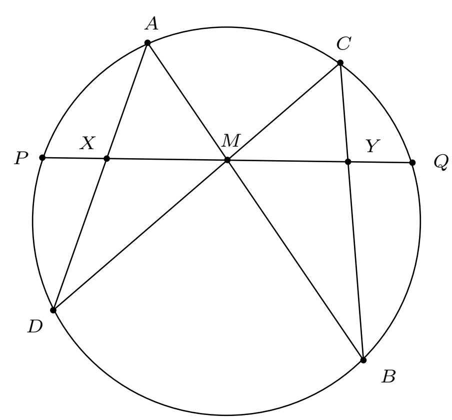

Figure 1.

2. The first proof: Thales' theorem

In Figure 2, the circumcircle of the triangle BOD intersects AB and CD again at E and F respectively, where O is the circumcenter of the cyclic quadrilateral $ACBD$. Since $\angle MFE = \angle DBE = \angle DCA$, $EF \parallel AC$. Let G be a point on OM such that $AG \parallel OE$. By Thales' Theorem, we have $\dfrac{MG}{MO} = \dfrac{MA}{ME} = \dfrac{MC}{MF}$. It follows that $CG \parallel OF$. Notice that $\angle GCB = \angle GCD + \angle DCB = \angle MFO + \angle DCB = \angle OBD + \angle DCB = 90^\circ$. Similarly, $\angle GAD = 90^\circ$. Since $GM \perp XY$, GX and GY are diameters of the circles (MAX) and (MCY) respectively. Therefore, $\angle MGX = \angle MAX = \angle MCY = \angle MGY$. The triangles MGX and MGY are congruent, and $MX = MY$. This proves the Butterfly Theorem.

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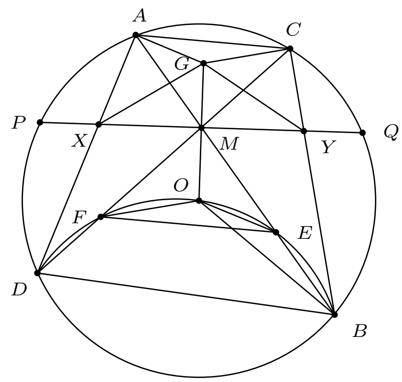

Figure 2.

3. The second proof: isogonal conjugates

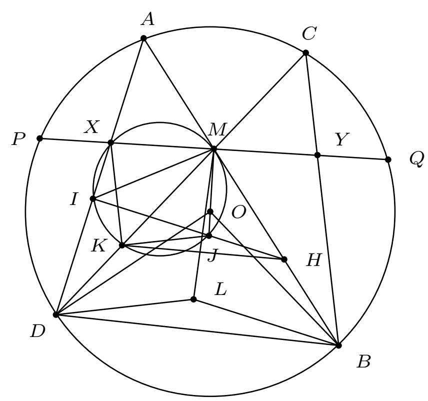

Figure 3.

Let O be the center of the circle. Denote by H and K the reflections of A and C through M. Since $MH \cdot MB = AM \cdot MB = CM \cdot MD = MK \cdot MD$, triangles MBD and MKH are oppositely similar (see Figure 3). Let L be the isogonal conjugate of O with respect to the triangle MBD. Denote by J the image of L under the dilative reflection that maps the triangle MBD onto the triangle MKH. HJ intersects AD at I. Since $\angle HMJ = \angle LMD = \angle HMO$, MJ passes through O. We have $\angle MHJ = \angle LDM = \angle BDO = \angle OBD = \angle MBL = \angle JKM = 90^\circ - \angle DAB$. From this, together with the fact that M is the midpoint of AH, it follows that $\angle XIJ = \angle XMJ = 90^\circ$, and $\angle JIM = \angle JKM = 90^\circ - \angle DAB$. Hence, points M, J, K, I, and X lie on a circle whose diameter is JX, and $\angle JKM + \angle DCB = 90^\circ$. Therefore, $KJ \perp KX$ and $KJ \perp BC$. It implies that $KX \parallel BC$. Triangles MXK and MYC are congruent, and $MX = MY$. This completes the second proof of the Butterfly Theorem.

4. The third proof: spiral similarity

Lemma. *The diagonals of a quadrilateral $ACBD$ that is inscribed in a circle (O) intersect at R. Let S be the center of the spiral similarity that maps AC onto DB. Then, SR is the common angle bisector of the angles ASB and CSD, and passes through O.*

Proof. Since S is the center of the spiral similarity that maps AC onto DB, the triangles SAD and SCB are directly similar. Therefore, the triangles SAC and SDB are also directly similar. It means that S is the center of the spiral similarity that maps AD onto CB (see Figure 4).

Since the triangles RCB and RAD are similar, $\dfrac{RC}{RA} = \dfrac{CB}{AD} = \dfrac{SC}{SA}$. Similarly, $\dfrac{RA}{RD} = \dfrac{AC}{DB} = \dfrac{SA}{SD}$. Hence, $\dfrac{RC}{RD} = \dfrac{SC}{SD}$, and SR is the angle bisector of $\angle CSD$. Similarly, SR is also the angle bisector of $\angle ASB$. Denote by (a, b) the directed angle from the line a to the line b (See [5, pp. 11–15]). Since the quadrilateral $ACBD$ is cyclic, we have

$$\begin{aligned}
(SC,\ SD) = &\ (SC,\ SA) + (SA,\ SD)\\
= &\ (CB,\ AD) + (AC,\ DB)\\
= &\ (BC,\ BD) + (DB,\ DA) + (CA,\ CB) + (BC, BD)\\
= &\ 2(BC,\ BD)\\
= &\ (OC,\ OD).
\end{aligned}$$

It follows that S, C, D and O are concyclic. From $OC = OD$, SO is the bisector of angle CSD. We deduce that SR passes through O. $\square$

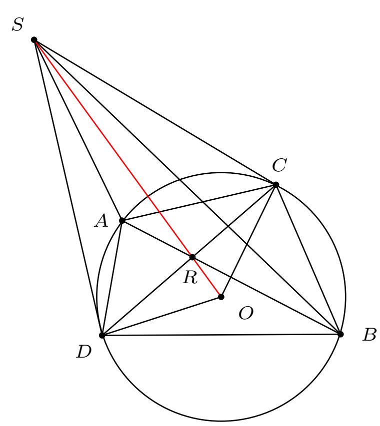

Figure 4

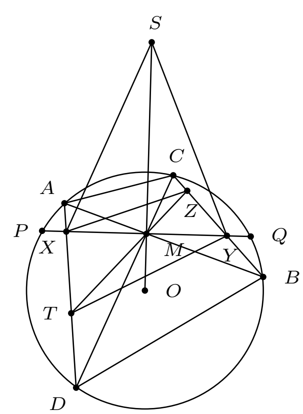

Figure 5

Proof of the butterfly theorem. Let Z and T be the points on the segments BC and AD respectively so that $\frac{ZC}{ZB} = \frac{XA}{XD}$ and $\frac{TA}{TD} = \frac{YC}{YB}$. As the triangles MAD and MCB are similar, $\angle CMZ = \angle AMX = \angle BMY = \angle DMT$ and $\angle MXT = \angle MZY$. Hence, ZT passes through M and the quadrilateral $XZYT$ is cyclic (see Figure 5). Let S be the center of the spiral similarity mapping AD onto CB. This spiral similarity maps X into Z, T into Y and XT onto ZY. Let (O) be the circumcircle of the cyclic quadrilateral $ACBD$. Then, by the Lemma, SM passes through O. Since M is the midpoint of PQ, $SM \perp XY$. Applying the Lemma to the cyclic quadrilateral $XZYT$, SM is the bisector of angle XSY. Combining with $SM \perp XY$, it follows that $MX = MY$ by symmetry. This completes the third proof of the Butterfly Theorem.

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Radii of Circles in Apollonius' Problem

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Abstract. The paper presents the relation for radii of the eight circles in Apollonius' problem for circles which are tangent to three given circles. Analogously, we derived the relations for radii of the 16 spheres which are tangent to four given spheres, with coordinates of their centers and with their radii.

1. Introduction

It is well known that for three given circles generically there are eight different circles that are tangent to them. The problem of ruler and compass constructability of these eight circles is well-known. There are famous Apollonius' and Gergonne's solutions to this problem [3]. Special cases of the three given circles are considered and a number of other problems is known [2]. The first case is when we consider three sides of the original triangle as three circles with infinite radii. The incircle and three excircles of the original triangle are four solutions to Apollonius' problem. The second case is when we have three excircles as a starting point. Three sides of the original triangle are three solutions to Apollonius' problem with infinite radii [1]. The nine-point circle is tangent externally to the three excircles, by Feuerbach theorem, and a relatively new object - the Apollonius circle is tangent internally to three exircles (for some results about this circle see [4]-[7]). To these five circles we can add three Jenkins circles which are tangent to three excircles, by adding two of them externally and the third one internally.

2. Main result

Let us assume that the three given circles are $K_1(O_1, r_1)$, $K_2(O_2, r_2)$, $K_3(O_3, r_3)$, Figure 1, with distances between centers $O_2O_3 = a$, $O_3O_1 = b$, $O_1O_2 = c$ and with the area $(O_1O_2O_3) = \Delta$, which is different from 0. The following theorem holds true:

Theorem 1. *Let us assume that the radii of the eight circles with centers S_i given in Figure 1* ((a), (b), (c) *and* (d)) *are p_j*, $(j = 1, 2, ..., 8)$. *Then*

$$\frac{1}{p_1} - \frac{1}{p_2} + \frac{1}{p_3} - \frac{1}{p_4} + \frac{1}{p_5} - \frac{1}{p_6} + \frac{1}{p_7} - \frac{1}{p_8} = 0. \tag{1}$$

Proof. Let us introduce the angles $\varphi_1 = \angle O_2SO_3$, $\varphi_2 = \angle O_3SO_1$, $\varphi_3 = \angle O_1SO_2$, where $S = S_1$ for Figure 1 (a), so as to obtain

$$\cos^2\varphi_1 + \cos^2\varphi_2 + \cos^2\varphi_3 - 2\cos\varphi_1\cos\varphi_2\cos\varphi_3 = 1. \tag{2}$$

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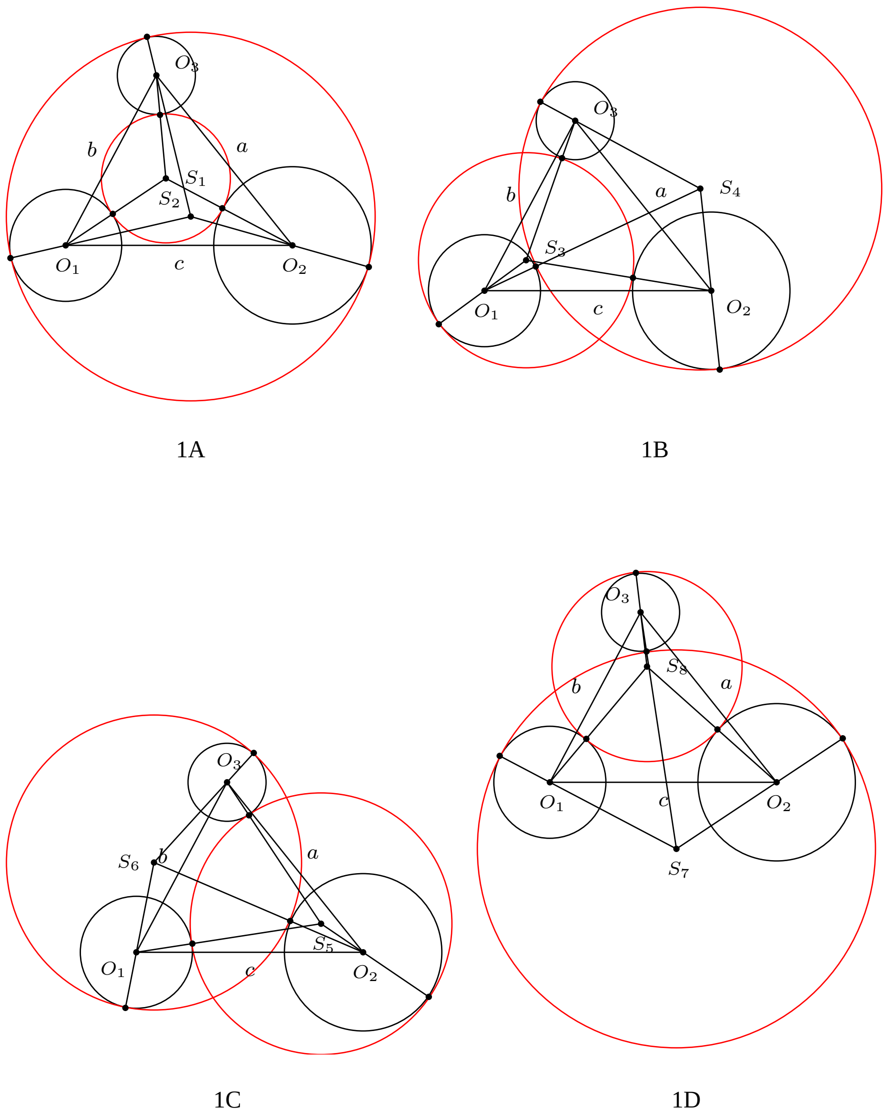


Figure 1. The eight different circles that are tangent to the three given circles.

If we substitute

$$t_1 = \sin^2 \frac{\varphi_1}{2}, \qquad t_2 = \sin^2 \frac{\varphi_2}{2}, \qquad t_3 = \sin^2 \frac{\varphi_3}{2},$$

from relation (2) we have

$$t_1^2 + t_2^2 + t_3^2 - 2(t_1t_2 + t_2t_3 + t_3t_1) + 4t_1t_2t_3 = 0. \tag{3}$$

If we generally denote the center of the circle and radius by S and p, then for the first unknown circle L_1 (see Figure 1 (a)) is $SO_1 = p + r_1$, $SO_2 = p + r_2$,

$SO_3 = p + r_3$ and

$$\cos\varphi_1 = \frac{(p+r_2)^2 + (p+r_3)^2 - a^2}{2(p+r_2)(p+r_3)} \implies t_1 = \frac{a^2 - (r_2 - r_3)^2}{4(p+r_2)(p+r_3)},$$

and analogously

$$t_2 = \frac{b^2 - (r_3 - r_1)^2}{4(p+r_3)(p+r_1)}, \qquad t_3 = \frac{c^2 - (r_1 - r_2)^2}{4(p+r_1)(p+r_2)}.$$

From relation (3) we now have relation (4):

$$\begin{aligned}
&(a^2 - (r_2 - r_3)^2)^2(p+r_1)^2 + (b^2 - (r_3 - r_1)^2)^2(p+r_2)^2\\
&+ (c^2 - (r_1 - r_2)^2)^2(p+r_3)^2\\
&- 2(a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(p+r_1)(p+r_2)\\
&- 2(a^2 - (r_2 - r_3)^2)(c^2 - (r_1 - r_2)^2)(p+r_1)(p+r_3)\\
&- 2(b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2)(p+r_2)(p+r_3)\\
&+ (a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2)\\
&= 0.
\end{aligned} \tag{4}$$

or in another form

$$F_1(p,\ r_1,\ r_2,\ r_3) = 0. \tag{5}$$

Equation (4) is of the second degree and is of the form

$$A_1 p^2 + B_1 p + C_1 = 0, \tag{6}$$

where

$$\begin{aligned}
A_1 =\ & 4(a^2(r_1 - r_2)(r_1 - r_3) + b^2(r_2 - r_3)(r_2 - r_1) + c^2(r_3 - r_1)(r_3 - r_2))\\
&\quad - 16\Delta^2\\
=\ & f(r_1, r_2, r_3), \qquad (7)\\
B_1 =\ & 2\{r_1(a^2 - (r_2 - r_3)^2)^2 + r_2(b^2 - (r_3 - r_1)^2)^2 + r_3(c^2 - (r_1 - r_2)^2)^2\\
&- (a^2 - (r_2 - r_3)^2)(b^2 - (r_3 - r_1)^2)(r_1 + r_2)\\
&- (b^2 - (r_3 - r_1)^2)(c^2 - (r_1 - r_2)^2)(r_2 + r_3)\\
&- (c^2 - (r_1 - r_2)^2)(a^2 - (r_2 - r_3)^2)(r_3 + r_1)\}\\
=\ & g(r_1, r_2, r_3), \qquad (8)\\
C_1 =\ & r_1^2 a^4 + r_2^2 b^4 + r_3^2 c^4 + a^2 b^2 c^2\\
&- a^2 b^2 (r_1^2 + r_2^2) - b^2 c^2 (r_2^2 + r_3^2) - c^2 a^2 (r_3^2 + r_1^2)\\
&+ a^2 (r_1^2 - r_2^2)(r_1^2 - r_3^2) + b^2 (r_2^2 - r_3^2)(r_2^2 - r_1^2)\\
&+ c^2 (r_3^2 - r_1^2)(r_3^2 - r_2^2)\\
=\ & h(r_1, r_2, r_3). \qquad (9)
\end{aligned}$$

For the second unknown circle L_2 (see Figure 1 (a)) we have $SO_1 = p - r_1$, $SO_2 = p - r_2$, $SO_3 = p - r_3$ and a corresponding equation in the form of equation (4), and

$$F_1(p,\ -r_1,\ -r_2,\ -r_3) = 0, \tag{10}$$

$$A_2 p^2 + B_2 p + C_2 = 0, \tag{11}$$

with

$$\begin{aligned} A_2 &= \ f_1(-r_1,\ -r_2,\ -r_3) = A_1, \\ B_2 &= \ g_1(-r_1,\ -r_2,\ -r_3) = B_1, \\ C_2 &= \ h_1(-r_1,\ -r_2,\ -r_3) = C_1, \end{aligned}$$

which implies that

$$A_1 p_1^2 + B_1 p_1 + C_1 = 0, \qquad A_1 p_2^2 - B_1 p_2 + C_1 = 0,$$

and

$$\frac{1}{p_1} - \frac{1}{p_2} = -\frac{B_1}{C_1}. \tag{12}$$

For the third circle L_3 (see Figure 1 (b)) we have $SO_1 = p - r_1$, $SO_2 = p + r_2$, $SO_3 = p + r_3$ and we get $F_1(p,\ -r_1,\ r_2,\ r_3) = 0$ with $A_3 p^2 + B_3 p + C_3 = 0$, $A_3 = f_1(-r_1,\ r_2,\ r_3)$, $B_3 = g_1(-r_1,\ r_2,\ r_3)$, $C_3 = h_1(-r_1\, r_2,\ r_3) = C_1$.

For the fourth circle L_4 (see Figure 1 (b)) we have $SO_1 = p + r_1$, $SO_2 = p - r_2$, $SO_3 = p - r_3$ and we get $F_1(p,\ r_1,\ -r_2,\ -r_3) = 0$ with $A_4 p^2 + B_4 p + C_4 = 0$, $A_4 = f_1(r_1,\ -r_2,\ -r_3) = A_3$, $B_4 = g_1(r_1,\ -r_2,\ -r_3) = -B_3$, $C_4 = h_1(r_1, -r_2, -r_3) = C_1$ and again we get

$$\frac{1}{p_3} - \frac{1}{p_4} = -\frac{B_3}{C_1}. \tag{13}$$

Analogously, we have

$$\frac{1}{p_5} - \frac{1}{p_6} = \ -\frac{B_5}{C_1}, \tag{14}$$

$$\frac{1}{p_7} - \frac{1}{p_8} = \ -\frac{B_7}{C_1}, \tag{15}$$

where

$$B_5 = g_1(r_1,\ -r_2,\ r_3), \qquad B_7 = g_1(r_1,\ r_2, -r_3).$$

Formula (1) follows from (12), (13), (14), (15) because

$$g_1(r_1,\ r_2,\ r_3) + g_1(-r_1,\ r_2,\ r_3) + g_1(r_1,\ -r_2,\ r_3) + g_1(r_1,\ r_2,\ -r_3) = 0.$$

□

Remark 1. If the index j of the circle with radius p_j is an even (odd) number, then $1/p_j$ (in formula (1)) has the sign $+$ ($-$).

Remark 2. In the first case of the three given circles mentioned in the introduction, we get the formula

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

where r, r_1, r_2, r_3 are the inradius and exradii of triangle ABC.

Remark 3. In the second case we get

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{q} = \frac{1}{m},$$

where p_1, p_2, p_3 are the radii of Jenkins circles, q is the radius of Apollonius' circle and $m = R/2$ is the radius of Euler circle or nine-point circle.

This formula can be proved independently since

$$p_1 = \frac{a}{b+c}q, \quad p_2 = \frac{b}{c+a}q, \quad p_3 = \frac{c}{a+b}q.$$

Remark 4. In the same way, the same result can be proved for the three given circles, provided that two of them are inside of the third one.

3. Positions of 8 circles

The radical circle of the three given circles $K_1(O_1, r_1)$, $K_2(O_2, r_2)$, $K_3(O_3, r_3)$, is the circle orthogonal to all of them, and pairs of circles (L_1, L_2), (L_3, L_4), (L_5, L_6), (L_7, L_8) – Figure 1, are inversive with respect to the radical circle. For this radical circle $K_0(S_0, r_0)$, Figure 2, the following holds true.

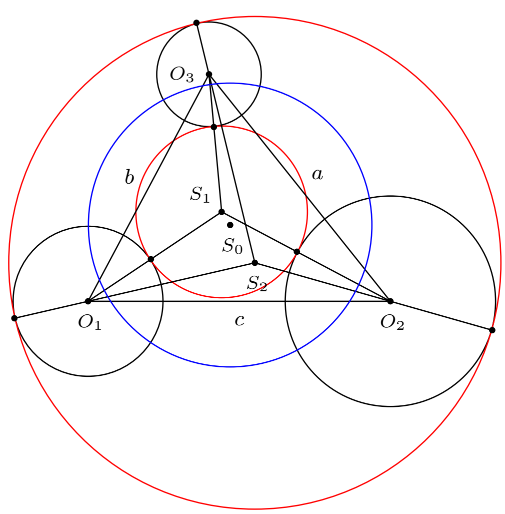


Figure 2. The radical circle of the three given circles with the circles L_1 and L_2, inversive to the radical circle, based on Figure 1 (a)

Proposition 2. (1) *The center $S_0(x_0 : y_0 : z_0)$ has barycentric coordinates with respect to triangle $O_1O_2O_3$*

$$\begin{aligned} x_0 =& \ a^2(b^2+c^2-a^2)+u_0, \\ y_0 =& \ b^2(c^2+a^2-b^2)+v_0, \\ z_0 =& \ c^2(a^2+b^2-c^2)+w_0, \end{aligned} \tag{16}$$

where

$$u_0 = \ (r_2^2+r_3^2-2r_1^2)a^2+(r^2-r_3^2)(b^2-c^2), \tag{17}$$

$$v_0 = \ (r_3^2+r_1^2-2r_2^2)b^2+(r^3-r_1^2)(c^2-a^2), \tag{18}$$

$$w_0 = \ (r_1^2+r_2^2-2r_3^2)a^2+(r^1-r_2^2)(a^2-b^2). \tag{19}$$

(2) *The radius of the radical circle is given by formula*

$$r_0^2 = \frac{C_1}{16\Delta^2}. \tag{20}$$

(3) *The coefficients B_1, B_3, B_5, B_7 are expressed in terms of the coordinates of S_0, i.e.,*

$$B_1 = \ -2(r_1x_0+r_2y_0+r_3z_0), \qquad B_3 = -2(r_1x_0-r_2y_0-r_3z_0), \tag{21}$$

$$B_5 = \ -2(-r_1x_0+r_2y_0-r_3z_0), \qquad B_7 = -2(-r_1x_0-r_2y_0+r_3z_0). \tag{22}$$

(4) *The line S_0O_0, where O_0 represents the circumcenter of triangle $O_1O_2O_3$, is orthogonal to the line $q : \ r_1^2x+r_2^2y+r_3^2z = 0$. This line passes through the midpoints of segments $M_{11}M_{12}$, $M_{21}M_{22}$ and $M_{31}M_{32}$, where M_{11} and M_{12} are the inner and outer centers of similarity of circles (K_2) and (K_3), which can analogously be applied to the other points.*

For the eight solutions $L_j(S_j, p_j)$ of Apollonius' problem, with $S_j(x_j : y_j : z_j)$ we have

Proposition 3. (1) *The coordinates of the centers S_j are as follows:*

$$\begin{aligned} x_1 &= 2p_1u_1+x_0, & y_1 &= 2p_1v_1+y_0, & z_1 &= 2p_1w_1+z_0, \\ x_2 &= -2p_2u_1+x_0, & y_2 &= -2p_2v_1+y_0, & z_2 &= -2p_2w_1+z_0, \end{aligned}$$

where

$$\begin{aligned} u_1 =& \ (r_2+r_3-2r_1)a^2+(r_2-r_3)(b^2-c^2) = u_1(r_1,r_2,r_3), \\ v_1 =& \ (r_3+r_1-2r_2)b^2+(r_3-r_1)(c^2-a^2) = v_1(r_1,r_2,r_3), \\ w_1 =& \ (r_1+r_2-2r_3)c^2+(r_1-r_2)(a^2-b^2) = w_1(r_1,r_2,r_3). \end{aligned}$$

$$\begin{aligned} x_3 &= 2p_3u_3+x_0, & y_3 &= 2p_3v_3+y_0, & z_3 &= 2p_3w_3+z_0, \\ x_4 &= -2p_4u_3+x_0, & y_4 &= -2p_4v_3+y_0, & z_4 &= -2p_4w_3+z_0, \end{aligned}$$

where

$$\begin{aligned} u_3(r_1,r_2,r_3) &= u_1(r_1,-r_2,-r_3), \\ v_3(r_1,r_2,r_3) &= v_1(r_1,-r_2,-r_3), \\ w_3(r_1,r_2,r_3) &= w_1(r_1,-r_2,-r_3). \end{aligned}$$

$$\begin{aligned} x_5 &= 2p_5u_5+x_0, & y_5 &= 2p_5v_5+y_0, & z_5 &= 2p_5w_5+z_0, \\ x_6 &= -2p_6u_5+x_0, & y_6 &= -2p_6v_5+y_0, & z_6 &= -2p_6w_5+z_0, \end{aligned}$$

where

$$u_5(r_1,r_2,r_3) = u_1(r_1,r_2,-r_3),$$
$$v_5(r_1,r_2,r_3) = v_1(r_1,r_2,-r_3),$$
$$w_5(r_1,r_2,r_3) = w_1(r_1,r_2,-r_3).$$

$$x_7 = 2p_7u_7 + x_0, \quad y_7 = 2p_7v_7 + y_0, \quad z_7 = 2p_7w_7 + z_0,$$
$$x_8 = -2p_8u_7 + x_0, \quad y_8 = -2p_8v_7 + y_0, \quad z_8 = -2p_8w_7 + z_0,$$

where

$$u_7(r_1,r_2,r_3) = u_1(r_1,-r_2,r_3),$$
$$v_7(r_1,r_2,r_3) = v_1(r_1,-r_2,r_3),$$
$$w_7(r_1,r_2,r_3) = w_1(r_1,-r_2,r_3).$$

(2) *There are collinear triplets of points* (S_0,S_1,S_2), (S_0,S_3,S_4), (S_0,S_5,S_6) *and* (S_0,S_7,S_8), *and*

$$S_0S_1 \perp q_1: \quad r_1x + r_2y + r_3z = 0,$$
$$S_0S_3 \perp q_3: \quad -r_1x + r_2y + r_3z = 0,$$
$$S_0S_5 \perp q_5: \quad r_1x - r_2y + r_3z = 0,$$
$$S_0S_7 \perp q_7: \quad r_1x + r_2y - r_3z = 0,$$

where q_1, q_3, q_5, q_7 *are the lines* $M_{12}M_{22}M_{32}$, $M_{12}M_{21}M_{31}$, $M_{11}M_{22}M_{31}$, *and* $M_{11}M_{21}M_{32}$ *respectively.*

(3)

$$\frac{1}{S_0S_1} - \frac{1}{S_2S_0} = \frac{2}{S_0V_1}, \tag{23}$$

where $U_1 = S_0S_1 \cap q_1$ *and* V_1 *and* U_1 *are inversive to each other with respect to the radical circle. Analogously, this is assumed for the other centers* S_j.

4. Three-dimensional case

In this case we have four spheres and a maximum of 16 spheres, each of which being tangent to all of the four given spheres. Analogous relations for radii of these 16 spheres will be found subsequently. Let us assume that the four spheres are $\Phi_1(O_1,r_1)$, $\Phi_2(O_2,r_2)$, $\Phi_3(O_3,r_3)$, $\Phi_4(O_4,r_4)$. We can take the basic tetrahedron ABCD to be tetrahedron $O_1O_2O_3O_4$ with $O_1 = A(1:0:0:0)$, $O_2 = B(0:1:0:0)$, $O_3 = C(0:0:1:0)$, $O_4 = D(0:0:0:1)$ given in the barycentric coordinates with mutual distances $AB = c$, $AC = b$, $AD = d$, $BC = a$, $BD = e$, $CD = f$. An important role in further investigations is played by Cayley-Menger determinant Δ_0 of tetrahedron $ABCD$ given as follows:

$$\Delta_0 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & d^2 \\ 1 & c^2 & 0 & a^2 & e^2 \\ 1 & b^2 & a^2 & 0 & f^2 \\ 1 & d^2 & e^2 & f^2 & 0 \end{vmatrix}. \tag{24}$$

The known result is that the volume V of tetrahedron $ABCD$ is given by formula

$$\Delta_0 = 288V^2.$$

If we apply Δ_{ij} to denote the algebraic cofactor of element with row-column position (i, j) in corresponding Cayley-Menger matrix, we obtain the following result.

Proposition 4. (1) *The center O of the circumscribed sphere of tetrahedron $ABCD$ (or circumcenter) has barycentric coordinates*

$$O(\Delta_{12} : \ \Delta_{13} : \ \Delta_{14} : \ \Delta_{15}). \tag{25}$$

(2) *The circumradius R of the upper circumsphere is given by*

$$R^2 = -\frac{\Delta_{11}}{2\Delta_0}. \tag{26}$$

(3) *If for point $P(x : y : z : t)$ we introduce two relevant expressions*

$$\tau = \tau(P) = \ x + y + z + t, \tag{27}$$

$$T = T(P) = \ a^2 yz + b^2 zx + c^2 xy + d^2 xt + e^2 yt + f^2 zt, \tag{28}$$

then we have

$$\tau(O) = \Delta_0, \qquad T(O) = \frac{1}{2}\Delta_0 \cdot \Delta_{11}. \tag{29}$$

Let us now introduce the radical sphere $\Phi_0(S_0, r_0)$, i.e., the sphere with property

$$S_0A^2 - r_1^2 = S_0B^2 - r_2^2 = S_0C^2 - r_3^2 = S_0D^2 - r_4^2 = r_0^2, \tag{30}$$

or the sphere which is orthogonal to the four given spheres Φ_1, Φ_2, Φ_3, Φ_4. Then we have

Proposition 5. (1) *This radical sphere corresponds to the equation*

$$T = \tau(r_1^2 x + r_2^2 y + r_3^2 z + r_4^2 t). \tag{31}$$

(2) *The center $S_0(x_0 : y_0 : z_0 : t_0)$ has coordinates*

$$x_0 = \ \Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52}, \tag{32}$$

$$y_0 = \ \Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}, \tag{33}$$

$$z_0 = \ \Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}, \tag{34}$$

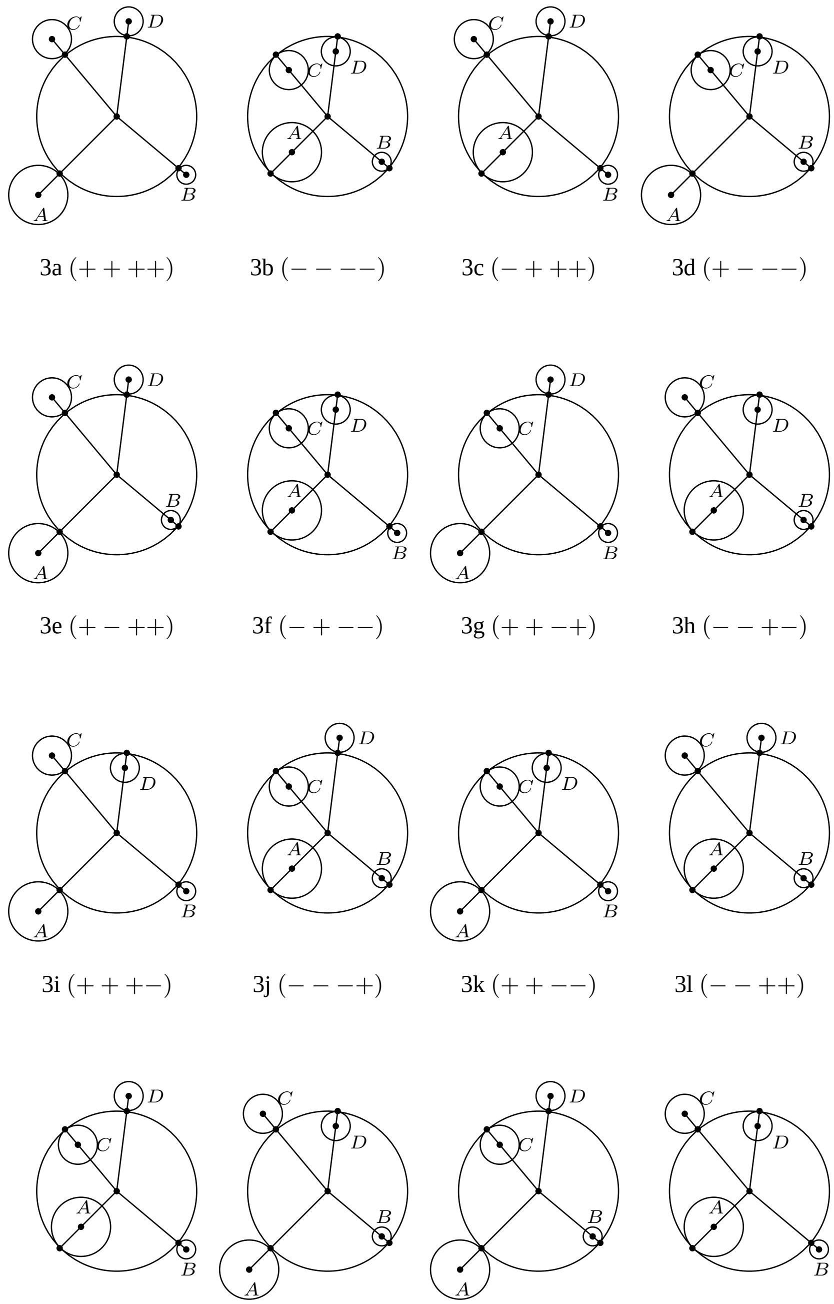
$$t_0 = \ \Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}. \tag{35}$$

(3) *For the radius r_0, the following formula holds.*

$$r_0^2 = R^2 - \left(r_1^2 x(M) + r_2^2 y(M) + r_3^2 z(M) + r_4^2 t(M)\right), \tag{36}$$

where M is the midpoint of the segment S_0O.

In Figure 3 we introduce corresponding ordered quadruplets. An appropriate number j in illustration (from (a) to (p)) denotes that sphere L_j is tangent to the four given spheres. The plus sign in the second position (given in brackets in each figure) means that sphere with center B is outside-externally tangent to sphere L_j, while the minus sign at the fourth position means that sphere with center D is inside sphere L_j – internally tangent, and similarly in other cases. For each of the 16 possible layouts, corresponding signs are given immediately under the figure,



3a $(++++)$ 3b $(----)$ 3c $(-+++)$ 3d $(+---)$

3e $(+-++)$ 3f $(-+--)$ 3g $(++-+)$ 3h $(--+-)$

3i $(+++-)$ 3j $(---+)$ 3k $(++--)$ 3l $(--++)$

3m $(-+-+)$ 3n $(+-+-)$ 3o $(+--+)$ 3p $(-++-)$

Figure 3. The 16 possible quadruplets- spheres each of which being tangent to all of the four given spheres

related to the respective spheres with centers A, B, C and D, depending on their position in relation to sphere L_j.

First of all, we will investigate sphere L_1. If we generally denote the center of the sphere and the radius by S and p, then for the first unknown sphere L_1 (see Figure 3) is

$$SO_1 = p + r_1, \quad SO_2 = p + r_2, \quad SO_3 = p + r_3, \quad SO_4 = p + r_4,$$

and equations of spheres $\Phi_1'(A, p+r_1)$, $\Phi_2'(B, p+r_2)$, $\Phi_3'(C, p+r_3)$, $\Phi_4'(D, p+r_4)$ are

$$T = \tau\{-(p+r_1)^2 x + (c^2 - (p+r_1)^2)y + (b^2 - (p+r_1)^2)z + (d^2 - (p+r_1)^2)t\}, \tag{37}$$

$$T = \tau\{(c^2 - (p+r_2)^2)x - (p+r_2)^2 y + (a^2 - (p+r_2)^2)z + (e^2 - (p+r_2)^2)t\}, \tag{38}$$

$$T = \tau\{(b^2 - (p+r_3)^2)x + (a^2 - (p+r_3)^2)y - (p+r_3)^2 z + (f^2 - (p+r_3)^2)t\}, \tag{39}$$

$$T = \tau\{(d^2 - (p+r_4)^2)x + (e^2 - (p+r_4)^2)y + (f^2 - (p+r_4)^2)z - (p+r_4)^2 t\}. \tag{40}$$

These equations determine the point $S_1(x_1 : y_1 : z_1 : t_1)$ which is the center of the sphere L_1 and radius $p = p_1$ of that sphere. For them we have

$$x_1 = x_0 + 2p(r_1\Delta_{22} + r_2\Delta_{32} + r_3\Delta_{42} + r_4\Delta_{52}), \tag{41}$$

$$y_1 = y_0 + 2p(r_1\Delta_{23} + r_2\Delta_{33} + r_3\Delta_{43} + r_4\Delta_{53}), \tag{42}$$

$$z_1 = z_0 + 2p(r_1\Delta_{24} + r_2\Delta_{34} + r_3\Delta_{44} + r_4\Delta_{54}), \tag{43}$$

$$t_1 = t_0 + 2p(r_1\Delta_{25} + r_2\Delta_{35} + r_3\Delta_{45} + r_4\Delta_{55}), \tag{44}$$

and these formulae lead us to the equation for $p = p_1$

$$F_1(p,\ r_1,\ r_2,\ r_3,\ r_4) = A_1 p^2 + B_1 p + C_1 = 0, \tag{45}$$

where

$$\begin{aligned} A_1 = {} & 2r_1(r_1\Delta_{22} + r_2\Delta_{32} + r_3\Delta_{42} + r_4\Delta_{52}) \\ & + 2r_2(r_1\Delta_{23} + r_2\Delta_{33} + r_3\Delta_{43} + r_4\Delta_{53}) \\ & + 2r_3(r_1\Delta_{24} + r_2\Delta_{34} + r_3\Delta_{44} + r_4\Delta_{54}) \\ & + 2r_4(r_1\Delta_{25} + r_2\Delta_{35} + r_3\Delta_{45} + r_4\Delta_{55}) + \Delta_0 \\ = {} & f(r_1, r_2, r_3, r_4), \end{aligned} \tag{46}$$

$$\begin{aligned} B_1 = {} & 2r_1(\Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52}) \\ & + 2r_2(\Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}) \\ & + 2r_3(\Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}) \\ & + 2r_4(\Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}) \\ = {} & 2(r_1x_0 + r_2y_0 + r_3z_0 + r_4t_0) \\ = {} & g(r_1, r_2, r_3, r_4), \end{aligned} \tag{47}$$

$$
\begin{aligned}
C_1 = \; & \frac{1}{2}\{r_1^2(\Delta_{12} + r_1^2\Delta_{22} + r_2^2\Delta_{32} + r_3^2\Delta_{42} + r_4^2\Delta_{52}) \\
& + r_2^2(\Delta_{13} + r_1^2\Delta_{23} + r_2^2\Delta_{33} + r_3^2\Delta_{43} + r_4^2\Delta_{53}) \\
& + r_3^2(\Delta_{14} + r_1^2\Delta_{24} + r_2^2\Delta_{34} + r_3^2\Delta_{44} + r_4^2\Delta_{54}) \\
& + r_4^2(\Delta_{15} + r_1^2\Delta_{25} + r_2^2\Delta_{35} + r_3^2\Delta_{45} + r_4^2\Delta_{55}) \\
& + (\Delta_{11} + r_1^2\Delta_{12} + r_2^2\Delta_{13} + r_3^2\Delta_{14} + r_4^2\Delta_{15})\} \\
= \; & h(r_1, r_2, r_3, r_4),
\end{aligned}
\tag{48}
$$

For the second unknown sphere L_2 (see Figure 3) we have $SO_1 = p - r_1$, $SO_2 = p - r_2$, $SO_3 = p - r_3$, $SO_4 = p - r_4$ as illustrated by the equation

$$
F_1(p,\ -r_1,\ -r_2,\ -r_3,\ -r_4) \equiv A_2p^2 + B_2p + C_2 \equiv A_1p^2 - B_1p + C_1 = 0. \tag{49}
$$

Now we get

$$
\frac{1}{p_1} - \frac{1}{p_2} = -\frac{B_1}{C_1}. \tag{50}
$$

This means that to the difference $1/p_1 - 1/p_2$ we can relate the ordered quadruple $(+,+,+,+)$ related to (r_1, r_2, r_3, r_4) since B_1 is linear with respect to all r_j. Since C_1 is the same for all 16 combinations $(\varepsilon_1 r_1,\ \varepsilon_2 r_2,\ \varepsilon_3 r_3,\ \varepsilon_4 r_4)$ for all $\varepsilon \in \{-1,\ 1\}$. When combining the signs of the ordered quadruplets, we obtain the following results given in the next theorem.

Theorem 6. *Let us assume that the radii of the sixteen spheres given in Figure 3 are* p_j $(j = 1, 2, ..., 16)$ *and the volume* V *is different from* 0*. Then*

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) = 0, \tag{51}
$$

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) - \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) = 0, \tag{52}
$$

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) - \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0, \tag{53}
$$

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) + \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0, \tag{54}
$$

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) = 0, \tag{55}
$$

$$
\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) = 0, \tag{56}
$$

$$
\begin{aligned}
& \left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \left(\frac{1}{p_3} - \frac{1}{p_4}\right) - \left(\frac{1}{p_5} - \frac{1}{p_6}\right) - \left(\frac{1}{p_7} - \frac{1}{p_8}\right) \\
& \quad - \left(\frac{1}{p_9} - \frac{1}{p_{10}}\right) + \left(\frac{1}{p_{11}} - \frac{1}{p_{12}}\right) + \left(\frac{1}{p_{13}} - \frac{1}{p_{14}}\right) + \left(\frac{1}{p_{15}} - \frac{1}{p_{16}}\right) = 0,
\end{aligned}
\tag{57}
$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)+\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)$$
$$-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{58}$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)+\left(\frac{1}{p_7}-\frac{1}{p_8}\right)$$
$$-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)-\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{59}$$

$$\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)$$
$$+\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)-\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0. \tag{60}$$

Corollary 7. *From the above formulae we can also obtain*

$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_5}-\frac{1}{p_6}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{61}$$
$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{62}$$
$$\left(\frac{1}{p_3}-\frac{1}{p_4}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)=0, \tag{63}$$
$$\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_7}-\frac{1}{p_8}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)=0, \tag{64}$$
$$\left(\frac{1}{p_5}-\frac{1}{p_6}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{11}}-\frac{1}{p_{12}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0, \tag{65}$$
$$\left(\frac{1}{p_7}-\frac{1}{p_8}\right)-\left(\frac{1}{p_9}-\frac{1}{p_{10}}\right)+\left(\frac{1}{p_{13}}-\frac{1}{p_{14}}\right)-\left(\frac{1}{p_{15}}-\frac{1}{p_{16}}\right)=0. \tag{66}$$

These formulae and formulae listed in Theorem 6 are all possible formulae of this type.

Corollary 8. *For the radius r_0 of the radical circle,*

$$r_0^2=-\frac{C_1}{\Delta_0}. \tag{67}$$

Proof. From formula (37) we have

$$T_1=\tau_1\left[(c^2y_1+b^2z_1+d^2t_1)=(p_1+r_1)^2\tau_1\right].$$

Since

$$\tau_1=\tau(S_1)=\tau(S_0)=\Delta_0,$$

for the coefficient C_1' without o_1, we have

$$C_1' = T_0 - \Delta_0(c^2 y_0 b^2 z_0 + d^2 t_0) + r_1^2 \Delta_0^2.$$

Now the desired formula follows from

$$\begin{aligned} C_1' &= \Delta_0 \cdot C_1, \\ r_0^2 &= \frac{1}{\tau_0}\left((c^2 y_0 + b^2 z_0 + d^2 t_0) - r_1^2 \tau_0 - \frac{T_0}{\tau_0}\right) \\ &= \frac{1}{\Delta_0}\left((c^2 y_0 + b^2 z_0 + d^2 t_0) - r_1^2 \Delta_0 - \frac{T_0}{\Delta_0}\right). \end{aligned}$$

□

Remark 5. If in the two-dimensional case, we introduce the determinant

$$\Delta_0 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix},$$

then we have $\Delta_0 = -16\Delta^2$. Consequently, $r_0^2 = -C_1/\Delta_0$ in both cases.

Corollary 9. *The centers S_1 and S_2 have coordinates*

$$\begin{aligned} x_1 &= x_0 + 2p_1 u_1, \quad y_1 = y_0 + 2p_1 v_1, \quad z_1 = z_0 + 2p_1 w_1, \quad t_1 = t_0 + 2p_1 \eta_1, \\ x_2 &= x_0 - 2p_2 u_1, \quad y_2 = y_0 - 2p_2 v_1, \quad z_2 = z_0 - 2p_2 w_1, \quad t_2 = t_0 - 2p_2 \eta_1, \end{aligned}$$

where

$$\begin{aligned} u_1 &= r_1 \Delta_{22} + r_2 \Delta_{32} + r_3 \Delta_{42} + r_4 \Delta_{52}, \\ v_1 &= r_1 \Delta_{23} + r_2 \Delta_{33} + r_3 \Delta_{43} + r_4 \Delta_{53}, \\ w_1 &= r_1 \Delta_{24} + r_2 \Delta_{34} + r_3 \Delta_{44} + r_4 \Delta_{54}, \\ \eta_1 &= r_1 \Delta_{25} + r_2 \Delta_{35} + r_3 \Delta_{45} + r_4 \Delta_{55}, \end{aligned}$$

with the property

$$u_1 + v_1 + w_1 + \eta_1 = 0.$$

Analogously to the planar case we can obtain coordinates of centers for all 16 spheres. There are eight planes, and each of them passes through six of the twelve inner or outer centers of mutual similarity of the given four spheres. As earlier, point S_0 with two centers is perpendicular to one of these eight planes.

Theorem 1 can be proved by the same technique used in the proof of Theorem 6.

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The Simson Triangle and Its Properties

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Abstract. Let ABC be a triangle and P_1, P_2, P_3 points on its circumscribed circle. The Simson triangle for P_1, P_2, P_3 is the triangle bounded by their Simson lines with respect to triangle ABC. We study some interesting properties of the Simson triangle.

1. Introduction

The following theorem is often called Simson's theorem. (see [2, p. 137, Theorem 192])

Theorem 1 (Wallace-Simson line)**.** *The feet of the perpendiculars to the sides of triangle from a point are collinear, if and only if the point is on the circumscribed circle of the triangle. This is Simson line* (*or Wallace-Simson line*)*.*

Definition (Simson triangle)**.** The Simson triangle is the triangle bounded by the Simson lines of three points on the circumscribed circle of a fixed triangle. It is degenerate if the three Simson lines are concurrent.

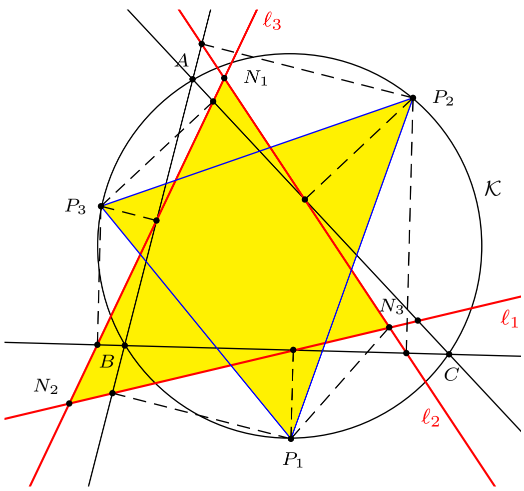

Figure 1

Theorem 2 (Orthopole, (see [2, p. 247, Theorem 406]))**.** *If perpendiculars are dropped on a line from the vertices of a triangle, the perpendiculars to the opposite sides from their feet are concurrent at a point called the orthopole of the line.*

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2. Notations

ABC is a triangle of reference.
The circumcircle $\mathcal{K}$ of $\triangle ABC$ has center O and radius R.
$P_1 \in \mathcal{K}, P_2 \in \mathcal{K}, P_3 \in \mathcal{K}$.
l_1, l_2, l_3 are the Simson lines of P_1, P_2, P_3 with respect to ABC.
$N_1N_2N_3$ is the Simson triangle bounded by these Simson lines:

$$N_1 = l_2 \cap l_3, \qquad N_2 = l_3 \cap l_1, \qquad N_3 = l_1 \cap l_2.$$

The circumcircle of $\triangle N_1N_2N_3$ is $\mathcal{K}_2$ with center O_2 and radius R_2.
The orthocenter of $\triangle ABC$ is H.
The orthocenter of $\triangle P_1P_2P_3$ is H_1.
The orthocenter of $\triangle N_1N_2N_3$ is H_2.
The center of nine-point circle for ABC is E.

We will use complex numbers in the proofs.
By u we shall denote the complex number, corresponding to point U.
Without loss of generality, we take the circumcircle $\mathcal{K}$ to be the unit circle. Then $R = 1$ and $O = 0$.
$a \cdot \bar{a} = b \cdot \bar{b} = c \cdot \bar{c} = p_1 \cdot \bar{p}_1 = p_2 \cdot \bar{p}_2 = p_3 \cdot \bar{p}_3 = 1$.
$h = a + b + c$; $e = 1/2(a + b + c)$; $h_1 = p_1 + p_2 + p_3$.

Lemma 3. *Let V and W be points on the unit circle. The orthogonal projection of a point P onto the line $\ell = VW$ is given by*

$$p_\ell = \frac{1}{2}(v + w + p - vw\bar{p}).$$

In particular, if P is also on the unit circle, then

$$p_\ell = \frac{1}{2}\left(v + w + p - \frac{vw}{p}\right).$$

Proof. Write the orthogonal projection as $p_\ell = (1-t)v + tw$ for some *real* number t. The vector $p - (1-t)v - tw$ is perpendicular to BC. This means that

$$(p - (1-t)v - tw)(\bar{v} - \bar{w}) + (\bar{p} - (1-t)\bar{v} - t\bar{w})(v - w) = 0.$$

Since v and w are on the unit circle, $\bar{v} = \frac{1}{v}$, $\bar{w} = \frac{1}{w}$. We have

$$(p - (1-t)v - tw)\left(\frac{1}{v} - \frac{1}{w}\right) + \left(\bar{p} - \frac{1-t}{v} - \frac{t}{w}\right)(v - w) = 0.$$

From this,

$$t = \frac{v - w - p + vw\bar{p}}{2(v - w)},$$

and the orthogonal projection is

$$p_\ell = (1-t)v + tw = \frac{1}{2}(v + w + p - vw\bar{p}).$$

If P is also on the unit circle, then $\bar{p} = \frac{1}{p}$, and $p_\ell = \frac{1}{2}\left(v + w + p - \frac{bc}{p}\right)$. $\square$

Proposition 4. *Let P be a point on the unit circumcircle of triangle ABC. The equation of its Simson line is*

$$2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p} = 0. \tag{1}$$

Proof. Let P be a point on the unit circumcircle. Its projections on the side lines of triangle ABC are, by Lemma 3,

$$\begin{aligned} p_a = & \ \frac{1}{2}\left(b+c+p-\frac{bc}{p}\right), \\ p_b = & \ \frac{1}{2}\left(c+a+p-\frac{ca}{p}\right), \\ p_c = & \ \frac{1}{2}\left(a+b+p-\frac{ab}{p}\right). \end{aligned}$$

The line joining p_b and p_c has equation

$$\begin{aligned} 0 = & \begin{vmatrix} z & p_b & p_c \\ \bar{z} & \bar{p}_b & \bar{p}_c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} z & \frac{1}{2}\left(c+a+p-\frac{ca}{p}\right) & \frac{1}{2}\left(a+b+p-\frac{ab}{p}\right) \\ \bar{z} & \frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{p}-\frac{p}{ca}\right) & \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{p}-\frac{p}{ab}\right) \\ 1 & 1 & 1 \end{vmatrix} \\ = & \ \frac{(b-c)(p-a)(2abcp\bar{z} - 2p^2z + p^3 + (a+b+c)p^2 - (bc+ca+ab)p - abc)}{4abcp^2} \\ = & \ \frac{(b-c)(p-a)(2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p})}{4abcp}. \end{aligned}$$

Therefore, the equation of the Simson line of P is given by (1) above. □

Proposition 5. *The intersection of the Simson lines of two points P, $Q \in \mathcal{K}$ is the point with coordinates*

$$\frac{1}{2}\left(p+q+a+b+c+\frac{abc}{pq}\right). \tag{2}$$

Proof. Let ℓ_p, ℓ_q be the Simson lines of two points P, Q on the unit circumcircle of ABC. Their intersection is given by the solution of

$$2abc\bar{z} - 2pz + p^2 + (a+b+c)p - (bc+ca+ab) - \frac{abc}{p} = 0, \tag{3}$$

$$2abc\bar{z} - 2qz + q^2 + (a+b+c)q - (bc+ca+ab) - \frac{abc}{q} = 0. \tag{4}$$

Subtracting (4) from (3), we obtain

$$-2(p-q)z + (p^2-q^2) + (a+b+c)(p-q) - abc\left(\frac{1}{p}-\frac{1}{q}\right) = 0.$$

Dividing by $2(p-q)$, we obtain z as given in (2) above. □

3. Simson triangle

Theorem 6. *The Simson triangle $N_1N_2N_3$ is directly similar to triangle $P_1P_2P_3$ (see Figure 1).*

Proof. For three points P_1, P_2, P_3 on $\mathcal{K}$, by Proposition 5, the pairwise intersections of their Simson lines are

$$n_1 = \frac{1}{2}\left(p_2 + p_3 + a + b + c + \frac{abc}{p_2p_3}\right),$$
$$n_2 = \frac{1}{2}\left(p_3 + p_1 + a + b + c + \frac{abc}{p_3p_1}\right),$$
$$n_3 = \frac{1}{2}\left(p_1 + p_2 + a + b + c + \frac{abc}{p_1p_2}\right),$$

the vertices of the Simson triangle. Note that

$$\begin{aligned} n_2 - n_3 &= \frac{1}{2}\left(p_3 + p_1 + a + b + c + \frac{abc}{p_3p_1}\right) - \frac{1}{2}\left(p_1 + p_2 + a + b + c + \frac{abc}{p_1p_2}\right) \\ &= \frac{1}{2}\left(p_3 - p_2 + \frac{abc(p_2 - p_3)}{p_1p_2p_3}\right) \\ &= \frac{abc - p_1p_2p_3}{2p_1p_2p_3}(p_2 - p_3). \end{aligned}$$

Since the factor $k := \frac{abc - p_1p_2p_3}{2p_1p_2p_3}$ is symmetric in p_1, p_2, p_3, we conclude that

$$N_2N_3 = k \cdot P_2P_3, \qquad N_3N_1 = k \cdot P_3P_1, \qquad N_1N_2 = k \cdot P_1P_2,$$

and the triangles $N_1N_2N_3$ and $P_1P_2P_3$ are directly similar. □

Corollary 7. *The Simson triangle $N_1N_2N_3$ has circumradius $\frac{|abc - p_1p_2p_3|}{2}$, and circumcenter at the midpoint of the segment joining the orthocenters H of ABC and H_1 of $P_1P_2P_3$ (see Figure 2).*

Proof. Since the Simson triangle is similar to $P_1P_2P_3$ with similarity factor $k = \frac{|abc - p_1p_2p_3|}{2}$, it clearly has circumradius k.

The orthocenters of triangles ABC and $P_1P_2P_3$ are the points

$$h = a + b + c \qquad \text{and} \qquad h_1 = p_1 + p_2 + p_3.$$

With these, we rewrite

$$n_1 = \frac{h_1 + h}{2} + \frac{1}{2}\left(\frac{abc}{p_2p_3} - p_1\right) = \frac{h_1 + h}{2} + \frac{1}{2}\left(\frac{abc - p_1p_2p_3}{p_2p_3}\right).$$

Therefore,

$$\left|n_1 - \frac{h_1 + h}{2}\right| = \frac{1}{2}\left|\frac{abc - p_1p_2p_3}{p_2p_3}\right| = k,$$

since $|p_2| = |p_3| = 1$. The same relation holds if n_1 is replaced by n_2 and n_3. This shows that the Simson triangle has circumcenter $\frac{h_1+h}{2}$, which is the midpoint of H_1 and H. It also confirms independently that the circumradius is k. □

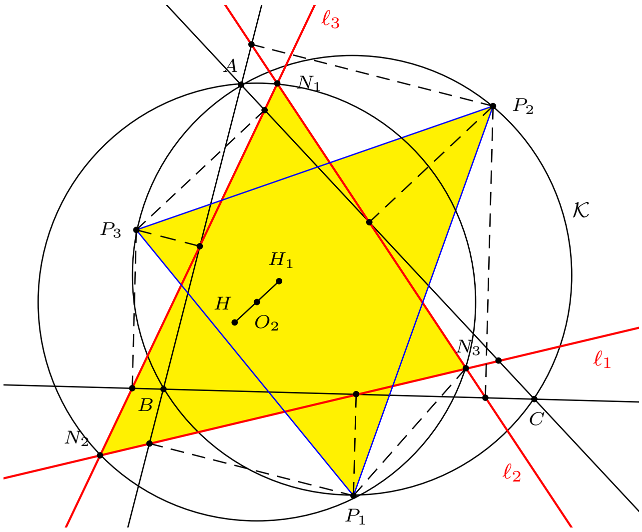

Figure 2

4. The Simson triangle and orthopoles

Lemma 8. *Let V and W be points on the unit circumcircle of triangle ABC, A_1 the orthogonal projection of A onto VW. The perpendicular from A_1 to BC has equation*

$$\bar{z} - \frac{z}{bc} + \frac{a-(v+w)}{2vw} + \frac{(a^2-bc)+a(v+w)-vw}{2abc} = 0. \tag{5}$$

Proof. By Lemma 3, the coordinates a_1 of A_1 and its complex conjugate are

$$a_1 = \frac{1}{2}\left(v+w+a-\frac{vw}{a}\right),$$
$$\bar{a}_1 = \frac{-a^2+a(v+w)+vw}{2avw}.$$

By Lemma 3 again, the coordinates a_2 of the orthogonal projection A_2 of A_1 onto BC, together with its complex conjugate, are

$$\begin{aligned} a_2 &= \frac{1}{2}(b+c+a_1-bc\bar{a}_1) \\ &= \frac{a^2bc-abc(v+w)+(a^2-bc+2ca+2ab)vw+avw(v+w)-v^2w^2}{4avw}, \\ \bar{a}_2 &= \frac{-a^2bc+abc(v+w)-(a^2-bc-2ca-2ab)vw-avw(v+w)+v^2w^2}{4abcvw}. \end{aligned}$$

The line A_1A_2 contains a point with coordinates z if and only if

$$\begin{aligned} 0 &= \begin{vmatrix} z & a_1 & a_2 \\ \bar{z} & \bar{a}_1 & \bar{a}_2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{f(a,b,c,v,w)g(a,b,c,v,w,z)}{8a^2bcv^2w^2}, \end{aligned}$$

where

$$\begin{aligned} f(a,b,c,v,w) &:= a^2bc - abc(v+w) - (a^2+bc-2ca-2ab)vw \\ &\quad - avw(v+w) + v^2w^2, \\ g(a,b,c,v,w,z) &:= 2abcvw\bar{z} - 2avwz + a^2bc - abc(v+w) \\ &\quad + (a^2-bc)vw + avw(v+w) - v^2w^2. \end{aligned}$$

Therefore, the equation of the perpendicular is $g(a,b,c,v,w,z) = 0$. Dividing by $2abcvw$, we obtain the equation (5). □

Now we consider the construction in Lemma 8 beginning with all three vertices of $\triangle ABC$. This results in the three lines

$$\begin{aligned} \bar{z} - \frac{z}{bc} + \frac{a-(v+w)}{2vw} + \frac{(a^2-bc)+a(v+w)-vw}{2abc} &= 0, \\ \bar{z} - \frac{z}{ca} + \frac{b-(v+w)}{2vw} + \frac{(b^2-ca)+b(v+w)-vw}{2abc} &= 0, \\ \bar{z} - \frac{z}{ab} + \frac{c-(v+w)}{2vw} + \frac{(c^2-ab)+c(v+w)-vw}{2abc} &= 0. \end{aligned}$$

The intersection of the last two lines is given by

$$\begin{aligned} &-\frac{z}{ca} + \frac{z}{ab} + \frac{b-(v+w)}{2vw} - \frac{c-(v+w)}{2vw} \\ &\quad + \frac{(b^2-ca)+b(v+w)-vw}{2abc} - \frac{(c^2-ab)+c(v+w)-vw}{2abc} = 0, \\ &-\frac{(b-c)z}{abc} + \frac{b-c}{2vw} + \frac{(b-c)(a+b+c+v+w)}{2abc} = 0, \end{aligned}$$

Multiplying by $\frac{abc}{b-c}$, we obtain

$$z = \frac{1}{2}\left(a+b+c+v+w+\frac{abc}{vw}\right).$$

Note that this is symmetric in a, b, c. This means that the three perpendiculars form A_1 to BC, B_1 to CA, and C_1 to AB are concurrent. The point of concurrency is the orthopole N of the line VW. By Proposition 5, this is also the same as the intersection of the Simson lines of V and W with respect to $\triangle ABC$ (see Figure 3 and [1, p.289, Theorem 697]). Applying this to the three side lines of the triangle $P_1P_2P_3$ for three points P_1, P_2, $P_3 \in \mathcal{K}$, we obtain the following theorem.

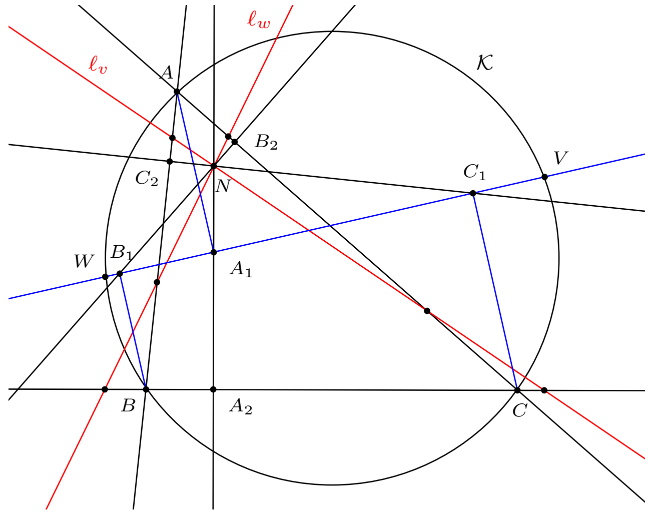

Figure 3

Theorem 9. *For three points P_1, P_2, P_3 on the circumcircle $\mathcal{K}$ of $\triangle ABC$, the orthopoles of the lines P_2P_3, P_3P_1, P_1P_2 coincide with the vertices N_1, N_2, N_3 of the Simson triangle bounded by the Simson lines of P_1, P_2, P_3 with respect to $\triangle ABC$.*

5. Examples

Example 1. *Let $\triangle P_1P_2P_3$ be an equilateral triangle. The circumcenter of the Simson triangle coincides with the center of nine-point circle for $\triangle ABC$.*

Proof. If $P_1P_2P_3$ is equilateral, its orthocenter coincides with the circumcenter. This means that $p_1 + p_2 + p_3 = 0$. The circumcenter of the Simson triangle is $\frac{1}{2}(a+b+c+p_1+p_2+p_3) = \frac{1}{2}(a+b+c)$, the center E of the nine-point circle of $\triangle ABC$. $\square$

Example 2. *Let $P_1 \in \mathcal{K}$ and let P_1E meet the circle $\mathcal{K}$ again in P_3 (E is the center of nine-point circle for $\triangle ABC$). Let $P_2 \in \mathcal{K}$ and $EP_2 \perp P_1P_3$. Then the circumcenter of the Simson triangle lies on the circle with center H (the orthocenter) and radius $\frac{1}{2}R$ (see Figure 4).*

Proof. Let P_4 be another point of $\mathcal{K}$ on the line P_2E.

$$P_1P_3 \perp P_2P_4 \Rightarrow (p_1 - p_3)\left(\frac{1}{p_2} - \frac{1}{p_4}\right) + \left(\frac{1}{p_1} - \frac{1}{p_3}\right)(p_2 - p_4) = 0$$
$$\Rightarrow (p_1 - p_3)(p_2 - p_4)(p_1p_3 + P_2p_4) = 0.$$

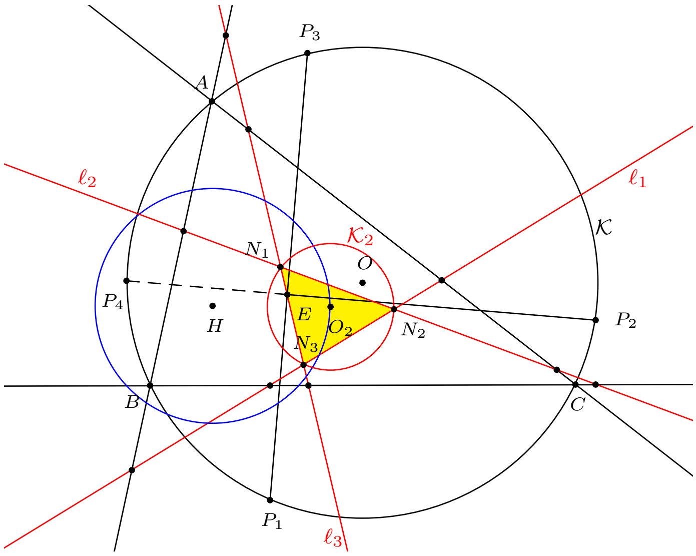

Figure 4

Therefore, $p_1p_3 + p_2p_4 = 0$ (see [3, p. 45]).

By Lemma 3 or [3, p. 45],

$$E \in P_1P_3 \Rightarrow p_1 + p_3 = e + p_1p_3\bar{e},$$
$$E \in P_2P_4 \Rightarrow p_2 + p_4 = e + p_2p_4\bar{e}.$$

Therefore,

$$p_1 + p_2 + p_3 + p_4 = 2e + p_1p_3 + p_2p_4 = 2e = a + b + c.$$

By Theorem 7, the circumcenter O_2 of the Simson triangle of $P_1P_2P_2$ has coordinates

$$o_2 = \frac{1}{2}(a + b + c + p_1 + p_2 + p_3) = a + b + c - \frac{p_4}{2},$$

and $|o_2 - h| = \left|\frac{p_4}{2}\right| = \frac{1}{2}$. Hence, O_2 lies on a circle with radius $\frac{1}{2}R$ and center H. □

Example 3. *Let A, B, C, A', B', C' be points on a circle $\mathcal{K}$. Construct the Simson triangle for A', B', C' with respect to $\triangle ABC$ and the Simson triangle for A, B, C with respect to $\triangle A'B'C'$. The six vertices of these two Simson triangles lie on a circle* (*see Figure* 5).

Proof. Let $N_1N_2N_3$ be the Simson triangle for A', B', C' with respect to $\triangle ABC$, and $N_1'N_2'N_3'$ that of A, B, C with respect to $\triangle A'B'C'$. By Theorem 7, the circumcircles of $N_1N_2N_3$ and $N_1'N_2'N_3'$ both have radius $\frac{|abc-a'b'c'|}{2}$, and center $\frac{1}{2}(a + b + c + a' + b' + c')$. Therefore the two circumcircles coincide. □

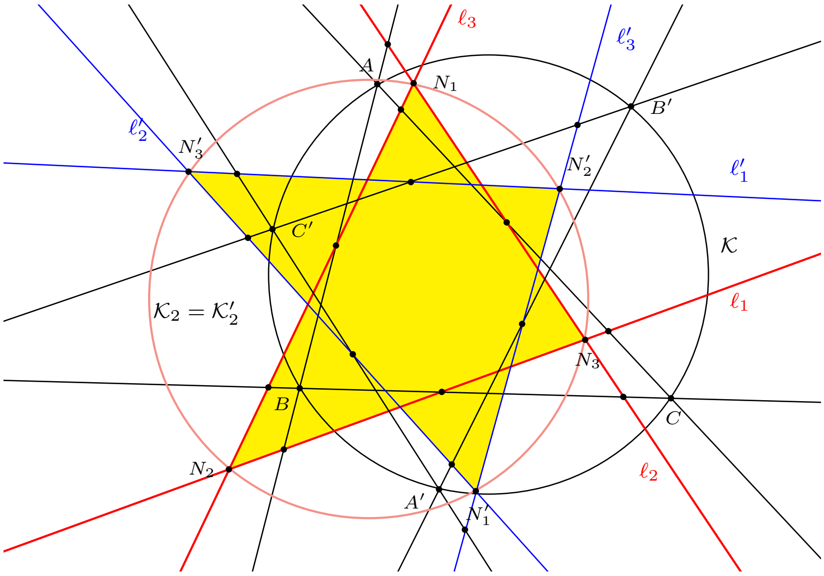

Figure 5

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The Periambic Constellation: Altitudes, Perpendicular Bisectors, and Other Radical Axes in a Triangle

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Abstract. Six circles may be constructed using a triangle's vertices as centers and its sides as radii. These circles determine twelve ordinary and three ideal radical axes, whose intersection points include the triangle's circumcenter and orthocenter, along with eight other ordinary points in interesting configurations. For instance, we show that the orthocenter, circumcenter, and two radical centers of the six circles form a parallelogram, and that six other radical centers (the intersection points of the altitudes and perpendicular bisectors) are the vertices of two congruent triangles which are inversely similar to the original. Underlying this "constellation" is a simple invariance property of three circles in which two are concentric.

1. Introduction

Consider a triangle ABC and the circles P_A, P_B, P_C having centers A, B, C and radii AB, BC, CA, respectively. Because these circles might whimsically be said to "walk around the triangle's perimeter," we call them *periambic circles*—*peri* suggesting "perimeter," and *ambic* echoing the Latin *ambire*, "go around." A second set of periambic circles Q_A, Q_B, Q_C, with centers A, B, C and radii AC, BA, CB, respectively, "walks" around triangle ABC in the opposite direction. We call P_A, P_B, P_C the *p-circles* and Q_A, Q_B, Q_C the *q-circles* (Figure 1).

The periambic circles give rise in pairs to $\binom{6}{2} = 15$ radical axes, many triples of which are concurrent in ordinary or ideal points. Among the ordinary points are the orthocenter and circumcenter of triangle ABC, along with eight others which may not yet be named in the literature. A rich geometric structure resides upon these ten points; underlying much of it are relatively simple results, including an invariance property of three circles in which two are concentric.

2. Definitions and notation

In triangle ABC, a, b, c are the sides opposite vertices A, B, C, respectively; R is the circumradius; and we define $\alpha = \angle BAC,\ \beta = \angle CBA$, and $\gamma = \angle ACB$. Positive angles are measured counterclockwise, and triangle ABC, labeled counterclockwise, has positive area Δ. If lines l_1 and l_2 intersect at V, we write $V = l_1 \wedge l_2$. Given distinct circles U_1 and U_2, we denote their radical axis by $\langle U_1, U_2\rangle$; note that $\langle U_1, U_2\rangle = \langle U_2, U_1\rangle$. The radical axes of three distinct circles are concurrent; we refer to this as the Radical Center Property.

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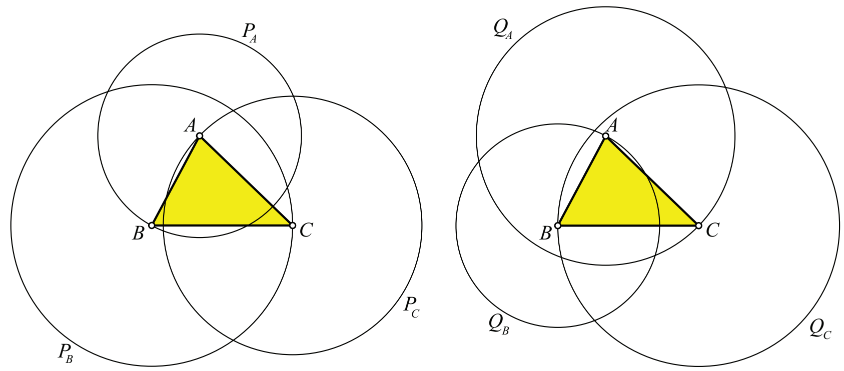

Figure 1. The p-circles P_A, P_B, P_C (left) and q-circles Q_A, Q_B, Q_C (right).

Definition 1 (p-lines and P)**.** *The p-lines of triangle ABC are*

$$p_a = \langle P_B, P_C \rangle$$
$$p_b = \langle P_C, P_A \rangle$$
$$p_c = \langle P_A, P_B \rangle.$$

The radical center of P_A, P_B, P_C is labeled P.

Definition 2 (q-lines and Q)**.** *The q-lines of triangle ABC are*

$$q_a = \langle Q_C, Q_B \rangle$$
$$q_b = \langle Q_A, Q_C \rangle$$
$$q_c = \langle Q_B, Q_A \rangle.$$

The radical center of Q_A, Q_B, Q_C is labeled Q.

We loosely characterize any set of points deriving exclusively from the p-circles as being of *gender* p; a set deriving from the q-circles is of *gender* q. While P and Q are clearly of genders p and q, respectively, there are other radical axes, concurrent in groups of three, defined by pairs of circles of opposite gender.

Definition 3. *The altitudes of triangle ABC are*

$$h_a = \langle P_C, Q_B \rangle$$
$$h_b = \langle P_A, Q_C \rangle$$
$$h_c = \langle P_B, Q_A \rangle\ ,$$

concurrent at the orthocenter H.

Definition 4. *The perpendicular bisectors of triangle ABC are*

$$o_a = \langle P_B, Q_C \rangle$$
$$o_b = \langle P_C, Q_A \rangle$$
$$o_c = \langle P_A, Q_B \rangle\ ,$$

concurrent at the circumcenter O.

Together, P, Q, H, and O comprise the *major periambic points* of triangle ABC.

A final set of points requiring names are the feet of the twelve just-defined radical axes on their respective sides of triangle ABC. The foot of a radical axis is labeled with the name of the axis, but with a capital rather than lowercase first letter; for instance, H_c is the foot of altitude h_c on side c. The twelve points P_a, H_a, O_a, Q_a, P_b, and so on will be called the *periambic feet*.

3. Constant-Distance Lemma

Given three circles of which two (but not all) are concentric, the radical axis of the concentric pair is the ideal line and the other two axes are parallel. While the radical center in this configuration is an ideal point, its ordinary parts possess a useful invariance property.

Lemma 1 (Constant-Distance Lemma)**.** *Let U_1 and U_2 be fixed concentric circles with center K_1 and radii r_1 and r_2, respectively. For any circle U_3 of variable radius r_3 and fixed center $K_3 \neq K_1$, the distance between $\langle U_1, U_3\rangle$ and $\langle U_2, U_3\rangle$ is constant.*

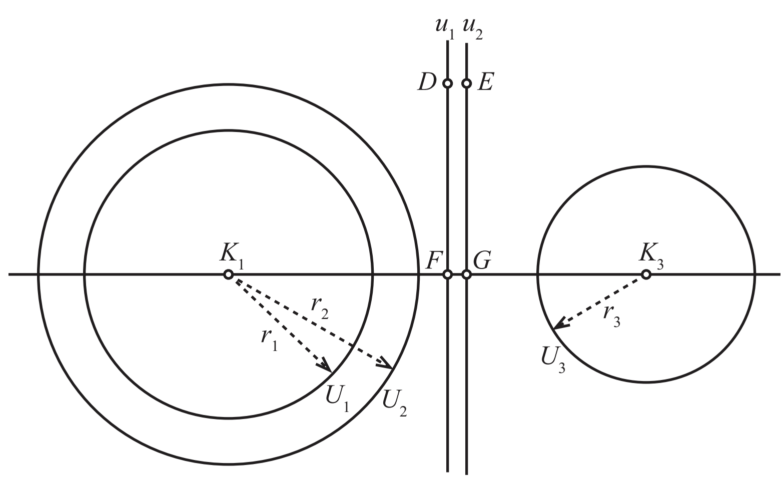

Figure 2. Fixed concentric circles U_1 and U_2, and variable circle U_3.

Proof. Let $u_1 = \langle U_1, U_3\rangle$ and $u_2 = \langle U_2, U_3\rangle$ meet K_1K_3 at F and G, respectively. Let D and E be points on u_1 and u_2, respectively, such that DE is parallel to K_1K_3 (Figure 2). The radical axis of two circles being the set of points having equal powers with respect to those circles, at D and E we have

$$DK_1^2 - r_1^2 = DK_3^2 - r_3^2$$
$$EK_1^2 - r_2^2 = EK_3^2 - r_3^2\ .$$

By the Pythagorean relation in triangles DK_1F, EK_1G, DFK_3, and EGK_3, these become

$$K_1F^2 + FD^2 - r_1^2 = FK_3^2 + DF^2 - r_3^2 \tag{1}$$

$$K_1G^2 + GE^2 - r_2^2 = EG^2 + GK_3^2 - r_3^2. \tag{2}$$

Subtracting (2) from (1), cancelling the identical quantities FD^2 and GE^2, and rearranging, we obtain

$$r_2^2 - r_1^2 = (FK_3^2 - K_1F^2) + (K_1G^2 - GK_3^2)\ .$$

Factoring each difference of squares on the right, and simplifying using $FK_3 + K_1F = K_1G + GK_3 = K_1K_3$, it follows that the directed distance from u_1 to u_2 is

$$FG = \frac{r_2^2 - r_1^2}{2\ K_1K_3}\ ,$$

a formula comprising known constants. □

The Constant-Distance Lemma has immediate consequences for the periambic radical axes.

Proposition 2. *On a given side of triangle ABC, the directed distance from the p-line to the altitude is equal to the directed distance from the perpendicular bisector to the q-line. That is, for any $x \in \{a, b, c\}$, $P_xH_x = O_xQ_x$.*

Proof. Consider, for instance, the directed distances P_aH_a and O_aQ_a. Let P_B and Q_B be the Constant-Distance Lemma's fixed concentric circles U_1 and U_2, respectively, and let P_C and Q_C represent two positionings of the variable third circle U_3, so that $r_2 = c$, $r_1 = a$, and $K_1K_3 = a$. By the lemma, the distance between $p_a = \langle P_B, P_C\rangle$ and $h_a = \langle P_C, Q_B\rangle = \langle Q_B, P_C\rangle$ is $(c^2 - a^2)/2a$, which is also the distance between $o_a = \langle P_B, P_C\rangle$ and $q_a = \langle Q_B, Q_C\rangle = \langle Q_C, Q_B\rangle$. Thus $P_aH_a = O_aQ_a$, and similarly $P_bH_b = O_bQ_b$ and $P_cH_c = O_cQ_c$. □

4. Central parallelogram

We may now state the most obvious feature of the major periambic points.

Theorem 3. *In any triangle, $POQH$ is a parallelogram.*

Proof. Let $T = p_c \wedge o_b$, $W = q_b \wedge h_c$, $X = q_c \wedge h_b$, and $Y = p_b \wedge o_c$ (Figure 3). The quadrilaterals $QWHX$ and $OTPY$ not only have parallel corresponding sides, they are in fact congruent parallelograms, because their projections onto sides b and c have equal lengths, by Proposition 2. Thus their diagonals HQ and PO are congruent and parallel, and $POQH$ is a parallelogram. □

We call $POQH$ the *central parallelogram* of triangle ABC. It may be shown that for an isosceles triangle ABC the central parallelogram is a rhombus, for an isosceles right triangle a square, and for an equilateral triangle a point.

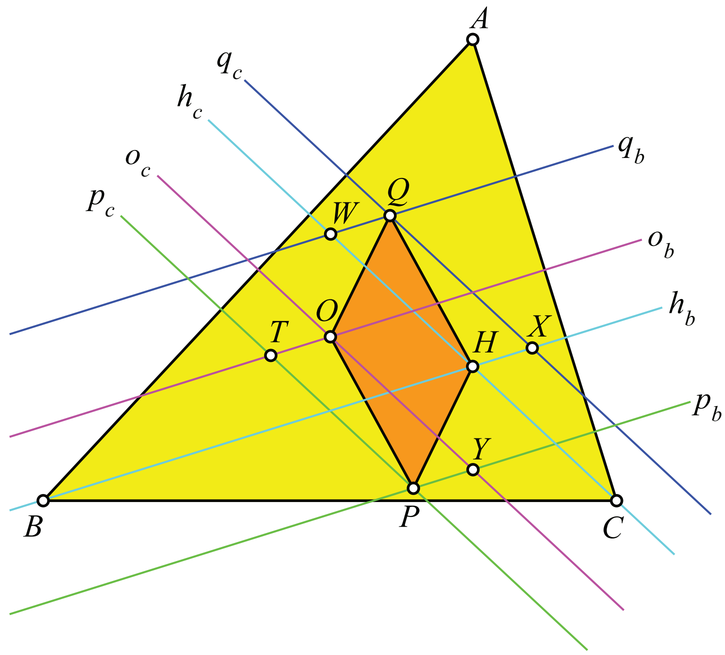

Figure 3. The central parallelogram $POQH$.

5. The minor periambic points

So far only two points, P and Q, have been defined as radical centers of triples of periambic circles. However, the number of triples of distinct periambic circles is the number of 3-letter words on the symbols P and Q, or 2^3. The eight radical axes thus defined and the names of their radical centers are listed in Table 1.

Description	**Triple of circles**	**Triple of radical axes**	**Radical center**
3 p-circles	$\{P_A, P_B, P_C\}$	$\{p_c, p_a, p_b\}$	P
2 p-circles, 1 q-circle	$\{Q_A, P_B, P_C\}$	$\{h_c, p_a, o_b\}$	A_p
	$\{P_A, Q_B, P_C\}$	$\{o_c, h_a, p_b\}$	B_p
	$\{P_A, P_B, Q_C\}$	$\{p_c, o_a, h_b\}$	C_p
1 p-circle, 2 q-circles	$\{P_A, Q_B, Q_C\}$	$\{o_c, q_a, h_b\}$	A_q
	$\{Q_A, P_B, Q_C\}$	$\{h_c, o_a, q_b\}$	B_q
	$\{Q_A, Q_B, P_C\}$	$\{q_c, h_a, o_b\}$	C_q
3 q-circles	$\{Q_A, Q_B, Q_C\}$	$\{q_c, q_a, q_b\}$	Q

Table 1. The eight triples of periambic circles and their radical centers.

The orthocenter and circumcenter are not products of the Radical Center Property, since the radical axes which define each point require six circles. We show that *triples of pairs* of opposite-gender periambic circles may determine two ordinary and four ideal points. Let P_A, P_B, and P_C be the first member of the first, second, and third pair, respectively. This reduces the problem to counting the permutations of Q_A, Q_B, and Q_C as second members in each pair, which is $3! = 6$. However, if in any pair of circles the subscripts match, then that pair is concentric, and their radical axis is an ideal line. To enumerate the ordinary points we must count, not the permutations of the q-circles, but their derangements, which is $3!(1 - 1/1! + 1/2! - 1/3!) = 2$. This analysis does not say anything about concurrency, but of course we know that the two cases produce H and O.

We call A_p, B_p, C_p, A_q, B_q, and C_q the *minor periambic points*, and triangle $A_pB_pC_p$ and triangle $A_qB_qC_q$ the *minor periambic triangles* (Figure 4). From

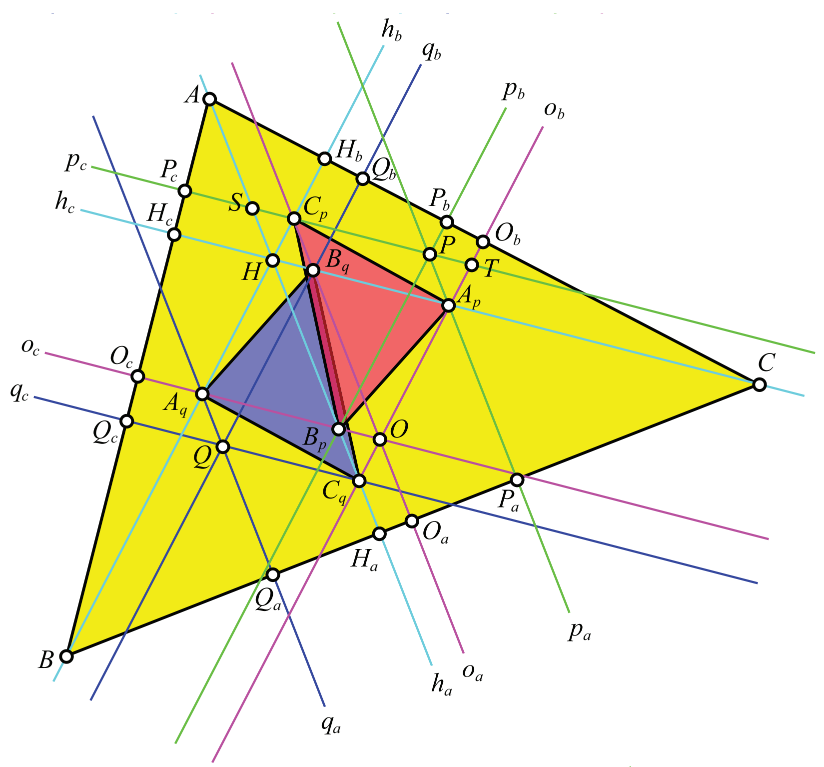

Figure 4. The minor periambic triangles $A_pB_pC_p$ and $A_qB_qC_q$.

Table 1 we see that the minor periambic points are entirely determined by intersections of non-parallel altitudes and perpendicular bisectors in triangle ABC. From the many aspects of their arrangement suggested by Figure 4, we begin with the following.

Theorem 4. *The minor periambic triangles are inversely similar to triangle ABC.*

Proof. Our "brute force" proof comprises lengthy mundane arithmetic; most of the coordinates and equalities below were computed in Mathematica [4]. Without loss of generality, place triangle ABC in the Cartesian plane with $A = (0,0)$, $B = (1,0)$, and $C = (c_1, c_2)$, where $c_2 > 0$. The altitudes and perpendicular bisectors are

$$\begin{aligned}
h_a &: y = \frac{1-c_1}{c_2}x & o_a &: y = \frac{1-c_1}{c_2}x + \frac{c_1^2+c_2^2-1}{2c_2} \\
h_b &: y = -\frac{c_1}{c_2}x + \frac{c_1}{c_2} & o_b &: y = -\frac{c_1}{c_2}x + \frac{c_1^2+c_2^2}{2c_2} \\
h_c &: x = c_1 & o_c &: x = \frac{1}{2}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
A_p &= h_c \wedge o_b = \left(c_1, \frac{c_2^2-c_1^2}{2c_2}\right) \\
B_p &= h_a \wedge o_c = \left(\frac{1}{2}, \frac{1-c_1}{2c_2}\right) \\
C_p &= h_b \wedge o_a = \left(\frac{1+2c_1-c_1^2-c_2^2}{2}, \frac{c_1-2c_1^2+c_1^3+c_1c_2^2}{2c_2}\right) .
\end{aligned}$$

The squares of the side lengths of triangle ABC are

$$\begin{aligned}
(AB)^2 &= 1 \\
(BC)^2 &= 1-2c_1+c_1^2+c_2^2 \\
(CA)^2 &= c_1^2+c_2^2 .
\end{aligned}$$

while the squares of the side lengths of triangle $A_pB_pC_p$ work out to be

$$\begin{aligned}
(A_pB_p)^2 &= -\frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} + \frac{1}{4c_2^2} - \frac{c_1}{2c_2^2} + \frac{3c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{c_1^4}{4c_2^2} + \frac{c_2^2}{4} \\
(B_pC_p)^2 &= -\frac{c_1}{2} + 2c_1^2 - 2c_1^3 + \frac{3c_1^4}{4} + \frac{1}{4c_2^2} - \frac{c_1}{c_2^2} + \frac{2c_1^2}{c_2^2} - \frac{5c_1^3}{2c_2^2} + \frac{2c_1^4}{c_2^2} - \frac{c_1^5}{c_2^2} \\
&\quad + \frac{c_1^6}{4c_2^2} - c_1c_2^2 + \frac{3c_1^2c_2^2}{4} + \frac{c_2^4}{4} \\
(C_pA_p)^2 &= \frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} - c_1^3 + \frac{3c_1^4}{4} + \frac{c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{3c_1^4}{4c_2^2} - \frac{c_1^5}{2c_2^2} + \frac{c_1^6}{4c_2^2} - \frac{c_2^2}{4} \\
&\quad - \frac{c_1c_2^2}{2} + \frac{3c_1^2c_2^2}{4} + \frac{c_2^4}{4} .
\end{aligned}$$

Computer calculations reveal that

$$\left(\frac{A_pB_p}{AB}\right)^2 = \left(\frac{B_pC_p}{BC}\right)^2 = \left(\frac{C_pA_p}{CA}\right)^2 = \rho^2 ,$$

where

$$\rho^2 = -\frac{1}{4} - \frac{c_1}{2} + \frac{c_1^2}{2} + \frac{1}{4c_2^2} - \frac{c_1}{2c_2^2} + \frac{3c_1^2}{4c_2^2} - \frac{c_1^3}{2c_2^2} + \frac{c_1^4}{4c_2^2} + \frac{c_2^2}{4} ,$$

hence triangles ABC and $A_pB_pC_p$ are similar.

To show that the similarity is inverse rather than direct, note first that the area Δ of triangle ABC is $c_2/2$. Taking A_p, B_p, and C_p to be vector endpoints, we compute the area Δ_p of triangle $A_pB_pC_p$ as

$$\begin{aligned}\Delta_p =& \frac{1}{2}\begin{vmatrix}\vec{B}_p - \vec{A}_p \\ \vec{C}_p - \vec{A}_p\end{vmatrix} = \frac{1}{2}\begin{vmatrix}\frac{1}{2} - c_1 & \frac{1 - c_1 + c_1^2 - c_2^2}{2c_2} \\ \frac{1 - c_1^2 - c_2^2}{2} & \frac{c_1 - c_1^2 + c_1^3 + c_1c_2^2 - c_2^2}{2c_2}\end{vmatrix} \\ =& \frac{1}{2}\left[\frac{c_2}{4} + \frac{c_1c_2}{2} - \frac{c_1^2c_2}{2} - \frac{1}{4c_2} + \frac{c_1}{2c_2} - \frac{3c_1^2}{4c_2} + \frac{c_1^3}{2c_2} - \frac{c_1^4}{4c_2} - \frac{c_2^3}{4}\right] \qquad (3) \\ =& -\rho^2\Delta .\end{aligned}$$

Since Δ_p is negative, we conclude that triangles $A_pB_pC_p$ and ABC are inversely similar. Much the same sequence of calculations shows that triangles $A_qB_qC_q$ and ABC are inversely similar as well. □

Proposition 5. *The minor periambic triangles are congruent, and radially symmetric around the midpoint of the Euler segment HO.*

Proof sketch. The congruence and radial symmetry arise because HA_pOA_q is a parallelogram whose diagonal A_pA_q is bisected by HO; the same is true for B_pB_q and C_pC_q in HB_pOB_q and HC_pOC_q, respectively. □

Proposition 5 implies that results proved about P, A_p, B_p, and C_p are automatically true for Q, A_q, B_q, and C_q, a fact which applies to the next result.

Proposition 6. *P, A_p, B_p, and C_p are concyclic.*

Proof. By construction, $\angle C_pPA_p = \alpha + \gamma$, while $\angle A_pB_pC_p = \beta$ by Theorem 4. With supplementary opposite angles, quadrilateral $PA_pB_pC_p$ is cyclic. □

6. A second look at area

The computer-calculated expression (3) for a minor periambic triangle's area could be described as unenlightening. We now derive a slightly less forbidding area formula having more obvious references to the configuration's geometry. The process begins with a lemma giving directed distances between parallel altitudes and perpendicular bisectors.

Lemma 7. *In directed distances and angles,*

$$\begin{aligned}O_aH_a &= R\sin(\gamma - \beta) \\ O_bH_b &= R\sin(\alpha - \gamma) \\ O_cH_c &= R\sin(\beta - \alpha) .\end{aligned}$$

Proof. To prove the first of these, multiply the identity $\sin(\gamma - \beta) = \sin\gamma\cos\beta - \sin\beta\cos\gamma$ by the successive terms in

$$R = \frac{c}{2\sin\gamma} = \frac{b}{2\sin\beta}$$

to obtain

$$\begin{aligned} R\sin(\gamma - \beta) &= \frac{c}{2\sin\gamma}\sin\gamma\cos\beta - \frac{b}{2\sin\beta}\sin\beta\cos\gamma \\ &= \tfrac{1}{2}(c\cos\beta - b\cos\gamma)\ . \end{aligned}$$

In triangles ABH_a and AH_aC we have

$$c\cos\beta = BH_a\ , \quad \text{and} \quad b\cos\gamma = H_aC\ ,$$

respectively. Thus, since $BH_a + H_aC = BC$ and $O_aH_a + H_aC = O_aC$, it follows that

$$\begin{aligned} R\sin(\gamma - \beta) &= \tfrac{1}{2}(BH_a - H_aC) \\ &= \tfrac{1}{2}\big((BC - H_aC) - H_aC\big) \\ &= \frac{BC}{2} - H_aC \\ &= O_aC - H_aC \\ &= O_aH_a\ . \end{aligned}$$

The proofs for O_bH_b and O_cH_c are similar. □

Proposition 8. *In a non-degenerate triangle ABC, the minor periambic triangles each have areas equal to*

$$\frac{R^2}{2}\left[\frac{\sin(\alpha-\gamma)\sin(\beta-\alpha)}{\sin\alpha} + \frac{\sin(\beta-\alpha)\sin(\gamma-\beta)}{\sin\beta} + \frac{\sin(\gamma-\beta)\sin(\alpha-\gamma)}{\sin\gamma}\right]. \quad (4)$$

Proof. Without loss of generality, position triangle ABC with its circumcenter at the pole of a polar coordinate system, and with BC parallel to the axis $\theta = 0$ (Figure 5). Let L and N be the feet of the perpendiculars from O to h_c and h_b, respectively, and let M be the foot of the perpendicular from B_p to o_a. To find the coordinates $(r_1, \theta_1),\ (r_2, \theta_2)$, and (r_3, θ_3) of A_p, B_p, and C_p, respectively, apply Lemma 7 in triangle OA_pL to get

$$r_1 = OA_p = \frac{OL}{\sin\alpha} = \frac{O_cH_c}{\sin\alpha} = \frac{R\sin(\beta-\alpha)}{\sin\alpha}\ .$$

In triangle OMB_p one has

$$r_2 = OB_p = \frac{MB_p}{\sin\beta} = \frac{O_aH_a}{\sin\beta} = \frac{R\sin(\gamma-\beta)}{\sin\beta}\ ,$$

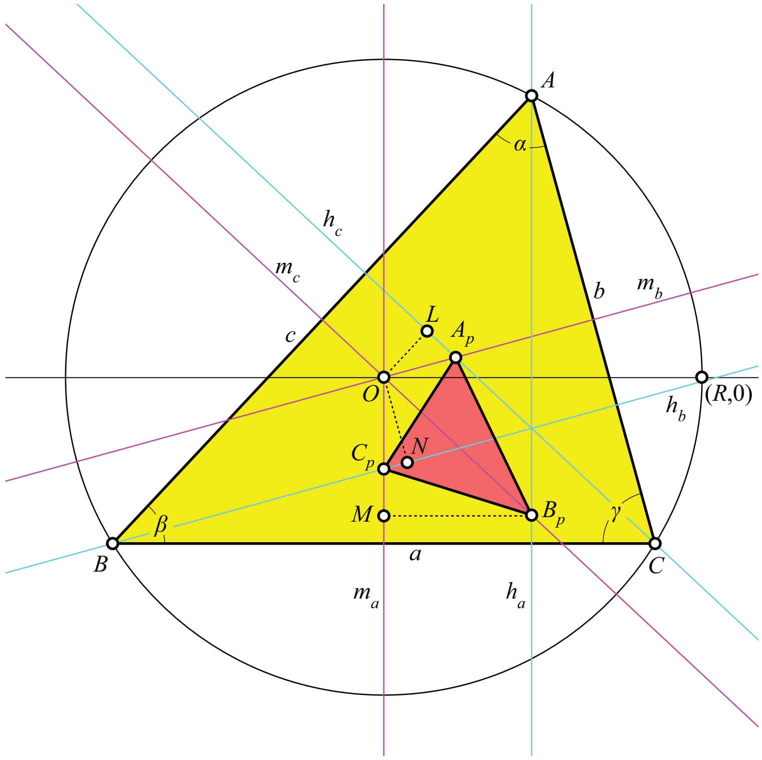

Figure 5. Polar coordinates for calculating the area of triangle $A_pB_pC_p$.

and in triangle OC_pN

$$r_3 = OC_p = \frac{ON}{\sin\gamma} = \frac{O_bH_b}{\sin\gamma} = \frac{R\sin(\alpha-\gamma)}{\sin\gamma}\ . \tag{5}$$

By construction, $\theta_1 = \alpha+\beta+\pi/2$, $\theta_2 = \beta-\pi/2$, and $\theta_3 = \pi/2$. (Note that $r_3 < 0$ in Equation (5) if $\alpha-\gamma < 0$.) In [1] the area of the triangle with vertices (r_1,θ_1), (r_2,θ_2), (r_3,θ_3) is given as

$$\tfrac{1}{2}\left[r_1r_2\sin(\theta_2-\theta_1)+r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)\right]\ ,$$

and we therefore obtain

$$\begin{aligned}\Delta_p = \tfrac{1}{2}\Bigg[&R^2\frac{\sin(\beta-\alpha)}{\sin\alpha}\frac{\sin(\gamma-\beta)}{\sin\beta}\sin(-\alpha-\pi)\\ &+R^2\frac{\sin(\gamma-\beta)}{\sin\beta}\frac{\sin(\alpha-\gamma)}{\sin\gamma}\sin(\pi-\beta)\\ &+R^2\frac{\sin(\alpha-\gamma)}{\sin\gamma}\frac{\sin(\beta-\alpha)}{\sin\alpha}\sin(\pi-\gamma)\Bigg]\ ,\end{aligned}$$

which simplifies to (4). Triangle $A_qB_qC_q$ has the same area as triangle $A_pB_pC_p$ by Proposition 5. □

Because the minor periambic triangles are inversely similar to triangle ABC, the formula (4) necessarily yields a non-positive value. This can be confirmed directly, if somewhat laboriously; one can for instance reduce (4) to a function of two variables, whose partial derivatives can be analyzed for local maxima, all of which are less than or equal to zero.

By definition and construction, triangle ABC has positive area, and we have proved that its minor periambic triangles have negative areas. However, the sign of the central parallelogram's area seems ambiguous at first sight. A consequence of the next result is that the central parallelogram, too, should be viewed as having negative area.

Proposition 9. *The area of the central parallelogram is equal to the sum of the areas of the minor periambic triangles.*

Proof. This is an instance of Theorem 107 in [3]: *Two triangles whose vertices lie on the sides of a given triangle at equal distances from their midpoints are equal in area.* In our case, triangles HPO and $A_pB_pC_p$ are inscribed in triangle TC_qS, where $S = h_a \wedge p_c$ and $T = o_b \wedge p_c$ (Figure 4). Because $P_aO_a = H_aQ_a$, we see that $PS = C_pT$, and so on; thus the two inscribed triangles have equal areas, and the area of triangle HPO is half the area of $PHQO$. □

7. Centroids

Centroids are at the heart of several interesting relationships between the minor periambic triangles and the central parallelogram. Investigation of these properties is aided by the following lemma concerning the distances between various periambic feet.

Lemma 10. *In absolute distances,*

$$P_aH_a \sin\alpha = O_bH_b \sin\beta$$
$$P_bH_b \sin\beta = O_cH_c \sin\gamma$$
$$P_cH_c \sin\gamma = O_aH_a \sin\alpha,$$

and also

$$P_aO_a \sin\alpha = O_cH_c \sin\gamma$$
$$P_bO_b \sin\beta = O_aH_a \sin\alpha$$
$$P_cO_c \sin\gamma = O_bH_b \sin\beta\ .$$

Proof. Each identity is derived by calculating a triangle side length in two ways. Referring to Figure 4, triangle A_pC_qH is clearly similar to triangle ABC; furthermore, the altitude from A_p to HC_q has length P_aH_a. Thus $P_aH_a/HA_p = \sin\beta$, or $HA_p = P_aH_a \sin\beta$. But also, the altitude from H to A_pC_q has length O_bH_b, so $HA_p = O_bH_b \sin\alpha$. This proves the first identity. The others are derived similarly, respectively using triangle A_qB_pH and sides A_qH and B_pA_q; triangle C_pHS and sides C_pH and HS; triangle A_pOB_q and sides OB_q and B_qA_p; triangle OC_qB_p and sides OC_q and C_qB_p; and triangle TOC_p and sides C_pT and TO. □

Proposition 11. *The centroids of triangles $A_pB_pC_p$ and HPO coincide.*

Proof. First, assume that triangle ABC is not isosceles. Theorem 276 in [3] states: *If the vertices of one triangle lie on the sides of a second, and divide them in a fixed ratio, the triangles have the same centroid.* We show that A_p, B_p, and C_p divide the sides of triangle TSC_q (Figure 4) in just such a fixed ratio. We calculate ratios in two ways using different altitudes in a triangle and absolute distances. By construction, triangles TSC_q and ABC are similar; thus, in triangle B_pC_pS,

$$C_pS = \frac{O_aH_a}{\sin\beta} \quad \text{and} \quad SB_p = \frac{P_cO_c}{\sin\beta}\ . \tag{6}$$

In triangle A_pTC_p,

$$TC_p = \frac{O_bH_b}{\sin\alpha} \quad \text{and} \quad A_pT = \frac{P_cH_c}{\sin\alpha}\ , \tag{7}$$

and in triangle $A_pB_pC_q$,

$$C_qA_p = \frac{P_aH_a}{\sin\gamma} \quad \text{and} \quad B_pC_q = \frac{P_bO_b}{\sin\gamma}\ . \tag{8}$$

The assumption that no two vertex angles in triangle ABC are equal allows us to form ratios incorporating non-zero segment lengths from Equations (6), (7), and (8). Applying substitutions chosen from the identities in Lemma 10, we find that

$$\begin{aligned} \frac{TC_p}{C_pS} &= \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha} \\ \frac{C_qA_p}{A_pT} &= \frac{P_aH_a\sin\alpha}{P_cH_c\sin\gamma} = \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha} \\ \frac{SB_p}{B_pC_q} &= \frac{P_cO_c\sin\gamma}{P_bO_b\sin\beta} = \frac{O_bH_b\sin\beta}{O_aH_a\sin\alpha}\ . \end{aligned} \tag{9}$$

It follows by [3, Theorem 276] that $A_pB_pC_p$ and TSC_q share the same centroid. Because C_pP, A_pC_q, and B_pH are each bisected by the midpoints of ST, TC_q, and C_qS, respectively, P, O, and H divide those sides in a constant ratio, namely the inverse of the ratio determined by A_p, B_p, and C_p. Thus triangles TSC_q, $A_pB_pC_p$, and HPO share the same centroid.

Now, if triangle ABC is isosceles (but not equilateral), there are three cases to consider:

(1) If $\alpha = \beta$, then the triangle is symmetric around the coincident lines $o_c = h_c$. Thus $O_aH_a = O_bH_b$, and the ratio $O_aH_a\sin\alpha/O_bH_b\sin\beta$ in Equation (9) is equal to 1. This means that triangle $A_pB_pC_p$ is the medial triangle of triangle TSC_q, and shares its centroid.

(2) If $\beta = \gamma$, we have o_a coincident with h_a, $O_aH_a = 0$, and $O_aH_a\sin\alpha/O_bH_b\sin\beta = 0$. Thus S is coincident with C_p, T with A_p, and C_q with B_p. Consequently, triangle $A_pB_pC_p$ is coincident with triangle TSC_q, with a trivially shared centroid.

(3) If $\gamma = \alpha$, then o_b is coincident with h_b and $O_bH_b = 0$. The reciprocal of the ratio in (9), namely $O_aH_a\sin\alpha/O_bH_b\sin\beta$, is therefore equal to

0, and thus S is coincident with B_p, T with C_p, and C_q with A_p. As in the previous case, triangles $A_pB_pC_p$ and TSC_q and their centroids are coincident.

Finally, if triangle ABC is equilateral, then the altitude, perpendicular bisector, and p- and q-lines perpendicular to a given side are coincident, and triangles $A_pB_pC_p$ and TSC_q are reduced to a pair of coincident points.

A similar proof applies for triangles $A_qB_qC_q$ and OQH. $\square$

Proposition 12. *The centroids of the minor periambic triangles lie on and trisect* PQ.

Proof. It has long been known (c.f. [2, Exercise 98]) that one of a parallelogram's diagonals divides it into two triangles whose centroids trisect the other diagonal. This means that the centroids of triangle PHO and triangle QOH trisect PQ, and the result follows directly by Proposition 11. $\square$

8. Trilinear coordinates

$P:$		$Q:$	
x	$\dfrac{b^2-2a^2}{a}\cot\beta+\dfrac{a^2}{c}\csc\beta$	x	$\dfrac{c^2-2a^2}{a}\cot\gamma+\dfrac{a^2}{b}\csc\gamma$
y	$\dfrac{c^2-2b^2}{b}\cot\gamma+\dfrac{b^2}{a}\csc\gamma$	y	$\dfrac{a^2-2b^2}{b}\cot\alpha+\dfrac{b^2}{c}\csc\alpha$
z	$\dfrac{a^2-2c^2}{c}\cot\alpha+\dfrac{c^2}{b}\csc\alpha$	z	$\dfrac{b^2-2c^2}{c}\cot\beta+\dfrac{c^2}{a}\csc\beta$
$A_p:$		$A_q:$	
x	$\cos\beta$	x	$\cos\gamma$
y	$\cos\alpha$	y	$-\cos 2\alpha$
z	$-\cos 2\alpha$	z	$\cos\alpha$
$B_p:$		$B_q:$	
x	$-\cos 2\beta$	x	$\cos\beta$
y	$\cos\gamma$	y	$\cos\alpha$
z	$\cos\beta$	z	$-\cos 2\beta$
$C_p:$		$C_q:$	
x	$\cos\gamma$	x	$-\cos 2\gamma$
y	$-\cos 2\gamma$	y	$\cos\gamma$
z	$\cos\alpha$	z	$\cos\beta$

Table 2. Relative trilinear coordinates $x:y:z$ for P, Q, and the minor periambic points.

Table 2 shows relative trilinear coordinates for P, Q, and the minor periambic points. Trilinears of the minor periambic points are easily derived from the altitudes and perpendicular bisectors that define them. For instance, C_p is the intersection of the perpendicular bisector o_a

$$x\sin(\beta-\gamma)+y\sin\beta-z\sin\gamma=0$$

(given in [6, Exercise 33]), and altitude h_b

$$x\cos\alpha-z\cos\gamma=0\ .$$

To calculate trilinear coordinates for P, we use the following information about the feet of the p-lines.

Lemma 13. *In directed distances,*

$$\begin{aligned} P_bA &= \frac{c^2}{2b}\ , & P_aC &= \frac{b^2}{2a}\ , & P_cB &= \frac{a^2}{2c}\ , \\ CP_b &= \frac{2b^2-c^2}{2b}\ , & BP_a &= \frac{2a^2-b^2}{2a}\ , & AP_c &= \frac{2c^2-a^2}{2c}\ . \end{aligned} \tag{10}$$

Proof. We prove the identity for P_bA. P_C passes through A; let G be the other intersection point of P_C with CA. Observe that $p_b=\langle P_C,P_A\rangle$ is the inversion of P_A in P_C, thus $GA\cdot P_bA=c^2$. Since $GA=2b$, we have $P_bA=c^2/2b$. Corresponding arguments produce the other equalities in row 1 of (10), and note that these quantities are always positive. Row 2 is obtained by subtracting the terms in row 1 from the respective sides of triangle ABC. □

Returning to P and its trilinears, assume first that triangle ABC is not a right triangle. Let $X_1=p_a\wedge CA$, $X_2=p_b\wedge AB$, and $X_3=p_c\wedge BC$. From triangle P_aCX_1 and Lemma 13 we have

$$CX_1=\frac{P_aC}{\cos\gamma}\ , \tag{11}$$

while the inversely similar right triangle PP_bX_1 yields

$$P_bX_1=P_bP\tan\gamma\ . \tag{12}$$

In directed distances, $CP_b+P_bX_1=CX_1$. From Lemma 13 and Equations (11) and (12), it follows that

$$\begin{aligned} P_bX_1 &= -CP_b+CX_1 \\ P_bP\tan\gamma &= \frac{c^2-2b^2}{2b}+\frac{P_aC}{\cos\gamma} \\ P_bP &= \frac{c^2-2b^2}{2b}\cot\gamma+\frac{b^2}{2a}\csc\gamma\ . \end{aligned} \tag{13}$$

This gives the distance from side b to P. The factor $1/2$ has been divided out of the relative coordinates for P in Table 2, since it occurs for P_aP and P_cP in similar derivations using X_2 and X_3.

If γ is a right angle, then P_bP is parallel to BC, and directly we have $P_bP=P_aC=b^2/2a$ in accordance with Equation (13).

The relative trilinears for Q are developed similarly.

9. Construction using isotomic points

The p- and q-circles were defined with radii equal to the side lengths of triangle ABC; for instance, P_B and Q_C each have radius a. Suppose B_a and C_a are isotomic conjugates on side a; that is, the segment B_aC_a is bisected by O_a [5]. Construct C_b and A_b on b such that C_aC_b and A_bB_a are parallel to c, and draw A_c and B_c on c so that A_bA_c and C_bB_c are parallel to a. Then C_b and A_b are isotomic conjugates on b, and A_c and B_c are isotomic conjugates on c. Furthermore, $A_bA/CA = B_cB/AB = C_aC/BC$ in this construction. We call the set of points $\{B_a, C_a, C_b, A_b, A_c, B_c\}$ a *proportional isotomic hexad.*

We define *isoperiambic circles* P'_x and Q'_x, $x \in \{A, B, C\}$, using centers and radii established above, as shown in Table 3. Radical axes p'_y, q'_y, and h'_y, $y \in \{a, b, c\}$, may then be defined by replacing P_x with P'_x and Q_x with Q'_x in Definitions 1–3. The points P', Q', and H' are the points of concurrency of the p'_y, q'_y, and h'_y, respectively; but note that the perpendicular bisectors and O remain unchanged when constructed with these new circles.

Circle	Center	Radius
P'_A	A	AB_c
P'_B	B	BC_a
P'_C	C	CA_b
Q'_A	A	AC_b
Q'_B	B	BA_c
Q'_C	C	CB_a

Table 3. Definitions of the isoperiambic circles on pairs of isotomic conjugates (B_a, C_a), (C_b, A_b), and (A_c, B_c).

Not surprisingly, the radical axes of the isoperiambic circles determine a parallelogram and pair of triangles (Figure 6). It can be shown that the "isoperiambic constellation" is identical to the periambic constellation of triangle ABC, but dilated around O.

Theorem 14. *Let B_a be a point on side a of triangle ABC. Construct the proportional isotomic hexad $\{B_a, C_a, C_b, A_b, A_c, B_c\}$ and radical axes p'_y, q'_y, and h'_y, $y \in \{a, b, c\}$, as described above. Let $A' = OA \wedge h'_a$, $B' = OB \wedge h'_b$, and $C' = OC \wedge h'_c$. Then the radical axes of the isoperiambic circles of triangle ABC are the radical axes of the periambic circles of triangle $A'B'C'$.*

Proof sketch. Let $d_1 = BB_a = CC_a$, $d_2 = BB_c = AA_c$, and $d_3 = AA_b = CC_b$. By construction, triangle AA_bA_c and triangle CC_aC_B are similar to triangle ABC,

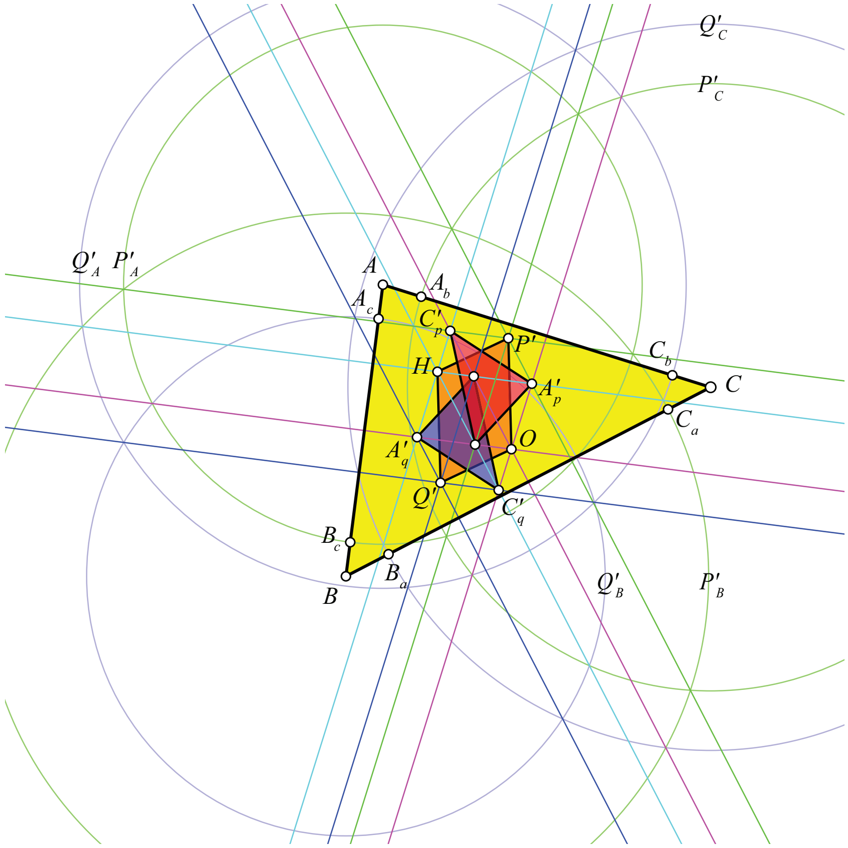

Figure 6. The isoperiambic constellation constructed on the proportional isotomic hexad $\{B_a, C_a, C_b, A_b, A_c, B_c\}$. Color key: green, p'-lines; blue, q'-lines; cyan, "altitudes" h'; magenta, perpendicular bisectors.

so

$$d_2 = \frac{cd_1}{a} \,,$$
$$d_3 = \frac{bd_1}{a} \,.$$

H'_a lies on $\langle P'_C, Q'_B \rangle$, so its equal powers with respect to these circles yields

$$BH_a'^2 - BA_c^2 = CH_a'^2 - CA_b^2 \tag{14}$$
$$BH_a'^2 - (c - d_2)^2 = CH_a'^2 - (b - d_3)^2 \tag{15}$$
$$BH_a'^2 - c^2 \left(1 - \frac{d_1}{a}\right)^2 = CH_a'^2 - b^2 \left(1 - \frac{d_1}{a}\right)^2 \,. \tag{16}$$

Set $\kappa = \left(1 - \frac{d_1}{a}\right)^2$. Substituting $BH'_a = BO_a + O_aH'_a$, $CH'_a = CO_a - O_aH'_a$, and $BO_a = CO_a$ in Equation (14), one obtains

$$O_aH'_a = \kappa\frac{c^2 - b^2}{2a} = \kappa O_aH_a \; . \tag{17}$$

Repeating this argument on sides b and c, one shows that, in the isoperiambic construction, the distance between each parallel perpendicular bisector and altitude in triangle ABC is dilated by a factor of κ around O, which remains invariant. One can show that triangle $A'B'C'$ is a dilation of triangle ABC from the same center and with the same scaling factor. For instance, let Q be the foot of the perpendicular to H'_a from O. In triangle H_bBC we have $\angle CBH_b = \pi - \gamma$; therefore $\angle LOO_b = \pi - \gamma$, and $\angle QOA' = \pi + \beta - \gamma$. From Equation (17), Lemma 7, and the relation

$$\frac{OQ}{OA'} = \cos\angle QOA' = -\sin(\beta - \gamma) = \sin(\gamma - \beta) \; ,$$

we see that

$$OA' = \frac{OQ}{\sin(\gamma - \beta)} = \frac{O_aH'_a}{\sin(\gamma - \beta)} = \frac{\kappa R\sin(\gamma - \beta)}{\sin(\gamma - \beta)} = \kappa R = \kappa OA \; .$$

Proceeding in this way, it may be shown that the radical axes of the periambic circles of triangle $A'B'C'$ are merely those of the isoperiambic circles of triangle ABC, dilated by a factor of κ around O. Details are left to the interested reader. $\square$

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Orthopoles, Flanks, and Vecten Points

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Abstract. Erect a square outwardly (or inwardly) from each side of the triangle. For the flank lines construct orthopoles. The lines from this orthopoles to corresponding vertexes of triangle concur in the Vecten point (or in the Inner Vecten point). This fact is showing together with a number of other interesting results, which include complex coordinates of Vecten points and orthopole.

1. Introduction

We will use complex numbers in the proofs.

Denote by a lowercase letter the complex coordinate of a point denoted by an uppercase letter.

Let ABC be a triangle. With no loss of generality, we can say that the circumcircle $\mathcal{C}$ is the union circle. Then $a.\bar{a} = b.\bar{b} = c.\bar{c} = 1$.

1.1. *The Outer Vecten point.*

Theorem 1. *Erect a square outwardly from each side of triangle ABC. Let $A_oB_oC_o$ be the triangle formed by the respective centers of the squares. The lines AA_o, BB_o, CC_o concur in X_{485}.* [1]

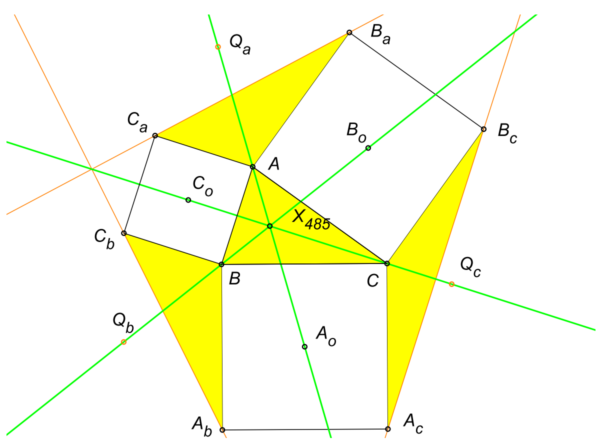

Figure 1. The Outer Vecten point

Point X_{485} is the Outer Vecten point.

Proof. Let the squares are: AC_aC_bB, BA_bA_cC, and CB_cB_aA with centers C_o, A_o, B_o respectively (Figure 1).

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Then $a_b = b + (b-c)i$, $b_c = c + (c-a)i$, $c_a = a + (a-b)i$, $\bar{a}_b = \frac{c+(b-c)i}{bc}$, $\bar{b}_c = \frac{a+(c-a)i}{ca}$, $\bar{c}_a = \frac{b+(a-b)i}{ab}$.

The coordinates of A_o, B_o, C_o and its complex conjugates are

$$a_o = \frac{1}{2}(a_b + c) = \frac{1}{2}((b+c) + (b-c)i) \tag{1}$$

$$\bar{a}_o = \frac{(b+c)+(b-c)i}{2bc}$$

$$b_o = \frac{1}{2}(b_c + a) = \frac{1}{2}((c+a) + (c-a)i)$$

$$\bar{b}_o = \frac{(c+a)+(c-a)i}{2ca}$$

$$c_o = \frac{1}{2}(c_a + b) = \frac{1}{2}((a+b) + (a-b)i)$$

$$\bar{c}_o = \frac{(a+b)+(a-b)i}{2ab}$$

The line AA_o, joining a and a_o has equation

$$\begin{aligned} 0 &= \begin{vmatrix} z & a & a_o \\ \bar{z} & \bar{a} & \bar{a}_o \\ 1 & 1 & 1 \end{vmatrix} = (\bar{a}_o - \bar{a})z - (a_o - a)\bar{z} + (a_o\bar{a} - \bar{a}_o a) \\ &= \frac{\bar{a}_o - \bar{a}}{a_o - a} z - \bar{z} + \frac{a_o\bar{a} - \bar{a}_o a}{a_o - a} \end{aligned} \tag{2}$$

Lines BB_o and CC_o have equations

$$\begin{aligned} 0 &= (\bar{b}_o - \bar{b})z - (b_o - b)\bar{z} + (b_o\bar{b} - \bar{b}_o b) \\ &= \frac{\bar{b}_o - \bar{b}}{b_o - b} \cdot z - \bar{z} + \frac{b_o\bar{b} - \bar{b}_o b}{b_o - b} \end{aligned} \tag{3}$$

$$\begin{aligned} 0 &= (\bar{c}_o - \bar{c})z - (c_o - c)\bar{z} + (c_o\bar{c} - \bar{c}_o c) \\ &= \frac{\bar{c}_o - \bar{c}}{c_o - c} \cdot z - \bar{z} + \frac{c_o\bar{c} - \bar{c}_o c}{c_o - c} \end{aligned} \tag{4}$$

The intersection of the last two lines is given by

$$\begin{aligned} 0 &= \left(\frac{\bar{b}_o - \bar{b}}{b_o - b} - \frac{\bar{c}_o - \bar{c}}{c_o - c}\right) \cdot z + \left(\frac{b_o\bar{b} - \bar{b}_o b}{b_o - b} - \frac{c_o\bar{c} - \bar{c}_o c}{c_o - c}\right) \\ &= \frac{-2(a-b)(b-c)(c-a) - i(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)}{2abc} \cdot z \\ &+ \frac{(a-b)(b-c)(c-a)(a+b+c) + i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2abc} \end{aligned}$$

$$z = \frac{(a-b)(b-c)(c-a)(a+b+c) + i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2(a-b)(b-c)(c-a) + i(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)}$$

Note that this is symmetric in a, b, c. This means that the three lines AA_o, BB_o, CC_o are concurrent. The point of concurrency is the Vecten point X_{485}. $\square$

Corollary 2. *Let the circumcircle of triangle ABC be the unit circle. The Outer Vecten point X_{485} has coordinates*

$$x_{485} = \frac{(a-b)(b-c)(c-a)(a+b+c) + i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2(a-b)(b-c)(c-a) + i(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)} \tag{5}$$

1.2. *The Inner Vecten point.*

Theorem 3. *Erect a square inwardly from each side of triangle ABC. Let $A'_oB'_oC'_o$ be the triangle formed by the respective centers of the squares. The lines AA'_o, BB'_o, CC'_o concur in X_{486}.* [1]

Point X_{486} is the Inner Vecten point.

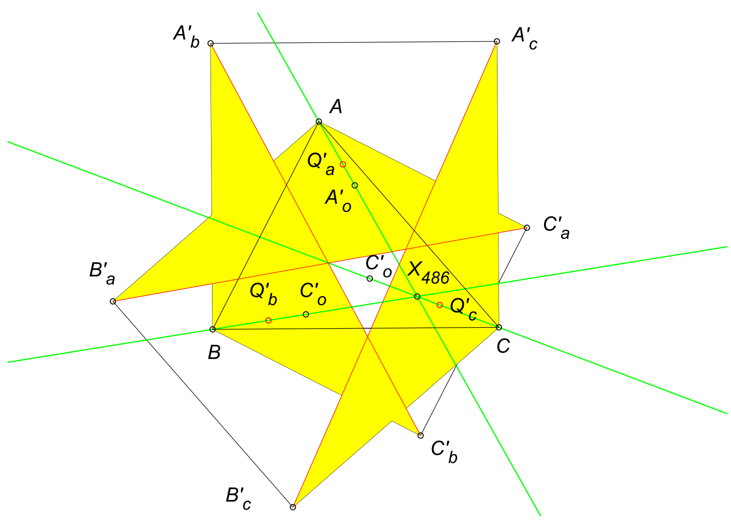

Figure 2. The Inner Vecten point

Proof. (See Figure 2) Let the squares are: $AB'_aB'_cC$, $CA'_cA'_bB$, and $BC'_bC'_aA$ with centers B'_o, A'_o, C'_o respectively.

Then $a'_b = b - (b-c)i$, $b'_c = c - (c-a)i$, $c'_a = a - (a-b)i$, $\bar{a}'_b = \frac{c-(b-c)i}{bc}$, $\bar{b}'_c = \frac{a-(c-a)i}{ca}$, $\bar{c}'_a = \frac{b-(a-b)i}{ab}$.

The coordinates of A'_o, B'_o, C'_o and its complex conjugates are

$$\begin{aligned} a'_o &= \frac{1}{2}(a'_b + c) = \frac{1}{2}((b+c) - (b-c)i) \\ \bar{a}'_o &= \frac{(b+c) - (b-c)i}{2bc} \\ b'_o &= \frac{1}{2}(b'_c + a) = \frac{1}{2}((c+a) - (c-a)i) \\ \bar{b}'_o &= \frac{(c+a) - (c-a)i}{2ca} \end{aligned} \tag{6}$$

$$\begin{aligned} c'_o =& \ \frac{1}{2}(c'_a + b) = \frac{1}{2}((a+b) - (a-b)i) \\ \bar{c}'_o =& \ \frac{(a+b)-(a-b)i}{2ab}. \end{aligned}$$

The line AA'_o, joining a' and a'_o has equation

$$0 = \begin{vmatrix} z & a & a'_o \\ \bar{z} & \bar{a} & \bar{a}'_o \\ 1 & 1 & 1 \end{vmatrix} = (\bar{a}'_o - \bar{a})z - (a'_o - a)\bar{z} + (a'_o\bar{a} - \bar{a}'_o a) \tag{7}$$

Lines BB'_o and CC'_o have equations

$$\begin{aligned} 0 &= (\bar{b}'_o - \bar{b})z - (b'_o - b)\bar{z} + (b'_o\bar{b} - \bar{b}'_o b) \\ &= \frac{\bar{b}'_o - \bar{b}}{b'_o - b}z - \bar{z} + \frac{b'_o\bar{b} - \bar{b}'_o b}{b'_o - b} \end{aligned} \tag{8}$$

$$\begin{aligned} 0 &= (\bar{c}'_o - \bar{c})z - (c'_o - c)\bar{z} + (c'_o\bar{c} - \bar{c}'_o c) \\ &= \frac{\bar{c}'_o - \bar{c}}{c'_o - c}z - \bar{z} + \frac{c'_o\bar{c} - \bar{c}'_o c}{c'_o - c} \end{aligned} \tag{9}$$

The intersection of the last two lines is given by

$$\begin{aligned} 0 &= \left(\frac{\bar{b}'_o - \bar{b}}{b'_o - b} - \frac{\bar{c}'_o - \bar{c}}{c'_o - c}\right) z + \left(\frac{b'_o\bar{b} - \bar{b}'_o b}{b'_o - b} - \frac{c'_o\bar{c} - \bar{c}'_o c}{c'_o - c}\right) \\ &= \frac{-2(a-b)(b-c)(c-a) + i(a(b-c)^2 + b(c-a)^2 c(a-b)^2)}{2abc} \cdot z \\ &+ \frac{(a-b)(b-c)(c-a)(a+b+c) - i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2abc} \end{aligned}$$

$$z = \frac{(a-b)(b-c)(c-a)(a+b+c) - i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2(a-b)(b-c)(c-a) - i(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)}$$

This is symmetric in a, b, c and means that the three lines AA'_o, BB'_o, CC'_o are concurrent. The point of concurrency is the Inner Vecten point X_{486}. □

Corollary 4. *Let the circumcircle of triangle ABC be the unit circle. The Inner Vecten point X_{486} has coordinates*

$$x_{486} = \frac{(a-b)(b-c)(c-a)(a+b+c) - i(a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2)}{2(a-b)(b-c)(c-a) - i(a(b-c)^2 + b(c-a)^2 + c(a-b)^2)} \tag{10}$$

1.3. *The Orthopole.*

Lemma 5. *Let V and W be points of the plane. The orthogonal projection P_1 of a point P onto the line VW is given by*

$$p_1 = \frac{(\bar{w} - \bar{v})p + (w - v)\bar{p} - (w\bar{v} - \bar{w}v)}{2(\bar{w} - \bar{v})} \tag{11}$$

Proof. The line WV contains a point with coordinates p_1 if and only if

$$0 = \begin{vmatrix} p_1 & \bar{p}_1 & 1 \\ w & \bar{w} & 1 \\ v & \bar{v} & 1 \end{vmatrix} = \frac{\bar{w}-\bar{v}}{w-v}p_1 - \bar{p}_1 + \frac{w\bar{v}-\bar{w}v}{w-v}$$

$P_1P \perp WV$:

$$0 = (p_1-p)(\bar{w}-\bar{v}) + (\bar{p}_1-\bar{p})(w-v) = \frac{\bar{w}-\bar{v}}{w-v}p_1 + \bar{p}_1 - \frac{\bar{w}-\bar{v}}{w-v}p - \bar{p}$$

Adding the last two expressions, receive

$$2\frac{\bar{w}-\bar{v}}{w-v}p_1 - \frac{\bar{w}-\bar{v}}{w-v}p - \bar{p} + \frac{w\bar{v}-\bar{w}v}{w-v} = 0$$

Dividing by $2\frac{\bar{w}-\bar{v}}{w-v}$, we obtain p_1, as given in (11) above. □

Remark. [7, Lemma 3] In case V and W be points on the unit circle,

$$p_1 = \frac{1}{2}(v+w+p-vw\bar{p})\,; \quad \bar{p}_1 = \frac{1}{2vw}(v+w-p+vw\bar{p}) \tag{12}$$

Lemma 6. *Let the circumcircle of triangle ABC be the unit circle. Let EF be a arbitrary line in the plane. Let A_1 be the orthogonal projection of A onto line EF. The perpendicular from A_1 to BC has equation*

$$\frac{z}{bc} - \bar{z} + \frac{a(bc(\bar{e}-\bar{f})-(e-f))}{2bc(e-f)} + \frac{bc(\bar{e}-\bar{f})-(e-f)}{2abc(\bar{e}-\bar{f})} + \frac{(bc(\bar{e}-\bar{f})+(e-f))(e\bar{f}-\bar{e}f)}{2bc(e-f)(\bar{e}-\bar{f})} = 0 \tag{13}$$

Proof. By Lemma 5, the coordinates a_1 of A_1 and its complex conjugate are

$$a_1 = \frac{1}{2(\bar{e}-\bar{f})}\big((\bar{e}-\bar{f})a + (e-f)\frac{1}{a} - (e\bar{f}-\bar{e}f)\big)$$

$$\bar{a}_1 = \frac{1}{2(e-f)}\big((\bar{e}-\bar{f})a + (e-f)\frac{1}{a} + (e\bar{f}-\bar{e}f)\big)$$

By (12), (Remark to Lemma 5), the coordinate a_2 of the orthogonal projection A_2 of A_1 onto BC, together with its conjugate, are

$$a_2 = \frac{1}{2}(b+c+a_1-bc\bar{a}_1)$$

$$\bar{a}_2 = \frac{1}{2bc}(b+c-a_1+bc\bar{a}_1)$$

The line A_1A_2 contains a point with coordinates z if and only if

$$0 = \begin{vmatrix} z & a_1 & a_2 \\ \bar{z} & \bar{a}_1 & \bar{a}_2 \\ 1 & 1 & 1 \end{vmatrix} = \frac{\bar{a}_1-\bar{a}_2}{a_1-a_2}z - \bar{z} + \frac{a_1\bar{a}_2-\bar{a}_1a_2}{a_1-a_2} = \frac{1}{bc}z - \bar{z}$$
$$+ \frac{(a^2(\bar{e}-\bar{f})+(e-f))(bc(\bar{e}-\bar{f})-(e-f)) + a(e\bar{f}-\bar{e}f)(bc(\bar{e}-\bar{f})+(e-f))}{2abc(e-f)(\bar{e}-\bar{f})} \tag{14}$$

This is equal to (13). □

Theorem 7 (*Orthopole*, [2, p 247, Theorem 406]). *If perpendiculars are dropped on any line from the vertices of a triangle, the perpendiculars to the opposite sides from their feet are concurrent at a point called the orthopole of the point.*

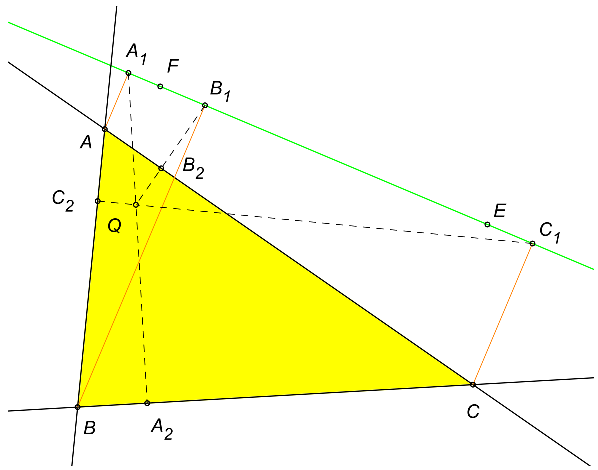

Figure 3. The Orthopole

Proof. Let EF be a arbitrary line of the plane. Let A_1, B_1, C_1 be the orthogonal projections from vertexes A, B, C respectively to the line EF (Figure 3).

We consider the construction in Lemma 6 beginning with all three vertices of triangle ABC. This results in the three lines

$$\frac{z}{bc}-\bar{z}+\frac{a(bc(\bar{e}-\bar{f})-(e-f))}{2bc(e-f)}+\frac{bc(\bar{e}-\bar{f})-(e-f)}{2abc(\bar{e}-\bar{f})}+\frac{(bc(\bar{e}-\bar{f})+(e-f))(e\bar{f}-\bar{e}f)}{2bc(e-f)(\bar{e}-\bar{f})}=0$$

$$\frac{z}{ca}-\bar{z}+\frac{b(ca(\bar{e}-\bar{f})-(e-f))}{2ca(e-f)}+\frac{ca(\bar{e}-\bar{f})-(e-f)}{2abc(\bar{e}-\bar{f})}+\frac{(ca(\bar{e}-\bar{f})+(e-f))(e\bar{f}-\bar{e}f)}{2ca(e-f)(\bar{e}-\bar{f})}=0$$

$$\frac{z}{ab}-\bar{z}+\frac{c(ab(\bar{e}-\bar{f})-(e-f))}{2ab(e-f)}+\frac{ab(\bar{e}-\bar{f})-(e-f)}{2abc(\bar{e}-\bar{f})}+\frac{(ab(\bar{e}-\bar{f})+(e-f))(e\bar{f}-\bar{e}f)}{2ab(e-f)(\bar{e}-\bar{f})}=0$$

The intersection of the last two lines is

$$\frac{z(b-c)}{abc}+\frac{abc(b-c)(\bar{e}-\bar{f})}{2abc(e-f)}-\frac{(b^2-c^2)(e-f)}{2abc(e-f)}-\frac{a(b-c)(\bar{e}-\bar{f})}{2abc(\bar{e}-\bar{f})}+\frac{(b-c)(e-f)(e\bar{f}-\bar{e}f)}{2abc(e-f)(\bar{e}-\bar{f})}=0$$

Multiplying by $\frac{abc}{b-c}$, we obtain

$$z=\frac{1}{2}(a+b+c)-\frac{abc(\bar{e}-\bar{f})}{2(e-f)}-\frac{e\bar{f}-\bar{e}f}{2(\bar{e}-\bar{f})}$$

Note that this is symmetric in a, b, c. This means that the three perpendiculars from A_1 to BC, B_1 to CA, and C_1 to AB are concurrent. The point of concurrency is the orthopole Q of the line EF with respect to triangle ABC. $\square$

Corollary 8. *Let the circumcircle of triangle ABC be the unit circle. The orthopole Q of the line EF with respect to triangle ABC has complex coordinate and conjugate*

$$q = \frac{1}{2}(a+b+c) - \frac{abc(\bar{e}-\bar{f})}{2(e-f)} - \frac{e\bar{f}-\bar{e}f}{2(\bar{e}-\bar{f})} \tag{15}$$

$$\bar{q} = \frac{ab+bc+ca}{2abc} - \frac{e-f}{2abc(\bar{e}-\bar{f})} + \frac{e\bar{f}-\bar{e}f}{2(e-f)}$$

2. Orthopoles and flank lines

Theorem 9. *Erect a square outwardly from each side of triangle ABC: AC_aC_bB, BA_bA_cC and CB_cB_aA with centers C_o, A_o, B_o respectively. For flank lines B_aC_a, C_bA_b, A_cB_c construct the orthopoles Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur in Vecten point X_{485}.*

The proof of the theorem relies on the following lemma.

Lemma 10. *The orthopole Q_a of the flank line B_aC_a with respect to triangle ABC and points A, A_o are collinear.*

Proof. (See Figure 1) With no loss of generality we shall say that the circumcircle of triangle ABC is the union circle. $b_a = a + (c-a)i$, $c_a = a + (a-b)i$, $\bar{b}_a = \frac{c+(c-a)i}{ac}$, $\bar{c}_a = \frac{b+(a-b)i}{ab}$

By Corollary 8 find the orthopole Q_a as substitute b_a, c_a for e, f in (15).

$$q_a = \frac{1}{2}(a+b+c) - \frac{abc(\bar{b}_a-\bar{c}_a)}{2(b_a-c_a)} - \frac{b_a\bar{c}_a-\bar{b}_ac_a}{2(\bar{b}_a-\bar{c}_a)}$$

$$q_a = \frac{1}{2}(a+b+c) - \frac{a(b+c)-2bc}{2(2a-b-c)} + \frac{b(c-a)^2+c(a-b)^2-i(a-b)(b-c)(c-a)}{2(a(b+c)-2bc)} \tag{16}$$

$$\bar{q}_a = \frac{ab+bc+ca}{2abc} + \frac{2a-b-c}{2(a(b+c)-2bc)} - \frac{b(c-a)^2+c(a-b)^2-i(a-b)(b-c)(c-a)}{2abc(2a-b-c)} \tag{17}$$

The points A, A_o, Q_a are collinear if and only if ([5, p. 65], [6, p. 37])

$$\frac{q_a-a}{a_o-a} = \eta \in \mathbb{R}^* \text{ i.e. } \frac{q_a-a}{a_o-a} = \frac{\bar{q}_a-\bar{a}}{\bar{a}_o-\bar{a}}$$

Use (16), (17), (1) and $\bar{a} = 1/a$.

$$\begin{aligned}&(q_a-a)(\bar{a}_o-\bar{a}) - (\bar{q}_a-\bar{a})(a_o-a)\\ =&\frac{(a-b)(c-a)(bc-a^2)(b-c)^2(i^2+1)}{2abc(b+c-2a)(2bc-ab-ca)} = 0\end{aligned}$$

□

Proof of Theorem 9. (See Figure 1) By Lemma 10 applied to all three vertices of triangle ABC: $Q_a \in AA_o$, $Q_b \in BB_o$ and $Q_c \in CC_o$. But from Theorem 1, the lines AA_o, BB_o, CC_o concur in the Vecten point X_{485}. $\square$

Theorem 11. *Erect a square inwardly from each side of triangle ABC : $BC'_bC'_aA$, $CA'_cA'_bB$ and $AB'_aB'_cC$ with centers C'_o, A'_o, B'_o respectively. For flank lines $B'_aC'_a$, $C'_bA'_b$, $A'_cB'_c$ construct the orthopoles Q'_a, Q'_b, Q'_c respectively. The lines AQ'_a, BQ'_b, CQ'_c concur in Inner Vecten point X_{486}.*

Lemma 12. *The orthopole Q'_a of the flank line $B'_aC'_a$ with respect to triangle ABC and points A, A'_o are collinear. (See Figure 2)*

Proof. The proof is the same as the proof of Lemma 10.

With no loss of generality we shall say that the circumcircle of triangle ABC is the union circle. $b'_a = a-(c-a)i$, $c'_a = a-(a-b)i$, $\bar{b}'_a = \frac{c-(c-a)i}{ca}$, $\bar{c}'_a = \frac{b-(a-b)i}{ab}$

$$q'_a = \frac{1}{2}(a+b+c) - \frac{a(b+c)-2bc}{2(2a-b-c)} + \frac{b(c-a)^2 + c(a-b)^2 + i(a-b)(b-c)(c-a)}{2(a(b+c)-2bc)} \tag{18}$$

$\square$

Proof of Theorem 11. (See Figure 2) By Lemma 12 applied to all three vertices of triangle ABC: $Q'_a \in AA'_o$, $Q'_b \in BB'_o$ and $Q'_c \in CC'_o$. But from Theorem 3, the lines AA'_o, BB'_o, CC'_o concur in the Inner Vecten point X_{486}. $\square$

Lemma 13. [4, p. 105] *Erect squares outwardly from sides of triangle ABC: AC_aC_bB, BA_bA_cC and CB_cB_aA with centers C_o, A_o, B_o.*

Erect a square $B_aB_{a1}C_{a1}C_a$ with center A_{o1} outwardly from B_aC_a. The points A_{o1}, A_o and A are collinear.

Proof. With no loss of generality, we can say that the circumcircle $\mathcal{C}$ of triangle ABC is the union circle. $b_a = a+(c-a)i$, $c_a = a+(a-b)i$, $\bar{b}_a = \frac{c+(c-a)i}{ac}$, $\bar{c}_a = \frac{b+(a-b)i}{ab}$

$$\begin{aligned}
b_{a1} &= |b_a + (b_a - c_a)i = 3a - b - c + (c-a)i \\
\bar{b}_{a1} &= \frac{3bc - ca - ab + (c-a)i}{abc} \\
a_{o1} &= \frac{1}{2}(c_a + b_{a1}) = \frac{1}{2}(4a - b - c - (b-c)i) \\
\bar{a}_{o1} &= \frac{4bc - ca - ab - a(b-c)i}{2abc}.
\end{aligned}$$

The points A, A_o, A_{o1} are collinear if and only if ([5, p. 65], [6, p. 37])

$$\frac{a_{o1}-a}{a_o-a} = \eta \in \mathbb{R}^* \text{ i.e. } \frac{a_{o1}-a}{a_o-a} = \frac{\bar{a}_{o1}-\bar{a}}{\bar{a}_o-\bar{a}}$$

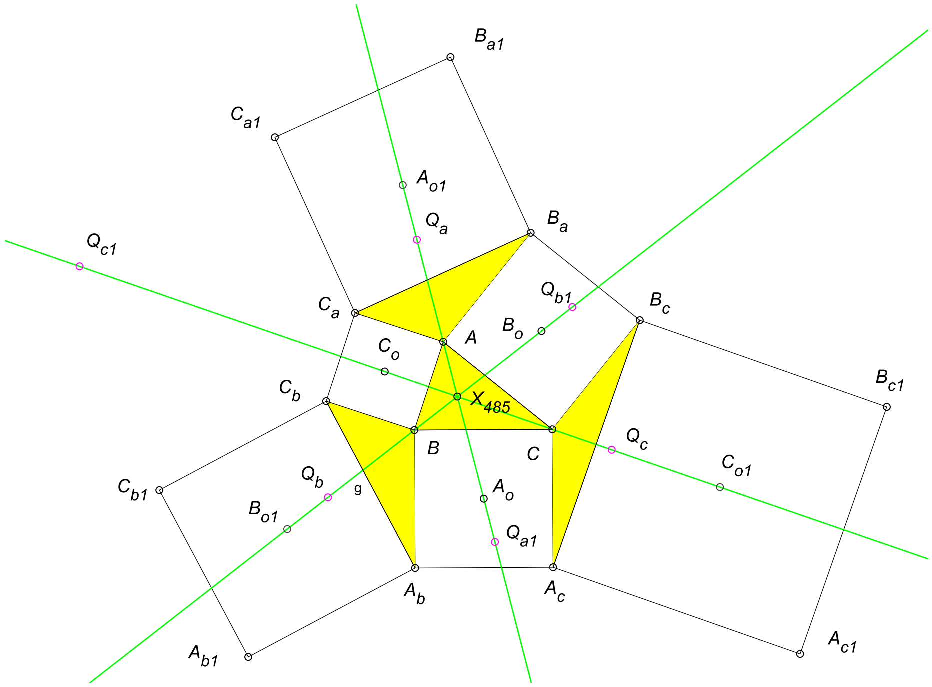

Figure 4.

$$\begin{aligned}&(a_{o1}-a)(\bar{a}_o-\bar{a})-(a_o-a)(\bar{a}_{o1}-\bar{a})\\=&\frac{2a-b-c-(b-c)i}{2}\cdot\frac{a(b+c)+a(b-c)i-2bc}{2abc}\\-&\frac{2bc-ca-ab-a(b-c)i}{2abc}\cdot\frac{b+c-2a+(b-c)i}{2}=0\end{aligned}$$

□

Theorem 14. *Erect a square outwardly from each side of triangle ABC: AC_aC_bB, BA_bA_cC and CB_cB_aA. Construct the orthopoles Q_{a1} for BC with respect to flank triangle AB_aC_a; Q_{b1} for CA with respect to flank triangle BC_bA_b; Q_{c1} for AB with respect to flank triangle CA_cB_c. The lines AQ_{a1}, BQ_{b1}, CQ_{c1} concur in Vecten point X_{485}.*

Proof. (See Figure 4) Let C_o, A_o, B_o be centers of the squares AC_aC_bB, BA_bA_cC, and CB_cB_aA respectively. Erect squares $B_aB_{a1}C_{a1}C_a$, $C_bC_{b1}A_{b1}A_b$, $A_cA_{c1}B_{c1}B_c$ outwardly from B_aC_a, C_bA_b, A_cB_c with centers A_{o1}, B_{o1}, C_{o1} respectively.

Triangle ABC is a flank triangle for triangle AB_aC_a. From Lemma 10 the orthopole Q_{a1} of the flank line BC with respect to triangle AB_aC_a and points A, A_{o1} are collinear. From Lemma 14 points A_{o1}, A_o and A are collinear. Follow Q_{a1}, A_{o1}, A, A_o are collinear.

By analogy Q_{b1}, B_{o1}, B, B_o and Q_{c1}, C_{o1}, C, C_o are collinear. From Theorem 1 the last three lines concur in the Vecten point X_{485}. □

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Two Interesting Integer Parameters of Integer-sided Triangles

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Abstract. When a triangle is described in terms of the segments into which its sides are divided by an inscribed circle, it permits determination of all integer sided triangles for which the area is an integer multiple of the perimeter. It is not possible to have integer-sided triangles with R/r an integer, where R and r are the radii of the circumcircle and incircle respectively, for right triangles, isosceles triangles, and triangles whose sides are in arithmetic progression. The exception is the equilateral triangle for which $R/r = 2$.

First consider integer-sided triangles for which the area A is an integer multiple m of the perimeter P.

$$A = mP. \tag{1}$$

For the particular case of right triangles (Figure 1), equation (1) becomes:

$$1/2ab = m(a + b + c). \tag{2}$$

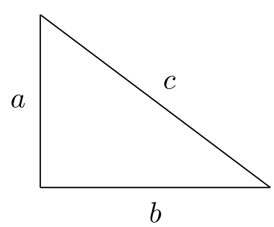

Figure 1.

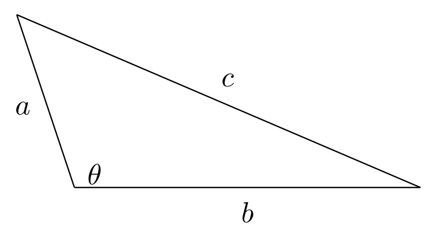

Figure 2.

Rearrangement gives

$$ab - 2m(a + b) = 2mc.$$

Squaring both sides of the equation, noting that $c^2 = a^2 + b^2$, gives

$$a^2b^2 + 4m^2(a + b)^2 - 4mab(a + b) = 4m^2(a^2 + b^2),$$

which can be simplified to

$$ab + 8m^2 - 4m(a + b) = 0.$$

This last equation can be factored to give

$$(a - 4m)(b - 4m) = 8m^2. \tag{3}$$

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The factors of $8m^2$ yield all right triangles satisfying equation (2). This solution was given by Goehl [1].

For a general triangle [2, 3, 3, 4, 5, 6], with $a \le b \le c$, equation (2) is generalized to:

$$\frac{1}{2}ab\sin\theta = m(a+b+c). \tag{4}$$

Rearrangement gives:

$$ab\sin\theta - 2m(a+b) = 2mc.$$

Squaring both sides of the equation, noting that $c^2 = a^2 + b^2 - 2ab\cos\theta$, gives

$$a^2b^2\sin^2\theta + 4m^2(a+b)^2 - 4mab(a+b)\sin\theta = 4m^2(a^2+b^2-2ab\cos\theta)$$

which can be simplified to

$$ab\sin^2\theta + 8m^2 - 4m(a+b)\sin\theta = -8m^2ab\cos\theta.$$

This last equation can be factored to give

$$(a\sin\theta - 4m)(b\sin\theta - 4m) = 8m^2(1-\cos\theta). \tag{5}$$

Note that, of course, equation (5) reduces to equation (3) when $\theta = 90°$.

In order to determine allowed values of θ, consider the circle inscribed in the triangle. Now

$$\begin{aligned} a =& \ \alpha + \beta, \\ b =& \ \alpha + \gamma, \\ c =& \ \beta + \gamma. \end{aligned}$$

Since $a \le b \le c$, $\alpha \le \beta \le \gamma$.

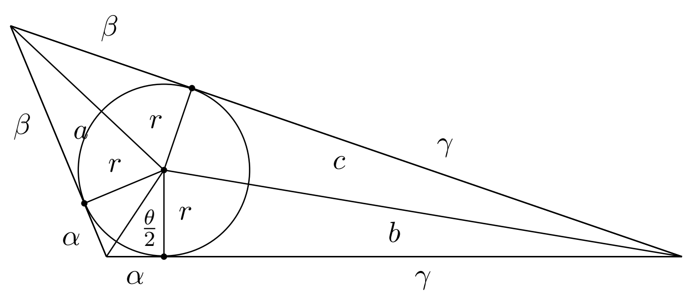

Figure 3.

From Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi perimeter and $P = 2s = 2(\alpha+\beta+\gamma)$.

The problem $A = mP$ is equivalent to $\sqrt{(\alpha+\beta+\gamma)(\gamma)(\beta)(\alpha(} = m\cdot 2(\alpha + \beta+\gamma)$. Simplification gives [5]

$$\alpha\beta\gamma = 4m^2(\alpha+\beta+\gamma). \tag{6}$$

Clearly there are no equilateral triangles as solutions because if $\alpha = \beta = \gamma$, equation (6)) implies that $\alpha^3 = 4m^2(3\alpha)$ or $\alpha^2 = 12m^2$. So α is not an integer. Note that the radius of the incircle r is equal to $2m$ [2, 6]. The combined area of the 6 small right triangles in the figure that have r as one side is equal to the area of the triangle:

$$A = \frac{1}{2}(r\alpha + r\alpha + r\beta + r\beta + r\gamma + r\gamma) = r(\alpha + \beta + \gamma) = \frac{rP}{2} = mP,$$

so $r = 2m$.

From equation (6), it follows that $\alpha^2\beta\gamma = \alpha r^2(\alpha + \beta + \gamma)$ and factorization yields

$$(\alpha\beta - r^2)(\alpha\gamma - r^2) = r^2(\alpha^2 + r^2). \tag{7}$$

This result also follows from identifying the angle in equation (5).

From the above figure, $\sin\frac{\theta}{2} = \frac{r}{\sqrt{\alpha^2+r^2}}$ and $\cos\frac{\theta}{2} = \frac{\alpha}{\sqrt{\alpha^2+r^2}}$. Now, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2r\alpha}{\alpha^2+r^2}$ and $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{\alpha^2-r^2}{\alpha^2+r^2}$. Substituting these values into equation (5) yields:

$$\left(a\left(\frac{2r\alpha}{\alpha^2+r^2}\right) - 2r\right)\left(b\left(\frac{2r\alpha}{\alpha^2+r^2}\right) - 4m\right) = 2r^2\left(1 - \frac{\alpha^2 - r^2}{\alpha^2 + r^2}\right).$$

Simplifying:

$$(a\alpha - (\alpha^2 + r^2))(b\alpha - (\alpha^2 + r^2)] = r^2(\alpha^2 + r^2).$$

Substituting $a = \alpha + \beta$, and $b = \alpha + \gamma$ yields:

$$(\alpha\beta - r^2)(\alpha\gamma - r^2) = r^2(\alpha^2 + r^2).$$

Since c is the largest side, θ is the largest angle. The largest value of the smallest parameter, α, results when $\alpha = \beta = \gamma$ and $\theta = 60^\circ$. So $\left(\tan\frac{\theta}{2}\right)_{\min} = \frac{r}{\sqrt{3}}$ 3 and $\frac{r}{\alpha} \geq \frac{1}{\sqrt{3}}$. Therefore allowed values for α are given by $1 \leq \alpha < \sqrt{3}r = 2\sqrt{3}m$.

From the figure, $\tan\frac{\theta}{2} = \frac{r}{\alpha}$ and $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan?^2\frac{\theta}{2}} = \frac{2r\alpha}{\alpha^2-r^2}$.

Of course, $m = 1$ corresponds to area = perimeter.

For $m = 1$, $r = 2m = 2$. So $1 \leq \alpha < 2\sqrt{3} \approx 3.46$, and the values for α are 1, 2, and 3. Equation (7) becomes $(\alpha\beta - 4)(\alpha\gamma - 4) = 4(\alpha^2 + 4)$.

For $\alpha = 1$, $\tan\theta = \frac{4}{1-4} = -\frac{4}{3}$, $\theta = 126.8698976^\circ$, and $(\beta - 4)(\gamma - 4) = 4(1 + 4)$. The three factorizations of 20, namely, $(4)(5)$, $(2)(10)$, and $(1)(20)$, all yield triangles:

$$\alpha,\ \beta,\ \gamma = \begin{cases} 1,\ 8,\ 9 \\ 1,\ 6,\ 14 \\ 1,\ 5, 24 \end{cases} \quad \text{and} \quad a,\ b,\ c = \begin{cases} 9,\ 10,\ 17 \\ 7,\ 15,\ 20 \\ 6,\ 25, 29 \end{cases}.$$

For $\alpha = 2$, $\theta = 90^\circ$, and $(\beta - 2)(\gamma - 2) = 4 + 4$. Both factorizations of 8, (2)(4) and (1)(8), yield triangles:

$$\alpha,\ \beta,\ \gamma = \begin{cases} 2,\ 4,\ 6 \\ 2,\ 3,\ 10 \end{cases} \quad \text{and} \quad a,\ b,\ c = \begin{cases} 6,\ 8,\ 10 \\ 6,\ 12, 13 \end{cases}.$$

For $\alpha = 3$, $\tan\theta = \frac{12}{9-4} = \frac{12}{5}$, $\theta = 67.38013505°$, and $(3\beta - 4)(3\gamma - 4) = 4(9+4)$. The three factorizations of 52, namely, $(4)(13)$, $(2)(26)$, and $(1)(52)$, yield no new solutions because $\alpha < \beta$.

The results for $m = 1$ and $m = 2$ are shown in Table 1.

A	β	γ	α	B	c	$a+b+c$	m	θ
1	5	24	6	25	29	60	1	126.8698976
1	6	14	7	15	20	42	1	126.8698976
1	8	9	9	10	17	36	1	126.8698976
2	3	10	5	12	13	30	1	90
2	4	6	6	8	10	24	1	90
1	17	288	18	289	305	612	2	151.9275131
1	18	152	19	153	170	342	2	151.9275131
1	20	84	21	85	104	210	2	151.9275131
1	24	50	25	51	74	150	2	151.9275131
1	32	33	33	34	65	132	2	151.9275131
2	9	88	11	90	97	198	2	126.8698976
2	10	48	12	50	58	120	2	126.8698976
2	12	28	14	30	40	84	2	126.8698976
2	13	24	15	26	37	78	2	126.8698976
2	16	18	18	20	34	72	2	126.8698976
3	6	72	9	75	78	162	2	106.2602047
3	7	32	10	35	39	84	2	106.2602047
3	8	22	11	25	30	66	2	106.2602047
3	12	12	15	15	24	54	2	106.2602047
4	5	36	9	40	41	90	2	90
4	6	20	10	24	26	60	2	90
4	8	12	12	16	20	48	2	90
6	7	8	13	14	15	42	2	67.38013505

Table 1.

An interesting special case is that of isosceles triangles. For isosceles triangles, let $\beta = \gamma$, then $\alpha\beta^2 = 4m^2(\alpha + 2\beta)$. The result is the quadratic equation: $\alpha^2 - 8m^2\beta - 4m^2\alpha = 0$. The solutions are:

$$\beta = \frac{2m}{\alpha}\left(2m \pm \sqrt{4m^2 + \alpha^2}\right). \tag{8}$$

The positive value is chosen. The restriction, $\alpha \leq \beta$, cannot be assumed.

For $m = 1$, $\beta = \frac{2}{\alpha}\left(2 + \sqrt{4 + \alpha^2}\right)$. To get integer solutions, $4 + \alpha^2$ must be the square of an integer, I^2.

So $I^2 - \alpha^2 = 4$, or $(I + \alpha)(I - \alpha) = 4$. The two possible factorizations, $(2)(2)$ and $(4)(1)$, result in no solutions. The first factorization gives $\alpha = 0$ while the second gives $\alpha = 2.5$.

For $m = 2$, $\beta = \frac{4}{\alpha}\left(4 + \sqrt{16 + \alpha^2}\right)$. To get integer solutions, $16 + \alpha^2$ must be the square of an integer, I^2.

So $I^2 - \alpha^2 = 16$, or $(I + \alpha)(I - \alpha) = 16$. The three possible factorizations, $(4)(4)$, $(8)(2)$, and $(16)(1)$, result in one solution. These factorizations give values for α of 0, 3, and 7.5. The one integer value for α also gives an integer value for β: 12.

The results for values of m from 2 through 6 are shown in Table 2. Note that there are no solutions for $m = 1$ and $m = 7$.

m	α	β	γ	a	b	c
2	3	12	12	15	15	24
3	8	12	12	20	20	24
4	6	24	24	30	30	48
5	15	15	24	39	39	30
6	5	60	60	65	65	120
6	9	36	36	45	45	72
6	16	24	24	40	40	48

Table 2.

Triangles with sides in arithmetic progression

Another interesting special case is that of triangles with sides in arithmetic progression. The problem has been studied before for consecutive integers [7]. Now consider the general case for an arithmetic progression $\alpha = \beta - \delta$ and $\gamma = \beta + \delta$. The sides become: $a = \alpha + \beta = 2\beta - \delta$, $b = \alpha + \gamma = 2\beta$, and $c = \beta + \gamma = 2\beta + \delta$. Now equation (6)) becomes

$$(\beta - \delta)\beta(\beta + \delta) = 4m^2(\beta - \delta + \beta + \beta + \delta)$$

or

$$(\beta - \delta)(\beta + \delta) = 12m^2.$$

To find all triangles with sides in arithmetic progression, find all factorizations of $12m^2$ that result in integer values of β and δ.

For example, for $m = 1$, the possible factorizations are $(12)(1)$, $(6)(2)$, and $(4)(3)$. Only the second factorization gives integer values: $\beta = 4$ and $\delta = 2$. The resulting triangle has sides $a = 6$, $b = 8$, and $c = 10$.

The results for values of m from 1 through 4 are shown in Table 3.

m	δ	α	β	γ	a	b	c
1	2	2	4	6	6	8	10
2	11	2	13	24	15	26	37
2	4	4	8	12	12	16	20
2	1	6	7	8	13	14	15
3	2	26	28	30	54	56	58
3	6	6	12	18	18	24	30
4	47	2	49	96	51	98	145
4	22	4	26	48	30	52	74
4	13	6	19	32	25	38	51
4	8	8	16	24	24	32	40
4	2	24	26	28	50	52	54

Table 3.

A second interesting parameter associated with a triangle is the ratio of the radii of its circumcircle and incircle. MacLeod [8] has shown that, for triangles with sides a, b, and c, this ratio is given by:

$$N = \frac{2abc}{(a+b-c)(a+c-b)(b+c-a)}. \tag{9}$$

He points out that triangles for which this ratio is an integer are ”relatively rare” and finds some of them. In fact, it will be shown that, with the exception of the equilateral triangle, no right triangles, isosceles triangles, or triangles whose sides are in arithmetic progression have an integer ratio.

First consider right triangles. Only primitive triangles need be considered. For such triangles, the sides can be represented by $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$, where m and n are relatively prime and of opposite parity. Then, from equation (9):

$$N = \frac{4mn(m^2-n^2)(m^2+n^2)}{(2n(m-n))(2n(m+n))(2m(m-n))}. \tag{10}$$

Simplification yields:

$$2n(m+n)N = m^2 + n^2. \tag{11}$$

This implies that n must divide m. The only possibility is $n = 1$. But then the left side of (11) would be even and the right side odd. Therefore there are no right triangles with integer N.

Next consider isosceles triangles. Letting $b = c$ in equation (9) gives $N = \frac{2b^2}{a(2b-a)}$. This may be written as a quadratic equation in a: $Na^2 - 2Nab + 2b^2 = 0$. This equation has solutions $a = \frac{b(N \pm I)}{N}$ where $I^2 = (N-1)^2 - 1$. Thus $N = 2$ and $I = 0$, and the only solution is $a = b = c$, the equilateral triangle.

Lastly consider triangles with sides in arithmetic progression. Letting $a = b + d$ and $c = b - d$ in equation (9) gives $N = \frac{2b(b^2-d^2)}{b^2-4d^2}$ or $\frac{b^2}{d^2} = \frac{2(2N-1)}{N-2}$. Thus $2(2N-1)(N-2) = J^2$ for some integer J. This may be rewritten as $(4N-5)^2 - (2J)^2 = (4N-5+2J)(4N-5+2J) = 9$. One factorization yields $J = 2$

and the non-integer $N = \frac{5}{2}$. The only other possible factorization yields $J = 0$ and $N = 2$, once again the equilateral triangle.

Conclusion

Insight is gained from the geometric approach to the solution of the problem of $A = mP$. Equation (5), in terms of the general angle, θ, leads immediately to equation (3) for right triangles and to equation (7) for the general case.

When a triangle is described in terms of the segments into which its sides are divided by an inscribed circle, equation (7) permits determination of all integer sided triangles for which the area is an integer multiple of the perimeter. The angle θ, opposite to the greatest side c, is shown to have a relatively small number of allowed values.

The ratio of the radii of circumcircle to incircle is considered.

The special cases of right triangles, isosceles triangles, and triangles with sides in arithmetic progression are solved generally for the two parameters.

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Heron Triangle and Rhombus Pairs With a Common Area and a Common Perimeter

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Abstract. By Fermat's method, we show that there are infinitely many Heron triangle and θ-integral rhombus pairs with a common area and a common perimeter. Moreover, we prove that there does not exist any integral isosceles triangle and θ-integral rhombus pairs with a common area and a common perimeter.

1. Introduction

We say that a Heron (resp. rational) triangle is a triangle with integral (resp. rational) sides and integral (resp. rational) area. And a rhombus is θ-integral (resp. θ-rational) if it has integral (resp. rational) sides, and both $\sin\theta$ and $\cos\theta$ are rational numbers. In 1995, R. K. Guy [5] introduced a problem of Bill Sands, that asked for examples of an integral right triangle and an integral rectangle with a common area and a common perimeter, but there are no non-degenerate such. In the same paper, R. K. Guy showed that there are infinitely many such integral isosceles triangle and rectangle pairs. In 2006, A. Bremner and R. K. Guy [1] proved that there are infinitely many such Heron triangle and rectangle pairs. In 2016, Y. Zhang [6] proved that there are infinitely many integral right triangle and parallelogram pairs with a common area and a common perimeter. At the same year, S. Chern [2] proved that there are infinitely many integral right triangle and θ-integral rhombus pairs. In a recent paper, P. Das, A. Juyal and D. Moody [3] proved that there are infinitely many integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and a common perimeter. By Fermat's method [4, p.639], we can give a simple proof of the following result, which is a corollary of Theorem 2.1 in [3].

Theorem 1. *There are infinitely many Heron triangle and θ-integral rhombus pairs with a common area and a common perimeter.*

But for integral isosceles triangle and θ-integral rhombus pair, we have

Theorem 2. *There does not exist any integral isosceles triangle and θ-integral rhombus pairs with a common area and a common perimeter.*

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2. Proofs of the theorems

Proof of Theorem 1. Suppose that the Heron triangle has sides (a, b, c), and the θ-integral rhombus has side p and intersection angle θ with $0 < \theta \leq \frac{\pi}{2}$. By Brahmagupta's formula, all Heron triangles have sides

$$(a, b, c) = ((v + w)(u^2 - vw),\ v(u^2 + w^2),\ w(u^2 + v^2)),$$

for positive integers u, v, w, where $u^2 > vw$. Noting the homogeneity of these sides, we can set $w = 1$, and u, v, p positive rational numbers. Now we only need to study the rational triangle and θ-rational rhombus pairs with a common area and a common perimeter. Then we have

$$\begin{cases} uv(v+1)(u^2 - v) = p^2 \sin^2\theta, \\ 2u^2(v+1) = 4p. \end{cases} \tag{1}$$

Since both $\sin\theta$ and $\cos\theta$ are rational numbers, we may set

$$\sin\theta = \frac{2t}{t^2+1}, \qquad \cos\theta = \frac{t^2-1}{t^2+1},$$

where $t > 1$ is a rational number. For $t = 1$, $\theta = \frac{\pi}{2}$, this is the case studied by R. K. Guy [5]. Thus we need only consider the case $t > 1$.

Eliminating p in equation (1), we have

$$\frac{u(v+1)(2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2)}{2(t^2+1)} = 0.$$

Let us study the rational solutions of the following equation

$$2t^2u^2v - tu^3v - 2t^2v^2 - tu^3 + 2u^2v - 2v^2 = 0. \tag{2}$$

Solving it for v, we get

$$v = \frac{\left(2t^2u - tu^2 + 2u \pm \sqrt{g(t)}\right)u}{4(t^2+1)},$$

where

$$g(t) = 4u^2t^4 - 4u(u^2+2)t^3 + u^2(u^2+8)t^2 - 4u(u^2+2)t + 4u^2.$$

Since v is a positive rational number, $g(t)$ should be a rational perfect square. So we need to consider the rational points on the curve

$$\mathcal{C}_1 : \qquad s^2 = g(t).$$

The curve $\mathcal{C}_1$ is a quartic curve with a rational point $P = (0, 2u)$. By Fermat's method [4, p. 639], using the point P we can produce another point $P' = (t_1, s_1)$, which satisfies the condition $t_1 s_1 \neq 0$. In order to construct such a point P', we put

$$s = rt^2 + qt + 2u,$$

where r, q are indeterminate variables. Then

$$s^2 - g(t) = \sum_{i=1}^{4} A_i t_i,$$

where the quantities $A_i = A_i(r, q)$ are given by

$$\begin{aligned} A_1 &= 4u^3 + 4qu + 8u, \\ A_2 &= -4u^2 + 4ru + q^2 - 8u^2, \\ A_3 &= 4u^3 + 2rq + 8u, \\ A_4 &= r^2 - 4u^2. \end{aligned}$$

The system of equations $A_3 = A_4 = 0$ in r, q has a solution given by

$$r = -2u, \quad q = u^2 + 2.$$

This implies that the equation

$$s^2 - g(t) = \sum_{i=1}^{4} A_i t^i = 0$$

has the rational roots $t = 0$ and

$$t = \frac{2u(u^2+2)}{3u^2-1}.$$

Then we have the point $P' = (t_1, s_1)$ with

$$\begin{aligned} t_1 &= \frac{2u(u^2+2)}{3u^2-1}, \\ s_1 &= -\frac{2u(u^6 - 4u^4 + 14u^2 + 3)}{(3u^2-1)^2}. \end{aligned}$$

Putting t_1 in equation (2) we get

$$v = \frac{u^2(u^2+2)}{4u^4+1}.$$

Hence, the rational triangle has rational sides

$$\begin{aligned} &(a, b, c) \\ =&\left(\frac{u^2(3u^2-1)(u^4+6u^2+1)}{(4u^2+1)^2},\ \frac{u^2(u^2+2)(u^2+1)}{4u^2+1},\ \frac{u^2(u^6+20u^4+12u^2+1)}{(4u^2+1)^2}\right). \end{aligned}$$

From the equation $2u^2(v+1) = 4p$ and $\sin\theta = \frac{2t}{t^2+1}$, we obtain the corresponding rhombus with side

$$p = \frac{(u^4+6u^2+1)u^2}{2(4u^2+1)},$$

and the intersection angle

$$\theta = \arcsin \frac{4u(u^2+2)(3u^2-1)}{(4u^2+1)(u^4+6u^2+1)}.$$

Since u, v, p are positive rational numbers, $0 < \sin\theta < 1$, $u^2 > v$ and $t_1 > 1$, we get the condition

$$u > \frac{\sqrt{3}}{3}.$$

Then for positive rational number $u > \frac{\sqrt{3}}{3}$, there are infinitely many rational triangle and θ-rational rhombus pairs with a common area and a common perimeter. Therefore, there are infinitely many such Heron triangle and θ-integral rhombus pairs. □

Examples. (1) If $u = 1$, we have a Heron triangle with sides $(8, 15, 17)$, and a θ-integral rhombus with side 10 and the smaller intersection angle $\arcsin \frac{3}{5}$, which have a common area 60 and a common perimeter 40.

(2) If $u = 2$, we have a Heron triangle with sides $(1804, 2040, 1732)$ and a θ-integral rhombus with side 1394 and the smaller intersection angle $\arcsin \frac{528}{697}$, which have a common area 1472064 and a common perimeter 5576.

Proof of Theorem 2. As before, we only need to consider the rational isosceles triangle and θ-rational rhombus pairs. As in [3], we may take the equal legs of the isosceles triangle to have length $u^2 + v^2$, with the base being $2(u^2 - v^2)$ and the altitude $2uv$, for some rational u, v. The area of the isosceles triangle is $2uv(u^2 - v^2)$, with a perimeter of $4u^2$.

Let p be the length of the side of the rhombus, and θ its smallest interior angle. For θ-rational rhombus, we have the perimeter $4p$ and area $p^2 \sin\theta$, where sin $\theta = \frac{2t}{t^2+1}$, for some $t \geq 1$.

If the rational isosceles triangle and θ-rational rhombus have the same area and perimeter, then

$$\begin{cases} 2uv(u^2 - v^2) = p^2 \sin\theta, \\ 4u^2 = 4p. \end{cases} \tag{3}$$

From equation (3), we obtain

$$\frac{2u(v(u-v)(u+v)t^2 - u^3 t + v(u-v)(u+v))}{t^2+1} = 0.$$

We only need to consider $v(u-v)(u+v)t^2 - u^3 t + v(u-v)(u+v) = 0$. If this quadratic equation has rational solutions in t, then its discriminant should be a rational perfect square, i.e.,

$$u^6 - 4u^4v^2 + 8u^2v^4 - 4v^6 = w^2.$$

Let $U = \frac{u}{v}$, $W = \frac{w}{v^3}$. We have

$$W^2 = U^6 - 4U^4 + 8U^2 - 4.$$

This is a hyperelliptic sextic curve of genus 2. The rank of the Jacobian variety is 1, and Magma's Chabauty routines determine the only finite rational points are

$$(U, W) = (\pm 1,\ \pm 1),$$

which lead to

$$(u, w) = (\pm v,\ \pm v^3),$$

then we get

$$v^3 t = 0.$$

So equation (3) does not have nonzero rational solutions. This means that there does not exist any integral isosceles triangle and θ-integral rhombus pair with a common area and a common perimeter. $\square$

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Isogonal Conjugates in an n-Simplex

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Abstract. As "one of the crown jewels of modern geometry", the symmedian point is not well known by mathematicians or mathematics educators. Locating a person in space without a GPS, or analysing data for a harbor mine-detecting algorithm are but some of the more recent applications to the symmedian point. In this article, we extend the notions of the symmedian point and isogonal conjugates to n-simplexes. Unlike another attempt, our generalisation preserves all the interesting properties of the symmedian point of a triangle. Also, its connection to the least squares point of a simplex is shown. In addition, the symmedian point is given in terms of a simple formula in barycentric coordinates.

1. Introduction

The *symmedian point* of a triangle is one of 11809 known points associated with the geometry of a triangle (see [4] where it is the sixth point on the list). One of the most important properties of the symmedian point is that it coincides with the point at which the sum of the squares of the perpendicular distances from the three sides of the triangle is minimum (the *least squares point*, LSP) [1]. For interesting practical applications to the symmedian point, see [5] and [10]. In [5], the symmedian point shows up in connection with locating a person in space without GPS, and in [10] the symmedian point comes in when analysing data from a test of harbor mine-detecting algorithm. Another interesting property of the symmedian point of a triangle is that, as shown in Figure 1 (b), if the triangle $A'B'C'$ is the pedal triangle of K (i.e., the triangle obtained by projecting K onto the sides of the original triangle), then the symmedian point of the triangle ABC and the centroid of the triangle $A'B'C'$ are concurrent.

The symmedian point is defined using the concept of isogonal lines. Two lines AR and AS through the vertex A of an angle are said to be isogonal if they are equally inclined from the sides that form $\angle A$. The lines that are isogonal to the medians of a triangle are called symmedian lines [3], pp. 75-76. Figure 1 (a) shows that the symmedian line through $\overline{AP}$ of the triangle ABC is obtained by reflecting the median $\overline{AM}$ through the corresponding angle bisector $\overline{AL}$. The symmedian lines intersect at a single point K known as the symmedian point, also called the Lemoine point.

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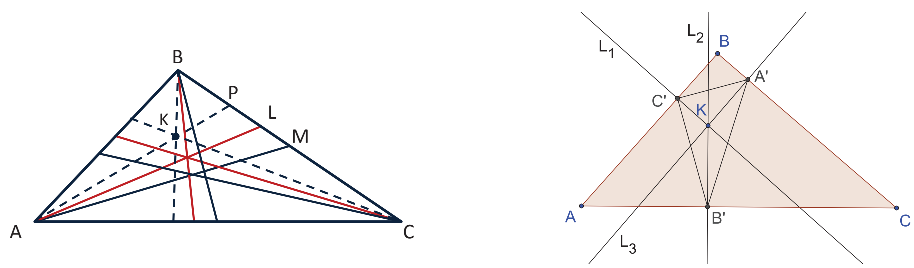
The existence of symmedian point of a triangle was proved by the French mathematician Emile Lemoine in 1873 ([3], Chapter 7). Later the symmedian point was defined by Marr for equiharmonic tetrahedrons in 1919 [7]. In [9], an alternate definition of the symmedian point for an *arbitrary* tetrahedron was given. Under the alternate definition, all the properties of the symmedian point for a triangle generalize naturally to a tetrahedron. In the present work we provide the definition and prove the existence of the symmedian point of an arbitrary n-simplex. Then we show that the symmedian point of a simplex coincides with the LSP of that simplex and the centroid of the corresponding pedal simplex. As it was shown in [6], a *least squares solution* method can be used to give an explicit formula for the location of the least squares point and hence the symmedian point in S_n.

The rest of this paper is organized as follows. In section 2, the existence of symmedian point of an n-simplex is proved. In section 3, it is shown that the symmedian point and LSP of an n-simplex are concurrent. In section 4, the concurrency of the symmedian point and the centroid of the corresponding pedal simplex is proved.



(a) (b)

Figure 1. *(a) The symmedian lines of triangle ABC intersect at the symmedian point K. (b) The symmedian point K of the ABC triangle coincides with the centroid $\widehat{C}$ of its pedal triangle, which is the $A'B'C'$ triangle formed by connecting the intersection points of the perpendicular lines L_1, L_2 and L_3 from K to the sides of the ABC triangle.*

2. Symmedian point of an n-simplex and barycentric coordinates

Let $v_0, v_1, \cdots v_n$ be the vertices of an n-simplex S_n in R^n, $n \geq 2$. Recall that a facet of S_n is an $(n-1)$-simplex formed by $(n-1)$ vertices of S_n. Recall also that the angle between two hyperplanes H_i and H_j, $\angle(H_i, H_j)$, is the supplement of the angle between their unit outer normal vectors. Denote by F_i the *facet* formed by the vertices in $\{v_l : l \neq i\}$. Let X and X^* be two points in S_n. Denote by (H_{ij}) and (H^*_{ij}) the hyperplanes formed by joining X and X^* respectively to v_l, $l \neq i, j$. The $(n-1)$-dimensional hyperplanes (H_{ij}) and (H^*_{ij}) are said to be *isogonal* conjugates if they are equally inclined from the facets F_i and F_j. (H_{ij}) is called the isogonal conjugate of (H^*_{ij}) and vice versa. Notice that by joining the point X to $(n-2)$ vertices of S_n, $\binom{n+1}{n-1} = \frac{n(n+1)}{2}$ different $(n-1)$-dimensional hyperplanes can be formed, and so there are $\frac{n(n+1)}{2}$ corresponding isogonal conjugate hyperplanes. There is no immediately obvious reason why these $\frac{n(n+1)}{2}$ conjugates should be concurrent. However, that this is always the case will follow from Lemma 2 below. Let M_{ij} be the midpoint of $\overline{v_i v_j}$. The $(n-1)$-hyperplane obtained from M_{ij} and $\{v_l : 0 \leq l \leq n, l \neq i, j\}$ will be called a *med-hyperplane* $(M_{ij} \cup H_{ij})$. The isogonal conjugate to a med-hyperplane $(M_{ij} \cup H_{ij})$ with respect to facets F_i and F_j is called a *symmedian hyperplane*. Taking the midpoints of the $\frac{n(n+1)}{2}$ sides of the simplex S_n, the following lemma shows all med-hyperplanes are concurrent at the centroid G_n of S_n [8].

Lemma 1. *All $\frac{n(n+1)}{2}$ med-hyperplanes planes $(M_{ij} \cup H_{ij})$; $0 \leq i, j, \leq n$, $i \neq j$ are concurrent.*

Proof. Let M_{ij} be the midpoint of the edge $\overline{v_i v_j}$ and $(M_{ij} \cup H_{ij})$ the corresponding med-hyperplane. We show that all med-hyperplanes pass through the centroid G_n of the of S_n given by $G_n = \frac{\sum_0^n v_l}{n+1}$ (see Figure 2 (a)). If $0 \leq k \leq n$, then G_n is located on the segment from an arbitrary vertex v_k to the centroid of the $(n-1)$-simplex $S_{n-1}^{(k)}$ obtained from the vertices in $\{v_l : 0 \leq l \leq n, l \neq k\}$. In fact, let $G_{n-1}^{(k)}$ be the centroid of $S_{n-1}^{(k)}$. Then

$$G_n = \frac{v_0 + \cdots + v_n}{n+1} \ = \ \frac{nG_{n-1}^{(k)}}{n+1} + \frac{v_k}{n+1} \ = \ \alpha G_{n-1}^{(k)} + \beta v_k,$$

where $\alpha + \beta = 1$. Thus G_n is the on the segment $\overline{G_{n-1}^{(k)} v_k}$. Since each med-hyperplane contains $\overline{G_{n-1}^{(k)} v_k}$ for some k, the result follows.$\square$

Lemma 2. *Consider the n-simplex S_n. Let $X \neq v_i, v_j$ be a point on the hyperplane (H_{ij}) formed by joining X to the vertices in $V_{ij} = \{v_l : 0 \leq l \leq n, l \neq i, j\}$. Let X^* be a point on the isogonal conjugate hyperplane (H_{ij}^*).*

(i): *If $\overline{XP_1}, \overline{XP_2}, \overline{X^*P_2^*}, \overline{X^*P_2^*}$, are the perpendiculars from X and X^* onto the facets F_j and F_i, respectively, then*

$$\frac{XP_1}{XP_2} \ = \ \frac{X^*P_2^*}{X^*P_1^*} \tag{1}$$

(ii): *If X is in (H_{ij}) and $\frac{XP_1}{XP_2} = \frac{X^*P_2^*}{X^*P_1^*}$, then X^* is in (H_{ij}^*), where (H_{ij}) and (H_{ij}^*) are isogonal conjugates.*

Proof. (i) We will show the triangles XP_1P_2 and $X^*P_1^*P_2^*$ are similar (see Figure 2 (b)). Let P_3 be the orthogonal projection of P_1 onto the $(n-2)$-hyperplane V_{ij} formed by the vertices $\{v_l : 0 \leq l \leq n, l \neq i, j\}$. Since $\overline{XP_1}$ is orthogonal to F_j, $\overline{XP_1}$ is orthogonal to V_{ij} which is contained in F_j. Thus, the plane formed by the points X, P_1, P_2 is orthogonal to V_{ij}. Similarly, the plane formed by X, P_2, P_3 is orthogonal to V_{ij}. This implies that the points X, P_1, P_2, P_3 are in the same two-dimensional plane. Furthermore, since $\overline{XP_3}$ is orthogonal to V_{ij}

and $\overline{XP_1}$ is orthogonal to F_j, $\angle(H_{ij}, F_j) = \angle(\overline{P_1P_3}, \overline{XP_3}) = \angle(\overline{P_1P_2}, \overline{XP_2})$ because the quadrilateral $P_2XP_1P_3$ is cyclic (its vertices are on the same circle). A similar argument yields $\angle(H^*_{ij}, F_i) = \angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_3})$, where P^*_3 is the projection of P^*_1 onto V_{ij}. Since (H_{ij}) and (H^*_{ij}) are isogonal conjugates, $\angle(H^*_{ij}, F_i) = \angle(H_{ij}, F_j)$, and so $\angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_2}) = \angle(\overline{XP_2}, \overline{P_1P_2})$. The similarity of the triangles now follows.

(ii) Since $\frac{XP_1}{XP_2} = \frac{X^*P_2*}{X^*P_1*}$ and $\angle(\overline{XP_1}, \overline{XP_2}) = \angle(\overline{X^*P_1}, \overline{X^*P^*_2})$, the triangles XP_1P_2 and $X^*P_1 * P^*_2$ are similar. It follows that $\angle(F_j, H_{ij}) = \angle(\overline{P_1P_3}, \overline{P_3X}) = \angle(\overline{P_1P_2}, \overline{P_2X}) = \angle(\overline{X^*P^*_1}, \overline{P^*_1P^*_2}) = \angle(\overline{X^*P^*_3}, \overline{P_3P^*_2}) = \angle(F_i, H^*_{ij})$, and so X^* is in the isogonal conjugate of (H_{ij}) $\square$

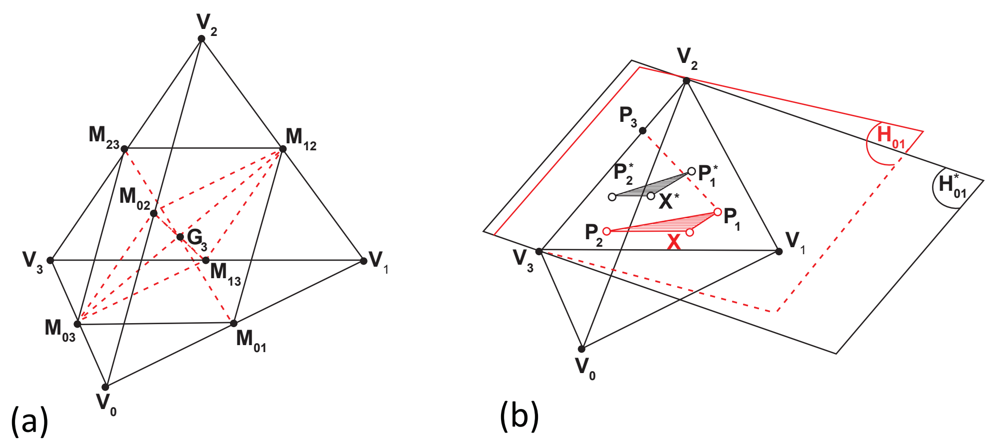


Figure 2. *(a) All six median planes obtained from a side of the 3-simplex $v_0v_1v_2v_3$ and the midpoint M_{ij} of its opposite side are concurrent. (b) A representation of the isogonal planes (H_{12}) and (H^*_{12}), and the perpendicular lines from X and X^* to the facets $v_1v_2v_3$ and $v_0v_2v_3$, respectively.*

Now we are ready to show the existence of the symmedian point of an n-simplex.

Theorem 3. *Let (H_{ij}) be the $\frac{n(n+1)}{2}$ $(n-1)$-hyperplanes constructed by joining a point X inside the simplex S_n to the vertices in $V_{ij} = \{v_l : 0 \le l \le n, l \ne i, j\}$. Then the corresponding $\frac{n(n+1)}{2}$ isogonal conjugate hyperplanes (H^*_{ij}) intersect at a point X^* inside S_n.*

Proof. Let X^* be the intersection of n isogonal conjugate hyperplanes $(H^*_{i_lj_l})$, $0 \le l \le n-1$, $i_l \ne j_l$. Let $(H_{i_kj_k})$, $k \notin \{0, 1, \cdots, n-1\}$ be a hyperplane passing through X. Let $(H^*_{i_kj_k})$ be the isogonal conjugate to $(H_{i_kj_k})$ with respect to some facets F^*_k and F_k, say. It suffices to show that $X^* \in (H^*_{i_kj_k})$. Let P_1 and P_2 be the orthogonal projections from X to F_k and F^*_k, and let P^*_1 and P^*_2 be the orthogonal projections from X^* to F_k and F_k^*, respectively. In view of Lemma 2 (ii), we need only show $\frac{X^*P_1^*}{X^*P_2^*} = \frac{XP_2}{XP_1}$. In fact, we have

$$\frac{X^*P_1^*}{X^*Q^*} = \frac{XQ}{XP_1}$$

and

$$\frac{X^*Q^*}{X^*P_2^*} = \frac{XP_2}{XQ},$$

where Q and Q^* are the orthogonal projections from X, and X^* onto a facet F_q, $q \leq l \leq n-1$. Multiplying the two left-hand sides and the two right-hand sides of the equalities yields

$$\begin{aligned} \frac{X^*P_1^*}{X^*P_2^*} &= \frac{X^*P_1^*}{X^*Q^*} \times \frac{X^*Q^*}{X^*P_2^*} \\ &= \frac{XQ}{XP_1} \times \frac{XP_2}{XQ} \\ &= \frac{XP_2}{XP_1}. \end{aligned}$$

The desired now follows. Similar argument shows that all the other isogonal conjugate hyperplanes pass through X^*.□
In view of Lemmas 1 and 2, the following definition becomes well-defined.

Definition. The symmedian point of an n-simplex is the isogonal conjugate of the intersection point of all its med-hyperplanes.

Now we briefly explore the relationship between *the barycentric coordinates* of a point X and its isogonal conjugate X^*. Recall that in general, if $x_0, \cdots, x_n$ are the vertices of a simplex S_n in an affine space $\mathcal{A}$ and if $(a_0 + \cdots + a_n)X = a_0x_0 + \cdots + a_nx_n$ and at least one of the $a_i^{,}s$ does not vanish, then we say that the coefficients $(a_0 : \cdots : a_n)$ are *barycentric coordinates* of X, where $X \in \mathcal{A}$ [11]. The barycentric coordinates are homogeneous:

$$(a_0, \cdots, a_n) = (\mu a_0 : \cdots : \mu a_n) \quad \mu \neq 0.$$

Analogous to the triangle case [2], and to the tetrahedron case [9], we have the following property for an n-simplex. Let X be a point in the nth dimensional Euclidean space E^n. Joining X to any n vertices, $n+1$ n-simplexes (H_i), $0 \leq i \leq n$ can be constructed. Let $X = (a_0 : a_2 : \cdots : a_n)$ and $X' = (a_0' : a_2' : \cdots : a_n')$ be the barycentric coordinates of X and X^*, respectively, with respect to S_n. Since *the volumes of these simplexes are proportional to the barycentric coordinates of* X, using lemma 2, we obtain

$$\frac{a_0'a_0}{(\delta H_0)^2} = \frac{a_1'a_1}{(\delta H_1)^2} = \cdots = \frac{a_n'a_n}{(\delta H_n)^2} = \mu,$$

where δH_i denote the volume of H_i (its content). It follows that

$$\begin{aligned} X' &= (a'_0 : a'_1 : \cdots : a'_n) \\ &= \left(\mu\frac{(\delta H_0)^2}{a_0} : \mu\frac{(\delta H_1)^2}{a_1} : \cdots : \mu\frac{(\delta H_n)^2}{a_n}\right) \\ &= \left(\frac{(\delta H_0)^2}{a_0} : \frac{(\delta H_1)^2}{a_1} : \cdots : \frac{(\delta H_n)^2}{a_n}\right). \end{aligned} \tag{2}$$

(2) gives an extension of isogonal conjugates to simplexes with a simple formula in barycentric coordinates. Applying (2) to the centroid $(1 : 1 : \cdots : 1)$, we obtain the symmedian point $\Big((\delta H_0)^2 : (\delta H_1)^2 : \cdots : (\delta H_n)^2\Big)$.

3. Concurrency of the Symmedian Point and the Least Squares Point

The least squares point LSP of a given simplex S_n is the point from which the sum of the squares of the perpendicular distances to the facets of S_n is minimized. We show that the symmedian point and the LSP of S_n are concurrent. We start with the following lemma.

Lemma 4. *For an n-simplex S_n, let M_{ij} be the midpoint of $\overline{v_iv_j}$ and $\overline{M_{ij}P}$ and $\overline{M_{ij}Q}$ be the perpendicular line segments from M_{ij} to F_i and F_j, respectively. Then*

$$\frac{M_{ij}Q}{M_{ij}P} = \frac{\delta(F_i)}{\delta(F_j)}, \tag{3}$$

where $\delta(F_k)$ represents the volume (or content) of F_k.

Proof. It is known (and easy to verify using a simple integral) that the volume of S_n is $\frac{1}{n!}$ times the"heights" of vertices, taken in any linear sequence, above the subspace containing the previous vertices [12]. Precisely, the volume $\delta(S_n)$ is given by

$$\delta(S_n) = \frac{1}{n!}h_n h_{n-1}\cdots h_1,$$

where h_1 is simply the distance between the first two vertices, h_2 is the height of the third vertex above the line containing those two vertices, h_3 is the height of the fourth vertex above the plane containing the first three vertices, and so on. Thus $\delta(M_{ij} \cup F_j) = \delta(M_{ij} \cup F_i)$, where $M_{ij} \cup F_k$ is the n-simplex constructed by joining M_{ij} to the vertices of F_k. Let $\overline{v_kH}$ be the perpendicular from v_k to F_k, $k \neq i, j$. Then

$$\begin{aligned} \frac{1}{n}M_{ij}Q \times \delta(F_j) &= \delta(M_{ij} \cup F_j) \\ &= \delta(M_{ij} \cup F_i) \\ &= \frac{1}{n}M_{ij}P \times \delta(F_i). \end{aligned}$$

This implies (3). $\square$

We're ready to prove the concurrency of the symmedian point and the LSP of S_n.

Theorem 5. *The symmedian point K of S_n coincides with its LSP.*

Proof. First, Lemma 2 (i) together with Lemma 4 imply

$$\frac{x_0}{\delta(F_0)} = \frac{x_1}{\delta(F_1)} = \cdots = \frac{x_n}{\delta(F_n)}, \tag{4}$$

where x_i is the distance from the symmedian point to F_i. If we denote $\delta(F_i)$ by d_i, then Lagrange's identity yields

$$\left(\sum_{i=0}^{n} x_i^2\right)\left(\sum_{i=0}^{n} d_i^2\right) - \left(\sum_{i=0}^{n} x_i d_i\right)^2 = \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} (d_i x_j - d_j x_i)^2. \tag{5}$$

Since the $d_i^{,}s$ are constant, and $\sum_{i=0}^{n} x_i d_i = n\delta(S_n)$, $\sum_{i=0}^{n} x_i^2$ is minimum if and only if the right hand side of (5) is zero. This occurs only when $x_i d_j = x_j d_i$. In view of (4), this occurs at the symmedian point K, and so the symmedian point coincides with the LSP. □

4. Concurrency of the Symmedian Point and the Centroid of the Corresponding Pedal n-simplex

In this section we show that the symmedian point of an n-simplex S_n coincides with the centroid of the corresponding pedal n-simplex Π_n.

Theorem 6. *The symmedian point of an n-simplex S_n coincides with the centroid of the corresponding pedal n-simplex Π_n.*

Proof. Let K be the symmedian point of S_n. Let the orthogonal projection from K onto a facet F_i intersect F_i at π_i. Let $\widehat{C}$ be the centroid of the pedal simplex Π_n of K. It is well known that $\widehat{C}$ minimizes the sum of the squares of the distances to the vertices $\pi_0, \pi_1, \cdots \pi_n$ of Π_n. Thus,

$$\sum_{i=0}^{n} (\widehat{C}\pi_i)^2 \le \sum_{i=0}^{n} (X\pi_i)^2 \text{ for any } X \in R^n. \tag{6}$$

Suppose $\widehat{C} \neq K$. Let the orthogonal projections from $\widehat{C}$ to the facet F_i of S_n be c_i. Since K is also the LSP of S_n

$$\sum_{i=0}^{n} (K\pi_i)^2 < \sum_{i=0}^{n} (\widehat{C}c_i)^2. \tag{7}$$

Since c_i is the orthogonal projection from $\widehat{C}$ ontoF_i, $\widehat{C}c_i \le \widehat{C}\pi_i$ for each i. So using (6) with $X = K$, we have

$$\sum_{i=0}^{n} (\widehat{C}c_i)^2 \le \sum_{i=0}^{n} (\widehat{C}\pi_i)^2 \le \sum_{i=0}^{n} (K\pi_i)^2,$$

which contradicts (7). So we must have $\widehat{C} = K$. □

Corollary 7. *The symmedian point and hence the LSP of S_n belongs to its interior.*

Summary. The symmedian point of S_n as defined above is a generalization of the symmedian point for a tetrahedron defined in [9]. This definition differs from the definition of the symmedian point of a tetrahedron given by Marr [7]. Marr's symmedian point of an equiharmonic tetrahedron $ABCD$ (that is, $AD \times BC = AB \times CD = AC \times BD$) is defined as the point of intersection of the lines joining the vertices to the symmedian points of the opposite faces. In [9], an example was given to show that the two symmedian points are in fact different. Our generalization, however, has the advantage of being the same as the least squares point of S_n, as is the case in any triangle. In addition, using a least squares solution technique, the location of the LSP, and hence the symmedian point, of S_n can be given explicitly [6].

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Classical Right-Angled Triangles and the Golden Ratio

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Abstract. In this article, we consider the family of classical right-angle triangles in 2-dimensional Euclidean space. We consider triangle with an arbitrary leg ratio k and show that at $k = 2^p$, where $p = \pm 1$, the area of all built-in triangles is linked to each other by the golden ratio φ. Keeping $k = 2^p$, we address changes in above triangles occurring at the planar similarity transformation, prove an invariancy of the area ratio between predecessor and successor triangle and show that evolution curve is a logarithmic spiral. Reason of such geometrical features is associated with the unique nature of φ providing parity between the linear and non-linear properties of geometry objects.

1. Theorem on area ratio

In 2-dimensional Euclidean space, consider arbitrary right-angled $\triangle ABC$ such that ratio $AC/BC = k$, where k is any rational number (Figure 1).

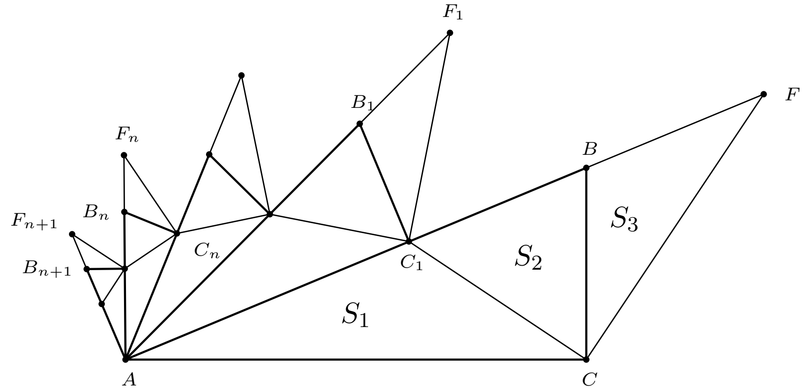

Figure 1. Family of triangles at planar similarity transformation. [1]

Mark off distance from the vertex B as $BC_1 = BC$ and $BF = BC$. Since $AC = k \cdot BC$, $AB = BC\sqrt{k^2+1}$, $AC_1 = AB - BC_1 = BC(\sqrt{k^2+1} - 1)$ and $AF = AB + BC = BC(\sqrt{k^2+1} + 1)$. Denote by S_1 the area of $\triangle AC_1C$, S_2 the area of $\triangle CC_1B$, S_3 the area of $\triangle BCF$, $S_1 + S_2 + S_3 = S$, the golden ratio [1] as φ, and

$$f_k = \frac{AF}{AC} = \frac{AC}{AC_1} = \frac{\sqrt{k^2+1}+1}{k}. \tag{1}$$

Then,

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[1]In Figure 1, successive changes in family of the right-angle triangles at planar similarity transformation are shown. The ordinate is original vertical edge in triangle ABC, while the abscissa is original horizontal edge in triangle ABC. Plot represents family of the similar right-angled triangles in 2-dimensional Euclidean space keeping the leg's ratio $k = 2$.

Theorem 1. *If in right-angled triangle ABC with leg ratio $\frac{AC}{BC} = 2^p$, where $p = \pm 1$, to mark off from the vertex B along hypotenuse AB the line segments BC_1 and BF with the length equal to length of BC, then the ratio*

$$\left(\frac{S}{S_2+S_3}\right)^p = \left(\frac{S_2+S_3}{S_1}\right)^p = \left(\frac{AF}{AC}\right)^p = \left(\frac{AC}{AC_1}\right)^p = \varphi^p. \tag{2}$$

First, we prove some lemmas.

Lemma 2. *The areas*

$$S_2 = S_3 = \frac{BC^2}{2}\sqrt{1-\frac{1}{k^2+1}}. \tag{3}$$

Proof. Since $\cos ABC = -\cos CBF = \frac{1}{\sqrt{k^2+1}}$, for $\triangle CC_1B$, using the law of cosines $C1C^2 = 2BC^2\left(1-\frac{1}{\sqrt{k^2+1}}\right)$. In the same way, in $\triangle BCF$, $CF^2 = 2BC^2\left(1+\frac{1}{\sqrt{k^2+1}}\right)$. Then, employing Heron's formula for area [2], we obtain (3). □

Lemma 3. *Area*

$$S_1 = \frac{BC^2}{2}\left(k-\sqrt{1-\frac{1}{k^2+1}}\right). \tag{4}$$

Proof. The area of $\triangle ABC = k\cdot BC^2/2$. From (3), we obtain (4). □

Lemma 4. *If $k=2$, the ratio*

$$\frac{S_2+S_3}{S_1} = \varphi. \tag{5}$$

Proof. From (3) and (4),

$$\begin{aligned}\frac{S_2+S_3}{S_1} &= 2\sqrt{1-\frac{1}{k^2+1}}\left(k-\sqrt{1-\frac{1}{k^2+1}}\right)^{-1}\\ &= \frac{1}{\sqrt{k^2+1}-1} = \frac{2}{k}\cdot\frac{\sqrt{k^2+1}+1}{k} = \frac{2}{k}\cdot f_k.\end{aligned} \tag{6}$$

So, at $k=2$, using (1), $\frac{S_2+S_3}{S_1} = f_k = \frac{\sqrt{5}+1}{2} = \varphi$. □

Lemma 5. *If $k=2$, the ratio*

$$\frac{S}{S_2+S_3} = \varphi. \tag{7}$$

Proof. Modify (7) as

$$\frac{S}{S_2+S_3} = 1+\frac{S_1}{S_2+S_3} = f_k. \tag{8}$$

From (5),

$$\frac{S_1}{S_2+S_3} = \frac{1}{f_k}\cdot\frac{k}{2}.$$

Then (8) becomes

$$1 + \frac{1}{f_k} \cdot \frac{k}{2} = f_k, \tag{9}$$

and at $k = 2$, the positive root of (9) is $f_k = \varphi$.

□

Obviously, all reasoning in Lemmas 4 and 5 is valid for $p = -1$ if we replace $k = 2$ with $k = \frac{1}{2}$, and rename the vertices A and B. Now, we can prove Theorem 1.

Proof of Theorem 1

In $\triangle ABF$, from (1), (5), and (7), immediately follows (2).

Corollary 6.

$$(SS_1)^p = (S_2 + S_3)^{2p}. \tag{10}$$

2. Evolution theorem

For the rest of this article, we will use $k = 2^p$. So, in $\triangle AB_1C_1$, from (2), $AC_1 = \frac{AC}{\varphi^p}$. From the point C_1 raise a perpendicular to AB, then on this perpendicular mark off the line segment B_1C_1 equal to $\frac{BC}{\varphi}$ and connect the points B_1 and A. Since $AB_1 = \frac{AB}{\varphi^p}$, triangle AB_1C_1 is similar to ABC as having three proportional edges and, obviously, the area of $\triangle AB_1C_1$ is $\frac{S}{\varphi^{2p}}$. Continuing this way, at an arbitrary step n, $n > 1$, in triangle AB_nC_n, $\frac{AC}{AC_n} = \varphi^{pn}$, and the area S_n of $\triangle AB_nC_n$ is $\frac{S}{\varphi^{2pn}}$. Further, we will call similar triangles AF_nC_n as the full ones with area S_n^F, $\triangle AC_{n+1}C_n$ as the core ones with area S_n^C in contrast to the transfer triangles, $C_nC_{n+1}B_n$ and $C_nF_nB_n$ with area S_n^T.

Theorem 7. *The full area S_{n+1}^F (S_n^F) of the first immediate successor (precursor) in the chain of similar triangles is exactly equal to the area S_n^C (S_{n+1}^C) of the core triangle in its immediate precursor (successor), i.e.*

$$S_{n+1}^F = S_n^C, \quad (S_n^F = S_{n+1}^C). \tag{11}$$

Proof. For the case $\partial S_n/\partial S < 0$, where S_n is the area of triangle AB_nC_n, n is the number of evolution steps.

$$S_{n+1}^F = \frac{S_n^F}{\varphi^2}. \tag{12}$$

From (7),

$$\frac{(S_n^F)^2}{4(S_n^T)^2} = \varphi^2. \tag{13}$$

Insert (13) into (12) and apply (10) for an arbitrary n, which is $4(S_n^T)^2 = S_n^F S_n^C$, then

$$S_{n+1}^F = 4\frac{S_n^F (S_n^T)^2}{(S_n^F)^2} = 4\frac{(S_n^T)^2}{S_n^F} = \frac{S_n^F S_n^C}{S_n^F} = S_n^C.$$

□

For the case $\partial S_n/\partial n > 0$, using the same logic as above,

$$S^C_{n+1} = \varphi^2 S^C_n = S^C_n \frac{(S^F_n)^2}{4(S^T_n)^2} = S^C_n \frac{(S^F_n)^2}{S^F_n S^C_n} = S^F_n.$$

Theorem 8. *At $\partial S_n/\partial n < 0$, the full area S^F_{n+1} of the first immediate successor in the chain of similar triangles is exactly equal to the area S^F_n of its immediate precursor decremented by the double area S^T_n of transfer triangle, i.e.*

$$S^F_{n+1} = S^F_n - 2S^T_n, \tag{14a}$$

while at $\partial S_n/\partial S > 0$, incremented by the double area S^T_n of transfer triangle multiplied by $\varphi + 1$, i.e.

$$S^F_{n+1} = S^F_n + 2(\varphi + 1)S^T_n. \tag{14b}$$

Proof. For the case $\partial S_n/\partial n < 0$, obviously,

$$S^C_n = S^F_n - 2S^T_n. \tag{15}$$

Then, by Theorem 7, replace in (15) S^C_n with S^F_{n+1}, and obtain (14a).

For the case $\partial S_n/\partial S > 0$, $S^F_{n+1} = S^C_{n+1} + 2S^T_{n+1}$. Since $S^C_{n+1} = S^F_n$ and $S^T_{n+1} = \varphi^2 S^T_n = (\varphi + 1)S^T_n$, we obtain (14b). □

3. Transformation curve

Theorem 9. *At planar similarity transformation, the plane curve dealing the path for the triangle's edge in the chain of similar triangles is a logarithmic spiral.*

Proof. In the case $\partial S_n/\partial n < 0$, the decrement $\Delta AC = \frac{AC}{\varphi} - AC = AC\left(\frac{1}{\varphi} - 1\right)$ occurred at the turn of AC to $\Delta\theta_n = \angle C_n AB_n = \arctan\frac{1}{k}$ (see Figure 1). Going to the differential values $d(AC) = AC\left(\frac{1}{\varphi} - 1\right)d\theta$, we have

$$AC = a \cdot \exp\left(\left(\frac{1}{\varphi} - 1\right)b\theta\right),$$

which is the equation of folding logarithmic spiral with the growth factor $b\left(\frac{1}{\varphi} - 1\right)$, $b = \frac{2\pi}{\arctan\frac{1}{2}}$, and $a = AC_0$.

For the case $\partial S_n/\partial n > 0$, $\Delta AC = AC(\varphi - 1)$ which yields the unfolding logarithmic spiral with the growth factor $b(\varphi - 1)$:

$$AC = a \cdot \exp\left((\varphi - 1)b\theta\right) = a \cdot \exp\left(\frac{b}{\varphi}\theta\right).$$

□

4. Conclusions and outlook

We have shown that in the right-angle triangle with leg's ratio $k = 2^p$, where $p = \pm 1$, the ratio of the edges and areas of built-in triangles is obeyed to the golden ratio relation. At planar similarity transformation at each step, the full area of the successor (predecessor) triangle is exactly equal to the area of the built-in core triangle of its predecessor (successor). It assumes that triangle's transformation takes such the way to preserve the key parameters of predecessor triangle irrelevant to the sign of $\frac{\partial S_n}{\partial n}$, i.e. the ratio $\frac{S^F_{n+1}}{S^C_n}\left(\frac{S^F_n}{S^C_{n+1}}\right) = 1$ and it is invariant under above transformation.

Saying about the directions for future investigations we note that it would be interesting to consider applicability of above results for the three and higher dimension case. So, we look forward to reporting this in our future work.

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The First Sharp Gyrotriangle Inequality in Möbius Gyrovector Space $(\mathbb{D}, \oplus, \otimes)$

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Abstract. In this paper we present two different inequalities (with their reverse inequalities) written in a common type by using classical vector addition "$+$" and classical multiplication "$\cdot$" in Euclidean geometry and by using Möbius addition "$\oplus$" and Möbius scalar multiplication "$\otimes$" in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$. It is known that this Möbius gyrovector space form the algebraic setting for the Poincaré disc model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

1. Introduction

There are many fundamental inequalities in mathematics and one of them is the famous "Triangle Inequality". In [7], Kato, Saito and Tamura presented the following "Sharp Triangle Inequality"in a Banach Space as follows:

Theorem 1. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,*

$$\begin{aligned}
&\left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\leq j\leq n} \|x_j\| \\
&\leq \sum_{j=1}^{n} \|x_j\| \\
&\leq \left\|\sum_{j=1}^{n} x_j\right\| + \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\leq j\leq n} \|x_j\|
\end{aligned}$$

holds.

Recently Mitani, Kato, Saito and Tamura [8] presented a new type Sharp Triangle Inequality as follows:

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Theorem 2. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X, $n \geq 2$ —small*

$$\|\sum_{j=1}^{n} x_j\| + \sum_{k=2}^{n}\left(k - \left\|\sum_{j=1}^{k}\frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_k^*\| - \|x_{k+1}^*\|)$$
$$\leq \sum_{j=1}^{n}\|x_j\|$$
$$\leq \|\sum_{j=1}^{n} x_j\| - \sum_{k=2}^{n}\left(k - \left\|\sum_{j=n-(k-1)}^{n}\frac{x_j^*}{\|x_j^*\|}\right\|\right)(\|x_{n-k}^*\| - \|x_{n-(k-1)}^*\|)$$

where $x_1^*, x_2^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$ and $x_0^* = x_{n+1}^* = 0$.

2. A sharp triangle inequality in Banach space

Theorem 3. *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,*

$$\begin{aligned}&\frac{\min_{1\leq j\leq n}\|x_j\|}{\max_{1\leq j\leq n}\|x_j\|}\left(\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|\right) + \left\|\sum_{j=1}^{n}x_j\right\| \\ &\leq \sum_{j=1}^{n}\|x_j\| \\ &\leq \frac{\max_{1\leq j\leq n}\|x_j\|}{\min_{1\leq j\leq n}\|x_j\|}\left(\sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|\right) + \left\|\sum_{j=1}^{n}x_j\right\|\end{aligned} \tag{1}$$

holds.

Proof. Let us assume

$$\|x_r\| := \min\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}$$

and

$$\|x_s\| := \max\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}.$$

Clearly,

$$\frac{\|x_r\|}{\|x_s\|} \leq 1 \leq \frac{\|x_s\|}{\|x_r\|}$$

holds true. Using the generalized triangle inequality in X, we get

$$0 \leq \sum_{j=1}^{n}\|x_j\| - \|\sum_{j=1}^{n}x_j\|.$$

Therefore, we immediately obtain

$$\begin{aligned}
&\frac{\|x_r\|}{\|x_s\|}\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) + \left\|\sum_{j=1}^{n}x_j\right\| \\
&\leq \sum_{j=1}^{n}\|x_j\| \\
&\leq \frac{\|x_s\|}{\|x_r\|}\left(\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|\right) + \left\|\sum_{j=1}^{n}x_j\right\|.
\end{aligned}$$

□

3. Gyrogroups and gyrovector spaces

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

in the complex plane $\mathbb{C}$ is given by the polar decomposition [6]

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}\left(z_0 \oplus z\right).$$

It induces the Möbius addition "$\oplus$" in the disc, allowing the Möbius transformation of the disc to be viewed as Möbius *left gyrotranslation*

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by rotation. Here $\theta \in \mathbb{R}$, $z_0 \in \mathbb{D}$ and Möbius substraction "$\ominus$" is defined by $a \ominus z = a \oplus (-z)$. Clearly $z \ominus z = 0$ and $\ominus z = -z$. The groupoid $(\mathbb{D},\oplus)$ is not a group since it is neither commutative nor associative, but it has a group-like structure. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of gyration,

$$gyr : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D},\oplus)$$

given by the equation

$$gyr\,[a,b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \tag{2}$$

where $Aut(\mathbb{D},\oplus)$ is the automorphism group of the groupoid $(\mathbb{D},\oplus)$. Therefore, the *gyrocommutative law* of Möbius addition $\oplus$ follows from the definition of gyration in (2),

$$a \oplus b = gyr\,[a,b]\,(b \oplus a)\,. \tag{3}$$

Coincidentally, the gyration $gyr[a,b]$ that repairs the breakdown of the commutative law of $\oplus$ in (3), repairs the breakdown of the associative law of $\oplus$ as well, giving rise to the respective *left* and *right gyroassociative laws*

$$\begin{aligned}
a \oplus (b \oplus c) &= (a \oplus b) \oplus gyr\,[a,b]\,c \\
(a \oplus b) \oplus c &= a \oplus (b \oplus gyr\,[b,a]\,c)
\end{aligned}$$

for all $a, b, c \in \mathbb{D}$.

Definition 1. *A groupoid* $(\mathbb{G},\oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms*
(G1) For each $a\in G$*, there is an element* $0\in G$ *such that* $0\oplus a=a$.
(G2) For each $a\in G$*, there is an element* $b\in G$ *such that* $b\oplus a=0$.
(G3) For all $a,b\in G$*, there exists a unique element* $gyr[a,b]c\in G$ *such that*

$$a\oplus(b\oplus c)=(a\oplus b)\oplus gyr[a,b]c$$

(G4) For all $a,b\in G$*,* $gyr\,[a,b]\in Aut(\mathbb{G},\oplus)$ *where* $Aut(\mathbb{G},\oplus)$ *is automorphism group.*
(G5) For all $a,b\in G$*,* $gyr\,[a,b]=gyr\,[a\oplus b,b]$.

Definition 2. *A gyrogroup* $(\mathbb{G},\oplus)$ *is gyrocommutative if its binary operation obeys the* gyrocommutative law
(G6) $a\oplus b=gyr\,[a,b]\,(b\oplus a)$ *for all* $a,b\in G$.

Clearly, with these properties, one can now readily check that the Möbius complex disc groupoid $(\mathbb{D},\oplus)$ is a gyrocommutative gyrogroup. For more details, we refer [1]-[4].

Now define the secondary binary operation $\boxplus$ in $\mathbb{G}$ by

$$a\boxplus b=a\oplus gyr\,[a,\ominus b]\,b.$$

The primary and secondary operations of $\mathbb{G}$ are collectively called the dual operations of gyrogroups. Let a,b be two elements of a gyrogroup $(\mathbb{G},\oplus)$. Then the unique solution of the equation

$$a\oplus x=b$$

for the unknown x is

$$x=\ominus a\oplus b$$

and the unique solution of the equation

$$x\oplus a=b$$

for the unknown x is

$$\begin{aligned} x &= b\boxminus a \\ &= b\boxplus(\ominus a). \end{aligned}$$

For further details see [3].

Identifying complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way:

$$\mathbb{C}\ni u=u_1+iu_2=(u_1,u_2)=\mathbf{u}\in\mathbb{R}^2.$$

Then the equations

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathrm{Re}(\overline{u}v) \\ \|\mathbf{u}\| &= |u|\,. \end{aligned} \tag{4}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc $\mathbb{R}^2_1=\left\{\mathbf{v}\in\mathbb{R}^2:\|\mathbf{v}\|<1\right\}$ of $\mathbb{R}^2$. In fact we get from that (4)

$$\begin{aligned} u \oplus v &= \frac{u+v}{1+\overline{u}v} \\ &= \frac{(1+u\overline{v})(u+v)}{(1+\overline{u}v)(1+u\overline{v})} \\ &= \frac{\left(1+\overline{u}v+u\overline{v}+|v|^2\right)u+\left(1-|u|^2\right)v}{1+\overline{u}v+u\overline{v}+|u|^2|v|^2} \qquad (5) \\ &= \frac{\left(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\|\mathbf{u}\|^2\right)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \mathbf{u}\oplus\mathbf{v} \end{aligned}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_1^2$.

Let $\mathbb{V}$ be any inner-product space and

$$\mathbb{V}_s = \{v \in \mathbb{V} : \|v\| < s\}$$

be the open ball of $\mathbb{V}$ with radius $s > 0$. Möbius addition in $\mathbb{V}_s$ is motivated by (5). It is given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{\left(1+\left(2/s^2\right)\mathbf{u}\cdot\mathbf{v}+\left(1/s^2\right)\|\mathbf{v}\|^2\right)\mathbf{u}+\left(1-\left(1/s^2\right)\|\mathbf{u}\|^2\right)\mathbf{v}}{1+(2/s^2)\,\mathbf{u}\cdot\mathbf{v}+(1/s^4)\,\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \qquad (6)$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$. Without loss of generality, we may assume that $s = 1$ in (6). However we prefer to keep s as a free positive parameter in order to exhibit the results that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands the whole of its real inner product space $\mathbb{V}$, and Möbius addition $\oplus$ reduces to vector addition $+$ in $\mathbb{V}$, i.e.,

$$\lim_{s\longrightarrow\infty} \mathbf{u}\oplus\mathbf{v} = \mathbf{u}+\mathbf{v}$$

and

$$\lim_{s\to\infty} \mathbb{V}_s = \mathbb{V}.$$

Möbius scalar multiplication "$\otimes$" is given by the equation

$$r\otimes\mathbf{v} = s\tanh\left(r\tanh^{-1}\left(\|\mathbf{v}\|/s\right)\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad (7)$$

where $r \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, $\mathbf{v} \neq 0$ and $r \otimes 0 = 0$.

Definition 3 (Real inner product gyrovector spaces)**.** *A real inner product space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:*
(1) G is a subset of a real inner product space $\mathbb{V}$ called the carrier of G, $G \subset \mathbb{V}$,

from which it inherits its inner product, "$\cdot$" and norm "$\|\cdot\|$" which are invariant under gyroautomorphisms, that is,

$$gyr[u,v]a \cdot gyr[u,v]b = a \cdot b$$

for all points $a, b, u, v \in G$.
(2) G admits a scalar multiplication $\otimes$, possessing the following properties. For all real numbers r, r_1, r_2 and all points $a \in G$.
(III.1) $1 \otimes a = a$
(III.2) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$
(III.3) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$
(III.4) $\frac{|r|\otimes a}{\|r\otimes a\|} = \frac{a}{\|a\|}$
(III.5) $gyr[u,v](r \otimes a) = r \otimes gyr[u,v]a$
(III.6) $gyr[r_1 \otimes v, r_2 \otimes v] = I$
(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one dimensional vectors

$$\|G\| = \{\pm\|a\| : a \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$, $a, b \in G$.
(III.7) $\|r \otimes a\| = |r| \otimes \|a\|$
(III.8) $\|a \oplus b\| \leq \|a\| \oplus \|b\|$.

Clearly, Möbius scalar multiplication possesses the properties above. For the proof of (III.8) in the complex unit disc $\mathbb{D}$ we refer [5].

Theorem 4. *A Möbius gyrogroup $(\mathbb{V}_s, \oplus)$ with Möbius scalar multiplication $\otimes$ in (7) forms a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, see* [1].

Definition 4. *The gyrodistance function in $(\mathbb{V}_s, \oplus, \otimes)$ is given by*

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{B} \ominus \mathbf{A}\|$$

for $\mathbf{A}, \mathbf{B} \in \mathbb{V}_s$, see [1].

The gyrodistance function in hyperbolic geometry gives rise to a gyrotriangle inequality which involves a gyroaddition $\oplus$. In contrast, the familiar hyperbolic distance function in the literature is designed so as to give rise to a triangle inequality which involves the addition $+$. The connection between the gyrodistance function and the standard hyperbolic distance function is described in [3].

4. A sharp gyrotriangle inequality and its reverse in Möbius gyrovector space $(\mathbb{D}, \oplus, \otimes)$

Lemma 5. *For all nonzero elements of $x_1, x_2, \cdots, x_n$ in $(\mathbb{D}, \oplus, \otimes)$*

$$\left|\bigoplus_{j=1}^{n} x_j\right| \leq \bigoplus_{j=1}^{n} |x_j| \tag{8}$$

holds, where $\bigoplus\limits_{j=1}^{n} |x_j| = |x_1| \oplus |x_2| \oplus \cdots \oplus |x_n|$, $|x_j| = |x_j \ominus 0|$ *and* $\oplus_{j=1}^{n} x_j = (\cdots((x_1 \oplus x_2) \oplus x_3) \oplus \cdots \oplus x_{n-1}) \oplus x_n$.

Proof. Firstly, the set

$$|\mathbb{D}| = \{\pm |a| : a \in \mathbb{D}\} \subset (-1, 1)$$

is a commutative group with the operation $\oplus$. Therefore, by the associativity property $\oplus$, we get

$$\left|\bigoplus_{j=1}^{n} u_j\right| = \left|\left(\bigoplus_{j=1}^{n-1} u_j\right) \oplus u_n\right| = \cdots = \left|u_1 \oplus \left(\bigoplus_{j=2}^{n} u_j\right)\right|$$

where $u_j \in [0,\ 1)$ for $1 \le j \le n$. For $x \in [0, 1)$, define the function

$$f_c(x) = x \oplus c = \frac{x + c}{1 + xc}$$

where c is a constant in $[0, 1)$. Since

$$f_c'(x) = \frac{1 - c^2}{(1 + xc)^2} > 0$$

holds true, we get that f_c must be a monotonically increasing function. Therefore,

$$\begin{aligned} |x_1 \oplus x_2 \oplus x_3| &= |(x_1 \oplus x_2) \oplus x_3| \\ &\le |x_1 \oplus x_2| \oplus |x_3| \\ &\le |x_1| \oplus |x_2| \oplus |x_3| \end{aligned}$$

by (III.8) and by the increasing property of $f_{|x_3|}$. Following this way, one can easily get

$$\begin{aligned} \left|\bigoplus_{j=1}^{n} x_j\right| &= \left|\left(\bigoplus_{j=1}^{n-1} x_j\right) \oplus x_n\right| \\ &\le \left|\bigoplus_{j=1}^{n-1} x_j\right| \oplus |x_n| \\ &\le |x_1| \oplus |x_2| \oplus \cdots \oplus |x_{n-1}| \oplus |x_n| \\ &= \bigoplus_{j=1}^{n} |x_j| \end{aligned}$$

by (III.8) and by the increasing properties of $f_{|x_n|}, f_{|x_{n-1}|}, \ldots, f_{|x_3|}$.

□

Theorem 6. *For all nonzero elements of $x_1, x_2, \cdots, x_n$ in $(\mathbb{D}, \oplus, \otimes)$*

$$\begin{aligned}
&\left(\frac{\min_{1\le j\le n}|x_j|}{\max_{1\le j\le n}|x_j|}\otimes\left(\bigoplus_{j=1}^{n}|x_j|\ominus\left|\bigoplus_{j=1}^{n}x_j\right|\right)\right)\oplus\left|\bigoplus_{j=1}^{n}x_j\right| \\
\le\ &\bigoplus_{j=1}^{n}|x_j| \\
\le\ &\left(\frac{\min_{1\le j\le n}|x_j|}{\max_{1\le j\le n}|x_j|}\otimes\left(\bigoplus_{j=1}^{n}|x_j|\ominus\left|\bigoplus_{j=1}^{n}x_j\right|\right)\right)\oplus\left|\bigoplus_{j=1}^{n}x_j\right|
\end{aligned}$$

holds.

Proof. Firstly, we have

$$\begin{aligned}
0 &\le \bigoplus_{j=1}^{n}|x_j| \boxminus \left|\bigoplus_{j=1}^{n}x_j\right| \\
&= \bigoplus_{j=1}^{n}|x_j| \ominus \left|\bigoplus_{j=1}^{n}x_j\right|
\end{aligned}$$

by (8). Let us assume

$$|x_r| := \min\{|x_1|, |x_2|, \cdots, |x_n|\}$$

and

$$|x_s| := \max\{|x_1|, |x_2|, \cdots, |x_n|\}.$$

Clearly we get

$$\frac{|x_r|}{|x_s|} \le 1 \le \frac{|x_s|}{|x_r|}.$$

Now define a function g_u on $\mathbb{R}^+$ by

$$\begin{aligned}
g_u(x) &= x \otimes u \\
&= \tanh(x \tanh^{-1} u)
\end{aligned}$$

where u is a constant in $[0, 1)$. Since

$$g_u'(x) = \frac{\tanh^{-1} u}{\cosh^2(x \tanh^{-1} u)} > 0,$$

we see that g_u is a monotonically increasing function on $\mathbb{R}^+$. Since $\frac{|x_r|}{|x_s|} \le 1 \le \frac{|x_s|}{|x_r|}$ holds, we get

$$g_w\left(\frac{|x_r|}{|x_s|}\right) \le g_w(1) \le g_w\left(\frac{|x_s|}{|x_r|}\right)$$

where $w := \bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \in [0,1)$. Hence we obtain

$$\begin{aligned}
&\frac{|x_r|}{|x_s|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \\
&\leq 1 \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \\
&\leq \frac{|x_s|}{|x_r|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right)
\end{aligned}$$

and this yields

$$\begin{aligned}
&\left(\frac{|x_r|}{|x_s|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \right) \oplus \left| \bigoplus_{j=1}^{n} x_j \right| \\
\leq\ & \bigoplus_{j=1}^{n} |x_j| \\
\leq\ & \left(\frac{|x_s|}{|x_r|} \otimes \left(\bigoplus_{j=1}^{n} |x_j| \ominus \left| \bigoplus_{j=1}^{n} x_j \right| \right) \right) \oplus \left| \bigoplus_{j=1}^{n} x_j \right| .
\end{aligned}$$

□

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A Cevian Locus and the Geometric Construction of a Special Elliptic Curve

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Abstract. Given a triangle ABC, we determine the locus $\mathscr{L}$ of points P, for which the affine mapping $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn, where $T_P(ABC) = DEF$ is the cevian triangle of P, $T_{P'}(ABC)$ is the cevian triangle of the isotomic conjugate P' of P, and K is the complement map, with respect to ABC. This completes the determination of the points P for which the inconic $\mathcal{I} = \mathsf{M}(\tilde{\mathcal{C}}_O)$ of P, tangent to the sides of ABC at the points D, E, F, is congruent to the circumconic $\tilde{\mathcal{C}}_O$ of ABC whose center is $O = T_{P'}^{-1} \circ K(Q)$, where $Q = K(P')$. We show that the locus $\mathscr{L}$ is an elliptic curve minus six points, whose j-invariant is $j = \frac{2^4 11^3}{5^2}$, and use the cevian geometry of ABC and P to give a synthetic construction of this elliptic curve.

1. Introduction

In previous papers [7], [10], [12] we have studied several conics defined for an ordinary triangle ABC relative to a given point P, not on the sides of ABC or its anticomplementary triangle, including the inconic $\mathcal{I}$ and circumconic $\tilde{\mathcal{C}}_O$. These two conics are defined as follows. Let DEF be the cevian triangle of P with respect to ABC (i.e., the diagonal triangle of the quadrangle $ABCP$). Let K denote the complement map and ι the isotomic map for the triangle ABC, and set $P' = \iota(P)$ and $Q = K(P') = K(\iota(P))$. Furthermore, let T_P be the unique affine map taking ABC to DEF, and $T_{P'}$ the unique affine map taking ABC to the cevian triangle for P'.

The inconic $\mathcal{I}$ for P with respect to ABC is the unique conic which is tangent to the sides of ABC at the traces (diagonal points) D, E, F. (See [7], Theorem 3.9 for a proof that this conic exists.) If $\mathcal{N}_{P'}$ is the nine-point conic of the quadrangle $ABCP'$ relative to the line at infinity l_∞, then the circumconic $\tilde{\mathcal{C}}_O$ is defined to be $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$. (See [10], Theorems 2.2 and 2.4; and [2], p. 84.) These two conics are generalizations of the classical incircle and circumcircle of a triangle, and coincide with these circles when the point $P = Ge$ is the Gergonne point of the triangle. In that case, the point $Q = I$ is the incenter. In general, the point Q is the center of the inconic $\mathcal{I}$. The center O of the circumconic $\tilde{\mathcal{C}}_O$ is given by the affine formula

$$O = T_{P'}^{-1} \circ K(Q),$$

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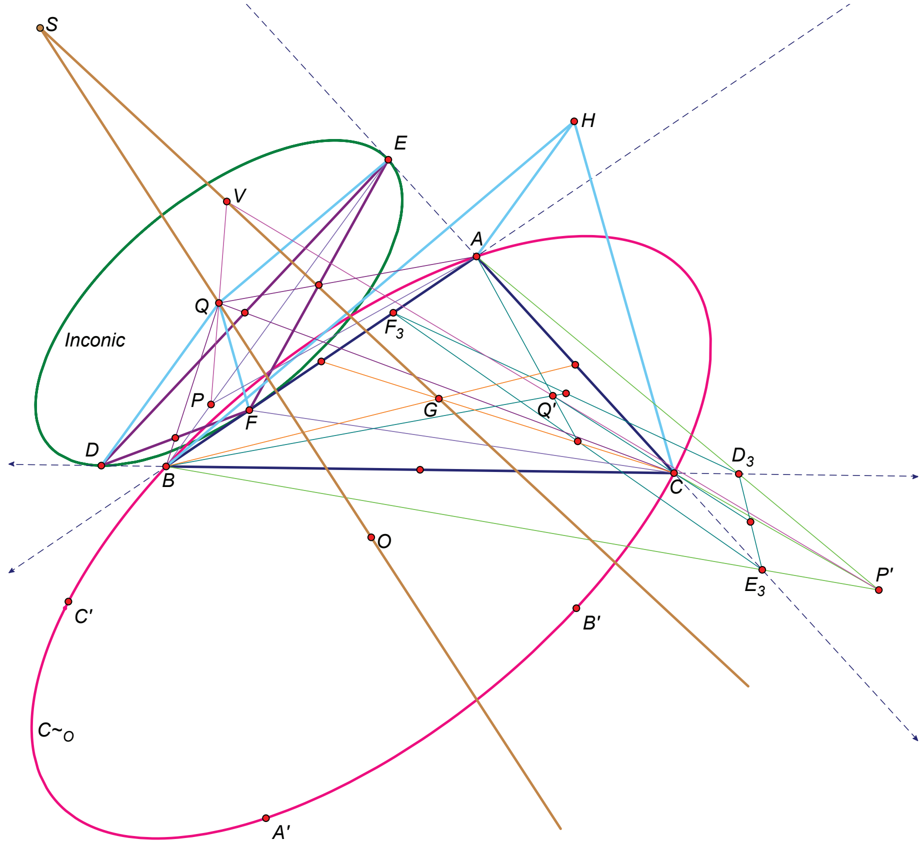

Figure 1. The conics $\tilde{\mathcal{C}}_O$ (strawberry) and $\mathcal{I}$ (green).

since the center of the conic $\mathcal{N}_{P'}$ turns out to be $K(Q)$.

We also showed in [10], Theorem 3.4, that the affine map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety or translation which maps the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$. If G is the centroid of ABC and $Q' = K(P)$, then the center of the map M is the point

$$S = OQ \cdot GV = OQ \cdot O'Q', \quad \text{where } V = PQ \cdot P'Q',$$

and $O' = T_P^{-1} \circ K(Q')$ is the generalized circumcenter for the point P'.

In [12] we showed that for a fixed triangle ABC the locus of points P, for which the map M is a translation, is an elliptic curve minus 6 points, and that this elliptic curve has infinitely many points defined over $\mathbb{Q}(\sqrt{2})$. Thus, there are infinitely many points P for which the conics $\mathcal{I}$ and $\tilde{\mathcal{C}}_O$ are congruent to each other. In this note we determine the remaining points P for which these two conics are congruent by determining (synthetically) the locus of points P for which the map M is a half-turn. We show, for example, that M being a half-turn is equivalent to the point P lying on the circumconic $\tilde{\mathcal{C}}_{O'}$, where $O' = T_P^{-1} \circ K(Q')$ is as above for the point

P', and this is also equivalent to the point $P' = \iota(P)$ lying on the circumconic $\tilde{\mathcal{C}}_O$. (By contrast, we showed in [12] that M is a translation if and only if the point P lies on $\tilde{\mathcal{C}}_O$.) This is interesting, since if $P = Ge$ is the Gergonne point of ABC, then $P' = Na$ is the Nagel point, which always lies *inside* the circumcircle $\tilde{\mathcal{C}}_O$. Given triangle ABC, the locus of all such points P turns out to be another elliptic curve (minus 6 points; see Theorem 9). As in [12], this elliptic curve can be constructed synthetically using a locus of affine maps defined for points on certain open arcs of a conic. In [12] the latter conic was a hyperbola, while here the conic needed to construct the elliptic curve is a circle. (See Figure 4.)

We adhere to the notation of [7]: $D_0E_0F_0$ is the medial triangle of ABC, with D_0 on BC, E_0 on CA, F_0 on AB (and the same for further points D_i, E_i, F_i); DEF is the cevian triangle associated to P; $D_2E_2F_2$ the cevian triangle for $Q = K \circ \iota(P)$; $D_3E_3F_3$ the cevian triangle for $P' = \iota(P)$; and G the centroid of ABC. As above, T_P and $T_{P'}$ are the unique affine maps taking triangle ABC to DEF and $D_3E_3F_3$, respectively. See [7] and [9] for the properties of these maps. Also, the generalized orthocenter for P with respect to ABC is the point $H = K^{-1}(O)$, which is also the intersection of the lines through the vertices A, B, C which are parallel, respectively, to the lines QD, QE, QF. Finally, the point Z is defined to be the center of the cevian conic $\mathcal{C}_P = ABCPQ$. (See [9].)

We also refer to the papers [7], [9], [10], and [11] as I, II, III, IV, respectively. See [1], [2], [3] for results and definitions in triangle geometry and projective geometry.

2. The locus of P for which M is a half-turn.

In this section we determine necessary and sufficient conditions for the map M to be a half-turn. We start with the following lemma.

Lemma 1. (a) *If the point P (not on a side of ABC or $K^{-1}(ABC)$) lies on the Steiner circumellipse $\iota(l_\infty)$ of ABC, then the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety with ratio $k = 4$, and is therefore not a half-turn.*

(b) *If the point P lies on a median of triangle ABC, but does not lie on the Steiner circumellipse $\iota(l_\infty)$ of ABC, then $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is not a half-turn.*

Proof. To prove (a), we use the result of I, Theorem 3.14, according to which P lies on $\iota(l_\infty)$ if and only if the maps T_P and $T_{P'}$ satisfy $T_P T_{P'} = K^{-1}$. If this condition holds, then because the map M is symmetric in P and P' (see III, Proposition 3.12b and IV, Lemma 5.2),

$$\mathsf{M} = T_{P'}K^{-1}T_P = T_{P'}T_PT_{P'}T_P = (T_{P'}T_P)^2 = (T_P^{-1}K^{-1}T_P)^2 = T_P^{-1}K^{-2}T_P.$$

The similarity ratio of the dilatation K^{-1} is -2, so the similarity ratio of K^{-2} is 4, which proves part (a).

For (b), suppose P lies on the median AG (G the centroid of ABC) and the map M is a half-turn. Then, since P' also lies on AG, we have that $D = D_0 = D_3$, so that

$$\mathsf{M}(A) = T_PK^{-1}T_{P'}(A) = T_PK^{-1}(D_3) = T_PK^{-1}(D_0) = T_P(A) = D = D_0$$

and the center S of M is the midpoint of AD_0. In particular, $\mathsf{M}(B)$ and $\mathsf{M}(C)$ are the reflections in S of B and C on the line $\ell = K^{-1}(BC)$. We claim that the line ℓ is tangent to the circumconic $\tilde{\mathcal{C}}_O$ at the point A. This is because the affine reflection ρ through the line $AG = AP$ in the direction of the line BC takes the triangle ABC to itself, and maps P to P, so it also takes the circumconic $\tilde{\mathcal{C}}_O$ to itself. Hence the tangent to $\tilde{\mathcal{C}}_O$ at A maps to itself, which implies that it must be parallel to BC (since the only other ordinary fixed line is AG, which lies on the center O of $\tilde{\mathcal{C}}_O$ and cannot be a tangent at an ordinary point). But ℓ is the unique line through A parallel to BC, so ℓ must be the tangent. (Also see III, Corollary 3.5.) It follows that the points $\mathsf{M}(B)$ and $\mathsf{M}(C)$, neither of which is A, must be exterior points of the conic $\tilde{\mathcal{C}}_O$. On the other hand, we claim that $D = D_0$ is an *interior* point of $\tilde{\mathcal{C}}_O$. This is because the segment BC, parallel to the tangent ℓ at A, is a chord of $\tilde{\mathcal{C}}_O$, and B and C lie on the same branch of $\tilde{\mathcal{C}}_O$, if the latter is a hyperbola (any tangent to a hyperbola separates the two branches). It follows that the segments $D\mathsf{M}(B)$ and $D\mathsf{M}(C)$ join the interior point D to exterior points, and so must each contain a point on $\tilde{\mathcal{C}}_O$. However, M is a half-turn mapping the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$, so that $\mathsf{M}(\mathcal{I}) = \tilde{\mathcal{C}}_O$. Hence, $\tilde{\mathcal{C}}_O$ must be inscribed in the triangle $\mathsf{M}(ABC) = D\mathsf{M}(B)\mathsf{M}(C)$, meaning that $\tilde{\mathcal{C}}_O$ touches all three extended sides of the triangle. But by what we just showed the intersections of $\tilde{\mathcal{C}}_O$ with the sides of $D\mathsf{M}(B)\mathsf{M}(C)$ lie on the segments joining the vertices. Hence, the point D lies on the two tangents $b = D\mathsf{M}(B)$ and $c = D\mathsf{M}(C)$ to $\tilde{\mathcal{C}}_O$, implying that D is an *exterior* point of $\tilde{\mathcal{C}}_O$. This contradiction proves the lemma. □

Proposition 2. *If the points P and P' are ordinary and do not lie on the sides or medians of triangles ABC and $K^{-1}(ABC)$, and H does not coincide with a vertex of ABC, the following are equivalent:*

(1) $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ *is a half-turn;*
(2) P *is on* $\tilde{\mathcal{C}}_{O'}$*;*
(3) P' *is on* $\tilde{\mathcal{C}}_O$*;*
(4) $T_P(P) = O'$*;*
(5) $T_{P'}(P') = O$*;*
(6) O' *lies on* $\mathcal{N}_P$*;*
(7) O *lies on* $\mathcal{N}_{P'}$*.*

Proof. First note that (4) $\iff$ (5): this follows on applying the affine reflection η from Part II to (4) (η is the harmonic homology with axis GZ and center $PP' \cdot l_\infty$), and using that

$$\eta(P) = P', \quad \eta(O) = O', \quad \text{and} \quad \eta \circ T_P = T_{P'} \circ \eta.$$

By III, Proposition 3.12 and IV, Lemma 5.2, M commutes with η, since η commutes with K and

$$\mathsf{M} \circ K^{-1} = (T_P \circ K^{-1}) \circ (T_{P'} \circ K^{-1}) = (T_{P'} \circ K^{-1}) \circ (T_P \circ K^{-1}).$$

Hence, $T_{P'} \circ K^{-1} \circ T_P = \eta \mathsf{M} \eta = \mathsf{M}$. This shows that the locus of points P for which M is a half-turn is invariant under $P \to P'$.

We now show that (1) is equivalent to (4) and (5). Namely, if M is a half-turn, then since $\mathsf{M}(O') = Q'$, we have to have $\mathsf{M}(Q') = O'$. But

$$\mathsf{M}(Q') = T_P K^{-1} T_{P'}(Q') = T_P K^{-1}(Q') = T_P(P),$$

and so $O' = T_P(P)$. Conversely, if $O' = T_P(P)$, then $\mathsf{M}(Q') = O'$, so $\mathsf{M}^2(O') = O'$, which implies that M must be a homothety with similarity ratio $k = \pm 1$, since M^2 fixes the point $O' \neq S$. However, k can't be $+1$, since in that case M is the identity and $O' = Q'$, impossible by the argument of III, Theorem 3.9. Therefore, $k = -1$, so M is a half-turn. (Note that $O' \neq S$, since otherwise $O = \eta(O') = \eta(S) = S$; but the points O, O', Q, Q' are distinct, by the proof of III, Theorem 3.9, as long as P does not lie on a median of ABC or on $\iota(l_\infty)$.) This shows that (1) $\iff$ (4) $\iff$ (5).

Furthermore, if (3) holds, then P' is on $\tilde{\mathcal{C}}_O$, so the latter conic lies on the vertices of quadrangle $ABCP'$ (since $\tilde{\mathcal{C}}_O$ is a circumconic), so the center O must lie on $\mathcal{N}_{P'}$, by definition of the 9-point conic. (See Part III, paragraph before Prop. 2.4.) Thus (3) implies (7). Also, (7) implies (3), because O being on $\mathcal{N}_{P'}$ implies O is the center of a conic on $ABCP'$. If O is not the midpoint of a side of ABC (which holds if and only if H is not a vertex), there is a unique such conic, namely $\tilde{\mathcal{C}}_O$. Hence P' lies on this conic. This shows that (3) $\iff$ (7). Similarly, (2) $\iff$ (6).

Now suppose that (3) holds. Then (7) holds, so $P' \in \tilde{\mathcal{C}}_O$. But $P' \in \mathcal{C}_P$, so P' is the fourth intersection of the circumconics $\mathcal{C}_P$ and $\tilde{\mathcal{C}}_O$, and therefore coincides with the point $\tilde{Z} = R_O K^{-1}(Z)$, by III, Theorem 3.14; here R_O is the half-turn about O and Z is the center of $\mathcal{C}_P$. Now, in the proof of III, Theorem 3.14 we showed that $T_{P'}(\tilde{Z}) = T_{P'}(P')$ lies on OP'. But $T_{P'}(P')$ lies on OQ (III, Proposition 3.12), so this forces $T_{P'}(P') = O$, i.e. (5), provided we can show that the line OP' is distinct from OQ.

However, if $OP' = OQ$, then $OP' = P'Q = QG$ so O, Q, G are collinear. Then $K^{-1}(O) = H$ is also on this line, so Q, H, P' are all on this line. We claim that these three points, Q, H, P', must all be distinct by our hypothesis on P. If $P' = Q = K(P')$ then $P' = G = P$, which can't hold because P is not on a median of ABC. If $Q = H$, then using the map $\lambda = T_{P'} \circ T_P^{-1}$ and III, Theorem 2.7 gives $Q = \lambda(H) = \lambda(Q) = P'$, so $P' = G$. Finally, if $P' = H$, then taking complements gives that $Q = O = T_{P'}^{-1}(K(Q))$ so $T_{P'}(Q) = K(Q)$, implying (by I, Theorem 3.7) that $P' = K(Q) = K^2(P')$ and therefore $P' = G = P$ once again. Therefore, the three distinct points Q, H, P', which all lie on the conic $\mathcal{C}_P$, are collinear, which is impossible. This shows that $OP' \cap OQ = O$, so $T_{P'}(P') = O$ and (3) $\Rightarrow$ (5) $\Rightarrow$ (1).

For the rest, it suffices to show that (1) $\Rightarrow$ (3). This is because of the symmetry of M in P and P': for example, (3) $\Rightarrow$ (1) $\Rightarrow$ (2) (switching P and P') and conversely, so (2) and (3) are equivalent, as are (6) and (7), and everything is equivalent to (1). Now assume (1). We will prove (7). Since (1) implies (5), we know $T_{P'}(P') = O$, so $T_{P'}(OP') = K(Q)O$ by the formula for O. But by III, Corollary 3.13(b) we know $K^{-1}(Z)$ lies on OP', so $T_{P'} \circ K^{-1}(Z) = Z$ (III, Prop. 3.10) lies on $K(Q)O$.

But Z also lies on $QN = K(P'O)$, which is parallel to OP'. This easily implies Z is an ordinary point and $QZ \parallel OP'$; for this, note $Q \neq Z$ since Z is the center of the conic $\mathcal{C}_P$, while Q is an ordinary point lying *on* $\mathcal{C}_P$. Also, $Z \neq K(Q)$, since $T_P \circ K^{-1}(Z) = Z$, while $T_P \circ K^{-1}(K(Q)) = Q$. Therefore, using the fact that the lines OP' and OQ are distinct, it follows that $QK(Q)Z$ and $P'K(Q)O$ are similar triangles. But $K(Q)$ is the midpoint of segment $P'Q$, so these triangles must be congruent. Hence, $K(Q)$ is also the midpoint of segment OZ, and O is the reflection of Z in the point $K(Q)$. Since $K(Q)$ is the center of $\mathcal{N}_{P'}$ and Z lies on $\mathcal{N}_{P'}$, this means O also lies on $\mathcal{N}_{P'}$, which is (7). $\square$

Corollary 3. *With the hypotheses of Proposition 2, the map* M *is a half-turn if and only if* $K^{-1}(S) = Z$, *which holds if and only if* $QZP'O$ *is a parallelogram.*

Proof. If M is a half-turn, the above argument shows that segments $QZ \cong OP'$; hence, $QZP'O$ is a parallelogram, and $P'Z \parallel QO$. Since $K(P') = Q$, this implies that line $P'Z$ is the same as the line $K^{-1}(QO) = P'H$ and Z is the midpoint of $P'H$, since the map K^{-1} doubles lengths of segments. But S is the midpoint of QO, so $K^{-1}(S) = Z$ is the midpoint of $P'H$. Conversely, if $K^{-1}(S) = Z$ then Z lies on $K^{-1}(OQ) = P'H$ (since S lies on OQ) and because $QZ \parallel P'O$, $QZP'O$ is a parallelogram with center $K(Q)$, the midpoint of $P'Q$. Hence, O lies on $\mathcal{N}_{P'}$ and M is a half-turn. $\square$

Remark. The condition of Corollary 3 is a necessary condition for M to be a half-turn, without the hypothesis that H not be a vertex. This follows from the last paragraph in the proof of Proposition 2, since that hypothesis is not used to prove that (1) $\Rightarrow$ (7). Lemma 1 then shows that $K^{-1}(S) = Z$ is a necessary condition for M to be a half-turn, without any extra hypotheses.

In the following proposition we will make use of the polarity induced by the conic $\mathcal{C}_P = ABCPQ = PQP'Q'H$.

Proposition 4. *If the hypotheses of Proposition 2 hold and the map* M *is a half-turn, then:*

(1) *The lines* $O'P$ *and* OP' *are tangents to the conic* $\mathcal{C}_P$ *at* P *and* P'.
(2) *The point* $V = PQ \cdot P'Q'$ *is the midpoint of segment* OO' *and line* $OO' = K^{-1}(PP')$.
(3) *On the line* GV, *the signed ratios* $\frac{GS}{SV} = \frac{5}{3}$ *and* $\frac{ZG}{GV} = \frac{5}{4}$.

Proof. (See Figure 2.) (1) As in the proof of Proposition 2 we have $P' = \tilde{Z} = R_O \circ K^{-1}(Z)$, so $K^{-1}(Z)$ lies on OP'; by symmetry, it also lies on $O'P$. We will show that pole of PP' is $K^{-1}(Z)$. Then (1) follows, since $K^{-1}(Z)$ is conjugate to both P and P', and so lies on the polars of P and P', which are the tangents to $\mathcal{C}_P$ at P and P'. Hence, $K^{-1}(Z)P = O'P$ and $K^{-1}(Z)P' = OP'$ are tangents to $\mathcal{C}_P$.

We do this by showing that $T_{P'}(P)$ and $T_P(P')$ are conjugate to $K^{-1}(Z)$. This implies that the polar of $K^{-1}(Z)$ is the join of $T_{P'}(P)$ and $T_P(P')$, which is PP' by II, Corollary 2.2(c). By symmetry it suffices to consider $T_{P'}(P)$. We use the

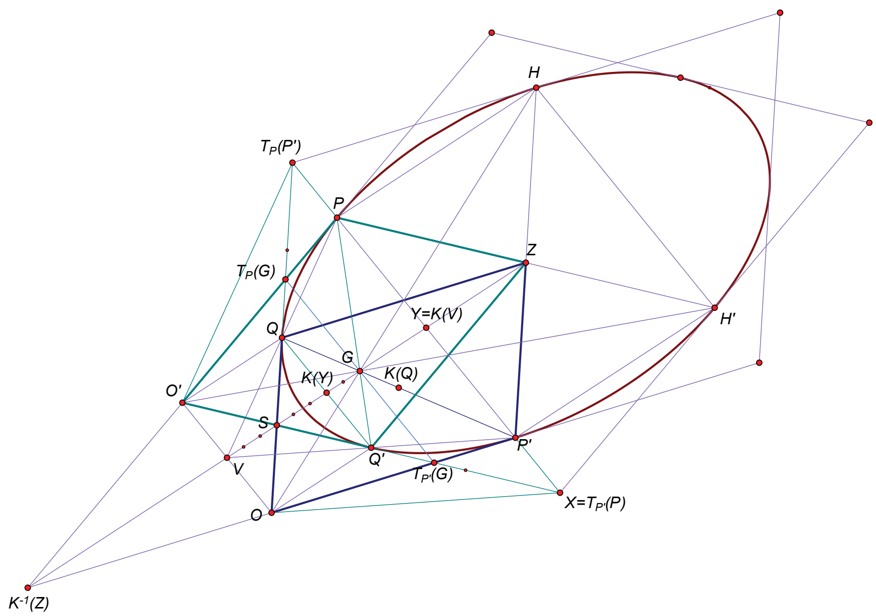

Figure 2. The parallelograms $QZP'O$ and $Q'ZPO'$ when M is a half-turn.

fact from Part IV, Prop. 3.10, that $O'Q'$ is tangent to $\mathcal{C}_P$ at Q'. Applying the map λ gives that $\lambda(O'Q')$ is tangent to $\lambda(\mathcal{C}_P) = \mathcal{C}_P$ at $\lambda(Q') = H'$, by II, Theorem 3.2 and III, Theorem 2.7. Using $T_P(P) = O'$ from Proposition 2 we know that

$$\lambda(O') = T_{P'} \circ T_P^{-1}(O') = T_{P'}(P),$$

so $T_{P'}(P)$ lies on the tangent to $\mathcal{C}_P$ at H' and is conjugate to H'. Also, P, Q', $K^{-1}(P)$ are collinear points, so applying the map $T_{P'}$ gives that $T_{P'}(P)$ is collinear with $T_{P'}(Q') = Q'$ and

$$T_{P'} \circ K^{-1}(P) = T_{P'} \circ K^{-1} \circ T_P(Q') = \mathsf{M}(Q') = O'.$$

Therefore, $T_{P'}(P)$ lies on the tangent $O'Q'$ and so is conjugate to the point Q'. Thus, the polar of $T_{P'}(P)$ is $Q'H'$, which lies on $K^{-1}(Z)$ since $Z, K(Q'), O' = K(H')$ are collinear, using the fact from Corollary 3 that $Q'ZPO'$ is a parallelogram and $K(Q')$ is the midpoint of the diagonal $Q'P$. This shows that $K^{-1}(Z)$ is conjugate to $T_{P'}(P)$, as desired. Note that this also shows that $T_{P'}(P) \neq T_P(P')$, since the polar of $T_P(P')$ is QH, and QH cannot be the same line as $Q'H'$, since the four points Q, H, Q', H' all lie on the conic $\mathcal{C}_P$.

(2) From the fact that OP' and OQ (IV, Prop. 3.10) are tangent to $\mathcal{C}_P$, it follows that $P'Q$ is the polar of O with respect to $\mathcal{C}_P$, and likewise, PQ' is the polar of O'. Hence, the pole of OO' is $P'Q \cdot PQ' = G$. On the other hand, the polar of G is the line VV_∞, where V_∞ is the infinite point on the line PP', by II, Proposition 2.3(a). Therefore, V lies on OO'. Since GV is the fixed line of the affine reflection η, V

must be the midpoint of segment OO'. Then V lies on the parallel lines $K^{-1}(PP')$ (II, Prop. 2.3(e)) and OO', so $K^{-1}(PP') = OO'$.

(3) From (2) we have $PP' = K(OO') = K^2(HH') = NN'$, where $N = K(O)$ and $N' = K(O')$ are the centers of the nine-point conics $\mathcal{N}_H$ and $\mathcal{N}_{H'}$. Thus, the line PP' is halfway between the parallel lines OO' and HH'. Taking complements, QQ' is halfway between $NN' = PP'$ and OO'. Also, the center S of the map M is located halfway between the lines QQ' and OO', since $\mathsf{M}(OO') = QQ'$. Let $X = T_{P'}(P) = O'Q' \cdot PP' = O'S \cdot PP'$ and $Y = K(V) = GV \cdot PP'$. Then triangles $VO'S$ and YXS are similar, with similarity ratio $1/3$, because S is the midpoint of segment $O'Q'$ and Q' is the midpoint of segment $O'X$, so that

$$|SX| = |SQ'| + |Q'X| = |SO'| + 2|SO'| = 3|SO'|.$$

Furthermore, V on OO' implies that $\mathsf{M}(V)$ is on QQ', so that $\mathsf{M}(V) = QQ' \cdot GV$ is the midpoint of $VY = VK(V)$ and therefore coincides with $K(Y)$. Then from $|SK(Y)| = |S\mathsf{M}(V)| = |SV|$, $|K(Y)G| = \frac{1}{3}|K(Y)Y|$, and $|K(Y)Y| = |VK(Y)| = |V\mathsf{M}(V)| = 2|SV|$ we find that

$$\begin{aligned}\frac{|GS|}{|SV|} &= \frac{|SK(Y)| + |K(Y)G|}{|SV|} = \frac{|SV| + |K(Y)Y|/3}{|SV|} \\ &= \frac{|SV| + 2|SV|/3}{|SV|} = \frac{5}{3}.\end{aligned}$$

Since S lies between G and V, this proves $\frac{GS}{SV} = \frac{5}{3}$. Now $\frac{ZG}{GV} = \frac{2GS}{8SV/3} = \frac{3}{4}\frac{GS}{SV} = \frac{5}{4}$. $\square$

Remark. The conditions of Proposition 4 are also sufficient for M to be a half-turn. We leave the verification of this for parts (1) and (2) to the reader. We will verify this for condition (3) in the next section.

3. Constructing the elliptic curve locus.

Now suppose a parallelogram $QZP'O$ is given, and with it: $K(Q)$ as the midpoint of QP'; G as the point for which the signed distance QG satisfies $QG = \frac{1}{3}QP'$; and $S = K(Z)$ (Corollary 3). Then Proposition 4 shows that the point V is determined by G and S. This determines, in turn, the points P and Q' uniquely, since P is the reflection in Q of the point V (see II, p. 26) and $Q' = K(P)$. Further, O' is also determined as the reflection of Q' in S, or as the reflection of O in V. Therefore, the parallelogram determines P, Q, P', Q' and $H = K^{-1}(O)$, and hence the conic $\mathcal{C}_P$ on these 5 points (by III, Theorem 2.8). Thus, any triangle ABC for which M is a half-turn with the given parallelogram $QZP'O$ must be inscribed in the conic $\mathcal{C}_P$. Furthermore, the affine maps T_P and $T_{P'}$ are also determined, since

$$T_P(PQQ') = O'QP \text{ and } T_{P'}(P'QQ') = OP'Q'. \tag{1}$$

Defining the maps T_P and $T_{P'}$ by (1), we will show that $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn about S.

Lemma 5. *Given collinear and distinct ordinary points G, V, Z and an ordinary point P not on GZ, if $Q' = K(P)$ and P' is the reflection of V in the point Q', and Q is the midpoint of PV, then:*

(a) *there is a unique conic $\mathcal{C}$ with center Z which lies on P, P', Q, and Q';*

(b) *with respect to any triangle ABC with vertices on $\mathcal{C}$ whose centroid is G, and whose vertices do not coincide with any of the points P, P', Q or Q', the point P' is the isotomic conjugate of P, and $\mathcal{C}$ coincides with the conic $\mathcal{C}_P$ for ABC.*

Proof. For (a), first note that PQ', $P'Q'$, and PQ do not lie on Z, since $PQ' \cdot GZ = G$ and $P'Q' \cdot GZ = PQ \cdot GZ = V$, which are distinct from Z by assumption. Suppose that Z is not on PP'. The point G, being $2/3$ of the way from P to the midpoint Q' of VP', is the centroid of triangle PVP'. Hence, $K(P') = Q$ and $VG = GZ$ is a median of triangle PVP', implying that GZ intersects PP' at the midpoint of segment PP'. Also, $QQ' \parallel PP'$, so $V_\infty = PP' \cdot QQ'$ is on the line at infinity.

Now let $\mathcal{C}$ be the conic with center Z, lying on the points P, Q, P'. This exists and is unique, since Z does not lie on $PQ, P'Q$ or PP'. With respect to this conic, Z is conjugate to V_∞, and so is the midpoint $GZ \cdot PP'$, since P and P' lie on $\mathcal{C}$. Since Z is not on PP', GZ is the polar of V_∞. Now, V_∞ lies on QQ', so the point $Q_m = GZ \cdot QQ'$, which is the midpoint of segment QQ', is conjugate to V_∞. Let Q^* be the second intersection of QQ' with C. Note that $Q^* \neq Q$; otherwise V_∞ would be conjugate to Q, so Q would lie on its polar GZ, implying that P also lies on GZ, which is contrary to assumption. Hence, V_∞ is conjugate to the midpoint of QQ^*, which must be Q_m. This implies that $Q^* = Q'$, so Q' lies on $\mathcal{C}$.

Now suppose Z is on PP'. Then Z is not on QQ', since $QQ' \parallel PP'$, so there is a unique conic $\mathcal{C}$ through P, Q, Q' with center Z. As above, the pole of GZ is V_∞, and switching the point pairs P, P' and Q, Q' in the argument above gives that P' lies on $\mathcal{C}$.

For (b), the triangle ABC determines the conic $\mathcal{C} = \mathcal{C}_P = ABCPQ'$, since P and Q' cannot lie on any of the sides of ABC and P does not lie on a median of ABC (see the proof of II, Theorem 2.1). We know that this conic has center Z, since ABC is inscribed in $\mathcal{C}$. Furthermore, the pole of GZ with respect to $\mathcal{C}$ is $V_\infty = l_\infty \cdot PP'$, as above. But the isotomic conjugate P^* of P with respect to ABC is the unique point $P^* \neq P$ on PV_∞ lying on the conic $\mathcal{C}_P$ (see II, p. 26), so that means $P^* = P'$. □

Lemma 6. *With the assumptions of Lemma 5, suppose the signed ratio $\frac{ZG}{GV} = \frac{5}{4}$. Then the tangent to the conic $\mathcal{C}$ at Q is $K(P'Z) = QK(Z)$.*

Proof. (See Figure 3.) As in the proof of Proposition 4, in triangle $PP'V$, the midpoint of PP' is the complement $Y = K(V)$ of V, and so the midpoint of QQ' is the complement $K(Y)$ of Y. Let $U = K^{-1}(V)$. Then $GV = 2GY = 4GK(Y)$ and $ZG = 5GK(Y)$, so $ZY = ZG - GY = 3GK(Y)$. In addition, $UG = 8GK(Y)$ so $UZ = UG - ZG = 3GK(Y) = ZY$; hence, Z is the midpoint of YU. Let H' be the reflection of P in Z. Then triangles $H'UZ$ and PYZ are

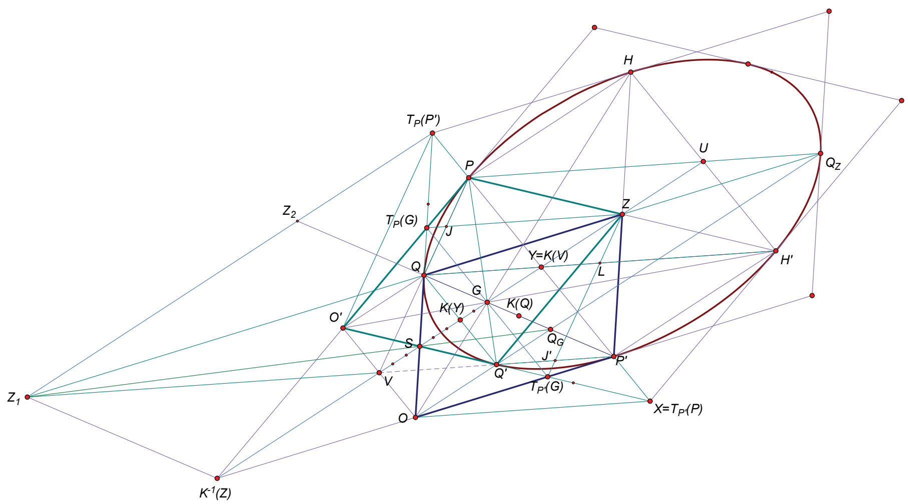

Figure 3. Proof of Lemma 3.2.

congruent so $H'U \parallel PY = P'Y$. Also, triangles PYZ and $PP'H'$ are similar (SAS), so $P'H' \parallel YZ = UY$. This implies $P'H'UY$ is a parallelogram and $P'H' = UY = VY$. Hence $P'H'YV$ is also a parallelogram and $H'Y \parallel P'V$. Since triangles QYP and $VP'P$ are similar (with similarity ratio $1/2$), $QY \parallel P'V$, so Y lies on $H'Q$.

Let Q_G and Q_Z be the reflections of Q in G and Z, respectively. Then $QQ_GQ_Z \sim QGZ$, so the line Q_GQ_Z is parallel to GZ and lies halfway between P' and GZ. Hence, Q' lies on Q_GQ_Z. Also, U lies on PQ_Z, since P, U, Q_Z are the reflections of H', Y, Q in Z, and we proved H', Y, Q are collinear in the previous paragraph. Moreover, $PQ_Z \parallel H'Q \parallel P'V$, since $P'H'YV$ is a parallelogram.

Let Z_1 be the reflection of Q_Z in Q. Then $Z_1QV \cong Q_ZQP$ (SAS), so $Z_1V \parallel PQ_Z \parallel P'V$, hence Z_1 lies on $P'V$. Now let J' be the midpoint of $P'Q'$ and L the midpoint of QH'. Then ZL is a midline in triangle PQH', so $ZL \parallel PQ = QV$. Since $P'Q' \parallel H'Q = QL$, and ZL lies in the conjugate direction to QH', it follows that Z, L, J' are collinear. Hence, $Z_1QV \sim Z_1ZJ'$, which implies, since $Z_1Q = 2QZ$, that $Z_1V = 2VJ' = 2 \cdot (\frac{3}{4}VP') = \frac{3}{2}VP'$. Let $Z^* = K^{-1}(Z)$. Then $Z^*G = 2 \cdot GZ = \frac{5}{2} \cdot VG$, by hypothesis, so $Z^*V = \frac{3}{2} \cdot VG$. Hence, triangles Z_1Z^*V and $P'GV$ are similar (SAS) and $Z_1Z^* \parallel P'G$. Let $S = Z_1Q_G \cdot GZ$. Since $Z_1Z^* \parallel Q_GG$, we have similar triangles Z_1Z^*S and Q_GGS. Moreover, $Z_1ZS \sim Z_1Q_ZQ_G$, since $SZ = GZ \parallel Q_GQ_Z$, with $Z_1Z = 3ZQ_Z$, so $Z_1S = 3SQ_G$. It follows that $Z^*S = 3SG$ and therefore $Z^*G = 4SG = 2GZ$. This implies $S = K(Z)$.

Let Z_2 be the intersection of $P'G$ with the line through Z_1 parallel to GZ. Also, let Z_∞ be the point at infinity on GZ. Then we have the following chain of perspectivities:

$$GVZK(Y) \overset{Q}{\overline{\overline{\wedge}}} P'VZ_1Q' \overset{Z_\infty}{\overline{\overline{\wedge}}} P'GZ_2Q_G \overset{Z_1}{\overline{\overline{\wedge}}} VGZ_\infty S.$$

The resulting projectivity on the line GZ is precisely the involution of conjugate points on GZ with respect to $\mathcal{C}$, because G and V are conjugate points (they are vertices of the diagonal triangle GVV_∞ of the inscribed quadrangle $PP'QQ'$) and the polar of Z is the line at infinity, which intersects GZ in Z_∞. This gives that $K(Y)$ is conjugate to S. But S is also conjugate to V_∞, since S lies on its polar GV. This implies that the polar of S is $K(Y)V_\infty = QQ'$. Thus, the tangent to $\mathcal{C}$ at Q is $QS = K(P'Z)$. □

Proposition 7. *Under the assumptions of Lemmas 5 and 6, for any triangle ABC with vertices on $\mathcal{C}$ whose centroid is G, and whose vertices do not coincide with any of the points P, P', Q or Q', the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn.*

Proof. By Lemmas 5 and 6, the tangent at Q to $\mathcal{C}_P$ goes through $K(Z)$. But the tangent at Q is OQ (IV, Prop. 3.10), hence the generalized insimilicenter S for ABC is $S = OQ \cdot GZ = K(Z)$, where Z is the center of $\mathcal{C}_P = \mathcal{C}$. Now the proposition follows from Corollary 3. □

Theorem 8. *Let G, V, Z be any distinct, collinear, and ordinary points with signed ratio $\frac{ZG}{GV} = \frac{5}{4}$, and P an ordinary point not on GZ. Define $Q' = K(P)$ (complement taken with respect to G) and let P' be the reflection of V in Q' and Q the midpoint of PV. Finally let $\mathcal{C}$ be the conic guaranteed by Lemma 5(a). For any point A on the arc $\mathscr{A} = PQQ'P'$ of $\mathcal{C}$ distinct from these four points, there is a unique pair of points $\{B, C\}$ on $\mathcal{C}$ (and then on the same arc), such that ABC has centroid G. For each such triangle, the map M is a half-turn, and this map is independent of A. Conversely, if ABC is inscribed in the conic $\mathcal{C}$ with centroid G, then $A \in \mathscr{A} - \{P, Q, Q', P'\}$.*

Proof. We start by showing that the hypotheses of the theorem can be satisfied for suitable points G, V, Z, P, for which the conic $\mathcal{C}$ is a circle. Start with a circle $\mathcal{C}$ with center Z. Pick points P', Q on $\mathcal{C}$ and O so that $QZP'O$ is a square. Let S be the midpoint of OQ and $G = SZ \cdot QP'$. Reflect P' and Q in GZ to obtain the points P and Q' on $\mathcal{C}$. Let Y on GZ be the midpoint of PP'. Also, let $V = PQ \cdot P'Q'$ on GZ. Since $G = QP' \cdot SZ$ is on the bisector of $\angle SQZ$ and $\frac{ZQ}{QS} = \frac{2}{1}$, we have $\frac{ZG}{GS} = \frac{2}{1}$, so $K(Z) = S$. This implies that $K(ZP') = SK(P')$ is parallel to ZP' and half the length, so $K(P') = Q$. In the same way, $K(P) = Q'$. Then in triangle PVQ' the segment VG bisects the angle at V, so $\frac{PV}{VQ'} = \frac{PG}{GQ'} = \frac{P'G}{GQ} = \frac{2}{1}$. Thus, $VP' = VP = 2 \cdot VQ'$. It follows that Q' is the midpoint of VP', Q is the midpoint of VP, and G is the centroid of VPP'. Hence, $Y = K(V)$. This shows that the hypotheses of Lemma 5 hold, so $\mathcal{C}$ is the conic of that lemma. By the same argument as in the proof of Proposition 4(2), using the fact that OQ and OP' are tangent to $\mathcal{C}$ at Q and P', respectively, and that GVV_∞ is a self-polar triangle with respect to $\mathcal{C}$, it follows that $K^{-1}(PP') = OO'$ and V is the midpoint of OO'.

Now, letting M be the half-turn about S, the same argument as in the proof of Proposition 4(3) gives that $\frac{ZG}{GV} = \frac{5}{4}$.

If G, V, Z, P are any points satisfying the hypotheses, then there is an affine map taking triangle VPP' to the corresponding triangle constructed in the previous paragraph, so that G (the centroid of VPP') and Z go to the similarly named points and the trapezoid $PP'Q'Q$ is mapped to the corresponding trapezoid for the circle. Then the image of the new conic $\mathcal{C}$ is the circle of the previous paragraph, so $\mathcal{C}$ must be an ellipse.

Given A on the arc $\mathscr{A} = PQQ'P'$ of $\mathcal{C}$, define $D_0 = K(A)$. Now P' and $K(P') = Q$ are on $\mathcal{C}$, as are P and $K(P) = Q'$. We claim that P and P' are the only two points R on $\mathcal{C}$ for which $K(R)$ is also on $\mathcal{C}$. This is because $K(\mathcal{C})$ is a conic with center $K(Z) = S$, meeting $\mathcal{C}$ at Q, Q', and lying on the point $K(Q)$. Note that the map K fixes all points on l_∞, so $K(\mathcal{C})$ induces the same involution on l_∞ that $\mathcal{C}$ does. It follows that there are exactly two points in $K(\mathcal{C}) \cap \mathcal{C}$. Since the point Q is on the given arc $\mathscr{A}$ and $K(Q)$, as the midpoint of segment QP', is interior to $\mathcal{C}$, it follows that the same is true for the point $D_0 = K(A)$, for any A on $\mathscr{A}$, while D_0 lies outside of $\mathcal{C}$ when A is on $\mathcal{C} - \mathscr{A}$. Now consider the reflection $\mathcal{C}'$ of the conic $\mathcal{C}$ in the point D_0. When D_0 lies inside $\mathcal{C}$, it also lies inside $\mathcal{C}'$, and therefore the two conics $\mathcal{C}, \mathcal{C}'$ overlap. Since reflection in D_0 fixes all the points on l_∞, the conic $\mathcal{C}'$ induces the same involution on l_∞ that $\mathcal{C}$ does. Therefore, they have exactly two points in common. Labeling these points as B and C, it is clear that D_0 is the midpoint of segment BC, and from this and $K(A) = D_0$ it follows that G is the centroid of ABC. On the other hand, if A lies outside of $\mathscr{A}$, then D_0 lies outside of $\mathcal{C}$, and in this case, $\mathcal{C}'$ does not intersect $\mathcal{C}$ (so there can be no triangle inscribed in $\mathcal{C}$ with centroid G). Applying the same argument to the points B and C instead of A shows that B and C are also on the arc $\mathscr{A}$. The next to last assertion follows from Proposition 7 and the comments preceding Lemma 5. $\square$

We can use the proof of Theorem 8 to give a construction of the locus $\mathscr{L}$ of points P, for a given triangle ABC, for which the map M is a half-turn. To do this, start with the construction of the points $Q_1 Z_1 P_1' O_1$ on circle $\mathcal{C}$, as in the first paragraph of the proof. Pick a point A_1 on the arc $\mathscr{A} = P_1 Q_1 Q_1' P_1'$, and determine the unique pair of points B_1, C_1 on $\mathscr{A}$ so that the centroid of $A_1 B_1 C_1$ is the point $G_1 = Z_1 S_1 \cdot Q_1 P_1'$, with S_1 the midpoint of segment $Q_1 O_1$. Then determine the unique affine map A for which $\mathsf{A}(A_1 B_1 C_1) = ABC$. The points $P = \mathsf{A}(P_1)$ and $P' = \mathsf{A}(P_1')$ describe the locus $\mathscr{L}$, as A runs over all affine maps with $A_1 \in \mathscr{A}$. This locus is shown in Figure 4, and turns out to be an elliptic curve $\mathcal{E}$ minus 6 points, as we show below. For the pictured triangle $A_1 B_1 C_1$ and its half-turn M_1, the map $\mathsf{M} = \mathsf{A} \circ \mathsf{M}_1 \circ \mathsf{A}^{-1}$ is a half-turn about the point $S = \mathsf{A}(S_1)$. Note that $\mathcal{E}$ is tangent to the sides of the anticomplementary triangle $K^{-1}(ABC)$ of ABC at the vertices.

An equation for the curve $\mathcal{E}$ can be found using barycentric coordinates. It can be shown (see [13]) that homogeneous barycentric coordinates of the points S and

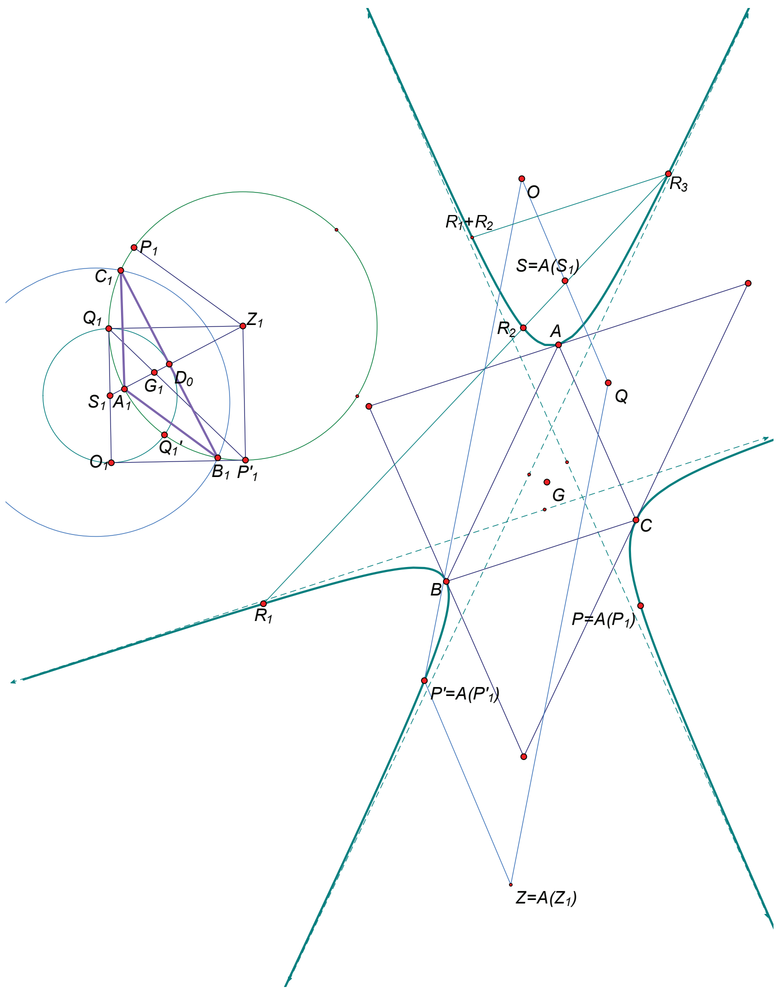

Figure 4. Elliptic curve locus of P with M a half-turn.

Z are

$$S = (x(y+z)^2, y(x+z)^2, z(x+y)^2), \quad Z = (x(y-z)^2, y(z-x)^2, z(x-y)^2),$$

where $P = (x, y, z)$. Using the remark after Corollary 3, we compute that the points $P = (x, y, z)$, for which M is a half-turn, satisfy $S = K(Z)$, so the coordinates of P satisfy the equation

$$\mathcal{E}: \quad x^2(y+z) + y^2(x+z) + z^2(x+y) - 2xyz = 0.$$

Note that $P \in \mathcal{E} \Rightarrow P' \in \mathcal{E}$. Setting $z = 1 - x - y$, where (x, y, z) are absolute barycentric coordinates, we get the affine equation for $\mathcal{E}$:

$$(5x-1)y^2 + (5x-1)(x-1)y - x^2 + x = 0. \tag{2}$$

This is the case $a = -5$ of the geometric normal form

$$(ax+1)y^2 + (ax+1)(x-1)y + x^2 - x = 0$$

which we consider in [13]. Rational points on $\mathcal{E}$ are $(x, y, z) = (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, which correspond to the vertices A, B, C. The points on $l_\infty \cap \mathcal{E}$ are $(x, y, z) = (0, 1, -1), (1, 0, -1)$, and $(1, -1, 0)$, which are the infinite points on the sides of ABC. No other points on the sides or medians of ABC or $K^{-1}(ABC)$ lie on the curve $\mathcal{E}$. Using (2) we can check directly that the curve $\mathcal{E}$ is tangent to $K^{-1}(ABC)$ at the points A, B, C and that it has no singular points. It follows from this that $\mathcal{E}$ is an elliptic curve, whose points form an abelian group under the addition operation given by the chord-tangent construction. (See [5], p. 67, or [14].) In Figure 4 the sum of the points R_1 and R_2 on $\mathcal{E}$ is the point $R_1 + R_2$, taking the point $A_\infty = BC \cdot l_\infty = (0, 1, -1)$ as the base point (identity for the addition operation on the curve). With the base point A_∞, the point A has order 2, while $B_\infty = AC \cdot l_\infty$ and $C_\infty = AB \cdot l_\infty$ have order 3, and the points B, C have order 6.

Note that if $P \in \iota(l_\infty)$ is a point on the Steiner circumellipse lying on $\mathcal{E}$, then $P' \in \mathcal{E} \cap l_\infty$, so that P' is one of the points $A_\infty = (0, 1, -1), B_\infty = (1, 0, -1), C_\infty = (1, -1, 0)$, whose isotomic conjugates are A, B, C. Other than the vertices of ABC, no points on the Steiner circumellipse lie on $\mathcal{E}$. Furthermore, the Steiner circumellipse is inscribed in the triangle $K^{-1}(ABC)$, while $\mathcal{E}$ is tangent to the sides of this triangle at A, B, C. By Proposition 7 and Theorem 8, any point P for which M is a half-turn has the property that the points $Q = K(P')$ and $Q' = K(P)$ are exterior to triangle ABC. It follows that P and P' are exterior to triangle $K^{-1}(ABC)$. Hence, *all* the points of $\mathcal{E} - \{A, B, C\}$ are exterior to triangle $K^{-1}(ABC)$, as pictured in Figure 4.

We now check that the hypotheses of Proposition 2 hold for all the points in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$. By the results of [12], the points for which the generalized orthocenter H is a vertex are contained in the union of three conics, $\overline{\mathcal{C}}_A \cup \overline{\mathcal{C}}_B \cup \overline{\mathcal{C}}_C$, which lie inside the Steiner circumellipse. By what we said above, none of the points in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ can lie on any of these conics, so H is never a vertex for these points. Hence, the hypotheses of Proposition 2 are satisfied for any P in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ and Corollary 3 implies that the map M for the point P is a half-turn. Thus, we have the following result.

Theorem 9. *The locus of points P, not lying on the sides of triangles ABC or $K^{-1}(ABC)$, for which $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a half-turn, coincides with the set of points whose barycentric coordinates lie in $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$.*

Every point P on $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ is a point for which $QZP'O$ is a parallelogram. This yields an affine map A, for which $\mathsf{A}(Q_1Z_1P_1'O_1) = QZP'O$, and implies by the proof of Theorem 8 that $\mathsf{A}^{-1}(ABC) = A_1B_1C_1$ is a triangle with centroid G_1, inscribed on the arc $\mathscr{A}$. Lemma 5 shows that P_1 is the isotomic conjugate of P_1' with respect to $A_1B_1C_1$; hence $P = \mathsf{A}(P_1)$. This shows that every point on $\mathcal{E}$ except the vertices and points at infinity is $P = \mathsf{A}(P_1)$ for some affine mapping A in the "locus" of maps with $A_1 \in \mathscr{A}$.

Now, each point A_1 on $\mathscr{A}$ yields two points on $\mathcal{E}$, since A maps both points P_1 and P_1' to points on $\mathcal{E}$. Alternatively, with a given triangle $A_1B_1C_1$, there are affine maps $\mathsf{A}, \widetilde{\mathsf{A}}$ for which $\mathsf{A}(A_1B_1C_1) = ABC$ and $\widetilde{\mathsf{A}}(A_1C_1B_1) = ABC$. We claim first that $\mathsf{A}(P_1) = -\widetilde{\mathsf{A}}(P_1)$, i.e., that $P = \mathsf{A}(P_1)$ and $\tilde{P} = \widetilde{\mathsf{A}}(P_1)$ are negatives on the curve $\mathcal{E}$ with respect to the addition on the curve. This is equivalent to the fact that the line $P\tilde{P}$ through these two points is parallel to BC. This is obvious from the fact that $\widetilde{\mathsf{A}} = \rho \circ \mathsf{A}$, where, as in the proof of Lemma 1, ρ is the affine reflection in the direction of the line BC, fixing the points on the median $AD_0 = AG$.

We claim now that the points on $\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$ are in $1-1$ correspondence with the collection of point-map pairs (A_1, A) and $(A_1, \widetilde{\mathsf{A}})$ for $A_1 \in \mathscr{A} - \{P_1, Q_1, P_1', Q_1'\}$. Suppose that (A_1, A_1) and (A_2, A_2) map to the same point P on $\mathcal{E}$, for $A_1, A_2 \in \mathscr{A} - \{P_1, Q_1, P_1', Q_1'\}$. Then $\mathsf{A}_1(A_1B_1C_1) = ABC = \mathsf{A}_2(A_2B_2C_2)$ or $\mathsf{A}_1(A_1B_1C_1) = ABC = \mathsf{A}_2(A_2C_2B_2)$, since the labeling of the points B_i, C_i can be switched; and for these maps, $\mathsf{A}_1(P_1) = P = \mathsf{A}_2(P_1)$. Then $\mathsf{A}_1^{-1}\mathsf{A}_2(A_2B_2C_2) = A_1B_1C_1$ or $A_1C_1B_1$ and $\mathsf{A}_1^{-1}\mathsf{A}_2(P_1) = P_1$. But then the map $\mathsf{A}_1^{-1}\mathsf{A}_2$ also fixes the points Q_1, P_1', Q_1', since G_1 is the centroid for both triangles. Hence, $\mathsf{A}_1^{-1}\mathsf{A}_2$ is the identity and $\mathsf{A}_1 = \mathsf{A}_2$, so that $A_1B_1C_1 = \mathsf{A}_1^{-1}(ABC) = A_2B_2C_2$ or $A_2C_2B_2$. Therefore, $(A_2, \mathsf{A}_2) = (A_1, \mathsf{A}_1)$.

Thus, we have proved the following.

Theorem 10. *Given a circle $\mathcal{C}$ with center Z_1, points Q_1 and P_1' on $\mathcal{C}$, and point O_1 for which $Q_1Z_1P_1'O_1$ is a square, then with the points G_1, P_1, S_1 as in Figure 4, the set of points*

$$\mathcal{E} - \{A, B, C, A_\infty, B_\infty, C_\infty\}$$

on the elliptic curve $\mathcal{E}$ coincides with the set of points $\mathsf{A}(P_1)$, where $A_1 \in \mathscr{A} = P_1Q_1Q_1'P_1'$ is a point on the arc $\mathscr{A}$ distinct from the points in $\{P_1, Q_1, Q_1', P_1'\}$, B_1, C_1 are the unique points on $\mathscr{A}$ for which $A_1B_1C_1$ has centroid G_1, and A is an affine map for which $\mathsf{A}(A_1B_1C_1) = ABC$ or $\mathsf{A}(A_1C_1B_1) = ABC$.

Note finally that the discriminant of (2) with respect to y is $D = (x-1)(5x-1)(5x^2-2x+1)$, so (2) is birationally equivalent to the curve

$$Y^2 = (X-1)(5X-1)(5X^2-2X+1).$$

Putting $X = \frac{u}{u-4}, Y = \frac{8v}{(u-4)^2}$ shows that this curve is, in turn, birationally equivalent (over $\mathbb{Q}$) to

$$v^2 = (u+1)(u^2+4), \tag{3}$$

which has j invariant $j = \frac{2^4 11^3}{5^2}$. This curve is curve (20A1) in Cremona's tables [4], and has the torsion subgroup $T = \{O, (-1,0), (0, \pm 2), (4, \pm 10)\}$ of order 6 and rank $r = 0$ over $\mathbb{Q}$. The curve $\mathcal{E}$ has infinitely many real points defined over quadratic extensions of $\mathbb{Q}$, including, for example,

$$P = (-4 + \sqrt{19}, -1, 3), \quad \left(\frac{9 + \sqrt{89}}{2}, -2, 1\right).$$

This shows that there are infinitely many points for which the map M is a half-turn. There are even infinitely many such points defined over the field $\mathbb{Q}(\sqrt{6})$, since the points $(u, v) = (2, 2\sqrt{6})$ and $(u, v) = (\frac{2}{3}, \frac{10\sqrt{6}}{9})$ have infinite order on (3).

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Golden Elliptical Orbits in Newtonian Gravitation

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Abstract. In spherical symmetry with radial coordinate r, classical Newtonian gravitation supports circular orbits and, for $-1/r$ and r^2 potentials only, closed elliptical orbits [1]. Various families of elliptical orbits can be thought of as arising from the action of perturbations on corresponding circular orbits. We show that one elliptical orbit in each family is singled out because its focal length is equal to the radius of the corresponding unperturbed circular orbit. The eccentricity of this special orbit is related to the famous irrational number known as the golden ratio. So inanimate Newtonian gravitation appears to exhibit (but not prefer) the golden ratio which has been previously identified mostly in settings within the animate world.

1. Introduction

In 1873, J. Bertrand [1] proved that the only spherically symmetric gravitational potentials that can support bound closed noncircular orbits are the Newton-Kepler $-1/r$ potential [12] and the isotropic Hooke r^2 potential [8], where r is the spherical radial coordinate with respect to the central mass that generates the potential. The elliptical orbits in these two potentials were already known to I. Newton [12].

Both potentials also support circular orbits and the elliptical orbits can be thought of as arising from such circular orbits perturbed by disturbances of any arbitrary amplitude. Given a circular orbit of radius r_o, an infinite family of elliptical orbits can thus be obtained with eccentricities in the range $0 < e < 1$, where e is related to the ratio of semiaxes b/a of the ellipses by

$$e = \sqrt{1 - \frac{b^2}{a^2}}\,. \tag{1}$$

In recent work [3], we found a new geometric property that characterizes each family of elliptical orbits and this property switches between the two potentials in a highly symmetric fashion: the circular radius r_o is the harmonic mean of the radii of the turning points $r_{\max} = a(1+e)$ and $r_{\min} = a(1-e)$ of the ellipses in a $-1/r$ potential; whereas r_o is the geometric mean of of the turning points $r_{\max} = a$ and $r_{\min} = b$ of the ellipses in an r^2 potential. For the reader's convenience, we summarize in Section 2 the derivation of these properties and we proceed in Section 3 to search in each family for special orbits with additional geometric properties. Interestingly, we find that Newtonian gravitation singles out

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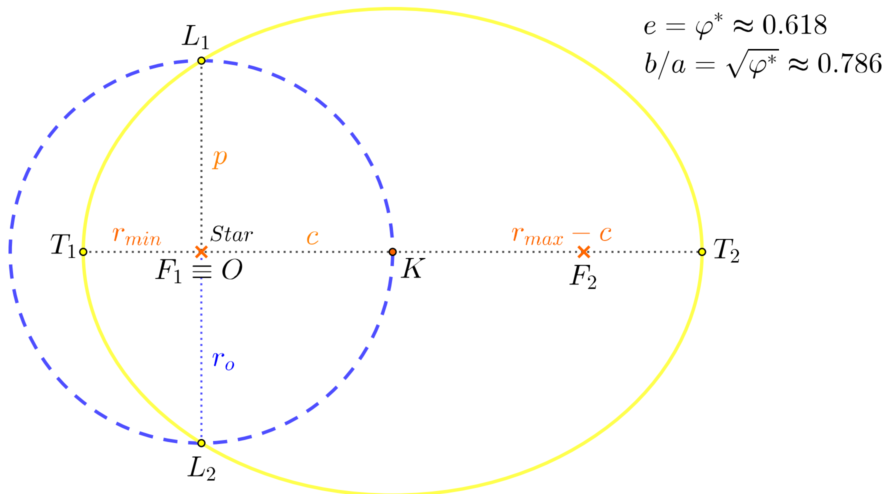

Figure 1. Schematic diagram of a circular orbit $O(r_o)$ and the associated golden elliptical orbit in a Newtonian gravitational potential of the form $-1/r$ due to a star located at focus F_1. This focus coincides with the center O of the circular orbit. Point K on $\odot O$ is the center of the ellipse; T_1 and T_2 are the turning points; and L_1 and L_2 are the endpoints of the latus rectum. The golden ellipse is singled out because of the equality $c = p = r_o$ and its eccentricity $e = \phi^* \approx 0.618$.

some elliptical orbits whose eccentricities are related to the golden ratio conjugate [14]

$$\varphi^* \equiv \frac{\sqrt{5} - 1}{2} \approx 0.618 \,. \tag{2}$$

The importance of the result lies in the fact that this famous irrational number is not introduced by the geometry of space as in the ubiquitous case of π in Euclidean spaces; the golden ratio conjugate is instead singled out by the dynamics of the noncircular orbits. We discuss our conclusions further in Section 4.

2. Known geometric properties of elliptical orbits

We apply the laws of energy and angular momentum conservation to a family of elliptical orbits that arise from disturbances acting on a given circular orbit of radius r_o. We distinguish two cases, a Newton-Kepler potential and an isotropic Hooke potential.

2.1. *Newton-Kepler $-1/r$ potential.* Consider an equilibrium orbit with radius $r = r_o$ in a $-1/r$ potential and assume that this orbit is perturbed to an elliptical shape with turning points $r_{\min} = r_o - A_1$ and $r_{\max} = r_o + A_2$, where $0 < A_1 < r_o$ and $A_2 > A_1$ (Figure 1). At the turning points T_1 and T_2 shown in Figure 1, the radial velocity is zero ($dr/dt = 0$) and the total energy per unit mass can then be written as [7]

$$\mathcal{E} = \frac{\mathcal{L}^2}{2r^2} - \frac{\mathcal{G}\mathcal{M}}{r}\,, \tag{3}$$

where $\mathcal{G}$ is the gravitational constant, $\mathcal{M}$ is the mass that generates the potential, and the specific angular momentum satisfies $\mathcal{L}^2 = \mathcal{G}\mathcal{M}r_o = \text{const.}$, hence eq. (3) can be written in the form

$$\frac{\mathcal{E}}{\mathcal{G}\mathcal{M}} = \frac{r_o}{2r^2} - \frac{1}{r} = \text{const.}\,, \text{ if } dr/dt = 0\,. \tag{4}$$

Applied to the turning points, this equation yields

$$\frac{1}{A_1} - \frac{1}{A_2} = \frac{2}{r_o}\,, \tag{5}$$

or equivalently

$$\frac{1}{r_{\min}} + \frac{1}{r_{\max}} = \frac{2}{r_o}\,. \tag{6}$$

This last equation shows that, in a $-1/r$ potential, the circular equilibrium radius r_o is the harmonic mean of the radii of the turning points $r_{\min}$ and $r_{\max}$ of the elliptical orbits [3].

2.2. *Isotropic Hooke r^2 potential.* Consider an equilibrium orbit with radius $r = r_o$ in a $\Omega^2 r^2/2$ potential ($\Omega =$const.) and assume that this orbit is perturbed to an elliptical shape with turning points $r_{\min} = b$ and $r_{\max} = a$, where a and $b < a$ are the semiaxes of the ellipse (Figure 2). Here the radii of the turning points take these special values because in the r^2 potential the gravitational force always points toward the center of the ellipse where the central mass is located.

At the turning points E_1 and E_2 shown in Figure 2, the radial velocity is zero ($dr/dt = 0$) and the total energy per unit mass can then be written as

$$\frac{\mathcal{E}}{\Omega^2/2} = \frac{r_o^4}{r^2} + r^2 = \text{const.}\,, \text{ if } dr/dt = 0\,, \tag{7}$$

where the constant specific angular momentum is given by $\mathcal{L} = \Omega r_o^2$ in this case.

Applied to the turning points, this equation yields

$$r_{\max} r_{\min} = ab = r_o^2\,. \tag{8}$$

This last equation shows that, in an isotropic r^2 potential, the circular equilibrium radius r_o is the geometric mean of the semiaxes a and b of the elliptical orbits [3].

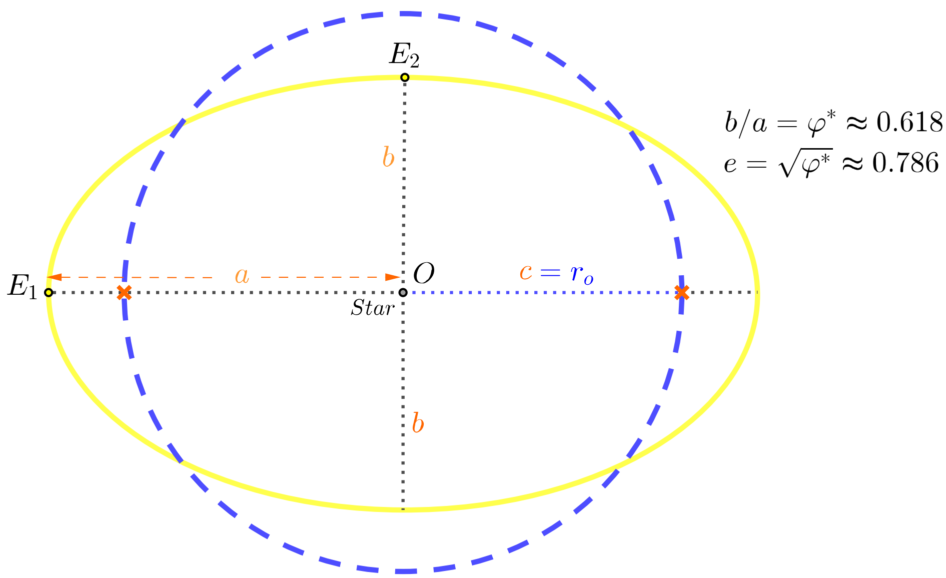

Figure 2. Schematic diagram of a circular orbit $O(r_o)$ and the associated golden elliptical orbit in an isotropic Hooke gravitational potential of the form r^2 due to a star located at center O. Unlike in the Newtonian case, the gravitational force always points toward O in this case. Points E_1 and E_2 are the endpoints of the semiaxes a and b of the ellipse. The golden ellipse is singled out because of the equality $c = r_o$, where c is the focal length, and its eccentricity $e = \sqrt{\phi^*} \approx 0.786$.

3. New geometric properties of elliptical orbits

The fundamental difference between the elliptical orbits in the above two cases is imposed by the dynamics: the gravitational force always points toward focus F_1 in Figure 1, whereas it always points toward center O in Figure 2. This difference is responsible for producing equations (6) and (8) and it also leads to several more geometric properties of the ellipses in each case, as we describe below.

3.1. *Newtonian ellipses.* Using the well-known equations of the ellipse with semiaxes a and b, viz.

$$a = (r_{\max} + r_{\min})/2\,, \tag{9}$$

and

$$b = \sqrt{r_{\max} r_{\min}}\,, \tag{10}$$

equation (6) takes the form

$$a r_o = b^2\,, \tag{11}$$

which indicates that, in Newtonian ellipses, the semiminor axis b is the geometric mean of a and r_o. More importantly, the semiaxes cannot be chosen independently in a particular family corresponding to a circular radius r_o. Since the semiaxes are also related to the eccentricity e (eq. (1)), each family of ellipses and the values of a and b are fully determined by the choice of e for fixed r_o.

Written in the equivalent form

$$r_o = \frac{b^2}{a} \equiv p\,, \tag{12}$$

equation (11) indicates that r_o is also equal to the semilatus rectum p.

Combining equations (12) and (1), we find that

$$r_o = a(1-e^2) = b\sqrt{1-e^2}\,, \tag{13}$$

and using the definition of the focal length c, i.e., $c = ae$, this equation takes the form

$$r_o = c\left(\frac{1-e^2}{e}\right). \tag{14}$$

Equation (13) shows that there are no ellipses in which one or the other semiaxis equals r_o. But equation (14) indicates that there exists a special ellipse in the family for which the focal length c equals r_o (Figure 1). The eccentricity of the special ellipse is found by setting $c = r_o$ in equation (14), in which case we find that

$$e^2 + e - 1 = 0\,, \tag{15}$$

whose solution is the golden ratio conjugate shown in equation (2) above. To summarize, the only special ellipse that is singled out by the Newton-Kepler dynamics for a given r_o is the golden ellipse with $e = \varphi^*$ and $c = p = r_o$, whereas all other ellipses in the family obey only $p = r_o$.

3.2. *Hookean ellipses.* We have already seen (eq. (8)) that in a family of Hookean ellipses, the radius r_o of the corresponding circular orbit is the geometric mean of the semiaxes a and b, which also implies that the areas of these ellipses are all equal to πr_o^2. Therefore, the semiaxes cannot be chosen independently within a particular family. Since the semiaxes are also related to the eccentricity e (eq. (1)), each family of ellipses and the values of a and b are fully determined by the choice of e for fixed r_o.

Using equations (8) and (1), we find that

$$r_o = a(1-e^2)^{1/4} = b(1-e^2)^{-1/4}\,, \tag{16}$$

and the semilatus rectum (eq. (12)) can be written as

$$p = r_o(1-e^2)^{3/4}\,. \tag{17}$$

Hence, these ellipses are very different than the Newtonian ellipses. None of the lengths a, b, or p can be equal to r_o.

Using the definition of the focal length $c = ae$, equation (16) takes the form

$$r_o = c\left[\frac{(1-e^2)^{1/4}}{e}\right], \tag{18}$$

which indicates that there exists a special ellipse in the family for which the focal length c equals r_o. The eccentricity of the special ellipse is found by setting $c = r_o$ in equation (18), in which case we find that

$$e^4 + e^2 - 1 = 0\,, \tag{19}$$

whose solution is $e = \sqrt{\varphi^*}$. This ellipse is plotted in Figure 2. Since $c = r_o$, an additional property of the golden ellipse is that its focal length is the geometric mean of its semiaxes, viz.

$$c^2 = ab\,. \tag{20}$$

4. Discussion

The main result of this work is the appearance of the golden ratio conjugate φ^* (eq. (2)) in Newtonian dynamics that predicts the existence of families of elliptical orbits only in $-1/r$ and r^2 spherically symmetric gravitational potentials [1]. In each case, the eccentricity e of the golden elliptical orbit is related to φ^* ($e = \varphi^*$ and $e = \sqrt{\varphi^*}$, respectively); these relations appear in the special case in which the focal length of the ellipse is equal to the radius of the corresponding circular orbit of the family ($c = r_o$ in Section 3 above).

A literature search shows that various authors use two different definitions of the golden ellipse: some authors [9, 10] define as golden the ellipse that is inscribed in a golden rectangle; others [13] construct the golden ellipse from golden right triangles in which case it is the eccentricity that is equal to the golden ratio conjugate. This latter definition is applicable to our Newtonian ellipses in which $b/a = \sqrt{\varphi^*}$. On the other hand, the former definition is applicable to our Hookean ellipses that have $b/a = \varphi^*$ and they can be inscribed in a golden rectangle.

Most of the appearances of φ^* in the animate world are well-known if not famous. The golden ratio has been identified in phyllotaxis of plants [4]; in human constructions, including the Parthenon and the Egyptian pyramids [11]; in the honeycombs of bees [6]; and recently in the shapes of red blood cells [15]. Lately, the golden ratio has also appeared in the inanimate world and our work concerning Newtonian dynamics falls in this category. Two more examples concern black holes [5] and cosmological theories [2], although the interested reader may easily track down more cases in solar neutrino mixing and quantum mechanics.

In the inanimate world, the golden ratio is usually introduced by the geometry of space or spacetime, but this is not the case in Newtonian dynamics. In our case, Newtonian gravitation allows for infinite families of elliptical orbits for a given circular equilibrium orbit, so the theory does not show a preference for ellipses with eccentricities $e = \varphi^*$. The golden ellipses are special only because they exhibit additional geometric properties, as we described in Section 3.1.

In our solar system, no planetary orbit has orbital eccentricity anywhere near the golden value of 0.618; our neighboring planets and the dwarf planets all move in fairly circular orbits with $e < 0.442$. Furthermore, we searched the exoplanet database (`//www.exoplanets.org`) that currently lists nearly 3,000 exoplanets with confirmed orbits and we found only three planets with $e = 0.610$, another two planets with $e = 0.630$, and no other planets with intermediate eccentricities.

So exoplanetary orbits appear to support the theoretical picture of no preference for the golden Newtonian elliptical orbits.

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On Some Elementary Properties of Quadrilaterals

Paris Pamfilos

Abstract. We study some elementary properties of convex quadrilaterals related to the idea of common harmonics of two pairs of points and detecting two similar parallelograms with diagonals parallel to those of the quadrilateral and such that the area of the quadrilateral is the mean geometric of their areas. We apply these results to the study of variation of area of quadrilaterals, whose angles remain fixed in measure.

1. The common harmonics

The common harmonics $\{X, Y\}$ of two pairs of collinear points $\{(A, B), (C, D)\}$ are points on their line, such that the resulting quadruples are simultaneously harmonic, i.e. their cross ratios ([4, p.70]) satisfy

$$\frac{XA}{XB} : \frac{YA}{YB} = \frac{XC}{XD} : \frac{YC}{YD} = -1.$$

A geometric view of the common harmonics results by considering them as the *limit points* of a non-intersecting pencil $\mathcal{P}$ of circles defined by the pairs $\{(A, B), (C, D)\}$ ([5, p. 109]). In fact the two circles $\{\alpha, \beta\}$, respectively on diameters $\{AB, CD\}$, define a pencil, and if this pencil is non-intersecting, then it has two limit points $\{X, Y\}$, which are harmonic conjugate with respect to the diametral

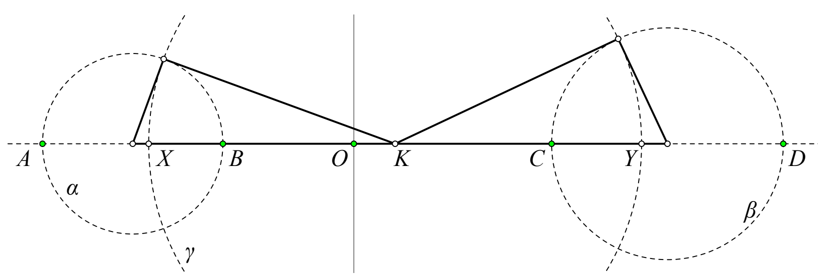

Figure 1. Common harmonics $\{X, Y\}$ of pairs $\{(A, B), (C, D)\}$

points $\{A, B\}$ of any circle of the pencil (see Figure 1). The common harmonics, as real points, exist precisely when the pencil is of non-intersecting type ([5, p. 118]). This case arrives when the intervals $\{(A, B), (C, D)\}$, as we say, *do not separate each other*, i.e. either are disjoint or one of them contains the other ([3, p. 284]). In the case the common harmonics exist, we can find them by intersecting their carrying line with a circle orthogonal to two circles of the aforementioned pencil. This, because the circles simultaneously orthogonal to those of the pencil

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$\mathcal{P}$ form another pencil $\mathcal{P}^{\perp}$ of intersecting type, all the circles of which pass through the points $\{X, Y\}$.

Lemma 1. *The common harmonics* $\{X, Y\}$ *of the pairs of collinear points* $\{(A, B), (C, D)\}$*, i.e. points for which their cross ratios satisfy* $(ABXY) = (CDXY) = -1$*, are given in coordinates by the following formula.*

$$x, y = \frac{1}{d+c-a-b}\left(cd - ab \pm \sqrt{(c-a)(c-b)(d-a)(d-b)}\right). \quad (1)$$

Proof. By the preceding remarks, in order to find quickly such an expression, it suffices to set two circles $\{\alpha, \beta\}$ on diameters $\{AB, CD\}$ and find the circle γ orthogonal to these two, with its center K on BC. Point K is the intersection of BC with the radical axis of the circles $\{\alpha, \beta\}$ and the intersection points of the circle γ with line BC define precisely the desired common harmonics $\{X, Y\}$ (see Figure 1). A short calculation, which I omit, shows that the corresponding coordinates are given by the stated formula. Here and in the sequel we use small letters $\{a, b, \dots\}$ for the line coordinates of points on a line represented with the corresponding capitals $\{A, B, \dots\}$. $\square$

2. The parallelogram associated to opposite vertices

It is well known, that the intersection points of an arbitrary line ε with four lines, passing through a common point O, define a cross ratio independent of the particular position of the line ε ([4, p.72], [2, p.251]). Thus, for every quadruple of lines, we can use such an arbitrary line ε, intersecting the quadruple at points $\{A, B, C, D\}$, and define the common harmonics $\{X, Y\}$ and the associated lines $\{OX, OY\}$, which we call the *common harmonics* of the line pairs $\{(OA, OB), (OC, OD)\}$. In particular, we may consider a convex quadrilateral $OBGC$ and from its vertex O draw parallels $\{OA, OD\}$ respectively to its sides $\{GC, GB\}$. Then define the common harmonics $\{OX, OY\}$ of the line pairs $\{(OA, OB), (OC, OD)\}$ (see Figure 1), and the parallelogram $OIGJ$ having its sides parallel to the lines$\{OX, OY\}$. In order to study this shape we use a coordinate system with origin at O and $x-$axis parallel to the diagonal BC of the

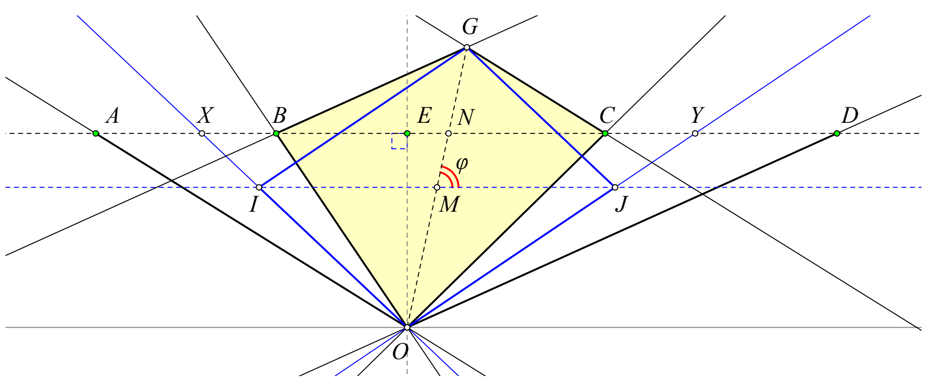

Figure 2. The parallelogram $OIGJ$

quadrilateral. In this system the $x-$coordinates of points $\{A, B, \dots\}$ are denoted by the corresponding small letters $\{a, b, \dots\}$, or occasionally by $\{A_x, B_y, \dots\}$ and the corresponding $y-$coordinates by $\{A_y, B_y, \dots\}$. A simple computation, which takes into account lemma-1 and which I omit, proves the first of the two next lemmata. In these E is the orthogonal projection of O on the diagonal BC.

Lemma 2. *The coordinates of G are*

$$G_x = \frac{cd - ba}{d-a}, \quad G_y = E_y \frac{c+d-a-b}{d-a},$$

the middle N of XY is on the diagonal OG and the diagonal IJ of the parallelogram $OJGI$ is parallel to the diagonal BC of $OBGC$.

Lemma 3. *The ratio of the areas of the quadrilaterals $\{OBGC, OIGJ\}$ depends only on the angles of the quadrilateral $OBGC$ and not on the length of its sides. This ratio is expressible through a cross ratio of the parallels to the four side-lines of the quadrilateral:*

$$\frac{|OIGJ|}{|OBGC|} = \sqrt{\frac{(c-a)(d-b)}{(c-b)(d-a)}} = \sqrt{(ABCD)}.$$

Proof. Using the angle ϕ between the diagonals and the formula for the area $|OBGC| = \frac{1}{2}|OG||BC|\sin(\phi)$, for the two quadrilaterals (see Figure 2), which have in common the diagonal OG, we see that the stated ratio is equal to the ratio of the other diagonals IJ/BC. Using the similarity of triangles $\{IOJ, XOY\}$, we see that $IJ = XY\frac{G_y}{2E_y}$. Hence, using lemmata 1 and 2 we get:

$$\begin{aligned}
\frac{IJ}{BC} &= \frac{XY \cdot G_y}{2 \cdot BC \cdot E_y} = \frac{XY}{2 \cdot BC} \cdot \frac{c+d-a-b}{d-a} \\
&= \frac{\frac{2}{d+c-a-b}\sqrt{(c-a)(c-b)(d-a)(d-b)}}{2(c-b)} \cdot \frac{c+d-a-b}{d-a} \\
&= \frac{\sqrt{(c-a)(c-b)(d-a)(d-b)}}{(c-b)(d-a)} = \sqrt{\frac{(c-a)(d-b)}{(c-b)(d-a)}}.
\end{aligned}$$

The last equality results from the definition of the expression of the cross ratio through the coordinates $(ABCD) = \frac{a-c}{b-c} : \frac{a-d}{b-d}$. This quantity expresses also the cross ratio of the ordered quadruple of lines (OA, OB, OC, OD), which remains fixed if the angles of the quadrilateral do not change. □

Starting with the opposite to O, vertex G of the quadrilateral, and doing the same work as before, leads to a parallelogram, which is identical to the previous one. The reason for this is apparent in Figure 3-I. The two configurations on either side of the diagonal BC of the quadrilateral are similar. The relations between triangles, more explicitly, being $\{OAB \sim GCA,\ OBC \sim GA'B',\ OCB \sim GB'B\}$. From the similarity follows that $\{(OX, GX'), (OY, GY')\}$ are pairs of parallel lines (see Figure 3-II), hence they coincide with the side-lines of the parallelogram $OIGJ$ of lemma 3. The following theorem recapitulates the results so far.

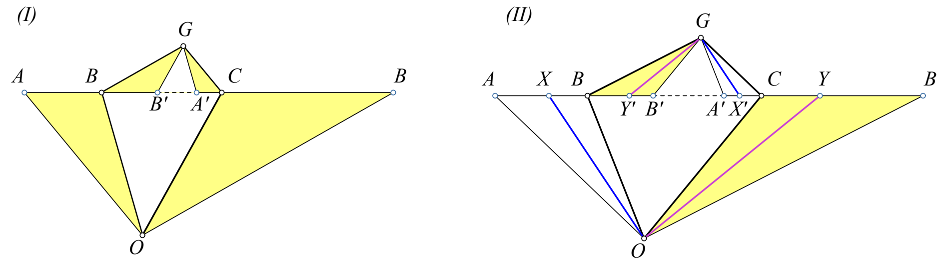

Figure 3. The same parallelogram relative to opposite vertices

Theorem 4. *Each pair of opposite vertices of a convex quadrilateral defines a parallelogram whose side-lines are the common harmonics of pairs of lines parallel to its sides and passing through the vertices of the pair. The diagonal joining the opposite vertices is also diagonal of the parallelogram. The other diagonals of the two quadrilaterals are parallel. In addition the ratio of the areas of the two quadrilaterals depends only on the angles of the convex quadrilateral.*

3. The two parallelograms

Repeating the construction of the previous section for the other pair of opposite points, we obtain a second parallelogram, one diagonal of which coincides now with the diagonal BC of the quadrangle of reference $OBGC$. Drawing the parallel ε' to the diagonal $\varepsilon = BC$ and doing the construction of the common

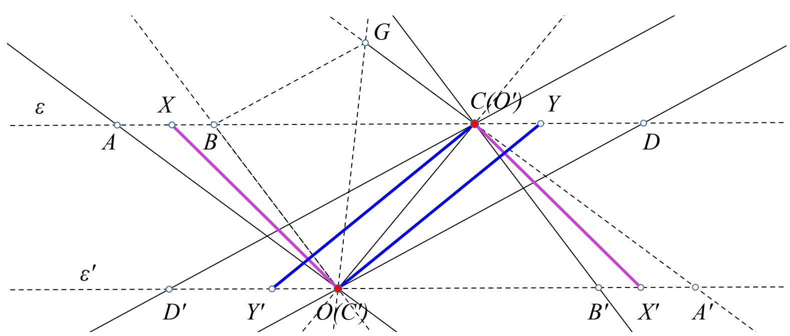

Figure 4. The common harmonics of adjacent vertices are pairwise parallel

harmonics this time on C (see Figure 4), we see that the parallels to the sides of the quadrilateral $OBGC$ from C form a quadruple of lines respectively parallel to those of the quadruple of lines through O. It follows that the common harmonics $\{CX', CY'\}$ of the pairs of lines $\{(CA', CB'), (CO, CD')\}$ are respectively parallel to the common harmonics $\{OX, OY\}$ constructed in the previous section. This leads to the following theorem.

Theorem 5. *The two parallelograms of pairs of opposite vertices of a general convex quadrilateral are homothetic w.r.t. to a point Q of the Newton line of the*

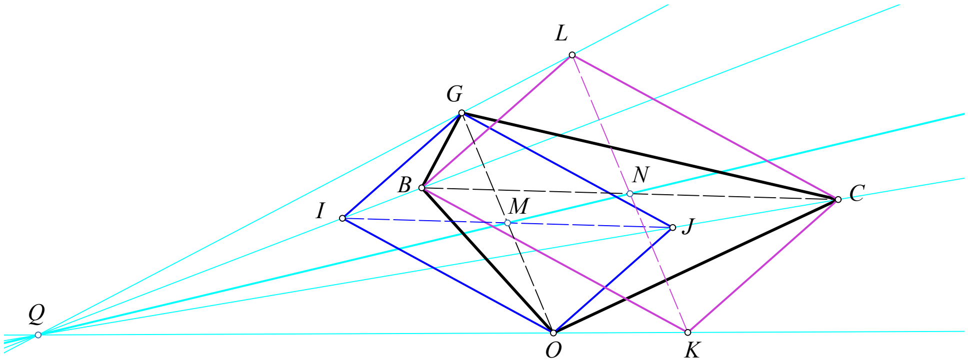

Figure 5. The two homothetic paralellolograms of a convex quadrilateral

quadrilateral. In addition, the area v *of the quadrilateral is the geometric mean* $v^2 = v_1 \cdot v_2$ *of the areas of the two parallelograms.*

Proof. By the preceding remarks, the two parallelograms have parallel sides. Besides, since their diagonals are parallel to the diagonals of the quadrilateral, they form the same angle. Thus, the parallelograms are similar and consequently also homothetic. Since their centers are the middles $\{M, N\}$ of the diagonals of the quadrilateral of reference, their homothety center lies on line MN, which is the Newton line of the quadrilateral (see Figure 5). For the last claim use lemma 3, by which the ratio v/v_1 is expressible through a cross ratio. Using the configuration of figure-4, we easily see that v/v_2 involves the inverse cross ratio of the previous one, so that the product $(v/v_1)(v/v_2) = 1$. $\square$

Remark. Using the ratio of lemma 3, we can express the ratio QM/QN of the homothety of the two parallelograms and through it also find arithmetically the location of Q on the Newton line (see Figure 5). In fact, $QM/QN = GO/LK = \sqrt{v_1/v_2} = \sqrt{(v_1/v)(v/v2)} = v_1/v$, the last quotient being expressible through a cross ratio as in the aforementioned lemma.

In the sequel we shall call these two the *associated parallelograms* of the convex quadrilateral.

4. Rhombi, rectangles and squares

For special quadrilaterals the corresponding associated parallelograms obtain characteristic shapes. The simplest case is the one of *orthodiagonal quadrilaterals*, i.e. quadrilaterals, whose the diagonals are orthogonal. Since the diagonals of the parallelograms are parallel to the diagonals of the quadrilateral of reference, we obtain the following characteristic property (see Figure 6).

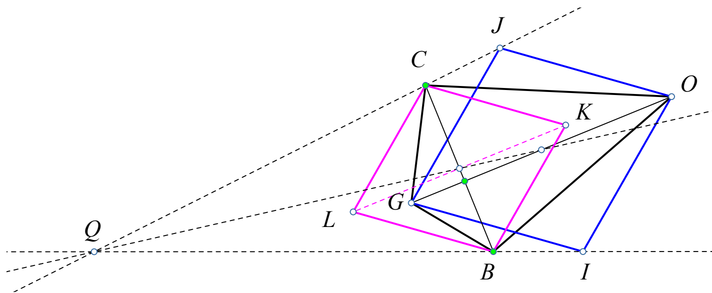

Figure 6. Rhombi as associated parallelograms of orthodiagonal quadrilaterals

Corollary 6. *For orthodiagonal quadrilaterals the associated paralleograms are rhombi.*

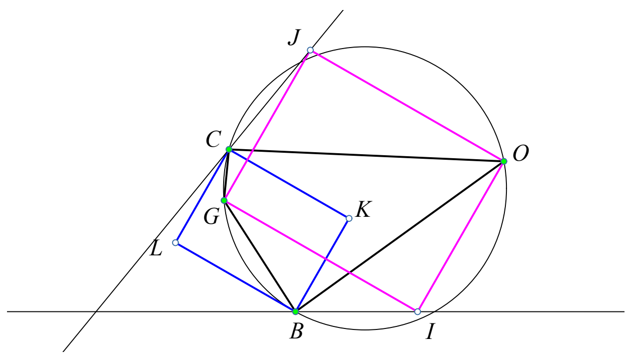

Figure 7. Rectangles characteristic for inscriptible quadrilaterals

Theorem 7. *For inscriptible in a circle quadrilaterals the associated parallelograms are rectangles.*

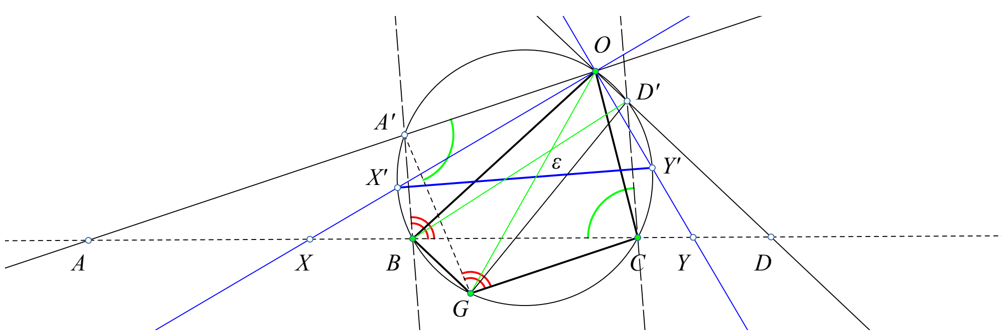

Figure 8. The orthogonal common harmonics $\{OX, OY\}$

Proof. For the proof repeat the construction of the common harmonics $\{OX, OY\}$ of the parallel to the sides through the vertex O of the quadrilateral $OBGC$, as in section 2. It suffices to show that $\{OX, OY\}$ are orthogonal. Let the paralells $\{OA, OD\}$ to sides $\{GC, GB\}$ intersect the circumcircles at points $\{A', D'\}$. We notice first that lines $\{BA', CD'\}$ are parallel (see Figure 8). This follows from a simple angle chasing argument: $\phi = \widehat{A'BC} = \widehat{A'GC}$ and $\psi = \widehat{BCD'} = \widehat{OA'G}$. Later because the two chords $\{OG, BD'\}$ are equal, since $BGD'O$ is an isosceles trapezium. From the isosceles trapezium $GA'OC$ we have also $\phi + \psi = \pi$, thereby proving the stated parallelity. Consider then the isosceles trapezium $BA'D'C$ and draw the diameter ε orthogonal to its parallel sides, intersecting the circle at points $\{X', Y'\}$. Later points are the middles of the arcs $\{\widehat{A'B}, \widehat{D'C}\}$, hence the lines $\{OX', OY'\}$ are orthogonal and they are also bisectors of the respective angles $\{\widehat{COD}, \widehat{AOB}\}$. Thus, these orthogonal lines define also the common harmonics of the pairs $\{(OA, OB), (OC, OD)\}$, thereby proving the claim. □

Combining the last corollary and theorem, we obtain also the following characterization.

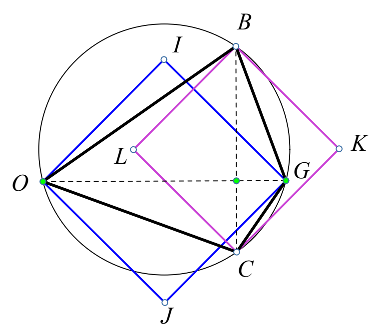

Figure 9. The case of squares

Corollary 8. *The associated parallelograms of orthodiagonal quadrilaterals, which are also inscriptible in a circle, are squares.*

5. The hyperbola arc

Next application of the associated parallelograms gives a generalization of the well known property of hyperbolas, according to which the parallelograms $OBGC$ with sides parallel to the asymptotes, where O is the center of the hyperbola and the opposite vertex G is a point of the hyperbola, have constant area ([7, p.192]). A more general problem would be to ask for the geometric locus of points G, which projected along the directions $\{\alpha, \delta\}$ on two intersecting at point O lines $\{\beta, \gamma\}$ form quadrilaterals $OBGC$ of constant area (see Figure 10). Next theorem handles this case in a slightly different formulation.

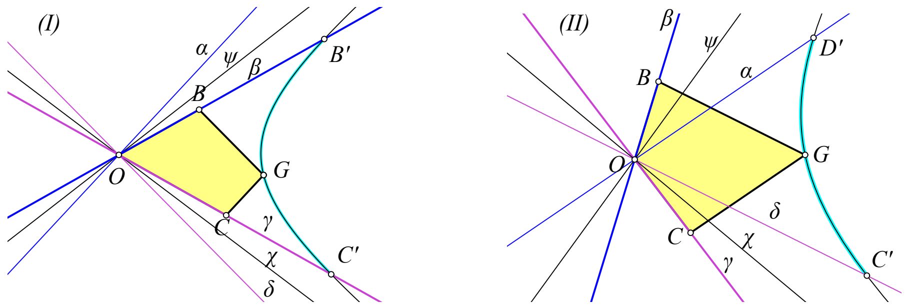

Figure 10. Locus of G for quadrilateral $OBGC$ of constant area

Theorem 9. *If the oriented convex quadrilateral $OBGC$ has sides pointing in fixed directions and its vertex O, as well as, its area is fixed, then the vertex G, opposite to the fixed one, lies on an arc of a hyperbola. The hyperbola is the one passing through G and having for asymptotes the common harmonics $\{\chi, \psi\}$ of the line pairs $\{(\alpha, \beta), (\gamma, \delta)\}$, where $\{\beta = OB, \gamma = OC\}$ and $\{\alpha, \delta\}$ are the parallels to $\{GC, GB\}$ from O.*

Proof. The proof follows directly from lemma-3, by drawing from G parallels to $\{\chi, \psi\}$ and defining the associated parallelogram $OIGJ$, as in figure-2. Since the area of the parallelogram is constant for G varying on the hyperbola and the area ratio $|OBGC|/|OIGJ|$ is constant, the area of $OIGJ$ remains also constant. The convexity requirement restricts the points G on an arc of the hyperbola determined by the lines $\{\beta, \gamma\}$, in the case the angle $\omega = \widehat{BOC}$ is acute (see Figure 10-I), and determined by the lines $\{\alpha, \delta\}$ in the case ω is obtuse (see Figure 10-II). □

6. Affine invariance

As is well known, affine transformations preserve the parallelity of lines ([1, p.191]). Since they are particular kinds of projective transformations, they preserve also the cross ratio of pencils of lines ([6, p.77]). This implies the *affine invariance* of the ratio of the areas of the two associated parallelograms of a convex quadrilateral, i.e. the invariance of this number for quadrilaterals, which map to each other by affine transformations. I take here the opportunity to make a short deviation into the way we can find a representative of a class of affinely equivalent convex quadrilaterals. The procedure is very simple and is illustrated by Figure 11. In this we fix a square $I_0J_0K_0L_0$ and for every other convex quadrilateral $q = ABCD$ we form its circumscribed parallelogram $IJKL$, with sides parallel to its diagonals. The correspondence of vertices $\{f_q(I) = I_0, f_q(J) = J_0, f_q(K) = K_0, f_q(L) = L_0\}$, extends to an affine transformation of the whole plane into itself, which we may denote by the same symbol f_q, and which maps q into the, inscribed in the square, quadrilateral $q' = f_q(q) = A'B'C'D'$. The general properties of affine transformations imply that the diagonals $\{A'C', B'D'\}$ are parallel to the sides of the square, hence equal, and the ratios $\{r_1, r_2\}$ on the diagonals, defined by their intersection

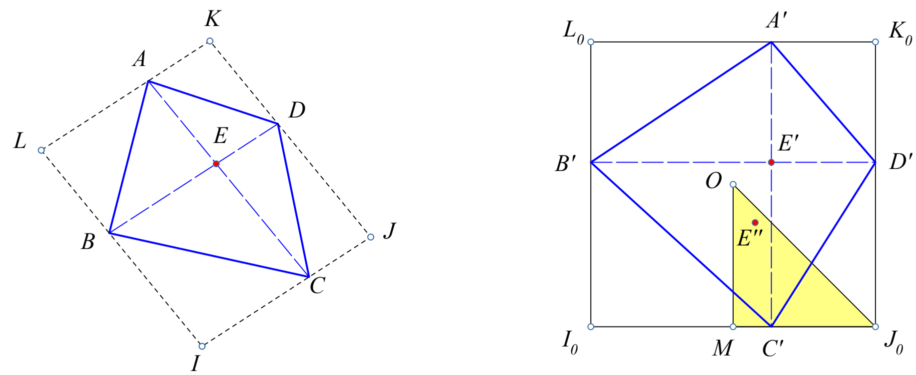

Figure 11. $A'B'C'D'$ representative of affine equivalents to $ABCD$

point, are preserved, i.e.

$$r_1 = \frac{EA}{EC} = \frac{E'A'}{E'C'}, \qquad r_2 = \frac{EB}{ED} = \frac{E'B'}{E'D'}.$$

Affine equivalent quadrilaterals map to equal corresponding inscribed in the square quadrilaterals $A'B'C'D'$. Later are characterized by the two ratios $\{r_1, r_2\}$. Using the symmetries of the square, for given $\{r_1, r_2\}$, we can in general construct 8 different but euclidean isometric inscribed in $I_0J_0K_0L_0$ quadrilaterals. This implies that the *moduli space*, i.e. the set of unique representatives for each class of affine equivalent quadrilaterals, can be set into bijective correspondence with the region defined by the triangle OMJ_0 shown in Figure 11 and representing the $1/8$ of the square. The points of this area determine a unique ordered pair of ratios (r_1, r_2) and a corresponding representative for the class of affinely equivalent quadrilaterals characterized by these two ratios. For example, point E'' in the figure represents $q = ABCD$ and results from E' by rotating it by 90° about the center O of the square. The boundary lines of the triangle have a special meaning. The point O represents the class of all parallelograms. The points of the segment OM represent the class of all quadrilaterals, whose one only of the diagonals is bisected by their intersection point. The points of the segment OJ_0 represent classes of quadrilaterals with diagonals divided by their intersection point into equal ratios, i.e. equilateral trapezia. Figure 12 shows the associated parallelograms of the affine representative $A'B'C'D'$ of $ABCD$. As noticed in section 4, they are rhombi, which in this case have one diagonal equal to the side of the square. The ratio of their areas, which is also equal to the area ratio of the original parallelograms, is equal to the ratio of their non-equal diagonals.

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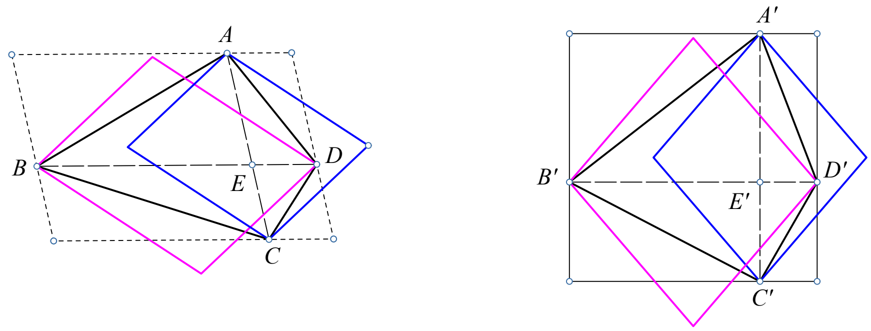

Figure 12. Affine representatives of associated parallelograms are rhombi

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Orthocenters of Simplices in Hyperbolic Spaces

Kenzi Satô

Abstract. We consider orthocenters of simplices of the hyperbolic space. Unlike the cases of the Euclidean spaces or the sphere, the similar condition does not always imply the existence of the orthocenter. In this paper, we give a characterization of the existence of the orthocenter.

1. Introduction

For a simplex of the Euclidean space, orthocenters exist only at most one point, and the existence of the orthocenter is equivalent to

$$(\mathbf{q}_i - \mathbf{q}_k) \cdot (\mathbf{q}_j - \mathbf{q}_\ell) = 0$$

for arbitrary pairwise distinct vertices $\mathbf{q}_i$, $\mathbf{q}_j$, $\mathbf{q}_k$, $\mathbf{q}_\ell$ (see, e.g., §1 of [2], (1.1) of [4]). For a simplex of the unit sphere of the Euclidean space, the existence of orthocenters is equivalent to

$$(\mathbf{p}_i^* \cdot \mathbf{p}_k^*)(\mathbf{p}_j^* \cdot \mathbf{p}_\ell^*) = (\mathbf{p}_i^* \cdot \mathbf{p}_\ell^*)(\mathbf{p}_j^* \cdot \mathbf{p}_k^*)$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$. Moreover, if there exist orthocenters, the uniqueness of the pair of antipodal orthocenters each other is equivalent that there exist at least two pairs of vertices $\{\mathbf{p}_i^*, \mathbf{p}_k^*\}$ and $\{\mathbf{p}_j^*, \mathbf{p}_\ell^*\}$ such that $\mathbf{p}_i^* \cdot \mathbf{p}_k^* \neq 0 \neq \mathbf{p}_j^* \cdot \mathbf{p}_\ell^*$. (Theorems 1 & 2 of [7]). In this paper we consider the orthocenter of the hyperbolic space. For a simplex of the Minkowski model of the hyperbolic space of the Minkowski space, orthocenters exist only at most one point, and if the orthocenter exists, then it holds that

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \langle \mathbf{p}_i^* | \mathbf{p}_\ell^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle \qquad \text{(a)}^*$$

for arbitrary pairwise distinct vertices $\mathbf{p}_i^*$, $\mathbf{p}_j^*$, $\mathbf{p}_k^*$, $\mathbf{p}_\ell^*$, where $\langle \cdot | \cdot \rangle$ means the pseudo-inner product of the Minkowski space. However, unlike two cases above, the condition (a)* does not always imply the existence of the orthocenter. The existence of it is equvalent to (a)* and the positivity of $-\langle \tilde{\mathbf{h}} \,|\, \tilde{\mathbf{h}} \rangle$ (Theorem 21), where $\tilde{\mathbf{h}}$ is a vector whose normalization is the orthocenter of the simplex of Minkowski model of the hyperbolic space (see the formula (73)). In the following figures, dashed curves, solid curves, dotted curves, and the black point are boundaries of Poincaré model of the 2-dimensional hyperbolic space, boundaries of simplices,

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pseudo-perpendiculars from vertices to the opposite faces, and the orthocenter, respectively (for the relation of Minkowski model and Poincaré model, see Remark 1).

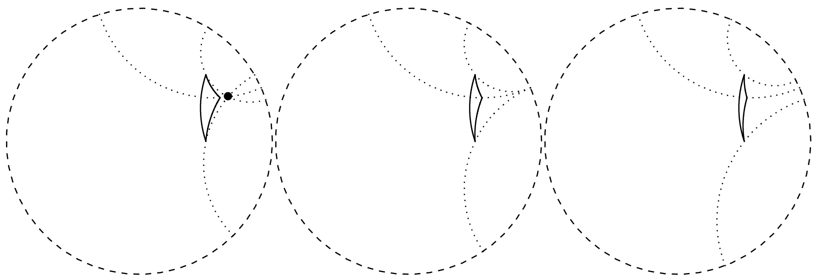

Figure 1.

2. Linear algebra.

We prepare three lemmas in this section. For $\mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \in \mathbb{R}^n$, $\langle\!\langle \mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \rangle\!\rangle \in \mathbb{R}^n$ is the unique vector such that

$$\langle\!\langle \mathbf{a}_0, \ldots, \mathbf{a}_{n-2} \rangle\!\rangle \cdot \mathbf{a}_{n-1} = \det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \tag{1}$$

for all $\mathbf{a}_{n-1} \in \mathbb{R}^n$, where $(\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}, \mathbf{a}_{n-1})$ is the matrix that has $\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}$, $\mathbf{a}_{n-1}$ as column vectors with Cartesian coordinate system [6]. The first lemma is the following.

Lemma 1. *For* $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1} \in \mathbb{R}^n$, *let*

$$\tilde{\mathbf{a}}_k = (-1)^{n-1-k} \langle\!\langle \mathbf{a}_0, \ldots \widehat{\mathbf{a}_k} \ldots, \mathbf{a}_{n-1} \rangle\!\rangle \tag{2}$$

for $k = 0, \ldots, n-1$, *where the circumflex indicates that the term below it has been omitted. Then we have*

$$(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1})^T = \mathrm{adj}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}), \tag{3}$$

$$\det(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1}) = (\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}))^{n-1}, \tag{4}$$

$$(-1)^{n-1-k} \langle\!\langle \tilde{\mathbf{a}}_0, \ldots, \widehat{\tilde{\mathbf{a}}_k}, \ldots, \tilde{\mathbf{a}}_{n-1} \rangle\!\rangle = (\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}))^{n-2} \mathbf{a}_k, \tag{5}$$

where adj $\mathbf{A}$ *is the transpose of the cofactor matrix of* $\mathbf{A}$.

Proof. The equation (3) comes from the first equation of Lemma 3.4 of [6] or from two equations

$$(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1})^T \cdot (\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) = \det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \cdot \mathbf{E}_n, \tag{6}$$

$$\mathrm{adj}\,\mathbf{A} \cdot \mathbf{A} = \det \mathbf{A} \cdot \mathbf{E}_n, \tag{7}$$

where (6) comes from (1) and (2), and (7) is an identity of an arbitrary square matrix $\mathbf{A}$ of degree n and the unit matrix $\mathbf{E}_n$ of degree n (notice that (6) and (7) imply (3) even if $\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) = 0$, because (3) becomes a polynomial identity if elements of matrices are replaced with variables. The same is true for the following equalities. See Chapter II, §5 of [5]). The equation (4) comes from (3) and the identity

$$\det(\mathrm{adj}\,\mathbf{A}) = (\det\mathbf{A})^{n-1}, \tag{8}$$

which comes from (7). The equation (3) means the identity of vectors $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$:

$$((-1)^{n-1}\langle\!\langle\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\rangle\!\rangle, \ldots, (-1)^{n-1-k}\langle\!\langle\mathbf{a}_0, \ldots \widehat{\mathbf{a}_k} \ldots, \mathbf{a}_{n-1}\rangle\!\rangle, \ldots, \langle\!\langle\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}\rangle\!\rangle)^T$$
$$= (\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_k, \ldots, \tilde{\mathbf{a}}_{n-1})^T = \mathrm{adj}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}).$$

So we also have the equation replaced $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$ with $\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1}$:

$$((-1)^{n-1}\langle\!\langle\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_{n-1}\rangle\!\rangle, \cdots, (-1)^{n-1-k}\langle\!\langle\tilde{\mathbf{a}}_0, \ldots \widehat{\tilde{\mathbf{a}}_k} \ldots, \tilde{\mathbf{a}}_{n-1}\rangle\!\rangle, \cdots, \langle\!\langle\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-2}\rangle\!\rangle)^T$$
$$= \mathrm{adj}(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1}),$$

which implies (5):

$$((-1)^{n-1}\langle\!\langle\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_{n-1}\rangle\!\rangle, \cdots, (-1)^{n-1-k}\langle\!\langle\tilde{\mathbf{a}}_0, \ldots \widehat{\tilde{\mathbf{a}}_k} \ldots, \tilde{\mathbf{a}}_{n-1}\rangle\!\rangle, \cdots, \langle\!\langle\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-2}\rangle\!\rangle)$$
$$= (\mathrm{adj}(\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1}))^T = \mathrm{adj}((\tilde{\mathbf{a}}_0, \ldots, \tilde{\mathbf{a}}_{n-1})^T)$$
$$= \mathrm{adj}(\mathrm{adj}(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})) = (\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}))^{n-2}(\mathbf{a}_0, \ldots, \mathbf{a}_k, \ldots, \mathbf{a}_{n-1}),$$

where the third equality comes from (3) and the last equality comes from the identity

$$\mathrm{adj}(\mathrm{adj}\,\mathbf{A}) = (\det\mathbf{A})^{n-2}\mathbf{A}, \tag{9}$$

which comes from (7), (8), and the identity replaced $\mathbf{A}$ of (7) with adj $\mathbf{A}$. □

$\langle\cdot|\cdot\rangle$ is the pseudo-inner product of $\mathbb{R}^n$, that is,

$$\langle\mathbf{x}|\mathbf{y}\rangle = \iota(\mathbf{x}) \,.\, \mathbf{y} \in \mathbb{R}, \tag{10}$$

for $\mathbf{x} = (\underline{\mathbf{x}}^T, x)^T$, $\mathbf{y} = (\underline{\mathbf{y}}^T, y)^T \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$, where

$$\iota(\mathbf{x}) = (\underline{\mathbf{x}}^T, -x)^T \in \mathbb{R}^n, \tag{11}$$

i.e.,

$$\langle\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{y}|\mathbf{x}\rangle = \underline{\mathbf{x}} \,.\, \underline{\mathbf{y}} - xy. \tag{12}$$

$\mathbb{R}^n$ with the pseudo-inner product $\langle\cdot\,|\,\cdot\rangle$ is called Minkowski space. The second lemma is the following.

Lemma 2. *Let* $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{n-1} \in \mathbb{R}^n$ *and* $m = 0, \ldots, n-1$. *Then, we have*

$$\iota(\langle\!\langle\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}\rangle\!\rangle) = -\langle\!\langle\iota(\mathbf{a}_0), \ldots, \iota(\mathbf{a}_{n-2})\rangle\!\rangle; \tag{13}$$

$$\begin{aligned}\det(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})\det(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}) &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1} \\ &= (-1)^{n-1}\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1}; \end{aligned} \tag{14}$$

$$\begin{aligned}\langle\langle\!\langle\mathbf{a}_0, \ldots, \mathbf{a}_{n-2}\rangle\!\rangle\,|\,\langle\!\langle\mathbf{b}_0, \ldots, \mathbf{b}_{n-2}\rangle\!\rangle\rangle &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2} \\ &= (-1)^{n-2}\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2}; \end{aligned} \tag{15}$$

if $\mathbf{b}_m \in \{\underline{\mathbf{0}}\} \times \mathbb{R}$,

$$\begin{aligned}\det(\mathbf{a}_k \,.\, \mathbf{b}_\ell)_{\ell=0,\ldots,m}^{k=0,\ldots,m} &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m} \\ &= (-1)^m\det(-\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m}, \end{aligned} \tag{16}$$

where $\underline{\mathbf{0}}$ *is the zero vector in* $\mathbb{R}^{n-1}$*, and we use the following notation:*

$$(F(k,\ell))_{\ell=\ell_0,\ldots,\ell_{h-1}}^{k=k_0,\ldots,k_{h-1}} = \left.\overbrace{\begin{pmatrix} & \vdots & \\ \cdots & F(k,\ell) & \cdots \\ & \vdots & \end{pmatrix}}^{\ell=\ell_0,\ldots,\ell_{h-1}}\right\} k=k_0,\ldots,k_{h-1}.$$

Proof. The latter equalities of (14), (15), and (16) are trivial, so it is enough to show the former equalities of them and (13). For the equation (13), we have

$$\begin{aligned}\langle -\iota(\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle)|\mathbf{a}'_{n-1}\rangle &= -\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle \,.\, \mathbf{a}'_{n-1}\\ &= -\det\left(\mathbf{a}_0,\ldots,\mathbf{a}_{n-2},\mathbf{a}'_{n-1}\right)\\ &= \det\left(\iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2}),\iota(\mathbf{a}'_{n-1})\right)\\ &= \langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle \,.\, \iota(\mathbf{a}'_{n-1})\\ &= \langle\langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle|\mathbf{a}'_{n-1}\rangle\end{aligned}$$

where $\mathbf{a}'_{n-1}$ is an arbitrary vector in $\mathbb{R}^n$. For (14), we have

$$\begin{aligned}\det(\mathbf{a}_0,\ldots,\mathbf{a}_{n-1})\det(\mathbf{b}_0,\ldots,\mathbf{b}_{n-1}) &= -\det(\iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-1}))\det(\mathbf{b}_0,\ldots,\mathbf{b}_{n-1})\\ &= -\det(\iota(\mathbf{a}_k)\,.\,\mathbf{b}_\ell)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1}\\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-1}^{k=0,\ldots,n-1}.\end{aligned}$$

For (15), we have

$$\begin{aligned}\langle\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle|\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle\rangle &= \iota(\langle\!\langle \mathbf{a}_0,\ldots,\mathbf{a}_{n-2}\rangle\!\rangle)\,.\,\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle\\ &= -\langle\!\langle \iota(\mathbf{a}_0),\ldots,\iota(\mathbf{a}_{n-2})\rangle\!\rangle\,.\,\langle\!\langle \mathbf{b}_0,\ldots,\mathbf{b}_{n-2}\rangle\!\rangle\\ &= -\det(\iota(\mathbf{a}_k)\,.\,\mathbf{b}_\ell)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2}\\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,n-2}^{k=0,\ldots,n-2},\end{aligned}$$

where the second equality comes from (13) and the third equality comes from Lemma 3.4 of [6]. For (16), we have

$$\begin{aligned}&\det(\mathbf{a}_k\,.\,\mathbf{b}_\ell)_{\ell=0,\ldots,m}^{k=0,\ldots,m}\\ &= b_m\det\begin{pmatrix}\underline{\mathbf{a}}_0\,.\,\underline{\mathbf{b}}_0+a_0b_0 & \cdots & \underline{\mathbf{a}}_0\,.\,\underline{\mathbf{b}}_{m-1}+a_0b_{m-1} & a_0\\ \vdots & & \vdots & \vdots\\ \underline{\mathbf{a}}_m\,.\,\underline{\mathbf{b}}_0+a_mb_0 & \cdots & \underline{\mathbf{a}}_m\,.\,\underline{\mathbf{b}}_{m-1}+a_mb_{m-1} & a_m\end{pmatrix}\\ &= b_m\det\begin{pmatrix}\underline{\mathbf{a}}_0\,.\,\underline{\mathbf{b}}_0-a_0b_0 & \cdots & \underline{\mathbf{a}}_0\,.\,\underline{\mathbf{b}}_{m-1}-a_0b_{m-1} & a_0\\ \vdots & & \vdots & \vdots\\ \underline{\mathbf{a}}_m\,.\,\underline{\mathbf{b}}_0-a_mb_0 & \cdots & \underline{\mathbf{a}}_m\,.\,\underline{\mathbf{b}}_{m-1}-a_mb_{m-1} & a_m\end{pmatrix}\\ &= -\det(\langle\mathbf{a}_k|\mathbf{b}_\ell\rangle)_{\ell=0,\ldots,m}^{k=0,\ldots,m},\end{aligned}$$

where $\mathbf{a}_k = ({\underline{\mathbf{a}}_k}^T, a_k)^T$ for $k = 0,\ldots, m$, $\mathbf{b}_\ell = ({\underline{\mathbf{b}}_\ell}^T, b_\ell)^T$ for $\ell = 0,\ldots, m-1$, and $\mathbf{b}_m = (\underline{\mathbf{0}}^T, b_m)^T$. □

The third lemma is the Cauchy-Schwarz inequality on Minkowski space.

Lemma 3. *For* $\mathbf{a} = (\underline{\mathbf{a}}^T, a)^T$, $\mathbf{b} = (\underline{\mathbf{b}}^T, b)^T \in \mathbb{R}^n$, *if* $|\underline{\mathbf{a}}| \leqq a$ *and* $|\underline{\mathbf{b}}| \leqq b$, *then we have*

$$-\langle \mathbf{a}|\mathbf{b}\rangle \geqq \sqrt{-\langle \mathbf{a}|\mathbf{a}\rangle}\sqrt{-\langle \mathbf{b}|\mathbf{b}\rangle}. \tag{17}$$

Moreover, if $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$ *and* $\mathbf{a}$ *and* $\mathbf{b}$ *are not parallel, i.e., if there does not exist* $c \in \mathbb{R} \setminus \{0\}$ *such that* $c\mathbf{a} = \mathbf{b}$, *then we also have*

$$-\langle \mathbf{a}|\mathbf{b}\rangle > \sqrt{-\langle \mathbf{a}|\mathbf{a}\rangle}\sqrt{-\langle \mathbf{b}|\mathbf{b}\rangle}. \tag{18}$$

Proof. We have

$$ab - \underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} \geqq ab - |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \geqq \sqrt{a^2 - |\underline{\mathbf{a}}|^2}\sqrt{b^2 - |\underline{\mathbf{b}}|^2}, \tag{19}$$

where the former and latter inequalities come from $\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} \leqq |\underline{\mathbf{a}}||\underline{\mathbf{b}}|$, and

$$(ab - |\underline{\mathbf{a}}||\underline{\mathbf{b}}|)^2 - (a^2 - |\underline{\mathbf{a}}|^2)(b^2 - |\underline{\mathbf{b}}|^2) = (b|\underline{\mathbf{a}}| - a|\underline{\mathbf{b}}|)^2 \geqq 0,$$

respectively. Moreover, if $\mathbf{a} \neq \mathbf{0} \neq \mathbf{b}$ (which means $a > 0$ and $b > 0$) and $\mathbf{a}$ and $\mathbf{b}$ are not parallel, one of two inequalities "$\geqq$" of (19) can be replaced with "$>$", because $\frac{b}{a}\underline{\mathbf{a}} \neq \underline{\mathbf{b}}$ implies either $\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} < |\underline{\mathbf{a}}||\underline{\mathbf{b}}|$ or $b|\underline{\mathbf{a}}| \neq a|\underline{\mathbf{b}}|$. In particular, if

$$\underline{\mathbf{a}} \,.\, \underline{\mathbf{b}} = |\underline{\mathbf{a}}||\underline{\mathbf{b}}| \tag{20}$$

and

$$b|\underline{\mathbf{a}}| = a|\underline{\mathbf{b}}|, \tag{21}$$

then we have $\underline{\mathbf{a}} = \underline{\mathbf{0}} = \underline{\mathbf{b}}$ or $\underline{\mathbf{a}} \neq \underline{\mathbf{0}} \neq \underline{\mathbf{b}}$. In the former case, we have $\frac{b}{a}\mathbf{a} = \frac{b}{a}(\underline{\mathbf{0}}^T, a)^T = (\underline{\mathbf{0}}^T, b)^T = \mathbf{b}$, which is a contradiction. In the latter case, there exists $c > 0$ such that $c\underline{\mathbf{a}} = \underline{\mathbf{b}}$ from (20) and we also have $c = \frac{b}{a}$ from (21), which is a contradiction. $\square$

Corollary 4. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, *if* $\langle \mathbf{a}|\mathbf{a}\rangle < 0$ *and* $\langle \mathbf{a}|\mathbf{b}\rangle = 0$ *then* $\langle \mathbf{b}|\mathbf{b}\rangle > 0$.

Proof. It is enough to show that $\langle \mathbf{a}|\mathbf{a}\rangle < 0$ and $\langle \mathbf{b}|\mathbf{b}\rangle \leqq 0$ imply $\langle \mathbf{a}|\mathbf{b}\rangle \neq 0$. If there exists $c \in \mathbb{R} \setminus \{0\}$ such that $c\mathbf{a} = \mathbf{b}$, we have $\langle \mathbf{a}|\mathbf{b}\rangle = c\langle \mathbf{a}|\mathbf{a}\rangle \neq 0$. Otherwise, from Lemma 3, we have one of the following four inequalities: $-\langle \mathbf{a}|\mathbf{b}\rangle > 0$, $-\langle \mathbf{a}|-\mathbf{b}\rangle > 0$, $-\langle -\mathbf{a}|\mathbf{b}\rangle > 0$, or $-\langle -\mathbf{a}|-\mathbf{b}\rangle > 0$. $\square$

3. Hyperbolic simplices.

Let

$$\mathbb{H}^{n-1} = \{\mathbf{x} = (\underline{\mathbf{x}}^T, x)^T \in \mathbb{R}^n : \langle \mathbf{x}|\mathbf{x}\rangle = -1, x > 0\},$$

which is Minkowski model of the hyperbolic space with constant curvature -1 (for hyperbolic geometry, see e.g., [1] or [8]).

Remark 1. The following figure is the relation of Minkowski model and Poincaré model, $\mathrm{int}(2 \cdot \mathbb{D}^{n-1}) = \{\underline{\mathbf{y}} \in \mathbb{R}^{n-1} : |\underline{\mathbf{y}}| < 2\}$, of the hyperbolic space (the correspondence of two models is $\underline{\mathbf{y}} = 2\underline{\mathbf{x}}/(x+1)$).

In hyperbolic space, an arbitrary point can be moved to another arbitrary point by a transformation preserving the pseudo-inner product.

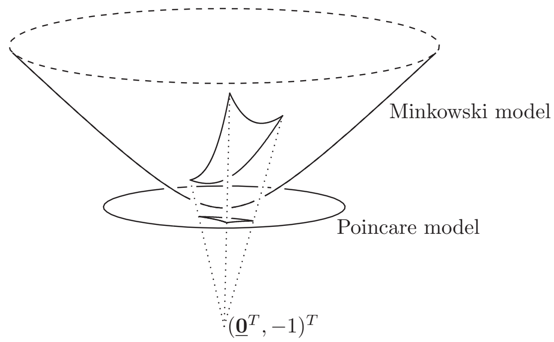

Figure 2.

Lemma 5. *For distinct* $\mathbf{a}$, $\mathbf{b} \in \mathbb{H}^{n-1}$, *the linear transformation* $\varphi(\,\cdot\,; \mathbf{a}, \mathbf{b})$ *of* $\mathbb{R}^n$ *defined by*

$$\varphi(\mathbf{x}; \mathbf{a}, \mathbf{b}) = \Big(\mathbf{x} + \langle \mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \frac{\langle \mathbf{x}\big|\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\rangle}{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}\big(\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\big)\Big) - \langle \mathbf{x}|\mathbf{a}\rangle\mathbf{b} + \frac{\langle \mathbf{x}\big|\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}\rangle}{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}\big(-\langle\mathbf{a}|\mathbf{b}\rangle\mathbf{b} - \mathbf{a}\big) \tag{22}$$

maps $\mathbf{a}$ *to* $\mathbf{b}$, *preserves the pseudo-inner product* $\langle\cdot|\cdot\rangle$ *of* $\mathbb{R}^n$, *and maps* $\mathbb{H}^{n-1}$ *onto itself. Remark that* $-\langle\mathbf{a}\,|\,\mathbf{b}\rangle > 1$ *from Lemma* 3 (*precisely, we have* $-\langle\mathbf{a}\,|\,\mathbf{b}\rangle = \cosh d_{\mathbb{H}^{n-1}}(\mathbf{a}, \mathbf{b})$).

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be the vectors of $\mathbb{R}\cdot\mathbf{a} + \mathbb{R}\cdot\mathbf{b}$ such that

$$\langle\mathbf{a}|\mathbf{v}\rangle = 0 = \langle\mathbf{b}|\mathbf{w}\rangle, \quad \langle\mathbf{b}|\mathbf{v}\rangle > 0 > \langle\mathbf{a}|\mathbf{w}\rangle, \quad \text{and} \quad \langle\mathbf{v}|\mathbf{v}\rangle = 1 = \langle\mathbf{w}|\mathbf{w}\rangle, \tag{23}$$

i.e.,

$$\mathbf{v} = \frac{\mathbf{b} + \langle\mathbf{a}|\mathbf{b}\rangle\mathbf{a}}{\sqrt{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}} \quad \text{and} \quad \mathbf{w} = \frac{-\langle\mathbf{a}|\mathbf{b}\rangle\mathbf{b} - \mathbf{a}}{\sqrt{\langle\mathbf{a}|\mathbf{b}\rangle^2 - 1}}. \tag{24}$$

(In other words, $\mathbf{v}$ (resp. $-\mathbf{w}$) is the tangent vector at $\mathbf{a}$ (resp. $\mathbf{b}$) of the geodesic between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{H}^{n-1}$ with $\langle\mathbf{v}|\mathbf{v}\rangle = 1$ (resp. $\langle\mathbf{w}|\mathbf{w}\rangle = 1$).) Then the transformation φ is represented by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$, and $\mathbf{w}$:

$$\varphi(\mathbf{x}; \mathbf{a}, \mathbf{b}) = \Big(\mathbf{x} + \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{v}\Big) - \langle\mathbf{x}|\mathbf{a}\rangle\mathbf{b} + \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{w}. \tag{25}$$

We have $\varphi(\mathbf{a}; \mathbf{a}, \mathbf{b}) = \Big(\mathbf{a} + (-\mathbf{a}) - \mathbf{0}\Big) - (-\mathbf{b}) + \mathbf{0} = \mathbf{b}$ and

$$\begin{aligned}
&\langle\varphi(\mathbf{x}; \mathbf{a}, \mathbf{b})\,|\,\varphi(\mathbf{y}; \mathbf{a}, \mathbf{b})\rangle \\
=\ &\Big\langle \mathbf{x} + \big(\langle\mathbf{x}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{x}|\mathbf{v}\rangle\mathbf{v}\big)\Big|\,\mathbf{y} + \big(\langle\mathbf{y}|\mathbf{a}\rangle\mathbf{a} - \langle\mathbf{y}|\mathbf{v}\rangle\mathbf{v}\big)\Big\rangle \\
&\quad + \langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\langle\mathbf{b}|\mathbf{b}\rangle + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle \\
=\ &\Big(\langle\mathbf{x}|\mathbf{y}\rangle + 2\big(\langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle - \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\big) + \big(\langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\cdot(-1) + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\cdot 1\big)\Big) \\
&\quad + \langle\mathbf{x}|\mathbf{a}\rangle\langle\mathbf{y}|\mathbf{a}\rangle\cdot(-1) + \langle\mathbf{x}|\mathbf{v}\rangle\langle\mathbf{y}|\mathbf{v}\rangle\cdot 1 \\
=\ &\langle\mathbf{x}|\mathbf{y}\rangle,
\end{aligned}$$

where the first equality comes from the pseudo-orthogonality of three terms of the right-hand side of (25) and the second equality comes from $\langle \mathbf{a} \,|\, \mathbf{a} \rangle = -1 = \langle \mathbf{b}|\mathbf{b} \rangle$ and (23). So $\varphi(\mathbb{H}^{n-1} \cup \iota(\mathbb{H}^{n-1}); \mathbf{a}, \mathbf{b}) = \mathbb{H}^{n-1} \cup \iota(\mathbb{H}^{n-1})$. The restriction $\varphi(\,\cdot\,; \mathbf{a}, \mathbf{b})|_{\mathbb{H}^{n-1} \cup \iota(\mathbb{H}^{n-1})}$ is homotopic to $\mathrm{id}_{\mathbb{H}^{n-1} \cup \iota(\mathbb{H}^{n-1})}$ by $\{\varphi(\,\cdot\,; \mathbf{a}, (1-t)\mathbf{a} \oplus_{\mathbb{H}^{n-1}} t\mathbf{b})|_{\mathbb{H}^{n-1} \cup \iota(\mathbb{H}^{n-1})} : t \in [0,1]\}$ where $(1-t)\mathbf{a} \oplus_{\mathbb{H}^{n-1}} t\mathbf{b}$ is the internally dividing point of the geodesic between $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{H}^{n-1}$ in the ratio of $t : (1-t)$ (we can represent it by

$$\begin{aligned} &(1-t)\mathbf{a} \oplus_{\mathbb{H}^{n-1}} t\mathbf{b} \\ &= \cosh\bigl(td_{\mathbb{H}^{n-1}}(\mathbf{a}, \mathbf{b})\bigr)\mathbf{a} + \sinh\bigl(td_{\mathbb{H}^{n-1}}(\mathbf{a}, \mathbf{b})\bigr)\mathbf{v} \\ &= \cosh\bigl((1-t)d_{\mathbb{H}^{n-1}}(\mathbf{a}, \mathbf{b})\bigr)\mathbf{b} - \sinh\bigl((1-t)d_{\mathbb{H}^{n-1}}(\mathbf{a}, \mathbf{b})\bigr)\mathbf{w}, \end{aligned} \tag{26}$$

precisely). $\mathbb{H}^{n-1}$ and $\iota(\mathbb{H}^{n-1})$ are disconnected, so $\varphi(\mathbb{H}^{n-1}; \mathbf{a}, \mathbf{b}) = \mathbb{H}^{n-1}$. $\square$

Let S be a hyperbolic simplex of $\mathbb{H}^{n-1}$ with vertices $\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^* \in \mathbb{H}^{n-1}$, i.e.,

$$S = \Bigl(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\Bigr) \cap \mathbb{H}^{n-1}, \tag{27}$$

where $\mathbb{R}^+$ is the set of non-negative real numbers. For an index $j = 0, \ldots, n-1$, if $\mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ satisfies $\langle \mathbf{p}_i^* | \mathbf{b} \rangle = 0$ for $i = 0, \ldots, \widehat{j}, \ldots, n-1$, then $\langle \mathbf{p}_j^* | \mathbf{b} \rangle \neq 0$ and $\langle \mathbf{b} | \mathbf{b} \rangle > 0$ from Corollary 4. Let $\mathbf{p}_j \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be one of the normalizations of such $\mathbf{b}$, i.e., let $\mathbf{p}_j$ be the unique vector such that

$$\langle \mathbf{p}_i^* | \mathbf{p}_j \rangle = 0 \quad \text{for} \quad i = 0, \ldots, \widehat{j}, \ldots, n-1; \quad \langle \mathbf{p}_j^* | \mathbf{p}_j \rangle > 0; \quad \text{and} \quad \langle \mathbf{p}_j | \mathbf{p}_j \rangle = 1. \tag{28}$$

Then,

$$S = \{\mathbf{x} \in \mathbb{H}^{n-1} : \langle \mathbf{x} | \mathbf{p}_i \rangle \geqq 0 \quad \forall i = 0, \ldots, n-1\}, \tag{29}$$

because of the following equivalence for $\mathbf{x} = \sum_{i=0}^{n-1} \alpha_i \mathbf{p}_i^* \in \mathbb{H}^{n-1}$:

$$\mathbf{x} \in S \iff \alpha_i \geqq 0 \quad \forall i \iff \alpha_i \langle \mathbf{p}_i^* | \mathbf{p}_i \rangle \geqq 0 \quad \forall i \iff \langle \mathbf{x} | \mathbf{p}_i \rangle \geqq 0 \quad \forall i.$$

Remark 2. From Lemma 5, we can assume that one vertex of S is $(\underline{\mathbf{0}}^T, 1)^T$, without loss of generality (the fact is similar to the spherical simplex, in particular, we can assume that one vertex of the spherical simplex S' of $\mathbb{S}^{n-1} = \{\mathbf{x}' \in \mathbb{R}^n : \mathbf{x}' \,.\, \mathbf{x}' = 1\}$ is $(\underline{\mathbf{0}}^T, 1)^T$, without loss of generality).

The symbols Δ^* and Δ mean

$$\Delta^* \left(k_0 \cdots k_\ell \begin{matrix} i_{\ell+1} \cdots i_m \\ j_{\ell+1} \cdots j_m \end{matrix} \right) = \det(-\langle \mathbf{p}_r^* | \mathbf{p}_s^* \rangle)_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_m}^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_m}, \tag{30*}$$

$$\Delta \left(k_0 \cdots k_\ell \begin{matrix} i_{\ell+1} \cdots i_m \\ j_{\ell+1} \cdots j_m \end{matrix} \right) = \det(\langle \mathbf{p}_r | \mathbf{p}_s \rangle)_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_m}^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_m}, \tag{30}$$

for integers $\ell, m = -1, \ldots, n-1$ with $\ell \leqq m$ and indices $k_0, \ldots, k_\ell$, $i_{\ell+1}, \ldots, i_m$, $j_{\ell+1}, \ldots, j_m = 0, \ldots, n-1$ (notice that left upper diagonal elements of matrices of the right-hand sides of the above, $-\langle \mathbf{p}_k^* \,|\, \mathbf{p}_k^* \rangle$ and $\langle \mathbf{p}_k \,|\, \mathbf{p}_k \rangle$ for $k = k_0, \ldots, k_\ell$, are equal to 1). For symbols above, we have following three lemmas, which are similar to lemmas of [6] and [7].

Lemma 6. *For an integer $m = -1,\ldots, n-1$ and pairwise distinct indices $k_0, \ldots, k_m = 0, \ldots, n-1$, we have the following inequalities:*

$$(-1)^m \Delta^*(k_0 \cdots k_m) \begin{cases} > 0 & (0 \leqq m \leqq n-1), \\ < 0 & (m = -1), \end{cases} \tag{31*}$$

$$\Delta(k_0 \cdots k_m) \begin{cases} > 0 & (-1 \leqq m \leqq n-2), \\ < 0 & (m = n-1). \end{cases} \tag{31}$$

Proof. It is enough to show the inequalities above for $k_r = r$ for $r = 0,\ldots, m$. For $m \geqq 0$, we can assume $\mathbf{p}_m^* = (\underline{\mathbf{0}}^T, 1)^T$, so we have

$$(-1)^m \Delta^*(0 \cdots m) = (-1)^m \det(-\langle \mathbf{p}_r^* | \mathbf{p}_s^* \rangle)_{s=0,\ldots,m}^{r=0,\ldots,m} = \det(\mathbf{p}_r^* \cdot \mathbf{p}_s^*)_{s=0,\ldots,m}^{r=0,\ldots,m} > 0.$$

where the second equality comes from (16). For $m = -1$, it is trivial:

$$(-1)^{-1} \Delta^*() = -1 < 0.$$

For $m \leqq n-2$, we can assume $\mathbf{p}_{n-1}^* = (\underline{\mathbf{0}}^T, 1)^T$, so $\mathbf{p}_0,\ldots, \mathbf{p}_m$ are in $\mathbb{R}^{n-1} \times \{0\}$. Hence, it is also trivial:

$$\Delta(0 \cdots m) = \det(\langle \mathbf{p}_r | \mathbf{p}_s \rangle)_{s=0,\ldots,m}^{r=0,\ldots,m} = \det(\mathbf{p}_r \cdot \mathbf{p}_s)_{s=0,\ldots,m}^{r=0,\ldots,m} > 0.$$

For $m = n-1$, we have

$$\Delta(0 \cdots n-1) = \det(\langle \mathbf{p}_r | \mathbf{p}_s \rangle)_{s=0,\ldots,n-1}^{r=0,\ldots,n-1} = -(\det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}))^2 < 0,$$

where the second equality comes from (14). □

Lemma 7. *Let*

$$\varepsilon^* = \operatorname{sgn} \det(\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*). \tag{32*}$$

Then we have

$$-\varepsilon^* = \operatorname{sgn} \det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}), \tag{32}$$

$$\mathbf{p}_k = (-1)^{n-1-k} \varepsilon^* \frac{\iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^* \rangle\!\rangle)}{\sqrt{(-1)^{n-2} \Delta^*(0 \cdots \widehat{k} \cdots n-1)}}, \tag{33*}$$

$$\mathbf{p}_k^* = (-1)^{n-1-k} (-\varepsilon^*) \frac{\iota(\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1} \rangle\!\rangle)}{\sqrt{\Delta(0 \cdots \widehat{k} \cdots n-1)}}, \tag{33}$$

$$\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle = \frac{\sqrt{(-1)^{n-1} \Delta^*(0 \cdots n-1)}}{\sqrt{(-1)^{n-2} \Delta^*(0 \cdots \widehat{k} \cdots n-1)}}, \tag{34*}$$

$$\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle = \frac{\sqrt{-\Delta(0 \cdots n-1)}}{\sqrt{\Delta(0 \cdots \widehat{k} \cdots n-1)}}, \tag{34}$$

for each index $k = 0,\ldots, n-1$.

Proof. From $(32)^*$, $\mathbf{p}_k$ and $(-1)^{n-1-k}\varepsilon^*\iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle)$ are parallel and have the same direction, so, we have $(33)^*$ from $\langle \mathbf{p}_k | \mathbf{p}_k\rangle = 1$ and

$$\begin{aligned}
&\langle \iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle) \,|\, \iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle)\rangle \\
&= \langle \langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle \,|\, \langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle\rangle \\
&= (-1)^{n-2} \det(-\langle \mathbf{p}_r^* | \mathbf{p}_s^*\rangle)_{s=0,\ldots,\widehat{k},\ldots,n-1}^{r=0,\ldots,\widehat{k},\ldots,n-1} \\
&= (-1)^{n-2}\Delta^*(0, \ldots, \widehat{k}, \ldots, n-1),
\end{aligned} \tag{35}$$

where the second equality comes from (15). We also have (32):

$$\begin{aligned}
&\operatorname{sgn}\det(\mathbf{p}_0, \ldots, \mathbf{p}_\ell, \ldots, \mathbf{p}_{n-1}) \\
&= \operatorname{sgn}\det(\ldots, (-1)^{n-1-\ell}\varepsilon^*\iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_\ell^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle), \ldots) \\
&= (\varepsilon^*)^n \operatorname{sgn}\det(\ldots, (-1)^{n-1-\ell}\iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_\ell^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle), \ldots) \\
&= -(\varepsilon^*)^n \operatorname{sgn}\det(\ldots, (-1)^{n-1-\ell}\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_\ell^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle, \ldots) \\
&= -(\varepsilon^*)^n \operatorname{sgn}(\det(\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*))^{n-1} \\
&= -\varepsilon^*,
\end{aligned}$$

where the first and fourth equalities come from $(33)^*$ and (4), respectively. From (32), $\mathbf{p}_k^*$ and $(-1)^{n-1-k}(-\varepsilon^*)\iota(\langle\!\langle \mathbf{p}_0, \ldots \widehat{\mathbf{p}_k} \ldots, \mathbf{p}_{n-1}\rangle\!\rangle)$ are parallel and have the same direction, so, we have (33) from $\langle \mathbf{p}_k^* | \mathbf{p}_k^*\rangle = -1$ and

$$\begin{aligned}
&-\langle \iota(\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}\rangle\!\rangle) \,|\, \iota(\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}\rangle\!\rangle)\rangle \\
&= -\langle \langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}\rangle\!\rangle \,|\, \langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}\rangle\!\rangle\rangle \\
&= \det(\langle \mathbf{p}_r | \mathbf{p}_s\rangle)_{s=0,\ldots,\widehat{k},\ldots,n-1}^{r=0,\ldots,\widehat{k},\ldots,n-1} \\
&= \Delta(0, \ldots, \widehat{k}, \ldots, n-1),
\end{aligned} \tag{36}$$

where the second equality comes from (15). We have $(34)^*$ from $(33)^*$ and

$$\begin{aligned}
&\langle \mathbf{p}_k^* | (-1)^{n-1-k}\varepsilon^*\iota(\langle\!\langle \mathbf{p}_0^*, \ldots, \widehat{\mathbf{p}_k^*}, \ldots, \mathbf{p}_{n-1}^*\rangle\!\rangle)\rangle \\
&= |\det(\mathbf{p}_0^*, \ldots, \mathbf{p}_{n-1}^*)| \\
&= \sqrt{(-1)^{n-1}\det(-\langle \mathbf{p}_r^* | \mathbf{p}_s^*\rangle)_{s=0,\ldots,n-1}^{r=0,\ldots,n-1}} \\
&= \sqrt{(-1)^{n-1}\Delta^*(0, \ldots, n-1)},
\end{aligned} \tag{37}$$

where the first and second equalities come from $(32)^*$ and (14), respectively. We also have (34) from (33) and

$$\begin{aligned}
&\langle (-1)^{n-1-k}(-\varepsilon^*)\iota(\langle\!\langle \mathbf{p}_0, \ldots, \widehat{\mathbf{p}_k}, \ldots, \mathbf{p}_{n-1}\rangle\!\rangle) | \mathbf{p}_k\rangle \\
&= |\det(\mathbf{p}_0, \ldots, \mathbf{p}_{n-1})| \\
&= \sqrt{-\det(\langle \mathbf{p}_r | \mathbf{p}_s\rangle)_{s=0,\ldots,n-1}^{r=0,\ldots,n-1}} \\
&= \sqrt{-\Delta(0, \ldots, n-1)},
\end{aligned} \tag{38}$$

where the first and second equalities come from (32) and (14), respectively. $\square$

Lemma 8. *Let* $\{k_0, \ldots, k_{n-1}\}$ *be a permutation of the set of all indices* $\{0, \ldots, n-1\}$*. Then we have*

$$\Delta(k_0 \cdots k_m) = \frac{(-1)^{n-2-m}\Delta^*(k_{m+1} \cdots k_{n-1})}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \prod_{r=0}^{m} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \qquad (39)^*$$

$$(-1)^m \Delta^*(k_0 \cdots k_m) = \frac{\Delta(k_{m+1} \cdots k_{n-1})}{-\Delta(0 \cdots n-1)} \prod_{r=0}^{m} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \qquad (39)$$

for an integer $m = -1, \ldots, n-1$*, and we also have*

$$\begin{aligned} &-\Delta\begin{pmatrix} k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \\ &= \frac{(-1)^{n-2-m}\Delta^*\begin{pmatrix} k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} \begin{matrix} j \\ i \end{matrix} \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \qquad (40)^* \end{aligned}$$

$$\begin{aligned} &(-1)^m \Delta^*\begin{pmatrix} k_0 \cdots k_{m-1} \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \\ &= \frac{-\Delta\begin{pmatrix} k_m \cdots \widehat{i} \cdots \widehat{j} \cdots k_{n-1} \begin{matrix} j \\ i \end{matrix} \end{pmatrix}}{-\Delta(0 \cdots n-1)} \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_{k_r} | \mathbf{p}_{k_r} \rangle^2, \qquad (40) \end{aligned}$$

for an integer $m = 0, \ldots, n-2$ *and distinct indices* i, $j = k_m, \ldots, k_{n-1}$ *(we use the notation* $k_m \cdots \widehat{k_s} \cdots \widehat{k_t} \cdots k_{n-1}$ *whether* $s < t$ *or not. If* $s > t$*, it means* $k_m \cdots \widehat{k_t} \cdots \widehat{k_s} \cdots k_{n-1}$*).*

Proof. (39) and (40) are natural consequences of (39)* and (40)*, so we only have to prove (39)* and (40)*. To prove them, it is enough to show the following four equations:

$$\Delta() = \frac{(-1)^{n-1}\Delta^*(0 \cdots n-1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot 1, \qquad (41)$$

$$\Delta(0 \cdots n-1) = \frac{(-1)^{-1}\Delta^*()}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \prod_{r=0}^{n-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \qquad (42)$$

$$\begin{aligned} -\Delta\begin{pmatrix} 0 \cdots n-3 \begin{matrix} n-2 \\ n-1 \end{matrix} \end{pmatrix} &= \frac{(-1)^0 \Delta^* \begin{pmatrix} n-1 \\ n-2 \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \\ &\quad \cdot \langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2} \rangle \langle \mathbf{p}^*_{n-1} | \mathbf{p}_{n-1} \rangle \prod_{r=0}^{n-3} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \qquad (43) \end{aligned}$$

$$\begin{aligned} \Delta\begin{pmatrix} 0 \cdots m-1 \begin{matrix} i \\ j \end{matrix} \end{pmatrix} &= \frac{(-1)^{n-2-m}(-1)^{j-i}\Delta^* \begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \cdot \\ &\quad \cdot \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2, \qquad (44) \end{aligned}$$

for m, i, $j = 0, \ldots, n-1$ with $\{i, j\} \subsetneq \{m, \ldots, n-1\}$. In particular, (41) implies (39)* for $m = -1$, (42) implies (39)* for $m = n-1$, (43) implies (40)* for $m = n-2$, and (44) implies (39)* for $0 \leqq m \leqq n-2$ and the sign inversion of (40)* for $0 \leqq m \leqq n-3$ because

$$(-1)^{j-i}\Delta^*\begin{pmatrix} m\cdots\widehat{i}\cdots n-1 \\ m\cdots\widehat{j}\cdots n-1\end{pmatrix} = \begin{cases} \Delta^*(m\cdots\widehat{i}\cdots n-1) & \text{if } i=j, \\ -\Delta^*\left(m\cdots\widehat{i}\cdots\widehat{j}\cdots n-1\,\begin{matrix} j \\ i\end{matrix}\right) & \text{if } i\neq j.\end{cases} \tag{45}$$

The equation (41) is obvious.

We will show (42) and (43) by (44) and afterwards we will show (44). We have (42) immediately:

$$\begin{aligned}\Delta(0\cdots n-1) &= -\langle \mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle^2\cdot\Delta(0\cdots n-2)\\ &= -\langle \mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle^2\cdot\frac{(-1)^0\Delta^*(n-1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\prod_{r=0}^{n-2}\langle\mathbf{p}^*_r|\mathbf{p}_r\rangle^2\\ &= \frac{(-1)^{-1}\Delta^*()}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\prod_{r=0}^{n-1}\langle\mathbf{p}^*_r|\mathbf{p}_r\rangle^2,\end{aligned}$$

where the first equality comes from (34) and the second equality comes from (44) for $m=i=j=n-2$. We also have (43) by

$$\begin{aligned} &-\Delta\left(0\cdots n-3\,\begin{matrix} n-2\\ n-1\end{matrix}\right)\\ &= \langle\langle\!\langle \mathbf{p}_0,\dots,\mathbf{p}_{n-2}\rangle\!\rangle|\langle\!\langle\mathbf{p}_0,\dots,\mathbf{p}_{n-3},\mathbf{p}_{n-1}\rangle\!\rangle\rangle\\ &= -\langle(-1)^0(-\varepsilon^*)\iota(\langle\!\langle\mathbf{p}_0,\dots,\mathbf{p}_{n-2}\rangle\!\rangle)|(-1)^1(-\varepsilon^*)\iota(\langle\!\langle\mathbf{p}_0,\dots,\mathbf{p}_{n-3},\mathbf{p}_{n-1}\rangle\!\rangle)\rangle\\ &= -\langle\sqrt{\Delta(0\cdots n-2)}\mathbf{p}^*_{n-1}|\sqrt{\Delta(0\cdots n-3,n-1)}\mathbf{p}^*_{n-2}\rangle\\ &= \sqrt{\Delta(0\cdots n-2)}\cdot\sqrt{\Delta(0\cdots n-3,n-1)}\cdot(-\langle\mathbf{p}^*_{n-1}|\mathbf{p}^*_{n-2}\rangle)\\ &= \frac{\sqrt{(-1)^0\Delta^*(n-1)}}{\sqrt{(-1)^{n-1}\Delta^*(0\cdots n-1)}}\prod_{r=0}^{n-2}\langle\mathbf{p}^*_r|\mathbf{p}_r\rangle\\ &\quad\cdot\frac{\sqrt{(-1)^0\Delta^*(n-2)}}{\sqrt{(-1)^{n-1}\Delta^*(0\cdots n-1)}}\prod_{\substack{r=0\\ r\neq n-2}}^{n-1}\langle\mathbf{p}^*_r|\mathbf{p}_r\rangle\cdot\Delta^*\begin{pmatrix} n-1\\ n-2\end{pmatrix}\\ &= \frac{(-1)^0\Delta^*\begin{pmatrix} n-1\\ n-2\end{pmatrix}}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle\mathbf{p}^*_{n-2}|\mathbf{p}_{n-2}\rangle\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle\prod_{r=0}^{n-3}\langle\mathbf{p}^*_r|\mathbf{p}_r\rangle^2,\end{aligned}$$

where the first equality comes from (15), the third equality comes from (33), and the fifth equality comes from (44) for $m=i=j=n-2$ and for $m=n-2$ and $i=j=n-1$. Now it remains only to show (44). To show it, we can assume $i\neq n-1\neq j$ and $\mathbf{p}^*_{n-1}=(\underline{\mathbf{0}}^T,1)^T$. Let

$$\mathbf{p}'^*_k = \frac{\mathbf{p}^*_k}{\sqrt{\mathbf{p}^*_k\cdot\mathbf{p}^*_k}} \tag{46}$$

for $k=0,\dots,n-1$ and $\mathbf{p}'_\ell\in\mathbb{R}^n$ be such that

$$\mathbf{p}'^*_k\cdot\mathbf{p}'_\ell = 0 \quad\text{for}\quad k=0,\dots\widehat{\ell}\dots,n-1;\quad \mathbf{p}'^*_\ell\cdot\mathbf{p}'_\ell>0;\quad\text{and}\quad \mathbf{p}'_\ell\cdot\mathbf{p}'_\ell=1; \tag{47}$$

for $\ell = 0,\ldots, n-1$ (notice that

$$\mathbf{p}'^*_\ell \cdot \mathbf{p}'_\ell = \frac{\langle \mathbf{p}^*_\ell | \mathbf{p}_\ell \rangle}{\sqrt{\mathbf{p}^*_\ell \cdot \mathbf{p}^*_\ell}}, \tag{48}$$

$$\mathbf{p}'_k \cdot \mathbf{p}'_\ell = \langle \mathbf{p}_k | \mathbf{p}_\ell \rangle, \tag{49}$$

for $k,\ \ell = 0,\ldots, n-2$, because $\mathbf{p}'^*_{n-1} = \mathbf{p}^*_{n-1} = (\underline{\mathbf{0}}^T, 1)^T$ and $\mathbf{p}'_\ell = \mathbf{p}_\ell \in \mathbb{R}^{n-1} \times \{0\}$). Then

$$S' = \left(\mathbb{R}^+ \cdot \mathbf{p}'^*_0 + \cdots + \mathbb{R}^+ \cdot \mathbf{p}'^*_{n-1}\right) \cap \mathbb{S}^{n-1} = \{\mathbf{x}' \in \mathbb{S}^{n-1} : \mathbf{x}' . \mathbf{p}'_\ell \geqq 0, \forall \ell = 0, \ldots, n-1\}$$

is a spherical simplex on the unit sphere $\mathbb{S}^{n-1}$ and we denote

$$\Delta'^*_{\mathbb{S}^{n-1}}\left(k_0 \cdots k_\ell \begin{matrix} i_{\ell+1} \cdots i_h \\ j_{\ell+1} \cdots j_h \end{matrix}\right) = \det(\mathbf{p}'^*_r \cdot \mathbf{p}'^*_s)^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_h}_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_h},$$

$$\Delta'_{\mathbb{S}^{n-1}}\left(k_0 \cdots k_\ell \begin{matrix} i_{\ell+1} \cdots i_h \\ j_{\ell+1} \cdots j_h \end{matrix}\right) = \det(\mathbf{p}'_r \cdot \mathbf{p}'_s)^{r=k_0,\ldots,k_\ell,i_{\ell+1},\ldots,i_h}_{s=k_0,\ldots,k_\ell,j_{\ell+1},\ldots,j_h}.$$

We have

$$\begin{aligned}
&\Delta'^*_{\mathbb{S}^{n-1}}\binom{m \cdots \widehat{i} \cdots n-1}{m \cdots \widehat{j} \cdots n-1} \\
&= \frac{\sqrt{\mathbf{p}^*_i \cdot \mathbf{p}^*_i}\sqrt{\mathbf{p}^*_j \cdot \mathbf{p}^*_j}}{\prod_{r=m}^{n-1}(\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \det(\mathbf{p}^*_k \cdot \mathbf{p}^*_\ell)^{k=m\cdots\widehat{i}\cdots n-1}_{\ell=m\cdots\widehat{j}\cdots n-1} \\
&= \frac{(-1)^{n-2-m}\sqrt{\mathbf{p}^*_i \cdot \mathbf{p}^*_i}\sqrt{\mathbf{p}^*_j \cdot \mathbf{p}^*_j}}{\prod_{r=m}^{n-1}(\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \Delta^*\begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix}
\end{aligned} \tag{50}$$

where the former and latter equalities come from (46) and (16), respectively, and we also have similarly

$$\Delta'^*_{\mathbb{S}^{n-1}}(0 \cdots n-1) = \frac{(-1)^{n-1}}{\prod_{r=0}^{n-1}(\mathbf{p}^*_r \cdot \mathbf{p}^*_r)} \Delta^*(0 \cdots n-1). \tag{51}$$

So, we have (44):

$$\begin{aligned}
&\Delta\left(0 \cdots m-1 \begin{matrix} i \\ j \end{matrix}\right) = \Delta'_{\mathbb{S}^{n-1}}\left(0 \cdots m-1 \begin{matrix} i \\ j \end{matrix}\right) \\
&= \frac{(-1)^{j-i}\Delta'^*_{\mathbb{S}^{n-1}}\binom{m \cdots \widehat{i} \cdots n-1}{m \cdots \widehat{j} \cdots n-1}}{\Delta'^*_{\mathbb{S}^{n-1}}(0 \cdots n-1)} \cdot (\mathbf{p}'^*_i \cdot \mathbf{p}'_i)(\mathbf{p}'^*_j \cdot \mathbf{p}'_j) \prod_{r=0}^{m-1}(\mathbf{p}'^*_r \cdot \mathbf{p}'_r)^2 \\
&= \frac{(-1)^{n-2-m}(-1)^{j-i}\Delta^*\begin{pmatrix} m \cdots \widehat{i} \cdots n-1 \\ m \cdots \widehat{j} \cdots n-1 \end{pmatrix}}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \prod_{r=0}^{m-1} \langle \mathbf{p}^*_r | \mathbf{p}_r \rangle^2,
\end{aligned}$$

where the first equality comes from (49), the second equality comes from Lemma 5.3 of [6] and (45) or from Remark 8 of [7] and (45), and the last equality comes from (48), (50), and (51). □

4. Orthocentric simplices

Hereafter, we consider orthocenters of simplices.

Definition 1. *The point* $\mathbf{n} \in \mathbb{H}^{n-1}$ *is called the orthocenter of* S *if* $\mathbf{n}$ *is a concurrent point of geodesics* C_i *for all* $i = 0, \ldots, n-1$*, where* C_i *is the pseudo-perpendicular to the opposite face*

$$\{\mathbf{x} \in S : \langle \mathbf{x} | \mathbf{p}_i \rangle = 0\} = \left(\mathbb{R}^+ \cdot \mathbf{p}_0^* + \cdots + \widehat{\mathbb{R}^+ \cdot \mathbf{p}_i^*} + \cdots + \mathbb{R}^+ \cdot \mathbf{p}_{n-1}^*\right) \cap \mathbb{H}^{n-1} \quad (52)$$

of the vertex $\mathbf{p}_i^*$*, in other words,* $\mathbf{n} \in \mathbb{R} \cdot \mathbf{p}_i^* + \mathbb{R} \cdot \mathbf{p}_i$ *for all* i*. A simplex having orthocenter is called an orthocentric simplex.*

Remark 3. If the orthocenter exists, it is unique (unlike in the case of the spherical simplex, it is impossible that $\mathbf{p}_i^* = \mathbf{p}_i$ for any index $i = 0, \ldots, n-1$, so, the concurrent point of $(\mathbb{R} \cdot \mathbf{p}_i^* + \mathbb{R} \cdot \mathbf{p}_i) \cap \mathbb{H}^{n-1}$ is unique, if it exists).

The conditions below hold an important role for orthocentric simplices.

Lemma 9. *The followings are equivalent:*

(a)* $\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta^* \begin{pmatrix} i \\ \ell \end{pmatrix} \Delta^* \begin{pmatrix} j \\ k \end{pmatrix}$ *holds for pairwise distict indices* i, j, k, $\ell = 0, \ldots, n-1$,

(a) $\Delta \begin{pmatrix} i \\ k \end{pmatrix} \Delta \begin{pmatrix} j \\ \ell \end{pmatrix} = \Delta \begin{pmatrix} i \\ \ell \end{pmatrix} \Delta \begin{pmatrix} j \\ k \end{pmatrix}$ *holds for pairwise distict indices* i, j, k, $\ell = 0, \ldots, n-1$.

Notice that the label (a)* was already used in Introduction. To prove the lemma above, we need the following lemma, which is similar to the case of simplices of the sphere.

Lemma 10. *We have*

$$\Delta^*(k_0 \cdots k_m) = \Delta^*(k_0 \cdots k_{m-1}) - \sum_{r=0}^{m-1} \Delta^* \begin{pmatrix} k_r \\ k_m \end{pmatrix} \Delta^* \begin{pmatrix} k_0 \cdots \widehat{k_r} \cdots k_{m-1} & k_m \\ & k_r \end{pmatrix}, \quad (53)^*$$

and the equation replaced Δ^* *with* Δ *of the equation above, named* (53)*, for an integer* $m = 0, \ldots, n-1$ *and pairwise distinct indices* $k_0, \ldots, k_m$*. If* (a)* *holds, then we have*

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{m-2} & i \\ & j \end{pmatrix} - \Delta^* \begin{pmatrix} i \\ k_{m-1} \end{pmatrix} \Delta^* \begin{pmatrix} k_0 \cdots k_{m-2} & k_{m-1} \\ & j \end{pmatrix}, \quad (54)^*$$

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m'-1} & i' \\ & j' \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i' \\ & j \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m'-1} & i \\ & j' \end{pmatrix}, \quad (55)^*$$

$$\Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} k & i \\ & j \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m'-1} & i' \\ & j' \end{pmatrix} = \Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & i \\ & j \end{pmatrix} \Delta^* \begin{pmatrix} k'_0 \cdots k'_{m'-1} k & i' \\ & j' \end{pmatrix}, \quad (56)^*$$

$$\Delta^*\begin{pmatrix}k_0\cdots k_{m-1}k\,{i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix}k'_0\cdots k'_{m'-1}\,{i' \atop j'}\end{pmatrix}=\Delta^*\begin{pmatrix}k_0\cdots k_{m-1}\,{i' \atop j}\end{pmatrix}\Delta^*\begin{pmatrix}k'_0\cdots k'_{m'-1}k\,{i \atop j'}\end{pmatrix}, \tag{57}^*$$

$$\Delta^*\begin{pmatrix}k_0\cdots k_{m-1}k\,{i \atop j}\end{pmatrix}\Delta^*\begin{pmatrix}k'_0\cdots k'_{m'-1}k'\,{i' \atop j'}\end{pmatrix}=\Delta^*\begin{pmatrix}k_0\cdots k_{m-1}k'\,{i' \atop j}\end{pmatrix}\Delta^*\begin{pmatrix}k'_0\cdots k'_{m'-1}k\,{i \atop j'}\end{pmatrix}, \tag{58}^*$$

for integers m, $m' = 0, \ldots, n-1$ and indices $k_0, \ldots, k_{m-1}$, $k'_0, \ldots, k'_{m'-1}$, i, j, i', j', k, and k' such that indices appearing in an identical $\Delta^*(\cdots)$ are pairwise distinct. If (a) *holds, then we have equations replaced Δ^* with Δ of five equations above, named* (54), (55), (56), (57), *and* (58), *respectively.*

Proof. The proof is essentially same to Lemmas 7, 8, 9 and Corollaries 10, 11 of [7]. (53)* and (54)* are Laplace expansions. We have (55)* by induction: (55)* for (m, m') with $m \geqq m' > 0$ comes from (55)* for $(m, m'-1)$ and $(m, 0)$, and (55)* for $(m, 0)$ with $m > 0$ comes from (55)* for $(m-1, 0)$. (56)* comes from (54)* and (55)*. (57)* and (58)* come from (54)*–(56)* and (54)*–(57)* where the proofs are divided into two cases of $k \neq i'$ and $k = i'$, and three cases of $k \neq i'$, $k' \neq i$, and $k = i'$ and $k' = i$, respectively. The proofs of (53)–(58) are similar. □

Remark that (53)* and (53) is valid with assumption neither (a)* nor (a). Lemma 10 is useful to prove the equivalence of (a)* and (a).

Proof of Lemma 9. The proof is essentially same to Lemma 12 of [7]. In particular, (a)* implies (58)*, so, for pairwise distinct indices i, j, i', and j', we have

$$\begin{aligned}&\Delta^*\begin{pmatrix}0\cdots\widehat{i}\cdots\widehat{j}\cdots n-1\,{j \atop i}\end{pmatrix}\Delta^*\begin{pmatrix}0\cdots\widehat{i'}\cdots\widehat{j'}\cdots n-1\,{j' \atop i'}\end{pmatrix}\\&\quad=\Delta^*\begin{pmatrix}0\cdots\widehat{i}\cdots\widehat{j'}\cdots n-1\,{j' \atop i}\end{pmatrix}\Delta^*\begin{pmatrix}0\cdots\widehat{i'}\cdots\widehat{j}\cdots n-1\,{j \atop i'}\end{pmatrix}.\end{aligned}$$

The equation above and (40)* imply (a): $\Delta\begin{pmatrix}i\\j\end{pmatrix}\Delta\begin{pmatrix}i'\\j'\end{pmatrix}=\Delta\begin{pmatrix}i'\\j\end{pmatrix}\Delta\begin{pmatrix}i\\j'\end{pmatrix}$. The proof of the converse (a) ⇒ (a)* is similar. □

In order to consider orthocenters, we have to define four kinds of values μ^*_k, μ_k, ν^*_k, ν_k. To define them, we need the following lemma.

Lemma 11. *Assume* (a)* (*which is equivalent to* (a)). *Then, we have*

$$\Delta^*\begin{pmatrix}i\\k\end{pmatrix}\Delta^*\begin{pmatrix}k\\j\end{pmatrix}\Delta^*\begin{pmatrix}i'\\j'\end{pmatrix}=\Delta^*\begin{pmatrix}i'\\k\end{pmatrix}\Delta^*\begin{pmatrix}k\\j'\end{pmatrix}\Delta^*\begin{pmatrix}i\\j\end{pmatrix}, \tag{59}^*$$

$$\Delta^*\begin{pmatrix}i\\k\end{pmatrix}\Delta^*\begin{pmatrix}k\\j\end{pmatrix}\Delta^*\begin{pmatrix}k\,{i' \atop j'}\end{pmatrix}=\Delta^*\begin{pmatrix}i'\\k\end{pmatrix}\Delta^*\begin{pmatrix}k\\j'\end{pmatrix}\Delta^*\begin{pmatrix}k\,{i \atop j}\end{pmatrix}, \tag{60}^*$$

for indices i, j, i', j', k such that indices appearing in an identical $\Delta^*(\cdots)$ are pairwise distinct. On the same assumption, we also have equations replaced Δ^* with Δ of the equations above, named (59) *and* (60), *respectively.*

Proof. They are proved essentially same to Lemma 15 of [7]. (59)* comes from (a)* where the proof is divided into two cases of $\{i, j\} \cap \{i', j'\} \neq \emptyset$ and $\{i, j\} \cap \{i', j'\} = \emptyset$. (60)* comes from (59)*. The proofs of (59) and (60) are similar. $\square$

Definition 2. *On the assumption* (a)* (*which is equivalent to* (a)), *for an index* $k = 0, \ldots, n-1$, *if there exists a pair of distinct indices* $i, j = 0, \ldots, \widehat{k}, \ldots, n-1$ *such that* $\Delta^* \begin{pmatrix} i \\ j \end{pmatrix} \neq 0$ *(resp.* $-\Delta^* \begin{pmatrix} k & \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \neq 0$, $-\Delta \begin{pmatrix} i \\ j \end{pmatrix} \neq 0$, $-\Delta \begin{pmatrix} k & \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \neq 0$*), we define*

$$\mu_k^* = \frac{\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} k \\ j \end{pmatrix}}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}} \qquad \left(\text{resp. } \nu_k^* = \frac{\Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} k \\ j \end{pmatrix}}{-\Delta^* \begin{pmatrix} k & \begin{matrix} i \\ j \end{matrix} \end{pmatrix}}, \right. \tag{61*}$$

$$\mu_k = -\frac{\left(-\Delta \begin{pmatrix} i \\ k \end{pmatrix}\right)\left(-\Delta \begin{pmatrix} k \\ j \end{pmatrix}\right)}{-\Delta \begin{pmatrix} i \\ j \end{pmatrix}}, \qquad \left. \nu_k = \frac{\left(-\Delta \begin{pmatrix} i \\ k \end{pmatrix}\right)\left(-\Delta \begin{pmatrix} k \\ j \end{pmatrix}\right)}{-\Delta \begin{pmatrix} k & \begin{matrix} i \\ j \end{matrix} \end{pmatrix}} \right) \tag{61}$$

which depends only the index $k = 0, \ldots, n-1$.

Remark 4. There exists μ_k^* and it is positive for each index k, because $\Delta^* \begin{pmatrix} i \\ j \end{pmatrix} = -\langle \mathbf{p}_i^* | \mathbf{p}_j^* \rangle > 1$ for arbitrary pair of distinct indices i, j from Lemma 3.

For four kinds of values above, we have the following three lemmas.

Lemma 12. *Assume* (a)*. *Then, we have*

$$(-1)^m \Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & \begin{matrix} i \\ j \end{matrix} \end{pmatrix} = \sqrt{\mu_i^*} \sqrt{\mu_j^*} \prod_{r=0}^{m-1} (\mu_{k_r}^* - 1), \tag{62}$$

for an integer $m = 0, \ldots, n-2$ *and pairwise distinct indices* $k_0, \ldots, k_{m-1}$, i, *and* j,

$$(-1)^m \Delta^*(k_0 \cdots k_m) = \sum_{s=0}^{m} \mu_{k_s}^* \prod_{\substack{r=0 \\ r \neq s}}^{m} (\mu_{k_r}^* - 1) - \prod_{r=0}^{m} (\mu_{k_r}^* - 1), \tag{63}$$

for an integer $m = -1, \ldots, n-1$ *and pairwise distinct indices* $k_0, \ldots, k_m$.

Proof. We have (62) by the following induction (it is trivial for $m = 0$ and $m = 1$):

$$\begin{aligned} (-1)^m \Delta^* \begin{pmatrix} k_0 \cdots k_{m-1} & \begin{matrix} i \\ j \end{matrix} \end{pmatrix} &= \frac{-\Delta^* \begin{pmatrix} k_{m-1} & \begin{matrix} i' \\ j' \end{matrix} \end{pmatrix}}{\Delta^* \begin{pmatrix} i' \\ j' \end{pmatrix}} \cdot (-1)^{m-1} \Delta^* \begin{pmatrix} k_0 \cdots k_{m-2} & \begin{matrix} i \\ j \end{matrix} \end{pmatrix} \\ &= \frac{\sqrt{\mu_{i'}^*} \sqrt{\mu_{j'}^*} (\mu_{k_{m-1}}^* - 1)}{\sqrt{\mu_{i'}^*} \sqrt{\mu_{j'}^*}} \cdot \sqrt{\mu_i^*} \sqrt{\mu_j^*} \prod_{r=0}^{m-2} (\mu_{k_r}^* - 1) \\ &= \sqrt{\mu_i^*} \sqrt{\mu_j^*} \prod_{r=0}^{m-1} (\mu_{k_r}^* - 1), \end{aligned}$$

where the first equality comes from (56)* with some distinct indices i', $j' = 0, \ldots \widehat{k_{m-1}} \ldots, n-1$. We also have (63) by the following induction (for $m=-1$, we have $(-1)^{-1}\Delta^*() = 0-1$):

$$
\begin{aligned}
(-1)^m \Delta^*(k_0 \cdots k_m) &= \sum_{s=0}^{m-1} \Delta^*\begin{pmatrix} k_s \\ k_m \end{pmatrix} \cdot (-1)^{m-1}\Delta^*\left(k_0 \cdots \widehat{k_s} \cdots k_{m-1} \begin{matrix} k_m \\ k_s \end{matrix}\right) \\
&\quad - (-1)^{m-1}\Delta^*(k_0 \cdots k_{m-1}) \\
&= \sum_{s=0}^{m-1} \sqrt{\mu^*_{k_s}}\sqrt{\mu^*_{k_m}} \cdot \sqrt{\mu^*_{k_m}}\sqrt{\mu^*_{k_s}} \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\mu^*_{k_r} - 1) \\
&\qquad - \Big(\sum_{s=0}^{m-1} \mu^*_{k_s} \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\mu^*_{k_r} - 1) - \prod_{r=0}^{m-1} (\mu^*_{k_r} - 1)\Big) \\
&= \Big(\sum_{s=0}^{m-1} \sqrt{\mu^*_{k_s}}\sqrt{\mu^*_{k_m}} \cdot \sqrt{\mu^*_{k_m}}\sqrt{\mu^*_{k_s}} \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\mu^*_{k_r} - 1) \\
&\quad - \sum_{s=0}^{m-1} \mu^*_{k_s} \prod_{\substack{r=0 \\ r\neq s}}^{m-1} (\mu^*_{k_r} - 1) + \mu^*_{k_m} \prod_{r=0}^{m-1} (\mu^*_{k_r} - 1)\Big) \\
&\quad - (\mu^*_{k_m} - 1) \prod_{r=0}^{m-1} (\mu^*_{k_r} - 1) \\
&= \sum_{s=0}^{m} \mu^*_{k_s} \prod_{\substack{r=0 \\ r\neq s}}^{m} (\mu^*_{k_r} - 1) - \prod_{r=0}^{m} (\mu^*_{k_r} - 1),
\end{aligned}
$$

where the first equality comes from (53)*. □

Lemma 13. *If* (a)* *holds, for each index* $k = 0, \ldots, n-1$, *there exists* ν_k *and we have*

$$1 - \nu_k = \frac{\mu^*_k - 1}{\langle \mathbf{p}^*_k | \mathbf{p}_k \rangle^2}. \tag{64}$$

Proof. Assume that there does not exist ν_k for some index k. Then, for arbitrary pair of distinct indices $i, j = 0, \ldots \widehat{k} \ldots, n-1$,

$$
\begin{aligned}
-\Delta\left(k \begin{matrix} i \\ j \end{matrix}\right) &= \frac{(-1)^{n-3}\Delta^*\left(0 \cdots \widehat{i} \cdots \widehat{j} \cdots \widehat{k} \cdots n-1 \begin{matrix} j \\ i \end{matrix}\right)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \langle \mathbf{p}^*_k | \mathbf{p}_k \rangle^2 \\
&= \frac{\sqrt{\mu^*_j}\sqrt{\mu^*_i} \prod_{\substack{\ell=0 \\ \ell\neq i,j,k}}^{n-1} (\mu^*_\ell - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \langle \mathbf{p}^*_i | \mathbf{p}_i \rangle \langle \mathbf{p}^*_j | \mathbf{p}_j \rangle \langle \mathbf{p}^*_k | \mathbf{p}_k \rangle^2
\end{aligned} \tag{65}
$$

is equal to 0, which means that there exists $\ell = 0, \ldots \widehat{i} \ldots \widehat{j} \ldots \widehat{k} \ldots, n-1$ such that $\mu^*_\ell = 1$. The indices i and j are arbitrary, so, there exist three indices ℓ, ℓ',

$\ell'' = 0, \ldots \widehat{k} \ldots, n-1$ such that all of

$$\mu^*_\ell = \frac{\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}}, \quad \mu^*_{\ell'} = \frac{\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}}, \quad \mu^*_{\ell''} = \frac{\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}\Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}}{\Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix}} \tag{66}$$

are equal to 1. This is a contradiction to $\Delta^*\begin{pmatrix}\ell'\\ \ell''\end{pmatrix}, \Delta^*\begin{pmatrix}\ell''\\ \ell\end{pmatrix}, \Delta^*\begin{pmatrix}\ell\\ \ell'\end{pmatrix} > 1$. Hence, for each k, there exists ν_k, so, there exist distinct indices $i, j = 0, \ldots \widehat{k} \ldots, n-1$ such that $-\Delta\begin{pmatrix}k\,{}^{i}_{j}\end{pmatrix} \neq 0$. Similarly to (65), we have

$$-\Delta\begin{pmatrix}i\\ j\end{pmatrix} = \frac{\sqrt{\mu^*_j}\sqrt{\mu^*_i}\prod\limits_{\substack{\ell=0\\ \ell\neq i,j}}^{n-1}(\mu^*_\ell - 1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\langle \mathbf{p}^*_i|\mathbf{p}_i\rangle\langle \mathbf{p}^*_j|\mathbf{p}_j\rangle. \tag{67}$$

Two equations (65) and (67) imply

$$1-\nu_k = \frac{-\Delta\begin{pmatrix}i\\ j\end{pmatrix}}{-\Delta\begin{pmatrix}k\,{}^{i}_{j}\end{pmatrix}} = \frac{\mu^*_k - 1}{\langle \mathbf{p}^*_k|\mathbf{p}_k\rangle^2}.$$

□

Lemma 14. *Assume* (a)*. *Then, we have*

$$\left(-\Delta\begin{pmatrix}i\\ j\end{pmatrix}\right)^2(1-\nu_i)(1-\nu_j) = \nu_i\nu_j, \tag{68}$$

for distinct indices i and j,

$$-\Delta\begin{pmatrix}k_0\cdots k_{m-1}\,{}^{i}_{j}\end{pmatrix}\prod_{r=0}^{m-1}(1-\nu_{k_r}) = -\Delta\begin{pmatrix}i\\ j\end{pmatrix}, \tag{69}$$

for an integer $m = 0, \ldots, n-2$ and pairwise distinct indices $k_0, \ldots, k_{m-1}$, i, and j,

$$\Delta(k_0\cdots k_m)\prod_{r=0}^{m}(1-\nu_{k_r}) = 1 - \sum_{r=0}^{m}\nu_{k_r}, \tag{70}$$

for an integer $m = -1, \ldots, n-1$ and pairwise distinct indices $k_0, \ldots, k_m$.

Proof. For (68), let k, ℓ, k', and ℓ' be indices such that $k \neq \ell \neq i \neq k \neq j$, $k' \neq \ell' \neq j \neq k' \neq i$, and

$$-\Delta\begin{pmatrix}i\,{}^{k}_{\ell}\end{pmatrix} \neq 0 \neq -\Delta\begin{pmatrix}j\,{}^{k'}_{\ell'}\end{pmatrix}.$$

Then we have

$$\left(-\Delta\begin{pmatrix}i\\ j\end{pmatrix}\right)(1-\nu_i)\cdot\left(-\Delta\begin{pmatrix}j\\ i\end{pmatrix}\right)(1-\nu_j)$$

$$
\begin{aligned}
&= \left(-\Delta\binom{i}{j}\right)\frac{-\Delta\binom{k}{\ell}}{-\Delta\left(i\,\substack{k\\ \ell}\right)}\cdot\left(-\Delta\binom{j}{i}\right)\frac{-\Delta\binom{k'}{\ell'}}{-\Delta\left(j\,\substack{k'\\ \ell'}\right)}\\
&= \frac{\left(-\Delta\binom{i}{\ell}\right)\left(-\Delta\binom{k}{j}\right)}{-\Delta\left(i\,\substack{k\\ \ell}\right)}\cdot\frac{\left(-\Delta\binom{j}{\ell'}\right)\left(-\Delta\binom{k'}{i}\right)}{-\Delta\left(j\,\substack{k'\\ \ell'}\right)}\\
&= \frac{\left(-\Delta\binom{i}{\ell}\right)\left(-\Delta\binom{k}{i}\right)}{-\Delta\left(i\,\substack{k\\ \ell}\right)}\cdot\frac{\left(-\Delta\binom{j}{\ell'}\right)\left(-\Delta\binom{k'}{j}\right)}{-\Delta\left(j\,\substack{k'\\ \ell'}\right)}\\
&= \nu_i\cdot\nu_j.
\end{aligned}
$$

The equation (69) comes from

$$
\begin{aligned}
-\Delta\left(k_0\cdots k_{\ell-1}k\,\substack{i\\ j}\right)(1-\nu_k) &= -\Delta\left(k_0\cdots k_{\ell-1}k\,\substack{i\\ j}\right)\frac{-\Delta\binom{i'}{j'}}{-\Delta\left(k\,\substack{i'\\ j'}\right)}\\
&= -\Delta\left(k_0\cdots k_{\ell-1}\,\substack{i\\ j}\right),
\end{aligned}
\tag{71}
$$

where the last equality comes from (56) with some distinct indices $i', j' = 0, \ldots \widehat{k} \ldots, n-1$. We also have (70) by

$$
\begin{aligned}
&\Delta(k_0\cdots k_m)\prod_{r=0}^{m}(1-\nu_{k_r})\\
&= \Bigg(\Delta(k_0\cdots k_{m-1})\prod_{r=0}^{m-1}(1-\nu_{k_r})\\
&\quad -\sum_{\ell=0}^{m-1}\left(-\Delta\binom{k_\ell}{k_m}\right)\left(-\Delta\left(k_0\cdots\widehat{k_\ell}\cdots k_{m-1}\,\substack{k_m\\ k_\ell}\right)\prod_{\substack{r=0\\ r\neq\ell}}^{m-1}(1-\nu_{k_r})\right)(1-\nu_{k_\ell})\Bigg)(1-\nu_{k_m})\\
&= \left((1-\sum_{r=0}^{m-1}\nu_{k_r})-\sum_{\ell=0}^{m-1}\left(-\Delta\binom{k_\ell}{k_m}\right)\left(-\Delta\binom{k_m}{k_\ell}\right)(1-\nu_{k_\ell})\right)(1-\nu_{k_m})\\
&= (1-\sum_{r=0}^{m-1}\nu_{k_r})(1-\nu_{k_m})-\sum_{\ell=0}^{m-1}\nu_{k_\ell}\nu_{k_m}\\
&= 1-\sum_{r=0}^{m}\nu_{k_r},
\end{aligned}
$$

where the first equality comes from (53), the second equality comes from the assumption of induction and (69), and the third equality comes from (68). □

If there exists ν_k^* (resp. μ_k) for an index $k = 0,\ldots, n-1$, it holds that

$$(\mu_k^* - 1)(\nu_k^* - 1) = \frac{-\Delta^* \begin{pmatrix} k \, {}^{i}_{j} \end{pmatrix}}{\Delta^* \begin{pmatrix} i \\ j \end{pmatrix}} \cdot \frac{\Delta^* \begin{pmatrix} i' \\ j' \end{pmatrix}}{-\Delta^* \begin{pmatrix} k \, {}^{i'}_{j'} \end{pmatrix}} = 1 \qquad (72)^*$$

$$(\text{resp.} \quad (1-\mu_k)(1-\nu_k) = \frac{-\Delta \begin{pmatrix} k \, {}^{i}_{j} \end{pmatrix}}{-\Delta \begin{pmatrix} i \\ j \end{pmatrix}} \cdot \frac{-\Delta \begin{pmatrix} i' \\ j' \end{pmatrix}}{-\Delta \begin{pmatrix} k \, {}^{i'}_{j'} \end{pmatrix}} = 1), \qquad (72)$$

with some pairs of distinct indices i, j; i', $j' = 0,\ldots \widehat{k} \ldots, n-1$. The following two lemmas are natural consequences of (64), (72)*, and (72).

Lemma 15. *If* (a)* *holds, for each index* $k = 0,\ldots, n-1$*, the followings are equivalent:*

(d)* $\mu_k^* \neq 1$,
(e)* *there exists* ν_k^*,
(d) *there exists* μ_k,
(e) $\nu_k \neq 1$.

Proof. The equivalence of (e) and (d)* comes from (64). The negation of (e)* means that

$$-\Delta^* \begin{pmatrix} k \, {}^{i}_{j} \end{pmatrix} = \Delta^* \begin{pmatrix} i \\ k \end{pmatrix} \Delta^* \begin{pmatrix} k \\ j \end{pmatrix} - \Delta^* \begin{pmatrix} i \\ j \end{pmatrix}$$

is equal to 0 for arbitrary distinct i, $j = 0,\ldots \widehat{k} \ldots, n-1$, which means $\mu_k^* = 1$. So (e)* and (d)* are equivalent. The negation of (d) means that (67) is equal to 0 for arbitrary distinct i, $j = 0,\ldots \widehat{k} \ldots, n-1$, which means the existence of $\ell = 0,\ldots \widehat{i} \ldots \widehat{j} \ldots, n-1$ such that $\mu_\ell^* = 1$. It is equivalent to

$\mu_k^* = 1$ or the existence of three indices $\ell, \ell', \ell'' \neq k$ such that $\mu_\ell^* = \mu_{\ell'}^* = \mu_{\ell''}^* = 1$.

However, the latter condition is impossible (see around (66) in the proof of Lemma 13), so it is equivalent to $\mu_k^* = 1$, that is, (d) is equivalent to (d)*. □

Corollary 16. *If* (a)* *holds, there exists at most one index* $i = 0,\ldots, n-1$ *such that* μ_i (*resp.* ν_i^*) *does not exist.*

Proof. Assume the nonentity of μ_i (resp. ν_i^*) for an index i. Then it implies $\nu_i = 1$, so, from (68), we have $\nu_j = 0 \neq 1$ for each $j = 0,\ldots \widehat{i} \ldots, n-1$, which means the existence of μ_j (resp. ν_j^*). □

Lemma 17. *On the assumption* (a)*, *we have*

$$1 - \mu_k = (\nu_k^* - 1)\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle^2, \qquad (64)^*$$

for each index $k = 0,\ldots, n-1$ *such that there exists* ν_k^* (*the existence of such* k *comes from Corollary* 16).

Proof. Neither μ_k^* nor ν_k is equal to 1 from Lemma 15, so, (64)* comes from (64), (72)*, and (72). □

5. Explicit representation of the orthocenter.

To represent the orthocenter, we need the following vector.

Lemma 18. *If* (a)* *holds, the vector*

$$\tilde{\mathbf{h}} = \frac{1}{\sqrt{\mu_k^*}}\Big(\mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle}\mathbf{p}_k\Big) = \frac{1}{\sqrt{\mu_k^*}}\Big(\mathbf{p}_k^* - (1-\nu_k)\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle \mathbf{p}_k\Big) \in \mathbb{R}^n \setminus \{\mathbf{0}\} \quad (73)$$

does not depend on $k = 0, \ldots, n-1$.

Proof. To show

$$\frac{1}{\sqrt{\mu_k^*}}(\mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle}\mathbf{p}_k) = \frac{1}{\sqrt{\mu_\ell^*}}(\mathbf{p}_\ell^* - \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell\rangle}\mathbf{p}_\ell)$$

for distinct indices k and ℓ, it is enough to compare the pseudo-inner products $\langle \mathbf{p}_i^* | \cdot \rangle$ of both sides for all indices $i = 0, \ldots, n-1$. □

Lemma 19. *If* (a)* *holds, we have*

$$\langle \tilde{\mathbf{h}} | \mathbf{p}_k^* \rangle = -\sqrt{\mu_k^*}, \quad (74)$$

$$\langle \tilde{\mathbf{h}} | \mathbf{p}_k \rangle = \frac{\nu_k}{\sqrt{\mu_k^*}}\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle, \quad (75)$$

for each index k,

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \tilde{\mathbf{h}} \rangle = \langle \mathbf{p}_i^* | \tilde{\mathbf{h}} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle, \quad (76)$$

for distinct indices i, j, *and* k, *and,*

$$-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle = \frac{\sum_{j=0}^{n-1} \mu_j^* \prod_{\substack{i=0 \\ i\neq j}}^{n-1} (\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} = \sum_{j=0}^{n-1} \nu_j. \quad (77)$$

Proof. (74), (75), and (76) are obvious. The former equality of (77) comes from

$$\begin{aligned}
-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle &= \frac{-1}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}}\Big\langle \mathbf{p}_k^* - \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle}\mathbf{p}_k \Big| \mathbf{p}_\ell^* - \frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell\rangle}\mathbf{p}_\ell \Big\rangle \\
&= \frac{-1}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}}\left(-\Delta^*\begin{pmatrix} k \\ \ell \end{pmatrix} + \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle}\frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell\rangle}\Delta\begin{pmatrix} k \\ \ell \end{pmatrix}\right) \\
&= 1 + \frac{\mu_k^* - 1}{\langle \mathbf{p}_k^* | \mathbf{p}_k\rangle}\frac{\mu_\ell^* - 1}{\langle \mathbf{p}_\ell^* | \mathbf{p}_\ell\rangle}\frac{-\Delta\begin{pmatrix} k \\ \ell \end{pmatrix}}{\sqrt{\mu_k^*}\sqrt{\mu_\ell^*}} \\
&= 1 + \frac{\prod_{i=0}^{n-1}(\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)} \\
&= \frac{\sum_{j=0}^{n-1} \mu_j^* \prod_{\substack{i=0 \\ i\neq j}}^{n-1} (\mu_i^* - 1)}{(-1)^{n-1}\Delta^*(0 \cdots n-1)}, \quad (78)
\end{aligned}$$

where k and ℓ are some distinct indices and the fourth and last equalities come from (67) and (63), respectively. The latter equality of (77) comes from

$$
\begin{aligned}
&\frac{\mu_j^* \prod_{i=0}^{n-1}{}^{i\neq j}(\mu_i^*-1)}{(-1)^{n-1}\Delta^*(0\cdots n-1)}\\
&= -\Delta(0\cdots n-1)\frac{\mu_j^*}{\langle \mathbf{p}_j^*|\mathbf{p}_j\rangle^2}\prod_{i=0}^{n-1}{}^{i\neq j}\frac{\mu_i^*-1}{\langle \mathbf{p}_i^*|\mathbf{p}_i\rangle^2}\\
&= -\Delta(0\cdots n-1)\Big((1-\nu_j)+\frac{1}{\langle \mathbf{p}_j^*|\mathbf{p}_j\rangle^2}\Big)\prod_{i=0}^{n-1}{}^{i\neq j}(1-\nu_i)\\
&= -\Delta(0\cdots n-1)\prod_{i=0}^{n-1}(1-\nu_i)+\Delta(0\cdots\widehat{j}\cdots n-1)\prod_{i=0}^{n-1}{}^{i\neq j}(1-\nu_i)\\
&= (\sum_{i=0}^{n-1}\nu_i-1)+(1-\sum_{i=0}^{n-1}{}^{i\neq j}\nu_i)\\
&= \nu_j,
\end{aligned}
\tag{79}
$$

where the first, second, third, and fourth equalities come from (39), (64), (34), and (70), respectively. $\square$

The following two theorems are main results.

Theorem 20. *If* (a)* *and* $\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle < 0$ *holds, then*

$$\mathbf{h} = \frac{\tilde{\mathbf{h}}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} \tag{80}$$

is in $\mathbb{H}^{n-1}$ *and it is the orthocenter of* S.

Proof. It is obvious that $\mathbf{h}\in\mathbb{H}^{n-1}$ or $-\mathbf{h}\in\mathbb{H}^{n-1}$. If$-\mathbf{h}\in\mathbb{H}^{n-1}$ then, for each k, $\langle\mathbf{h}|\mathbf{p}_k^*\rangle = -\langle-\mathbf{h}|\mathbf{p}_k^*\rangle \geqq 1$ from Lemma 3, which is a contradiction to

$$\langle\mathbf{h}|\mathbf{p}_k^*\rangle = \frac{\langle\tilde{\mathbf{h}}|\mathbf{p}_k^*\rangle}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} = \frac{-\sqrt{\mu_k^*}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} < 0.$$

From (73) and Definition 1, $\mathbf{h}$ is the orthocenter. $\square$

Remark 5. For an index $k=0,\ldots,n-1$ such that there exists μ_k, let

$$\tilde{\mathbf{h}}_k^* = \mathbf{p}_k - \frac{1-\mu_k}{\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle}\mathbf{p}_k^* = \mathbf{p}_k - (\nu_k^*-1)\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle\mathbf{p}_k^* \in \mathbb{R}^n\setminus\{\mathbf{0}\}. \tag{81}$$

Then we have

$$\tilde{\mathbf{h}}_k^* = \frac{-(1-\mu_k)\sqrt{\mu_k^*}}{\langle\mathbf{p}_k^*|\mathbf{p}_k\rangle}\tilde{\mathbf{h}}, \qquad \Delta\binom{k}{i}\tilde{\mathbf{h}}_\ell^* = \Delta\binom{\ell}{i}\tilde{\mathbf{h}}_k^*, \qquad \mu_k\tilde{\mathbf{h}}_\ell^* = \Delta\binom{\ell}{k}\tilde{\mathbf{h}}_k^*, \tag{82}$$

for k, ℓ, and i are pairwise distinct indices such that there exist μ_k and μ_ℓ. In particular, we have the first equation directly:

$$\tilde{\mathbf{h}}_k^* = \mathbf{p}_k - \frac{1-\mu_k}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\mathbf{p}_k^* = \frac{1-\mu_k}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\left(\frac{\mu_k^*-1}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\mathbf{p}_k - \mathbf{p}_k^*\right) = \frac{-(1-\mu_k)\sqrt{\mu_k^*}}{\langle \mathbf{p}_k^* | \mathbf{p}_k \rangle}\tilde{\mathbf{h}},$$

and we also have the second and third equations by comparing the pseudo-inner products of both sides and $\mathbf{p}_j$ for each index $j = 0, \ldots, n-1$.

Theorem 21. *The followings are equivalent:*

(b) *there exists the orthocenter,*

(c)* (a)* *and* $\sum_{j=0}^{n-1} \mu_j^* \prod_{i=0}^{\substack{i\neq j\\ n-1}} (\mu_i^* - 1) > 0$ *hold,*

(c) (a) *and* $\sum_{j=0}^{n-1} \nu_j > 0$ *hold.*

Proof. The equivalence (c)* $\Leftrightarrow$ (c) comes from (31)*, (77), and Lemma 9. The implication (c) $\Rightarrow$ (b) comes from (77) and Theorem 20. For the implication (b) $\Rightarrow$ (c)*, assume that there exists the orthocenter $\mathbf{n}$. Then we can represent $\mathbf{n} = A_k^*\mathbf{p}_k^* + A_k\mathbf{p}_k$ with some A_k^*, $A_k \in \mathbb{R}$ for each k, so we have

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{n} \rangle = \langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle \tag{83}$$

for distinct indices i, $j = 0, \ldots \widehat{k} \ldots, n-1$, because the both sides are equal to $A_k^* \langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle$. So we also have (a)*, in particular, we have

$$\langle \mathbf{p}_i^* | \mathbf{p}_k^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \frac{\langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle}{\langle \mathbf{p}_j^* | \mathbf{n} \rangle} \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle = \frac{\langle \mathbf{p}_i^* | \mathbf{n} \rangle \langle \mathbf{p}_j^* | \mathbf{p}_\ell^* \rangle}{\langle \mathbf{p}_j^* | \mathbf{n} \rangle} \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle = \langle \mathbf{p}_i^* | \mathbf{p}_\ell^* \rangle \langle \mathbf{p}_j^* | \mathbf{p}_k^* \rangle.$$

Moreover, (76) and (83) imply that $\langle \mathbf{p}_i^* | \tilde{\mathbf{h}} \rangle / \langle \mathbf{p}_i^* | \mathbf{n} \rangle$ does not depend on the index $i = 0, \ldots, n-1$, so $\tilde{\mathbf{h}}$ and $\mathbf{n}$ are parallel (notice that neither $\tilde{\mathbf{h}}$ nor $\mathbf{n}$ is equal to $\mathbf{0}$). Hence, $-\langle \mathbf{n} | \mathbf{n} \rangle > 0$ implies $-\langle \tilde{\mathbf{h}} | \tilde{\mathbf{h}} \rangle > 0$, which means that $\sum_{j=0}^{n-1} \mu_j^* \prod_{i=0}^{\substack{i\neq j\\ n-1}} (\mu_i^* - 1) > 0$. $\square$

6. Appendix.

Similarly to the story of the sphere [7], we consider the simplex replaced a vertex with the orthocenter of an orthocentric simplex. If we repeat this operation, a new vertex does not appear except original vertices and the orthocenter of the original simplex. We consider the generic case, i.e., there exists the orthocenter, neither $\Delta^*\begin{pmatrix} k \begin{matrix} i \\ j \end{matrix} \end{pmatrix}$, $\Delta\begin{pmatrix} i \\ j \end{pmatrix}$, nor $\Delta\begin{pmatrix} k \begin{matrix} i \\ j \end{matrix} \end{pmatrix}$ is equal to 0 for an arbitrary trio of pairwise disjoint indices i, j, and k, all of μ_k^*, ν_k^*, μ_k, and ν_k exist, neither of them is equal to 0, and neither of them is equal to 1 for an arbitrary index k (notice that $\Delta^*\begin{pmatrix} i \\ j \end{pmatrix}$ cannot be equal to 0. See Remark 4).

Let $S' = \mathcal{M}(S)$ be the simplex replaced the last vertex with the orthocenter of an orthocentric simplex S, i.e.,

$$\mathbf{p}'^*_k = \mathbf{p}^*_k \quad \text{for} \quad k = 0, \ldots, n-2, \quad \mathbf{p}'^*_{n-1} = \mathbf{h}. \tag{84}$$

Then, it holds that

$$\mathbf{p}'_k = \frac{\mathbf{p}_k - \frac{\Delta\binom{k}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1}}{\sqrt{-\mu_k(\frac{1}{\nu_k} + \frac{1}{\nu_{n-1}})}}, \qquad \mathbf{p}'_{n-1} = \operatorname{sgn}\nu_{n-1} \cdot \mathbf{p}_{n-1}. \tag{85}$$

In particular, the pseudo-inner product

$$\Big\langle \mathbf{p}_k - \frac{\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}\Big|\mathbf{p}_k - \frac{\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}\Big\rangle = -\mu_k(\frac{1}{\nu_k} + \frac{1}{\nu_{n-1}})$$

is positive from Corollary 4 and

$$\begin{aligned}
&\Big\langle \mathbf{h}\Big|\mathbf{p}_k - \frac{\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}\Big\rangle \\
&= \Big\langle \frac{\tilde{\mathbf{h}}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\Big|\mathbf{p}_k - \frac{\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}\Big\rangle \\
&= \Big\langle \frac{\mathbf{p}^*_{n-1} - (1-\nu_{n-1})\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\Big|\mathbf{p}_k - \frac{\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\mu_{n-1}}\mathbf{p}_{n-1}\Big\rangle \\
&= \frac{\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle\Delta\begin{pmatrix}k\\n-1\end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\Big((0 - \frac{1}{\mu_{n-1}}) - (1-\nu_{n-1})(1 - \frac{1}{\mu_{n-1}})\Big) \\
&= 0.
\end{aligned}$$

So we have the former equation of (85). We also have the latter equation of (85) from

$$\langle\mathbf{p}'^*_{n-1}|\operatorname{sgn}\nu_{n-1}\cdot\mathbf{p}_{n-1}\rangle = \frac{\operatorname{sgn}\nu_{n-1}}{\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\langle\tilde{\mathbf{h}}|\mathbf{p}_{n-1}\rangle = \frac{|\nu_{n-1}|\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} > 0,$$

where the second equality comes from (75). From (76), we can define μ'^*_k, ν'^*_k, μ'_k, and ν'_k for each $k = 0, \ldots, n-1$. Similar to the Appendix of [7], we have

$$\nu'_k = \frac{-\nu_k}{\nu_{n-1}}, k = 0, \ldots, n-2 \quad \text{and} \quad \nu'_{n-1} = \frac{1}{\nu_{n-1}}, \tag{86}$$

from

$$\Delta'\binom{i}{j} = \frac{-\Delta\begin{pmatrix} i \\ j \end{pmatrix}/\nu_{n-1}}{\sqrt{-\mu_i(\frac{1}{\nu_i}+\frac{1}{\nu_{n-1}})}\sqrt{-\mu_j(\frac{1}{\nu_j}+\frac{1}{\nu_{n-1}})}}$$

and

$$\Delta'\binom{i}{n-1} = \frac{\Delta\begin{pmatrix} i \\ n-1 \end{pmatrix}/|\nu_{n-1}|}{\sqrt{-\mu_i(\frac{1}{\nu_i}+\frac{1}{\nu_{n-1}})}}$$

for pairwise distinct i, $j = 0,\ldots, n-2$. On the other hand, it is obvious that

$$\nu'^*_k = \nu^*_k \quad \text{for} \quad k = 0, \ldots, n-2. \tag{87}$$

We also have

$$\nu'^*_{n-1} = 1 + \frac{1}{\mu'^*_{n-1} - 1} = 1 - \sum_{k=0}^{n-1} \nu^*_k, \tag{88}$$

whose latter equality comes from

$$\begin{aligned}
\mu'^*_{n-1} &= \Delta'^*\binom{i}{n-1}\Delta'^*\binom{n-1}{j}/\Delta'^*\binom{i}{j} \\
&= \frac{\Delta^*\begin{pmatrix} i \\ n-1 \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\frac{\Delta^*\begin{pmatrix} n-1 \\ j \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}/\Delta^*\begin{pmatrix} i \\ j \end{pmatrix} \\
&= \frac{1}{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle} = \frac{(-1)^{n-1}\Delta^*(0\cdots n-1)}{\sum\limits_{k=0}^{n-1}\mu^*_k\prod\limits_{\substack{\ell=0 \\ \ell\neq k}}^{n-1}(\mu^*_\ell - 1)} \\
&= 1 - \frac{\prod\limits_{\ell=0}^{n-1}(\mu^*_\ell - 1)}{\sum\limits_{k=0}^{n-1}\mu^*_k\prod\limits_{\substack{\ell=0 \\ \ell\neq k}}^{n-1}(\mu^*_\ell - 1)} \\
&= 1 - \frac{1}{\sum\limits_{k=0}^{n-1}\nu^*_k},
\end{aligned}$$

where i and j are arbitrary distinct indices of $0,\ldots, n-2$, and the second equality comes from

$$\Delta'^*\binom{i}{n-1} = \frac{-\left\langle \mathbf{p}^*_i \middle| \mathbf{p}^*_{n-1} - \frac{\mu^*_{n-1}-1}{\langle\mathbf{p}^*_{n-1}|\mathbf{p}_{n-1}\rangle}\mathbf{p}_{n-1}\right\rangle}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}} = \frac{\Delta^*\begin{pmatrix} i \\ n-1 \end{pmatrix}}{\sqrt{\mu^*_{n-1}}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}.$$

S' is also orthocentric from (76) and

$$\begin{aligned}\sum_{j=0}^{n-1}\mu_j'^*\prod_{i=0}^{\substack{i\neq j\\ n-1}}(\mu_i'^*-1) &= \Big(\prod_{i=0}^{n-1}(\mu_i'^*-1)\Big)\cdot\sum_{j=0}^{n-1}\nu_j'^*\\ &= \Big(\prod_{i=0}^{n-2}(\mu_i^*-1)\Big)\frac{-1}{\sum\limits_{j=0}^{n-1}\nu_j^*}\cdot\Big(\sum_{j=0}^{n-2}\nu_j^*+(1-\sum_{j=0}^{n-1}\nu_j^*)\Big)\\ &= \Big(\prod_{i=0}^{n-2}(\mu_i^*-1)\Big)\frac{-1}{\sum\limits_{j=0}^{n-1}\nu_j^*}\cdot\frac{-1}{\mu_{n-1}^*-1}>0,\end{aligned}$$

where the last inequality comes from

$$\Big(\prod_{i=0}^{n-1}(\mu_i^*-1)\Big)\cdot\sum_{j=0}^{n-1}\nu_j^* = \sum_{j=0}^{n-1}\mu_j^*\prod_{i=0}^{\substack{i\neq j\\ n-1}}(\mu_i^*-1)>0.$$

The calculations above are useful to consider the following two cases of changing twice a vertex of an orthocentric simplex to the orthocenter. The former case is $S'=\mathcal{M}(S)$ and $S''=\mathcal{M}(S')$. Then the vertices of S'' are

$$\mathbf{p}_k''^*=\mathbf{p}_k'^*=\mathbf{p}_k^* \quad \text{for} \quad k=0,\dots,n-2, \quad \mathbf{p}_{n-1}''^*=\mathbf{h}'=\mathbf{p}_{n-1}^*. \tag{89}$$

In particular, $\mathbf{h}'=\mathbf{p}_{n-1}^*$ comes from

$$\begin{aligned}\sqrt{\mu_{n-1}'^*}\tilde{\mathbf{h}}' &= \mathbf{p}_{n-1}'^*-(1-\nu_{n-1}')\langle\mathbf{p}_{n-1}'^*|\mathbf{p}_{n-1}'\rangle\mathbf{p}_{n-1}'\\ &= \frac{\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}-(1-\frac{1}{\nu_{n-1}})\langle\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\\ &= \frac{\sqrt{\mu_{n-1}^*}\tilde{\mathbf{h}}+(1-\nu_{n-1})\langle\mathbf{p}_{n-1}^*|\mathbf{p}_{n-1}\rangle\mathbf{p}_{n-1}}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}}\\ &= \frac{\mathbf{p}_{n-1}^*}{\sqrt{\mu_{n-1}^*}\sqrt{-\langle\tilde{\mathbf{h}}|\tilde{\mathbf{h}}\rangle}},\end{aligned}$$

where the third equality comes from (75). The latter case is: $S'=\mathcal{M}(S)$,

$$\mathbf{p}_k''^*=\mathbf{p}_k'^* \quad \text{for} \quad k=0,\dots,n-3, \quad \mathbf{p}_{n-2}''^*=\mathbf{p}_{n-1}'^*, \quad \mathbf{p}_{n-1}''^*=\mathbf{p}_{n-2}'^*,$$

and $S'''=\mathcal{M}(S'')$. Then the vertices of S''' are

$$\mathbf{p}_k'''^*=\mathbf{p}_k''^*=\mathbf{p}_k'^*=\mathbf{p}_k^*, \quad \mathbf{p}_{n-2}'''^*=\mathbf{p}_{n-2}''^*=\mathbf{p}_{n-1}'^*=\mathbf{h}, \quad \mathbf{p}_{n-1}'''^*=\mathbf{h}''=\mathbf{p}_{n-1}^*. \tag{90}$$

In particular, $\mathbf{h}''=\mathbf{p}_{n-1}^*$ comes from the impossibility of $\mathbf{h}''=-\mathbf{p}_{n-1}^*$ and

$$\begin{aligned}\sqrt{\mu_{n-1}''^*}\tilde{\mathbf{h}}'' &= \mathbf{p}_{n-1}''^*-\frac{\mathbf{p}_{n-1}''}{(\nu_{n-1}''^*-1)\langle\mathbf{p}_{n-1}''^*|\mathbf{p}_{n-1}''\rangle}\\ &= \mathbf{p}_{n-2}'^*-\frac{\mathbf{p}_{n-2}'}{(\nu_{n-2}'^*-1)\langle\mathbf{p}_{n-2}'^*|\mathbf{p}_{n-2}'\rangle}\end{aligned}$$

$$= \mathbf{p}'^*_{n-2} - \frac{\sqrt{-\mu_{n-2}(\frac{1}{\nu_{n-2}} + \frac{1}{\nu_{n-1}})}\mathbf{p}'_{n-2}}{(\nu'^*_{n-2} - 1)\langle \mathbf{p}'^*_{n-2} | \sqrt{-\mu_{n-2}(\frac{1}{\nu_{n-2}} + \frac{1}{\nu_{n-1}})}\mathbf{p}'_{n-2}\rangle}$$

$$= \mathbf{p}^*_{n-2} - \frac{\mathbf{p}_{n-2} - \frac{\Delta\binom{n-2}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1}}{(\nu^*_{n-2} - 1)\langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2}\rangle}$$

$$= \frac{-(\tilde{\mathbf{h}}^*_{n-2} - \frac{\Delta\binom{n-2}{n-1}}{\mu_{n-1}}\mathbf{p}_{n-1})}{(\nu^*_{n-2} - 1)\langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2}\rangle}$$

$$= \frac{-\Delta\begin{pmatrix} n-2 \\ n-1 \end{pmatrix} \cdot \frac{\tilde{\mathbf{h}}^*_{n-1} - \mathbf{p}_{n-1}}{\mu_{n-1}}}{(\nu^*_{n-2} - 1)\langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2}\rangle}$$

$$= \frac{-\Delta\begin{pmatrix} n-2 \\ n-1 \end{pmatrix} \cdot \frac{\mathbf{p}^*_{n-1}}{\nu_{n-1}\langle \mathbf{p}^*_{n-1} | \mathbf{p}_{n-1}\rangle}}{(\nu^*_{n-2} - 1)\langle \mathbf{p}^*_{n-2} | \mathbf{p}_{n-2}\rangle},$$

where the fifth equality comes from (81), the sixth equality comes from (82), and the last equality comes from (72) and (81).

From two cases above, if we denote $\mathbf{p}^*_n = \mathbf{h}$, then orthocenters of the simplex with vertices $\mathbf{p}^*_0, \ldots \widehat{\mathbf{p}^*_k} \ldots, \mathbf{p}^*_n$ are $\mathbf{p}^*_k$, for $k = 0, \ldots, n-1$. Moreover, for $k = 0, \ldots, n$, from (87), ν^*_k does not change if S is replaced with the simplex with vertices $\mathbf{p}^*_0, \ldots \widehat{\mathbf{p}^*_i} \ldots, \mathbf{p}^*_n$ for any $i = 0, \ldots \widehat{k} \ldots, n-1$. If we denote ν'^*_{n-1} for $S' = \mathcal{M}(S)$ by ν^*_n, we have

$$\nu^*_0 + \cdots + \nu^*_{n-1} + \nu^*_n = 1, \tag{91}$$

from (88) (see the equations (2) of [2], (4) of [3], and (24) of [7]).

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Properties of a Pascal Points Circle in a Quadrilateral with Perpendicular Diagonals

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Abstract. The theory of a convex quadrilateral and a circle that forms Pascal points is a new topic in Euclidean geometry. The theory deals with the properties of the Pascal points on the sides of a convex quadrilateral and with the properties of circles that form Pascal points.

In the present paper, we shall continue developing the theory, and we shall define the concept of the "Pascal points circle".

We shall prove four theorems regarding the properties of the points of intersection of a Pascal points circle with a quadrilateral that has intersecting perpendicular diagonals.

1. Introduction: General concepts and Fundamental Theorem of the theory of a convex quadrilateral and a circle that forms Pascal points

First, we shall briefly survey the definitions of some essential concepts of the theory of a convex quadrilateral and a circle that forms Pascal points on its sides, and then we shall present this theory's Fundamental Theorem (see [1], [2], [3]). The theory considers the situation in which $ABCD$ is a convex quadrilateral for which there exists a circle ω that satisfies the following two requirements:

(i) Circle ω passes through point E, the point of intersection of the diagonals, and through point F, the point of intersection of the extensions of sides BC and AD.

(ii) Circle ω intersects sides BC and AD at interior points (points M and N, respectively, in Figure 1).

The Fundamental Theorem of the theory holds in this case.

The Fundamental Theorem.
Let there be: a convex quadrilateral; a circle that intersects a pair of opposite sides of the quadrilateral, that passes through the point of intersection of the extensions of these sides, and that passes through the point of intersection of the diagonals.
In addition, let there be four straight lines, each of which passes both through the point of intersection of the circle with a side of the quadrilateral and through the point of intersection of the circle with the extension of a diagonal.
Then there holds: the straight lines intersect at two points that are located on the other pair of opposite sides of the quadrilateral.
(In Figure 2, straight lines h and g intersect at point P on side AB, and straight lines i and j intersect at point Q on side CD).

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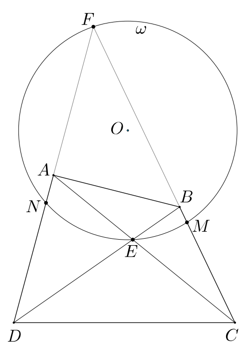

Figure 1

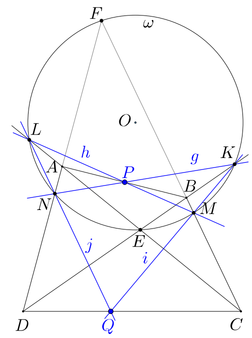

Figure 2

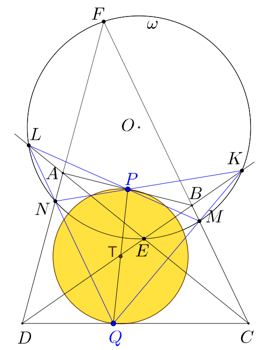

Figure 3

The Fundamental Theorem is proven using the general Pascal's Theorem (see [1]).

Definitions

Because the proof of the properties of the points of intersection P and Q is based on Pascal's Theorem,

(I) points P and Q are termed *Pascal points on sides AB and CD of the quadrilateral*;

(II) the circle that passes through points of intersection E and F and through two opposite sides is termed *a circle that forms Pascal points on the sides of the quadrilateral*.

We define a new concept: Pascal points circle.

(III) We shall call a circle whose diameter is segment PQ (see Figure 3) a *Pascal points circle*.

2. Properties of a quadrilateral with perpendicular intersecting diagonals, a circle that forms Pascal points, and a Pascal points circle.

Theorem 1.

Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of the sides BC and AD; ω_{EF} is the circle whose diameter is segment EF. Then,

(a) *circle ω_{EF} forms Pascal points on sides AB and CD* (see Figure 4)*;*
there are an infinite number of circles that form Pascal points on sides AB and CD;

(b) *for every circle, ω, that intersects sides BC and AD at points M and N, respectively, and forms Pascal points P and Q on sides AB and CD, respectively, there holds:*

the point of intersection, T, of the tangents to circle ω at points M and N is the middle of segment PQ.

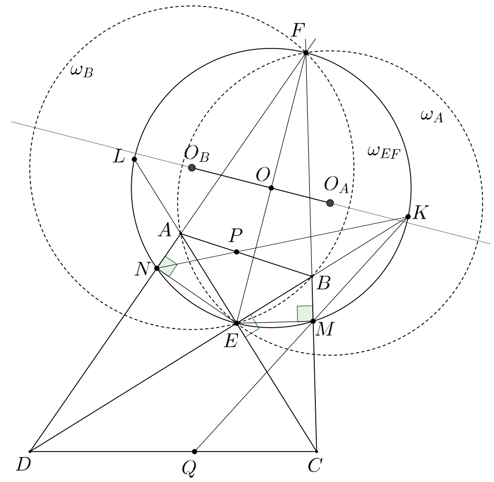

Figure 4.

Proof.
(a) Let us show that circle ω_{EF} intersects sides BC and AD at internal points.
In circle ω_{EF}, angle $\measuredangle EMF$ equals $90°$. Therefore, in right triangle $\triangle BCE$, segment EM is an altitude to hypotenuse BC, and hence it follows that the foot of altitude EM (point M in Figure 4) is an interior point of side BC. Similarly, we prove that point N (the base of the altitude to hypotenuse AD in right triangle $\triangle ADE$) is an internal point of side AD.
Based on the fundamental theorem, since circle ω_{EF} intersects sides BC and AD at internal points, this circle necessarily forms Pascal points on sides AB and CD. It is clear that if there is even one circle that passes through points E and F and also through internal points of sides BC and AD, then there must be an infinite number of such circles. Therefore, in our case, there are an infinite number of circles that pass through points E and F and through internal points of sides BC and AD. All these circles form Pascal points on sides AB and CD.

(b) Let us employ the following property that holds true for a convex quadrilateral (whose diagonals are not necessarily perpendicular) and a circle, ω, that forms Pascal points P and Q on sides AB and CD.
We denote: M and N are the intersection points of circle ω with sides BC and AD, respectively, and K and L are the intersection points of circle ω with the extensions of diagonals BD and AC, respectively (see Figure 5).
It thus holds that the four points P, Q, T, and R (P and Q are the two Pascal points, T is the point of intersection of the tangents to the circle at points M and

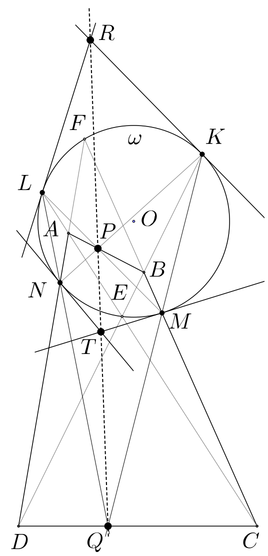

Figure 5

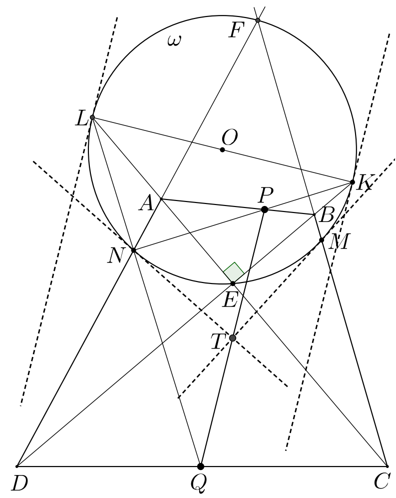

Figure 6

N, and R is the point of intersection of the tangents to the circle at points K and L) constitute a harmonic quadruple, in other words, there holds: $\dfrac{QT}{TP} = \dfrac{QR}{RP}$ (see [3, Theorem 1].)

In our case, the quadrilateral has perpendicular diagonals.
Therefore, $\measuredangle KEL = 90°$ and segment KL is a diameter of ω. Therefore, the tangents to circle ω at points K and L are parallel to each other (see Figure 6). In this case, their point of intersection, R, is at infinity, and ratio $\dfrac{QR}{RP}$ equals 1. Hence it also holds that $\dfrac{QT}{TP} = 1$, or $QT = TP$. In other words, point T is the middle of segment PQ. $\square$

Theorem 2.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD; ω is a circle that passes through points E and F and intersects sides BC and AD at points M and N, respectively; P and Q are Pascal points formed using ω on sides AB and CD, respectively; σ_{PQ} is a circle whose diameter is segment PQ (a Pascal points circle); T is the center of circle σ_{PQ} (see Figure 7).
Then,
(a) *Sides BC and AD each have at least one common point with circle σ_{PQ}. In other words:*

(1) *In the case that the center, O, of circle ω does not belong to straight lines BF and AF, circle σ_{PQ} intersects sides BC and AD at two points each. Two of these four points of intersection are N and M; the other two points are denoted as V and W* (see Figure 7).

(2) *When center O lies on straight line BF, circle σ_{PQ} is tangent to side BC at point M. In this case, point V coincides with M.*

(3) *When center O lies on straight line AF, circle σ_{PQ} is tangent to side AD at point N. In this case, point W coincides with N.*

(b) *Points V, T, and W lie on the same straight line. This property holds even in cases when point V coincides with point M or point W coincides with point N.*
(c) *Circles ω and σ_{PQ} are perpendicular to each other.*

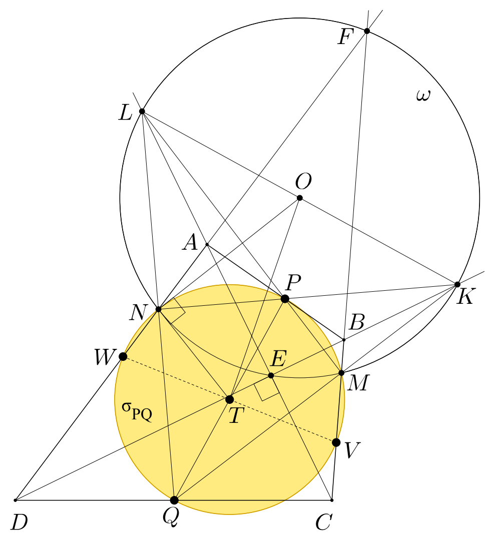

Figure 7.

Proof.
(a) We shall use the method of complex numbers in plane geometry. (The principles of the method and a system of formulas that we use in the proofs appear, for example, in [6, pp. 154-181]; some isolated formulas may be found in [4], [5]).
Let us choose a system of coordinates such that circle ω is the unit circle (O is the origin, and the radius is $OE = 1$). In this system, the equation of circle ω is $z \cdot \overline{z} = 1$, where z is the complex coordinate of some point Z that belongs to circle ω, and $\overline{z}$ is the conjugate of z.
We denote the complex coordinates of points K, L, M and N by k, l, m and n, respectively. These points are located on unit circle ω, therefore there holds: $\overline{k} = \dfrac{1}{k}$, $\overline{l} = \dfrac{1}{l}$, $\overline{m} = \dfrac{1}{m}$, and $\overline{n} = \dfrac{1}{n}$.
Point P is the point of intersection of straight lines KN and LM. Let us express the complex coordinate of P (and its conjugate) using the coordinates of points K,

L, M, and N. We shall use the following formula:
Let $A(a)$, $B(b)$, $C(c)$, and $D(d)$ be four points on the unit circle, and let $S(s)$ be the point of intersection of straight lines AB and CD. Then the coordinate s and its conjugate $\overline{s}$ satisfy:

$$\overline{s} = \frac{a+b-c-d}{ab-cd} \quad \textit{and} \quad s = \frac{bcd+acd-abd-abc}{cd-ab} \tag{I}$$

In our case, segment KL is a diameter of circle ω. Therefore, $k=-l$, and the expressions for p and $\overline{p}$ are:

$$\overline{p} = \frac{n+k-m-l}{nk-ml} = \frac{2l+m-n}{l\,(m+n)} \quad \textit{and} \quad p = \frac{2mn+nl-ml}{m+n}.$$

Now, let us find complex coordinate t of point T (which is the point of intersection of the tangents to the unit circle at points M and N). We use the following formula:
Let $S(s)$ be the point of intersection of the tangents to the unit circle at points $A(a)$ and $B(b)$, which are located on the circle. Then coordinate s and its conjugate $\overline{s}$ satisfy:

$$s = \frac{2ab}{a+b} \quad \textit{and} \quad \overline{s} = \frac{2}{a+b}. \tag{II}$$

In our case, we obtain for coordinate t and its conjugate $\bar{t}$ the following:

$$t = \frac{2mn}{m+n} \quad \textit{and} \quad \bar{t} = \frac{2}{m+n}.$$

Point T is the center of circle σ_{PQ}. Therefore, the equation of circle σ_{PQ} is

$$(z-t)\left(\overline{z}-\overline{t}\right) = r_{\sigma_{PQ}}^2,$$

where $r_{\sigma_{PQ}}$ is the radius of the circle, and z is the complex coordinate of some point Z that belongs to the circle.
Let us find the square of the radius of circle σ_{PQ}. Point P lies on the circle, therefore the following equality holds: $(p-t)\left(\overline{p}-\overline{t}\right) = r_{\sigma_{PQ}}^2$.
Let us substitute the expressions for p and $\overline{p}$ in the left-hand side of the equality. We obtain:

$$\begin{aligned}&\left(\frac{2mn+nl-ml}{m+n} - \frac{2mn}{m+n}\right)\left(\frac{2l+m-n}{l\,(m+n)} - \frac{2}{m+n}\right)\\ &= \frac{nl-ml}{m+n}\cdot\frac{m-n}{l\,(m+n)} = -\left(\frac{m-n}{m+n}\right)^2.\end{aligned}$$

In other words, there holds $r_{\sigma_{PQ}}^2 = -\left(\dfrac{m-n}{m+n}\right)^2$. Therefore, the equation for circle σ_{PQ} is

$$\left(z - \frac{2mn}{m+n}\right)\left(\overline{z} - \frac{2}{m+n}\right) = -\left(\frac{m-n}{m+n}\right)^2. \tag{1}$$

Now let us find the equations of straight lines BC and AD and, subsequently, their points of intersection with circle σ_{PQ}.

We use the following formula of a straight line passing through two points $A(a)$ and $B(b)$ belonging to the unit circle:

$$z + ab\overline{z} = a + b. \tag{III}$$

In accordance with this formula, the equation of straight line BC (which passes through points $F(f)$ and $M(m)$ that belong to the unit circle) shall be $z + fm\overline{z} = f + m$. Hence:

$$\overline{z} = -\frac{1}{fm}z + \frac{f+m}{fm}. \tag{2}$$

We substitute the expression for $\overline{z}$ from (2) into (1) and obtain:

$$\begin{aligned}
&\left(z - \frac{2mn}{m+n}\right)\left(-\frac{1}{fm}z + \frac{f+m}{fm} - \frac{2}{m+n}\right) + \left(\frac{m-n}{m+n}\right)^2 = 0,\\
&-\frac{1}{fm}z^2 + \left(\frac{f+m}{fm} - \frac{2}{m+n} + \frac{2mn}{fm\,(m+n)}\right)z\\
&-\frac{2mn\,(f+m)}{fm\,(m+n)} + \frac{4mn}{(m+n)^2} + \frac{(m-n)^2}{(m+n)^2} = 0.
\end{aligned}$$

This leads to the following quadratic equation:

$$(m+n)\,z^2 - \left(3mn - fm + fn + m^2\right)z + fmn + 2m^2n - fm^2 = 0. \tag{3}$$

The solutions of this equation are:

$$\begin{aligned}
z_{1,2} &= \frac{3mn - fm + fn + m^2 \pm \sqrt{\begin{array}{c}\left(3mn - fm + fn + m^2\right)^2\\ -4\,(m+n)\left(fmn + 2m^2n - fm^2\right)\end{array}}}{2\,(m+n)}\\
&= \frac{3mn - fm + fn + m^2 \pm \sqrt{(m-n)^2\,(m+f)^2}}{2\,(m+n)}.
\end{aligned}$$

Equation (3) is a quadratic equation with complex coefficients.
It follows that if expression $(m-n)^2\,(m+f)^2$ does not equal 0, then (3) will have two solutions. In the present case it necessarily holds that $m \neq -f$, and hence it follows that points F and M are not the ends of the diameter of circle ω. This means that the center, O, of the circle does not belong to straight line MF (the line BF). In this case the two solutions of the equation are:

$$z_1 = \frac{3mn - fm + fn + m^2 + (m-n)\,(m+f)}{2\,(m+n)} = \frac{2mn + 2m^2}{2\,(m+n)} = m,$$

and

$$z_2 = \frac{3mn - fm + fn + m^2 - (m-n)\,(m+f)}{2\,(m+n)} = \frac{2mn - fm + fn}{m+n}.$$

It is clear that the first solution is the complex coordinate of point M, and the second solution is the coordinate of another point that belongs to straight line BC (denoted by V).

In other words: $v = \dfrac{2mn - fm + fn}{m+n}$.
Similarly, one can prove that in the case where the center, O, of circle ω does not lie on straight line AF, circle σ_{PQ} will intersect straight line AD at two points: 1) at point N, and 2) at some other point (designated as W) whose complex coordinate can be expressed as $w = \dfrac{2mn - fn + fm}{m+n}$.
If $(m-n)^2(m+f)^2 = 0$ holds, then Equation (3) has a single solution. For two different points M and N located on unit circle ω, there holds $m \neq n$, therefore necessarily there holds $m = -f$. In other words, points F and M are the ends of a diameter of circle ω, and therefore center O of the circle belongs to straight line BF.
In this case, the only solution of the equation is:

$$z = \frac{3mn - fm + fn + m^2}{2(m+n)} = \frac{3mn + m^2 - mn + m^2}{2(m+n)} = \frac{2mn + 2m^2}{2(m+n)} = m.$$

In other words, in this case, line BF is tangent to circle σ_{PQ} at point M.
Similarly, we can prove that when the center, O, of ω belongs to line AF, line AF will be tangent to circle σ_{PQ} at point N.

(b) Let us prove that points V, T, and W lie on the same straight line (see Figure 7).
We shall use the following formula, which gives the relation between the coordinates of any three collinear points $A(a)$, $B(b)$, and $C(c)$:

$$a\left(\overline{b} - \overline{c}\right) + b\left(\overline{c} - \overline{a}\right) + c\left(\overline{a} - \overline{b}\right) = 0. \tag{IV}$$

According to this formula, points V, T, and W are collinear provided the following equality holds:

$$v\left(\overline{t} - \overline{w}\right) + t\left(\overline{w} - \overline{v}\right) + w\left(\overline{v} - \overline{t}\right) = 0. \tag{4}$$

Let us first calculate the conjugates of coordinates v and w:

$$\overline{v} = \overline{\frac{2mn - fm + fn}{m+n}} = \frac{\dfrac{2}{mn} - \dfrac{1}{fm} + \dfrac{1}{fn}}{\dfrac{1}{m} + \dfrac{1}{n}} = \frac{2f - n + m}{f(m+n)}$$

and similarly

$$\overline{w} = \frac{2f - m + n}{f(m+n)}.$$

We substitute the expressions for t, $\overline{t}$, v, $\overline{v}$, w, and $\overline{w}$ into (4), to obtain:

$$\begin{aligned} &\frac{2mn - fm + fn}{m+n}\left(\frac{2}{m+n} - \frac{2f - m + n}{f(m+n)}\right) \\ &+ \frac{2mn}{m+n}\left(\frac{2f - m + n}{f(m+n)} - \frac{2f - n + m}{f(m+n)}\right) \\ &+ \frac{2mn - fn + fm}{m+n}\left(\frac{2f - n + m}{f(m+n)} - \frac{2}{m+n}\right) = 0. \end{aligned}$$

After simplifying the left-hand side, we have:

$$\begin{aligned}&\frac{2mn-fm+fn}{m+n}\cdot\frac{m-n}{f\left(m+n\right)}+\frac{2mn}{m+n}\cdot\frac{2n-2m}{f\left(m+n\right)}\\&+\frac{2mn-fn+fm}{m+n}\cdot\frac{m-n}{f\left(m+n\right)}=0,\\&\frac{m-n}{f\left(m+n\right)^2}\cdot\underbrace{\left(2mn-fm+fn-4mn+2mn-fn+fm\right)}_{=0}=0.\end{aligned}$$

We have thus obtained $0=0$.
In other words, (IV) is satisfied, and therefore points V, T, and W must be on the same straight line, and segment VW is a diameter of circle σ_{PQ}.

(c) In (a), we proved that circles ω and σ_{PQ} intersect at points M and N, and therefore $r_\omega=OM$ and $r_{\sigma_{PQ}}=TM$. Let us find the distance, OT, between the centers of circles ω and σ_{PQ}:

$$OT^2=(t-0)\left(\bar{t}-\overline{0}\right)=\left(\frac{2mn}{m+n}-0\right)\left(\frac{2}{m+n}-0\right)=\frac{4mn}{\left(m+n\right)^2}.$$

Now we calculate the sum $r_\omega^2+r_{\sigma_{PQ}}^2$:

$$r_\omega^2+r_{\sigma_{PQ}}^2=1-\left(\frac{m-n}{m+n}\right)^2=\frac{\left(m+n\right)^2-\left(m-n\right)^2}{\left(m+n\right)^2}=\frac{4mn}{\left(m+n\right)^2}.$$

Therefore, the equation $r_\omega^2+r_{\sigma_{PQ}}^2=OT^2$ holds , and, specifically, $OM^2+TM^2=OT^2$ holds.
It thus follows that angle $\measuredangle OMT$ is a right angle, and therefore line OM is tangent to circle σ_{PQ}, and line TM is tangent to ω.
We obtained that the tangents to circles ω and σ_{PQ} at the point of their intersection, M, are perpendicular to each other.
Therefore the circles are perpendicular to each other. $\square$

Conclusions from Theorem 2.
(1) We obtained that the two segments PQ and VW are diameters of circle σ_{PQ}. Therefore their lengths are equal, and they bisect each other (at point T). It follows that quadrilateral $PVQW$ is a rectangle (see Figure 8).

Note: Rectangle $PVQW$ (in which two opposite vertices are Pascal points) is usually different from the rectangle inscribed in quadrilateral $ABCD$ in such a manner that its sides are parallel to diagonals AC and BD, which are perpendicular to each other. (In Figure 8 rectangle $PXYZ$ is inscribed in the quadrilateral and its sides are parallel to the diagonals of the quadrilateral.)

(2) For any quadrilateral, $ABCD$, with perpendicular diagonals and any circle, ω_i, that forms a pair of Pascal points P_i and Q_i on sides AB and CD, one can define a rectangle that is inscribed in quadrilateral $ABCD$ as follows:
We construct a Pascal points circle $\sigma_{P_iQ_i}$ that intersects sides BC and AD at points V_i and W_i (in addition to points N_i and M_i). Points P_i, V_i, Q_i, and W_i define a rectangle inscribed in quadrilateral $ABCD$.

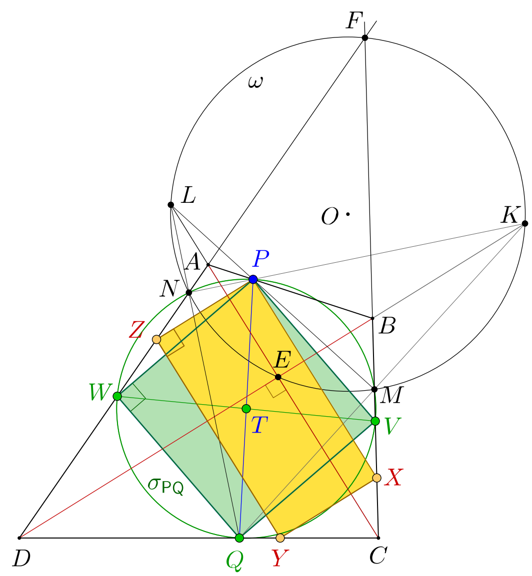

Figure 8.

(3) Let ω be a circle that passes through points E and F, intersects sides BC and AD at points M and N, respectively, and forms Pascal points P and Q on sides AB and CD. T is the point of intersection of the tangents to circle ω at points M and N.
In this case, the circle whose center is at point T and whose radius is segment TM is the Pascal points circle σ_{PQ}.

Explanation: In Theorem 1 we proved that the tangents to circle ω at points M and N intersect in the middle of segment PQ (at point T). In Theorem 2 we proved that Pascal points circle σ_{PQ} passes through points M and N.

Theorem 3.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD; ω_{EF} is a circle whose diameter is segment EF; Circle ω_{EF} intersects sides BC and AD at points M_0 and N_0, respectively, and forms Pascal points P_0 and Q_0 on sides AB and CD; $\sigma_{P_0Q_0}$ is the Pascal points circle *of points P_0 and Q_0. Then:*
(a) *Circle $\sigma_{P_0Q_0}$ intersects the sides of quadrilateral $ABCD$ at 8 points, as follows:*
It intersects side AB at points P_0 and M_1, side BC at M_0 and V_0, side CD at points Q_0 and N_1, side AD at N_0 and W_0. (In Figure 9, one can observe the fours points of intersection mentioned in Theorem 2, N_0, M_0, V_0, and W_0, and also two additional points of intersection, M_1 and N_1).
(b) *Chords V_0N_0, W_0M_0, Q_0M_1, and P_0N_1 of the circle intersect at point E.*

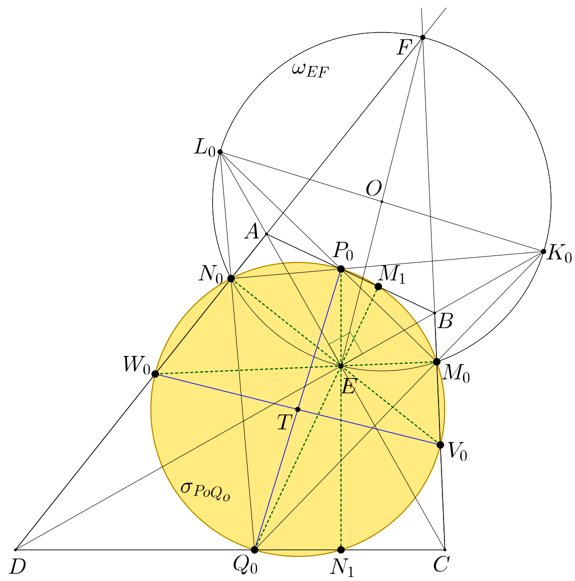

Figure 9.

Proof.
The center, O, of circle ω_{EF} does not belong to straight lines FB and FA. Therefore, from Theorem 2, circle $\sigma_{P_0Q_0}$ intersects each of the sides BC and AD at two points (at points M_0 and V_0, and at points N_0 and W_0, respectively). Therefore, it remains to be proven that $\sigma_{P_0Q_0}$ intersects each of the other two sides at two points.
We will first prove one additional property that holds for the points of intersection of circle $\sigma_{P_0Q_0}$ with sides BC and AD. We will show that chords V_0N_0 and W_0M_0 both pass through point E.
We choose a system of coordinates such that circle ω_{EF} is the unit circle (O is the origin and the radius, OE, equals 1).
From formula (IV) in the proof of Theorem 2, points $N_0\,(n)$, $E\,(e)$ and $V_0\,(v)$ are collinear provided the following equality holds:

$$n\,(\overline{e}-\overline{v})+e\,(\overline{v}-\overline{n})+v\,(\overline{n}-\overline{e})=0.$$

For e, $\overline{e}$, v, and $\overline{v}$ there holds: $e=-f$ (because segment EF is the diameter of the unit circle), $\overline{e}=-\dfrac{1}{f}$, $v=\dfrac{2mn-fm+fn}{m+n}$, and $\overline{v}=\dfrac{2f-n+m}{f\,(m+n)}$ (see the proof of Theorem 2).
We substitute these expressions in the left-hand side of the formula above, and

obtain:

$$
\begin{aligned}
& n\left(\overline{e}-\overline{v}\right)+e\left(\overline{v}-\overline{n}\right)+v\left(\overline{n}-\overline{e}\right)\\
&= n\left(-\frac{1}{f}-\frac{2f-n+m}{f\left(m+n\right)}\right)-f\left(\frac{2f-n+m}{f\left(m+n\right)}-\frac{1}{n}\right)+\frac{2mn-fm+fn}{m+n}\left(\frac{1}{n}+\frac{1}{f}\right)\\
&= n\cdot\frac{-2m-2f}{f\left(m+n\right)}-f\cdot\frac{fn-n^2+mn-fm}{fn\left(m+n\right)}+\frac{\left(2mn-fm+fn\right)\left(f+n\right)}{fn\left(m+n\right)}\\
&= \frac{0}{fn\left(m+n\right)}\\
&= 0.
\end{aligned}
$$

In other words, the equality holds and therefore the points N_0, E, and V_0 are collinear (see Figure 9).
Similarly, we also prove that points M_0, E, and W_0 are collinear.
To find the remaining two points of intersection, we follow the following path:
The first stage is to find two points that can be candidates for the intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD. The second stage is to prove that these two points are really the points of intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD.
It is reasonable to assume that the property satisfied for the four points of intersection of circle $\sigma_{P_0Q_0}$ with sides BC and AD shall also hold for the four points of intersection of circle $\sigma_{P_0Q_0}$ with sides AB and CD. Therefore, at the first stage we shall choose points M_1 and N_1 to be our candidates, which are the intersection points of line AB with line Q_0E and line CD with line P_0E, respectively.
At the second stage, we shall prove that the points M_1 and N_1 belong to circle $\sigma_{P_0Q_0}$.
Using formula (IV) we obtain the equation of straight line AB.
The formula holds for three collinear points $A(a)$, $B(b)$, and $C(c)$. If we replace the coordinate of point C by the coordinate of some point $Z(z)$ that belongs to straight line AB, we obtain the equation of AB:

$$a\left(\overline{b}-\overline{z}\right)+b\left(\overline{z}-\overline{a}\right)+z\left(\overline{a}-\overline{b}\right)=0.$$

This can be put in the form:

$$\overline{z}=\frac{\overline{a}-\overline{b}}{a-b}z+\frac{a\overline{b}-\overline{a}b}{a-b}. \tag{V}$$

Let us express the coordinates of $A(a)$ and $B(b)$ (and their conjugates) using the coordinates of points F, E, K, L, M, and N, which lie on the unit circle. We shall use the formulas (I) from the proof of Theorem 2.
In our case, segments K_0L_0 and EF are diameters of circle ω_{EF}. Therefore, there holds that $k=-l$ and $e=-f$. The following expressions are therefore obtained:

$$
\begin{aligned}
a &= \frac{2nl+fl-fn}{n+l}, & \overline{a} &= \frac{2f+n-l}{f\left(n+l\right)},\\
b &= \frac{2ml+fl+fm}{l-m}, & \overline{b} &= \frac{2f+m+l}{f\left(m-l\right)}.
\end{aligned}
$$

We substitute these expressions into (V) to obtain:

$$\overline{z}=\frac{\dfrac{2f+n-l}{f\,(n+l)}-\dfrac{2f+m+l}{f\,(m-l)}}{\dfrac{2nl+fl-fn}{n+l}-\dfrac{2ml+fl+fm}{l-m}}z+\frac{\dfrac{2nl+fl-fn}{n+l}\cdot\dfrac{2f+m+l}{f\,(m-l)}-\dfrac{2f+n-l}{f\,(n+l)}\cdot\dfrac{2mk+fl+fm}{l-m}}{\dfrac{2nl+fl-fn}{n+l}-\dfrac{2ml+fl+fm}{l-m}}.$$

After simplifying, we obtain:

$$\overline{z}=\frac{fm-fn-2fl-ml-nl}{fl\,(fm+fn+2mn+ml-nl)}z+\frac{2fnl+2fml+2f^2l+2mnl-f^2n+f^2m+nl^2-ml^2}{fl\,(fm+fn+2mn+ml-nl)}. \tag{5}$$

Similarly, we can obtain the equation of line QE: we replace the letters a and b in (V) with the letters e and q to obtain: $\overline{z}=\dfrac{\overline{q}-\overline{e}}{q-e}z+\dfrac{q\overline{e}-\overline{q}e}{q-e}$.

In our case there holds: $e=-f$ and $\overline{e}=-\dfrac{1}{f}$.

Point Q is the point of intersection of straight lines KM and LN. In addition, in our case there holds that $k=-l$. Therefore, from the formulas (I) for q and $\overline{q}$, we obtain the following expressions: $q=\dfrac{2mn+ml-nl}{m+n}$ and $\overline{q}=\dfrac{2l+n-m}{l\,(m+n)}$.

We substitute these expressions in the equation of straight line QE, and obtain:

$$\overline{z}=\frac{\dfrac{2l+n-m}{l\,(m+n)}+\dfrac{1}{f}}{\dfrac{2mn+ml-nl}{m+n}+f}z+\frac{\dfrac{2mn+ml-nl}{m+n}\cdot\left(-\dfrac{1}{f}\right)+\dfrac{2l+n-m}{l\,(m+n)}\cdot f}{\dfrac{2mn+ml-nl}{m+n}+f},$$

and after simplifying:

$$\overline{z}=\frac{2fl+fn-fm+ml+nl}{fl\,(2mn+ml-nl+fm+fn)}z+\frac{2f^2l+f^2n-f^2m-2mnl-ml^2+nl^2}{fl\,(2mn+ml-nl+fm+fn)}. \tag{6}$$

By equating the right-hand sides of (5) and (6), we obtain an expression for complex coordinate z_{M_1} of the intersection point of straight lines AB and QE:

$$z_{M_1}\underset{\text{we denote}}{=}m_1=\frac{f^2n-f^2m-2lmn-fml-fnl}{fm-fn-2fl-ml-nl}.$$

The expression for the conjugate of m_1 is: $\overline{m}_1=\dfrac{ml-nl-2f^2-fm-fn}{f\,(nl-ml-2mn-fm-fn)}$.

We now prove that point M_1 belongs to circle $\sigma_{P_0Q_0}$.

From formula (1) in the proof of Theorem 2, the equation of circle $\sigma_{P_0Q_0}$ is:

$$\left(z-\frac{2mn}{m+n}\right)\left(\overline{z}-\frac{2}{m+n}\right)=-\left(\frac{m-n}{m+n}\right)^2.$$

Let us substitute the expressions for m_1 and $\overline{m}_1$ in the equation of the circle. We obtain:

$$\begin{aligned}&\left(\frac{f^2n-f^2m-2lmn-fml-fnl}{fm-fn-2fl-ml-nl}-\frac{2mn}{m+n}\right)\\&\times\left(\frac{ml-nl-2f^2-fm-fn}{f\left(nl-ml-2mn-fm-fn\right)}-\frac{2}{m+n}\right)\\&=\ -\left(\frac{m-n}{m+n}\right)^2\end{aligned}$$

Let us check if this equality is a true statement.

Observe the left-hand side of the equality. After adding fractions and collecting similar terms, we obtain:

$$\begin{aligned}&\frac{fn^2-fm^2-lm^2-ln^2+2mnl-2m^2n-2mn^2}{\left(fm-fn-2fl-ml-nl\right)\left(m+n\right)}\\&\times\frac{lm^2-ln^2-fm^2-fn^2+2fmn-2fnl+2fml}{\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)}.\end{aligned}$$

After factoring the expressions in the numerators, we obtain:

$$\begin{aligned}&\frac{(n-m)\left(fm+fn-nl+ml+2mn\right)}{\left(fm-fn-2fl-ml-nl\right)\left(m+n\right)}\cdot\frac{(m-n)\left(ml+nl-fm+fn+2fl\right)}{\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)}\\&=\frac{-\left(m-n\right)^2\left(-fm-fn+nl-ml-2mn\right)\left(-ml-nl+fm-fn-2fl\right)}{\left(fm-fn-2fl-ml-nl\right)\left(nl-ml-2mn-fm-fn\right)\left(m+n\right)^2}\\&=-\left(\frac{m-n}{m+n}\right)^2.\end{aligned}$$

We have obtained an identical expression on both sides of the equality. Therefore the last equality is a true statement, and therefore point M_1 belongs to circle $\sigma_{P_0Q_0}$. Since point M_1 belongs to line AB, it follows that circle $\sigma_{P_0Q_0}$ intersects straight line AB at point M_1.

Similarly, we can prove that circle $\sigma_{P_0Q_0}$ intersects straight line CD at point N_1.

In summary, we have shown that the four chords V_0N_0, W_0M_0, Q_0M_1, and P_0N_1 of circle $\sigma_{P_0Q_0}$ pass through point E, which is, therefore, their point of intersection. □

Conclusions from Theorems 1-3.

In proving Theorems 1-3 we considered a quadrilateral whose two opposite sides, BC and AD, are not parallel, and we did not require any additional conditions concerning the remaining opposite sides.

In the case that sides AB and CD also intersect (we denote the point of their intersection by G), there will be circles that pass through points E and G and form Pascal points on sides BC and AD. In this case, for these circles Theorems similar to Theorems 1-3 shall hold (the proofs of these theorems are similar to the proofs

of Theorems 1-3).
According to these theorems, the circle whose diameter is segment EG (we denote it by ψ_{EG}) satisfies the following properties:

(a) The circle ψ_{EG} forms Pascal points on sides BC and AD (for now we denote these points by $\overline{P}$ and $\overline{Q}$, respectively).

(b) The circle ψ_{EG} and the circle whose diameter is segment $\overline{P}\,\overline{Q}$ are perpendicular to each other, and they intersect at the points at which circle ψ_{EG} intersects sides AB and CD (for now we denote these points by $\overline{M}$ and $\overline{N}$, respectively).

(c) The circle whose diameter is segment $\overline{P}\,\overline{Q}$ intersects sides AB and CD at points $\overline{V}$ and $\overline{W}$ (in addition to points $\overline{M}$ and $\overline{N}$). The four points $\overline{P}$, $\overline{V}$, $\overline{Q}$, and $\overline{W}$ define a rectangle inscribed in quadrilateral $ABCD$.

Theorem 4.
Let $ABCD$ be a quadrilateral with perpendicular diagonals in which E is the point of intersection of the diagonals, F is the point of intersection of the extensions of sides BC and AD, and G is the point of intersection of the extensions of the sides AB and CD; ω_{EF} is a circle whose diameter is segment EF which forms Pascal points P_0 and Q_0 on sides AB and CD, respectively; $\sigma_{P_0Q_0}$ is the Pascal points circle *of points P_0 and Q_0, which intersects the sides of quadrilateral $ABCD$ at the following eight points: P_0, Q_0, M_0, N_0, V_0, W_0, M_1, and N_1* (see Theorem 3)*; ψ_{EG} is the circle whose diameter is segment EG. Then:*
(a) *The circle ψ_{EG} intersects sides AB and CD at points M_1 and N_1, respectively.*
(b) *Circles ψ_{EG} and $\sigma_{P_0Q_0}$ are perpendicular to each other.*
(c) *Points V_0 and W_0 are the Pascal points formed by circle ψ_{EG} on sides BC and AD, respectively.*
(d) *The angle between diameters EF and EG of circles ω_{EF} and ψ_{EG} is equal to one of the two angles between diameters P_0Q_0 and V_0W_0 of circle $\sigma_{P_0Q_0}$* (in Figure 10, there holds: $\measuredangle FEG = \measuredangle V_0EQ_0$).

Proof.
(a) In circle $\sigma_{P_0Q_0}$, inscribed angle $\measuredangle P_0M_1Q_0$ rests on diameter P_0Q_0. It therefore holds that $\measuredangle P_0M_1Q_0 = 90°$, and therefore also $\measuredangle EM_1G = 90°$. Hence it follows that point M_1 belongs to the circle whose diameter is EG (circle ψ_{EG}).
Similarly, $\measuredangle P_0N_1Q_0 = 90°$. Therefore $\measuredangle EN_1G = 90°$ and therefore $N_1 \in \psi_{EG}$.

(b) Inscribed angles $\measuredangle P_0N_1M_1$ and $\measuredangle P_0Q_0M_1$ rest on the same arc, $\widehat{P_0M_1}$, in circle $\sigma_{P_0Q_0}$ (see Figure 11). Therefore $\measuredangle P_0Q_0M_1 = \measuredangle P_0N_1M_1$.
In addition, for angle $\measuredangle TQ_0M_1$ (which is another name for the angle $P_0Q_0M_1$) there holds: $\measuredangle TQ_0M_1 = \measuredangle TM_1Q_0$ (because they are the base angles of isosceles triangle TQ_0M_1). Therefore:

$$\measuredangle TM_1Q_0 = \measuredangle P_0N_1M_1. \tag{7}$$

Similarly, in circle ψ_{EG} there holds that $\measuredangle EN_1M_1 = \measuredangle EGM_1$, and also $\measuredangle O_1GM_1 = \measuredangle O_1M_1G$. Therefore:

$$\measuredangle O_1M_1G = \measuredangle EN_1M_1. \tag{8}$$

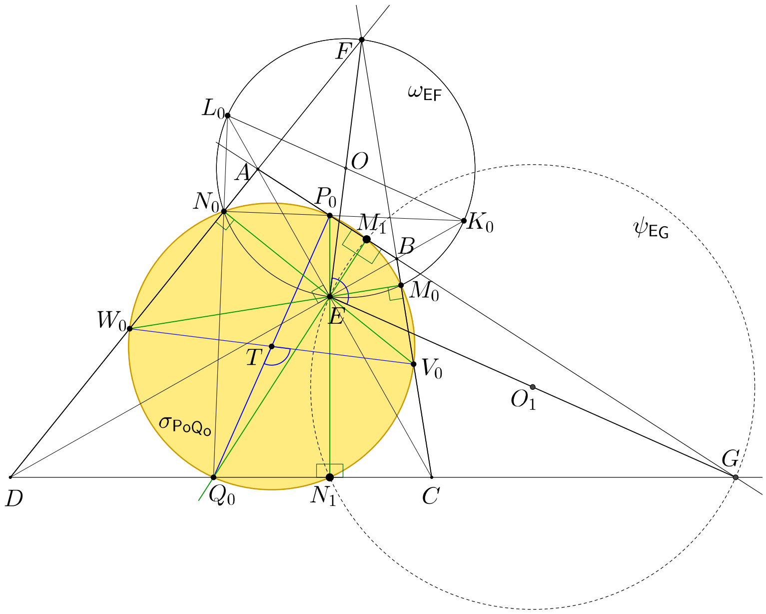

Figure 10.

Since angles $\measuredangle P_0N_1M_1$ and $\measuredangle EN_1M_1$, which appear in the right-hand side of equalities (7) and (8), are the same angle, therefore $\measuredangle TM_1Q_0 = \measuredangle O_1M_1G$.
Now, consider angle $\measuredangle TM_1O_1$:

$$\begin{aligned}
\measuredangle TM_1O_1 &= \measuredangle TM_1Q_0 + \measuredangle Q_0M_1O_1 \\
&= \measuredangle TM_1Q_0 + (\measuredangle Q_0M_1G - \measuredangle O_1M_1G) \\
&= \measuredangle TM_1Q_0 + 90^\circ - \measuredangle O_1M_1G \\
&= 90^\circ.
\end{aligned}$$

We obtained that $\measuredangle TM_1O_1 = 90^\circ$, and therefore M_1T is tangent to circle ψ_{EG}, and M_1O_1 is tangent to circle $\sigma_{P_0Q_0}$. Hence it follows that circles $\sigma_{P_0Q_0}$ and ψ_{EG} are perpendicular to each other.

(c) Circles $\sigma_{P_0Q_0}$ and ψ_{EG} intersect at an additional point: N_1. Therefore the tangent to circle ψ_{EG} at point N_1 also passes through the center, T, of circle $\sigma_{P_0Q_0}$.
We obtained that the tangents to circle ψ_{EG} at points M_1 and N_1 intersect at point T. Points M_1 and N_1 are the points of intersection of circle ψ_{EG} with sides AB and CD. Therefore, from Conclusion 3 of Theorem 2, we have that the circle whose center is point T and whose radius is segment TM_1 is the *Pascal points circle* of the points formed by circle ψ_{EG} on sides AB and CD.
On the other hand, in Theorem 3, we have proven that Pascal points circle $\sigma_{P_0Q_0}$ passes through points M_1 and N_1, and its center is at point T.
Therefore the *Pascal points circle* of the points formed by circle ψ_{EG} is circle $\sigma_{P_0Q_0}$.

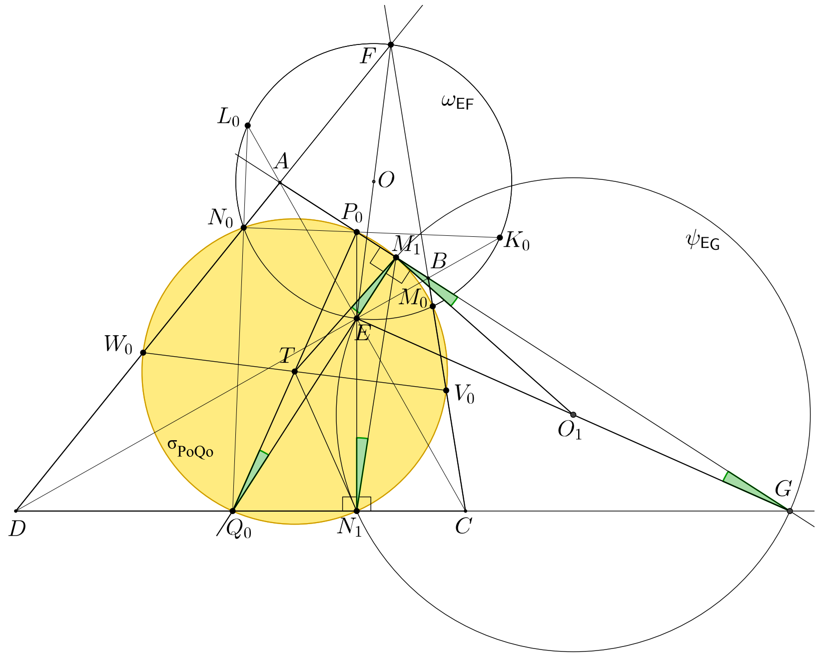

Figure 11.

In Theorem 3 we saw that circle $\sigma_{P_0Q_0}$ intersects side BC at points M_0 and V_0 and also intersects side AD at points N_0 and W_0.
Of the four chords that connect a point on side BC with a point on side AD (M_0W_0, N_0V_0, M_0N_0, and V_0W_0), only V_0W_0 passes through T, the center of the circle $\sigma_{P_0Q_0}$. In other words, only V_0W_0 is a diameter of circle $\sigma_{P_0Q_0}$.
Therefore points V_0 and W_0 are Pascal points formed by circle ψ_{EG} on sides BC and AD.

(d) Let us prove that straight line FE is perpendicular to diameter V_0W_0.
From Theorem 3, we have that segments W_0M_0 and V_0N_0 pass through point E. In circle $\sigma_{P_0Q_0}$, angles $\measuredangle W_0M_0V_0$ and $\measuredangle V_0N_0W_0$ are inscribed angles resting on diameter V_0W_0 (see Figure 10). Therefore, they are right angles.
We obtained that segments W_0M_0 and V_0N_0 in triangle FV_0W_0 are altitudes to sides FV_0 and FW_0, respectively, and that E is their point of intersection. It follows that straight line FE contains the third altitude (the altitude to side W_0V_0) of triangle FV_0W_0, and therefore $EF \perp V_0W_0$.
Similarly, one can prove that $EG \perp P_0Q_0$.
In summary, segments EF and EG are perpendicular to diameters V_0W_0 and P_0Q_0, respectively, of circle $\sigma_{P_0Q_0}$, and therefore angle $\measuredangle FEG$ is equal to one of the angles between diameters V_0W_0 and P_0Q_0. □

Conclusion from Theorems 2-4.
In a quadrilateral, $ABCD$, in which diagonals are perpendicular and intersect at

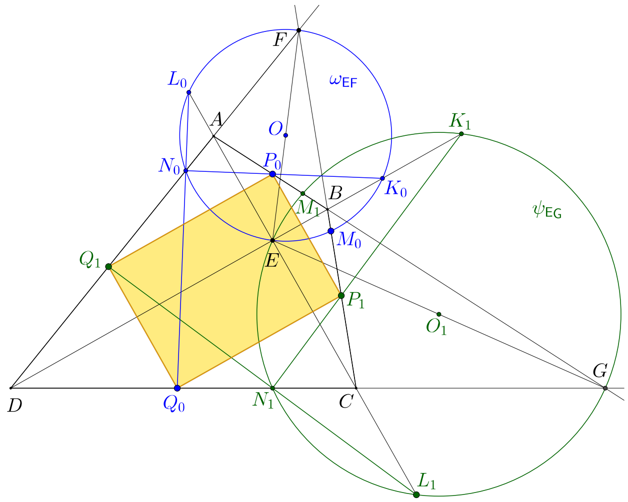

Figure 12.

point E, and the extensions of the opposite sides intersect at points F and G, there holds: the Pascal points formed by circles ω_{EF} and ψ_{EG} are the vertices of a rectangle inscribed in the quadrilateral (see Figure 12).

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On the Circumscribing Ellipse of Three Concentric Ellipses

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Abstract. Consider three coplanar non-degenerate line segments OA, OB, and OC, where only two of them can be collinear. Three concentric ellipses are then formed, say c_1', c_2', and c_3', where (OA, OB), (OB, OC) and (OC, OA) are being respectively the corresponding three pairs of their defining conjugate semi-diameters. Then, there exist another concentric ellipse c^* which circumscribes (i.e. being tangent to) all the ellipses c_i', $i = 1, 2, 3$. Moreover, the common tangent line on each common (contact) point between each c_i' and their tangent ellipse c^*, is parallel to the line segment (from the bundle of OA, OB, and OC) which does not belong to the pair of conjugate semi-diameters which forms each time the specific c_i'. The above result is derived through synthetic methods of the Projective Plane Geometry. Moreover, certain geometric properties (concerning, among others, the orthoptic circle of c^* or the existence of an involution between two bundles of rays of c^*), as well as the study of some special cases, are also discussed. A series of figures clarify the performed geometric constructions.

1. Introduction

Consider the following problem:

Problem 1. *Let A, B, C and O be four non-collinear points on a plane e, forming three line segments OA, OB and OC, where two of them can coincide. If the pairs (OA, OB), (OB, OC) and (OC, OA) are considered as the pairs of conjugate semi-diameters of three ellipses c_1, c_2 and c_3 respectively, then determine a new concentric (to c_i) ellipse c^* which circumscribes all c_i, $i = 1, 2, 3$.*

The above problem, depicted in Figure 1, was proved in [5] were the authors utilized synthetic methods of Projective Geometry, through their so-called "Four Ellipses Theorem". In particular, the proof of the Four Ellipses Theorem were delivered in [5] through the help of two lemmas, Lemma 1 which proved first, and its generalization Lemma 2. Note here that, the existence of total two circumscribing ellipses of c_i, $i = 1, 2, 3$, was proven in [14] utilizing an analytic-geometric methodology.

In the present work, the proof of the "Four Ellipses Theorem", i.e. the determination of the concentric and circumscribing ellipse of all c_i, $i = 1, 2, 3$, as described in Problem 1, is attained as a direct application of Lemma 1, i.e. without the use

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of the generalized Lemma 2, using, again, exclusively methods of Synthetic Plane Projective Geometry. For the facilitation of the development of this new proof, the presentation of the subject is being done in two stages. In the first stage, the basic elements of Lemma 1 are presented, supplemented by some new and extra properties that refer either to the ellipses of our theorem or to other parts of the topic. In the second stage, the new proof of the main theorem is developed, adding a study of the special case which is mentioned in the theorem.

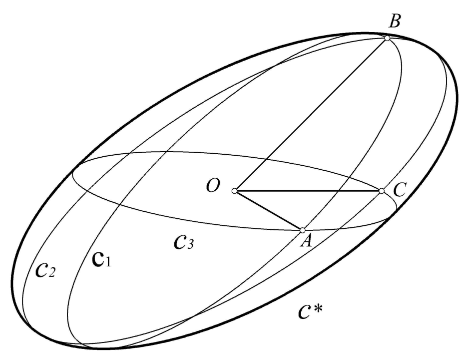

Figure 1. Illustration of Problem 1.

Note that, Problem 1 has appeared as an intermediate property (concerning these four ellipses) in a proof by G. A. Peschka [10] of the Karl Pohlke's Fundamental Theorem of Axonometry [6, pg. 250]. In Peschka's proof of the Pohlke's theorem, methods of the three-dimensional Euclidean Space were used, although Problem 1 is, by its description, a two-dimensional problem; see also [7, pg. 244] and [12]. In particular, a parallel projection was considered of an appropriate sphere onto the plane e where the concentric ellipses c_i, $i = 1, 2, 3$, lie. Figure 2 illustrates the above projection method.

Also, note that Evelyn et al. in [2] proved a more general form of "Four Ellipses Theorem" applying methods of the Analytic Plane Geometry with the use of suitable equations. Indeed, in their *"Double Contact Theorem"*, they stated that: *If three conics having a point (not on any of these conics) lying on a common (distinct) chord of each pair of the three conics, then there exist a conic which has a double contact with each of the three conics*. However, for the special case of Problem 1, the present paper provides a methodology of constructing the requested circumscribing ellipse as well as an investigation, where certain properties are derived. For the topic of concentric tangent ellipses see [3] among others.

In particular, in Section 2 we provide an outline of Lemma 1, proved in [5], while new properties concerning the construction are presented. Our main theorem (accompanying with the new corresponding geometric construction), is proven in Section 3, and certain special cases and properties are also presented in Section 4.

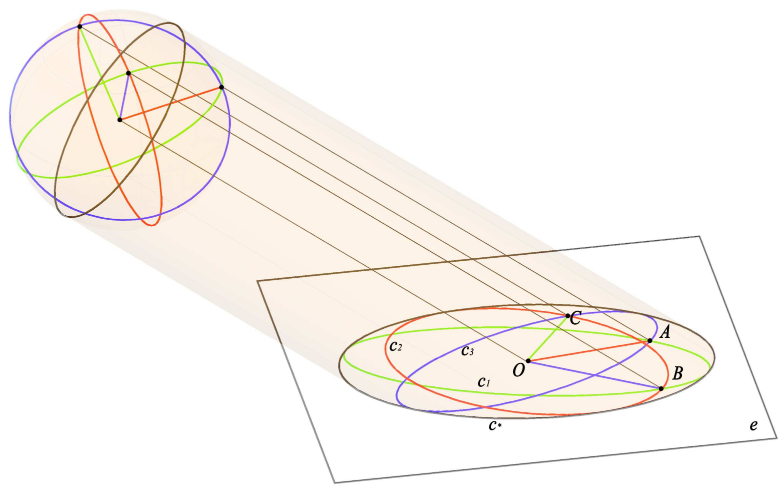

Figure 2. A parallel-projected sphere S onto the plane e where c_i, $i = 1, 2, 3$, lie.

2. A helpful construction

In this section a certain geometric construction is given, which shall be used for the proof of our main result in the next section, and is formulated through the following theorem. Since its proof is based on the properties of a projective transformation called homology, the plane —in which our problem is restricted— is considered to be the augmented Euclidean plane.

Proposition 1. *Let circle $c(O, \beta := |OB|)$ and an arbitrary point D' on the plane of c. We consider an arbitrary variable ellipse c_1' having (OD', OH) as a pair of conjugate semi-diameters, where $H \in c$ an arbitrary point of circle c* (see Figure 3)*. Then, it holds that:*

i. *The concentric and tangent to c ellipse c', which has point D' as one of its foci, is circumscribed (i.e. being tangent) to every variable ellipse c_1' as the point $H \in c$ varies.*
ii. *If $0 < \beta' < \alpha'$ correspond to the semi-axes of the circumscribing ellipse c', then*
$$\alpha'^2 + \beta'^2 = |OD'|^2 + 2\beta^2. \tag{1}$$
iii. *If $0 < \beta_1' < \alpha_1'$ correspond to the semi-axes of the varying ellipse c_1', then*
$$\alpha_1'^2 + \beta_1'^2 = |OD'|^2 + |OH|^2 = |OD'|^2 + \beta^2 = \text{const.} \tag{2}$$

Proof. The proof of this proposition corresponds to the proof of Lemma 1 in [5] where the following procedure was adopted: Consider a homology f with the line spanned by BB' be f's homology axis, and the pair of points (A', A) be its homology pair of points. Since the requested ellipse c' is always tangent to the variable ellipse c_1', then the corresponding homologue curves, under homology f, should

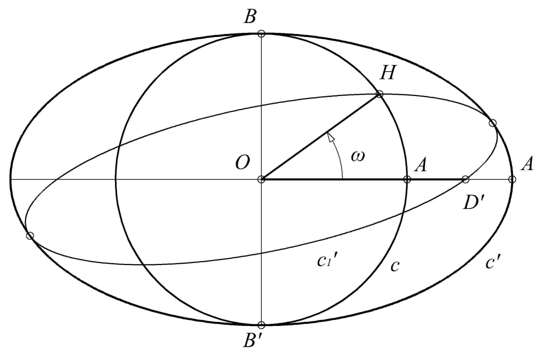

Figure 3. Illustration of Proposition 1.

be tangent with each other, i.e. circle c and ellipse c_1 have two (common) contact points, say A_1 and A_2; see Figure 4. Therefore, in order to complete the proof of this proposition, the variable ellipse c_1 should have $\alpha_1 = \beta$ as its major axis, as proved in [5, Lemma 1].

The following is essential for understanding of the procedure that finally yields $\alpha_1 = \beta$: The conjugate semi-diameters $(OD', OH = OL')$ of ellipse c_1', forming an angle $\omega \in (0, \pi/2)$, are homologue, under f, to the corresponding conjugate semi-diameters (OD, OL) of the ellipse c_1, which forms an angle $\varphi \in (0, \pi/2)$. For the ellipse c', consider the notations, $\alpha' := |OA'|$, $\beta' := \beta = |OB|$ and $\gamma' := |OD'|$. The following relations were proved; see [5, Lemma 1]:

$$|OD| = \frac{\beta\gamma'}{\alpha'}, \tag{3}$$

$$|OL| = \frac{\beta\sqrt{k}}{\alpha'}, \quad k := \alpha'^2 \sin^2\omega + \beta^2 \cos^2\omega, \tag{4}$$

$$\sin\varphi = \frac{\alpha' \sin\omega}{\sqrt{k}}, \text{ and} \tag{5}$$

$$\alpha_1 = \beta, \tag{6}$$

where α_1 is the major semi-axis of ellipse c_1.

The following mappings, given in (7) and (8), clarify the notations, where $0 < \beta < \alpha$ are the principal semi-axes of an ellipse in general and f being the orthogonal homology:

$$\text{ellipse } c'(\alpha', \beta') \quad \xmapsto{\text{orth. homology } f} \quad \text{circle } c(O, |OB| = \beta), \tag{7}$$

$$\text{ellipse } c_1'(\alpha_1', \beta_1') \quad \xmapsto{\text{orth. homology } f} \quad \text{ellipse } c_1(\alpha_1 = \beta, \beta_1). \tag{8}$$

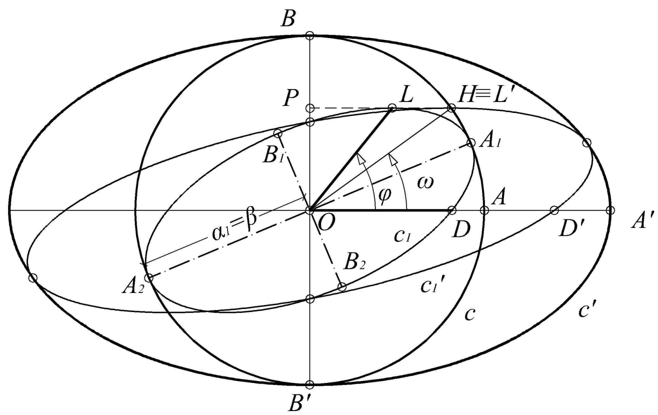

Figure 4. Illustration of the mappings (7) and (8).

We shall finally show that (1) and (2) hold. Indeed, for the ellipse c' with $0 < \beta' = \beta < \alpha'$ we derive that

$$|OA'|^2 + |OB|^2 = {\alpha'}^2 + \beta^2 = \left(\beta^2 + |OD'|^2\right) + \beta^2 = |OD'|^2 + 2\beta^2, \qquad (9)$$

while for the ellipse c_1' with $0 < \beta_1' < \alpha_1'$, Apollonius' theorem yields that

$$ {\alpha_1'}^2 + {\beta_1'}^2 = |OD'|^2 + |OH|^2 = |OD'|^2 + \beta^2 = \text{const.}, \qquad (10)$$

and theorem has been proved. □

The following remarks are new supplementary properties (not included in [5]) that derived from the proof of Proposition 1.

Remark 1. Alternatively, we can rewrite Proposition 1 as follows: *Let ellipse c' with principal semi-axes $0 < |OB'| = \beta' < |OA'| = \alpha'$ and a focus point D'. Let $\gamma' := |OD'|$. If H is an arbitrary point of the minor circle $c(O, \beta = |OB|)$ of the ellipse c', then the ellipse c_1' formed by its pair of conjugate semi-diameters (OD', OH) is tangent with the ellipse c'*; see Figures 3 and 5.

Remark 2. Notice that the angle φ between the conjugate semi-diameters OD and OL of c_1 can be written in a simpler form, i.e.

$$\tan\varphi = \frac{\alpha'}{\beta}\tan\omega, \quad \omega, \phi \in [0, \pi/2]. \qquad (11)$$

Indeed, (5) yields that

$$\tan^2\varphi = \frac{\sin^2\varphi}{1-\sin^2\varphi} = \frac{\left(\alpha'\frac{\sqrt{k}}{k}\sin\omega\right)^2}{1-\left(\alpha'\frac{\sqrt{k}}{k}\sin\omega\right)^2} = \frac{{\alpha'}^2\sin^2\omega}{k-{\alpha'}^2\sin^2\omega} = \frac{{\alpha'}^2\sin^2\omega}{\beta^2\cos^2\omega} = \frac{{\alpha'}^2}{\beta^2}\tan^2\omega,$$

and hence (11) holds.

Remark 3. An alternative method, other than the homology method, of determining point $L \in c_1$, homologue to $L' \in c_1'$, can be considered using (11). Figure 5 illustrates this method. In particular, if we consider circle $c(O, |OA|)$ as a trigonometric circle with $\overline{OA} = 1$, then $\tan\omega = \overline{AS}$. The line spanned by OS has a common point with the tangent line of ellipse c' at point, say J. From (11) or, alternatively, from $\tan\varphi / \tan\omega = \alpha'/\beta$ where $\alpha' = |OA'|$ and $\beta = |OA|$, and from the similarity of triangles (OAS) and $(OA'J)$, we obtain that $\tan\varphi = \overline{A'J}$. Therefore, if we set $A'J := AT$, the line spanned by OT forms angle φ, and hence the perpendicular line from L' to line spanned by OB intersects with OT at the requested point L.

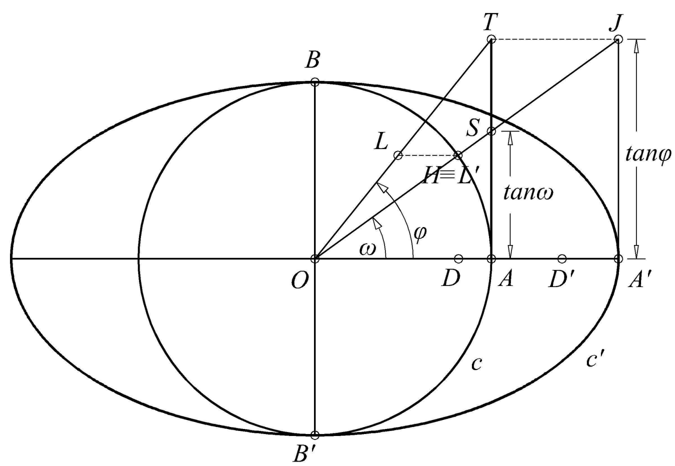

Figure 5. Alternative determination of point $L \in c_1$ with the help of the trigonometric circle (O,OA).

Remark 4. For the calculation of the minor semi-axis β_1 of c_1, Apollonius' theorem $\mu^2 + \nu^2 = \alpha_1^2 + \beta_1^2$ implies that $\beta_1 = \sqrt{\mu^2 + \nu^2 - \beta^2}$, where $\mu := |OD|$ and $\nu := |OL|$, and by substitution of

$$|OD| = \nu = \frac{\beta\gamma'}{\alpha'}, \tag{12}$$

(see [5, Lemma 1]) together with the fact that ${\alpha'}^2 = \beta^2 + {\gamma'}^2$, we finally obtain

$$\beta_1 = \frac{\beta}{\alpha'}\sqrt{\left(\beta^2 - \alpha'^2\right)\cos^2\omega + {\gamma'}^2} = \frac{\beta\gamma'}{\alpha'}\sin\omega, \tag{13}$$

Moreover, (13) and (12) yield the new property

$$\beta_1 = |OD|\sin\omega, \tag{14}$$

which means that the distance OD^* of point D from the corresponding variable $OH \equiv OL'$ is equal with the length of the minor semi-axis β_1 of the variable ellipse c_1 which is tangent to the circle c; see also Figure 6. Therefore, the length of β_1 can be determined from the beginning, i.e. when point $H \equiv L$ is considered.

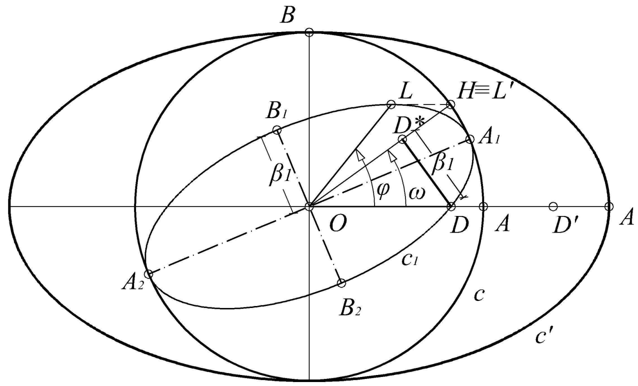

Figure 6. The c_1's minor semi-axis $\beta_1 = |DD^*|$ with $DD^* \perp OH$.

Remark 5. The following properties, which were not appeared in [5, Lemma 1], are hold; see Figure 7:

1. The orthogonal homology between ellipse c' and circle c has as homologue points the fixed pair (A', A). The absolute invariant of f is then $\lambda := \overline{OA}/\overline{OA'} = |OA|/|OA'| = \beta/\alpha'$. Therefore, the homologue ellipses c_1' and c_1 are intersect with each other onto the homology axis BB' of f.
2. Circle c and ellipse c_1 are orthogonally homologue to each other, under a well known homology f_1, with the corresponding homology axis spanned by the major axis A_1A_2 of ellipse c_1, which is also a diameter of circle c. The line spanned by the minor axis B_1B_2 of c_1 intersects circle c into two points, T and U. The orthogonal homology f_1 adopts (T, B_2) as its defining pair of points, with its absolute invariant being $\lambda_1 := \overline{OT}/\overline{OB}_2 = |OT|/\beta = \beta_1/\beta$, and via (13), $\lambda_1 = (\gamma'/\alpha)\sin\omega = \big(|OD'|/|OA'|\big)\sin\omega$.
3. If c_1' denotes a fixed ellipse, ellipses c_1' and c' are then homologue to each other under some homology f_1'.

The first two properties are trivial, and thus we shall now focus in the proof of the above third property: Let a point $S' \in c'$. The homologue point of S', under orthogonal homology f, is point $S \in c$, where the homology axis of f is spanned by BB', and (A', A) is the f's defining pair of homology points. The homologue point of S, under an orthogonal homology f_1, is point $S_1 \in c_1$, where f_1 is defined with homology axis spanned by A_1A_2 and (T, B_2) is considered as the homology pair of points. Moreover, the homologue point of S_1, under the inverse f^{-1} of the orthogonal homology f (defined as above), is point $S_1' \in c_1'$. Since the pairs (S', S) and (S_1', S_1) are homologue to each other, under homology f, the corresponding lines (spanned by) $S'S_1'$ and SS_1 are having a common point, say Q, onto the homology axis of f (spanned by BB'). As SS_1 is parallel to the minor axis B_1B_2 of ellipse c_1, the homologue under f line (spanned by) $S'S_1'$ is then parallel to the corresponding homologue fixed line (spanned by) B_1B_2, i.e. to $OB_2'T'$, where

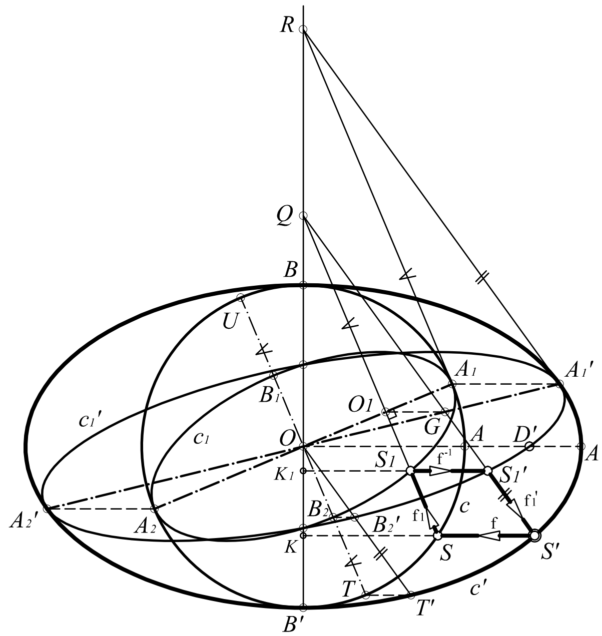

Figure 7. The consecutive homology mappings $c' \overset{f}{\mapsto} c \overset{f_1}{\mapsto} c_1 \overset{f^{-1}}{\mapsto} c_1' \overset{f_1'}{\mapsto} c'$.

the direction of $OB_2'T'$ remains invariant as S' moves along ellipse c'. Moreover, since the common tangent line (spanned by) A_1R at point A_1, between circle c and ellipse c_1, is perpendicular to the diameter A_1A_2 (i.e. parallel to B_1B_2), the homologue $S'S_1'$ of SS_1, under f, is then parallel to the common tangent line RA_1' at point A_1' of both homologue to c_1 and c (under f) ellipses c' and c_1'. As the pairs (A_1', A_1) and (A_2', A_2) are belonging to homology f, then the lines (spanned by) $A_1'A_2'$ and A_1A_2 are homologue. Therefore, if the line (spanned by) SS_1Q intersects with the line (spanned by) A_1A_2 at point, say O_1, then the corresponding homologue under f line (spanned by) $S'S_1'Q$ intersects with the line of $A_1'A_2'$ at, the homologue to O_1, point G. As a result, $GO_1 \parallel S_1'S_1 \parallel S'S$, since the homologue points are correspond to each other under f (see Figure 7). Therefore,

$$\frac{|GS_1'|}{|GS'|} = \frac{|O_1S_1|}{|O_1S|} = \frac{|OB_2|}{|OT|} = \lambda_1 = \text{const.},$$

for each angle ω and thus for specific (each time) ellipses c_1 and c_1'. From the above discussion it is concluded that ellipse c' and each of the variable ellipses c_1', tangent to c', are homologue with each other under homology, say f_1', which is defined by the homologue axis (spanned by) $A_1'A_2'$ and having absolute invariant ratio $\lambda_1' = |GS'|/|GS_1'| = |OT|/|OB_2| = \lambda_1^{-1} = \big(|OA'|/|OD'|\big)\csc\omega = \text{const.}$ The direction of this homology is parallel to the common tangent line (spanned by) RA_1' between the two ellipses c' and c_1' at their contact point A_1'. Conclusively, the two homologies f_1 and f_1' have:

- Inverse absolute invariant ratios λ_1 and λ'_1, i.e. $\lambda'_1 = 1/\lambda_1$.
- The axes A_1A_2 and $A'_1A'_2$ of homologies f_1 and f'_1 respectively are homologue lines under f.
- Homologue directions, determined by line (spanned by) OB_2T and its homologue OB'_2T', also under f.
- The three axes BB', A_1A_2 and $A'_1A'_2$ of homologies f, f_1 and f'_1 respectively are intersecting at point O.
- It holds that $\lambda \times \lambda_1 \times \lambda^{-1} \times \lambda'_1 = 1$.

The homology mappings in (7) and (8) can now be completed in Figure 8 with the cyclic representation of homologies according to Figure 7.

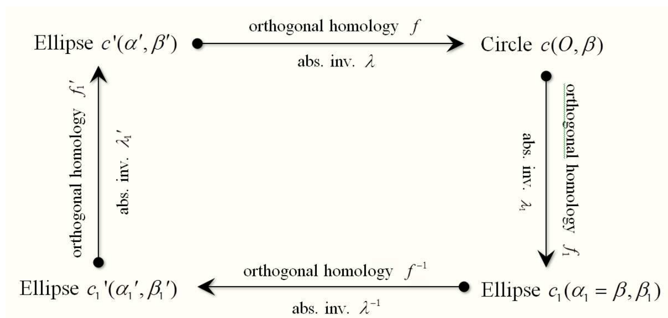

Figure 8. The cyclic action of homologies $f \rightarrow f_1 \rightarrow f^{-1} \rightarrow f'_1$ with their corresponding absolute invariants λ, λ_1, λ^{-1} and λ'_1.

3. Circumscribing ellipse of three concentric ellipses

We present now the plane-geometric proof of Problem 1, which is proved through the construction in Proposition 1.

Theorem 2. *Consider three given arbitrary coplanar and non-degenerated line segments OA, OB and OC, where only two of them can lie on the same line. Let c'_1, c'_2, and c'_3, denote the ellipses defined by the pairs of conjugate semi-diameters (OA, OB), (OB, OC), and (OC, OA), respectively. Then, there always exists an ellipse c^*, concentric to c'_i, $i = 1, 2, 3$, which circumscribes all c'_i ellipses, i.e. being tangent at two points with each one of them. Moreover, the common tangent line on each contact point between each c'_i and their tangent ellipse c^*, is parallel to the line segment which does not belong to the pair of conjugate semi-diameters that form each time the specific c'_i* (see Figure 9).

Proof. Consider a bundle of three arbitrary coplanar line segments OA, OB and OC, where $O \notin \{A, B, C\}$, i.e. OA, OB and OC are non-degenerated line segments. Two cases are then distinguished regarding the collinearity of those segments:

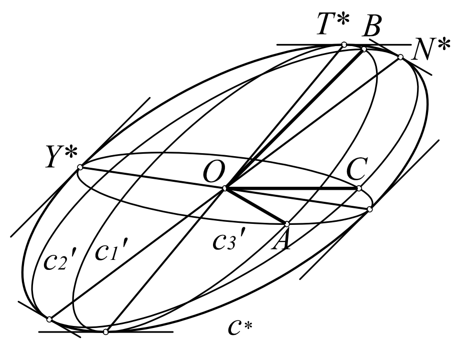

Figure 9. Contact points and the corresponding tangent lines between c^* and c_i', $i = 1, 2, 3$.

• *General case.* Assume that none of the OA, OB and OC, are collinear; see Figure 10. Let c_1' be an ellipse defined by its conjugate semi-diameters (OA, OB) via through Rytz's construction which is presented below. Figure 11 depicts the requested principal axes G_1D_1 and G_2D_2 of c_1'. For the Rytz's construction see also [8, pg. 69], [4, pg. 183] and [13] among others. In particular, we consider the following steps:

i. First we form the perpendicular line segment OP to OD, such that $|OP| = |OA|$.
ii. We then form the line segment PB, and from the middle point L of PB we draw a circle of radius LO. This circle intersects with PB at the points, say T and S.The principal axes of the requested ellipse c_1 are thus spanned by the line segment OT and OS.
iii. We then place $|BT| = \beta_1$ onto the line spanned by OS, and the apexes G_1 and D_2 of the c_1's minor axis are determined as $|OG_2| = |OD_2| = |BT| = \beta_1$.
iv. Similarly, the apexes G_1 and D_1 of the c_1's major axis are determined by placing $|OS| = |OG_1| = |OD_1| = \alpha_1$ onto the line spanned by OT.

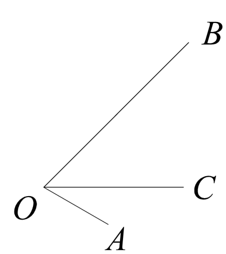

Figure 10. The given bundle of the three line segments OA, OB and OC.

Consider the major circle c of ellipse c_1' (i.e. the concentric circle which corresponds to ellipse's major radius) with diameter G_1D_1, as in Figure 12, which

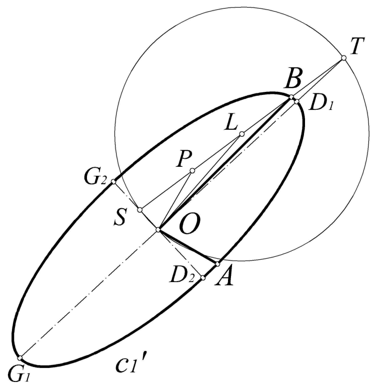

Figure 11. The principal semi-axes G_1D_1 and G_2D_2 of c_1' as obtained through Rytz's construction.

is the homologue of c_1', according to the orthogonal homology f with homology pair of points (G_2, G_2') and homology axis G_1D_1. In this homology, the points A and B of ellipse c_1' correspond to the points A' and B' respectively of circle c. Similarly, the third point C is having C' as its homologue point. Therefore, the given triplet (OA, OB, OC) is having (OA', OB', OC') as its corresponding homologue triplet.

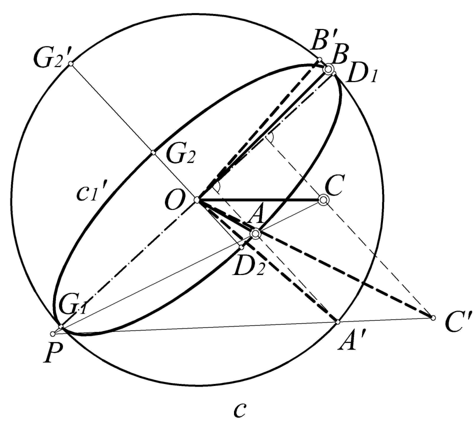

Figure 12. Construction of the homologue triplet (OA', OB', OC') of (OA, OB, OC).

Since the pair of line segments (OA, OB) is the pair of conjugate semi-diameters of ellipse c_1', then, according from a known property of homology, the homologue pair (OA', OB') of (OA, OB) is then a pair of perpendicular radii of circle c. Figure 13 presents a detail of Figure 12 in which included only the final triplet (OA', OB', OC') as well as the circle c.

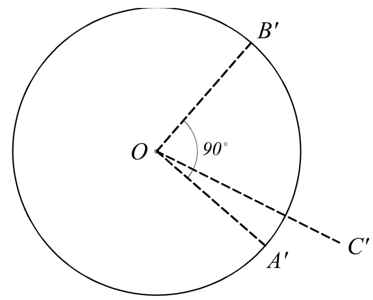

Figure 13. Triplet (OA', OB', OC'); detail from Figure 12.

It is then sufficient to construct an ellipse c' which it would be tangent to circle c, while it would be also tangent with two ellipses having (OB', OC') and (OC', OA') as pairs of their conjugate semi-diameters. With the construction of ellipse c' (presented in Step I below), the homologue of c' (under homology f) ellipse c^* (presented in Step II below), it would be simultaneously tangent with the three given ellipses c_1', c_2', and c_3'. Note that the following proof is based only to Proposition 1, which corresponds to [5, Lemma 1].

Step I. (Construction of the ellipse c'). Consider the perpendicular diameter of circle c to OC' which is intersecting c in two points T' and R'. We construct an ellipse with minor semi-axis OT' and focus point C'. Trivially, $T'C'$ equals with the major semi-axis OS' of the ellipse; see Figure 14. We shall show that the above constructed ellipse is the requested c'. Indeed, since C' is considered to be the focus point of c', and points A' and B' are belonging to circle c (Figure 14), then from Proposition 1 it is clear that the two ellipses defined by the pairs of conjugate semi-diameters (OB', OC') and (OC', OA'), depicted in Figure 15, are both tangent to ellipse c'; see also Figure 3.

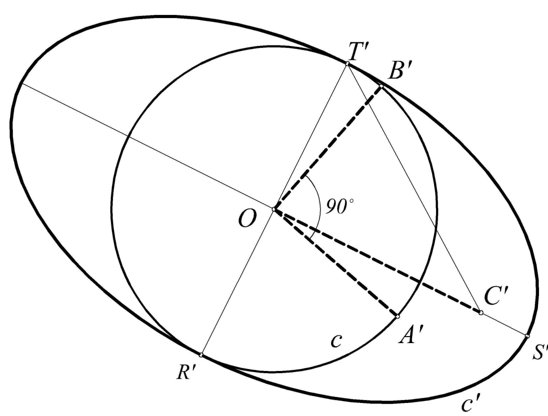

Figure 14. Construction of ellipse c'.

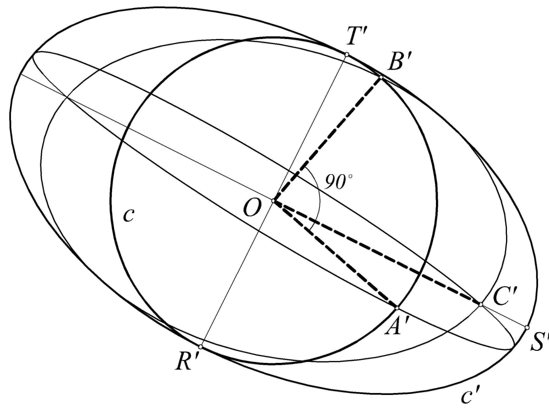

Figure 15. The ellipses defined by (OB', OC') and (OC', OA') are both tangent to ellipse c'.

Step II. (Construction of the ellipse c^*, homologue to c'). Figure 16 depicts the construction of the requested ellipse c^*, which is homologue (under orthogonal homology f) to ellipse c'. The corresponding homology axis is considered to be the line spanned by G_1D_1 while the pair of homology points is (C, C'); see Figure 12. Indeed, the pair of principal semi-axes (OS', OT') of ellipse c' adopts (OS^*, OT^*) as its homologue pair of conjugate semi-diameters of the homologue ellipse c^*. The points S^* and T^* are obtained through the known construction of homologue points:

- The perpendicular line to the homology axis G_1D_1 that passes through S', intersects line segment OC at the requested point S^*.
- For the point T^* we consider the line spanned by $S'T'$ which intersects with the homology axis at a point, say Q. Thus, the perpendicular line to homology axis that passes through T', intersects with S^*Q at the requested point T^*.
- Similarly, the point R^* of ellipse c^*, which is homologue to point R', is determined.

In order to complete the proof, the contact points between the circumscribing ellipse c^* and the given ellipse c_1' have to be determined, as well as the corresponding tangent lines on them. Figure 17 depicts the contact point T^* between the ellipses c_1' and c^* as well as their corresponding common tangent line t^* at T^*. Note that the second contact point R^* is diametrical to T^*. For the derivation of T^* we work as follows (recall Figure 16): Since ellipse c' and the circle c are in contact at point T', then the corresponding homologue curves c^* and c_1' are in contact at the already known point T^*, which is homologue of T'. Moreover, since the tangent line t' at T' is parallel to OC' (Figure 16), the homologue of this tangent line is then also parallel to OC which is the homologue line of OC'. The tangent line r^* of c^* at

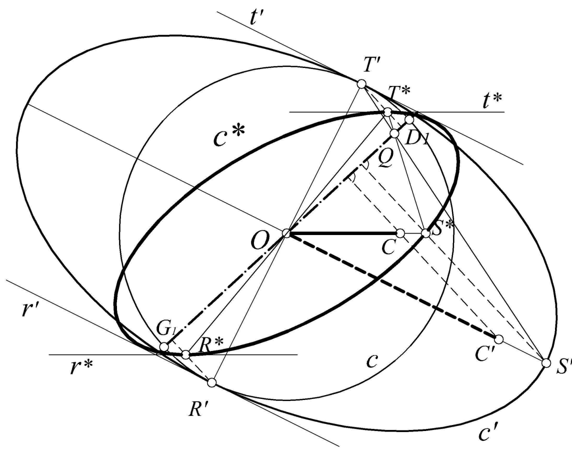

Figure 16. construction of the homologue to c' ellipse c^*.

point R^* can be determined in the same way. Figure 17 is a detail of Figure 16 where ellipse c_1', homologue to circle c, has been included.

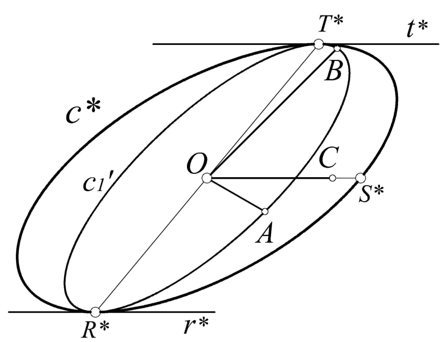

Figure 17. Contact points T^* and R^*, between c_1' and c^*, and their corresponding tangent lines t^* and r^*.

Similarly to the above described procedure, the contact points and the corresponding tangent lines between the requested circumscribing ellipse c^* and the other two ellipses c_1' and c_2' can be determined. In the final Figure 9 depicts all the ellipses c_i', $i = 1, 2, 3$, their common tangent ellipse c^*, as well as all the six contact points between them together with their corresponding tangent lines on them.

• *Degenerate case.* Assume that two line segments, say AO and OB, out of given three (OA, OB, OC) are being collinear with $A \neq B$; see Figure 18.

Then, two ellipses can be defined, say c_2' and c_3', when the pairs (OB, OC) and (OC, OA) are considered, respectively, to be their conjugate semi-diameters. These two ellipse are then tangent with each other at their common diametrical points T^* and C, and their corresponding common tangent lines on these points

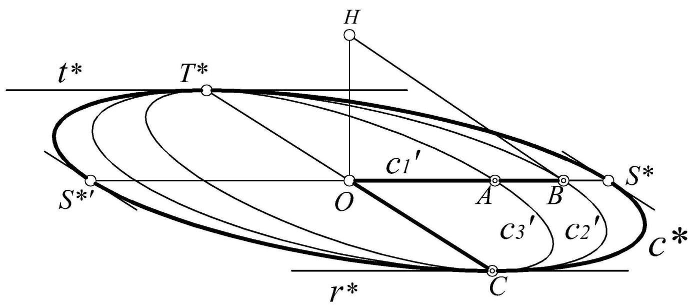

Figure 18. Degenerate circumscribing ellipse c^* for the degenerate case of ellipse c_1'.

are $t^* \parallel r^* \parallel OA$. In such a case, there exist an infinite number of real ellipses c^* circumscribing both c_2' and c_3' having two contact points on each of them. These contact points can only be T^* and C; see Figure 18. However, we can define one unique ellipse c^* as a solution for this special case of collinearity by assuming the collinear segments (OA, OB) as a degenerate pair of conjugate semi-diameters of a degenerate third ellipse, say c_1'. Indeed, in every ellipse, the Apollonius theorem can be applied, i.e.

$$|OA|^2 + |OB|^2 \;=\; \alpha^2 + \beta^2 \;\text{ and} \tag{15}$$

$$|OA||OB| \sin\varphi \;=\; \alpha\beta, \tag{16}$$

where $0 < \beta < \alpha$ denote the principal semi-axes of the ellipse defined by its pair of conjugate semi-diameters (OA, OB), and $\varphi := \measuredangle(OA, OB) \in [0, \pi/2]$. For the special degenerate case of ellipse c_1', the second relation (16) implies that

$$\alpha\beta = 0, \tag{17}$$

since $\varphi = 0$ which is due to the collinearity of OA and OB. However, as —in principle— $\beta < |OA| < \alpha$ and $\beta < |OB| < \alpha$ are both hold in the case of the non-degenerate ellipse where $|OA| > 0$ and $|OB| > 0$, we derive clearly that $\alpha > 0$, due to the fact that (17) should hold with $\alpha > \beta$. Therefore, (17) yields $\beta = 0$, and (15) is then written as

$$|OA|^2 + |OB|^2 = \alpha^2, \tag{18}$$

which means that the major semi-axis α of the degenerate ellipse c_1' equals with the hypotenuse HB of the orthogonal triangle (HOB), where $OA = OH$; see Figure 18. Consider the line segments $OS^* = OS^{*\prime} = HB$ onto OA, and admit as the solution ellipse c^* only one circumscribing ellipse around c_2' and c_3', with contact points T^* and C, which also includes the points S^* and $S^{*\prime}$ of the degenerate third ellipse c_1'. It is then clear that ellipse c^* has (OC, OS^*) as a pair of conjugate semi-diameters, since $OS^* \parallel r^*$. Therefore, at the points S^* and $S^{*\prime}$ the corresponding tangent lines of c^* are being parallel to OC, preserving the corresponding result of Theorem 2. Similarly, the tangent lines t^* and r^* at the contact points T^* and C are parallel to OA and OB. $\square$

Recall the notion of the involution transform between two bundles of rays both sharing their common centers; see [1, 11] among others.

Corollary 3. *Consider the contact points T^*, N^* and Y^* of each ellipse c_i, $i = 1, 2, 3$* (as in Theorem 2) *with their common circumscribing ellipse c^** (see Figure 9). *The six rays spanned by the pairs (OT^*, OC), (ON^*, OA) and (OY^*, OB) are then correspond to two bundles of rays, i.e. (OT^*, ON^*, OY^*) and (OC, OB, OA), being in involution having center point O.*

Proof. The tangent line on each contact point T^*, N^* and Y^* of the ellipse c^* with one the concentric ellipses c_1, c_2 and c_3, is parallel each time to one of the three given line segments OC, OA and OB respectively (recall Theorem 2). Therefore, the rays spanned by (OT^*, ON^*, OY^*) correspond to rays being in involution with the rays spanned by (OC, OA, OB) respectively, due to the known fact the conjugate diameters of an ellipse (c^* in our case) are being in involution with each other; see also [4, pg. 175]. Figure 19 clarifies the above discussion, where the corresponding Fregier point of the formed involution (related to a certain circle passing through O) is also depicted, confirming the existence of the involution. For the Fregier point see also [9] among others. □

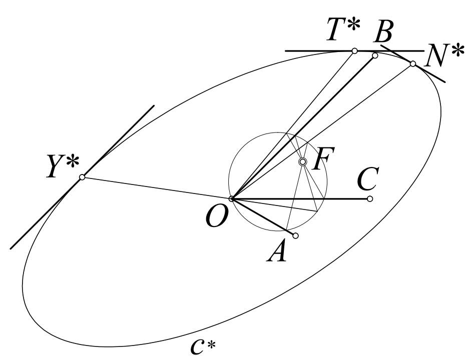

Figure 19. The rays spanned by (OT^*, ON^*, OY^*) are being in involution with the rays spanned by the given bundle (OC, OA, OB).

4. Properties and special cases

In the following we present some interesting properties concerning the orthoptic circle of ellipse c^*. Recall the notion of the *orthoptic circle* c_{M} (also known as *directive circle*, or *Fermat-Apollonius circle*, or *Monge circle*) of a given ellipse c, with principal semi-axes $0 < \beta < \alpha$. It is well known that is the concentric circle to c, having radius $\sqrt{\alpha^2 + \beta^2}$; see [1, pg. 261] among others. The characteristic property of c_{M} is that it is formed by the intersecting points between all the pairs of orthogonal (to each other) tangent lines of the ellipse c or, vice-versa, is the concentric to c circle c_{M} where from every point of c_{M} a pair of tangent lines to c is always orthogonal.

The following results are based on the relation

$$\alpha^{*2} + \beta^{*2} = |OA|^2 + |OB|^2 + |OC|^2, \tag{19}$$

where $0 < \beta^* < \alpha^*$ denote the principle semi-axes of c^*, which proved in [5, Theorem 3.1]. Note that (19) still holds for the special collinearity case of the above Theorem 2. An alternative interpretation of relation (19), is given with the following:

Corollary 4. *Consider three concentric and coplanar ellipses, say* c_i, $i = 1, 2, 3$ (recall Problem 1 or Theorem 2). *Then, the sum of the areas of the three orthoptic circles* $c_{\mathrm{M},i}$ *of* c_i, $i = 1, 2, 3$, *is two times the area of the orthoptic circle* c^*_{M} *of an ellipse* c^* *that circumscribes* (according to Theorem 2) *all* c_i, $i = 1, 2, 3$, *i.e.* $2A(c^*_{\mathrm{M}}) = \sum_{i=1}^{3} A(c_{\mathrm{M},i})$. *For the special case where one of the ellipses is a circle, say* c_1, *the sum of the areas of the two orthoptic circles of the other two non-circular ellipses is two times the area of the major circle* c^*_{maj} *of the circumscribed ellipse* c^*, *i.e.* $2A\big(c^*_{\mathrm{maj}}\big) = A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$.

Proof. Denote with OA, OB and OC the three common conjugate semi-diameters of c_i, $i = 1, 2, 3$, meaning that each two of them form each one of three given ellipses c_i, $i = 1, 2, 3$. Therefore, according to [5, Theorem 3.1], property (19) holds true. Utilizing Apollonius' theorem, i.e. $\alpha^2 + \beta^2 = \mu^2 + \nu^2$, where $\alpha, \beta > 0$ being the principal semi-diameters of an ellipse while $\mu, \nu > 0$ being a pair conjugate semi-diameters of the ellipse, relation (19) can be written as

$$\begin{aligned} 2\big(\alpha^{*2} + \beta^{*2}\big) &= \big(|OA|^2 + |OB|^2\big) + \big(|OB|^2 + |OC|^2\big) + \big(|OC|^2 + |OA|^2\big) \\ &= \sum_{i=1}^{3} \big(\alpha_i^2 + \beta_i^2\big), \end{aligned} \tag{20}$$

where $0 < \beta_i < \alpha_i$, $i = 1, 2, 3$, denote the principal semi-axes of c_i, $i = 1, 2, 3$, respectively. Therefore, multiplying (20) with π, and recalling the notion of the orthoptic circle given earlier, it holds that $2A(c^*_{\mathrm{M}}) = A(c_{\mathrm{M},1}) + A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$.

For the special case where, for instance, c_1 is being a circle, the minor semi-axes β^* of c^* must coincide with β_1, since circle c_1 must be the minor circle of its circumscribing ellipse c^*, Therefore, (20) yields $2\alpha^{*2} + 2\beta_1^2 = 2\beta_1^2 + \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2$, i.e. $2\alpha^{*2} = \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2$, and hence the special case $2A\big(c^*_{\mathrm{maj}}\big) = A(c_{\mathrm{M},2}) + A(c_{\mathrm{M},3})$ has been proved. □

A special case of relation (19) is given in the following:

Corollary 5. *Consider three line segments* OA, OB *and* OC, *as in Theorem 2, that can be freely rotated around their common point* O. *Then, all the circumscribed ellipses* c^* (derived via Theorem 2, as OA, OB and OC rotated independently around their common point O) *have the same orthoptic circle or, equivalently, the diagonals of the corresponding bounding rectangle of all* c^* (as well as on every rectangle that circumscribes c^*) *are of fixed length, which are the diameters of the orthoptic circle* c^*_{M}.

Proof. The orthoptic circle c^*_{M} of the variable common tangent ellipse c^* should have radius $\sqrt{\alpha^{*2}+\beta^{*2}}$, where $0<\beta^*<\alpha^*$ denote the c^*'s principal semi-axes. However, according to (19) it holds that each orthoptic circle c^*_{M} has the same radius since line segments OA, OB and OC preserve their length by assumption as they are rotated around O. Figure 20 depicts the common orthoptic circle c^*_{M} of c^* for two cases of OA, OB and OC. □

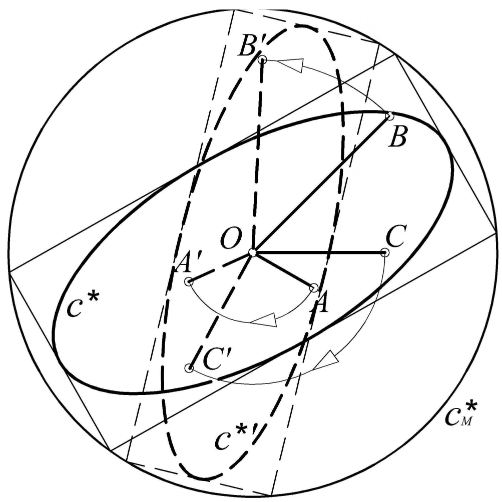

Figure 20. Common orthoptic circle c^*_{M} of two circumscribing ellipses c^* and $c^{*\prime}$.

Remark 6. It is clear that every rectangle that circumscribes each one c^*, an in Corollary 5, has a diagonal of fixed length, which is the diameter of the orthoptic circle c^*_{M}.

Remark 7. Note that, if we consider a fixed point, say Q, on the orthoptic circle c^*_{M}, as in Corollary 5, it is clear that all the rectangles that have point Q as one of their vertices and circumscribing each one c^*, would also have another fixed common vertex, which is diametrical to Q with respect to the orthoptic circle c^*_{M}.

Corollary 5 can be alternatively stated in the following form:

Corollary 6. *Consider a bundle of three line segments, that can be freely rotated around their* (pivoting) *common point and, therefore, forming three circles, say* κ_1, κ_2 *and* κ_3. *The area* $A\big(c^*_{\mathrm{M}}\big)$ *of the orthoptic circle* c^*_{M} *of the circumscribed ellipse* c^* (according to Theorem 2) *is always constant as the bundle rotates around its pivot, and equal to the sum of the areas of* κ_1, κ_2 *and* κ_3, *i.e.* $A\big(c^*_{\mathrm{M}}\big) = A\big(c^*_{\mathrm{maj}}\big)+A\big(c^*_{\mathrm{min}}\big)=\sum_{i=1}^{3}A(\kappa_i)$, *where* c^*_{maj} *and* c^*_{min} *denote the principle circles* (major and minor) *of the circumscribing ellipse* c^* *respectively. For the special case when two of the given three line segments remain always orthogonal and of the same length, the area of the ring between the major and minor circle of the* *circumscribing ellipse c^* is always equal to the area of the circle κ_n formed by the third, say $n \in \{1,2,3\}$, rotating line segment, i.e. $A\big(c^*_{\mathrm{maj}}\big) - A\big(c^*_{\mathrm{min}}\big) = A(\kappa_n)$, $n \in \{1,2,3\}$.*[1]

Proof. The non-special case is derived straightforward via (19). For the special case, assuming that $\beta := |OA| = |OB|$ and $OA \perp OB$, (19) yields $\alpha^{*2} + \beta^2 = 2\beta^2 + |OC|^2$, i.e. $\alpha^{*2} - \beta^2 = |OC|^2$, due to the fact that $\beta^* = \beta$ since OA and OB form a circle of radius β centered at point O, which must coincide with the minor circle of its circumscribing ellipse c^*. This completes the proof of the special case. $\square$

Corollary 7. *Consider three line segments OA, OB and OC, as in Theorem 2, that can be rotated around their common point O. Assume now that two of the given three line segments remain always orthogonal and of the same length. It then holds that all the circumscribing ellipses c^** (recall Theorem 2) *are the same* (up to rotation)*, i.e. they all have the same length of principal axes* (see Figure 21)*.*

Proof. Without loss of generality, consider that OA and OB are being always orthogonal and of equal length, as they rotate around O, i.e. $\beta := |OA| = |OB| =$ const. and $\measuredangle(OA, OB) := \pi/2 =$ const. It is then concluded that each common tangent ellipse c^* (derived via Theorem 2 at every position of OA, OB and OC) should have a minor semi-axis of the same length as the two orthogonal line segments, i.e. $\beta^* = \beta$, since (OA, OB) always form a circle of radius β centered at O when it considered as a pair of conjugate semi-diameters. This circle trivially corresponds to the minor circle of the circumscribing ellipse c^*, i.e. the circle of the minor radius of c^*. Relation (19) now yields $\alpha^{*2} = \beta^2 + |OC|^2 =$ const., and hence the principal semi-axes α^* and β^* of c^* are preserving their length or, alternatively, c^* is always preserving its shape as OA, OB and OC are rotated around O. Figure 21 depicts two cases of circumscribing ellipses, say c^* and $c^{*\prime}$, where $OA \perp OB$ and OC are rotated around O to $OA' \perp OB'$ and OC'. $\square$

Discussion

Consider the problem of determining a concentric ellipse c^* which circumscribes three coplanar and concentric ellipses c_i, $i = 1, 2, 3$. Each one of the above ellipses is defined by two conjugate semi-diameters, taken from a bundle of three given coplanar line segments assumed that only two of them may coincide.

In the present paper the authors prove the existence of a circumscribing ellipse c^*, providing also a construction methodology of c^* in terms of plane Projective Geometry. This was achieved utilizing only a form of Lemma 1 in [5], since a different approach was adopted. Note that the construction of c^* provided in [5] were performed through a generalization of [5, Lemma 1]. Moreover, certain properties arising from the new process were also revealed and discussed. In addition, the degenerate case where two of the given three line segments can coincide were also studied. Finally, some geometric properties concerning the orthoptic circle of c^*

[1]For example, if $OA \perp OB$ and $|OA| = |OB|$, then κ_3 is the circle (O, OC).

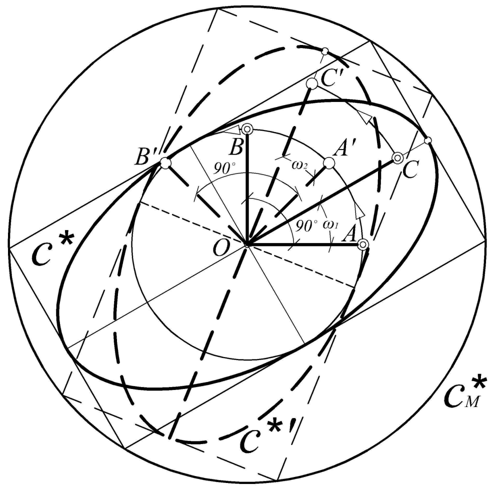

Figure 21. Two circumscribing ellipses c^* and $c^{*\prime}$ with the same orthoptic circle c^*_{M}.

were given. The provided figures illustrate the corresponding geometric constructions of the proofing process.

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Author Index