FORUM GEOMETRICORUM

A Journal on Classical Euclidean Geometry and Related Areas

published by

Department of Mathematical Sciences
Florida Atlantic University

Volume 18
2018

http://forumgeom.fau.edu

ISSN 1534-1188

Table of Contents

Forum Geometricorum
Volume 18 (2018) 1–6.

FORUM GEOM
ISSN 1534-1178

The Relativity of Conics and Circles

Stefan Liebscher and Dierck-E. Liebscher

Abstract. Foci are defined for a *pair* of conics. They are the *six* vertices of the quadrilateral of common tangents. To be circle is a derived property of a pair of conics, too.

1. Introduction

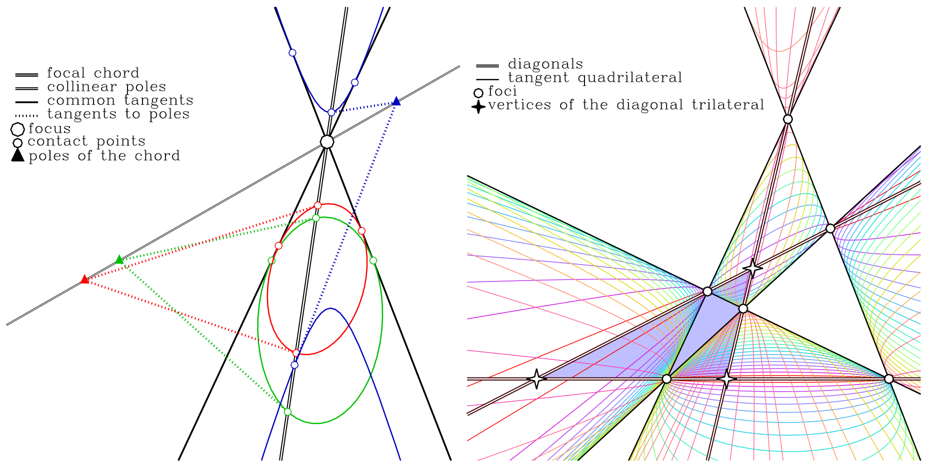
The beloved properties of conics, which we start to discover at school, manifest themselves as projective *relations between two conics*, as soon as we use the embedding in the non-Euclidean Cayley-Klein geometries. Non-Euclidean geometry enables us to find real constructions of objects which otherwise lie in infinity or are imaginary.

Our school-level understanding of geometry is a play with ruler and compass on the Euclidean plane. Conics constitute the last topic to be tackled in this manner and the first that leads beyond its limits. The most prominent property of conics is the existence of foci.

In non-Euclidean geometry, the scale on the drawing plane is not constant. So, how can we define motions and determine congruency? We start by generating the group of motions by defining reflections [1]. Reflections define perpendiculars, and in Cayley-Klein geometries, a conic provides the reflections simply by requiring its own invariance [1]. Reflection of tangents to this absolute conic yield other tangents again, and the reflection of a point is the intersection of the reflection of its two tangents to the absolute conic. Two lines are perpendicular if one is reflected by the other onto itself. Such lines contain the others pole. The pole becomes the common intersection of all perpendiculars of its polar.

Geometries can now be classified by their absolute conics. In Euclidean geometry, this conic is a double line at infinity with complex fixed points, so it is not prominent. The line at infinity contains the poles of all lines and is the polar of all points. In Minkowski geometry, the fixed points are real, and become prominent in their role as directions of the world lines of light signals [2]. In Galilei geometry, these two fixed points coincide [5]. The Beltrami-Klein model of the hyperbolic plane uses a regular conic. To adapt to our Euclidean habits, it has the form of a circle. As we shall see, a conic acquires the properties of a circle when declared absolute.

Publication Date: January 16, 2018. Communicating Editor: Paul Yiu.



Left: Given a set of conics with two common tangents. The poles of any line passing the vertex of the tangents are concurrent with this vertex. Right: Two conics define four common tangents and generate a pencil of conics. Any two conics of the pencil have the same quadrilateral of common tangents with the same six vertices and three diagonals (dashed lines). The vertices are foci by this pole-collinearity property. If any conic of the pencil is declared absolute, the foci obtain the familiar metric properties of a focus, too.

Figure 1. Confocal conics for real common tangents

2. Foci as properties of pairs of conics

The focus of a conic is characterized by a particular property of its pencil of rays: The pole (with respect to the conic considered) of any line through the focus lies on the perpendicular to the line in the focus. In other words, the poles of a focal ray with respect to the two conics (the considered and the absolute) are collinear with the focus. This statement does not refer to the task of the absolute conic to define perpendicularity. It solely uses the pole-polar relation of two conics.

Foci thus become properties of pairs of conics instead of a single conic and the metric. Let us consider a tangent from the focus to one of the conics, so that its pole is the contact point. The pole by the other conic can be collinear with this contact point and the focus only when it lies on the tangent, too. That is, the line is tangent to both conics. A focus must be an intersection of tangents common to the conics, whether real or not. Consequently, the intersections of the common tangents are the foci (Fig. 1, left).

The foci of a pair of conics are the vertices of the quadrilateral of common tangents. A pair of conics has six foci, more precisely three pairs of opposite foci. Fig. 1, right, shows a pencil of confocal conics when all four common tangents are real lines. The diagonal lines of the tangent quadrilateral form a self-polar triangle: Its vertices and edges are pole-polar pairs (to *all* conics of the pencil). Its vertices are also the diagonal points of the quadrangle of intersections of any two conics of the pencil. Thus, the diagonal triangle is self-dual, too [3].

The consideration of metric properties of foci requires the promotion of one conic of the pencil as absolute, i.e. as generating the metric. For any other conic of the pencil and any focus, we obtain the three familiar properties:
(1) All poles of a line through a focus are collinear with the focus. The line connecting the focus with the poles of the reference line is perpendicular to the latter, independent of which conic of the pencil is taken as absolute.
(2) The lines through a focus are reflected in the tangents of any conic of the pencil onto lines though the opposite focus, independent of which conic of the pencil is taken as absolute.
(3) Yet more, the focus itself is reflected in the tangents of a chosen conic of the pencil onto the points of a circle around the opposite focus, independent of which conic of the pencil is taken as absolute. (This adds the gardener's rule to outline an ellipse using a rope attached to two pegs in the foci of the ellipse.)

3. Being circle as a relation between conics

Circles are usually understood by metric considerations, in particular the constant distance from some center. In non-Euclidean geometry, we refer to simpler projective properties. When symmetry is defined by an absolute conic, a circle should be symmetric with respect to all reflections on the lines through some center, i.e. the diameters. The tangents in the intersection of a diameter with the circle should be perpendicular to the diameter, i.e. the pole of a diameter with respect to the circle coincides with its pole relative to the absolute conic.

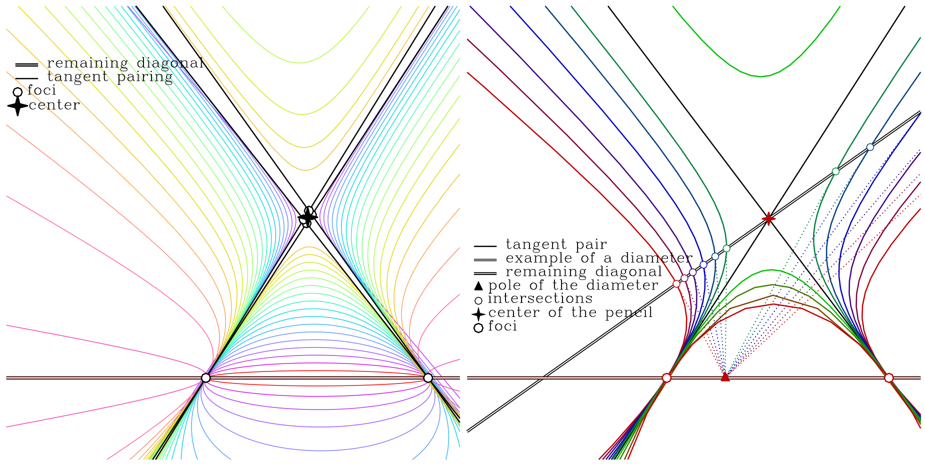
Again, we argue with the diameters tangent to the circle. The pole of such a diameter with respect to the circle is the point of contact.

When the poles with respect to the circle and to the absolute conic have to coincide, the point of contact with the circle is equally the point of contact with the absolute conic. A conic is a circle if it touches the absolute conic twice. The center of the circle is the intersection of the two tangents in the points of contact.

A set of concentric circles has a common center and a common pair of points of contact. This is the reason why any of the circles of the pencil can play the role of the absolute conic without any change in the pencil. The defining structure is a pair of tangents with a pair of contact points (Fig. 2). It is a quadrilateral formed by the two tangents and the double line connecting the contact points. Any two conics that show such a structure are circle with respect to each other. Two conics become circles to each other if two pairs of their foci collide and thus become the (common) midpoint of the circles. The third pair becomes the pair of contact points. In short: Two conics are circles to each other if they touch twice.

4. Confocal conics

This is an example to demonstrate the scope of projective phrasing. In Euclidean formulation, it is observed for two confocal ellipses that the tangents of the inner one intersect the outer ellipse in two points. When the tangent is reflected at the intersection point by the outer ellipse, it becomes the second tangent to the inner ellipse from this point. The projective phrasing shows without toil that a hyperbola



Left: When two pairs of foci coincide, the other pair becomes a pair of contact points common to all conics of the pencil. Its connection is the polar of the center (formed by the collision of the two pairs) with respect to all conics of the pencil. Right: The poles of a line passing the center coincide. We found the pencil of concentric circles.

Figure 2. Concentric circles for real common tangents and contact points

confocal to the ellipse yields the same, and that the unreflected absolute conic of the Euclidean plane can be replaced by any other conic confocal with the two just considered.

Given two conics, $\mathcal{K}$ and $\mathcal{L}$, they define their foci as vertices of the quadrilateral of common tangents. A third conic $\mathcal{C}$ is confocal to this pair, if it belongs to the pencil $\kappa\mathcal{K} + \lambda\mathcal{L}$, i.e. if it is tangent to the common quadrilateral.

Proposition: Given the conic $\mathcal{C}$ which determines perpendicularity. Any tangent t_1 to $\mathcal{K}$ is reflected at the intersection with a confocal with $\mathcal{K}$ and $\mathcal{C}$ conic $\mathcal{L}$ by the tangent to $\mathcal{L}$ into a tangent t_2 to $\mathcal{K}$. Proof: Any point $Q \notin \mathcal{K}$ determines two tangents $t_{1,2}$ to $\mathcal{K}$, whether real or not. Some conic $\mathcal{C}$ is chosen to determine perpendicularity. If the numbers $\alpha_{1,2} = \langle t_{1,2}, \mathcal{C}t_{1,2}\rangle$ have the same sign, we obtain the angular bisectors of the tangents $t_{1,2}$ at Q as

$$w_\pm \propto \sqrt{\alpha_2}\, t_1 \pm \sqrt{\alpha_1}\, t_2$$

Both bisectors are tangents to conics confocal to the pair $\{\mathcal{K}, \mathcal{C}\}$, in particular

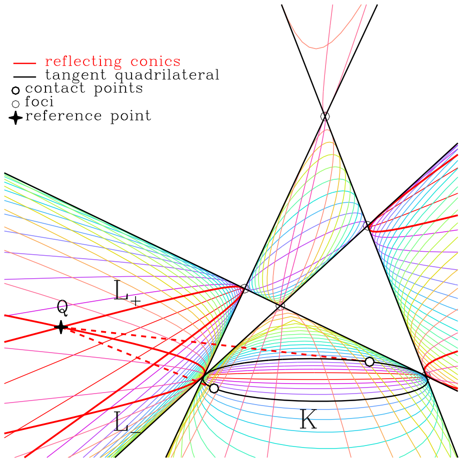
$$\mathcal{L}_\pm = \langle w_\pm, \mathcal{C}w_\pm\rangle\mathcal{K} - \langle w_\pm, \mathcal{K}w_\pm\rangle\mathcal{C}.$$

The contact points $\mathcal{L}_\pm w_\pm$ of the bisectors conicide with their intersection because expansion yields

$$\langle t_{1,2}, \mathcal{L}_\pm w_\pm\rangle = 0.$$

We obtained for the point Q two conics $\mathcal{L}_\pm$ which provide the required reflection of t_1 in t_2. $\Box$

The two conics are determined by the quadrilateral of tangents of the pair $\{\mathcal{K}, \mathcal{C}\}$. In addition, because the two conics $\mathcal{L}_\pm$ confocal to $\{\mathcal{K}, \mathcal{C}\}$ do not change when



In a pencil of confocal conics, the tangents from any point Q to any conic $\mathcal{K}$ are reflected into each other by both conics $\mathcal{L}_{\pm}$ of the pencil that pass this point. At the intersection points of two conics $\mathcal{L}_{\pm}$ of the pencil, the two tangents to $\mathcal{L}_{\pm}$ are perpendicular if the absolute conic is anyone of the pencil, too.

Figure 3. Tangents of confocal conics

another conic of the pencil is chosen as absolute, the angular bisectors are independent of such a choice.

We conclude: In a pencil of confocal conics, a tangent to a conic $\mathcal{K}$ of the pencil from a point Q on any other conic of the pencil is reflected into the second tangent from Q to $\mathcal{K}$ at the tangents of the (two) conics of the pencil in Q, independent of which (third) conic of the pencil is promoted to serve as the metric-determining.

5. Summary

The definition of foci in non-Euclidean (precisely metric-projective) geometries reveals a structure basically independent of the particular explicit metric properties. The structure can be understood as a pure relation between conics.

(1) The quadrilateral of tangents common to two conics defines six points, which reveal the familiar properties of foci. The two poles of a line through a focus are collinear with the focus.

(2) The four lines of a quadrilateral define a pencil of conics touching the four lines. The intersection point of the four lines are the foci for any pair of conics of this pencil. For *any* line through a focus, the poles with respect to all the conics of the pencil are *collinear*.

(3) The diagonal lines of the quadrilateral form a self-polar triangle: Its vertices and edges are pole-polar pairs (to all conics of the pencil). Its vertices are also the diagonal points of the quadrangle of intersections of any two conics of the pencil. Thus, the diagonal triangle is self-dual, too.

(4) Independent of which conic of the pencil is taken as absolute, the three cited characteristics of foci are present.

References

[1] F. Bachmann (1959): *Der Aufbau der Geometrie aus dem Spiegelungsbegriff* (Geometry of Reflections 1971), Springer, Heidelberg.
[2] D.-E. Liebscher (2005): *The Geometry of Time*, Wiley-VCH, Weinheim.
[3] S. Liebscher (2017): *Projektive Geometrie der Ebene. Ein klassischer Zugang mit interaktiver Visualisierung*, Springer, Heidelberg; see also `stefan-liebscher.de/geometry/`.
[4] J. Richter-Gebert (2011): *Perspectives on Projective Geometry, A Guided Tour Through Real and Complex Geometry*, Springer, XXII, 571p. 380 illus., 250 illus. in color.
[5] I. M. Yaglom (1979): *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer, Heidelberg (Princip otnositel'nosti Galileia i neevklidova geometrija 1969).

Stefan Liebscher: TNG Technology Consulting, Munich, Germany
E-mail address: `contact@stefan-liebscher.de`

Dierck-E. Liebscher: Leibniz-Institut für Astrophysik Potsdam, Germany
E-mail address: `deliebscher@aip.de`

Forum Geometricorum
Volume 18 (2018) 7–16.

FORUM GEOM
ISSN 1534-1178

Revisiting the Quadrisection Problem of Jacob Bernoulli

Carl Eberhart

Abstract. Two perpendicular segments which divide a given triangle into 4 regions of equal area is called a *quadrisection* of the triangle. Leonhard Euler proved in 1779 that every scalene triangle has a quadrisection with its triangular part on the middle leg. We provide a complete description of the quadrisections of a triangle. For example, there is only one isosceles triangle which has exactly two quadrisections.

1. Introduction

In 1687, Jacob Bernoulli [1] published his solution to the problem of finding two perpendicular lines which divide a given triangle into four equal areas. He gave a general algebraic solution which required finding a root of a polynomial of degree 8 and worked this out numerically for one scalene triangle.

The question of whether Bernoulli's polynomial equation of degree 8 has the needed root in all cases is not answered completely. Leonhard Euler's [2], in 1779, wrote a 22 page paper in which he gives a complete solution using trigonometry.

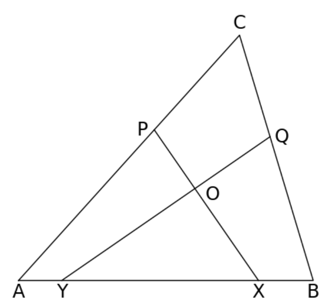


Figure 1.

Euler states his solution in a theorem which we paraphrase.

Theorem 1 (Euler 1779)**.** *Given a scalene triangle ΔABC with AB the side of middle length, there is a quadrisection XP and YQ intersecting in a point O in the interior of the triangle so that X and Y lie on side AB and triangle XOY is one of the 4 areas of the quadrisection. The other areas of the quadrisection are quadrangles.*

Publication Date: January 17, 2018. Communicating Editor: Paul Yiu.

I want to thank my colleagues Dr. Fred Halpern and Aaron Raden for helpful discussions during the writing of this paper.

Euler does not claim that the triangular portion of a quadrisection must lie on the side of middle length. Also, he does not appear to discuss whether there is more than one quadrisection of a triangle, except to note that an equilateral triangle has 3 quadrisections. In fact, we will see there are lots of triangles with quadrisections where the triangular portion lies on the shortest side, but no triangles having a quadrisection with the triangular portion on the longest side.

This paper was written in latex using an account on cloud.sagemath.com. Some animations designed to augment this paper can be found at

http://www.ms.uky.edu/~carl/sagelets/arcsoftriangles.html

2. Initial analysis.

Take any triangle T. We can scale and position it in the plane so that two vertices A and B have coordinates $(0,0)$ and $(1,0)$ respectively and the third vertex $C = (h, ht)$ is chosen so that it is in the quadrant $h \geq 1/2$ and $ht > 0$. Under these assumptions, there is only one choice for C, $(1/2, \sqrt{3}/2)$, if T is the equilateral triangle.

If T is an isoceles triangle, then there are two possibilites: (1) if the vertex angle is greater than $\pi/3$, then $C = (1/2, ht)$ with $ht < \sqrt{3}/2$ or $C = (h, \sqrt{2\,h - h^2})$ with $1/2 < h < 2$, (2) if the vertex angle is less than $\pi/3$, then $C = (1/2, ht)$ with $ht > \sqrt{3}/2$ or $C = (h, \sqrt{1 - h^2})$ with $1/2 < h < 1$.

If T is a scalene triangle, then depending on how we chose AB, C will be in one of three open regions R_1, R_2, R_3 respectively:

(1) If AB is the longest leg, then then $h^2 + ht^2 < 1$,

(2) If AB is the middle leg, then $h^2 + ht^2 > 1$ and $(h-1)^2 + ht^2 < 1$, or

(3) If AB is the shortest leg, then $(h-1)^2 + ht^2 > 1$

Since AB can be any one of the 3 sides of T, we know each of the three regions above together with its boundary of isosceles triangles contains a unique copy of each triangle up to similarity. Denote these sets by $\overline{R_1}, \overline{R_2}, \overline{R_3}$ respectively.

We can use inversion about a circle[1] to match up a triangle in one region with its similar versions in the other regions. Inversion about the unit circle interchanges points in region 2 with points in region 1, and inversion about the unit circle centered at $(1,0)$ interchanges points in region 2 with points in the unbounded region 3. For example, let C be a point in region 2, and let $C^{'}$ be it's inverse about the unit circle centered at $(1,0)$. Then $C^{'}$ is in region 1, and $B, C, C^{'}$ are collinear with C between B and $C^{'}$. Further $BC\,BC^{'} = 1$. Using this, we see that ΔABC is similar with $\Delta C^{'}BA$. By the same reasoning, letting C^{*} denote the inverse of C about the unit circle, we get that ΔABC is similar to $\Delta AC^{*}B$.

Terminology: In what follows, when we refer to a triangle $C = (h, ht)$, we are referring to ΔABC with $A = (0,0),\ B = (1,0)$, where $h \geq 1/2$ and $ht > 0$.

[1] Regarding points as vectors, the **inverse** of $p = (x,y)$ about the circle of radius 1 centered at $c = (h,k)$ is $p' = c + (p-c)/(p-c)^2 = (h + (x-h)/d, k + (y-k)/d)$, where $d = (x-h)^2 + (y-k)^2$. https://en.wikipedia.org/wiki/Inversive_geometry

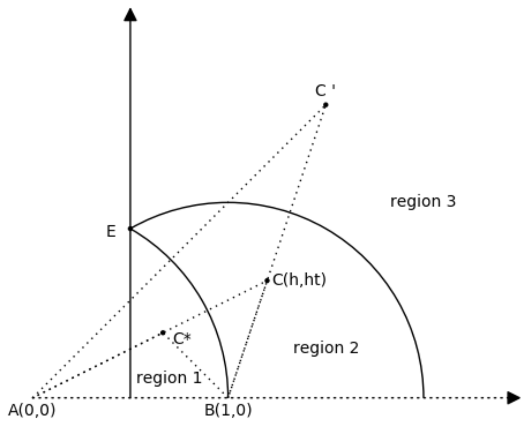


Figure 2.

3. The equations for quadrisection of a triangle.

Regarding points as vectors, we write $X = x\,B$, $Y = (1-y)\,B$, $P = s\,C$, and $Q = (1-r)B + r\,C$ for $x, y, r, s \in [0,1]$. Now since area $\Delta XPY = s/2\,ht\,x =$ area $\Delta ABC = ht/2$, we see that $s = 1/(2x)$. Similarly, $r = 1/(2y)$. Note that $1/(2x) < 1$, so $1/2 < x$ and similarly for y. So $1 < x + y$.

There are two equations which determine x and y:

(a) The area of triangle ΔXOY is one fourth of the total area of the triangle.

Writing $O = (x_0, y_0)$, this is the area equation $4\,y_0(x+y-1) = ht$. We can calculate y_0 by writing $O = u\,P + (1-u)\,X = v\,Q - (1-v)\,Y$ for some $u, v \in [0,1]$. Expand this out to get two linear equations in u, v. Solve for u to get $u = \dfrac{(x+y)-1}{(x+\frac{s}{r}y)-s}$. Substitute this into $O = u\,P + (1-u)\,X)$ and calculate $O_1 = y_0 = ht\,\dfrac{1-(x+y)}{1-\frac{x}{s}-\frac{y}{r}} = ht\,\dfrac{1-(x+y)}{1-2\,(x^2+y^2)}$. Substitute this into the area equation, divide both sides by ht and simplify to get the

Area Equation: (Aeq) $(x^2+y^2) + 4\,(x\,y - x - y) + 5/2 = 0$

This equation has two solutions for $y = y(x)$ in terms of x, but the one we want is $y(x) = 2 - 2x + \sqrt{12x^2 - 16x + 6}/2$. Note $y(\sqrt{2}/2) = 1$ and $y(1) = \sqrt{2}/2$. Also $y(5/6) = 5/6$.

Note: In the duration, $y(x) = 2 - 2x + \sqrt{12x^2 - 16x + 6}/2$.

(b) XP and YQ are perpendicular. This means the dot product $(Q-Y)\cdot(P-X) = 0$, or $(h\,s - x,\ ht\,s)\cdot(h\,r - r + y,\ ht\,r) = 0$.

Substitute $s = 1/(2\,x)$, $r = 1/(2\,y)$ and simplify to get the

Perpendicularity Equation: (Peq) $\big(x^2 - h/2\big)\big(y^2 - (1-h)/2\big) = (ht/2)^2$

To use Peq(h, ht) to find all the quadrisections of a particular triangle T, substitute $y = y(x)$ into Peq(h, ht) and solve the resulting equation in x for each (h, ht) (with $h \geq 1/2, ht > 0$) which is similar to T. The total number of solutions is the number of quadrisections.

4. Peq viewed as a 1-parameter family of circular arcs

If instead of setting the values for h and ht in Peq(h, ht) , set the value of x between $\sqrt{2}/2$ and 1, and then set $y = y(x)$, then **Aeq** is satisfied and we have fixed the base YX on AB of the triangular part of a quadrisection. Then Peq(x,y(x)) is a quadratic equation in h and ht, which if we rewrite in the form

$$Peq(x, y(x)): \qquad ht^2 + \left(h - (x^2 - y(x)^2 + 1/2)\right)^2 = \left(x^2 + y(x)^2 - 1/\right)^2$$

we recognize as the circle $Cir(x)$ in the h, ht plane with center $X(x) = (c(x), 0)$ and radius $r(x) = x^2 + y(x)^2 - 1/2$, where $c(x) = x^2 - y(x)^2 + 1/2$.

The arc $Arc(x)$ of the circle $Cir(x)$ that lies in the quadrant $h \geq 1/2, ht > 0$ consists of all triangles (h, ht) with a quadrisection with base $YX = [(1 - y(x), 0), (x, 0)]$. $Arc(\sqrt{2}/2)$ lies in the unit circle, and forms the lower boundary of Region 2, and $Arc(1)$ lies in the unit circle centered at $(1, 0)$ and forms the upper boundary of Region 2. Let $(1/2, z(x))$ be the terminal point on $Arc(x)$, and let $\theta(x)$ be the radian measure of $\angle(1/2, z(x))(c(x), 0)(2, 0)$. So $\theta(x) = \arccos((1/2 - c(x))/r(x))$, and $z(x) = \sqrt{r(x)^2 - (1/2 - c(x))^2}$.

Let **Arcs** denote the union of all arcs $Arc(x)$. The triangles $(h, ht) \in$ **Arcs** are precisely the triangles which have a quadrisection with the triangular portion on $[(0, 0), (1, 0)]$.

4.1. *A useful mapping.* Let $D = \{(x, \theta) | x \in [\sqrt{2}/2, 1], \theta \in [0, \theta(x)]\}$.
We define a mapping F from D onto **Arcs** by

$$F(x, \theta) = (c(x) + r(x)\cos(\theta), r(x)\sin(\theta)).$$

F maps each segment $[(x, 0), (x, \theta(x))]$ in D 1-1 onto the corresponding arc $Arc(x)$.

A small table of values

x	$y(x)$	$c(x)$	$r(x)$	$\theta(x)$	$F(x, \theta(x)) = (1/2, z(x))$
$\sqrt{2}/2$	1	0	1	$\pi/3$	$(1/2, \sqrt{3}/2)$
5/6	5/6	1/2	8/9	$\pi/2$	$(1/2, 8/9)$
1	$\sqrt{2}/2$	1	1	$2\pi/3$	$(1/2, \sqrt{3}/2)$

The Jacobian determinant of F is $|J_F(x, \theta)| = c'(x)r(x)\cos(\theta) + r'(x)r(x)$. This vanishes on the curve $J_0 = \{p(x) | x \in [\sqrt{2}/2, 1]\}$, where

$$p(x) = (x, \arccos(-r'(x)/c'(x)).$$

$D \setminus J_0$ is the union of two disjoint relatively open sets U, and V in D, with $(\sqrt{2}/2, 0) \in U$. Let $D_1 = U \cup J_0$ and $D_2 = V \cup J_0$. See the diagram. D_1 and D_2 are both topological closed disks, with boundaries $\partial D_1 = S_1 \cup S_2 \cup S_3 \cup J_0 \cup C_1$, and $\partial D_2 = J_0 \cup S_1 \cup C_2$ as shown in the diagram. It follows from the definition of F that it is 1-1 on each of D_1 and D_2.

Note that $F(S_1 \cup S_2) = Arc(1)$, $F(S_4) = Arc(\sqrt{2}/2)$, $F(S_3) = [(1,0),(2,0)]$ (not corresponding to any triangles), and

$$F(C_1) = F(C_2) = [(1/2, \sqrt{3}/2), (1/2, 8/9)].$$

$F(J_0)$ is the concave up portion of the upper boundary of $F(D_1)$. Each arc $Arc(x)$ for $x \in [5/6, 1]$ is tangent to it, so $F(J_0)$ is the envelope[2] of those arcs.

Let R_4 denote the closed disk with boundary $F(J_0) \cup F(C_1) \cup F(S_4) \cup F(S_3) \cup F(S_2)$, and R_5 the closed disk with boundary $F(J_0) \cup F(S_1) \cup F(C_1)$. Then $R_4 \supset$ **Arcs**, and $R_5 \subset \overline{R_3}$ with $R_5 \cap R_4 = F(J_0)$

Lemma 2. *The transformation F maps D_1 homeomorphically onto the closed disk $R_4 \supset \overline{R_1}$, and D_2 homeomorphically onto R_5.*

Proof. F maps the boundaries ∂D_1 and ∂D_2 onto the boundaries R_4 and R_5, respectively. Consequently, it follows from the Brouwer invariance of domain theorem [3], that $F(D_i)$ is the closed disk enclosed by $F(\partial D_i)$ for $i = 1, 2$. □

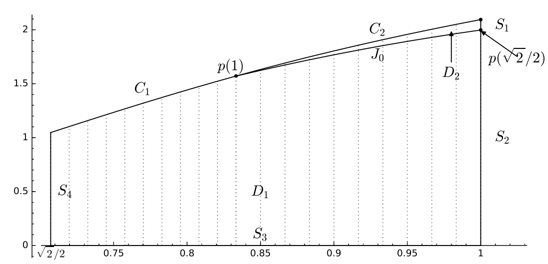


Figure 3.

Note that the orientation of ∂D_2 is the reverse of the orientation of $F(D_2)$. This is because $|dF| < 0$ on D_2. F folds D_2 over along J_0 and fits it onto $F(D_2)$.

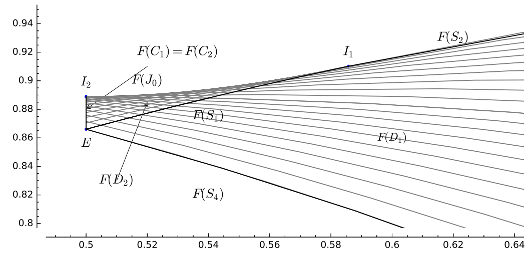


Figure 4.

[2] https://en.wikipedia.org/wiki/Envelope_(mathematics)
[3] https://en.wikipedia.org/wiki/Invariance_of_domain

Since $F(D_1)$ contains R_1, we have another proof of Euler's theorem. In fact, since $F(D_2) \cap R_1 = \emptyset$, we have a stronger version.

Theorem 3. *Each scalene triangle T has* ***exactly*** *one quadrisection whose triangular portion lies on the middle leg of T and* ***no*** *quadrisection whose triangular portion lies on the longest leg of T.*

Proof. Scale and position T so that its vertices are $(0,0),(1,0),C$ with $C = (h, ht) \in R_1$. Then since $R_1 \subset F^(D_1)$, $(x,\theta) = F^{-1}(h, ht)$ gives a quadrisection of T with its triangular portion on its middle leg. But also $F(D_2) = R_4 \subset \overline{R_2}$, and $\overline{R_2} \cap R_1 = \emptyset$ so T has only the 1 quadrisection with it's triangular portion on the middle leg. and no quadrisection with its triangular portion on the shortest leg. $\square$

4.2. *Counting the quadrisections of a triangle.*

Definition. Let Quads(T) denote the number of quadrisections of the triangle T. If (h, ht) is a point with $h \geq 1/2, ht > 0$ such that T is similar to $\Delta A(0,0)B(1,0)C(h,ht)$, then we write Quads$(h, ht)$=Quads$(T)$.

Theorem 4. ***(Quadrisection theorem for triangles)*** *Let T be a triangle. Let $C = (h, ht)$ be the unique triangle in $\overline{R_2}$ similar to T. Let $C' = (h', ht')$ be the inverse of C about the unit circle with center $(1,0)$.*

(1) *If $C' \in F(D_2) \setminus F(J_0)$ then Quads(T)= 3.*
(2) *If $C' \in F(J_0)$ and $C' \neq I_1$, then Quads(T)= 2.*
(3) *Otherwise (that is, if $C' \notin F(D_2)$ or $C' = I_1$), then Quads(T)= 1.*

Proof. Assume case 1. $F(D_2) \setminus F(J_0)$ is doubly covered, once by $D_2 \setminus J_0$ and once by D_1. So T has two quadrisections with the triangular part on a shortest side. Since T also has a quadrisection with the trianglular part on its middle side, Quads(T)= 3.

Assume case 2. $F(J_0 \setminus \{p(\sqrt{2}/2)\})$ is covered once by F. So T has a single quadrisection with the triangular portion on the shortest side. Since T also has a quadrisection with the trianglular part on its middle side, Quads(T)= 2

Assume case 3. If $C' \notin F(D_2)$ then T has only the quadrisection with triangular part on a middle side. If $C' = C = I_1 = F(p(\sqrt{2}/2))$, then $Arc(1)$ is the only arc that contains C'. So Quads(T)= 1. $\square$

The isoseceles triangles I_1 and I_2 occupy interesting positions amongst the isosceles triangles. The vertex angle of I_1 is greater than $\pi/3$ and it has only 1 quadrisection. Any other isosceles triangle with these 2 properties has a larger vertex angle. I_2 is interesting because it is the only isosceles triangle with exactly 2 quadrisections. But also, one of the quadrisections is rational, that is, the vertices of the triangle and the endpoints of the segments forming the quadrisection are rational.

Question 1. *Is there another triangle with rational vertices and a rational quadrisection?*

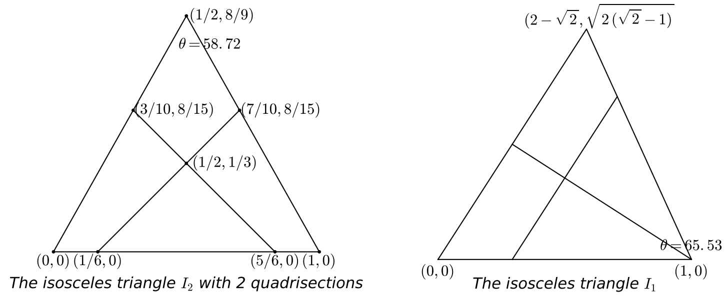


Figure 5.

5. The space of triangles

Let Υ be the set $\{(h, ht) \mid 1/2 \le h < 2,\ ht > 0,\ \text{with } \sqrt{1-h^2} \le ht \le \sqrt{h(2-h)}\}$, with the subspace topology from the Cartesian plane. As observed earlier, every triangle is similar to exactly one triangle $[A(0,0), B(1,0), C(h,ht)]$ with $(h, ht) \in \Upsilon$. So it is natural to call Υ the space of triangles. In that space, we see that the reflection of $T(J_0)$ about the circle of radius 1 centered at $(1,0)$ is an arc S_2 of scalene triangles from I_1 to the reflection of I_2 which is $C = (175/337, 288/337)$. All these have Quads(C)$=2$, except Quads(I_2)$=1$. This arc separates Υ into two relatively open sets U and V, with the equilateral triangle $E = (1/2, \sqrt{3}/2) \in U$. Quads($C$)$=3$ for each $C \in U$, and Quads(C)$=1$ for each $C \in V$.

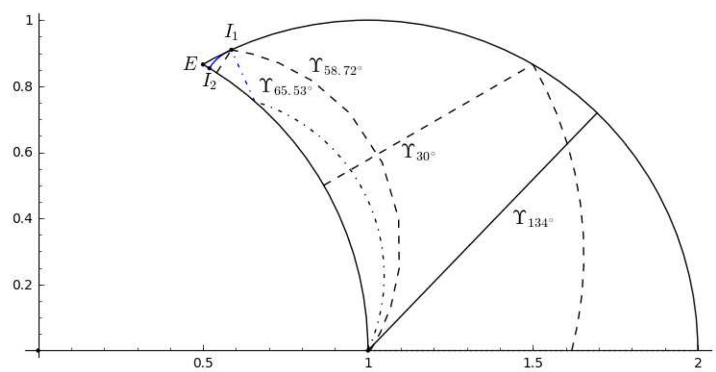


Figure 6.

Note that the vast majority of triangles have only one quadrisection, and the only ones that have more than one quadrisection are very near the equilateral triangle. It also gives a useful way to visualize the position of certain classes of triangles in the space, such as the isosceles triangles, which form the boundary of the space. Where does the class $\Upsilon_{\pi/2}$ of right triangles sit in Υ? Since the points $(1, ht)$ with $ht > 1$ lie in region 3, they invert onto the points $(1, 1/ht)$, so it is simply the vertical segment $((1,0),(1,1)]$ in Υ. Topologically, it separates the space Υ into two pieces. The right hand piece consists of all triangles with an obtuse angle; the left hand piece all triangles with all angles acute. Similarly, the class Υ_θ for all θ with $\pi/2 \leq \theta < \pi$ is a segment in Υ separating it into two pieces. If $0 < \theta \leq \pi/2$, then one works out that Υ_θ consists of a radial segment in Υ together with a circular arc in Υ which separates Υ into three pieces.

There are two angles of particular interest:
1) the vertex angle θ of I_2, which we have calculated as $\theta \approx 58.72°$. Any triangle with an angle at least θ has only the one quadrisection.
2) the vertex angle of I_1, which we have calculated as $\theta \approx 65.53°$

6. Further Questions.

This half-lune model for the space of triangles is reminiscent of the Poincare model for the hyperbolic plane. Since no two triangles are similar in the hyperbolic plane, there are more triangles to quadrisect. This suggests that the quadrisection problem may be more delicate to solve completely in the hyperbolic plane.

Question 2. *What is the analogue for Theorem 3 in the hyperbolic plane?*

Our discussion of the quadrisection problem for triangles naturally leads to the question of determining the quadrisections of any convex polygon. Investigating this question leads to the following conjecture.

Conjecture. A convex $2\,n+1$-gon R has at most $2\,n+1$ quadrisections. Further, if R is 'sufficiently close to' the regular $2\,n+1$-gon, then R has $2\,n+1$ quadrisections.

7. Historical notes

7.1. *Bernoulli's solution.* Bernoulli worked in a time before the use of Cartesian geometry had become widespread, so it is not surprising that he did not coordinatise and normalize the problem. In any case, he did obtain a method for constructing triangles and their quadrisections. However, it is not made clear that the construction produces all possible triangles.

Using Bernoulli's labeling of the triangle, let $AC = a$, $CB = b$, $BA = c$, $KB = d$, $KC = e$, $KA = f$, $CD = x$, $AF = y$. Note that Bernoulli's x and y are our y and x. Bernoulli derives versions of the area and perpendicularity equations:

Bernoulli's equations: $y^2 = 4\,a\,y - 4\,x\,y - 2\,\frac{1}{2}\,a^2 + 4\,a\,x - x^2$ and $y^2 = \pm\frac{1}{2}a\,f + \dfrac{a^2\,d^2}{4\,x^2 \pm 2\,a\,e}$.

If we normalize these equations by letting $a = 1$, then $d = ht$, $f = h$, $e = 1 - h$, and Bernoulli's equations are our **Aeq** and Peq(h, ht) . He then reduces his equation to a polynomial equation in one variable of degree 8. If we normalize by letting $a = 1$, then the polynomial 'simplifies' to

$$p(x) = x^8 - 8x^7 + (3b^2 - 3c^2 + 17)x^6 - 2(b^2 - c^2 + 5)x^5$$
$$- \tfrac{1}{4}(3b^4 - 6b^2c^2 + 3c^4 + 38b^2 - 24c^2 + 17)x^4$$
$$+ (b^4 - 2b^2c^2 + c^4 + 12b^2 - 6c^2 + 5)x^3 + \tfrac{1}{4}(4b^4 - 5b^2c^2 + c^4 - 7b^2 - 1)x^2$$

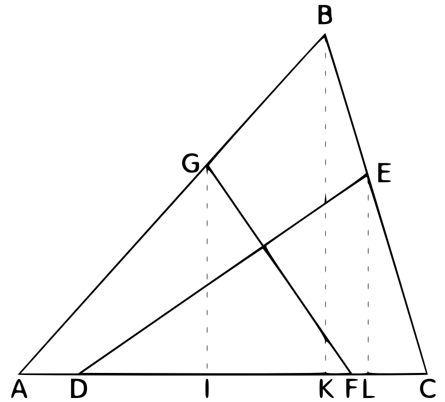


Figure 7. Bernoulli diagram

$-\frac{1}{2}(4b^4 - 5b^2c^2 + c^4 + 5b^2 - 2c^2 + 1)x$
$+\frac{3}{4}b^4 - \frac{3}{4}b^2c^2 + \frac{3}{4}b^2 - \frac{1}{8}c^2 + \frac{1}{16}c^4 + \frac{1}{16} = 0.$

Bernoulli describes a method for constructing simultaneously a triangle and a quadrisection using the area and perpendicularity equations, and illustrates it by constructing a triangle with $a = 484, b = 490, c = 495, x = 386$. Checking this with his normalized polynomial, we get $p(386/484)\,484 = 2.85$, which is not 0 but relatively close to 0. The correct value rounded to two decimals for this x is $x = 368.86$. This triangle is close enough to equilateral that it has 3 quadrisections.

7.2. *Euler's clarification.* Euler chooses to use angles $\alpha, \beta, \gamma, \phi$ in his analysis.

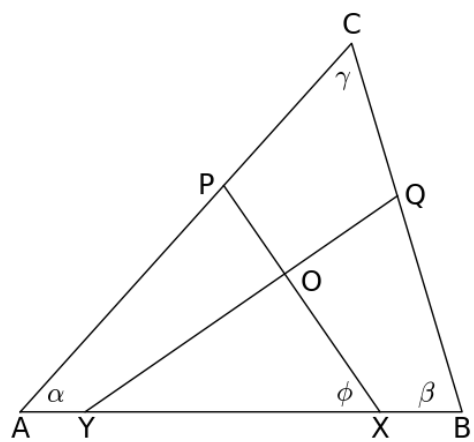


Figure 8. Euler's diagram

Using the diagram, let $AX = x$, $YB = y$, $f = \cot\alpha, g = \cot\beta$,and $t = \tan\phi$. Then he shows that $x = k\sqrt{f + 1/t}$, $y = k\sqrt{g + t}$, where k^2 denotes the area of the triangle. Next he obtains a single equation in the unknown t:

Euler's equation $\sqrt{f+\frac{1}{t}}+\sqrt{g+t}-\sqrt{2\,(f+g)}=\sqrt{\frac{1+t^2}{2\,t}}$

Euler then makes use of the equation to construct a lengthy direct proof of his theorem that every scalene triangle has a quadrisection with its triangular part on the middle side.

He also shows how to use his equation to estimate the value ϕ to the nearest second for the right triangle with sides $2, 1, \sqrt{5}$ and also calculates $x = 1.5146$. This right triangle, as with all right triangles, has only one quadrisection, and the correct value for x rounded to 5 decimals is $x = 1.51443$, so his estimate is pretty close.

7.3. *An explanation of interest.* I became aware of the triangle quadrisection problem when looking up biographical information about Jacob Bernoulli in connection with a fictional story (*An Elegant Solution*, by Paul Robertson, 2013) about a young Leonard Euler and the Bernoulli brothers, Johann and Jacob. Several entries mentioned this as his contribution to geometry. It is interesting that both mathematicians in this story wrote papers on this same subject.

References

[1] Jacob Bernoulli, Solutio algebraica problematis de quadrisectione trianguli scaleni, per duas normales rectas, *Collected Works, No. XXIX*, (1687) 228–335; `http://dx.doi.org/10.3931/e-rara-3584`.

[2] L. Euler, Dilucidationes Problemate Geometrico De Quadrisecione Trianguli a Jacobo Bernoulli, 1779; *Euler Archive, Index Number E729*; `http://eulerarchive.maa.org`.

Carl Eberhart: 4050 Rochelle Dr. Dallas, Texas 75220, USA
E-mail address: `carl.eberhart@gmail.com, carl.eberhart@uky.edu`

Forum Geometricorum
Volume 18 (2018) 17–36.

FORUM GEOM
ISSN 1534-1178

Areas and Shapes of Planar Irregular Polygons

C. E. Garza-Hume, Maricarmen C. Jorge, and Arturo Olvera

Abstract. We study analytically and numerically the possible shapes and areas of planar irregular polygons with prescribed side-lengths. We give an algorithm and a computer program to construct the cyclic configuration with its circumcircle and hence the maximum possible area. We study quadrilaterals with a self-intersection and prove that not all area minimizers are cyclic. We classify quadrilaterals into four classes related to the possibility of reversing orientation by deforming continuously. We study the possible shapes of polygons with prescribed side-lengths and prescribed area.

In this paper we carry out an analytical and numerical study of the possible shapes and areas of general planar irregular polygons with prescribed side-lengths. We explain a way to construct the shape with maximum area, which is known to be the cyclic configuration. We write a transcendental equation whose root is the radius of the circumcircle and give an algorithm to compute the root. We provide an algorithm and a corresponding computer program that actually computes the circumcircle and draws the shape with maximum area, which can then be deformed as needed. We study quadrilaterals with a self-intersection, which are the ones that achieve minimum area and we prove that area minimizers are not necessarily cyclic, as mentioned in the literature ([4]).

We also study the possible shapes of polygons with prescribed side-lengths and prescribed area. For areas between the minimum and the maximum, there are two possible configurations for quadrilaterals and an infinite number for polygons with more than four sides.

The work was motivated by the study of two Codices, ancient documents from central Mexico written around 1540 by a group called the Acolhua ([9],[10] or online [3]). The codices contain drawings of polygonal fields, with their side-lengths in one section and their areas in another. We had to find out if the proposed areas were correct and to confirm the location of some of the fields.

The fields are not drawn to scale and it is not known how the Acolhua computed (or measured) areas. There are no angles or diagonals therefore we could not compute the actual areas but one thing we could do was compute maximum and minimum possible areas for the given side-lengths and say the areas in the documents were feasible if they were between those two values. Maxima are easy to

Publication Date: January 17, 2018. Communicating Editor: Paul Yiu.
We thank Ana Pérez Arteaga and Ramiro Chávez Tovar for computational support, figures and animations. This work was supported by Conacyt Project 133036-F..

find for quadrilaterals by applying Brahmagupta's formula ([5], [6]) but we could not find in the literature a formula for the maximum possible area of polygons with more than four sides. There are some results in [8] but not a formula. We could also not find a complete treatment of minima in the literature.

The study of the documents also required an analysis of the possible shapes of polygons with prescribed side-lengths and area to try to localize the fields. For polygons with more than four sides there is no easily available, explicit formula for this. We explain the mathematics of the problem, provide an algorithm to find the possible shapes and a computer implementation in javascript of the algorithm.

Observe that finding areas of polygons is just an application of calculus *if* the coordinates of the vertices are known but the problem we are addressing here is, what can be done when only side-lengths are known, not angles, diagonals or coordinates. The question is relevant in surveying, architecture, home decorating, gardening and carpentry to name but a few.

For some quadrilaterals it is possible to change orientation by deforming continuously and in others it is necessary to perform a reflection. This observation led to a classification of quadrilaterals into four classes.

The case of triangles is special because they are the only rigid polygons, that is, the shape and area are determined by the side-lengths. Also, all triangles can be inscribed in a circle whose center is located at the intersection of the perpendicular bisectors. If the side-lengths are called a, b, c the area is given by Heron's Formula,

$$\mathrm{A}_H(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where $s = \frac{1}{2}(a+b+c)$ is the semi-perimeter. In what follows we will discuss polygons with more than three sides.

1. Computing the shape with maximum area for any polygon

In this section we will describe how to compute the maximum possible area of a planar polygon with n sides and how to construct the shape with maximum area given its ordered side-lengths. It is well-known that no side-length can be larger than the sum of the others if they are to form a closed polygon. We will call this the compatibility condition. It was proved in [4] that the configuration with maximum possible area is given by the cyclic polygon, that is, the configuration that can be inscribed in a circle. The problem is that we do not know *which* circle; we do not know its center or its radius. We will solve this problem in the present section.

We will first give an analytical solution for quadrilaterals and then describe an iterative method that works for any polygon and its numerical implementation.

Analytic solution for n=4: Circle-Line construction. For $n = 4$, Bretschneider's formula gives the (unoriented) area $\mathcal{A}$ of a quadrilateral with side-lengths a, b, c, d and opposite angles α and γ as

$$\mathcal{A} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2(\frac{\alpha+\gamma}{2})}, \tag{2}$$

where $s = (a+b+c+d)/2$, see Figure 1. Observe that $\mathcal{A}$ is always non-negative.

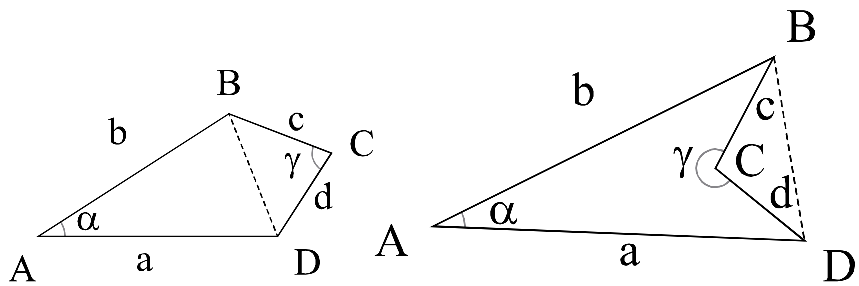


Figure 1. Two examples of a quadrilateral $ABCD$.

The maximum value of this expression is

$$\mathcal{A}_{max} = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{3}$$

and it is attained when the sum of either pair of opposite angles is π radians, that is, when the quadrilateral es cyclic. But this does not give directly the coordinates of the shape with maximum area.

There is a formula for the vertices of the shape with maximum area which is described in [6] SI Appendix p.414, for a quadrilateral with given side-lengths a, b, c, d and prescribed area $\mathcal{A}_c$. We give here the geometric idea.

As explained in [6], vertex A is placed at the origin and vertex D at $(a, 0)$; it remains to find the coordinates of vertices B and C. The coordinates of vertex B are (u, v) and of vertex C, (w, z) (see Figure 2).

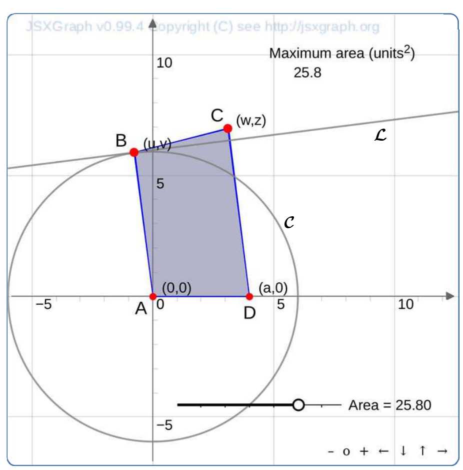


Figure 2. Shape with maximum area of a quadrilateral with side-lengths 4,6,4,7 arbitrary units.

After some lengthy computations with analytic geometry which we include in Appendix A, it is shown that vertex B is given by the intersection of a circle $\mathcal{C}$ of radius b and center at the origin, and a line $\mathcal{L}$ which depends on the side-lengths and on the prescribed area $\mathcal{A}_c$. The maximum area is attained when the line is tangent to the circle. The coordinates of vertex B are given explicitly and the coordinates of vertex C and found from the lengths of the remaining sides.

Figure 2 shows an example of maximum area produced using a program we wrote using routines from JSXGraph which can be found under "Circle-line construction" in www.fenomec.unam.mx/ipolygons. The value of the maximum area can be computed from equation (3). In this example, in arbitrary units $a = 4$, $b = 6$, $c = 4$, $d = 7$ therefore $s = \frac{1}{2}(4+6+4+7) = 21/2$ and the maximum possible area is approximately 25.8 square units. The program allows the computation of the *shape* with maximum area.

The construction also shows that if $\mathcal{A}_c > \mathcal{A}_{max}$ then the line and circle do not intersect and hence no quadrilateral exists with those side-lengths and area. If $\mathcal{A}_{min} < \mathcal{A}_c < \mathcal{A}_{max}$ then $\mathcal{L}$ intersects $\mathcal{C}$ twice giving two possible shapes for the given quadrilateral data. Here $\mathcal{A}_{min}$ represents the minimum possible area. We will say more about it later.

We recommend playing with the computer program to understand the construction and its implications.

Iterative method for any $n \geq 3$ and its numerical implementation. For $n > 4$ there is no explicit formula for the center or radius r of the circumcircle of the configuration with maximum area.

We describe here an iterative method that works for any $n \geq 3$ for finding the circumcircle and thus the maximum area of the polygon and the coordinates of its vertices.

Observe first that the center of the circumcircle of a cyclic polygon can be inside, outside or on the boundary of the polygon, as illustrated in Figure 3, and this determines how the circumcircle is computed.

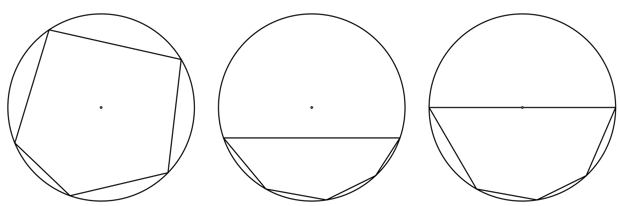


Figure 3. Position of the center of the circumcircle.

Now consider n side-lengths $a_1, a_2, \ldots, a_n$ that satisfy the compatibility condition, placed in clockwise order in the shape of maximum area.

Referring to Figure 4, by the cosine rule

$$a_j^2 = 2r^2 - 2r^2 \cos\theta_j$$

therefore

$$\cos\theta_j = \frac{2r^2 - a_j^2}{2r^2}.$$

It is not known *a priori* if the center of the circumcircle will be inside or outside the polygon. If it is inside the polygon (Figure 4, left), then $\sum_{j=1}^{n} \theta_j = 2\pi$ and

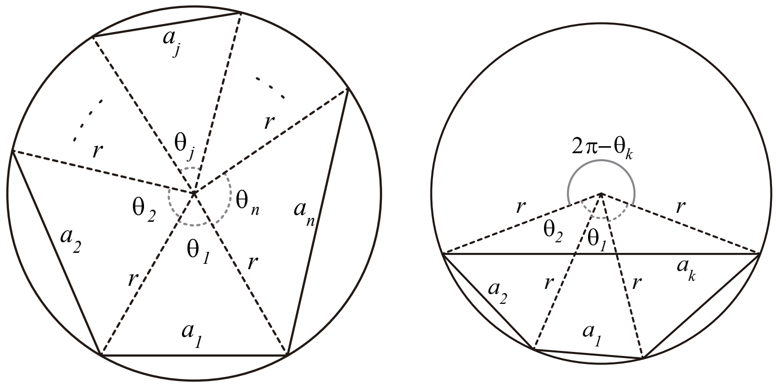


Figure 4. Determination of r when the center of the circumcircle is inside (left) and outside (right) the polygon.

therefore

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) = 2\pi.$$

From this observation we define a function $F_1(r)$ whose root is the radius of the circumcircle when the center is inside the polygon:

$$F_1(r) = \sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - 2\pi = 0. \tag{4}$$

We have used the normal convention that Arccos refers to the principal branch and arccos refers to the secondary branch which is given by $2\pi -$ Arccos.

If the center of the circumcircle is outside the polygon (Figure 4, right) the angle subtended at the center by the longest side is equal to the sum of the angles subtended by the other sides, or equivalently, the inverse cosine of the longest side has to be taken in the secondary branch. We define another function, $F_2(r)$ whose root is the radius of the circumcircle when the center is outside the polygon. If we call a_k the longest side,

$$\begin{aligned} F_2(r) &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) + \text{arccos}(1 - \frac{a_k^2}{2r^2}) - 2\pi \\ &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) + 2\pi - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) - 2\pi \\ &= \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) \\ &= 0. \end{aligned} \tag{5}$$

The radius of the circumcircle, r_M, is the root of $F_1(r)$ in the first case and of $F_2(r)$ in the second. If the center of the circumcircle is on the boundary of the polygon then both expressions coincide. One would normally look for the root numerically using a Newton method. However, in this case the graphs of F_1 and F_2 are almost vertical at the root and a Newton method is not adequate. A bisection method or quasi-Newton will work as long as one can find an interval $[r_l, r_r]$ that contains the root, that is, two values, r_l, r_r such that the function has opposite signs at r_l and r_r. The value r_l is easy to find because the inverse cosine is only defined between -1 and +1. Therefore, for all j,

$$-1 \leq 1 - \frac{a_j^2}{2r^2}$$

and so $r \geq a_j/2$ and the minimum positive value of r for which F_1 and F_2 are defined is $r_l = \max_j(a_j/2)$. Finding r_r is not so easy because when one side of the polygon approaches the sum of the others, the radius of the circumcircle tends to infinity.

The graphs of F_1 from equation (4) and F_2 from equation (5) for side-lengths 6,7,8,9,10 units are shown in Figure 5 and for side-lengths 9,9,9,9,29 in Figure 6.

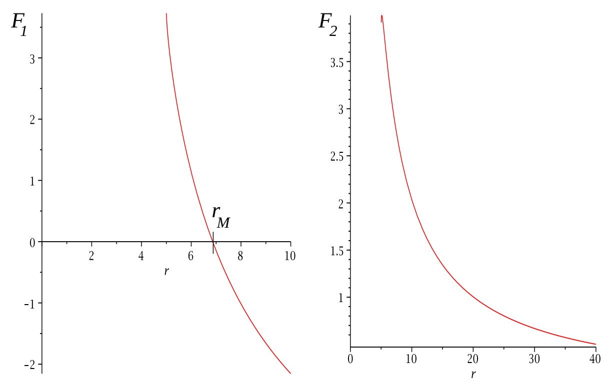


Figure 5. Graph of F_1 (left) and F_2 (right) for side-lengths 6,7,8,9,10. Only F_1 has a root, which is r_M, the radius of the circumcircle.

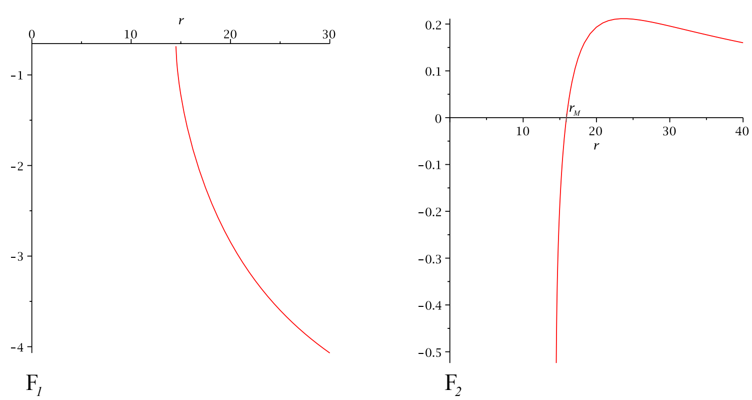


Figure 6. Graph of F_1 (left) and F_2 (right) for side-lengths 9,9,9,9,29. Only F_2 has a root, r_M, the radius of the circumcircle.

If F_1 has a root then the center is inside the polygon and F_2 does not have a root. Otherwise, F_2 has a root and the center is outside the polygon (see the proof in Appendix B.)

Once the radius r_M of the circumcircle $\mathcal{C}_{r_M}$ is calculated, then the area of the polygon with maximum area can be computed for instance by triangulating as in Figure 4 . The area of each triangle can be computed from Heron's formula and we obtain, if r_M was computed from F_1,

$$A = \sum_{i=1}^{n} \mathrm{A}_H(a_i, r_M, r_M)$$

and if r_M was computed from F_2,

$$A = \sum_{\substack{i=1 \\ i \neq k}}^{n} \mathrm{A}_H(a_i, r_M, r_M) - \mathrm{A}_H(a_k, r_M, r_M).$$

To draw the polygon with maximum area first draw $\mathcal{C}_{r_M}$ with center at the origin. Then it is easiest to start with the longest side, a_k, place its right end at the point $P = (r_M, 0)$ and mark the two intersections of $\mathcal{C}_{r_M}$ with the circle of center P and radius a_k. Call i_1 the intersection in the lower semicircle and i_2 the one in the upper semicircle (see Figure 7.) If $F_1(r_M) = 0$ then the left end of side a_k is placed at i_1 and if $F_2(r_M) = 0$ then it is placed at i_2. The remaining vertices can be found by marking on $\mathcal{C}_{r_M}$ the side-lengths in clock-wise order. The coordinates of all the vertices can be calculated from this construction since they are intersections of $\mathcal{C}_{r_M}$ with circles of radius a_j and center at the previous vertex. The figure can then be rotated or translated as needed.

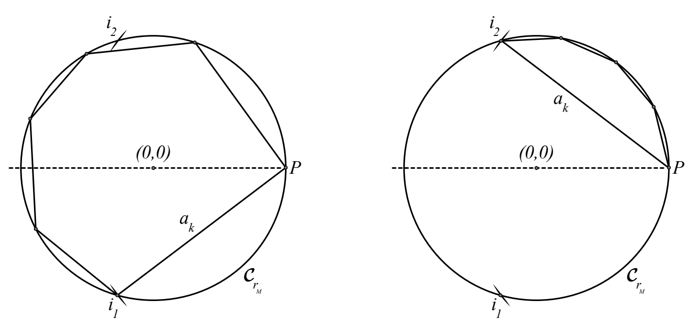


Figure 7. Generic construction of the shape with maximum area when F_1 has root (left) and when F_2 has root (right).

An implementation of this using javascript can be found in www.fenomec.unam.mx/ipolygons under "Build your own polygon".

Figure 8 shows the shapes corresponding to figures 5 and 6.

2. Note on oriented area

Now that we have the coordinates of the vertices, we could choose to compute the maximum area from the Shoelace formula ([7]), which is just an application

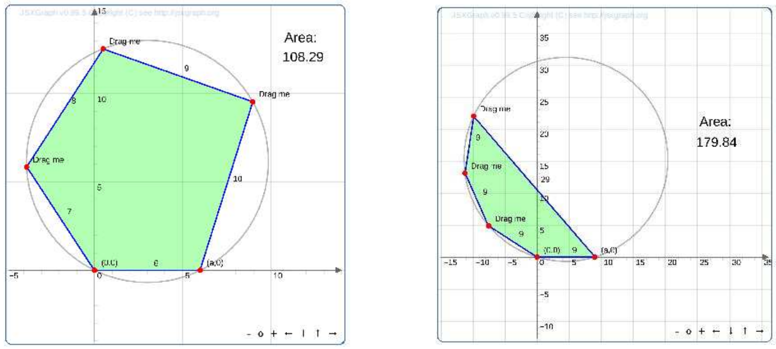


Figure 8. Shapes with maximum area for pentagon with sides 6,7,8,9,10 (left) and 9,9,9,9,29 (right).

of Green's Theorem. We already have the area from triangulation and Heron's formula but that is an unoriented area, it does not take into account the orientation of the sides. The Shoelace Formula does take orientation into account. For a polygon with vertices $A_0 = (x_0, y_0)$, $A_1 = (x_1, y_1)$, …, $A_n = (x_n, y_n)$ this formula can be written as

$$\text{Area} = \frac{1}{2}[(x_0y_1 + x_1y_2 + x_2y_3 + \cdots + x_ny_0) - (x_1y_0 + x_2y_1 + x_3y_2 + \cdots + x_0y_n)] \tag{6}$$

which is more easily remembered when written as in [7], shown in Figure (9).

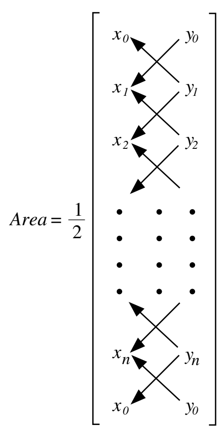


Figure 9. Shoelace formula

Arrows going up represent the products in the first bracket and arrows going down the products in the second bracket of (6). This gives a positive area if sides are taken in the positive direction, that is, counter-clockwise and negative otherwise.

We have actually used clockwise orientation as the positive one to be consistent with the data bases of the codices we created before.

The concept of oriented area is needed in the following sections.

3. Minima of irregular quadrilaterals

A quadrilateral is feasible if its prescribed area is between the maximum and the minimum possible values. To study minima of quadrilaterals one has to decide first whether to accept figures with self-intersections. If not, then the minimum will be attained by the smallest triangle into which the quadrilateral degenerates as the angles are moved (see Figure 10 for an example). This has been studied in [2] for any polygon. If self-intersections are admitted then one has to include the notion of signed area which was mentioned in the previous section.

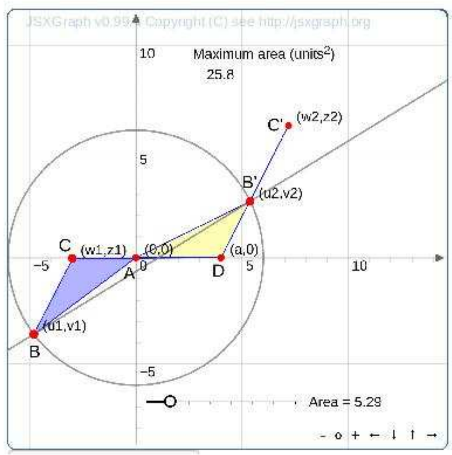


Figure 10. Two possible minima without self-intersection, $ABCD$ and $AB'C'D$, for quadrilateral with side-lengths 4,6,4,7. The side-lengths are the same as in Figure 2.

Decomposing the quadrilaterals in Figure 1 into two triangles we obtain

$$\mathcal{A} = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\gamma$$

which gives area $\triangle ABD +$ area $\triangle BCD$ if angle γ is less than π radians and area $\triangle ABD -$ area $\triangle BCD$ if γ is more than π radians because sine is then negative.

For quadrilaterals with self-intersections (see Figure 11) Bretschneider's Formula (equation (1)) yields the absolute value of the difference of the areas of the triangles ABE and EDC because triangle BED cancels out and because the formula computes the square root of the square of the area, which is the absolute value. We call this figure a quadrilateral with self-intersection or bow-tie because it arises by deforming a convex quadrilateral and its area is the *difference* of the areas of the two parts.

The minimum value of (2) is

$$\mathcal{A}B_{min} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd} \tag{7}$$

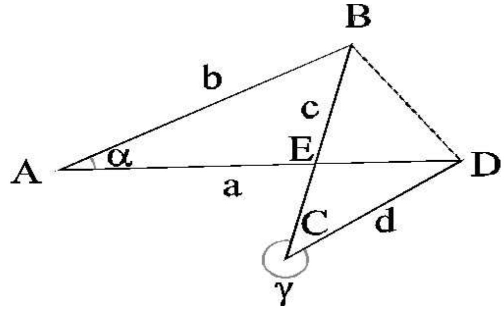


Figure 11. Quadrilateral with self-intersection or bow-tie ABCD.

but not all figures can achieve this theoretical minimum. If the expression inside the radical is negative, then $\mathcal{A}B_{min}$ is not defined over the reals and the minimum area, $\mathcal{A}_{min}$, is zero. When $\mathcal{A}B_{min}$ *is* achieved, $\alpha+\gamma = 2\pi$ hence $\cos^2(\frac{\alpha+\gamma}{2}) = 1$ and the self-intersecting quadrilateral is cyclic. See appendix C for the proof. In reference [4] it is said that minima are cyclic but this is not always the case. It is only true when $(s-a)(s-b)(s-c)(s-d)-abcd \geq 0$. If $(s-a)(s-b)(s-c)(s-d) - abcd\cos^2(\frac{\alpha+\gamma}{2})$ becomes zero for some $\alpha+\gamma < 2\pi$ then that configuration achieves the minimum area and it is NOT cyclic.

Minimum areas of quadrilaterals can also be studied using the construction given in [6]. A quadrilateral with given side-lengths and prescribed area $\mathcal{A}_c$ is obtained from the intersection of the circle and line from section 1. Let us analyze what happens when A_c goes from ∞ to 0 for a quadrilateral with fixed side-lengths. The slope of the line is $-P/Q$ where $Q = 4a\mathcal{A}_c$ hence the line is horizontal for infinite area and gradually becomes more inclined until it becomes vertical at area $\mathcal{A}_c = 0$, except when $P = 0$, in which case the line is always horizontal[1]. If the sides do not satisfy the compatibility condition and hence no quadrilateral exists then the line never intersects the circle. Otherwise, the line becomes tangent, giving the configuration with maximum area, then intersects the circle twice, giving the two possible shapes of the quadrilateral with the prescribed area and then two things can happen: the line can become vertical before leaving the circle in which case $\mathcal{A}B_{min}$ is not achieved and the minima are not cyclic (see Figure 12) or the line can again become tangent to the circle, leave, and then become vertical. This other point of tangency gives the cyclic configuration with area $\mathcal{A}B_{min}$ (see Figure 13.) We strongly recommend going to “Circle-line construction” in www.fenomec.unam.mx/ipolygons and playing with different side-lengths.

If $(s-a)(s-b)(s-c)(s-d) - abcd > 0$ then $\mathcal{A}_{min}$ is given by equation (7) otherwise, $\mathcal{A}_{min} = 0$. The coordinates of the vertices of the quadrilateral with minimum area are obtained by letting $\mathcal{A}_c = \mathcal{A}_{min}$ in the circle-line construction.

As far as we know there is no known formula for the minimum area of a polygon with more than four sides and self-intersections.

[1] If $a = c$ and $b = d$ then $P = 0$ and there is an infinite number of configurations with area 0.

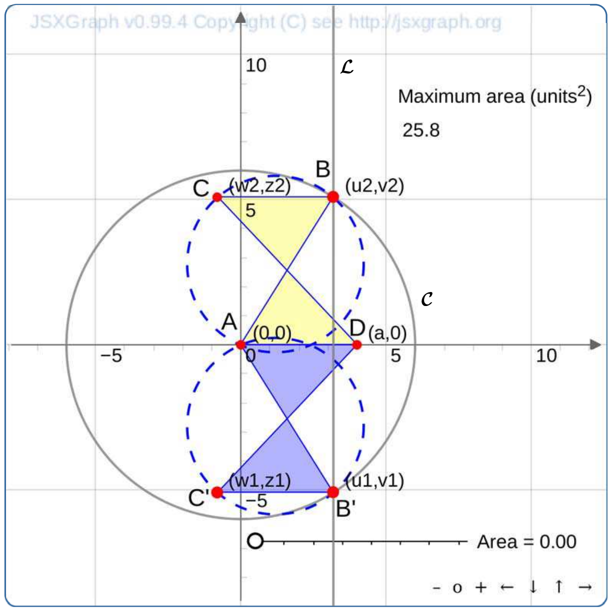


Figure 12. Two configurations with area zero of bow-tie with side-lengths 4,6,4,7. These minima are not cyclic.

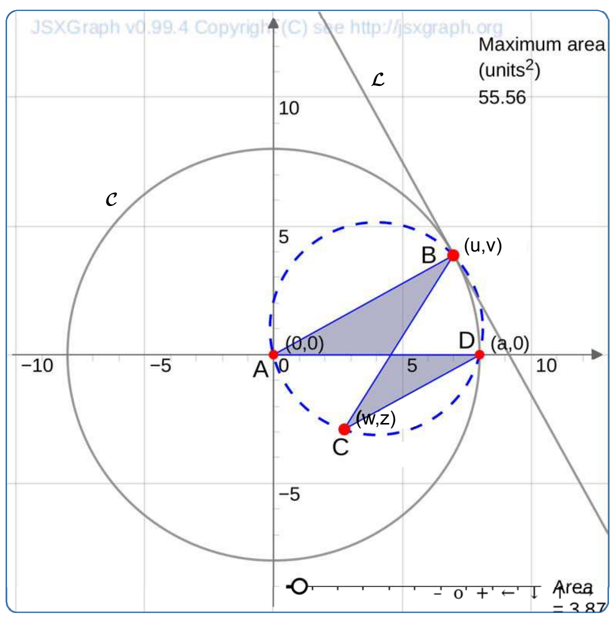


Figure 13. Bow-tie with side-lengths 8,8,8,6 and minimum area. This configuration is cyclic.

4. Polygonal shapes determined by angles and areas

In this section we consider a polygon with fixed side-lengths and study how its shape depends on the prescribed area.

4.1. *Case n=4.* Again the method from [6] described in Section 2 provides the coordinates of the vertices of the quadrilateral with any prescribed area between the maximum and the minimum. You can view an animation of this under "Circle-line construction (movie)" in www.fenomec.unam.mx/ipolygons.

Another way to compute the possible areas and shapes of a quadrilateral if coordinates are not needed is by using angle-area graphs. Referring to Figure 1, if angle α is given then by the cosine rule:

$$a^2 + b^2 - 2ab\cos\alpha = \overline{BD}^2 = c^2 + d^2 - 2cd\cos\gamma$$

and therefore

$$\gamma(\alpha) = \arccos\left(\frac{-a^2 - b^2 + c^2 + d^2 + 2ab\cos\alpha}{2cd}\right), \tag{8}$$

where the inverse cosine can be in the principal or secondary branch.

Renaming constants and using (8), (1) can be written as

$$\mathcal{A}(\alpha) = \sqrt{C_1 - C_2\cos^2\left(\frac{\alpha + \gamma(\alpha)}{2}\right)}. \tag{9}$$

where $C_1 = (s-a)(s-b)(s-c)(s-d)$ and $C_2 = abcd$.

To find the shape if side-lengths and area are given we can plot α *vs* area from equation (9) and read the values of the angle for a given area. We show examples of this in Figures 14 and 15, where we use Bretschneider, unoriented area.

The first hump of Figure 14 corresponds to α between 0 and π and the second to α between π and 2π. Points I, III, IV correspond to quadrilaterals in the upper half-plane; points II, V, VI to quadrilaterals in the lower half-plane. Points VII and VIII correspond to degenerate quadrilaterals. There are two possible configurations for any area between the minimum and the maximum in each half-plane or equivalently, two with clockwise orientation and two with counterclockwise orientation.

Figure 15 shows only values of α between 0 and π. Again each point on the angle-area graph corresponds to a different shape. Observe again that for a fixed orientation there is a unique shape with maximum and minimum area and two different shapes for areas in between. Also, there are two possible areas for each value of angle α; this is because angle γ can have two possible values depending on the branch of the inverse cosine.

If a more accurate value of α is required one can solve for α from equation (9) given a prescribed area $\mathcal{A}$ to obtain

$$\alpha + \gamma(\alpha) = 2\arccos\left(\sqrt{\frac{C_1 - \mathcal{A}^2}{C_2}}\right).$$

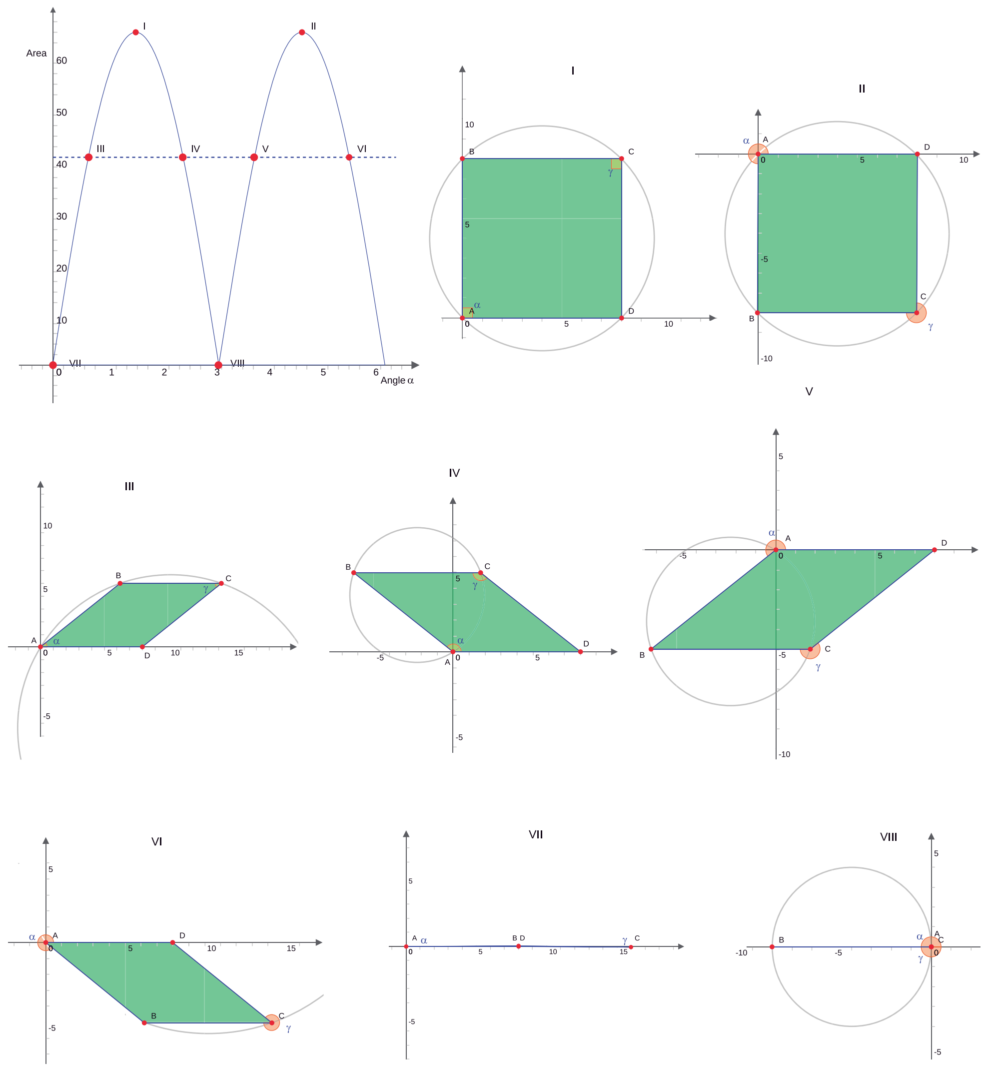


Figure 14. Angle-area graph and corresponding shapes for quadrilateral with side-lengths 8,8,8,8. Figures I and II correspond to the shapes with maximum area, I with sides in clockwise order and II with sides in counterclockwise order. Figures III to VI have the same area, III and VI in clockwise order, V and VI in counterclockwise order. Figures VII and VIII are degenerate cases with area 0.

and then use a Newton method to solve for α.

If we use oriented (Shoelace) area, angle-area graphs have a positive and a negative region. The negative region comes from changing the orientation of the sides. In some quadrilaterals it is possible to go from one orientation to the other by continuously deforming the shape, without reflecting. In others it is not.

This is best studied by identifying 0 and 2π and thus working on the cylinder. If the angle-area graph is disconnected on the cylinder then the two orientations are

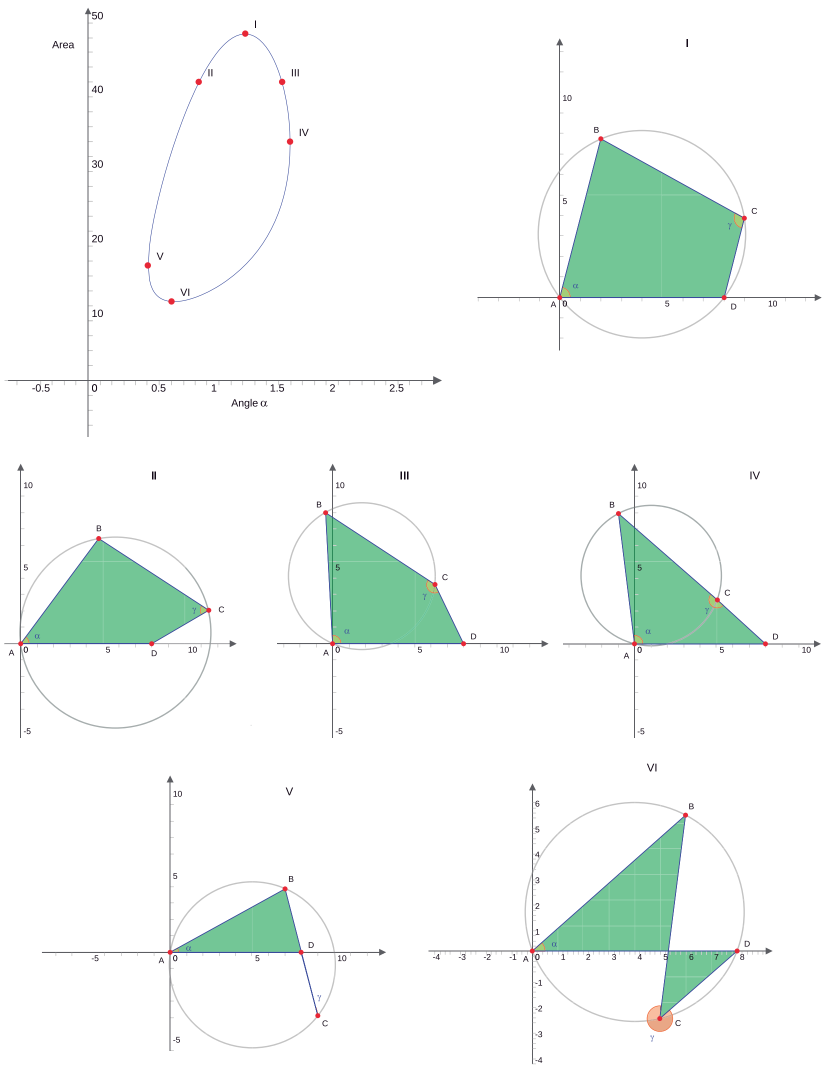


Figure 15. Angle-area graph and shapes for quadrilateral with side-lengths 8,8,8,4. Figure I show the configuration with maximum area; figure V a configuration with minimum area without self-intersection and figure VI the configuration with minimum area and self-intersection.

disjoint. Angle-area graphs on the cylinder provide a way of classifying quadrilaterals into four different classes depending on whether the graph consists of (i) one curve without self-intersection, (ii) one curve with self-intersection, (iii) two curves not homotopic to 0, (iv) two curves homotopic to 0. We show examples in Figure 16. In the left-most figure we used the same side-lengths as in Figure 14 but now the second hump has negative area. In this case it is possible to go from one orientation to the other by deforming continuously.

In the right-most figure we used the same side-lengths as in Figure 15. It is not possible to go from one orientation to the other by deforming continuously.

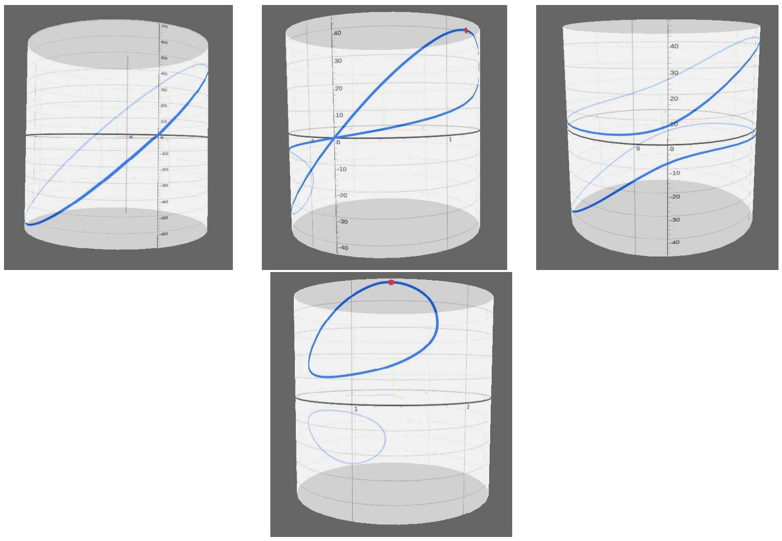


Figure 16. Angle-area graphs on the cylinder showing the four possible classes. From left to right side-lengths 8,8,8,8; 8,7,6,5; 8,5,8,6 and 8,8,8,4.

4.2. *Case n=5.* One can decompose a pentagon into a quadrilateral and a triangle and therefore there are two free angles plus the branches of the inverse cosine. The angle-area graph is now a surface and $\mathcal{A}_c = constant$ is a plane and therefore the curve of intersection between the two corresponds to an infinite number of configurations with the same side-lengths and the same area. In Figure 17 we show a portion of an angle-area graph located near the maximum area. It would be interesting to study complete angle-area graphs of pentagons.

4.3. *Other polygons.* For $n > 4$ there is an infinite number of configurations with any area between the minimum and the maximum. There is no formula for the shape but we wrote a program using JSXGraph that allows one to change the shape and compute the resulting area. We start with the configuration with maximum area, as computed numerically in Section 1. The user can then deform the shape and the area is computed from the Shoelace formula (go to "Build your own polygon" in www.fenomec.unam.mx/ipolygons). In Figure 18 we show an example of a polygon with ten sides in the configuration with maximum area and three other configurations with the same smaller area.

5. Conclusions

In this paper we study the possible areas and shapes of irregular polygons with prescribed side-lengths. We give a way to compute the maximum possible area of any feasible irregular polygon and draw the shape with that area. For quadrilaterals

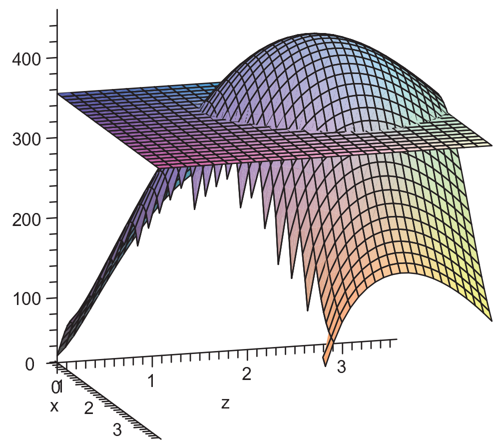


Figure 17. Angle-area graph for a pentagon with side-lengths 10,30,19,12,15 and plane area=355.

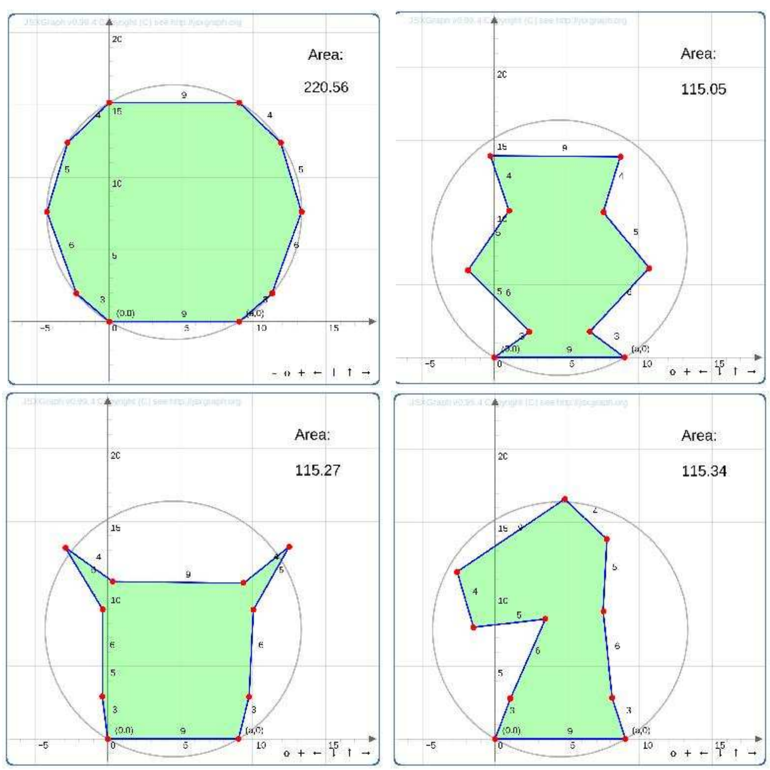


Figure 18. Shape with maximum area for side-lengths 9,3,6,5,4,9,4,5,6,3 units and some other possible shapes with area 115 units squared.

we use the Circle-Line Construction. For general polygons we find the circumcircle of the configuration with maximum area.

To study areas of quadrilaterals with self-intersections we use oriented area. We show that not all minima of quadrilaterals are cyclic and explain the fact both in terms of Bretschneider's Formula and the Circle-Line Construction.

The use of angle-area graphs on the cylinder led to a a classification of quadrilaterals into four different classes which are related to the possibility of reversing orientation by continuously deforming the quadrilaterals.

We provide a computer program in javascript which makes use of JSXGraph to draw the configuration of maximum area of a polygon with given sides. The figure can then be modified to attain any area between the maximum and the minimum.

These results have been useful to us in the study of the codices.

Appendix A: Circle-line construction

Consider Figure 19:

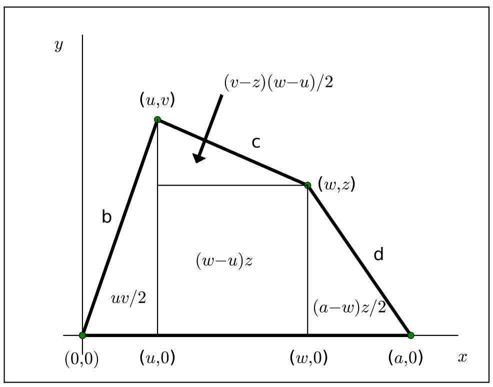


Figure 19. Coordinates of the quadrilateral and area in terms of (u, v) and (w, z) for Appendix A.

By Pythagoras' Theorem, the coordinates of the vertices of the quadrilateral satisfy

$$u^2 + v^2 = b^2, \quad (w-u)^2 + (v-z)^2 = c^2, \quad (a-w)^2 + z^2 = d^2. \tag{10}$$

The area, $\mathcal{A}_c$, of the quadrilateral using coordinates is calculated subdividing it in triangles and rectangles and is given by

$$2\mathcal{A}_c = vw + z(a-u). \tag{11}$$

Simplification of equations (10) and (11) give the unknown coordinates (w, z) in terms of (u, v) as

$$w = \frac{M(a-u) + 2\mathcal{A}_c v}{(a-u)^2 + v^2} \quad \text{and} \quad z = \frac{2\mathcal{A}_c(a-u) - Mv}{(a-u)^2 + v^2}, \tag{12}$$

where $2M = a^2 + c^2 - b^2 - d^2$. Substitution of (12) into (11) gives the coordinates (u, v) as the intersection of a line and a circle given respectively by

$$\mathcal{L} : Pu + Qv = R, \quad \mathcal{C} : u^2 + v^2 = b^2. \tag{13}$$

where

$$\begin{aligned} P &= 2a(a^2 - M - d^2) = a(a^2 + b^2 - c^2 - d^2),\ Q = 4\mathcal{A}_c a, \\ R &= (M - a^2)^2 + a^2 b^2 + 4\mathcal{A}_c^2 - d^2(a^2 + b^2). \end{aligned}$$

Therefore, for fixed side-lengths, the intersection of the line and the circle depends on $\mathcal{A}_c$. If it is too big for the given side-lengths there is no intersection; if it is the maximum possible area, $\mathcal{A}_{max}$ the line is tangent to the circle and if $\mathcal{A}_c$ is smaller then $\mathcal{L}$ intersects $\mathcal{C}$ twice giving two possible coordinates (u_1, v_1) and (u_2, v_2) and thus two different shapes for the given quadrilateral data.

The configuration with maximum area occurs when the line is tangent to the circle. The y coordinate of the intersection of the line and the circle in (13) is

$$v = \frac{R - Pu}{Q} = \sqrt{b^2 - u^2}.$$

This leads to the equation

$$u^2(P^2 + Q^2) + u(-2RP) + R^2 - Q^2 b^2 = 0.$$

The double root of this quadratic gives the $x-$coordinate of vertex B of the quadrilateral and the $y-$ coordinate is found from the equation of the line and therefore the coordinates (u, v) of vertex B are

$$u = \frac{RP}{Q^2 + P^2}, \qquad v = \frac{R}{Q} - \frac{RP^2}{Q(Q^2 + P^2)}, \tag{14}$$

where $\mathcal{A}_c = \mathcal{A}_{max}$. The coordinates (w, z) of vertex C are

$$w = \frac{M(a - u) + 2\mathcal{A}_{max} v}{a^2 + b^2 - 2au}, \;\; z = \frac{2A_{max}(a - u) - Mv}{a^2 + b^2 - 2au}. \tag{15}$$

Appendix B: Proof that either F_1 or F_2 have a root

Recall that

$$F_1(r) = \sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - 2\pi \tag{16}$$

and

$$F_2(r) \;\; = \;\; \sum_{j \neq k}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r^2}) - \text{Arccos}(1 - \frac{a_k^2}{2r^2}) \tag{17}$$

where a_k is the longest side and that r_l is the smallest positive r for which F_1 and F_2 are defined.

The derivative of F_1 is given by $\sum_j \frac{-2a_j}{r\sqrt{4r^2 - a_j^2}}$ and is therefore negative for all r. Since $\lim_{r\to\infty} F_1(r) = -2\pi$ then F_1 will have a root as long as $F_1(r_l) > 0$. In this case the center of the circumcircle is inside the polygon of maximum area.

If $F_1(r_l) < 0$ then we will show first that $F_2(r_l) < 0$ and then that F_2 has a root if the side-lengths satisfy the compatibility condition. Since $F_1(r_l) < 0$ then

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r_l^2}) < 2\pi$$

but

$$\sum_{j=1}^{n} \text{Arccos}(1 - \frac{a_j^2}{2r_l^2}) \quad = \quad F_2(r_l) + 2\text{Arccos}(1 - \frac{a_k^2}{2r_l^2})$$

therefore $F_2(r_l) < 2\pi - 2\text{Arccos}(1 - \frac{a_k^2}{2r_l^2}) = 0$, since $r_l = a_k/2$ and $\text{Arccos}(-1) = \pi$.

Using the asymptotic expansion in [1],

$$\arcsin(1-z) = \frac{\pi}{2} - (2z)^{1/2}[1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^{2m}(2m+1)m!} z^m],$$

since $\arccos(x) = \pi/2 - \arcsin(x)$, taking $z = a_j^2/2r^2$ and keeping only the first term in z we obtain

$$\lim_{r \to \infty} F_2 \simeq \sum_{j \neq k}^{n} \frac{a_j}{r} - \frac{a_k}{r}$$

which is positive if $a_k < \sum_{j \neq k} a_j$, therefore F_2 will have at least one root when the side-lengths satisfy the compatibility condition.

Appendix C: Proof that quadrilaterals with self-intersection are cyclic if an only if opposite angles add up to 2π radians

Referring to Figure 20 (left), if the quadrilateral is cyclic then since angles α and $2\pi - \gamma$ are subtended by the same arc BD they are equal and therefore $\alpha + \gamma = 2\pi$.

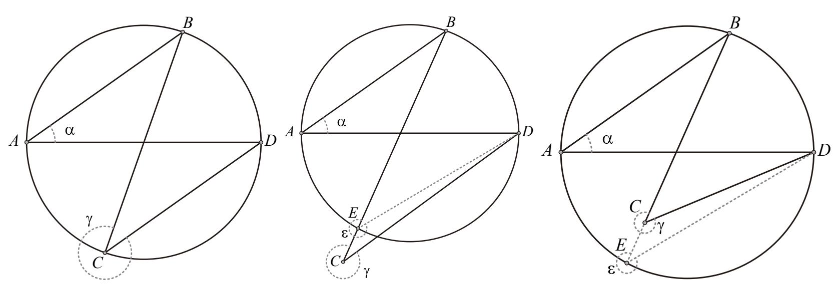


Figure 20. Cyclic bow-tie.

On the other hand, if opposite angles add up to 2π radians then we show by contradiction that the quadrilateral must be cyclic. Draw the circumcircle of ABD, see Figure 20 (middle and right). If vertex C is not on the circle then call E the intersection of the line BC with the circle. By hypothesis $\alpha + \gamma = 2\pi$ but $\alpha + \varepsilon$ also equals 2π because $ABED$ is cyclic. Therefore C has to coincide with E.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1970).

[2] Böröczky, K., The minimum area of a simple polygon with given side lengths, *Periodica Mathematica Hungarica*, 39, Numbers 1-3 (2000) 33–49.

[3] Codex Vergara, Gallica, gallica.bnf.fr searching for Codex Vergara or directly at gallica.bnf.fr/ark:/12148/btv1b84528032/f21.item.r=codex vergara

[4] H. Demir, Maximum area of a Region Bounded by a Closed Polygon with Given Sides, *Math. Mag.*, 39 (1966) 228–231.

[5] C. E. Garza-Hume, M. C. Jorge y Jorge, and A. Olvera, Quadrilaterals and Bretschneider's Formula, submitted to *Mathematics Teacher*.

[6] M. C. Jorge y Jorge, B. Williams, C. E. Garza-Hume, and A. Olvera, Mathematical accuracy of Aztec land surveys assessed from records in the Codex Vergara, *PNAS*, 108 (2011) no. 37.

[7] B. Polster, *The Shoelace Book*, AMS, 2006.

[8] D. P. Robbins Areas of Polygons Inscribed in a Circle, *Discrete Comput. Geom.*, 12 (1994) 223–236.

[9] B. J. Williams, and H. Harvey, *The Códice de Santa María Asunción: Households and Lands in Sixteenth Century Tepetlaoztoc*, Salt Lake City, USA. University of Utah Press (1997).

[10] B. J. Williams, and F. Hicks, *El Códice Vergara. Edición Facsimilar con Comentario: pintura indígena de casas, campos y organización social de Tepetlaoztoc a mediados del siglo XVI.* Mexico City, Mexico. Universidad Nacional Autónoma de México & ADABI de México A.C. (2011)

C. E. Garza-Hume: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `clara@mym.iimas.unam.mx`

Maricarmen C. Jorge: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `mcj@mym.iimas.unam.mx`

Arturo Olvera: IIMAS, UNAM, Circuito Escolar, CU, CDMX 04510, MEXICO
E-mail address: `aoc@mym.iimas.unam.mx`

Forum Geometricorum
Volume 18 (2018) 37–38.

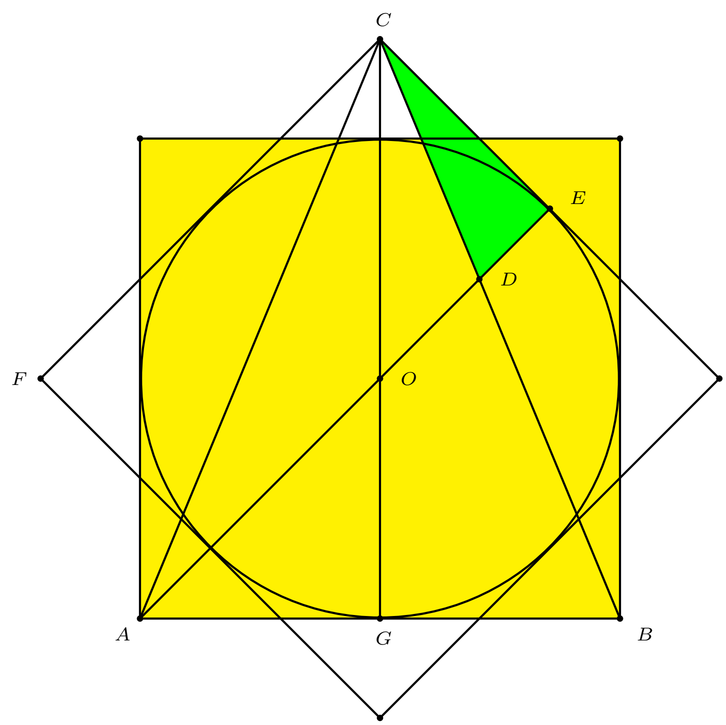
FORUM GEOM
ISSN 1534-1178

Irrationality of $\sqrt{2}$: Yet Another Visual Proof

Samuel G. Moreno and Esther M. García–Caballero

Abstract. Another visual proof of the irrationality of $\sqrt{2}$.

Two identical right triangles intersect as shown in the figure resulting in a smaller right triangle which happens to be, as is easy to check, similar to the original ones.



Since $\overline{AE}$ and $\overline{CF}$ are parallel line segments, $\angle EAC = \angle ACF$. By symmetry, $\angle ACF = \angle DCE$ so that triangles DCE and CAE are similar. Moreover, $\angle ADB = \angle CDE$, but by similarity $\angle CDE = \angle ACE$. Since $\angle ACE = \angle CBG = \angle DBA$, it follows that triangle ADB is isosceles.

The ratio of the lengths of the legs in the right triangles ACE and CDE is $(\overline{AO} + \overline{OE})/\overline{CE} = (\overline{OC} + \overline{OE})/\overline{OE} = \sqrt{2} + 1$. If $\sqrt{2}$ is rational, so is $\sqrt{2} + 1$, and thus $\overline{AE} = m$ and $\overline{CE} = n$ for some positive integers m, n. Since ABD is

Publication Date: January 17, 2018. Communicating Editor: Paul Yiu.

isosceles, then $\overline{DE} = \overline{AE} - \overline{AD} = \overline{AE} - \overline{AB} = \overline{AE} - 2\overline{CE} = m - 2n$. This process may be repeated indefinitely, triggering an infinite decreasing sequence of positive integers $m > n > m - 2n > 5n - 2m > \cdots$. But this is impossible. Thus, $\sqrt{2}$ cannot be rational.

Reference

[1] A. Bogomolny, Square root of two is irrational,
`http://www.cut-the-knot.org/ proofs/sq_root.shtml`.

Samuel G. Moreno: Departamento de Matemáticas, E.P.S. Linares, Campus Científico Tecnológico, Universidad de Jaén, Avenida Universidad s/n, 23700 - LINARES, SPAIN
E-mail address: `samuel@ujaen.es`

Esther M. García–Caballero: Departamento de Matemáticas, E.P.S. Linares, Campus Científico Tecnológico, Universidad de Jaén, Avenida Universidad s/n, 23700 - LINARES, SPAIN
E-mail address: `emgarcia@ujaen.es`

Forum Geometricorum
Volume 18 (2018) 39–42.

FORUM GEOM
ISSN 1534-1178

Two Hinged Regular n-sided Polygons

Manfred Pietsch

Abstract. We consider two regular polygons of the same type hinged at a common vertex C. If we further generate two regular polygons of the same type, each on a segment with one neighboring vertex of C from each polygon as endpoints, then these two new polygons must have a common center. The proof is based on two propositions about properties of equidiagonal quadrilaterals.

Given any two regular n-sided polygons hinged together at a common vertex C. Let A and B (respectively D and E) be the adjacent vertices of C in both polygons.

Theorem 1. *The regular n-sided polygons over the line segments BD and AE have a common center* (see Figure 1).

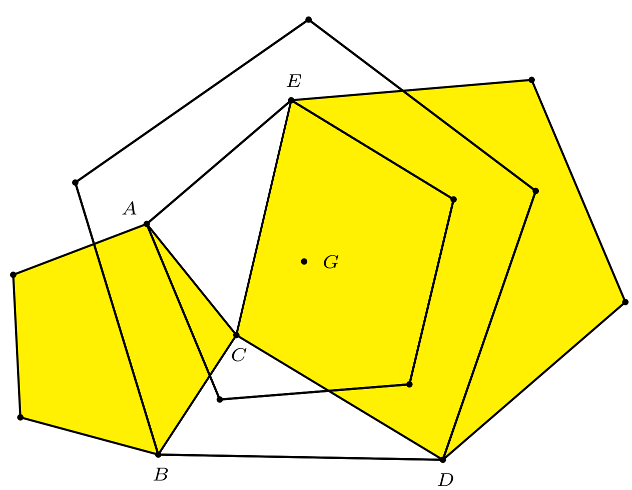


Figure 1.

Proposition 2. *Given two similar isosceles triangles ABC and DEC hinged at the vertex C,*

(1) $\overline{BE} = \overline{AD}$,

(2) *the circumferences of the triangles ABC and DEC concur at the intersection point of AD and BE* (see Figure 2).

Publication Date: January 24, 2018. Communicating Editor: Paul Yiu.

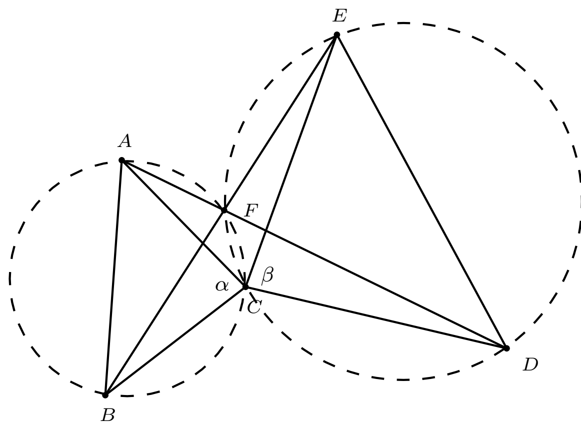


Figure 2.

Proof. If BE and AD intersect at C, the statements (1) and (2) are evident.

Let therefore BE and AD intersect at a point $F \neq C$.

(1) The triangles ACD and BCE are congruent:

- $AC = BC$ and $CD = CE$,
- $\angle DCA = \angle ECB$ as $\alpha = \beta$.

Thus, $BE = AD$.

(2) The congruence implies that the triangle BCE can be interpreted as the result of a rotation of triangle ACD around C through α. Therefore the angles between BE and AD must be identical to α: $\angle AFB = \angle DFE = \alpha (= \beta)$.

$\angle AFB = \alpha$ implies that C and F are located on the circumcircle of triangle ABC. Analogously, C and F are also on the circumcircle of triangle DEC. Thus F is located on both circumferences, which was to be proved. □

Proposition 3. *Given a quadrilateral $ABDE$ whose diagonals AD and BE are of equal length and intersect at a point F. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the circumcircles of triangles AFE and BDF respectively. The perpendicular bisectors of BD and AE intersect $\mathcal{K}_1$ and $\mathcal{K}_2$ at the same point* (see Figure 3).

Proof. If $\mathcal{K}_1$ and $\mathcal{K}_2$ touch each other at point F, then the above statement is evident.

Let therefore $\mathcal{K}_1$ and $\mathcal{K}_2$ intersect at F and another point G (see Figure 4).

The triangles GAD and GEB are congruent:

- $AD = BE$ (given),
- $\angle GBE = \angle GDA$ as both angles are peripheral angles of the chord FG on $\mathcal{K}_2$,
- analogously, $\angle BEG = \angle DAG$ (peripheral angles over FG on $\mathcal{K}_1$).

The congruence implies that $BG = DG$ and $EG = AG$, which was to be proved. □

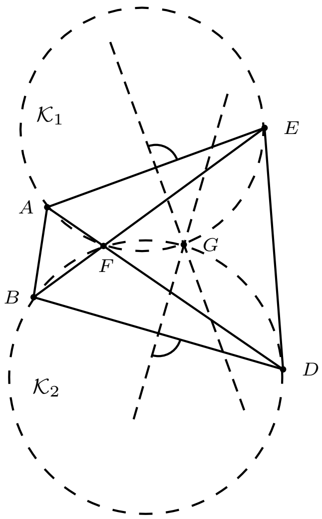


Figure 3

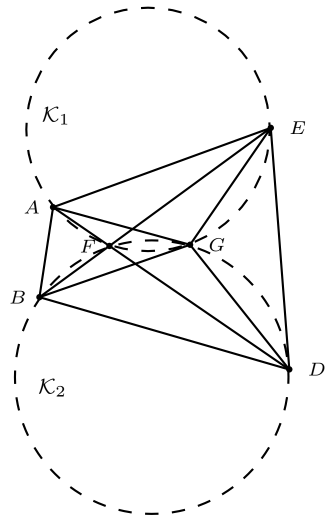


Figure 4

Proof of Theorem 1. In any two regular n-sided polygons hinged together at a common vertex C, the adjacent vertices of C in both polygons form with C two similar isosceles triangles ABC and DEC with $\angle ACB = \angle DCE$ (respectively $\alpha = \beta$); see Figure 5.

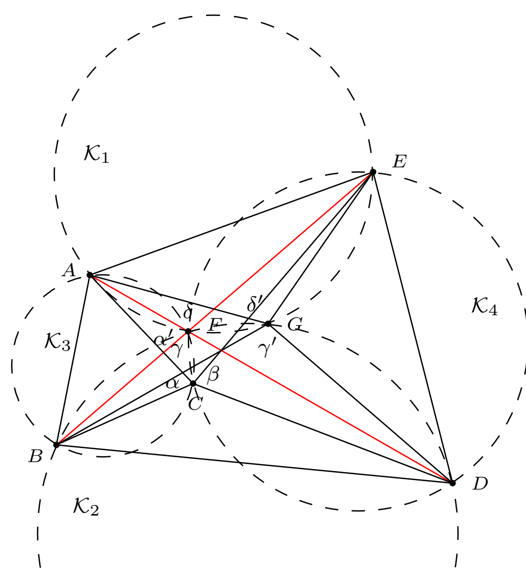


Figure 5.

According to Proposition 2 the quadrilateral $ABDE$ has two diagonals AD and BE with equal length and intersecting at a point F lying on both circumcircles of ABC and EDC. Thus the inscribed angles α and $\alpha'(= \angle AFB)$ must be equal. This implies $\gamma = 180° - \alpha = \delta$, wherein $\gamma = \angle BFD$ and $\delta = \angle EFA$.

As γ and γ' (respectively δ and δ') are inscribed angles in $\mathcal{K}_2$ (respectively $\mathcal{K}_1$), we obtain

$$\gamma' = 180° - \alpha = \delta' \qquad (*)$$

Proposition 3 guarantees that the triangles BDG and EAG are isosceles. Therefore, they are similar isosceles triangles.

Because of (*) the interior angles α (respectively β) of the original polygons fit to the central angles γ' (respectively δ') of the generated polygons. These must also be regular n-sided polygons and have the common center G. □

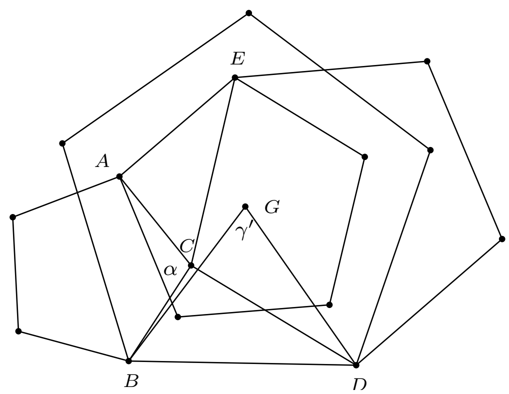


Figure 6.

Manfred Pietsch: Stiftisches Gymnasium, Altenteich 14, 52349 Düren, Germany
E-mail address: manfredpietsch@googlemail.com

Forum Geometricorum
Volume 18 (2018) 43–44.

FORUM GEOM
ISSN 1534-1178

A Remark on the Arbelos and the Regular Star Polygon

Hiroshi Okumura

Abstract. We give a condition that a regular star polygon $\{\frac{n}{2}\}$ can be constructed from an arbelos.

We consider to construct a regular star polygon $\{\frac{n}{2}\}$ from an arbelos. Let us consider an arbelos made by the three circles α, β, γ with diameters AO, BO, AB, respectively, for a point O on the segment AB. Circles of radius $ab/(a+b)$ are called Archimedean circles, where a and b are the radii of α and β, respectively. We call the perpendicular to AB at O the axis.

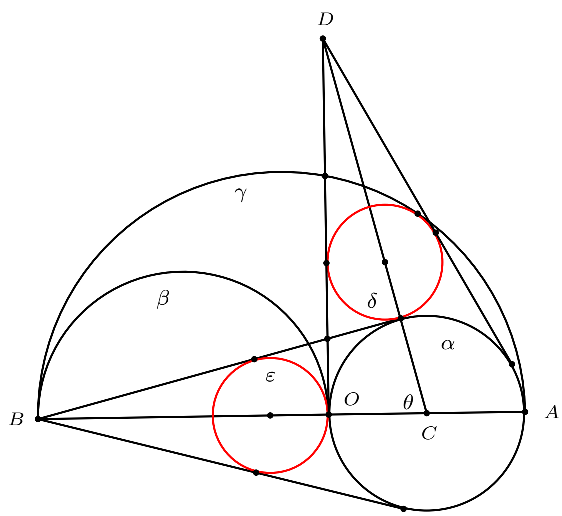


Figure 1.

The circle touching γ internally, α externally and the axis is Archimedean [1], which we describe by δ (see Figure 1). While the circle touching β internally and the tangents of α from B is also Archimedean [1], which is denoted by ε. Therefore the figure made by δ, α and their tangents are congruent to the figure made by ε, α and their tangents. We assume that C is the center of α, D is the external center of similitude of α and δ, and $\theta = \angle BCD$. The congruence of the two figures implies that we can construct a regular star polygon $\{\frac{n}{2}\}$ with center C and adjacent vertices B and D if $\theta = 2\pi/n$, while $\cos\theta = a/(a+2b)$. Therefore

Publication Date: January 24, 2018. Communicating Editor: Paul Yiu.

we can construct a regular star polygon $\{\frac{n}{2}\}$ *with center* C *and adjacent vertices* B *and* D *if and only if*

$$\frac{a}{a+2b} = \cos\frac{2\pi}{n}.$$

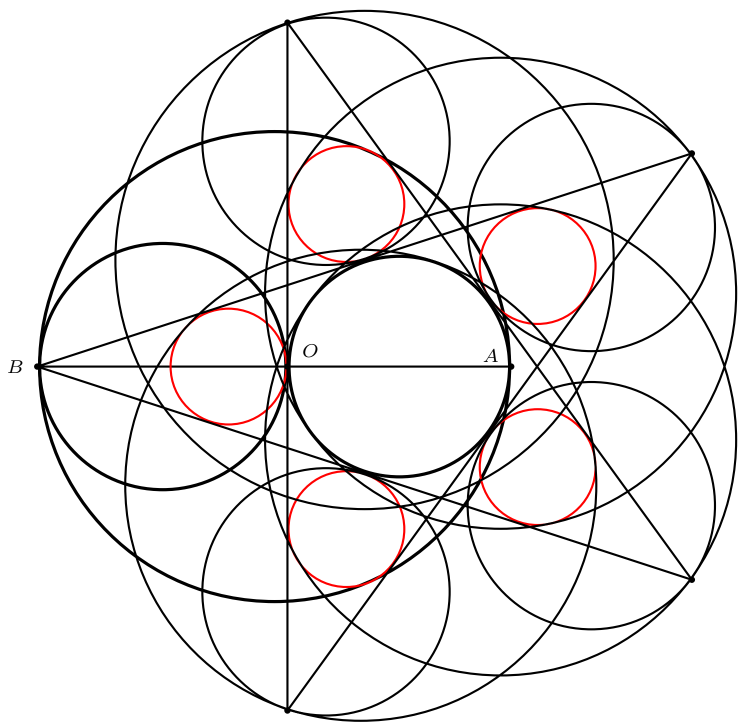


Figure 2. $\left\{\frac{5}{2}\right\}, \quad b = \sqrt{5}a/2$

Figure 2 shows the case $n = 5$. The distance between D and the point of contact of α and δ equals BO by the congruence. A problem stating this fact can be found in [2].

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

[2] T. Hermann, Twin segments in the arbelos, Solution to Problem 10895, *Amer. Math. Monthly*, 110 (2003) 63–64.

Hiroshi Okumura: Maebashi Gunma 371-0123, Japan
E-mail address: hokmr@protonmail.com

Forum Geometricorum
Volume 18 (2018) 45–46.

FORUM GEOM
ISSN 1534-1178

Revisiting the Infinite Surface Area of Gabriel's Horn

Lubomir P. Markov

Abstract. We show that the integral which gives the surface area of Gabriel's horn can be calculated in a simple way, thus eliminating the need for a comparison theorem to prove its divergence.

Gabriel's horn is defined as the solid obtained by revolving the region

$$\mathcal{R} = \left\{(x, y) :\ x \in [1,\ +\infty), 0 \leq y \leq \frac{1}{x}\right\}$$

about the x-axis. This object has been of enduring interest because is has the curious property of having finite volume, yet infinite surface area. A straightforward application of the disk method easily shows the horn's volume to be equal to π. (An interesting "wedding cake" version has been considered by Julian Fleron [2], the volume of which is $\frac{1}{6}\pi^3$). Regarding the surface area S of Gabriel's horn, one easily sees that it is given by the integral

$$S = 2\pi \int_1^\infty \frac{\sqrt{1+x^4}}{x^3} dx, \tag{1}$$

which needs to be proven divergent. The standard approach found in many books (see e.g. [1]) is to use comparison properties of improper integrals, and to show that

$$\int_1^\infty \frac{\sqrt{1+x^4}}{x^3} dx \geq \int_1^\infty \frac{1}{x} dx. \tag{2}$$

In his paper, Fleron mentions that the integral in (1) "cannot be evaluated readily", notes that it can be solved with a computer algebra system, and proceeds to perform the usual estimate (2). The purpose of our short note is to show that (1) can in fact be solved quite easily, thereby providing a direct evaluation of S:

Publication Date: January 24, 2018. Communicating Editor: Paul Yiu.

Take $2\displaystyle\int \frac{\sqrt{1+x^4}}{x^3}dx$ and make the substitution $x^2 = \tan\theta$. Then $2xdx = \sec^2\theta d\theta$, and the integral becomes

$$\begin{aligned}\int \frac{\sec\theta}{\tan^2\theta}\sec^2\theta d\theta &= \int \sec\theta\csc^2\theta d\theta \qquad \text{integration by parts}\\ &= -\sec\theta\cot\theta + \int \sec\theta\tan\theta\cot\theta d\theta\\ &= -\csc\theta + \ln|\sec\theta+\tan\theta| + C.\end{aligned}$$

It is now obvious that

$$2\pi\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}dx = \pi\left[-\csc\theta + \ln|\sec\theta+\tan\theta|\right]_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} = \infty. \tag{3}$$

Returning to the original variable gives the solution:

$$2\int \frac{\sqrt{1+x^4}}{x^3}dx = \ln\left[\sqrt{x^4+1}+x^2\right] - \frac{\sqrt{x^4+1}}{x^2} + C, \tag{4}$$

and of course we see again that $2\pi\displaystyle\int_1^\infty \frac{\sqrt{1+x^4}}{x^3}dx = \infty$. In addition, the solved integral allows for the calculation of the surface area of a finite piece of Gabriel's horn, defined on any $[a,\ b] \subset [1,\ \infty)$.

References

[1] E. Ellis and D. Gulick, *Calculus: One and Several Variables*, Harcourt Brace Jovanovich, Fort Worth, 1991.

[2] J. Fleron, Gabriel's wedding cake, *College Math. J.*, 30 (1999) 35–38.

Lubomir P. Markov: Department of Mathematics and CS, Barry University, 11300 NE Second Avenue, Miami Shores, Florida 33161 USA

E-mail address: `lmarkov@mail.barry.edu`

Forum Geometricorum
Volume 18 (2018) 47–55.

FORUM GEOM
ISSN 1534-1178

Integer Sequences and Circle Chains Inside a Circular Segment

Giovanni Lucca

Abstract. We derive the conditions for inscribing, inside a circular segment, a chain of mutually tangent circles having the property that the ratio between the largest circle radius and the radius of any other circle of the chain is an integer number.

1. Introduction

If we consider a straight line intersecting a circle $\mathcal{C}$, the two areas bounded by the common chord and one of the two arcs are named circular segments. Inside each one of the circular segments it is possible to inscribe infinite chains of mutually tangent circles which are also tangent to the outer circle and to the common chord; a generic example of chain is shown in Figure 1.

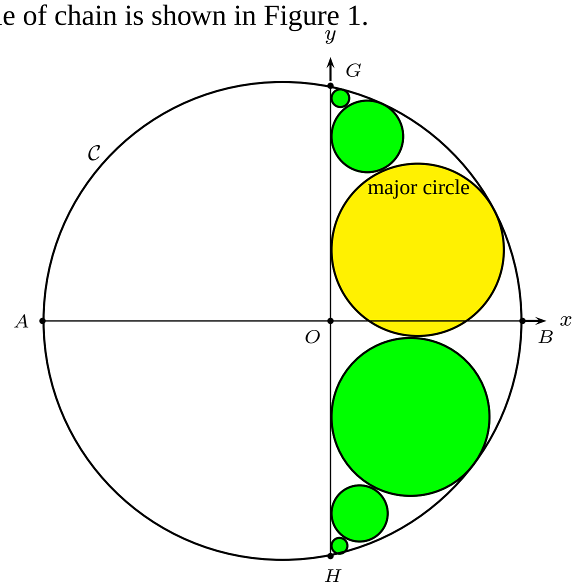


Figure 1. Example of a circle chain inscribed inside a circular segment

In a previous paper [2], we derived some geometrical properties and formulas related the circles forming the chain. Here, we follow a different approach, based on the inversion technique, and investigate about some connections that can be found between the circle chains and certain integer sequences.

Publication Date: January 26, 2018. Communicating Editor: Paul Yiu.

2. Some definitions and useful expressions

For the following, it is convenient to define the *major circle* in the chain (see Figure 1) as the one having the largest radius and label it by index 0; thus, we can subdivide a generic chain into two sub-chains: an up chain starting from the major circle and converging to point G and a down chain starting from the major circle and converging to point H.

Moreover, let $2(a+b)$ be the diameter of the outer circle and $2b$ the length of segment OB. By setting up a coordinate system with origin at O, we have A, B, G and H with coordinates $(-2a,0)$, $(2b,0)$, $(0,2\sqrt{ab})$ and $(0,-2\sqrt{ab})$ respectively.

In this paper, we want to investigate the conditions under which the ratios τ_k, $k=1,2,\dots$, between the radius of the major circle and the generic k-th circle are integers for both the up and down sub-chains. In other words, what are the conditions (provided they exist) for the radii of the circles of the chain to be submultiples of the major circle radius?

From [2], we report that the radius r_i and center coordinates $(X_i,\ Y_i)$ are related by the following formula:

$$r_i = X_i = -\frac{Y_i^2}{4a} + b, \qquad i = 0,\ \pm 1,\ \pm 2,\ \dots. \tag{1}$$

Note that, for the major circle ordinate Y_0, the following relation must hold:

$$-2a\left(-1+\sqrt{1+\frac{b}{a}}\right) \le Y_0 \le 2a\left(-1+\sqrt{1+\frac{b}{a}}\right). \tag{2}$$

In correspondence of two particular values for Y_0, we have two symmetrical dispositions for the up and down chains:

- if $Y_0 = 0$, we have that $r_0 = b$ and the major circle is bisected by the x-axis (central symmetry);
- if $Y_0 = \pm 2a\left(-1+\sqrt{1+\frac{b}{a}}\right)$, we have that $r_0 = 2a\left(-1+\sqrt{1+\frac{b}{a}}\right)$ and two equal major circles, (one for the up chain and one for the down chain), both tangent to x-axis, exist (bi-central symmetry).

3. Circle radii expressions by inversion

In [2], the expressions for the center coordinates (X_i, Y_i) and radius r_i of the generic i-th circle of the chain have been obtained in form of continued fractions. Here we want to get them in a closed form. According to a hint of F. J. García Capítan (in a personal communication), such a result can be obtained by means of the inversion technique.

In particular, we are interested in obtaining a closed form formula for the ratio between the major circle radius and the generic k-th circle radius.

To this aim, it is worthwhile to remark some points and to write some equations that shall be used in the following:

- Outer circle $\mathcal{C}$ equation:

$$x^2 + y^2 - 2(b-a)x - 4ab = 0 \tag{3}$$

- Inversion circle $\mathcal{C}_{\text{inv}}$ (center in H and radius HG) having equation:
$$x^2 + y^2 + 4\sqrt{ab}y - 12ab = 0 \tag{4}$$
and center coordinates $(X_{\text{inv}},\ Y_{\text{inv}})$ and radius R_{inv} given respectively by:
$$X_{\text{inv}} = 0, \quad Y_{\text{inv}} = -2\sqrt{ab}, \qquad R_{\text{inv}} = 4\sqrt{ab}. \tag{5}$$

- The inversive image of the y-axis with respect to $\mathcal{C}_{\text{inv}}$ is still the y-axis while the image of the outer circle $\mathcal{C}$ is the straight line s:
$$(a-b)x - 2\sqrt{ab}y + 4ab = 0. \tag{5}$$
The straight line s is the radical axis relevant to the outer circle C and to the inversion circle $\mathcal{C}_{\text{inv}}$ (see Figure 2). For convenience, in Figure 2 we have also added the bisector β of the angle between the y-axis and the straight line s.

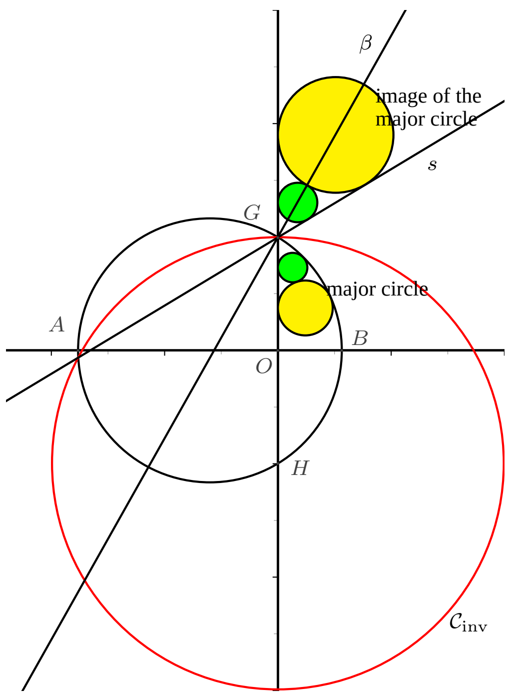


Figure 2. Some circles of the chain with their inversive images

- The cosine of the angle ψ between the y-axis and the straight line s is given by:
$$\cos\psi = \frac{a-b}{a+b} \tag{6}$$
and from (6) one can deduce:
$$\sin\frac{\psi}{2} = \sqrt{\frac{b}{a+b}}. \tag{7}$$

Now, let us consider, the major circle having center coordinates and radius given respectively by:

$$(X_0,\ Y_0) = \left(-\frac{Y_0^2}{4a} + b,\ Y_0\right), \tag{8a}$$

$$r_0 = -\frac{Y_0^2}{4a} + b. \tag{8b}$$

The inversive image of the major circle, by applying the formulas given in [4], is still a circle having center coordinates and radius given by:

$$(X_0',\ Y_0') = \left(\frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right),\ -2\sqrt{ab} + \frac{16ab}{Y_0 + 2\sqrt{ab}}\right), \tag{9a}$$

$$r_0' = \frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right). \tag{9b}$$

The inversive images of the circles inscribed in the circular segment are still mutually tangent circles that are also tangent to the y-axis and to the straight line s (see Figure 2); moreover, their centers lie on the bisector β. In Figure 3, we show three among these circles: the image of the major circle $\mathcal{C}_0'$, the image of the first circle of the up chain $\mathcal{C}_1'$ and the image of the first circle of the down chain $\mathcal{C}_{-1}'$.

In particular by looking at Figure 3, we can easily show, by similitude, that the ratio between the radius of the generic circles $\mathcal{C}_i'$ and the radius of the image of the major circle $\mathcal{C}_0'$ is:

$$\frac{r_k'}{r_0'} = \left(\frac{1 - \sin\frac{\psi}{2}}{1 + \sin\frac{\psi}{2}}\right)^i. \tag{10}$$

From formula (10) and by taking into account (7) and (9b), one has:

$$r_i' = X_i' = \left(\frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}\right)^i \frac{16ab}{(Y_0 + 2\sqrt{ab})^2}\left(-\frac{Y_0^2}{4a} + b\right), \tag{11a}$$

$$Y_i' = -\frac{a}{\sqrt{ab}}\left(\frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}\right)^i \frac{(Y_0 - 2\sqrt{ab})4b}{Y_0 + 2\sqrt{ab}} + 2\sqrt{ab}. \tag{11b}$$

From [4], one can write the following relation:

$$\frac{r_0}{r_i} = \frac{r_0}{r_i'}\left|\frac{(X_i' - X_{\mathrm{inv}})^2 + (Y_i' - Y_{\mathrm{inv}})^2 - (r_i')^2}{R_{\mathrm{inv}}}\right|. \tag{12}$$

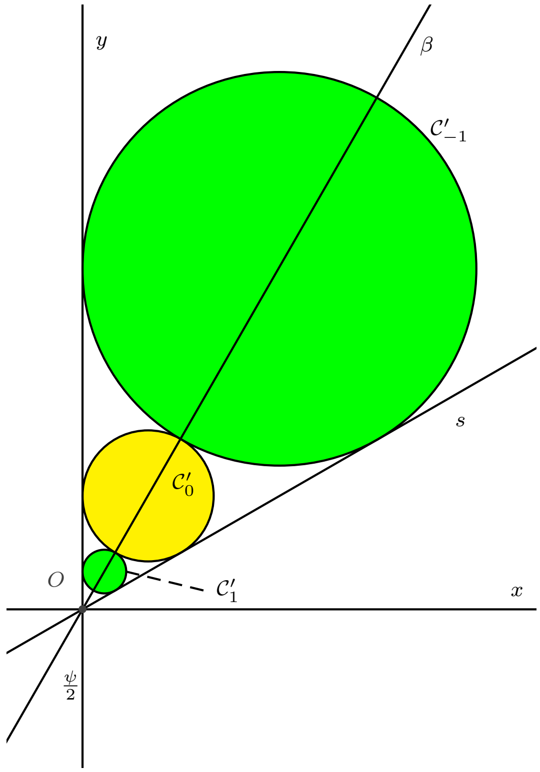


Figure 3. Inversive images of circles $\mathcal{C}_{-1}$, $\mathcal{C}_0$ and $\mathcal{C}_1$

Such a formula, by taking into account (4b), (11a) and (11b), and after some algebra, becomes:

$$\frac{r_0}{r_i} = \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^i \frac{(Y_0 - 2\sqrt{ab})^2}{16ab} + \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{-i} \frac{(Y_0 + 2\sqrt{ab})^2}{16ab} - \frac{2(Y_0^2 - 4ab)}{16ab} \tag{13a}$$

or equivalently,

$$\frac{r_0}{r_i} = \left(\left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{i/2} \frac{(Y_0 - 2\sqrt{ab})}{4\sqrt{ab}} - \left(\frac{1-\sqrt{\frac{b}{a+b}}}{1+\sqrt{\frac{b}{a+b}}}\right)^{-i/2} \frac{(Y_0 + 2\sqrt{ab})}{4\sqrt{ab}}\right)^2. \tag{13b}$$

Equation (13b) simply shows that the ratio r_0/r_i is always a square.
Equations (13a) or (13b) define the sequence $\{\tau_i\} = \{r_0/r_i\}$ for $i = 0, \pm1, \pm2, \ldots$
On this sequence we shall focus our attention in the following sections.

4. A recursive formula for $\{\tau_k\}$

If we look at (13a), we note that it is a Binet-like formula of the type:

$$\tau_i = A\omega^i + B\omega^{-i} + C \tag{14}$$

where, in our case, is:

$$\omega = \frac{1 - \sqrt{\frac{b}{a+b}}}{1 + \sqrt{\frac{b}{a+b}}}, \tag{15}$$

$$A = \frac{(Y_0 - 2\sqrt{ab})^2}{16ab}, \qquad B = \frac{(Y_0 + 2\sqrt{ab})^2}{16ab}, \qquad C = -\frac{2(Y_0^2 - 4ab)}{16ab}. \tag{16}$$

In [1], J. Kocik has demonstrated that a generic Binet-like relation can be expressed in an equivalent way by means of a non-homogeneous recursive relation of the type:

$$\tau_{i+2} = \left(\omega + \frac{1}{\omega}\right)\tau_{i+1} - \tau_i + C\left(2 - \omega - \frac{1}{\omega}\right) \tag{17}$$

that, by taking into account (15) and the expression for C in (16), becomes:

$$\tau_{i+2} = 2\left(1 + 2\frac{b}{a}\right)\tau_{i+1} - \tau_i + \left[\frac{1}{2}\left(\frac{Y_0}{a}\right)^2 - 2\frac{b}{a}\right]. \tag{18}$$

5. Conditions for $\{\tau_i\}$ to be an integer sequence

In the general case, the sequence $\{\tau_i\}$ is composed of real numbers. Here we want to find the conditions for which the sequence $\{\tau_i\}$ is entirely composed of integers. To this aim, it is convenient to introduce the following variables:

$$u = \frac{Y_0}{a}, \quad v = \frac{b}{a} \quad \text{with the condition } u \leq 2|\sqrt{1+v} - 1|. \tag{19}$$

The condition for the variable u comes from formula (2).

By considering the new variables u and v, equation (13a) becomes:

$$\tau_i = \left(\frac{1 - \sqrt{\frac{v}{1+v}}}{1 + \sqrt{\frac{v}{1+v}}}\right)^i \frac{(u - 2\sqrt{v})^2}{16v} + \left(\frac{1 - \sqrt{\frac{v}{1+v}}}{1 + \sqrt{\frac{v}{1+v}}}\right)^{-i} \frac{(u + 2\sqrt{v})^2}{16v} - \frac{2(u^2 - 4v)}{16v}. \tag{20}$$

From the general point of view, it is possible to impose, by means of (20), that the ratios τ_1 and τ_{-1} are equal to two given real numbers λ and μ ($\lambda > 1$ and $\mu > 1$) respectively; that yields the following system:

$$\begin{cases} \left(\frac{1-\sqrt{\frac{v}{1+v}}}{1+\sqrt{\frac{v}{1+v}}}\right) \frac{(u-2\sqrt{v})^2}{16v} + \left(\frac{1+\sqrt{\frac{v}{1+v}}}{1-\sqrt{\frac{v}{1+v}}}\right) \frac{(u+2\sqrt{v})^2}{16v} - \frac{2(u^2-4v)}{16v} = \lambda, \\ \left(\frac{1+\sqrt{\frac{v}{1+v}}}{1-\sqrt{\frac{v}{1+v}}}\right) \frac{(u-2\sqrt{v})^2}{16v} + \left(\frac{1-\sqrt{\frac{v}{1+v}}}{1+\sqrt{\frac{v}{1+v}}}\right) \frac{(u+2\sqrt{v})^2}{16v} - \frac{2(u^2-4v)}{16v} = \mu \end{cases} \tag{21}$$

having the solution

$$\begin{cases} u &= \sqrt{\lambda} - \sqrt{\mu}, \\ v &= \frac{(\sqrt{\lambda}+\sqrt{\mu})^2}{4} - 1. \end{cases} \tag{22}$$

Thus, if $\lambda = m$ and $\mu = n$, (m, n being integers), we have that by choosing a, b and Y_0 satisfying the relations:

$$\frac{Y_0}{a} = \sqrt{m} - \sqrt{n}, \qquad \frac{b}{a} = \frac{(\sqrt{m}+\sqrt{n})^2 - 4}{4}. \tag{23}$$

we also have that:

$$\tau_1 = m, \qquad \tau_{-1} = n. \tag{24}$$

Relations (23) only imply that τ_1 and τ_{-1} are integers, but do not imply that τ_i and τ_{-i}, $i = 2, 3, \ldots$ are also integers.

Nevertheless, if m and n are integers satisfying the following relation:

$$mn = K^2 \tag{25}$$

(that is their product is equal to a square integer) we have that the coefficients of the recursion (18) are always integer numbers and their expressions are:

$$\begin{aligned} 2\left(1 + 2\tfrac{b}{a}\right) &= -2 + m + 2K + n, \\ \tfrac{1}{2}\left(\tfrac{Y_0}{a}\right)^2 - 2\tfrac{b}{a} &= -2K + 2. \end{aligned} \tag{26}$$

Therefore, being $\tau_0 = 1$, $\tau_1 = m$, $\tau_{-1} = n$ and taking into account (26), we have, from (18) that τ_2 and τ_{-2} too are integers. Finally, due to the fact that the expressions in (26) represent integer numbers and being (18) a recursive relation, we deduce that τ_i is an integer for any index i, and conclude that (23) together (25) are sufficient conditions in order to be $\{\tau_i\}$ an integer sequence.

6. Symmetrical chains

Two particular cases relevant to conditions (23) and (25) are represented by:

(a) $m = n$.
(b) $m = 1$, $n = K^2$ or $m = K^2$, $n = 1$.

In case (a) one finds the chains with central symmetry while in case (b) the chains with bi-central symmetry.

In both cases the *up* and *down* sequences are equal and some of them are classified in OEIS (The *On Line Encyclopedia of Integer Sequences*) [3]. We reported them in Tables I and II.

Table I: sequences listed in OEIS and related to chains having central symmetry

Values of m and n	**Classification of the sequence according to** OEIS
$m = n = 2$	**A055997**
$m = n = 3$	**A171640**
$m = n = 4$	**A055793**
$m = n = 9$	**A055792**
$m = n = 10$	**A247335**

Table II: sequences listed in OEIS and related to chains having bi-central symmetry

Values of m and n	**Classification of the sequence according to** OEIS
$m = 1, n = 4$ **or** $n = 1, m = 4$	**A081068**
$m = 1, n = 25$ **or** $n = 1, m = 25$	**A008844**
$m = 1, n = 81$ **or** $n = 1, m = 81$	**A046172**
$m = 1, n = 169$ **or** $n = 1, m = 169$	**A006051**

7. Examples

Let us show some examples.

(a) Case of non symmetrical chains:

If we choose $m = 2$ and $n = 8$, we have from (23):

$$\frac{Y_0}{a} = -\sqrt{2}, \quad \frac{b}{a} = \frac{7}{2}.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\text{up}}\} = \ 1,\ 2,\ 25,\ 392,\ 6241,\ \ldots,$$
$$\{\tau_{\text{down}}\} = \ 1,\ 8,\ 121,\ 1922,\ 30625,\ \ldots.$$

(b) Case of chains with central symmetry:

If we choose $m = 2$ and $n = 2$, we have from (23):

$$\frac{Y_0}{a} = 0, \quad \frac{b}{a} = 1.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\text{up}}\} = \ \{\tau_{\text{down}}\} = 1,\ 2,\ 9,\ 50,\ 289,\ 1682,\ \ldots.$$

This is sequence **A055997** in OEIS.

(c) Case of chains with bi-central symmetry:

If we choose $m = 1$ and $n = 4$, we have from (23):

$$\frac{Y_0}{a} = -1, \quad \frac{b}{a} = \frac{5}{4}.$$

The generated integer sequences up and down are respectively:

$$\{\tau_{\text{up}}\} = \{\tau_{\text{down}}\} = 1,\ 4,\ 25,\ 169,\ 1156,\ \ldots.$$

This is sequence **A081068** in OEIS.

References

[1] J. Kocik, Lens sequences, `arXiv: 0710.3226v1[math.NT] 17 Oct. 2007`; `http://arxiv.org/pdf/0710.3226.pdf`

[2] G. Lucca, Circle chains inside a circular segment, *Forum Geom.*, 9 (2009) 173–179.

[3] N. J. A. Sloane (editor), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

[4] E. W. Weisstein, *Inversion*, from MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/Inversion.html`.

GIOVANNI LUCCA: VIA CORVI 20, 29122 PIACENZA, ITALY

E-mail address: vanni_lucca@inwind.it

Forum Geometricorum
Volume 18 (2018) 57–62.

FORUM GEOM
ISSN 1534-1178

On the Extrema of Some Distance Ratios

Michel Bataille

Abstract. In the plane, given a line ℓ and two distinct points A, B with B not on ℓ, we use inversion to study the extremal values of the ratio MA/MB when M varies on ℓ. The method also holds when an arbitrary curve $\mathcal{C}$ is substituted for ℓ, and we dwell upon the case when $\mathcal{C}$ is a circle.

1. Introduction

Fixing distinct points A, B not on a line ℓ, the construction of the points H, K of ℓ such that $HA/HB \le MA/MB \le KA/KB$ for all points M on ℓ has already received some attention. Two recent articles justify the same construction by using coordinates ([2]) or by a synthetic proof ([3]). Here we propose a totally different construction which readily leads to the exact values of HA/HB and KA/KB. In a second part, we extend the method, examining in particular the case when ℓ is replaced by a circle.

2. The construction

As in [2] and [3], we assume that AB and ℓ are not perpendicular; in contrast to B, the point A is allowed to be on ℓ. Let $\mathbf{I}$ denote the inversion in the circle with center B and radius BA. The line ℓ inverts into a circle Γ passing through B, and for any point M on ℓ, the point $M' = \mathbf{I}(M)$ is the point of intersection other than B of the line BM and Γ (see Figure 1).

As for distances, we have

$$MA = \frac{AB^2 \cdot M'A}{BM' \cdot BA} = AB \cdot M'A \cdot \frac{MB}{AB^2} = \frac{M'A \cdot MB}{AB}$$

(since $\mathbf{I}(A) = A$ and $BM \cdot BM' = BA^2$) and therefore

$$\frac{MA}{MB} = \frac{M'A}{AB}.$$

Thus, as M varies on ℓ, the ratio $\frac{MA}{MB}$ is extremal if and only if $M'A$ is and therefore $\frac{MA}{MB}$ attains its minimum (resp. maximum) at a unique point H (resp. K) of ℓ: if the diameter of Γ through A intersects Γ at H' and K' with $AH' < AK'$, the desired points H and K are $H = \mathbf{I}(H')$ and $K = \mathbf{I}(K')$.

In passing, note that the line $K'H'$ inverts into a circle $\mathbf{I}(K'H')$ through B, A, H, K; since $\ell = \mathbf{I}(\Gamma)$ and because Γ is cut orthogonally by its diameter $K'H'$, HK is a diameter of this circle $\mathbf{I}(K'H')$ passing though A and B. This result leads to the construction presented in [2] and [3].

Publication Date: February 5, 2018. Communicating Editor: Paul Yiu.

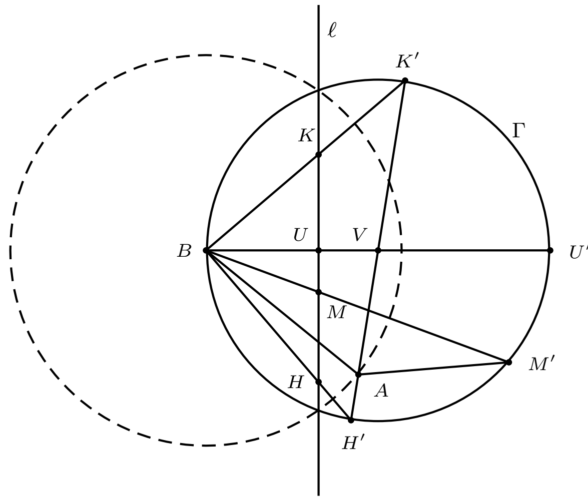


Figure 1.

3. The values of $\frac{HA}{HB}$ and $\frac{KA}{KB}$

We need the following preliminary result:

Lemma 1. *A is exterior or interior to Γ according as A and B are on the same side of ℓ or not.*

Proof. We introduce the orthogonal projection U of B onto ℓ and the midpoint V of the line segment BU' (where $U' = \mathbf{I}(U)$), that is, the center of Γ (Figure 1). With these notations, we have

$$4AV^2 + BU'^2 = 2(AB^2 + AU'^2), \qquad BU' = \frac{BA^2}{BU}, \qquad AU' = \frac{BA \cdot AU}{BU}$$

and a short calculation then gives

$$4AV^2 - 4BV^2 = \frac{2AB^2}{BU^2}(BU^2 + AU^2 - AB^2) = 4AB^2 \cdot \frac{AU}{BU}\cos(\angle AUB).$$

Thus, $AV < BV$ is equivalent to $\angle AUB > 90°$, and the announced property follows. □

For simplicity we set $\lambda = \frac{HA}{HB} = \min\{\frac{MA}{MB} : M \in \ell\}$ and $\mu = \frac{KA}{KB} = \max\{\frac{MA}{MB} : M \in \ell\}$ and now prove the following theorem:

Theorem 2. *If $d(X,\ell)$ denotes the perpendicular distance from the point X to the line ℓ, and $\varepsilon = +1$ if A, B are on the same side of ℓ, and $\varepsilon = -1$ if not, we have*

(i) $\lambda\mu = \frac{d(A,\ell)}{d(B,\ell)}$ *and* $\mu - \varepsilon\lambda = \frac{AB}{d(B,\ell)}$,

(ii) *the values of* λ *and* μ *are given by*

$$\lambda = \left| \frac{AB - \sqrt{AB^2 + 4\varepsilon d(A,\ell)d(B,\ell)}}{2d(B,\ell)} \right|,$$

$$\mu = \frac{AB + \sqrt{AB^2 + 4\varepsilon d(A,\ell)d(B,\ell)}}{2d(B,\ell)}.$$

Proof. Let r be the radius of Γ. First, suppose that A and B are on either side of ℓ (Figure 1). Then $AH' = \lambda AB$, $AK' = \mu AB$. Hence, A being interior to Γ, $(\lambda+\mu)AB = AH' + AK' = 2r$. Since $2r = BU' = \frac{BA^2}{BU} = \frac{AB^2}{d(B,\ell)}$, we obtain $\lambda + \mu = \frac{2r}{AB} = \frac{AB}{d(B,\ell)}$. Exchanging the roles of A and B, we see that $\lambda' = \min\{\frac{MB}{MA} : M \in \ell\} = \frac{1}{\mu}$ and $\mu' = \max\{\frac{MB}{MA} : M \in \ell\} = \frac{1}{\lambda}$ satisfy $\lambda' + \mu' = \frac{AB}{d(A,\ell)}$, that is, $\frac{\lambda+\mu}{\lambda\mu} = \frac{AB}{d(A,\ell)}$. The result (i) is readily deduced. Now, λ and μ are the roots of the quadratic equation $x^2 d(B,\ell) - xAB + d(A,\ell) = 0$ with $\lambda < \mu$ and (ii) follows.

Second, suppose that A and B are on the same side of ℓ (Figure 2). Then $AK' - AH' = 2r$ so that, as above, $\mu - \lambda = \frac{AB}{d(B,\ell)} = \lambda\mu\frac{AB}{d(A,\ell)}$. Thus (i) remains true. This time, $-\lambda$ and μ are the roots of $x^2 d(B,\ell) - xAB - d(A,\ell) = 0$ and (ii) holds.

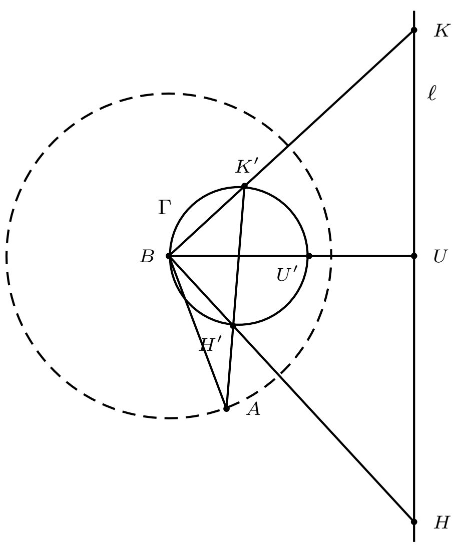


Figure 2

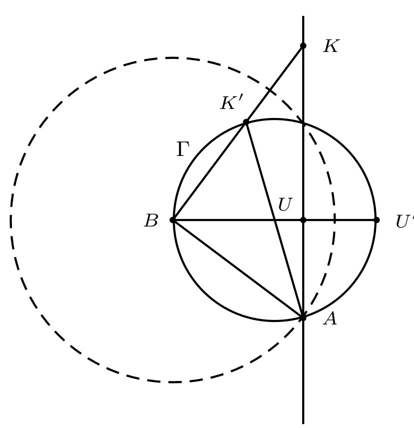


Figure 3

Lastly, if A is on ℓ, the formulas continue to hold: they give the expected values $\lambda = 0$ and $\mu = \frac{AB}{d(B,\ell)} = \frac{AB}{BU}$: indeed, because AK' is a diameter of Γ, we have $\angle ABK = 90^\circ$, hence BU is the altitude from B in the right-angled ABK and so $\frac{KA}{KB} = \frac{AB}{BU}$ (Figure 3). $\square$

4. An extension and a consequence

We quickly examine the discarded case when AB is perpendicular to ℓ. Of course, if A and B are symmetrical in ℓ, then $\frac{MA}{MB} = 1$ for all points M of ℓ. Otherwise, supposing for example that A, B are on the same side of ℓ with $AU < BU$, then $\frac{MA}{MB}$ has no maximum when M varies on ℓ, but $\sup\left\{\frac{MA}{MB} : M \in \ell\right\} = 1$ while

$$\frac{MA^2}{MB^2} = \frac{AU^2 + UM^2}{BU^2 + UM^2} = \frac{UA^2}{UB^2} + \frac{MU^2(UB^2 - UA^2)}{BU^2(BU^2 + UM^2)} \geq \frac{UA^2}{UB^2}$$

so that $\min\left\{\frac{MA}{MB} : M \in \ell\right\} = \frac{UA}{UB}$. The other cases can be treated in a similar way. It is readily checked that if we replace max by $\sup$ and $\min$ by $\inf$ in the definition of λ and μ, the results (i) and (ii) of the theorem continue to hold.

Taking this extension of the theorem into account, we can deduce a nice corollary about conics:

Corollary 3. *Let $\mathcal{K}$ be the conic with focus F, associated directrix ℓ and eccentricity e. For P on $\mathcal{K}$, let $\lambda(P) = \inf\left\{\frac{MF}{MP} : M \in \ell\right\}$ and $\mu(P) = \sup\left\{\frac{MF}{MP} : M \in \ell\right\}$. Then we have*

$$\lambda(P) = |\mu(P) - e|. \tag{1}$$

Proof. Since $PF = ed(P, \ell)$, part (i) of the theorem gives $\mu(P) - \lambda(P) = e$ if $\mathcal{K}$ is a parabola or an ellipse ($e \leq 1$).

If $\mathcal{K}$ is a hyperbola, then $\mu(P) - \lambda(P) = e$ or $\mu(P) + \lambda(P) = e$ depending on the branch of $\mathcal{K}$ containing P, and so $\lambda(P) = |\mu(P) - e|$. □

Note that relation (1) does not characterize the points of $\mathcal{K}$ ((1) remains valid for a point P of the reflection of $\mathcal{K}$ in ℓ).

5. Generalizing the method

In the plane, let $\mathcal{C}$ be any curve and A, B two fixed points with B not on $\mathcal{C}$. Let $\mathcal{C}' = \mathbf{I}(\mathcal{C})$ where $\mathbf{I}$ denotes the inversion in the circle with center B and radius BA. Mimicking the calculations of the second paragraph above, we readily obtain that for M on $\mathcal{C}$, we have $\frac{MA}{MB} = \frac{M'A}{AB}$ where $M' = \mathbf{I}(M)$. Thus, the ratio $\frac{MA}{MB}$ attains an extremum on $\mathcal{C}$ if and only if $M'A$ does on $\mathcal{C}'$.

This can be applied when $\mathcal{C}$ is a circle. In that case, the ratio $\frac{MA}{MB}$ is constant as M describes $\mathcal{C}$ if and only if A is the center of the circle $\mathbf{I}(\mathcal{C})$, that is, if and only if A, B are inverse in $\mathcal{C}$; otherwise the ratio $\frac{MA}{MB}$ attains its maximum (resp. minimum) at a unique point of the circle $\mathcal{C}$, which can be constructed with the help of the inverse of $\mathcal{C}$ in the same way as in the case of the line. The reader will fill in the details or find them in [1].

6. An alternative construction when $\mathcal{C}$ is a circle

It is worth mentioning a close but simpler construction that avoids the construction of $\mathbf{I}(\mathcal{C})$. We suppose that A, B are not inverse in the circle $\mathcal{C}$ and introduce the inversion $\mathbf{J}$ with center B leaving $\mathcal{C}$ globally invariant. Let $A' = \mathbf{J}(A)$. For any point M of $\mathcal{C}$, $M' = \mathbf{J}(M)$ is on $\mathcal{C}$ as well and

$$MA = \frac{|p| \cdot M'A'}{BM' \cdot BA'}$$

where p is the power of B with respect to $\mathcal{C}$ (Figures 4 and 5). Since $|p| = BM \cdot BM'$, we see that

$$\frac{MA}{MB} = \frac{A'M'}{A'B}. \tag{2}$$

If the diameter of $\mathcal{C}$ passing through A' intersects the circle in H' and K' with $A'H' < A'K'$, it follows that $H = \mathbf{J}(H')$ and $K = \mathbf{J}(K')$ satisfy

$$\frac{HA}{HB} = \min\left\{\frac{MA}{MB} : M \in \mathcal{C}\right\} \quad \text{and} \quad \frac{KA}{KB} = \max\left\{\frac{MA}{MB} : M \in \mathcal{C}\right\}.$$

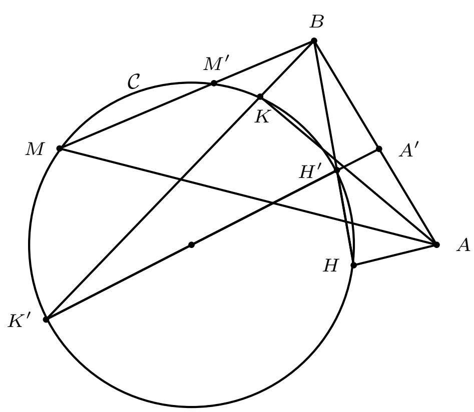


Figure 4

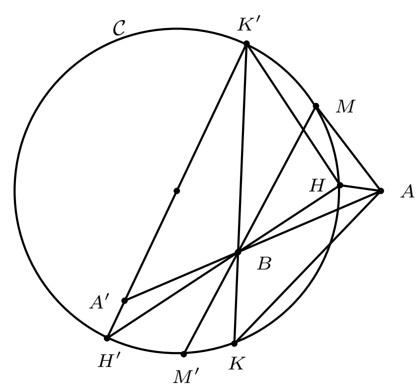


Figure 5

To conclude, we examine a nice application in the case when A is a fixed point on the circle $\mathcal{C}$ and B is not on $\mathcal{C}$ (Figure 6). Of course, we consider only the point K of $\mathcal{C}$ where $\frac{MA}{MB}$ attains its maximum. Let r be the radius of $\mathcal{C}$. From (2) we have

$$\frac{KA}{KB} = \frac{A'K'}{A'B} = \frac{2r}{A'B}.$$

Now, let k be a positive real number. Then, $\max\left\{\frac{MA}{MB} : M \in \mathcal{C}\right\} = k$ if and only if $A'B = \frac{2r}{k}$, that is, if and only if B belongs to the conchoid of $\mathcal{C}$ for pole A and constant $\frac{2r}{k}$ (a limaçon of Pascal). In the particular case $k = 1$, the limaçon is a cardioid (Figure 7) and we obtain the following, perhaps new, characterization of the cardioid:

Proposition 4. *Let $\mathcal{L}$ be the cardioid with cusp at A and fixed circle $\mathcal{C}$. Then a point B, not on $\mathcal{C}$, is on $\mathcal{L}$ if and only if* $\max\left\{\frac{MA}{MB} : M \in \mathcal{C}\right\} = 1$.

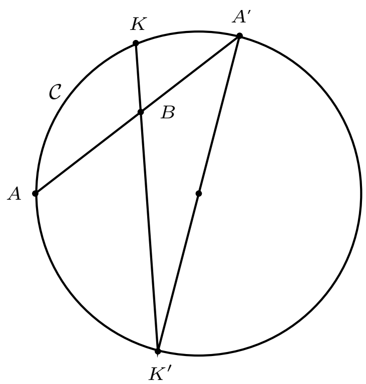


Figure 6

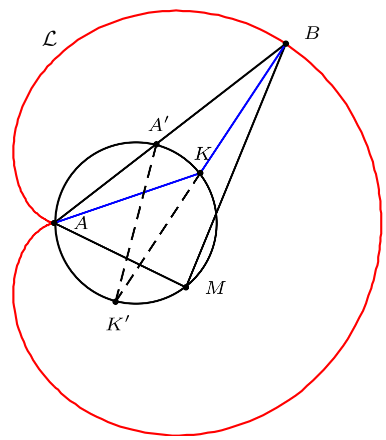


Figure 7

References

[1] M. Bataille, Problem 4084, *Crux Math.*, 42 (2016) 408.
[2] A. Bialostocki and R. Ely, Points on a line that maximize and minimize the ratio of the distances to two given points, *Forum Geom.*, 15 (2015) 177–178.
[3] G. Nicollier, Extremal distance ratios, *Math. Gazette*, 100 (2016) No 547, 129–130.

Michel Bataille: 6 square des Boulots, 76520 Franqueville-Saint-Pierre, France
E-mail address: `michelbataille@wanadoo.fr`

Forum Geometricorum
Volume 18 (2018) 63–70.

FORUM GEOM
ISSN 1534-1178

Twins of Hofstadter Elements

Apostolos Hadjidimos

Abstract. In this article the Hofstadter triangles, transversals, points and sectrices are considered and, based on these known elements, new ones closely related to Hofstadter's elements called "*twins*" are defined, analyzed and studied.

1. Introduction

We begin with a scalene acute triangle ΔABC. Let a, b, c be its sides opposite the vertices A, B, C and α, β, γ be its respective angles in radians, where without loss of generality we assume that $\alpha < \gamma < \beta \left(< \frac{\pi}{2}\right)$. Let us consider three pairs of straight semi-lines such that the elements of each pair have origins two of the vertices (poles) and each passes through the other vertex of ΔABC and both elements rotate about their poles in opposite directions (counterclockwise and clockwise, respectively) at constant rates analogous to the angles they sweep. The intersection points of the elements of the three pairs of semi-lines for any $x \in [0, 1]$ define the "*Hofstadter triangles*". It is known that the vertices of the Hofstadter triangles draw straight-lines called "*Hofstadter $x-$transversals*" and the vertices of Hofstadter triangle joined with the opposite vertices of ΔABC are concurrent at a point called "*Hofstadter point H_x*" [1]. One more characteristic of what we call $x-$sectrices (a term borrowed from "*Maclaurin's sectrices*" [2]) as x increases from 0 to 1 is that they pass through some well-known points of the triangle, like the vertices of the Morley triangle [3], the incenter, etc. In this article we prove that the orthogonal projections (simply projections from now on) of the vertices of the Hofstadter points H_x, for a certain $x \in [0, 1]$, onto the opposite sides form a "*twin*" Hofstadter triangle. The vertices of the latter triangle joined with the opposite vertices of ΔABC by straight-line segments, the "*twin*" Hofstadter $x-$transversals, are concurrent at a "*twin*" Hofstadter point O_x. More interesting results are found when the "*twin*" Hofstadter triangles, $x-$transversals and points are analyzed and studied in detail. Moreover, as x increases from 0 to 1 each twin Hofstadter point sweeps an arc of a curve called Hofstadter x-sectrix associated with the point considered.

Publication Date: February 5, 2018. Communicating Editor: Paul Yiu.
The author thanks his friend and colleague Dr. Michael Tzoumas for drawing the figures in the text.

2. Twin Hofstadter's triangles, transversals, points

Let the two straight semi-lines of the ordered pair $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ have origins the vertices B and C and pass through C and B, respectively, rotate about B counterclockwise and about C clockwise at constant rates proportional to β and γ, so that for any $x \in [0,1]$ the angles swept by them are βx and γx, respectively. (*Note:* The two letters in the subscript denote, for $x = 0$, the side of the triangle on which both elements of the pair lie, the first letter denotes the corresponding origin and pole, and the numbers 1 and 2 denote counterclockwise and clockwise rotation.) Let the two straight semi-lines of $(\mathcal{E}_{\mathrm{BC1}}, \mathcal{E}_{\mathrm{CB2}})$ intersect at the point A_x, a vertex of the Hofstadter triangle for $x \in [0,1]$ whose projection onto BC is A'_x. (*Note:* The points A_0 and A'_0 will be defined later.) Similarly, using a cyclic notation and properties we consider the other two pairs of straight semi-lines $(\mathcal{E}_{\mathrm{CA1}}, \mathcal{E}_{\mathrm{AC2}})$ and $(\mathcal{E}_{\mathrm{AB1}}, \mathcal{E}_{\mathrm{BA2}})$, their corresponding $x-$transversals as B_x and C_x sweep the interior of the triangle ΔABC and also their projections B'_x and C'_x onto the sides CA and AB, respectively.

The various definitions given so far enable us to determine new elements called "*twins*", and prove some properties associated with the particular Hofstadter triangles, the $x-$transversals and the Hofstadter points H_x. First, we present a known proposition for a Hofstadter triangle and then we state and prove a new one for its "*twin*" triangle.

Theorem 1 ([1]). *Under the notation and the assumptions so far let $\Delta A_xB_xC_x$ be the Hofstadter triangle corresponding to a certain $x \in [0,1]$. Then, the straight-line segments AA_x, BB_x, CC_x are concurrent at a point H_x whose trilinear coordinates are*

$$\begin{array}{c} \frac{\sin(x\alpha)}{\sin((1-x)\alpha)} : \frac{\sin(x\beta)}{\sin((1-x)\beta)} : \frac{\sin(x\gamma)}{\sin((1-x)\gamma)} \ \text{ for } \ x \in (0,1), \\ \frac{\alpha}{a} : \frac{\beta}{b} : \frac{\gamma}{c} \ \text{ for } \ x = 0, \\ \frac{a}{\alpha} : \frac{b}{\beta} : \frac{c}{\gamma} \ \text{ for } \ x = 1. \end{array} \tag{1}$$

Theorem 2. *Under the assumptions of Theorem 1, the vertices of the twin Hofstadter triangle $\Delta A'_xB'_xC'_x$, for a certain $x \in [0,1]$, joined with the corresponding vertices A, B, C of ΔABC define the concurrent straight-line segments (x-transverals) AA'_x, BB'_x, CC'_x. (Note: The point of concurrency is called the twin Hofstadter x-point O_x.)*

Proof. We begin with the proof of the statement for any $x \in (0,1]$. From the right triangles $\Delta BA_xA'_x$ and $\Delta CA_xA'_x$ for which $A_xA'_x$ is a common side we obtain that

$$(A_xA'_x) = (BA'_x)\tan(\beta x), \ \ (A_xA'_x) = (A'_xC)\tan(\gamma x). \tag{2}$$

Solving for (BA'_x) and (A'_xC) we immediately get that

$$a = (BA'_x) + (A'_xC) = \frac{(A_xA'_x)}{\tan(\beta x)} + \frac{(A_xA'_x)}{\tan(\gamma x)} \ \ \Rightarrow \ \ (A_xA'_x) = \frac{\sin(\beta x)\sin(\gamma x)}{\sin((\beta+\gamma)x)}a. \tag{3}$$

From the right equation of (3) and from each of (2) we find the two expressions

$$(BA'_x) = \frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a, \ \ (A'_xC) = \frac{\sin(\beta x)\cos(\gamma x)}{\sin((\beta+\gamma)x)}a. \tag{4}$$

Also, from the right equation of (3) as well as from their cyclic expressions, we can readily obtain that

$$(A_xA'_x) = \frac{\sin(\beta x)\sin(\gamma x)}{\sin((\beta+\gamma)x)}a, \ \ (B_xB'_x) = \frac{\sin(\gamma x)\sin(\alpha x)}{\sin((\gamma+\alpha)x)}b, \ \ (C_xC'_x) = \frac{\sin(\alpha x)\sin(\beta x)}{\sin((\alpha+\beta)x)}c. \tag{5}$$

Now, from the expressions in (4) and their cyclic ones we directly have that

$$\frac{(BA'_x)}{(A'_xC)}\cdot\frac{(CB'_x)}{(B'_xA)}\cdot\frac{(AC'_x)}{(C'_xB)} = \frac{\frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a}{\frac{\sin(\beta x)\cos(\gamma x)}{\sin((\beta+\gamma)x)}a}\cdot\frac{\frac{\sin(\alpha x)\cos(\gamma x)}{\sin((\gamma+\alpha)x)}b}{\frac{\sin(\gamma x)\cos(\alpha x)}{\sin((\gamma+\alpha)x)}b}\cdot\frac{\frac{\sin(\beta x)\cos(\alpha x)}{\sin((\alpha+\beta)x)}c}{\frac{\sin(\alpha x)\cos(\beta x)}{\sin((\alpha+\beta)x)}c} = 1. \tag{6}$$

By virtue of (6) and the Ceva Theorem [4], AA'_x, BB'_x, CC'_x are concurrent for any $x \in (0,1]$. (*Note:* For $x = 1$, the points A_1, B_1, C_1 are the three vertices A, B, C of ΔABC.) To find an analogous statement for $x = 0$ we find the following limits

$$\lim_{x\to 0^+} BA'_x =: BA_0, \ \ \lim_{x\to 0^+} A'_xC =: A_0C \tag{7}$$

The above limiting expressions are obtained if we take limits of the first expression in (4) in which case we have

$$\begin{aligned}
&\lim_{x\to 0^+}(BA'_x) = \lim_{x\to 0^+}\frac{\sin(\gamma x)\cos(\beta x)}{\sin((\beta+\gamma)x)}a \\
&= \frac{\gamma\cdot\lim_{x\to 0^+}\frac{\sin(\gamma x)}{\gamma x}\cdot\lim_{x\to 0^+}\cos(\beta x)}{\beta\cdot\lim_{x\to 0^+}\frac{\sin(\beta x)}{\beta x}\cdot\lim_{x\to 0^+}\cos(\gamma x)+\gamma\cdot\lim_{x\to 0^+}\frac{\sin(\gamma x)}{\gamma x}\cdot\lim_{x\to 0^+}\cos(\beta x)}\cdot a \\
&= \frac{\gamma\cdot 1\cdot 1}{\beta\cdot 1\cdot 1+\gamma\cdot 1\cdot 1}a = \frac{\gamma}{\beta+\gamma}a =: (BA'_0) \equiv (BA_0).
\end{aligned} \tag{8}$$

Similarly, we can obtain that $\lim_{x\to 0^+}(A'_xC) = \frac{\beta}{\beta+\gamma}a =: (A'_0C) \equiv (A_0C)$ and then, cyclicly, we can have the corresponding expressions for all the other limits of the relevant line segments. Therefore, applying again the Ceva Theorem we have that

$$\frac{(BA_0)}{(A_0C)}\cdot\frac{(CB_0)}{(B_0A)}\cdot\frac{(AC_0)}{(C_0B)} = \frac{\frac{\gamma}{\beta+\gamma}a}{\frac{\beta}{\beta+\gamma}a}\cdot\frac{\frac{\alpha}{\gamma+\alpha}b}{\frac{\gamma}{\gamma+\alpha}b}\cdot\frac{\frac{\beta}{\alpha+\beta}c}{\frac{\alpha}{\alpha+\beta}c} = 1, \tag{9}$$

and so the three line segments AA_0, BB_0, CC_0 are concurrent. □

Definition 1. *For any $x \in [0,1]$ the three line segments AA'_x, BB'_x, CC'_x are called twin Hofstadter $x-$transversals of AA_x, BB_x, CC_x, respectively, while the point O_x of their concurrency is called twin Hofstadter point of H_x.*

Theorem 3. *Let O_x, $x \in [0,1]$, be the point of concurrency of AA'_x, BB'_x, CC'_x of Theorem 2 and h_{BCx}, h_{CAx}, h_{ABx} be the altitudes of the triangles ΔO_xBC, ΔO_xCA, ΔO_xAB from their common vertex O_x to BC, CA, AB, respectively.*

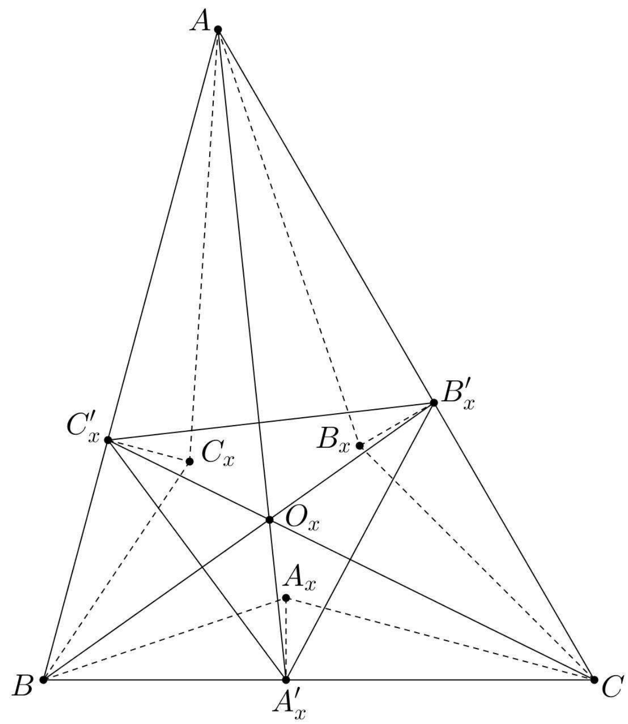


Figure 1. *Acute Case* ($\alpha = \frac{\pi}{4} < \gamma = \frac{\pi}{3} < \beta = \frac{5\pi}{6},\ x = \frac{1}{4}$)

Then, the trilinear coordinates of the twins O_x of the Hofstadter points H_x are given as follows:

$$\frac{\tan(\alpha x)}{a} : \frac{\tan(\beta x)}{b} : \frac{\tan(\gamma x)}{c} \quad \text{for } x \in (0,1),$$
$$\frac{\alpha}{a} : \frac{\beta}{b} : \frac{\gamma}{c} \quad \text{for } x = 0,$$
$$\frac{1}{\cos(\alpha)} : \frac{1}{\cos(\beta)} : \frac{1}{\cos(\gamma)} \text{ or } \frac{1}{a(b^2+c^2-a^2)} : \frac{1}{b(c^2+a^2-b^2)} : \frac{1}{c(a^2+b^2-c^2)} \quad \text{for } x = 1. \tag{10}$$

Proof. The pairs of triangles $\Delta ABA'_x$, $\Delta AA'_xC$ and $\Delta O_xBA'_x$, $\Delta O_xA'_xC$ have the same altitudes and bases the segments BA'_x and A'_xC, respectively. So, the ratios of their areas will be proportional to the lengths of their bases. In view of this we successively obtain

$$\frac{(BA'_x)}{(A'_xC)} = \frac{(ABA'_x)}{(AA'_xC)} = \frac{(O_xBA'_x)}{(O_xA'_xC)} = \frac{(ABA'_x) - (O_xBA'_x)}{(AA'_xC) - (O_xA'_xC)} = \frac{(O_xAB)}{(O_xCA)}. \tag{11}$$

By virtue of (4), the leftmost ratio above becomes $\frac{(BA'_x)}{(A'_xC)} = \frac{\tan(\gamma x)}{\tan(\beta x)}$ while the rightmost one in (11) equals $\frac{(O_xAB)}{(O_xCA)} = \frac{c\,h_{\mathrm{ABx}}}{b\,h_{\mathrm{CAx}}}$. From the two extreme ratios in (11) and their equivalent expressions just obtained we readily get $\frac{h_{\mathrm{ABx}}}{h_{\mathrm{CAx}}} = \frac{b\tan(\gamma x)}{c\tan(\beta x)}$. Considering also the other two cyclic expressions to the one just found we eventually have that

$$\frac{h_{\mathrm{ABx}}}{h_{\mathrm{CAx}}}=\frac{b\tan(\gamma x)}{c\tan(\beta x)},\;\;\frac{h_{\mathrm{CAx}}}{h_{\mathrm{BCx}}}=\frac{a\tan(\beta x)}{b\tan(\alpha x)},\;\;\frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}}=\frac{c\tan(\alpha x)}{a\tan(\gamma x)}. \tag{12}$$

From any two of the equalities in (12) we can obtain

$$\frac{h_{\mathrm{BCx}}}{\frac{\tan(\alpha x)}{a}}=\frac{h_{\mathrm{CAx}}}{\frac{\tan(\beta x)}{b}}=\frac{h_{\mathrm{ABx}}}{\frac{\tan(\gamma x)}{c}} \tag{13}$$

from which the first trilinear coordinates of $O_x\ \forall\, x\in(0,1)$ in (10) are immediately obtained. Note, however, that for $x=0$ one may use the limiting expression in relation (8), as $x\to 0^+$, as well as its cyclic ones, to obtain the second trilinear coordinates for O_0 in (8). For $x=1$ we begin with the denominator of h_{BCx} which is then $\frac{\sin(\alpha)}{a\cos(\alpha)}=\frac{1}{2R\cos(\alpha)}$, where R is the radius of the circumscribed circle to ΔABC. From the last expression and its cyclic ones we readily obtain the first set of relations in (8). To find expressions in terms of the sides of ΔABC we use the cosine formula $a^2=b^2+c^2-2bc\cos(\alpha)$ and its cyclic ones. Solving for $\cos(\alpha)$ we have $\cos(\alpha)=\frac{b^2+c^2-a^2}{2bc}$ from which the second set of expressions for $x=1$ in (8) are produced. □

Remark 1. It should be pointed out that the twin point O_0 and the Hofstadter point H_0 coincide, while the twin point O_1 of H_1 is the orthocenter of the triangle ΔABC.

In our analysis so far it has been assumed that the given triangle is acute. If ΔABC is a right triangle, with $\beta=\frac{\pi}{2}$, or an obtuse triangle, $\beta\in\left(\frac{\pi}{2},\pi\right)$, then some of the formulas of Theorems 2 and 3 and whatever is derived from them change.

(1) Right triangle:

- For $x\in[0,1)$ all formulas obtained in the proofs of Theorems 2 and 3 remain the same. However, as $x\to 1^-$ then

$$\lim_{x\to 1^-}(\beta x)=\frac{\pi}{2},\;\lim_{x\to 1^-}\sin(\beta x)=1,\;\lim_{x\to 1^-}\cos(\beta x)=0,\;\lim_{x\to 1^-}\tan(\beta x)=+\infty.$$

  So, formulas (2) and the first of (3) do not have any meaning as they stand. However, we observe that if in (3) we take limits as $x\to 1^-$ we very easily find that $(A_xA'_x)=\frac{\sin(\gamma)}{\sin(\alpha)}a=c$. Similarly, if we take limits in (4) as $x\to 1^-$ we obtain

$$\lim_{x\to 1^-}(BA'_x)=0\;\Leftrightarrow\;A'_x\equiv B,\;\;\lim_{x\to 1^-}(A'_xC)=(BC)=a.$$

- The applications of Ceva's Theorem in (6) and (9) remain unchanged for any $x\in[0,1)$. However for $x=1$, when $\cos(\alpha)$ becomes zero, first we simplify by $\cos(\beta x)$ as $x\to 1^-$ and then we take limits as $x\to 1^-$; hence the end result is the same as before. Note that the point of concurrency is the vertex B, which is the orthocenter of the triangle, as was expected.

- We also have $\lim_{x\to 0^+}(BA'_x) = \lim_{x\to 0^+}\tan(\gamma x)a = 0$ meaning that $A'_x \equiv B$ something which was expected since for $\lim_{x\to 0} A'_x \equiv A_x \equiv B$ as was proved before.
- Due to the fact that $\lim_{x\to 1^-}\tan(\beta x) = +\infty$ it implies that $h_{\mathrm{BC1}} = h_{\mathrm{AB1}} = 0$. However, from the last equality in (13) we have that

$$\lim_{x\to 1^-}\frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}} = \frac{c\tan(\alpha)}{a\tan(\gamma)} = \frac{c\sin(\alpha)\cos(\gamma)}{a\sin(\gamma)\cos(\alpha)} = \frac{2R\cos(\gamma)}{2R\cos(\alpha)} = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{c}$$

meaning that the asymptotic convergence rate of the ratio of the altitudes from O_x to the sides BC and AB, $\frac{h_{\mathrm{BCx}}}{h_{\mathrm{ABx}}}$, as $x \to 1^-$ is $\frac{a}{c}$.
- As $x \to 1^-$, the first set of trilinear coordinates can be written as

$$\frac{\sin(\alpha x)}{a}\cos(\beta x)\cos(\gamma x) : \frac{\sin(\beta x)}{b}\cos(\gamma x)\cos(\alpha x) : \frac{\sin(\gamma x)}{c}\cos(\alpha x)\cos(\beta x).$$

Then, taking limits as $x \to 1^-$, we observe that the two extreme coordinates become 0, due to $\lim_{x\to 1^-}\cos(\beta x) = \cos(\frac{\pi}{2}) = 0$, while the one in the middle becomes $\frac{1}{2R}\cos(\alpha)\cos(\gamma) = \frac{ca}{b^3}$. Multiplying all three coordinates found by b^2 we have that the trilinear coordinates in the third set of relations (10) become

$$0 : \frac{ac}{b} : 0 \quad \text{for} \quad x = 1. \tag{14}$$

Note that these trilinear coordinates give also the distances of the point $O_1 \equiv B$ from the three sides a, b, c of the right triangle ΔABC, respectively.

(2) Obtuse triangle:
- Recall that $\beta \in \left(\frac{\pi}{2}, \pi\right)$. Since $\alpha x < \gamma x < \beta x$, then for $\beta x \in (0, \frac{\pi}{2}) \Leftrightarrow x \in (0, \frac{\pi}{2\beta})$, everything will be the same as in the acute case.
- For $\beta x = \frac{\pi}{2} \Leftrightarrow x = \frac{\pi}{2\beta}$ the points $A_{\frac{\pi}{2\beta}}$ and $C_{\frac{\pi}{2\beta}}$ will be on the straight semi-lines $\mathcal{E}_{\mathrm{BC1}}$ and $\mathcal{E}_{\mathrm{BA2}}$ which will be perpendicular to the sides BC and AB, respectively. This means that the two points $A'_{\frac{\pi}{2\beta}}$ and $C'_{\frac{\pi}{2\beta}}$ will coincide with the vertex B. So, the three straight-line segments $AA'_{\frac{\pi}{2\beta}}$, $BB'_{\frac{\pi}{2\beta}}$, $CC'_{\frac{\pi}{2\beta}}$ will have as a point of concurrency the twin point $O_{\frac{\pi}{2\beta}}$ of the Hofstadter point $H_{\frac{\pi}{2\beta}}$. The various elements associated with $O_{\frac{\pi}{2\beta}}$ will be found as the limiting ones of the acute case as $x \to \left(\frac{\pi}{2\beta}\right)^-$.
- For $x \in \left(\frac{\pi}{2\beta}, 1\right)$, the angles $\beta x = \widehat{CBA_{\frac{\pi}{2\beta}}} = \widehat{ABC_{\frac{\pi}{2\beta}}} \in \left(\frac{\pi}{2}, \pi\right)$ will be obtuse meaning that the projections $A'_{\frac{\pi}{2\beta}}$ and $C'_{\frac{\pi}{2\beta}}$ of $A_{\frac{\pi}{2\beta}}$ and $C_{\frac{\pi}{2\beta}}$ will lie onto the extensions of CB and AB, respectively, and so strictly outside the triangle ΔABC. Then, all the expressions for $(BA'_x), \tan(\beta x), \cos(\beta x)$ are exactly the same as the ones in the acute case except that now all of them are negative.

The point O_x of concurrency of AA'_x, BB'_x, CC'_x will lie strictly outside the triangle and specifically in the vertically opposite angle of β.

- For $x = 1$ the various elements of interest are found from the corresponding ones in the previous item as $x \to 1^-$ and will be the orthocenter O_1 of the triangle ΔABC.

Remark 2. Summarizing, in the obtuse case there are three situations that one should have in mind. If $\beta x < \frac{\pi}{2}\,(x < \frac{\pi}{2\beta})$, then, all three points A'_x, B'_x, C'_x will lie strictly within the corresponding sides of the triangle ΔABC and Figure 2 will look like Figure 1. If $\beta x = \frac{\pi}{2}\,(x = \frac{\pi}{2\beta})$, then the points A'_x, C'_x and O_x will coincide with the vertex B. If $\pi > \beta x > \frac{\pi}{2}\,(\frac{\pi}{\beta} > x > \frac{\pi}{2\beta})$, then the points A'_x, C'_x will lie on the extensions of the sides CB and AB, respectively, while O_x will lie in the vertically opposite angle of β of the triangle ΔABC as all these are depicted in Figure 2.

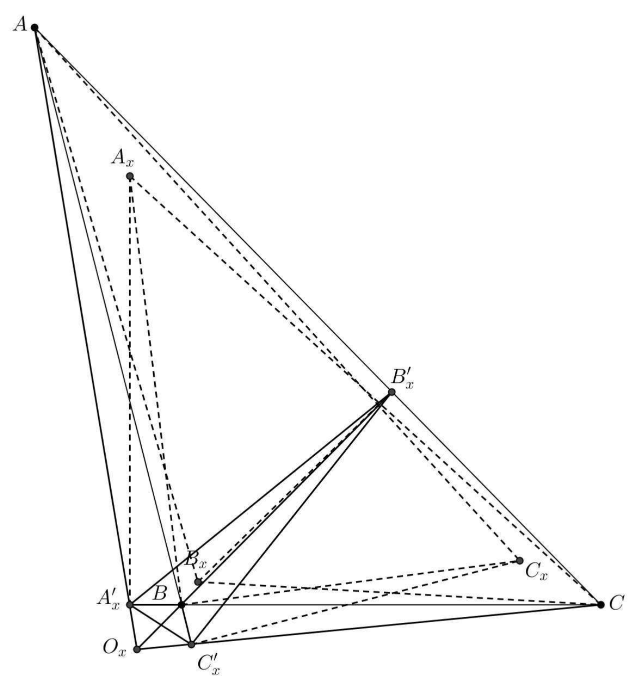


Figure 2. *Obtuse Case* ($\alpha = \frac{\pi}{6} < \gamma = \frac{\pi}{4} < \beta = \frac{5\pi}{8},\ x = \frac{13}{14}$)

Remark 3. The exhaustive investigation of all three cases of the scalene triangle ΔABC (acute, right, obtuse) concludes our detailed analysis of the twin Hofstadter elements (triangles, transversals, points, sectrices). The cases of an isosceles or an equilateral triangle are trivial and are, therefore, omitted.

References

[1] C. Kimberling, Hofstadter points, *Nieuw Archief voor Wiskunde*, 12 (1994) 109–114.

[2] C. Maclaurin, *Sectrix of Maclaurin;* https://en.wikipedia.org/wiki/Sectrix_of_Maclairin

[3] C. O. Oakley and J. C. Baker, The Morley trisector theorem, *Amer. Math. Monthly*, 85 (1978) 737–745.

[4] A. S. Posamentier and Ch. T. Salkind, *Challenging Problems in Geometry.* Dover Publishing Co., second revised edition, 1996.

Apostolos Hadjidimos: Department of Electrical & Computer Engineering, University of Thessaly, GR-382 21 Volos, Greece

E-mail address: hadjidim@gmail.com;hadjidim@e-ce.uth.gr

Forum Geometricorum
Volume 18 (2018) 71–77.

FORUM GEOM
ISSN 1534-1178

Primitive Heronian Triangles With Integer Inradius and Exradii

Li Zhou

Abstract. It is well known that primitive Pythagorean triangles have integer inradius and exradii. We investigate the generalization to prim- itive Heronian triangles. In particular, we study the special cases of isosceles triangles and triangles with sides in arithmetic progression. We also give two families of primitive Heronian triangles, one decomposable and one indecomposable, which have integer inradii and exradii. When realized as lattice triangles, these two families have incenters and excenters at lattice points as well. Finally we pose two problems for further research.

1. Introduction

A Pythagorean triangle ABC is a right triangle with three sides a, b, $c \in \mathbb{N}$. It is primitive if $\gcd(a, b, c) = 1$.

Now suppose that ABC is a primitive Pythagorean triangle with $a^2 + b^2 = c^2$. Since $\{0, 1\}$ is the complete set of quadratic residues modulo 4, a and b must have opposite parity and c must be odd. Hence $s = \frac{1}{2}(a + b + c) \in \mathbb{N}$ and its area $T = \frac{1}{2}ab \in \mathbb{N}$. Let r, r_a, r_b, r_c be its inradius and exradii opposite A, B, C, respectively. Then r, r_a, r_b, $r_c \in \mathbb{N}$ (see Figure 1).

Conversely, if a right triangle ABC has any three of r, r_a, r_b, r_c in $\mathbb{N}$, then

$$\begin{aligned} a =& \ r + r_a = r_c - r_b \in \mathbb{N}, \\ b =& \ r + r_b = r_c - r_a \in \mathbb{N}, \\ c =& \ r_a + r_b = r_c - r \in \mathbb{N}, \end{aligned}$$

so the triangle is Pythagorean.

This motivates us to consider the more general case of Heronian triangles. A Heronian triangle ABC is a triangle with integer sides and integer area, that is, a, b, c, $T \in \mathbb{N}$. It is primitive if $\gcd(a, b, c) = 1$. Furthermore, it is *indecomposable* if h_a, h_b, h_c are not in $\mathbb{N}$, where h_a, h_b, h_c are the altitudes on a, b, c, respectively. Otherwise, it is called decomposable. Finally, we denote by I, I_a, I_b, I_c the incenter and the excenters opposite A, B, C, respectively.

2. Isosceles triangles

First, we consider the special case of isosceles triangles. For examples:

- If $(a, b, c) = (5, 5, 6)$, then $(T, r, r_a, r_b, r_c) = \left(12, \frac{3}{2}, 4, 4, 6\right)$.
- If $(a, b, c) = (13, 13, 10)$, then $(T, r, r_a, r_b, rc) = \left(60, \frac{10}{3}, 12, 12, \frac{15}{2}\right)$.

We have the following theorem.

Publication Date: February 7, 2018. Communicating Editor: Paul Yiu.

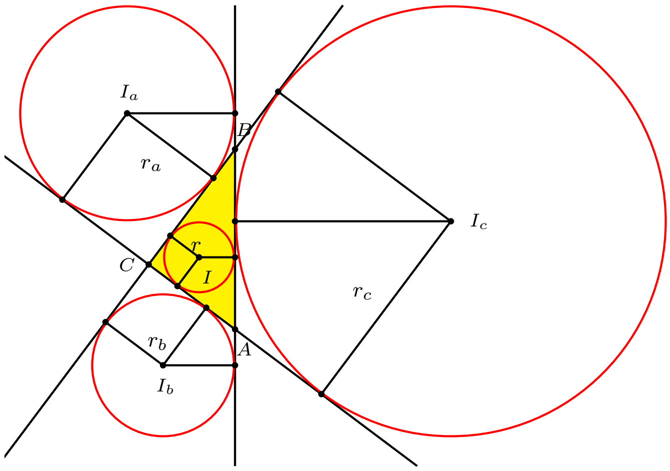


Figure 1. A right triangle ABC with its incircle and excircles

Theorem 1. *Suppose that ABC is a primitive Heronian triangle with $a = b$. Then $r_a = r_b \in \mathbb{N}$, but r and r_c cannot both be integers.*

Proof. Note that

$$s = a + \frac{c}{2}, \quad s - a = s - b = \frac{c}{2}, \quad s - c = a - \frac{c}{2},$$

so $T = \frac{c}{4}\sqrt{4a^2 - c^2}$, which implies that $4a^2 - c^2 = m^2$ for some $m \in \mathbb{N}$. Hence $-c^2 \equiv m^2 \pmod 4$, thus $c = 2d$ and $m = 2n$ for some $d, n \in \mathbb{N}$, with $\gcd(d, n) = 1$. Then $T = dn$ and $r_a = r_b = \frac{T}{s-a} = n$. Also,

$$r = \frac{T}{s} = \frac{dn}{a+d}, \qquad r_c = \frac{T}{s-c} = \frac{dn}{a-d}.$$

For contradiction, assume that $r, r_c \in \mathbb{N}$. Then

$$r_c - r = \frac{2d^2n}{a^2 - d^2} = \frac{2d^2}{n} \in \mathbb{N},$$

which forces $n \in \{1, 2\}$. Since $n^2 = (a + d)(a - d)$, neither $n = 1$ nor $n = 2$ can yield $d \in \mathbb{N}$. This contradiction completes the proof. $\square$

3. Sides in arithmetic progression

For another special case, we consider triangles with sides in arithmetic progression. For examples:

- If $(a, b, c) = (13, 14, 15)$, then $(T, r, r_a, r_b, r_c) = \left(84, 4, \frac{21}{2}, 12, 14\right)$.
- If $(a, b, c) = (51, 52, 53)$, then $(T, r, r_a, r_b, r_c) = \left(1170, 15, \frac{130}{3}, 45, \frac{234}{5}\right)$.

We have the following theorem.

Theorem 2. *Suppose that ABC is a primitive Heronian triangle with $d = b - a = c - b > 0$. Then $r, r_b \in \mathbb{N}$. But except for $(a, b, c) = (3, 4, 5)$, r_a and r_c cannot both be integers.*

Proof. Note that

$$s = \frac{3b}{2}, \quad s-a = \frac{b}{2}+d, \quad s-b = \frac{b}{2}, \quad s-c = \frac{b}{2}-d,$$

so $T = \frac{b}{4}\sqrt{3(b^2-4d^2)}$, which implies that $b^2-4d^2 = 3m^2$ for some $m \in \mathbb{N}$. Hence $b^2 \equiv 3m^2 \pmod 4$, thus $b = 2e$ and $m = 2n$ for some e, $n \in \mathbb{N}$, with $\gcd(e,n) = 1$. Then $T = 3en$, $r = \frac{T}{s} = n$, and $r_b = \frac{T}{s-b} = 3n$. Also,

$$r_a = \frac{T}{s-a} = \frac{3en}{e+d}, \quad r_c = \frac{T}{s-c} = \frac{3en}{e-d}.$$

Assume that $r_a,\ r_c \in \mathbb{N}$. Then

$$r_a + r_c = \frac{6e^2n}{e^2-d^2} = \frac{2e^2}{n} \in \mathbb{N},$$

which forces $n \in \{1,\ 2\}$.

If $n = 1$, then $3 = 3n^2 = (e+d)(e-d)$, so $e = 2$ and $d = 1$, that is, $(a,b,c) = (3,4,5)$. If $n = 2$, then $12 = (e+d)(e-d)$. Since $\gcd(e,d) = 1$, we cannot have $(e+d, e-d) = (6,2)$. The only remaining possibilities $(e+d, e-d) = (12,1)$ or $(4,3)$ cannot yield $d \in \mathbb{N}$. $\square$

4. Triangles with r, r_a, r_b, $r_c \in \mathbb{N}$

It is possible for a primitive Heronian (non-Pythagorean) triangle to have all r, r_a, r_b, $r_c \in \mathbb{N}$. For example, if $(a,b,c) = (7,15,20)$, then $(T,r,r_a,r_b,r_c) = (42,2,3,7,42)$. Note that this triangle has $h_a = \frac{2T}{a} = 12$, so is decomposable. We show that there are infinitely many such decomposable ones.

Theorem 3. *There are infinitely many primitive and decomposable Heronian (non-Pythagorean) triangles with r, r_a, r_b, $r_c \in \mathbb{N}$.*

Proof. For $n > 1$, let

$$\begin{aligned} a &= 4n^2, \\ b &= 4n^3 - 2n^2 + 1 = (2n+1)(2n^2-2n+1), \\ c &= 4n^3 + 2n^2 - 1 = (2n-1)(2n^2+2n+1). \end{aligned}$$

Since b is odd and $a+b-c = 2$, the triangles are primitive for all $n \geq 1$. Also, from $c+2 = a+b$ we get

$$c^2 - a^2 - b^2 = 2(ab - 2c - 2) = 2(ab - 2a - 2b + 2) \geq 0,$$

with equality if and only if $n = 1$. Therefore, for all $n > 1$, the triangles are obtuse and thus non-Pythagorean. Now,

$$\begin{aligned} s &= 4n^3 + 2n^2 = 2n^2(2n+1), \\ s-a &= 4n^3 - 2n^2 = 2n^2(2n-1), \\ s-b &= 4n^2 - 1 = (2n-1)(2n+1), \\ s-c &= 1; \\ T &= 2n^2(2n-1)(2n+1); \\ h_a &= \frac{2T}{a} = (2n-1)(2n+1); \\ r &= \frac{T}{s} = 2n-1, \\ r_a &= \frac{T}{s-a} = 2n+1, \\ r_b &= \frac{T}{s-b} = 2n^2, \\ r_c &= \frac{T}{s-c} = 2n^2(2n-1)(2n+1) = T, \end{aligned}$$

completing the proof. □

Notice that $n = 1$ yields the Pythagorean $(a, b, c) = (4, 3, 5)$ and $n = 2$ yields $(a, b, c) = (16, 25, 39)$.

This naturally leads to the question of whether there are such indecomposable triangles.

Theorem 4. *There are infinitely many primitive and indecomposable Heronian* (*non-Pythagorean*) *triangleswith* r, r_a, r_b, $r_c \in \mathbb{N}$.

Proof. For $n > 1$, let

$$\begin{aligned} a &= 25n^2 + 5n - 5 = 5(5n^2 + n - 1), \\ b &= 25n^3 - 5n^2 - 7n + 3 = (5n+3)(5n^2 - 4n + 1), \\ c &= 25n^3 + 20n^2 - 2n - 4 = (5n-2)(5n^2 + 6n + 2). \end{aligned}$$

Since a is odd and $a+b-c=2$, the triangles are primitive. Similarly, $c+2=a+b$ implies that $c^2-a^2-b^2>0$. Moreover,

$$\begin{aligned}
s =&\ 25n^3+20n^2-2n-3=(5n+3)(5n^2+n-1),\\
s-a =&\ 25n^3-5n^2-7n+2=(5n-2)(5n^2+n-1),\\
s-b =&\ 25n^2+5n-6=(5n-2)(5n+3),\\
s-c =&\ 1;\\
T =&\ (5n-2)(5n+3)(5n^2+n-1);\\
r =&\ \frac{T}{s}=5n-2,\\
r_a =&\ \frac{T}{s-a}=5n+3,\\
r_b =&\ \frac{T}{s-b}=5n^2+n-1,\\
r_c =&\ \frac{T}{s-c}=(5n-2)(5n+3)(5n^2+n-1)=T.
\end{aligned}$$

Finally,

$$\begin{aligned}
h_a =&\ \frac{2T}{a}=\frac{2(5n-2)(5n+3)}{5}\notin\mathbb{N},\\
h_b =&\ \frac{2T}{b}=\frac{2(5n-2)(5n^2+n-1)}{5n^2-4n+1}=10n+6-\frac{2}{5n^2-4n+1}\notin\mathbb{N},\\
h_c =&\ \frac{2T}{c}=\frac{2(5n+3)(5n^2+n-1)}{5n^2+6n+2}=10n-4+\frac{2}{5n^2+6n+2}\notin\mathbb{N}.
\end{aligned}$$

□

The beginning case of $n=2$ yields $(a,b,c)=(105,169,272)$.

5. Embedding as lattice triangles

In [1], Paul Yiu discovered and proved that all Heronian triangles are lattice triangles. That is, they can be embedded so that the coordinates of the three vertices are all integers. Now we wonder whether for some, I, I_a, I_b, I_c can be lattice points as well.

Theorem 5. *There are infinitely many primitive Heronian* (*non-Pythagorean*) *triangles, when realized as lattice triangles, have I, I_a, I_b, I_c at lattice points as well.*

Proof. (1) The decomposable family in Theorem 3 with

$$a=4n^2,\quad b=(2n+1)(2n^2-2n+1),\quad c=(2n-1)(2n^2+2n+1)$$

can be realized at

$$\begin{aligned} A =& \ (-2n(n-1)(2n+1),\ (2n-1)(2n+1)), \\ B =& \ (4n^2,\ 0), \\ C =& \ (0,\ 0). \end{aligned}$$

Then

$$\begin{aligned} I =& \ (s-c,\ r) = (1,\ 2n-1), \\ I_a =& \ (s-b,\ r_a) = ((2n-1)(2n+1),\ 2n+1), \\ I_b =& \ (a-s,\ r_b) = (-2n^2(2n-1),\ 2n^2), \\ I_c =& \ (s,\ r) = (2n^2(2n+1),\ 2n^2(2n-1)(2n+1)). \end{aligned}$$

For the beginning value of $n = 2$, the points are

$$\begin{aligned} &A = (-20,\ 15), \quad B = (16,\ 0), \quad C = (0,\ 0); \\ &I = (1,\ 3), \quad I_a = (15,\ 5), \quad I_b = (-24,\ 8), \text{ and } I_c = (40,\ 120). \end{aligned}$$

(2) The indecomposable family in Theorem 4 with

$$a = 5(5n^2+n-1), \quad b = (5n+3)(5n^2-4n+1), \quad c = (5n-2)(5n^2+6n+2)$$

can be realized at

$$\begin{aligned} A =& \ (2n(2n-1)(5n+3),\ (n-1)(3n-1)(5n+3)) \\ =& \ (2n(2n-1)r_a,\ (n-1)(3n-1)r_a), \\ B =& \ (-4(5n^2+n-1),\ -3(5n^2+n-1)) \\ =& \ (-4r_b,\ -3r_b), \\ C =& \ (0,0). \end{aligned}$$

Then

$$\begin{aligned} I =& \ \frac{aA+bB+cC}{a+b+c} = (3n-2,\ 4n+1), \\ I_a =& \ \frac{-aA+bB+cC}{-a+b+c} = ((-4n+1)r_a,\ (-3n+2)r_a), \\ I_b =& \ \frac{aA-bB+cC}{a-b+c} = ((4n-1)r_b,\ (3n-2)r_b), \\ I_c =& \ \frac{aA+bB-cC}{a+b-c} = ((3n-2)r_a r_b,\ (-4n+1)r_a r_b). \end{aligned}$$

For the beginning value of $n = 2$, the points are

$$\begin{aligned} &A = (156,\ 65), \quad B = (-84,\ -63), \quad C = (0,\ 0); \\ &I = (4,\ -7), \quad I_a = (-91,\ -52), \quad I_b = (147,\ 84), \text{ and } I_c = (1092,\ -1911). \end{aligned}$$

□

6. Further problems

The triangles given in the proofs of Theorem 3 and Theorem 4 are all obtuse, which suggests the following question.

Problem 1. Are there infinitely many acute primitive Heronian triangles with r, r_a, r_b, $r_c \in \mathbb{N}$?

On the other extreme, there are also primitive Heronian triangles with $r, r_a, r_b, r_c \notin \mathbb{N}$. For examples:

- If $(a, b, c) = (1921, 2929, 3600)$, then
$$(T, r, r_a, r_b, r_c) = \left(2808000, \frac{8640}{13}, \frac{4875}{4}, \frac{6500}{3}, \frac{22464}{5}\right).$$
- If $(a, b, c) = (2525, 2600, 2813)$, then
$$(T, r, r_a, r_b, r_c) = \left(3011652, \frac{47804}{63}, \frac{39627}{19}, \frac{81396}{37}, \frac{44289}{17}\right).$$

We propose the following problem.

Problem 2. Prove that there are infinitely many primitive Heronian triangles with r, r_a, r_b, $r_c \notin \mathbb{N}$.

Reference

[1] P. Yiu, Heronian triangles are lattice triangles, *Amer. Math. Monthly*, 108 (2001) 261–263.

Li Zhou: Department of Mathematics, Polk State College, Winter Haven, FL 33881 USA
E-mail address: `lzhou@polk.edu`

Forum Geometricorum
Volume 18 (2018) 79–82.

FORUM GEOM
ISSN 1534-1178

A Family of Triangles for which Two Specific Triangle Centers Have the Same Coordinates

Francisco Javier García Capitán

Abstract. It is well known that different points can have same ETC search numbers. We consider the family of triangles for which X_{3635} and X_{15519} have the same barycentric coordinates. This family includes triangle $(6, 9, 13)$. We give a geometrical construction and a locus point of view for this family of triangles.

1. Introduction

The two triangle centers in question are $X(3635)$ and $X(15519)$ in the Encyclopedia of Triangle Centers [2]. Randy Hutson [1] points out that they have the same ETC search number, while the barycentrics for $X(3635)$ are

$$(6a - b - c : \cdots : \cdots)$$

and those of $X(15519)$ are

$$((-a + b + c)(3a - b - c)^2 : \cdots : \cdots).$$

When we calculate the cross product of the coordinates of these two points we get an expression of the form

$$\{Q(a, b, c)(b - c),\ Q(a, b, c)(c - a),\ Q(a, b, c)(a - b)\}$$

where

$$Q(a, b, c) = 146abc + \sum_{\text{cyclic}} (3a^3 - 23a^2(b + c)). \tag{1}$$

The reason for the apparently strange behavior of $X(15519)$ and $X(3635)$ is the fact that $Q(6, 9, 13) = 0$.

The relation (1) can be written in terms of R, r, s as

$$5s^2 + 8r^2 - 80rR = 0,$$

which in turn is equivalent to

$$45GI^2 = 17r^2. \tag{2}$$

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

2. Construction

This leads to an easy construction of the triangles satisfying (1). We first consider the following lemma.

Lemma 1. [1] *Let ABC be a triangle with incenter I. Call J the orthogonal projection of I on the parallel to BC through A, and M the midpoint of BC. Then AM meets IJ at J_0, the inverse of J with respect to the incircle.*

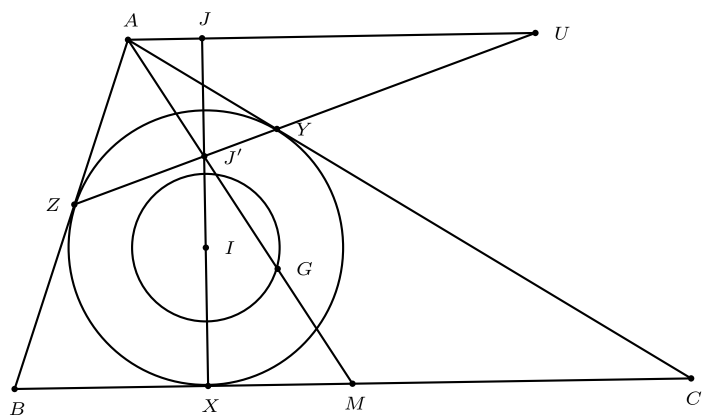


Figure 1.

Proof. Since M is the midpoint of BC and AJ is parallel to BC, AM is the harmonic conjugate of AJ with respect to AB and AC.

Let XYZ be the pedal triangle of I. If $J' = AM \cap YZ$ and $U = AJ \cap YZ$, then J' is the harmonic conjugate of U with respect to Z, Y, and therefore J' lies on the polar of U with respect to the incircle.

By the reciprocal property of pole and polar, since J' lies on the polar of U, the polar of J' goes through U, that is J is the inverse of J' with respect to the incircle. □

Now we can solve the construction problem of triangle ABC from I, G, and X (the pedal of I on BC)

Construction 2. *Given I, G, and X, construct*
(1) *D, the othogonal projection of G on the tangent to (I, IX) at X,*
(2) *L, the point on line DG such that $DG : GL = 1 : 2$,*
(3) *J, the intersection of the parallel to BC through L and line XI,*
(4) *J', inversion of J with respect to circle (I, IX),*
(5) *A, the intersection of lines GJ' and JL, and finally,*
(6) *$B,\ C$, the intersections of BC and tangents to (I, IX) from A.*

[1]This lemma was proposed by the author in the 11-th Mathematics and Computer Science National Contest "Grigore Moisil", held in Urziceni, Romania, February 2-4, 2018.

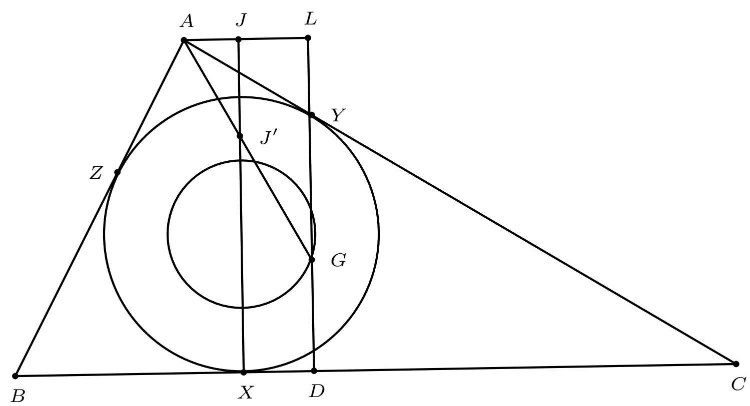


Figure 2.

Now to construct the triangles satisfying (2), we start from the circle (I, IX) and the tangent at some point X, the future line BC. If we take $r = IX$, we can perform a compass and ruler construction for the distance $g = \sqrt{\frac{17}{45}}r$ taking into account that

$$3g = \sqrt{\frac{17}{5}}r = \sqrt{\frac{4^2+1^2}{2^2+1^2}}r.$$

3. Locus

For any point G on circle (I, g) we can construct a triangle ABC satisfying (1).

What is the locus of A when G varies on circle (I, g)?

In Figure 3, X' is the reflection of X in I, and X'' the reflection of X in X'.

The curve looks like a Nichomedes conchoid with pole X' and having the parallel to BC at X'' as asymptote.

However if we take X' as the origin, and AX', $X'X''$ as axes, the cartesian equation of the curve becomes

$$(y-2r)^2x^2 = y^2\left(\sqrt{\frac{17}{5}}r + r - y\right)\left(\sqrt{\frac{17}{5}}r - r + y\right),$$

or

$$(y-2r)^2x^2 = y^2(3g + r - y)(3g - r + y),$$

while the equation of a Nichomedes conchoid is of the form

$$(y-b)^2x^2 = y^2(a + b - y)(a - b + y).$$

References

[1] R. Hutson, Advanced Plane Geometry group, message 4270, December 18, 2017;
`https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/4270`.

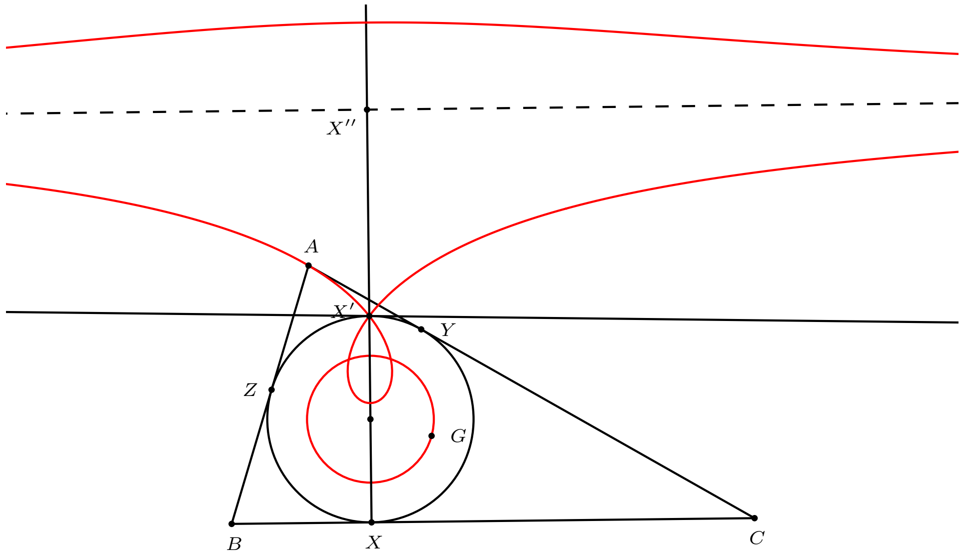


Figure 3.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: `garciacapitan@gmail.com`

Forum Geometricorum
Volume 18 (2018) 83–86.

FORUM GEOM
ISSN 1534-1178

A New Proof of Erdős-Mordell Inequality

Robert Bosch

Abstract. In this note we show a new proof of Erdős-Mordell inequality. The new idea is to consider three interior points to the triangle, the resulting inequality becomes Erdős-Mordell inequality when the three before mentioned points coincide.

1. Introduction

The famous Erdős-Mordell inequality states: From a point O inside a given triangle ABC the perpendiculars OP, OQ, OR are drawn to its sides. Prove that $OA+OB+OC \geq 2(OP+OQ+OR)$. Equality holds if and only if the triangle ABC is equilateral and the point O is its center. This inequality was proposed by Paul Erdős to the journal American Mathematical Monthly in 1935. Later in 1937 were published solutions by L. J. Mordell and D. F. Barrow in the same journal. A complete and extensive survey on the history of the problem can be found in [1]. There are numerous proofs in the literature to the inequality, but always considering a single point inside the triangle. So, it is natural to explore what happen for two or three interior points. More precisely, to find similar inequalities to Erdős-Mordell for more than one interior point. For two points we can find the following result in the journal Crux Mathematicorum [3].

Problem 982. (Proposed by George Tsintsifas, Thessaloniki, Greece.)
Let P and Q be interior points of triangle $A_1A_2A_3$. For $i=1,2,3$, let $PA_i = x_i$, $QA_i = y_i$, and let the distances from P and Q to the side opposite A_i be p_i and q_i, respectively. Prove that

$$\sqrt{x_1y_1}+\sqrt{x_2y_2}+\sqrt{x_3y_3} \geq 2(\sqrt{p_1q_1}+\sqrt{p_2q_2}+\sqrt{p_3q_3}).$$

When $P=Q$, this reduces to the well-known Erdős-Mordell inequality.

When we consider three interior points to the triangle we only found the following result: Let ABC be a triangle and let P, Q, R be three points inside it so that $QR \perp BC$, $RP \perp CA$ and $PQ \perp AB$. Let QR meet BC at D, RP meet CA at E and PQ meet AB at F. Prove that

$$PA+QB+RC \geq PE+PF+QF+QD+RD+RE. \tag{1}$$

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

This one is the motivation for this note since if P, Q, R coincide ($P = Q = R$), then we get Erdős-Mordell inequality. So, a solution to this problem lead to a new proof of the famous inequality. This problem was proposed as Problem 6 in a problem session from Computational Geometry course in *AwesomeMath Summer Program* 2017. During the camp, I was not aware of a solution, but in January 2018 I came back to work on this one, obtaining a satisfactory and elegant solution, my proof uses the well-known geometric lemma that is common to many different proofs of Erdős-Mordell inequality (see Lemma 1 in the next section). The lemma provides three inequalities relating the lengths of the sides of ABC and the distances from O to the vertices and to the sides. Which one has proved to be useful and today can be considered a classical result in geometric inequalities. There are different proofs for the above lemma, in [1], Claudi Alsina and Roger B. Nelsen construct a trapezoid, also in [2, p.202], the authors of the book consider the cyclic quadrilateral and orthogonal projections, obtaining the same lemma but in trigonometrical form, both are simply equivalent by Law of Sines. To tackle Problem 6 from AMSP, the last version is adequate since the inner triangle PQR has the same interior angles that triangle ABC. In this way we are relating both triangles, this task result impossible, or at least hard for the sides. Also we shall use an algebraic inequality whose proof is simple.

2. Lemmas

Lemma 1. *Given a triangle ABC and an interior point P, draw perpendiculars PY and PZ to the sides AC and AB respectively. Then we have*

$$PA \geq PY \cdot \frac{\sin C}{\sin A} + PZ \cdot \frac{\sin B}{\sin A}.$$

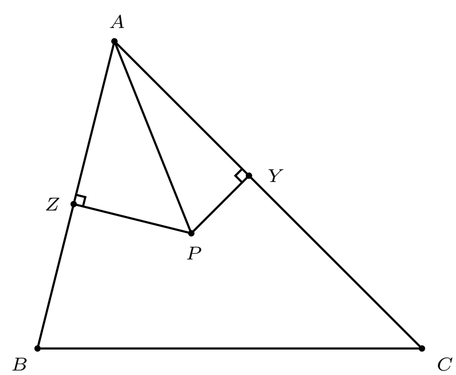


Figure 1.

Proof. See [2, page 202]. □

Lemma 2. *Let x, y, z be positive real numbers. The following inequality holds:*

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq x + y + z.$$

Proof. For a positive real number α, clearly $\alpha + \frac{1}{\alpha} \geq 2$, since $(\alpha - 1)^2 \geq 0$. Now we have

$$\frac{yz}{x} + \frac{zx}{y} = z\left(\frac{y}{x} + \frac{x}{y}\right) \geq 2z.$$

Similarly,

$$\begin{aligned} \frac{zx}{y} + \frac{xy}{z} &\geq\ 2x, \\ \frac{yz}{x} + \frac{xy}{z} &\geq\ 2y. \end{aligned}$$

Summing up the three inequalities the conclusion follows. □

3. Main result

Now we are ready to prove inequality (1).

Let ABC be a triangle and let P, Q, R be three points inside it so that $QR \perp BC$, $RP \perp CA$ and $PQ \perp AB$. Let QR meet BC at D, RP meet CA at E and PQ meet AB at F. Prove that

$$PA + QB + RC \geq PE + PF + QF + QD + RD + RE.$$

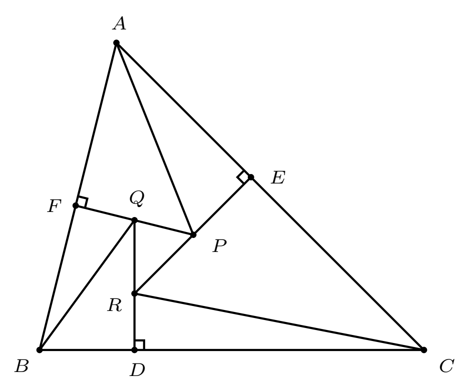


Figure 2.

By Lemma 1, the following inequalities are valid:

$$\begin{aligned} PA &\geq PE \cdot \frac{\sin C}{\sin A} + PF \cdot \frac{\sin B}{\sin A}, \\ QB &\geq QF \cdot \frac{\sin A}{\sin B} + QD \cdot \frac{\sin C}{\sin B}, \\ RC &\geq RD \cdot \frac{\sin B}{\sin C} + RE \cdot \frac{\sin A}{\sin C}. \end{aligned}$$

Quadrilateral $AFPE$ is cyclic, so $\angle FPE = 180^\circ - A$, and then $\angle QPR = A$. By analogy $\angle PQR = B$ and $\angle QRP = C$. Now, by the Law of Sines in triangle

PQR, and sum of segments, the above inequalities can be rewritten as

$$\begin{aligned} PA \geq\ & PE \cdot \frac{PQ}{QR} + PF \cdot \frac{RP}{QR} = PE \cdot \frac{PQ}{QR} + \frac{PQ \cdot RP}{QR} + \frac{QF \cdot RP}{QR}, \\ QB \geq\ & QF \cdot \frac{QR}{RP} + QD \cdot \frac{PQ}{RP} = QF \cdot \frac{QR}{RP} + \frac{QR \cdot PQ}{RP} + \frac{RD \cdot PQ}{RP}, \\ RC \geq\ & RD \cdot \frac{RP}{PQ} + RE \cdot \frac{QR}{PQ} = RD \cdot \frac{RP}{PQ} + \frac{RP \cdot QR}{PQ} + \frac{PE \cdot QR}{PQ}. \end{aligned}$$

Summing up and using Lemma 2 we obtain

$$\frac{PQ \cdot RP}{QR} + \frac{QR \cdot PQ}{RP} + \frac{RP \cdot QR}{PQ} \geq PQ + QR + RP.$$

Next, note that

$$\begin{aligned} \frac{PE \cdot PQ}{QR} + \frac{PE \cdot QR}{PQ} &\geq 2PE, \\ \frac{QF \cdot RP}{QR} + \frac{QF \cdot QR}{RP} &\geq 2QF, \\ \frac{RD \cdot PQ}{RP} + \frac{RD \cdot RP}{PQ} &\geq 2RD. \end{aligned}$$

Finally,

$$\begin{aligned} PA + QB + RC \geq\ & PQ + QR + RP + 2(PE + QF + RD) \\ =\ & PE + PF + QF + QD + RD + RE. \end{aligned}$$

References

[1] C. Alsina and R. B. Nelsen, A visual proof of the Erdős-Mordell inequality, *Forum Geom.*, 7 (2007) 99–102.

[2] T. Andreescu, O. Mushkarov, and L. Stoyanov, *Geometric Problems on Maxima and Minima*, Birkhäuser, 2006.

[3] G. Tsintsifas, Problem 982, *Crux Math.*, 10 (1984) 291.

Robert Bosch: Archimedean Academy, 12425 SW 72nd St, Miami, Florida 33183, USA

E-mail address: bobbydrg@gmail.com

Forum Geometricorum
Volume 18 (2018) 87–92.

FORUM GEOM
ISSN 1534-1178

Parabola Conjugate to Rectangular Hyperbola

Paris Pamfilos

Abstract. We study the conjugation, naturally defined in a bitangent pencil of conics, and determine the triangles, which define corresponding pencils, for which the parabola member is conjugate to the rectangular hyperbola member of the pencil.

1. The bitangent pencil of conics

A *bitangent* family or pencil of conics is created by quadratic equations in the form of linear combinations

$$\alpha \cdot (f(x) \cdot g(x)) + \beta \cdot h(x)^2 = 0, \qquad \text{with } \alpha + \beta = 1, \tag{1}$$

where $\{f(x) = g(x) = h(x) = 0, x \in \mathbb{R}^2\}$ are equations of lines in general position and $\{\alpha, \beta\}$ are real constants ([1, II.p.194]). The family consists of conics,

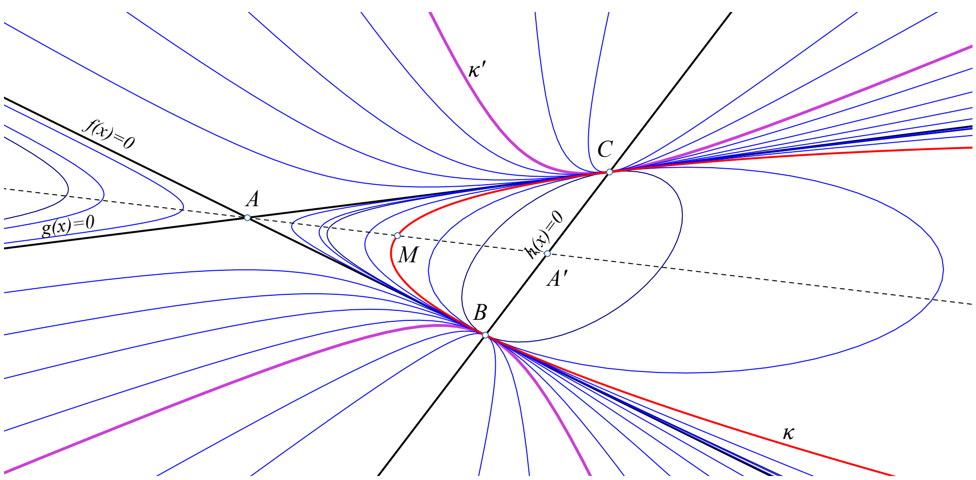


Figure 1. Bitangent family of conics

which are tangent to the lines $\{f = 0, g = 0\}$ at their intersections $\{B, C\}$ with the line $h = 0$. Thus, BC is a chord also common to all member-conics of the pencil, and the intersection point A of the two tangents, common to all members, is the pole of the line BC again with respect to all these members (see Figure 1). Since the median line AA' of the triangle ABC is the conjugate diameter of BC with respect to all members of the family, the centers of these conics lie on this line. Some general facts of a slightly more general kind of such families, in which the product of lines $f(x) \cdot g(x) = 0$ is replaced by a general conic $c(x) = 0$,

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

are discussed in [7]. Every bitangent family has a unique parabola member, and a unique rectangular hyperbola member. The parabola κ passes through the middle M of the median AA' of the triangle ABC and the rectangular hyperbola κ' has its center at the projection of the orthocenter of the triangle ABC on this median line ([4], [6]). The aim of the article is to show that these two exceptional members of a bitangent pencil are *conjugate*, in a sense to be explained right below, only in the case the triangles ABC, formed by the three lines, are such that the median satisfies $BC = \sqrt{2} \cdot AA'$.

2. The perspectivities group of a triangle

A triangle ABC produces a set of three *harmonic perspectivities* (or *harmonic homologies* [8, I, p.223], [3, p.248]), which, together with the identity transformation, build a group of projective transformations having the vertices of the triangle for fixed points. The harmonic perspectivity f_B relative to the vertex B say, is

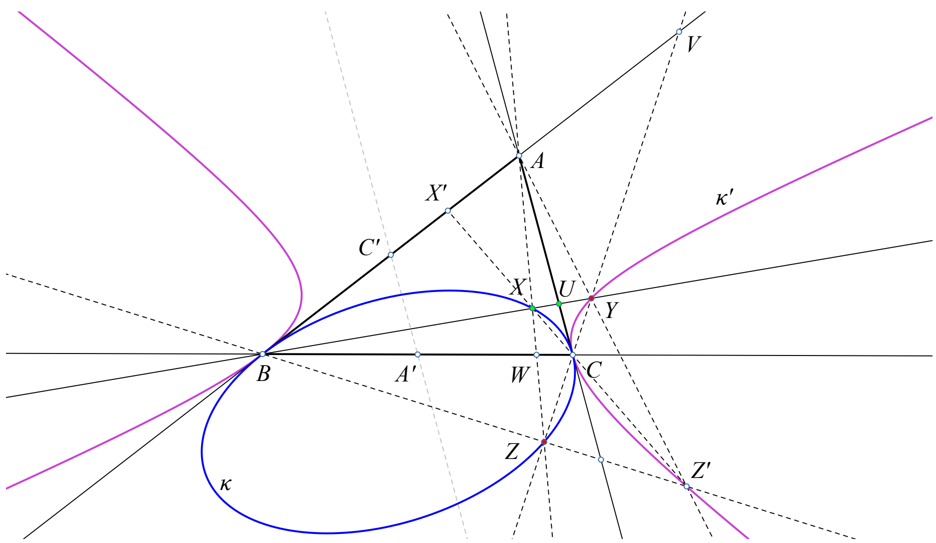


Figure 2. Harmonic perspectivity $Y = f_B(X)$ relative to B

defined by corresponding to a point X the point $Y = f_B(X)$ on line BX, so that the pairs $(B, U) \sim (X, Y)$ are harmonic, point U being the intersection of BX with AC (see Figure 2). The thus defined projective transformation f_B, leaves B and AC pointwise fixed and maps the line $A'C'$, of middles of sides $\{BC, BA\}$, to the line at infinity. Analogously are defined the maps $\{F_A, F_C\}$, and, from the properties of harmonic pencils ([5, p.88]), results the commutativity and the group property of these transformations (see Figure 2):

$$f_A^2 = 1, \quad f_A \circ f_B = f_B \circ f_A = f_C,$$

and the analogous properties for cyclic permutations of the letters $\{A, B, C\}$. Consider now the bitangent at $\{B, C\}$ family of conics $\mathcal{F}$ and the perspectivity $f_A(X) = Z$ and a member-conic κ. Since BC is the polar of point A with respect to κ, the map f_A leaves κ invariant. It is easy to see that the two other maps $\{f_B, f_C\}$ send κ to another member κ' of the family $\mathcal{F}$ (see Figure 2). In fact, if κ' is the family member through $Y = f_B(X)$ and $Z' = f_C(X)$, then points $\{X, Y, Z', Z = F_B(Z')\}$ define a complete quadrilateral ([5, p.100]), from whose harmonicity properties results that $Z' = f_A(Y)$. Consequently the same member κ' passes through $\{Y = f_B(X), Z' = f_C(X)\}$ and coincides with the image conic of κ under either of these maps: $\kappa' = f_B(\kappa) = f_C(\kappa)$. The last equality can be interpreted also by the conjugacy $f_C = h \circ f_B \circ h^{-1}$, where $h = h^{-1}$ is the affine reflection ([3, p.203]), defined by the lines $\{AA', BC\}$. Since these are conjugate diameters for each conic of the bitangent pencil $\mathcal{F}$, the transformation h leaves every such conic invariant.

We call two member-conics of the bitangent pencil $\mathcal{F}$, like $\{\kappa, \kappa'\}$, *conjugate*. Each of them results from the other by applying the perspectivity f_B or f_C. The property to be proved below is the following.

Theorem 1. *The parabola member of the family $\mathcal{F}$ is conjugate to the rectangular hyperbola member, if and only if, the triangle ABC and its median AA' satisfy $BC = \sqrt{2} \cdot AA'$.*

3. Hyperbola conjugate to ellipse

Using the notation and conventions adopted in the previous sections, we study the case in which the member-conic κ of the pencil $\mathcal{F}$ has a conjugate κ', which is a hyperbola. This is always the case for the parabola member of the pencil as well as for any other conic member κ contained in the same angular region, defined by the two lines $\{AB, AC\}$, in which lies the segment BC. The arguments used below are valid for all kinds of such conics and we can safely work assuming that κ is an ellipse, ultimately passing to the limiting case of the parabola. The following observations lead to the proof of the theorem.

(1) The directions of the asymptotes of the hyperbola κ' can be determined using the side-lines of the complementary triangle $A'B'C'$ of ABC (see Figure 3). In fact, if $\{B_1, B_2\}$ are the intersections of κ with line $A'C'$, send to infinity by f_B, then, because of the affine symmetry h, the points $\{C_1 = h(B_1), C_2 = h(B_2)\}$ are on the line $A'B'$, send to infinity by f_C. Using the commutativity properties of the maps $\{f_A, f_B, f_C, h\}$, we see immediately that $\{(BB_1, CC_2), (CC_1, BB_2)\}$ are pairs of lines parallel to asymptotic directions of the hyperbola κ'. In addition, it is seen that $\{(B_1, C_2, A), (C_1, B_2, A)\}$ are triples of collinear points, and that $\{B_1C_1, B_2C_2\}$ are parallel to the line BC, which implies that the intersections $I = (BB_2, CC_2)$ and $J = (BC_1, CB_1)$ are on line AA'.

(2) Consider now the intersections $\{E_1, E_2\}$ of κ' with line $B'C'$, send to infinity by f_A. We may assume that $\{AE_1, AE_2\}$ are parallel, respectively, to the asymptotic directions $\{BB_1, CC_1\}$ of κ'. For E_1 this means that

$$f_A(E_1) = f_B(B_1) \quad \Leftrightarrow \quad E_1 = f_A(f_B(B_1)) = f_C(B_1),$$

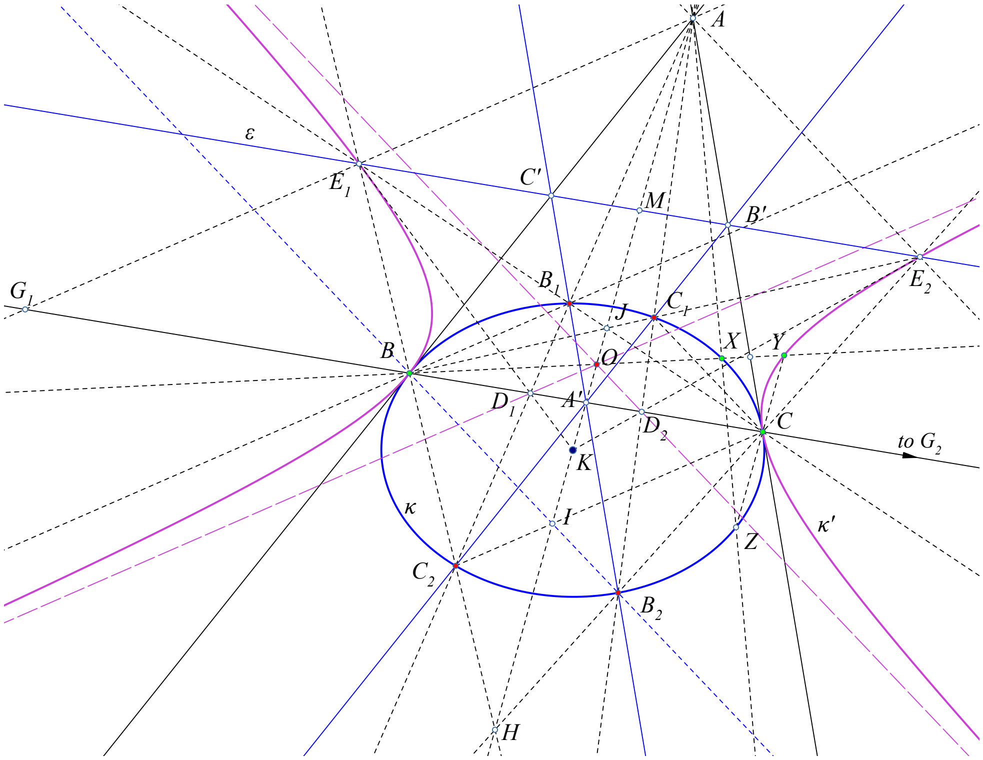


Figure 3. The conjugate to κ, hyperbola κ', and its asymptotes

so that $\{C, B_1, E_1\}$ are collinear points. Analogously is also seen that $\{B, C_1, E_2\}$ are collinear points.

(3) The polars of every point G on BC with respect to the conics $\{\kappa, \kappa'\}$ are the same, since they must pass through A and the harmonic conjugate G' of G with respect to (B, C). In particular, the intersection points $\{D_1 = (BC, AB_1), G_1 = (BC, AE_1)\}$ are harmonic conjugate with respect to (B, C) and the polar of one passes through the other i.e. AD_1 is the polar of G_1 and AG_1 is the polar of D_1 with respect to either conic. Analogous property holds also for the intersection points $\{D_2 = (BC, B_2C_2), G_1 = (BC, AE_2)\}$.

(4) Now, since the polar AG_1 of D_1 with respect to κ' is parallel to an asymptote, intersecting the hyperbola at E_1, the point D_1 must itself be on that asymptote and the tangent to the hyperbola κ' at E_1 must pass through D_1 ([2, p.281]). Hence if O is the center of the hyperbola, then OD_1 is the asymptote parallel to AG_1. Analogously, point $D_2 = (AB_1, BC)$ is on the tangent to κ' at E_2 and line OD_2 is the other asymptote.

(5) By the symmetry of the configuration with respect to the affine reflection h, we have that $\{E_2 = h(E_1), D_2 = h(D_1), G_2 = h(G_1)\}$ define segments $\{D_1D_2, E_1E_2, G_1G_2\}$ having their middles on AA'.

(6) By the parallelism of $\{AG_1, BB_1, OD_1, CC_2\}$ follows that these parallels intersect on E_1B a harmonic quadruple. Also, since the polar of D_1 passes through E_1, the polar of E_1 with respect to κ must pass through D_1, hence it coincides with the asymptote OD_1 of κ'. Analogously the polar with respect to κ of E_2 is the asymptote OD_2 of κ'. It follows then that O is the pole of $C'B'$ with respect to κ.

(7) Since the tangent E_1D_1 of κ' at E_1 passes through the intersection point D_1 of the diagonals of the trapezium BB_1CC_2 and the intersection point E_1 of its non-parallel sides, it passes also through the middles of the parallel sides $\{BC_1, CC_2\}$, hence also through the center K of κ. Analogous property holds also for the tangent E_2D_2 to κ'. It follows that line $\varepsilon = B'C'$ is the polar of K with respect to the hyperbola κ'.

(8) The pencil $D_1(OKA'A)$ consists of the sides of the triangle AD_1G_1, its median D_1E_1 and the parallel D_1O to its base, hence it is harmonic. It follows that $(O, K) \sim (A', A)$ are harmonic pairs.

In the case under consideration, in which κ is a parabola and κ' is a rectangular hyperbola, the center K of κ lies at infinity and, by (8), the conjugate O to it with respect to (A, A') must be coincident with the middle of $B'C'$ (see Figure

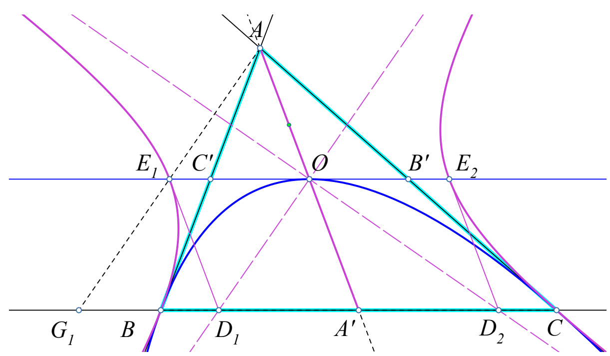


Figure 4. The case of parabola conjugate to a rectangular hyperbola

4). Taking line coordinates on BC with origin at G_1 and setting $\{x = G_1B, y = G_1D_1, BC = d\}$, the harmonicity relation becomes

$$\frac{G_1B}{G_1C} = -\frac{D_1B}{D_1C} \qquad \Leftrightarrow \qquad \frac{x}{x+d} = \frac{y-x}{x+d-y}.$$

Taking into account the parallelism of $\{AG_1, D_1J\}$ and the orthogonality of the triangle JD_1D_2, we find that $\{x, y\}$ satisfy also the equation

$$2x - 4y + d = 0.$$

From these equations, eliminating x, we deduce the equation $8y^2 - d^2 = 0$, which, because of $y = D_1A' = A'J = AA'/2$, is equivalent to the necessity condition of the theorem. The sufficiency of the condition is an easy reversing of the sequence of arguments and is left as an exercise.

References

[1] M. Berger, P. Pensu, J.-P. Berry, and X. Saint-Raymond, *Problems in Geometry*, Springer, Berlin 1984.
[2] J. Casey, *A treatise on the analytical geometry of the point, line, circle and conic sections*, Hodges Figgis and Co., Dublin, 1893.
[3] H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., John Wiley, New York, 1969.
[4] N. Dergiades, Conics tangent at the vertices to two sides of a triangle, *Forum Geom.*, 10 (2010) 41–53.
[5] H. Eves, *A survey of Geometry*, Allyn and Bacon, Inc., Boston, 1963.
[6] P. Pamfilos, Bitangent Rectangular Member;
`http://users.math.uoc.gr/~pamfilos/eGallery/Gallery.html`, 2007.
[7] P. Pamfilos, Conic homographies and bitangent pencils, *Forum Geom.*, 9 (2009) 229–257.
[8] O. Veblen and J. W. Young, *Projective Geometry I*, Ginn and Company, Boston 1938.

Paris Pamfilos: University of Crete, Greece
E-mail address: `pamfilos@uoc.gr`

Forum Geometricorum
Volume 18 (2018) 93–97.

FORUM GEOM
ISSN 1534-1178

A Toroidal Approach to the Doubling of the Cube

Gerasimos T. Soldatos

Abstract. A doubling of the cube is attempted as a problem equivalent to the doubling of a horn torus. Both doublings are attained through the circle of Apollonius.

According to Eratosthenes [1], a plague was sent by god Apollo to the Aegean island of Delos around 430 B.C., and when its citizens consulted the oracle of Delphi to learn how to defeat the plague, the answer was to double the altar of Apollo, which was a cube. And, a cube, K, with twice the volume of a given cube of side length 1, has volume equal to 2, which implies that in order to construct K, side length equal to $\sqrt[3]{2}$ has to be constructed first. Several solutions to this problem have been advanced since antiquity, but not within the context of classical geometry, because as Wantzel [2] proved much later, in 1837, the number $\sqrt[3]{2}$ is not constructible with unruled straightedge and compass. This article adds one more solution to the Delian problem by relating it to Euclidean horn torus metrics as follows.

Let the volume of horn torus $\overline{\mathbb{T}}$ be $2\pi^2R^3$, where R is the radius underlying $\overline{\mathbb{T}}$. Let $\mathbb{T}$ be the horn torus whose volume is half the volume of $\overline{\mathbb{T}}$, that is, $2\pi^2x^3 = \pi^2R^3$, where x is the radius defining $\mathbb{T}$. It follows that $2x^3 = R^3$, that is, the volume of the cube $\overline{\mathbb{C}}$ with edge length R is twice the volume of the cube $\mathbb{C}$ with edge x. Consequently, the problem of doubling the cube translates into the problem of doubling the horn torus. The problem of the constructibility of $\sqrt[3]{2}$ remains but it can be circumvented if this toroidal approach to the doubling of the cube is combined with a similar approach to the quadrature by the present author [3] in this *Journal* as follows.

The latter approach enables the construction of the line length z that squares the circle underlying torus $\overline{\mathbb{T}}$. But then

$$z = R\sqrt{\pi} \implies z^3\sqrt{\pi} = \pi^2R^3.$$

Letting

$$z^3\sqrt{\pi} = 2\pi^2x^3 \implies z^3 = 2(x^3\pi\sqrt{\pi}). \tag{1}$$

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

If, now, y is the square edge that squares the circle characterizing torus $\mathbb{T}$ so that $y = x\sqrt{\pi} \implies y^3 = x^3\pi\sqrt{\pi}$, inserting this expression in (1) yields

$$z^3 = 2y^3. \tag{2}$$

Hence, in view of (2), to double the cube $\mathbb{C}$ when torus $\mathbb{T}$ with radius x is given, we can start by finding square edge y, and from y proceed to identify square edge z and subsequently, the line segment R as side length of cube $\overline{\mathbb{C}}$ and as radius of the horn torus $\overline{\mathbb{T}}$ whose volume is twice that of $\mathbb{T}$: So,

Problem. Given a horn torus $\mathbb{T}$ with defining circle and tube circle radius x in the 3-dimensional Euclidean space, find horn torus $\overline{\mathbb{T}}$ whose defining and tube circle radius R is such that $R^3 = 2x^3 \implies \pi^2R^3 = 2\pi^2x^3$.

Analysis: For educational purposes, consider first this problem from the viewpoint of the square edges y and z, squaring the circles with radiuses x and R, respectively. Suppose that from this viewpoint, this problem has been solved as in Figure 1, with $z = y + \nu$ being subsequently the cube edge doubling the cube with edge equal to y so that $z = y\sqrt[3]{2}$. It follows that

$$z^2 = y^2\left(\sqrt[3]{2}\right)^2. \tag{3}$$

Since triangle $M\Gamma N$ is a right-angled one with Γ being the vertex of the right angle and with altitude $\Gamma\Xi = y$, we have also that

$$z^2 = y^2 + \beta^2. \tag{4}$$

Equating (3) and (4) yields

$$\beta^2 = y^2\left[\left(\sqrt[3]{2}\right)^2 - 1\right]. \tag{5}$$

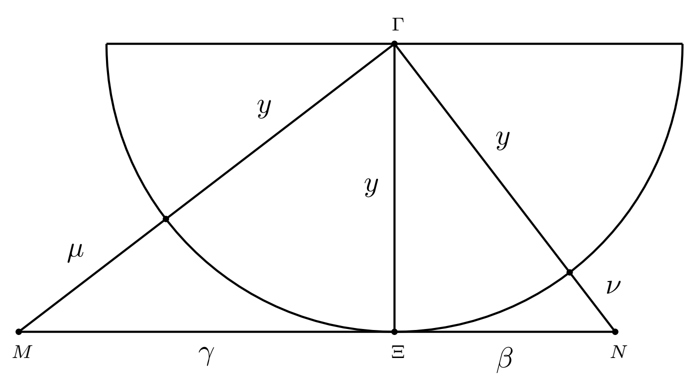


Figure 1.

By the power of point theorem in connection with point N and given that y is also the radius of circle $(\Gamma, \Gamma\Xi = y)$, we have

$$\beta^2 = \nu(\nu + 2y) \tag{6}$$

too, which when equated with (5), gives the quadratic equation

$$\nu^2 + 2y\nu - y^2 \left[\left(\sqrt[3]{2} \right)^2 - 1 \right] = 0.$$

Solving this equation for $\nu > 0$ gives

$$\nu = y \left(\sqrt[3]{2} - 1 \right) \tag{7}$$

and hence

$$z = y + \nu = \sqrt[3]{2}. \tag{8}$$

Note that since ν is as in (7), the sum in (8) obtains regardless the length of y but can be reconciled with the assumption that $z = y\sqrt[3]{2}$ only if $y = 1$. It appears that to double a cube presupposes the normalization of its edge to the value of one. Indeed, since $\beta + \gamma$ is the hypotenuse MN of the right triangle $M\Gamma N$,

$$(\beta + \gamma)^2 = (\mu + y)^2 + z^2. \tag{9}$$

Also, note that from the right triangle $M\Xi\Gamma$,

$$\gamma^2 = (\mu + y)^2 - y^2 \implies (\mu + y)^2 = \gamma^2 + y^2. \tag{10}$$

Inserting (8) and (10) in (9) gives

$$(\beta + \gamma)^2 = \gamma^2 + y^2 + (\sqrt[3]{2})^2$$

which in view of (5) becomes

$$y^2 \left[\left(\sqrt[3]{2} \right)^2 - 1 \right] + 2\beta\gamma = y^2 + (\sqrt[3]{2})^2. \tag{11}$$

By the power of point theorem, $y^2 = \beta\gamma$, which when inserted in (11), produces the relation

$$y^2 \left[\left(\sqrt[3]{2} \right)^2 - 1 \right] + 2y^2 = y^2 + \left(\sqrt[3]{2} \right)^2 \implies y^2 = 1 \implies y = 1.$$

That is, if one tried to solve the original problem of the doubling of a cube one should presuppose that its edge is equal to one, indeed.

But, this is not necessary when one more datum to the Analysis is added as for instance is done herein borrowing from torus geometry as follows. Figure 2 presents a version of Apollonius' definition of circle where x, the circle radius of torus $\mathbf{T}$, appears as well. Triangles $\Sigma T\Psi$ and $\Sigma T'\Phi$ are similar isosceles triangles with $TT' = T'T'' = \Phi\Psi = x$ and $\Sigma T' = \Sigma\Phi = y$, satisfying the proportions

$$\frac{\Sigma\Phi}{\Phi\Psi} = \frac{y}{x} = \frac{\Sigma\Omega}{\Psi\Omega} = \frac{\Sigma\Upsilon}{\Psi\Upsilon} = \frac{\Sigma\Upsilon'}{\Psi\Upsilon'}. \tag{12}$$

Points like Υ, Υ' and Ω on the periphery of the circle $(O, O\Upsilon = \varrho)$ satisfy the same ratio of distances in (12) with regard to points Σ and Ψ. Presumably, $\Sigma\Upsilon$ is tangent to this circle at point Υ, $T\Psi \parallel T'\Phi$, $T''\Psi \parallel T'\Omega$, and $\angle\Sigma\Upsilon\Phi = \angle\Phi\Upsilon\Psi$. Methodologically, since $y^2 = \pi x^2 \implies y = x\sqrt{\pi}$ and $z^2 = \pi R^2 \implies z = R\sqrt{\pi}$, it follows that $y/x = z/R$. So, given the endpoints Σ and Ψ of length $y + x = \Sigma\Psi$, with Φ being the point at which $\Sigma\Phi = y$ is extended by $x = \Phi\Psi$, lengths z and R could be searched in terms of the locus of points like Υ and

Ω whose distance from Σ and Ψ satisfies the property $y/x = z/R$. This locus defines Apollonius' circle with candidates for z lengths like $\Sigma\Upsilon$, $\Sigma\Upsilon'$ and $\Sigma\Omega$, and candidates for R lengths like $\Psi\Upsilon$, $\Psi\Upsilon'$ and $\Psi\Omega$, respectively. Actually, it is $\Sigma\Upsilon = z$ and $\Sigma\Upsilon = R$ as follows: From the relation $y/x = \Sigma\Upsilon/\Psi\Upsilon \implies \Sigma\Upsilon = \Psi\Upsilon(y/x) \implies$

$$\Sigma\Upsilon^2 = \Psi\Upsilon^2(y^2/x^2) \tag{13}$$

while power of point theorem in connection with point Σ gives

$$\Sigma\Upsilon^2 = y(y + 2\varrho). \tag{14}$$

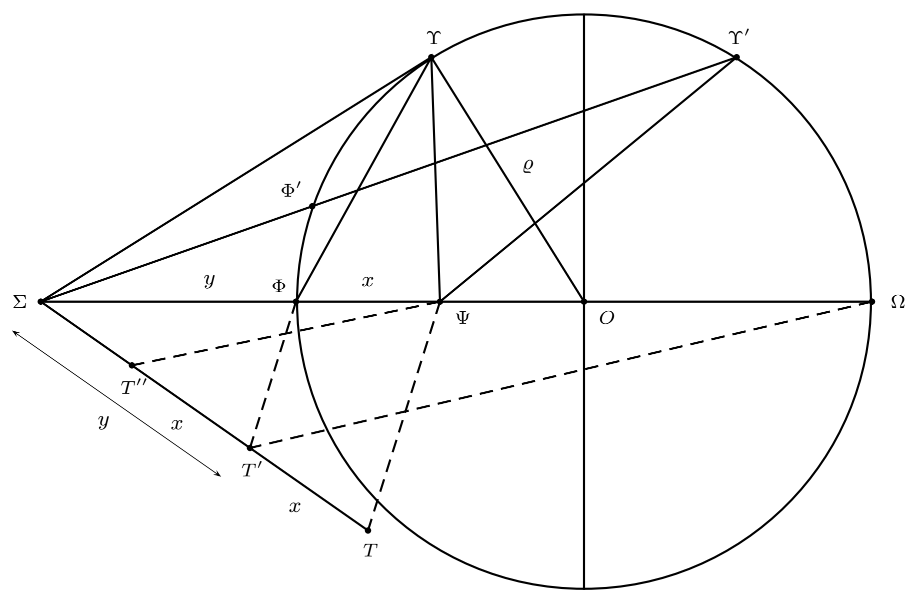


Figure 2.

Equating (13) and (14) results in the expression

$$\Psi\Upsilon^2 = \frac{x^2(y + 2\varrho)}{y}. \tag{15}$$

But, we should also have by construction that

$$\Sigma\Upsilon^2 = y(y + 2\varrho) = \pi\Psi\Upsilon^2 \implies \Psi\Upsilon^2 = \frac{y(y + 2\varrho)}{\pi}. \tag{16}$$

Combining (15) and (16) yields $y^2 = \pi x^2$, which is true. It could not be that $\Sigma\Omega = z$ and $\Psi\Omega = R$, because from Thales' intercept theorem,

$$\frac{\Sigma T''}{\Sigma\Psi} = \frac{y - x}{y + x} = \frac{T'T'' = x}{\Psi\Omega} \implies \Psi\Omega = x\frac{y + x}{y - x},$$

which in view of $y^2 = \pi x^2$, becomes

$$\Psi\Omega = x\frac{\sqrt{\pi}+1}{\sqrt{\pi}-1}$$

and hence, $\Sigma\Omega^2 = \pi\Psi\Omega^2 = \pi x^2\frac{(\sqrt{\pi}+1)^2}{(\sqrt{\pi}-1)^2}$. $\Sigma\Omega$ is square and cube edge that does not double the cube with edge y and $\Psi\Omega$ is a circle radius of some horn torus that does not double torus $\mathbb{T}$. Consider finally a case like $\Sigma\Upsilon' = z$ and $\Psi\Upsilon' = R$. From the power of point theorem,

$$\Sigma\Phi'(\Sigma\Upsilon') = \Sigma\Phi(\Sigma\Omega) \implies \Sigma\Phi' z = y(y+2\varrho)$$

or setting $z = R\sqrt{\pi}$ and solving for R,

$$R = \frac{y(y+2\varrho)}{\Sigma\Phi'\sqrt{\pi}}.$$

That is, points in general like $\Upsilon' \neq \Upsilon$ do not solve the problem of the doubling of the cube either.

Construction: Given circle radius $x = \Phi\Psi$, construct square edge $y = \Sigma\Phi$ as in Soldatos [3], draw line segment $y + x = \Sigma\Phi + \Phi\Psi = \Sigma\Psi$, form the circle of Apollonius (O, ϱ) defined by the ratio of distances y/x with respect to points Σ and Ψ, draw from Σ tangent to this circle, and connect the tangency point Υ with point Ψ: The line segment $\Psi\Upsilon = R$ (while $\Sigma\Upsilon = z$).

Proof. The construction reproduces Figure 2. And, the proof reproduces the argument starting with relation (13), leading at the same time to the following corollary. $\square$

Corollary: The line segment $\Sigma\Upsilon = z$.

References

[1] Eratosthenes, in T. L. Heath, *A History of Greek Mathematics*, vol. 1, Oxford University Press, 1951 [1921], 244–245.

[2] G. T. Soldatos, A toroidal approach to the Archimedian quadrature, *Forum Geom.*, 17 (2017) 17–19.

[3] P. L. Wantzel, Recherches sur les moyens de reconnaîtres un Problème de Géométrie peut se résoudre avec la règle et le compas, *Journal de Mathematiques Pures et Appliquees*, 1 (1837) 366–372.

Gerasimos T. Soldatos: Taburishanska 73, 27505 Svitlovodsk, Ukraine
E-mail address: soldgera@yahoo.com

Forum Geometricorum
Volume 18 (2018) 99–102.

FORUM GEOM
ISSN 1534-1178

Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by Okumura and Watanabe

Hiroshi Okumura and Saburou Saitoh

Abstract. From the viewpoint of the division by zero ($0/0 = 1/0 = z/0 = 0$) and the division by zero calculus, we will show some surprising new phenomena for geometrical properties with the concrete example which was given by the paper "The Twin Circles of Archimedes in a Skewed Arbelos" of H. Okumura and T. Watanabe, *Forum Geom.*, 4 (2004) 229–251.

1. Introduction

Let V_z be the point with coordinates $(0,\ 2\sqrt{ab}/z)$ for real numbers z, and $a,\ b > 0$ in the plane. H. Okumura and M. Watanabe gave the following theorem in [7]:

Theorem (Theorem 7 of [7]). *The circle touching the circle $\alpha : (x-a)^2+y^2 = a^2$ and the circle $\beta : (x+b)^2 + y^2 = b^2$ at points different from the origin O and passing through $V_{z_{\pm 1}}$ is represented by*

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{1}$$

for a real number $z \neq \pm 1$. The common external tangents of α and β can be expressed by the equations

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{2}$$

Anyhow the authors give the exact representation with a parameter of the general circles touching with two circles touching each other. The common external tangent may be considered a circle (as we know we can consider circles and lines as same ones in complex analysis or with the stereographic projection), however, they stated in the proof of the theorem that the common external tangents are obtained by the limiting $z \to \pm 1$. However, its logic will have a problem. Following our recent new concept of the division by zero calculus, we will consider the case $z = \pm 1$ for the singular points in the general parametric representation of the touching circles. Then, we will see interesting phenomena that we can discover a new circle in the context.

Publication Date: February 12, 2018. Communicating Editor: Paul Yiu.

2. The division by zero calculus

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n, \tag{3}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{4}$$

(Here, as convention, we consider as $0^0 = 1$.)

For the correspondence (4) for the function $f(z)$, we will call it the *division by zero calculus*. By considering the derivatives in (3), we can define any order derivatives of the function f at the singular point a.

We have considered our mathematics around an isolated singular point for analytic functions, however, we do not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the value at the singular point of the function are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We will discuss the equation (1) from this viewpoint at the singular points $z = \pm 1$ that has a clear geometric meaning.

The division by zero ($0/0 = 1/0 = z/0 = 0$) is trivial and clear in the natural sense of the generalized division (fraction), since we know the Moore-Penrose generalized inverse for the elementary equation $az = b$. Therefore, the division by zero calculus above and its applications are important. See the references [10, 1, 5, 11, 2, 3, 6, 8, 4, 9] for the details and the related topics.

However, in this paper we do not need any information and results in the division by zero, we need only the definition (4) of the division by zero calculus.

3. Results

First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (1), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0,$$

respectively.

Secondly, multiplying (1) by $(z^2 - 1)$, we immediately obtain surprisingly (2) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (1), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

4. On the circle appeared

Let us consider the circle ζ expressed by

$$x^2 + \frac{b-a}{2}x + y^2 - \sqrt{ab}y - ab = 0.$$

Then ζ meets the circle α in two points

$$P_a\left(2r_A,\ 2r_A\sqrt{\frac{a}{b}}\right),\quad Q_a\left(\frac{2ab}{9a+b},\ -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_A = ab/(a+b)$ (see Figure 1). Also ζ meets the circle β in points

$$P_b\left(-2r_A,\ 2r_A\sqrt{\frac{b}{a}}\right),\quad Q_b\left(\frac{-2ab}{a+9b},\ -\frac{6b\sqrt{ab}}{a+9b}\right).$$

Notice that P_aP_b is the external common tangent of α and β expressed by (2) with the minus sign. The lines P_aQ_a and P_bQ_b intersect at the point $R(0,\ -\sqrt{ab})$, which lies on the remaining external common tangent of α and β. Furthermore, the circle ζ is orthogonal to the circle with center R passing through the origin.

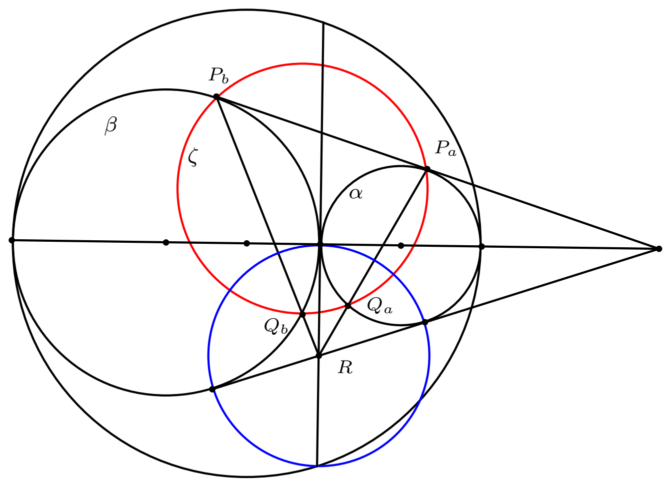


Figure 1.

5. Conclusion

By the division by zero calculus, we are able to obtain definite meaningful results simply. The new result of Section 4has never been expected thereby will have a special interest.

References

[1] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, *Int. J. Appl. Math.*, 27 (2014) 191–198; DOI: 10.12732/ijam.v27i2.9.

[2] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, *Advances in Linear Algebra & Matrix Theory*, 6 (2016) 51–58;
published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt, http://dx.doi.org/10.4236/alamt.2016.62007.

[3] T. Matsuura and S. Saitoh, Division by zero calculus and singular integrals; submitted for publication.

[4] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, in *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics.
[5] H. Michiwaki, S. Saitoh and M. Yamada, Reality of the division by zero $z/0 = 0$, *International J. of Applied Physics and Math.*, 6 (2015) 1–8.
`http://www.ijapm.org/show-63-504-1.html`.
[6] H. Michiwaki, H. Okumura and S. Saitoh, Division by zero $z/0 = 0$ in Euclidean Spaces, *International Journal of Mathematics and Computation*, 28 (2017), 1–16.
[7] H. Okumura and M. Watanabe, The twin circles of Archimedes in a skewed arbelos, *Forum Geom.*, 4 (2004) 229–251.
[8] H. Okumura, S. Saitoh and T. Matsuura, Relations of 0 and ∞, *Journal of Technology and Social Science*, 1 (2017) 70–77.
[9] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations, in *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics.
[10] S. Saitoh, Generalized inversions of Hadamard and tensor products for matrices, *Advances in Linear Algebra & Matrix Theory*, 4 (2014) 87–95;
`http://www.scirp.org/journal/ALAMT/`.
[11] S. Saitoh, A reproducing kernel theory with some general applications, in
T. Qian/L. Rodino, (eds.): *Mathematical Analysis, Probability and Applications - Plenary Lectures: Isaac 2015, Macau, China*, Springer Proceedings in Mathematics and Statistics, 177 (2016) 151–182 (Springer).

Hiroshi Okumura: Maebashi Gunma 371-0123, Japan
E-mail address: `hokmr@protonmail.com`

Saburou Saitoh: Institute of Reproducing Kernels Kawauchi-cho, 5-1648-16, Kiryu 376-0041, Japan
E-mail address: `kbdmm360@yahoo.com.jp`

Forum Geometricorum
Volume 18 (2018) 103–113.

FORUM GEOM
ISSN 1534-1178

On the Cyclic Quadrilaterals with the Same Varignon Parallelogram

Sándor Nagydobai Kiss

Abstract. The circumcenters and anticenters of the quadrilaterals cyclic $PQRS$ with the same Varignon parallelogram describe an hyperbola Γ. We propose the characterization of this hyperbola. The vertices of $PQRS$ one by one move also on hyperbolas which are the translations of hyperbola Γ.

1. Introduction

Let A, B, C, D be the midpoints of the sides PQ, QR, RS, SP of a quadrilateral $PQRS$. The quadrilateral $ABCD$ is called the *Varignon parallelogram* of $PQRS$. Note with θ the angle ABC, $\theta \in (0, \pi)$ and let $AB = 2q$, $BC = 2p$, where $p > 0$, $q > 0$ (Figure 1). Let O be the center of symmetry of the parallelogram $ABCD$. If the triplet (p, q, θ) is given and the parallelogram $ABCD$ is fixed, then there are an infinite number of quadrilaterals $PQRS$ with the same Varignon parallelogram $ABCD$ [1]. The construction of a such quadrilateral is the following: let X be an arbitrary point in the plan of parallelogram $ABCD$ and $X \notin AB$; construct the anticomplementary triangle of the triangle ABX and note its vertices with P_X, Q_X, R_X (let A be the midpoint of P_XQ_X, B the midpoint of Q_XR_X and X the midpoint of P_XR_X); the point S_X will be the symmetric of R_X with respect to C. Call $P_XQ_XR_XS_X$ the *quadrilateral generated by the points* A, B and X. Denote the set of this quadrilaterals by $\mathcal{G}_{ABX}$.

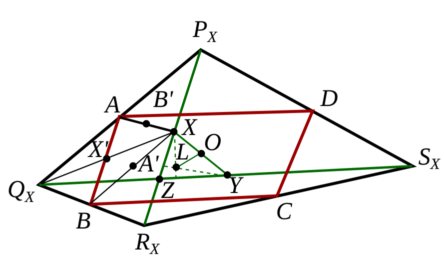


Figure 1

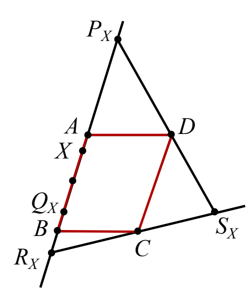


Figure 2

Let A', B', X' be the midpoints of the segments BX, XA, AB (Figure 1). The points P_X, Q_X, R_X are the symmetric of B, X, A with respect to the points B', X', A' respectively. From this property results an other construction of the

Publication Date: February 16, 2018. Communicating Editor: Paul Yiu.

quadrilateral $P_XQ_XR_XS_X$: consider for example the symmetric of X with respect to the midpoint of AB (i.e. the point Q_X); R_X will be the symmetric of Q_X with respect to B, S_X the symmetric of R_X with respect to C, P_X the symmetric of S_X with respect to D. With this method is possible the construction of the quadrilateral $P_XQ_XR_XS_X$ for the points $X \in AB$ (Figure 2).

In this paper we assume that the parallelogram $ABCD$ is fixed and for all points X from the plane of $ABCD$ the quadrilaterals $P_XQ_XR_XS_X$ have $ABCD$ as Varignon parallelogram. Denote this set by $\mathcal{G}_X$. Therefore

$$\mathcal{G}_X = \mathcal{G}_{ABX} \cup \mathcal{G}_{BCX} \cup \mathcal{G}_{CDX} \cup \mathcal{G}_{DAX}.$$

$\mathcal{G}_X$ is the set of all quadrilaterals $P_XQ_XR_XS_X$ generated by the point X.

Remark 1.1. If two quadrilaterals in $\mathcal{G}_X$ have a common vertex, then they coincide.

We pose the following question: *for what positions of the point X will the quadrilateral $P_XQ_XR_XS_X$ be cyclic*?

2. The set of quadrilaterals $P_XQ_XR_XS_X$

Attach to the parallelogram $ABCD$ an oblique coordinate system xOy with origin at O, x- and y-axes parallel to BC and AB respectively (Figure 3). The coordinates of the points A, B, C, D are $A = (-p, q)$, $B = (-p, -q)$, $C = (p, -q)$, $D = (p, q)$.

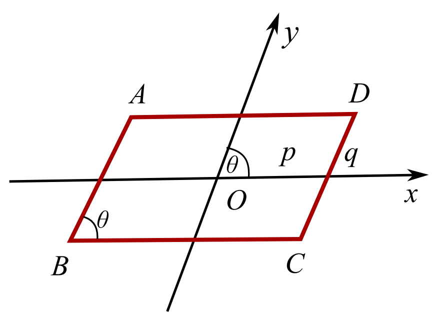


Figure 3

If $X = (\alpha, \beta)$, then

$$\begin{aligned}
2A' &= X + B = (\alpha - p,\ \beta - q),\\
2B' &= X + A = (\alpha - p,\ \beta + q),\\
2X' &= A + B = (-2p,\ 0),\\
P_X &= 2B' - B = X + A - B = (\alpha - p,\ \beta + q) - (-p, q) = (\alpha,\ \beta + 2q),\\
Q_X &= 2X' - X = A + B - X = (-2p,\ 0) - (\alpha,\ \beta) = (-\alpha - 2p,\ -\beta),\\
R_X &= 2A' - A = X + B - A = (\alpha - p,\ \beta - q) - (-p,\ q) = (\alpha,\ \beta - 2q).
\end{aligned}$$

If $S_X = 2C - R = (2p, -2q) - (\alpha, \beta - 2q) = (-\alpha + 2p, -\beta)$, then $2D = P_X + S_X$, i.e. D is the midpoint of segment $P_X S_X$. Consequently, $ABCD$ is the Varignon parallelogram of the quadrilateral $P_X Q_X R_X S_X$.

The point X is the midpoint of diagonal $P_X R_X$. Let Y be the midpoint of the other diagonal $Q_X S_X$. Since

$$Y = \frac{1}{2}(Q_X + S_X) = \frac{1}{2}(-\alpha - 2p,\ -\beta) + \frac{1}{2}(-\alpha + 2pm\ - \beta) = (-\alpha,\ -\beta),$$

the points X and Y are symmetric with respect to O (Figure 1).

Lemma 1. (a) $\mathcal{G}_{ABX} = \mathcal{G}_{CDX}$;
(b) $\mathcal{G}_{BCX} = \mathcal{G}_{DAX}$.

Proof. (a) Let $P_X Q_X R_X S_X \in \mathcal{G}_{ABX}$ and $P'_X Q'_X R'_X S'_X \in \mathcal{G}_{CDX}$. We have

$$S'_X = C + D - X = (p,\ -q) + (p,\ q) - (\alpha,\ \beta) = (-\alpha + 2p,\ -\beta) = S_X,$$

i.e. the quadrilaterals $P_X Q_X R_X S_X$ and $P'_X Q'_X R'_X S'_X$ coincide. □

Corollary 2. $\mathcal{G}_X = \mathcal{G}_{ABX} \cup \mathcal{G}_{BCX} = \mathcal{G}_{CDX} \cup \mathcal{G}_{DAX}$.

Lemma 3. *If X and Y are symmetric with respect to O, then*
(a) $\mathcal{G}_{ABX} = \mathcal{G}_{DAY}$;
(b) $\mathcal{G}_{BCX} = \mathcal{G}_{CDY}$.

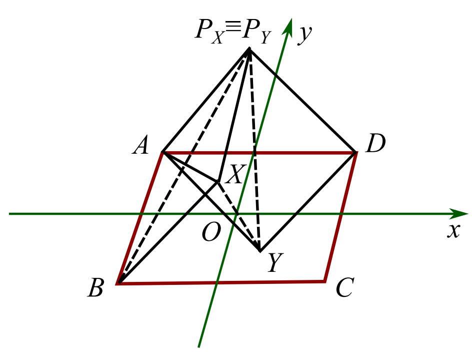


Figure 4

Proof. (a) Let $P_X Q_X R_X S_X \in \mathcal{G}_{ABX}$ and $PYQYRYSY \in \mathcal{G}_{DAY}$. We have

$$P_X = \ X + A - B = (\alpha,\ \beta) + (-p,\ q) - (-p,\ -q) = (\alpha,\ \beta + 2q),$$
$$P_Y = \ A + D - Y = (-p,\ q) + (p,\ q) - (-\alpha,\ -\beta) = (\alpha,\ \beta + 2q) = P_X,$$

i.e. the quadrilaterals $P_X Q_X R_X S_X$ and $P_Y Q_Y R_Y S_Y$ coincide (Figure 4). □

Corollary 4. *If X and Y are symmetric with respect to O, then $\mathcal{G}_X = \mathcal{G}_Y$.*

Henceforth we denote the quadrilaterals $P_X Q_X R_X S_X$ more simply by $(PQRS)_X$ or $PQRS$.

3. Special cases

I. If $p = q$, then $ABCD$ is rhombus. If $X \in BD$ (first bisector of xOy) then $\alpha = \beta$ and

$$P = (\alpha,\ \alpha + 2p), \quad Q = (-\alpha - 2p,\ -\alpha),$$
$$R = (\alpha,\ \alpha - 2p), \quad S = (-\alpha + 2p,\ -\alpha).$$

Since $\overrightarrow{PQ} = -2(\alpha + p,\ \alpha + p)$, $\overrightarrow{BD} = 2(p,\ p)$, $\overrightarrow{RS} = -2(\alpha - p,\ \alpha - p)$, results then $PQ \parallel BD \parallel RS$. In oblique coordinate system the length of segment EF determined by two points $E = (x_1, y_1)$ and $F = (x_2,\ y_2)$ is

$$EF = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\theta}.$$

In our case

$$PS^2 = 4(\alpha - p)^2 + 4(\alpha + p)^2 + 8(\alpha - p)(\alpha + p)\cos\theta = QR^2;$$

therefore, the quadrilateral $PQRS$ is an isosceles trapezoid, so it is cyclic (Figure 5).

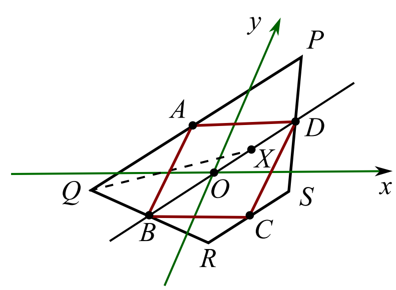


Figure 5

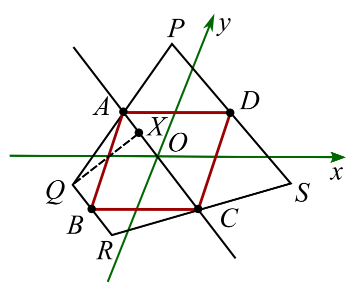


Figure 6

If $X \in AC$ (second bisector of xOy), then $\alpha = -\beta$ and $P = (\alpha,\ -\alpha + 2p)$, $Q = (-\alpha - 2p,\ \alpha)$, $R = (\alpha,\ -\alpha - 2p)$, $S = (-\alpha + 2p,\ \alpha)$. Since $\overrightarrow{PS} = -2(\alpha - p,\ -\alpha + p)$, $\overrightarrow{AC} = 2(p,\ -p)$, $\overrightarrow{RQ} = -2(\alpha + p,\ -\alpha - p)$, these give $PS \parallel AC \parallel RQ$ and

$$PQ^2 = 4(\alpha + p)^2 + 4(\alpha - p)^2 - (\alpha + p)(\alpha - p)\cos\theta = RS^2.$$

The quadrilateral $PQRS$ is also an isosceles trapezoid (Figure 6).

Conclusion: if $p = q$ and the point X is on the first or second bisector of the coordinate system xOy, then the quadrilateral $PQRS$ is cyclic.

II. If $p \neq q$ and $X \equiv D$, then $P = (p,\ 3q)$, $Q = (-3p,\ -q)$, $R = (p,\ -q) = C = S$, so the quadrilateral $PQRS$ will be a triangle (Figure 7).

If $X \in \{A,\ B,\ C\}$, then the quadrilateral $PQRS$ turn into triangles, too, consequently $PQRS$ is cyclic. In the following we suppose that $p \neq q$.

III. If $X \equiv O$, then $P_O = (0,\ 2q)$, $Q_O = (-2p,\ 0)$, $R_O = (0,\ -2q)$, $S_O = (2p,\ 0)$, so the quadrilateral $P_OQ_OR_OS_O$ is a parallelogram which has $ABCD$ as Varignon parallelogram (Figure 8). Call $P_OQ_OR_OS_O$ the *zero parallelogram*.

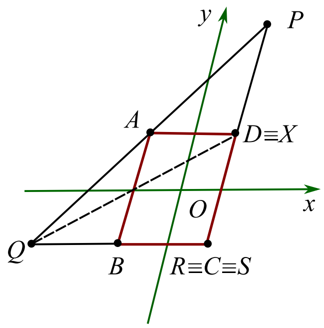


Figure 7

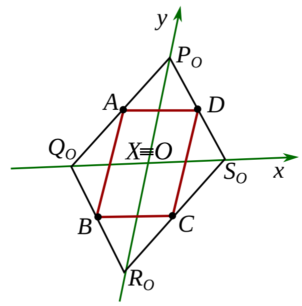


Figure 8

4. The coordinates of the centers U

The circumcenter U of quadrilateral $PQRS$ is the symmetric of the anticenter L with respect to O (the center of symmetry of the parallelogram $ABCD$). The anticenter of $PQRS$ is the orthocenter of triangle XYZ, where Z is the point of intersection of diagonals PR and QS (Figure 1). In the oblique coordinate system xOy the equation of the line perpendicular to the line with equation $lx+my+n=0$ and which through the point $(x',\ y')$ is

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ l-m\cos\theta & m-l\cos\theta & 0 \end{vmatrix}=0.$$

The equation of XZ, respectively ZY , are $x=\alpha$, respectively $y=-\beta$. The equation of the altitudes of triangle XYZ from X, respectively Y, are

$$\begin{vmatrix} x & y & 1 \\ \alpha & \beta & 1 \\ -\cos\theta & 1 & 0 \end{vmatrix}=0 \Longleftrightarrow x+y\cos\theta=\alpha+\beta\cos\theta,$$

$$\begin{vmatrix} x & y & 1 \\ -\alpha & -\beta & 1 \\ 1 & -\cos\theta & 0 \end{vmatrix}=0 \Longleftrightarrow x\cos\theta+y=-(\alpha\cos\theta+\beta).$$

We determine the coordinates of the anticenter:

$$\Delta=\begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{vmatrix}=1-\cos^2\theta=\sin^2\theta\neq 0, \quad \text{if } \theta\in(0,\pi),$$

$$\Delta_x=\begin{vmatrix} \alpha+\beta\cos\theta & \\ -(\alpha\cos\theta+\beta) & 1 \end{vmatrix}=\alpha+2\beta\cos\theta+\alpha\cos^2\theta,$$

$$\Delta_y=\begin{vmatrix} 1 & \alpha+\beta\cos\theta \\ \cos\theta & -(\alpha\cos\theta+\beta) \end{vmatrix}=-(\beta+2\alpha\cos\theta+\beta\cos^2\theta).$$

Therefore, the coordinates of U are

$$\begin{aligned} x_U = & -x_L = -\frac{\Delta_x}{\Delta} = -\frac{\alpha + 2\beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta} \quad \text{and} \\ y_U = & -y_L = -\frac{\Delta_y}{\Delta} = \frac{\beta + 2\alpha\cos\theta + \beta\cos^2\theta}{\sin^2\theta}. \end{aligned}$$

5. The distances of U to the vertices P, Q, R, S

Theorem 5.

$$\begin{aligned} UP = & \frac{2}{\sin\theta}\sqrt{(\alpha + \beta\cos\theta)^2 + q^2\sin^2\theta} = UR, \\ UQ = & \frac{2}{\sin\theta}\sqrt{(\alpha\cos\theta + \beta)^2 + p^2\sin^2\theta} = US. \end{aligned} \tag{1}$$

Proof. We have

$$UP^2 = (\alpha - x_U)^2 + (\beta + 2q - y_U)^2 + 2(\alpha - x_U)(\beta + 2q - y_U)\cos\theta,$$

where

$$\begin{aligned} \alpha - x_U = & \alpha + \frac{\alpha + 2\beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta} = \frac{2(\alpha + \beta\cos\theta)}{\sin^2\theta} \quad \text{and} \\ \beta - y_U = & \beta - \frac{\beta + 2\alpha\cos\theta + \beta\cos^2\theta}{\sin^2\theta} = -\frac{2(\alpha + \beta\cos\theta)\cos\theta}{\sin^2\theta}. \end{aligned}$$

Consequently

$$\begin{aligned} UP^2 = & \frac{4}{\sin^4\theta}\Big\{(\alpha + \beta\cos\theta)^2 + \big[q\sin^2\theta - (\alpha + \beta\cos\theta)\cos\theta\big]^2 \\ & + 2(\alpha + \beta\cos\theta)\big[q\sin^2\theta - (\alpha + \beta\cos\theta)\cos\theta\big]\cos\theta\Big\} \\ = & \frac{4}{\sin^2\theta}\big((\alpha + \beta\cos\theta)^2 + q^2\sin^2\theta\big) = UR^2. \end{aligned}$$

Analogously we obtain

$$UQ^2 = (\alpha + 2p + x_U)^2 + (\beta + y_U)^2 + 2(\alpha + 2p + x_U)(\beta + y_U)\cos\theta,$$

where

$$\begin{aligned} \alpha + x_U = & \alpha - \frac{\alpha + 2\beta\cos\theta + \alpha\cos 2\theta}{\sin^2\theta} = -\frac{2(\alpha\cos\theta + \beta)\cos\theta}{\sin^2\theta}, \\ \beta + y_U = & \beta + \frac{\beta + 2\alpha\cos\theta + \beta\cos 2\theta}{\sin^2\theta} = \frac{2(\alpha\cos\theta + \beta)}{\sin^2\theta} \quad \text{and} \\ UQ^2 = & \frac{4}{\sin^4\theta}\Big\{(\alpha\cos\theta + \beta)^2 + \big[p\sin^2\theta - (\alpha\cos\theta + \beta)\cos\theta\big]^2 \\ & + 2(\alpha\cos\theta + \beta)\big[p\sin^2\theta - (\alpha\cos\theta + \beta)\cos\theta\big]\cos\theta\Big\} \\ = & \frac{4}{\sin^2\theta}\big((\alpha\cos\theta + \beta)^2 + p^2\sin^2\theta\big) = US^2. \end{aligned}$$

□

6. Conditions for a quadrilateral to be cyclic

Theorem 6. *The following statements are equivalent:*

(a) *The quadrilateral $PQRS$ is cyclic.*

(b) *The perpendicular bisectors of sides PQ, QR, RS, SP are concurrent in a point U* (see [1] Theorem 4).

(c) *There is a point V in the plane of $PQRS$ that $VP = VQ = VR = VS$.*

Theorem 7. *If $X \notin \{A,\ B,\ C,\ D\}$, then among the quadrilaterals with the same Varignon parallelogram $ABCD$ a quadrilateral $(PQRS)_X$ is cyclic if and only if*

$$\alpha^2 - \beta^2 = p^2 - q^2. \tag{2}$$

Proof. The quadrilateral $(PQRS)_X$ is cyclic if and only if $UP = UQ$

$$\begin{aligned}&\Leftrightarrow (\alpha + \beta\cos\theta)^2 + q^2\sin^2\theta = (\alpha\cos\theta + \beta)^2 + p^2\sin^2\theta\\ &\Leftrightarrow \alpha^2 - \beta^2 = p^2 - q^2.\end{aligned}$$

□

Remark 6.1. If we start from the anticomplementary triangle of triangle BCX or CDX or DAX we obtain the same condition (2).

Consider the equation $x^2 - y^2 = p^2 - q^2$, which represents a rectangular hyperbola [2]. Note this hyperbola with Γ. So the quadrilateral $PQRS$ is cyclic if and only if the point X, the midpoint of diagonal PR, is on the hyperbola Γ. Henceforth from among all quadrilaterals with the same Varignon parallelogram $ABCD$ we consider only the quadrilaterals cyclic.

Remark 6.2. The points $A,\ B,\ C,\ D$ and Y are on the hyperbola Γ.

Theorem 8. *The centers U and the anticenters L of all quadrilaterals cyclic with the same Varignon parallelogram describe the same rectangular hyperbola Γ.*

Proof. Indeed,

$$\begin{aligned}x_U^2 - y_U^2 =& \frac{1}{\sin^4\theta}\left((\alpha + 2\beta\cos\theta + \alpha\cos^2\theta)^2 - (\beta + 2\alpha\cos\theta + \beta\cos^2\theta)^2\right)\\ =& \frac{1}{\sin^4\theta}\left((\alpha + 2\beta\cos\theta + \alpha\cos^2\theta - \beta - 2\alpha\cos\theta - \beta\cos^2\theta\right)\\ &\cdot\left(\alpha + 2\beta\cos\theta + \alpha\cos^2\theta + \beta + 2\alpha\cos\theta + \beta\cos^2\theta\right)\\ =& \frac{1}{\sin^4\theta}(\alpha - \beta)(1 - \cos\theta)^2(\alpha + \beta)(1 + \cos\theta)^2\\ =& \alpha^2 - \beta^2\\ =& p^2 - q^2.\end{aligned}$$

□

Remark 6.3. The Euler center and the anticenter of a cyclic quadrilateral coincide (see [1] Theorem 5), results that the Euler center is on the hyperbola Γ too.

In this paper we propose the characterization of hyperbola Γ.

7. The axes and the foci of hyperbola Γ

The asymptotes of hyperbola Γ are the first and the second bisector of the coordinate system xOy (Figure 9). Note these asymptotes with h and h'. The axes of symmetry of Γ are the interior and exterior bisectors of the right angle formed by h and h'. Note these axes of symmetry with a and a' (let a be the transverse axis). We propose to determine the coordinates of vertices A_1 and A_2 of Γ . First, we write the equation of the line A_1A_2.

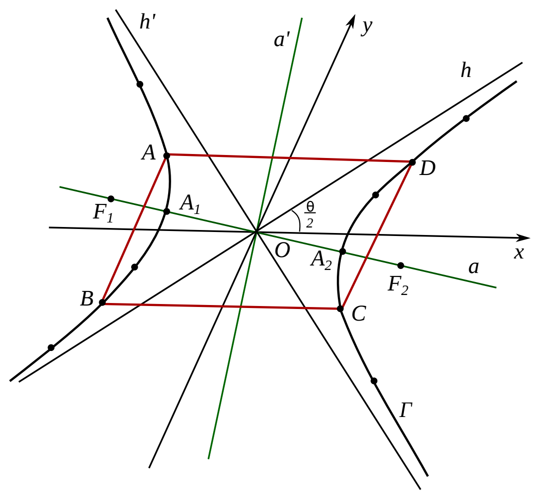


Figure 9

If a line OT forms an angle γ with the axe Ox, then the equation of OT in the oblique coordinates system xOy is $y = \frac{\sin\gamma}{\sin(\theta-\gamma)}x$. The line A_1A_2 forms an angle $\frac{\theta}{2} - \frac{\pi}{4}$ with the axe Ox. The equation of A_1A_2 is

$$y = \frac{\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}{\sin\left(\theta - \frac{\theta}{2} + \frac{\pi}{4}\right)}x = \frac{\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}{\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}x = \frac{\sin\frac{\theta}{2} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}x = -\frac{\cos\theta}{1+\sin\theta}x.$$

With the notation $m = -\frac{\cos\theta}{1+\sin\theta}$, the equation of A_1A_2 is $y = mx$.

Theorem 9. *The major axis of the hyperbola Γ is*

$$A_1A_2 = 2\sqrt{|p^2 - q^2|\sin\theta}. \tag{3}$$

Proof. We determine the coordinates of the vertices A_1 and A_2:

$$x^2 - m^2x^2 = p^2 - q^2 \iff x^2 = \frac{p^2-q^2}{1-m^2}.$$

The roots are

$$x_1 = -\sqrt{\frac{|p^2-q^2|}{1-m^2}} \quad \text{and} \quad x_2 = \sqrt{\frac{|p^2-q^2|}{1-m^2}},$$

since $1 - m^2 = 1 - \left(\frac{\cos\theta}{1+\sin\theta}\right)^2 = \frac{2\sin\theta}{1+\sin\theta} > 0$ if $\theta \in (0, \pi)$. Therefore

$$A_1 = (x_1,\ mx_1) = \left(-\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}, \frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}\right),$$
$$A_2 = (x_2,\ mx_2) = \left(\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|},\ -\frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{2\sin\theta}|p^2-q^2|}\right).$$

We calculate the length of segments $OA_1 = OA_2$:

$$\begin{aligned} OA_1^2 = OA_2^2 &= \frac{p^2-q^2}{1-m^2} + m^2\frac{p^2-q^2}{1-m^2} + 2m\frac{p^2-q^2}{1-m^2}\cos\theta \\ &= \frac{p^2-q^2}{1-m^2}(1+m^2+2m\cos\theta) \\ &= \frac{p^2-q^2}{1-m^2}\left(2 - \frac{2\sin\theta}{1+\sin\theta} - \frac{2\cos^2\theta}{1+\sin\theta}\right) \\ &= (p^2-q^2)\sin\theta. \end{aligned}$$

It follows that $A_1A_2 = 2\cdot OA_1 = 2\sqrt{|p^2-q^2|\sin\theta}$. □

Theorem 10. *The coordinates of the foci F_1 and F_2 of the hyperbola Γ are*

$$F_1 = \left(-\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}, \frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}\right),$$
$$F_2 = \left(\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|},\ -\frac{\cos\theta}{1+\sin\theta}\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}\right).$$

Proof. Since $-F_1 = F_2 = (x,\ mx)$ and

$$OF_1 = OF_2 = \sqrt{2}\cdot OA_1 = \sqrt{2}\sqrt{|p^2-q^2|\sin\theta},$$

we have

$$\begin{aligned} x^2 + m^2x^2 + 2mx^2\cos\theta &= 2|p^2-q^2|\sin\theta \\ \Longleftrightarrow (1+m^2+2m\cos\theta)x^2 &= 2|p^2-q^2|\sin\theta. \end{aligned}$$

Therefore,

$$\frac{2\sin^2\theta}{1+\sin\theta}x^2 = 2|p^2-q^2|\sin\theta \Longleftrightarrow x_{1,2} = \pm\sqrt{\frac{1+\sin\theta}{\sin\theta}|p^2-q^2|}.$$

□

8. The loci of the vertices P, Q, R, S

Theorem 11. *The loci of the vertices P, Q, R, S are the hyperbolas Γ_P, Γ_Q, Γ_R, Γ_S, the translations of hyperbola Γ by the vectors $(0,\ 2q)$, $(-2p,\ 0)$, $(0,\ -2q)$, $(2p,\ 0)$ respectively.*

Proof. The coordinates of the point P are $x = \alpha$ and $y = \beta + 2q$. Since the point $(\alpha,\ \beta)$ describes the hyperbola Γ, with equation $\alpha^2 - \beta^2 = p^2 - q^2$, the locus of P will be the hyperbola Γ_P with equation $x^2 - (y - 2q)^2 = p^2 - q^2$. The hyperbola Γ_P is the translation of Γ by the vector $(0,\ 2q)$. The equations of hyperbolas Γ_P, Γ_Q, Γ_R, Γ_S are

$$\begin{array}{rlcl}
\Gamma_P: & x^2 - (y-2q)^2 = p^2 - q^2 & \iff & x^2 - y^2 + 4qy - p^2 - 3q^2 = 0;\\
\Gamma_Q: & (x+2p)^2 - y^2 = p^2 - q^2 & \iff & x^2 - y^2 + 4px + 3p^2 + q^2 = 0;\\
\Gamma_R: & x^2 - (y+2q)^2 = p^2 - q^2 & \iff & x^2 - y^2 - 4qy - p^2 - 3q^2 = 0;\\
\Gamma_S: & (x-2p)^2 - y^2 = p^2 - q^2 & \iff & x^2 - y^2 - 4px + 3p^2 + q^2 = 0.
\end{array}$$

□

Remark 8.1. The centers of symmetry of hyperbolas Γ_P, Γ_Q, Γ_R, Γ_S are the vertices of zero parallelogram $P_O Q_O R_O S_O$.

9. Orthodiagonal quadrilaterals

If $\theta = \frac{\pi}{2}$, then the diagonals PR and QS are perpendiculars (Figure 10). In this case the coordinates of vertices A_1 and A_2 of the hyperbola Γ are $A_1 = \left(-\sqrt{|p^2 - q^2|},\ 0\right)$, $A_2 = \left(\sqrt{|p^2 - q^2|},\ 0\right)$, The coordinates of the foci are $F_1 = \left(-\sqrt{2}\sqrt{|p^2 - q^2|},\ 0\right)$, $F_2 = \left(\sqrt{2}\sqrt{|p^2 - q^2|},\ 0\right)$.

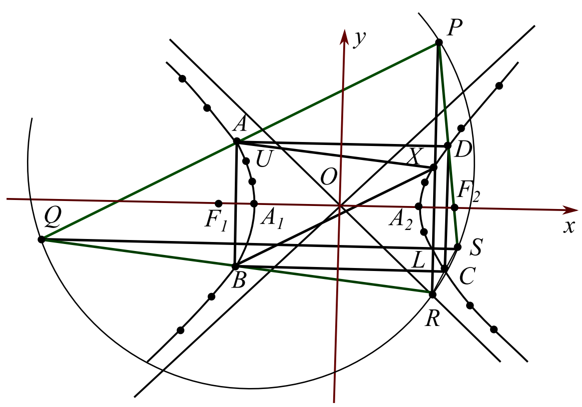


Figure 10

The center of circle circumscribed to the quadrilateral $PQRS$ is $U = (-\alpha,\ \beta)$, which is the symmetric of anticenter $(\alpha,\ -\beta)$, the point of intersection of diagonals PR and QS. In conclusion, in case of orthodiagonal quadrilaterals the point of intersection of its diagonals is also on the hyperbola Γ.

An open question is the geometric determination of the vertices and the foci of hyperbola Γ. Further it remains untackled if among the quadrilateral cyclic $(PQRS)_X$ could there be one with minimum area?

References

[1] S. N. Kiss and O. T. Pop, On the four concurrent circles, *GJARCMG*, 3 (2014) 91–101.
[2] S. N. Kiss, The equations of the conics in oblique coordinates systems, *Int. J. Geom.*, 3 (2014) 29–36.

Sándor Nagydobai Kiss: 'Constantin Brâncuşi' Technology Lyceum, Satu Mare, Romania
E-mail address: d.sandor.kiss@gmail.com

Forum Geometricorum
Volume 18 (2018) 115–118.

FORUM GEOM
ISSN 1534-1178

An Extension of Miquel's Six-Circles Theorem

Gábor Gévay

Abstract. We extend the classical theorem of Miquel from 6 to $2n$ circles ($n \geq 3$). As a by-product of the proof of our theorem, we obtain the nice corollary that the product of the $2n$ cross ratios of the quadruples of points determining the circles is equal to 1. Moreover, the theorem can also be formulated in an equivalent form, which extends Miquel's Triangle Theorem to an arbitrary n-sided polygon.

Miquel's Six-Circles Theorem is a well known old theorem [5]. In Johnson's book [4], it is formulated in the following form.

Theorem 1 (Miquel)**.** *If the circles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$ are concurrent at a point O, the circles $A_1B_2B_3$, $A_2B_3B_1$, $A_3B_1B_2$ are concurrent at a point P.*

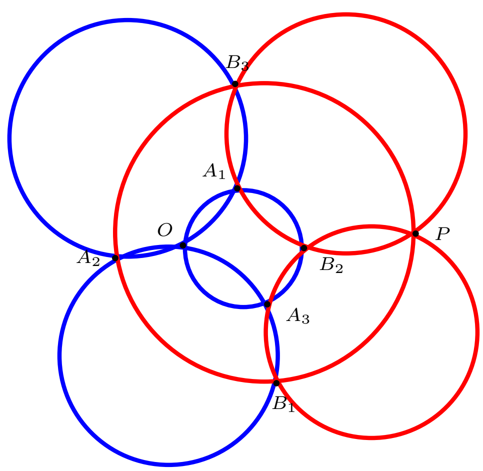


Figure 1. Miquel's Six-Circles Theorem

This formulation motivates the following extension.

Theorem 2. *Let $n \geq 3$ be an integer. Suppose that the circles $A_1B_1A_2$, $A_2B_2A_3$, $\dots$, $A_nB_nA_1$ are concurrent at a point O, and the circles $B_1A_2B_2$, $B_2A_3B_3$, $\dots$, $B_{n-1}A_nB_n$ are concurrent at a point O'. Then the circle $B_nA_1B_1$ also passes through O'.*

Publication Date: February 16, 2018. Communicating Editor: Paul Yiu.
This work is supported by the OTKA grant NN-114614, Hungary.

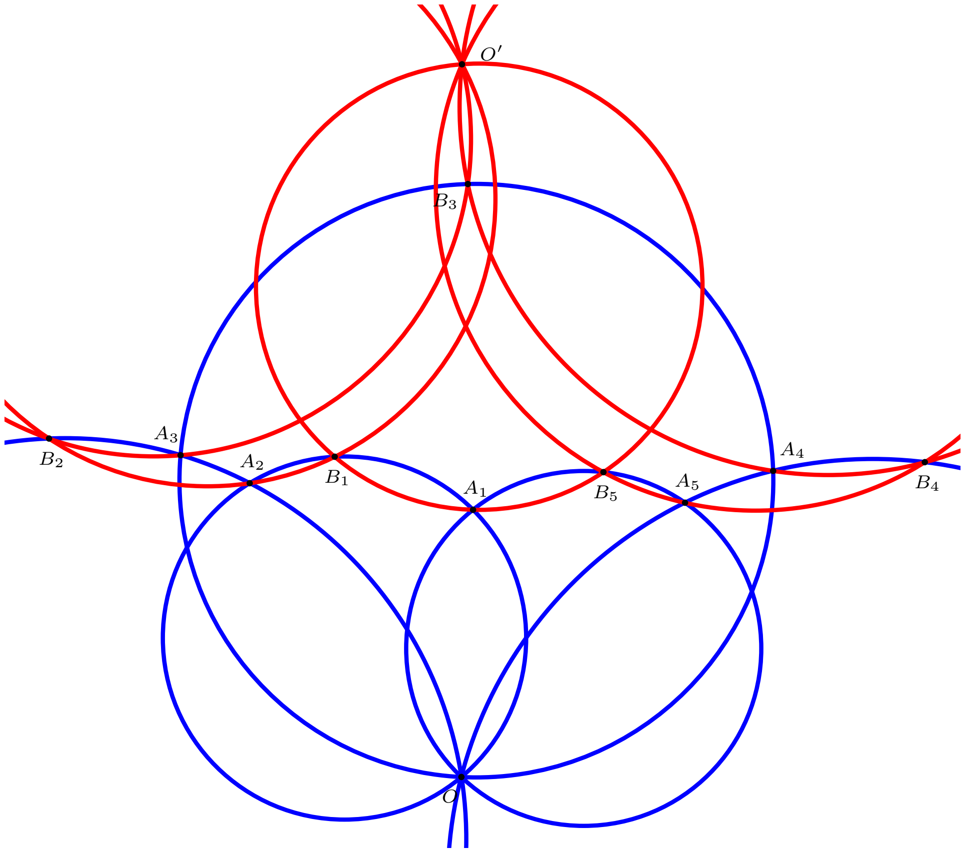


Figure 2. Example for Theorem 2 with $n = 5$

It is directly seen that for $n = 3$ it returns Miquel's original theorem; hence the latter theorem is an extension of the former, indeed.

In the proof, we shall use complex cross ratios. For a point X, we denote its *affix*, i.e. the corresponding complex number, by the same lowercase letter x. Given an ordered quadruple of points (X, Y, Z, W), we assign to it the cross ratio in the following form:

$$\frac{x-y}{x-w} : \frac{z-y}{z-w}.$$

We note that this is slightly different from the standard way used in the literature [3, 6], which is motivated by the particular arrangement of the points in our case; this choice is justified by the lemma given below. We shall need this lemma in the proof of our theorem.

Lemma 3. *Four points lie on the same circle if and only if their cross ratio is real.*

This is a well known result, see e.g. [3, 6]. Note that it is valid on the inversive plane, i.e. on the completed Euclidean plane $\mathbb{E}^2 \cup \{\infty\}$, where ∞ is the (unique) point at infinity [1]; the circles are meant in a generalized sense, i.e. they can also be (straight) lines.

Proof of Theorem 2. For $i \in \{1, \dots, n\}$, we denote the cross ratio of the quadruple (O, A_i, B_i, A_{i+1}) and $(O', B_i, A_{i+1}, B_{i+1})$ by γ_i and γ_i', respectively, where the indices are taken modulo n. Using the corresponding affixes, these cross ratios can be written as follows:

$$\gamma_i = \frac{o - a_i}{o - a_{i+1}} : \frac{b_i - a_i}{b_i - a_{i+1}} \text{ and } \gamma_i' = \frac{o' - b_i}{o' - b_{i+1}} : \frac{a_{i+1} - b_i}{a_{i+1} - b_{i+1}}, \tag{1}$$

respectively. Form the product

$$\begin{aligned}\prod_{i=1}^{n} \gamma_i &= \left(\frac{o-a_1}{o-a_2} : \frac{b_1-a_1}{b_1-a_2}\right)\left(\frac{o-a_2}{o-a_3} : \frac{b_2-a_2}{b_2-a_3}\right)\cdots\left(\frac{o-a_n}{o-a_1} : \frac{b_n-a_n}{b_n-a_1}\right)\\ &= \left(1 : \frac{(b_1-a_1)(b_2-a_2)\cdots(b_n-a_n)}{(b_1-a_2)(b_2-a_3)\cdots(b_n-a_1)}\right)\\ &= \frac{(b_1-a_2)(b_2-a_3)\cdots(b_n-a_1)}{(b_1-a_1)(b_2-a_2)\cdots(b_n-a_n)}\\ &= \frac{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)(b_n-a_1)}{(b_1-a_1)(b_2-a_2)\cdots(b_{n-1}-a_{n-1})(b_n-a_n)}.\end{aligned} \tag{2}$$

Likewise, form the product

$$\begin{aligned}\prod_{i=1}^{n-1} \gamma_i' &= \left(\frac{o'-b_1}{o'-b_2} : \frac{a_2-b_1}{a_2-b_2}\right)\left(\frac{o'-b_2}{o'-b_3} : \frac{a_3-b_2}{a_3-b_3}\right)\cdots\left(\frac{o'-b_{n-1}}{o'-b_n} : \frac{a_n-b_{n-1}}{a_n-b_n}\right)\\ &= \frac{o'-b_1}{o'-b_n} : \frac{(a_2-b_1)(a_3-b_2)\cdots(a_n-b_{n-1})}{(a_2-b_2)(a_3-b_3)\cdots(a_n-b_n)}\\ &= \frac{o'-b_1}{o'-b_n} : \frac{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)}{(b_2-a_2)(b_3-a_3)\cdots(b_n-a_n)}.\end{aligned} \tag{3}$$

Again we take the product $\left(\prod_{i=1}^{n} \gamma_i\right)\left(\prod_{i=1}^{n-1} \gamma_i'\right)$:

$$\begin{aligned}&\frac{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)(b_n-a_1)}{(b_1-a_1)(b_2-a_2)\cdots(b_{n-1}-a_{n-1})(b_n-a_n)}\\ &\quad\cdot\left(\frac{o'-b_1}{o'-b_n} : \frac{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)}{(b_2-a_2)(b_3-a_3)\cdots(b_n-a_n)}\right)\\ =&\frac{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)(b_n-a_1)}{(b_1-a_1)(b_2-a_2)\cdots(b_{n-1}-a_{n-1})(b_n-a_n)}\\ &\quad\cdot\frac{o'-b_1}{o'-b_n}\cdot\frac{(b_2-a_2)(b_3-a_3)\cdots(b_n-a_n)}{(b_1-a_2)(b_2-a_3)\cdots(b_{n-1}-a_n)}\\ =&\frac{b_n-a_1}{b_1-a_1}\cdot\frac{o'-b_1}{o'-b_n} = \frac{b_n-a_1}{b_1-a_1} : \frac{o'-b_n}{o'-b_1} = \frac{a_1-b_n}{a_1-b_1} : \frac{o'-b_n}{o'-b_1},\end{aligned} \tag{4}$$

which is the reciprocal of the cross ratio of the quadruple (O', B_n, A_1, B_1).

Considering the condition of our theorem, and using Lemma 3, we see that the cross ratios $\gamma_1, \dots, \gamma_n$ and $\gamma_1', \dots, \gamma_{n-1}'$ are real. Hence the products (2) and (3) are also real, and so is (4). It follows, again by the lemma, that the quadruple (O', B_n, A_1, B_1) is cyclic. □

As a consequence of the proof, we obtain the following corollary.

Corollary 4. *For the arrangement of points and circles described in Theorem 2, the following relation holds:*

$$\prod_{i=1}^{n} \gamma_i \gamma_i' = 1.$$

Recall that Miquel's Six-Circles Theorem has an equivalent version which is called Miquel's Triangle Theorem (it is also called the Pivot Theorem [1]; it also has interesting relationships in triangle geometry, see [7]). The two theorems are equivalent in terms of inversive geometry; indeed, the equivalence is based on the fact that choosing any point of intersection X of the circles, and applying an inversion with center X, the circles meeting at X transform to the side lines of a triangle.

Theorem 5 (Miquel)**.** *If $A_1A_2A_3$ is a triangle and B_1, B_2, B_3 are any three points on the lines A_2A_3, A_3A_1, A_1A_2, respectively, then the three circles $A_1B_2B_3$, $A_2B_3B_1$, $A_3B_1B_2$ have a common point.*

Likewise, Theorem 2 can also be given in the following equivalent form which, on the other hand, is an extension of Miquel's Triangle Theorem.

Theorem 6. *For an integer $n \geq 3$, let $A_1 \dots A_n$ be a polygon, and let $B_1, \dots,$ B_n be any points on the lines $A_1A_2, \dots, A_nA_1$, respectively. Suppose that the circles $B_1A_2B_2, \dots, B_{n-1}A_nB_n$ meet in the point P. Then the circle $B_nA_1B_1$ also passes through the point P.*

Note that in this extended case the equivalence is guaranteed by an inversion with center O.

Finally, we note that by a very recent result, Miquel's Six-Circles Theorem can also be extended in a different direction, as follows.

Theorem 7 ([2])**.** *Let α and β be two circles. Let $n > 2$ be an even number, and take the points $A_1, \dots, A_n$ on α, and $B_1, \dots, B_n$ on β, such that each quadruple $(A_1, B_1, A_2, B_2), \dots, (A_{n-1}, B_{n-1}, A_n, B_n)$ is cyclic. Then the quadruple (A_n, B_n, A_1, B_1) is also cyclic.*

References

[1] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, MAA, 1967.
[2] G. Gévay, Resolvable configurations, submitted for publication.
[3] L. Hahn, *Complex Numbers and Geometry*, MAA, 1994.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, Inc., New York, 1960.
[5] A. Miquel, Théorèmes sur les intersections des cercles et des sphères, *Journal de mathématiques pures et appliquées 1re série*, 3 (1838) 517–522.
[6] H. Schwerdtfeger, *Geometry of Complex Numbers*, Dover Publications, Inc., New York, 1979.
[7] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf.

Gábor Gévay: Bolyai Institute, University of Szeged, Szeged, Hungary
E-mail address: gevay@math.u-szeged.hu

Forum Geometricorum
Volume 18 (2018) 119–127.

FORUM GEOM
ISSN 1534-1178

A Curvature Invariant Inspired by Leonhard Euler's Inequality $R \geq 2r$

Nicholas D. Brubaker, Jasmine Camero, Oscar Rocha Rocha,
Roberto Soto, and Bogdan D. Suceavă

Abstract. It is of major interest to point out natural connections between the geometry of triangles and various other areas of mathematics. In the present work we show how Euler's classical inequality between circumradius and inradius inspires, by using a duality between triangle geometry and three-dimensional hypersurfaces lying in $\mathbb{R}^4$, the definition of a curvature invariant. We investigate this invariant by relating it to other known curvature invariants.

1. A historical motivation

Leonhard Euler believed that one could not define a good measure of curvature for surfaces. He wrote in 1760 that [5]: "la question sur la courbure des surfaces n'est pas susceptible d'une réponse simple, mais elle exige à la fois une infinité de déterminations." Euler's main objection to the very idea of the curvature of surfaces was that if one can approximate a planar curve with a circle (as Isaac Newton described), then for a surface it would be impossible to perform a similar construction. For example, Euler wrote in his investigation, what would we do with a saddle surface if we intend to find the sphere that best approximates it at a saddle point? On which side of the saddle should an approximating sphere lie? There are two choices and both appear legitimate. Today we see that Euler's counterexample corresponds to the case of a surface with negative Gaussian curvature at a given point, but in Euler's period no definition of curvature for a surface was available. Since Leonhard Euler contributed so much to the knowledge of mathematics in his time, one might expect that he was responsible for introducing and investigating a measure for the curvature of surfaces. However, this is not what happened. Instead, Leonhard Euler wrote a profound paper in which he obtained a well-known theorem about normal sections and stopped right there.

Moreover, Euler's paper [5] was extremely influential in the mathematical world of his times. When Jean-Baptiste Meusnier came to see his mentor, Gaspard Monge, for the first time, Monge instructed him to think of a problem related to the curvature of curves lying on a surface. Monge asked Meusnier to study Euler's paper [5], yet before reading it, in a stroke of genius the following night, Meusiner

Publication Date: April 24, 2018. Communicating Editor: Paul Yiu.

proved a theorem that today bears his name, and he did so by using a different geometric idea than Euler's. Decades afterwards, C.F. Gauss [7] introduced a measure for the curvature of surfaces that was complemented by the very inspired Sophie Germain [8], who introduced the mean curvature.

Perhaps the most direct way to explain the two curvature invariants for surfaces is the following. Consider a smooth surface S lying in $\mathbb{R}^3$, and an arbitrary point $P \in S$. Consider N_P, the normal to the surface at P, and the family of all planes passing through P that contain the line through P with the same direction as N_P. These planes yield a family of curves on S called *normal sections*. We now determine the curvature $\kappa(P)$ of the normal sections, viewed as planar curves. Then $\kappa(P)$ has a maximum, denoted κ_1, and a minimum, denoted κ_2. The curvatures κ_1 and κ_2 are called the *principal curvatures*. Using these principal curvatures, one may define the Gaussian curvature $K(P)$ as, $K(P) = \kappa_1(P) \cdot \kappa_2(P)$, and the mean curvature $H(P)$ as, $H(P) = \frac{1}{2}\left[\kappa_1(P) + \kappa_2(P)\right]$. Note that in the 18th century these constructions were not possible. They are related in a very profound way to the birth of non-Euclidean geometry and the revolution that took place in geometry at the beginning of the 19th century.

Taking into account this important historical context, we would like to investigate the following question. Was it possible for Leonhard Euler to have determined himself a curvature invariant, perhaps a concept for which all the algebra that he needed was in place, but in which it was just a matter of interpretation to define it? We are here in the territory of speculations, of alternative history, and only our surprise while we read and re-read Euler's original works leads us to such a question. However, this question makes a lot of sense as Euler had many contributions in geometry - from plane Euclidean geometry to the geometry of surfaces - and one of them in particular could have been tied to his investigations on curvature. Hence, was there any geometric property that was actually obtained by Euler which leads us to a property related to curvature?

2. Notations in the geometry of hypersurfaces

In order better approach the problem, we turn our attention to some classical ideas from the differential geometry of smooth hypersurfaces, a concept that was extensively explored after the second half of the 19th century. Let $\sigma : U \subset \mathbb{R}^3 \to \mathbb{R}^4$ be a hypersurface given by the smooth map σ and let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to 3. Consider $\{\sigma_1(p), \sigma_2(p), \sigma_3(p), N(p)\}$, the orthonormal Gauss frame of the hypersurface, where N denotes the normal vector field to the hypersurface at every point.

The quantities similar to κ_1 and κ_2 in the geometry of surfaces are the principal curvatures, denoted λ_1, λ_2, and λ_3. They are introduced as the eigenvalues of the so-called Weingarten linear map, as we will describe below. Since our discussion is focused on the curvature quantities of three-dimensional smooth hypersurfaces in $\mathbb{R}^4$, we start by introducing these quantities. Similar to the geometry of surfaces, the curvature invariants in higher dimensions can also be described in terms of the principal curvatures. The mean curvature at the point p is

$$H(p) = \frac{1}{3}[\lambda_1(p) + \lambda_2(p) + \lambda_3(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = \lambda_1(p)\lambda_2(p)\lambda_3(p).$$

In Riemannian geometry, a third important curvature quantity is the scalar curvature ([2], p.19) denoted by $scal(p)$, which intuitively sums up all the sectional curvatures on all the faces of the trihedron formed by the tangent vectors in the Gauss frame [4]:

$$scal(p) = \sec(\sigma_1 \wedge \sigma_2) + \sec(\sigma_2 \wedge \sigma_3) + \sec(\sigma_3 \wedge \sigma_1) = \lambda_1\lambda_2 + \lambda_3\lambda_1 + \lambda_2\lambda_3.$$

The last equality is due to the Gauss' equation of the hypersurface $\sigma(U)$ in the ambient space $\mathbb{R}^4$ endowed with the Euclidean metric.

To provide a brief description of how the Weingarten map is computed, we will review its construction here. Denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. They are the outcome of the following dot products:

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \quad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The quantities $\lambda_1, \lambda_2, \lambda_3$ are always real; here we explain why. The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $h^i_j(p)_{1 \leq i,j \leq 3}$ the matrix associated to the Weingarten map, that is:

$$L_p(\sigma_i(p)) = \sum_{k=1}^{3} h^k_i(p)\sigma_k(p),$$

Weingarten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are all real, at every point p of the hypersurface.

In the last decades several very interesting inequalities with geometric invariants have been obtained and investigated by many authors, see e.g. [2] for a thorough presentation of these developments.

3. A duality in triangle geometry

After this brief presentation of a few concepts pertaining to differential geometry, we will turn our attention to the classical geometry of a triangle. The duality described here is discussed also in [11]. Consider a triangle ΔABC in the Euclidean plane, with sides of lengths a, b, c.

Consider three arbitrary real numbers $x, y, z > 0$. By using the substitutions $a = y+z, b = z+x$, and $c = x+y$, one can immediately see that there exists a non-degenerate triangle in the Euclidean plane with sides of lengths a, b, c. The reason why this is true is that $a+b = x+y+2z > x+y = c$, hence the triangle inequality is always satisfied. Similarly for the other two triangle inequalities involving a, b, c.

The converse of this claim is also true. Namely, if a, b, c are the sides of a triangle in the Euclidean plane, then the system given through the equations $a = y+z$, $b = z+x$, and $c = x+y$, has a unique solution $x = s-a$, $y = s-b$, $z = s-c$, where we denote by s the semiperimeter of the triangle, namely $s = \frac{1}{2}(a+b+c)$.

Some classical facts in advanced Euclidean geometry can be proved by substituting a, b, c, instead of the variables x, y, z,. This technique is called Ravi substitution.

We can view these three positive real numbers x, y, z in a different way. We could view them as principal curvatures in the geometry of a three-dimensional smooth hypersurface lying in the four dimensional Euclidean space. To every triangle with sides a, b, c there corresponds a triple $x, y, z > 0$, and these numbers can be interpreted as principal curvatures, given pointwise, at some point p lying on the hypersurface. We call this correspondence a Ravi transformation and we would like to see whether this duality leads us to some interesting geometric interpretations of facts from triangle geometry into the geometry of hypersurfaces into $\mathbb{R}^4$ endowed with the canonical Euclidean product.

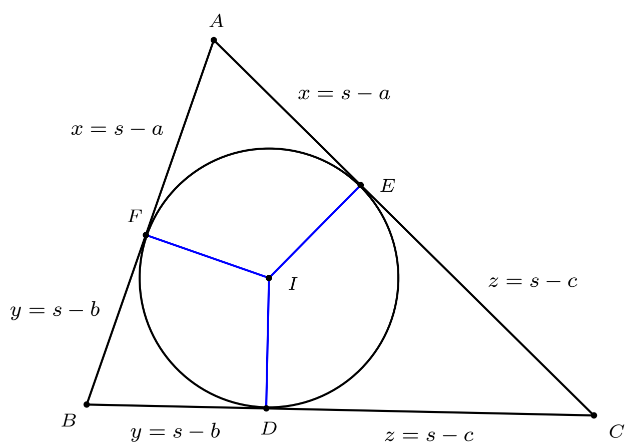


Figure 1

To better see how we use Ravi's substitutions, we recall here a few useful formulae for a triangle ΔABC lying in the Euclidean plane. Denote the area of the triangle by $\mathcal{A}$, its circumradius by R , its inradius by r, its semiperimeter by s, and its perimeter by $\mathcal{P}$. Then Heron's formula is $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz(x+y+z)}$. The inradius can be obtained as

$$r = \frac{\mathcal{A}}{s} = \sqrt{\frac{xyz}{x+y+z}},$$

and the circumradius can be obtained as

$$R = \frac{abc}{4\mathcal{A}} = \frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}}.$$

Euler's inequality was originally obtained in 1763 (see [6]) when he tried to construct a triangle by giving its incenter, its orthocenter, its barycenter and its circumcenter. That investigation generated several important consequences and ushered a new era in the investigations on the geometry of the triangle. Euler's inequality states that in any triangle in the Euclidean plane, the circumradius and the inradius satisfy $R \geq 2r$.

For a direct proof, using the formulae derived above, we note that by using the Ravi substitutions we need to show that the following holds:

$$\frac{(y+z)(z+x)(x+y)}{4\sqrt{xyz(x+y+z)}} \geq 2\sqrt{\frac{xyz}{x+y+z}},$$

or, by a direct computation:

$$(y+z)(z+x)(x+y) \geq 8xyz.$$

This last inequality holds true since $x + y \geq 2\sqrt{xy}$, and two other similar inequalities, multiplied term by term, conclude the argument. Actually we notice that the inequality $(y+z)(z+x)(x+y) \geq 8xyz$ is equivalent, via Ravi transformation, to Euler's inequality $R \geq 2r$.

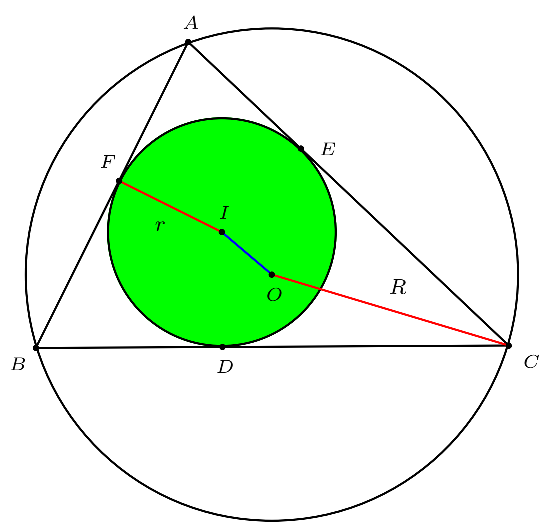


Figure 2

It is natural to consider a duality between a triangle ΔABC with lengths of sides a, b, c, and corresponding $x = s - a$, $y = s - b$, $z = s - c$, and a convex three-dimensional smooth hypersurface such that a point p lies on it, and the principal curvatures stand for $\lambda_1(p) = x$, $\lambda_2(p) = y$, $\lambda_3(p) = z$. The convexity condition is needed to insure that $x, y, z > 0$ in an open neighborhood around p. For further details about this idea see [11].

4. Constructing a curvature invariant inspired by Euler's inequality

We recall here a class of curvature invariants introduced in [9], namely the sectional absolute mean curvatures defined by

$$\bar{H}_{ij}(p) = \frac{|\lambda_i(p)| + |\lambda_j(p)|}{2}.$$

By using the same method as above Euler's inequality, $R \geq 2r$, transforms, as we have seen before, into

$$(y+z)(z+x)(x+y) \geq 8xyz,$$

which admits the interpretation in terms of curvature invariants as follows:

$$\bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{31} \geq K.$$

This inequality between the curvature invariants of a three dimensional smooth hypersurface in $\mathbb{R}^4$ was obtained and investigated in [9]. We recall here the following.

Definition. [3] Let $\sigma : U \subset \mathbb{R}^3 \to \mathbb{R}^4$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if all the principal curvatures satisfy $|k_1| = |k_2| = |k_3|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

In conclusion, a curvature invariant inspired by this result obtained by Leonhard Euler is

Definition.

$$E(p) = \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{31} = \frac{1}{8}(y+z)(z+x)(x+y).$$

Building on the development from [1], in [9] the following is proved.

Theorem 1. *Let $M^3 \subset \mathbb{R}^4$ be a smooth hypersurface and k_1, k_2, k_3 be its principal curvatures in the ambient space $\mathbb{R}^4$ endowed with the canonical metric. Let $p \in M$ be an arbitrary point. Denote by A the amalgamatic curvature (for its definition see* [3, 1]*), by H the mean curvature, $\bar{H}$ the absolute mean curvature, and K the Gauss-Kronecker curvature. Introduce the sectional absolute mean curvatures defined by*

$$\bar{H}_{ij}(p) = \frac{|k_i(p)| + |k_j(p)|}{2}.$$

Then the following inequalities hold true at every point $p \in M$:

$$A \cdot \bar{H}_{12} \cdot \bar{H}_{23} \cdot \bar{H}_{13} \geq \bar{H} \cdot |K| \geq A \cdot |K|.$$

Equality holds for absolutely umbilical points.

In conclusion, the inequality in the theorem can be written as follows.

Corollary 2.

$$A \cdot E \geq \bar{H} \cdot |K| \geq A \cdot |K|.$$

This relation naturally relates the amalgamatic curvature, the Gaussian curvature, and the curvature invariant E, which we introduced inspired by Euler's inequality for triangles in Euclidean geometry, $R \geq 2r$. In particular, note that we have the following.

Corollary 3.

$$E(p) \geq |K(p)| \tag{1}$$

at any point p of $M^3 \subset \mathbb{R}^4$. The equality holds at a point p if and only if the point is absolutely umbilical.

It may be of interest to note that this inequality may be viewed as an extension of the inequality $H^2(p) \geq K(p)$ from the geometry of surfaces.

We conclude this section by pointing out that all the algebra was definitely in place in Euler's times, and the only new part was the geometric interpretation in terms of curvature invariants. This vision was greatly enhanced after the major revisitation of curvature invariants described in Bang-Yen Chen's comprehensive monograph [2], which best represents a research direction with many developments (see e.g. [10, 12], among many other works), which inspires also the context of our present investigation.

5. Hypersurfaces of dimension n lying in the Euclidean space $\mathbb{R}^{n+1}$

In this section we inquire whether it is possible to extend the inequality (1) to hypersurfaces in the general case.

To establish our notations, let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n. Consider $\{\sigma_1(p), \sigma_2(p), ..., \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then we have

$$g_{ij}(p) = \langle \sigma_i(p), \sigma_j(p) \rangle, \qquad h_{ij}(p) = \langle N(p), \sigma_{ij}(p) \rangle.$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $(h^i_j(p))_{1 \leq i,j \leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p)) = h^k_i(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $k_1(p), k_2(p), ..., k_n(p)$ of this algebraic equation. The mean curvature at the point p is

$$H(p) = \frac{1}{n}[k_1(p) + ... + k_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = k_1(p)k_2(p)...k_n(p).$$

Definition. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if all the principal curvatures satisfy $|k_1| = |k_2| = ... = |k_n|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

Definition. Define the curvature invariant

$$E(p) = (2)^{-\binom{n}{2}} \prod_{i \neq j} (|k_i| + |k_j|).$$

With this definition we prove the following.

Proposition 4. *Let $M^n \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $k_1, k_2, \ldots, k_n$ be its principal curvatures in the ambient space $\mathbb{R}^{n+1}$ endowed with the canonical Euclidean metric. Let $p \in M$ be an arbitrary point. Denote by K the Gauss-Kronecker curvature. Then the following inequality holds true at every point $p \in M$:*

$$E(p) \geq |K(p)|^{\frac{n-1}{2}}.$$

Equality holds if p is an absolutely umbilical point.

Proof: Denote by $x_i = |k_i|$, for all $i = 1, 2, ..., n$. Then we have $\binom{n}{2}$ inequalities of the following form:

$$x_i + x_j \geq 2\sqrt{x_i x_j},$$

with equality if and only if $x_i = x_j$. By multiplying term by term these $\binom{n}{2}$ inequalities we obtain the stated result. □

It may be of interest to note that this inequality represents an extension of the inequality $H^2(p) \geq K(p)$ that holds true in the geometry of surfaces. To see this, let $n = 2$ in the statement above.

References

[1] B. Brzycki, M. D. Giesler, K. Gomez, L. H. Odom, and B. D. Suceavă, A ladder of curvatures for hypersurfaces in the Euclidean ambient space, *Houston Journal of Mathematics*, 40 (2014) 1347–1356.

[2] B.-Y. Chen, *Pseudo-Riemannian geometry, $\delta-$invariants and applications*, World Scientific, 2011.

[3] C. T. R. Conley, R. Etnyre, B. Gardener, L. H. Odom and B. D. Suceavă, New Curvature Inequalities for Hypersurfaces in the Euclidean Ambient Space, *Taiwanese Journal of Mathematics*, 17 (2013) 885–895.

[4] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.

[5] L. Euler, Recherches sur la courbure des surfaces, *Memoires de l'academie des sciences de Berlin* (written in 1760, published in 1767), 16: 119–143.

[6] L. Euler, Solutio facilis problematum quorundam geometricorum difficillimorum, *Novi Commentarii academiae scientiarum Petropolitanae* 11 (1767) 103–123.

[7] C. F. Gauss, *Disquisitiones circa superficies curvas*, Typis Dieterichianis, Goettingen, 1828.

[8] S. Germain, Mémoire sur la courbure des surfaces, *Journal für die reine und andewandte Mathematik, Herausgegeben von A. L. Crelle*, Siebenter Band, pp. 1–29, Berlin, 1831.

[9] B. D. Suceavă, The amalgamatic curvature and the orthocurvatures of three dimensional hypersurfaces in $\mathbb{E}^4$, *Publicationes Mathematicae*, 87 (2015) 35–46.

[10] B. D. Suceavă, The spread of the shape operator as conformal invariant, *J. Inequal. Pure Appl. Math.*, 4 (2003) no. 4, Article 74, 8 pp. (electronic).

[11] B. D. Suceavă, A geometric interpretation of curvature inequalities on hypersurfaces via Ravi substitutions in the euclidean plane, *Mathematical Intelligencer*, (to appear)

[12] B. D. Suceavă and M. B. Vajiac, Remarks on Chen's fundamental inequality with classical curvature invariants in Riemannian Spaces, *Annals Sti. Univ. Al. I. Cuza, Math.*, 54 (2008) 27–37.

Nicholas D. Brubaker: California State University Fullerton, 800 N. State College Blvd, 154 McCarthy Hall, Fullerton, California 92831-6850, U.S.A.
E-mail address: `nbdubaker@fullerton.edu`

Jasmine Camero: California State University Fullerton, 800 N. State College Blvd, 154 McCarthy Hall, Fullerton, California 92831-6850, U.S.A.
E-mail address: `jasminecamero@csu.fullerton.edu`

Oscar Rocha Rocha: California State University Fullerton, 800 N. State College Blvd, 154 McCarthy Hall, Fullerton, California 92831-6850, U.S.A.
E-mail address: `oscrocha167@csu.fullerton.edu`

Roberto Soto: California State University Fullerton, 800 N. State College Blvd, 154 McCarthy Hall, Fullerton, California 92831-6850, U.S.A.
E-mail address: `rcsoto@fullerton.edu`

Bogdan D. Suceavă: California State University Fullerton, 800 N. State College Blvd, 154 McCarthy Hall, Fullerton, California 92831-6850, U.S.A.
E-mail address: `bsuceava@fullerton.edu`

Forum Geometricorum
Volume 18 (2018) 129–133.

FORUM GEOM
ISSN 1534-1178

Constructing a Triangle from Two Vertices and the Symmedian Point

Michel Bataille

Abstract. Given three noncollinear points A, B, K, we propose a simple straightedge-and-compass construction of C such that K is the symmedian point of triangle ABC. Conditions for the construction to be possible and the number of solutions are studied in details.

1. Introduction

The problem of constructing a triangle ABC given two vertices A, B and one of the familiar centers (centroid, incenter, orthocenter, circumcenter), which appears in Wernick's famous list published in 1982 ([6, 7]), is easily solved. The case when the selected center is the symmedian point K being not listed, this article aims at offering a simple construction of the third vertex C from A, B, K and at studying the conditions on the three given points for the existence of a solution. In what follows, we always assume that the point K does not lie on the line AB.

2. The construction

The construction of a suitable point C will rest upon a not-very-well-known property of the symmedians, stated and proved in [1]. To keep this paper self-contained, it is presented here with a different proof.

Proposition 1. *Let ABC be a triangle and let B' be the reflection of B in A. The symmedian through A is the tangent at A to the circumcircle of $\Delta CAB'$.*

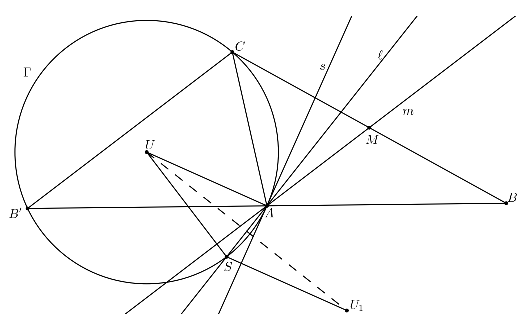


Figure 1

Publication Date: May 17, 2018. Communicating Editor: Paul Yiu.

Proof. Let M be the midpoint of BC and let s be the symmedian through A, that is, the reflection of the median $m = AM$ in the internal bisector ℓ of $\angle BAC$ (Figure 1). Let U be the center of the circle Γ through C, A, B' and S the point of intersection other than A of ℓ and Γ. Since ℓ is the external bisector of $\angle CAB'$, we have $SC = SB'$ and it follows that US is perpendicular to $B'C$, hence to $m = AM$. As a result, s is perpendicular to U_1S, where U_1 is the reflection of U in ℓ, hence to UA since UAU_1S is a rhombus. □

Turning to our problem, consider the three points A, B, K and let B' be the reflection of B in A and A' be the reflection of A in B. Let γ_A (resp. γ_B) denote the unique circle passing through B' (resp. A') and tangent to AK at A (resp. tangent to BK at B). The point K is the symmedian point of ΔABC if and only if the lines AK and BK are the symmedians through A and B, respectively. Proposition 1 shows that this will occur if and only if the circles γ_A and γ_B both pass through C. Thus, the construction of the circles γ_A and γ_B readily yields the desired third vertex C as a common point of these circles, provided that this common point exists. Clearly, our problem has at most two solutions (see Figure 2, where two solutions C_1 and C_2 are obtained).

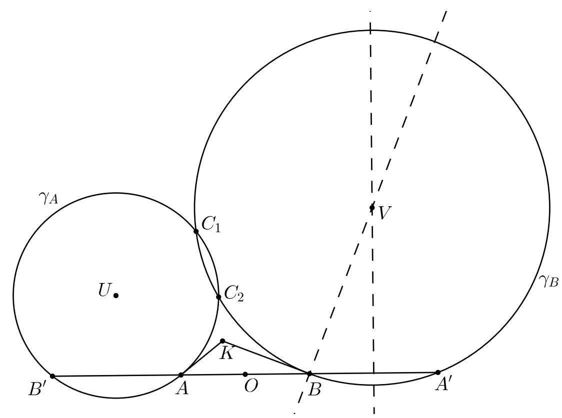


Figure 2

3. The discussion

From the construction above, the number of solutions for C depends on the relative position of the circles γ_A and γ_B. The way this position is related to the location of the points A, B, K is described in the following result:

Theorem 2. *The number of solutions for C is 0, 1, or 2, according as the sum $KA + KB$ is greater than, equal to, or less than $\frac{2AB}{\sqrt{3}}$.*

Proof. Let U and V be the centers of γ_A and γ_B, and let O denote the midpoint of AB (Figure 2). In a suitable system of axes with origin at O, we have $A(-\frac{c}{2}, 0),\ B(\frac{c}{2}, 0)$ (where $c = AB$) and we set $K(m, n)$. Expressing that V is the point of intersection of the perpendicular bisector of BA' (with equation $x = c$) and the perpendicular to BK at B (whose equation $\left(m - \frac{c}{2}\right) x + ny = \frac{c}{2}\left(m - \frac{c}{2}\right)$

is easily obtained), we find $V(c, v)$ where $v = \frac{c}{2n}\left(\frac{c}{2} - m\right)$. Similarly, $U(-c, u)$ where $u = \frac{c}{2n}\left(\frac{c}{2} + m\right)$ and a short calculation yields

$$UV^2 = \frac{c^2(m^2 + 4n^2)}{n^2}, \quad UA^2 = \frac{c^2((2m + c)^2 + 4n^2)}{16n^2}, \quad VB^2 = \frac{c^2((2m - c)^2 + 4n^2)}{16n^2}. \tag{1}$$

The condition $UV > |VB - UA|$ always holds because

$$UV^2 = 4c^2 + |v - u|^2 = 4c^2 + VB^2 - \frac{c^2}{4} + UA^2 - \frac{c^2}{4} - 2uv > VB^2 + UA^2 - 2UA \cdot VB.$$

Thus, the circles γ_A and γ_B are secant (resp. tangent) if and only if $UV < UA + VB$ (resp. $UV = UA + VB$).

Now, from (1), and with some algebra, we see that $UV < UA + VB$ is equivalent to

$$28n^2 + 4m^2 - c^2 < \sqrt{(4n^2 + 4m^2 + c^2)^2 - 16c^2m^2},$$

which itself is equivalent to

$$4m^2 + 28n^2 < c^2 \quad \text{or} \quad 3m^2 + 12n^2 < c^2,$$

and finally to $3m^2 + 12n^2 < c^2$.

The latter condition means that K is interior to the ellipse $\mathcal{E}$ with foci A and B and major axis $\frac{2c}{\sqrt{3}}$. Clearly, the circles γ_A and γ_B are tangent if and only if K lies on $\mathcal{E}$. Since $\mathcal{E}$ is the set of all points P such that $PA + PB = \frac{2c}{\sqrt{3}}$, the proof is complete. □

4. Properties of the solutions

First, we examine the case when the problem has two solutions C_1 and C_2 (Figure 2) and prove the following

Proposition 3. *If $C_1 \neq C_2$ and K is the symmedian point of both ΔABC_1 and ΔABC_2, then the line C_1C_2 is a median of each triangle.*

Proof. We just observe that the midpoint O of AB has the same power with respect to γ_A and γ_B (since $OA \cdot OB' = \frac{3c^2}{4} = OB \cdot OA'$) and deduce that O is on the radical axis C_1C_2 of these two circles. □

In the case when the solution C is unique, the triangle ABC has a feature which is worth mentioning:

Proposition 4. *If K is the symmedian point of ΔABC for a unique point C, then $CA^2 + CB^2 = 2AB^2$.*

In other words, the triangle is a C-root-mean-square triangle (see [2, 4] for numerous properties of such triangles).

Proof. Again, O lies on the radical axis of γ_A and γ_B, hence on the common tangent to γ_A and γ_B at C. It follows that $\frac{3c^2}{4} = OC^2$; but, CO being the length of the median from C, we also have $4CO^2 = 2(CA^2 + CB^2) - c^2$ and a short calculation gives the desired relation. □

Note in passing that when K traverses the ellipse $\mathcal{E}$, the corresponding unique solution C traverses the circle with center O and radius $\frac{c\sqrt{3}}{2}$ (less the common points with the line AB, of course).

5. Further remarks

To conclude, we draw some interesting corollaries of the results above.
In an arbitrary triangle ABC, not only does the symmedian point K satisfy $KA + KB \leq \frac{2AB}{\sqrt{3}}$, but it also satisfies the similar inequalities associated with the pairs of vertices (B, C) and (C, A):

Corollary 5. *The symmedian point K of any triangle ABC is located in the common part of the interiors of the three ellipses with foci A and B, with foci B and C, with foci C and A, and with respective major axes $\frac{2AB}{\sqrt{3}}$, $\frac{2BC}{\sqrt{3}}$, $\frac{2CA}{\sqrt{3}}$ (boundary included).*

It is known that K is always interior to the circle with diameter GH where G is the centroid and H the orthocenter (see [3]). However, the common part interior to the three ellipses can be much smaller (Figure 3).

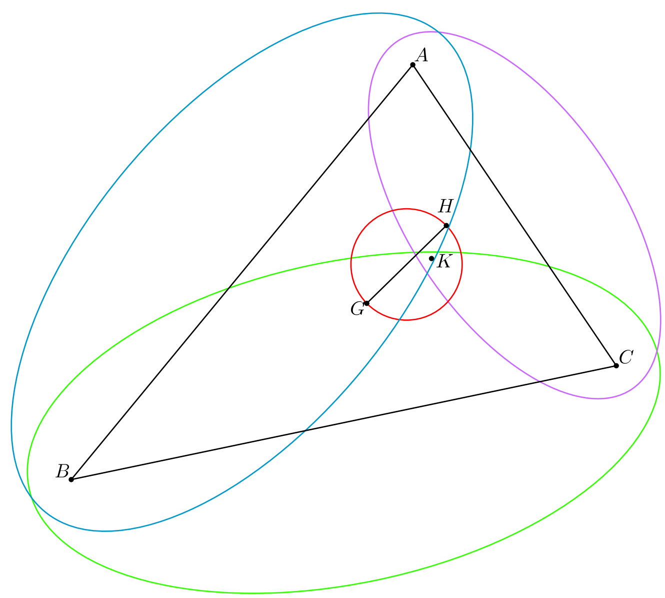


Figure 3

Lastly, we deduce a nice geometric inequality, perhaps difficult to prove directly:

Corollary 6. *Let m_a, m_b, m_c be the lengths of the medians of a triangle ABC with sides $a = BC, b = CA, c = AB$. Then, the following inequality holds:*

$$\max\{bm_c + cm_b, cm_a + am_c, am_b + bm_a\} \leq \frac{a^2 + b^2 + c^2}{\sqrt{3}}. \tag{2}$$

Interestingly, substituting $m_a, m_b, m_c, \frac{3a}{4}, \frac{3b}{4}, \frac{3c}{4}$ for a, b, c, m_a, m_b, m_c, respectively, leaves the inequality unchanged, meaning that the inequality is its own median-dual ([5] p. 109). No hope then to derive it from a known inequality through median duality!

Proof. We know that the symmedian point K of ΔABC is the center of masses of (A, a^2), (B, b^2), (C, c^2), that is, $(a^2 + b^2 + c^2)K = a^2A + b^2B + c^2C$. It follows that $(a^2 + b^2 + c^2)\overrightarrow{AK} = b^2\overrightarrow{AB} + c^2\overrightarrow{AC}$ and so

$$\begin{aligned}(a^2 + b^2 + c^2)^2 AK^2 &= b^4c^2 + c^4b^2 + (b^2c^2)(2\overrightarrow{AB} \cdot \overrightarrow{AC}) \\ &= b^2c^2(b^2 + c^2 + b^2 + c^2 - a^2) \\ &= 4b^2c^2m_a^2.\end{aligned}$$

As a result, $KA = \frac{2bcm_a}{a^2+b^2+c^2}$ and similarly we obtain $KB = \frac{2cam_b}{a^2+b^2+c^2}$. The inequality $KA + KB \leq \frac{2c}{\sqrt{3}}$ now rewrites as $bm_a + am_b \leq \frac{a^2+b^2+c^2}{\sqrt{3}}$. Cyclically, the numbers $bm_c + cm_b$ and $cm_a + am_c$ are also less than or equal to $\frac{a^2+b^2+c^2}{\sqrt{3}}$. $\square$

Note that equality holds if and only if ΔABC is a root-mean-square triangle, that is, if and only if one of the relations $2a^2 = b^2 + c^2,\ 2b^2 = c^2 + a^2,\ 2c^2 = a^2 + b^2$ is satisfied.

References

[1] M. Bataille, Characterizing a symmedian, *Crux Math.*, 43 (2017) 148.
[2] M. Bataille, On the centres of root-mean-square triangles, *Crux Math.*, 44 (2018) 46–51.
[3] C. J. Bradley and G. C. Smith, The locations of triangle centers, *Forum Geom.*, 6 (2006) 57–70.
[4] J. C. Fisher, Recurring Crux configurations, *Crux Math.*, 37 (2011) 304–307.
[5] D. S. Mintrinović, J. E. Pečarić, and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer, 1989.
[6] P. Schreck, P. Mathis, V. Marinković, and P. Janicić, Wernick's List: A Final Update, *Forum Geom.*, 16 (2016) 69–80.
[7] W. Wernick, Triangle Constructions with Three Located Points, *Math. Mag.*, 55 (1982) 227–230.

Michel Bataille: 6 square des Boulots, 76520 Franqueville-Saint-Pierre, France
E-mail address: michelbataille@wanadoo.fr

Forum Geometricorum
Volume 18 (2018) 135–140.

FORUM GEOM
ISSN 1534-1178

The "Circle" of Apollonius in Hyperbolic Geometry

Eugen J. Ionaşcu

Abstract. In Euclidean geometry the circle of Apollonius is the locus of points in the plane from which two collinear adjacent segments are perceived as having the same length. In Hyperbolic geometry, the analog of this locus is an algebraic curve of degree four which can be bounded or "unbounded". We study this locus and give a simple description of this curve using the Poincaré half-plane model. In the end, we give the motivation of our investigation and calculate the probability that three collinear adjacent segments can be seen as of the same positive length under some natural assumptions about the setting of the randomness considered.

1. Introduction

There are at least four circles known as Apollonius circles in the history of classical geometry. We are only concerned with only one of them: "the set of all points whose distances from two fixed points are in a constant ratio ρ ($\rho \neq 1$) " (see [1], [5]). In Euclidean geometry this is equivalent to asking for the locus of points P satisfying $\angle APB \equiv \angle BPC$, given fixed collinear points A, B and C. This same locus is the focus of our investigation in Hyperbolic geometry.

We are going to use the half-plane model, $\mathbb{H}$ (viewed in polar coordinates as the upper-half, i.e., $\theta \in (0, \pi)$), to formulate our answer to this question (see Anderson [2] for the terminology and notation used). Without loss of generality we may assume that the three points are on the y-axis: $A(0, a)$, $B(0, b)$ and $C(0, c)$, with real positive numbers a, b and c such that $a > b > c$.

Theorem 1. *Given points A, B and C as above, the set of points $P(x, y)$ in the half-plane $\mathbb{H}$, characterized by the equality $\angle APB \equiv \angle BPC$ in $\mathbb{H}$ is the curve given in polar coordinates by*

$$r^4(2b^2 - a^2 - c^2) = 2r^2(a^2c^2 - b^4)\cos(2\theta) + b^2(2a^2c^2 - a^2b^2 - c^2b^2). \quad (1)$$

Moreover,

(i) if $b = \left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$, this curve is half of the hyperbola of equation

$$r^2\cos(2\theta) + b^2 = 0, \ \ \theta \in (\frac{\pi}{4}, \frac{3\pi}{4}),$$

Publication Date: May 17, 2018. Communicating Editor: Paul Yiu.

(ii) if $b=\sqrt{ac}$ *this curve is the semi-circle* $r=b$*,* $\theta\in(0,\pi)$*,*

(iii) if $b=\left(\frac{a^{-2}+c^{-2}}{2}\right)^{-\frac{1}{2}}$*, this curve is half of the lemniscate of equation*

$$r^2+b^2\cos(2\theta)=0,\ \ \theta\in(\frac{\pi}{4},\frac{3\pi}{4}).$$

In most of the textbooks the Circle of Apollonius is discussed in conjunction with the Angle Bisector Theorem: "*The angle bisector in a triangle divides the opposite side into a ratio equal to the ratio of the adjacent sides.*" Once one realizes that the statement can be equally applied to the exterior angle bisector, then the Circle of Apollonius appears naturally (Figure 1), since the two angle bisectors are perpendicular.

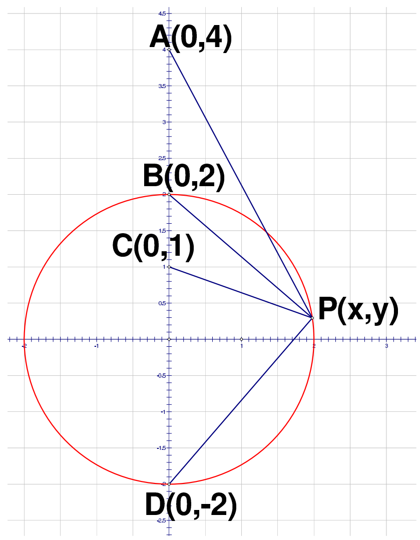


Figure 1. The Circle $x^2+y^2=4$

For instance, an easy exercise in algebra shows that the circle of equation $x^2+y^2=4$ is equivalent to

$$\frac{\sqrt{x^2+(y-4)^2}}{\sqrt{x^2+(y-1)^2}}=\frac{BA}{BC}=\frac{DA}{DC}=2,$$

taking $A(0,4)$, $B(0,2)$, $C(0,1)$ and $D(0,-2)$. Similar calculations can be employed to treat the general situation, i.e., taking $A(0,a)$, $B(0,b)$ and $C(0,c)$ with real positive numbers a, b and c such that $a>b>c$. Then we can state the well-known result:

Theorem 2 (Apollonius)**.** *Given points* A*,* B *and* C *as above, the set of points* $P(x,y)$ *in the plane characterized by the equality* $\angle APB\equiv\angle BPC$ *is*

(i) the line of equation $y=(a+c)/2$ *if* $b=(a+c)/2$*;*

(ii) the circle of equation $x^2+y^2=b^2$*, if* $b<(a+c)/2$ *and* $b^2=ac$*.*

Let us observe that the statement of this theorem does not reduce the generality since the y-coordinate of point D can be shown to be $y_D = (2ac - bc - ab)/(a + c - 2b)$ and one can take the origin of coordinates to be the midpoint of $\overline{BD}$. This turns out to happen precisely when $b^2 = ac$.

It is interesting that each of the special cases in Theorem 1, can be accomplished using integer values of a, b and c. This is not surprising for the Diophantine equation $b^2 = ac$ since one can play with the prime decomposition of a and c to get ac a perfect square. For the equation $2b^2 = a^2 + c^2$ one can take a Pythagorean triple and set $a = |m^2 + 2mn - n^2|$, $c = |m^2 - 2mn - n^2|$ and $b = m^2 + n^2$ for $m, n \in \mathbb{N}$. Perhaps it is quite intriguing for some readers that the last Diophantine equation $2a^2c^2 = a^2b^2 + c^2b^2$ is satisfied by the product of some quadratic forms, namely

$$\begin{aligned} a &= (46m^2 + 24mn + n^2)(74m^2 + 10mn + n^2), \\ b &= (46m^2 + 24mn + n^2)(94m^2 + 4mn - n^2), \\ \text{and } \ c &= (94m^2 + 4mn - n^2)(74m^2 + 10mn + n^2), \ \text{ for } \ m, n \in \mathbb{Z}. \end{aligned}$$

Another surprising fact is that the locus is a circle both in Theorem 1 and Theorem 2 if $b^2 = ac$.

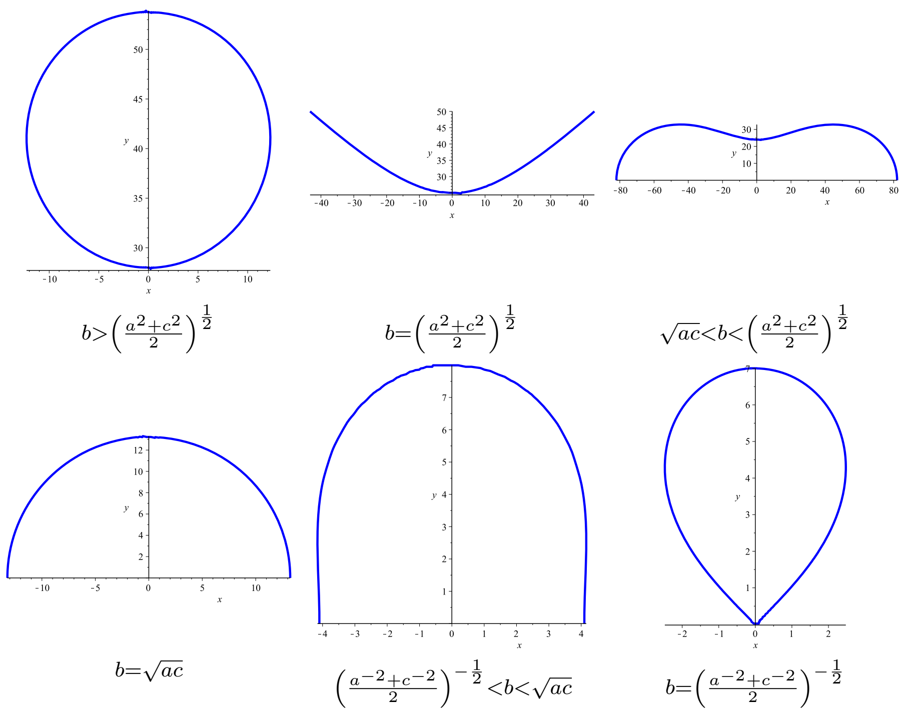


Figure 2. The curve in polar coordinates (a=35, c=5), in $\mathbb{H}$

In Figure 2, we included all of the possible shapes of the locus in Theorem 1 except for the case $b < \left(\frac{a^{-2}+c^{-2}}{2}\right)^{-\frac{1}{2}}$ which is similar to the case $b > \left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$. We notice a certain symmetry of these cases showing that the hyperbola ($b =$

$\left(\frac{a^2+c^2}{2}\right)^{\frac{1}{2}}$) is nothing else but a lemniscate in hyperbolic geometry. With this identification, it seems like the curves we get, resemble all possible shapes of the intersection of a plane with a torus.

In the next section we will prove Theorem 1.

2. Proof of Theorem 1

Let us consider a point P of coordinates (x, y) with the given property as in Figure 3 which is not on the line $\overline{AC}$, ($x \neq 0$). Then the Hyperbolic lines determined by P and the three points A, B and C are circles orthogonal on the x-axis. We denote their centers by $A'(a', 0)$, $B'(b', 0)$ and $C'(c', 0)$. The point A' can be

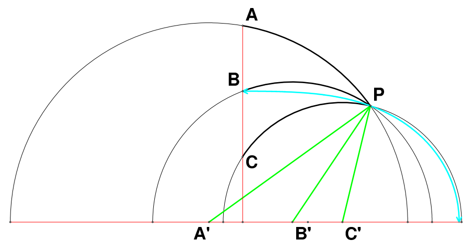


Figure 3. The point P and the lines determined by it with A, B and C

obtained as the intersection of the perpendicular bisector of $\overline{PA}$ and the x-axis. Similarly we obtain the other two points B' and C'. The equation (Y as a function of X) of the perpendicular bisector of $\overline{PA}$ is $Y - \frac{y+a}{2} = -\frac{x}{y-a}(X - \frac{x}{2})$ and so $a' = \frac{x^2+y^2-a^2}{2x}$. Similar expressions are then obtained for b' and c', i.e., $b' = \frac{x^2+y^2-b^2}{2x}$ and $c' = \frac{x^2+y^2-c^2}{2x}$. This shows that the order of the points A', B' and C' is reversed ($a' < b' < c'$). The angle between the Hyperbolic lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PA}$ is defined by the angle between the tangent lines to the two circles at P, which is clearly equal to the angle between the radii corresponding to P in each of the two circles. So, $m_{\mathbb{H}}(\angle APB) = m(\angle A'PB')$ and $m_{\mathbb{H}}(\angle BPC) = m(\angle B'PC')$. This equality is characterized by the proportionality given by the Angle Bisector Theorem in the triangle $PA'C'$:

$$\frac{PA'}{PC'} = \frac{A'B'}{B'C'} \Leftrightarrow \frac{\sqrt{(x^2-y^2+a^2)^2+4x^2y^2}}{\sqrt{(x^2-y^2+c^2)^2+4x^2y^2}} = \frac{a^2-b^2}{b^2-c^2}.$$

Using polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, we observe that $x^2 - y^2 = r^2\cos 2\theta$ and $2xy = r^2\sin 2\theta$. Hence the above equality is equivalent to

$$(r^4 + 2a^2r^2\cos 2\theta + a^4)(b^2 - c^2)^2 = (r^4 + 2c^2r^2\cos 2\theta + c^4)(a^2 - b^2)^2.$$

One can check that a factor of $(a^2 - c^2)$ can be simplified out and in the end we obtain (1).

3. Four points "equally" spaced and our motivation

Our interest in this locus was motivated by the Problem 11915 in this Monthly ([4]). This problem stated: *Given four (distinct) points A, B, C and D in (this) order on a line in Euclidean space, under what conditions will there be a point P off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure? (Figure 4)*

It is not difficult to show, using two Apollonius circles, that the existence of such a point P is characterized by the inequality involving the cross-ratio

$$[A,B;C,D] = \tfrac{BC}{BA}/\tfrac{DC}{DA} < 3. \tag{2}$$

We were interested in finding a similar description for the same question in Hyperbolic space. It is difficult to use the same idea of the locus that replaces the Apollonius circle in Euclidean geometry due to the complicated description as in Theorem 1. Fortunately, we can use the calculation done in the proof of Theorem 1 and formulate a possible answer in the new setting. Given four points A, B, C and D in (this) order on a line in the Hyperbolic space, we can use an isometry to transform them on the line $x = 0$ and having coordinates $A(0,a)$, $B(0,b)$, $C(0,c)$ and $D(0,d)$ with $a > b > c > d$. The the existence of a point P off the line $x = 0$, such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure in the Hyperbolic space is equivalent to the existence of P in Euclidean space corresponding to the points A', B', C' and D' as constructed in the proof of Theorem 1. Therefore, the answer to the equivalent question posed in the Problem 11915 but in Hyperbolic geometry, is in terms of a similar inequality

$$[A',B';C',D'] = \tfrac{B'C'}{B'A'}/\tfrac{D'C'}{D'A'} < 3 \Leftrightarrow \frac{(b^2-c^2)(a^2-d^2)}{(a^2-b^2)(c^2-d^2)} < 3. \tag{3}$$

Now we can use (2) and (3) to compute the following natural corresponding geometric probability: *if two points B and C are randomly selected (uniform distribution with respect to the arc-length in hyperbolic geometry) on the segment $\overline{AD}$ (B being the closest to A), what is the probability that a point P off the line $\overline{AD}$ exists, such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure? (Figure 4)*

In Euclidean geometry this turns out to be equal to

$$P_e = \frac{15 - 16\ln 2}{9} \approx 0.4345$$

The inequality (3) gives us the similar probability in the Hyperbolic space:

$$P_h = \frac{2\sqrt{5}\ln(2+\sqrt{5}) - 5}{5\ln 2} \approx 0.4201514924$$

where the uniform distribution here means that it is calculated with respect to the measure $\frac{1}{y}dy$ along the y-axis.

Our probability question is even more natural in the setting of spherical geometry. Due to the infinite nature of both Euclidean and Hyperbolic spaces the

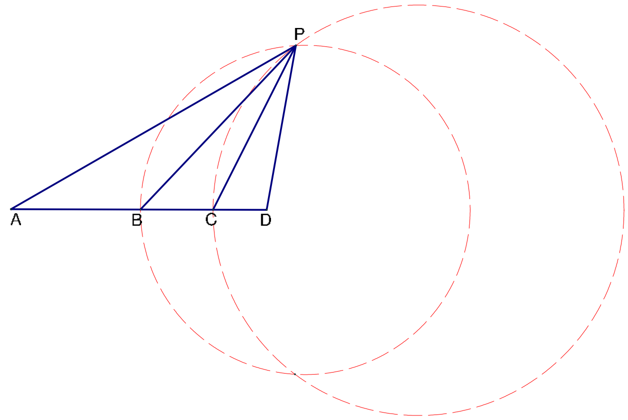


Figure 4. The point P and the lines determined by it with A, B , C and D

geometric probability question makes sense only in limiting situations. In the case of spherical geometry we can simply ask (and leave it as an open question):

Given a line in spherical geometry and four points on it, chosen at random with uniform distribution, what is the probability that the points look equidistant from a point on the sphere that is not on that line?

References

[1] C. V. Durell, *Modern Geometry: The Straight Line and Circle*, London: Macmillan, p. 16, 1928.
[2] J. W. Anderson, *Hyperbolic Geometry*, 2nd ed.-Springer undergraduate mathematics series, Springer-Verlag London Limited, 2005.
[3] B. Hopkins, ed., *Resources for Teaching Discrete Mathematics*, Mathematical Association of America, Washington DC, 2009.
[4] M. E. Kidwell and M. D. Meyerson, Problem 11915, *Amer. Math. Monthly*, 123 (2016) 613; solution, 125 (2018) 467–468.
[5] C. S. Ogilvy, *Excursions in Geometry*, New York: Dover, pp. 14-23, 1990.

Eugen J. Ionaşcu: Mathematics Department, Columbus State University, Columbus, Georgia 31907, USA

E-mail address: `math@ejionascu.ro`

Forum Geometricorum
Volume 18 (2018) 141–154.

FORUM GEOM
ISSN 1534-1178

A Geometric Inequality for Cyclic Quadrilaterals

Mark Shattuck

Abstract. In this paper, we establish an inequality involving the cosines of the arc measures determined by four arbitrary points along the circumference of a circle. Our proof is analytic in nature and as a consequence of it, we obtain some inequalities involving the interior angle measures of a triangle. Extending our arguments yields a sharpened version of the aforementioned inequality and also the best possible positive constant for which it holds.

1. Introduction

A variety of inequalities have been shown for convex quadrilaterals [4], in particular, in the bicentric case (see, e.g., [5, 6]). Perhaps the most well-known of these and simplest states that $\frac{r}{R}$ is at most $\frac{1}{\sqrt{2}}$, where r and R denote the radii of the incircle and circumcircle of a bicentric quadrilateral $ABCD$ (see [1, p. 132]). A refinement of this inequality was shown by Yun [7] and is given by

$$\frac{r\sqrt{2}}{R} \le \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \le 1, \tag{1}$$

which was proved again in different ways by Josefsson [3] and later Hess [2]. Note that the middle quantity in (1) may be rewritten in terms of the measures of arcs subtended by the sides of quadrilateral $ABCD$ as

$$\frac{1}{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right),$$

where $2a = m(\overset{\frown}{AB})$, etc.

In this paper, we find a lower bound for this quantity (see Theorem 16 below) that applies to all cyclic quadrilaterals and that differs from (and is incomparable to) the bound given in (1) in the bicentric case. For the sake of clarity of the proofs, we first establish the following bound in the next two sections.

Theorem 1. *Let $ABCD$ be a cyclic quadrilateral, with $2a = m(\overset{\frown}{AB})$, $2b = m(\overset{\frown}{BC})$, $2c = m(\overset{\frown}{CD})$ and $2d = m(\overset{\frown}{DA})$. If $a = \max\{a,b,c,d\}$, then we have*

$$2\cos a+\cos b+\cos c+\cos d \le \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right), \tag{2}$$

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.

with equality if and only if $ABCD$ is a square.

We assume that all arc measures in Theorem 1 and elsewhere are taken in the same direction as the labeling of the vertices of $ABCD$. To prove (2), we proceed analytically, treating separately the acute and obtuse cases of a. We then extend our arguments in finding a sharpened form of (2), along with the best positive constant for which it holds. That is, we show that one may replace the weights of $\frac{1}{5}(2,1,1,1)$ for the respective cosine terms on the left-hand side of (2) (after dividing both sides of (2) by 5) with $\frac{1}{\alpha+3}(\alpha,1,1,1)$, where $\alpha = \frac{12-3\sqrt{2}}{4+\sqrt{2}} \approx 1.43$ is as small as possible. As corollaries to our approach, we obtain some related geometric inequalities involving triangles and bicentric quadrilaterals.

2. The acute case of the inequality

To prove (2), we divide into cases based on whether or not a is acute, treating in this section the acute case. Let

$$\begin{aligned} &f(a,b,c,d)\\ &=\frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(2\cos a+\cos b+\cos c+\cos d) \end{aligned}$$

and

$$S=\{(a,b,c,d)\in\mathbb{R}^4 : a+b+c+d=\pi,\ \text{where } 0\le b,c,d\le a\le \pi/2\}.$$

We will regard S as a closed subset of the metric space M consisting of all points in $\mathbb{R}^4$ such that $a+b+c+d=\pi$. When speaking of the boundary or interior of S, it will be in reference to M. Then the acute case of (2) is equivalent to showing $f \ge 0$ for all points in S. The next four lemmas show that f is non-negative for all points along the boundary of S.

Lemma 2. *We have $f > 0$ for all points (a,b,c,d) in S such that $abcd = 0$.*

Proof. We show that $h \ge 0$, where

$$h(a,b,c,d)=\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(\cos a+\cos b+\cos c+\cos d),$$

with $0 \le a,b,c,d \le \pi/2$, $abcd = 0$ and $a+b+c+d=\pi$, which implies the stated result for f. To show $h \ge 0$, we may assume $d = 0$, by symmetry. Thus, we must show

$$j(a,b,c)=\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b}{2}\right)\right)-(\cos a+\cos b+\cos c+1)\ge 0, \quad (3)$$

where $a+b+c=\pi$ and $0 \le a,b,c \le \pi/2$. We first show that inequality (3) holds for the boundary values. By symmetry of the a and c variables, we need only consider the following cases: (i) $a=0$, (ii) $b=0$, (iii) $a=\pi/2$, (iv) $b=\pi/2$. If (i) or (ii) holds, then the other two variables must be $\pi/2$ and (3) is obvious in either case. If (iii), then we must show $\sqrt{2}\left(\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)+\cos\left(\frac{b}{2}\right)\right)\geq\cos b+\cos c+1$, where $b+c=\pi/2$, i.e.,

$$2\cos\left(\frac{b}{2}\right)\geq\frac{1}{\sqrt{2}}(\sin b+\cos b+1)=\sin\left(b+\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}},\qquad 0\leq b\leq\pi/2.$$

The last inequality follows from observing that the function $k(b)=2\cos\left(\frac{b}{2}\right)-\sin\left(b+\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}$ is decreasing on $(0,\pi/2)$ with $k(\pi/2)=0$. If (iv), then we need $\sqrt{2}\cos\left(\frac{a-c}{2}\right)\geq\cos a+\cos c=2\cos\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)$, which holds with equality since $a+c=\pi/2$.

To complete the proof of (3), we must consider any possible interior local extreme points of j. Making use of Lagrange multipliers and equating the a- and the c- partial derivative equations implies (I) $a+c=\pi/2$ or (II) $a=c$. Note that if (I) holds, then $b=\pi/2$, which has already been considered, so assume (II). Using the cosine double angle formula, we must show in this case that

$$\sqrt{2}(\cos y+1)-4\cos^2 x-2\cos^2 y+2\geq 0,\tag{4}$$

where $2x+y=\pi/2$ and $0<x,y<\pi/4$. For (4), we show equivalently $\sqrt{2}(\cos y+1)\geq 2\sin y+2\cos^2 y$, i.e.,

$$\sqrt{2}\cos y+2\left(\sin^2 y-\sin y\right)\geq 2-\sqrt{2},\qquad 0<y<\pi/4.$$

This last inequality holds since the two sides are equal when $y=\pi/4$, with the left-hand side decreasing as one may verify. $\square$

By (3) and the fact that $\cos(\angle A)+\cos(\angle B)+\cos(\angle C)=1+\frac{r}{R}$ in a triangle ABC, we obtain the following bound on the ratio $\frac{r}{R}$ in an acute or right triangle.

Corollary 3. *Let ABC be a non-obtuse triangle. Then*

$$2+\frac{r}{R}\leq\sqrt{2}\left(\cos\left(\frac{A-C}{2}\right)+\cos\left(\frac{B}{2}\right)\right),$$

with equality if and only if ABC is a right triangle with hypotenuse AC.

Lemma 4. *We have $f>0$ for all points (a,b,c,d) in S such that $a=\pi/2$.*

Proof. Let

$$h(b,c,d)=\frac{5\sqrt{2}}{4}\left(\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(\cos b+\cos c+\cos d).$$

We must show $h>0$ for all $b,c,d\geq 0$ such that $b+c+d=\pi/2$. We may assume $bcd\neq 0$, by the previous lemma. To show $h>0$, we apply Lagrange multipliers and equate the b- and d-partial derivative equations to obtain $\sin b-\sin d=\frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$, i.e., $2\sin\left(\frac{b-d}{2}\right)\cos\left(\frac{b+d}{2}\right)=\frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$. So we must have (i) $b+d=2\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$ or (ii) $b=d$. If (i) holds, then $\frac{\pi}{4}-\frac{c}{2}=\frac{b+d}{2}$ so that $\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)=\frac{5\sqrt{2}}{8}$, which implies $h>0$ in this case. So assume (ii) holds and we must show $h(b,c,b) > 0$, where $b, c > 0$ and $2b + c = \frac{\pi}{2}$. Substituting $c = \frac{\pi}{2} - 2b$ into $h(b,c,b)$, we need for

$$q(b) = \left(\frac{5\sqrt{2}}{4} - 2\right)\cos b - \sin 2b + \frac{5\sqrt{2}}{4} > 0, \qquad 0 < b < \pi/4.$$

This holds since $2 - \frac{5\sqrt{2}}{4} < \frac{1}{4}$ implies $q(b) > -\frac{1}{4} + \frac{5\sqrt{2}}{4} - 1 = \frac{5}{4}\left(\sqrt{2} - 1\right) > 0$. $\square$

As a consequence of the previous lemma, we obtain the following trigonometric inequality for acute triangles.

Corollary 5. *If ABC is an acute triangle, then*

$$\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) < \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{\pi - A}{4}\right) + \cos\left(\frac{B - C}{4}\right)\right).$$

The next two lemmas concern the case when one of the other variables equals a.

Lemma 6. *We have $f \geq 0$ for all points (a,b,c,d) in S such that $a = c$.*

Proof. We must show

$$h(b,c,d) = \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{b-d}{2}\right) + 1\right) - (\cos b + 3\cos c + \cos d) \geq 0, \qquad (5)$$

where $0 \leq b, d \leq c \leq \pi/2$ and $b + 2c + d = \pi$. By Lemmas 2 and 4, we may assume $b, d > 0$ and $c < \pi/2$ in (5). Suppose first that b or d equals c, say d. Then $h(b,c,c) \geq 0$ if and only if $\frac{5\sqrt{2}}{4}\left(\cos\left(\frac{b-c}{2}\right) + 1\right) \geq \cos b + 4\cos c$, where $b + 3c = \pi$ and $0 < b \leq c < \pi/3$, i.e.,

$$\frac{5\sqrt{2}}{4}\left(\sin(2c) + 1\right) \geq 4\cos c - \cos(3c), \qquad \pi/4 \leq c < \pi/3. \qquad (6)$$

Dividing both sides of (6) by $\cos c$, we need to show

$$k(c) = \frac{5\sqrt{2}}{4}\left(2\sin c + \sec c\right) + 4\cos^2 c - 7 \geq 0, \qquad \pi/4 \leq c < \pi/3.$$

This last inequality follows from noting

$$\frac{k'(c)}{\cos c} = \frac{5\sqrt{2}}{4}\left(2 + \sec^2 c\tan c\right) - 8\sin c > 0$$

for $\pi/4 < c < \pi/3$, with $k(\pi/4) = 0$.

By Lagrange's method, any interior extreme points (b,c,d) within the set over which we are minimizing h must satisfy (i) $b + d = 2\theta$, where $\theta = \cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$, or (ii) $b = d$. If (i) holds, then we have $c = \frac{\pi}{2} - \theta$ and thus

$$\begin{aligned}
&\frac{5\sqrt{2}}{4}\left(1 + \cos\left(\frac{b-d}{2}\right)\right) - (\cos b + \cos d) \\
&= \frac{5\sqrt{2}}{4} + 2\cos\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right) - (\cos b + \cos d) = \frac{5\sqrt{2}}{4} > \frac{3\sqrt{14}}{8} \\
&= 3\cos c,
\end{aligned}$$

which implies (5) in this case. If (ii) holds, then we need to show $\frac{5\sqrt{2}}{2} \geq 2\cos b + 3\cos c$, where $b + c = \pi/2$ and $0 < b \leq c$. Note that $2\cos b + 3\cos c = 2\sin c + 3\cos c = \sqrt{13}\sin(c+\lambda)$, where $\lambda = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$. Then $\lambda > \pi/4$ implies $\sqrt{13}\sin(c+\lambda) \leq \sqrt{13}\sin\left(\frac{\pi}{4}+\lambda\right) = \frac{5\sqrt{2}}{2}$ for $\pi/4 \leq c < \pi/2$, as desired, which establishes (5) and completes the proof. Note that there is equality in (5) iff $a = b = c = d = \pi/4$, i.e., iff $ABCD$ is a square. □

Lemma 7. *We have $f \geq 0$ for all points (a,b,c,d) in S such that $a = b$.*

Proof. We must show

$$h(a,c,d) = \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{a-d}{2}\right)\right) - (3\cos a + \cos c + \cos d) \geq 0, \tag{7}$$

where $0 \leq c, d \leq a \leq \pi/2$ and $2a + c + d = \pi$. By Lemmas 2 and 4, we may assume $c, d > 0$ and $a < \pi/2$ in (7). By Lemma 6, we may also assume $a > c$ and $a > d$. Thus, we need only check points (a,c,d) corresponding to any possible interior extrema of the function h. Such points must satisfy $\sin c - \sin d = \frac{5\sqrt{2}}{8}\left(\sin\left(\frac{a-d}{2}\right) - \sin\left(\frac{a-c}{2}\right)\right)$, i.e., $\sin\left(\frac{c-d}{2}\right)\cos\left(\frac{c+d}{2}\right) = \frac{5\sqrt{2}}{8}\sin\left(\frac{c-d}{4}\right)\cos\left(\frac{2a-c-d}{4}\right)$, from which we get

$$\text{(i) } \frac{5\sqrt{2}}{16}\cos\left(\frac{2a-c-d}{4}\right) = \cos\left(\frac{c-d}{4}\right)\cos\left(\frac{c+d}{2}\right), \text{ or (ii) } c = d.$$

Since $c + d = \pi - 2a$ and $\pi/4 < a < \pi/2$, we have $c + d < \pi/2$ and thus

$$\cos\left(\frac{c-d}{4}\right)\cos\left(\frac{c+d}{2}\right) > \frac{1}{2} > \frac{5\sqrt{2}}{16} \geq \frac{5\sqrt{2}}{16}\cos\left(\frac{2a-c-d}{4}\right),$$

whence (i) is not possible. So assume (ii), in which case we must show

$$k(a) = \frac{5\sqrt{2}}{2}\cos\left(a - \frac{\pi}{4}\right) - \sqrt{13}\sin(a+\lambda) \geq 0, \qquad \pi/4 < a < \pi/2, \tag{8}$$

where $\lambda = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$. Note that $k(\pi/4) = 0$, with

$$k'(a) = -\frac{5\sqrt{2}}{2}\sin\left(a - \frac{\pi}{4}\right) + \sqrt{13}\sin\left(a - \left(\frac{\pi}{2} - \lambda\right)\right) > 0, \qquad \pi/4 < a < \pi/2,$$

since $\sqrt{13} > \frac{5\sqrt{2}}{2}$ and $\frac{\pi}{4} > \frac{\pi}{2} - \lambda$ implies $\sqrt{13}\sin\left(a - \left(\frac{\pi}{2} - \lambda\right)\right) > \frac{5\sqrt{2}}{2}\sin\left(a - \frac{\pi}{4}\right)$. This implies (8), which completes the proof. □

We now must consider any possible local minima of f located within the interior of S. Treating these cases will yield the following result.

Theorem 8. *We have $f \geq 0$ for all points in S, with equality if and only if $a = b = c = d = \pi/4$.*

Proof. Let (a, b, c, d) denote an interior local extreme point of f in S. From the a- and the c-partial derivative equations, such points must satisfy

$$2\sin a - \sin c = \frac{5\sqrt{2}}{4}\sin\left(\frac{a-c}{2}\right). \tag{9}$$

Given $0 < a < \pi/2$, define the function

$$h_a(x) = \frac{5\sqrt{2}}{4}\sin\left(\frac{x-a}{2}\right) - \sin x + 2\sin a, \qquad 0 < x < a.$$

Below, we show that the equation $h_a(x) = 0$ has no solution. Hence, neither does (9), which implies f has no interior extreme points. Thus, the minimum value of f on the compact set S must be achieved along its boundary, and by Lemmas 2, 4, 6 and 7, we have $f \geq 0$ for all points along the boundary. Thus, $f \geq 0$ on all of S, as desired. From the proofs of Lemmas 6 and 7, there is equality as stated.

We now show $h_a(x)$ has no solution. First suppose $0 < x < \frac{a}{2}$. Then

$$\begin{aligned} h_a(x) >&\ -\frac{5\sqrt{2}}{4}\sin\left(\frac{a}{2}\right) - \sin\left(\frac{a}{2}\right) + 4\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right) \\ >&\ \left(2\sqrt{2} - 1 - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{2}\right) > 0, \end{aligned}$$

since $a < \pi/2$ implies $\cos(a/2) > \sqrt{2}/2$. If $a/2 \leq x < a$, then

$$\begin{aligned} h_a(x) &> -\frac{5\sqrt{2}}{4}\sin\left(\frac{a}{4}\right) + \sin a = \left(4\cos\left(\frac{a}{2}\right)\cos\left(\frac{a}{4}\right) - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{4}\right) \\ &> \left(2 - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{4}\right) > 0, \end{aligned}$$

since $\cos(a/2),\ \cos(a/4) > \sqrt{2}/2$. □

3. The obtuse case

Assume now $2a' = m(\widehat{AB})$ is at least π, as shown below. In this case, we replace a' with $\pi - a$, where $a = b + c + d$ is acute (or possibly right).

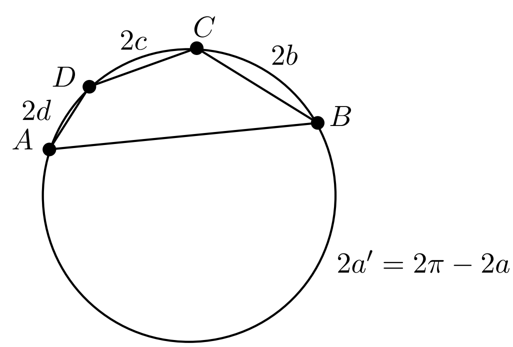


Figure 1. Case when $2a' = m(\widehat{AB})$ exceeds π.

Define the set T by

$$T = \{(a, b, c, d) \in \mathbb{R}^4 : a = b + c + d, \text{ where } b, c, d \geq 0 \text{ and } a \leq \pi/2\}.$$

We regard T as a closed subset of the metric space comprising all points in $\mathbb{R}^4$ such that $a = b + c + d$. Define the function g on $\mathbb{R}^4$ by

$$g(a, b, c, d) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right) + 2\cos a - (\cos b + \cos c + \cos d).$$

Then the task of establishing (2) in the case when $m(\widehat{AB})$ is at least π is equivalent to showing that g can assume only positive values on T. We first consider the case of a point in T where $c = 0$.

Lemma 9. *We have $g > 0$ for all points (a, b, c, d) in T such that $c = 0$.*

Proof. We must show

$$h(a, b, d) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right) + 2\cos a - (\cos b + \cos d + 1) > 0, \tag{10}$$

where $a = b + d$, $0 \leq a \leq \pi/2$ and $b, d \geq 0$. By Lemma 4, we may assume $a < \pi/2$. If $b = 0$ or $d = 0$, say $b = 0$, then $a = d$ and (10) reduces in this case to

$$\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{a}{2}\right)\right) + \cos a - 2 > 0, \qquad 0 \leq a < \pi/2.$$

The last inequality can be shown by considering the first two derivatives of the left-hand side.

We now check h at any possible interior extreme points (a, b, d). From the b- and d-partial derivative equations, we get $\sin b - \sin d = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$, i.e., $2\sin\left(\frac{b-d}{2}\right)\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$. This implies such points (a, b, d) satisfy (i) $\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{8}$ or (ii) $b = d$. If (i) holds, then $a = b + d$ implies $\cos a = \frac{9}{16}$, $\sin\left(\frac{a}{2}\right) = \frac{\sqrt{14}}{8}$ and $\cos\left(\frac{b-d}{2}\right) \geq \frac{5\sqrt{2}}{8}$. Since $\cos b + \cos d \leq 2\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{4}$, inequality (10) follows in this case. If (ii) holds, then (10) reduces to

$$k(b) = \frac{5\sqrt{2}}{4}(\sin b + 1) + 2\cos 2b - 2\cos b - 1 > 0, \qquad 0 < b < \pi/4. \tag{11}$$

To show (11), first note that

$$\frac{k'(b)}{\cos b} = \frac{5\sqrt{2}}{4} - 8\sin b + 2\tan b.$$

Since the function $8\sin b - 2\tan b$ is increasing on $(0, \pi/4)$, one has that $k'(b)$ changes sign once on $(0, \pi/4)$, from positive to negative. Inequality (11) now follows from observing that $k(0)$ and $k(\pi/4)$ are both positive, which completes the proof. $\square$

Lemma 10. *We have $g > 0$ for all points (a, b, c, d) in T such that $d = 0$.*

Proof. We must show

$$h(a,b,c) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + \cos\left(\frac{b}{2}\right)\right) + 2\cos a - (\cos b + \cos c + 1) > 0, \tag{12}$$

where $a = b + c$, $0 \le a \le \pi/2$ and $b, c \ge 0$. Again, we may assume $a < \pi/2$. By the previous lemma, we may also assume $c > 0$. On the other hand, if $b = 0$, then $a = c$ and (12) reduces to $\frac{5\sqrt{2}}{4}(\sin a + 1) + \cos a - 2 > 0$, where $0 \le a < \pi/2$, which follows from observing $\max\{\sin a, \cos a\} \ge \sqrt{2}/2$ for first quadrant a.

We now consider possible internal extreme points (a, b, c) of h and apply the Lagrange method with constraint $-a + b + c = 0$. From the b- and c-partial derivative equations, we get

$$2\sin\left(\frac{b-c}{2}\right)\cos\left(\frac{a}{2}\right) = \frac{5\sqrt{2}}{4}\cos\left(\frac{\pi}{4} + \frac{c}{2}\right)\cos\left(\frac{\pi}{4} - \frac{a}{2}\right). \tag{13}$$

We show that (13) cannot hold, which will imply the desired result. Note first that (13) implies $b > c$ and hence $0 < c < a/2$. Also, observe that the ratio $\sin\left(\frac{\pi}{4} - \frac{c}{2}\right) / \sin\left(\frac{a}{2} - \frac{c}{2}\right)$ for a fixed a is minimized when $c = 0$. Thus, we have

$$\begin{aligned}\frac{\cos\left(\frac{\pi}{4} + \frac{c}{2}\right)}{\sin\left(\frac{b}{2} - \frac{c}{2}\right)} \cdot \frac{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)} &> \frac{\sin\left(\frac{\pi}{4} - \frac{c}{2}\right)}{\sin\left(\frac{a}{2} - \frac{c}{2}\right)} \cdot \frac{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)} > \frac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{a}{2}\right)} \cdot \frac{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)} \\ &= \frac{\sin\left(\frac{a}{2}\right) + \cos\left(\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)} = \frac{1}{2}\left(\sec\left(\frac{a}{2}\right) + \csc\left(\frac{a}{2}\right)\right).\end{aligned}$$

Since the function $\sec\left(\frac{a}{2}\right) + \csc\left(\frac{a}{2}\right)$ is decreasing on $(0, \pi/2)$, we have $\sec\left(\frac{a}{2}\right) + \csc\left(\frac{a}{2}\right) \ge 2\sqrt{2}$ and thus

$$\frac{\cos\left(\frac{\pi}{4} + \frac{c}{2}\right)}{\sin\left(\frac{b}{2} - \frac{c}{2}\right)} \cdot \frac{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)} > \sqrt{2},$$

whence (13) cannot hold, which completes the proof. □

Theorem 11. *We have $g > 0$ for all points in T.*

Proof. By Lemmas 4, 9 and 10, we have already shown that $g > 0$ for all points along the boundary of the compact set T. To show that $g > 0$ on all of T, we need to check g at any possible interior extrema. To do so, we apply Lagrange multipliers to g with constraint $b + c + d - a = 0$. Equating the a- and the c- partial derivative equations gives

$$2\sin a - \sin c = \frac{5\sqrt{2}}{4}\cos\left(\frac{a+c}{2}\right). \tag{14}$$

Let $\theta = 2\sin^{-1}\left(5\sqrt{2}/16\right)$ and a be fixed where $0 < a < \theta$. Then (14) cannot hold for $0 < c < a < \theta$, upon considering the function $h_a(x) = \sin x + \frac{5\sqrt{2}}{4}\cos\left(\frac{x+a}{2}\right) - 2\sin a$ for each a and showing $h_a(x) > 0$ for $0 < x < a$ (by showing $h_a(0) > 0$, $h_a(a) > 0$ with $h_a''(x) < 0$ for $0 < x < a$). Thus, there are no interior extreme points in T for which $a < \theta$. Henceforth, let us assume $a \ge \theta$.

Equating the b- and d-partial derivative equations gives

$$\sin b - \sin d = \frac{5\sqrt{2}}{4} \sin\left(\frac{b-d}{2}\right),$$

which implies (i) $b = d$ or (ii) $\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{8}$. First assume (i). We will show that $g > 0$ for all interior points in T such that $b = d$ and $a \geq \theta$. In this case, we must show $j(a, b, c) > 0$, where

$$j(a,b,c) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) + 2\cos a - 2\cos b - \cos c$$

and $a = 2b + c$. By the concavity of the cosine function on $(0, \pi/2)$, we have $2\cos b + \cos c \leq 3\cos\left(\frac{2b+c}{3}\right) = 3\cos\left(\frac{a}{3}\right)$, so to show $j > 0$, it suffices to show $k(a, c) > 0$, where

$$k(a,c) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) + 2\cos a - 3\cos\left(\frac{a}{3}\right).$$

Since $a \geq \theta$, we have $\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) > \frac{5}{2}$, so clearly $k(a, c) > 0$ for $\theta \leq a < \cos^{-1}(1/4)$. On the other hand, if $a \geq \cos^{-1}(1/4)$, then $\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) > 2.85$, whereas $3\cos\left(\frac{a}{3}\right) < 2.72$, which again implies $k(a, c) > 0$.

So assume (ii) holds and let $\lambda = 2\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$. Then $a - c = b + d = \lambda$ and we show that (14) has no solution in this case. Suppose to the contrary that (14) does hold for some a and c, where $a > \lambda$ and $c = a - \lambda$. Then

$$\frac{5\sqrt{2}}{4}\cos\left(\frac{a+c}{2}\right) = \frac{5\sqrt{2}}{4}\cos\left(a - \frac{\lambda}{2}\right) = \frac{25}{16}\cos a + \frac{5\sqrt{7}}{16}\sin a,$$

so that (14) holds if and only if

$$\left(2 - \frac{5\sqrt{7}}{16}\right)\sin a - \frac{25}{16}\cos a = \sin c = \sin(a-\lambda) = \sin a\cos\lambda - \cos a\sin\lambda,$$

i.e.,

$$q(a) = \left(2 - \frac{5\sqrt{7}}{16} - \cos\lambda\right)\sin a - \left(\frac{25}{16} - \sin\lambda\right)\cos a = 0. \tag{15}$$

Since the coefficients inside the parentheses are both positive quantities, $q(a)$ is an increasing function of a, where $\lambda < a < \pi/2$. Since $q(\lambda) > 0$, it follows that (15) and hence (14) cannot hold if (ii) does, which completes the proof. $\square$

Theorem 1 above now follows from combining Theorems 8 and 11, which cover the acute and obtuse cases of a, respectively, upon replacing a by $\pi - a$ in Theorem 11. $\square$

We obtain as a corollary to Theorem 1 the following variant of inequality (1).

Corollary 12. *Let $ABCD$ be a cyclic quadrilateral, with $2a = m(\overset{\frown}{AB})$, $2b = m(\overset{\frown}{BC})$, $2c = m(\overset{\frown}{CD})$ and $2d = m(\overset{\frown}{DA})$. If AB is the longest side length of $ABCD$, then*

$$\begin{aligned}&\frac{\sqrt{2}}{5}(2\cos a + \cos b + \cos c + \cos d)\\ &\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \leq 1. \quad (16)\end{aligned}$$

Proof. We have

$$\begin{aligned}&\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\\ &= \sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\sin\frac{A}{2} + \cos\frac{A}{2}\sin\frac{B}{2} + \cos\frac{B}{2}\cos\frac{A}{2}\\ &= \sin\left(\frac{A+B}{2}\right) + \cos\left(\frac{A-B}{2}\right) = \sin\left(\frac{b+2c+d}{2}\right) + \cos\left(\frac{b-d}{2}\right)\\ &= \sin\left(\frac{\pi-(a-c)}{2}\right) + \cos\left(\frac{b-d}{2}\right) = \cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right),\end{aligned}$$

so that the right inequality in (16) is clear. The left inequality then follows from Theorem 1. Note that there is equality in the right inequality iff $ABCD$ is a rectangle and in the left iff $ABCD$ is a square. □

4. A sharpened version of the inequality

To sharpen inequality (2), we seek the smallest positive constant δ such that

$$\begin{aligned}f_\delta(a,b,c,d) = &\ \frac{(\delta+3)\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)\\ &- (\delta\cos a + \cos b + \cos c + \cos d) \geq 0 \quad (17)\end{aligned}$$

for all points (a,b,c,d) in S and

$$\begin{aligned}g_\delta(a,b,c,d) = &\ \frac{(\delta+3)\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)\\ &+ \delta\cos a - (\cos b + \cos c + \cos d) \geq 0 \quad (18)\end{aligned}$$

for all points in T. Theorem 1 above then corresponds to the $\delta = 2$ case. Upon dividing (17) by $\delta+3$, one sees that (17) amounts to comparing a certain weighted average of the individual cosine terms with the quantity $\frac{\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)$. A similar interpretation applies to (18). From this, one sees that if inequalities (17) and (18) hold for some $\delta_0 > 0$, then they hold for all $\delta > \delta_0$. Note that $\delta = 1$ is too small since in the case of a bicentric quadrilateral, the inequalities would be reversed (see Theorem 17 below). This leaves open the question of finding the best possible δ for which (17) and (18) hold where $1 < \delta < 2$.

Taking (a,b,c,d) equal to the origin in (18) implies that the best possible δ is at least $\frac{12-3\sqrt{2}}{4+\sqrt{2}} \approx 1.43$, which we will denote by α. In fact, by modifying appropriately the proofs of Theorems 8 and 11 above, one can show that (17) and (18) indeed hold when $\delta = \alpha$. In the proofs of the following lemmas, we carry out the most extensive modifications that are required.

Lemma 13. *Let*

$$u_a(x) = \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{x-a}{2}\right) - \sin x + \alpha \sin a, \qquad 0 < x < a,$$

where $\pi/4 < a < \pi/2$. *Then* $u_a(x) > 0$ *for* $0 < x < a$.

Proof. First assume $0 < x < a/2$. Then one may verify that $u_a'(0) < 0$ and $u_a''(x) > 0$. Note further that

$$u_a'(a/2) = \frac{(\alpha+3)\sqrt{2}}{8}\cos(a/4) - \cos(a/2)$$

is an increasing function of a, which is negative at $a = 86^\circ$ and positive at $a = 87^\circ$. First assume $45^\circ < a \le 86^\circ$. Then $u_a'(x) < 0$ for $0 < x < a/2$ and

$$u_a(a/2) = \alpha \sin a - \sin(a/2) - \frac{(\alpha+3)\sqrt{2}}{4}\sin(a/4) > 0, \qquad \pi/4 < a < \pi/2,$$

as one may verify, which implies $u_a(x) > 0$ for $0 < x < a/2$ in this case. If $86^\circ < a < 90^\circ$, then to establish the desired result in this case, it suffices to show

$$r(x) = \alpha \sin(86^\circ) - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(45^\circ - \frac{x}{2}\right) > 0, \qquad 0 < x < \pi/4. \tag{19}$$

Since $r'(43^\circ) < 0$, $r'(44^\circ) > 0$ and $r''(x) > 0$, the function $r(x)$ must achieve its minimum value somewhere on the interval $[43^\circ, 44^\circ]$. If $43^\circ \le x \le 44^\circ$, then

$$r(x) > \alpha \sin(86^\circ) - \sin(44^\circ) - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(45^\circ - \frac{43^\circ}{2}\right) > 0,$$

which implies (19).

Now suppose $a/2 \le x < a$. In this case, we fix x and consider $u_a(x)$ as a function of a, where $x < a \le 2x$. First assume $x \le \pi/4$ and note that $\frac{d^2}{da^2}u_a(x) < 0$ for all a. Since $u_x(x) > 0$ and

$$u_{2x}(x) = \alpha \sin(2x) - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin(x/2) > 0, \qquad 0 < x \le \pi/4,$$

as one may verify, it follows that $u_a(x) > 0$ for $x < a \le 2x$ in this case. On the other hand, if $x > \pi/4$, then $u_a(x) > 0$ for $x < a < \pi/2$ follows from observing $u_x(x) > 0$, $\frac{d^2}{da^2}u_a(x) < 0$ and

$$u_{\pi/2}(x) = \alpha - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{\pi}{4} - \frac{x}{2}\right) > 0, \qquad \pi/4 < x < \pi/2,$$

which completes the proof. $\square$

Lemma 14. *We have* $v \geq 0$ *for the function*

$$v(a,b,c) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{b-c}{2}\right)\right) + \alpha\cos a - (\cos b + \cos c + 1),$$

where $a = b + c$, $0 \leq a \leq \pi/2$ *and* $b, c \geq 0$.

Proof. By Lemma 4 (suitably extended to f_α), we may assume $a < \pi/2$. If b or c equals zero, say b, then $a = c$ and $v \geq 0$ reduces in this case to

$$r(a) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{a}{2}\right)\right) + (\alpha-1)\cos a - 2 \geq 0, \quad 0 \leq a \leq \pi/2. \tag{20}$$

Note that $\alpha > 1$ implies $r''(a) < 0$. Since $r(0) = 0$ (by definition of α) and $r(\pi/2) = \frac{\alpha-1}{2} > 0$, inequality (20) follows, with equality only for $a = 0$.

We now check v at any possible interior extreme points (a, b, c). From the b- and c-partial derivative equations, we get (i) $\cos\left(\frac{b+c}{2}\right) = \frac{(\alpha+3)\sqrt{2}}{8}$ or (ii) $b = c$. If (i) holds, then $a = b + c$ implies $\cos a \approx 0.23$ and $\sin(a/2) \approx 0.62$. Then we have in this case

$$\begin{aligned} v(a,b,c) &= \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{a}{2}\right) + 2\cos\left(\frac{b+c}{2}\right)\cos\left(\frac{b-c}{2}\right) \\ &\quad + \alpha\cos a - (\cos b + \cos c + 1) \\ &= \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{a}{2}\right) + \alpha\cos a - 1 > 0. \end{aligned}$$

If (ii) holds, then $v > 0$ is equivalent to

$$s(b) = \frac{(\alpha+3)\sqrt{2}}{4}(\sin b + 1) + \alpha\cos(2b) - 2\cos b - 1 > 0, \qquad 0 < b < \pi/4.$$

One can show that $s'(a)$ has one sign change on the interval $(0, \pi/4)$, from positive to negative. Since $s(0) = 0$ and $s(\pi/4) > 0$, this implies $s(b) > 0$ for $0 < b < \pi/4$, which completes the proof. $\square$

Lemma 15. *We have* $w > 0$ *for the function*

$$w(a,b,c) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) + \alpha\cos a - 2\cos b - \cos c,$$

where $a = 2b + c$, $b, c > 0$ *and* $\pi/4 < a < \pi/2$.

Proof. By the concavity of cosine on $(0, \pi/2)$, to show $w > 0$, it suffices to show

$$r(a) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + 1\right) + \alpha\cos a - 3\cos\left(\frac{a}{3}\right) > 0, \quad \pi/4 < a < \pi/2. \tag{21}$$

A direct computation reveals $r'''(a) > 0$ for $\pi/4 < a < \pi/2$ and that $r''(a)$ changes sign once on this interval. One may also verify $r'(\pi/4) < 0$ and $r'(\pi/2) < 0$, which implies $r'(a) < 0$ for all a. Inequality (21) now follows from observing $r(\pi/2) > 0$. $\square$

One then gets the following strengthened version of Theorem 1.

Theorem 16. *We have*

$$\alpha\cos a+\cos b+\cos c+\cos d\leq\frac{(\alpha+3)\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right),\tag{22}$$

where $\alpha=\frac{12-3\sqrt{2}}{4+\sqrt{2}}$, *for all* $0\leq b,c,d\leq a$ *such that* $a+b+c+d=\pi$, *with equality if and only if* $a=b=c=d=\pi/4$ *or* $\pi-a=b=c=d=0$. *Furthermore,* α *is the smallest constant for which* (22) *holds.*

Proof. We make appropriate modifications, which we briefly describe, to the proof of Theorem 1 above using f_α and g_α in place of f and g. Note that in the proofs of Theorems 8 and 11 above, one would use, respectively, Lemmas 13 and 15 where needed. One would also substitute Lemma 14 for 9. Note that the proof of Lemma 2 implies that this result also applies to the function f_α. In other proofs, one would proceed as before, but instead with the function f_α in the acute and g_α in the obtuse case, which may at times require a bit more analysis than previously. Observe further that the strict inequality for g in the statements of Lemmas 9 and 10 and Theorem 11 should be replaced by the inclusive inequality $g_\alpha\geq 0$, where there is equality when all arguments are zero. Finally, the statements of the lemmas and theorems in the acute case will remain unchanged when considering the sharpened version of the inequality. □

Note that Corollary 12 above may also be strengthened by replacing the leftmost expression $\frac{\sqrt{2}}{5}(2\cos a+\cos b+\cos c+\cos d)$ with the larger quantity $\frac{\sqrt{2}}{\alpha+3}(\alpha\cos a+\cos b+\cos c+\cos d)$. For bicentric quadrilaterals, the inequality is in fact reversed when α is replaced by 1.

Theorem 17. *Let ABCD be a bicentric quadrilateral, with* $2a=m(\overset{\frown}{AB})$, $2b=m(\overset{\frown}{BC})$, $2c=m(\overset{\frown}{CD})$ *and* $2d=m(\overset{\frown}{DA})$. *Then*

$$\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)\leq\cos a+\cos b+\cos c+\cos d,\tag{23}$$

with equality if and only if $ABCD$ *is a kite.*

Proof. Replacing the right-hand side of (23) by

$$2\cos\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)+2\cos\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right),$$

and rearranging, we show equivalently

$$\left(\cos\left(\frac{b+d}{2}\right)-\frac{\sqrt{2}}{2}\right)\cos\left(\frac{b-d}{2}\right)\geq\left(\frac{\sqrt{2}}{2}-\cos\left(\frac{a+c}{2}\right)\right)\cos\left(\frac{a-c}{2}\right).\tag{24}$$

Since $ABCD$ has an inscribed circle, we have $AB+CD=BC+DA$, which implies $\sin a+\sin c=\sin b+\sin d$. This may be rewritten as $\sin\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)=$

$\sin\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right)$, i.e., $\cos\left(\frac{a-c}{2}\right)=\tan\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right)$, since $a+b+c+d=\pi$. Substituting this into (24), and letting $x=\frac{b+d}{2}$, gives

$$\cos x-\sqrt{2}/2\geq(\sqrt{2}/2-\sin x)\tan x,\qquad 0<x<\pi/2. \tag{25}$$

Rearranging (25) gives $\sec x-\frac{\sqrt{2}}{2}\tan x\geq\frac{\sqrt{2}}{2}$ for $0<x<\pi/2$. To show this, let $f(x)=\sec x-\frac{\sqrt{2}}{2}\tan x$. Then $f'(x)=\sec^2 x\left(\sin x-\frac{\sqrt{2}}{2}\right)$ so that $f(x)$ is minimized when $x=\pi/4$, with $f(\pi/4)=\sqrt{2}/2$. This implies (25) and hence (23). Note that there is equality in (23) iff $x=\frac{b+d}{2}=\frac{\pi}{4}$, i.e., $a+c=\frac{\pi}{2}=b+d$. Since $ABCD$ is bicentric, we then have $a-c=b-d$ or $a-c=d-b$, which implies $a=b, c=d$ or $a=d, b=c$. Thus, there is equality iff $ABCD$ is a kite, which completes the proof. □

References

[1] O. Bottema, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, 1969.

[2] A. Hess, Bicentric quadrilaterals through inversion, *Forum Geom.* **13** (2013), 11–15.

[3] M. Josefsson, A new proof of Yun's inequality for bicentric quadrilaterals, *Forum Geom.* **12** (2012), 79–82.

[4] M. Josefsson, A few inequalities in quadrilaterals, *International J. Geom.* **4(1)** (2015), 11–15.

[5] O. T. Pop, Identities and inequalities in a quadrilateral, *Octogon Math. Mag.* **17(2)** (2009), 754–763.

[6] M. Radic, Certain inequalities concerning bicentric quadrilaterals, hexagons and octogons, *J. Inequal. Pure and Appl. Math.* **6(1)** (2005), Art. 1.

[7] Z. Yun, Euler's inequality revisited, *Math. Spectrum* **40** (2008), 119–121.

Mark Shattuck: Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996, USA

E-mail address: shattuck@math.utk.edu

Forum Geometricorum
Volume 18 (2018) 155–159.

FORUM GEOM
ISSN 1534-1178

Harmonic Mean and Division by Zero

Hiroshi Okumura and Saburou Saitoh

Dedicated to Professor Josip Pečarić on the occasion of his 70th birthday

Abstract. We examine the harmonic mean from the view point of the division by zero and show that the harmonic mean of a and 0 is $2a$ or a or zero from several examples. We further show that a result on an Archimedean circle of the arbelos still holds in the singular case by using the method of the division by zero.

1. Introduction

The simplest introduction of our division by zero is given strictly by the division by zero calculus: For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n,$$

we **define** the value of the function f at $z = a$ by

$$f(a) = C_0.$$

In particular, for the fundamental function

$$f(z) = \frac{a}{z},$$

we have $f(0) = 0$. In this paper, we need only this property - division by zero. For many motivations and applications of the division by zero calculus, see the recent papers [2, 3, 4, 5]. In this paper, without any information and background for the division by zero calculus we can discuss our mathematics **based on this assumption**. We will be able to see that this definition has a good sense from even this paper.

We recall that the harmonic mean $H(a, b)$ for non zero real numbers a, b is given by

$$H(a, b) = \frac{2ab}{a+b}, \tag{1}$$

or

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}. \tag{2}$$

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.

In this paper, we consider the case $b = 0$. When we use the representation (1), then we have $H(a, 0) = 0$, however, when we use the representation (2), we have $H(a, 0) = 2a$, by the division by zero. In this paper, we would like to show that the following result may also happen:

$$H(a, 0) = a.$$

Our interpretation for the last equation is: the zero term does not exist, while the harmonic mean of one quantity merely coincides with itself. We will show such an interesting example in section 3, which is a new phenomenon on Euclidean geometry.

2. Triangle case

We consider the triangle with vertices $(0, 0)$, $(1, 0)$ and point of intersection of the lines $y = bx$ and $y = -ax + a$ (see Figure 1). The point of intersection of the two line has coordinates $(a/(a + b), ab/(a + b)) = (a/(a + b), H/2)$. For the line $y = bx$, $b = 0$ implies $y = 0$ and we have $H = 0$. However, if we use the equation $y/b = x$ to express the same line, then $b = 0$ implies $x = 0$ by the division by zero. Therefore we have $H = 2a$.

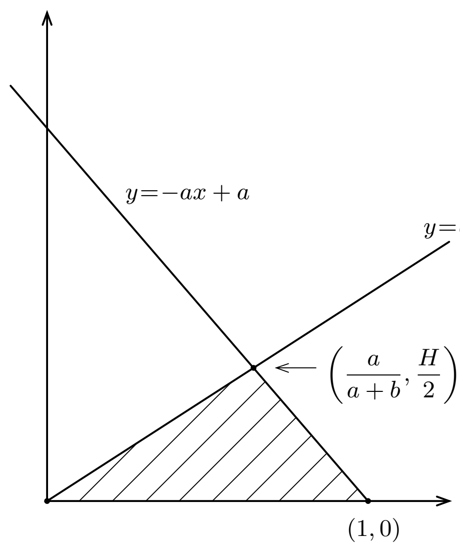


Figure 1.

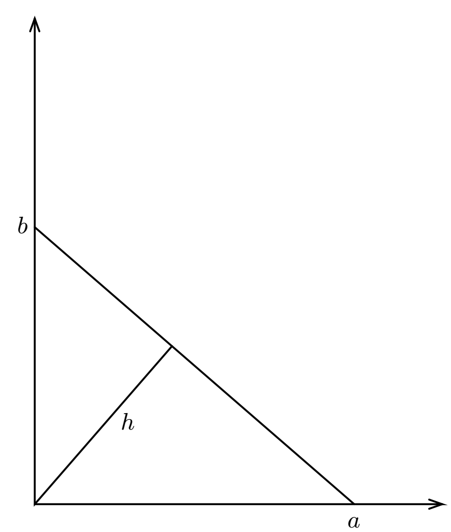


Figure 2.

We consider the line passing through the two points $(a, 0)$ and $(0, b)$ $(a, b > 0)$ and the distance h from the origin to the line (see Figure 2). The distance is given by

$$h^2 = \frac{H(a^2, b^2)}{2} = \frac{a^2 b^2}{a^2 + b^2}.$$

If we consider that the line is expressed by the equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

then $b = 0$ implies $x = a$ by the division by zero, i.e., $h = a$. Therefore $H(a^2, 0) = 2a^2$. Meanwhile, the equation $bx + ay = ab$ implies $y = 0$ if $b = 0$, i.e., $h = 0$. Therefore $H(a^2, 0) = 0$.

3. Semicircle case

For a fixed $a > 0$ and for $b > 0$, we consider the semicircle given by

$$x^2 - (a+b)x + y^2 = 0 \tag{3}$$

in the region $y \geq 0$. Let E be the point in common of the line $x = a$ and the semicircle, and let F be the foot of perpendicular from the point $(a, 0)$ to the line joining E and the point $((a+b)/2, 0)$ (see Figure 3). Then

$$EF = H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}.$$

The fact was observed by C. Dodge [1, p. 203] as a fact of the arbelos.

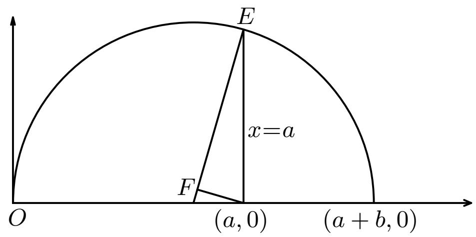


Figure 3.

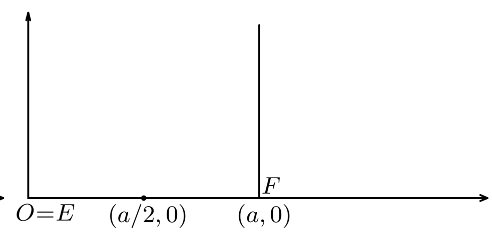


Figure 4.

If $b = 0$, (3) gives

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2.$$

This means $H = 0$.

However, from

$$\frac{x^2}{b} - \left(\frac{a}{b} + 1\right)x + \frac{y^2}{b} = 0,$$

we have $x = 0$ if $b = 0$ (see Figure 4). Since the lines $x = 0$ and $x = a$ are parallel, they meet in the origin [4], i.e., $E = O$, which means $H = a$.

4. An Archimedean circle of the arbelos

We consider the following theorem, which is stated as a result for an Archimedean circle of the arbelos denoted by W_5 in [1] (see Figure 5).

Theorem 1. *Let α and β be externally touching circles of radii a and b, respectively, with an external common tangent t. If c is the distance between t and the point of tangency of α and β, we have $c = H(a, b)$, i.e.,*

$$c = \frac{2ab}{a+b}, \tag{4}$$

or

$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b}. \tag{5}$$

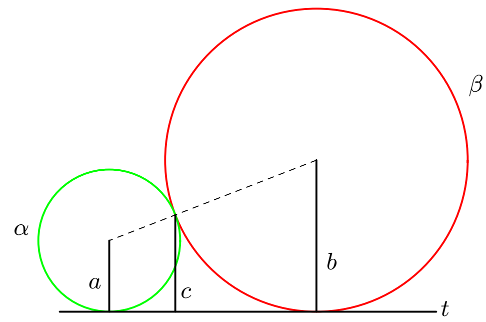


Figure 5.

A circle or a line has an equation $e(x^2+y^2)+2fx+2gy+d=0$, and its radius equals $\sqrt{(g^2+f^2-de)/e^2}$, if it is a circle. Therefore by the division by zero we have that the radius of a line equals 0 [4].

We now consider the case $b=0$ in Theorem 1 by the division by zero. We fix the circle α and use a rectangular coordinate system with origin at the point of tangency of α and t such that $(0,a)$ are the coordinates of the center of α. Since the distance between the points of tangency of α and β to t equals $2\sqrt{ab}$, we can assume that β is expressed by the equation $(x-2\sqrt{ab})^2+(y-b)^2=b^2$. The equation can also be stated as in the following three ways:

$$x^2+y^2-4\sqrt{ab}x+2b(2a-y)=0,$$

$$\frac{x^2+y^2}{\sqrt{b}}-4\sqrt{a}x+2\sqrt{b}(2a-y)=0,$$

and

$$\frac{x^2+y^2}{b}-4\sqrt{\frac{a}{b}}x+2(2a-y)=0.$$

Applying the division by zero, we obtain the following three equations in the case $b=0$, respectively

$$x^2+y^2=0, \quad x=0, \quad y=2a.$$

The last three equations show that β coincides with the origin, the y-axis, and the tangent of α parallel to t, respectively (see Figures 6, 7, 8). Notice that $\tan(\pi/2)=0$ by the division by zero calculus [4]. Therefore we can consider that the y-axis touches α and t. Hence the second case, β being the y-axis, totally makes sense.

In the first case and the second case, we get $b=c=0$ by the division by zero. Therefore (4) holds. In the third case, we have $b=0$ and $c=2a$. Therefore (5) holds. Therefore Theorem 1 still holds in those three cases. The last observation also shows that even if we consider $c=2a$ in the second case, which is the distance from the furthest point on α to t, then (5) also holds.

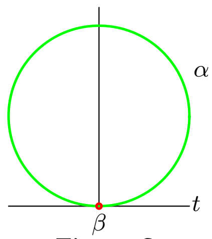


Figure 6.

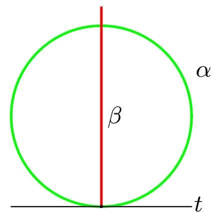


Figure 7.

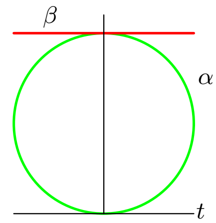


Figure 8.

If α and β are both points or both lines or a line and a point, we have $a = b = c = 0$. Hence Theorem 1 still holds in those cases.

References

[1] C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.*, 72 (1999) 202–213.

[2] M. Kuroda, H. Michiwaki, S. Saitoh, and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, *Int. J. Appl. Math.*, 27 (2014) 191–198.

[3] H. Michiwaki, S. Saitoh, and M. Yamada, Reality of the division by zero $z/0 = 0$, *Int. J. of Appl. Physics and Math.*, 6 (2015) 1–8.

[4] H. Michiwaki, H. Okumura and S. Saitoh, Division by zero $z/0 = 0$ in Euclidean spaces, *Int. J. Mathematics and Computation*, 28 (2017) 1–16.

[5] S. Saitoh, A reproducing kernel theory with some general applications, in Qian, Tao, Rodino, Luigi G. (Eds.): *Mathematical Analysis, Probability and Applications* - Plenary Lectures: Isaac 2015, Macau, China, Springer Proceedings in Mathematics and Statistics, 177 (2016) 151–182 (Springer).

Hiroshi Okumura: Maebashi Gunma 371-0123, Japan
E-mail address: `hokmr@yandex.com`

Saburou Saitoh: Institute of Reproducing Kernels, Kawauchi-cho, 5-1648-16, Kiryu 376-0041, Japan
E-mail address: `kbdmm360@yahoo.com.jp`

Forum Geometricorum
Volume 18 (2018) 161–184.

FORUM GEOM
ISSN 1534-1178

Rectangles Circumscribing a Quadrangle

Paris Pamfilos

Abstract. In this article we study the circumscribed rectangles about a quadrangle and in particular their extrema with respect to the area. We show that there are two such extrema represented by two similar rectangles. In addition we study the four similarities interchanging these extremal rectangles and show the existence of two twin quadrangles with given extremal rectangles.

1. Circumscribing rectangles

In this article we study the configuration created by an arbitrary non-orthodiagonal quadrangle $ABCD$ and the circles $\{\alpha, \beta, \gamma, \delta\}$ on diameters the sides of the quadrangle (See Figure 1). These circles carry the vertices of all the rectangles

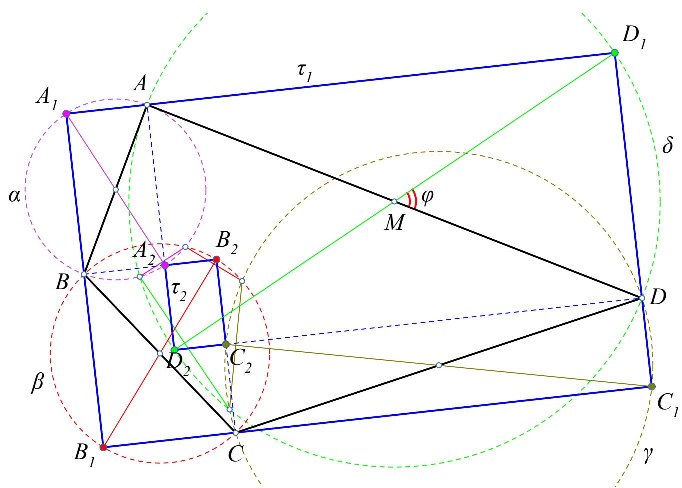


Figure 1. Rectangles circumscribing the quadrangle $ABCD$

that circumscribe the quadrangle, and the figure shows also two prominent such rectangles $\{\tau_1, \tau_2\}$, which are similar to each other. They represent two extrema of the signed area function $f(\phi)$ of the circumscribing rectangles. The big one τ_1 can be considered to have positive area and is the greatest, w.r.t. to the area, rectangle circumscribing $ABCD$. The small one τ_2 corresponds then to the minimal extremal value of the signed area function, which in section 3 is proved to be a periodic sinuisoidal function with two extrema, as seen in the corresponding graph in figure 2. In the graph are also noticed the two extrema, which occur at points $\{\phi = 0, \phi = \pi\}$, corresponding to diametral points on each one of the circles $\{\alpha, \beta, \gamma, \delta\}$. I will refer in the sequel to the two prominent rectangles as the

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.

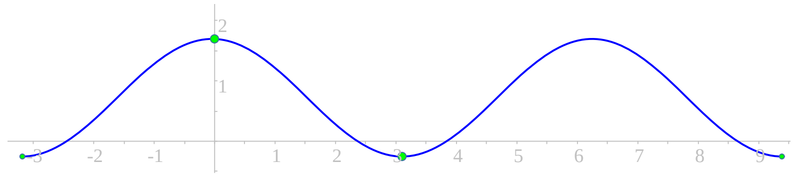


Figure 2. Graph of the area function

extremal rectangles of $ABCD$, distinguishing also between the *maximal* τ_1 with area E_b and the *minimal* one τ_2 with area E_s. The restriction on non-orthodiagonal quadrangles, i.e. those that have their diagonals not orthogonal, is equivalent with the condition, that not three of the circles $\{\alpha, \beta, \gamma, \delta\}$ are concurrent at a point, which in turn will be seen below to be equivalent with the non-degeneration of the minimal rectangle τ_2 or, equivalently, the condition $E_s \neq 0$.

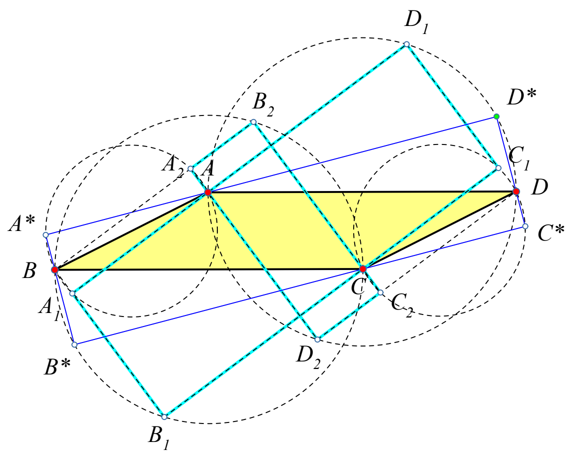


Figure 3. Circumscribing but not enclosing

At this point it should be stressed the difference of *circumscribing* to *enclosing*. The rectangles we deal with *circumscribe*, i.e. have their side-lines passing through the vertices of the quadrangle of reference $ABCD$, but may not *enclose* it in their inner domain. This is illustrated by figure 3, in the case in which the quadrangle of reference $ABCD$ is a parallelogram. Neither of the two extremal rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$ encloses $ABCD$, as it does the rectangle $A^*B^*C^*D^*$. The set of enclosing rectangles is a subset of the circumscribing ones. Thus, the maximal enclosing may coincide with the maximal circumscribing, but it can also be different from this, having E_b as an upper bound for its area. The problem of maximal *enclosing* rectangles has more interesting computational aspects, as can be seen e.g. in [10], where it is handled for the special case of rectangles enclosing paralellograms. Instead, the configuration of rectangles *circumscribing* a quadrangle is connected with interesting geometric structures, as will be hopefully seen in the following sections.

2. Four similar kites

In the case of non-orthodiagonal quadrangles $q = ABCD$, the other than the vertices of q intersection points of the circles $\{\alpha \cap \beta, \beta \cap \gamma, \gamma \cap \delta, \delta \cap \alpha\}$ can be easily identified with the projections of the vertices of q on its diagonals. These points define a similar to q quadrangle $q' = A'B'C'D'$ (See Figure 4).

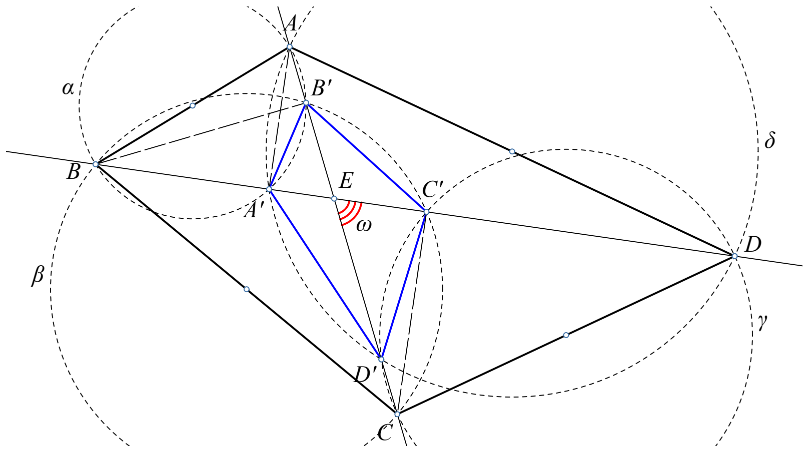


Figure 4. Projecting the vertices on the diagonals

Lemma 1. *The quadrangle $q' = A'B'C'D'$ is similar to q and inversely oriented to it. The similarity ratio of q' to q is equal to $\cos(\omega)$, where $0 < \omega < \pi/2$ is the angle of the diagonals of q.*

Proof. By the figure, which shows that triangles $\{ABC, A'B'C'\}$ are similar. In fact, $\widehat{C'A'B'} = \widehat{CAB}$ by the cyclic quadrangle $AB'A'B$, and $\widehat{A'C'B'} = \widehat{ACB}$ by the cyclic quadrangle $BB'C'C$. Analogously is seen that $\{BCD, B'C'D'\}$ are similar. The similarity ratio is the ratio of the diagonals $A'C'/AC = \cos(\omega)$. The reversing of the orientation results from the fact that the two quadrangles have the same lines as diagonals, but their roles are interchanged, the diagonal carrying $\{B, D\}$ now carrying $\{A', C'\}$, etc. $\square$

This quadrangle is of significance for the location of the extremal circumscribing rectangles, since, as will be seen below, the vertices of these rectangles lie on the medial lines of the sides of q'. In fact, each side of q', together with its medial line, which defines a diameter of the corresponding circle, creates a *kite* inscribed in the corresponding circle. Figure 5 shows the kite $D_1A'D_2D'$, inscribed in the circle δ. It is created from the diameter D_1D_2 of the circle δ, which is orthogonal to its chord $A'D'$, which is a side of q'. The chord is non degenerate $(A' \neq D')$, precisely under the general assumption made, that the quadrangle is non orthodiagonal, equivalently, that the three circles $\{\alpha, \gamma, \delta\}$ are not concurrent at a point. There are then three other similarly defined kites, corresponding to the other sides of q'. This similarity is an instance of a more general one concerning the four

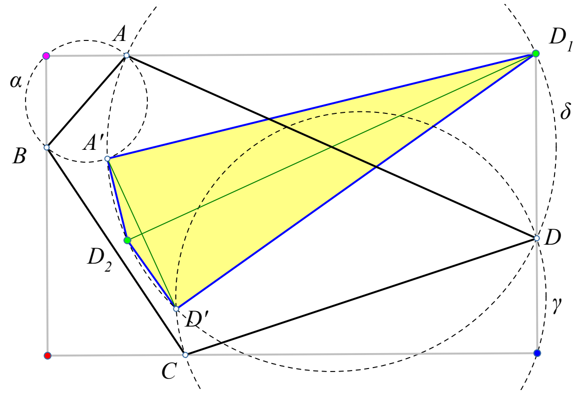


Figure 5. Kites carrying the vertices of the extremal rectangles

cyclic quadrangles defined by any circumscribed rectangle $\sigma_1 = A_1B_1C_1D_1$ and its *antipodal* $\sigma_2 = A_2B_2C_2D_2$, created by the diametral points of the vertices of σ_1, on the respective circles $\{\alpha, \beta, \gamma, \delta\}$ (See Figure 6). Next lemma lists some

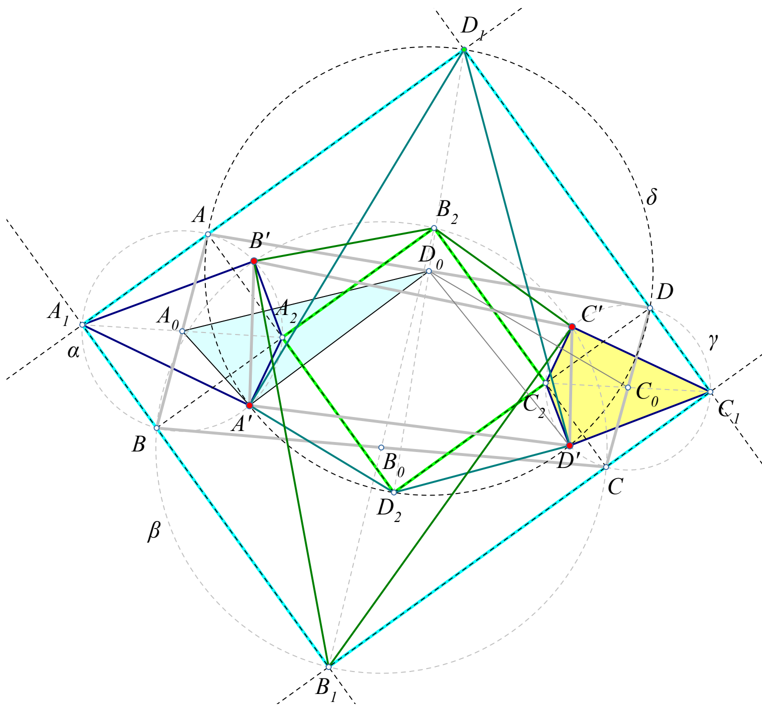


Figure 6. Circumscribed $\sigma_1 = A_1B_1C_1D_1$, "antipodal" $\sigma_2 = A_2B_2C_2D_2$

fundamental properties of this figure.

Lemma 2. *With the notation and the conventions introduced in this section, and denoting by* $\{A_0, B_0, C_0, D_0\}$ *the middles of the sides* $\{AB, BC, CD, DA\}$*, the following are valid properties.*

(1) *The circumscribing rectangles* $\{\sigma_1, \sigma_2\}$ *have parallel sides.*
(2) *The triangles* $\{A'D_1A_1, A'D_2A_2, A'D_0A_0\}$ *are similar, the same property holding for the cyclic permutations of the letters* $\{A, B, C, D\}$.
(3) *The cyclic quadrangles* $\{D_1A'D_2D', A_1A'A_2B'\}$ *are similar, the same property holding for the cyclic permutations of the letters* $\{A, B, C, D\}$.
(4) *The quadrangles of nr-3 have two right angles and the other two are equal to* $\pi/2 \pm \omega$*, where* $\omega \leq \pi/2$ *is the angle of the diagonals of* $ABCD$.

Proof. Nr-1 results from the right angles $\{\widehat{D_2AD_1}, \widehat{A_2AA_1}\}$. This implies that $\{A, A_2, D_2\}$ are collinear and their line is parallel to A_1B_1. Analogously is seen the parallelity of the other pairs of sides.

Nr-2 is a standard exercise in elementary euclidean geometry ([7, p.290]). The map $D_1 \mapsto A_1$ of the circle δ onto circle α can be described by a similarity $A_1 = f_A(D_1)$ with center, or *invariant point* ([5, p.72]) at A angle $\phi_A = \widehat{D_0A'A_0} = \widehat{BAD}$ and ratio $k_A = A'A_0/A'D_0$.

Nr-3 results by showing that the similarity f_A of *nr-2* maps one quadrangle onto the other $f_A(D_1A'D_2D') = A_1A'B'A_1$, which follows by a simple angle chasing argument.

Nr-4 follows from a simple angle chasing argument, since $\widehat{A'D_1D'} = \widehat{A'D_0D'}/2$, which is $\pi - 2(\phi + \psi)$, where $\phi = \widehat{AD_0A_0}$ and $\psi = \widehat{C_0D_0D}$. The claim follows from the fact that $\{A_0D_0, D_0C_0\}$ are parallel to the diagonals. □

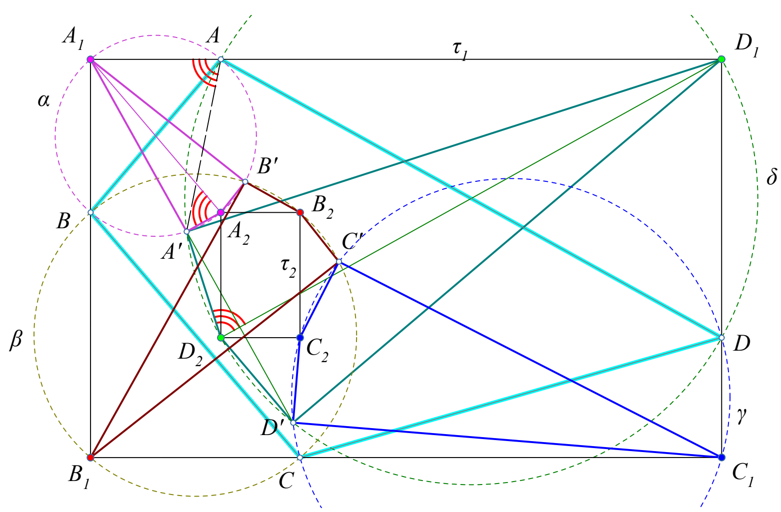


Figure 7. The four similar kites of the quadrangle $ABCD$

Figure 7 shows the four similar kites, resulting from the previous lemma in the case the diagonals of the cyclic quadrangles like $A_1A'A_2B'$ are orthogonal. In section 4 we will show that the rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$ of this figure are the extremal circumscribed rectangles of q.

Lemma 3. *In the special case in which* $\{D_1, D_2\}$ *are vertices of the kite* $D_1A'D_2D'$, *the corresponding circumscribing rectangles* $\{\tau_1, \tau_2\}$ *are similar, the similarity ratio, being equal to the ratio* $D'D_2/D'D_1$ *of the sides of the kite.*

Proof. The similarity of the rectangles for the claimed particular position of $D_1 \in \delta$ is implied from the similarity of the kites. In fact, for that position of D_1 on δ, the triangles $\{C'B_2C_2, C'B_1C_1\}$ are similar and their similarity ratio is equal to the ratio $C'B_2/C'B_1$ of the sides of a kite, which is the same for all four kites. This shows that $B_2C_2/B_1C_1 = C'B_2/C'B_1$ and an analogous argument shows that the last ratio is also equal to C_2D_2/C_1D_1. □

These preliminary remarks, made in this and the previous section, show the existence and the way to construct the two similar extremal rectangles circumscribing the quadrangle $ABCD$. It remains to justify their names and show actually their extremal property. This will be done in section 4, after a short study of the function of the signed area of the circumscribing rectangle.

3. The area function

In order to study the signed area function of the circumscribed rectangles of the quadrangle $q = ABCD$, it suffices to consider their half, defined by a diagonal of them. Figure 8 shows the right angled triangle XYZ, which is such a half of a

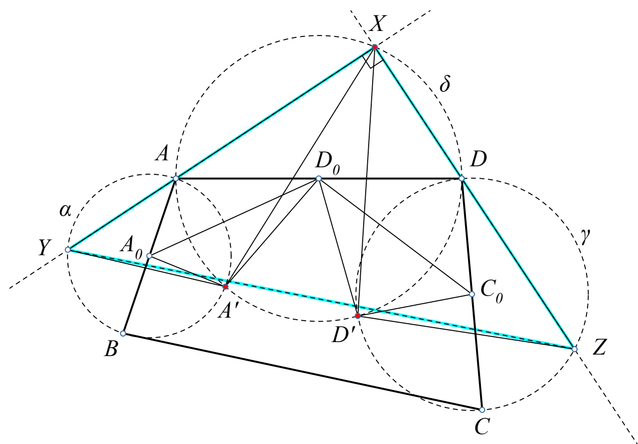


Figure 8. A half of the rectangle circumscribing $ABCD$

circumscribing rectangle. The study of its area can be done by using the naturally defined similarity transformations introduced in the previous section. In fact, the correspondence $X \mapsto Y$, for X varying on the circle δ, can be expressed by the similarity $Y = f_A(X)$ with center at A', angle $\phi_A = \widehat{D_0A'A_0} = \widehat{A}$ and ratio $k_A = A'A_0/A'D_0 = AB/AD$, points $\{D_0, A_0, C_0\}$ being the middles respectively of the sides $\{AD, AB, CD\}$. This similarity can be conveniently described using complex numbers, in the form

$$Y = A' + k_A \cdot e^{i\phi_A}(X - A') \qquad \Rightarrow \qquad Y - X = (k_A \cdot e^{i\phi_A} - 1) \cdot (X - A').$$

Analogously, the correspondence $X \mapsto Z$ can be described by the similarity f_D centerred at D', in the form

$$Z = D' + k_D \cdot e^{i\phi_D}(X - D') \qquad \Rightarrow \qquad Z - X = (k_D \cdot e^{i\phi_D} - 1) \cdot (X - D'),$$

where $\phi_D = \widehat{D_0 D' C_0} = \widehat{D}$ and $k_D = D'C_0/D'D_0 = DC/AD$. Using further complex numbers ([4, p.70], [8, p.48]), the signed area of the triangle $\sigma = XYZ$ can be expressed by the formula

$$\begin{aligned} [XYZ] &= \frac{1}{2} Im((\overline{Y - X})(Z - X)) \\ &= \frac{1}{2} Im((\overline{(k_A e^{i\phi_A} - 1)(X - A')})(k_D e^{i\phi_D} - 1)(X - D')) \\ &= \frac{1}{2} Im((k_A e^{-i\phi_A} - 1)(k_D e^{i\phi_D} - 1)\overline{(X - A')}(X - D')). \end{aligned}$$

Identifying D_0 with the origin and $\{A, D\}$ on the x-axis and symmetric w.r.t. to D_0, we arive for the signed area at an expression of the form

$$[XYZ] = N - Im(M(\overline{A'}X + D'\overline{X})), \tag{1}$$

where, since $\overline{X}X = |D_0 A|^2$, the numbers $\{M, N\}$ are respectively the complex and real constants

$$\begin{aligned} M &= \frac{1}{2}(k_A e^{-i\phi_A} - 1)(k_D e^{i\phi_D} - 1), \\ N &= Im(M(\overline{X}X + \overline{A'}D')) = Im(M(|D_0 A|^2 + \overline{A'}D')). \end{aligned}$$

Setting X in polar form, $X = |D_0 A| e^{i\psi}$, in formula (1), we see, after a short calculation, that the signed area $[XYZ]$ is a sinuisoidal periodic function of the polar angle ψ, as this was suggested by figure 2. A second conclusion from this formula results by replacing X with $X' = -X = |D_0 A| e^{i(\psi + \pi)}$, which corresponds to the construction of the circumscribed rectangle, starting this time with the diametral point X' of X w.r.t. to the circle δ. The result is

$$[XYZ] + [X'Y'Z'] = 2N, \tag{2}$$

which shows that the signed areas corresponding to diametral points sum to a constant. Regarding the value of the real constant N, it can be determined by considering particular positions of the circumscribed rectangles. These are positions $\{D_{AC}, D_{AB}\}$ of D_1 (See Figure 9), for which the sides of $\sigma_1 = A_1 B_1 C_1 D_1$ become parallel and not identical to a diagonal of $ABCD$. Then it is trivial to see that $\sigma_2 = A_2 B_2 C_2 D_2$ degenerates and the area of the σ_1 becomes, up to sign, equal to $|AC||BD| \sin(\omega) = 2|ABCD|$, where $\omega < \pi/2$ the angle of the diagonals. Selecting the orientation of the triangle ABD and of one such rectangle to be positive, we see that the area of the rectangle σ_1 varying continiusly in dependence of $D_1 \in \delta$, is positive for D_1 varying on the greater arc of δ, defined by its chord $A'D'$, and negative for D_1 on the small arc defined by that chord. As a consequence, the maximal and minimal circumscribing rectangles $\{\tau_1, \tau_2\}$

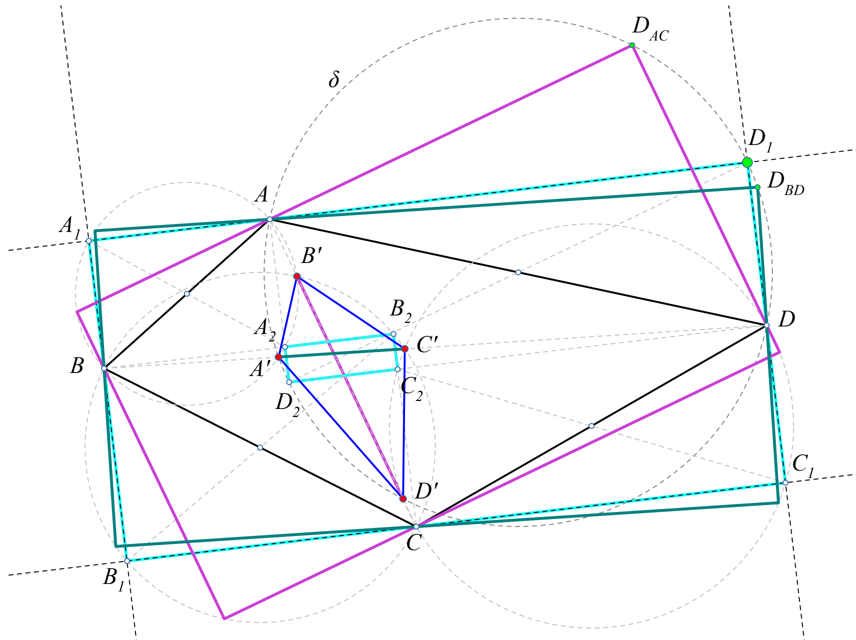


Figure 9. Positions for which $\sigma_2 = A_2B_2C_2D_2$ degenerates

have opposite orientations, their areas $\{E_b, E_s\}$ have opposite signs and we can set $N = |ABCD|/2 > 0$, which leads to the formula

$$E_b + E_s = 2|ABCD|. \tag{3}$$

There is an interesting configuration, which should be considered here, concerning the case in which the aforementioned arcs of δ are equal. This is the case of self-intersecting quadrangles, whose signed area $|ABCD| = 0$. Since the

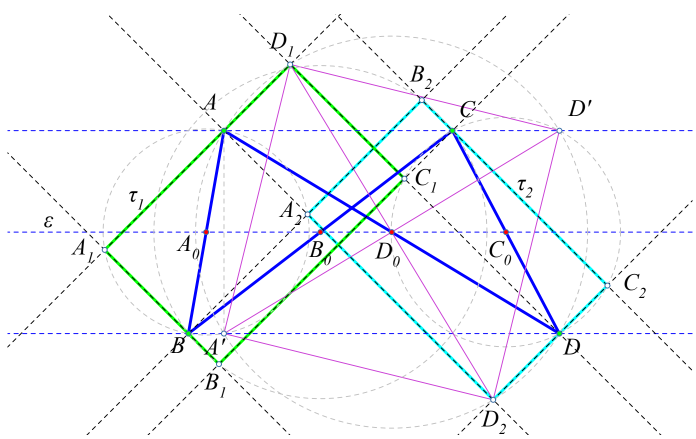


Figure 10. Side-middles on a line ε

area of a quadrangle $q = ABCD$ is a multiple of the area of the corresponding

Varignon parallelogram $q_0 = A_0B_0C_0D_0$, with vertices the middles of the sides of $ABCD$, this condition is equivalent with the degeneration of q_0 or, equivalently, the collinearity of these middles. Figure 10 shows such a case, together with the two inversely oriented extremal rectangles, which, as is easily seen, now are congruent. The corresponding kites, like $D_1A'D'D_2$, are squares, the diagonals $\{AC, BD\}$ are parallel and their angle can be considered to be $\omega = 0$. In this case the quadrangle $q' = A'B'C'D'$, of the projections of the vertices of q on its diagonals, is the reflected of q w.r. to the line ε carrying the middles, hence congruent to it. Finally, it can be easily verified that the ratio of the sides of the extremal rectangles D_1C_1/D_1A_1 is equal to the ratio of the diagonals AC/BD, which is something shown below (corollary 9) to be generally valid for all quadrangles.

We summarize the results so far in the following theorem.

Theorem 4. *Under the notation and conventions of this section the following are valid properties.*

(1) *The signed area function of the circumscribed rectangle is a sinuisoidal periodic function of the polar angle ψ of a side of the rectangle.*
(2) *The sum of the signed areas of a circumscribed rectangle and its diametral, obtained for $\psi + \pi$, is constant.*
(3) *The areas of the two extremal rectangles have opposite signs and their sum is $E_b + E_s = 2|ABCD|$.*
(4) *The two extremal rectangles are congruent precisely when $|ABCD| = 0$, equivalently, when the middles of the sides of $ABCD$ are collinear.*

4. The extremal rectangles

The question of the extremal circumscribed rectangles can be settled using further their half, defined by a diagonal of them, as in the preceding section, on the ground of figure 8. To this figure refers also next theorem, establishing the connection of the extremals with the kites introduced in section 2 for non-orthodiagonal quadrangles.

Theorem 5. *The area of the triangle XYZ obtains an extremal value, precisely when the triangle $A'XD'$ is isosceles, having $|XA'| = |XD'|$.*

Proof. Since by theorem 4 the extremals have non zero areas, in the task to locate them, we can use absolute values. Using then the expressions for the sides of XYZ of the preceding section, we see that the product of these sides is

$$\begin{aligned} |X - Y||X - Z| &= |(k_A e^{i\phi_A} - 1)(X - A')||(k_D e^{i\phi_D} - 1)(X - D')| \\ &= S|X - A'||X - D'|, \end{aligned}$$

the last expression involving the constant $S = |k_A \cdot e^{i\phi_A} - 1| \cdot |k_D \cdot e^{i\phi_D} - 1|$. Thus, the maximal rectangle occurs precisely, when the product $|X - A'||X - D'|$ becomes maximal, for $\{A', D'\}$ fixed and X variable on the circle δ. The proof of the theorem follows then from the following lemma, which should be well known. □

Lemma 6. *For two fixed points $\{A, B\}$ on the circle δ and a variable point X on it, the product $|XA||XB|$ takes an extremal value, precisely when $|XA| = |XB|$. This occurs at the ends $\{O, O'\}$ of the diameter of δ, which is orthogonal to AB.*

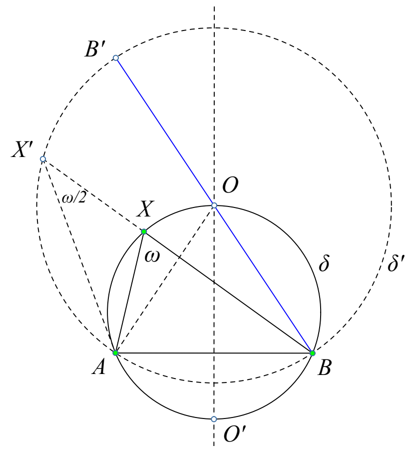


Figure 11. Maximizing the product $|XA||XB|$

Proof. The proof follows immediately from figure 11. Taking $X'X = XA$ on the extension of BX, we see that point X' varies on the circle δ' viewing the segment AB under the angle $\widehat{AX'B} = \widehat{AXB}/2$, whose center O is on the circle δ. Thus, the product $|XA||XB|$ is, up to sign, the power of X w.r. to the circle δ', which becomes maximal, when X coincides with the center O of δ'. If X varies on the other arc defined by the chord AB, then the local maximum occurs analogously at the diametral O' of O. $\square$

Corollary 7. *The similarity ratio of the minimal rectangle to the maximal one is equal, up to sign, to $\tan\left(\frac{\pi}{4} - \frac{\omega}{2}\right)$, where $\omega < \pi/2$ is the angle of the diagonals of the quadrangle of reference $ABCD$.*

Proof. In fact, as was noticed in lemma 3, the similarity ratio of the two extremal rectangles equals, up to sign, the ratio of the sides of the kites $|D'D_2|/|D'D_1| = \tan(\chi/2)$, where $\chi = \widehat{A'D_1D'} = \widehat{A'D_0D'}/2$ (See Figure 12). But the last angle can be readily seen to be equal to $\widehat{A'KD'} = \pi - 2(\phi + \psi) = \pi - 2\omega$, where $\{\phi = \widehat{ADB}, \psi = \widehat{DAC}\}$. $\square$

Combining this with theorem 4, we see that

Corollary 8. *The two extremal rectangles of a non-orthodiagonal quadrangle q are congruent, if and only if the middles of the sides of q are collinear, equivalently its signed area is zero, equivalently its characteristic kites are squares.*

Corollary 9. *The ratio of sides of an extremal rectangle is equal to the ratio of the diagonals of the quadrangle of reference $q = ABCD$.*

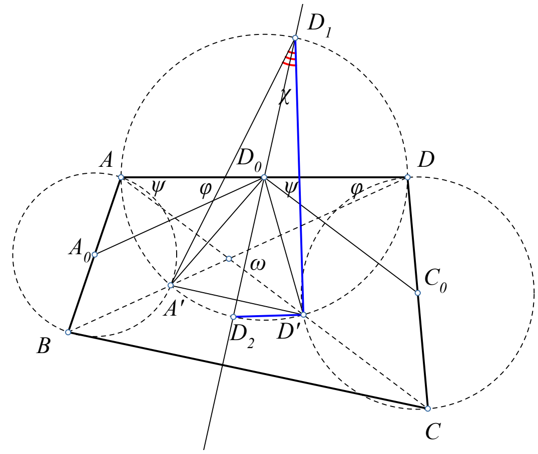


Figure 12. The similarity ratio of the extremal rectangles

Proof. In the proof of the previous theorem we saw that the ratio of the sides of an extremal rectangle can be expressed using complex numbers (see figure-8):

$$\frac{|XY|}{|XZ|} = \frac{|k_A \cdot e^{i\phi_A} - 1|}{|k_D \cdot e^{i\phi_D} - 1|},$$

resulting from the formulas there, in the case of an extremum, for which $|XA'| = |XD'|$. Multiplying this with $|D_0A'| = |D_0D'|$, we see that this ratio expresses the length ratio

$$\begin{aligned}\frac{|k_A \cdot e^{i\phi_A} - 1||D_0A'|}{|k_D \cdot e^{i\phi_D} - 1||D_0D'|} &= \frac{|k_A \cdot e^{i\phi_A}(D_0 - A') - (D_0 - A')|}{|k_D \cdot e^{i\phi_D}(D_0 - D') - (D_0 - D')|}\\ &= \frac{|A_0 - A' - (D_0 - A')|}{|C_0 - D' - (D_0 - D')|} = \frac{|A_0D_0|}{|C_0D_0|},\end{aligned}$$

which is precisely the claimed ratio of the diagonals. □

Corollary 10. *The extremal rectangles of a quadrangle $ABCD$ are squares, if and only if, the quadrangle has equal diagonals.*

Figure 13 shows cases of quadrangles $ABCD$, whose extremal circumscribed rectangles are squares.

Corollary 11. *The area of the minimal rectangle is zero precisely when the quadrangle of reference $ABCD$ is orthodiagonal.*

Corollary 12. *The positive areas $\{E_b, E_s\}$, respectively of the maximal and minimal rectangles, satisfy the relations*

$$E_b + E_s = |AC||BD| \qquad \textit{and} \qquad E_b - E_s = 2E,$$

where E is the non-negative area of the quadrangle of reference $ABCD$.

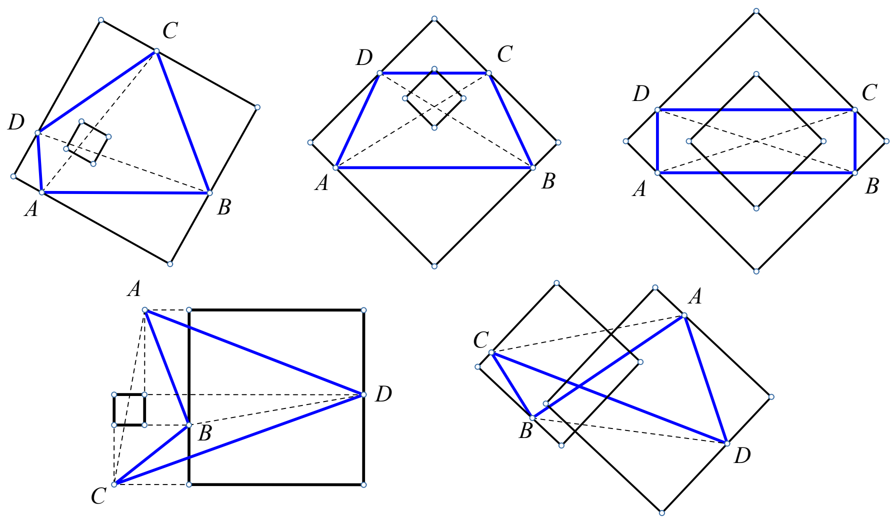


Figure 13. Quadrangles for which the extremal rectangles are squares

Proof. The second relation is already proved in theorem 4, but will be considered here from another view point. In fact, selecting two homologous sides A_1B_1, A_2B_2 of the similar rectangles, the ratio of the areas is

$$\frac{E_s}{E_b} = \frac{|A_2B_2|^2}{|A_1B_1|^2} \quad \Rightarrow \quad \frac{E_s + E_b}{E_b} = \frac{|A_2B_2|^2 + |A_1B_1|^2}{|A_1B_1|^2} = \frac{|AC|^2}{|A_1B_1|^2}.$$

Analogously, using the ratio of the two other homologous sides $\{B_1C_1, B_2C_2\}$, we have

$$\frac{E_s}{E_b} = \frac{|B_2C_2|^2}{|B_1C_1|^2} \quad \Rightarrow \quad \frac{E_s + E_b}{E_b} = \frac{|B_2C_2|^2 + |B_1C_1|^2}{|B_1C_1|^2} = \frac{|BD|^2}{|B_1C_1|^2}.$$

Since $|A_1B_1||B_1C_1| = E_b$, multiplying the two expressions and simplifying, we get

$$(E_s + E_b)^2 = |AC|^2|BD|^2 \qquad \Leftrightarrow \qquad E_b + E_s = |AC||BD|.$$

On the other side, by corollary 7 and setting $t = \tan(\omega/2)$, where $\omega \leq \pi/2$ is the angle of the diagonals of $ABCD$, the ratio of the areas must also be equal to

$$\begin{aligned} \frac{E_s}{E_b} &= \tan\left(\frac{\pi}{4} - \frac{\omega}{2}\right)^2 = \left(\frac{1-t}{1+t}\right)^2 = \frac{1-\sin(\omega)}{1+\sin(\omega)} \\ \Rightarrow &\quad E_b - E_s = (E_b + E_s)\sin(\omega), \end{aligned}$$

which, combined with the first relation, proves the second one. □

Corollary 13. *The positive areas of the extremal rectangles of the quadrangle $ABCD$, whose angle of diagonals $\{AC, BD\}$ is $0 \leq \omega \leq \pi/2$, are respectively*

$$E_s = \frac{1-\sin(\omega)}{2}|AC||BD|, \qquad E_b = \frac{1+\sin(\omega)}{2}|AC||BD|.$$

Notice that the method used here can be applied also to prove the existence of squares circumscribing a quadrilateral ([1, p.8], [9], see also corollary 29).

5. Two similar orthogonally lying rectangles

Motivated by the configuration of the two extremal rectangles, we study the general case of rectangles $\{\tau_1 = A_1B_1C_1D_1, \tau_2 = A_2B_2C_2D_2\}$, which are similar and have corresponding sides orthogonal or, equivalently, the rotation angle of the similarity has a measure of $\pi/2$. The focus here is on the description of the similarities carrying τ_1 onto τ_2. In general there are two *direct* or *spiral* ([12, p.36,II], [6, p.136]) similarities, and two *antisimilarities* or *dilative reflections* ([12, p.49,II], [6, p.175]) doing this operation. A complete treatment of similarities, including methods to find their centers and other defining them characteristics, can be found in [2, ch.IV], where the direct similarities and the antisimilarities are called respectively *stretch rotations* and *strech reflections*.

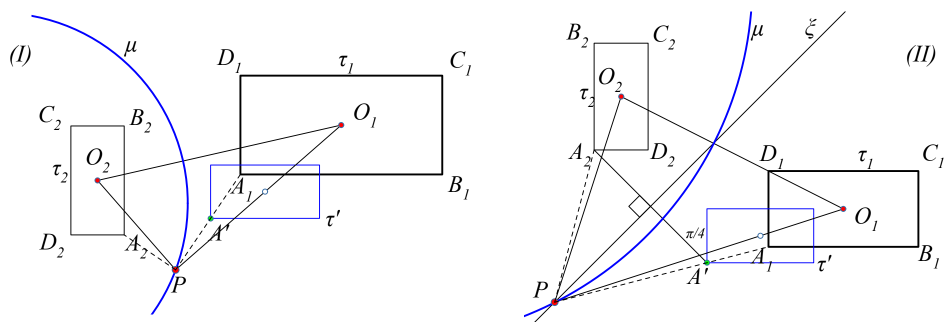


Figure 14. Similarities mapping $A_1B_1C_1D_1$ to $A_2B_2C_2D_2$

Figure 14 gives a short account of the way the similarities are defined in our case. In (I) we have a direct similarity, which per definition is a composition of a homothety and a rotation about the same center P. In (II) we have an antisimilarity, which per definition again is a composition of a homothety and a reflection on a line ξ through the homothety center P. In both cases point P is on the Apollonian circle μ ([1, p.15]), defined as the geometric locus of points X, such that the ratio $XO_2/XO_1 = k$, where k is the homothety ratio and $\{O_1, O_2\}$ are the centers of the similar rectangles, assumed to be different. The figure shows also the intermediate rectangle τ', resulting by applying to τ_1 only the homothety part of the similarity, so that $PA'/PA_1 = k$.

In our configuration, starting from the two similar rectangles $\{\tau_1, \tau_2\}$, the two *direct* similarities $\{f_1, f_2\}$ preserve the orientation and are defined by the correspondence of the vertices suggested by the equations (See Figure 15)

$$f_1(A_1B_1C_1D_1) = D_2A_2B_2C_2 \quad \text{and} \quad f_2(A_1B_1C_1D_1) = B_2C_2D_2A_2.$$

The two *antisimilarities* $\{g_1, g_2\}$, carrying τ_1 onto τ_2 reverse the orientation of the rectangles and correspond their vertices in the order suggested by the equations

$$g_1(A_1B_1C_1D_1) = A_2D_2C_2B_2 \quad \text{and} \quad g_2(A_1B_1C_1D_1) = C_2B_2A_2D_2.$$

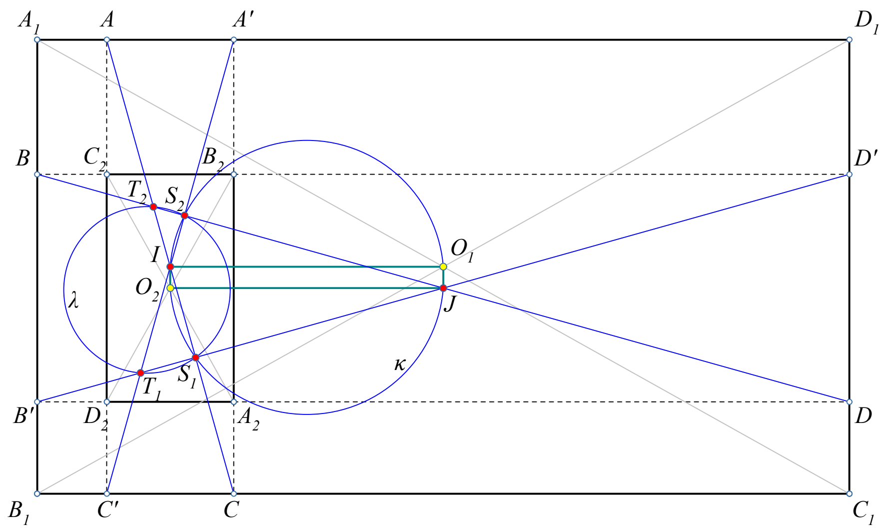


Figure 15. Similarities interchanging $\{\tau_1, \tau_2\}$

The figure shows the locations $\{S_1, S_2\}$ of the two direct- and $\{T_1, T_2\}$ of the two anti-similarities. There are also two rectangles $\{\rho_1 = BB'D'D, \rho_2 = AA'CC'\}$, proved below to be also similar to each other and having the intersections of their diagonals coincident with the similarity centers. They are defined through the intersections of the sides of the given rectangles and will be called *companion rectangles* of $\{\tau_1, \tau_2\}$. There is a sort of symmetry here, since taking the companion rectangles of $\{\rho_1, \rho_2\}$, we come back to $\{\tau_1, \tau_2\}$.

Theorem 14. *With the notation and conventions of this section, the following are valid properties.*

(1) *The centers $\{J, I\}$ of the companion rectangles and the centers $\{O_1, O_2\}$ of $\{\tau_1, \tau_2\}$ are vertices of a rectangle.*
(2) *The two companion rectangles $\{\rho_1, \rho_2\}$ are similar.*
(3) *The similarity centers $\{S_1, S_2\}$ lie on the circumcircle κ of the rectangle IO_1JO_2 symmetrically w.r. to its diameter O_1O_2.*
(4) *The similarity centers $\{S_1, S_2\}$ lie also on respective diagonals of ρ_1, being thus the second intersections of κ with respective diagonals of $\{\rho_1, \rho_2\}$.*

Proof. Nr-1 is obvious. *Nr-2* follows from the assumed similarity of $\{\tau_1, \tau_2\}$. In fact,

$$\frac{BB'}{B'D} = \frac{CC'}{CA'} \Leftrightarrow \frac{BB'}{CC'} = \frac{B'D}{CA'}.$$

Nr-3 follows from *nr-1* and the fact, that the direct similarities must map the center O_1 to O_2, so that $\widehat{O_1S_iO_2}$ is a right angle. This shows them to lie on κ. The fact that $S_iO_2/S_iO_1 = k$ is the similarity ratio, makes them symmetric w.r. to the diameter O_1O_2 of κ.

Nr-4 follows from *nr-3*, since this implies that $\widehat{S_1JS_2}$ is bisected by JO_2, which is parallel to BD'. On the other side we have also that $BB'/BD' = k$. This implies that $\{S_1, J, D'\}$ are collinear. Analogously is seen that $\{S_2, J, D\}$ are collinear. By the similarity of $\{\rho_1, \rho_2\}$ follows then that the similarity centers are respectively the intersections $S_1 = (B'D, AC), S_2 = (BD, A'C')$. □

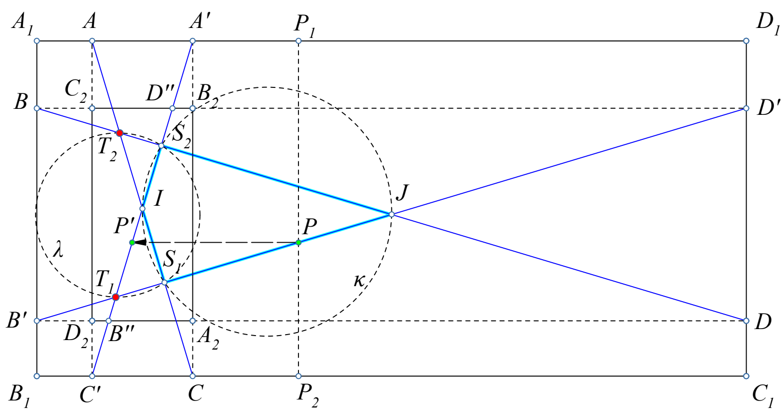


Figure 16. The centers $\{T_1, T_2\}$ of the antisimilarities

Theorem 15. *Continuing with the notation and conventions adopted so far, the following are valid properties.*

(1) *The centers $\{T_1, T_2\}$ of the antisimilarities $\{g_1, g_2\}$ are the intersection points of the opposite sides of the cyclic quadrangle IS_1JS_2.*
(2) *Points $\{T_1, T_2\}$ are diametral points of a circle λ, which is orthogonal to the circumcircle κ of IS_1JS_2 and passes through $\{S_1, S_2\}$.*

Proof. For *nr-1* we work with the similarity g_1, showing that the intersection point $T_1 = (A'C', B'D')$ is its similarity center. The proof for the similarity center T_2 of g_2 is completely analogous. We start by showing that $B'D'$ maps under g_1 onto line $A'C'$. In fact, since g_1, by definition, maps line D_1C_1 onto line B_2C_2, point $D' \in D_1C_1$ will map to some point $D'' \in B_2C_2$. Analogously, since line A_1B_1 maps onto line A_2D_2, point $B' \in A_1B_1$ will map to some point $B'' \in A_2B_2$. By the preservation of ratios by similarities follows

$$\frac{D'D_1}{D'C_1} = \frac{B_2A'}{B_2C} = \frac{D''B_2}{D''C_2} = \frac{B_2A'}{B_2C} \quad \Rightarrow \quad \frac{B_2D''}{B_2A'} = \frac{D''C_2}{B_2C} = \frac{D''C_2}{C_2C'}.$$

This implies that D'' is on $A'C'$ and analogously B'' is also on this line, hence g_1 maps line $D'B'$ onto line $A'C'$, as claimed. Now, projecting P to points $\{P_1, P_2\}$ on the parallels $\{A_1D_1, B_1C_1\}$ and working analogously with the ratios PP_1/PP_2 of the varying point $P \in D'B'$ and its image $P' = g_1(P)$ (See Figure 16), we see that line PP' is always parallel to A_1D_1, hence T_1 is the fixed point of g_1.

Nr-2 is a trivial consequence of the previous properties, since $\{\widehat{T_2S_2I}, \widehat{T_1S_1I}\}$ are right angles. It is also a general property for cyclic quadrangles. This is handled in detail in [11], where the circle λ on diameter KL is called the *orthocycle* of the cyclic quadrangle IS_1JS_2, here coinciding also with an Apollonian circle of the segment O_1O_2. □

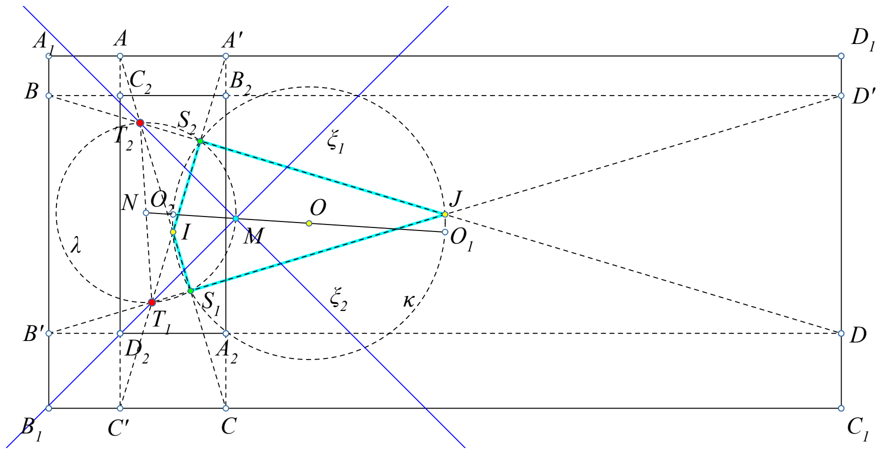


Figure 17. The axes $\{T_1M, T_2M\}$ of the antisimilarities $\{g_1, g_2\}$

Regarding the *axes* $\{\xi_1, \xi_2\}$ of the reflections involved in the definition of the antisimilarities, the following theorem describes their location. The proof uses the fact that such an axis passes through the center of the antisimilarity and is bisecting the angle of a line through its center and its image-line under the antisimilarity (See Figure 17).

Theorem 16. *Continuing with the notation and conventions adopted so far, the following are valid properties.*

(1) *The axes of the antisimilarities $\{g_1, g_2\}$ are respectively bisectors of the angles $\{\widehat{JT_1I}, \widehat{IT_2J}\}$.*
(2) *These lines intersect orthogonally at a point M on line O_1O_2, which passes also through the center N of the circle λ.*
(3) *The four points $\{T_1, T_2, I, J\}$ define an orthocentric quadruple, i.e. each triple of them defines a triangle whose orthocenter is the fourth point.*

Proof. Nr-1 derives directly from the definition of the antisimilarity and the fact proved in the previous theorem, that line $B'D'$ maps under g_1 onto line $A'C'$. This shows that the bisector $\xi_1 = T_1M$ of the angle $\widehat{S_1T_1S_2}$ is the axis of g_1. Analogously is seen that the bisector $\xi_2 = T_2M$ of the angle $\widehat{S_1T_2S_2}$ is the axis of g_2.

Nr-2, the part of orthogonality $\xi_1 \perp \xi_2$, results by a simple angle chasing argument and is left as an exercise ([3, p.21]). Because $\xi_1 = T_1M$ is a bisector of the angle $\widehat{S_1T_1S_2}$, point M is the middle of the arc S_1S_2 of the circle λ. This implies the other claims of this *nr*.

Nr-3 derives from the fact that $\{T_1S_2, T_2S_1\}$ are two altitudes of triangle T_2T_1J, intersecting at I. □

We call the ordered orthocentric quadruple (T_1, T_2, I, J) the *associated quadruple* of $\{\tau_1, \tau_2\}$. By the previous theorems, the four similarities carrying τ_1 onto τ_2 can be completely determined by the data of this quadruple. Also, given such a quadruple, and setting I as the orthocenter of the triangle T_1T_2J, we can define a double infinity of configurations like the one of figure 17, by selecting arbitrarily the position of one vertex of τ_1, like the point C_1 say. In fact, using the quadruple, we can easily determine the circle κ and its diameter O_1O_2. Then, reflecting the arbitrary point C_1 on the lines $\{O_1I, O_1J\}$ we define respectively the points $\{D_1, B_1\}$ and from these the rectangle $\tau_1 = A_1B_1C_1D_1$. Then, we define the direct similarity f_1 with center at S_1, angle $\pi/2$ and ratio S_1O_2/S_1O_1 and through it the rectangle $\tau_2 = f_1(\tau_1) = A_2B_2C_2D_2$. The two rectangles $\{\tau_1, \tau_2\}$ define by the procedures of this section an associated quadruple coinciding with the given one. The four similarities mapping τ_1 to τ_2 are in all cases the same. Also their companion rectangles are all similar to each other and are characterized by the angle of their diagonals, which is $\widehat{T_1JT_2}$. We summarize these facts in the form of the next corollary.

Corollary 17. *Every ordered orthocentric quadruple (T_1, T_2, I, J) with point I selected as orthocenter of the triangle T_1T_2J, defines a diameter O_1O_2 on the circle with diameter IJ and a double infinity of similar rectangles $\{\tau_1, \tau_2\}$, centerred correspondingly at $\{O_1, O_2\}$ and with sides parallel to $\{O_1I, O_1J\}$, such that the associated quadruple is the given one and all their companion rectangles $\{\rho_1, \rho_2\}$ are similar, having the same angle of diagonals $\widehat{T_1JT_2}$.*

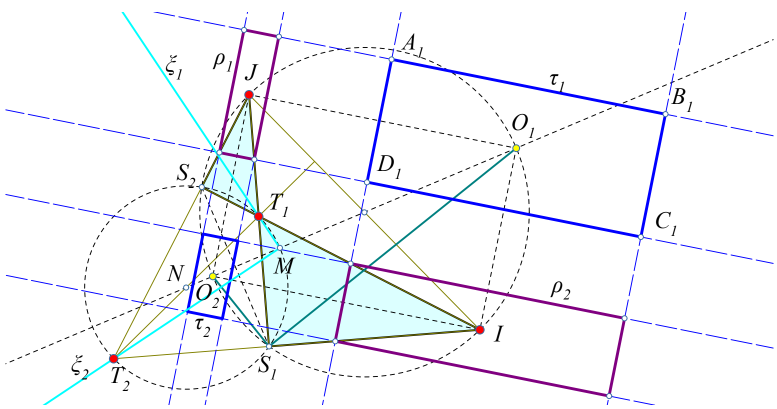


Figure 18. Similar rectangles $\{\tau_1, \tau_2\}$ and their companions $\{\rho_1, \rho_2\}$

Since an unordered quadruple defines six ordered pairs of the type (T_1, T_2, I, J) we obtain six possibilities to construct such double infinities of similar rectangles and their companions. Figure 18 shows a case in which the quadrangle IS_1JS_2 is self intersecting. Notice that the four similarities and their inverses, interchanging $\{\tau_1, \tau_2\}$, do not map the companions $\{\rho_1, \rho_2\}$ to each other. Latter rectangles are interchanged by four other similarities and their inverses, creating an analogous

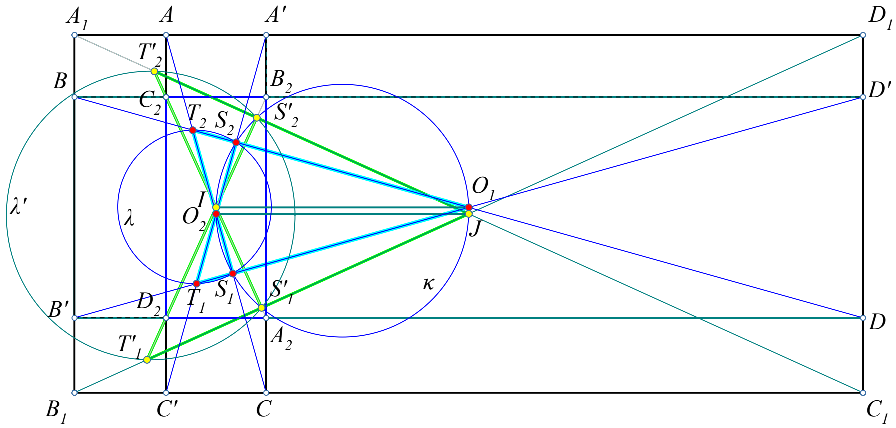


Figure 19. Similarities interchanging the companions

orthocentric quadruple (T_1', T_2', O_1, O_2) (See Figure 19). The figure reflects the aforementioned symmetry, by which, taking the companions of $\{\rho_1, \rho_2\}$, we come back to $\{\tau_1, \tau_2\}$. The similarity centers $\{T_1', T_2', S_1', S_2'\}$, in this case, are on an Apollonian circle λ' of the segment IJ, which, like λ, is orthogonal to κ carrying $\{S_1, S_2, S_1', S_2'\}$. It is easily seen that also $\{T_1, T_2, T_1', T_2'\}$ are on a circle κ'.

6. The case of the quadrangle and its twin

Leaving aside, for a while, the special case of the parallelogram and considering a generic non-orthodiagonal quadrangle $q = ABCD$, we formulate first the consequences of the results of the previous section for the pair of its extremal rectangles $\{\tau_1, \tau_2\}$.

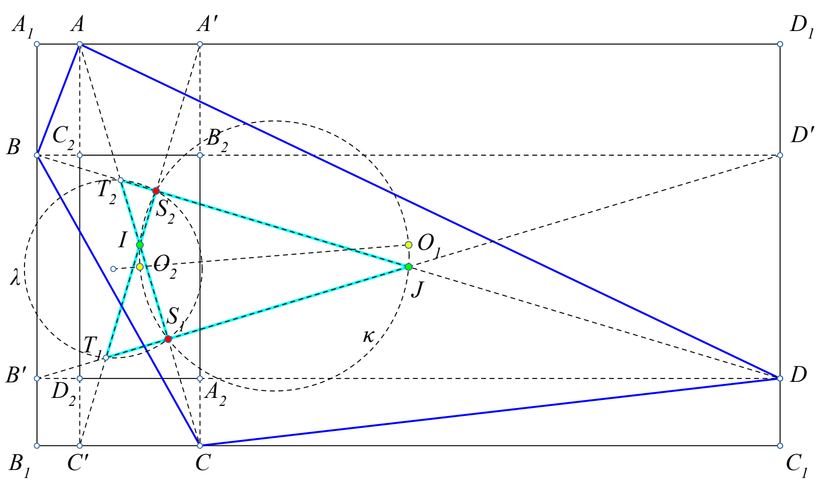


Figure 20. Similarity centers of extremal rectangles of $q = ABCD$

Theorem 18. *With the notation and conventions introduced so far, the following are valid properties for the generic quadrangle* $q = ABCD$.

(1) *The centers* $\{O_1, O_2\}$ *of the rectangles, respectively,* $\{\tau_1, \tau_2\}$ *and the middles* $\{I, J\}$ *of the diagonals of* q *define a rectangle* IO_1JO_2, *with sides parallel to the sides of the extremal rectangles.*
(2) *The direct similarity centers* $\{S_1, S_2\}$ *lie on the circumcircle* κ *of the previous rectangle with diameter* IJ *and define a kite* $S_1O_1S_2O_2$, *which is similar to the kites carrying the vertices of* $\{\tau_1, \tau_2\}$.
(3) *The similarity centers* $\{S_1, S_2\}$ *lie also on respective diagonals of* q, *being thus the second intersections of* κ *with the diagonals of* q.
(4) *The centers* $\{T_1, T_2\}$ *of the antisimilarities are the intersection points of the opposite sides of the cyclic quadrangle* IS_1JS_2.
(5) *Points* $\{T_1, T_2\}$ *are diametral points of a circle* λ, *which is orthogonal to the circumcircle* κ *of* IS_1JS_2, *contains also the centers* $\{S_1, S_2\}$ *and coincides with the Apollonian circle of the segment* O_1O_2 *for the ratio* $k = \tan(\pi/4 - \omega/2)$, *where* $0 < \omega < \pi/2$ *is the angle of the diagonals of* q.

Corollary 19. *The centers of the two direct similarities, relating the extremal rectangles, are the projections of the middles of the diagonals of the quadrangle on the other diagonals.*

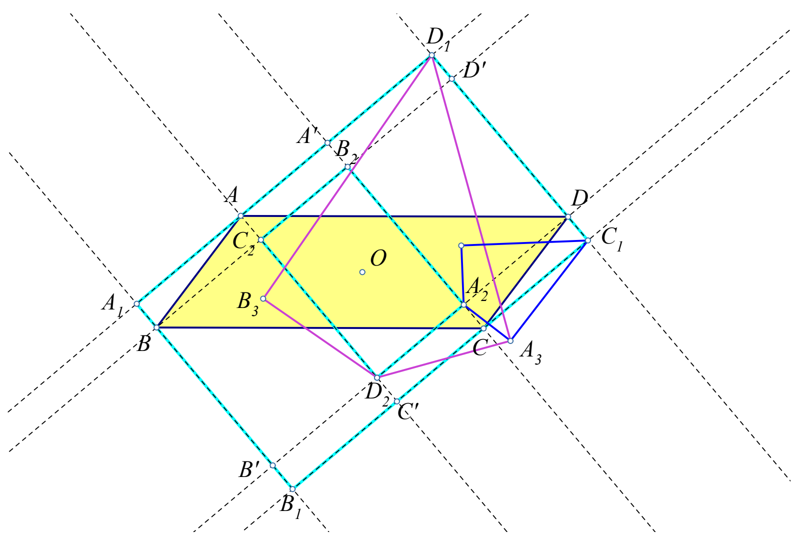


Figure 21. The case of parallelograms

Figure 21 shows the left aside case of a parallelogram $q = ABCD$, displaying its two extremal rectangles and two of the characteristic kites, defining them. All the similarity centers of the direct- as well the anti-similarities here coincide with the center of the parallelogram. The proof of the next corollary is left as an exercise.

Corollary 20. *In the case of a parallelogram $q = ABCD$, the extremal rectangles have their centers, as well as, the centers of the similarities, coinciding with the center O of q. The ratio of the similarities is, as in the generic case, expressible through the sides of the kites, $k = |D_2A_3|/|D_1A_3| = \tan(\pi/4 - \omega/2)$, where $0 < \omega < \pi/2$ is the angle of the diagonals of q.*

Given the non-orthodiagonal quadrangle $q = ABCD$, we can define a *twin* quadrangle $q' = A'B'C'D'$, which, like q, is also inscribed simultaneously in the associated extremal rectangles $\{\tau_2, \tau_1\}$ of q and has the same companion rectangles. This is seen in figure 22, which among other properties shows that the new

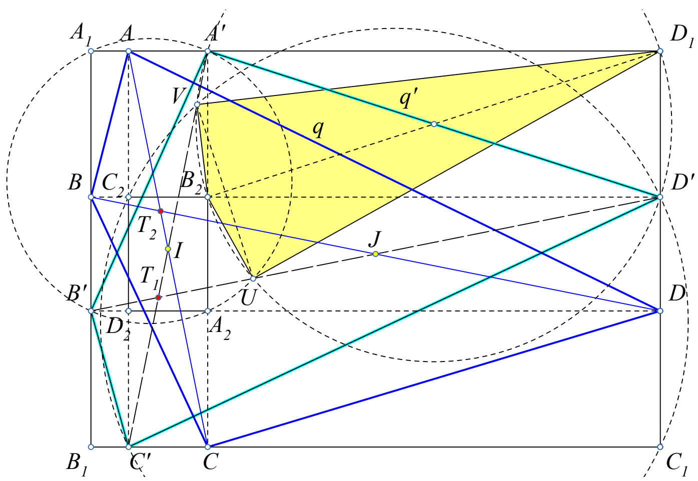


Figure 22. The "twin" quadrangle $A'B'C'D'$ of $ABCD$

quadrangle has diagonals of the same length and same angle between them with the original one, hence also the same area. In addition, the two rectangles are also extremal w.r. to the new one and the orthocentric quadruple (T_1, T_2, I, J) plays the same role for q' as it does for q. The characteristic property of q' is that its diagonals coincide with the diagonals of the companion rectangles $\{AA'CC', BB'DD'\}$, which are different from the diagonals of q. Figure 22 displays also a characteristic kite for q' carrying vertices of its own extremal rectangles and seen to be identical with D_1 and B_2. In fact, the two circles on diameters, respectively, $\{A'D', A'B'\}$ intersect at a point U of $B'D'$, which defines the altitude $A'U$ of triangle $A'B'D$. Analogously the circles on diameters $\{A'D', D'C'\}$ intersect at point V defining the altitude DV of triangle $A'D'C'$. From the similarity of the rectangles $\{AA'CC', BB'DD'\}$ follows that

$$\widehat{B_2D_1V} = \widehat{B_2A'V} = \widehat{B_2D'U} = \widehat{B_2D_1U},$$

which proves that VD_1UB_2 is a kite of q', like those carrying the vertices of the extremal rectangles, considered in section 2. Next theorem summarizes these observations.

Theorem 21. *For every non-orthodiagonal quadrangle q, the associated twin quadrangle q' shares with q the same extremal rectangles and the same associated orthocentric quadruple.*

Notice that the kites of q', like the VD_1UB_2 of figure 22, though not identical to those of q, they are nevertheless similar to them, since their similarity type is completely determined by the angle of the diagonals of q', which is the same with that of q. Notice also that the "twin" relation is reflective, so that the twin of q' is the original quadrangle q. Also, using corollary 19, we deduce easily the follow-

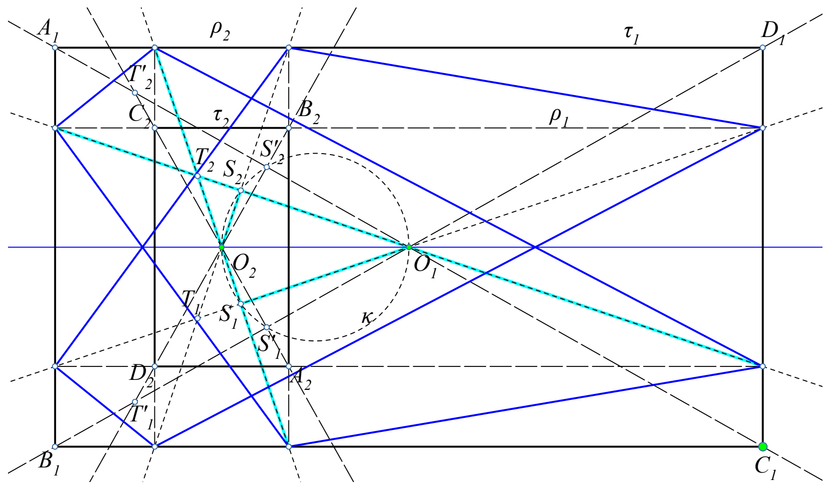


Figure 23. Quadrangle with congruent twin

ing characterizations of the particular class of quadrangles, which have congruent twins (See Figure 23).

Corollary 22. *The twin quadrangle q' of a non-orthodiagonal, non-parallelogrammic quadrangle q is congruent to q precisely when the projections of the middles of the diagonals of q on the other diagonals are symmetric w.r. to the Newton line, joining these middles, equivalently, the sides of the extremal rectangles are respectively parallel and orthogonal to the Newton line, equivalently, the centers of these rectangles coincide with the middles of the diagonals of q, equivalently the twin q' is the reflection of q on the Newton line.*

An easy testing, which I omit, of the various possibilities to define a quadrangle simultaneous inscribed in two given similar and orthogonally lying rectangles, shows the following corollary.

Corollary 23. *Given two similar and orthogonally lying rectangles $\{\tau_1, \tau_2\}$, there is precisely one pair of twin quadrangles having them for extremal.*

7. The associated orthodiagonals

An additional feature of the existence of the two extremal rectangles is the existence of a couple of orthodiagonal quadrangles inscribed simultaneously in these two rectangles. Next corollary, whose easy proof is left as an exercise, summarizes their characteristics.

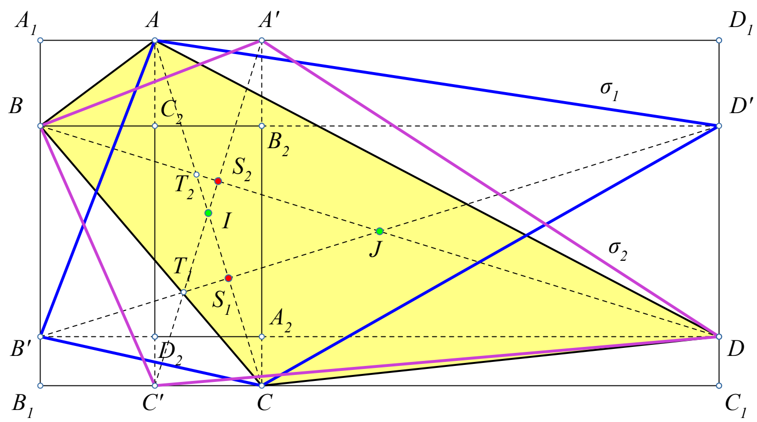


Figure 24. The two orthodiagonals $\{\sigma_1, \sigma_2\}$ defined by a non-orthodiagonal $ABCD$

Corollary 24. *Every non-orthodiagonal quadrangle $q = ABCD$ defines two orthodiagonal quadrangles $\{\sigma_1 = AB'CD', \sigma_2 = A'BC'D\}$, which are inscribed in the two extremal quadrangles $\{\tau_1, \tau_2\}$ of q. The quadrangle $\sigma_1(\sigma_2)$ shares with q the diagonal $AC(BD)$ and its other diagonal $|B'D'| = |BD|(|A'C'| = |AC|)$. The intersection points of the diagonals of $\{\sigma_1, \sigma_2\}$ coincide with the similarity centers $\{S_1, S_2\}$, and the areas of these orthodiagonals are equal to the difference $E_b - E$. In the case q is a parallelogram, $\{\rho_1, \rho_2\}$ are rhombi lying symmetric w.r. to its center O.*

An easily proved consequece of these observations is the following corollary.

Corollary 25. *The sum of squares of the sides of the generic non-orthodiagonal quadrangle q is equal to the corresponding sum of squares of its twin quadrangle q'.*

8. The case of orthodiagonals

In the case $ABCD$ is an orthodiagonal quadrangle the following theorem lists several related facts, which more or less are well known and their proof is left as an exercise on the ground of figure 25. In this, points $\{I, J\}$ are the middles of the diagonals intersecting at point K, points $\{M, L\}$ are the other than K intersections respectively of the circles $\{\beta \cap \delta, \alpha \cap \gamma\}$ and $\{H, N\}$ are respectively the centers of the maximal circumscribing $A_1B_1C_1D_1$ and the variable circumscribing rectangle $A^*B^*C^*D^*$.

Theorem 26. *Every orthodiagonal quadrangle $ABCD$ has the following properties.*

(1) *All circumscribing rectangles are similar to each other and the maximal one $A_1B_1C_1D_1$ has its sides parallel to the diagonals of $ABCD$.*

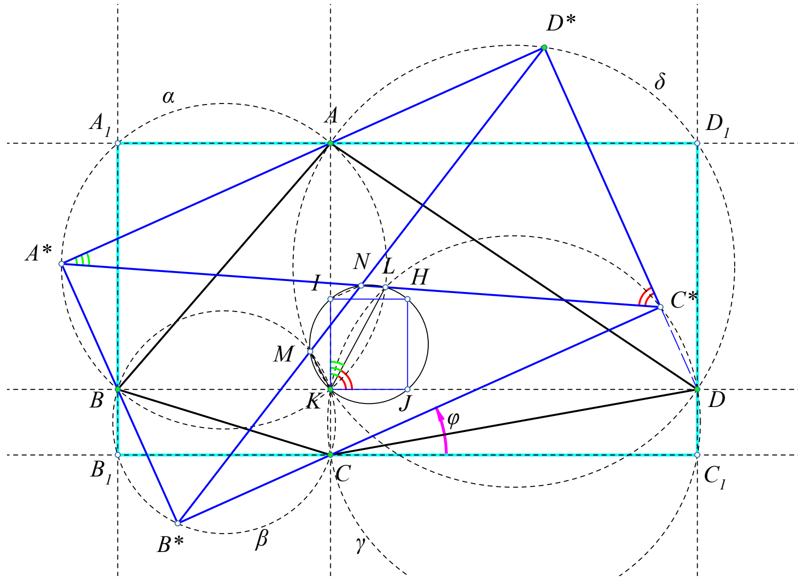


Figure 25. Circumsribing rectangles of an orthodiagonal quadrangle $ABCD$

(2) *The similarity center of two such rectangles is at point K and the similarity ratio of the variable rectangle $A^*B^*C^*D^*$ to the maximal one $A_1B_1C_1D_1$ is $\cos(\phi)$, where ϕ is the angle between two homologous sides of these two rectangles.*

(3) *$KJHI$ is a rectangle and points $\{L, M, N\}$ lie on its circumcircle.*

Corollary 27. *A quadrangle is orthodiagonal, if and only if all its circumscribed rectangles are similar to the rectangle of its diagonals, equivalently, if all its circumscribed rectangles are similar to each other.*

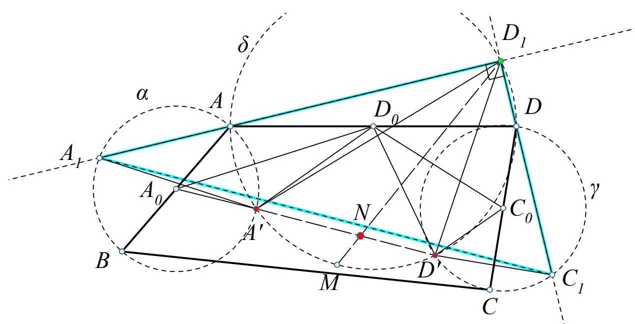


Figure 26. The ratio of sides of circumscribing rectangles

Proof. Here the necessity part follows from the previous theorem. The sufficiency is also easily deduced from figure 26 and using the arguments of section 3, by which the ratio of the sides of the circumscribed rectangle is

$$\frac{D_1A_1}{D_1C_1} = S \cdot \frac{D_1A'}{D_1D'} = S \cdot \frac{NA'}{ND'}.$$

Here S is a constant and N is the trace on $A'D'$ of the bisector of the angle $\widehat{A'D_1D'}$. In the case the quadrangle of reference is non-orthodiagonal, the last ratio cannot be constant for D_1 varying on δ, hence the proof of the sufficiency of the corollary. □

A refinement of the previous argument, considering three positions of N for which the ratio NA'/ND' has the constant value k or $1/k$, proves also next corollarylaries.

Corollary 28. *A quadrangle is orthodiagonal, if and only if it has three similar circumscribed rectangles. In this case all its circumscribed rectangles are similar.*

Corollary 29. *A quadrangle has equal and orthogonal diagonals, if and only if it has three circumscribed squares. In this case all its circumscribed rectangles are squares.*

References

[1] N. Altshiller-Court, *College Geometry*, Dover Publications Inc., New York, 1980.
[2] W. Barker and R. Howe, *Continuous Symmetry, From Euclid to Klein*, Amer. Math. Soc., 2007.
[3] E. Catalan, *Theoremes et problemes de Geometrie Elementaire*, Carilian-Coeury, Paris, 1852.
[4] G. Chang and T. Sederberg, *Over and Over Again*, Math. Assoc. America, 1997.
[5] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons Inc., New York, 1961.
[6] R. Deaux, *Introduction to the Geometry of Complex Numbers*, Dover, New York, 1956.
[7] F. G. M., *Exercices de Geometrie*, Librairie Generale, Paris, 1952.
[8] L. S. Hahn, *Complex Numbers and Geometry*, Math. Assoc. America, 1994.
[9] C. M. Hebbert, The inscribed and circumscribed squares of a quadrilateral and their significance in kinematic geometry, *Annals of Math.*, 16 (1914) 38–42.
[10] R. E. Horton, Extreme rectangles about a parallelogram, *Amer. Math. Monthly*, 57 (1950) 35–36.
[11] P. Pamfilos, Orthocycles, bicentrics and orthodiagonals, *Forum Geom.*, 7 (2007) 73–86.
[12] I. M. Yaglom, *Geometric Transformations I, II, III*, Math. Assoc. America, 1962.

Paris Pamfilos: University of Crete, Greece
E-mail address: pamfilos@uoc.gr

Forum Geometricorum
Volume 18 (2018) 185–194.

FORUM GEOM
ISSN 1534-1178

The Blundon Theorem in an Acute Triangle and Some Consequences

Mihály Bencze and Marius Drăgan

Abstract. The purpose of this article is to give an analoque of Blundon theorem in an acute triangle and using this result to obtain the best inequality of the type

$$\sum \sqrt{\frac{b+c-a}{a}} \geq f\left(R, r\right)$$

where f is a homogenous function.

Let $C\left(O, r\right)$ and $C\left(I, r\right)$ be two circles such that $I \in int\, C\left(O, r\right)$ and $OI = \sqrt{R^2 - 2Rr}$.

For any triangle ABC with sides $BC = a$, $CA = b$, $AB = c$, and semiperimeter $s = \frac{a+b+c}{2}$, we denote by $C\left(O, R\right)$ the circumcircle and $C\left(I, r\right)$ the incircle.

Theorem of the present paper is an analogue in an acute triangle of Theorem 2 of Blundon [3].

Also Theorem represents the best improvement of the type

$$\sum \sqrt{\frac{b+c-a}{a}} \geq f\left(R, r\right),$$

where $f(R, r)$ is a homogeneous function of the inequality $\sum \sqrt{\frac{b+c-a}{a}} \geq 3$. See [1, p. 159-165], which is known as the Rădulescu - Maftei Theorem and which in [1] has 2 solutions, one elementary and other based on the Lagrange multiplier Theorem.

Main Results

Lemma 1. *In any triangle ABC are true the following equalities*

1). $a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$

2). $ab + bc + ca = s^2 + r^2 + 4Rr$

3). $a^2b^2 + b^2c^2 + c^2a^2 = \left(s^2 + r^2 + 4Rr\right)^2 - 16Rrs^2$

Lemma 2. *In any triangle ABC is true the following equality:*

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.

$$\prod \cos A = \frac{s^2 - r^2 - 4Rr - 4R^2}{4R^2}$$

Proof. In the following we will denote $x = a^2 + b^2 + c^2$. From the cosine theorem it follows that:

$$\prod \cos A = \frac{\prod \left(b^2 + c^2 - a^2\right)}{8(\prod a)^2} = \frac{\prod \left(x - 2a^2\right)}{8(\prod a)^2} =$$

$$= \frac{x^2 - 2\sum a^2 x + 4\sum a^2 b^2 x - 8(\prod a)^2}{8(\prod a)^2} = \frac{s^2 - r^2 - 4Rr - 4R^2}{4R^2}$$

□

Theorem 3. *In any acute triangle is true the following inequality:*

$$s > 2R + r$$

Proof. As in any acute triangle is true the inequality: $\prod \cos A > 0$ according with Lemma 2 it follows the inequality from the statement. □

Theorem 4 (Blundon)**.** *In any triangle ABC is true the following inequality: $s_1 \leq s \leq s_2$ where*

$$s_1 = \sqrt{2R^2 + 10Rr - r^2 - 2\sqrt{R\left(R - 2r\right)^3}}\,,\ s_2 = \sqrt{2R^2 + 10Rr - r^2 + 2\sqrt{R\left(R - 2r\right)^3}}$$

represent the semiperimeter of two issoscels triangle $A_1B_1C_1$ and $A_2B_2C_2$ with the sides

$$a_1 = 2\sqrt{R^2 - (r - t)^2},\ b_1 = c_1 = \sqrt{2R\left(R + r - t\right)}$$

$$a_2 = 2\sqrt{R^2 - (r + t)^2},\ b_2 = c_2 = \sqrt{2R\left(R + r + t\right)}$$

where $t = OI = \sqrt{R^2 - 2Rr}$.

Lemma 5. *Let $A_3B_3C_3$ be a triangle with $C\left(O, R\right)$ the circumscrible and $C\left(I, r\right)$ the incircle and with the semiperimeter $s_3 = 2R + r$. Then the sides of triangle $A_3B_3C_3$ is unique determinated by the equalities:*

$$a_3 = 2R$$

$$b_3 = R + r + \sqrt{R^2 - 2Rr - r^2}$$

$$c_3 = R + r - \sqrt{R^2 - 2Rr - r^2}$$

where A_3 is a right angle.

Proof. We have the following equalities:

$$a+b+c=2s$$

$$ab+bc+ca=s^2+r^2+4Rr$$

$$abc=4Rrs$$

or

$$a+b+c=4R+2r$$

$$ab+bc+ca=4R^2+8Rr+2r^2 \tag{1}$$

$$abc=4Rr\left(2R+r\right)$$

From (1) it follows that a,b,c are the solutions of the equation:

$$u^3-\left(4R+2r\right)u^2+\left(4R+8Rr+2r^2\right)u-4Rr\left(2R+r\right)=0 \tag{2}$$

The equation (2) may be written as:

$$\left(u-2R\right)\left[u^2-\left(2R+2r\right)u+4Rr+2r^2\right]=0$$

which has the solutions from the statement. □

Theorem 6. *In any acute triangle with $C\left(O,R\right)$ the circumscribed and $C\left(I,r\right)$ the inscribed are true the following inequalities:*

$$s_1\le s\le s_2 \textit{ if } 2\le\frac{R}{r}<\sqrt{2}+1$$

and

$$s_3\le s\le s_2 \textit{ if } \frac{R}{r}\ge\sqrt{2}+1$$

where s_1,s_2 are the semiperimeter of two issosceles triangle $A_1B_1C_1, A_2B_2C_2$ with the sides from Theorem 2 and s_3 is the semiperimeter of the right triangle $A_3B_3C_3$ from Lemma 3.

Proof. We denote $\frac{R}{r}=x$. We consider two cases:

Case 1. $2\le x<\sqrt{2}+1$

We will prove that $s_1>s_3$ or in an equivalent form:

$$2x^2+10x-1-2\sqrt{x\left(x-2\right)^3}-\left(2x+1\right)^2=2\left[-\sqrt{x\left(x-2\right)^3}-\left(x^2-3x+1\right)\right]>0$$

or

$$-\left(x^2-3x+1\right)>\sqrt{x\left(x-2\right)^3} \tag{3}$$

But $x^2 - 3x + 1 < 0$ as $x < \sqrt{2} + 1 < \frac{3+\sqrt{5}}{2}$. After squaring in (3) we obtain:

$$\left(x^2 - 3x + 1\right)^2 > x\left(x-2\right)^3 \text{ or } -x^2 + 2x + 1 > 0 \text{ or}$$

$$\left(\sqrt{2} - 1 - x\right)\left(x - \left(\sqrt{2}+1\right)\right) > 0$$

inequality which is true. It results that $s_3 < s_1 \leq s_2$.

But as $s_1 \leq s \leq s_2$ and $s \geq s_3$ it follows that $s_1 \leq s \leq s_2$.

Case 2a. $\sqrt{2} + 1 \leq x < \frac{3+\sqrt{5}}{2}$ or $x^2 - 3x + 1 < 0$.

We will prove that $s_1 \leq s_3$ or in an equivalent form:

$$2x^2 + 10x - 1 - 2\sqrt{x\left(x-2\right)^3} < \left(2x+1\right)^2 \text{ or } -\left(x^2 - 3x + 1\right) \leq \sqrt{x(x-2)^3} \tag{4}$$

After squaring and performing some calculation the inequality (4) may be written as

$$\left(x - \left(\sqrt{2}-1\right)\right)\left(x - \left(\sqrt{2}+1\right)\right) \geq 0$$

inequality which is true.

We will prove that $s_3 < s_2$ or in an equivalent form:

$$\left(2x+1\right)^2 < 2x^2 + 10x - 1 - 2\sqrt{x\left(x-2\right)^3} \quad \text{or} \quad x^2 - 3x + 1 < \sqrt{x\left(x-1\right)^3} \tag{5}$$

The inequality (5) is true as $x^2 - 3x + 1 < 0$. It results that $s_1 \leq s_3 < s_2$. But as $s_1 \leq s \leq s_2$ and $s \geq s_3$ it follows that $s_3 \leq s \leq s_2$.

Case 2.b. $x \geq \frac{3+\sqrt{5}}{2}$ or $x^2 - 3x + 1 \geq 0$.

We will prove that

$$s_1 < s_3 \text{ or } -\left(x^2 - 3x + 1\right) < \sqrt{x\left(x-2\right)^3}$$

inequality which is true.

We will prove that

$$s_3 < s_2 \text{ or } x^2 - 3x + 1 < \sqrt{x\left(x-2\right)^3}$$

or in an equivalent form

$$\left[x - \left(\sqrt{2}-1\right)\right]\left[x - \left(\sqrt{2}+1\right)\right] > 0$$

It results that $s_1 < s_3 < s_2$. But as $s_1 \leq s \leq s_2$ and $s \geq s_3$ it follows that $s_3 \leq s \leq s_2$.

It results in the cases 2a and 2b that $s_3 \leq s \leq s_2$ which is equivalent with the inequality from the statement. $\square$

Lemma 7. *In any triangle ABC is true the equalities:*

1). $\sum \frac{s-a}{a} = \frac{s^2+r^2-8Rr}{4Rr}$

2). $\sum \frac{(s-a)(s-b)}{ab} = \frac{2R-r}{2R}$

Proof.

$$\sum \frac{s-a}{a} = \frac{s\sum bc - 3abc}{abc} = \frac{s\left(s^2+r^2+4Rr\right) - 12Rr}{abc} = \frac{s^2+r^2-8Rr}{4Rr} =$$

$$= \sum \frac{(s-a)(s-b)}{ab} = \frac{s^2\left(\sum a\right) - 2s\left(s^2+r^2+4Rr\right) + 12Rrs}{abc} = \frac{2R-r}{2R}$$

□

Theorem 8 (A refinement of Rădulescu - Maftei Theorem)**.** *In any triangle ABC is true the following inequality:*

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{2R-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}} + \sqrt{\frac{r}{R}}$$

if $\frac{R}{r} \geq \sqrt{2}+1$ *or*

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}}$$

if $2 \leq \frac{R}{r} < \sqrt{2}+1$.

Proof. We denote $t = \sum \sqrt{\frac{s-a}{a}}$. By squaring we obtain

$$t^2 = \sum \frac{s-a}{a} + 2\sqrt{\frac{\sum (s-a)(s-b)}{ab} + 2\sqrt{\frac{(s-a)(s-b)(s-c)}{abc}}}$$

From Lemma 4, 1) and 2) it follows that:

$$\left(t^2 - \frac{s^2+r^2-8Rr}{4Rr}\right)^2 = 4\left(\frac{2R-r}{2R} + 2\sqrt{\frac{r}{4R}}t\right)$$

We consider the function $f:(0,+\infty) \to R$

$$f(u) = u^4 - \frac{s^2+r^2-8Rr}{2Rr}u^2 - 8\sqrt{\frac{r}{4R}}u + \left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 - \frac{4R-2r}{R}$$

We have $f(t) = 0$. We will prove that

$$\left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2 < \frac{4R-2r}{R}$$

or in an equivalent form:

$$s^2 < 8Rr - r^2 + 4\sqrt{Rr^2(4R-2r)}$$

But as $s^2 \le s_2^2$. It will be sufficient to prove that

$$s_2^2 = 2R^2 + 10Rr - r^2 - 2\sqrt{R\left(R-2r\right)^3} < 8Rr - r^2 + 4\sqrt{Rr^2\left(4R-2r\right)} \quad (6)$$

We denote $x = \frac{R}{r}$. The inequality (6) may be written as:

$$2x^2 + 10x - 1 - 2\sqrt{x\left(x-2\right)^3} < 8x - 1 + 4\sqrt{x\left(4x-2\right)}$$

or

$$x^2 + x < \sqrt{x\left(x-2\right)^3} + 2\sqrt{x\left(4x-2\right)} \quad (7)$$

After squaring the inequality (7) we will obtain:

$$x^4 + 2x^3 + x^2 < x\left(x^3 - 6x^2 + 12x - 8\right) + 16x^2 - 8x + 4x\sqrt{\left(x-2\right)^3\left(4x-2\right)}$$

or

$$8x^3 - 27x^2 + 16x < 4x\sqrt{\left(x-2\right)^3\left(4x-2\right)}$$

or

$$8x^2 - 27x + 16 < 4\sqrt{\left(x^3 - 6x^2 + 12x - 8\right)\left(4x-2\right)} \quad (8)$$

If

$$8x^2 - 27x + 16 \le 0$$

the inequality (8) is true. For $8x^2 - 27x + 16 > 0$ we will square (8) and we will obtain:

$$64x^4 + 729x^2 + 256 - 432x^3 + 256x^2 - 864x < 64x^4 - 416x^3 + 960x^2 - 896x + 256$$

or

$$16x^3 - 25x^2 - 32x > 0 \text{ or } 16x^2 - 25x - 32 > 0$$

But $8x^2 - 27x + 16 > 0$. It results that $x > \frac{27+\sqrt{217}}{16} > \frac{25+\sqrt{2673}}{32}$ or $16x^2 - 2x - 32 > 0$.

We denote $a_2 = \frac{s^2+r^2-8Rr}{2Rr}$, $a_1 = 8\sqrt{\frac{r}{4R}}$, $a_0 = \frac{4R-2r}{R} - \left(\frac{s^2+r^2-8Rr}{4Rr}\right)^2$. The equation $f\left(u\right) = 0$ may be written as: $u^4 - a_2u^2 - a_1u - a_0 = 0$ with $a_0, a_1, a_2 > 0$ or $1 - \frac{a_2}{u^2} - \frac{a_1}{u^3} - \frac{a_0}{u^4} = 0$. But $g : (0, +\infty) \to R$, $g\left(u\right) = 1 - \frac{a_2}{u^2} - \frac{a_1}{u^3} - \frac{a_0}{u^4}$ is an increasing function. It results that t is the only positive root of equation $f\left(u\right) = 0$.

It result that if exists a unique continue function $u : [s_1, s_2] \to R$ such that $f\left(u\left(s\right)\right) = 0$, $\left(\forall\right) s \in [s_1, s_2]$. From implicite Theorem it follows that u is derivable on interval $\left(s_1, s_2\right), u : [s_1, s_2] \to R$ which verify the condition:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)^2=4\left(\frac{2R-r}{2R}+2\sqrt{\frac{r}{4R}}u(s)\right),\ \ (\forall)\,s\in[s_1,s_2] \tag{9}$$

After we derivate the equality (9) we will obtain:

$$\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)\left(u(s)\,u'(s)-\frac{s}{4Rr}\right)=\sqrt{\frac{r}{R}}u'(s)\,,(\forall)\,s\in[s_1,s_2]$$

or in an equivalent form:

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)$$

or

$$\left(u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}\right)u'(s)=\frac{s}{4Rr}\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right),(\forall)\,s\in[s_1,s_2]$$

From:

$$u^2(s)=\sum\frac{s-a}{a}+2\sum\sqrt{\frac{(s-a)(s-b)}{ab}}\geq\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}}=$$

$$=\frac{s^2+r^2-8Rr}{4Rr}+6\sqrt[3]{\frac{r}{4R}},(\forall)\,s\in[s_1,s_2]$$

it results that:

$$u^3(s)-\frac{s^2+r^2-8Rr}{4Rr}u(s)-\sqrt{\frac{r}{R}}=u(s)\left(u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}\right)-\sqrt{\frac{r}{R}}\geq\sqrt{6\sqrt[3]{\frac{r}{4R}}}\cdot6\sqrt[3]{\frac{r}{4R}}-$$

$$-\sqrt{\frac{r}{R}}=\left(3\sqrt{6}-1\right)\sqrt{\frac{r}{R}}>0,\ \ (\forall)\,s\in[s_1,s_2]$$

and $u^2(s)-\frac{s^2+r^2-8Rr}{4Rr}>0,\,(\forall)\,s\in[s_1,s_2]$. It results that u is an increasing function on interval $[s_1,s_2]$.

From Theorem 3 it follows that $s_1\leq s$, for $2\leq\frac{R}{r}<\sqrt{2}+1$ which implies that $u(s_1)\leq u(s)$.

Replacing the sides a_1,b_1,c_1 of the $A_1B_1C_1$ triangle from Theorem 2 we will obtain:

$$\sum\sqrt{\frac{b+c-a}{a}}\geq\sqrt{\frac{R-r-d}{r}}+2\sqrt{\frac{R+d}{R}}\text{ if }2\leq\frac{R}{r}<\sqrt{2}+1$$

From Theorem 3 it follows that $s_3\leq s$ if $\frac{R}{r}\geq\sqrt{2}+1$ which implies that $u(s_3)\leq u(s)$

By replacing the sides a_3, b_3, c_3 from Lemma 3 it follows that:

$$\sum\sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{2R^2-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{r}{R}} \text{ if } \frac{R}{r} \geq \sqrt{2}+1$$

$\square$

Lemma 9. *In any triangle ABC is true the following inequality:*

$$\sqrt{\frac{2R-2\sqrt{R^2-2Rr-r^2}}{R+r+\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{2R+2\sqrt{R^2-2Rr-r^2}}{R+r-\sqrt{R^2-2Rr-r^2}}}+\sqrt{\frac{r}{R}} \geq 3 \text{ if } \frac{R}{r} \geq \sqrt{2}+1 \tag{10}$$

Proof. We denote $d_2 = \sqrt{x^2-2x-1}$. By squaring the inequality (10) we will obtain:

$$2\sqrt{\frac{(2x-2d_2)(2x+2d_2)}{(x+1+d_2)(x+1-d_2)}}+\frac{(2x-2d_2)(x+1-d_2)+(2x+2d_2)(x+1+d_2)}{(x+1+d_2)(x+1-d_2)} \geq \left(3-\frac{1}{\sqrt{x}}\right)^2$$

or

$$2\sqrt{\frac{4(x^2-x^2+2x+1)}{x^2+2x+1-x^2+2x+1}}+$$

$$+\frac{2x^2+2x-2xd_2-2xd_2-2d_2+2x^2-4x-2+2x^2+2x+2xd_2+2xd_2+2d_2+2x^2-4x-2}{x^2+2x+1-x^2+2x+1} \geq$$

$$\geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$\frac{8x^2-4x-4}{4x+2}+2\sqrt{2} \geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x-2+2\sqrt{2} \geq 9+\frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x-11+2\sqrt{2} \geq \frac{1}{x}-\frac{6}{\sqrt{x}}$$

or

$$2x^2+\left(2\sqrt{2}-11\right)x \geq 1-6\sqrt{x}$$

or

$$2x^2 + \left(2\sqrt{2} - 11\right) x + 6\sqrt{x} - 1 \geq 0$$

We consider the function $f : \left[\sqrt{2} + 1, +\infty\right) \to R$

$$f(x) = 2x^2 + \left(2\sqrt{2} - 11\right) x + 6\sqrt{x} - 1$$

with the derivate

$$f'(x) = 4x + 2\sqrt{2} - 11 + \frac{3}{\sqrt{x}} = 4\left(x - \sqrt{2} - 1\right) + 6\sqrt{2} - 7 + \frac{3}{\sqrt{x}} \geq 0$$

It results that f is an increasing function on interval $\left[\sqrt{2} + 1, +\infty\right)$ which implies that $f(x) > f\left(\sqrt{2} + 1\right)$.

After performing some calculation we obtain $f\left(\sqrt{2} + 1\right) > 0$. □

Lemma 10. *In any triangle ABC is true the following inequality:*

$$\sqrt{\frac{R - r - d}{r}} + 2\sqrt{\frac{R + d}{R}} \geq 3, \text{ if } 2 \leq \frac{R}{r} \leq 8 \tag{11}$$

Proof. We denote $\frac{R}{r} = x, d_x = \frac{\sqrt{R(R-2r)}}{r} = \sqrt{x(x-2)}$. The inequality (11) may be written as:

$$\sqrt{x - 1 - d_x} + 2\sqrt{\frac{x + d_x}{x}} \geq 3$$

By squaring we will obtain:

$$\frac{4x + 4d_x}{x} \geq 9 + x - 1 - d_x - 6\sqrt{x - 1 - d_x}$$

or

$$6\sqrt{x - 1 - d_x} \geq 8 - d_x + x - \frac{4x + 4d_x}{x}$$

or

$$6\sqrt{x - 1 - d_x} \geq \frac{4x - xd_x + x^2 - 4d_x}{x}$$

or

$$6\sqrt{x - 1 - d_x} \geq \frac{(x + 4)(x - d_x)}{x}$$

or

$$36x^2 (x - 1 - d_x) \geq \left(x^2 + 8x + 16\right) 2 (x - d_x - 1) x$$

or

$$2x (x - d_x - 1)\left(18x - x^2 - 8x - 16\right) \geq 0 \text{ and as } x - d_x - 1 > 0$$

It will be sufficient to prove that:

$$x^2 - 10x + 16 \leq 0 \text{ or } (x-2)(x-8) \leq 0 \text{ or } x \leq 8$$

□

Theorem 11. *(The inequality Rădulescu-Maftei) In any acute triangle is true the following inequality:*

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3$$

Proof. It results from Theorem 4, Lemma 5 and 6. □

Theorem 12. *In any triangle ABC with $2 \leq \frac{R}{r} \leq 8$ is true the following inequality:*

$$\sqrt{\frac{b+c-a}{a}} + \sqrt{\frac{c+a-b}{b}} + \sqrt{\frac{a+b-c}{c}} \geq 3$$

Proof. According with the proof of Theorem 4 it follows that $u : [s_1, s_2] \to R$ is an increasing function. But $s_1 \leq s$. It results that $u(s) \geq u(s_1)$ or

$$\sum \sqrt{\frac{b+c-a}{a}} \geq \sqrt{\frac{R-r-d}{r}} + 2\sqrt{\frac{R+d}{R}} \geq 3$$

according with Lemma 6. □

References

[1] Drăgan, M., Maftei, I.V., Rădulescu, S., *Câteva consideraţii asupra inegalităţii Rădulescu-Maftei,* Inegalităţi matematice (Extinderi şi generalizări), (159-165) EDP 2012.

[2] Rădulescu, S., Drăgan, M., Maftei, I.V., *Cateva inegalităţi cu radicali şi geometria triunghiului,* Arhimede (31-37), Nr. 1-12, 2009.

[3] Blundon, W.J., *Inequalities associated with a triangle,* Canadian Math. Bull. (1965), 615-626.

[4] Mitrinovic, D.S., Pecaric, J.E. and Volonec, V., Recent Advances in Geometric inequalities

[5] Octogon Mathematical Magazine (1997-2014)

Mihály Bencze: Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania
E-mail address: `benczemihaly@yahoo.com`

Marius Drăgan: 61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania
E-mail address: `marius.dragan2005@yahoo.com`

Forum Geometricorum
Volume 18 (2018) 195–201.

FORUM GEOM
ISSN 1534-1178

Side Disks of a Spherical Great Polygon

Purevsuren Damba and Uuganbaatar Ninjbat

Abstract. Take a circle and mark $n \in \mathbb{N}$ points on it designated as vertices. For any arc segment between two consecutive vertices which does not pass through any other vertex, there is a disk centered at its midpoint and has its end points on the boundary. We analyze intersection behavior of these disks and show that the number of disjoint pairs among them is between $\frac{(n-2)(n-3)}{2}$ and $\frac{n(n-3)}{2}$ and their intersection graph is a subgraph of a triangulation of a convex n-gon.

1. Introduction

An *intersection graph* of a set of figures is a graph such that there is a unique vertex associated to each figure, and two vertices are adjacent if and only if the corresponding figures are intersecting. Huemer and Perez-Lantero [4] showed that the intersection graph of a set of disks with the sides of a convex n-gon as their diameters (which are called *side disks*) is planar (see Theorem 4 in [4]). This result has a direct combinatorial consequence: the number of disjoint pairs among these disks is at least $\frac{(n-3)(n-4)}{2}$ which follows from the fact that every planar graph with n vertices has at most $3(n-2)$ edges (see Corollary 11.1(b) in [3]).

We believe that the problem of analyzing intersection patterns of side disks is of considerable interest because of the geometrical challenges resulting from its unusual conclusion, i.e. it reflects on disjointness of geometrical figures. In Euclidean geometry a search for new results by replacing line segments with conic sections is often rewarding as illustrated in the following well known results: Pappus's hexagon theorem vs. Pascal's theorem, and Ceva's theorem vs. Haruki's theorem (see Chap. 6 in [1]). Accordingly, in Sect. 2 instead of a convex n-gon we consider a circle partitioned into $n \in \mathbb{N}$ arc segments. The concept of side disk naturally extends to this setting: for each arc segment there is a unique disk centered at its midpoint and is having its two end points on its boundary. When $n = 5$ the resulting configuration already appears in Miquel's five circles theorem (see Chap. 5 in [1]). In Theorem 3 we show that for $n \geq 3$ there are at least $\frac{(n-2)(n-3)}{2}$ and at most $\frac{n(n-3)}{2}$ disjoint pairs of side disks for the partitioned circle with n arc segments. We also verify that these bounds are tight for all $n \geq 3$ and the intersection graph of these disks is a subgraph of a triangulation of a convex polygon (see Theorem 4).

Publication Date: June 1, 2018. Communicating Editor: Paul Yiu.
Financial support from the National University of Mongolia (P2016-1225) is acknowledged.

Throughout this paper we use the following conventions. For any points X, Y and Z in the plane the line passing through X, Y is denoted as XY-line, their connecting line segment is denoted as XY, and $|XY|$ is its length. $\measuredangle XYZ$ is the angle between XY and YZ measured in the clockwise direction. For any disk ω, its boundary circle is denoted as $\partial(\omega)$ and when there is no ambiguity we identify a given disk (or circle) with its center X and call it X-disk (or X-circle), etc. For any plane regions ω and τ, $(\omega \cap \tau)$ is the region in their intersection, and $\omega \subset \tau$ means the former is included (strictly) in the latter, i.e. every point in ω is in τ but not vice versa. For a point X and region τ, $X \in \tau$ means X is located in τ and $X \notin \tau$ means the opposite.

2. The main results

Let C_n be a circle partitioned into $n \in \mathbb{N}$ arc segments by marking n points on it. We identify each marked point as vertex and each arc segment between two consecutive vertices which does not pass through any other vertex as a side. Then, C_n is a spherical polygon with vertices at a great circle and we call it as *spherical great polygon*; for more on spherical polygons see e.g. Chap. 6.4 in [2]. The case where each side has the same length is denoted as $C_n^{\star}$. For each side of C_n, there is a unique disk centered at its midpoint and is having its two end points on the boundary. This is the *side disk* of that side and two side disks are *neighbouring* if their corresponding sides are adjacent.

Notice that each side of C_n divides its disk into two parts, one of which intersects with the region enclosed by C_n. We call this as *inner part* and the other as *outer*, and as a convention we include the corresponding arc of C_n to the inner part of the side disk, but not to the outer. Then, convexity implies that outer parts of two side disks of C_n do not intersect. We shall prove two lemmas.

Lemma 1. *Let ω be a given disk and A, B, C be points on $\partial(\omega)$ such that AC-arc is a segment of AB-arc. If ω_1 and ω_2 are the side disks of AB-arc and AC-arc, respectively, then $(\omega \cap \omega_2) \subset (\omega \cap \omega_1)$ and any point in $(\omega \cap \omega_2)$ except A is in the interior of ω_1.*

Proof. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Since AC-arc is contained in AB-arc, O_2 must be on the AO_1-arc not passing through B (see Fig. 1). Since O_1 is the mid-point of AB-arc, AO_1-arc is always less than a half of $\partial(\omega)$. Thus, $\measuredangle O_1O_2A > \frac{\pi}{2}$ and $\triangle AO_2O_1$ is an obtuse triangle with $|AO_1| > |AO_2|$. Let O_3 be the point on AO_1 with $|AO_3| = |AO_2|$, and ω_3 be the disk centered at O_3 and is having A on its boundary (see the dashed disk in Fig. 1). Since A, O_3 and O_1 are collinear and $|AO_1| > |AO_3|$, we have $\omega_3 \subset \omega_1$ and $A = \partial(\omega_3) \cap \partial(\omega_1)$. Then, we can conclude that $(\omega \cap \omega_3) \subset (\omega \cap \omega_1)$, and the only point in $(\omega \cap \omega_3)$ which is on $\partial(\omega_1)$ is A. On the other hand, ω_2 is a rotation of ω_3 around A in the direction to move its center from an interior point of ω, O_3, to a boundary point, O_2. Thus, we must have $(\omega \cap \omega_2) \subset (\omega \cap \omega_3)$, which implies $(\omega \cap \omega_2) \subset (\omega \cap \omega_3) \subset (\omega \cap \omega_1)$. Finally, from our proof its clear that any point in $(\omega \cap \omega_2)$ except A must be in the interior of ω_1. $\square$

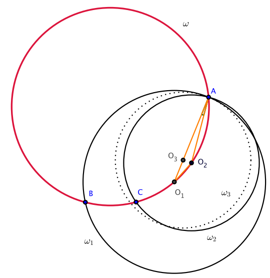


Figure 1. Illustration for Lemma 1

Lemma 2. *Let ω be a given disk and A, B, C, D be four points marked subsequently on $\partial(\omega)$. Further let ω_{ab} be the side disk corresponding to the AB-arc, and let ω_{bc}, ω_{cd} and ω_{da} be defined analogously (see Fig. 2). Let X be the intersection point of $\partial(\omega_{da})$ and $\partial(\omega_{cd})$, other than D; and Y, Z and T be defined analogously for the pairs $\partial(\omega_{cd})$ and $\partial(\omega_{bc})$, $\partial(\omega_{bc})$ and $\partial(\omega_{ab})$, and $\partial(\omega_{ab})$ and $\partial(\omega_{da})$, respectively. Then,*

(a) $X, Y \notin \omega_{ab}$, $Y, Z \notin \omega_{da}$, $Z, T \notin \omega_{cd}$ *and* $X, T \notin \omega_{bc}$*; and*
(b) *Quadrilateral $XYZT$ is a rectangle.*

Proof. To prove Lemma 2 (a), it suffices to show that $Z \notin \omega_{cd}$ as a similar argument applies to the others. Consider Fig. 2 and let the dashed disk ω_{bd} be the disk corresponding to BD-arc. It is well known, and can easily be proven that $\partial(\omega_{bd})$

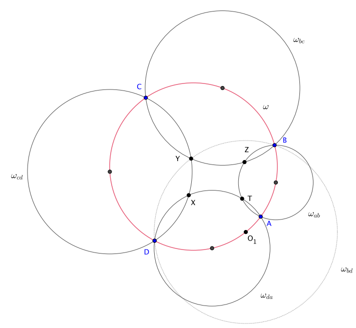


Figure 2. Illustration for Lemma 2 (a)

passes through Y, which is the incenter of $\triangle BCD$ (see below). Notice that all conditions of Lemma 1 are met for ω, ω_{bd} and ω_{ab}. Thus, $Z \in (\omega \cap \omega_{ab})$ must be located in the interior of ω_{bd}. But then $Z \notin \omega_{cd}$ as the only point which is in $(\omega_{cd} \cap \omega_{bd} \cap \omega_{bc})$ is Y, and Y and Z are distinct points as $Y \in \partial(\omega_{bd})$ while $Z \notin \partial(\omega_{bd})$. This proves Lemma 2 (a).

Let H, G, F and W be the centers of ω_{ab}, ω_{bc}, ω_{cd} and ω_{da}, respectively. We claim that X, Y, Z and T are the incenters of $\triangle ADC$, $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$, respectively (see Fig. 3). Notice that since F and W are the centers of two circles intersecting at D and X, FW is a perpendicular bisector of DX and $\measuredangle XFD = 2\measuredangle WFD$. Since W is the midpoint of AD-arc, we also have $\measuredangle WFD = \frac{1}{2}\measuredangle AFD$, which implies $\measuredangle XFD = \measuredangle AFD$. Thus, points F, X and A are collinear. Since F is the midpoint of DC-arc, $\measuredangle DAF = \measuredangle FAC$, hence AF is a bisector of $\angle DAC$. Similar argument shows that W, X and C are collinear

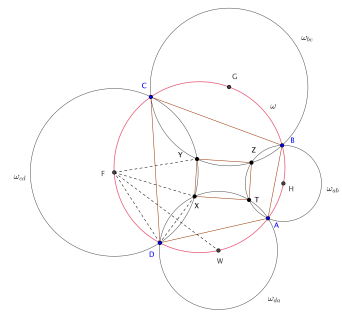


Figure 3. Illustration for Lemma 2 (b)

and CW is a bisector of $\angle DCA$. Thus, X is the incenter of $\triangle ADC$. By the same token, we may conclude that Y, Z and T are the incenters of $\triangle DCB$, $\triangle CBA$ and $\triangle BAD$, respectively. Then, the result in Lemma 2 (b) follows from Problem 6.13 in [5]. $\square$

Remark. From Lemma 1 it follows that the inner part of a side disk of C_n is always contained in C_n. Since outer parts of side disks of C_n are disjoint, this implies that *two side disks of C_n with $n \geq 2$ intersect if and only if they intersect in the region enclosed by C_n.* To our knowledge, the only widely known result directly related to Lemma 2 is Miquel's four circles theorem which states that when ω_{ab}, ω_{bc}, ω_{cd} and ω_{da} are not necessarily centered on $\partial(\omega)$, X, Y, Z, T are concyclic (see [7]; p.151). The result used in the last step of proving Lemma 2 (b) is often referred to as the Japanese theorem (see [6]).

For C_n, let $d(C_n)$ be the number of disjoint pairs among its side disks. Our main result is as follows.

Theorem 3. *For $n \geq 3$, $\frac{(n-2)(n-3)}{2} \leq d(C_n) \leq \frac{n(n-3)}{2}$.*

Proof. Since $d(C_3) = 0$, as all side disks are neighbouring, we assume $n \geq 4$. We prove the lefthand inequality in three steps.

STEP 1: Let us prove that $1 \leq d(C_4)$.

Let A, B, C, D be the points marked on C_4 and F, G, H, W be the centers of its four side disks. Further let X, Y, Z, T be points other than A, B, C, D in which pairs of neighbouring side disks intersect by their boundaries (see Fig. 4). By Lemma 2 (b), we know that $XYZT$ is a rectangle. Let $E = XZ \cap YT$, i.e. the intersection of the diagonals of $XYZT$. We claim that $E = WG \cap FH$. Since

$|FY| = |FX|$, $|HZ| = |HT|$ and $XYZT$ is a rectangle, points F, H and the midpoints of the sides XY and ZT are collinear. Similarly, G, W and the midpoints of ZY and TX are collinear. Thus, FH and GW intersect in a point where two bimedians of $XYZT$ intersect, which must be E. This proves our claim.

Since $\measuredangle XEY + \measuredangle YEZ = \pi$ one of these two angles (summands) must be at most $\frac{\pi}{2}$, and without loss of generality we may assume that $\measuredangle XEY \leq \frac{\pi}{2}$. Then, we claim that the side disks centered at F and H are disjoint. To see this, it suffices to prove that E is located outside of these side disks, as then we have $|FE| > r_F$ and $|EH| > r_H$, hence, $|FH| = |FE| + |EH| > r_F + r_H$, where r_F and r_H are the radii of the disks to be shown as disjoint. Let us then prove that $E \notin F$-disk and a similar argument shows that $E \notin H$-disk. By Lemma 2 (a), XY-line separates F and rectangle $XYZT$. Since E is an interior point of $XYZT$, we can conclude that XY-line strictly separates E and F.

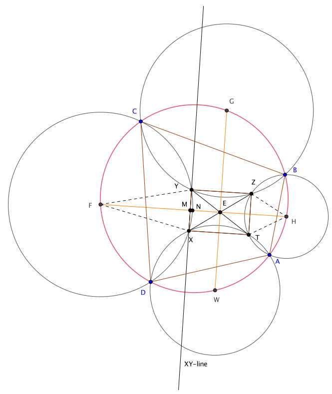


Figure 4. Side disks of C_4

Let M be the midpoint of XY and $N = \partial(F\text{-disk}) \cap FH$. It is clear that N and E lie on the same half-plane with respect to XY-line, as both are strictly separated from F by the line. We know that FH passes through the midpoints of XY and ZT, thus it must be orthogonal to XY. Recall that, both E and N lie on FH. Then, $\measuredangle XEY \leq \frac{\pi}{2}$ together with the observation that $\measuredangle XNY > \frac{\pi}{2}$ imply that $|ME| > |MN|$.[1] Then, $|FE| = |FM| + |ME| > |FM| + |MN| = r_F$. Thus, E is outside of F-disk. This proves our last claim and completes STEP 1.

<u>STEP 2</u>: Consider C_n and its side disks, labeled as $\omega_1, ..., \omega_n$ in the clockwise direction. For any $i, j = 1, 2, ..., n$, let (ω_i, ω_j) be the set of disks strictly between ω_i and ω_j, in the clockwise direction. We shall prove that if ω_i and ω_j intersect, then any disk in (ω_i, ω_j) is disjoint from any one in (ω_j, ω_i).

We can assume that ω_i and ω_j are non-neighbouring as the result is trivial otherwise. Let ω_i, ω_j be side disks of AB-arc and CD-arc, respectively. Then, AB-arc

[1] Take the circle centered at F. It is clear that XY is strictly shorter than its diameter. Then, for any $N^\star$ lying on the minor arc connecting X and Y, we have $\measuredangle XN^\star Y > \frac{\pi}{2}$.

and CD-arc are disjoint and we can also assume that A, B, C, D are located subsequently in the clockwise order. Let ω_{bc} and ω_{da} be the side disks of BC-arc and DA-arc, respectively (see Fig. 5). Then by STEP 1 there must be a disjoint fair among ω_i, ω_j, ω_{bc} and ω_{da}. But since the former two intersect, it must the latter two which are disjoint. By Lemma 1, we know that when restricted to the region enclosed by C_n, ω_{bc} contains all disks in (ω_i, ω_j), and similarly, ω_{da} contains all disks in (ω_j, ω_i). This implies that, none of the disks in (ω_i, ω_j) intersects with a

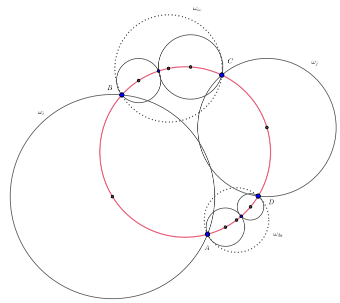


Figure 5. ω_i and ω_j intersect

disk in (ω_j, ω_i) in the region enclosed by C_n. But since two side disks intersect only in that region, we can conclude that any disk in (ω_i, ω_j) is disjoint from any one in (ω_j, ω_i). This completes STEP 2.

STEP 3: Let us prove that for $n \geq 4$, $\frac{(n-2)(n-3)}{2} \leq d(C_n)$.

Take P_n, a convex n-gon, and label its vertices with the side disks of C_n such that two disks of C_n are neighbouring if and only if their associated vertices in P_n are adjacent. Draw all $\frac{n(n-1)}{2} - n$ diagonals of P_n, and colour them with

- Red if the side disks corresponding to the end vertices intersect, and
- Blue if otherwise.

By STEP 2 we know that two red diagonals never cross in P_n. The maximal set of non-crossing diagonals of P_n gives a triangulation of it, and every triangulation involves $n-3$ diagonals (see Theorem 1.8 in [2]). Thus, the number of red diagonals is at most $n-3$, and the number of blue diagonals is at least $\frac{n(n-1)}{2} - n - (n-3) = \frac{(n-2)(n-3)}{2}$. This immediately implies that $\frac{(n-2)(n-3)}{2} \leq d(C_n)$, and completes STEP 3. The lefthand inequality in Theorem 3 is proved. Finally, since two neighbouring disks are never disjoint we have $d(C_n) \leq \frac{n(n-3)}{2}$. $\square$

Remark. It is easy to show that the upper bound in Theorem 3 is attained on $C_n^{\star}$, i.e. it is tight. Let $C_n^{\triangle}$ be a spherical great n-gon such that one of its side disks intersects with all the others, and any two of the other side disks intersect only if they are neighbouring. It is easy to show that this construction is well defined and $d(C_n^{\triangle}) = \frac{(n-2)(n-3)}{2}$. Thus, the lower bound is also tight for $n \geq 3$.

We can now characterize the intersection graph of side disks of C_n. Recall that a planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same (exterior) face (see Chap. 11 in [3]).

Theorem 4. *The intersection graph of side disks of C_n for $n \geq 3$ is a subgraph of a triangulation of a convex n-gon. In particular, it is outerplanar.*

Proof. Let $G(C_n)$ be the intersection graph. The result is obvious when $n = 3$. For $n \geq 4$, in STEP 2 of proof of Theorem 3 we showed that $G(C_n)$ can be drawn with no crossing edges. So, it is a subgraph of triangulation of the convex polygon with vertices at the centers of the side disks, hence outerplanar. $\square$

References

[1] M. Chamberland, *Single Digits: In Praise of Small Numbers*, Princeton University Press, 2015.

[2] S. L. Devadoss and J. O'Rourke, *Discrete and Computational Geometry*, Princeton University Press, 2011.

[3] F. Harary, *Graph Theory*, Addison-Wesley, MA, 1969.

[4] C. Huemer and P. Pérez-Lantero, The intersection graph of the disks with diameters the sides of a convex n-gon, (2016), Online: arXiv:1410.4126v3

[5] V. V. Prasolov, *Problems in Plane and Solid Geometry: Vol. 1 Plane geometry*, 3rd eds., 2001. Translated and edited by D. Leites and online at: `http://students.imsa.edu/~tliu/Math/planegeo.pdf`

[6] W. Reyes, An application of Thébault's theorem, *Forum Geom.*, 2 (2002), 183–185.

[7] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin Books, New York, 1991.

Purevsuren Damba: Mathematics Department, The National University of Mongolia, Ulaanbaatar, Mongolia

E-mail address: `purevsuren@smcs.num.edu.mn`

Uuganbaatar Ninjbat: Mathematics Department, The National University of Mongolia, Ulaanbaatar, Mongolia

E-mail address: `uugnaa.ninjbat@gmail.com`

Forum Geometricorum
Volume 18 (2018) 203–218.

FORUM GEOM
ISSN 1534-1178

Parallelograms Inscribed in Convex Quadrangles

Nikolaos Dergiades

Abstract. In this article we construct the inscription in a quadrangle of a parallelogram similar to a given one and we study the parallelograms inscribed in quadrangles with special emphasis on determining the classes of quadrangles which allow the inscription of infinite many parallelograms of a given similarity type. As special cases we obtain the characterization of quadrangles allowing the inscription of infinite many rhombi of a fixed similarity type, of infinite many rectangles of a fixed similarity type and of quadrangles allowing the inscription of infinite many squares.

1. A simple case of inscribed parallelograms

In an arbitrary convex quadrilateral $q = ABCD$ we can easily inscribe infinite many parallelograms with sides parallel to the diagonals $\{AC, BD\}$ of the quadrilateral of reference q (See Figure 1). The vertex A_1 of such a parallelogram is

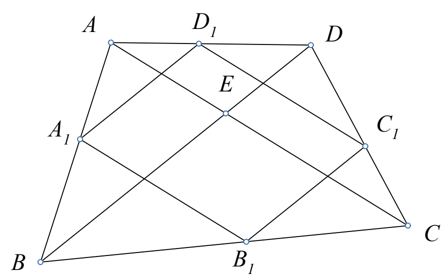


Figure 1. Parallelograms with sides parallel to the diagonals

selected on the side AB. Then, drawing a parallel to the diagonal AC we define its intersection point B_1 with BC. Drawing further from B_1 a parallel to BD we define its intersection C_1 with CD and continuing this way, we obtain, using Thales theorem, an inscribed parallelogram $A_1B_1C_1D_1$ with sides parallel to the diagonals of q.

It is easy to see that among these parallelograms there exist one that is a rhombus satisfying the following property:

Lemma 1. *The rhombus $A_1B_1C_1D_1$, inscribed in the quadrangle $q = ABCD$, with sides parallel to the diagonals of q is constructible with ruler and compasses.*

Publication Date: June 6, 2018. Communicating Editor: Paul Yiu.

The author thanks Paris Pamfilos for his many improvements in the text, the synthetic proof in Theorem 4, and simplification of proofs in Theorems 5 and 12.

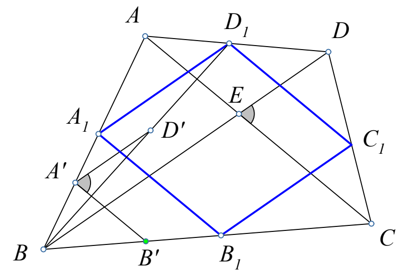


Figure 2. Inscribing a rhombus $A_1B_1C_1D_1$ in the quadrangle $ABCD$

Proof. In fact, from an arbitrary point B' on the side BC, draw the parallel to CA, that meets BA at the point A' (See Figure 2). Rotate then B' about A' by the angle of the diagonals $\widehat{CED}$ to get the point D'. The line BD' meets AD at the point D_1, from which with parallels to the diagonals of q we construct the other vertices of $A_1B_1C_1D_1$, seen to be a rhombus by applying the Thales theorem

$$\frac{A_1D_1}{A'D'} = \frac{BA_1}{BA'} = \frac{A_1B_1}{A'B'}.$$

□

2. The inscription in a quadrangle of a parallelogram

Given a convex quadrangle $q = ABCD$ with positive orientation and a parallelogram $A_0B_0C_0D_0$ with $\widehat{A_0C_0B_0} = \alpha$, $\widehat{B_0A_0C_0} = \gamma$ and $C_0B_0A_0 = \beta$, it is very interesting the problem of inscribing in q a parallelogram $A_1B_1C_1D_1$ similar to $A_0B_0C_0D_0$.

(1) $\beta = \pi/2$, that is $A_0B_0C_0D_0$ is a rectangle.

(a) $\alpha \neq \gamma$.

The rotation of line AD about A towards B by the angle α gives the line L_1. The rotation of line BC, about B towards A by the angle γ gives the line L_2. The perpendicular from A to line L_2 meets the line BC at A' and the parallel from A' to line L_2 meets the line L_1 at the point A'' (See Figure 3). Similarly the perpendicular from B to line L_1 meets the line AD at B' and the parallel from B' to line L_1 meets the line L_2 at the point B''. Let C_1 be the intersection of line $A''B''$ with the line CD. The parallels from C_1 to $\{L_1, L_2\}$ meet the lines $\{AD, BC\}$ respectively at the points $\{D_2, B_2\}$ and the perpendicular from B_2 to L_2 meets AB at the point A_1. From Thales theorem we have

$$\frac{AA_1}{A_1B} = \frac{A'B_2}{B_2B} = \frac{A''C_1}{C_1B''} = \frac{AD_2}{D_2B'},$$

which means that A_1D_2 is parallel to BB', which is perpendicular to L_1. Hence $\widehat{A_1D_2C_1} = \widehat{A_1B_2C_1} = \pi/2$ and the circle on diameter A_1C_1 passes through $\{B_2, D_2\}$ and meets again the lines $\{BC, AD\}$

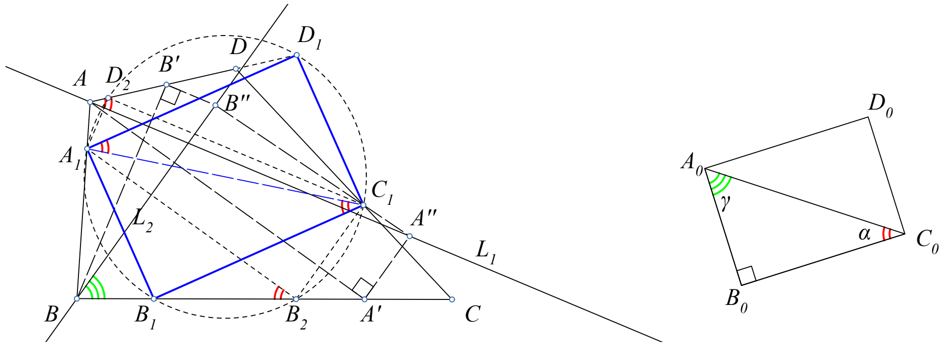


Figure 3. The inscription in a quadrangle of a rectangle

respectively at points $\{B_1, D_1\}$. Since A_1B_2 is perpendicular to L_2, the angle $\widehat{BB_2A_1} = \alpha$. Since $\widehat{B_1C_1A_1} = \widehat{B_1B_2A_1} = \alpha$, $\widehat{C_1A_1D_1} = \widehat{C_1D_2D_1} = \alpha$ and $\widehat{C_1B_1A_1} = \widehat{A_1D_1C_1} = \pi/2$, the quadrangle $A_1B_1C_1D_1$ is a rectangle similar to $A_0B_0C_0D_0$.

Note: If the line $A''B''$ is parallel to CD then the problem has no solution. If the line $A''B''$ coincides with line CD, then the problem has infinite many solutions. We'll see these cases later on.

(b) $\alpha = \gamma = \pi/4$, that is $A_0B_0C_0D_0$ is a square.
The construction is obviously similar to the previous one.

(2) $\beta \neq \pi/2$.

(a) $\alpha \neq \gamma$, that is $A_0B_0C_0D_0$ is not a special parallelogram.
The construction differs slightly from the previous one.
The rotation of line AD about A towards B by the angle α gives the line L_1. The rotation of line BC, about B towards A by the angle γ gives the line L_2. The rotation of the parallel line from A to BC, about A towards B by angle α gives the line AA', where A' is on BC, such that $\widehat{BA'A} = \alpha$. Similarly the rotation of the parallel line from B to AD, about B towards A by angle γ gives the line BB', where B' is on AD, such that $\widehat{AB'B} = \gamma$. The parallel from A' to L_2 meets L_1 at the point A'' and the parallel from B' to L_1 meets L_2 at the point B'' (See Figure 4). Let C_1 be the intersection of line $A''B''$ with the line CD. The parallels from C_1 to $\{L_1, L_2\}$ meet the lines $\{AD, BC\}$ respectively at the points $\{D_2, B_2\}$ and the parallel from B_2 to AA' meets AB at the point A_1. From Thales theorem we have

$$\frac{AA_1}{A_1B} = \frac{A'B_2}{B_2B} = \frac{A''C_1}{C_1B''} = \frac{AD_2}{D_2B'},$$

which means that A_1D_2 is parallel to BB'. It is then easy to see that $\widehat{A_1D_2C_1} = \widehat{A_1B_2C_1} = \pi - \alpha - \gamma = \beta$. If B_1 is the second intersection of the circumcircle of the triangle $A_1B_2C_1$ with the

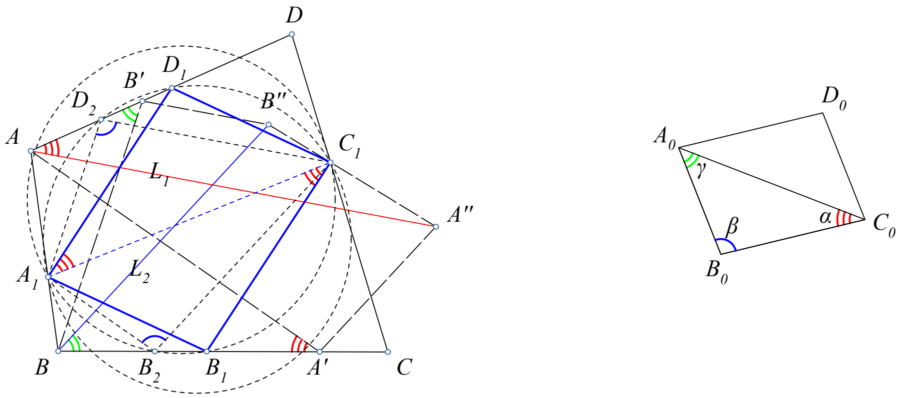


Figure 4. The inscription in a quadrangle of a general parallelogram

line BC and D_1 is the second intersection of the circumcircle of triangle $A_1D_2C_1$ with line AD, then we have $\widehat{A_1C_1B_1} = \widehat{A_1B_2B} = \widehat{AA'B} = \alpha$, $\widehat{C_1A_1D_1} = \widehat{C_1D_2D_1} = \alpha$ and $\widehat{C_1B_1A_1} = \widehat{A_1D_1C_1} = \beta$, which means that $A_1B_1C_1D_1$ is a parallelogram similar to $A_0B_0C_0D_0$.

(b) $\alpha = \gamma$, that is, $A_0B_0C_0D_0$ is a rhombus.
The construction is similar to the previous one.

3. Similar quadrangles inscribed in the quadrangle

The situation described in the preceding section changes radically, when we require from the quadrangle to have infinite many inscribed quadrangles of a *fixed similarity type*. This implies severe restrictions to the circumscribing quadrilateral, which we study in this section. To fix the ideas, we assume that the quadrangle $q_1 = A_1B_1C_1D_1$ is inscribed in the quadrangle $q = ABCD$, when the vertices $\{A_1, B_1, C_1, D_1\}$ are respectively on the sides $\{AB, BC, CD, DA\}$. Next theorem expresses a basic feature of the configuration under consideration.

Theorem 2. *If in the quadrangle $q = ABCD$, there are inscribed two directly similar quadrangles $\{q_1, q_2\}$, then the following properties hold.*

(1) *There are infinite many other quadrangles $\{q_t\}$ directly similar to q_1 and inscribed in q.*
(2) *The similarity center P of the quadrangles $\{q_1, q_2\}$ defines through its projections on the sides of q its pedal quadrangle q_0, which is the minimal similar to q_1 quadrangle.*
(3) *Every pair of the family of directly similar inscribed quadrangles $\{q_1, q_2\}$ has the same similarity center P.*

Proof. The claims are consequences of well known facts about direct similarities, an exposition of which can be found in [7, vol.II] and [5, p.74]. The center of the similarity is the Miquel point P ([3, p.79], [4, p.131]) of one the resulting, inscribed in $ABCD$, triangles, $A_1B_1C_1$ say (See Figure 5), and is identified with

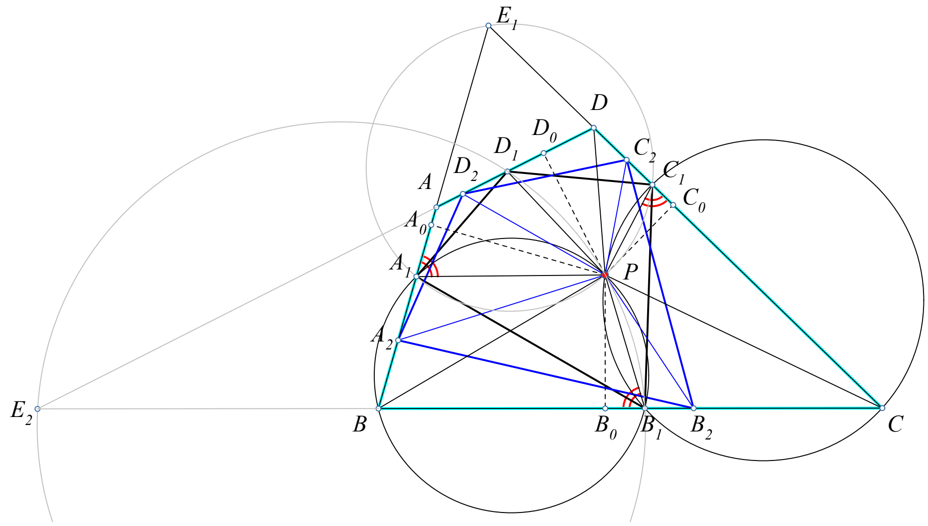


Figure 5. The similarity center P of $\{q_1 = A_1B_1C_1D_1, q_2 = A_2B_2C_2D_2\}$

the intersection P of the circles $\{(A_1BB_1), (B_1CC_1), (E_1A_1C_1)\}$. A simple angle chasing argument shows that the angles ([4, p. 142])

$$\widehat{BPC} = \widehat{A_1B_1C_1} + \widehat{A_1E_1C_1}.$$

And a similar argument shows that $\widehat{CPE_1} = \widehat{A_1C_1B_1} + \widehat{E_1BC}$. Thus point P is the same for the two quadrangles $\{q_1, q_2\}$ and the triangles $\{PA_1A_2, PB_1B_2, PC_1C_2\}$ are similar. Analogous work with the triangle E_2CD and the inscribed in it triangles $B_1C_1D_1$ and $B_2C_2D_2$, shows that P is also the similarity center for these two triangles and the triangles $\{B_1PB_2, C_1PC_2, D_1PD_2\}$ are also similar. The claims of the theorem follow easily from these remarks. □

Notice that, as a parameter $t \in \mathbb{R}$ in the family of inscribed quadrangles $\{q_t\}$ we can consider the signed distance of one of its vertices, B_t say, from the corresponding projection B_0 of P. All the quadrangles q_t result by defining the similar and similarly oriented right triangles $\{PA_0A_t, PB_0B_t, PC_0C_t, PD_0D_t\}$.

Corollary 3. *A necessary and sufficient condition for the possibility, to inscribe in a quadrangle q infinite many, directly similar to each other quadrangles q_t, is the existence of a point P, whose pedal quadrangle with respect to q is similar to q_t.*

4. Similar parallelograms inscribed in a quadrangle

By theorem 2, if we have two similar parallelograms inscribed in the convex quadrangle $q = ABCD$, then we have infinite many similar to those two parallelograms, inscribed in q. We can fix the parallelogram and its pivot point P and consider the circumscribed quadrangles q_t of the parallelogram, which are all similar to the antipedal $q_1 = A_1B_1C_1D_1$ of P with respect to the parallelogram (See

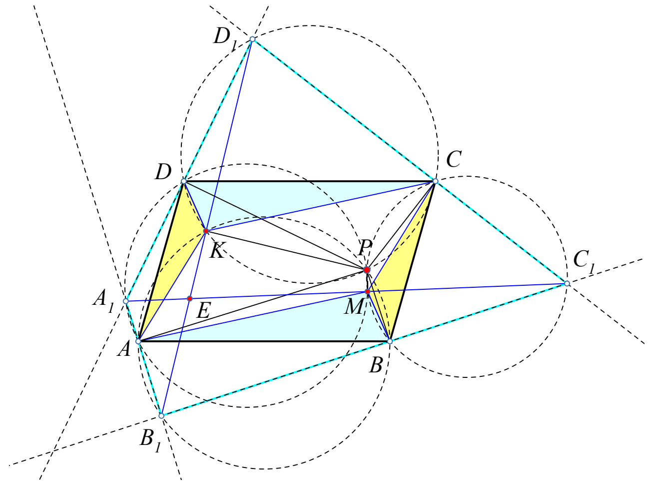


Figure 6. Angles of diagonals of antipedal with respect to a parallelogram

Figure 6). Next theorem relates the angles of the parallelogram to those of the diagonals of the antipedal.

Theorem 4. *Let $q_1 = A_1B_1C_1D_1$ be the antipedal quadrangle of a parallelogram $q = ABCD$ with respect to the point P. Then, the angles of the diagonals of q_1 are equal to the angles of the parallelogram q.*

Proof. Consider the intersections of the pairs of circles $M = (PBC) \cap (PAD)$ and $K = (PDC) \cap (PAB)$. Points $\{K, M\}$ are symmetric with respect to the center of the parallelogram and define equal triangles $\{ADK, CBM\}$ (See Figure 6). This is seen by examininig the angles of these triangles. For example, the angle $\widehat{DKA}$, by taking the adjacents to it angles, which sum to 2π, is seen to be equal to the sum

$$\widehat{DKA} = \widehat{DCP} + \widehat{PBA} = \widehat{CPB} = \widehat{CMB}.$$

Analogously is seen that $\widehat{DKC}$ is equal to $\widehat{AMB}$, so that the points are symmetric as stated. It is easily seen also that the diagonals of q_1 pass respectively through $\{K, M\}$ and make there respectively with $\{PK, PM\}$ right angles, so that the quadrangle $PKEM$, formed by their intersection, is cyclic. But the angle of the diagonals

$$\widehat{KEM} = \pi - \widehat{KPM} \text{ and } \widehat{KPM} = 2\pi - \widehat{KPC} - \widehat{MPC} = \widehat{KDC} + \widehat{MBC} = \widehat{ABC}.$$

□

Theorem 5. *With the notation and conventions of the preceding theorem, the pivot point P of $q_1 = A_1B_1C_1D_1$ is viewing each side of q_1 by an angle, which is the sum of the angles viewing that side from the other two vertices.*

Proof. This follows by splitting the angle (See Figure 7)

$$\widehat{B_1PC_1} = \widehat{B_1PB} + \widehat{BPC_1} = \widehat{B_1AB} + \widehat{BCC_1}.$$

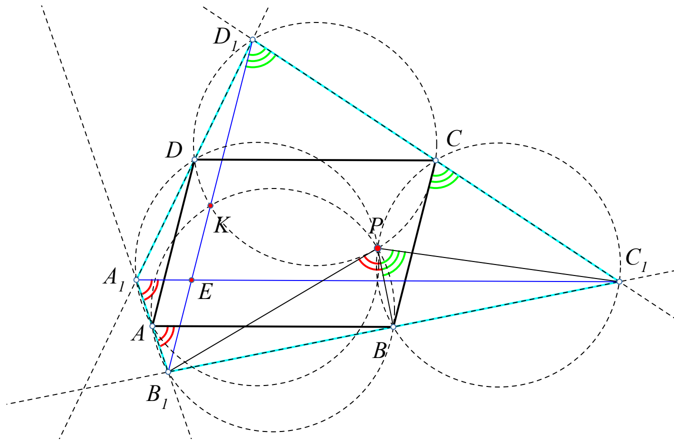


Figure 7. How the pivot of $ABCD$ is viewing the sides of $A_1B_1C_1D_1$

By the previous theorem, fixing q_1 and pivoting $ABCD$, so that its sides become parallel to the diagonals of q_1, we have that the sum in the last formula is equal to $\widehat{B_1A_1C_1} + \widehat{B_1D_1C_1}$. This proves the claim for the side B_1C_1, the proof for the other sides being completely analogous. □

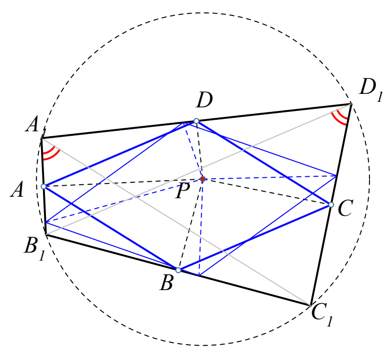


Figure 8. In cyclic quadrangles the pivoting parallelogram is that of Varignon

Corollary 6. *If $q_1 = A_1B_1C_1D_1$ is a cyclic quadrangle, then the pivoting parallelogram $q = ABCD$ of it is the one of Varignon, defined by the middles of the sides of q_1 and the pivot P is the center of the circumcircle.*

Proof. By theorem 5 the pivot P is then viewing B_1C_1 under the ange $\widehat{B_1PC_1} = 2\widehat{B_1AC_1}$, which identifies P with the center of the circumcircle (See Figure 8). □

The converse is also true and its proof is left as an exercise.

Corollary 7. *If the pivoting parallelogram inside the quadrangle q_1 is similar to the Varignon parallelogram of the middles of the sides, then the quadrangle q_1 is cyclic.*

Next corollary is also easy to show using the figure 9, and is left also as an

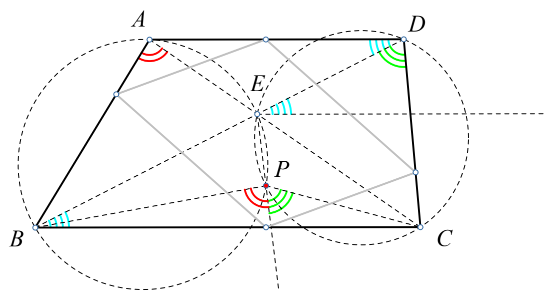


Figure 9. The pivot P of similar parallelograms inscribed in a trapezium

exercise.

Corollary 8. *The pivot P of the similar parallelograms pivoting inside a trapezium is the second intersection point of the circles $\{(EAB), (EDC)\}$ through the intersection of the diagonals $E = (AC, BD)$ and the end points of the non-parallel sides.*

In the case $q = ABCD$ is a parallelogram, the points $\{E, P\}$ in figure 9 coincide and the pivot is the center of the parallelogram.

Corollary 9. *The pivot of similar parallelograms inside a parallelogram $q = ABCD$ is the center P of the parallelogram q.*

5. Alternative approach for the pivot of similar parallelograms

It is possible to construct special parallelograms, from the family of those pivoting about a point P in the quadrangle $q = ABCD$. A particular one, of importance for the following development is given by the next theorem.

Theorem 10. *If the distances of the intersection point E of the diagonals of the quadrilateral $q = ABCD$ from the vertices are correspondingly $\{a, b, c, d\}$, then there is a parallelogram $q_1 = A_1B_1C_1D_1$ of the family of similar parallelograms pivoting inside q, such that*

$$\frac{BB_1}{B_1C} = \frac{bd}{ac}.$$

Proof. Let the parallel ε from E to BC intersect AB at B' and C' be the symmetric of B' with respect to E (See Figure 10). Let the circle $\kappa = (AB'C)$ intersect ε at B'' and the circle $\lambda = (BC'D)$ intersect the line ε at C''. Consider also the intersection of lines $F = (BC'', CB'')$. Then $B_1 = (EF, BC)$ is the required point.

For the proof consider the power of E with respect to λ : $EC' \cdot EC'' = EB \cdot ED = bd$. Similarly, the power of E with respect to κ is $EB' \cdot EB'' = EA \cdot EC = ac$. It follows that the ratio

$$\frac{BB_1}{B_1C} = \frac{C''E}{EB''} = \frac{bd/EC'}{ac/EB'} = \frac{bd}{ac}.$$

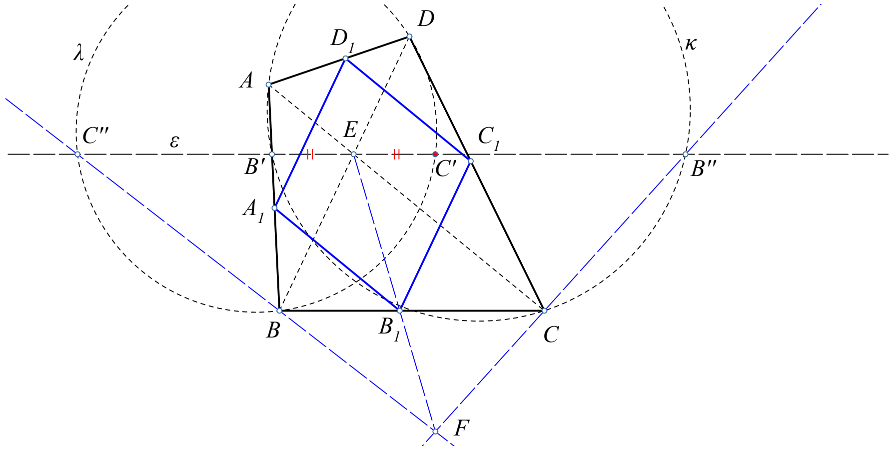


Figure 10. Special inscribed parallelogram

□

Theorem 11. *With the notation and conventions of the preceding theorem, the circles* $(AA_1D_1), (BB_1A_1), (CC_1B_1)$ *and* (DD_1C_1) *pass through the same point* P*, which is the pivot of the similar parallelograms inscribed in* q*.*

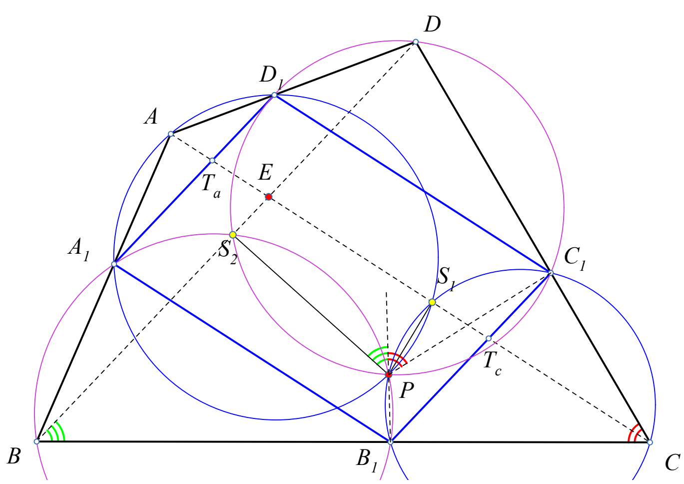


Figure 11. The pivot determined by the special inscribed parallelogram

Proof. Using $\{EA = a, EB = b, EC = c, ED = d\}$, the assumption $\frac{BB_1}{B_1C} = \frac{bd}{ac}$ and denoting the intersections of lines by $\{T_a = (A_1D_1, AC), T_c = (B_1C_1, AC)\}$ (See Figure 11), we have

$$\frac{AA_1}{AB} = \frac{B_1C}{BC} = \frac{ac}{ac+bd} = \frac{AT_a}{AE} = m \quad \Rightarrow \quad AT_a = am.$$

Similarly we get

$$A_1T_a = B_1T_c = bm, \qquad T_aD_1 = T_cC_1 = dm, \qquad T_cC = cm.$$

Let S_1 be the second intersection of the circle (AA_1D_1) with AC. Then, we have the relations

$$AT_a \cdot T_aS_1 = A_1T_a \cdot T_aD_1 \quad \Rightarrow \quad T_aS_1 = \frac{bdm}{a} \quad \Rightarrow$$

$$\begin{aligned} S_1T_c &= (AC - AT_a - T_cC) - T_aS_1 = (a+c)(1-m) - \frac{bdm}{a} \\ &= (a+c)\frac{bd}{ac+bd} - \frac{bdc}{ac+bd} = \frac{bdm}{c} \quad \Rightarrow \quad S_1T_c \cdot T_cC = B_1T_c \cdot T_cC_1. \end{aligned}$$

This means that the circle (B_1CC_1) also passes through point S_1. Similarly we show that the circles $\{(BB_1A_1), (DD_1C_1)\}$ intersect a second time at the point S_2 on the diagonal BD. Let now P be the second intersection point of the circles $\{(BB_1A_1), (CC_1B_1)\}$. Then, we have $\widehat{C_1PS_2} = \widehat{CBD} + \widehat{DCB} = \pi - \widehat{C_1DS_2}$, i.e. P is on the circle $(DS_2C_1) = (DD_1C_1)$. Similarly we prove that P is on the circle (AA_1D_1). $\square$

6. Similar rectangles inscribed in a quadrangle

By theorem 2 the condition to have infinite many similar rectangles inscribed in a quadrangle is equivalent with the existence of a point whose pedal quadrangle is a rectangle. Next theorem supplies some basic implications for such a case.

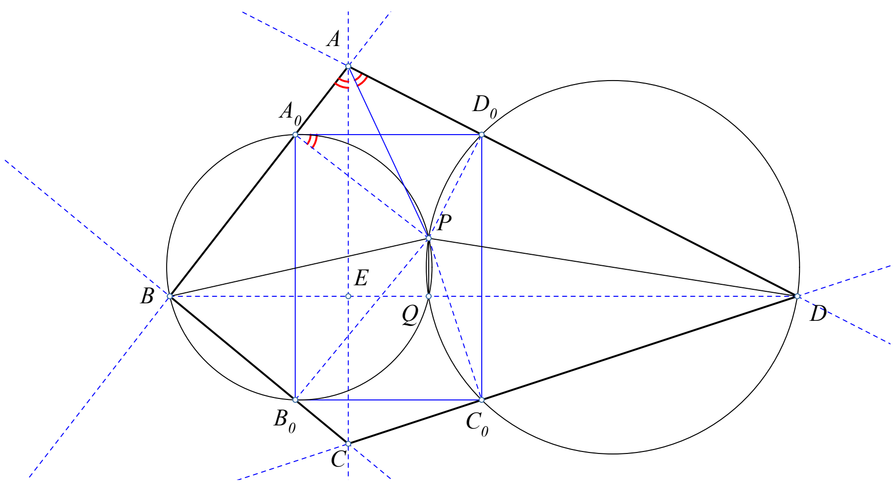


Figure 12. Quadrangle q possessing a pedal q_0, which is a rectangle

Theorem 12. *If the pedal quadrangle $q_0 = A_0B_0C_0B_0$ of a point P, relative to the quadrangle $q = ABCD$ is a rectangle, then*

(1) *The sides of q_0 are parallel to the diagonals of q, i.e. q is orthodiagonal.*
(2) *The intersection point E of the diagonals of q is isogonal conjugate to P.*

Proof. For *nr-1* consider the intersection Q of the circles $\{PC_0DD_0), (PB_0BA_0)\}$ (See Figure 12). Since $A_0B_0C_0D_0$ is a rectangle, the figure consisting of it and the two circles is symmetric with respect to the line joining the middles of the opposite sides $\{A_0B_0, C_0D_0\}$, hence line PQ is parallel to these sides too. By the inscribed quadrangles in these circles, the angles at Q are both right, so that $\{B, Q, D\}$ are collinear, hence BD is orthogonal to A_0B_0. Analogously is seen the orthogonality of AC to A_0D_0.

Nr-2 follows from *nr-1* and a simple angle chasing argument, showing that the angles $\widehat{PAD_0} = \widehat{A_0AE}$. Handling analogously the angles of the rays from the vertices of q to points $\{E, P\}$, we see indeed the isogonal property of $\{E, P\}$. □

Corollary 13. *Under the conditions of the preceding theorem, the points* $\{E, P\}$ *are also isogonal with respect to the triangles* $\{E_1BC, E_1DA, E_2AB, E_2CD\}$, *for the intersections of opposite sides* $\{E_1 = (AB, CD), E_2 = (AD, BC)\}$.

7. Similar rhombi inscribed in a quadrangle

In this section we specialize from similar parallelograms to similar rhombi inscribed in a quadrangle. Next lemma establishes a property of the triangle used below.

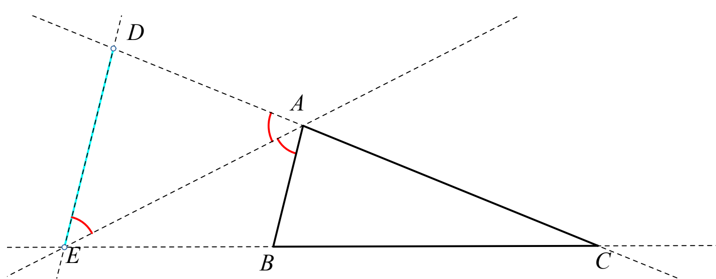


Figure 13. A property of the triangle

Lemma 14. *If* AE *is the external bisector of the triangle* ABC *with* $AB < AC$, *and the parallel from* E *to* AB *meets* AC *at* D, *then we have*

$$\frac{1}{AB} - \frac{1}{AC} = \frac{1}{ED}.$$

Proof. The triangle DEA is isosceles $DA = DE$. Hence we have (See Figure 13)

$$\frac{DE}{AB} = \frac{DC}{AC} = \frac{DA + AC}{AC} = \frac{DE + AC}{AC} = \frac{DE}{AC} + 1 \quad \text{or} \quad \frac{1}{AB} - \frac{1}{AC} = \frac{1}{ED}.$$

□

Lemma 15. *Let* $q = ABCD$ *be a quadrangle whose triangle of diagonal points of the corresponding complete quadrilateral is* EE_1E_2 *(See Figure* 14*). Then, the following properties are equivalent.*

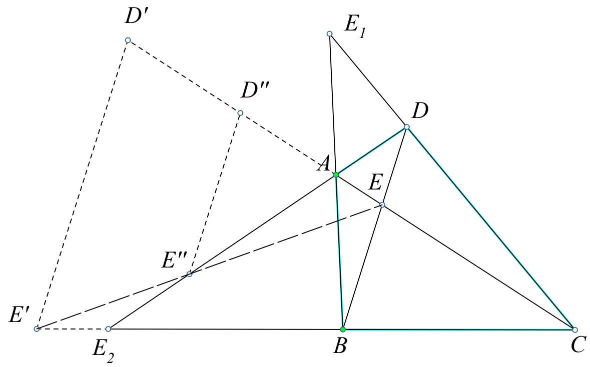


Figure 14. Bisecting the angle of the diagonals

(1) *The triangle* EE_1E_2 *has a right angle at* $E = (AC, BD)$.
(2) *The lines* $\{EE_1, EE_2\}$ *are bisectors of the angle of the diagonals* $\{AC, BD\}$ *of* q.
(3) *The harmonic means of the segments on the diagonals are equal:* $\frac{1}{EA} + \frac{1}{EC} = \frac{1}{EB} + \frac{1}{ED}$.

Proof. The equivalence of *nr-1* and *nr-2* follows from the general fact that the diagonals of a quadrilateral and the lines $\{EE_1, EE_2\}$ form a harmonic pencil ([1, p.101]). Thus, if two of the arrays of the pencil make a right angle, then they bisect the angle of the other two ([1, p.102]).

The equivalence of *nr-2* and *nr-3* results by considering the intersections $\{E', E''\}$ of the bisector of the angle $\widehat{AEB}$ respectively with $\{BC, AD\}$. The parallels from $\{E', E''\}$ to the diagonal BD intersect the other diagonal AC respectively at the points $\{D', D''\}$. Applying lemma 14 respectively to the triangles $\{EBC, AED\}$ we obtain

$$\frac{1}{EB} - \frac{1}{EC} = \frac{1}{E'D'} = \frac{1}{ED'} \qquad \text{and} \qquad \frac{1}{AE} - \frac{1}{ED} = \frac{1}{ED''}.$$

Thus, the assumed condition is equivalent to

$$\frac{1}{EA}+\frac{1}{EC} = \frac{1}{EB}+\frac{1}{ED} \quad \Leftrightarrow \quad \frac{1}{EA}-\frac{1}{ED} = \frac{1}{EB}-\frac{1}{EC} \quad \Leftrightarrow \quad \frac{1}{ED''} = \frac{1}{ED'},$$

which means that $\{E', E''\}$ must coincide with E_2. □

Theorem 16. *The necessary and sufficient condition to inscribe in a quadrangle infinite many similar rhombi, is that the bisectors of the angle of the diagonals pass through the intersections of the opposite sides of the quadrangle.*

Proof. Let again $\{EA = a, EB = b, EC = c, ED = d\}$ denote the segments to the vertices from the intersection point $E = (AC, BD)$ of the diagonals of the quadrangle $q = ABCD$ (See Figure 15). By theorem 10 we have a parallelogram $q_1 = A_1B_1C_1D_1$ of the infinite family of inscribed parallelograms, which has

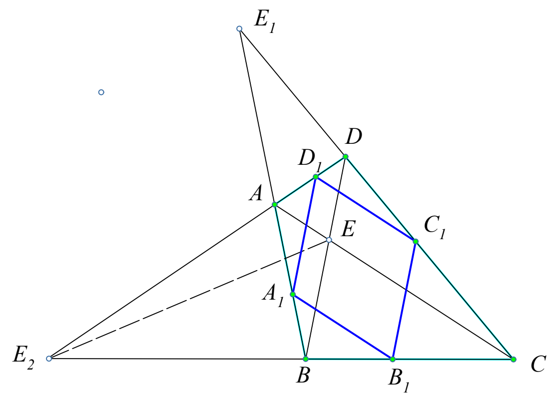


Figure 15. Condition for infinite many pivoting rhombi

sides parallel to the diagonals of q and satisfies $\frac{BB_1}{B_1C} = \frac{bd}{ac}$. For this parallelogram we obtain the relation

$$\frac{B_1C_1}{BD} = \frac{B_1C}{BC} \quad \Rightarrow \quad B_1C_1 = \frac{ac(b+d)}{ac+bd} \quad \text{and similarly} \quad A_1B_1 = \frac{bd(a+c)}{ac+bd}.$$

The parallelogram being a rhombus iff $A_1B_1 = B_1C_1$ combined with the preceding relations gives the necessary and sufficient condition ([6])

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d},$$

which, by lemma 15, is the necessary and sufficient condition for the bisectors, to pass respectively through points $\{E_1, E_2\}$. □

Theorem 17. *If a quadrangle $q = ABCD$ has infinite many inscribed rhombi, then for the rhombus $A_1B_1C_1D_1$ of this family, whose sides are parallel to the diagonals of q we have $BB_1 : B_1C = BD : AC$.*

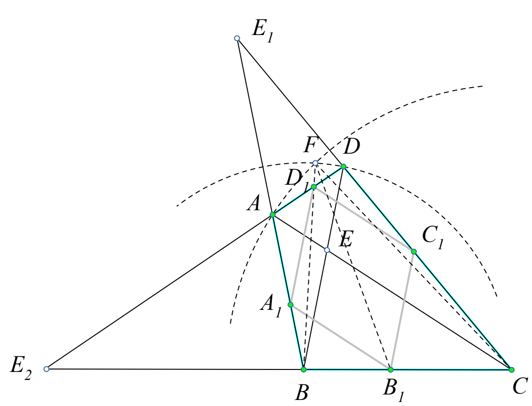


Figure 16. Constructing the pivoting rhombus with side parallel to the diagonals

Proof. By theorem 16 and lemma 15, we know that such a quadrangle q is characterized by the condition $\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{d}$, hence $bd:ac=BD:AC$. Hence from the known relation for this $q_1:\ BB_1:B_1C=bd:ac$ we get $BB_1:B_1C=BD:AC$. Thus, the point B_1 can be constructed using the bisector FB_1 of the triangle FBC, taking F with $FB=BD$ and $FC=AC$ (See Figure 16). □

8. Impossibility to inscribe similar parallelograms

From the theory exposed so far it is clear that if we are given a quadrangle and a parallelogram $\{q=ABCD, q_0=A_0B_0C_0D_0\}$ and we want to inscribe in q infinite many parallelograms $q_t=A_tB_tC_tD_t$ similar to q_0, with A_t on AB, B_t on BC etc., then the following theorem holds.

Theorem 18. *Let E denote the intersection of the diagonals of q and $EA=a,\ \ EB=b,\ \ EC=c,\ \ ED=d$. Then it is impossible to inscribe in q infinite many parallelograms similar to q_0 if one of the following conditions is valid.*

(1) $A_0B_0:B_0C_0\ \neq\ \left(\frac{1}{a}+\frac{1}{c}\right):\left(\frac{1}{b}+\frac{1}{d}\right)$.

(2) $\widehat{BEC}\ \neq\ \widehat{A_0B_0C_0}$.

9. Impossibility to inscribe a given parallelogram

We saw in section 2 that the inscription in a quadrangle $q=ABCD$ of a parallelogram $q_1=A_1B_1C_1D_1$, similar to $q_0=A_0B_0C_0D_0$ is impossible if the point $D_1=(A''B'',CD)$ is at infinity, i.e. the line $A''B''$ is parallel to line CD. Using homogeneous barycentric coordinates with respect to the triangle ABC and considering the point D variable with coordinates (x,y,z) we can ask for the geometric locus of D, so that the line $A''B''$ is parallel to the line CD. It turns out that this locus is a conic, so that *if the point D is on that conic, then the problem has no solution*. In order to reduce the complexity, we don't give further explanations and stay in the case $\alpha=\gamma$, i.e. the case in which q_0 is a rhombus, for which we note by $\beta=\widehat{A_0B_0C_0}$. Then the following theorem is valid.

Theorem 19. *It is impossible to inscribe in q a rhombus $q_1=A_1B_1C_1D_1$ similar to q_0, with $\widehat{A_0B_0C_0}=\beta$, when $D_1=(A''B'',CD)$ is at inifinity, which is equivalent with the condition that $D(x,y,z)$ lies on the conic represented in the aforementioned barycentric coordinates by the equation*

$$\sin(\beta)(a^2yz+b^2zx+c^2xy)+(x+y)(y+z)S=0,$$

where $\{a,b,c,S\}$ are the lengths of the sides and S is the double area of ABC.

Proof. It is left to the reader as a result of computer aided calculations, such as in [2]. □

In order to construct this conic we need to know 5 points on it. Since

$$C_c:\ a^2yz+b^2zx+c^2xy=0,\quad P_{bc}:\ y+z=0,\quad P_{ab}:\ x+y=0,$$

are respectively the equations of the circumcircle of ABC and the lines which are parallel from $\{A,C\}$ to $\{BC,BA\}$, it is easy to see that four of these points are

$\{A, X, C, Z\}$, where $\{P_{bc}, P_{ab}\}$ meet respectively the circle C_c. A fifth special point D' on the conic is constructed as follows.

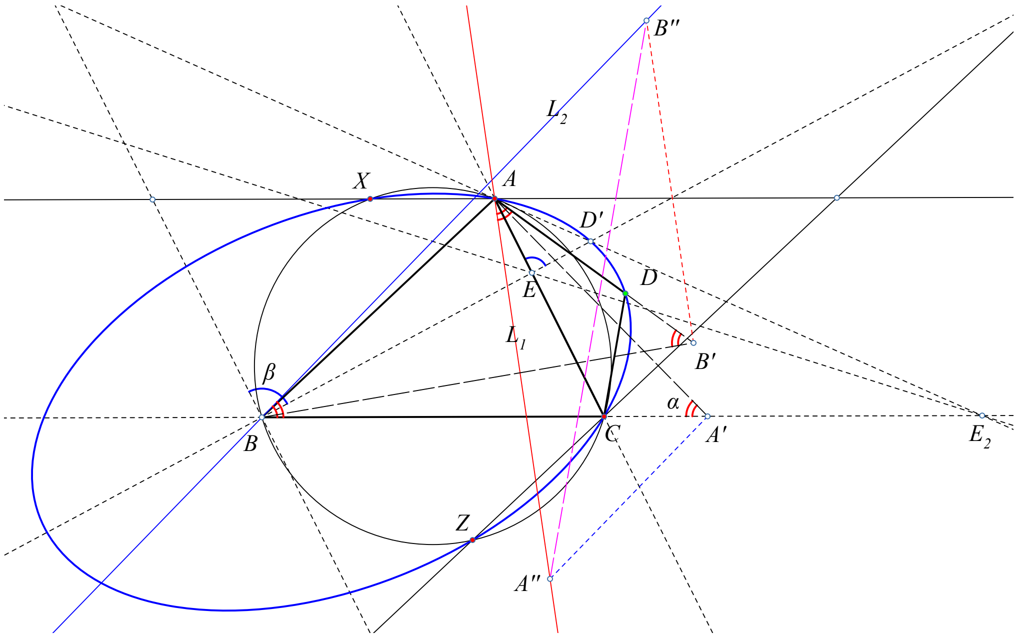


Figure 17. The conic locus of D, such that $A''B''$ is parallel to CD

The parallel from B to AC is rotated about B towards A by the angle β to give line BED', where E is the intersection with AC (See Figure 17). The bisector of the angle $\widehat{CED'}$ meets the line BC at the point E_2 and the line AE_2 meets the line BE at the required point D'. This lies on the conic, because in the quadrangle $ABCD'$ we can inscribe infinite many rhombi similar to $A_0B_0C_0D_0$. Now we consider the conic through the points $\{A, X, Z, C, D'\}$ and take on this a point D, such that the quadrangle $ABCD$ is convex. Since $\alpha = \gamma$ we have $\alpha = (\pi - \beta)/2$ and as in section 2 we construct lines $\{L_1, L_2\}$ and the line $A''B''$, that now is seen to be parallel and different from the line CD, for every point D on this conic, except for the point D', where lines $A''B''$ and CD coincide.

If the angle $\beta = \pi/2$, then the rhombus is a square, the line BD' is perpendicular to AC, the equation of the conic and the baricentric coordinates of D' are

$$a^2yz + b^2zx + c^2xy + (x+y)(y+z)S = 0,$$
$$D' = (2(a^2+b^2-c^2)S : -(a^2+b^2-c^2)(b^2+c^2-a^2) : 2(b^2+c^2-a^2)S).$$

Hence we have the following final result.

Theorem 20. *Given a convex quadrangle $q = ABCD$ and a rhombus $r_0 = A_0B_0C_0D_0$ with $\widehat{A_0B_0C_0} = \beta$, then*

(1) *If the point D is not on the conic described previously, we can inscribe in q only one rhombus similar to r_0.*

(2) *If the point D is on the conic and $D = D'$, we can inscribe in q infinite many rhombi similar to r_0.*
(3) *If the point D is on the conic and $D \neq D'$, it is impossible to inscribe in q a rhombus similar to r_0.*

References

[1] H. Eves, *A survey of Geometry*, Allyn and Bacon, Inc., Boston, 1963.
[2] F. J. García Capitán, Coordenadas Baricentricas, 2013; `http://garciacapitan.esy.es/baricentricas/`.
[3] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, New Library, Washington, 1995.
[4] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.
[5] T. Lalesco, *La Geometrie du triangle*, Librairie Vuibert, Paris, 1952.
[6] D. Laurain, Advanced Geometry post. https://groups.yahoo.com/neo/groups /AdvancedPlaneGeometry/conversations/messages/4574, 2018.
[7] I. M. Yaglom, *Geometric Transformations I, II, III*, The Mathematical Association of America, 1962.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: `ndergiades@yahoo.gr`

Forum Geometricorum
Volume 18 (2018) 219–222.

FORUM GEOM
ISSN 1534-1178

“A Study of Risāla al-Watar wa’l Jaib” (“The Treatise on the Chord and Sine”): Revisited

Mohammad K. Azarian

Abstract. The purpose of this article is three-fold. First, we use Jamshīd al-Kāshī’s cubic equation and *Mathematica* to calculate $\sin 1^\circ$ to over $11,200,000$ decimal digits of accuracy; an amazing improvement over the 17 digits of accuracy that al-Kāshī found by pencil and paper in 1426 [2]. Then, we conjecture that al-Kāshī’s cubic equation can be used to calculate $\sin 1^\circ$ to any desired accuracy. Second, we set the record straight about the number of correct digits that al-Kāshī obtained. Third, we correct some statements that we made in [2].

1. Preliminaries

As we studied in detail [2], *Risāla al-watar wa’l jaib* (“The Treatise on the Chord and Sine”), is one of the three most significant mathematical achievements of the Iranian mathematician and astronomer Ghiyāth al-Dīn Jamshīd Mas’ūd al-Kāshī (1380-1429) that deals chiefly with the calculation of sine and chord of one-third of an angle with known sine and chord. As we discussed in [2], al-Kāshī applied Ptolemy’s theorem to an inscribed quadrilateral to obtain his famous cubic equation, and then he used his cubic equation to calculate $\sin 1^\circ$ to 17 correct decimal digits as a root of his cubic equation.

The purpose of this article is three-fold. In Section 2, we use Jamshīd al-Kāshī’s cubic equation that we discussed in [2] and *Mathematica* to calculate $\sin 1^\circ$ to over $11,200,000$ decimal digits of accuracy; an astonishing advancement over the 17 digits of accuracy that al-Kāshī found by pencil and paper in 1426 [2]. Also, in Section 2, we analyze our sexagesimal result in *Mathematica*. Moreover, we conjecture that al-Kāshī’s cubic equation can be used to calculate $\sin 1^\circ$ to any desired accuracy. In Section 3, we set the record straight about the number of correct digits that al-Kāshī achieved. Finally, in Section 4, we correct some statements that we made in [2].

2. Al-Kāshī’s cubic equation and *Mathematica*

We recall [2] that al-Kāshī used Ptolemy’s theorem to obtain the following two forms of his famous cubic equation

Publication Date: June 6, 2018. Communicating Editor: Paul Yiu.

$$(i)\ \ x = \frac{a + x^3}{b}$$
$$(ii)\ \ x = \frac{4}{3}x^3 + \frac{1}{3}\sin 3^\circ.$$

In Section 4.2 of [2], we used (ii), his algorithm, and performed the calculation in decimal system to obtain the same level of accuracy for approximation of $\sin 1^\circ$ that was used for the value of $\sin 3^\circ$, namely, 17 decimal digits. In the following two lines of *Mathematica* program, we use the value of $\sin 3^\circ$ form the built-in function in *Mathematica* and al-Kāshī's cubic equation (ii) to obtain $11,200,000$ decimal digits of accuracy for $\sin 1^\circ$.

```
y = x /. NSolve[x - (4/3)*x^3 == (1/3) Sin[Pi/60], x, Reals, 11200000][[2]];
y - N[Sin[Pi/180], 11200000]
```

We note that the limitation of the accuracy of our calculation here is due to the weakness of the author's desktop computer's computational capability, and not al-Kāshī's cubic equation. Therefore, we can state the following conjecture.

Conjecture 2.1. Jamshīd al-Kāshī's cubic equation (ii) can be used to calculate $\sin 1^\circ$ to any desired accuracy in the decimal system, provided we use at least the same degree of accuracy for the value of $\sin 3^\circ$.

Remark 2.2. In Section 4.1 of [2], we used al-Kāshī's cubic equation (i), his algorithm, and $\sin 3^\circ$ with 9 digits of accuracy in sexagesimal system as he himself did to find an approximation for $\sin 1^\circ$ with 9 sexagesimal digits of accuracy as well. In the following four lines of *Mathematica* program, we use al-Kāshī's cubic equation (i) and the value of $\sin 3^\circ$ from the built-in function in *Mathematica* to obtain $10,912$ sexagesimal digits of accuracy for $\sin 1^\circ$.

```
$RecursionLimit = Infinity;
x[1] = 2;
x[n_] := N[x[n - 1]^3/(3*60^2) + 40*Sin[Pi/60], 19403]
RealDigits[x[19403]/120, 60, 10912] -RealDigits[Sin[Pi/180], 60, 10912]
```

Again, the limitation of the accuracy of our calculation here is due to the weakness of the author's desktop computer's computational capability in sexagesimal system. It is interesting to note that, as in [2] for every additional correct sexagesimal digit of $\sin 3^\circ$, we obtain an extra correct sexagesimal digit for $\sin 1^\circ$.

3. Let's set the record straight

In [2] we used al-Kāshī's cubic equation and his algorithm to find the value of $\sin 1^\circ$ in sexagesimal system as

$$1; 2, 49, 43, 11, 14, 44, 16, 26, 17.$$

This is equivalent to

$$\sin 1^\circ = 0.017452406437283510371 2,$$

in the decimal system, where the first 17 digits are correct. However, the last sexagesimal digit is not the same as the actual value of $\sin 1^\circ$. The actual value is 18 and not 17. Therefore, using al-Kāshī's cubic equation, the value of $\sin 3^\circ$ with 9 sexagesimal digits of accuracy, and his algorithm, we obtain only 9 sexagesimal digits of accuracy for $\sin 1^\circ$ and not 10.

As we stated in [2], the Arabic and the Persian manuscripts of *Risāla al-watar wa'l jaib* have been translated and/or commented on by various historians of mathematics and astronomy into English, French, German, and Russian. The number of correct sexagesimal digits of $\sin 1^\circ$ in these papers varies from 8 to 10 sexagesimal digits. Likewise, the number of correct decimal digits of $\sin 1^\circ$ in these papers ranges from 16 to 22 decimal digits. For example, Rosenfeld and Youschkevitch [3], stated that al-Kāshī obtained 10 sexagesimal digits and 18 decimal digits of accuracy. Aaboe [1], however, reported that al-Kāshī obtained only 8 sexagesimal digits of accuracy. Moreover, in the *Encyclopedia of Islam* [4], 16 decimal digits of accuracy is reported.

4. Corrections/comments regarding [2]

In this section, we correct some of the statements that we made in [2] as follows.

4.1. Footnote 7, p. 231. The author was deceived by claims in the literature that 'Abd al-'Alī Bīrjandī (d. 1528) was a student and colleague of both Jamshīd al-Kāshī and his cousin Mu'īn al-Dīn al-Kāshī. While Bīrjandī certainly looked at both al-Kāshī and his cousin as role models and they were a source of deep inspiration for him, he did not benefit from direct instruction from either one of them. One simple argument would be their age differences. Al-Kāshī died in 1429 while Bīrjandī died in 1528. Also, the village that Bīrjandī was born is actually *Bujd* and not *Wujd*.

4.2. Footnote 19, p. 238. The last line must be divided by 60. That is, the last line should be

$$1 \cdot 60^{-1} + 2 \cdot 60^{-2} + 49 \cdot 60^{-3} + 43 \cdot 60^{-4} + \cdots + 26 \cdot 60^{-9} + 17 \cdot 60^{-10}.$$

4.3. As we stated in Section 3, al-Kāshī's cubic equation (i) and his algorithm gave him only 9 sexagesimal digits of accuracy, which is equivalent to 17 decimal digits. In [2, line 17], the author erred in stating that al-Kāshī obtained 10 sexagesimal digits of accuracy.

4.4. Line 3, p. 240. (6) should be replaced by (13).

References

[1] A. Aaboe, Al-Kashi's Iteration Method for the Determination of $\sin 1^\circ$, *Scripta Mathematica*, 20 (1954) 24–29.

[2] M. K. Azarian, A Study of Risa-la al-Watar wa'l Jaib ("The Treatise on the Chord and Sine"), *Forum Geom.*, 15 (2015) 229–242. Mathematical Reviews, MR 3418854 (Reviewed), Zentralblatt MATH, Zbl 1328.01015.

[3] B. A. Rosenfeld and A. P. Youschkevitch, Ghiyath al-din Jamshid Masud al-Kashi (or al-Kashani), *Dictionary of Scientific Biography*, 7 (1981) 255–262.

[4] J. Vernet, *al-Kāshī*, Encyclopedia of Islam, Second Edition. edited by: P. Bearman, Th. Bianquis, C.E. Bosworth, E. van Donzel, W.P. Heinrichs. Brill Online, 2016. Reference: Mohammad K. Azarian [aff. University of Evansville]. 17 May 2016.

Mohammad K. Azarian: Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA

E-mail address: azarian@evansville.edu

Forum Geometricorum
Volume 18 (2018) xx–xx.

FORUM GEOM
ISSN 1534-1178

Algebraic Equations of All Involucre Conics in the Configuration of the c-Inscribed Equilateral Triangles of a Triangle

Blas Herrera

Abstract. Let $\triangle ABC$ be a triangle with side length $c = AB$; here we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_jB_jC_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_jB_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, and from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

1. Introduction

Many configurations linking conics and equilateral triangles with the triangle have been described by different geometers in the past; here we give a new one. Let $\triangle ABC$ be a triangle with side length $c = AB$; in this work we want to present the determination of the existence and quantity of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_jB_jC_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_jB_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

Readers can find the construction of the c-inscribed equilateral triangles [3]. And from the kinematic point of view we are considering a well known result of planar kinematics: we consider the motion of the point X of an equilateral triangle $\triangle PQX$, where P and Q slide along straight (non-parallel) lines. It is well known that, in the general case, the trajectory of X is an ellipse (for each of the two possible orientations of $\triangle PQX$). Therefore, we consider nothing else than a special case of the well known elliptic motion or Cardan motion [1], [2]. Nevertheless, in this work, through long but straightforward calculations, we present not the well known kinematic point of view, but the algebraic equations of the special case of all the conics which are linked with the c-equilateral triangles which are sliding on a triangle $\triangle ABC$. More precisely, let $\triangle ABC$ be a triangle with side length $c = AB$, let be their equilateral triangles, of side length c, which are sliding on the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$. In the next section we present the algebraic equations

Draft: June 1, 2018. Communicating Editor: Paul Yiu.

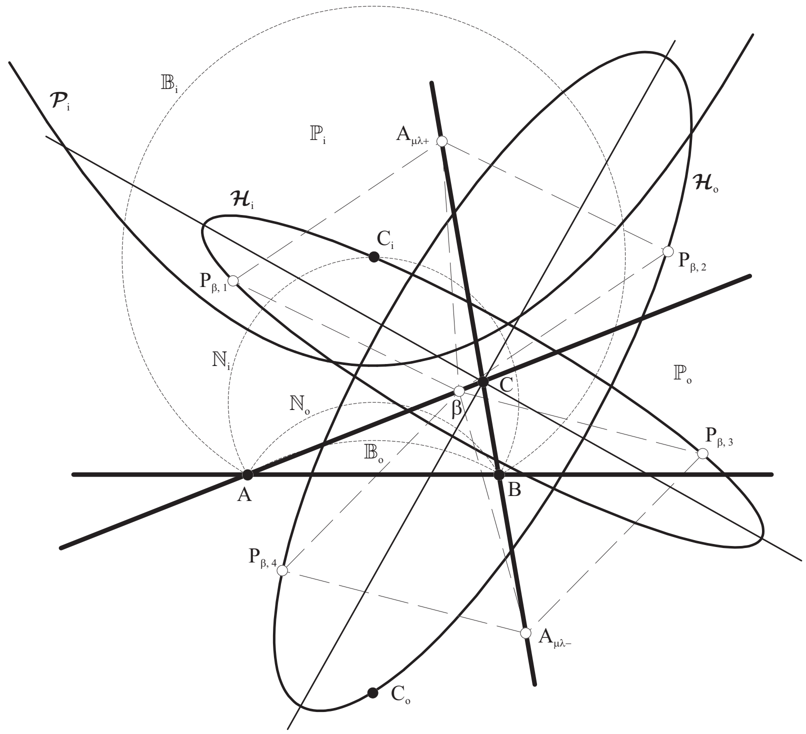


Figure 1. The conics linked with the c-equilateral triangles which are sliding on a triangle.

of all the conics which are linked with these c-equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (see Figure 1). And with these configuration equations, we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. (see Figures 2, 3).

Of course, the triangle $\triangle ABC$ has its other two similar configurations for its other two sides $b = AC$, $a = BC$.

1.1. *Elements of the configuration.* In the following we fix, with precision, the notation and the elements involved with the configuration (for the case of the side $\overline{AB}$).

Lemma 1. *Let $\triangle ABC$ be a triangle in the affine euclidean plane $\mathbb{A}^2$, with its length side $c = AB$, and (see Figures 1, 2, 3):*

1.- Let $\{T_{\beta,k} = \triangle P_{\beta,k} A_{\beta,k} \beta\}_{k=1}^{k=4}$ be its four β-sliding equilateral triangles: i.e. β is an arbitrary point with $\beta \in \overleftrightarrow{CA}$, $A_{\beta,k} \in \overleftrightarrow{BC}$, and $T_{\beta,k}$ has its length sides equal to $c = AB$ (see Figure 1, and the proof of Lemma 2) [this is a special case of the well known elliptic motion].

2.- Let $\mathbb{N}_o$ and $\mathbb{N}_i$ be the circumcircle of $\triangle ABC_o$ and $\triangle ABC_i$, the outer equilateral triangle and the inner equilateral triangle of $\overline{AB}$, respectively. Let $\mathbb{B}_o$ and

$\mathbb{B}_i$ *be the circumference of center* C_o *with radius* $\overline{C_oA}$ *and the circumference of center* C_i *with radius* $\overline{C_iA}$*, respectively.*

3.- Let $\mathcal{P}_i$ *be the parabola such that* $\overleftrightarrow{AB}$ *and* C_i *are its: directrix and focus, respectively. Let* $\mathbb{P}_i$ *and* $\mathbb{P}_o$ *be the convex arc-connected region and the non-convex arc-connected region, respectively, of* $\mathbb{S} \setminus \mathcal{P}_i$*, where* $\mathbb{S}$ *is the semiplane, with side* $\overleftrightarrow{AB}$*, containing* C_i *and* $\triangle ABC$*.*

4.- Let $\mathcal{H}_k$ *be the locus of the points* $P_{\beta,k}$ *when* $T_{\beta,k}$ *is being sliding on the pair of the straight lines* $\overleftrightarrow{BC}$*,* $\overleftrightarrow{CA}$*: i.e. when* β *is being sliding on* $\overleftrightarrow{CA}$ *(see Figure 1, and Corollary 6) [this is a special case of the well known elliptic motion].*

5.- Let $\{\mathbb{T}_j\}_{j=1}^{j=m}$ *be the* c*-inscribed equilateral triangles of* $\triangle ABC$*: i.e.* $\mathbb{T}_j = \triangle A_jB_jC_j$ *with* $A_j \in \overleftrightarrow{BC}$*,* $B_j \in \overleftrightarrow{CA}$*,* $C_j \in \overleftrightarrow{AB}$*, and* $c = A_jB_j$ *(see Figures 2, 3).*

The number m will be determined in the Corollary 8.

Without loss generality we can assume, along of this paper, that $AB = c = 1$; also we can consider a Cartesian system of coordinates (x, y) such that $A = (0, 0)$, $B = (1, 0)$, $C = (a, b)$ with $b > 0$.

2. Results

Lemma 2. *Equations (1), (2), (3), (4) are algebraic formulae of* $\{P_{\beta,k}\}_{k=1}^{k=4}$*.*

Proof. Let $\beta_\lambda = (\lambda a, \lambda b)$, $\lambda \in \mathbb{R}$ with $\beta_\lambda = \beta$, be an arbitrary point on $\overleftrightarrow{CA}$. Let $A_\mu = (1 + \mu a - \mu, \mu b)$, $\mu \in \mathbb{R}$, be an arbitrary point on $\overleftrightarrow{BC}$.

With an easy calculation, we can observe that there are two points A_μ with $\beta_\lambda A_\mu = 1$, which depend on λ, and with $\mu = \mu_{\lambda\pm}$ where:

$$\mu_{\lambda\pm} = \tfrac{1}{(a-1)^2+b^2}(\lambda\left(a^2 - a + b^2\right) - a + 1 \pm \sqrt{(a-1)^2 + (2-\lambda)\lambda b^2}).$$

We will put: $\Phi = \sqrt{(a-1)^2 + b^2} = BC$, $\Psi_\lambda = \sqrt{(a-1)^2 + (2-\lambda)\lambda b^2}$ and $\Lambda_\lambda = \lambda\left(a^2 - a + b^2\right)$.

Points $A_{\mu_{\lambda\pm}}$ exist if and only if $\Psi_\lambda \geq 0 \Leftrightarrow \lambda \in \left[1 - \frac{\Phi}{b}, 1 + \frac{\Phi}{b}\right]$. Points $A_{\mu_{0\pm}}$exist, with $\lambda = 0$, and $\mu_{0-} = \frac{1}{\Phi^2}\left(-a + 1 - |a-1|\right)$, $\mu_{0+} = \frac{1}{\Phi^2}(-a + 1 + |a-1|)$; and we have:

$$\text{if } a - 1 \geq 0 \Rightarrow\ A_{\mu_{0-}} = \tfrac{1}{\Phi^2}\left(b^2 - (a-1)^2, 2b(1-a)\right), A_{\mu_{0+}} = B;$$
$$\text{if } a - 1 < 0 \Rightarrow\ A_{\mu_{0-}} = B, A_{\mu_{0+}} = \tfrac{1}{\Phi^2}\left(b^2 - (a-1)^2, 2b(1-a)\right).$$

It happens that $B = A_{\mu_{0-}} = A_{\mu_{0+}} \Leftrightarrow \mu = 0 \Leftrightarrow 1 = a \Leftrightarrow \angle ABC$ is a right angle. Also $A_{\mu_{1\pm}}$ exist with $\lambda = 1 \Rightarrow \mu_{1-} = 1 - \frac{1}{\Phi}$ and $\mu_{1+} = 1 + \frac{1}{\Phi}$, thus $A_{\mu_{1-}} = \left(a - \frac{a-1}{\Phi}, b - \frac{b}{\Phi}\right)$ and $A_{\mu_{1+}} = \left(a + \frac{a-1}{\Phi}, b + \frac{b}{\Phi}\right)$.

In general, to any $\lambda \in \left[1 - \frac{\Phi}{b}, 1 + \frac{\Phi}{b}\right] \setminus \{0\}$, we have that

$$A_{\mu_{\lambda\pm}} = \left(\tfrac{b^2 + (a-1)(\Lambda_\lambda \pm \Psi_\lambda)}{\Phi^2}, b\tfrac{-a+1+\Lambda_\lambda \pm \Psi_\lambda}{\Phi^2}\right).$$

And the two middle points $M_{\mu_{\lambda\pm}}$, of the segments $\overline{A'_{\mu_{\lambda\pm}}B'_{\lambda}}$, are:

$$\begin{aligned} M_{\mu_{\lambda+}} &= \left(\tfrac{(\lambda+1)b^2+(a-1)(2\Lambda_\lambda+\Psi_\lambda)}{2\Phi^2}, b\tfrac{(\lambda+1)(-a+1)+2\Lambda_\lambda+\Psi_\lambda}{2\Phi^2}\right), \\ M_{\mu_{\lambda-}} &= \left(\tfrac{(\lambda+1)b^2+(a-1)(2\Lambda_\lambda-\Psi_\lambda)}{2\Phi^2}, b\tfrac{(\lambda+1)(-a+1)+2\Lambda_\lambda-\Psi_\lambda}{2\Phi^2}\right). \end{aligned}$$

Then, making a calculation, it follows that: $C_{\mu_{\lambda++}} = \left(C_{\mu_{\lambda++x}}, C_{\mu_{\lambda++y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda++x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)+\left(a-1-\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda++y}} &= \tfrac{b(\lambda+1)(1-a)-(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(b-\sqrt{3}+\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{1}$$

and also that $C_{\mu_{\lambda+-}} = \left(C_{\mu_{\lambda+-x}}, C_{\mu_{\lambda+-y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda+-x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(-\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)+\left(a-1+\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda+-y}} &= \tfrac{b(\lambda+1)(1-a)+(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(b+\sqrt{3}-\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{2}$$

and also that $C_{\mu_{\lambda-+}} = \left(C_{\mu_{\lambda-+x}}, C_{\mu_{\lambda-+y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda-+x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)-\left(a-1-\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C''_{\mu_{\lambda-+y}} &= \tfrac{b(\lambda+1)(1-a)-(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(-b+\sqrt{3}-\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{3}$$

and also that $C_{\mu_{\lambda--}} = \left(C_{\mu_{\lambda--x}}, C_{\mu_{\lambda--y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda--x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(-\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)-\left(a-1+\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda--y}} &= \tfrac{b(\lambda+1)(1-a)+(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(-b-\sqrt{3}+\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}. \end{aligned} \tag{4}$$

Where $C_{\mu_{\lambda++}}$, $C_{\mu_{\lambda+-}}$, $C_{\mu_{\lambda-+}}$, $C_{\mu_{\lambda--}}$, are the vertices of the equilateral triangles $\triangle C_{\mu_{\lambda++}}A_{\mu_{\lambda+}}\beta_\lambda$, $\triangle C_{\mu_{\lambda+-}}A_{\mu_{\lambda+}}\beta_\lambda$, $\triangle C_{\mu_{\lambda-+}}A_{\mu_{\lambda-}}\beta_\lambda$, $\triangle C_{\mu_{\lambda--}}A_{\mu_{\lambda-}}\beta_\lambda$, respectively.

In the case of $\lambda = 0$ it happens that: if $a-1>0$ then $C_{\mu_{0++}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0+-}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; and if $a-1<0$ then $C_{\mu_{0-+}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0--}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; and if $a-1=0$ then $C_{\mu_{0++}} = C_{\mu_{0-+}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0+-}} = C_{\mu_{0--}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Therefore, finally we have: $\beta = \beta_\lambda$, $A_{\beta,1} = A_{\beta,2} = A_{\mu_{\lambda+}}$, $A_{\beta,3} = A_{\beta,4} = A_{\mu_{\lambda-}}$, $P_{\beta,1} = C_{\mu_{\lambda++}}$, $P_{\beta,2} = C_{\mu_{\lambda+-}}$, $P_{\beta,3} = C_{\mu_{\lambda-+}}$, $P_{\beta,4} = C_{\mu_{\lambda--}}$. $\square$

With $g_{P,\theta}$ the rotation with angle of amplitude θ and with center point P: by construction, we have that $g_{P_{\beta,1},-\frac{\pi}{3}}(A_{\beta,1}) = g_{P_{\beta,2},\frac{\pi}{3}}(A_{\beta,2}) = g_{P_{\beta,3},-\frac{\pi}{3}}(A_{\beta,3}) = g_{P_{\beta,4},\frac{\pi}{3}}(A_{\beta,4}) = \beta$; and we have that the equilateral triangles $\triangle P_{\beta,1}A_{\beta,1}\beta$, $\triangle P_{\beta,2}A_{\beta,2}\beta$, $\triangle P_{\beta,3}A_{\beta,3}\beta$, $\triangle P_{\beta,4}A_{\beta,4}\beta$ are not coincident.

Along all the paper we will put $\Phi = \sqrt{(a-1)^2+b^2}$, $\Lambda_\lambda = \lambda\left(a^2-a+b^2\right)$ and $\Psi_\lambda = \sqrt{(a-1)^2+(2-\lambda)\lambda b^2}$.

The following lemma also is a *Definition*:

Lemma 3. *Let $\mathcal{H}_i$, $\mathcal{H}_o$ be the two conics determined by: both conics have the center point C, the outer and inner bisectors of the angle $\angle ACB$ are the axes of the both conics, the two points $C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $C_{\mu_{1++}} = \left(a + \frac{a-1-b\sqrt{3}}{2\Phi}, b + \frac{b+(a-1)\sqrt{3}}{2\Phi}\right)$ belong to $\mathcal{H}_i$, the two points $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $C_{\mu_{1+-}} = \left(a + \frac{a-1+b\sqrt{3}}{2\Phi}, b + \frac{b+(1-a)\sqrt{3}}{2\Phi}\right)$ belong to $\mathcal{H}_o$.*

Then we have:

$\mathcal{H}_i$ is circumference $\Leftrightarrow C \in \mathbb{B}_i$, in this case $\mathcal{H}_i$ has equation (5);

$\mathcal{H}_o$ is circumference $\Leftrightarrow C \in \mathbb{B}_o$, in this case $\mathcal{H}_o$ has equation (6);

if $a = \frac{1}{2}$, it happens that $b \neq \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is ellipse, in this case $\mathcal{H}_i$ has equation (7);

if $a = \frac{1}{2}$, it happens that $b = 1 + \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is circumference, in this case $\mathcal{H}_i$ has equation (8);

if $a = \frac{1}{2}$, it happens that $b = \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is a pair of coincident straight lines parallel to $\overleftrightarrow{AB}$, in this case $\mathcal{H}_i$ has equation (9);

if $a = \frac{1}{2}$, it happens that $b \neq \frac{\sqrt{3}}{6} \Leftrightarrow \mathcal{H}_o$ is ellipse, in this case $\mathcal{H}_o$ has equation (10);

if $a = \frac{1}{2}$, it happens that $b = 1 - \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_o$ is circumference, in this case $\mathcal{H}_o$ has equation (11);

if $a = \frac{1}{2}$, it happens that $b = \frac{\sqrt{3}}{6} \Leftrightarrow \mathcal{H}_o$ is a pair of coincident straight lines orthogonal to $\overleftrightarrow{AB}$, in this case $\mathcal{H}_i$ has equation (12);

if $a \neq \frac{1}{2}$, it happens that $C \notin \mathbb{B}_i \Leftrightarrow \mathcal{H}_i$ is not circumference, in this case $\mathcal{H}_i$ has equation (13);

if $a \neq \frac{1}{2}$, it happens that $C \notin \mathbb{B}_o \Leftrightarrow \mathcal{H}_o$ is not circumference, in this case $\mathcal{H}_o$ has equation (14).

Proof. Let $\mathcal{H}$ be a conic, its equation is $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + Dxy + Ex + Fy + G = 0$; and $M = \begin{pmatrix} G & \frac{1}{2}E & \frac{1}{2}F \\ \frac{1}{2}E & \xi & \frac{1}{2}D \\ \frac{1}{2}F & \frac{1}{2}D & \zeta \end{pmatrix}$ is its associated matrix with its algebraic parameters: $T = trace\,(M_{00})$, $U = \det M_{11} + \det M_{22}$, where $M_{00} = \begin{pmatrix} \beta & \frac{1}{2}D \\ \frac{1}{2}D & \zeta \end{pmatrix}$, $M_{11} = \begin{pmatrix} G & \frac{1}{2}F \\ \frac{1}{2}F & \zeta \end{pmatrix}$ and $M_{22} = \begin{pmatrix} G & \frac{1}{2}E \\ \frac{1}{2}E & \xi \end{pmatrix}$.

From now, we consider that the inner and outer bisectors of the angle $\angle ACB$ are the axes of $\mathcal{H}$.

By requiring that C is the center $\mathcal{H}$, we have two possibilities: the bisectors of $\angle ACB$ are parallel to the coordinate axes, or they are not; and these two possibilities are equivalent to $a = \frac{1}{2}$ or $a \neq \frac{1}{2}$. Therefore, if $a = \frac{1}{2}$ then the axes of $\mathcal{H}$ are parallel to the coordinate axes; but if $a \neq \frac{1}{2}$ then the axes of $\mathcal{H}$ are not parallel to the coordinate axes unless $\mathcal{H}$ is circumference.

Now, whatever the value of a, we first consider that $\mathcal{H}$ is circumference with center point C. By imposing the circumference $\mathcal{H}$ passes through the point $C_i =$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have

$$\mathcal{H}^i \equiv x^2 + y^2 - 2ax - 2by - 1 + a + b\sqrt{3} = 0. \tag{5}$$

By imposing the circumference $\mathcal{H}$ passes through the point $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ we have

$$\mathcal{H}^o \equiv x^2 + y^2 - 2ax - 2by - 1 + a - b\sqrt{3} = 0. \tag{6}$$

And if $\mathcal{H}^i = \mathcal{H}_i$ passes through the point $C_{\mu_{1++}}$ then $a^2 + b^2 - a - b\sqrt{3} = 0$; also if $\mathcal{H}^o = \mathcal{H}_o$ passes through the point $C''_{\mu_{1+-}}$ then $a^2 + b^2 - a + b\sqrt{3} = 0$.

The equations of $\mathbb{B}_i$ and $\mathbb{B}_o$ are $\mathbb{B}_i \equiv x^2 + y^2 - x - y\sqrt{3} = 0$ and $\mathbb{B}_0 \equiv x^2 + y^2 - x + y\sqrt{3} = 0$ respectively. Accordingly, if the conics $\mathcal{H}_i$, $\mathcal{H}_o$ are circumferences then $C \in \mathbb{B}_i$, $C \in \mathbb{B}_o$ respectively.

The inverse assertion is true, i.e. if the conics $\mathcal{H}_i$ and $\mathcal{H}_o$ have their centers $C \in \mathbb{B}_i$ and $C \in \mathbb{B}_o$, respectively, then they are circumferences; we will prove later this inverse assertion.

Now, regardless if $\mathcal{H}$ is a circumference or not, we consider the case of $a = \frac{1}{2}$. Then we have that the axes of $\mathcal{H}$ are parallel to the coordinate axes; which implies that the matrix A_{00} has the eigenvectors $\{(1,0), (0,1)\}$, which algebraically imposes that $D = 0$. Then $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + Ex + Fy + G = 0$. But, $\mathcal{H}$ is conic with center, then $\mathcal{H}$ is not a parabola; and then, algebraically, the value 0 is discarded as eigenvalue of A_{00}, so $\xi \neq 0$ and $\zeta \neq 0$. Therefore, without loss generality, we can consider that $\zeta = 1$, and we have that $\mathcal{H} \equiv \xi x^2 + y^2 + Ex + Fy + G = 0$. Moreover C is center point of $\mathcal{H}$; then, with $a = \frac{1}{2}$, analytically we have that $\begin{cases} \frac{1}{2}\xi + \frac{1}{2}E = 0 \\ b + \frac{1}{2}F = 0 \end{cases}$, which implies that $\mathcal{H} \equiv -Ex^2 + y^2 + Ex - 2by + G = 0$. If $\mathcal{H}$ also passes through the point C_i then we denote the conic as $\mathcal{H}^i$, and has the following equation: $\mathcal{H}^i \equiv -\left(-3 + 4b\sqrt{3} - 4G\right)x^2 + y^2 + \left(-3 + 4b\sqrt{3} - 4G\right)x - 2by + G = 0$. Similarly, if $\mathcal{H}$ also passes through the point C_o then we denote the conic as $\mathcal{H}^o$, and has the following equation: $\mathcal{H}^o \equiv -\left(-3 - 4b\sqrt{3} - 4G\right)x^2 + y^2 + \left(-3 - 4b\sqrt{3} - 4G\right)x - 2by + G = 0$.

We impose now that $\mathcal{H}^i$ passes through the point $C''_{\mu_{1++}}$, with $a = \frac{1}{2}$ and $b \neq \frac{1}{2}\sqrt{3}$, then we have that $\mathcal{H}^i = \mathcal{H}_i$ with

$$\begin{aligned} &\mathcal{H}_i \equiv 48b^2\left(2b - \sqrt{3}\right)^2 x^2 + 4\left(6b + \sqrt{3}\right)^2 y^2 - 48b^2\left(2b - \sqrt{3}\right)^2 x \\ &-8b\left(6b + \sqrt{3}\right)^2 y + 3\left(8b^3 + 12b^2\sqrt{3} - 6b - \sqrt{3}\right)\left(2b + \sqrt{3}\right) = 0. \end{aligned} \tag{7}$$

With $b \neq \frac{1}{2}\sqrt{3}$, the conic $\mathcal{H}_i$ has the algebraic parameters: $\det A = -192b^2(2b - \sqrt{3})^4(6b + \sqrt{3})^4 \neq 0$, $\det A_{00} = 192b^2\left(2b - \sqrt{3}\right)^2\left(6b + \sqrt{3}\right)^2 > 0$, $T = 48b^2(2b - \sqrt{3})^2 + 4(6b + \sqrt{3})^2$ and $T \det A < 0$; therefore $\mathcal{H}_i$ is a real non-degenerate ellipse. If $b = 1 + \frac{1}{2}\sqrt{3}$, i.e. $C \in \mathbb{B}_i$, then

$$\mathcal{H}_i \equiv x^2 + y^2 - x - \left(2 + \sqrt{3}\right)y + 1 + \sqrt{3} = 0, \tag{8}$$

and $\mathcal{H}_i$ is a circumference of radius 1. If $b=\frac{1}{2}\sqrt{3}$then the conic $\mathcal{H}_i$ degenerates to

$$\mathcal{H}_i \equiv y^2-\sqrt{3}y+\frac{3}{4}=0, \tag{9}$$

with $\xi=0$, whose algebraic parameters are: $\det A=0$, $\det A_{00}=0$, $U=0$; therefore $\mathcal{H}_i$ is the pair of coincident straight lines $\left\{y=\frac{1}{2}\sqrt{3}\right\}$, $\left\{y=\frac{1}{2}\sqrt{3}\right\}$.

We impose now that $\mathcal{H}^o$ passes through the point $C''_{\mu_{1+-}}$,with $a=\frac{1}{2}$ and $b\neq\frac{1}{6}\sqrt{3}$, then we have that $\mathcal{H}^o=\mathcal{H}_o$ with

$$\begin{aligned}&\mathcal{H}_o \equiv 48b^2\left(2b+\sqrt{3}\right)^2x^2+4\left(6b-\sqrt{3}\right)^2y^2-48b^2\left(2b+\sqrt{3}\right)^2x\\&-8b\left(6b-\sqrt{3}\right)^2y++3\left(8b^3-12b^2\sqrt{3}-6b+\sqrt{3}\right)\left(2b-\sqrt{3}\right)=0.\end{aligned} \tag{10}$$

With $b\neq\frac{1}{6}\sqrt{3}$, the conic $\mathcal{H}_o$ has the algebraic parameters: $\det A=-192b^2(2b+\sqrt{3})^4\,(6b-\sqrt{3})^4\neq 0$, $\det A_{00}=192b^2\left(2b+\sqrt{3}\right)^2\left(6b-\sqrt{3}\right)^2>0$, $T=48b^2(2b+\sqrt{3})^2+4(6b-\sqrt{3})^2$ and $T\det A<0$; therefore $\mathcal{H}_o$ is a real non-degenerate ellipse. If $b=1-\frac{1}{2}\sqrt{3}$, i.e. $C\in\mathbb{B}_o$, then

$$\mathcal{H}_o \equiv x^2+y^2-x-\left(2-\sqrt{3}\right)y+1-\sqrt{3}=0, \tag{11}$$

is a circumference of radius 1. If $b=\frac{1}{6}\sqrt{3}$ then the conic $\mathcal{H}_o$ degenerates to

$$\mathcal{H}_o \equiv x^2-x+\frac{1}{4}=0, \tag{12}$$

with $\zeta=0$, whose algebraic parameters are: $\det A=0$, $\det A_{00}=0$, $U=0$; therefore $\mathcal{H}_o$ is the pair of coincident straight lines $\left\{x=\frac{1}{2}\right\}$, $\left\{x=\frac{1}{2}\right\}$.

We have just seen that if $a=\frac{1}{2}$ and $C\in\mathbb{B}_i$, $C\in\mathbb{B}_o$ then $\mathcal{H}_i$, $\mathcal{H}_o$ are circumferences, respectively. Let's see what is also true with $a\neq\frac{1}{2}$. If $C\in\mathbb{B}_o$, then is easy to calculate that C is at the same distance from C_o to $C''_{\mu_{1+-}}$. But the three points C_o, $C''_{\mu_{1+-}}$ and $C\in\mathbb{B}_o$, are not aligned. To prove the above assertion: if we impose that $\left(\left(a+\frac{a-1+b\sqrt{3}}{2\Phi},b+\frac{b+(1-a)\sqrt{3}}{2\Phi}\right)-(a,b)\right)-\alpha\left(\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)-(a,b)\right)=(0,0)$, then this implies that $\alpha=-\frac{a-1+b\sqrt{3}}{\Phi(2a-1)}$ and $3a^2+3b^2-3a+b\sqrt{3}=0$. But moreover $a^2+b^2-a+b\sqrt{3}=0$, then necessarily $C=(0,0)=A$ or $C=(1,0)=B$, in contradiction. Therefore: C is at the same distance from C_o to $C''_{\mu_{1+-}}$, they are three not aligned points, and C is center of symmetry of the conic $\mathcal{H}_o$ which passes through C_o and $C''_{\mu_{1+-}}$; then $\mathcal{H}_o$ is a circumference. Similarly: if $C\in\mathbb{B}_i$, then is easy to calculate that C is at the same distance from C_i to $C''_{\mu_{1++}}$. But the three points C_i, $C''_{\mu_{1++}}$ and $C\in\mathbb{B}_i$, are not aligned. To prove the above assertion: if we impose that $\left(\left(a+\frac{a-1-b\sqrt{3}}{2\Phi},b+\frac{b+(a-1)\sqrt{3}}{2\Phi}\right)-(a,b)\right)-\alpha\left(\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)-(a,b)\right)=(0,0)$, then this implies that $\alpha=\frac{-a+1+b\sqrt{3}}{\Phi(2a-1)}$ and $3a^2+3b^2-3a-b\sqrt{3}=0$. But moreover $a^2+b^2-a-b\sqrt{3}=0$, then necessarily $C=(0,0)=A$ or $C=(1,0)=B$, in contradiction. Therefore: C is at the same distance from C_i to $C''_{\mu_{1++}}$, they are three not aligned points, and C is center of symmetry of the conic $\mathcal{H}_i$ which passes through C_i and $C''_{\mu_{1++}}$; then $\mathcal{H}_i$ is a circumference.

Now we consider the case of $a \neq \frac{1}{2}$.

The axes of $\mathcal{H}$ are not parallel to the coordinate axes unless that any straight line through C is axis of $\mathcal{H}$, and then $\mathcal{H}$ is circumference. Algebraically, $\mathcal{H}$ is not circumference if and only if the vector $\{(1,0),(0,1)\}$ are not eigenvectors of M_{00}, then $\mathcal{H}$ is not circumference if and only if $D \neq 0$.

We consider that $\mathcal{H}$ is not circumference (if $\mathcal{H}$ is circumference then is already a studied case) and we can assume that $D = 1$, so that $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + xy + Ex + Fy + G = 0$. The vectors $\overrightarrow{u}_+ = \left(\beta - \gamma + \sqrt{(\beta-\gamma)^2+1}, 1\right)$, $\overrightarrow{u}_- = \left(\beta - \gamma - \sqrt{(\beta-\gamma)^2+1}, 1\right)$ are eigenvectors with eigenvalues $\frac{1}{2}\beta + \frac{1}{2}\gamma + \frac{1}{2}\sqrt{(\beta-\gamma)^2+1}$, $\frac{1}{2}\beta + \frac{1}{2}\gamma - \frac{1}{2}\sqrt{(\beta-\gamma)^2+1}$, of M_{00}, respectively. Then $\overrightarrow{u}_+$, $\overrightarrow{u}_-$, are direction vectors of the axes of $\mathcal{H}$. The inner and outer bisectors of the angle $\angle ACB$ are directed by $\overrightarrow{v}_+ = (a,b)\frac{1}{\sqrt{a^2+b^2}} + (a-1,b)\frac{1}{\sqrt{(a-1)^2+b^2}}$, $\overrightarrow{v}_- = (a,b)\frac{1}{\sqrt{a^2+b^2}} - (a-1,b)\frac{1}{\sqrt{(a-1)^2+b^2}}$. Then we have the proportionality $\overrightarrow{u}_+ = \epsilon\overrightarrow{v}_+$, $\overrightarrow{u}_- = \varepsilon\overrightarrow{v}_-$, which implies, with $a \neq \frac{1}{2}$, that $\beta = \frac{a^2-b^2-a}{b(2a-1)} + \zeta$; so: $\mathcal{H} \equiv \left(\frac{a^2-b^2-a}{b(2a-1)} + \zeta\right)u^2 + \zeta y^2 + xy + Ex + Fy + G = 0$. By imposing that (a,b) is the center $\mathcal{H}$ we have: $\zeta = -\frac{2a^3-2a^2-b^2}{2ab(2a-1)} - \frac{E}{2a}$, $F = -\frac{a^2+b^2}{a(2a-1)} + \frac{Eb}{a}$, which implies that $\xi = -\frac{1}{2}\frac{b+E}{a}$. Therefore:

$$\mathcal{H} \equiv -\frac{1}{2}\frac{b+E}{a}x^2 - \frac{2a^3-2a^2-b^2+Eb(2a-1)}{2ab(2a-1)}y^2 + xy + Ex + \left(-\frac{a^2+b^2}{a(2a-1)} + \frac{Eb}{a}\right)y + G = 0.$$

And by imposing that $\mathcal{H} = \mathcal{H}^o$ passes through the point $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and by computing we obtain

$$G_o = \frac{3a^3-3a^2+ab^2-2b^2-\sqrt{3}ab-2\sqrt{3}b^3+Eb\left(6a-2-4a^2+4\sqrt{3}ab-2\sqrt{3}b\right)}{4ab(2a-1)},$$

for the conic $\mathcal{H}^o$.

Also, by imposing that $\mathcal{H} = \mathcal{H}^i$ passes through the point $C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and by computing we obtain

$$G_i = \frac{3a^3-3a^2+ab^2-2b^2+\sqrt{3}ab+2\sqrt{3}b^3+Eb\left(6a-2-4a^2-4\sqrt{3}ab+2\sqrt{3}b\right)}{4ab(2a-1)},$$

for the conic $\mathcal{H}^i$.

Now, considering that $a \neq \frac{1}{2}$, and also that $a^2+b^2-a-b\sqrt{3} \neq 0$ because $C \notin \mathbb{B}_i$ -$\mathcal{H}$ is not circumference-; by imposing that $\mathcal{H}^i = \mathcal{H}_i$ passes through the point $C_{\mu_{1++}}$ we have $E_i = b\frac{a^2+b^2+a-b\sqrt{3}}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}$. The above expression of E_i

implies that:

$$\begin{aligned}&G_i=\frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)},\\&\mathcal{H}_i\equiv -b\frac{a^2+b^2-a-\sqrt{3}b+1}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}x^2-\frac{a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2}{b\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}y^2\\&+xy+b\frac{a^2+b^2+a-\sqrt{3}b}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}x-\frac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}y+G_i=0.\end{aligned} \tag{13}$$

Similarly, considering that $a\neq\frac{1}{2}$, and also that $a^2+b^2-a+b\sqrt{3}\neq 0$ because $C\notin\mathbb{B}_0$ -$\mathcal{H}$ is not circumference- by imposing that $\mathcal{H}^o=\mathcal{H}_o$ passes through the point $C_{\mu_{1+-}}$ we have $E_o=b\frac{a^2+b^2+a+\sqrt{3}b}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}$. The above expression of E_o implies that:

$$\begin{aligned}&G_o=\frac{\left(3a^3-6a^2+a^2\sqrt{3}b+3a+3ab^2-6b^2-\sqrt{3}b+\sqrt{3}b^3\right)\left(3a-\sqrt{3}b\right)}{12b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)},\\&\mathcal{H}_o\equiv -b\frac{a^2+b^2-a+\sqrt{3}b+1}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}x^2-\frac{a^4-2a^3+a^2b^2+a^2+a^2\sqrt{3}b-ab^2-\sqrt{3}ab+b^2}{b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}y^2\\&+xy+b\frac{a^2+b^2+a+\sqrt{3}b}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}x-\frac{a^3-a^2+ab^2+\sqrt{3}ab-2b^2}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}y+G_o=0.\end{aligned} \tag{14}$$

□

And, as a result of the above lemmas, in short we have the following algebraic equations

Theorem 4. *The conics $\mathcal{H}_i$, $\mathcal{H}_o$ have the equations:*

$$\begin{aligned}&\mathcal{H}_i\equiv -b^2(a^2+b^2-a-\sqrt{3}b+1)x^2-(a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b\\&-ab^2+\sqrt{3}ab+b^2)y^2+b(a^2+b^2-a-b\sqrt{3})\,(2a-1)\,xy+b^2(a^2+b^2+a\\&-\sqrt{3}b)x-b(a^3-a^2+ab^2-\sqrt{3}ab-2b^2)y+\tfrac{1}{12}(3a^3-6a^2-a^2\sqrt{3}b+3a\\&+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3)(3a+b\sqrt{3})=0;\end{aligned} \tag{15}$$

$$\begin{aligned}&\mathcal{H}_o\equiv -b^2(a^2+b^2-a+\sqrt{3}b+1)x^2-(a^4-2a^3+a^2b^2+a^2+a^2\sqrt{3}b\\&-ab^2-\sqrt{3}ab+b^2)y^2+b(a^2+b^2-a+\sqrt{3}b)\,(2a-1)\,xy+b^2(a^2+b^2+a\\&+\sqrt{3}b)x-b(a^3-a^2+ab^2+\sqrt{3}ab-2b^2)y+\tfrac{1}{12}(3a^3-6a^2+a^2\sqrt{3}b+3a\\&+3ab^2-6b^2-\sqrt{3}b+\sqrt{3}b^3)(3a-\sqrt{3}b)=0.\end{aligned} \tag{16}$$

Now, the following Proposition 5 and the Corollary 6 are consequence of a special case of the well known elliptic motion; but we can get these results with algebraic arguments using the above Theorem 4:

Proposition 5. *The two conics $\mathcal{H}_i$, $\mathcal{H}_o$:*

1.- are ellipses if and only if $C\notin\mathbb{N}_i$ and $C\notin\mathbb{N}_o$, respectively.

2.- are a pair of coincident straight lines if and only if $C\in\mathbb{N}_i$ and $C\in\mathbb{N}_o$, respectively; and they are outer and the inner bisectors of the angle $\angle ACB$, respectively.

3.- are circumferences if and only if $C\in\mathbb{B}_i$ and $C\in\mathbb{B}_o$, respectively.

Proof. The affirmation 3 has previously been shown in the proof of the previous theorem. Then here we consider the affirmations 1 and 2 in the case $C \notin \mathbb{B}_i$ and $C \notin \mathbb{B}_o$.

The equations of $\mathbb{N}_i$ and $\mathbb{N}_o$ are: $\mathbb{N}_i \equiv x^2 + y^2 - x - \frac{1}{3}\sqrt{3}y = 0$ and $\mathbb{N}_0 \equiv x^2 + y^2 - x + \frac{1}{3}\sqrt{3}y = 0$.

If $a = \frac{1}{2}$ then the affirmations 1 and 2 have previously been shown in the proof of the previous theorem.

Let us consider then $a \neq \frac{1}{2}$.

The algebraic parameters of $\mathcal{H}_o$ are:

$$\det A = \frac{\left(3a^2+3b^2-3a+\sqrt{3}b\right)^4}{144b\left(a^2+b^2-a+\sqrt{3}b\right)^3(2a-1)^3},\ \det A_{00} = \frac{\left(3a^2+3b^2-3a+\sqrt{3}b\right)^2}{12\left(a^2+b^2-a+\sqrt{3}b\right)^2(2a-1)^2},$$
$$T = -\frac{2a^2b^2-2ab^2+2b^2+b^4+\sqrt{3}b^3+a^4-2a^3+a^2+a^2\sqrt{3}b-\sqrt{3}ab}{b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)};$$

where $b \neq 0$, $2a-1 \neq 0$ and $a^2+b^2-a+\sqrt{3}b \neq 0$ because $C \notin \mathbb{B}_o$. Moreover if $C \notin \mathbb{N}_0$ then $\det A \neq 0$, $\det A_{00} > 0$, and $T \det A = -\frac{1}{144}\frac{\varphi(a,b)\left(3a^2+3b^2-3a+\sqrt{3}b\right)^4}{b^2\left(a^2+b^2-a+\sqrt{3}b\right)^4(2a-1)^4}$ with $\varphi(a,b) = 2a^2b^2 - 2ab^2 + 2b^2 + b^4 + \sqrt{3}b^3 + a^4 - 2a^3 + a^2 + a^2\sqrt{3}b - \sqrt{3}ab$. We have that $\varphi(1,1) > 0$ and the equation $\varphi(a,b) = 0$ has the four roots: $a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 2ib\sqrt{5} - 4b^2 - 2\sqrt{3}b}$, $a = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2ib\sqrt{5} - 4b^2 - 2\sqrt{3}b}$, $a = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 2ib\sqrt{5} - 4b^2 - 2\sqrt{3}b}$, $a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2ib\sqrt{5} - 4b^2 - 2\sqrt{3}b}$, which are real roots only if $b = 0$. Therefore $T \det A < 0$ and then $\mathcal{H}_o$ is a real non-degenerate ellipse. But, if $C \in \mathbb{N}_0$ then $\det A = 0$, $\det A_{00} = 0$, and

$$U = \frac{-\sqrt{3}\left(3b^3+2\sqrt{3}b^2+a^2b^2\sqrt{3}-2\sqrt{3}ab^2+3a^2b-3ab+a^2\sqrt{3}+a^4\sqrt{3}-2a^3\sqrt{3}\right)\left(3a^2+3b^2-3a+\sqrt{3}b\right)^2}{36b^2\left(a^2+b^2-a+\sqrt{3}b\right)^2(2a-1)^2}$$

with $U = 0$. Therefore if $C \in \mathbb{N}_0$ then $\mathcal{H}_o$ are two coincident straight lines; and, by construction of $\mathcal{H}_o$, they are the inner bisectrix of the angle $\angle ACB$.

The algebraic parameters of $\mathcal{H}_i$ are:

$$\det A = \frac{\left(3a^2+3b^2-3a-\sqrt{3}b\right)^4}{144b\left(a^2+b^2-a-\sqrt{3}b\right)^3(2a-1)^3},\ \det A_{00} = \frac{\left(3a^2+3b^2-3a-\sqrt{3}b\right)^2}{12\left(a^2+b^2-a-\sqrt{3}b\right)^2(2a-1)^2},$$
$$T = -\frac{2a^2b^2-2ab^2+2b^2+b^4-\sqrt{3}b^3+a^4-2a^3+a^2-a^2\sqrt{3}b+\sqrt{3}ab}{b\left(a^2+b^2-a-\sqrt{3}b\right)(2a-1)}.$$

where $b \neq 0$, $2a-1 \neq 0$ and $a^2+b^2-a-\sqrt{3}b \neq 0$ because $C \notin \mathbb{B}_i$. Moreover if $C \notin \mathbb{N}_i$ then $\det A \neq 0$, $\det A_{00} > 0$, and $T \det A = -\frac{1}{144}\frac{\psi(a,b)\left(3a^2+3b^2-3a-\sqrt{3}b\right)^4}{b^2\left(a^2+b^2-a-\sqrt{3}b\right)^4(2a-1)^4}$ with $\psi(a,b) = 2a^2b^2 - 2ab^2 + 2b^2 + b^4 - \sqrt{3}b^3 + a^4 - 2a^3 + a^2 - a^2\sqrt{3}b + \sqrt{3}ab$. We have that $\psi(1,1) > 0$ and the equation $\psi(a,b) = 0$ has the four roots: $a = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 2ib\sqrt{5} - 4b^2 + 2\sqrt{3}b}$, $a = \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2ib\sqrt{5} - 4b^2 + 2\sqrt{3}b}$, $a = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 2ib\sqrt{5} - 4b^2 + 2\sqrt{3}b}$, $a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2ib\sqrt{5} - 4b^2 + 2\sqrt{3}b}$, which are real roots only if $b = 0$. Therefore $T \det A < 0$ and then $\mathcal{H}_i$ is a real non-degenerate ellipse. But, if $C \in \mathbb{N}_i$ then $\det A = 0$, $\det A_{00} = 0$, and

$$U = \frac{\sqrt{3}\left(3b^3-2\sqrt{3}b^2-a^2b^2\sqrt{3}+2\sqrt{3}ab^2+3a^2b-3ab-a^2\sqrt{3}-a^4\sqrt{3}+2a^3\sqrt{3}\right)\left(3a^2+3b^2-3a-\sqrt{3}b\right)^2}{36b^2\left(a^2+b^2-a-\sqrt{3}b\right)^2(2a-1)^2}$$

with $U = 0$. Therefore if $C \in \mathbb{N}_i$ then $\mathcal{H}_i$ are two coincident straight lines; and, by construction of $\mathcal{H}_i$, they are the outer bisectrix of the angle $\angle ACB$. $\square$

With the Lemma 2 and the Theorem 4, we can prove algebraically the following:

Corollary 6. *Let* $\{T_{\beta,k} = \triangle P_{\beta,k} A_{\beta,k} \beta\}_{k=1}^{k=4}$, *then:*
1.- $\mathcal{H}_i = \mathcal{H}_1 \cup \mathcal{H}_3$ *is the geometrical locus of the vertices* $P_{\beta,1}$ *and* $P_{\beta,3}$.
2.- $\mathcal{H}_o = \mathcal{H}_2 \cup \mathcal{H}_4$ *is the geometrical locus of the vertices* $P_{\beta,2}$ *and* $P_{\beta,4}$.

Proof. With a very lengthy and straightforward calculation we can check the result in all its parts and implications.

Let us see a case, let us see, for example, the explicit calculations that show that $C_{\mu_{\lambda++}} = P_{\beta,1} \in \mathcal{H}_i$. Using the Equations (1), (15) we must to prove that

$$\begin{array}{l}
-4\Phi^4 b \frac{a^2+b^2-a-\sqrt{3}b+1}{1} \left(C_{\mu_{\lambda++x}}\right)^2 \\
-4\Phi^4 \frac{\left(a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2\right)}{b} \left(C_{\mu_{\lambda++y}}\right)^2 \\
+4\Phi^4 \left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)\, C_{\mu_{\lambda++x}} C_{\mu_{\lambda++y}} \\
+4\Phi^4 b \frac{a^2+b^2+a-\sqrt{3}b}{1} C_{\mu_{\lambda++x}} - 4\Phi^4 \frac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{1} C_{\mu_{\lambda++y}} \\
+4\Phi^4 \frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b} = 0.
\end{array}$$

Then: with a very lengthy and straightforward calculation we have

$$\begin{array}{l}
\Psi_1 = -4\Phi^4 b \frac{a^2+b^2-a-\sqrt{3}b+1}{1} \left(C_{\mu_{\lambda++x}}\right)^2 \\
-4\Phi^4 \frac{\left(a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2\right)}{b} \left(C_{\mu_{\lambda++y}}\right)^2,
\end{array}$$

where

$$\begin{array}{l}
\Psi_1 = -4b - 3\sqrt{3}a - 4\lambda b^3 + \sqrt{3}b^2 - 80ba^4 + 100ba^3 + 4ab^3 - 70ba^2 + 26ba \\
-11a^2b^3 + 4b^4\sqrt{3} + 6\Psi_\lambda b^3 - 6ba^6 + 34ba^5 + 24b^5\lambda - 18b^5\lambda^2 + 22a^3b^3 \\
-12a^4b^3 - 10a^2b^5 + 22b^5a + 3b^6\sqrt{3} - 10b^5\Psi_\lambda - 4b^7\lambda - 2b^7\lambda^2 - 56b^5a\lambda \\
-158\lambda^2a^4b^3 - 78a^2\lambda^2b^5 - 18\Psi_\lambda ab^3 + 8a^2\lambda b^3 - 94a^2\lambda^2b^3 + 20a\lambda^2b^3 \\
-4\sqrt{3}a^3b^2 - 10\Psi_\lambda\lambda b^3 - 8\sqrt{3}b^4\lambda + 6\sqrt{3}b^4\lambda^2 + 2\Psi_\lambda\sqrt{3}b^4 - 3a^4\sqrt{3}b^2 \\
+3a^2b^4\sqrt{3} + 12a^2\Psi_\lambda b^3 + 88a^5\lambda^2b^3 + 56a^3\lambda^2b^5 + 4a^5b^3\lambda - 4a^3b^5\lambda \\
+10a^2\sqrt{3}b^2 - 6\sqrt{3}ab^2 + 12a\lambda b^3 + 168\lambda^2b^3a^3 + 64\lambda^2b^5a - 9b^4\sqrt{3}a \\
-32b^3\lambda a^3 - 4a\sqrt{3}b^2\lambda - 8\sqrt{3}a^4b^2\lambda - 16\sqrt{3}a^3b^2\lambda - 48\sqrt{3}a^2b^4\lambda \\
+16\sqrt{3}a^2b^2\lambda + 86a^4\sqrt{3}b^2\lambda^2 - 58a^3\sqrt{3}b^2\lambda^2 + 76a^2\sqrt{3}b^4\lambda^2 + 18a^2\sqrt{3}b^2\lambda^2 \\
-2\sqrt{3}b^2\lambda^2a - 10\Psi_\lambda a^2\sqrt{3}b^2 - 52\Psi_\lambda a^2\lambda b^3 + 4\Psi_\lambda a\sqrt{3}b^2 + 32\sqrt{3}ab^4\lambda
\end{array}$$

$$
\begin{aligned}
&-34\sqrt{3}ab^4\lambda^2+4\Psi_\lambda\sqrt{3}b^2\lambda+2\Psi_\lambda\sqrt{3}b^4\lambda-8a^6\sqrt{3}b^2\lambda+20a^5\sqrt{3}b^2\lambda\\
&-16a^4\sqrt{3}b^4\lambda+16a^6\sqrt{3}b^2\lambda^2-60a^5\sqrt{3}b^2\lambda^2+32a^4\sqrt{3}b^4\lambda^2-2a^4\Psi_\lambda\sqrt{3}b^2\\
&-16a^4\Psi_\lambda\lambda b^3+8a^3\Psi_\lambda\sqrt{3}b^2+40a^3\sqrt{3}b^4\lambda-80a^3\sqrt{3}b^4\lambda^2-8b^6\sqrt{3}a^2\lambda\\
&+16b^6a^2\sqrt{3}\lambda^2+40b^3\Psi_\lambda a^3\lambda-8b^5\Psi_\lambda a^2\lambda+20b^6\sqrt{3}a\lambda-20b^6\sqrt{3}a\lambda^2\\
&-2b^6\Psi_\lambda\sqrt{3}\lambda-8b^5\Psi_\lambda a\lambda-106b\Psi_\lambda a^4\lambda+110b\Psi_\lambda a^3\lambda-54b\Psi_\lambda a^2\lambda\\
&+10b\Psi_\lambda\lambda a-8ba^6\Psi_\lambda\lambda+48ba^5\Psi_\lambda\lambda-3b^3-16b^5-b^7+54\Psi_\lambda a^2\sqrt{3}b^2\lambda\\
&-24\Psi_\lambda a\sqrt{3}b^2\lambda-24a^6\lambda^2b^3-24a^4\lambda^2b^5+12a^4\lambda b^3+2a^5\sqrt{3}b^2+32a^2b^5\lambda\\
&+8b^7\lambda^2a-2b^6\sqrt{3}a-4b^7a\lambda-8b^7a^2\lambda^2+4b^5\Psi_\lambda a+6b^5\Psi_\lambda\lambda-8b^6\sqrt{3}\lambda\\
&+6b^6\sqrt{3}\lambda^2+2b^6\Psi_\lambda\sqrt{3}-82b\lambda^2a^6+88b\lambda^2a^5-52b\lambda^2a^4+16b\lambda^2a^3\\
&-2b\lambda^2a^2-8ba^8\lambda^2+40ba^7\lambda^2-56\Psi_\lambda\sqrt{3}b^2\lambda a^3-8\Psi_\lambda\sqrt{3}b^4a\lambda\\
&+6a^2\Psi_\lambda\sqrt{3}b^4\lambda+26a^4\Psi_\lambda\sqrt{3}b^2\lambda-4a^5\Psi_\lambda\sqrt{3}b^2\lambda+4a^3\Psi_\lambda\sqrt{3}b^4\lambda\\
&+4b^6\Psi_\lambda\sqrt{3}a\lambda-8b\lambda a^2-\tfrac{45}{b}a^4+\tfrac{18}{b}a^3-\tfrac{3}{b}a^2-\tfrac{45}{b}a^6+\tfrac{60}{b}a^5-\tfrac{3}{b}a^8\\
&+\tfrac{18}{b}a^7+2\Psi_\lambda a^2\sqrt{3}\lambda-12\Psi_\lambda\sqrt{3}\lambda a^3+28a^4\Psi_\lambda\sqrt{3}\lambda-32a^5\Psi_\lambda\sqrt{3}\lambda\\
&-8a^5\Psi_\lambda\sqrt{3}+2\Psi_\lambda a^2\sqrt{3}+12a^4\Psi_\lambda\sqrt{3}-8a^3\Psi_\lambda\sqrt{3}+38b^3\Psi_\lambda a\lambda+2a^6\Psi_\lambda\sqrt{3}\\
&+17a^2\sqrt{3}+36b\lambda a^3+6b\Psi_\lambda a-40\sqrt{3}a^3+50a^4\sqrt{3}-14ba^2\Psi_\lambda+56ba^5\lambda\\
&+6ba^3\Psi_\lambda-64ba^4\lambda-35a^5\sqrt{3}+13a^6\sqrt{3}+6ba^4\Psi_\lambda+4ba^7\lambda-4ba^5\Psi_\lambda\\
&-24ba^6\lambda-2a^7\sqrt{3}-2b^4a^2\Psi_\lambda\sqrt{3}+18a^6\Psi_\lambda\sqrt{3}\lambda-4a^7\Psi_\lambda\sqrt{3}\lambda+2b^4a^3\sqrt{3},
\end{aligned}
$$

and we have

$$
\Psi_2=4\Phi^4\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)\,C_{\mu_{\lambda++x}}C_{\mu_{\lambda++y}},
$$

where

$$
\begin{aligned}
\Psi_2=\;&3b+\sqrt{3}a+6\lambda b^3+\sqrt{3}b^2-5ba^4-20ba^3+32ab^3+30ba^2-16ba\\
&-40a^2b^3-14\Psi_\lambda b^3-4ba^6+12ba^5-28b^5\lambda+18b^5\lambda^2+16a^3b^3+4a^2b^5\\
&-12b^5a-b^6\sqrt{3}+2b^5\Psi_\lambda-2b^7\lambda+2b^7\lambda^2+80b^5a\lambda+158\lambda^2a^4b^3\\
&+78a^2\lambda^2b^5+42\Psi_\lambda ab^3-48a^2\lambda b^3+94a^2\lambda^2b^3-20a\lambda^2b^3-16\sqrt{3}a^3b^2\\
&+10\Psi_\lambda\lambda b^3+8\sqrt{3}b^4\lambda-6\sqrt{3}b^4\lambda^2+2\Psi_\lambda\sqrt{3}b^4+9a^4\sqrt{3}b^2+3a^2b^4\sqrt{3}\\
&-28a^2\Psi_\lambda b^3-88a^5\lambda^2b^3-56a^3\lambda^2b^5-4a^5b^3\lambda+4a^3b^5\lambda+14a^2\sqrt{3}b^2\\
&-6\sqrt{3}ab^2-4a\lambda b^3+2\Psi_\lambda\sqrt{3}b^2-168\lambda^2b^3a^3-64\lambda^2b^5a-b^4\sqrt{3}a\\
&+80b^3\lambda a^3+4a\sqrt{3}b^2\lambda+8\sqrt{3}a^4b^2\lambda+16\sqrt{3}a^3b^2\lambda+48\sqrt{3}a^2b^4\lambda\\
&-16\sqrt{3}a^2b^2\lambda-86a^4\sqrt{3}b^2\lambda^2+58a^3\sqrt{3}b^2\lambda^2-76a^2\sqrt{3}b^4\lambda^2\\
&-18a^2\sqrt{3}b^2\lambda^2+2\sqrt{3}b^2\lambda^2a+26\Psi_\lambda a^2\sqrt{3}b^2+52\Psi_\lambda a^2\lambda b^3-12\Psi_\lambda a\sqrt{3}b^2\\
&-32\sqrt{3}ab^4\lambda+34\sqrt{3}ab^4\lambda^2-4\Psi_\lambda\sqrt{3}b^2\lambda-2\Psi_\lambda\sqrt{3}b^4\lambda+8a^6\sqrt{3}b^2\lambda\\
&-20a^5\sqrt{3}b^2\lambda+16a^4\sqrt{3}b^4\lambda-16a^6\sqrt{3}b^2\lambda^2+60a^5\sqrt{3}b^2\lambda^2-32a^4\sqrt{3}b^4\lambda^2\\
&+8a^4\Psi_\lambda\sqrt{3}b^2+16a^4\Psi_\lambda\lambda b^3-24a^3\Psi_\lambda\sqrt{3}b^2-40a^3\sqrt{3}b^4\lambda+80a^3\sqrt{3}b^4\lambda^2\\
&+8b^6\sqrt{3}a^2\lambda-16b^6a^2\sqrt{3}\lambda^2-40b^3\Psi_\lambda a^3\lambda+8b^5\Psi_\lambda a^2\lambda-8b^4\Psi_\lambda a\sqrt{3}\\
&-20b^6\sqrt{3}a\lambda+20b^6\sqrt{3}a\lambda^2+2b^6\Psi_\lambda\sqrt{3}\lambda+8b^5\Psi_\lambda a\lambda+106b\Psi_\lambda a^4\lambda\\
&-110b\Psi_\lambda a^3\lambda+54b\Psi_\lambda a^2\lambda-10b\Psi_\lambda\lambda a+8ba^6\Psi_\lambda\lambda-48ba^5\Psi_\lambda\lambda-8b^3+5b^5\\
&-54\Psi_\lambda a^2\sqrt{3}b^2\lambda+24\Psi_\lambda a\sqrt{3}b^2\lambda+24a^6\lambda^2b^3+24a^4\lambda^2b^5-30a^4\lambda b^3\\
&-2a^5\sqrt{3}b^2-50a^2b^5\lambda-8b^7\lambda^2a+2b^6\sqrt{3}a+4b^7a\lambda+8b^7a^2\lambda^2-4b^5\Psi_\lambda a\\
&-6b^5\Psi_\lambda\lambda+8b^6\sqrt{3}\lambda-6b^6\sqrt{3}\lambda^2+82b\lambda^2a^6-88b\lambda^2a^5+52b\lambda^2a^4-16b\lambda^2a^3
\end{aligned}
$$

$$+2b\lambda^2a^2+8ba^8\lambda^2-40ba^7\lambda^2+56\Psi_\lambda\sqrt{3}b^2\lambda a^3+8\Psi_\lambda\sqrt{3}b^4a\lambda-6a^2\Psi_\lambda\sqrt{3}b^4\lambda$$
$$-26a^4\Psi_\lambda\sqrt{3}b^2\lambda+4a^5\Psi_\lambda\sqrt{3}b^2\lambda-4a^3\Psi_\lambda\sqrt{3}b^4\lambda-4b^6\Psi_\lambda\sqrt{3}a\lambda+2b\lambda a^2$$
$$-2\Psi_\lambda a^2\sqrt{3}\lambda+12\Psi_\lambda\sqrt{3}\lambda a^3-28a^4\Psi_\lambda\sqrt{3}\lambda+32a^5\Psi_\lambda\sqrt{3}\lambda-38b^3\Psi_\lambda a\lambda$$
$$-7a^2\sqrt{3}-12b\lambda a^3+2b\Psi_\lambda a+20\sqrt{3}a^3-30a^4\sqrt{3}-10ba^2\Psi_\lambda-32ba^5\lambda$$
$$+18ba^3\Psi_\lambda+28ba^4\lambda+25a^5\sqrt{3}-11a^6\sqrt{3}-14ba^4\Psi_\lambda-4ba^7\lambda+4ba^5\Psi_\lambda$$
$$+18ba^6\lambda+2a^7\sqrt{3}+8b^4a^2\Psi_\lambda\sqrt{3}-18a^6\Psi_\lambda\sqrt{3}\lambda+4a^7\Psi_\lambda\sqrt{3}\lambda-2b^4a^3\sqrt{3},$$

and we have

$$\Psi_3=4\Phi^4b\frac{a^2+b^2+a-\sqrt{3}b}{1}C_{\mu_{\lambda++x}}-4\Phi^4\frac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{1}C_{\mu_{\lambda++y}},$$

where

$$\Psi_3=-2\lambda b^3+12ba^4-8ba^3-24ab^3+2ba^2+24a^2b^3+8\Psi_\lambda b^3$$
$$+2ba^6-8ba^5+4b^5\lambda-16a^3b^3+6a^4b^3+6a^2b^5-8b^5a+8b^5\Psi_\lambda$$
$$+6b^7\lambda-24b^5a\lambda-24\Psi_\lambda ab^3+40a^2\lambda b^3-4\Psi_\lambda\sqrt{3}b^4+16a^2\Psi_\lambda b^3$$
$$-8a\lambda b^3-2\Psi_\lambda\sqrt{3}b^2-48b^3\lambda a^3-16\Psi_\lambda a^2\sqrt{3}b^2+8\Psi_\lambda a\sqrt{3}b^2$$
$$-6a^4\Psi_\lambda\sqrt{3}b^2+16a^3\Psi_\lambda\sqrt{3}b^2+8b^4\Psi_\lambda a\sqrt{3}+10b^3+12b^5+2b^7$$
$$+18a^4\lambda b^3+18a^2b^5\lambda-2b^6\Psi_\lambda\sqrt{3}+6b\lambda a^2+8a^5\Psi_\lambda\sqrt{3}-2\Psi_\lambda a^2\sqrt{3}$$
$$-12a^4\Psi_\lambda\sqrt{3}+8a^3\Psi_\lambda\sqrt{3}-2a^6\Psi_\lambda\sqrt{3}-24b\lambda a^3-8b\Psi_\lambda a+24ba^2\Psi_\lambda$$
$$-24ba^5\lambda-24ba^3\Psi_\lambda+36ba^4\lambda+8ba^4\Psi_\lambda+6ba^6\lambda-6b^4a^2\Psi_\lambda\sqrt{3}.$$

And with all these, simplifying, we arrive to

$$\Psi_1+\Psi_2+\Psi_3=-\frac{1}{3}\frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3ab^2+3a-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+\sqrt{3}b\right)\left(a^2-2a+1+b^2\right)^2}{b},$$

and then $\Psi_1+\Psi_2+\Psi_3+4\Phi^4\frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b}=0$, finishing the calculation. □

Now, in the following, with the Theorem 4, we present the determination and the construction with ruler and compass, with Equations (15), (16), of $\mathbb{T}_j=\triangle A_jB_jC_j$ the c-inscribed equilateral triangles of $\triangle ABC$ (Figures 2, 3). Of course the following proposition 7 also is consequence of a special case of the well known elliptic motion; but, with our approach, we give the algebraic formulae (17), (18):

Proposition 7. *The conic $\mathcal{H}_o$:*

1.- with $C\in\mathbb{N}_o$, is a pair of coincident straight lines which intersect in one point with $\overleftrightarrow{AB}$. (Figure 3c)

2.- with $C\notin\mathbb{N}_o$, is an ellipse which intersect in two points with $\overleftrightarrow{AB}$. (Figures 2, 3a, 3b, 3d)

The algebraic formula of the above intersections is (17).

The conic $\mathcal{H}_i$:

3.- with $C=C_i$, is a pair of coincident straight lines parallel to $\overleftrightarrow{AB}$, and if $C\in\mathbb{N}_i\setminus C_i$ then is a pair of coincident straight lines which intersect in one point with $\overleftrightarrow{AB}$. (Figure 3d)

4.- with $C\in\mathbb{P}_o\setminus\mathbb{N}_i$, $C\in\mathcal{P}_i\backslash\mathbb{N}_i$, $C\in\mathbb{P}_i\setminus\mathbb{N}_i$, is an ellipse which: intersect in two points (Figures 2, 3c), is tangent (Figure 3b), not intersect (Figure 3a), respectively with $\overleftrightarrow{AB}$.

The algebraic formula of the above intersections is (18).

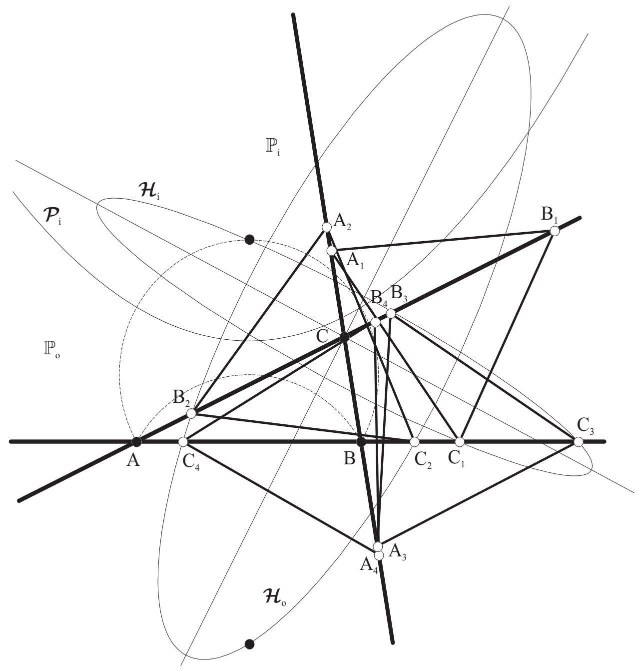


Figure 2. $\mathcal{H}_o$, $\mathcal{H}_i$ and $\mathbb{T}_j = \triangle A_jB_jC_j$ the c-inscribed equilateral triangles of $\triangle ABC$.

Proof. The equation of $\mathcal{P}_i$ is $x^2 - x + 1 = \sqrt{3}y$, the equation of $\mathbb{P}_i$ is $x^2 - x + 1 < \sqrt{3}y$, and the equation of $\mathbb{P}_o$ is $x^2 - x + 1 > \sqrt{3}y$ with $y > 0$.

With Equation (16), we have that:

$$\begin{aligned} &\mathcal{H}_o \cap \overleftrightarrow{AB} = \left(\frac{1}{6} \frac{3b^3+3\sqrt{3}b^2+3ba^2+3ba\pm\sqrt{3}\sqrt{\Delta_1}}{\left(a^2+b^2-a+\sqrt{3}b+1\right)b}, 0 \right), \\ &\Delta_1 = \left(\sqrt{3}b + a^2 - a + 1\right)\left(3b^2 + \sqrt{3}b - 3a + 3a^2\right)^2, \end{aligned} \tag{17}$$

and moreover $3b^2+3a^2-3a+\sqrt{3}b = 0 \Leftrightarrow C \in \mathbb{N}_o \Rightarrow \mathcal{H}_o \cap \overleftrightarrow{AB} = (\frac{1}{2}\frac{b^2+\sqrt{3}b+a^2+a}{a^2+b^2-a+\sqrt{3}b+1}, 0)$. Note that $a^2 + b^2 - a + \sqrt{3}b + 1 > 0$ and $\sqrt{3}b + a^2 - a + 1 > 0$.

With Equation (15), we have that:

$$\begin{aligned} &\mathcal{H}_i \cap \overleftrightarrow{AB} = \left(\frac{1}{6} \frac{3b^3-3\sqrt{3}b^2+3ba^2+3ba\pm\sqrt{3}\sqrt{\Delta_2}}{\left(a^2+b^2-a-\sqrt{3}b+1\right)b}, 0 \right), \\ &\Delta_2 = \left(-\sqrt{3}b + a^2 - a + 1\right)\left(3b^2 - \sqrt{3}b - 3a + 3a^2\right)^2, \end{aligned} \tag{18}$$

and $a^2+b^2-a-\sqrt{3}b+1 = 0 \Leftrightarrow C = C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; and $3b^2-\sqrt{3}b-3a+3a^2 = 0 \Leftrightarrow C \in \mathbb{N}_i$. And moreover $-\sqrt{3}b+a^2-a+1 = 0 \Leftrightarrow C \in \mathcal{P}_i$; and $-\sqrt{3}b+a^2-a+1 > 0 \Leftrightarrow C \in \mathbb{P}_o$. Also $C \in \mathbb{N}_i \cup \mathcal{P}_i \Rightarrow \mathcal{H}_i \cap \overleftrightarrow{AB} = \left(\frac{1}{2}\frac{b^2-\sqrt{3}b+a^2+a}{a^2+b^2-a-\sqrt{3}b+1}, 0\right)$.

So these calculations together Proposition 5 prove the result. $\square$

Accordingly, through the above we can arrive to the following:

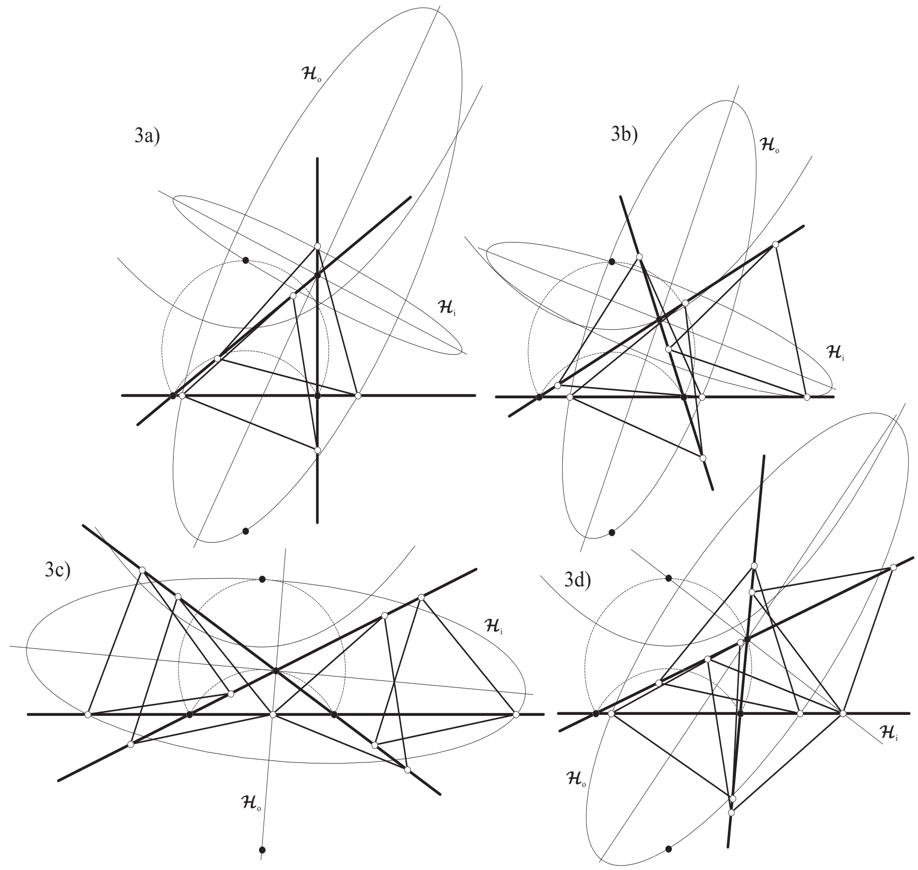


Figure 3. Several cases for $\mathcal{H}_o$, $\mathcal{H}_i$ and $\mathbb{T}_j = \triangle A_j B_j C_j$.

Corollary 8. *On every triangle $\triangle ABC$ exists its c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ with $m = 4$, $m = 3$, $m = 2$, if $C \in \mathbb{P}_o$, $C \in \mathcal{P}_i$, $C \in \mathbb{P}_i$, respectively. (Figures 2, 3)*

Proof. Let the triangle $\mathbb{T}_j = \triangle A_j B_j C_j$, then by construction and with Lemma 1, necessarily $C_j \in \{P_{\beta,k}\}_{k=1}^{k=4}$, and with Corollary 6 we have that $C_j = P_{\beta,j} \in \left(\mathcal{H}_i \cap \overleftrightarrow{AB}\right) \cup \left(\mathcal{H}_o \cap \overleftrightarrow{AB}\right)$.

If $\mathbb{T}_j$ exists then by Lemma 1 $C_1 = P_{\beta,1}$, $C_3 = P_{\beta,3}$, $C_2 = P_{\beta,2}$ and $C_4 = P_{\beta,4}$.

If $C \in \mathbb{P}_i \setminus \mathbb{N}_i$ or $C \in C_i$, then, by Proposition 7, $\mathbb{T}_1 = \triangle A_1 B_1 C_1$and $\mathbb{T}_3 = \triangle A_3 B_3 C_3$ do not exist.

If $C \in \mathbb{P}_i \cup \mathbb{N}_i$ then the triangles $\mathbb{T}_1 = \triangle A_1 B_1 C_1$and $\mathbb{T}_3 = \triangle A_3 B_3 C_3$ do not exist. This claim is true because in this case $C_1 = C_3 = P_{\beta,1} = C_{\mu_{\lambda++}} = P_{\beta,3} = C_{\mu_{\lambda-+}} = \mathcal{H}_i \cap \overleftrightarrow{AB}$, and $\mathcal{H}_i$ is the outer bisectrix of the angle $\angle ACB$ (Proposition 5). And, by continuity, exists two straight lines r', r'' at the both sides of the $\mathcal{H}_i$ which are parallel to $\mathcal{H}_i$ and also they are the outer bisectors of two triangles $\triangle ABC'$ and $\triangle ABC''$, respectively, which have the same $\mathbb{N}_i$, but which have $C' \in \mathbb{P}_i \setminus \mathbb{N}_i$, $C'' \in \mathbb{P}_i \setminus \mathbb{N}_i$. Therefore, by Proposition 7, these bisectors r', r'' intersect to $\overleftrightarrow{AB}$ in two points at distance greater than 1 from the straight line $\overleftrightarrow{AC'}$ and from the straight line $\overleftrightarrow{AC''}$. So, this implies that $\mathcal{H}_i \cap \overleftrightarrow{AB}$ also is a point at distance greater than 1 from the straight line $\overleftrightarrow{AC}$.

If $C \in \mathcal{P}_i \setminus \mathbb{N}_i$ then $\mathcal{H}_i$ is ellipse tangent to $\overleftrightarrow{AB}$, then only one of the two triangles $\mathbb{T}_1$, $\mathbb{T}_3$ exist; then by continuity the same is true the case that $C \in \mathcal{P}_i \cap \mathbb{N}_i$.

If $C \in \mathbb{P}_o$, the two triangles $\mathbb{T}_1$, $\mathbb{T}_3$ exists and, by Lemma 1, they are not coincident.

The triangles $\mathbb{T}_2 = \triangle A_2B_2C_2$, $\mathbb{T}_4 = \triangle A_4B_4C_4$, by Lemma 1, they are not coincident, and by Corollary 1 and Proposition 7, they always exist. □

Remark 9. *If $\mathbb{T}_j$ exists then it is constructible with ruler and compass because the points $\left(\mathcal{H}_i \cap \overleftrightarrow{AB}\right) \cup \left(\mathcal{H}_o \cap \overleftrightarrow{AB}\right)$ are contructible with ruler and compass; this claim is true because Formulae* (**??**), (**??**) *of Proposition 7 are quadratic rationals of the numbers a, b, which have been already constructed, they are the coordinates of C. The other points A_j, B_j are trivially obtained as intersection of the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ with the circumference of radius c and center point C_j. Then, with Formulae* (**??**), (**??**) *we can construct $\mathbb{T}_j$ with ruler and compass, nevertheless readers can found much more elegant constructions in [3].*

References

[1] O. Bottema and B. Roth, *Theoretical Kinematics*, Dover Publications, Miami, U.S.A., 2012.
[2] E. A. Dijksman, *Motion Geometry of Mechanisms*, Cambridge Univ. Press, Cambridge, Great Britain, 1976.
[3] S. Ochoński, Equilateral triangles whose vertices belong to three given straight lines. *The Journal of Polish Society for Geometry and Engineering Graphics*, 19 (2009) 15–26.

Blas Herrera: Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda Països Catalans 26, 43007, Tarragona, Spain.
E-mail address: blas.herrera@urv.cat

Forum Geometricorum
Volume 18 (2018) 239–244.

FORUM GEOM
ISSN 1534-1178

A Construction of the Golden Ratio in an Arbitrary Triangle

Tran Quang Hung

Abstract. We have known quite a lot about the construction of the golden ratio in the special triangles. In this article, we shall establish a construction of the golden ratio in arbitrary triangle using two symmedians and give a synthetic proof for this.

Given triangle ABC inscribed in circle (ω) center O.
(i) the symmedians AD and CF.
(ii) the ray DF meets (ω) at the point P.
(iii) the perpendicular line from P to OA meets AB and AC at Q and R respectively (See Figure 1).

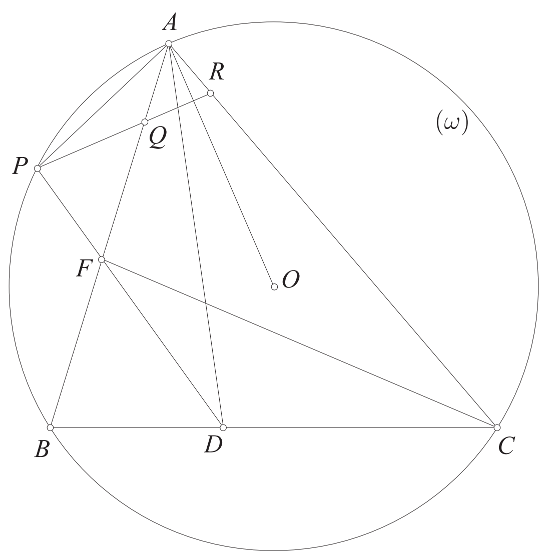


Figure 1

Proposition 1. *Q divides PR in the golden ratio.*

We give three lemmas to prove this proposition.

Lemma 2. *Given convex cyclic quadrilateral $ABCD$. Two diagonals AC and BD intersect at P. Then*

Publication Date: June 20, 2018. Communicating Editor: Paul Yiu.
The author is grateful to Professor Paul Yiu for his help in the preparation of this article.

$$\frac{PA}{PC} = \frac{AB \cdot AD}{CB \cdot CD}.$$

Proof. By inscribed angles are equal in the cyclic quadrilateral, we have the similar triangles $\triangle PAB \sim \triangle PDC$ and $\triangle PAD \sim \triangle PBC$. From this, we get the ratios

$$\frac{PA}{PB} = \frac{AD}{BC} \tag{1}$$

and

$$\frac{PB}{PC} = \frac{AB}{CD}. \tag{2}$$

From (1) and (2), we obtain

$$\frac{PA}{PC} = \frac{AB \cdot AD}{CB \cdot CD}.$$

This finishes the proof. □

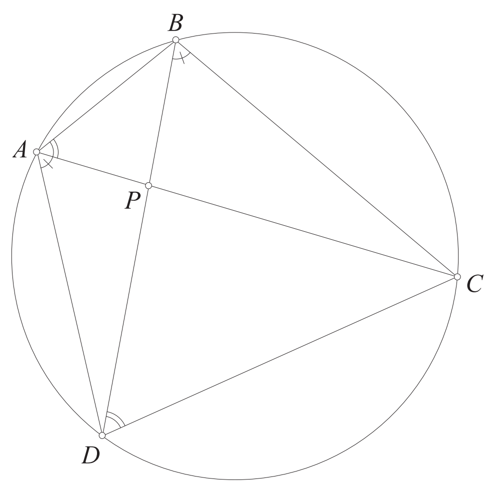


Figure 2

Lemma 3 (Ptolemy's Theorem [1])**.** *For a cyclic quadrilateral, the sum of the products of the two pairs of opposite sides equals the product of the diagonals.*

Using concept of homogeneous barycentric coordinates [9], we give and prove the following lemma

Lemma 4. *Let ABC be a triangle inscribed in circle (ω). P is a point inside triangle ABC. P has homogeneous barycentric coordinates $(x : y : z)$. DEF is cevian triangle of P. Ray EF meets (ω) at Q. Then*

$$\frac{CA}{yQB} = \frac{BC}{xQA} + \frac{AB}{zQC}.$$

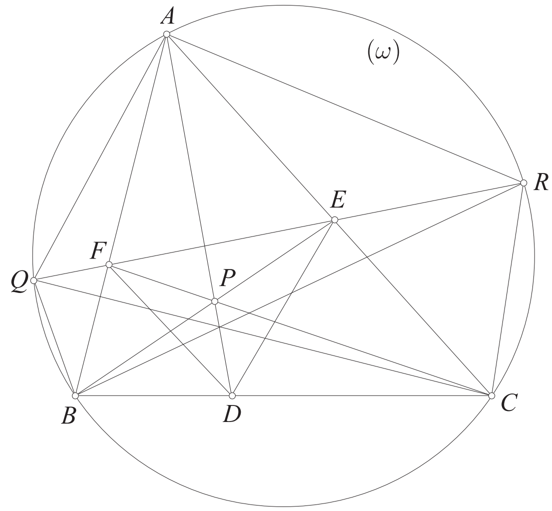


Figure 3

Proof. Because P has homogeneous barycentric coordinates $(x : y : z)$ so $E(x : 0 : z)$ and $F(x : y : 0)$. Thus we have the ratio

$$\frac{EA}{EC} = \frac{z}{x}. \tag{3}$$

Extend ray QE to meet (ω) again at R. From Lemma 2, we have

$$\frac{EA}{EC} = \frac{AQ \cdot AR}{CQ \cdot CR}. \tag{4}$$

From (3) and (4), we deduce

$$\frac{AQ \cdot AR}{CQ \cdot CR} = \frac{z}{x}.$$

Thus,

$$zQC = x\frac{AQ \cdot AR}{CR}. \tag{5}$$

Similarly, we have the identity

$$\frac{AQ \cdot AR}{BQ \cdot BR} = \frac{y}{x}.$$

Thus,

$$yQB = x\frac{AQ \cdot AR}{BR}. \tag{6}$$

Using (5) and (6) and Lemma 3, we consider the expression

$$\begin{aligned}\frac{CA}{yQB}-\frac{AB}{zQC}&=\frac{CA}{x\frac{AQ\cdot AR}{BR}}-\frac{AB}{x\frac{AQ\cdot AR}{CR}}\\&=\frac{CA\cdot RB-AB\cdot RC}{xAQ\cdot AR}\\&=\frac{BC\cdot RA}{xAQ\cdot AR}\\&=\frac{BC}{xQA}.\end{aligned}$$

Therefore

$$\frac{CA}{yQB}=\frac{BC}{xQA}+\frac{AB}{zQC}.$$

This completes the proof of Lemma 4 (See Figure 4). □

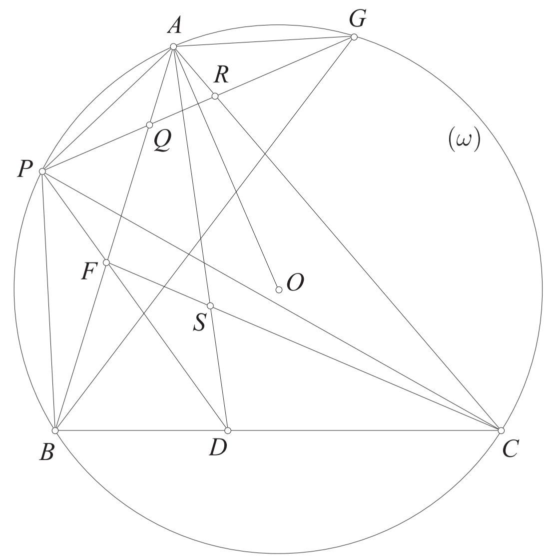


Figure 4

Proof of Proposition 1. The line PR meets (ω) again at G. From the similar triangles $\triangle QAP \sim \triangle QGB$ and $\triangle RAG \sim \triangle RPC$. We have the ratios

$$\frac{QA}{QP}=\frac{GA}{PB},\ \frac{RP}{RA}=\frac{PC}{GA}.$$

Therefore,

$$\frac{PR}{PQ}\cdot\frac{QA}{RA}=\frac{PC}{PB}. \tag{7}$$

Because QR is perpendicular to OA so QR is antiparallel line, this follows from $\angle AQR = \angle ACB$, thus $\triangle AQR \sim \triangle ACB$. We get

$$\frac{QA}{RA} = \frac{AC}{AB}. \tag{8}$$

From (7) and (8), we deduce

$$\frac{PR}{PQ} = \frac{AB \cdot PC}{CA \cdot PB}. \tag{9}$$

Let symmedians AD and CF meet at S then $S(BC^2 : CA^2 : AB^2)$ see [9]. Apply Lemma 4 for triangle ABC with S and ray DF meet (ω) at P, we have

$$\frac{BC}{BC^2 \cdot PA} = \frac{AB}{AB^2 \cdot PC} + \frac{CA}{CA^2 \cdot PB}.$$

This is equivalent to

$$\frac{1}{BC \cdot PA} = \frac{1}{AB \cdot PC} + \frac{1}{CA \cdot PB}. \tag{10}$$

Note that, by Lemma 3,

$$BC \cdot PA = AB \cdot PC - CA \cdot PB. \tag{11}$$

From (10) and (11) we have

$$\frac{1}{AB \cdot PC - CA \cdot PB} = \frac{1}{AB \cdot PC} + \frac{1}{CA \cdot PB}$$

This means

$$1 - \frac{CA \cdot PB}{AB \cdot PC} - \left(\frac{AB \cdot PC}{CA \cdot PB} - 1\right) = 1. \tag{12}$$

From (9) and (12), we obtain

$$\frac{PR}{PQ} - \frac{PQ}{PR} = 1.$$

This enough to show that the ratio

$$\frac{PR}{PQ} = \frac{\sqrt{5}+1}{2}$$

which is such the golden ratio. This completes the proof of Proposition 1.

References

[1] A. Bogomolny, Ptolemy's Theorem, Interactive Mathematics Miscellany and Puzzles, `http://www.cut-the-knot.org/proofs/ptolemy.shtml`

[2] T. O. Dao, Q. H. Ngo, and P. Yiu, Golden sections in an isosceles triangle and its circumcircle, *Global Journal of Advanced Research on Classical and Modern Geometries*, 5 (2016) 93–97.

[3] D. Paunić and P. Yiu, Regular polygons and the golden section, *Forum Geom.*, 16 (2016) 273–281.

[4] K. Hofstetter, A simple construction of the golden section, *Forum Geom.*, 2 (2002) 65–66.

[5] G. Odom and J. van de Craats, Elementary Problem 3007, *Amer. Math. Monthly*, 90 (1983) 482; solution, 93 (1986) 572.
[6] M. Pietsch, The golden ratio and regular polygons, *Forum Geom.*, 17 (2017) 17–19.
[7] Q. H. Tran, The golden section in the inscribed square of an isosceles right triangle, *Forum Geom.*, 15 (2015) 91–92.
[8] Q. H. Tran, Another simple construction of the golden ratio in an isosceles triangle, *Forum Geom.*, 17 (2017) 287–288.
[9] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at
http://math.fau.edu/Yiu/Geometry.html

Tran Quang Hung: High school for Gifted students, Hanoi University of Science, Hanoi National University, Hanoi, Vietnam
E-mail address: analgeomatica@gmail.com

Forum Geometricorum
Volume 18 (2018) 245–250.

FORUM GEOM
ISSN 1534-1178

New Constructions of Triangle from $\alpha, b - c, t_A$

Martina Štěpánová

Abstract. We give two constructions of a triangle given its internal angle α, the length t_A of the angle bisector of α, and the difference $b - c$ of the side lengths b and c.

1. Introduction

In 2016 Paris Pamfilos published in this journal his solution of the following construction problem [3]. Let ABC be a triangle and let $a = |BC|$, $b = |CA|$, $c = |AB|$ denote the lengths of its sides. Let α be the interior angle at A, and $t_A = |AH|$, where H is the intersection of the side BC and the angle bisector of α. The problem is to construct the triangle ABC given α, $b - c$, t_A.

The key-point of Pamfilos' solution is the detection of a parabola which goes hand in hand with ABC and which is unambiguously constructible from the given data. In this paper, we present two other solutions. We think that our compass-and-straightedge constructions are simpler and shorter. We give two different verifications of the first construction. In the first verification of the first construction and in the second construction, any parabola does not play any role. In the second verification of the first construction, a parabola is considered, but not constructed at all.

In ABC, two cases can occur: either $b - c = 0$ or $b - c \neq 0$. If $b - c = 0$, then ABC is isosceles and its construction is trivial, since the side BC is perpendicular to the line-segment AH. If $b - c \neq 0$, then we assume without loss of generality that $b > c$.

2. First construction

First steps of the first construction are the same as in [3]. We recall not only them but also their justifications.

First steps of the first construction

Let M be the midpoint of the side BC. Consider a line parallel to AH that passes through M and denote its intersections with AB and AC by D and E, respectively (see Figure 1). Next, consider a line parallel to AH that passes through C and denote its intersection with AB by G.

Publication Date: June 20, 2018. Communicating Editor: Paul Yiu.

It is easily seen that $\angle AED = \angle ADE = \angle ACG = \angle AGC = \frac{\alpha}{2}$. Hence, triangles AED and ACG are isosceles and

$$|AD| = |AE| = |BD| - |AB| = \frac{|BG|}{2} - |AB| = \frac{b+c}{2} - c = \frac{b-c}{2}. \qquad (1)$$

So, the first steps of the first construction are the following:

1 Draw a line-segment AH.
2 Draw lines m, n such that they form with AH angles $\frac{\alpha}{2}$.
3 Draw the isosceles triangle AED such that D lies on m, E lies on n, and $|AD| = |AE| = \frac{b-c}{2}$ (see Figure 1).

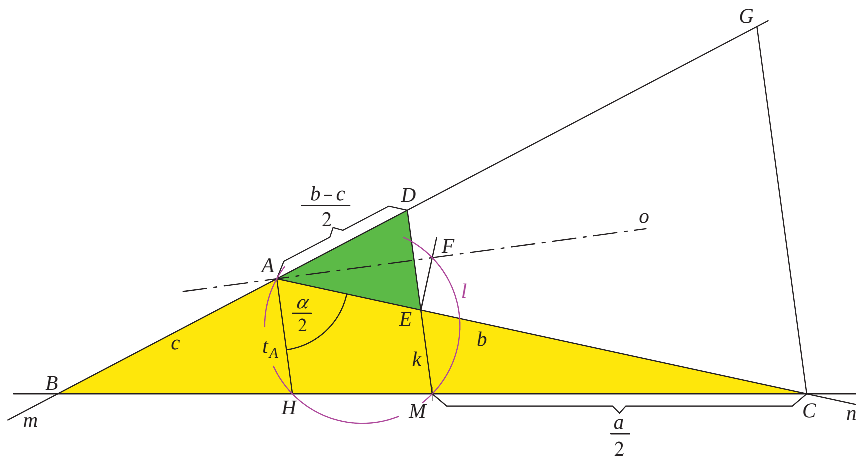


Figure 1.

2.1. *First approach.* Let F be the intersection of the perpendicular bisector of the line-segment DE with the line which is perpendicular to the line AE and passes through E. It is obvious that $\angle AFE = \angle AED = \frac{\alpha}{2}$, and therefore

$$|ED| = (b-c) \cdot \cos\frac{\alpha}{2}, \qquad |EF| = \frac{b-c}{2 \cdot \tan\frac{\alpha}{2}} . \qquad (2)$$

Denote by k the length of the line-segment ME. Since the triangles AHC and EMC are similar, we obtain

$$\frac{t_A}{b} = \frac{k}{\frac{b+c}{2}}, \qquad \text{and, after rearranging,} \qquad t_A = \frac{2bk}{b+c} . \qquad (3)$$

From the similarity of triangles BMD, BHA and from (2), we have

$$\frac{k + (b-c) \cdot \cos\frac{\alpha}{2}}{\frac{b+c}{2}} = \frac{t_A}{c},$$

and from (3), we obtain

$$k = \frac{t_A}{c} \cdot \frac{b+c}{2} - (b-c) \cdot \cos\frac{\alpha}{2} = \frac{bk}{c} - (b-c) \cdot \cos\frac{\alpha}{2}.$$

Therefore,

$$k\left(\frac{b}{c}-1\right)=(b-c)\cdot\cos\frac{\alpha}{2},$$

and thus

$$k=c\cdot\cos\frac{\alpha}{2}. \tag{4}$$

Set up a Cartesian coordinate system with A at the origin and the positive direction of the x-axis which coincides with the ray AC. Then (using (1), (2) and (4))

$$A=[0;\,0]\,,\qquad C=[b;\,0]\,,\qquad E=\left[\frac{b-c}{2};\,0\right],$$

$$F=\left[\frac{b-c}{2};\,\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right],\qquad H=\left[t_A\cdot\cos\frac{\alpha}{2};\,-t_A\cdot\sin\frac{\alpha}{2}\right],$$

$$M=\left[\frac{b-c}{2}+k\cdot\cos\frac{\alpha}{2};\,-k\cdot\sin\frac{\alpha}{2}\right]=\left[\frac{b-c}{2}+c\cdot\cos^2\frac{\alpha}{2};\,-c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right].$$

Next,

$$\overrightarrow{FM}=\left(c\cdot\cos^2\frac{\alpha}{2};\,-c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}-\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right),$$

$$\overrightarrow{CM}=\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2};\,-c\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right).$$

It follows that

$$\begin{aligned}\overrightarrow{FM}\cdot\overrightarrow{CM}&=c\cdot\cos^2\frac{\alpha}{2}\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2}\right)+\\&\quad+\left(c\cdot\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\frac{b-c}{2\cdot\tan\frac{\alpha}{2}}\right)\cdot c\cdot\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\\&=c\cdot\cos^2\frac{\alpha}{2}\left(\frac{b-c}{2}-b+c\cdot\cos^2\frac{\alpha}{2}+c\cdot\sin^2\frac{\alpha}{2}+\frac{b-c}{2}\right)=\\&=c\cdot\cos^2\frac{\alpha}{2}\,(b-c-b+c)=0.\end{aligned}$$

Hence, the lines $CM=HM$ and FM are perpendicular, and, in consequence, the point M is the intersection of the line ED and the circle with HF as a diameter.

2.2. *Second approach.* Consider a parabola $\mathcal{P}$ to which the lines AE, AD, ED, BC are tangent. This parabola $\mathcal{P}$ is determined unambiguously ([1], p. 212). According to Lambert's theorem, the circumcircle of a triangle formed by three tangent lines to the parabola passes through the focus of the parabola. Thus, the circumcircles of the triangles AED, ABC and EMC pass through a single point (the focus of the parabola), which will be denoted by F (see Figure 2). It follows that $\angle BFC=\angle BAC=\alpha$ and $\angle MFC=\angle MEC=\frac{\alpha}{2}$. Hence, the focus F lies on the perpendicular bisector of the line-segment BC and the angles FMC, FMB are right. Since $\angle FEC=\angle FMC$, the angles FEC and FEA are also right. It implies that the line-segment AF is a diameter of the circumcircle of the isosceles triangle AED (and the point F is the same point as the point F in the first approach).

Since the angle FMH is right, the point M is the intersection of the line ED and the circle with diameter HF.

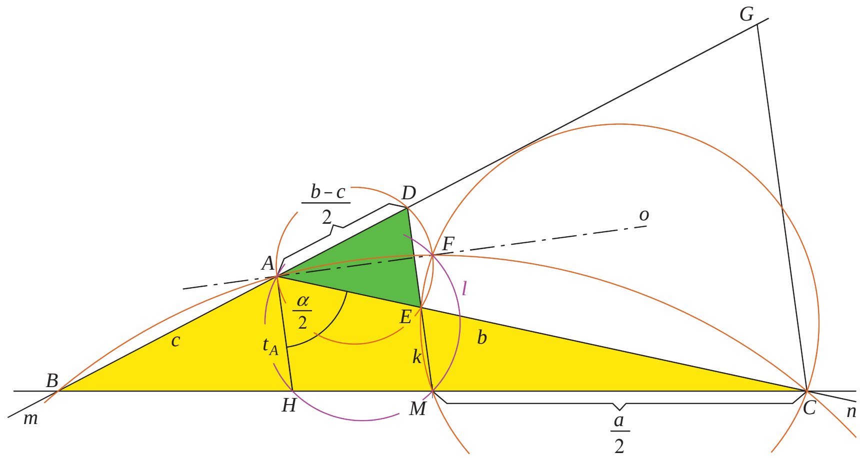


Figure 2.

Based on our previous considerations, we can now proceed with the remaining steps of our construction:

4a Draw the intersection F of the perpendicular bisector of DE with the line which is perpendicular to AE and passes through E, or,

4b draw the intersection F of the perpendicular bisector of DE with the circumcircle of the triangle AED.

5 Draw the intersection M of the line ED and the circle l with FH as a diameter (only the intersection M that lies in the half-plane determined by the line n and the point H can be taken into account).

6 Draw the intersection B of lines m and HM, and the intersection C of lines n and HM.

Note that the perpendicular bisector of DE is simultaneously the line passing through the point A and perpendicular to the line AH (and, of course, also the bisector of the angle EAD).

Comments on the solution published in [3]

We proceed with several comments on the solution presented in [3]. Paris Pamfilos considered the intersection of the perpendicular bisectors of the line-segments ED and BC. Clearly, this intersection is our point F (see Figure 3). Using Lambert's theorem he proved that the parabola $\mathcal{P}'$ with focus at F and whose tangent at its vertex is ED is the above-mentioned parabola $\mathcal{P}$, i.e. $\mathcal{P}'=\mathcal{P}$. He constructed the point F as the intersection of the perpendicular bisector of ED and the circumcircle of the triangle AED. Thus, the beginning of his solution coincides with our steps 1, 2, 3, 4b. But then, he constructed the line BC as the tangent to the parabola $\mathcal{P}'$ passing through H in a too complicated and time-consuming manner.

It is well known that the foot of the perpendicular line to any tangent of a parabola passing through its focus lies on the tangent at the vertex ([2], p. 34). Hence, the point M must lie on the circle with FH as a diameter. So, the construction can be completed using our very easy steps 5 and 6. Again, it is not necessary to construct any parabola.

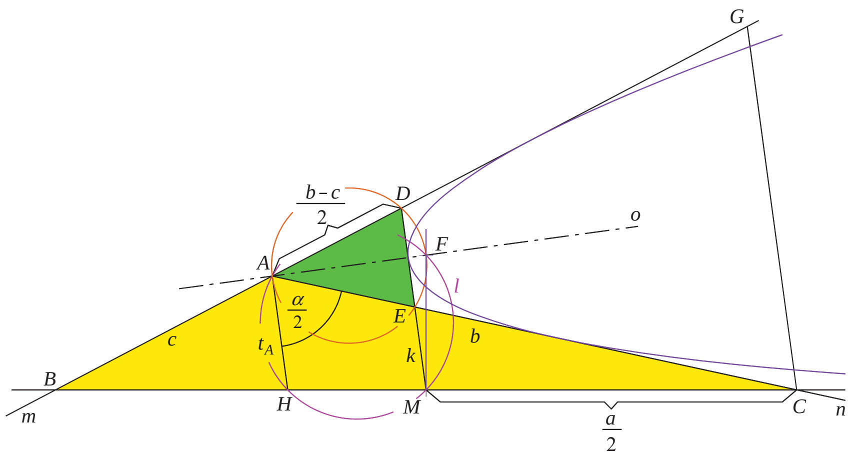


Figure 3.

3. Second construction

Let D' be a point on the ray opposite to the ray AB such that $|AD| = b - c$, i.e. $|BD'| = b$ (see Figure 4). Through this point D' draw a line parallel to AH and denote its intersections with lines AB, AC and BC by D', E' and M', respectively. Next, let

$$k' = \left|M'E'\right|, \qquad q = \frac{|D'E'|}{2}.$$

Since triangles $BM'D'$ and BHA are similar, we have

$$\frac{k' + 2q}{b} = \frac{t_A}{c}. \tag{5}$$

Analogously, from the similarity of triangles HCA and $M'CE'$, we obtain

$$\frac{t_A}{b} = \frac{k'}{c}, \qquad \text{and equivalently,} \qquad \frac{1}{b} = \frac{k'}{t_A \cdot c}. \tag{6}$$

Therefore (from (6) and (5)),

$$\frac{k'(k' + 2q)}{t_A \cdot c} = \frac{t_A}{c},$$

$$k'^2 + 2qk' - t_A^2 = 0.$$

This quadratic equation in the one unknown k' has two roots. But only the root $k' = -q + \sqrt{t_A^2 + q^2}$ is positive. If we consider a right triangle with legs of the lengths t_A and q, then $k' = -q + \sqrt{t_A^2 + q^2}$ is equal to the difference of the length $\sqrt{t_A^2 + q^2}$ of its hypotenuse and the length q of its leg.

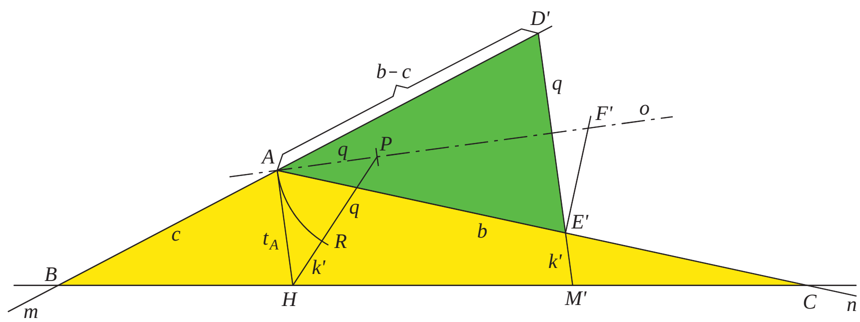


Figure 4.

Therefore, the steps of our second construction are the following (see Figure 4):

1 Draw a line-segment AH.
2 Draw lines m, n such that they form with AH angles $\frac{\alpha}{2}$.
3 Draw the isosceles triangle $AE'D'$ such that D' lies on m, E' lies on n, and $|AD'| = |AE'| = b - c$ (see Figure 4).
4 Draw the point P on the perpendicular bisector o of $D'E'$ such that $|AP| = \frac{|E'D'|}{2}$.
5 Draw the point R on the line-segment HP such that $|PR| = \frac{|E'D'|}{2}$.
6 Draw the point M' on the ray opposite to the ray $E'D'$ such that $|E'M'| = |RH|$.
7 Draw the intersection B of lines m and HM', and the intersection C of lines n and HM'.

Remark. If we consider the above-mentioned parabola $\mathcal{P}$, then the point E' is a point of $\mathcal{P}$ (therefore, AC touches $\mathcal{P}$ at E'). This follows from the fact that the sum of the lengths of subtangent and subnormal at any point of a parabola (except its vertex) is bisected by the focus of $\mathcal{P}$. Indeed, by comparing Figures 1 and 4, we observe that $|AF'| = 2\,|AF|$, where F' is the intersection of o and the line which is perpendicular to AC and passes through E'.

References

[1] M. Berger, *Geometry II*, Springer Verlag, Heidelberg, 1987.

[2] G. Glaeser, H. Stachel, and B. Odehnal, *The Universe of Conics. From the ancient Greeks to 21st century developments*, Springer Verlag, Berlin, Heidelberg, 2016.

[3] P. Pamfilos, The Triangle Construction $\{\alpha,\ b-c,\ t_A\}$, *Forum Geom.*, 16 (2016) 115–117.

Martina Štěpánová: Department of Mathematics Education, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail address: stepanov@karlin.mff.cuni.cz

Forum Geometricorum
Volume 18 (2018) 251–253.

FORUM GEOM
ISSN 1534-1178

A New Proof of Pitot Theorem by AM-GM Inequality

Robert Bosch

Abstract. In this note we show a new proof of Pitot theorem by Arithmetic Mean - Geometric Mean inequality. Our novel idea is to consider the before mentioned theorem as an extremal case of a general geometric inequality. This new approach cannot be found in the literature.

1. Introduction

The Pitot theorem states:

A quadrilateral $ABCD$ is tangential (has an inscribed circle) if and only if $AB + DC = AD + BC$.

To avoid reproduction we recommend to read the third and fourth paragraph of the excellent paper [1]. In this one the reader can find a brief survey on the theorem. Clearly we are interested in the converse of Pitot theorem, according to Martin Josefsson there are four proofs in the literature, providing references in each case. In this note we show still a new one, by means of the AM-GM inequality, since the well-known theorem can be considered an extremal case of a simple and general geometric inequality involving a quadrilateral, a circle, and tangents.

2. Lemma

Let $ABCD$ be a convex but not tangential quadrilateral. The extensions of DA and CB intersect at E, and the extensions of BA and CD intersect at F. Then the excircle of triangle EAB cut the side DC or the excircle of triangle FAD cut the side BC.

Proof. In Figure 1, we draw the angle-bisectors of angles E and F respectively. Clearly points on the first line are equidistant from the sides AD and BC, similar for the second one, where the points are equidistant from sides AB and DC. Now, suppose the excircle of triangle EAB, is interior to the quadrilateral $ABCD$, hence its center O_1 is in the upper halfplane determined by the $F-$angle-bisector. But, O_2, the center of the second excircle is the intersection of the lines AO_1 and the $F-$angle-bisector, clearly this point is located in the right halfplane determined by the $E-$angle-bisector. Finally the excircle of triangle FAD cut the side BC. $\square$

Publication Date: June 20, 2018. Communicating Editor: Paul Yiu.
The author would like to thank the referee for his valuable suggestions.

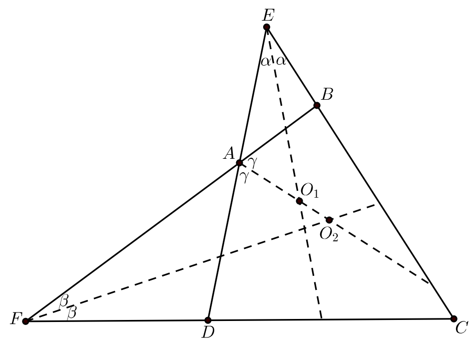


Figure 1

3. Converse of Pitot theorem

In this section we shall prove an stronger result than the converse of Pitot theorem. Let us see:

Geometric Inequality:

Let $ABCD$ be a quadrilateral with a circle that is tangent to the sides AB, AD and BC respectively, and cut the side DC. Then

$$AB + DC \geq AD + BC.$$

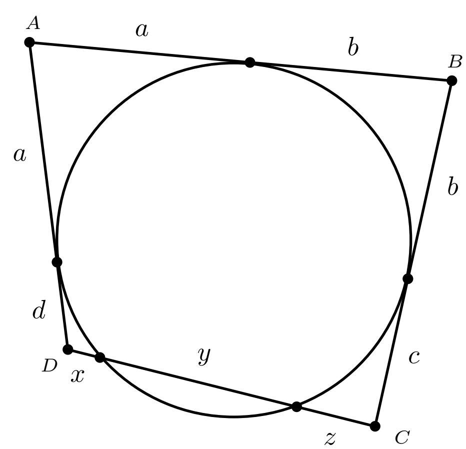


Figure 2

Proof. This circle exists by the Lemma.

See Figure 2, where the notation is completely justified by the *two tangent theorem*, that two tangents to a circle from an external point are of equal length. Now, the inequality to be proved becomes

$$x + y + z \geq c + d.$$

But, by Power of a Point theorem we have

$$\begin{aligned} c^2 &= z(y+z), \\ d^2 &= x(x+y). \end{aligned}$$

So, all that we need to prove is

$$\sqrt{z(y+z)} + \sqrt{x(x+y)} \leq x + y + z,$$

and this is a direct consequence of AM-GM inequality, since

$$\begin{aligned} \sqrt{z(y+z)} &\leq \frac{z+y+z}{2} = \frac{2z+y}{2}, \\ \sqrt{x(x+y)} &\leq \frac{x+x+y}{2} = \frac{2x+y}{2}, \end{aligned}$$

equality holds if and only if $y = 0$. Completing the proof of the inequality.

The geometric meaning of this result is that the circle is tangent to the side DC, i.e. the quadrilateral $ABCD$ is tangential. More precisely, to prove the converse of Pitot theorem, suppose the opposite sides add to the same number, and for sake of contradiction, the circle cut a side, then the inequality is strict, obtaining a contradiction. Done. □

Bibliography.

[1] M. Josefsson, More characterizations of tangential quadrilaterals, *Forum Geom.*, 11 (2011) 65–82.

Robert Bosch: Archimedean Academy, 12425 SW 72nd St, Miami, Florida 33183, USA
E-mail address: bobbydrg@gmail.com

Forum Geometricorum
Volume 18 (2018) 255–273.

FORUM GEOM
ISSN 1534-1178

A Model of Continuous Plane Geometry that is Nowhere Geodesic

John Donnelly

Abstract. We construct a model $\mathbb{M}_1$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, the model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the Ruler Postulate and Protractor Postulate from Birkhoff's set of axioms for the euclidean plane.

1. Introduction

In his talk given at the international congress of mathematicians in Paris in 1900, David Hilbert presented a list of problems. The fourth problem in this list is the following:

Problem of the straight line as the shortest distance between two points:

Another problem relating to the foundations of geometry is this : If from among the axioms necessary to establish ordinary euclidean geometry, we exclude the axiom of parallels, or assume it as not satisfied, but retain all other axioms, we obtain, as is well known, the geometry of Lobachevsky (hyperbolic geometry). We may therefore say that this is a geometry standing next to euclidean geometry. If we require further that that axiom be not satisfied whereby, of three points on a straight line, one and only one lies between the other two, we obtain Riemann's (elliptic) geometry, so that this geometry appears to be the next after Lobachevsky's. If we wish to carry out a similar investigation with respect to the axiom of Archimedes, we must look upon this as not satisfied, and we arrive thereby at the non-Archimedean geometries which have been investigated by Veronese and myself. The more general question now arises : Whether from other suggestive standpoints geometries may not be devised which, with equal right, stand next to euclidean geometry. Here I should like to direct your attention to a theorem which has, indeed, been employed by many authors as a definition of a straight line, viz., that the straight line is the shortest distance between two points. The essential content of this statement reduces to the theorem of Euclid that in a triangle the sum of two sides is always greater than the third side-a theorem which, as is easily seen, deals solely with elementary concepts, i. e., with such as are derived directly from the axioms, and

Publication Date: June 20, 2018. Communicating Editor: Paul Yiu.

is therefore more accessible to logical investigation. Euclid proved this theorem, with the help of the theorem of the exterior angle, on the basis of the congruence theorems. Now it is readily shown that this theorem of Euclid cannot be proved solely on the basis of those congruence theorems which relate to the application of segments and angles, but that one of the theorems on the congruence of triangles is necessary. We are asking, then, for a geometry in which all the axioms of ordinary euclidean geometry hold, and in particular all the congruence axioms except the one of the congruence of triangles (or all except the theorem of the equality of the base angles in the isosceles triangle), and in which, besides, the proposition that in every triangle the sum of two sides is greater than the third is assumed as a particular axiom.

One finds that such a geometry really exists and is no other than that which Minkowski constructed in his book, Geometrie der Zahlen [1] *and made the basis of his arithmetical investigations. Minkowski's is therefore also a geometry standing next to the ordinary euclidean geometry; it is essentially characterized by the following stipulations:*

1. The points which are at equal distances from a fixed point 0 lie on a convex closed surface of the ordinary euclidean space with 0 as a center.

2. Two segments are said to be equal when one can be carried into the other by a translation of the ordinary euclidean space.

In Minkowski's geometry the axiom of parallels also holds. By studying the theorem of the straight line as the shortest distance between two points, I arrived [2] *at a geometry in which the parallel axiom does not hold, while all other axioms of Minkowski' s geometry are satisfied. The theorem of the straight line as the shortest distance between two points and the essentially equivalent theorem of Euclid about the sides of a triangle, play an important part not only in number theory but also in the theory of surfaces and in the calculus of variations. For this reason, and because I believe that the thorough investigation of the conditions for the validity of this theorem will throw a new light upon the idea of distance, as well as upon other elementary ideas, e. g., upon the idea of the plane, and the possibility of its definition by means of the idea of the straight line, the construction and systematic treatment of the geometries here possible seem to me desirable.*

[Translated for the BULLETIN, with the author's permission, by Dr. MARY WINSTON NEWSON. The original appeared in the Göttinger Nachrichten, 1900, pp. 253-297, and in the Archiv der Mathernatik una Physik, 3d ser., vol. 1 (1901), pp. 44-63 and 213-237.]

There are various interpretations of Hilbert's fourth problem, some interpretations dealing with convex subsets of the Euclidean plane. The first main contribution to this problem was given by Hilbert's student G. Hamel. Hamel reduced the problem to metrics on convex subsets of Euclidean spaces [10]. In particular, if $\mathcal{S}$ is a convex subset of a Euclidean space, then the restriction to $\mathcal{S}$ of the ambient Euclidean metric is a metric on $\mathcal{S}$ satisfying the condition that the restriction to $\mathcal{S}$ of

[1] Leipzig, 1896.

[2] *Math Annalen*, Vol. 46, p.91.

the Euclidean straight lines are geodesics for the restriction to $\mathcal{S}$ of the Euclidean metric. One form of Hilbert's problem asks for a characterization of all metrics on $\mathcal{S}$ for which the Euclidean lines in $\mathcal{S}$ are geodesics [10].

Other interpretations deal with metrics in projective geometry [11]. Solutions to other interpretations are given by H. Busemann [2] and Z. I. Szabo [11]. There are many results related to this problem. For more information, see [2], [10], [11].

The contents of this paper were inspired by Hilbert's fourth problem. However, in this paper we look at the triangle inequality from the opposite point of view. In particular, we investigate plane geometry in which the straight line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We give a model $\mathbb{M}_1$ of plane geometry in which all of the incidence axioms, betweenness axioms, congruence axioms (with the exception of Side-Angle-Side), and even the euclidean parallel postulate hold, yet in which the straight geodesic line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We refer to such a model as *nowhere geodesic* since the geodesic line segment $\overline{AB}$ connecting two points A and B is never the shortest path from A to B. We prove that the following holds in $\mathbb{M}_1$:

Given any convex polygon $\mathcal{P}$, then for each pair of points A and B inside $\mathcal{P}$ and for any $\epsilon > 0$, there exists a path q from A to B such that the arc length along q is less than ϵ and such that q has nonempty intersection with the exterior of $\mathcal{P}$.

Thus, by going "outside" of $\mathcal{P}$, we can find shorter and shorter paths in $\mathbb{M}_1$ from A to B.

The model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the ruler postulate and protractor postulate of Birkhoff [1], [7], [8], [9]. More specifically, all lines in $\mathbb{M}_1$ have a bijective correspondence with $\mathbb{R}$, and there is a bijective correspondence between all angles with fixed vertex V that are on a given halfplane of line $\overleftrightarrow{VP}$ and the open interval $(0, \pi)$.

We note that if all of the incidence axioms, betweenness axioms, and congruence axioms (with the possible exception of Side-Angle-Side) hold, and if in addition the exterior angle theorem, pons asinorum, and angle addition hold, then we can prove that the triangle inequality holds [4], [5], [7], [8]. When constructing the model $\mathbb{M}_1$, we use usual euclidean angle measure, so that both the exterior angle theorem and angle addition hold. However, due to the fact that distance is altered in $\mathbb{M}_1$, then the Pons Asinorun fails in the model.

In Taxicab geometry, both the exterior angle theorem and angle addition hold, but the pons asinorum fails. However, we still have that the general triangle inequality holds [3], [8]. Thus, in Taxicab Geometry, the straight line segment $\overline{AB}$ connecting two points A and B is still a shortest path from A to B.

2. Hilbert's Axioms

In this section we state the axioms of plane geometry given by Hilbert (as communicated by R. Hartshorne in [5]). We will show that the model $\mathbb{M}_1$ satisfies all of these axioms, with the exception of Side-Angle-Side.

The Incidence Axioms

(1) Given any two distinct points A and B, then there exists a unique line $\overleftrightarrow{AB}$ passing through A and B.
(2) Given any line l, then there exist at least two distinct points A and B on l.
(3) There exist three distinct noncollinear points A, B and C.

The Betweenness Axioms

(1) If B is between A and C (written $A - B - C$), then A, B, and C are three distinct collinear points. In this case we also have $C - B - A$.
(2) Given any two distinct points A and B, then there exists a point C such that $A - B - C$.
(3) Given three distinct points on a line, then exactly one of the three points is between the other two points.
(4) (Pasch) Let A, B, and C be three distinct noncollinear points, and let l be a line not passing through any of A, B, or C. If l passes through a point D lying between A and B, then either l passes through a point H lying between A and C, or else l passes through a point K lying between B and C, but not both.

We note that Betweenness Axiom (4) (i.e. Pasch) is logically equivalent to the Plane Separation Postulate stated below [4],[5],[7],[8]. Since Pasch and the Plane Separation Postulate are logically equivalent, then we will remove Pasch as an axiom and replace it with the Plane Separation Postulate.

The Plane Separation Postulate

Given any line l, then the set of points not lying on l can be divided into two nonempty subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ with the following properties:

(1) Two points A and B not on l belong to the same set ($\mathcal{H}_1$ or $\mathcal{H}_2$) if and only if segment $\overline{AB}$ does not intersect l.
(2) Two points C and D not on l belong to opposite sets ($C \in \mathcal{H}_1$ and $D \in \mathcal{H}_2$) if and only if segment $\overline{CD}$ intersects l at a point H such that $C - H - D$.

The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *halfplanes* (or *sides*) of the line l, and l is called an *edge* of each of the halfplanes $\mathcal{H}_1$ and $\mathcal{H}_2$. In case (1), we say that A and B are on the *same side* of l. In case (2), we say that C and D are on *opposite sides* of l. When quoting the Plane Separation Postulate, we will abbreviate it by PSP.

The Congruence Axioms for Line Segments

(1) Given a line segment $\overline{AB}$, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$.
(2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Every line segment is congruent to itself.

(3) (Segment Addition) Given three points A, B, and C such that $A - B - C$, and given three points D, E, and F such that $D - E - F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

The Congruence Axioms for Angles

(1) Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, then there exists a unique ray $\overrightarrow{DE}$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle EDF$.
(2) For any three angles α, β, and γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

The following is usually assumed as an axiom when working with Hilbert's axiom system for plane geometry. However, it does not hold in the model $\mathbb{M}_1$.

Side-Angle-Side

Given triangles $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

When referring to Side-Angle-Side, then we abbreviate it as *SAS*. It is well-known that if SAS holds, then one can prove the Exterior Angle Theorem, the Pons Asinorum, and Angle Addition as theorems, and consequently, one can prove that the Triangle Inequality also holds.

The following version of the Euclidean Parallel Postulate is by John Playfair (although it had previously been mentioned by Proclus) (page 39 of [5]):

Given a point P and a line l not passing through P, then there exists a unique line q such that q passes through P and is parallel to l.

3. The Ruler and Protractor Postulates

In this section we state two of the axioms given by Birkhoff in his development of plane geometry [1]. In particular, we state the ruler postulate and protractor postulate. Unlike the axioms of Hilbert given above, both the ruler postulate and protractor postulate incorporate the use of the real numbers through the concepts of distance and angle measure. Both of these postulates are satisfied by the model $\mathbb{M}_1$.

The Ruler Postulate

Let $\mathcal{P}$ denote the set of points in the plane. There exists a function $q : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ such that for each line l, there exists a bijection $f : l \to \mathbb{R}$ with the property that for all points P and Q on l, $q(P,Q) = |f(P) - f(Q)|$.

For all $P, Q \in \mathcal{P}$, we call $q(P,Q)$ the *distance* from P to Q. If for a line l, a bijection $f : l \to \mathbb{R}$ is such that for all $P, Q \in l$, $q(P,Q) = |f(P) - f(Q)|$, then f is called a *coordinate system* for l. We note that $q(P,Q)$ is not necessarily a metric on $\mathcal{P}$, and that $q(P,Q)$ does not necessarily satisfy the triangle inequality. In fact, we will define a distance function on the model $\mathbb{M}_1$ that is not a metric and that in general does not satisfy the triangle inequality.

The Protractor Postulate

There exists a function m from the set of all angles to the open interval $(0, \pi)$ such that

(1) For each ray $\overrightarrow{PQ}$ on the edge of halfplane $\mathcal{H}$ (where $\mathcal{H}$ is a halfplane of line $\overleftrightarrow{PQ}$), and for each $r \in (0, \pi)$, there exists a unique ray $\overrightarrow{PR}$, with $R \in \mathcal{H}$, such that $m(\angle QPR) = r$.

(2) If T is a point in the interior of $\angle QPR$, then $m(\angle QPT) + m(\angle TPR) = m(\angle QPR)$.

Given an angle $\angle ABC$, then $m(\angle ABC)$ is called the *measure* of angle $\angle ABC$, and is denoted by $m\angle ABC$.

4. The Model $\mathbb{M}_1$

To construct the model $\mathbb{M}_1$, we start with the Cartesian plane $\mathbb{R}^2$, and we define points and lines in $\mathbb{M}_1$ to be exactly the same as points and lines in $\mathbb{R}^2$. In particular, a point in $\mathbb{M}_1$ is given by an ordered pair of numbers (x, y), and a line is the set of all points in the plane satisfying an equation in one of the forms $y = mx + b$ or $x = k$. We also define angle measure in $\mathbb{M}_1$ to be exactly the same as euclidean angle measure in $\mathbb{R}^2$. Moreover, we define betweenness in $\mathbb{M}_1$ to be exactly the same as betweenness in euclidean geometry: point B is between points A and C in $\mathbb{M}_1$ (denoted $A - B - C$) if and only if B is between points A and C in $\mathbb{R}^2$. More specifically, if we let $d(X, Y)$ denote the euclidean distance from point X to point Y, then point B is between points A and C in both $\mathbb{M}_1$ and $\mathbb{R}^2$ if the following conditions hold:

(1) A, B, and C are distinct collinear points
(2) $d(A, C) = d(A, B) + d(B, C)$

Since points, lines, and betweenness in $\mathbb{M}_1$ are exactly the same as points, lines, and betweenness in euclidean geometry, then it follows immediately that the incidence and betweenness axioms hold in $\mathbb{M}_1$. Moreover, since points, lines, and betweenness in $\mathbb{M}_1$ are exactly the same as points, lines, and betweenness in euclidean geometry, then polygons in $\mathbb{M}_1$ are exactly the same as polygons in euclidean geometry. Also, since segments and polygons in $\mathbb{M}_1$ are the same as segments and polygons in euclidean geometry, then a polygon $\mathcal{P}$ is convex in $\mathbb{M}_1$ if and only if $\mathcal{P}$ is convex in euclidean geometry. Similarly, since angle measure in $\mathbb{M}_1$ is the same as euclidean angle measure, then it follows immediately that the congruence axioms for angles hold in $\mathbb{M}_1$. We note that since the Exterior Angle Theorem and Angle Addition are based solely on angle measure and not on distance, and since both of these statements hold in euclidean geometry, then they both hold in $\mathbb{M}_1$. However, to construct $\mathbb{M}_1$, we will need to alter distance in the plane, and as a consequence the Pons Asinorum does not hold in $\mathbb{M}_1$. Furthermore, since we are altering distance, then we will need to prove that the congruence axioms for line segments hold in $\mathbb{M}_1$. To define distance in $\mathbb{M}_1$, we start with the set of points (u, v) in the plane such that $u, v \in \mathbb{Q}$. That is, we start with the cartesian product $\mathbb{Q}^2$. Since $\mathbb{Q}^2$ is countably infinite, then we can enumerate the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$. Thus, we often will treat the points in $\mathbb{Q}^2$ as a sequence (P_j). Using this enumeration on the set of all points (P_j) in $\mathbb{Q}^2$, we can enumerate all (non-ordered) pairs of distinct points from $\mathbb{Q}^2$ in the following way: Given the pairs of points $(P_l, P_t), (P_h, P_k) \in \mathbb{Q}^2 \times \mathbb{Q}^2$, where $P_l \neq P_t$ and $P_h \neq P_k$, then

assume that k is the largest of the four subscripts l, t, h, and k. If k is strictly larger than the other subscripts l, t, and h, then (P_l, P_t) comes before (P_h, P_k) in the enumeration. If $k = t$, then we compare l and h. In this case, (P_l, P_t) comes before (P_h, P_k) if and only if $l < h$. Similarly, if $k = l$, then (P_l, P_t) comes before (P_h, P_k) if and only if $t < h$. For example, the first ten pairs are enumerated in the following way:
(P_1, P_2), (P_1, P_3), (P_2, P_3), (P_1, P_4), (P_2, P_4), (P_3, P_4), (P_1, P_5), (P_2, P_5), (P_3, P_5), (P_4, P_5),

Note that the order in which the points in each pair are written is irrelevant when enumerating the pairs of points this way. In particular, (P_l, P_t) is considered the same as (P_t, P_l) for the purposes of enumeration.

Again, we let $d(X, Y)$ denote the euclidean distance from a point X to a point Y. We let $n(X, Y)$ denote the distance in the model $\mathbb{M}_1$ from a point X to a point Y. For each pair of distinct points P_r and P_t in $\mathbb{Q}^2$, with $r < t$, we construct a sequence $(D_{r,t}(m))$ of points in $\mathbb{R}^2$ such that for each $m \geq 1$, $d(P_r, D_{r,t}(m)) \geq m$ and $d(P_t, D_{r,t}(m)) \geq m$, and such that there exists a path from P_r to P_t passing through $D_{r,t}(m)$ whose arc length in $\mathbb{M}_1$ is equal to $\dfrac{n(P_r, P_t)}{2^m}$.

Let P_1 and P_2 be the first two points in the sequence (P_j). If X and Y are points on the line $\overleftrightarrow{P_1P_2}$, then we define $n(X, Y) = d(X, Y)$. In particular, $n(P_1, P_2) = d(P_1, P_2)$.

Let l_1 be a line perpendicular to $\overleftrightarrow{P_1P_2}$. Since l_1 is unbounded in euclidean geometry, then there exists a point $D_{1,2}(1)$ on l_1 such that

(1) $D_{1,2}(1) \notin \overleftrightarrow{P_1P_2}$
(2) $d(P_1, D_{1,2}(1)) \geq 1$ and $d(P_2, D_{1,2}(1)) \geq 1$

In particular, the three points $D_{1,2}(1)$, P_1, and P_2 are not collinear. We define $n(P_1, D_{1,2}(1)) = n(P_2, D_{1,2}(1)) = \dfrac{1}{4} n(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(1)}$ and $\overline{P_2D_{1,2}(1)}$ is $\dfrac{1}{2} n(P_1, P_2)$.

We now show how to compute the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_1D_{1,2}(1)}$ between a point $P \in \overleftrightarrow{P_1D_{1,2}(1)}$ and either of the points P_1 or $D_{1,2}(1)$. We have three cases.

First assume that $P - P_1 - D_{1,2}(1)$. In this case, $n(P, P_1) = d(P, P_1)$ and $n(P, D_{1,2}(1)) = n(P, P_1) + n(P_1, D_{1,2}(1)) = d(P, P_1) + \dfrac{1}{4} n(P_1, P_2)$.

Next, assume that $P_1 - P - D_{1,2}(1)$. In this case, there exist $r_1, r_2 \in (0, 1)$ such that $d(P_1, P) = r_1 d(P_1, D_{1,2}(1))$ and $d(P, D_{1,2}(1)) = r_2 d(P_1, D_{1,2}(1))$. We define $n(P_1, P) = r_1 n(P_1, D_{1,2}(1))$ and $n(P, D_{1,2}(1)) = r_2 n(P_1, D_{1,2}(1))$. Note that we use the same constants of proportionality r_1 and r_2 for both euclidean geometry and $\mathbb{M}_1$.

Finally, assume that $P_1 - D_{1,2}(1) - P$. In this case, $n(D_{1,2}(1), P) = d(D_{1,2}(1), P)$ and $n(P_1, P) = n(P_1, D_{1,2}(1)) + n(D_{1,2}(1), P) = \dfrac{1}{4} n(P_1, P_2) + d(D_{1,2}(1), P)$.

We can now use these definitions of length in $\mathbb{M}_1$ together with the use of segment addition and constants of proportionality to define the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_1D_{1,2}(1)}$ between any two points X_1 and X_2 on $\overleftrightarrow{P_1D_{1,2}(1)}$.

If $X_1 - X_2 - P_1 - D_{1,2}(1)$, then $n(X_1, X_2) = d(X_1, X_2)$. Similarly, if $P_1 - D_{1,2}(1) - X_1 - X_2$, then $n(X_1, X_2) = d(X_1, X_2)$.

Assume that $P_1 - X_1 - X_2 - D_{1,2}(1)$. In this case, there exists $r_3 \in (0, 1)$ such that $d(X_1, X_2) = r_3 d(P_1, D_{1,2}(1))$. We define $n(X_1, X_2) = r_3 n(P_1, D_{1,2}(1))$.

Next, assume that $X_1 - P_1 - D_{1,2}(1) - X_2$. In this case, we define $n(X_1, X_2)$ = $n(X_1, P_1) + n(P_1, D_{1,2}(1)) + n(D_{1,2}(1), X_2)$ = $d(X_1, P_1) + n(P_1, D_{1,2}(1)) + d(D_{1,2}(1), X_2)$.

Next, assume that $X_1 - P_1 - X_2 - D_{1,2}(1)$. In this case, we define $n(X_1, X_2)$ = $n(X_1, P_1) + n(P_1, X_2)$.

Finally, assume that $P_1 - X_1 - D_{1,2}(1) - X_2$. Similar to the previous case, we define $n(X_1, X_2)$ = $n(X_1, D_{1,2}(1)) + n(D_{1,2}(1), X_2)$.

One can use the same exact method to define the distance in $\mathbb{M}_1$ along the line $\overleftrightarrow{P_2D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_2D_{1,2}(1)}$.

Let $\mathcal{S}_{1,2}(1)$ denote the set of lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1D_{1,2}(1)}$, and $\overleftrightarrow{P_2D_{1,2}(1)}$. Let l_2 be a line not in $\mathcal{S}_{1,2}(1)$. Since $\mathcal{S}_{1,2}(1)$ is finite, and since l_2 is unbounded and has infinitely many points, then there exists a point $D_{1,2}(2)$ on l_2 such that

(1) $d(P_1, D_{1,2}(2)) \geq 2$ and $d(P_2, D_{1,2}(2)) \geq 2$
(2) $D_{1,2}(2)$ is not on any of the lines in $\mathcal{S}_{1,2}(1)$.

Note that $D_{1,2}(2) \neq D_{1,2}(1)$, and that no three of P_1, P_2, $D_{1,2}(1)$, or $D_{1,2}(2)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(2)$ and either of P_1 and P_2 has not yet been defined.

We define $n(P_1, D_{1,2}(2)) = n(P_2, D_{1,2}(2)) = \dfrac{1}{8} n(P_1, P_2) = \dfrac{1}{8} d(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1D_{1,2}(2)}$ and $\overline{P_2D_{1,2}(2)}$ is $\dfrac{1}{4} n(P_1, P_2)$.

Using a method similar to the one given above to define distance in $\mathbb{M}_1$ along $\overleftrightarrow{P_1D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can similarly define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,2}(2)}$ or between any two points on the line $\overleftrightarrow{P_2D_{1,2}(2)}$.

Let $\mathcal{S}_{1,2}(2)$ denote the set of lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1D_{1,2}(1)}$, $\overleftrightarrow{P_2D_{1,2}(1)}$, $\overleftrightarrow{P_1D_{1,2}(2)}$, $\overleftrightarrow{P_2D_{1,2}(2)}$, and $\overleftrightarrow{D_{1,2}(1)D_{1,2}(2)}$. Let l_3 be a line not in $\mathcal{S}_{1,2}(2)$. Since $\mathcal{S}_{1,2}(2)$ is finite, and since l_3 is unbounded and has infinitely many points, then there exists a point $D_{1,2}(3)$ on l_3 such that

(1) $d(P_1, D_{1,2}(3)) \geq 3$ and $d(P_2, D_{1,2}(3)) \geq 3$
(2) $D_{1,2}(3)$ is not on any of the lines in $\mathcal{S}_{1,2}(2)$.

Note that $D_{1,2}(3)$ is distinct from $D_{1,2}(1)$ and $D_{1,2}(2)$, and that no three of P_1, P_2, $D_{1,2}(1)$, $D_{1,2}(2)$, or $D_{1,2}(3)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(3)$ and either of P_1 and P_2 has not yet been defined.

We define $n(P_1, D_{1,2}(3)) = n(P_2, D_{1,2}(3)) = \dfrac{1}{16} n(P_1, P_2) = \dfrac{1}{16} d(P_1, P_2)$. We see that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1 D_{1,2}(3)}$ and $\overline{P_2 D_{1,2}(3)}$ is $\dfrac{1}{8} n(P_1, P_2)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ along $\overleftrightarrow{P_1 D_{1,2}(1)}$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can similarly define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,2}(3)}$ or between any two points on the line $\overleftrightarrow{P_2 D_{1,2}(3)}$.

Assume that there exist k distinct points $D_{1,2}(1), D_{1,2}(2), \ldots, D_{1,2}(k) \in \mathbb{R}^2$ such that:

(1) No three of the points $P_1, P_2, D_{1,2}(1), D_{1,2}(2), \ldots, D_{1,2}(k)$ are collinear.
(2) For each $t = 1, \ldots k$, $d(P_1, D_{1,2}(t)) \geq t$ and $d(P_2, D_{1,2}(t)) \geq t$
(3) For each $t = 1, \ldots k$, $n(P_1, D_{1,2}(t)) = n(P_2, D_{1,2}(t)) = \dfrac{1}{2^{t+1}} n(P_1, P_2)$
(4) For each $t = 1, \ldots k$, we use a method similar to the one used above to define distance in $\mathbb{M}_1$ between two points on the line $\overleftrightarrow{P_1 D_{1,2}(t)}$ or between any two points on the line $\overleftrightarrow{P_2 D_{1,2}(t)}$

We note that for each $t = 1, \ldots k$, the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1 D_{1,2}(t)}$ and $\overline{P_2 D_{1,2}(t)}$ is $\dfrac{1}{2^t} n(P_1, P_2)$.

Let $\mathcal{S}_{1,2}(k)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1 P_2}$
(2) For each $t = 1, \ldots k$, $\overleftrightarrow{P_1 D_{1,2}(t)}$ and $\overleftrightarrow{P_2 D_{1,2}(t)}$
(3) For each $t, h \in \{1, \ldots k\}$, with $t < h$, $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$

Let l_{k+1} be a line not in $\mathcal{S}_{1,2}(k)$. Since $\mathcal{S}_{1,2}(k)$ is finite, and since l_{k+1} is unbounded and has infinitely many points, then there exists a point $D_{1,2}(k+1)$ on l_{k+1} such that

(1) $d(P_1, D_{1,2}(k+1)) \geq k+1$ and $d(P_2, D_{1,2}(k+1)) \geq k+1$
(2) $D_{1,2}(k+1)$ is not on any of the lines in $\mathcal{S}_{1,2}(k)$.

Note that $D_{1,2}(k+1)$ is distinct from all of the points $D_{1,2}(1), D_{1,2}(2), \ldots, D_{1,2}(k)$, and that no three of $P_1, P_2, D_{1,2}(1), D_{1,2}(2), \ldots, D_{1,2}(k), D_{1,2}(k+1)$ are collinear. Thus, the distance in $\mathbb{M}_1$ between $D_{1,2}(k+1)$ and either of P_1 and P_2 has not yet been defined. We define $n(P_1, D_{1,2}(k+1)) = n(P_2, D_{1,2}(k+1)) = \dfrac{1}{2^{k+2}} n(P_1, P_2)$.

Again, using a method similar to the one used above, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,2}(k+1)}$ or between any two points on the line $\overleftrightarrow{P_2 D_{1,2}(k+1)}$.

We note that the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1 D_{1,2}(k+1)}$ and $\overline{P_2 D_{1,2}(k+1)}$ is $\dfrac{1}{2^{k+1}} n(P_1, P_2)$.

Thus, there exists a sequence $(D_{1,2}(m))$ of distinct points in $\mathbb{R}^2$ such that

(1) No three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear

(2) For each $m \geq 1$, $d(P_1, D_{1,2}(m)) \geq m$ and $d(P_2, D_{1,2}(m)) \geq m$
(3) For each $m \geq 1$, $n(P_1, D_{1,2}(m)) = n(P_2, D_{1,2}(m)) = \dfrac{1}{2^{m+1}} n(P_1, P_2)$.
(4) For each $m \geq 1$, we use a method similar to the one used above to define distance in $\mathbb{M}_1$ between two points on the line $\overleftrightarrow{P_1 D_{1,2}(m)}$ or between any two points on the line $\overleftrightarrow{P_2 D_{1,2}(m)}$.

We note that for each $m \geq 1$, the arc length in $\mathbb{M}_1$ from P_1 to P_2 along the segments $\overline{P_1 D_{1,2}(m)}$ and $\overline{P_2 D_{1,2}(m)}$ is $\dfrac{1}{2^m} n(P_1, P_2)$.

Let P_3 denote the third point in the sequence (P_j). If P_3 is on $\overleftrightarrow{P_1 P_2}$ or if there exists $m \geq 1$ such that P_3 is on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then $n(P_1, P_3)$ has already been defined. More generally, if P_3 is on $\overleftrightarrow{P_1 P_2}$ or if there exists $m \geq 1$ such that P_3 is on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then for each pair of points $X, Y \in \overleftrightarrow{P_1 P_3}$, the distance $n(X, Y)$ has already been defined. On the other hand, if P_3 is not on $\overleftrightarrow{P_1 P_2}$ and if for each $m \geq 1$, P_3 is not on $\overleftrightarrow{P_1 D_{1,2}(m)}$ then $n(P_1, P_3)$ has not yet been defined. In this case, we define $n(P_1, P_3) = d(P_1, P_3)$, and in general, given $X, Y \in \overleftrightarrow{P_1 P_3}$, we define $n(X, Y) = d(X, Y)$. In either case, $n(P_1, P_3)$ has been defined, and more generally, distance along $\overleftrightarrow{P_1 P_3}$ has been defined.

Let $\mathcal{S}_{1,3}(1)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$
(3) For each t, h, with $h \geq 3$ and $t < h$, $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 3\}$ are on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$. In particular, $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)} \notin \mathcal{S}_{1,3}(1)$. Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(1)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(1)$ with the line $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$. Since $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$, then there exists a point $D_{1,3}(1)$ on $\overleftrightarrow{D_{1,2}(1) D_{1,2}(2)}$ such that:

(1) $d(P_1, D_{1,3}(1)) \geq 1$ and $d(P_3, D_{1,3}(1)) \geq 1$
(2) $D_{1,3}(1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(1)$.

Since $D_{1,3}(1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(1)$, then neither $n(P_1, D_{1,3}(1))$ nor $n(P_3, D_{1,3}(1))$ has yet been defined. We define $n(P_1, D_{1,3}(1)) = n(P_3, D_{1,3}(1)) = \dfrac{1}{4} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(1)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(1)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2D_{1,3}(1)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \geq 3$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$, we define $n(X,Y) = d(X,Y)$.

Let $\mathcal{S}_{1,3}(2)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1D_{1,2}(m)}$, $\overleftrightarrow{P_2D_{1,2}(m)}$, and $\overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$
(3) $\overleftrightarrow{P_1D_{1,3}(1)}$, $\overleftrightarrow{P_2D_{1,3}(1)}$, and $\overleftrightarrow{P_3D_{1,3}(1)}$
(4) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \notin \{2,3\}\}$ are on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Moreover, by the construction above, we have that the point $D_{1,3}(1)$ is not on line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Thus, $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)} \notin \mathcal{S}_{1,3}(2)$.

Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(2)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(2)$ with the line $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$. Since $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$, then there exists a point $D_{1,3}(2)$ on $\overleftrightarrow{D_{1,2}(2)D_{1,2}(3)}$ such that:

(1) $d(P_1, D_{1,3}(2)) \geq 2$ and $d(P_3, D_{1,3}(2)) \geq 2$
(2) $D_{1,3}(2)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(2)$.

Since $D_{1,3}(2)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(2)$, then neither $n(P_1, D_{1,3}(2))$ nor $n(P_3, D_{1,3}(2))$ has yet been defined. We define $n(P_1, D_{1,3}(2)) = n(P_3, D_{1,3}(2))$ $= \dfrac{1}{8}n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1D_{1,3}(2)}$ and $\overline{P_3D_{1,3}(2)}$ is $\dfrac{1}{4}n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,3}(2)}$ or between any two points on the line $\overleftrightarrow{P_3D_{1,3}(2)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2D_{1,3}(2)}$ we define $n(X,Y) = d(X,Y)$. Moreover, for each $m \neq 2, 3$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(2)}$, we define $n(X,Y) = d(X,Y)$.

Let $\mathcal{S}_{1,3}(3)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1D_{1,2}(m)}$, $\overleftrightarrow{P_2D_{1,2}(m)}$, $\overleftrightarrow{D_{1,2}(m)D_{1,3}(1)}$, and $\overleftrightarrow{D_{1,2}(m)D_{1,3}(2)}$
(3) $\overleftrightarrow{P_1D_{1,3}(1)}$, $\overleftrightarrow{P_2D_{1,3}(1)}$, $\overleftrightarrow{P_3D_{1,3}(1)}$, $\overleftrightarrow{P_1D_{1,3}(2)}$, $\overleftrightarrow{P_2D_{1,3}(2)}$, $\overleftrightarrow{P_3D_{1,3}(2)}$, and $\overleftrightarrow{D_{1,3}(1)D_{1,3}(2)}$

(4) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$.

Since no three points in the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \geq 1\}$ are collinear, then none of the points from the set $\{P_1, P_2\} \cup \{D_{1,2}(m) \mid m \notin \{3, 4\}\}$ are on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Moreover, by the constructions above, we have that neither of the points $D_{1,3}(1)$ or $D_{1,3}(2)$ are on line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Thus, $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)} \notin \mathcal{S}_{1,3}(3)$.

Since there are only a countably infinite number of lines that are elements in $\mathcal{S}_{1,3}(3)$, then there are at most a countably infinite number of points of intersection of any of the lines in $\mathcal{S}_{1,3}(3)$ with the line $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$. Since $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$ is unbounded, and since there exist an uncountably infinite number of points between any two distinct points on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$, then there exists a point $D_{1,3}(3)$ on $\overleftrightarrow{D_{1,2}(3)D_{1,2}(4)}$ such that:

(1) $d(P_1, D_{1,3}(3)) \geq 3$ and $d(P_3, D_{1,3}(3)) \geq 3$
(2) $D_{1,3}(3)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(3)$.

Since $D_{1,3}(3)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(3)$, then neither $n(P_1, D_{1,3}(3))$ nor $n(P_3, D_{1,3}(3))$ has yet been defined. We define $n(P_1, D_{1,3}(3)) = n(P_3, D_{1,3}(3))$ $= \dfrac{1}{16} n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1D_{1,3}(3)}$ and $\overline{P_3D_{1,3}(3)}$ is $\dfrac{1}{8} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1D_{1,3}(3)}$ or between any two points on the line $\overleftrightarrow{P_3D_{1,3}(3)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2D_{1,3}(3)}$ we define $n(X, Y) = d(X, Y)$. Moreover, for each $m \neq 3, 4$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m)D_{1,3}(3)}$, we define $n(X, Y) = d(X, Y)$.

Assume that there exist r distinct points $D_{1,3}(1)$, $D_{1,3}(2)$, $D_{1,3}(3)$, …, $D_{1,3}(r)$ (where $r \geq 3$) such that

(1) For each $j = 1, \ldots r$, $D_{1,3}(j)$ is a point on the line $\overleftrightarrow{D_{1,2}(j)D_{1,2}(j+1)}$.
(2) No three of the points $D_{1,3}(1)$, $D_{1,3}(2)$, $D_{1,3}(3)$, …, $D_{1,3}(r)$ are collinear.
(3) For each $j = 1, \ldots r$, $D_{1,3}(j)$ is not a point on any of the following lines:
   (a) $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, and $\overleftrightarrow{P_2P_3}$
   (b) For each $m \geq 1$, $\overleftrightarrow{P_1D_{1,2}(m)}$ and $\overleftrightarrow{P_2D_{1,2}(m)}$
   (c) For each $k \neq j$, $\overleftrightarrow{P_1D_{1,3}(k)}$, $\overleftrightarrow{P_2D_{1,3}(k)}$, and $\overleftrightarrow{P_3D_{1,3}(k)}$
   (d) For each $m \geq 1$, and for each $k \neq j$, $\overleftrightarrow{D_{1,2}(m)D_{1,3}(k)}$
   (e) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(j)D_{1,2}(j+1)}$
   (f) Any line in the form $\overleftrightarrow{D_{1,3}(t)D_{1,3}(h)}$, where $t \neq j$ and $h \neq j$.

(4) For each $j = 1, \ldots r$, $d(P_1, D_{1,3}(j)) \geq j$ and $d(P_3, D_{1,3}(j)) \geq j$
(5) For each $j = 1, \ldots r$, $n(P_1, D_{1,3}(j)) = n(P_3, D_{1,3}(j)) = \dfrac{1}{2^{j+1}} n(P_1, P_3)$.
(6) For each $j = 1, \ldots r$, distance in $\mathbb{M}_1$ is defined between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(j)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(j)}$ using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$.

Let $\mathcal{S}_{1,3}(r+1)$ denote the set of lines in any of the following forms:

(1) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$
(2) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$
(3) For each $j = 1, \ldots r$, $\overleftrightarrow{P_1 D_{1,3}(j)}$, $\overleftrightarrow{P_2 D_{1,3}(j)}$, and $\overleftrightarrow{P_3 D_{1,3}(j)}$
(4) For each $m \geq 1$, and for each $j = 1, \ldots r$, $\overleftrightarrow{D_{1,2}(m) D_{1,3}(j)}$
(5) Any line in the form $\overleftrightarrow{D_{1,2}(t) D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)}$
(6) Any line in the form $\overleftrightarrow{D_{1,3}(t) D_{1,3}(h)}$, where $t, h \in \{1, \ldots r\}$.

By the definition of the set $\mathcal{S}_{1,3}(r+1)$, we see that $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)} \notin \mathcal{S}_{1,3}(r+1)$. Let $D_{1,3}(r+1)$ be a point on $\overleftrightarrow{D_{1,2}(r+1) D_{1,2}(r+2)}$ such that

(1) $d(P_1, D_{1,3}(r+1)) \geq r+1$ and $d(P_3, D_{1,3}(r+1)) \geq r+1$
(2) $D_{1,3}(r+1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(r+1)$

Since $D_{1,3}(r+1)$ is not a point on any of the lines in $\mathcal{S}_{1,3}(r+1)$, then neither $n(P_1, D_{1,3}(r+1))$ nor $n(P_3, D_{1,3}(r+1))$ has yet been defined. We define $n(P_1, D_{1,3}(r+1)) = n(P_3, D_{1,3}(r+1)) = \dfrac{1}{2^{r+2}} n(P_1, P_3)$.

Thus, the arc length of the path from P_1 to P_3 along the segments $\overline{P_1 D_{1,3}(r+1)}$ and $\overline{P_3 D_{1,3}(r+1)}$ is $\dfrac{1}{2^{r+1}} n(P_1, P_3)$.

Using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$, one can define distance in $\mathbb{M}_1$ between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(r+1)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(r+1)}$.

Also, for any $X, Y \in \overleftrightarrow{P_2 D_{1,3}(r+1)}$ we define $n(X, Y) = d(X, Y)$. Moreover, for each $m \neq r+1, r+2$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m) D_{1,3}(r+1)}$, we define $n(X, Y) = d(X, Y)$.

Thus, there exists a sequence $(D_{1,3}(j))$ of distinct points in $\mathbb{R}^2$ such that

(1) For each $j \geq 1$, $D_{1,3}(j)$ is a point on the line $\overleftrightarrow{D_{1,2}(j) D_{1,2}(j+1)}$.
(2) No three of the points in $(D_{1,3}(j))$ are collinear.
(3) For each $j \geq 1$, $D_{1,3}(j)$ is not a point on any of the following lines:
  (a) $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_1 P_3}$, and $\overleftrightarrow{P_2 P_3}$
  (b) For each $m \geq 1$, $\overleftrightarrow{P_1 D_{1,2}(m)}$ and $\overleftrightarrow{P_2 D_{1,2}(m)}$
  (c) For each $k \neq j$, $\overleftrightarrow{P_1 D_{1,3}(k)}$, $\overleftrightarrow{P_2 D_{1,3}(k)}$, and $\overleftrightarrow{P_3 D_{1,3}(k)}$
  (d) For each $m \geq 1$, and for each $k \neq j$, $\overleftrightarrow{D_{1,2}(m) D_{1,3}(k)}$

(e) Any line in the form $\overleftrightarrow{D_{1,2}(t)D_{1,2}(h)}$ (where $t, h \geq 1$, with $t < h$), other than the line $\overleftrightarrow{D_{1,2}(j)D_{1,2}(j+1)}$

(f) Any line in the form $\overleftrightarrow{D_{1,3}(t)D_{1,3}(h)}$, where $t \neq j$ and $h \neq j$.

(4) For each $j \geq 1$, $d(P_1, D_{1,3}(j)) \geq j$ and $d(P_3, D_{1,3}(j)) \geq j$

(5) For each $j \geq 1$, $n(P_1, D_{1,3}(j)) = n(P_3, D_{1,3}(j)) = \dfrac{1}{2^{j+1}} n(P_1, P_3)$.

(6) For each $j \geq 1$, distance in $\mathbb{M}_1$ is defined between any two points on the line $\overleftrightarrow{P_1 D_{1,3}(j)}$ or between any two points on the line $\overleftrightarrow{P_3 D_{1,3}(j)}$ using a method similar to the one used above to define distance in $\mathbb{M}_1$ between any two points on $\overleftrightarrow{P_1 D_{1,2}(1)}$.

Also, for each $j \geq 1$, and for any $X, Y \in \overleftrightarrow{P_2 D_{1,3}(j)}$ we define $n(X, Y) = d(X, Y)$. Moreover, for each $m \neq j, j+1$, and for any $X, Y \in \overleftrightarrow{D_{1,2}(m) D_{1,3}(j)}$, we define $n(X, Y) = d(X, Y)$.

Using the enumeration defined above on the set of unordered pairs of points (P_r,P_t), one can repeat the above process to construct a sequence $(D_{r,t}(m))$ of points in $\mathbb{R}^2$ for each pair of distinct points P_r and P_t in $\mathbb{Q}^2$, with $r < t$, such that for each $m \geq 1$, $d(P_r, D_{r,t}(m)) \geq m$ and $d(P_t, D_{r,t}(m)) \geq m$, and such that there exists a path from P_r to P_t passing through $D_{r,t}(m)$ whose arc length in $\mathbb{M}_1$ is equal to $\dfrac{n(P_r, P_t)}{2^m}$.

If l is a line in $\mathbb{R}^2$ that is not used in this construction, then we define the distance in $\mathbb{M}_1$ between any two points X and Y on l to be the usual euclidean distance $d(X, Y)$.

5. A Proof that $\mathbb{M}_1$ Satisfies the Ruler Postulate

In this section, we give a proof that $\mathbb{M}_1$ satisfies the Ruler Postulate. We leave it to the reader to check that in general, if a model satisfies the Ruler Postulate, then that model also satisfies the congruence axioms for segments.

Given a line l, then it follows by the way that distance is defined in $\mathbb{M}_1$ that either for each pair of points X and Y on l, $n(X, Y) = d(X, Y)$, or else there exists a segment $\overline{BH}$ on l (with $\overleftrightarrow{BH} = l$) such that:

(1) the euclidean length $d(B, H)$ of $\overline{BH}$ has been altered to define the new length $n(B, H)$ of $\overline{BH}$ in $\mathbb{M}_1$

(2) For each pair of points U and V in segment $\overline{BH}$, there exists $r \in [0, 1]$ such that both $n(U, V) = rn(B, H)$ and $d(U, V) = rd(B, H)$

(3) For each pair of points X and Y on l such that $X - Y - B - H$, $n(X, Y) = d(X, Y)$

(4) For each pair of points X and Y on l such that $B - H - X - Y$, $n(X, Y) = d(X, Y)$

(5) For each point X on l such that $X - B - H$, $n(X, B) = d(X, B)$

(6) For each point Y on l such that $B - H - Y$, $n(H, Y) = d(H, Y)$

For all other possible pairs of points X and Y on l, we employ segment addition to compute $n(X,Y)$. For example, if $X - B - Y - H$, then $n(X,Y) = n(X,B) + n(B,Y)$.

First assume that for each pair of points X and Y on l, $n(X,Y) = d(X,Y)$. Since euclidean distance d satisfies the Ruler Postulate (page 31 of [8]), then there exists a bijection $f : l \to \mathbb{R}$ such that for each pair of points X and Y on l, $d(X,Y) = |f(X) - f(Y)|$. In this case, we use f as a coordinate system for l, and for each pair of points X and Y on l, we see that $n(X,Y) = d(X,Y) = |f(X) - f(Y)|$.

Next, assume that there exists a segment $\overline{BH}$ on l (with $\overleftrightarrow{BH} = l$) such that conditions (1) through (6) above are satisfied. We will define a bijection $g : l \to \mathbb{R}$ such that for each pair of points X and Y on l, $n(X,Y) = |g(X) - g(Y)|$.

Again, since euclidean distance d satisfies the Ruler Postulate, then there exists a coordinate system $f : l \to \mathbb{R}$ such that $f(B) = 0$, $f(H) > 0$, and for each point X on l such that $X - B - H$, $f(X) < 0$ [7].

Let $g(B) = 0$ and $g(H) = n(B,H)$. For each point X on l such that $X - B - H$, let $g(X) = f(X)$. Let $h = n(B,H) - d(B,H) = g(H) - f(H)$. For each point Y on l such that $B - H - Y$, let $g(Y) = f(Y) + h = f(Y) + g(H) - f(H)$. Let $t = \dfrac{n(B,H)}{d(B,H)}$. For each point X on l such that $B - X - H$, let $g(X) = tf(X)$.

We note that $g(B) = 0 = (t)(0) = tf(B)$, and that $g(H) = n(B,H) = (d(B,H))\left(\dfrac{n(B,H)}{d(B,H)}\right) = td(B,H) = t|f(B) - f(H)| = t|0 - f(H)| = tf(H)$. We also note that $g(H) = f(H) + g(H) - f(H) = f(H) + h$. We first prove that g is a bijection.

Proof that g is a bijection. Let $k \in \mathbb{R}$. First assume that $k = 0$. By definition of g, we have that $g(B) = 0$. Suppose that there exists a point X on l, with $X \neq B$ such that $g(X) = 0$. Since $g(H) > 0$, then $X \neq H$. If $X - B - H$, then $g(X) < 0$, a contradiction. If $B - X - H$, then $f(X) > 0$ and moreover, $g(X) = tf(X) > 0$, again a contradiction. If $B - H - X$, then $f(X) > f(H)$, which implies that $g(X) = f(X) + h > f(H) + h = g(H) > 0$, also a contradiction. (As above, $h = n(B,H) - d(B,H) = g(H) - f(H)$.) Thus, B is the only point such that $g(B) = 0$.

Assume that $k < 0$. Since $f : l \to \mathbb{R}$ is a bijection then there exists a unique point X_k on l such that $X_k - B - H$ and $f(X_k) = k$. By definition of g, we have that $g(X_k) = f(X_k) = k$. As just argued in the previous case, if $X = B$, $B - X - H$, $X = H$, or $B - H - X$, then $g(X) \geq 0$. Thus, X_k is the only point such that $g(X_k) = k$.

Assume that $k \geq n(B,H)$. Again, let $h = n(B,H) - d(B,H)$. Since $k \geq n(B,H)$, then $k - h \geq n(B,H) - h$. Therefore, $k - h \geq n(B,H) - (n(B,H) - d(B,H))$, which implies that $k - h \geq d(B,H)$. Since f is a coordinate system for l, then there exists a unique point Y_k on l such that $f(Y_k) = k - h$. Since $f(Y_k) = k - h \geq d(B,H)$, then either $Y_k = H$ or else $B - H - Y_k$. Thus, there exists a unique point Y_k on l such that either $Y_k = H$ or else $B - H - Y_k$, and such that $g(Y_k) = f(Y_k) + h = k$.

Finally, assume that $0 < k < n(B,H)$. Again, let $t = \dfrac{n(B,H)}{d(B,H)}$. Since f is a coordinate system for l, then there exists a unique point W_k on l such that $f(W_k) = \dfrac{k}{t}$. Since $0 < k < n(B,H)$ and since $t > 0$, then $0 = \dfrac{0}{t} < \dfrac{k}{t} < \dfrac{n(B,H)}{t} = \dfrac{n(B,H)}{\left(\dfrac{n(B,H)}{d(B,H)}\right)} = d(B,H)$. Since $f(B) = 0$, $f(H) = d(B,H)$, and $f(W_k) = \dfrac{k}{t}$, then $f(B) < f(W_k) < f(H)$, which implies that $B - W_k - H$. By definition of g, we see that $g(W_k) = tf(W_k) = (t)\left(\dfrac{k}{t}\right) = k$. Thus, there exists a unique point W_k such that $B - W_k - H$ and $g(W_k) = k$.

In any case, we see that there exists a unique point X on l such that $g(X) = k$. Hence, g is a bijection. □

We next prove that g is a coordinate system for l.

Proof that g is a coordinate system for l. We now check that for each pair of points X and Y on l, $n(X,Y) = |g(X) - g(Y)|$.

First assume that $X - Y - B - H$. In this case, we have that $n(X,Y) = d(X,Y)$, $g(X) = f(X)$, and $g(Y) = f(Y)$. Thus, $n(X,Y) = d(X,Y) = |f(X) - f(Y)| = |g(X) - g(Y)|$.

Similarly, if $X - B - H$, then we have that $n(X,B) = d(X,B)$, $g(X) = f(X)$, and $g(B) = f(B) = 0$. Thus, $n(X,B) = d(X,B) = |f(X) - f(B)| = |g(X) - g(B)|$.

Assume that $B - H - X - Y$. In this case, we have that $n(X,Y) = d(X,Y)$, $g(X) = f(X) + h$, and $g(Y) = f(Y) + h$, where, as above, $h = n(B,H) - d(B,H)$. Thus, $n(X,Y) = d(X,Y) = |f(X) - f(Y)| = |f(X) - f(Y) + h - h| = |(f(X) + h) - (f(Y) + h)| = |g(X) - g(Y)|$.

Similarly, if $B - H - Y$, then we have that $n(H,Y) = d(H,Y)$, $g(H) = f(H) + h$, and $g(Y) = f(Y) + h$. Thus, $n(H,Y) = d(H,Y) = |f(H) - f(Y)| = |f(H) - f(Y) + h - h| = |(f(H) + h) - (f(Y) + h)| = |g(H) - g(Y)|$.

Next, assume that X and Y are on segment $\overline{BH}$. As above, we let $t = \dfrac{n(B,H)}{d(B,H)}$. In particular, $n(B,H) = td(B,H)$. Also, we have that $g(X) = tf(X)$ and $g(Y) = tf(Y)$. Moreover, there exists $r \in [0,1]$ such that both $n(X,Y) = rn(B,H)$ and $d(X,Y) = rd(B,H)$. Thus, we see that $n(X,Y) = rn(B,H) = rtd(B,H) = td(X,Y) = t|f(X) - f(Y)| = |tf(X) - tf(Y)| = |g(X) - g(Y)|$.

For any other possible combination of points X and Y on l, we use segment addition. For example, assume that $X - B - Y - H$. In this case, $g(X) < g(B) < g(Y)$, which implies that $|g(X) - g(B)| + |g(B) - g(Y)| = |g(X) - g(Y)|$. Thus, we see that $n(X,Y) = n(X,B) + n(B,Y) = |g(X) - g(B)| + |g(B) - g(Y)| = |g(X) - g(Y)|$. The cases where $B - X - H - Y$ or $X - B - H - Y$, as well as the cases where either $B - H - Y$ with $X = B$, or else $X - B - H$ with $Y = H$ are similar and are left to the reader.

Hence, g is a coordinate system for l. □

6. Convex Polygons

In this section we extend some results from previous sections to all of $\mathbb{R}^2$. In particular, we prove that given any two points $A, B \in \mathbb{R}^2$ and given any $\epsilon \in (0, 1)$, then there exists a path q from A to B whose arc length in $\mathbb{M}_1$ is less than ϵ. We begin with the following lemmas.

Lemma 1. *Let $D = (a, b)$ be a point in $\mathbb{R}^2$ such that $b \notin \mathbb{Q}$. Let l denote the vertical line $x = a$. Let $\delta \in (0, 1)$. Then there exists a point $T = (a, q)$ such that $q \in \mathbb{Q}$, $d(D, T) < \delta$, and $n(D, T) < \delta$.*

Proof. Given the vertical line l, then it follows by the way that distance in $\mathbb{M}_1$ is constructed above that either for each pair of points X and Y on l, $n(X, Y) = d(X, Y)$, or else there exists a segment $\overline{HB}$ (with $\overleftrightarrow{HB} = l$) such that:

(1) For each pair of points X and Y on l such that either $X - Y - H - B$ or $H - B - X - Y$, we have that $n(X, Y) = d(X, Y)$
(2) For each pair of points U and V on segment $\overline{HB}$, there exists $r \in [0, 1]$ such that $n(U, V) = rn(H, B)$ and $d(U, V) = rd(H, B)$

First assume that for each pair of points X and Y on l, $n(X, Y) = d(X, Y)$. In this case, it follows by the density of the rational numbers that there exists $q_0 \in \mathbb{Q}$ such that $|b - q_0| < \delta$. In this case the point $T_0 = (a, q_0)$ is such that $n(D, T_0) = d(D, T_0) = |b - q_0| < \delta$.

Next assume that there exists a segment $\overline{HB}$ (with $\overleftrightarrow{HB} = l$) such that:

(1) For each pair of points X and Y on l such that either $X - Y - H - B$ or $H - B - X - Y$, we have that $n(X, Y) = d(X, Y)$
(2) For each pair of points U and V on segment $\overline{HB}$, there exists $r \in [0, 1]$ such that $n(U, V) = rn(H, B)$ and $d(U, V) = rd(H, B)$

Let $H = (a, k)$. Since D cannot equal both H and B simultaneously, then we may assume without loss of generality that $D \neq H$. Furthermore, we may assume that $k < b$. The proof when $b < k$ is similar and is left to the reader.

First assume that D is a point on segment $\overline{HB}$. Thus, either $D = B$ or else $B - D - H$. Let $\rho \in (0, 1)$ be such that $(\rho)(d(D, H)) < \delta$ and $(\rho)(n(D, H)) < \delta$. Again, by the density of the rational numbers that there exists $q_1 \in \mathbb{Q}$ such that $k < q_1 < b$ and $|b - q_1| < (\rho)(d(D, H))$. Let $T_1 = (a, q_1)$. Thus, we see that $d(D, T_1) = |b - q_1| < (\rho)(d(D, H)) < \delta$. Therefore, $n(D, T_1) < (\rho)(n(D, H)) < \delta$.

Finally, assume that $D - H - B$. Again, by the density of the rational numbers, there exists $q_2 \in \mathbb{Q}$ such that $k < q_2 < b$ and $|b - q_2| < \delta$. Let $T_2 = (a, q_2)$. Thus, we see that $n(D, T_2) = d(D, T_2) = |b - q_2| < \delta$. The case where $H - B - D$ is similar and is left to the reader. □

The proof of the following lemma is similar to the proof of Lemma 1 and is left to the reader.

Lemma 2. *Let $D = (a, b)$ be a point in $\mathbb{R}^2$ such that $a \notin \mathbb{Q}$. Let l denote the horizontal line $y = b$. Let $\delta \in (0, 1)$. Then there exists a point $T = (q, b)$ such that $q \in \mathbb{Q}$, $d(D, T) < \delta$, and $n(D, T) < \delta$.*

Lemma 3. *Let $\delta \in (0,1)$. Given a point $D \in \mathbb{R}^2$, then there exists a point $T \in \mathbb{Q}^2$ and a path p from D to T such that $d(D,T) < \delta$ and such that the arc length in $\mathbb{M}_1$ along p is at most δ.*

Proof. If $D \in \mathbb{Q}^2$, then we let $T = D$. In this case p consists of the single point D and the length along p is 0.

Assume that $D \notin \mathbb{Q}^2$. Let $D = (a,b)$. If $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$, then it follows by Lemma 1 that there exists a point $T_1 = (a,q_1) \in \mathbb{Q}^2$ such that $d(D,T_1) < \delta$ and $n(D,T_1) < \delta$. In this case p is segment $\overline{DT_1}$.

If $a \notin \mathbb{Q}$ and $b \in \mathbb{Q}$, then it follows by Lemma 2 that there exists a point $T_2 = (q_2,b) \in \mathbb{Q}^2$ such that $d(D,T_2) < \delta$ and $n(D,T_2) < \delta$. In this case p is segment $\overline{DT_2}$.

Assume that $a,b \notin \mathbb{Q}$. By Lemma 1 there exists a point $W = (a,q_3)$ such that $q_3 \in \mathbb{Q}$, $d(D,W) < \frac{\delta}{2}$, and $n(D,W) < \frac{\delta}{2}$. By Lemma 2 there exists a point $T_3 = (q_4,q_3) \in \mathbb{Q}^2$ such that $d(W,T_3) < \frac{\delta}{2}$ and $n(W,T_3) < \frac{\delta}{2}$. In this case we let p be the union of segments $\overline{DW} \cup \overline{WT_3}$. Thus, the arc length along p from D to T_3 is less than δ. Moreover, by applying the triangle inequality in euclidean geometry, we have that $d(D,T_3) < d(D,W) + d(W,T_3) < \dfrac{\delta}{2} + \dfrac{\delta}{2} = \delta$. □

Theorem 4. *Given any convex polygon $\mathcal{P}$, then for each pair of points A and B inside $\mathcal{P}$ and for any $\epsilon > 0$, there exists a path q from A to B such that the arc length along q is less than ϵ and such that q has nonempty intersection with the exterior of $\mathcal{P}$.*

Proof. Let $\mathcal{P}$ be a convex polygon. Let A and B denote two points in the interior of $\mathcal{P}$. Let $\epsilon > 0$. Since $\mathcal{P}$ is a convex polygon in both $\mathbb{M}_1$ and euclidean geometry, then there exists a euclidean circle $\mathcal{C}$ that bounds $\mathcal{P}$. Let V and r denote the center and radius of $\mathcal{C}$, respectively. Let $\delta > 0$ be such that $\delta < \min\left\{\dfrac{r - d(A,V)}{2}, \dfrac{r - d(B,V)}{2}, \dfrac{\epsilon}{3}\right\}$. If $A \in \mathbb{Q}^2$, then let $W_A = A$. If $A \notin \mathbb{Q}^2$, then it follows by Lemma 3 that there exists $W_A \in \mathbb{Q}^2$ such that $d(A,W_A) < \delta$ and such that there exists a path q_A from A to W_A whose arc length in $\mathbb{M}_1$ is less than δ. Similarly, if $B \in \mathbb{Q}^2$, then let $W_B = B$, and if $B \notin \mathbb{Q}^2$, then there exists $W_B \in \mathbb{Q}^2$ such that $d(B,W_B) < \delta$ and such that there exists a path q_B from B to W_B whose arc length in $\mathbb{M}_1$ is less than δ. In either case, it follows by the triangle inequality in euclidean geometry that $d(V,W_A) < r$ and $d(V,W_B) < r$. Thus, both W_A and W_B are in the interior of $\mathcal{C}$. It also follows that the respective arc lengths of q_A and q_B in $\mathbb{M}_1$ are both less than $\frac{\epsilon}{3}$. By the construction above, there exists a sequence (D_j) of points in $\mathbb{R}^2$ such that for each $j \geq 1$,

(1) $d(W_A,D_j) \geq j$ and $d(W_B,D_j) \geq j$

(2) $n(W_A,D_j) = n(W_B,D_j) = \dfrac{1}{2^{j+1}} n(W_A,W_B)$

Let $t \in \mathbb{Z}^+$ be such that $t > 3r$ and $\dfrac{1}{2^t} n(W_A,W_B) < \dfrac{\epsilon}{3}$. Thus, D_t is a point such that

(1) $d(W_A, D_t) \geq t > 3r$ and $d(W_B, D_t) \geq t > 3r$
(2) The arc length in $\mathbb{M}_1$ from W_A to W_B along the path $p = \overline{W_A D_t} \cup \overline{W_B D_t}$ is less than $\frac{1}{2^t} n(W_A, W_B)$, and therefore less than $\frac{\epsilon}{3}$.

Thus we see that the arc length in $\mathbb{M}_1$ from A to B along the path $q = q_A \cup p \cup q_B$ is less than ϵ.

If $d(V, D_t) \leq r$, then it follows by the triangle inequality in euclidean geometry that $d(W_A, D_t) < d(W_A, V) + d(V, D_t) < r + r = 2r$, a contradiction. Therefore, it must be the case that D_t is in the exterior of $\mathcal{C}$, and therefore in the exterior of $\mathcal{P}$. Since D_t is a point on the path q, then q has nonempty intersection with the exterior of $\mathcal{P}$. □

References

[1] G. D. Birkhoff, A set of postulates for plane geometry based on scale and protractor, *Ann. of Math.*, 33 (1932) 329–345.

[2] H. Busemann, *The Geometry of Geodesics*, Academic Press Inc., New York (1955)

[3] J. Donnelly, The equivalence of side-angle-side and side-side-side in the absolute plane, *Journal of Geometry*, 106 (2015) 541–550.

[4] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History*, W. H. Freeman and Company, New York (1993)

[5] R. Hartshorne, *Geometry: Euclid and Beyond*, Springer-Verlag, New York (2000)

[6] D. Hilbert, *Grundlagen der Geometrie*, B. G. Teubner, Leipzig (1899)

[7] G. E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, Springer-Verlag, New York (1986)

[8] R. Millman and G. Parker, *Geometry: a Metric Approach with Models*, Springer-Verlag, New York (1991)

[9] E. E. Moise, *Elementary Geometry from an Advanced Standpoint*, Addison-Wesley, Reading (1990)

[10] A. Papadopoulos, On Hilbert's fourth problem, arXiv:1312.3172

[11] Z. I. Szabo, Hilbert's Fourth Problem, I, *Advances in Mathematics*, 59 (1986) 185–301.

John Donnelly: Department of Mathematics, University of Southern Indiana, 8600 University Boulevard, Evansville, Indiana 47712, USA

E-mail address: `jrdonnelly@usi.edu`

Forum Geometricorum
Volume 18 (2018) 275–296.

FORUM GEOM
ISSN 1534-1178

A Model of Nowhere Geodesic Plane Geometry in which the Triangle Inequality Fails Everywhere

John Donnelly

Abstract. Continuing with the results from an earlier paper, we construct a model $\mathbb{M}_2$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, in the model $\mathbb{M}_2$, the triangle inequality always fails for any triple of noncollinear points.

1. Introduction

In [2] the author constructs a model $\mathbb{M}_1$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), yet in which the geodesic line segment connecting any two points A and B is never the shortest path from A to B. Moreover, it is shown in [2] that the model $\mathbb{M}_1$ is continuous in the sense that it satisfies both the ruler postulate and protractor postulate of Birkhoff [1], [4]. However, depending on the order in which we choose points while constructing $\mathbb{M}_1$, there might exist a specific triangle $\triangle ABC$ in $\mathbb{M}_1$ that satisfies the triangle inequality in the sense that the sum of the lengths of any two sides of $\triangle ABC$ is greater than the length of the third side. However, given any two of the vertices of $\triangle ABC$, and given $\epsilon > 0$, then we can always find a path connecting those two vertices whose arc length in $\mathbb{M}_1$ is less than ϵ.

In this paper, we construct a model $\mathbb{M}_2$ of plane geometry that satisfies all of Hilbert's axioms for the euclidean plane (with the exception of Sided-Angle-Side), and which is Nowhere Geodesic, yet in which no triangle satisfies the triangle inequality. We note that the model $\mathbb{M}_2$ is not continuous in the sense that neither the ruler postulate nor the protractor postulate hold in $\mathbb{M}_2$. We refer the reader to [2] and [3] for a list of Hilbert's axioms for the euclidean plane.

2. The Model $\mathbb{M}_2$

In this section we construct the model $\mathbb{M}_2$. The points in $\mathbb{M}_2$ are precisely the points in $\mathbb{Q}^2$. A line in $\mathbb{M}_2$ is the intersection of $\mathbb{Q}^2$ with a line in $\mathbb{R}^2$ whose equation is either $y = mx + b$, where $m, b \in \mathbb{Q}$, or else $x = k$, where $k \in \mathbb{Q}$. We define betweenness in $\mathbb{M}_2$ to be exactly the same as betweenness in the euclidean

Publication Date: June 20, 2018. Communicating Editor: Paul Yiu.

plane. As in [2], we will replace PASCH with PSP as a betweenness axiom, and will show that $\mathbb{M}_2$ satisfies PSP. It follows immediately that betweenness axioms (1) - (3) hold in $\mathbb{M}_2$. We leave it to the reader to check that the incidence axioms hold in $\mathbb{M}_2$.

We prove that PSP holds in $\mathbb{M}_2$. We will essentially prove part (3) of the following lemma while showing that PSP holds in $\mathbb{M}_2$.

Lemma 1. *Given two distinct lines l and q in $\mathbb{M}_2$, then exactly one of the following is true:*

(1) *neither l nor q is vertical, and they have the same slope and are therefore parallel*
(2) *l and q are both vertical, and are therefore parallel*
(3) *l and q intersect at a unique point $P \in \mathbb{Q}^2$*

Proof of PSP: Let l be a line in $\mathbb{M}_2$. First assume that l is not vertical, and that l has equation $y = mx + b$, where $m, b \in \mathbb{Q}$. Let $\mathcal{H}_1 = \{(c, d) \in \mathbb{Q}^2 \mid d > mc + b\}$ and let $\mathcal{H}_2 = \{(c, d) \in \mathbb{Q}^2 \mid d < mc + b\}$. It follows immediately that $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonempty, disjoint subsets of $\mathbb{Q}^2$ whose union is precisely the set of points in $\mathbb{Q}^2$ that are not on l. Let $B_1 = (u_1, v_1)$ and $B_2 = (u_2, v_2)$ be two points in $\mathbb{Q}^2$ such that B_1 and B_2 are not on l.

Assume that $\overleftrightarrow{B_1B_2}$ is not vertical. Thus, $u_1 \neq u_2$. Let $\overleftrightarrow{B_1B_2}$ have equation $y = rx + s$, where $r = \dfrac{v_2 - v_1}{u_2 - u_1} \in \mathbb{Q}$ and $s \in \mathbb{Q}$.

First assume that $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$. Since $r, s, m, b \in \mathbb{Q}$, then both of $\dfrac{s - b}{m - r}$ and $r\left(\dfrac{s - b}{m - r}\right) + s$ are in $\mathbb{Q}$. Thus, we see that the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point

$$\left(\frac{s - b}{m - r}, r\left(\frac{s - b}{m - r}\right) + s\right) \in \mathbb{Q}^2$$

If $m = r$, then $\overleftrightarrow{B_1B_2}$ and l are parallel, which implies that either both of B_1 and B_2 are above l or else both of B_1 and B_2 are below l, a contradiction. Assume that $m \neq r$, and consequently, that $\overleftrightarrow{B_1B_2}$ and l are not parallel.

Assume that $m > r$. Since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m - r)(u_1) < s - b$, and therefore that $u_1 < \dfrac{s - b}{m - r}$. Since $B_2 \in \mathcal{H}_2$, then $ru_2 + s = v_2 < mu_2 + b$. This implies that $(m - r)(u_2) > s - b$, and therefore that $u_2 > \dfrac{s - b}{m - r}$. Thus, $u_1 < \dfrac{s - b}{m - r} < u_2$. Since $\dfrac{s - b}{m - r}$ is between u_1 and u_2, and since $y = rx + s$ is linear, then $\left(\dfrac{s - b}{m - r}, r\left(\dfrac{s - b}{m - r}\right) + s\right)$ is between B_1 and B_2 on the segment $\overline{B_1B_2}$.

Now assume that $m < r$. Again, since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m - r)(u_1) < s - b$, and therefore that $u_1 > \dfrac{s - b}{m - r}$. Since $B_2 \in \mathcal{H}_2$, then $ru_2 + s = v_2 < mu_2 + b$. This implies that $(m - r)(u_2) > s - b$.

and therefore that $u_2 < \dfrac{s-b}{m-r}$. Thus, $u_1 > \dfrac{s-b}{m-r} > u_2$. Thus, we again have that $\left(\dfrac{s-b}{m-r}, r\left(\dfrac{s-b}{m-r}\right)+s\right)$ is between B_1 and B_2 on the segment $\overline{B_1B_2}$.

Now assume that B_1 and B_2 are both in $\mathcal{H}_1$. Since $u_1 \neq u_2$, then we may assume without loss of generality that $u_1 < u_2$. The proof for the case $u_1 > u_2$ is similar and is left to the reader.

Again, if $m = r$, then $\overleftrightarrow{B_1B_2}$ and l are parallel, which implies that $\overleftrightarrow{B_1B_2}$ does not intersect l.

Assume that $m > r$. Since $B_2 \in \mathcal{H}_1$, then $ru_2 + s = v_2 > mu_2 + b$. This implies that $(m-r)(u_2) < s-b$, and therefore that $u_1 < u_2 < \dfrac{s-b}{m-r}$.

Assume that $m < r$. Since $B_1 \in \mathcal{H}_1$, then $ru_1 + s = v_1 > mu_1 + b$. This implies that $(m-r)(u_1) < s-b$, and therefore that $u_2 > u_1 > \dfrac{s-b}{m-r}$.

In either case, since $y = rx + s$ is linear, then segment $\overline{B_1B_2}$ does not intersect l.

If B_1 and B_2 are both in $\mathcal{H}_2$, then one can use a similar argument to show that segment $\overline{B_1B_2}$ does not intersect l.

Now assume that $\overleftrightarrow{B_1B_2}$ is vertical. Thus, $u_1 = u_2$, and $\overleftrightarrow{B_1B_2}$ has equation $x = u_1$.

Assume that $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$. In this case, the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point $(u_1, mu_1 + b) \in \mathbb{Q}^2$. Since $B_1 \in \mathcal{H}_1$ and $B_2 \in \mathcal{H}_2$, then $v_1 > mu_1 + b > v_2$. Thus, it follows that $(u_1, mu_1 + b)$ is between that points B_1 and B_2 on the segment $\overline{B_1B_2}$.

Assume that $B_1, B_2 \in \mathcal{H}_1$. We may assume without loss of generality that $v_1 > v_2$. Again, the lines $\overleftrightarrow{B_1B_2}$ and l intersect at the point $(u_1, mu_1 + b) \in \mathbb{Q}^2$. Since $B_1, B_2 \in \mathcal{H}_1$ and $v_1 > v_2$, then $v_1 > v_2 > mu_1 + b$. Thus, the point $(u_1, mu_1 + b)$ is not between B_1 and B_2. If B_1 and B_2 are both in $\mathcal{H}_2$, then one can use a similar argument to show that segment $\overline{B_1B_2}$ does not intersect l.

Now assume that l is vertical, and that l has equation $x = k$, where $k \in \mathbb{Q}$. Let $\mathcal{H}_3 = \{(c,d) \in \mathbb{Q}^2 \mid c > k\}$ and let $\mathcal{H}_4 = \{(c,d) \in \mathbb{Q}^2 \mid c < k\}$. Again, we leave it to the reader to confirm that $\mathcal{H}_3$ and $\mathcal{H}_4$ are nonempty, disjoint subsets of $\mathbb{Q}^2$ whose union is precisely the set of points in $\mathbb{Q}^2$ that are not on l. Let $B_3 = (u_3, v_3)$ and $B_4 = (u_4, v_4)$ be two points in $\mathbb{Q}^2$ such that B_3 and B_4 are not on l.

Assume that $B_3 \in \mathcal{H}_3$ and $B_4 \in \mathcal{H}_4$. In this case, $\overleftrightarrow{B_3B_4}$ can not be vertical. Let $\overleftrightarrow{B_3B_4}$ have equation $y = px + d$, where $p, d \in \mathbb{Q}$. In this case, the lines $\overleftrightarrow{B_3B_4}$ and l intersect at the point $(k, pk + d) \in \mathbb{Q}^2$. Since $B_3 \in \mathcal{H}_3$ and $B_4 \in \mathcal{H}_4$, then $u_3 > k > u_4$. Thus, it follows that $(k, pk + d)$ is between that points B_3 and B_4 on the segment $\overline{B_3B_4}$.

Finally, assume that $B_3, B_4 \in \mathcal{H}_3$. If $\overleftrightarrow{B_3B_4}$ is vertical, then segment $\overline{B_3B_4}$ and line l do not intersect. Assume that $\overleftrightarrow{B_3B_4}$ is not vertical. Again, we assume that $\overleftrightarrow{B_3B_4}$ has equation $y = px + d$, where $p, d \in \mathbb{Q}$. We may assume without loss of generality that $u_3 > u_4$. Since $B_3, B_4 \in \mathcal{H}_3$, then $u_3 > u_4 > k$. Thus, it follows that $(k, pk+d)$ is not between that points B_3 and B_4 on the segment $\overline{B_3B_4}$. If B_3 and B_4 are both in $\mathcal{H}_4$, then one can use a similar argument to show that segment $\overline{B_3B_4}$ does not intersect l. Hence, PSP holds in $\mathbb{M}_2$.

3. Angle Measure in $\mathbb{M}_2$

We define angle measure in $\mathbb{M}_2$ to be the same as euclidean angle measure. It follows immediately that Congruence Axiom (2) for Angles holds in $\mathbb{M}_2$. Moreover, given an angle $\angle BAC$ in $\mathbb{M}_2$ and given a ray $\overrightarrow{DF}$ in $\mathbb{M}_2$, then there exists a unique ray $\overrightarrow{DE}$ in $\mathbb{R}^2$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle FDE$. It remains only to show that ray $\overrightarrow{DE}$ is not only a ray in $\mathbb{R}^2$, but more specifically is a ray in $\mathbb{M}_2$. To show this, we show that either $\overleftrightarrow{DE}$ is vertical or else that the slope of line $\overleftrightarrow{DE}$ is a rational number.

Since both the slope of a line and euclidean angle measure are preserved under horizontal or vertical translation, then we may assume that D is the origin (0,0). Similarly, we may assume that the vertex A of $\angle BAC$ is the origin (0,0).

If the slope of a ray $\overrightarrow{DK}$ is $s \in \mathbb{Q}$, then the slope of the ray that we get by rotating $\overrightarrow{DK}$ clockwise around D through an angle of $\dfrac{\pi}{2}$ is $-\dfrac{1}{s} \in \mathbb{Q}$. Thus, we can assume that ray $\overrightarrow{DF}$ is either in the first quadrant or is equal to the positive x-axis. If $\overrightarrow{DF}$ is in the second, third, or fourth quadrant, then we rotate both rays $\overrightarrow{DF}$ and $\overrightarrow{DE}$ clockwise around D through an angle of $\dfrac{\pi}{2}$, π, or $\dfrac{3\pi}{2}$, respectively. In any of these three cases, the ray we get by rotating $\overrightarrow{DF}$ will be a ray in the first quadrant with a rational slope. Moreover, if the slope of the ray we get by rotating $\overrightarrow{DE}$ is a rational number, then the slope of $\overrightarrow{DE}$ is also a rational number. Since angle measure in euclidean geometry is preserved by rotation, then the angle measure of the new angle is precisely the same as the angle measure of the original angle $\angle FDE$. Similarly, if $\overrightarrow{DF}$ is the positive y-axis, the negative x-axis, or the negative y-axis, then we again rotate both rays $\overrightarrow{DF}$ and $\overrightarrow{DE}$ clockwise around D through an angle of $\dfrac{\pi}{2}$, π, or $\dfrac{3\pi}{2}$, respectively. In any of these three cases, the ray we get by rotating $\overrightarrow{DF}$ will be the positive x-axis.

Finally, we assume that $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are above the line $\overleftrightarrow{DF}$. The case where the ray $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are below the line $\overleftrightarrow{DF}$ is similar, and is left to the reader.

Let $\angle BAC$ and $\angle FDE$ be two angles in $\mathbb{R}^2$ that are congruent. As above, we assume that the vertices A and D of angles $\angle BAC$ and $\angle FDE$, respectively, are the origin (0,0). Moreover, by rotating $\overrightarrow{AB}$ if necessary, we may assume that $\overrightarrow{AB}$ is the positive x-axis. By using the reflection in the x-axis and reflecting $\overrightarrow{AC}$ to its mirror image $\overrightarrow{AC'}$, if necessary, then we may assume that $\overrightarrow{AC}$ is above the x-axis. In particular, if the slope of $\overrightarrow{AC}$ is $s \in \mathbb{Q}$, then the slope of $\overrightarrow{AC'}$ is

$-s \in \mathbb{Q}$. Also, since the reflection in a line preserves euclidean angle measure, then $m\angle BAC = m\angle BAC'$. We denote the common angle measure of $\angle BAC$ and $\angle FDE$ by λ. We assume that ray $\overrightarrow{DF}$ has rational slope, and is therefore a ray in $\mathbb{M}_2$. We will show that ray $\overrightarrow{AC}$ is in $\mathbb{M}_2$ if and only if ray $\overrightarrow{DE}$ is in $\mathbb{M}_2$.

If $\lambda = \dfrac{\pi}{2}$, then $\overrightarrow{AC}$ is vertical, and the slope of $\overrightarrow{DE}$ is the negative reciprocal of the slope of $\overrightarrow{DF}$, which implies that the slope of $\overrightarrow{DE}$ is rational. In this case, $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are both in $\mathbb{M}_2$.

Assume that $\lambda \neq \dfrac{\pi}{2}$.

If $\overrightarrow{DF}$ is horizontal (and therefore the positive x-axis), then $\angle BAC$ and $\angle FDE$ are the same angle. In this case, $\overrightarrow{DE}$ and $\overrightarrow{AC}$ are the same ray, and therefore have the same rational slope.

Assume that $\overrightarrow{DF}$ is not horizontal. As above, we may assume that $\overrightarrow{DF}$ is in the first quadrant, and that $\overrightarrow{DE}$ is in the halfplane of line $\overleftrightarrow{DF}$ consisting of all points that are above the line $\overleftrightarrow{DF}$.

Let $\rho = m\angle BDF$.

First assume that $\overrightarrow{AC}$ is in $\mathbb{M}_2$. Since $\lambda \neq \dfrac{\pi}{2}$, then $\overrightarrow{AC}$ is not vertical. Thus, it must be the case that the slope of $\overrightarrow{AC}$ is rational.

If $\overrightarrow{DE}$ is vertical (i.e. the positive y-axis), then it has no slope, but instead has equation $x = 0$, and is therefore a ray in $\mathbb{M}_2$.

If $\overrightarrow{DE}$ is horizontal (i.e. the negative x-axis), then it has slope 0, and is therefore a ray in $\mathbb{M}_2$.

Assume that $\overrightarrow{DE}$ is neither the positive y-axis nor the negative x-axis. Since $\overrightarrow{AB}$ is the positive x-axis, then the slope of $\overrightarrow{AC}$ is Tan(λ), the slope of $\overrightarrow{DF}$ is Tan(ρ), and the slope of $\overrightarrow{DE}$ is Tan(λ + ρ). Since the slopes of $\overrightarrow{AC}$ and $\overrightarrow{DF}$ are rational, then it follows by the formula

$$\text{Tan}(\lambda + \rho) = \frac{\text{Tan}(\lambda) + \text{Tan}(\rho)}{1 - \text{Tan}(\lambda)\text{Tan}(\rho)}$$

that Tan$(\lambda + \rho)$ is rational. Thus, the slope of $\overrightarrow{DE}$ is rational.

Thus, in any of these cases, we have that $\overrightarrow{DE}$ is a ray in $\mathbb{M}_2$.

Next, assume that $\overrightarrow{DE}$ is in $\mathbb{M}_2$.

First assume that $\overrightarrow{DE}$ is vertical (i.e. the positive y-axis). In this case, we use the reflection in the line $y = x$ to reflect both $\overrightarrow{DF}$ and $\overrightarrow{DE}$ to their mirror images $\overrightarrow{DF'}$ and $\overrightarrow{DE'}$, respectively. Since the ray $\overrightarrow{DE}$ is the positive y-axis, then $\overrightarrow{DE'}$= $\overrightarrow{AB}$. Since the reflection in a line preserves angle measure, then the reflection of the angle $\angle FDE$ is the unique angle above the x-axis whose initial ray is $\overrightarrow{AB}$ and whose angle measure is λ. But this is precisely the angle $\angle BAC$. Thus, we see that $\angle F'DE'$ = $\angle BAC$. More specifically, $\overrightarrow{DE'}$ = $\overrightarrow{AB}$ and $\overrightarrow{DF'}$= $\overrightarrow{AC}$. If the slope of $\overrightarrow{DF}$ is $s \in \mathbb{Q}$, then the slope of $\overrightarrow{DF'}$ is $\dfrac{1}{s} \in \mathbb{Q}$. Since $\overrightarrow{DF'} = \overrightarrow{AC}$, then it follows that $\overrightarrow{AC}$ is in $\mathbb{M}_2$.

Now assume that that $\overrightarrow{DE}$ is not vertical. Thus, in this case the slope of $\overrightarrow{DE}$ is rational. Moreover, since $\overrightarrow{DE}$ is not vertical, then $\lambda + \rho \neq \dfrac{\pi}{2}$.

As above, we have that the slope of $\overrightarrow{AC}$ is Tan(λ), the slope of $\overrightarrow{DF}$ is Tan(ρ), and the slope of $\overrightarrow{DE}$ is Tan($\lambda + \rho$). Since the slopes of $\overrightarrow{DF}$ and $\overrightarrow{DE}$ are rational, then it follows by the formula

$$\text{Tan}(\lambda) = \frac{\text{Tan}(\lambda+\rho) - \text{Tan}(\rho)}{1 + \text{Tan}(\lambda+\rho)\text{Tan}(\rho)}$$

that the slope of $\overrightarrow{AC}$ is also rational. Thus, $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$.

Let $\angle HKG$ be an angle in $\mathbb{M}_2$, and let $\overrightarrow{DF}$ be a ray in $\mathbb{M}_2$. As stated above, there exists a unique ray $\overrightarrow{DE}$ in $\mathbb{R}^2$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle HKG \cong \angle FDE$. If, as above, we let $\overrightarrow{AB}$ denote the positive x-axis, then there exists a unique ray $\overrightarrow{AC}$ in $\mathbb{R}^2$ above the x-axis such that $\angle BAC \cong \angle HKG$. Since $\angle HKG$ is an angle in $\mathbb{M}_2$, then it follows that both rays $\overrightarrow{KH}$ and $\overrightarrow{KG}$ are in $\mathbb{M}_2$. Applying the previous argument to the angles $\angle BAC$ and $\angle HKG$, it follows that $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$. Applying the previous argument to the angles $\angle BAC$ and $\angle FDE$, we see that since $\overrightarrow{AC}$ is a ray in $\mathbb{M}_2$, then $\overrightarrow{DE}$ is a ray in $\mathbb{M}_2$. Thus, $\angle FDE$ is an angle in $\mathbb{M}_2$.

Hence, it follows that Congruence Axiom (i) for Angles holds in $\mathbb{M}_2$.

4. Distance in $\mathbb{M}_2$

In this section we define distance in $\mathbb{M}_2$. We do this in such a way so that not only is $\mathbb{M}_2$ nowhere geodesic, but moreover, so that no triangle in $\mathbb{M}_2$ satisfies the triangle inequality. To define distance, we use the enumeration on the points in $\mathbb{Q}^2$. In particular, we start by defining distance between P_1 and P_2. We then define the distance between each subsequent point P_k in the sequence (P_j) and all of the points $P_1, \ldots, P_{k-1}$ that come before P_k in (P_j). As we define these distances, we also construct a subsequence $(K_{i,j})$ of points from (P_j), which is used to ensure that $\mathbb{M}_2$ is nowhere geodesic. The distance between a point P_t in (P_j) and a corresponding point $K_{t,j}$ in $(K_{i,j})$ is defined to be sufficiently small so that the segments in the form $\overline{P_t K_{t,i}}$ can be used to construct shorter and shorter paths in $\mathbb{M}_2$. On the other hand, certain distances in $\mathbb{M}_2$ are defined to be larger and larger as we use points further out in the sequence (P_j). Thus, we can think of distances in $\mathbb{M}_2$ as being defined with two opposite goals in mind. One goal is to have certain distances get larger and larger without bound, and the other goal is to have certain distances getting smaller and smaller and closer to 0. We denote the euclidean distance between point A and point B by $d(A, B)$. We denote the taxicab distance between point A and point B by $t(A, B)$ We denote the distance in $\mathbb{M}_2$ between point A and point B by $g(A, B)$.

We start with the following lemma.

Lemma 2. *Let $\{A_1, \ldots, A_k\} \subseteq \mathbb{Q}^2$ be a nonempty finite subset of $\mathbb{Q}^2$. Let P_i and P_l be two distinct points in $\mathbb{Q}^2$. Assume that $T_0, T_1, \ldots, T_m$ are points on $\overline{P_i P_l}$ such that*

(1) *$T_0 = P_i$ and $T_m = P_l$*
(2) *If $m \geq 2$, then for each $d \in \{1, \ldots, m-1\}$, we have that $T_{d-1} - T_d - T_{d+1}$*
(3) *If $m \geq 2$, then for each $d \in \{1, \ldots, m-1\}$, we have that T_d is the point of intersection of $\overline{P_i P_l}$ with one of the lines $\overleftrightarrow{A_h A_b}$ where $1 \leq h, b \leq k$ and $\overleftrightarrow{A_h A_b} \neq \overleftrightarrow{P_i P_l}$*
(4) *For each $d \in \{1, \ldots, m\}$, none of the lines $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$ and $\overleftrightarrow{A_h A_b} \neq \overleftrightarrow{P_i P_l}$, intersect int$\left(\overline{T_{d-1} T_d}\right)$*

Then for each $d \in \{1, \ldots, m\}$, there exists a triangle $\triangle T_{d-1} T_d Q_d$ such that

(1) *No line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_{d-1} T_d Q_d)$.*
(2) *No segment in the form $\overline{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_{d-1} T_d Q_d)$.*

Proof. We will prove this for $d = 1$. The proofs for the other possible values of d are similar, and are left to the reader. Let $\mathcal{H}$ denote a halfplane of $\overleftrightarrow{P_i P_l}$. If none of the points $A_1, \ldots, A_k$ are in $\mathcal{H}$, then we let X be any point in $\mathcal{H} \cap \mathbb{Q}^2$. Even though none of the points $A_1, \ldots, A_k$ are in $\mathcal{H}$, it might still be the case that one of the lines $\overleftrightarrow{A_h A_b}$ intersects int$(\angle T_1 T_0 X)$.

First assume that no line in the form $\overleftrightarrow{A_z A_w}$, where $1 \leq z, w \leq k$, passes through T_0 and intersects int$(\angle T_1 T_0 X)$.

Assume that no line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$\left(\overrightarrow{T_0 X}\right)$. Let Q_1 be any point on ray $\overrightarrow{T_0 X}$ such that $Q_1 \neq T_0$. If there exists a line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, that intersects int$(\triangle T_0 T_1 Q_1)$, then it follows by the Line-Triangle Theorem together with Lemma 1 that $\overleftrightarrow{A_h A_b}$ must intersect the interior of at least one of the sides $\overline{T_0 T_1}$ or $\overline{T_0 Q_1}$, a contradiction [4]. Thus, no line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_0 T_1 Q_1)$. In particular, no segment in the form $\overline{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_0 T_1 Q_1)$.

Next assume that there exists a line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, that intersects int$\left(\overrightarrow{T_0 X}\right)$. Let $L_1, \ldots, L_r$ denote all the points of intersection of int$\left(\overrightarrow{T_0 X}\right)$ with lines in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$. Since there exists only a finite number of lines in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, then there exist only a finite number of such points $L_1, \ldots, L_r$ of intersection. Let L_u denote the point on int$\left(\overrightarrow{T_0 X}\right)$ such that none of the other points from $L_1, \ldots, L_r$ are between T_0 and L_u. That is, for each $j \neq u$, L_j is not between T_0 and L_u. Since there exist only a finite number of points $L_1, \ldots, L_r$ of intersection, then such a point L_u exists. One can now use an argument similar to the one given above to show that no line in the form $\overleftrightarrow{A_h A_b}$, where $1 \leq h, b \leq k$, intersects int$(\triangle T_1 T_0 L_u)$.

Now assume that there exists a line in the form $\overleftrightarrow{A_z A_w}$, where $1 \leq z, w \leq k$, that passes through T_0 and intersects int$(\angle T_1 T_0 X)$. Since none of the points

$A_1, \ldots, A_k$ are in $\mathcal{H}$, then it follows that all points on $\overleftrightarrow{A_z A_w}$ in $\operatorname{int}(\angle T_1T_0X)$ are on the opposite side of $\overleftrightarrow{P_iP_l}$ as $A_1, \ldots, A_k$. Let $\mathcal{S}$ denote the set of all angles in the form $\angle T_1T_0B$ such that B is in $\operatorname{int}(\angle T_1T_0X)$ and such that B is on a line $\overleftrightarrow{A_zA_w}$, where $1 \le z, w \le k$, that passes through T_0. Let D be a point in $\operatorname{int}(\angle T_1T_0X)$ such that D is on a line $\overleftrightarrow{A_zA_w}$, where $1 \le z, w \le k$, that passes through T_0 and such that $\angle T_1T_0D$ is the smallest angle in $\mathcal{S}$. We now apply arguments similar to those given above, using ray $\overrightarrow{T_0D}$ in the place of ray $\overrightarrow{T_0X}$, to show that triangle $\triangle T_1T_0D$ is such that no line in the form $\overleftrightarrow{A_hA_b}$, where $1 \le h, b \le k$ and $\overleftrightarrow{A_hA_b} \ne \overleftrightarrow{P_iP_l}$, intersects $\operatorname{int}(\triangle T_1T_0D)$.

Now assume that at least one of the points $A_1, \ldots, A_k$ is in $\mathcal{H}$. Let α denote the smallest angle in the form $\angle T_1T_0A_t$, where A_t is from the points $A_1, \ldots, A_k$ that are in $\mathcal{H}$. Since there are only a finite number of points from $A_1, \ldots, A_k$ that are in $\mathcal{H}$, then α is well defined. We now apply arguments similar to those given above, using ray $\overrightarrow{T_0A_t}$ in the place of ray $\overrightarrow{T_0X}$, to show that triangle $\triangle T_1T_0A_t$ is such that no line in the form $\overleftrightarrow{A_hA_b}$, where $1 \le h, b \le k$, intersects $\operatorname{int}(\triangle T_1T_0A_t)$. $\square$

The proof of the following lemma follows from the definitions of betweenness in euclidean geometry, taxicab geometry, and $\mathbb{M}_2$, and is left to the reader.

Lemma 3. *Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$ be three points in $\mathbb{Q}^2$. Then $A - B - C$ in euclidean geometry if and only if $A - B - C$ in taxicab geometry. Moreover, $A - B - C$ in taxicab geometry if and only if $A - B - C$ in $\mathbb{M}_2$.*

The proof of the following lemma follows from the definition of distance in taxicab geometry, and is left to the reader.

Lemma 4. *Let $A, B \in \mathbb{Q}^2$. Then $t(A, B) \in \mathbb{Q}$.*

Define $g(P_1, P_2) = 1$. Let $K_{1,2}$ be any point in $\mathbb{Q}^2$ such that $K_{1,2}$ is not on $\overleftrightarrow{P_1P_2}$. Define $g(P_1, K_{1,2}) = g(P_2, K_{1,2}) = \dfrac{1}{4}g(P_1, P_2) = \dfrac{1}{4}$. Note that the arc length in $\mathbb{M}_2$ of the path from P_1 to P_2 along $\overline{P_1, K_{1,2}}$ and $\overline{P_2, K_{1,2}}$ is $\dfrac{1}{2}g(P_1, P_2)$.

First assume that $P_3 = K_{1,2}$. In this case the distances $g(P_1, P_3)$ and $g(P_2, P_3)$ are already defined. In particular, we have from above that $g(P_1, P_3) = g(P_2, P_3)$ $= \dfrac{1}{4}g(P_1, P_2) = \dfrac{1}{4}$.

By Lemma 2, there exists a triangle $\triangle P_1P_3Q_0$ such that neither of the lines $\overleftrightarrow{P_1P_2}$ or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_0$. Let $K_{1,3}$ be a point in $\operatorname{int}(\triangle P_1P_3Q_0) \cap \mathbb{Q}^2$. Note that $K_{1,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. Therefore, neither $\overline{P_1K_{1,3}}$ nor $\overline{P_3K_{1,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \dfrac{1}{4}g(P_1, P_3) = \dfrac{1}{16}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\dfrac{1}{2}g(P_1, P_3) = \dfrac{1}{8}$.

Similarly, there exists a triangle $\triangle P_2P_3\hat{Q}_0$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3\hat{Q}_0$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3\hat{Q}_0) \cap \mathbb{Q}^2$. Note that $K_{2,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. Therefore, neither $\overline{P_2K_{2,3}}$ nor $\overline{P_3K_{2,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \dfrac{1}{4}g(P_2, P_3) = \dfrac{1}{16}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3) = \dfrac{1}{8}$.

Note that each of the points $K_{i,j}$ and $K_{i,j,t}$ defined below will at some step in the process be realized as one of the points P_i in the sequence (P_j) of points in $\mathbb{Q}^2$.

Now assume that $P_3 \neq K_{1,2}$. We have several cases when defining distance in $\mathbb{M}_2$ between P_3 and either of P_1 or P_2. First assume that P_3 is a point on $\overleftrightarrow{P_1P_2}$. We have three cases: $P_1 - P_3 - P_2$, $P_3 - P_1 - P_2$, or $P_1 - P_2 - P_3$.

Assume that $P_1 - P_3 - P_2$. In this case, since $P_1 - P_3 - P_2$ in both $\mathbb{M}_2$ and in taxicab geometry, then there exist $r_1, r_2 \in (0, 1) \cap \mathbb{Q}$ such that

(1) $r_1 + r_2 = 1$
(2) $t(P_1, P_3) = r_1 t(P_1, P_2)$
(3) $t(P_3, P_2) = r_2 t(P_1, P_2)$

Note that since $P_1 - P_3 - P_2$, then $t(P_1, P_2) = t(P_1, P_3) + t(P_3, P_2)$, which is consistent with $r_1 + r_2 = 1$.

Define $g(P_1, P_3) = r_1 g(P_1, P_2) = r_1$ and $g(P_3, P_2) = r_2 g(P_1, P_2) = r_2$.

By Lemma 2, there exists a triangle $\triangle P_1P_3Q_1$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_1$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_1) \cap \mathbb{Q}^2$. Note that $K_{1,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. Therefore, neither $\overline{P_1K_{1,3}}$ nor $\overline{P_3K_{1,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \dfrac{1}{4}g(P_1, P_3) = \dfrac{r_1}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\dfrac{1}{2}g(P_1, P_3)$.

Similarly, there exists a triangle $\triangle P_2P_3Q_2$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3Q_2$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_2) \cap \mathbb{Q}^2$. Note that $K_{2,3}$ is not on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. Therefore, neither $\overline{P_2K_{2,3}}$ nor $\overline{P_3K_{2,3}}$ are on any of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \dfrac{1}{4}g(P_2, P_3) = \dfrac{r_2}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3)$.

Now assume that $P_1 - P_2 - P_3$. In this case, we define $g(P_2, P_3) = 4$. In particular, we define $g(P_2, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Thus, we define $g(P_2, P_3)$ so that $g(P_2, P_3) > 2s$, where $s = g(P_1, P_2)+g(P_1, K_{1,2})+g(P_2, K_{1,2})$. By Lemma 2, there exists a triangle $\triangle P_2P_3Q_3$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_2P_3Q_3$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_3)\cap \mathbb{Q}^2$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \frac{1}{4}g(P_2, P_3) = 1$.

We define $g(P_1, P_3)$ by means of segment addition. In particular, $g(P_1, P_3)$= $g(P_1, P_2) + g(P_2, P_3) = 1 + 4 = 5$. Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,2}}$, $\overline{K_{1,2}P_2}$, $\overline{P_2K_{2,3}}$, and $\overline{K_{2,3}P_3}$ is $\frac{1}{2}g(P_1, P_3) = \frac{5}{2}$.

Finally, assume that $P_3-P_1-P_2$. This case is similar to the previous case where $P_1 - P_2 - P_3$. In this case, we define $g(P_1, P_3) = 4$, and $g(P_2, P_3)$= $g(P_1, P_3) + g(P_2, P_1) = 4 + 1 = 5$. As above, there exists a triangle $\triangle P_1P_3Q_4$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_4$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_4) \cap \mathbb{Q}^2$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \frac{1}{4}g(P_1, P_3) = 1$. Again, note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{1,2}}$, $\overline{K_{1,2}P_1}$, $\overline{P_1K_{1,3}}$, and $\overline{K_{1,3}P_3}$ is $\frac{1}{2}g(P_2, P_3) = \frac{5}{2}$.

Now assume that P_1, P_2, and P_3 are not collinear. We have several cases, depending on whether $\overleftrightarrow{P_1K_{1,2}}$ or $\overleftrightarrow{P_2K_{1,2}}$ intersect either of the segments $\overline{P_1P_3}$ or $\overline{P_2P_3}$ at any points other than P_1 or P_2.

First assume that neither of the lines $\overleftrightarrow{P_1K_{1,2}}$ nor $\overleftrightarrow{P_2K_{1,2}}$ intersect either of the segments $\overline{P_1P_3}$ or $\overline{P_2P_3}$ at any points other than P_1 or P_2. In this case, we define $g(P_1, P_3) = 4$. Again, we define $g(P_1, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

By Lemma 2, there exists a triangle $\triangle P_1P_3Q_5$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_5$. Let $K_{1,3}$ be a point in int$(\triangle P_1P_3Q_5) \cap \mathbb{Q}^2$. We define $g(P_1, K_{1,3}) = g(P_3, K_{1,3}) = \frac{1}{4}g(P_1, P_3) = 1$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\frac{1}{2}g(P_1, P_3) = 2$.

We define $g(P_2, P_3) = 16$. Again, we define $g(P_2, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Applying Lemma 2, there exists a triangle $\triangle P_2P_3Q_6$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3}}$, or $\overleftrightarrow{P_3K_{1,3}}$ intersect the interior of $\triangle P_2P_3Q_6$. Let $K_{2,3}$ be a point in int$(\triangle P_2P_3Q_6) \cap \mathbb{Q}^2$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \frac{1}{4}g(P_2, P_3) = 4$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\frac{1}{2}g(P_2, P_3) = 8$.

Next assume that no three of P_1, P_2, P_3, and $K_{1,2}$ are collinear.

Assume that $\overleftrightarrow{P_2K_{1,2}}$ crosses $\overline{P_1P_3}$ at a point $I_{1,3}$, but that $\overleftrightarrow{P_1K_{1,2}}$ does not cross $\overline{P_2P_3}$. In this case, we define $g(P_1, I_{1,3}) = 4$. In particular, we define $g(P_1, I_{1,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

As above, applying Lemma 2, there exists a triangle $\triangle P_1I_{1,3}Q_7$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1I_{1,3}Q_7$. Let $K_{1,3,1}$ be a point in $\mathrm{int}(\triangle P_1I_{1,3}Q_7)\cap\mathbb{Q}^2$. We define $g(P_1, K_{1,3,1}) = g(I_{1,3}, K_{1,3,1}) = \frac{1}{4}g(P_1, I_{1,3}) = 1$. Note that the arc length in $\mathbb{M}_2$ of the path from P_1 to $I_{1,3}$ along segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}I_{1,3}}$ is $\dfrac{1}{2}g(P_1, I_{1,3}) = 2$.

We define $g(I_{1,3}, P_3) = 16$. Again, we define $g(I_{1,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

Again, applying Lemma 2, there exists a triangle $\triangle P_3I_{1,3}Q_8$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,1}}$ intersect the interior of $\triangle P_3I_{1,3}Q_8$. Let $K_{1,3,2}$ be a point in $\mathrm{int}(\triangle P_3I_{1,3}Q_8)\cap\mathbb{Q}^2$. We define $g(I_{1,3}, K_{1,3,2}) = g(P_3, K_{1,3,2}) = \frac{1}{4}g(P_3, I_{1,3}) = 4$. Note that the arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{1,3}$ along segments $\overline{P_3K_{1,3,2}}$ and $\overline{K_{1,3,2}I_{1,3}}$ is $\dfrac{1}{2}g(P_3, I_{1,3}) = 8$. Also, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,3,1}I_{1,3}}$, $\overline{I_{1,3}K_{1,3,2}}$ and $\overline{K_{1,3,2}P_3}$ is $\dfrac{1}{2}g(P_1, P_3)$.

We define $g(P_2, P_3) = 64$. As above, we define $g(P_2, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

Again, applying Lemma 2, there exists a triangle $\triangle P_2P_3Q_9$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$, $\overleftrightarrow{P_3K_{1,3,2}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,2}}$ pass through the interior of triangle $\triangle P_2P_3Q_9$. Let $K_{2,3}$ be a point in the interior of triangle $\triangle P_2P_3Q_9$. We define $g(P_2, K_{2,3}) = g(P_3, K_{2,3}) = \frac{1}{4}g(P_2, P_3) = 16$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along the segments $\overline{P_2K_{2,3}}$ and $\overline{P_3K_{2,3}}$ is $\dfrac{1}{2}g(P_2, P_3) = 32$.

We refer to the segment $\overline{P_1P_2}$ as the *initial segment*. We refer to segments such as $\overline{P_1I_{1,3}}$, $\overline{I_{1,3}P_3}$, and $\overline{P_2P_3}$ as *expansion segments*, since the length of each of these segments is defined by expanding to more than twice the length of all previous segments added together. Similarly, we refer to segments such as $\overline{P_1K_{1,2}}$, $\overline{P_2K_{1,2}}$, and $\overline{P_1K_{1,3}}$ $\overline{P_3K_{1,3}}$ as *contraction segments*, since the length of each of these segment is the contraction of the length of a previous segment.

Assume that $\overleftrightarrow{P_1K_{1,2}}$ crosses $\overline{P_2P_3}$ at a point $I_{2,3}$, but that $\overleftrightarrow{P_2K_{1,2}}$ does not cross $\overline{P_1P_3}$. In this case, we define $g(P_1, P_3) = 4$. As above, we define $g(P_1, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

By applying Lemma 2, there exists a triangle $\triangle P_1P_3Q_{10}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ pass through the interior of triangle $\triangle P_1P_3Q_{10}$. Let $K_{1,3}$ be a point in the interior of triangle $\triangle P_1P_3Q_{10}$. We define $g(P_1, K_{1,3}) =$

$g(P_3, K_{1,3}) = \frac{1}{4}g(P_1, P_3) = 1$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3}}$ and $\overline{P_3K_{1,3}}$ is $\dfrac{1}{2}g(P_1, P_3) = 2$.

We define $g(P_2, I_{2,3}) = 16$. Again, we define $g(P_2, I_{2,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_2I_{2,3}Q_{11}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3}}$, and $\overleftrightarrow{P_3K_{1,3}}$ pass through the interior of triangle $\triangle P_2I_{2,3}Q_{11}$. Let $K_{2,3,1}$ be a point in the interior of triangle $\triangle P_2I_{2,3}Q_{11}$. We define $g(P_2, K_{2,3,1}) = g(I_{2,3}, K_{2,3,1}) = \frac{1}{4}g(P_2, I_{2,3}) = 4$. Note that the arc length in $\mathbb{M}_2$ of the path from P_2 to $I_{2,3}$ along segments $\overline{P_2K_{2,3,1}}$ and $\overline{K_{2,3,1}I_{2,3}}$ is $\dfrac{1}{2}g(P_2, I_{2,3}) = 8$.

We define $g(I_{2,3}, P_3) = 64$. As above, we define $g(I_{2,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3I_{2,3}Q_{12}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3}}$, $\overleftrightarrow{P_3K_{1,3}}$, $\overleftrightarrow{P_2K_{2,3,1}}$, and $\overleftrightarrow{I_{2,3}K_{2,3,1}}$ pass through the interior of triangle $\triangle P_3I_{2,3}Q_{12}$. Let $K_{2,3,2}$ be a point in the interior of triangle $\triangle P_3I_{2,3}Q_{12}$. We define $g(I_{2,3}, K_{2,3,2}) = g(P_3, K_{2,3,2}) = \frac{1}{4}g(P_3, I_{2,3})$ = 16. Note that the arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{2,3}$ along segments $\overline{P_3K_{2,3,2}}$ and $\overline{K_{2,3,2}I_{2,3}}$ is $\dfrac{1}{2}g(P_3, I_{2,3}) = 32$. Also, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along segments $\overline{P_2K_{2,3,1}}$, $\overline{K_{2,3,1}I_{2,3}}$, $\overline{I_{2,3}K_{2,3,2}}$ and $\overline{K_{2,3,2}P_3}$ is $\dfrac{1}{2}g(P_2, P_3)$.

Assume that $\overleftrightarrow{P_2K_{1,2}}$ crosses $\overline{P_1P_3}$ at a point $I_{1,3}$, and that $\overleftrightarrow{P_1K_{1,2}}$ crosses $\overline{P_2P_3}$ at a point $I_{2,3}$.

In this case, we define we define $g(P_1, I_{1,3}) = 4$. Again, we define $g(P_1, I_{1,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_1I_{1,3}Q_{13}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, and $\overleftrightarrow{P_2K_{1,2}}$ pass through the interior of triangle $\triangle P_1I_{1,3}Q_{13}$. Let $K_{1,3,1}$ be a point in the interior of triangle $\triangle P_1I_{1,3}Q_{13}$. We define $g(P_1, K_{1,3,1}) = g(I_{1,3}, K_{1,3,1}) = \frac{1}{4}g(P_1, I_{1,3}) = 1$.

We define $g(I_{1,3}, P_3) = 16$. Again, we define $g(I_{1,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3I_{1,3}Q_{14}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, and $\overleftrightarrow{I_{1,3}K_{1,3,1}}$ pass through the interior of triangle $\triangle P_3I_{1,3}Q_{14}$. Let $K_{1,3,2}$ be a point in the interior of triangle $\triangle P_3I_{1,3}Q_{14}$. We define $g(I_{1,3}, K_{1,3,2}) = g(P_3, K_{1,3,2}) = \frac{1}{4}g(P_3, I_{1,3})$ = 4.

We define $g(P_2, I_{2,3}) = 64$. Again, we define $g(P_2, I_{2,3})$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. There exists a triangle $\triangle P_2I_{2,3}Q_{15}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}, K_{1,3,2}}$, and $\overleftrightarrow{P_3K_{1,3,2}}$ pass through the interior of triangle $\triangle P_2 I_{2,3} Q_{15}$. Let $K_{2,3,1}$ be a point in the interior of triangle $\triangle P_2 I_{2,3} Q_{15}$. We define $g(P_2, K_{2,3,1}) = g(I_{2,3}, K_{2,3,1}) = \frac{1}{4} g(P_2, I_{2,3}) = 16.$

We define $g(I_{2,3}, P_3) = 256$. Again, we define $g(I_{2,3}, P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed.

There exists a triangle $\triangle P_3 I_{2,3} Q_{16}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1P_3}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1, K_{1,3,1}}$, $\overleftrightarrow{I_{1,3}K_{1,3,1}}$,$\overleftrightarrow{I_{1,3}, K_{1,3,2}}$, $\overleftrightarrow{P_3K_{1,3,2}}$ $\overleftrightarrow{P_2K_{2,3,1}}$, and $\overleftrightarrow{I_{2,3}K_{2,3,1}}$ pass through the interior of triangle $\triangle P_3 I_{2,3} Q_{16}$. Let $K_{2,3,2}$ be a point in the interior of triangle $\triangle P_3 I_{2,3} Q_{16}$. We define $g(I_{2,3}, K_{2,3,2}) = g(P_3, K_{2,3,2}) = \frac{1}{4} g(P_3, I_{2,3})$ = 64.

The arc length in $\mathbb{M}_2$ of the path from P_1 to $I_{1,3}$ along segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}I_{1,3}}$ is $\dfrac{1}{2} g(P_1, I_{1,3})$. The arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{1,3}$ along segments $\overline{P_3K_{1,3,2}}$ and $\overline{K_{1,3,2}I_{1,3}}$ is $\dfrac{1}{2} g(P_3, I_{1,3})$. Also, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,3,1}I_{1,3}}$, $\overline{I_{1,3}K_{1,3,2}}$ and $\overline{K_{1,3,2}P_3}$ is $\dfrac{1}{2} g(P_1, P_3)$. The arc length in $\mathbb{M}_2$ of the path from P_2 to $I_{2,3}$ along segments $\overline{P_2K_{2,3,1}}$ and $\overline{K_{2,3,1}I_{2,3}}$ is $\dfrac{1}{2} g(P_2, I_{2,3})$. The arc length in $\mathbb{M}_2$ of the path from P_3 to $I_{2,3}$ along segments $\overline{P_3K_{2,3,2}}$ and $\overline{K_{2,3,2}I_{2,3}}$ is $\dfrac{1}{2} g(P_3, I_{2,3})$. Also, the arc length in $\mathbb{M}_2$ of the path from P_2 to P_3 along segments $\overline{P_2K_{2,3,1}}$, $\overline{K_{2,3,1}I_{2,3}}$, $\overline{I_{2,3}K_{2,3,2}}$ and $\overline{K_{2,3,2}P_3}$ is $\dfrac{1}{2} g(P_2, P_3)$.

Now assume that P_1, P_3, and $K_{1,2}$ are collinear. This case is similar to the case above where P_1, P_2, and P_3 are collinear. Note that in this present case, we are still assuming that P_1, P_2, and P_3 are noncollinear. We have the three possibilities $P_1 - P_3 - K_{1,2}$, $P_1 - K_{1,2} - P_3$, or $P_3 - P_1 - K_{1,2}$.

Assume that $P_1 - P_3 - K_{1,2}$. In this case, since $P_1 - P_3 - K_{1,2}$ in both $\mathbb{M}_2$ and in taxicab geometry, then there exist $r_1, r_2 \in (0,1) \cap \mathbb{Q}$ such that

(1) $r_1 + r_2 = 1$
(2) $t(P_1, P_3) = r_1 t(P_1, K_{1,2})$
(3) $t(P_3, K_{1,2}) = r_2 t(P_1, K_{1,2})$

Define $g(P_1, P_3) = r_1 g(P_1, K_{1,2})$ and $g(P_3, K_{1,2}) = r_2 g(P_1, K_{1,2})$.

There exists a triangle $\triangle P_1 P_3 Q_{17}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1 P_3 Q_{17}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1 P_3 Q_{17}) \cap \mathbb{Q}^2$. We define $g(P_1, K_{1,3,1}) = g(P_3, K_{1,3,1}) = \dfrac{1}{4} g(P_1, P_3) = \dfrac{r_1 g(P_1, K_{1,2})}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$ and $\overline{P_3K_{1,3,1}}$ is $\dfrac{1}{2} g(P_1, P_3)$.

Similarly, there exists a triangle $\triangle K_{1,2} P_3 Q_{18}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{P_3K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2} P_3 Q_{18}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2} P_3 Q_{18}) \cap \mathbb{Q}^2$. We define $g(K_{1,3,2}, K_{1,2}) = g(P_3, K_{1,3,2}) =$

$\dfrac{1}{4}g(K_{1,2},P_3) = \dfrac{r_2 g(P_1,K_{1,2})}{4}$. Thus, the arc length in $\mathbb{M}_2$ of the path from $K_{1,2}$ to P_3 along the segments $\overline{K_{1,2}K_{1,3,2}}$ and $\overline{P_3K_{1,3,2}}$ is $\dfrac{1}{2}g(K_{1,2},P_3)$.

Since $P_1 - P_3 - K_{1,2}$, and since P_1, P_2, and $K_{1,2}$ are not collinear, then P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. We define $g(P_2,P_3)$ to be a positive integer such that $g(P_2,P_3) > 2s$, where s is the sum of all previously constructed segments.

Now assume that $P_1 - K_{1,2} - P_3$. In this case, we define $g(K_{1,2},P_3) = 4$. In particular, we define $g(K_{1,2},P_3)$ to be a positive integer that is at least twice as large as the sum of all segments previously constructed. Thus, we define $g(K_{1,2},P_3)$ so that $g(K_{1,2},P_3) > 2s$, where $s = g(P_1,P_2) + g(P_1,K_{1,2}) + g(P_2,K_{1,2})$.

We define $g(P_1,P_3)$ by means of segment addition. In particular, $g(P_1,P_3)$= $g(P_1,K_{1,2}) + g(K_{1,2},P_3)$.

By Lemma 2, there exists a triangle $\triangle P_1K_{1,2}Q_{19}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1K_{1,2}Q_{19}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1K_{1,2}Q_{19}) \cap \mathbb{Q}^2$. We define $g(K_{1,2},K_{1,3,1}) = g(P_1,K_{1,3,1}) = \dfrac{1}{4}g(K_{1,2},P_1)$.

By Lemma 2, there exists a triangle $\triangle K_{1,2}P_3Q_{20}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{K_{1,2}K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2}P_3Q_{20}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2}P_3Q_{20}) \cap \mathbb{Q}^2$. We define $g(K_{1,2},K_{1,3,2}) = g(P_3,K_{1,3,2}) = \dfrac{1}{4}g(K_{1,2},P_3)$.

Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$, $\overline{K_{1,2}K_{1,3,1}}$, $\overline{K_{1,3,2}K_{1,2}}$, and $\overline{K_{1,3,2}P_3}$ is $\dfrac{1}{2}g(P_1,P_3)$.

Again, we have that P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. As above, we define $g(P_2,P_3) > 2s$, where s is the sum of all previously constructed segments.

Finally, assume that $P_3 - P_1 - K_{1,2}$. This case is similar to the previous case where $P_1 - K_{1,2} - P_3$.

In this case, we define $g(P_1,P_3) = 4$. Thus, we define $g(P_1,P_3)$ so that $g(P_1,P_3) > 2s$, where $s = g(P_1,P_2) + g(P_1,K_{1,2}) + g(P_2,K_{1,2})$.

We define $g(P_3,K_{1,2})$ by means of segment addition. In particular, $g(P_3,K_{1,2})$= $g(P_3,P_1) + g(P_1,K_{1,2})$.

As above, there exists a triangle $\triangle P_1P_3Q_{21}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, or $\overleftrightarrow{P_2K_{1,2}}$ intersect the interior of $\triangle P_1P_3Q_{21}$. Let $K_{1,3,1}$ be a point in int$(\triangle P_1P_3Q_{21}) \cap \mathbb{Q}^2$. We define $g(P_1,K_{1,3,1}) = g(P_3,K_{1,3,1}) = \dfrac{1}{4}g(P_1,P_3) = 1$. Again, note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to P_3 along the segments $\overline{P_1K_{1,3,1}}$ and $\overline{K_{1,3,1}P_3}$ is $\frac{1}{2}g(P_1,P_3)$.

By Lemma 2, there exists a triangle $\triangle K_{1,2}P_1Q_{22}$ such that none of the lines $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_1K_{1,2}}$, $\overleftrightarrow{P_2K_{1,2}}$, $\overleftrightarrow{P_1K_{1,3,1}}$, or $\overleftrightarrow{P_3K_{1,3,1}}$ intersect the interior of $\triangle K_{1,2}P_1Q_{22}$. Let $K_{1,3,2}$ be a point in int$(\triangle K_{1,2}P_1Q_{22}) \cap \mathbb{Q}^2$. We define $g(K_{1,2},K_{1,3,2}) = g(P_1,K_{1,3,2}) = \dfrac{1}{4}g(K_{1,2},P_1)$. Thus, the arc length in $\mathbb{M}_2$ of the path from P_1

to $K_{1,2}$ along the segments $\overline{P_1K_{1,3,2}}$ and $\overline{K_{1,3,2}K_{1,2}}$ is $\frac{1}{2}g(P_1, K_{1,2})$. Note that in this case, the arc length in $\mathbb{M}_2$ of the path from P_1 to $K_{1,2}$ along the segments $\overline{P_3K_{1,3,1}}$, $\overline{K_{1,3,1}P_1}$, $\overline{P_1K_{1,3,2}}$, and $\overline{K_{1,3,2}K_{1,2}}$ is $\dfrac{1}{2}g(P_3, K_{1,2})$.

Again, we have that P_3 is not a point on $\overleftrightarrow{P_2K_{1,2}}$. As above, we define $g(P_2, P_3) > 2s$, where s is the sum of all previously constructed segments.

The cases where P_2, P_3, and $K_{1,2}$ are collinear are similar to the cases where P_1, P_3, and $K_{1,2}$ are collinear, and are left to the reader.

Assume that there exist n points $P_1, \ldots P_n$ in $\mathbb{Q}^2$ (with $n \geq 3$) such that

(1) For each $i, j \in \{1, \ldots, n\}$, $g(P_i, P_j)$ is defined and is a rational number. More specifically, $g(P_i, P_j)$ is defined by using one of the methods given below.

(2) For each $i, j \in \{1, \ldots, n\}$, with $i \neq j$, if segment $\overline{P_iP_j}$ is not intersected by any previously constructed line, then there exists a point $K_{i,j}$ such that $g(P_i, K_{i,j}) = g(P_j, K_{i,j}) = \dfrac{1}{4}g(P_i, P_j)$, and such that $K_{i,j}$ is not on any other previously constructed line or previously constructed segment. Moreover, no other previously constructed line or previously constructed segment intersects the interior of either segment $\overline{P_iK_{i,j}}$ or $\overline{P_jK_{i,j}}$.

(3) Given $i, j \in \{1, \ldots, n\}$, with $i \neq j$, if there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_j$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_iP_j}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_hP_b}$, $\overleftrightarrow{P_hK_{w,u}}$, or $\overleftrightarrow{P_hK_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_dI_{d+1}}$), then for each $d \in \{0, \ldots, t\}$, there exist a point $K_{i,j,d}$ such that $g(I_d, K_{i,j,d}) = g(I_{d+1}, K_{i,j,d}) = \dfrac{1}{4}g(I_dI_{d+1})$, and such that $K_{i,j,d}$ is not on any other previously constructed line or previously constructed segment. Moreover, no other previously constructed line or previously constructed segment intersects the interior of either segment $\overline{I_dK_{i,j,d}}$ or $\overline{I_{d+1}K_{i,j,d}}$.

If P_j is one of the previously constructed points $K_{i,h}$, $K_{i,h,t}$, or I_d, then some (although not necessarily all) of the distances $g(P_i, P_j)$ involving P_i, along with other previously constructed points, will already be defined. However, in general, even if P_j = $K_{i,h}$, P_j = $K_{i,h,t}$, or P_j = I_d, then we will still need to define many of the distances $g(P_b, P_j)$ involving P_b, where $P_b \neq P_i$, using the methods given below. For the cases that follow, we assume that P_j is not one of the previously constructed points $K_{i,h}$, $K_{i,h,t}$, or I_d, so that $g(P_i, P_j)$ has not yet been defined. We also assume that $\overline{P_iP_j}$ is not the initial segment $\overline{P_1P_2}$, whose length is already defined to be 1. We now give the methods that must be used to compute $g(P_i, P_j)$. We assume that $i < j$.

(i) Assume that P_i and P_j are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, and that the interior of segment $\overline{P_iP_j}$ is not intersected by any previously constructed line. In this case we define

$g(P_i, P_j)$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments.

(ii) Assume that P_i and P_j are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, but that there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_j$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_i P_j}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_h P_b}$, $\overleftrightarrow{P_h K_{w,u}}$, or $\overleftrightarrow{P_h K_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_d I_{d+1}}$). In this case, for each $d \in \{0, \ldots, t\}$, we define $g(I_d, I_{d+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. We then define $g(P_i, P_j)$ using segment addition. In particular, we define $g(P_i, P_j)$ to be the sum $g(P_i, P_j) = \sum_{d=0}^{t} g(I_d, I_{d+1})$.

(iii) If there exists P_k (where $k < j$) such that $P_i - P_j - P_k$, and there are no other points P_b (where $b < j$) that are between either P_i and P_j or else between P_j and P_k, then $g(P_i, P_j)$ and $g(P_j, P_k)$ are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_j)$ and $t(P_j, P_k)$. That is, if $t(P_i, P_j) = r_1 t(P_i, P_k)$ and $t(P_j, P_k) = r_2 t(P_i, P_k)$, then $g(P_i, P_j) = r_1 g(P_i, P_k)$ and $g(P_j, P_k) = r_2 g(P_i, P_k)$. Note that r_1 and r_2 are positive rational numbers.

(iv) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_i - P_j - K_{i,h}$, such that $K_{i,h}$ was constructed previous to P_j, and such that there are no other points P_h (where $h < j$) that are between either P_i and P_j or else between P_j and $K_{i,h}$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

(v) If there exists P_k (where $k < j$) such that $P_j - P_i - P_k$, and there are no other points P_b (where $b < j$) that are between P_i and P_j, then we define $g(P_i, P_j)$ using one the methods above given in cases (i) through (iv), and we define $g(P_j, P_k)$ by means of segment addition.

(vi) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_j - P_i - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_j. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

(vii) We define $g(P_i, P_j)$ and $g(P_j, K_{i,h})$ similarly if there exists a point $K_{i,h}$ such that $P_j - K_{i,h} - P_i$ and such that $K_{i,h}$ was constructed previous to P_j. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

It follows by the way that distance is defined in $\mathbb{M}_2$ together with the way that the points $K_{i,j}$ and $K_{i,j,t}$ are chosen using Lemma 2 that there exists at most one contraction segment on a given line l. The remaining parts of l consist of either the initial segment $\overline{P_1 P_2}$ or else expansion segments.

Let P_{n+1} be the next point in the sequence of points (P_j) from $\mathbb{Q}^2$ that comes immediately after the points $P_1, \ldots P_n$. Let $i \in \{1, \ldots, n\}$. We assume that there are no other points P_h (where $h \leq n$) that are between P_i and P_{n+1}. If such points existed, then we would first define $g(P_h, P_{n+1})$, and then define $g(P_i, P_{n+1})$ by means of segment addition. There are several cases when defining the distance

$g(P_i, P_{n+1})$ in $\mathbb{M}_2$. They are similar to the cases given above, but are included here for the sake of completeness.

Case (i): If $P_{n+1} = K_{i,h}$, where $K_{i,h}$ is a previously constructed point used when constructing a contraction segment with P_i, then $g(P_i, P_{n+1})$ has already been defined. In this case, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \frac{1}{4}g(P_i, P_{n+1})$.

Assume that $P_{n+1} \neq K_{i,h}$, where $K_{i,h}$ is any of the previously constructed points used when constructing a contraction segment with P_i.

Case(ii): Assume that P_i and P_{n+1} are not collinear with any other previously constructed points in the form P_k (where $k \leq n$), $K_{i,h}$, or $K_{i,h,t}$, and that the interior of segment $\overline{P_i P_{n+1}}$ is not intersected by any previously constructed line. In this case we define $g(P_i, P_{n+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. As above, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \frac{1}{4}g(P_i, P_{n+1})$.

Case (iii): Assume that P_i and P_{n+1} are not collinear with any other previously constructed points in the form P_k (where $k < j$), $K_{i,h}$, or $K_{i,h,t}$, but that there exist points $I_0, I_1, I_2, \ldots, I_{t+1}$ such that $I_0 = P_i$, $I_{t+1} = P_{n+1}$, for each $d \in \{1, \ldots, t\}$, I_d is the point of intersection of $\overline{P_i P_{n+1}}$ with a previously constructed line (in one of the forms $\overleftrightarrow{P_h P_b}$, $\overleftrightarrow{P_h K_{w,u}}$, or $\overleftrightarrow{P_h K_{w,u,t}}$), for each $d \in \{1, \ldots, t\}$, $I_{d-1} - I_d - I_{d+1}$, and for each $d \in \{0, \ldots, t\}$, no previously constructed line intersects int($\overline{I_d I_{d+1}}$). In this case, for each $d \in \{0, \ldots, t\}$, we define $g(I_d, I_{d+1})$ to be a positive integer that is strictly greater that twice the sum of the lengths of all previously constructed segments. We then define $g(P_i, P_{n+1})$ using segment addition. In particular, we define $g(P_i, P_{n+1})$ to be the sum $g(P_i, P_{n+1}) = \sum_{d=0}^{t} g(I_d, I_{d+1})$. For each $d \in \{0, \ldots, t\}$, we apply Lemma 2 to get a point $K_{i,n+1,d}$ such that $g(I_d, K_{i,n+1,d}) = g(K_{i,n+1,d}, I_{d+1}) = \frac{1}{4}g(I_d, I_{d+1})$.

Case (iv): If there exists P_k (where $k \leq n$) such that $P_i - P_{n+1} - P_k$, and there are no other points P_b (where $b \leq n$) that are between P_k and P_{n+1}, then $g(P_i, P_{n+1})$ and $g(P_{n+1}, P_k)$ are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_{n+1})$ and $t(P_{n+1}, P_k)$. That is, if $t(P_i, P_{n+1}) = r_1 t(P_i, P_k)$ and $t(P_{n+1}, P_k) = r_2 t(P_i, P_k)$, then $g(P_i, P_{n+1}) = r_1 g(P_i, P_k)$ and $g(P_{n+1}, P_k) = r_2 g(P_i, P_k)$. Note that r_1 and r_2 are both positive rational numbers. As above, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4}g(P_i, P_{n+1})$ and such that $g(P_k, K_{i,n+1,2}) = g(P_{n+1}, K_{i,n+1,2}) = \frac{1}{4}g(P_k, P_{n+1})$.

Case(v): If there exists a point $K_{i,h}$ such that $P_i - P_{n+1} - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, and there are no other points P_b (where $b \leq n$) that are between P_{n+1} and $K_{i,h}$, then $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$

are defined using the same constants of proportionality r_1 and r_2 that are used when defining the taxicab distances $t(P_i, P_{n+1})$ and $t(P_{n+1}, K_{i,h})$. That is, if $t(P_i, P_{n+1}) = r_1 t(P_i, K_{i,h})$ and $t(P_{n+1}, K_{i,h}) = r_2 t(P_i, K_{i,h})$, then $g(P_i, P_{n+1}) = r_1 g(P_i, K_{i,h})$ and $g(P_{n+1}, K_{i,h}) = r_2 g(P_i, K_{i,h})$. Again, both r_1 and r_2 are positive rational numbers. Again, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(P_i, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_{n+1}, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_{n+1})$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

Case(vi): If there exists P_k (where $k \leq n$) such that $P_{n+1} - P_i - P_k$, then we define $g(P_i, P_{n+1})$ using one the methods above given in cases (ii), (iii), (iv), or (v), and we define $g(P_{n+1}, P_k)$ by means of segment addition. As above, we apply Lemma 2 to get a point $K_{i,n+1}$ such that $g(P_i, K_{i,n+1}) = g(P_{n+1}, K_{i,n+1}) = \frac{1}{4} g(P_i, P_{n+1})$. Note that a point $K_{i,k}$ such that $g(P_i, K_{i,k}) = g(P_k, K_{i,k}) = \frac{1}{4} g(P_i, P_k)$ has already been previously constructed.

Case(vii): If there exists a point $K_{i,h}$ such that $P_{n+1} - P_i - K_{i,h}$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, then we define $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$ similar to case (vi). Again, we apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(P_i, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(P_i, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_i, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_i)$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

Case(viii): If there exists a point $K_{i,h}$ such that $P_{n+1} - K_{i,h} - P_i$ and such that $K_{i,h}$ was constructed previous to P_{n+1}, then we again define $g(P_i, P_{n+1})$ and $g(P_{n+1}, K_{i,h})$ similar to case (vi). In this case the roles of P_i and $K_{i,h}$ are reversed from Case (vii). We apply Lemma 2 to get points $K_{i,n+1,1}$ and $K_{i,n+1,2}$ such that $g(K_{i,h}, K_{i,n+1,1}) = g(P_{n+1}, K_{i,n+1,1}) = \frac{1}{4} g(K_{i,h}, P_{n+1})$ and such that $g(K_{i,h}, K_{i,n+1,2}) = g(P_i, K_{i,n+1,2}) = \frac{1}{4} g(K_{i,h}, P_i)$. This is similar if we replace $K_{i,h}$ with $K_{i,h,t}$.

In any of the above cases, we see that $g(P_i, P_{n+1})$, and where appropriate $g(P_{n+1}, K_{i,h})$ or $g(P_{n+1}, K_{i,h,t})$, are defined. In this way we can recursively define distance between any two points in the model $\mathbb{M}_2$. We also see that between any two points P_i and P_j in $\mathbb{M}_2$, there exists a path whose arc length in $\mathbb{M}_2$ is $\frac{1}{2} g(P_i, P_j)$. By repeatedly applying Lemma 2, we can construct paths from P_i and P_j whose arc lengths in $\mathbb{M}_2$ are arbitrarily small. Thus, there is no shortest path in $\mathbb{M}_2$ from P_i to P_j, and consequently, $\mathbb{M}_2$ is nowhere geodesic.

We next show that the congruence axioms for line segments hold in $\mathbb{M}_2$.

It follows immediately by the way that distance $g(P_i, P_j)$ (and consequently congruence) is defined in $\mathbb{M}_2$ that congruence axioms (ii) and (iii) for line segments hold in $\mathbb{M}_2$.

We now prove congruence axiom (1) for line segments. Let $A, B, C \in \mathbb{Q}^2$, and let $r = \overrightarrow{CT}$ denote a ray in $\mathbb{M}_2$ originating at C. We will prove that there exists a unique point D on r such that $g(A,B) = g(C,D)$. If $A = B$, then we let $C = D$. Assume that $A \neq B$, and therefore that $g(A,B)$ is a strictly positive rational number.

We now construct a subsequence of points (P_{j_t}) from (P_j) that are on r. We define the interior of r to be all points on r other than the point C. Let $C = P_{j_0}$. Let P_{j_1} denote the first point from the sequence (P_j) that is in the interior of r. Let P_{j_2} denote the first point from the sequence (P_j) such that $C - P_{j_1} - P_{j_2}$. Note by our choices of the points P_{j_1} and P_{j_2} that it must be the case that P_{j_2} comes after P_{j_1} in the sequence (P_j). Assume that we have chosen points P_{j_0}, P_{j_1}, P_{j_2}, …, P_{j_t} from (P_j) such that

(1) For each $i = 0, \ldots, t-1$, we have that P_{j_i} comes before $P_{j_{i+1}}$ in the sequence (P_j).
(2) For each $i = 0, \ldots, t-2$, we have $P_{j_i} - P_{j_{i+1}} - P_{j_{i+2}}$.

In particular, for each $i = 1, \ldots, t-1$, $P_{j_{i+1}}$ is the first point from the sequence (P_j) such that $C - P_{j_i} - P_{j_{i+1}}$.

Let $P_{j_{t+1}}$ denote the first point from the sequence (P_j) such that $C - P_{j_t} - P_{j_{t+1}}$. Again, it must be the case that P_{j_t} comes before $P_{j_{t+1}}$ in the sequence (P_j).

Thus, we have a subsequence (P_{j_t}) of (P_j) such that

(1) For each $i \geq 1$, we have that P_{j_i} comes before $P_{j_{i+1}}$ in the sequence (P_j).
(2) For each i, we have $P_{j_i} - P_{j_{i+1}} - P_{j_{i+2}}$.

In particular, for each $i \geq 1$, $P_{j_{i+1}}$ is the first point from the sequence (P_j) such that $C - P_{j_i} - P_{j_{i+1}}$.

Let P_{j_b} denote the first point from the subsequence (P_{j_t}) such that $g(C, P_{j_b}) \geq g(A,B)$. Since there exists at most one contraction segment on line $\overleftrightarrow{CT}$, then it follows by the way that the lengths of expansion segments are constructed that such a point P_{j_b} exists. If $g(C, P_{j_b}) = g(A,B)$, then we let $D = P_{j_b}$. Assume that $g(C, P_{j_b}) > g(A,B)$. If $b = 1$, then let $v = 0$. Assume that $b \geq 2$. In this case, we let $v = \displaystyle\sum_{n=1}^{b-1} g(P_{j_{n-1}}, P_{j_n}) < g(A,B)$. Let $u = \dfrac{g(A,B) - v}{g(P_{j_{b-1}}, P_{j_b})} \in \mathbb{Q} \cap (0,1)$.

We leave it to the reader to check, using density of $\mathbb{Q}^2$ in $\mathbb{R}^2$, and therefore on the line $\overleftrightarrow{CT}$, that there exists a unique point D such that $D \in \mathbb{Q}^2$, such that D is between $P_{j_{b-1}}$ and P_{j_b}, and such that $t(P_{j_{b-1}}, D) = u(t(P_{j_{b-1}}, P_{j_b}))$.

It follow by the way that the subsequence (P_{j_t}) was constructed that D comes after P_{j_b} in the sequence (P_j). It now follows immediately by the way that distance is defined in $\mathbb{M}_2$ that D is the unique point in the interior of segment $\overline{P_{j_{b-1}} P_{j_b}}$ such that $g(P_{j_{b-1}}, D) = u(g(P_{j_{b-1}}, P_{j_b})) = g(A,B) - v$.

Moreover, it now follows immediately by the way that distance is defined in $\mathbb{M}_2$ (via segment addition) that D is the unique point on ray r such that $g(C,D) = v + u(g(P_{j_{b-1}}, P_{j_b})) = v + (g(A,B) - v) = g(A,B)$.

Thus, it follows that congruence axiom (1) for line segments holds in $\mathbb{M}_2$.

Lemma 5. *Given a triangle $\triangle P_iP_jP_t$, then exactly one of the following is true:*

(1) *At least one of the sides $\overline{P_iP_j}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined.*
(2) *At least one of the sides $\overline{P_iP_j}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined.*
(3) *The segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments such that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\dfrac{1}{2}g(P_i,P_j)$.*

Proof. By the way that the points $K_{i,j}$ and $K_{i,j,t}$ are chosen using Lemma 2, it can not be the case that all three sides of a triangle $\triangle P_iP_jP_t$ are subsegments of contraction segments. Suppose that $\overline{P_iP_j}$ is a subsegment of a contraction segment $\overline{P_{z_1}K_1}$, $\overline{P_iP_t}$ is a subsegment of a contraction segment $\overline{P_{z_2}K_2}$, and $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$. For this to happen, P_i is the point of intersection of $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$, and P_t is the point of intersection of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$. We may assume that K_3 is chosen and consequently $\overline{P_{z_3}K_3}$ is constructed using Lemma 2 after both of K_1 and K_2 are chosen and $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$ are constructed. However, when choosing K_3 to construct the contraction segment $\overline{P_{z_3}K_3}$, we apply Lemma 2 and choose K_3 so that no line containing a previously constructed contraction segment intersects $\overline{P_{z_3}K_3}$. This, contradicts the above statement that P_i is the point of intersection of $\overline{P_{z_1}K_1}$ and $\overline{P_{z_2}K_2}$, and P_t is the point of intersection of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$.

A similar argument applies if segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are subsegments of contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$, respectively, and the length of $\overline{P_iP_j}$ is defined before $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$ are constructed. In this case K_2 and K_3 are chosen using Lemma 2 so that no previously constructed line such as $\overleftrightarrow{P_iP_j}$ intersects either of $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$. However, in this case we have that P_i is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_2}K_2}$, and P_j is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_3}K_3}$, respectively, a contradiction. Thus, if segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are subsegments of contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$, respectively, then $\overline{P_iP_j}$ must contain, as a subsegment, an expansion segment $\overline{P_uP_v}$ whose length is defined after $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$ have been constructed. More generally, the length of $\overline{P_uP_v}$ is defined after the lengths of the other two sides of $\triangle P_iP_jP_t$ have been defined as proportions of the length of the contraction segments $\overline{P_{z_2}K_2}$ and $\overline{P_{z_3}K_3}$.

Moreover, a similar argument applies if the segment $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$, respectively, and the lengths of $\overline{P_iP_j}$ and $\overline{P_iP_t}$ are defined before $\overline{P_{z_3}K_3}$ is constructed. In this case K_3 is chosen using Lemma 2 so that no previously constructed lines such as $\overleftrightarrow{P_iP_j}$ or $\overleftrightarrow{P_iP_t}$ intersect $\overline{P_{z_3}K_3}$. However, in this case we have that P_j is the point of intersection of $\overleftrightarrow{P_iP_j}$ and $\overline{P_{z_3}K_3}$, and P_t is the point of intersection of $\overleftrightarrow{P_iP_t}$ and $\overline{P_{z_3}K_3}$, respectively, a contradiction. Thus, if segment $\overline{P_jP_t}$ is a subsegment of a contraction segment $\overline{P_{z_3}K_3}$, then one of $\overline{P_iP_j}$ or $\overline{P_iP_t}$ must contain, as a subsegment, an expansion segment $\overline{P_uP_v}$ whose length is defined after $\overline{P_{z_3}K_3}$ has been constructed. More generally, the length of $\overline{P_uP_v}$ is defined after the lengths of the other two sides of $\triangle P_iP_jP_t$ have been defined.

If $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are themselves contraction segments, then by the way that contraction segments are defined, we have that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\frac{1}{2}g(P_i, P_j)$. Note that this is the only case where $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments and the length of $\overline{P_iP_j}$ is defined before $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are constructed.

Next assume that there exist points K_4, K_5, and K_6 in the interiors of sides $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$, respectively, such that each of the segments $\overline{P_iK_4}$, $\overline{P_iK_5}$, and $\overline{P_jK_6}$ are either themselves contraction segments or else contain a contraction segment as a subsegment. In this case, each of $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$ contain as a subsegment at least one expansion segment. By the way that contraction segments are constructed using Lemma 2, it must be the case that these expansion segments are constructed after the contraction segments are constructed. Thus, in this case one of the sides contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

A similar result occurs if there exist two points K_5 and K_6 in the interiors of sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$, respectively, such that each of the segments $\overline{P_iK_5}$ and $\overline{P_jK_6}$ are either themselves contraction segments or else contain a contraction segment as a subsegment, and the length of $\overline{P_iP_j}$ is defined before the contraction segments are constructed. In this case one of the sides $\overline{P_iP_t}$ or $\overline{P_jP_t}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

Finally, if there exists a point K_6 in the interior of side $\overline{P_jP_t}$, respectively, such that the segment $\overline{P_jK_6}$ is either itself a contraction segment or else contains a contraction segment as a subsegment, and the lengths of $\overline{P_iP_t}$ and $\overline{P_iP_j}$ are defined before the contraction segment is constructed. In this case, the side $\overline{P_jP_t}$ contains as a subsegment an expansion segment whose length is defined after the lengths of the other two sides are defined.

The only remaining possibility is that one of the sides $\overline{P_iP_j}$, $\overline{P_iP_t}$, and $\overline{P_jP_t}$ is an expansion segment whose length is defined after the lengths of the other two sides are defined. □

Assume that we are given triangle $\triangle P_iP_jP_t$. Assume that one of the sides $\overline{P_iP_j}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined. In this case, $g(P_i, P_j)$ is strictly larger than twice the sum of all segments whose lengths have been defined previously. More precisely, $g(P_i, P_j) > 2s$, where s denotes the sum of the lengths of all segments whose lengths have been defined previously. Since $g(P_i, P_t)$ and $g(P_t, P_j)$ are comprised of segments whose lengths are defined prior to defining the length of $\overline{P_iP_j}$, then $s \geq g(P_i, P_t) + g(P_t, P_j)$. Thus, $g(P_i, P_j) > 2s \geq g(P_i, P_t) + g(P_t, P_j)$.

Next assume that at least one of the sides $\overline{P_iP_j}$ contains a subsegment $\overline{P_uP_v}$ such that $\overline{P_uP_v}$ is an expansion segment whose length is defined after the lengths of the other two sides $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are defined. Similar to the previous case above, we have that $g(P_u, P_v) > 2s$, where s denotes the sum of the lengths of all segments whose lengths have been defined previously. Again, since $g(P_i, P_t)$ and $g(P_t, P_j)$ are comprised of segments whose lengths are defined prior to defining the length of $\overline{P_uP_v}$, then $s \geq g(P_i, P_t) + g(P_t, P_j)$. Thus, $g(P_i, P_j) \geq g(P_u, P_v) > 2s \geq g(P_i, P_t) + g(P_t, P_j)$.

Next assume that $\overline{P_iP_t}$ and $\overline{P_jP_t}$ are contraction segments such that the arc length in $\mathbb{M}_2$ along the segments $\overline{P_iP_t}$ and $\overline{P_jP_t}$ is $\dfrac{1}{2}g(P_i, P_j)$. In this case, $g(P_i, P_j) > \frac{1}{2}g(P_i, P_j) = g(P_i, P_t) + g(P_t, P_j)$.

Thus, in any of these cases, the triangle inequality fails for $\triangle P_iP_jP_t$ in $\mathbb{M}_2$.

References

[1] G. D. Birkhoff, A set of postulates for plane geometry based on scale and protractor, *Ann. of Math.*, 33 (1932) 329–345.

[2] J. Donnelly, A model of continuous plane geometry that is nowhere geodesic, *Forum Geom.*, 18 (2018) 255–273.

[3] R. Hartshorne, *Geometry: Euclid and Beyond*, Springer-Verlag, New York (2000)

[4] G. E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, Springer-Verlag, New York (1986)

John Donnelly: Department of Mathematics, University of Southern Indiana, 8600 University Boulevard, Evansville, Indiana 47712, USA

E-mail address: `jrdonnelly@usi.edu`

Forum Geometricorum
Volume 18 (2018) 297–306.

FORUM GEOM
ISSN 1534-1178

Is The Mystery of Morley's Trisector Theorem Resolved?

Omid Ali Shahni Karamzadeh

Dedicated to all those mathematicians who have been enthusiastic about the above theorem and have published a proof to the theorem. In particular, to John Conway, whose proof prevailed upon me to return to this theorem after more than half a century. I should also remember Brian Stonebridge, who has published a few different proofs of Morley's Theorem and sadly has recently passed away. He was so enthusiastic about the theorem that even while confined to bed with hip fractures, he presented a new proof to the theorem and shortly before his death, dispatched it to me.

Abstract. In this paper we discuss with some reasons why the above possible mystery in the title is resolved. In particular, we prove Morley's Theorem is, in fact, a natural consequence of an overlooked simple result concerning a general property of angle bisectors as loci. We try to show with emphasis why one cannot expect a simpler proof of Morley's Theorem than a particular given proof of the theorem in the literature. Finally, in view of the above overlooked fact, we observe that two known problems in the literature can naturally be generalized and turned into two corollaries with no needed proofs. More applications of this overlooked fact is also observed.

Let us first recall that if the adjacent trisectors of a triangle $\triangle ABC$ intersect at points X, Y, and Z as in Figure (a), then the triangle $\triangle XYZ$ is called the Morley triangle of triangle $\triangle ABC$. Frank Morley, an English mathematician, while studying some properties of cardioid in [15], has given the following incredible theorem (note, the theorem is proved by considering the locus of the centre of a cardioid which touches the sides of a given triangle).

Morley's Theorem (1899). In every triangle the Morley triangle is equilateral.

Some authors believe this theorem is one of the most surprising and mysterious twentieth century results in Euclidean geometry, see for example [2], [7], and [21]. Based on the criteria: the place of the theorem in the literature, the quality of the proofs, the unexpectedness of the result, Morley's Theorem is in a list of "The

---

Publication Date: September 7, 2018. Communicating Editor: Paul Yiu.

Hundred Greatest Theorems in Mathematics". [1] Although, Morley in [15] did not give an elementary proof to his theorem, but soon after only a decade an elementary proof appeared in [16]. Since then many other elementary proofs of Morley's Theorem have been published, [2] where there are 27 different proofs of the theorem (note, still there are many more). Béla Bollobás, Alain Connes, J, John Conway, Edsger Dijksta, D.J. Newmann, Roger Penrose, M.R.F. Smyth, Brian Stonebridge (just to mention a few) are among the authors who have given some of the above 27 different proofs. These proofs are mostly trigonometric, analytic, algebraic, or contain some unexplained elements in their reasoning, and those which are backwards, usually start with an equilateral triangle $\triangle DEF$ and construct a triangle $\triangle ABC$ with arbitrary angles 3α, 3β and 3γ, where $\alpha+\beta+\gamma=60°$, such that triangle $\triangle DEF$ is its Morley triangle. Thus the triangle $\triangle ABC$ would be similar to a given triangle and the proof is finished, see for example [19],[6],[10],[17] and one of the two proofs of Bollobás, among the above 27 proofs, for five such proofs. In some of these backwards proofs, the authors usually construct three triangles around the equilateral triangle $\triangle DEF$ with some specified angles without any explanations for the specification of these angles. Of course these proofs are just fine, but unfortunately with lack of motivation and for this reason, some authors rightly criticize these kind of proofs, see for example [3] and [13, P 74]. Some of the authors who use trigonometry, give a proof to the theorem, by finding a nice symmetric formula for the length of a side of Morley triangle of a triangle $\triangle ABC$ (note, this length equals to $8Rsin\frac{A}{3}sin\frac{B}{3}sin\frac{C}{3}$, where R denotes the circumradius of the triangle $\triangle ABC$, see [24]). The reader should be reminded that, using elementary Cartesian analysis, this length was also obtained by C.N. Mills. His complete proof occupied some twenty sheets of papers, see [24, The Editor Note]. John Conway in [6] claims that "of all the many proofs of this theorem his is indisputably the simplest". By modifying this proof in [12], it is shown that this modified proof (still, called Conway's proof), which also avoids using similarity, is simpler than Conway's, and therefore it was, comparatively, the simplest proof of Morley's Theorem, for the time being, at that time. In [12], it is also admitted that one does not know what happens tomorrow, with regard to the simplicity of the proof, see the last comment in [12]. Although, in [8] the authors already claim, in the title of the article, that Morley's Theorem is no longer mysterious, but we would like to offer here some detailed reasoning for this true claim (at least, to us). In particular, our main aim is to observe and claim with emphasis that the mystery of Morley's Theorem, is indeed, resolved. To this end, by just looking more carefully, at the simple proposition in [12], which is called Bisector Proposition in [8], we may infer that Morley's Theorem is nothing but a straightforward corollary of this simple proposition. Surprisingly, it seems this proposition, which is a natural generalization of the classical property of the bisectors (i.e., a point lies on the bisector of an angle if and only if it is equidistant from the sides of the angle), is

[1] See `http://pirate.shu.edu/ kahlnath/Top100.html`.

[2] See, e.g., `http:// www.cut-the-knot.org/triangle/Morley/sb.shml`, [2-4, 7-14]

overlooked in all these years, before its appearance in [12]. Let us cite a comment which is made in [13, Last line, P 73]: with all the various proofs in the literature, still a geometric, concise and logically transparent proof of Morley's Theorem is desirable. A similar comment is also given in [21, L 3, P 1]. In what follows, we aim to emphasize on the effectiveness of, Bisector Proposition as a tool, and on its unique role in connection with Morley's Theorem. We also believe with some reasoning that this demanded proof in [13], [21], which in our opinion, could have been the modified proof in [12], at that time, is already in [8], with some improvement. We should also emphasize that this proof is geometric, concise, transparant, and simpler than the proof in [12], which in turn, as shown there, is simpler than Conway's, where to many authors including Conway himself, Conway's was the simplest possible until 2014, see [6]. Although, by reading [12] and [8] carefully, this claim needs no proof, but one feels this important fact (at least to us) should be better recorded. Since we are trying to give our reasoning with scrupulous care, the reader might notice some unnecessary details in our arguments. We should recall here that each vertex of the Morley triangle of a given triangle, which is the intersection point of a pair of trisectors, lies trivially on the bisector of the angle formed by the intersection of two other trisectors (e.g., Z lies on the bisector of the angle formed by the intersection of trisectors BX and AY, see Figure (a)). Consequently, each vertex of the Morley triangle takes the position of point A in the next proposition. Hence, the conditions (1), (3), in the following proposition, immediately give us a natural criterion , as in the corollary which follows the proposition, for the Morley triangle of any triangle, to be equilateral. In fact, what Morley's Theorem asserts is the fact that this natural criterion holds in every triangle.

The next proposition, says the point A lies on the bisector of the angle $\angle xOy$ if, and only if, any one of the conditions (2) or (3) in the proposition implies the other one. We notice, in the classical fact, the condition (2) is stated, in the particular case, when B, C are the feet of perpendiculars from the point A to the arms of angle $\angle xOy$, and the condition (3) is stated, in the particular case, that the angles $\angle OBA$, $\angle OCA$, are both right angle, in which case, they are both equal and supplementary angles (i.e., condition (3) below is naturally satisfied in the particular case).

Bisector Proposition. Suppose that A is a point inside the angle $\angle xOy$ and B, C are two points on the arms Ox and Oy, respectively. Then if any two of the following hold, so does the third.

(1) A lies on the bisector of the angle $\angle xOy$.
(2) $AB = AC$.
(3) Angles $\angle OBA$ and $\angle OCA$ are either equal or supplementary angles.

We should remind the reader that, it is manifest that, the angles $\angle OBA$ and $\angle OCA$ satisfy condition (3) in the above proposition if, and only if, their adjacent angles satisfy this condition, too. This evident observation is in fact, what is used when Bisector Proposition is invoked in [12], to deal with the proof of the corollary which follows. This corollary which is given right after the proof of the above proposition in [12], is not stated there as a corollary. Indeed, it is given there informally to refute a claim by Cain in [3], that specification of the angles in Conway's proof is not justified. We should remind the reader that the following corollary which gives "a necessary and sufficient condition" for the Morley triangle of any triangle to be equilateral, leaves no unexplained steps whatsoever (unlike, the other backwards proofs) in the proof of Morley's Theorem in [8].

The next corollary, which relies only on the above proposition, is as good as Morley's Theorem itself, for it is the only result in Euclidean geometry, so far, which gives a natural "necessary and sufficient condition", for the Morley triangle of any triangle to be equilateral, before knowing the validity of Morley's Theorem.

Corollary 1. In the triangle $\triangle ABC$, in Figure (a), let $\angle A = 3\alpha$, $\angle B = 3\beta$, and $\angle C = 3\gamma$. If the intersection of the adjacent trisectors of the angles of this triangle are X, Y, and Z as in Figure (a). Then the triangle $\triangle XYZ$ (i.e., the Morley triangle of triangle $\triangle ABC$) is equilateral if, and only if, the base angles of the triangles $\triangle AZY$, $\triangle BXZ$, and $\triangle CXY$ consist of three equal pairs (i.e., $\angle 1 = \angle 4 = 60° + \gamma$, $\angle 2 = \angle 5 = 60° + \beta$, and $\angle 3 = \angle 6 = 60° + \alpha)$, see Figure (a).

The reader must be reminded that the values of the above pairs of angles are determined, with a purely geometric method (thanks to Bisector Proposition), for the first time in the history of Morley's Trisector Theorem, in [12]. In any configuration related to Morley's Theorem we have seven smaller triangles inside the original one. Why should we only be asking for the values of angles of the Morley's triangle?, see also Newmann's comments in his proof among the above 27 proofs. In this regards, let us briefly recall the role of Bisector Proposition in determining the values of the angles of all these seven triangles. Let us assume the Morley triangle, of a triangle , $\triangle ABC$ say, to be equilateral and try to find the values of angles $\angle 1$ up to $\angle 6$, in a relevant configuration, as in Figure (a). We may easily form a system of 6 linear equations in terms of these values. For the sake of completeness we may write down these equations: $\angle 1 + \angle 2 = 180° - \alpha$, $\angle 2 + \angle 3 = 180° - \gamma$, $\angle 3 + \angle 4 = 180° - \beta$, $\angle 4 + \angle 5 = 180° - \alpha$, $\angle 5 + \angle 6 = 180° - \gamma$, $\angle 6 + \angle 1 = 180° - \beta$. Manifestly, these equations are not independent. Indeed, any one of these equations can be derived from the other five, and therefore any one of the equations can be removed without affecting the solution of the system. Consequently, we have a system of five linear equations with six unknown values of angles, whose unique solution cannot usually be determined, by ordinary elimination method. Although, there is no unanimity of opinion, on any reason, for the mysteriousness of Morley's Theorem, perhaps the latter observation seems to be a good possible reason as to why this theorem is called mysterious, by some authors. It should be emphasized that Bisector Proposition plays an effective and indispensable role in showing that the above system has a unique solution, and conversely due to Bisector Proposition again, this solution without any further calculation whatsoever for the angles of the Morley's triangle, immediately shows that the Morley's triangle is equilateral, see the proof of the above corollary, in [12].

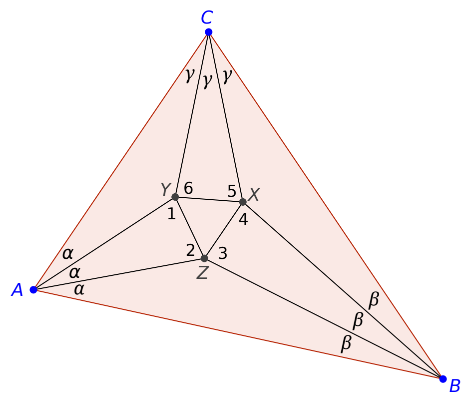


Figure (a)

Corollary 2 (Morley's Theorem). The Morley triangle of every triangle is equilateral.

If we follow the proof in [8], we notice that Bisector Proposition, which is responsible for the proof of Corollary 1, is also responsible for every single step in this proof, without using any extra mathematics whatsoever. In fact, Bisector Proposition shows in [8], one can simultaneously construct any triangle with its trisectors in a few steps, in a motivated simple way, without drawing any extra single line, to automatically, obtaining an equilateral triangle as its Morley triangle (note, in this proof unlike other backwards proofs, one does not start with an equilateral triangle). In fact, it may be considered as drawing the trisectors in the triangle, in a motivated way, to get an equilateral triangle, directly, as the Morley triangle of the triangle (once again, it should be emphasized that in this drawing not a single extra line is drawn).

Envoi: some authors, use the words mystery, miraculous, magic and some other synonyms to these words, in connection with Morley's Theorem and are wondering why the ancient Greeks did not discover this theorem. They generally blame the concept of trisectors for this failure. They believe the ancient Greeks, apart from having problems with the existence of trisectors (i.e., trisecting an angle, in general, with just a compass and an unmarked straightedge), expected to know some properties of trisectors similarly to those of bisectors. But it seems, if only the ancient Greeks noticed that, each vertex of the Morley triangle lies clearly on the bisector of the angle formed by a pair of trisectors, whose intersection is not that vertex, they might have been motivated to search for new properties of bisectors.

Moreover, if they could have also guessed the Bisector Proposition, which genuinely and naturally belongs to their era, then they could have easily discovered Morley's Theorem. Although at the same time, one should admit that, it seems the ancient Greeks have not ever dealt with any result in geometry, whose related possible configuration, is not constructible with Euclidean instruments. However, I do believe the fact that Bisector Proposition (a natural elementary geometric fact, which also seems to be a key factor related to Morley' Theorem) is gone unnoticed in all these years, is more mysterious than the missing of Morley's Theorem itself, by the ancient Greeks! Therefore, we should admit, it seems as though, all these years we were waiting for the Bisector Proposition to appear, to give us a natural, transparent geometric proof of Morley's Theorem. Incidentally, this proof, see [8], needs only the simplest possible configuration (i.e., a figure consisting merely of the triangle itself and its trisectors). Moreover, it also uses only the most elementary tools (i.e., SAS, ASA and RHS). Shouldn't we ask that: Can a proof of this theorem be expected, in the future, which avoids less elementary tools? Shouldn't we admit that Bisector Proposition has indeed resolved the mystery of Morley's Theorem? To re-emphasizing on the effectiveness of Bisector Proposition as a useful tool, in general, not only with regards to Morley's Theorem, I could not help recalling the following well known old problems to show how, by invoking Bisector Proposition, the two problems can be generalized and turned into two natural evident corollaries of this proposition. Some more applications of the proposition are also observed.

Problem 1 ([20, Problems 9, 18, P 236, 267], [1, Problem 24]): Let P be the center of the square constructed on the hypotenuse AC of the right triangle $\triangle ABC$. Prove that BP bisects angle $\angle ABC$.

Corollary 3 . Let P be the center of the square constructed on the side AC of the triangle $\triangle ABC$. Then BP bisects the angle $\angle ABC$ if, and only if, the triangle $\triangle ABC$ is either a right-angled triangle, with $\angle B$ as the right angle, or an isosceles one with the apex B.

Proof. It is really evident by Bisector Proposition (just note, $PA = PC$).

Terence Tao in [22, pp. 52-55], has given a trigonometric solution with almost three pages of justifications for his solution to the next problem. But, in view of Bisector Proposition, we observe briefly, this problem does not really need any solution at all. Naturally, Bisector Proposition was not available to Tao , at that time, because the proposition was already overlooked then. Similarly to Problem 1, this problem, can also be generalized and stated as an immediate corollary of Bisector Proposition. However, in what follows, we present a proof for this corollary with some unnecessary details, just to repeat the effectiveness of Bisector Proposition as a useful tool.

Problem 2 ([22, Problem 4.2, p. 52]): In a triangle $\triangle BAC$ the bisector of the angle at B meets AC at D; the angle bisector of C meets AB at E. These bisectors meet at O. Suppose that $OD = OE$. Prove that either $\angle A = 60°$ or that the triangle $\triangle BAC$ is isosceles (or both).

Corollary 4. In a triangle $\triangle ABC$ let the bisectors of angles $\angle B$, $\angle C$, intersect the opposite sides at D and E, respectively, and take O to be the intersection of these bisectors. Then $OD = OE$ if, and only if, either the triangle $\triangle BAC$ is isosceles with apex A or $\angle A = 60°$.

Proof. Just draw any configuration which satisfies our assumption, then the proof may go as follows : Let us first assume that, $OD = OE$. Since the point O lies on the bisector of $\angle A$, then by part (3), of Bisector Proposition, we infer that either $\angle AEO = \angle ADO$, in which case, it implies $\angle \frac{C}{2} + \angle B = \angle \frac{B}{2} + \angle C$, i.e., the triangle $\triangle BAC$ is isosceles with apex A, or $\angle ADO$ and $\angle AEO$ are supplementary angles, in which case, it implies $\angle A$ and $\angle DOE$ must also be supplementary angles. Hence $\angle A = 60°$ (note, $\angle DOE = \angle COB = 90° + \angle \frac{A}{2}$). Conversely, let the triangle $\triangle BAC$ be isosceles with $\angle B = \angle C$, then clearly $OD = BD - BO = CE - CO = OE$. Finally, if $\angle A = 60°$, then $\angle A + \angle DOE = 180°$, for $\angle DOE = \angle COB = 90° + \frac{A}{2} = 120°$. Consequently, the angles $\angle ADO$ and $\angle AEO$ must also be supplementary angles. Since O lies on the bisector of angle $\angle A$, then in view of, parts (1), (3), of Bisector Proposition, we immediately infer that $OD = OE$, and we are done.

Let us assume that the above triangle $\triangle ABC$ is non-isosceles and $\angle A \neq 60°$. Then one may digress for a moment and ask a natural question of what happens to the comparability of OD and OE (i.e., considering any shape for the triangle, which one has a greater length?), where D, E, and O are the same points as in the above corollary. Before answering this question we need the following lemma which is also a consequence of Bisector Proposition and it is somewhat a complement to it.

Lemma (Bisector Proposition Extended). Suppose that A is a point on the bisector of the angle $\angle xOy$ and B, C are two points on the arms Ox and Oy, respectively with $\angle OBA = \alpha$ and $\angle OCA = \beta$. Then $AB > AC$ if, and only if, either $\alpha > \beta$ with $\alpha + \beta > 180°$ or $\alpha < \beta$ with $\alpha + \beta < 180°$.

Proof. Draw any configuration which satisfies our assumption. Let us assume that $AB > AC$, then by Bisector Proposition $\alpha \neq \beta$ and $\alpha + \beta \neq 180°$. Take the point B' to be the symmetric point of the point B with respect to the bisector of the angle $\angle xOy$. Now either $\alpha > \beta$, in which case, B' manifestly lies on Oy between O and C (note, $\angle OB'A = \angle OBA = \alpha > \beta = \angle OCA$). Consequently in the triangle $\triangle AB'C$, $AB' = AB > AC$ implies that $\beta > 180° - \alpha$, i.e., we have $\alpha > \beta$ with $\alpha + \beta > 180°$. Or we may have $\alpha < \beta$, in which case, B' cannot lie between O and C. Consequently in the triangle $\triangle ACB'$, $AB' = AB > AC$

implies that $180° - \beta > \alpha$ and we are done. The converse is evident.

The following corollary which is an immediate consequence of the previous lemma settles the above question. Corollary 4, which is also clearly a consequence of the next corollary (it's briefly observed), together with this corollary, give a nontrivial universal fact about any triangle. Motivated by this, we cannot help reciting a comment, with emphasis, as in [11, p. 49], that when it comes to deducing results in mathematics just from the definition of an object, nothing can hold a candle to the triangle.

Corollary 5. In a triangle $\triangle ABC$ let the bisectors of angles $\angle B$, $\angle C$, intersect the opposite sides at D and E, respectively, and take O to be the intersection of these bisectors. Then $OE > OD$ if, and only if, either $\angle A < 60°$ with $\angle C < \angle B$ or $\angle A > 60°$ with $\angle C > \angle B$.

Proof. Let us put $\angle AEO = \alpha$ and $\angle ADO = \beta$. Then clearly $\angle A + \angle DOE = 90° + \frac{3\angle A}{2}$. Thus in the quadrilateral $AEOD$ it is manifest that $\angle A < 60°$ (resp., $\angle A > 60°$) if and only if $\alpha + \beta > 180°$ (resp., $\alpha + \beta < 180°$). We also notice that $\alpha > \beta$ (resp., $\alpha < \beta$) if and only if $\angle B > \angle C$ (resp., $\angle B < \angle C$). Finally, it remains to invoke the above lemma to complete the proof.

Remark. It goes without saying that, in the above corollary, $OD > OE$ if and only if $\angle A < 60°$ with $\angle B < \angle C$ or $\angle A > 60°$ with $\angle B > \angle C$. Considering this, we should remind the reader that Corollary 4, is now an immediate consequence of the above corollary, too. We may also recall that in any triangle $\triangle ABC$, $\angle B > \angle C$ if, and only if, the length of the bisector of $\angle B$ is shorter than that of the bisector of $\angle C$. In case $\angle A \leq 60°$ the latter well-known fact is a consequence of our above observations (note, let $\triangle ABC$ be a triangle with $\angle B > \angle C$ and consider any related configuration to Corollaries 4, 5. Then $BD = BO + OD$ and $CE = CO + OE$, where clearly $BO < CO$ and whenever $\angle A \leq 60°$, then in view of Corollaries 4, 5, we have $OD \leq OE$, which implies that $BD < CE$ and we are done). It is worth noting that although the validity of $BD = BO + OD < CO + OE = CE$ does not depend on the measure of $\angle A$. However, one should emphasize that in contrast to the previous case that $BO < CO$, $OD \leq OE$ and consequently $BD = BO + OD < CO + OE = CE$, the case $\angle A > 60°$, reveals a new information in this regard, namely, $BO < CO$, and although $OD > OE$, but still of course $BD = BO + OD < CE = CO + OE = CE$.

After all these discussions and recollection about Bisector Proposition and some of its consequences, everyone is expected to clearly comprehend the proposition, remember it forever; and use it as naturally as the classical fact about Bisectors, like the time we were dealing with geometric facts in our school days. In particular, as a consequence of this proposition, everyone should also be certain visually, that the Morley triangle of a triangle $\triangle ABC$ is equilateral if and only if $\angle 1 = \angle 4$, $\angle 2 =$

$\angle 5$, and $\angle 3 = \angle 6$, see Figure (a) (i.e., anyone with some adequate knowledge of Euclidean geometry including Bisector Proposition could have easily guessed a natural equivalent form of Morley's Theorem). Therefore, one may claim Morley's Theorem is, in essence, a fact related to properties of bisectors. In particular, it is a consequence of a missing property of bisectors, i.e., Bisector Proposition. Let us, in what follows, give some more justifications for the previous claim: Manifestly every trisector of a triangle is, in fact, a bisector of an angle in any configuration related to Morley's Theorem (indeed, of the two trisectors of an angle, $\angle A$ say, in the triangle $\triangle ABC$, clearly the one adjacent to side AB (resp., AC) is the bisector of the angle whose arms are AB (resp., AC) and the other trisector, respectively). Moreover every vertex of the Morley's triangle also lies on the bisector of the angle formed by the intersection of a pair of trisectors, whose intersection is not that vertex. Considering all these evident observations and the natural and simple Bisector Proposition, including its consequences, shouldn't we ask, **where is the mystery, then?**

References

[1] E. D. J. Barbeau, M. S. Klamkin, and W. O. J. Moser, *Five Hundred Mathematical Challenges*, MAA, 1995.

[2] A. Bremner, J. R. Goggins, M. J. T. Guy, and R. K. Guy, On rational Morley triangle, *Acta Arithmetica*, XCIII.2 (2000) 177–187.

[3] A. Cain, Deus ex Machina and the Aesthetics of proofs, *The Mathematical Intelligencer*, 32 (2010), No. 3, 7–11.

[4] J. M. Child, Proof of "Morley's theorem", *Math. Gazzette*, 11 (1923) 171.

[5] A. Connes, A new proof of Morley's theorem, les relations entre les mathematiques et la physique theorique, *Hautes Sci., Bures*, 1998, pp. 43-46, MR 99m:51027.

[6] J. Conway, On Morley's Trisector Theorem, *The Mathematical Intelligencer*, 36(3) (2014) 3.

[7] R. L. Francis, Modern mathematical milestone: Morley's mystery, *Missouri Journal of Mathematical Sciences*, 14 (2002), 1–3.

[8] I. Gorjian, O. A. S. Karamzadeh, and M. Namdari, Morley's theorem is no longer mysterious!, *The Mathematical Intelligencer*, 37 (2015) No.4, 6–7.

[9] R. K. Guy, The lighthouse theorem, Morley & Malfatti-a budget of paradoxes, *Amer. Math. Monthly*, 114 (2007) 97–14.

[10] Y. Hashimoto, A short proof of Morley's theorem, *Elem. Math.*, 62 (2007) 121–121. 0013-6018/07/030121-1

[11] O. A. S. Karamzadeh, The Mathematics of Mathematics Houses (The Snaky Connection), *The Mathematical Intelligencer*, 34 (2012) No. 4, 46–52.

[12] O. A. S. Karamzadeh, O. A. S. Karamzadeh, Is John Conway's proof of Morley's theorem the simplest and free of a deus ex machina? *The Mathematical Intelligencer*, 36 (2014) No.3, 4–7.

[13] Spiridon A. Kuruklis, Trisectors like bisectors with equilaterals instead of points, *CUBO A Mathematical Journal*, 16, No 01, (71110); June 2014.

[14] A. Letac, Solution (Morley's triangle), problem No. 490 [Sphinx:revue mensuelle des questions recreatives, Brussels, 8(1938), 106] Sphinx, 9 (1939), 46.

[15] F. Morley and F. V. Morley, *Inversive Geometry*, Ginn, Boston, 1933. (Reissued by Chelsea, Bronx, New York, 1954 and by Dover Books on Mathematics, 2013).

[16] M. T. Navansiengar, Solution to Morley's problem, *Mathematical Questions and Solutions, from Edu. Times*, New Series 15 (1909) 45.

[17] D. J. Newmann, The Morley's miracle, *The Mathematical Intelligencer*, 18 (1996) No.1, 31–32.

[18] C. O. Oakley and J. C. Baker, The Morley trisector theorem, *Amer. Math. Monthly*, 85 (1978) 737–745.

[19] R. Penrose, Morley's trisector theorem, *Eureka*, 16 (1953) 6–7.
[20] A. H. Schoenfeld, *Mathematical Problem Solving*, Academy Press, 1985.
[21] B. Stonebridge, A simple geometric proof of Morley's trisector theorem, *Applied Probability Trust*, 2009.
[22] T. Tao, *Solving Mathematical Problems, A Personal Perspective*, Oxford University Press, 2006.
[23] G. Wanner, Elementare Beweise des Satzes von Morley, *Elem. Math.*, 59 (2009) 144–150.
[24] R. J. Webster, Morley's triangle theorem, *Math. Mag.*, 43 (1970) 209–210.

Omid Ali Shahni Karamzadeh: Department of mathematics, Shahid Chamran university of Ahvaz, Ahvaz-Iran
E-mail address: `karamzadeh@ipm.ir`

Forum Geometricorum
Volume 18 (2018) 307–308.

FORUM GEOM
ISSN 1534-1178

A Synthetic Proof of the Equality of Iterated Kiepert Triangles $\mathcal{K}(\phi, \psi) = \mathcal{K}(\psi, \phi)$

Floor van Lamoen

Abstract. We use Miquel's Theorem to prove the equality $\mathcal{K}(\phi, \psi) = \mathcal{K}(\psi, \phi)$ of iterated Kiepert triangles.

1. Introduction

The Kiepert triangle $\mathcal{K}(\phi) = A_\phi B_\phi C_\phi$ is the triangle formed by the apices of isosceles triangles with base angles $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ erected on sides BC, AC, and BC of triangle ABC respectively. The iterated Kiepert triangle $\mathcal{K}(\phi, \psi)$ is found by subsequently erecting isosceles triangles with base angles ψ to $\mathcal{K}(\phi)$. Paul Yiu and the author showed by calculating homogeneous barycentric coordinates that $\mathcal{K}(\phi, \psi) = \mathcal{K}(\psi, \phi)$ ([2], section 3). In this paper we will show that the result can be acquired by synthetic means, by application of Miquel's Theorem.

2. Application of Miquel's Theorem

We recall Miquel's theorem, in the wording of [1, Theorem 4], for instance, which contains the well known corollaries added to the original theorem [3].

Theorem 1 (Miquel). *Let $A_1B_1C_1$ be a triangle inscribed in triangle ABC. There is a pivot point P such that $A_1B_1C_1$ is the image of the pedal triangle of P after a rotation about P followed by a homothety with center P. All inscribed triangles directly similar to $A_1B_1C_1$ have the same pivot point.*

It is well known that this theorem can be proven syntheticly.

A consequence of this theorem is that if two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are directly similar, then each triangle $A_3B_3C_3$ with $A_3 \in A_1A_2$, $B_3 \in B_1B_2$, and $C_3 \in C_1C_2$ such that the ratios of directed distances fulfill

$$A_1A_3 : A_3A_2 = B_1B_3 : B_3B_2 = C_1C_3 : C_3C_2$$

is directly similar to these as well, as A_1A_2, B_1B_2, and C_1C_2 bound a triangle.

Remark. This result is valid for general similar figures and is often referred to as "Fundamental Theorem of Directly Similar Figures".

Publication Date: September 7, 2018. Communicating Editor: Paul Yiu.

Now we consider triangle ABC, with Kiepert vertices A_ϕ, C_ϕ, A_ψ, C_ψ and iterated Kiepert vertices $B_{\phi,\psi}$ and $B_{\phi,\phi}$. M_a, M_c, and $M_{b,\phi}$ are the midpoints of BC, AB, and $A_\phi C_\phi$ respectively. See Figure 1. Triangles $A_\phi BC$ and $C_\phi AB$ are directly similar, so triangle $M_{b,\phi}M_cM_a$ is directly similar to these and hence an isosceles triangle with base angle ϕ as well. So triangles $M_aM_{b,\phi}M_c$ and $A_\phi B_{\phi,\phi}C_\phi$ are directly similar, and clearly

$$M_aA_\psi : A_\psi A_\phi = M_{b,\phi}B_{\phi,\psi} : B_{\phi,\psi}B_{\psi,\phi} = M_cC_\psi : C_\psi C_\phi.$$

This shows that triangle $A_\psi B_{\phi,\psi}C_\psi$ is also isosceles with base angle ϕ. We have derived that $B_{\phi,\psi} = B_{\psi,\phi}$. With similar reasoning for the A- and C-vertices we conclude the proof that $\mathcal{K}(\phi,\psi) = \mathcal{K}(\psi,\phi)$.

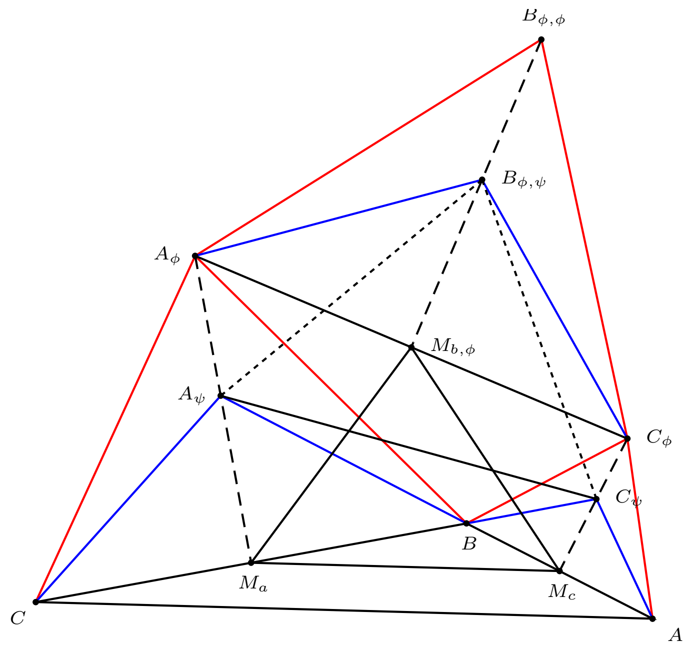


Figure 1

References

[1] F. M. van Lamoen, Inscribed squares, *Forum Geom.*, 4 (2004) 207–214.
[2] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.
[3] A. Miquel, Mḿoire de Géométrie, *Journal de mathématiques pures et appliquées de Liouville*, 1 (1838) 485–487.

Floor van Lamoen: Ostrea Lyceum, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: fvanlamoen@planet.nl

Forum Geometricorum
Volume 18 (2018) 103–114.

FORUM GEOM
ISSN 1534-1178

Geometric Inequalities in Pedal Quadrilaterals

Şahlar Meherrem, Gizem Günel Açıksöz, Serenay Şen, Zeynep Sezer, and Güneş Başkes

Abstract. The aim of this paper is to investigate the general properties of the pedal quadrilateral of a point P with respect to a convex and closed quadrilateral $ABCD$. In particular, we present an analogue of Erdős-Mordell Inequality, stating that for any triangle ABC and a point P inside ABC, the sum of the distances from P to the sides is less than or equal to half of the sum of the distances from P to the vertices of ABC, for an inscribed quadrilateral.

1. Introduction

Let ABC be a triangle and P be a point in plane. If the feet of the perpendiculars drawn from P to sides of the triangle $[BC]$, $[CA]$, $[AB]$ are respectively X, Y and Z, then triangle XYZ is called the pedal triangle of P with respect to ABC and P is called the pedal point ([3, 7]). If the point P lies on the circumcircle of the triangle ABC, then the points X, Y and Z are collinear. The line passing through the points X, Y and Z is known as the *Simson line* of P ([3]).

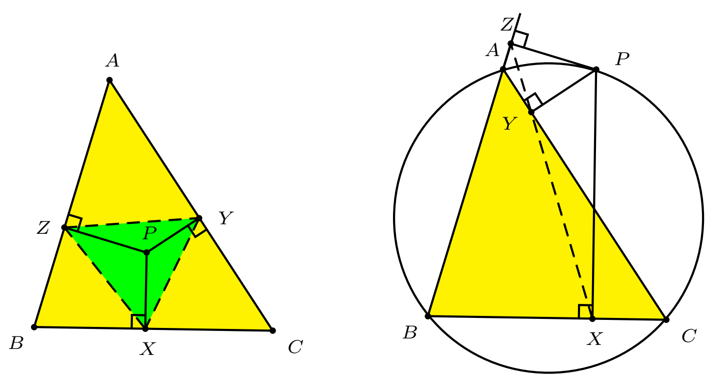


Figure 1

A similar notion has been generalized for polygons with n sides ([3]). Let $A_1A_2A_3\cdots A_n$ be a polygon and P be a point inside $A_1A_2A_3\cdots A_n$. If the feet of the perpendiculars from P to the line segments $[A_1A_2]$, $[A_2A_3]$, ..., $[A_nA_1]$ are respectively H_1, H_2, ..., H_n, then the polygon $H_1H_2\cdots H_n$ is called the *pedal polygon* of P with respect to $A_1A_2A_3\cdots A_n$ and P is called the pedal point ([3]). Although there are lots of investigations on pedal triangles, so little investigations have been studied on pedal quadrilaterals in the literature ([3, 5, 6, 7, 9, 11]).

Publication Date: April 20, 2018. Communicating Editor: Paul Yiu.

In this paper, we study on the general properties of the pedal quadrilateral of a point P with respect to a convex and closed quadrilateral $ABCD$. We investigate some interesting geometric inequalities on pedal quadrilaterals. In particular, we state an analogue of Erdős-Mordell Inequality ([1, 2, 4, 8, 10]) for quadrilaterals.

2. Geometric inequalities in pedal quadrilaterals

Proposition 1. *Let $ABCD$ be an inscribed quadrilateral and P be a point inside $ABCD$. If the feet of perpendiculars from P to the line segments $[AB]$, $[BC]$, $[CD]$ and $[AD]$ are respectively K, M, N, L and the radius of the circumcircle of $ABCD$ is R then,*

$$|KM| = \frac{|AC|\cdot|PB|}{2R}, \qquad |KL| = \frac{|BD|\cdot|PA|}{2R},$$

$$|MN| = \frac{|BD|\cdot|PC|}{2R}, \qquad |LN| = \frac{|AC|\cdot|PD|}{2R}.$$

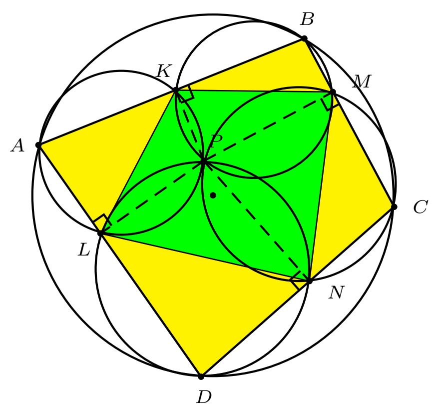


Figure 2

Proof. As seen in Figure 2 and since $\angle BKP = \angle PLD = \angle PNC = \angle PMB = 90°$, the quadrilaterals $BKPM$, $PLDN$, $PNCM$ and $PLAK$ are inscribed quadrilaterals. By the law of sines, we have

$$|PB| = \frac{KM}{\sin KPM} \quad \text{and} \quad 2R = \frac{|AC|}{\sin(\pi - \angle KPM)} = \frac{|AC|}{\sin KPM}.$$

Hence, we get

$$|KM| = \frac{|AC|\cdot|PB|}{2R}.$$

It can be easily seen that

$$|KL| = \frac{|BD|\cdot|PA|}{2R}, \qquad |MN| = \frac{|BD|\cdot|PC|}{2R}, \qquad |LN| = \frac{|AC|\cdot|PD|}{2R}.$$

□

Proposition 2. *Let $ABCD$ be a quadrilateral and P be a point inside $ABCD$.If the feet of perpendiculars from P to the line segments $[AB]$, $[BC]$, $[CD]$ and $[AD]$ are respectively A_1, B_1, C_1, D_1, then*

$$r_1 + r_2 + r_3 + r_4 \leq \frac{M \cdot C}{2\sqrt{2}},$$

where r_1, r_2, r_3 and r_4 are respectively the radii of the incircles of the triangles AA_1D_1, A_1BB_1, CB_1C_1, DC_1D_1, and

$$M = \max\left\{\frac{1}{\sqrt{1-\cos A} + \sqrt{2}}, \frac{1}{\sqrt{1-\cos B} + \sqrt{2}}, \frac{1}{\sqrt{1-\cos C} + \sqrt{2}}, \frac{1}{\sqrt{1-\cos D} + \sqrt{2}}\right\},$$

and C is the perimeter of the quadrilateral $ABCD$. The equality holds if $ABCD$ is chosen as a square and P is chosen at the center of the square.

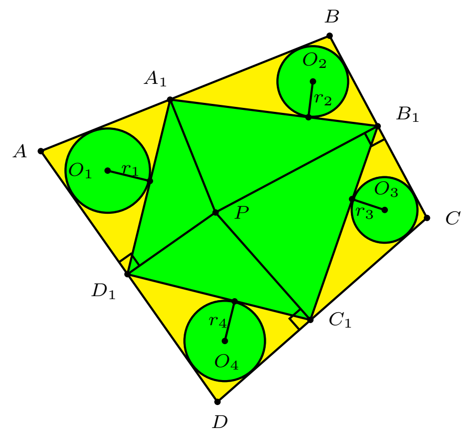


Figure 3

Proof. By the law of cosines, we have

$$|A_1B_1| = \sqrt{|A_1B|^2 + |BB_1|^2 - 2|A_1B||BB_1|\cos B}. \tag{1}$$

If $S_{A_1BB_1}$ is the area of the triangle A_1BB_1, then

$$S_{A_1BB_1} = \frac{|A_1B| \cdot |BB_1| \sin B}{2}. \tag{2}$$

Since $2S_{A_1BB_1} = (|A_1B| + |BB_1| + |A_1B_1|)r_2$, we get

$$r_2 = \frac{2S_{A_1BB_1}}{|A_1B| + |BB_1| + |A_1B_1|}. \tag{3}$$

By the equalities (2) and (3),

$$r_2 = \frac{|A_1B| \cdot |BB_1| \sin B}{|A_1B| + |BB_1| + |A_1B_1|}. \tag{4}$$

It follows from Arithmetic-Geometric Mean Inequality that

$$|A_1B|^2 + |BB_1|^2 \geq 2 \cdot |A_1B| \cdot |BB_1|. \tag{5}$$

If the inequalities (1) and (5) are considered together, then

$$\begin{aligned}|A_1B_1| =& \sqrt{|A_1B|^2+|BB_1|^2-2|A_1B|\cdot|BB_1|\cos B}\\ \geq& \sqrt{2|A_1B|\cdot|BB_1|-2|A_1B|\cdot|BB_1|\cos B}\\ \geq& \sqrt{2}\,\sqrt{|A_1B|\cdot|BB_1|}\sqrt{1-\cos B} \end{aligned} \tag{6}$$

It is also obvious from Arithmetic-Geometric Mean Inequality that

$$|A_1B|+|BB_1| \geq 2\sqrt{|A_1B|\cdot|BB_1|}. \tag{7}$$

Therefore, by the inequalities (6) and (7), we observe that

$$|A_1B|+|BB_1|+|A_1B_1| \geq 2\sqrt{|A_1B|\cdot|BB_1|}+\sqrt{2}\,\sqrt{|A_1B|\cdot|BB_1|}\sqrt{1-\cos B}. \tag{8}$$

Hence, it follows from the inequalities (4) and (8) that

$$\begin{aligned}r_2 \leq& \ |A_1B|\cdot|BB_1|\frac{\sin B}{\sqrt{2}\cdot\sqrt{|A_1B|\cdot|BB_1|}\sqrt{1-\cos B}+2\sqrt{|A_1B|\cdot|BB_1|}}\\ \leq& \ |A_1B|\cdot|BB_1|\cdot\frac{\sin B}{\sqrt{2}\sqrt{|A_1B|\cdot|BB_1|}\sqrt{1-\cos B}+\sqrt{2})}\\ \leq& \ \sqrt{|A_1B|\cdot|BB_1|}\cdot\frac{\sin B}{\sqrt{2}(\sqrt{1-\cos B}+\sqrt{2})}\\ \leq& \ \frac{|A_1B|+|BB_1|}{2\sqrt{2}(\sqrt{1-\cos B}+\sqrt{2})} \end{aligned} \tag{9}$$

Let $M = \max\left\{\frac{1}{\sqrt{1-\cos A}+\sqrt{2}}, \frac{1}{\sqrt{1-\cos B}+\sqrt{2}}, \frac{1}{\sqrt{1-\cos C}+\sqrt{2}}, \frac{1}{\sqrt{1-\cos D}+\sqrt{2}}\right\}$. By the inequality (9),

$$r_2 \leq \frac{M\cdot(|A_1B|+|BB_1|)}{2\sqrt{2}}. \tag{10}$$

Similarly, we get

$$\left.\begin{aligned} r_1 \leq& \frac{M\cdot(|A_1A|+|AD_1|)}{2\sqrt{2}},\\ r_3 \leq& \frac{M\cdot(|C_1C|+|CB_1|)}{2\sqrt{2}},\\ r_4 \leq& \frac{M\cdot(|C_1D|+|DD_1|)}{2\sqrt{2}}. \end{aligned}\right\} \tag{11}$$

If the inequalities (10) and (11) are added side by side, it is easy to see that

$$r_1+r_2+r_3+r_4 \leq \frac{M\cdot C}{2\sqrt{2}}$$

where C is the perimeter of the quadrilateral $ABCD$.

Now suppose that $ABCD$ is a square with side length $2a$ and P is at the center of the $ABCD$ (Figure 4). If the area of the triangle D_1AA_1 is $S_{D_1AA_1}$ then,

$$S_{D_1AA_1} = \frac{2a+a\sqrt{2}}{2}r_1 = \frac{a^2}{2}.$$

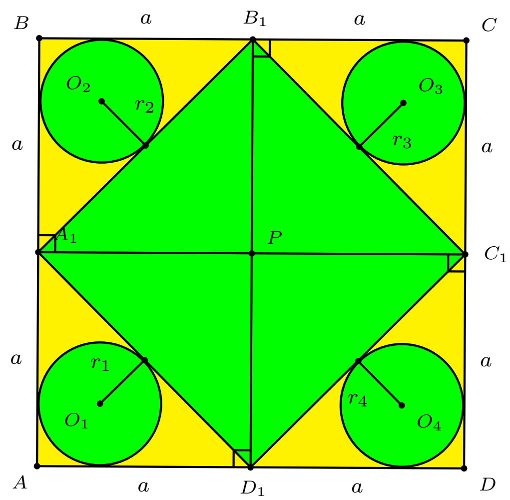


Figure 4

Hence, we get $r_1 = \frac{a}{2+\sqrt{2}}$. Since $r_1 = r_2 = r_3 = r_4$, $C = 8a$ and $M = \frac{1}{\sqrt{2}+1}$, then the following equality holds:

$$r_1 + r_2 + r_3 + r_4 = 4 \cdot \frac{a}{2+\sqrt{2}} = \frac{8a \cdot \left(\frac{1}{\sqrt{2}+1}\right)}{2\sqrt{2}} = \frac{C \cdot M}{2\sqrt{2}}.$$

□

Proposition 3. *Let $A_1A_2A_3 \cdot A_n$ be a polygon and P be a point inside $A_1A_2A_3 \cdots A_n$. If the feet of perpendiculars from P to the line segments $[A_1A_2]$, $[A_2A_3]$, $\ldots$, $[A_nA_1]$ are respectively $H_1, H_2, \ldots, H_n$, then*

$$r_1 + r_2 + \cdots + r_n \leq \frac{M \cdot C}{2\sqrt{2}}$$

where r_1, r_2, $\ldots$, r_n are respectively the radii of the incircles of the triangles $A_1H_1H_n$, $H_1A_2H_2, \ldots, H_{n-1}A_nH_n$, and

$$M = \max\left\{\frac{1}{\sqrt{1-\cos A_1}+\sqrt{2}}, \frac{1}{\sqrt{1-\cos A_2}+\sqrt{2}}, \ldots, \frac{1}{\sqrt{1-\cos A_n}+\sqrt{2}}\right\}$$

and C is the perimeter of the polygon $A_1A_2 \cdots A_n$. The equality holds if $A_1A_2 \cdots A_n$ is chosen as a square and P is chosen as the centroid of the square.

This can be easily shown as in Proposition 2.

Proposition 4. *Let $ABCD$ be a circumscribed quadrilateral with area S. If the center and the radius of the incircle of $ABCD$ are respectively the point P and r, and the feet of perpendiculars from P to the line segments $[AB]$, $[BC]$, $[CD]$ and*

$[AD]$ are respectively K, L, M and N, then

$$\frac{S_{\text{pedal}}}{S} \leq 1 - \frac{1}{4r}\sqrt{a^2+b^2+c^2+d^2},$$

where $a = |BL|$, $b = |LC|$, $c = |MD|$, $d = |AN|$, and S_{pedal} is the area of the pedal quadrilateral $KLMN$. The equality holds if $ABCD$ is a square.

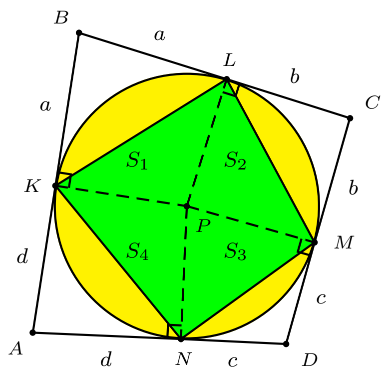


Figure 5

Proof. As seen in Figure 5, if S_1, S_2, S_3 and S_4 are respectively the areas of triangles KPL, LPM, MPN and NPK, then we get

$$\begin{aligned} S_1 &= a \cdot r - \frac{a^2 \sin B}{2}, \\ S_2 &= b \cdot r - \frac{b^2 \sin C}{2}, \\ S_3 &= c \cdot r - \frac{c^2 \sin D}{2}, \\ S_4 &= d \cdot r - \frac{d^2 \sin A}{2}. \end{aligned}$$

Hence, it is obvious that

$$S_1+S_2+S_3+S_4 = r(a+b+c+d) - \frac{1}{2}(a^2\sin B + b^2 \sin C + c^2 \sin D + d^2 \sin A).$$

Since we also have $S = (a+b+c+d)r$, we get

$$\begin{aligned} \frac{S_{\text{pedal}}}{S} &= \frac{S_1+S_2+S_3+S_4}{S} \\ &= 1 - \frac{a^2\sin B + b^2 \sin C + c^2 \sin D + d^2\ \sin A}{2r(a+b+c+d)}. \end{aligned} \tag{12}$$

By the Cauchy-Schwarz Inequality, we get

$$\begin{aligned} a+b+c+d = &\ a\cdot 1+b\cdot 1+c\cdot 1+d\cdot 1 \\ \leq &\ \sqrt{a^2+b^2+c^2+d^2}\cdot\sqrt{1^2+1^2+1^2+1^2} \\ = &\ 2\sqrt{a^2+b^2+c^2+d^2}. \end{aligned}$$

Thus,

$$-\frac{1}{a+b+c+d} \leq -\frac{1}{2\sqrt{a^2+b^2+c^2+d^2}}.$$

By the equality (12),

$$\begin{aligned} \frac{S_{\text{pedal}}}{S} = &\ 1-\frac{a^2\sin B+b^2\sin C+c^2\sin D+d^2\sin A}{2r(a+b+c+d)} \\ \leq &\ 1-\frac{a^2+b^2+c^2+d^2}{4r\sqrt{a^2+b^2+c^2+d^2}} \\ = &\ 1-\frac{1}{4r}\sqrt{a^2+b^2+c^2+d^2}. \end{aligned}$$

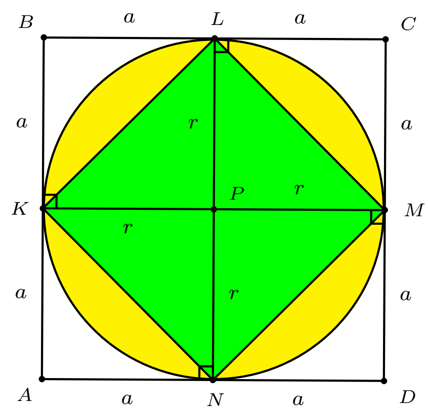


Figure 6

Now suppose that $ABCD$ is a square and $|AB|=|BC|=|CD|=|AD|=2a$. As seen in Figure 6, since $r=a$, $S=4a^2$, $S_{\text{pedal}}=2a^2$, we have

$$\frac{S_{\text{pedal}}}{S}=\frac{2a^2}{4a^2}=\frac{1}{2}=1-\frac{1}{4a}\sqrt{a^2+a^2+a^2+a^2}=1-\frac{1}{4r}\sqrt{a^2+b^2+c^2+d^2}.$$

□

Proposition 5. *Let P be a point inside a polygon $A_1A_2A_3\cdots A_n$ where $|A_1A_2|=a_1$, $|A_2A_3|=a_2,\ldots,|A_nA_1|=a_n$. If the lengths of the perpendiculars from P to the sides of the polygon $A_1A_2A_3\cdots A_n$ are respectively $h_1, h_2, \ldots, h_n$, then*

$$\frac{S}{C^2}\geq\frac{1}{2}\cdot\frac{1}{\frac{a_1}{h_1}+\cdots+\frac{a_n}{h_n}}$$

where S and C are respectively the area and the perimeter of $A_1A_2A_3\cdots A_n$. The equality holds if $a_1 = a_2 = \cdots = a_n = a$ and $h_1 = h_2 = \cdots = h_n = h$.

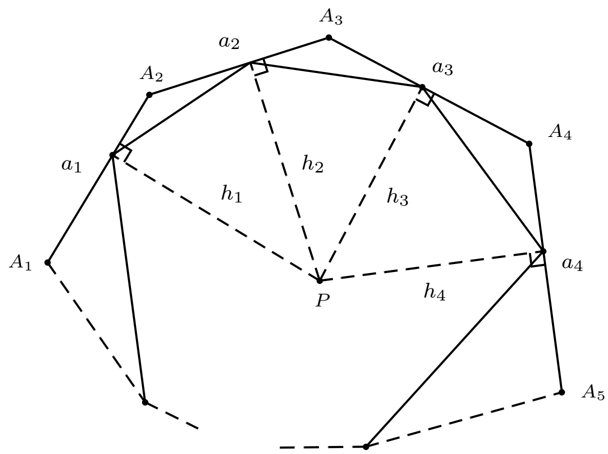


Figure 7

Proof. As seen in Figure 7, it is obvious that $2S = a_1h_1 + a_2h_2 + \cdots + a_nh_n$. By the Cauchy-Schwarz Inequality, we get

$$(a_1h_1+a_2h_2+\cdots+a_nh_n)\left(\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}\right) \geq (\sqrt{a_1^2}+\sqrt{a_2^2}+\cdots+\sqrt{a_n^2})^2.$$

Since

$$2S\left(\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}\right) \geq (\sqrt{a_1^2}+\sqrt{a_2^2}+\cdots+\sqrt{a_n^2})^2$$

we have

$$2S \geq \frac{(a_1+a_2+\cdots+a_n)^2}{\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}} = \frac{C^2}{\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}}.$$

Hence,

$$\frac{S}{C^2} \geq \frac{1}{2}\cdot\frac{1}{\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}}.$$

Now suppose that $a_1 = a_2 = \cdots = a_n = a$, and $h_1 = h_2 = \cdots = h_n = h$. Since,

$$S = n\cdot\frac{ah}{2}, \qquad C = na,$$

we get

$$\frac{S}{C^2} = \frac{nah}{2n^2a^2} = \frac{h}{2na} = \frac{1}{2}\cdot\frac{1}{\frac{a}{h}+\frac{a}{h}+\cdots+\frac{a}{h}} = \frac{1}{2}\cdot\frac{1}{\frac{a_1}{h_1}+\frac{a_2}{h_2}+\cdots+\frac{a_n}{h_n}}.$$

□

Proposition 6. *Let P be a point inside a polygon $A_1A_2A_3\cdots A_n$ where $|A_1A_2| = a_1$, $|A_2A_3| = a_2, \ldots, |A_nA_1| = a_n$. If the lengths of the perpendiculars from P to the sides of the polygon $A_1A_2A_3\cdots A_n$ are respectively $h_1, h_2, \ldots, h_n$, then*

$$\frac{1}{a_1h_1}+\frac{1}{a_2h_2}+\cdots+\frac{1}{a_nh_n}\geq\frac{n^2}{2S}$$

where S is the area of $A_1A_2A_3\cdots A_n$. The equality holds if $a_1 = a_2 = \cdots = a_n = a$ and $h_1 = h_2 = \cdots = h_n = h$.

Proof. By the Cauchy-Schwarz Inequality, we have

$$(a_1h_1+a_2h_2+\cdots+a_nh_n)\left(\frac{1}{a_1h_1}+\frac{1}{a_2h_2}+\cdots+\frac{1}{a_nh_n}\right)\geq(\underbrace{1+1+\cdots+1}_{n\text{ times}})^2.$$

Since $2S = a_1h_1 + a_2h_2 + \cdots + a_nh_n$, it is easy to see that

$$\frac{1}{a_1h_1}+\frac{1}{a_2h_2}+\cdots+\frac{1}{a_nh_n}\geq\frac{n^2}{2S}.$$

Now suppose that $a_1 = a_2 = \cdots = a_n = a$ and $h_1 = h_2 = \cdots = h_n = h$. Since $S = n\cdot\frac{ah}{2}$, we have

$$\frac{1}{a_1h_1}+\frac{1}{a_2h_2}+\cdots+\frac{1}{a_nh_n}=n\cdot\frac{1}{ah}=\frac{n^2}{nah}=\frac{n^2}{2S}.$$

□

Theorem 7. *Let $ABCD$ be an inscribed quadrilateral and P be a point inside $ABCD$. If the feet of perpendiculars from P to the line segments $[AB]$, $[BC]$, $[CD]$ and $[AD]$ are A_1, B_1, C_1, D_1 respectively, then*

$$|PA|+|PB|+|PC|+|PD|>4\sqrt[4]{|PA_1|\cdot|PA_2|\cdot|PA_3|\cdot|PA_4|}.$$

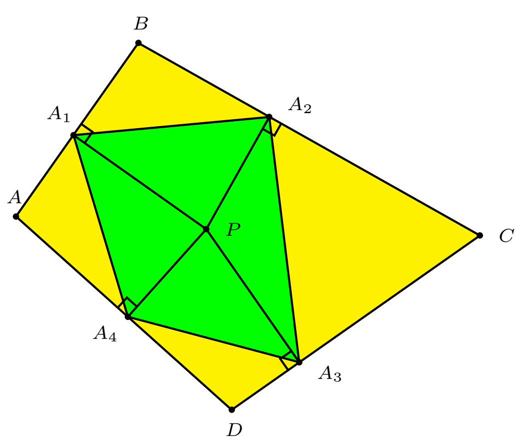


Figure 8

Proof. Let $|PA_1| = h_1$, $|PA_2| = h_2$, $|PA_3| = h_3$ and $|PA_4| = h_4$. Then by the law of cosines, we have

$$\begin{aligned}
|A_1A_2| &= \sqrt{h_1^2 + h_2^2 - 2h_1h_2\cos(\pi - B)},\\
|A_2A_3| &= \sqrt{h_2^2 + h_3^2 - 2h_2h_3\cos(\pi - C)},\\
|A_3A_4| &= \sqrt{h_3^2 + h_4^2 - 2h_3h_4\cos(\pi - D)},\\
|A_4A_1| &= \sqrt{h_4^2 + h_1^2 - 2h_1h_4\cos(\pi - A)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|A_1A_2|^2 &= h_1^2 + h_2^2 - 2h_1h_2\cos(\pi - B)\\
&= h_1^2 + h_2^2 - 2h_1h_2\cos(A + C + D - \pi)\\
&= h_1^2 + h_2^2 + 2h_1h_2\cos(A + C + D)\\
&= h_1^2 + h_2^2 + 2h_1h_2\cos(A + C)\cos D - 2h_1h_2\sin(A + C)\sin D\\
&= (h_2\sin D - h_1\sin(A + C))^2 + (h_2\cos D + h_1\cos(A + C))^2.
\end{aligned}$$

Since $A + C = \pi$, we have $|A_1A_2|^2 = (h_2\sin D)^2 + (h_2\cos D - h_1)^2$.

Now suppose that $(h_2\cos D - h_1)^2 = 0$. Then, $\cos D = \frac{h_1}{h_2}$. Since $\angle A_1PA_2 = \angle D$, by the law of cosines,

$$\begin{aligned}
|A_1A_2|^2 &= h_1^2 + h_2^2 - 2h_1h_2\cos D\\
&= h_1^2 + h_2^2 - 2h_1h_2 \cdot \frac{h_1}{h_2}\\
&= h_2^2 - h_1^2.
\end{aligned}$$

Hence, we get $h_2^2 = |A_1A_2|^2 + h_1^2$, implying that $\angle PA_1A_2 = 90°$, which is a contradiction. Therefore, $(h_2\cos D - h_1)^2 \neq 0$. Then, it is easy to verify that

$$|A_1A_2| > h_2\sin D \tag{13}$$

Similarly, we have

$$|A_2A_3| > h_3\sin A \tag{14}$$
$$|A_3A_4| > h_4\sin B \tag{15}$$
$$|A_1A_4| > h_1\sin C \tag{16}$$

Since PA_1BA_2, PA_2CA_3, PA_3DA_4 and PA_4AA_1 are inscribed quadrilaterals, if we apply the law of sines to the triangles AA_1A_4, BA_1A_2, CA_2A_3, DA_3A_4, then we respectively get,

$$|PA| = \frac{|A_1A_4|}{\sin A} \tag{17}$$

$$|PB| = \frac{|A_1A_2|}{\sin B} \tag{18}$$

$$|PC| = \frac{|A_2A_3|}{\sin C} \tag{19}$$

$$|PD| = \frac{|A_3A_4|}{\sin D} \tag{20}$$

If the inequalities (13), (14), (15) and (16) are respectively considered together with the inequalities (17), (18), (19) and (20), then

$$\begin{aligned}
|PB| &= \frac{|A_1A_2|}{\sin B} > \frac{h_2 \sin D}{\sin B},\\
|PC| &= \frac{|A_2A_3|}{\sin C} > \frac{h_3 \sin A}{\sin C},\\
|PD| &= \frac{|A_3A_4|}{\sin D} > \frac{h_4 \sin B}{\sin D},\\
|PA| &= \frac{|A_1A_4|}{\sin A} > \frac{h_1 \sin C}{\sin A}.
\end{aligned}$$

Therefore, it is easy to see that

$$\begin{aligned}
|PA| + |PB| + |PC| + |PD| &> \frac{h_1 \sin C}{\sin A} + \frac{h_2 \sin D}{\sin B} + \frac{h_3 \sin A}{\sin C} + \frac{h_4 \sin B}{\sin D}\\
&> 4\sqrt[4]{\frac{h_1 \sin C}{\sin A} \cdot \frac{h_2 \sin D}{\sin B} \cdot \frac{h_3 \sin A}{\sin C} \cdot \frac{h_4 \sin B}{\sin D}}\\
&= 4\sqrt[4]{|PA_1| \cdot |PA_2| \cdot |PA_3| \cdot |PA_4|}
\end{aligned}$$

by Arithmetic-Geometric Mean Inequality. □

References

[1] C. Alsina and R. B. Nelsen, A visual proof of the Erdős-Mordell inequality, *Forum Geom.*, 7 (2007) 99–102.

[2] L. Bankoff, An elementary proof of the Erdős-Mordell theorem, *Amer. Math. Monthly*, 65 (1958), 521.

[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. America, 1967.

[4] P. Erdős, Problem 3740, *Amer. Math. Monthly*, 42 (1935) 396.

[5] D. Ferrarello, M. F. Mammana, and M. Pennisi, Pedal poygons, *Forum Geom.*, 13 (2013) 153–164.

[6] R. Honsberger, *Episodes In Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. America, 1995.

[7] R. A. Johnson, *Advanced Euclidean Geometry*, Dover Publications, 2007.

[8] D. K. Kazarinoff, A simple proof of the Erdős-Mordell inequality for triangles, *Michigan Math. Journal*, 4 (1957) 97–98.

[9] M. F. Mammana, B. Micale, and M. Pennisi, Orthic quadrilaterals of a convex quadrilateral, *Forum Geom.*, 10 (2010) 79–91.

[10] L. J. Mordell and D. F. Barrow, (1937). Solution to Problem 3740, *Amer. Math. Monthly*, 44 (1937) 252–254.
[11] M. Şahin, *Matematik Olimpiyatlarına Hazırlık Geometri - I*, Palme Yayınları, Ankara, 2000.

Şahlar Meherrem: Yasar University, Faculty of Science and Letters, Department of Mathematics, Izmir, Turkey
E-mail address: sahlar.meherrem@yasar.edu.tr

Gizem Günel Açıksöz: Ozel Ege Lisesi, Izmir, Turkey
E-mail address: gizem.aciksoz@egelisesi.k12.tr

Serenay Şen : Ozel Ege Lisesi, Izmir, Turkey
E-mail address: serenay.sen3@gmail.com

Zeynep Sezer : Ozel Ege Lisesi, Izmir, Turkey
E-mail address: gunes_baskes@hotmail.com

Güneş Başkes: Ozel Ege Lisesi, Izmir, Turkey
E-mail address: zeynepsezer99@gmail.com

Forum Geometricorum
Volume 18 (2018) 321–347.

FORUM GEOM
ISSN 1534-1178

Self-Pivoting Convex Quadrangles

Paris Pamfilos

Abstract. In this article we examine conditions for the existence of pivoting circumscriptions of a convex quadrangle q_1 by another quadrangle q similar to q_1, pivoting about a fixed point. There are 8 types of such circumscriptions and we formulate the restrictions on the quadrangles of each type, imposed by the requirement of such a circumscription. In most cases these restrictions imply that the quadrangles are cyclic of a particular type, in one case are harmonic quadrilaterals, and in two cases are trapezia of a special kind.

1. Possible configurations

Given two convex quadrangles $\{q_1 = A_1B_1C_1D_1, q_2 = A_2B_2C_2D_2\}$, we study the circumscription of a quandrangle $q = ABCD$, which is similar to q_2, about the quadrangle q_1. Disregarding, for the moment, the right proportions of sides, and focusing only on angles, we can easily circumscribe to q_1 quadrangles q with the same *angles* as those of q_2 and in the same succession. For this, it suffices to consider the circles $\{\kappa_i, i = 1, 2, 3, 4\}$, the points of each viewing corresponding

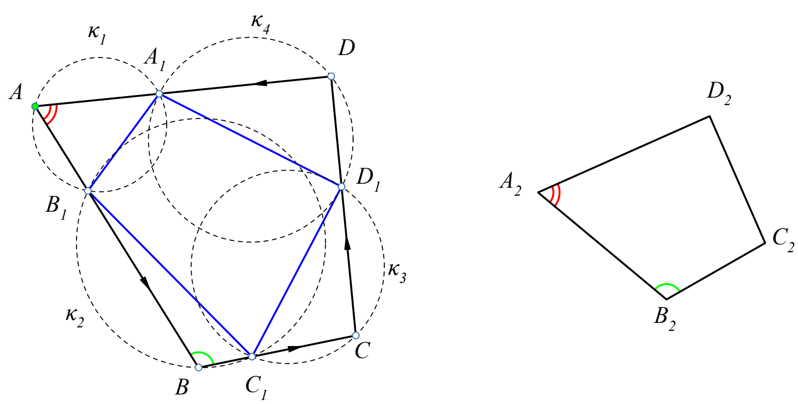


Figure 1. Circumscribing q_2 about q_1

sides of q_1 under corresponding angles of q_2 or their supplement. Then, taking an arbitrary point A on κ_1, we draw the line AB_1 intersecting κ_2 at B, then draw the line BC_1 intersecting κ_3 at C and so on (See Figure 1). It is trivial to see that the procedure closes and defines a circumscribed quadrangle with the same angles as q_2. Below we refer to this construction as *"the standard circumscription procedure"* (SCP). By this, the resulting quadrangles q have the same angles as q_2

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

but in general not the right proportions of sides, that would make them similar to q_2.

There are here several possible configurations, that must be considered, since a vertex of q, A say, can be opposite or *"view"* any of the four sides of q_1. Each vertex of q_2 and each side of q_1 give, in principle, two possibilities for circumscription, corresponding to equally or oppositely oriented quadrangles. Fixing the side A_1B_1 and considering all possibilities for the opposite to it angle of q together with the possible orientations, we see that there are in total 8 possibilities. We call each of them a *"circumscription type of* q_2 *about* q_1*"* and denote it by $q_2(q_1, X)^{\pm}$. In this notation X stands for the vertex of q_2 opposite to the side A_1B_1 of q_1 and the sign denotes the equal or inverse orientation of the circumscribing to the circumscribed quadrangle. Thus, in figure-1 the symbol denoting the case would be $q_2(q_1, A)^+$.

In general there are 8 different types of circumscription, each type delivering infinite many circumscribed quadrangles q with the same angles as q_2. In some special cases the number of types can be smaller, as f.e. in the case of a circumscribing square, in which there is only one type. We stress again the fact, that if we stick to a type and repeat the procedure SCP, starting from arbitrary points $A \in \kappa_1$, we obtain in general quadrangles with the same angles as q_2 but varying ratios of side-lengths. There is an exceptional case in this procedure, which produces circumscribed quadrangles q of the same similarity type, for every starting position of A on κ_1. This is the *"pivotal case"* examined in the next section.

2. The pivotal case

This type of circumscription, besides the quadrangle $q_1 = A_1B_1C_1D_1$, which we circumscribe, it is characterized by an additional point, the *"pivot"*, not lying on the side-lines of q_1. The pivot, for a generic convex quadrangle, defines four circles $\kappa_1 = (A_1B_1P)$, $\kappa_2 = (B_1C_1P)$, $\kappa_3 = (C_1D_1P)$, and $\kappa_4 = (D_1A_1P)$ (See

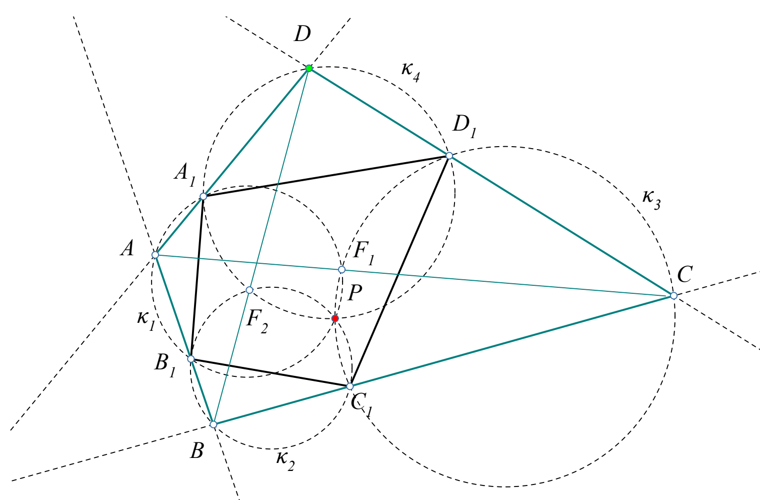


Figure 2. The pivotal case

Figure 2), and through them a circumscribed quadrangle $q = ABCD$, that can be non convex, even degenerate in the case P is the *Miquel Point* of the complete quadrilateral generated by q_1 ([8, p.139]). Having the circles $\{\kappa_i\}$, a circumscribed quadrilateral can be defined by starting from an arbitrary point A on κ_1 and applying the SCP procedure. Measuring the angles at the second intersection point F_1 of the circles $\{\kappa_1, \kappa_3\}$

$$\widehat{AF_1P} = \widehat{PB_1B} \quad \text{and} \quad \widehat{CF_1P} = \widehat{PC_1B},$$

we see that F_1 is on the diagonal AC of q. Analogously we see that the intersection point F_2 of the other pair of opposite lying circles $\{\kappa_2, \kappa_4\}$ is on the diagonal BD of q. This implies that the triangles created by the diagonals of q are of fixed similarity type, independent of the position of A on κ_1. Thus, we obtain an infinity of pairwise similar quadrangles circumscribing q_1 in a way that justifies the naming *"pivoting"*.

At this point we should notice that the *pivotal* attribute is a characteristic of a particular *type of circumscription of* q_2 *about* q_1. The same q_2 can have a pivotal type of circumscription about q_1 and at the same time have also another type of circumscription, which is non-pivotal. Figure 3 shows such a case. The quadrangle

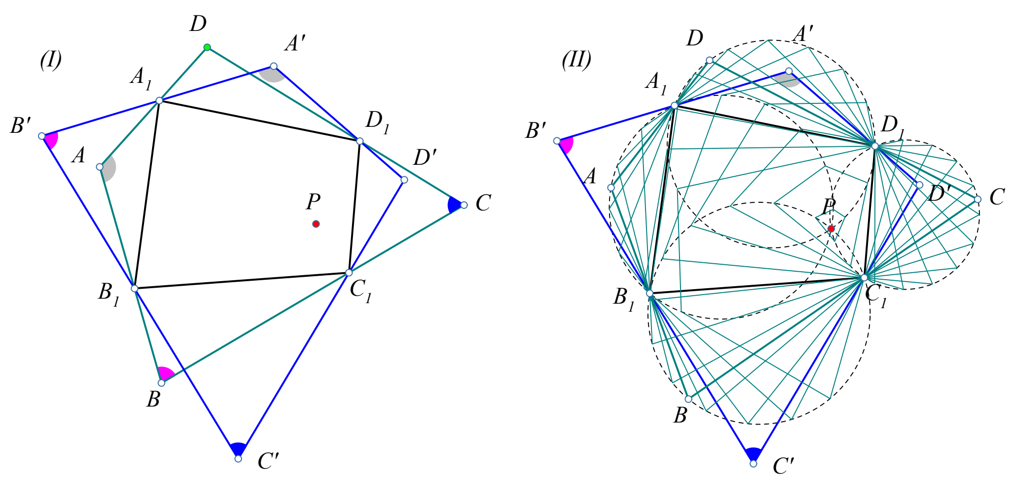


Figure 3. A pivoting circumscription and a non-pivoting one

$q = q_2 = ABCD$ is pivotal about q_1 with shown pivot P. This circumscription is described by the symbol $q_2(q_1, A)^+$. The other quadrangle $q' = A'B'C'D'$, circumscribing q_1, is similar to $q = q_2$, but it is not pivotal. Opposite to side A_1B_1 has the angle $\widehat{B}$ of q_2, its symbol being $q_2(q_1, B)^+$. The characteristic property of the pivotal circumscription is formulated in the next theorem.

Theorem 1. *The convex quadrangle q_2 is pivotal of a given type for q_1, if and only if the pedal quadrangle of q_2 w.r. to some point P is similar to q_1 and q_2 has the given type w.r to that pedal.*

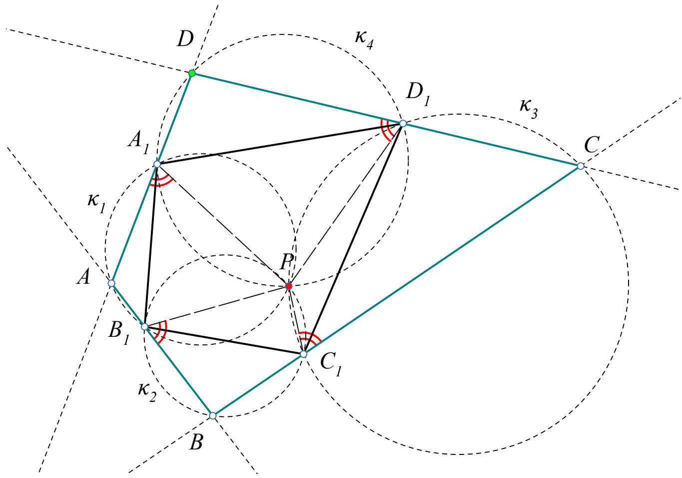


Figure 4. Pivotal q about its pedal (similar to) q_1

Proof. The proof follows from the equalities of the angles shown in figure 4, which results easily from the inscribed quadrangles in the circles $\{\kappa_i\}$. These angles vary with the location of the point A on κ_1 and become right when A obtains the diametral position of P on κ_1, showing that q_1 is then the pedal of $q_2 = ABCD$ w.r. to P. The inverse is equally trivial. If q_1 is the pedal of P, then, per definition, the segments $\{PA_1, PB_1, \ldots\}$ are orthogonal to the sides $\{DA, AB, \ldots\}$ and the circles $\{\kappa_i\}$ are defined and carry the vertices of quadrangles $ABCD$, pairwise similar and pivoting about P. □

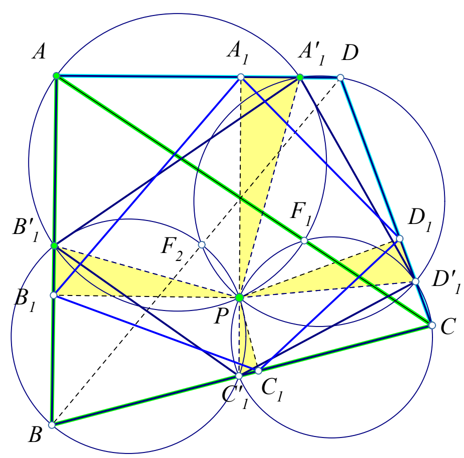


Figure 5. Pedal and dual viewpoint

There is an interesting dual view of the pivotal circumscription, which we should notice here (See Figure 5), but which we do not follow further in the sequel. If

$q_1 = A_1B_1C_1D_1$ is the pedal of $q = ABCD$ w.r. to some point P, then turning the projecting lines $\{PA_1, PB_1, \ldots\}$ by the same angle ϕ, we obtain points $\{A_1', B_1', \ldots\}$ on the sides of q, defining the quadrangle $q_1' = A_1'B_1'C_1'D_1'$, similar to q_1 and inscribed in q. Taking the obvious similarity f to draw q_1' back to q_1 and applying it to q, we find $q' = f(q)$ similar to q and circumscribing q_1. Thus, pivotal circumscriptions about a point are dual and correspond, in this sense, to some pivotal inscriptions about the same point and with the same similarity types of the involved quadrangles.

3. Self-pivoting quadrangles

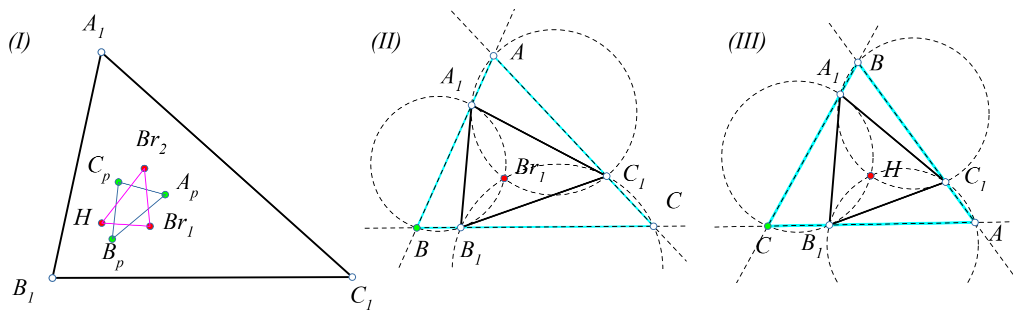


Figure 6. Triangle self-pivoting about a Brocard point and the orthocenter

By *"self-pivoting"* is meant a convex quadrangle q that allows pivotal circumscriptions by quadrangles q' similar to q. The corresponding problem for triangles leads to the two well known Brocard points $\{Br_1, Br_2\}$ ([7, p.94], [9, p.117]), the orthocenter H and three additional points $\{A_p, B_p, C_p\}$ (See Figure 6-I), which, for a generic triangle, are the 6 internal pivots of circumscribed triangles ABC similar to the triangle of reference $A_1B_1C_1$. Thus, every triangle is self-pivoting in all possible ways, which means that the pivoting triangle ABC may have opposite to A_1B_1 any of the three angles of the triangle and the orientation of the circumscribing can be the same or the opposite of the circumscribed one.

The situation for quadrangles is quite different. A quadrangle may not allow a pivotal circumscription by quadrangles similar to itself or allow the pivotal circumscription for some types of circumscription only. In the sections to follow we investigate the possibility of self pivotal circumscription by one of the 8 types of circumscription, which, in principle, could be possible and are schematically displayed in figure 7. The symbols used are abbreviations of those introduced in the preceding section, q denoting the similarity type of a convex quadrangle. Thus, the symbol $q(A)^+$ denotes a pivoting configuration, in which the circumscribed and the circumscribing quadrangle have the same orientation, are both of the similarity type of the quadrangle $q = ABCD$, and the circumscribing has its vertex A lying opposite to the side AB of the circumscribed quadrangle.

Looking a bit closer to the different types, reveals some similarities between them, implying similarities of the corresponding circumscription structures. For

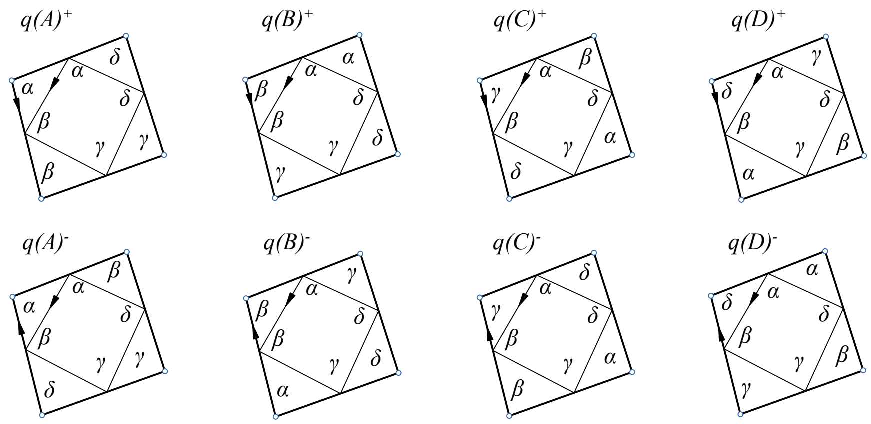


Figure 7. Types of self-pivoting quadrangles $q' \sim q$ about q

instance, the first two types $\{q(A)^+, q(B)^+\}$ are the only types for which each vertex X of the circumscribing is viewing a side VW of the circumscribed by an angle ϕ, which is also adjacent to the side VW of the circumscribed quadrangle. We could call this phenomenon an *"adjacent angle side viewing"* of the vertex X and use for it the acronym ASV. Thus, for these two types all four vertices have the ASV property. The next two $\{q(C)^+, q(D)^+\}$ are the only circumscription types which have no vertices with the ASV property. Finally, for all other types the ASV property is valid for precisely two vertices lying oppositely. The following discussion shows that self-pivoting quadrangles with the same number of ASV vertices have some common geometric properties, e.g. the types $\{q(C)^+, q(D)^+\}$ consist of trapezia of a particular kind (section 7).

4. The types $q(A)^+$ and $q(B)^+$

The symbol $q(A)^+$ resp. $q(B)^+$, represents the pivotal circumscription of a quadrangle by one similar to itself and equally oriented, opposite to side AB having the angle $\widehat{A}$ resp. $\widehat{B}$. Figure 8 shows an example of the type $q(B)^+$. The characteristic of $q(B)^+$ is that the circle κ_1 carrying the vertices B is tangent to B_1C_1, the circle κ_2 carrying the vertices C is tangent to C_1D_1 and so on. The type $q(A)^+$ would change the meaning and orientation of these circles, i.e. κ_1' would carry the vertices of angles $\widehat{A}$ and would be tangent to A_1D_1, κ_2' would carry the vertices of angles $\widehat{B}$ and would be tangent to A_1B_1 and so on. For these two types, the circumscribing quadrangle q can take the position and become identical with the circumscribed q_1. In the case $q(B)^+$, shown in the figure, the tangency of the circles to corresponding line-sides of the quadrangle implies that the angles $\{\widehat{BA_1B_1}, \widehat{CB_1C_1}, \ldots\}$ are equal. For the same reason also the lines from the pivot P to the vertices, make with the sides equal angles

$$\widehat{PA_1B_1} = \widehat{PB_1C_1} = \widehat{PC_1D_1} = \widehat{PD_1A_1} = \omega.$$

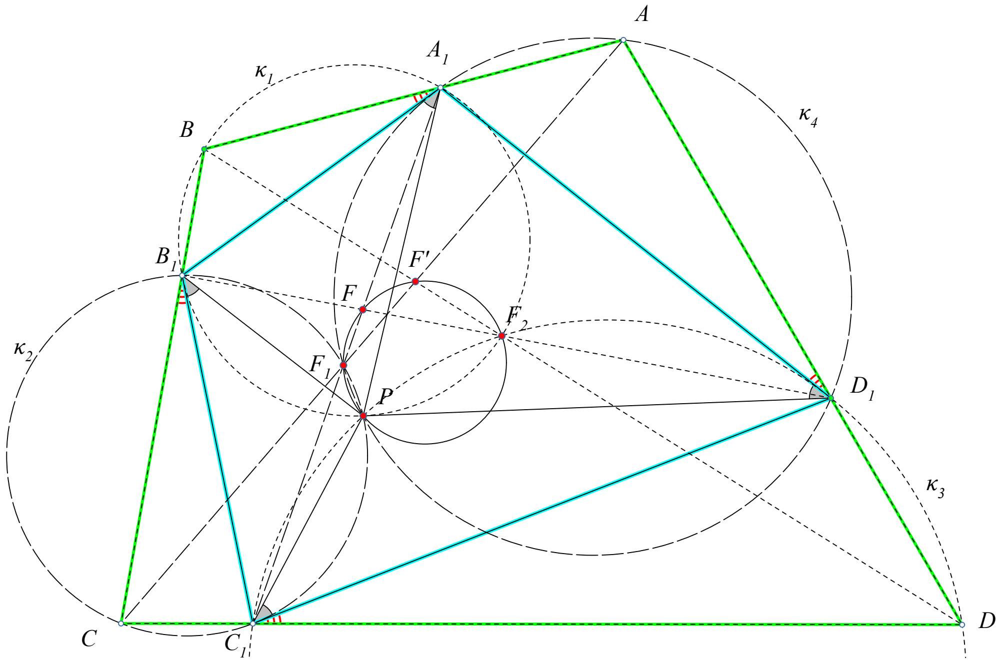


Figure 8. Self-pivoting of type $q(B)^+$

This is reminiscent of the way the Brocard points of a triangle are defined. In fact, if the quadrangle $ABCD$ is *cyclic*, then the property of self-pivoting of type $q(A)^+$ or $q(B)^+$ is known to be equivalent with the harmonicity of the quadrangle ([6]). In this case the two types occur similtaneously, i.e. if the quadrangle is harmonic, then it is also simultaneously pivotal of type $q(A)^+$ and $q(B)^+$. And if it is cyclic and pivotal of one of these two types, then it is pivotal of the other type too. The corresponding pivots are then the two, so called, *"Brocard points"* of the harmonic quadrangle ([13]) and the angles ω for the two pivotal circumscriptions coincide with the, so called, *"Brocard angle"* of the harmonic quadrangle. Next theorem gives a related characterization of the harmonic quadrangle, without to assume that it is cyclic, but deducing this property from the possibility to have simultaneously the two types of pivoting.

Theorem 2. *A convex quadrangle $q_1 = A_1B_1C_1D_1$ is harmonic, if and only if it is simultaneously self-pivoting of type $q(A)^+$ and $q(B)^+$.*

Proof. By the preceding remarks, it suffices to show that, if the quadrangle is simultaneously self-pivoting of type $q(A)^+$ and $q(B)^+$, then it is cyclic. For this we use the remark also made in section 2, that the diagonal AC of a pivoting quadrangle passes through a fixed point F_1 lying on the diagonal A_1C_1 of the circumscribed quadrangle q_1. This is seen in figure 9-I for the pivoting of type $q(A)^+$. The crucial step in the proof is to show that the pivoting of type $q(B)^+$ has the corresponding F_1' on A_1C_1 identical with F_1. This is seen by considering a particular position of the $q(A)^+$ pivoting, seen in figure 9-II. For this position the diagonal

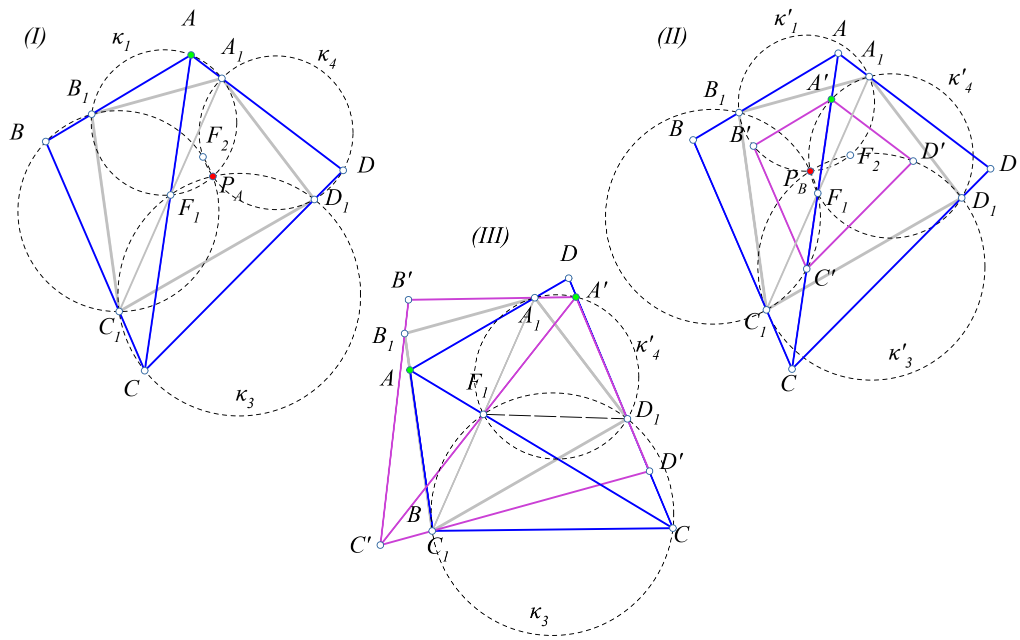


Figure 9. Simultaneous self-pivoting of types $q(A)^+$ and $q(B)^+$

AC of the pivoting intersects the circle κ'_4 of the $q(B)^+$ pivoting at a point A'. This point, taken as vertex of the $q(B)^+$ pivoting quadrangle $q' = A'B'C'D'$, shows that the diagonals $\{A'C', AC\}$ of the quadrangles q' and $q = ABCD$ coincide. Thus, both diagonals pass through the same fixed point of A_1C_1, thereby proving the coincidence $F'_1 = F_1$.

The last step, in proving that q is cyclic, follows by considering also special positions for the pivoting quadrangles of the two types. These positions are seen in figure 9-III. The quadrangle q is now taken so that its vertex B coincides with C_1, the vertex A obtaining then a position on B_1C_1. For the pivoting quadrangle q' we choose then the position for which A' is the intersection of CD with the circle κ'_4, carrying the vertices A' from which A_1D_1 is seen under the angle $\widehat{A_1}$. It is easily verified that the points $\{D, A', D_1, D', C\}$ are then collinear. Besides, the circles $\{\kappa'_4, \kappa_3\}$ pass both through F_1 and the quadrangles $\{A'A_1F_1D_1, D_1F_1C_1C\}$ are cyclic. Considering the angles of these quadrangles, we see that $\{A'B', CB\}$ are parallel

$$\widehat{A'} + \widehat{C} = \widehat{D_1F_1C_1} + \widehat{D_1F_1A_1} = \pi,$$

which is equivalent with $\widehat{A} + \widehat{C} = \pi$ and proves the theorem. □

In the rest of this section we describe a general procedure, which, starting with an arbitrary triangle, produces, under some restrictions, general self-pivoting quadrangles of types $q(A)^+$ or/and $q(B)^+$, not necessarily cyclic. We call the corresponding angle $\omega = \widehat{PA_1B_1} = \widehat{PB_1C_1} = \ldots$ the *"Brocard Angle"* of the

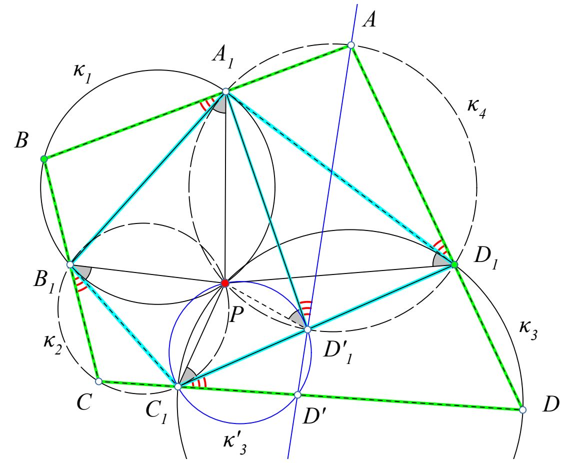


Figure 10. A second self-pivoting of the same type $q(B)^+$

pivotal circumscription (See Figure 10). First we formulate a simple lemma showing that a self-pivoting quadrangle of this two types is related to other self-pivoting quadrangles of the same type and the same Brocard angle. The lemma, formulated for brevity as a property of the circle κ_4 and the type $q(B)^+$, holds analogously for the type $q(A)^+$, and in both cases the circle κ_4 can be replaced, with the necessary adaptation, by any one of the circles $\{\kappa_i\}$.

Lemma 3. *If the circle κ_4, of a self-pivoting quadrangle q_1 of type $q(B)^+$, which is tangent to the side A_1B_1, has a second intersection D_1' with the opposite side C_1D_1, so that $q_1' = A_1B_1C_1D_1'$ is convex, then this is also a self-pivoting quadrangle of the same type, with the same angles as q_1, and the same Brocard angle.*

Proof. Referring to the case of figure 10, all that is needed here is to show that the points $\{D, D_1', A\}$ are collinear, which follows from a trivial angle chasing argument. The quadrangle $q_1' = A_1B_1C_1D_1'$ has the same angles with q_1, the two angles at $\{B_1, C_1\}$ being identical and the angles at $\{A_1, D_1'\}$ being those at $\{A_1, D_1\}$ permuted. □

Corollary 4. *The two quadrangles $\{q_1, q_1'\}$ of the preceding lemma, if they exist, they are completely determined by the triangle A_1B_1P and the position of C_1 on the tangent ε of the circumcircle κ_1 at B_1.*

Proof. In fact, the given point C_1 determines the circle κ_2. The circle κ_4 is determined by its property to be tangent to A_1B_1 at A_1 and the points $\{D_1', D_1\}$ are the intersections of circle κ_4 with the tangent of κ_2 at C_1. The angle $\widehat{PA_1B_1}$ of the triangle is the Brocard angle of the circumscription of this type. □

As is suggested by the lemma and its corollary, not every triangle $\tau = A_1B_1P$ allows the determination of two points $\{D_1, D_1'\}$, defining the two corresponding self-pivoting quadrangles. Depending on the angles of the given triangle τ and especially on the angle $\omega = \widehat{PA_1B_1}$, destined to play the role of the Brocard

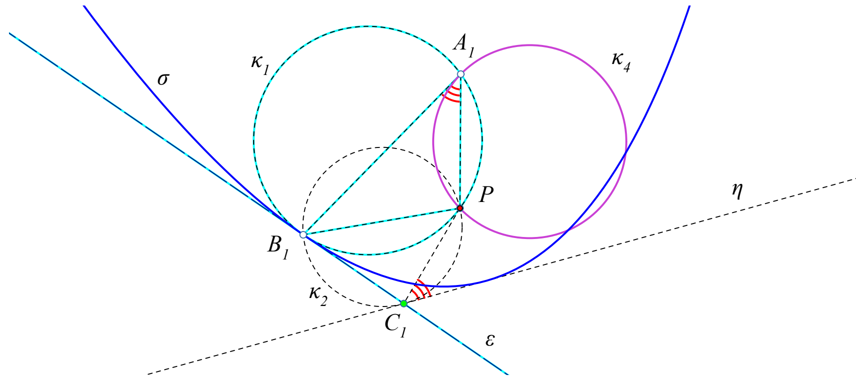


Figure 11. Existence of self-pivoting quadrangles

angle, we may have one, two or none of these points in existence. In fact, consider the points C_1 varying on the tangent ε at B_1 to the circumcircle κ_1 of τ (See Figure 11). Then, by the well known Newton's method to generate conics ([15, p.259], [5, I,p.44]), the lines η, making with PC_1 the constant angle ω, envelope a parabola σ with focus at P and tangent to ε. If this parabola has κ_4 totally in its inner region, then its tangents cannot have common points with κ_4, i.e. there are no points $\{D_i\}$. If σ touches κ_4, then there is only one line η intersecting the circle κ_4, i.e. the points $\{D_1, D_1'\}$ coincide and we have one solution only. Finally, if

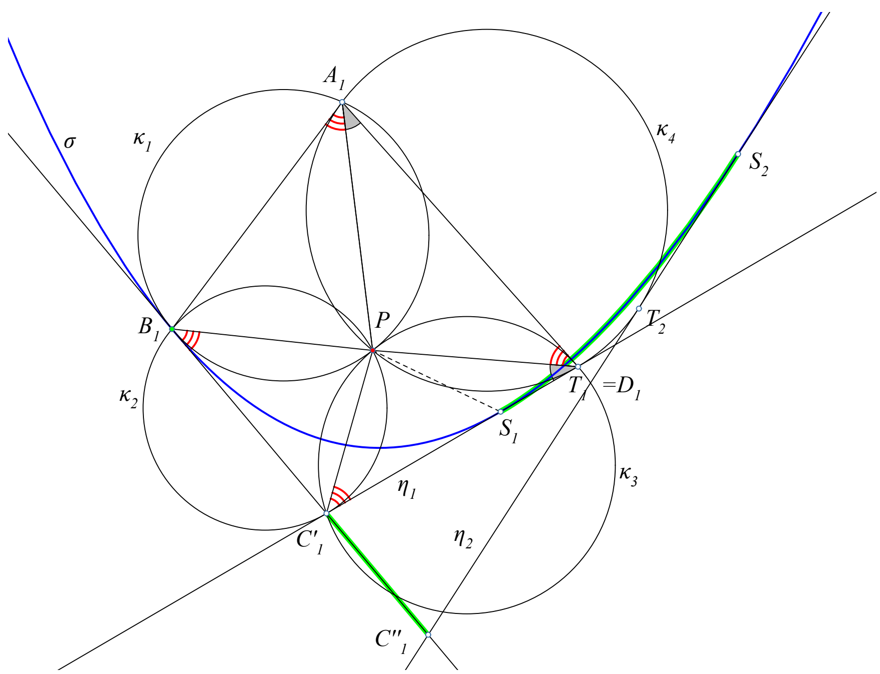


Figure 12. Case in which κ_4 intersects the parabola

the circle and the parabola intersect at two points, the common tangents to $\{\sigma, \kappa_4\}$ define corresponding contact points $\{T_1, T_2\}$ of the circle, contact points $\{S_1, S_2\}$ of the parabola and two points $\{C_1', C_1''\}$ on ε (See Figure 12). The tangents $\eta = C_1S$ of σ intersect κ_4 only if C_1 is in the interval $C_1 \in [C_1', C_1''] \subset \varepsilon$. For the points $C_1 \in (C_1', C_1'')$ of the open interval we have two solutions, whereas at the end points $\{C_1', C_1''\}$ of this interval we have one solution only. Figure 12 shows the special self-pivoting quadrangle having $C_1 = C_1'$ and D_1 identical with the corresponding contact point T_1 of the common tangent η_1 with κ_4. Notice that the circle κ_4 cannot be completely outside of the parabola, since it passes through the focus P of it. Hence, the cases considered above exhaust the possibilities that may occur, concerning the existence or not of self-pivoting quadrangles of type $q(B)^+$, defined from a triangle AB_1P using this recipe.

5. The Brocard angle

In the previous section we defined the Brocard angle of a pivotal circumscription of a quadrangle of the types $q(A)^+$ and $q(B)^+$. This angle is intimately related to the aforementioned generation of parabolas, resulting by varying an angle of fixed measure ([4, p.28]). Figure 13-I shows a consequence of this relation. Here we

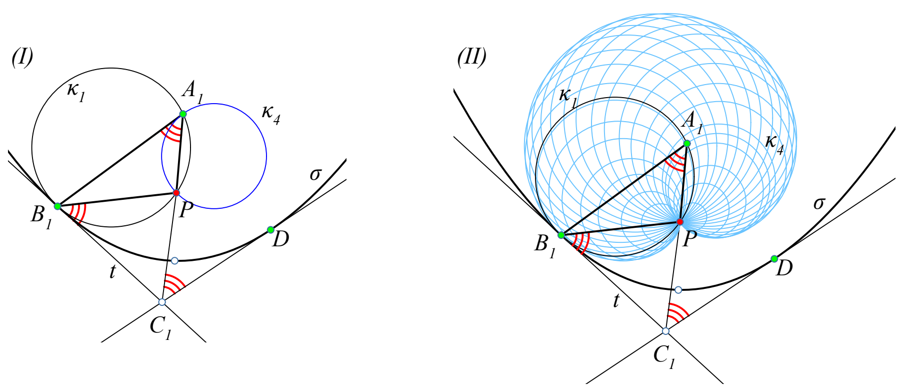


Figure 13. The circles $\{\kappa_4\}$ for $A_1 \in \kappa_1$ generate a cardioid

start with a parabola σ and the tangent t at a point B_1 of it. For every point C_1 on t the, other than t, tangent to the parabola from C_1 makes with the line PC_1 the same constant angle $\omega = \widehat{PA_1B_1}$. The circle κ_1 is defined by its property to be tangent to σ at B_1 and pass through the focus P of the parabola. The points A_1 of κ_1 view the segment PB_1 under the fixed angle $\omega = \widehat{PA_1B_1}$. If there were a pivotal quadrangle of type $q(B)^+$ with Brocard angle ω, then this would produce such a figure, with the circle κ_4 intersecting the parabola. By its definition, the circle κ_4 is tangent to A_1B_1 at A_1 and passes through P. By the discussion in the previous section, in order to have a pivotal circumscription with Brocard angle ω, it is necessary and sufficient to have a place of A_1 on κ_1, such that the corresponding circle κ_4 intersects the parabola. By varying the position of A_1 on κ_1, we obtain a one-parameter family of circles κ_4, which will be seen below to envelope a cardioid tangent to σ at B_1 ([1], [11, p.34], [10, p.118], [14, p.73]). Then, the condition of existence of a circle κ_4 intersecting the parabola appears to be equivalent with the existence of an intersection point of the cardioid and the parabola, other than B_1.

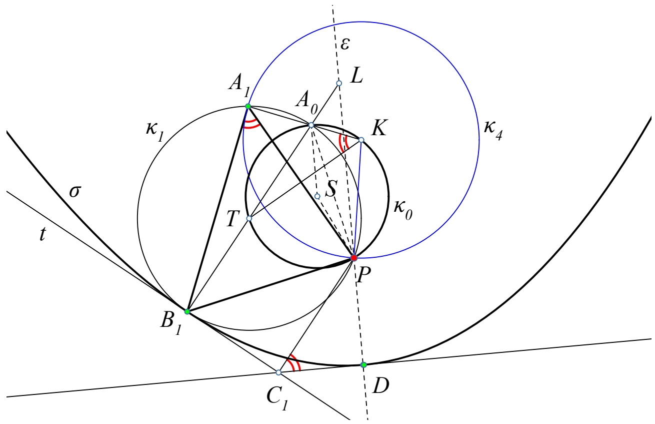


Figure 14. Seeking the position of the cardioid relative to the parabola σ

In the sequel we determine the position of the cardioid, relative to the parabola, showing that the value $\pi/4$ of the Brocard angle ω plays an important role, separating the intersecting from the non-intersecting cases. We start with the case case $\omega = \widehat{PA_1B_1} > \pi/4$, the case $\omega < \pi/4$ allowing a similar handling. For this, consider the tangent at the vertex D of the parabola σ, intersecting the tangent t at the point C_1 (See Figure 14). Let also L be the intersection point of the axis ε of the parabola with the line B_1A_0, where A_0 is the diametral point of B_1 on κ_1. The quadrangle LB_1C_1D is cyclic and, since all tangents of σ intersect line t at points C_1 making with PC_1 the angle ω, the lines $\{LB_1, PC_1\}$ are parallel. Using this and measuring the angles at A_0, we find that $\widehat{A_0PL} = 2\omega - \pi/2$. On the other side, the center K of the circle κ_4 is on the medial line of PA_1 and since κ_4 is tangent to B_1A_1 at A_1, the line A_1K passes through A_0. From the aforementioned tangency follows also that the angle $\widehat{A_0KT} = \omega$, point T being the center of κ_1. Thus, the center K of κ_4 is viewing the fixed segment A_0T under the fixed angle ω. This implies that, as A_1 changes its position on κ_1, the center K, of the corresponding circle, moves on a circle κ_0 passing through the points $\{A_0, T, P\}$. Thus, all circles κ_4 represent one of the standard ways to generate a cardiod, by fixing a point P on a circle κ_0 and for every other point K on κ_0 drawing the circle $\kappa_4(K, |KP|)$ ([11, p.35]). The cardioid is then the envelope of all these circles $\{\kappa_4\}$. It is also well known that the line, which is simultaneously tangent to the cardioid and the generating it circle κ_4, is the symmetric of the tangent of the circle κ_4 at P w.r. to the tangent of the circle κ_0 at the center K of κ_4. Later applied to κ_1, which represents a special position of κ_4, shows that the line t is also tangent to the cardioid. Concerning the location of the center S of the circle κ_0, we notice that the angle $\widehat{A_0SP} = 2\pi - 4\omega$, which implies that $\widehat{SPA_0} = 2\omega - \pi/2 = \widehat{A_0PL}$.

A short calculation shows also that the radius of κ_0 is

$$|PS| = \frac{|PB_1|}{(2\sin(\omega))^2} = \frac{|PD|}{4\sin(\omega)^4}.$$

From this follows immediately that

$$|PS| < |PD| \quad \text{precisely when} \quad \omega > \pi/4. \tag{1}$$

On the other side, the cardioid is intimately related to the parabola, since, as is well

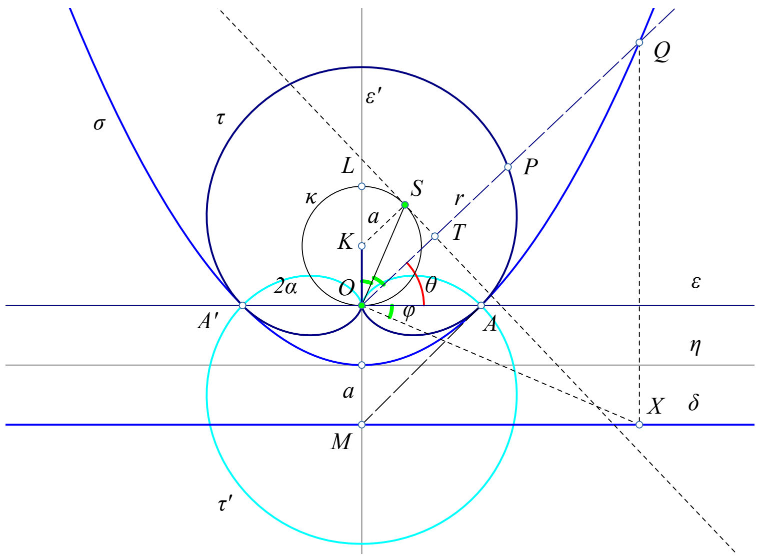


Figure 15. The associated cardioid τ of the parabola σ

known, the cardioid is the inverse of a parabola, the center of inversion taken at the focus of the parabola ([16, p.91,p.131]). Figure 15 shows the result of inversion of the parabola with equation $x^2 = 4ay$, relative to the axis $\{\eta, \varepsilon'\}$, consisting of the tangent at the vertex and the axis of the parabola. The inversion is done w.r. to the focus O and the circle with diameter the *latus rectum* AA' of the parabola and produces the cardioid τ'. Of interest for our discussion is the reflection τ of τ' w.r. to the latus-rectum line ε. This cardioid τ is tangent to the parabola at $\{A, A'\}$. It is also generated as the envelope of the circles $\{\kappa_4(S, |SO|)\}$, for the points S of the circle $\kappa(K, a)$, where K is the symmetric of the vertex of the parabola w.r. to its focus O. All these facts are consequences of straightforward calculations, which I leave as an exercise. I call τ the "*associated to the parabola cardioid*". A short calculation w.r. to the system with axes $\{\varepsilon, \varepsilon'\}$ shows also that corresponding points $\{P, Q\}$ on $\{\tau, \sigma\}$ with the same polar angle θ are represented by

$$r = OP = 2a(1+\sin(\theta)), \quad r' = OQ = \frac{2a}{1-\sin(\theta)} \quad \Rightarrow \quad r' - r = \frac{2a\sin(\theta)^2}{1-\sin(\theta)}. \tag{2}$$

This shows that the cardioid τ is completely inside the inner domain of the parabola, containing the focus, touching the parabola only at the points $\{A, A'\}$.

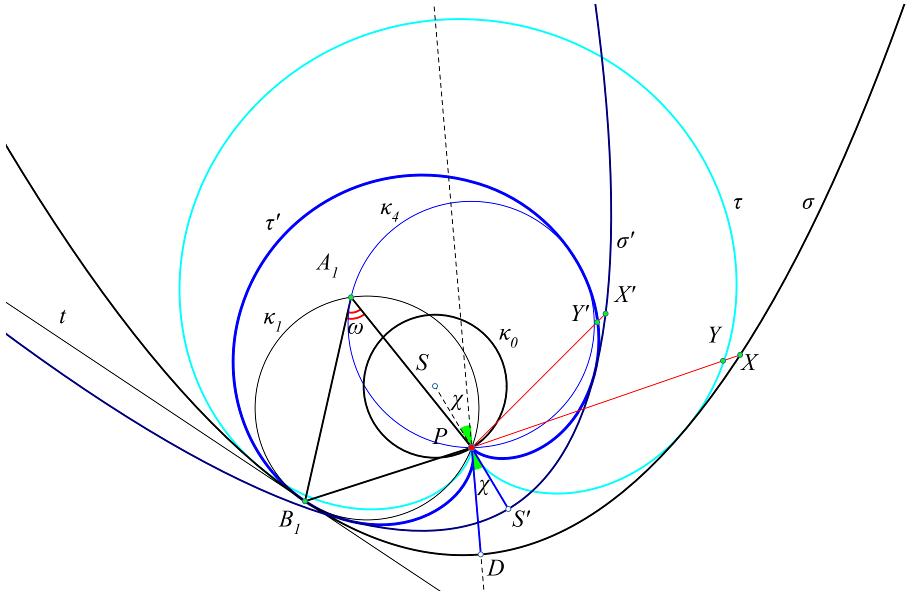


Figure 16. The two parabolas and their associated cardioids

Figure 16 combines the results of the preceding discussion and illustrates the arguments for the proof of the next theorem. The cardioid generated by the circles $\{\kappa_4\}$ is now denoted by τ', whereas τ denotes the associated cardioid of the parabola σ. The two cardioids are connected with a similarity f with center P, mapping τ onto τ'. Point P and the symmetric S' of the center S of the circle κ_0 w.r. to P are respectively the focus and the vertex of a parabola σ', whose associated cardioid is τ'. The similarity f has its center at P, its angle is $\chi = \widehat{DPS'}$ and its ratio is $k = PS'/PD$. It is easily seen, that this similarity maps the parabola σ onto σ' and τ onto τ'.

With this preparation, we can now prove that the cardioid τ' has no other than B_1 intersection point with the parabola σ. In fact, assume that X is a point common to σ and τ'. Consider then the point Y on the position radius PX, lying on τ. From equation 2 we know that Y is between the points $\{P, X\}$. Consider now points $\{X' = f(X), Y' = f(Y)\}$. The order relation is preserved by f. Thus, Y' is again between the points $\{P, X'\}$. But since X is an intersection point $X \in \sigma \cap \tau'$, its image $X' = f(X)$ must be a point lying between $\{P, Y'\}$, which is not possible if $X \neq B_1$. Thus, we have proved that if $\omega > \pi/4$, then there is no other than B_1 point of intersection of the cardioid τ' with the parabola σ. An analogous argument for $\omega < \pi/4$ shows that there is indeed such an intersection of the corresponding τ' with σ. Finally for $\omega = \pi/4$, the cardioid τ' coincides with the associated cardioid of σ and for this Brocard angle the corresponding self-pivoting quadrangle is easily seen to be a square. Summarizing the previous arguments, we arive at the proof of the following theorem.

Theorem 5. *The Brocard angle of a self-pivoting convex quadrangle of the type $q(A)^+$ or $q(B)^+$ is less or equal to $\pi/4$, the equality being valid only for the square.*

6. Types $q(A)^+$ and $q(B)^+$ with three equal angles

Here we continue the analysis of figure 12, of a self-pivoting quadrangle of type $q(B)^+$, taking into account general properties of parabolas ([2, p.21], [4, p.28], [17, p.133]), and noticing that the triangles $\{B_1C_1'P, PC_1'S_1\}$ are similar. Thus, in the case the circle κ_4 and the parabola σ are tangent, then all four points $\{S_i, T_j\}$ coincide, the three triangles $\{B_1PC_1, C_1PD_1, D_1PA_1\}$ are similar and three of the angles of the quadrangle are equal. The proof of the following theorem, formulated for the type $q(B)^+$, gives a recipe to construct such self-pivoting quadrangles with three equal angles. The analogous theorem for $q(A)^+$ is also valid. The necessary formulation, as well as minor adaptations and changes for its proof, are left as an exercise.

Theorem 6. *The case, of a self-pivoting quadrangle of type $q(B)^+$, in which the circle κ_4 is tangent to the parabola σ and the triangles $\{B_1PC_1, C_1PD_1, D_1PA_1\}$ are similar, is, up to similarity, completely determined by the angle ϕ of the triangles at the pivot point P.*

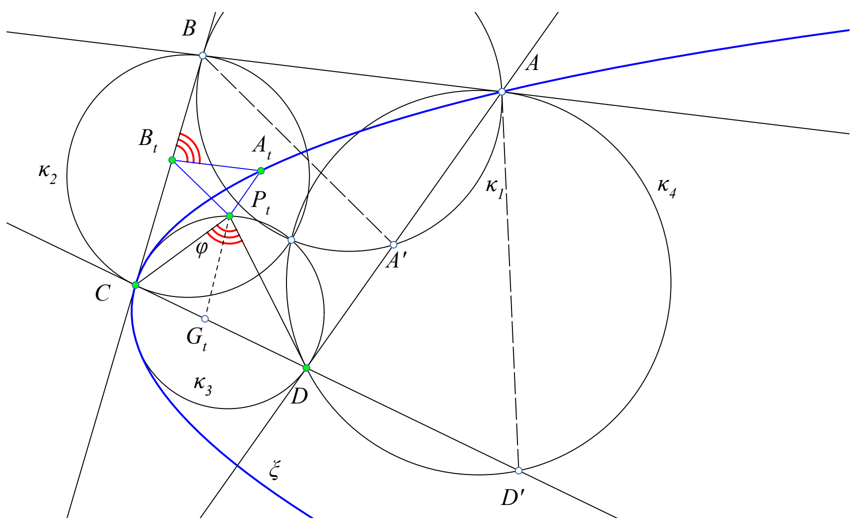


Figure 17. Self-pivoting quadrangle of type $q(B)^+$ with three equal angles

Proof. Figure 17 shows the recipe to construct such a self-pivoting quadrangle of type $q(B)^+$, up to similarity and from a given angle ϕ at P. The construction starts with a circle κ_3 and a chord of it CD, viewed from points of the circle under the angle ϕ or its supplement. Taking such a point $P_t \in \kappa_3$, we define the similarity transformation f_t ([3, ch.IV]) with center at P_t, rotation angle ϕ and similarity ratio $k_t = P_tC/P_tD$. The image $B_t = f_t(C)$ of C under the similarity defines the triangle $P_tB_tC \sim P_tCD$ and applying f_t once more we obtain $A_t = f_t(B_t) = f_t^2(C)$ and the triangle $P_tA_tB_t \sim P_tB_tC$. As P_t varies on the circle κ_3, the corresponding point $A_t = f_t^2(C)$ moves on a parabola ξ. This is an immediate consequence of two facts. The first, following from a simple angle chasing argument, is that the line A_tB_t makes the angle ϕ with the tangent CB of κ_3 at C. The second fact, following directly from the definitions, is that the ratio $A_tB_t/B_tC = k_t$.

Combining these two facts, we can find a simple parametrization of the variable point A_t, showing that it moves on a parabola. For this we use the bisector P_tG_t of the angle ϕ at P_t, which divides the side CD at the ratio $k_t = t/(a-t)$, where $a = CD$. Then setting $x = CB_t$ and $y = B_tA_t$, we see that

$$\frac{x}{a} = k_t = \frac{t}{a-t}, \quad \frac{y}{x} = k_t \quad \Rightarrow \quad x = a\frac{t}{a-t}, \quad y = a\frac{t^2}{(a-t)^2},$$

which is a parametrization of a parabola in oblique axes. The point A is an intersection of the parabola with the tangent DA of κ_3 at D and point B is the intersection of CB with the parallel to A_tB_t, making with BC the angle ϕ. From its definition follows that the quadrangle $ABCD$ is self-pivoting of type $q(B)^+$ with pivot P, coinciding with the second intersection of the circles $\{\kappa_3, \kappa_2\}$, where κ_2 is the circle tangent to CD at C, passing through B. □

It is easy to see that if $\phi = \pi/2$, then the quadrangle $ABCD$ is a square. In general, since the three angles of the quadrangle are equal to $\pi - \phi$, the magnitude of the angle is restricted by the inequalities for the fourth angle at D

$$\widehat{D} = 2\pi - 3(\pi - \phi) = 3\phi - \pi \quad \text{and} \quad 0 < \widehat{D} < \pi \qquad \Rightarrow \qquad 60^\circ < \phi < 120^\circ.$$

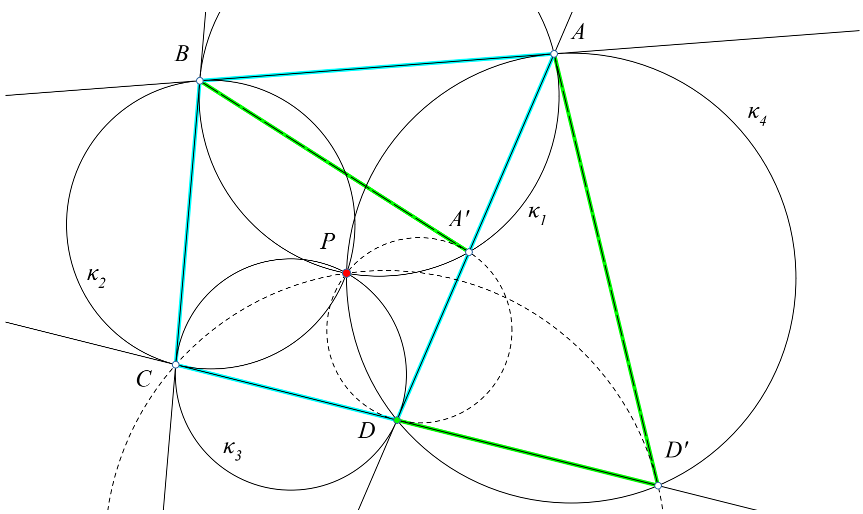


Figure 18. The similar quadrangles $ABCD \sim BCDA' \sim D'ABCD$

Figure 18 shows another characteristic of this class of quadrangles, determined, up to similarity from an angle ϕ satisfying the above restriction. The other in-

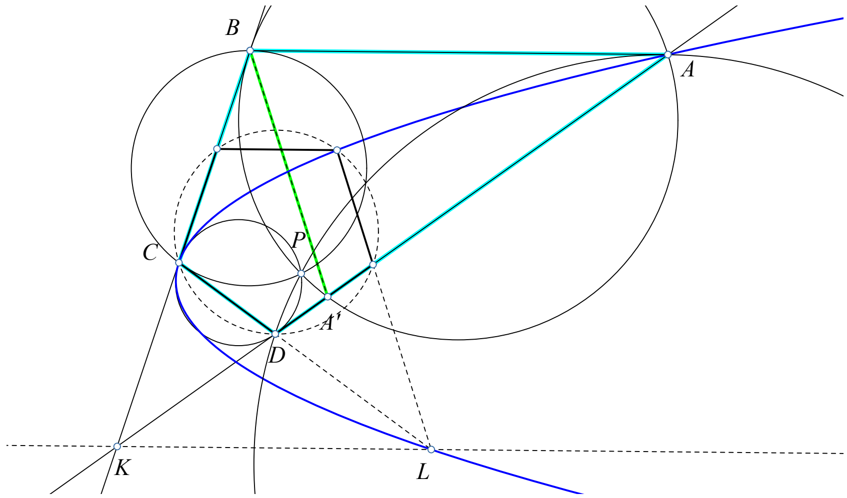


Figure 19. Self-pivoting quadrangle from a regular pentagon

tersection points $\{A', D'\}$ of the two circles $\{\kappa_1, \kappa_4\}$ correspondingly with sides $\{AD, DC\}$ define similar quadrangles $ABCD \sim BCDA' \sim D'ABC.$ It is also easily verified that the new quadrangles are self-pivoting about the same point P with the original one. Figure 19 shows one more example of a self-pivoting quadrangle of the type $q(B)^+$. It is the one with three angles equal to $108°$, the angle of the regular pentagon. By the previous restriction for the angles and the discussion so far, follows that the rectangle and the pentagon are the only regular polygons, whose angles may appear in this kind of self-pivoting quadrangles. Notice that in this case the axis of the parabola ξ is parallel to the line KL, which is parallel to a diagonal of the pentagon.

7. Self-pivoting quadrangles of types $q(C)^+$ and $q(D)^+$

As noticed in section 3, these two types of self-pivoting quadrangles have similar arrangements of vertices, leading to identical geometric properties. For this reason we confine our discussion to one of them, the type $q(C)^+$, and make some remarks on the differences for the related type $q(D)^+$, at the end of the section. In the type $q(C)^+$ the circumscribing q, which is similar to the circumscribed q_1, has opposite to A_1B_1 the angle $\widehat{C}$ and the orientation of the two quadrangles is the same. Next theorem shows that the requirement of self-pivoting of this type is quite restrictive for the quadrangle.

Theorem 7. *A convex self-pivoting quadrangle of the type $q(C)^+$ is necessarily a trapezium of a special kind, for which the circles $\{\kappa_2, \kappa_4\}$ are equal. The trapezium in this case defines another isosceles trapezium inscribed in κ_2 called the "core" of q. Each isosceles trapezium is the core of two, in general, different trapezia, which are self-pivoting of type $q(C)^+$.*

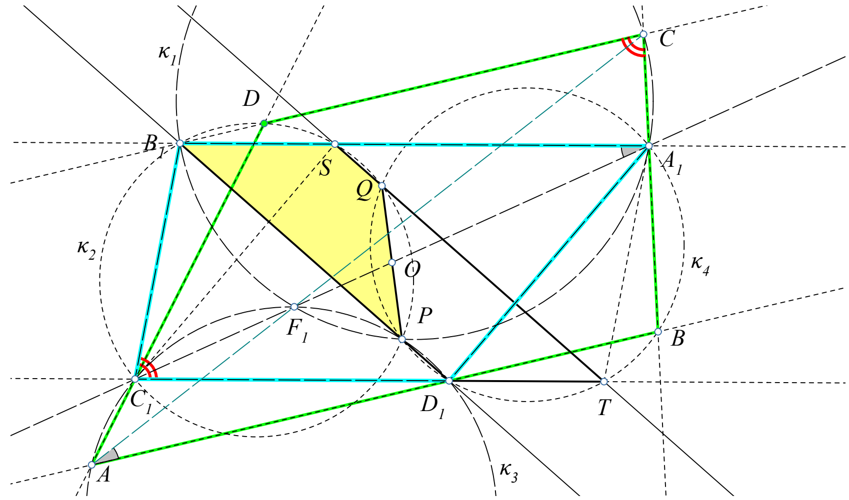


Figure 20. Self-pivoting quadrangle of the type $q(C)^+$ is a trapezium

Proof. Figure 20 shows a quadrangle of this type together with the corresponding isosceles trapezium $\tau = QSB_1P$, which is the *core* of the quadrangle q_1. In order to define it, start with q_1 and consider the second intersection points $\{S, T\}$ of $\{\kappa_2, \kappa_4\}$ respectively with $\{A_1B_1, C_1D_1\}$. A simple angle chasing shows that lines $\{C_1S, A_1T\}$ are respectively tangent to the circles $\{\kappa_3, \kappa_1\}$. For A_1T this follows from the equality of angles

$$\widehat{CB_1A_1} = \widehat{CF_1A_1} = \widehat{C_1F_1A} = \widehat{C_1D_1A} = \widehat{BD_1T} = \widehat{BA_1T}.$$

Analogously is seen the other tangency. This implies that $C_1TA_1S \sim ABCD$ is a position of the pivoting quadrangle $ABCD$. The similarity of the triangles $A_1B_1C_1 \sim ABC$ implies then that $\widehat{B_1A_1C_1} = \widehat{A_1C_1T}$, which shows that the lines $\{A_1B_1, C_1D_1\}$ are parallel. From this follows immediately that $SC_1D_1A_1$, $B_1C_1TA_1$ are parallelograms and $\{SB_1C_1, D_1TA_1\}$ are equal triangles. This implies in turn that the circles $\{\kappa_2, \kappa_4\}$ are equal, since the equal segments $\{C_1S, D_1A_1\}$ are respectively seen from $\{B, T\}$ under equal angles.

Combining these facts, we see that the whole figure can be reproduced from the two glued, equal to the core, isosceli trapezia $\{B_1SQP, QPD_1T\}$. In fact, if we start from these two equal trapezia and their circumcircles $\{\kappa_2, \kappa_4\}$, then the missing vertices $\{A_1, C_1\}$ can be defined as intersections of these circles respectively with the lines $\{BS, D_1T\}$. Figure 21 shows how the self-pivoting trapezium is defined from the isosceles trapezium B_1SQP. In fact, starting with the arbitrary isosceles trapezium B_1SQP we define its symmetric QPD_1T w.r to the middle O of the non parallel side PQ. The missing vertices lie on respective circumcircles of the trapezia and are symmetric w.r. to O. Thus, there result two acceptable solutions $\{A_1B_1C_1D_1, A'_1B_1C'_1D_1\}$ i.e. self-pivoting quadrangles of the type $q(C)^+$ about the point P. □

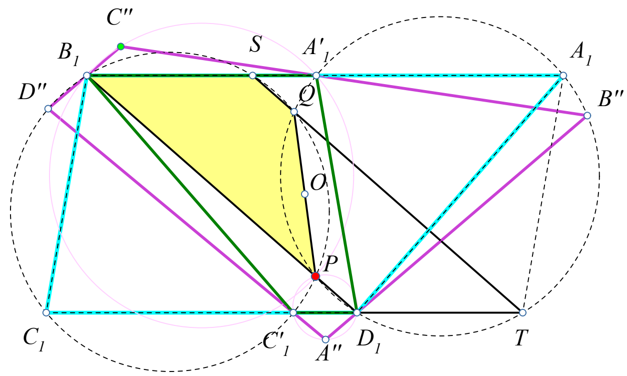


Figure 21. The two cases defined by the trapezium $ABCD$

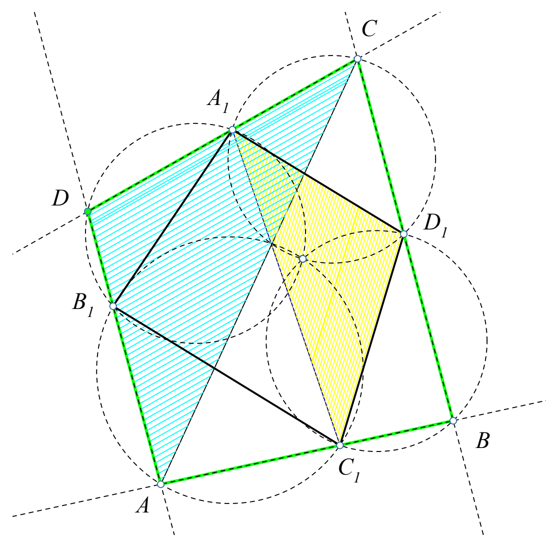


Figure 22. Self pivoting quadrangle of the type $q(D)^+$

The arguments used in the discussion of the type $q(C)^+$ apply almost verbatim to the case of $q(D)^+$ and lead to a similar result. The quadrangle is again a trapezium of the kind, referred to in the previous theorem.

Theorem 8. *A convex self-pivoting quadrangle of the type $q(D)^+$ is necessarily a trapezium of the special kind, considered in theorem 7.*

Figure 22 shows a characteristic case of a self-pivoting of type $q(D)^+$. The only difference is in the arrangement of parallel sides. Here the parallel sides are $\{A_1D_1, B_1C_1\}$, whereas in the previous case the parallels are $\{A_1B_1, C_1D_1\}$.

Corollary 9. *The only self-pivoting convex quadrangle w.r. to all types of positive pivoting is the square.*

Proof. By *positive* pivoting we mean the types $\{q(A)^+, q(B)^+, qC)^+, q(D)^+\}$. If the quadrangle q_1 is of the first two types simultaneously, then, by theorem 2 it is harmonic. If it is also simultaneously of the two last types, then, by the theorems of this section, it is also a parallelogram. But the only harmonic parallelogram is the square. □

8. Self-pivoting of types $\{q(A)^-, q(B)^-, q(C)^-, q(D)^-\}$

These four types of self-pivoting quadrangles, have some common traits, noticed in section 3. They are precisely the types that have exactly two opposite vertices with the ASV property. The main consequence of this is given by the next theorem.

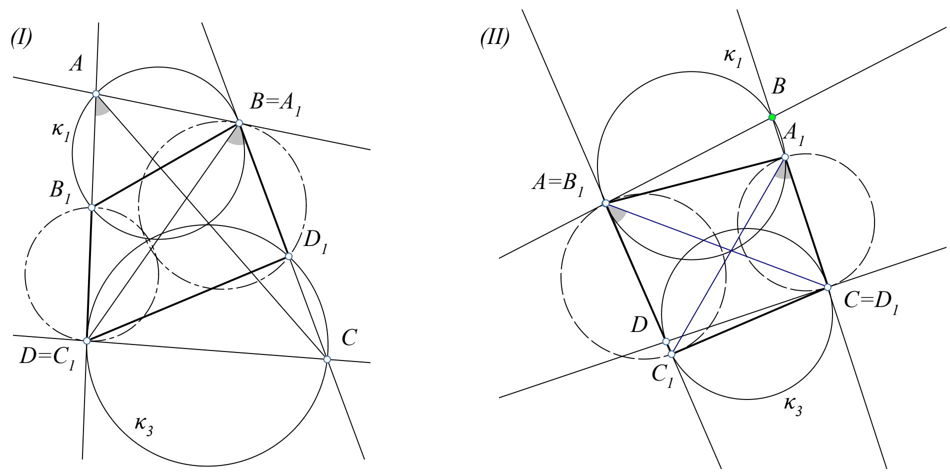


Figure 23. $\{q(A)^-, q(B)^-\}$ have two opposite vertices with the ASV property

Theorem 10. *The self-pivoting quadrangles of the types* $q(A)^-, q(B)^-$, $q(C)^-$, $q(D)^-$ *are cyclic.*

Proof. The proof relies on the fact, that the vertices of the circumscribing pivoting quadrangle $q = ABCD$ with the ASV property move on circles which are tangent to two opposite sides of the circumscribed quadrangle $q_1 = A_1B_1C_1D_1$ (See Figure 23). Working with the type $q(A)^-$, we notice that there is a position of the circumscribing q, for which we have identification of two opposite vertices with corresponding two vertices of the circumscribed q_1. The vertices are $B = A_1$ and $D = C_1$ (See Figure 23-I). Consequently also the side-lines $\{A_1D_1 = BC, B_1C_1 = AD\}$. This is valid also in the case of type $q(B)^-$ (See Figure 23-II), and also in the remaining two types. This identification of vertices and side-lines is, more generally, valid also in the case the two quadrangles $\{q, q_1\}$ have the same angles and the correct arrangement of the angles, according to the circumscription type under consideration, even if they are not similar, as is the case with the two configurations for $q(A)^-$ and $q(B)^-$ in figure 23. If however the quadrangles $\{q, q_1\}$ are similar, then the side CD resp. C_1D_1 is viewed from the vertices $\{A, A_1 = B\}$ resp. $\{A_1, A = B_1\}$ under the same angle. This proves that the quadrangles are cyclic for the two types $\{q(A)^-, q(B)^-\}$, the proof for the remaining two types being exactly the same. $\square$

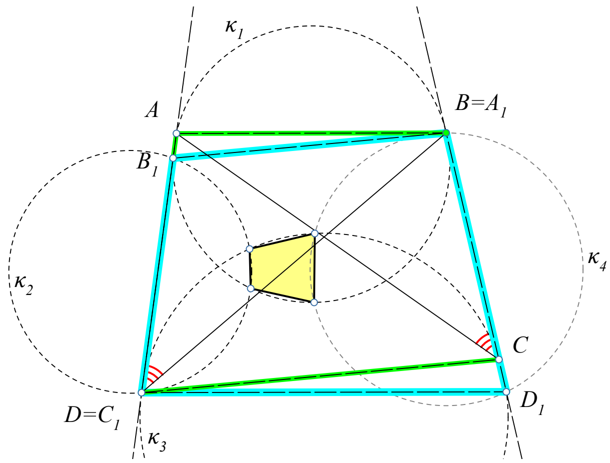


Figure 24. Trying to draw a quadrangle of type $q(A)^-$

Figure 24 shows how to draw a cyclic quadrangle q_1 and a particular circumscribing q of the type $q(A)^-$ allowing a quick construction of the circles $\{\kappa_i\}$. The condition on the angles implies that (A_1B_1, CD) and (AB, C_1D_1) are pairs of parallels. It is also easily seen, that drawing in an arbitrary cyclic quadrangle q_1 parallels $\{A_1A, C_1C\}$ to opposite sides we obtain a new cyclic quarangle $q = ABCD$, with the same angles as q_1. Figure 24 shows such a general construction from an arbitrary cyclic quadrangle q_1. In order to obtain a self-pivoting one we must succeed to have four points coincident. These are the second intersections of adjacent circle pairs $\{\kappa_1 \cap \kappa_2, \kappa_2 \cap \kappa_3, \ldots\}$, their quadrangle shown also in the figure. Next theorem shows how this is done. The proof again is given in detail only for the type $q(A)^-$, the other cases allowing a completely analogous handling.

Theorem 11. *The self-pivoting quadrangles of type $q(A)^-, q(B)^-, q(C)^-, q(D)^-$ are necessarily cyclic quadrangles. Given a cyclic quadrangle q_0, there is, up to similarity, precisely one self-pivoting quadrangle q_1 of each of these types, with the same angles and the same succession of angles as q_0.*

Proof. The first part of the theorem follows from the preceding discussion. To show the second part for the type $q(A)^-$, we consider the arbitrary fixed convex cyclic quadrangle $q_0 = A_0B_0C_0D_0$ and the family of other cyclic quadrangles $q' = ABC_0D_0$, produced by varying one of its sides (AB) parallel to itself (See Figure 25). In order to locate the self-pivoting among all these quadrangles we exploit the fact that, for this type of self-pivoting, the circles $\{\kappa_1, \kappa_3, \kappa_4\}$ are respectively tangent to $\{A_0D_0, B_0C_0, AB'\}$ at the points $\{A, C_0, A\}$, where AB' is parallel to C_0D_0. In this case the self-pivoting quadrangle results when, for varying A on $\varepsilon = A_0D_0$, the second intersection point P of the circles $\{\kappa_1, \kappa_4\}$ takes a

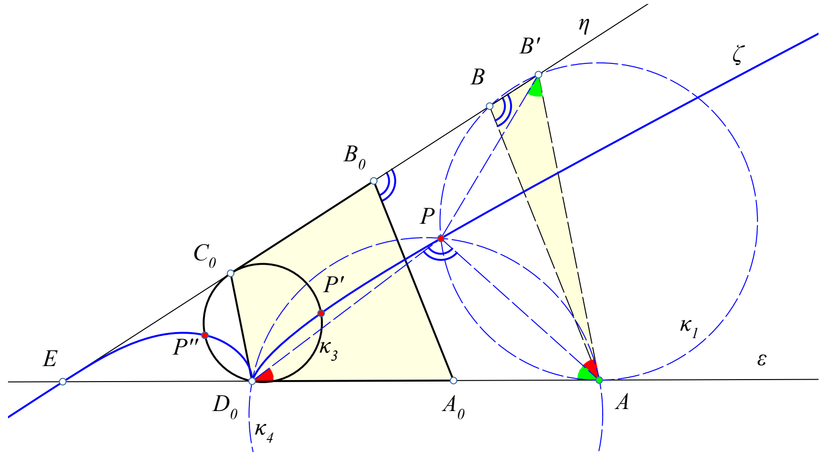


Figure 25. Seeking a self-pivoting quadrangle of type $q(A)^-$ with given angles

position on the circle κ_3. But as the point A varies on ε, the lemmata below show that the corresponding point P describes a cubic curve ζ having a singular point at D_0 and intersecting the circle κ_3 at two other points $\{P', P''\}$, leading to two similar quadrangles with the desired properties. The determination of the whole quadrangle from the point P is trivial, since the angle

$$\delta = \widehat{A_0D_0C_0} = \widehat{D_0PA} = \widehat{APB'},$$

remains constant for all positions of P. Hence, having the position of P, we can draw the angle $\widehat{D_0PA} = \delta$ and find the position of A, from which the parallel to A_0B_0 determines the whole quadrangle.

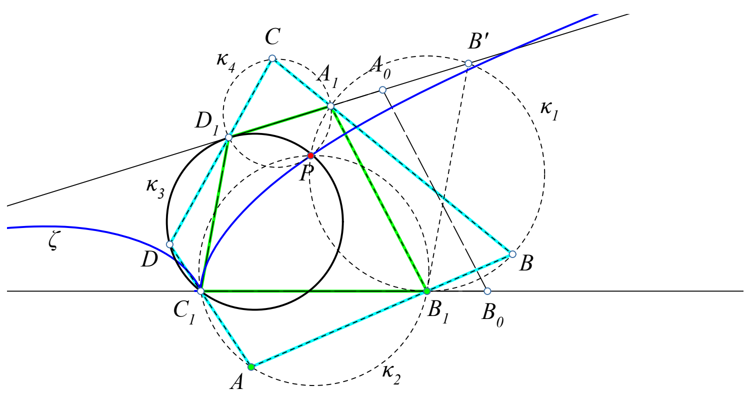


Figure 26. The cubic determining the self-pivoting $q_1 = A_1B_1C_1D_1$ of type $q(B)^-$

The proof for the other types is the same. Figure 26 shows e.g. the corresponding cubic for the type $q(B)^-$. In this case the cubic is the geometric locus of the second intersection points $\{P\}$ of the circles $\{\kappa_1, \kappa_2\}$, circle κ_1 being tangent to

B_1C_1 at B_1 and passing through A_1, as B_1 moves on line C_1B_0. The circle κ_2 is tangent to the parallel B_1B' to C_1D_1 and passes through C_1. Here we start again from an arbitrary cyclic quadrilateral $A_0B_0C_1D_1$ and draw parallels A_1B_1 to A_0B_0 seeking the position of B_1, which defines the self-pivoting quadrangle q_1. The figure shows the appropriate position of the pivot P, which is the intersection of the cubic with κ_3. It shows also a pivoting quadrangle $q = ABCD \sim q_1$ circumscribing q. □

In the rest of this section we discuss in detail the case of the cubic ζ related to the type $q(A)^-$, the arguments for the cases $\{q(B)^+, \dots\}$ and the cubics related to these types being completely analogous.

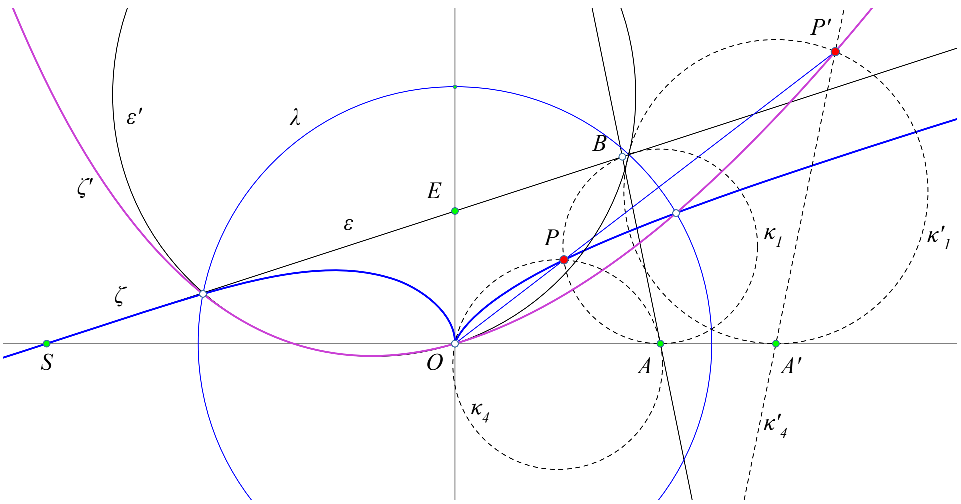


Figure 27. The cubic carrying the points P

Lemma 12. *Under the notation and conventions of theorem 11, for the type* $q(A)^-$, *the intersection point* P *of the circles* $\{\kappa_1, \kappa_4\}$ *describes a cubic curve.*

Proof. Referring to figure 25, we use cartesian coordinates with origin O at the vertex D_0 of the quadrangle q_0 and identify the $x-$axis with the line D_0A_0. The line AB moves parallel to itself with A on the $x-$axis. Also B is on the fixed line $\varepsilon = A_0B_0$. The point P is on the circle κ_1, which is tangent to the $x-$axis at A and passes through B. It is also on the circle κ_4, which is tangent to AB at A and passes through the origin O. Now we consider the inversion f w.r. to a fixed circle λ centered at O (See Figure 27). By this the image $\varepsilon' = f(\varepsilon)$ is a fixed circle, $\kappa_1' = f(\kappa_1)$ is a circle tangent to the $x-$axis and intersecting the circle ε' under a fixed angle, and $\kappa_4' = f(\kappa_4)$ is a line passing through $A' = f(A)$ and having a fixed direction. Thus the inverted $P' = f(P)$ is the intersection of a line κ_4' and a circle κ_1' passing though A' tangent there to the $x-$axis and cutting the fixed circle ε' under a fixed angle. Next lemma shows that the geometric locus ζ' of such points P' is a parabola passing through the origin and having the form

$$(px + qy)^2 - rx - sy = 0.$$

This, taking again the inverted in the form $(x, y) = f(x', y') = k(x', y')/(x'^2 + y'^2)$, with a constant k, shows that the inverted of the parabola $\zeta = f(\zeta')$ satisfies the cubic equation

$$k(px + qy)^2 - (rx + sy)(x^2 + y^2) = 0.$$

□

Lemma 13. *A point A moves on the fixed line ε, and the line η through A has a fixed direction. The circle $\kappa(X, r)$ is tangent to ε at A and intersects a fixed circle $\lambda(K, r_0)$ passing through $O \in \varepsilon$ under a fixed angle ϕ. Then, the geometric locus*

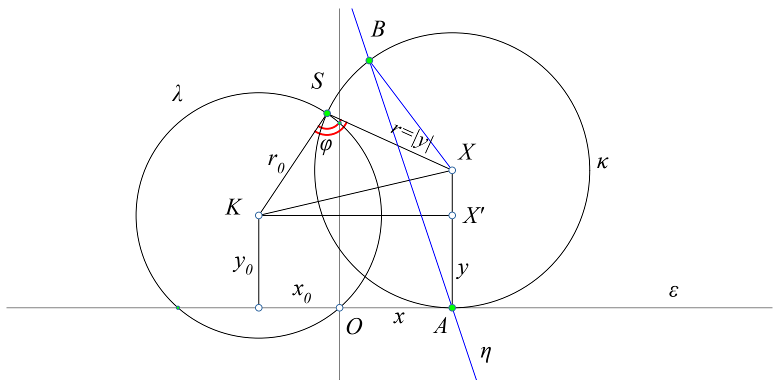


Figure 28. The parabola of points B

of the second intersection point B of η and κ is a parabola through the point O.

Proof. Using the point O as origin and the line ε as $x-$axis of a cartesian coordinate system, in which $K = (x_0, y_0)$, we find that the center X of the variable circle κ satisfies the parabola equation (See Figure 28)

$$x^2 - 2x_0x - 2(y_0 - r_0\cos(\phi))y = 0.$$

On the other side the intersection point $B(x', y')$ of κ and the line η in fixed direction can be expressed in terms of $X(x, y)$ and a fixed unit vector $e = (e_1, e_2)$ by the equations

$$B = X + ye \qquad \Leftrightarrow \qquad x' = x + ye_1, \quad y' = y + ye_2.$$

This means that B describes the affine transformation of a parabola, which is also a parabola with the desired properties. □

Lemma 14. *Under the notation and conventions of this section, the two quadrangles, created from the intersection points $\{P', P''\}$ of the cubic ζ, of theorem 11 for the type $q(A)^-$, with the circle κ_3, are similar and inversely oriented. The points $\{P', P''\}$ are isogonal conjugate w.r. to the angle $\widehat{C_1ED_1}$.*

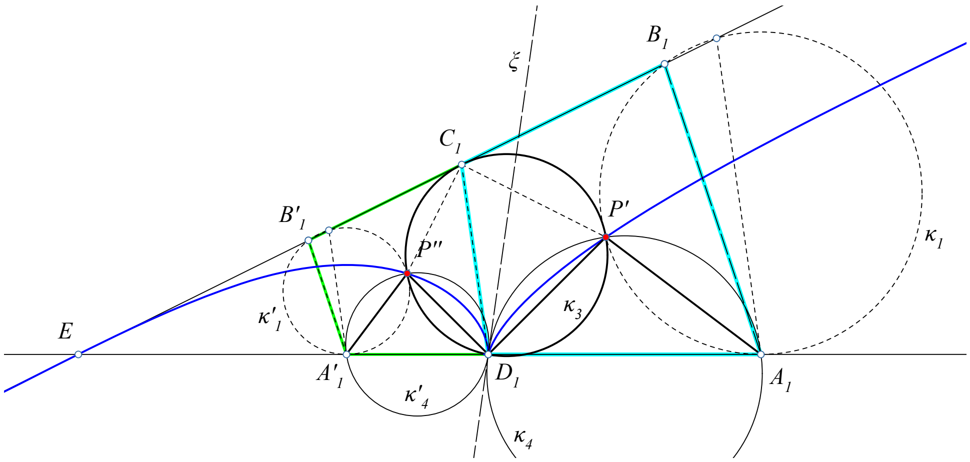


Figure 29. The two similar quadrangles of type $q(A)^-$

Proof. The fact that there are precisely two such intersection points follows from the geometric definition of the points P of the cubic ζ. The cubic has a singular point at D_1 and its limit tangents at D_1 from the two branches coincide with the line ξ, which is symmetric to D_1C_1 w.r. to D_1A_1 (See Figure 29). The circles $\{\kappa_4, \kappa'_4, \ldots\}$ for the various positions of P are members of the pencil of circles tangent to ξ at D_1. This implies geometrically that there are two intersection points of the cubic with κ_3 lying on either sides of ξ. It is also easily seen geometrically that the two resulting solutions are similar and inversely oriented. That there are no more intersection points follows from the fact that the cubic and the circle κ_3 are inverses under f respectively of a parabola and a line, which can have no more than two intersection points. □

9. Self-pivoting quadrangles of type $q(C)^-$

In the previous section we saw that a self-pivoting quadrangle, whose type coincides with one of $\{q(A)^-, q(B)^-, q(C)^-, q(D)^-\}$, is necessarily cyclic. We saw further that, given a convex cyclic quadrangle q_0, there is, up to similarity, precisely one self-pivoting quadrangle q_1 of each one of these types, with the same angles and the same succession of angles as q_0. Next theorem shows that this q_1, in the case of the type $q(C)^-$, satisfies a stronger condition.

Theorem 15. *The only self-pivoting quadrangles of type $q(C)^-$ are the harmonic quadrangles.*

Proof. Constructing the self-pivoting of this type by the method of the previous section, we start with an arbitrary cyclic quadrangle $q_0 = A_0B_1C_1D_0$, with fixed given angles, and an arbitrary point D_1 moving on the line C_1D_0. From D_1 we draw the parallel D_1A_1 to D_0A_0 and consider the variable circles κ_3 tangent to D_1A_1 at D_1 and passing through C_1 and κ_4 tangent to C_1D_1 at D_1 and passing through A_1. Their intersection point P describes, as D_1 moves on C_1D_0, a cubic ζ.

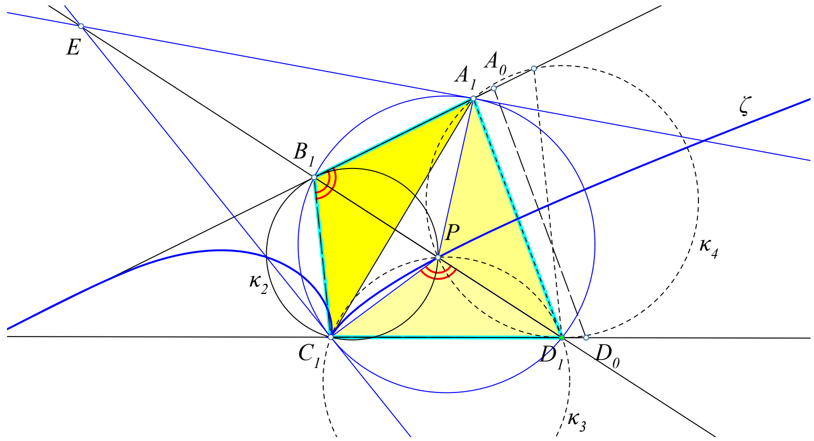


Figure 30. Constructing a quadrangle of type $q(C)^-$

This is exactly the same situation as in the preceding section. The difference from that is now that the similar triangles $\{PC_1D_1, PD_1A_1\}$ have their angle at P equal to $\widehat{B_1}$, while in the previous case this angle was equal to $\widehat{C_1}$. The equality of the angles implies now that the intersection point P of the cubic with the fixed circle κ_2, which is tangent to B_1A_1 at B_1 and passes through C_1, is on the diagonal B_1D_1. This follows trivially from the equality of the angles $\widehat{B_1PC_1} = \pi - \widehat{C_1B_1A_1}$. Then, an equally simple angle chasing argument shows that the two similar triangles $\{PC_1D_1, PD_1A_1\}$ are also similar to the triangle $A_1B_1C_1$, which is a characteristic property of the harmonic quadrangles ([12]). □

There are several partial results that could be produced as corollaries of the previous disccussion. Corollaries concerning self-pivoting of quadrangles w.r. to combinations of some types, or/and corollaries concerning such combinations and some particular kinds of quadrangles, like rectangles, parallelograms, trapezia etc, all left as exercises for the interested reader.

References

[1] A. Akopyan, Geometry of the cardioid, *Amer. Math. Monthly*, 121 (2013) 1–6.
[2] A. Akopyan and A. Zaslavsky, *Geometry of Conics*, American Mathematical Society, New York, 2012.
[3] W. Barker and R. Howe, *Continuous Symmetry, From Euclid to Klein*, American Mathematical Society, 2007.
[4] W. H. Besant, *Elementary Conics*, George Bell and Sons, London, 1901.
[5] J. Booth, *A treatise on some new geometrical methods*, I, II, Longmans, London, 1873.
[6] F. Brown, Brocard points for a quadrilateral, *Math. Gazette*, 9:83?85, 1917.
[7] W. Gallatly, *The Modern Geometry of the Triangle*, Francis Hodgsonn, London, 1913.
[8] R. Johnson, *Advanced Euclidean Geometry*, Dover Publications, New York, 1960.
[9] T. Lalesco, *La Geometrie du triangle*, Librairie Vuibert, Paris, 1952.
[10] D. Lawrence, *A Catalog of Special Plane Curves*, Dover Publications, New York, 1972.
[11] E. Lockwood, *A Book of Curves*, Cambridge university press, Cambridge, 1961.
[12] P. Pamfilos, The associated harmonic quadrilateral, *Forum Geom.*, 14 (2014) 15–29.

[13] C. Pohoata, Harmonic quadrilaterals revisited, *Gazeta Matematica*, seria A, 29 (2011) 15–35.
[14] R. Proctor, *The Geometry of Cycloids*, Longmans, Green and Co., London, 1878.
[15] G. Salmon, *A treatise on Conic Sections*, Longmans, Green and Co., London, 1917.
[16] H. Wieleitner, *Spezielle Ebene Kurven*, Goeschensche Verlagshandlung, Leipzig, 1908.
[17] C. Zwikker, *The Advanced Geometry of Plane Curves and Their Applications*, Dover, New York, 1963.

Paris Pamfilos: University of Crete, Greece
E-mail address: pamfilos@uoc.gr

Forum Geometricorum
Volume 18 (2018) 349–351.

FORUM GEOM
ISSN 1534-1178

Orthopoles and Variable Flanks

Floor van Lamoen

Abstract. We extend Zaharinov's result on the perspector of the triangle of flank line orthopoles to variable flanks. The locus of perspectors is the Kiepert hyperbola.

1. Introduction

Zaharinov [3] has studied the orthopoles of the a-, b-, c-sides of the of the A-, B-, and C-flanks respectively. If we attach squares to the sides of triangle ABC we find at each vertex of ABC a flank triangle [2]. He found that these orthopoles form a triangle perspective with ABC, with the Vecten point as perspector. In this paper we extend the results to variable flanks, replacing the attached squares by similar rectangles, which Čerin [1] used to find various loci. We show that the orthopoles of variable flanks form triangles perspective with ABC, the locus of perspectors being the Kiepert hyperbola.

2. Orthopoles of flank lines

Consider rectangles ABC_bC_a, BCA_cA_b, and CAB_aB_c attachted to the sides of triangle ABC and satisfying $\angle BAC_b = \angle CBA_c = \angle ACB_a = \phi$. We will calculate barycentrics for the orthopole of line A_bC_b, the B-flank line (see Figure 1).

We have that

$$A_b = (-a^2 : S_C + S_\phi : S_B),$$
$$C_b = (S_B : S_A + S_\phi : -c^2).$$

Lines perpendicular to A_bC_b are parallel to the B-median of ABC as the orthocenter and centroid are friends (see [2]). So these lines meet the line at infinitiy $\mathcal{L}^\infty$ in the point $(1 : -2 : 1)$.

The perpendicular ℓ_A through A to A_bC_b hence has equation $\ell_A : \quad y + 2z = 0$, while the equation for A_bC_b is given by

$$(S^2 + (c^2 + S_B)S_\phi)x + S^2 y + (S^2 + (a^2 + S_B)S_\phi)z = 0.$$

If we denote by $\overline{\phi}$ the complement of ϕ, the latter equation can be rewritten as

$$(S_B + c^2 + S_{\overline{\phi}})x + S_{\overline{\phi}}y + (S_B + a^2 + S_{\overline{\phi}})z = 0,$$

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

noting that $S^2 = S_{\phi\overline{\phi}}$.
So the point A' where ℓ_A and A_bC_b intersect has coordinates

$$A' = (-S_B - a^2 + S_{\overline{\phi}} : \ -2(S_B + c^2 + S_{\overline{\phi}}) : \ S_B + c^2 + S_{\overline{\phi}}).$$

Similarly, if ℓ_C is the perpendicular through C to A_bC_b, then the point C' where ℓ_C and A_bC_b meet has coordinates

$$C' = (S_B + a^2 + S_{\overline{\phi}} : \ -2(S_B + a^2 + S_{\overline{\phi}}) : \ -S_B - c^2 + S_{\overline{\phi}}).$$

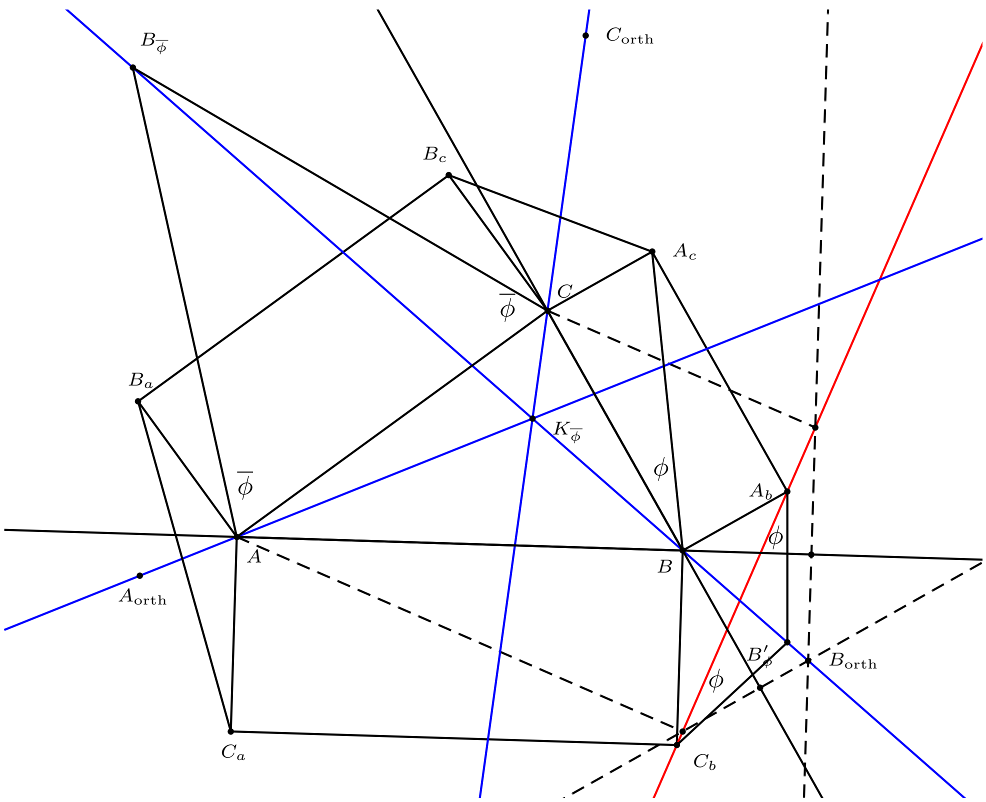


Figure 1.

Now the line through A' perpendicular to BC is given by

$$\begin{aligned}(S_B + a^2)(S_B + c^2 + S_{\overline{\phi}})x + ((S_B + a^2)S_{\overline{\phi}} + S^2)y \\ + ((S_B + a^2)S_{\overline{\phi}} + S_B(S_C + 3a^2) + a^2(b^2 + c^2))z = 0.\end{aligned}$$

With a similar result for the line through C' perpendicular to AB we find as point of intersection for these two lines the orthopole of the B-flank line

$$\begin{aligned} B_{\mathrm{orth}} = \; & \Big(S^2(2a^2-b^2+2c^2)(S_C+S_{\overline{\phi}}) \\ & : \; -4S^2(2a^2-b^2+2c^2)(a^2+c^2+S_{\overline{\phi}}) \\ & : \; S^2(2a^2-b^2+2c^2)(S_A+S_{\overline{\phi}})\Big) \\ = \; & (S_C+S_{\overline{\phi}} : \; -4(a^2+c^2+S_{\overline{\phi}}) : \; S_A+S_{\overline{\phi}}). \end{aligned}$$

By symmetry this shows that the triangle of the three orthopoles of the flank lines is perspective to ABC, the Kiepert perspector $K_{\overline{\phi}}$ being the perspector. For variable flanks the line will hence run through the Kiepert hyperbola. As $K_{\overline{\phi}}$ and K_ϕ are friends, we know that the line connecting B, $K_{\overline{\phi}}$, and B_{orth} will also pass through the apices of isosceles triangles erected on AC and A_bC_b with base angles $\overline{\phi}$ and ϕ respectively. Naturally, the orthopole of BC and the Kiepert ϕ-perspector, both with respect to the B-flank, join this line, to complete the friendly symmetry.

References

[1] Z. Čerin, Loci related to variable flanks, *Forum Geom.*, 2 (2002) 105–113.
[2] F. M. van Lamoen, Friendship among triangle centers, *Forum Geom.*, 1 (2001) 1–6.
[3] T. Zaharinov, Orthopoles, flanks, and Vecten points, *Forum Geom.*, 17 (2017) 401–410.

Floor van Lamoen: Ostrea Lyceum, Fruitlaan 3, 4462 EP Goes, The Netherlands
E-mail address: `fvanlamoen@planet.nl`

Forum Geometricorum
Volume 18 (2018) 353–359.

FORUM GEOM
ISSN 1534-1178

Simple Proofs of Feuerbach's Theorem and Emelyanov's Theorem

Nikolaos Dergiades and Tran Quang Hung

Abstract. We give simple proofs of Feuerbach's Theorem and Emelyanov's Theorem with the same idea of using the anticomplement of a point and homogeneous barycentric coordinates.

1. Proof of Feuerbach's Theorem

Feuerbach's Theorem is known as one of the most important theorems with many applications in elementary geometry; see [1, 2, 3, 5, 6, 8, 9, 10, 12].

Theorem 1 (Feuerbach, 1822)**.** *In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and is externally tangent to the excircles.*

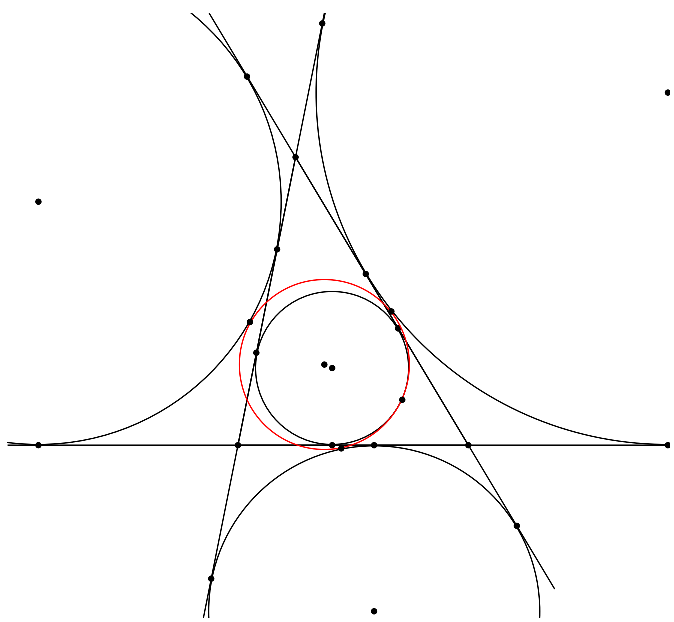


Figure 1

We establish a lemma to prove Feuerbach's Theorem.

Lemma 2. *The circumconic* $p^2yz + q^2zx + r^2xy = 0$ *is tangent to line at infinity* $x + y + z = 0$ *if and only if*

$$(p + q + r)(-p + q + r)(p - q + r)(p + q - r) = 0$$

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

and the point of tangency is the point $Q = (p : q : r)$ *or* $Q_a = (-p : q : r)$ *or* $Q_b = (p : -q : r)$ *or* $Q_c = (p : q : -r)$ *according to which factor of the above product is zero.*

Proof. We shall prove that the system

$$\begin{cases} p^2yx + q^2zx + r^2xy = 0 \\ x + y + z = 0 \end{cases}$$

has only one root under the above condition.

Substituting $x = -y - z$, we get

$$p^2yz - (q^2z + r^2y)(y + z) = 0.$$

The resulting quadratic equation for y/z is

$$r^2y^2 + (q^2 + r^2 - p^2)yz + q^2z^2 = 0.$$

This quadratic equation has discriminant

$$\begin{aligned} D = &\ (q^2 + r^2 - p^2)^2 - 4(qr)^2 \\ = &\ -(p + q + r)(-p + q + r)(p - q + r)(p + q - r). \end{aligned}$$

Thus (1) has only one root if and only if the discriminant $D = 0$.

If $\varepsilon_1 p + \varepsilon_2 q + \varepsilon_3 r = 0$ for $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 = \pm 1$, then $D = 0$, which means that the circumconic $p^2yz + q^2zx + r^2xy = 0$ or $\frac{p^2}{x} + \frac{q^2}{y} + \frac{r^2}{z} = 0$ has a double point on the line at infinity and that the point $Q = (\varepsilon_1 p,\ \varepsilon_2 q,\ \varepsilon_3 r)$ lies on the line at infinity and also on the circumconic because $\dfrac{p^2}{\varepsilon_1 p} + \dfrac{q^2}{\varepsilon_2 q} + \dfrac{r^2}{\varepsilon_3 r} = \varepsilon_1 p + \varepsilon_2 q + \varepsilon_3 r = 0$. Hence this point Q is the double point on the line at infinity of the circumconic.

That leads to a proof of the lemma. □

Remark. The conic which is tangent to the line at infinity must be a parabola. Thus our lemma is exactly a characterization of the circumparabola of a triangle.

Proof of Feuerbach's Theorem. Let G be the centroid of triangle ABC, and the incircle of ABC touch BC, CA, and AB at D, E, and F respectively. The homothety $\mathcal{H}(G, -2)$ transforms a point P to its anticomplement P', which divides PG in the ratio $PG : GP' = 1 : 2$ (see [11]). We consider the anticomplement of the points D, E, and F, which are X, Y, and Z respectively (see Figure 2). Under the homothety $\mathcal{H}(G, -2)$, the nine-point circle of ABC transforms to circumcircle (Γ) of ABC. Thus, it is sufficient to show that the circumcircle (ω) of triangle XYZ is tangent to (Γ).

Let M be the midpoint of BC. Then $\mathcal{H}(G, -2)$ maps M to A. Thus, $AX = 2DM = |b - c|$, and AX is also tangent to (ω). We deduce that the power of A with respect to (ω) is $AX^2 = (b - c)^2$. Similarly, the power of B with respect to (ω) is $(c - a)^2$, and the power of C with respect to (ω) is $(a - b)^2$.

Using the equation of a general circle in [12], we have the equation of (ω) as follows:

$$a^2yz + b^2zx + c^2xy - (x + y + z)((b - c)^2x + (c - a)^2y + (a - b)^2z) = 0.$$

Hence, the line $L:\ (b-c)^2x+(c-a)^2y+(a-b)^2z=0$ is the radical axis of (Γ) and (ω).

Analogously with the excircles, we can find that the radical axes of the circumcircle with the homothetic of the excircles under $\mathcal{H}(G,-2)$ are the lines

$$\begin{aligned} L_a: &\quad (b-c)^2x+(c+a)^2y+(a+b)^2z=0,\\ L_b: &\quad (b+c)^2x+(c-a)^2y+(a+b)^2z=0,\\ L_c: &\quad (b+c)^2x+(c+a)^2y+(a-b)^2z=0. \end{aligned}$$

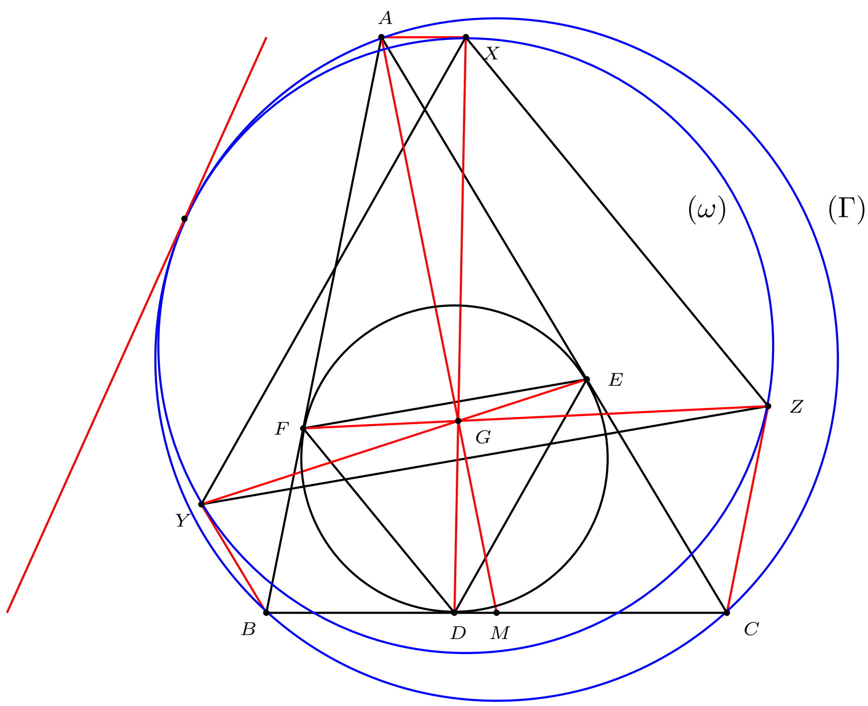


Figure 2

In order to prove Feuerbach's theorem, we shall prove that the lines L, L_a, L_b, and L_c are tangent to the circumcircle (Γ) of ABC: The isogonal conjugate of L, L_a, L_b, and L_c are respectively the conics

$$\begin{aligned} LL: &\quad (a(b-c))^2yz+(b(c-a))^2zx+(c(a-b))^2xy=0,\\ LL_a: &\quad (a(b-c))^2yz+(b(c+a))^2zx+(c(a+b))^2xy=0,\\ LL_b: &\quad (a(b+c))^2yz+(b(c-a))^2zx+(c(a+b))^2xy=0,\\ LL_c: &\quad (a(b+c))^2yz+(b(c+a))^2zx+(c(a-b))^2xy=0. \end{aligned}$$

Using Lemma 2, we easily check that the conics LL, LL_a, LL_b, and LL_c are tangent to the line at infinity $x+y+z=0$, which is the isogonal conjugate of circumcircle (Γ).

In particular, for LL, the point of tangency is the point $Q=(a(b-c):b(c-a):c(a-b))$, and in order to find the Feuerbach point (which is the contact point of nine-point circle with the incircle), we find the isogonal conjugate of Q. This is $\left(\frac{a}{b-c}:\ \frac{b}{c-a}:\ \frac{c}{a-b}\right)$, the anticomplement of X_{11}.

This completes the proof.

2. Proof of Emelyanov's Theorem

Continuing with the idea of using the anticomplement of a point [11] and barycentric coordinates [12], we give a simple proof for Lev Emelyanov's theorem [4, 7].

Theorem 3 (Emelyanov, 2001)**.** *The circle passing through the feet of the internal bisectors of a triangle contains the Feuerbach point of the triangle.*

Lemma 4. *Let ABC be a triangle and the point P $(x:y:z)$ in homogeneous barycentric coordinates, with cevian triangle $A'B'C'$. If M is the midpoint of BC, then signed length of MA' is*

$$MA' = \frac{z-y}{2(y+z)}BC.$$

Proof. Because $A'B'C'$ is the cevian triangle of P, $A'=(0:y:z)$. Using signed lengths of segments, we have $BA' = \frac{z}{y+z}BC$ and $CA' = \frac{y}{y+z}CB$. Therefore, we get the signed length of MA':

$$MA' = \frac{BA'+CA'}{2} = \frac{1}{2}\left(\frac{z}{y+z}BC + \frac{y}{y+z}CB\right) = \frac{z-y}{2(y+z)}BC.$$

We are done. □

Proof of Emelyanov's Theorem. In the triangle ABC, let A_1, B_1, and C_1 be the feet of the internal bisectors of the angles A, B, and C respectively. Let (γ) be the circumcircle of the triangle $A_1B_1C_1$, we must prove that (γ) contains the Feuerbach point $F_{\rm e}$.

Let the circle (γ) meet the sides BC, CA and AB again at A_2, B_2 and C_2 respectively. From Carnot's theorem [12], we have that the triangle $A_2B_2C_2$ is the cevian triangle of a point $I^* = (x:y:z)$ such that $BC_1\cdot BC_2 = BA_1\cdot BA_2$ and $CA_1\cdot CA_2 = CB_1\cdot CB_2$. From these, we get

$$\frac{ac}{a+b}\cdot\frac{xc}{x+y} = \frac{za}{y+z}\cdot\frac{ca}{b+c} \quad \text{and} \quad \frac{ba}{b+c}\cdot\frac{ya}{y+z} = \frac{xb}{z+x}\cdot\frac{ab}{c+a}$$

or

$$\begin{array}{lll} a_1 = a(a+b), & b_1 = (a-c)(a+b+c), & c_1 = -c(b+c), \\ a_2 = -a(c+a), & b_2 = b(b+c), & c_2 = (b-a)(a+b+c). \end{array}$$

We have the system

$$\begin{cases} a_1yz + b_1zx + c_1xy = 0 \\ a_2yz + b_2zx + c_2xy = 0 \end{cases}$$

$$\Longleftrightarrow \frac{yz}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{zx}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{xy}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

$$(x : y : z) = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right)^{-1}$$

This gives

$$\begin{aligned} I^* = \ & \big(abc + (a+b+c)(-a^2+b^2+c^2) \\ & : \ abc + (a+b+c)(a^2-b^2+c^2) \\ & : \ abc + (a+b+c)(a^2+b^2-c^2)\big)^{-1}. \end{aligned}$$

I^* is also the cyclocevian of I which is the point $X(1029)$ [5].

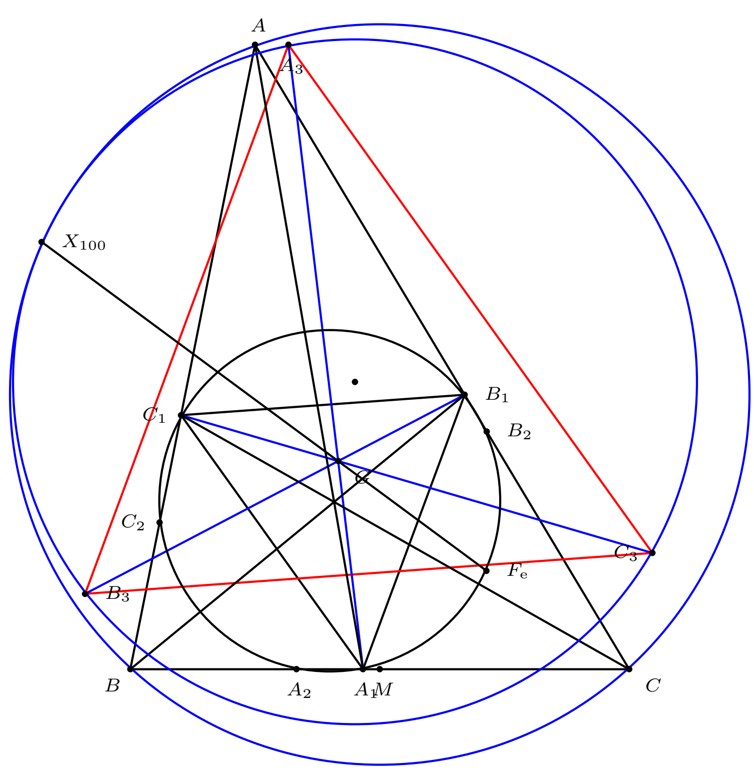


Figure 3

Let M be the midpoint of BC. Using Lemma 4, the signed length of MA_1 is

$$MA_1 = \frac{c-b}{2(b+c)}BC.$$

and the signed length of MA_2 is

$$MA_2 = \frac{\left(\frac{1}{abc+(a+b+c)(a^2+b^2-c^2)} - \frac{1}{abc+(a+b+c)(a^2-b^2+c^2)}\right)}{2\left(\frac{1}{abc+(a+b+c)(a^2-b^2+c^2)} + \frac{1}{abc+(a+b+c)(a^2+b^2-c^2)}\right)}BC.$$

An easy simplification leads to

$$MA_2 = \frac{(c^2-b^2)(a+b+c)}{2a(c+a)(a+b)}BC.$$

Let G be the centroid of ABC: We consider the anticomplement of the points A_1, B_1, and C_1, which are the points A_3, B_3, and C_3. Under the homothety $\mathcal{H}(G,-2)$, in order to prove that (γ) contains the Feuerbach point of ABC, we shall show that the circumcircle (Ω) of the triangle $A_3B_3C_3$ contains the anticomplement of the Feuerbach point. Indeed, as in our above proof of Feuerbach's theorem, we showed that anticomplement of Feuerbach point is

$$X(100) = \left(\frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}\right).$$

Because the anticomplement of M is A, the power of A with respect to circumcircle of triangle $A_3B_3C_3$ is

$$p = 2MA_1 \cdot 2MA_2 = \frac{(c-b)}{(b+c)}BC \cdot \frac{(c^2-b^2)(a+b+c)}{a(c+a)(a+b)}BC = \frac{a(a+b+c)(b-c)^2}{(c+a)(a+b)}.$$

Similarly, the powers of B and C with respect to circumcircle of triangle $A_3B_3C_3$ are

$$q = \frac{b(a+b+c)(c-a)^2}{(b+c)(b+a)} \quad \text{and} \quad r = \frac{c(a+b+c)(a-b)^2}{(c+a)(b+c)}$$

respectively.

Using the equation of a general circle in [12] again, we have the equation of (Ω):

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0.$$

Using the coordinates of $X(100)$, we easily check the expression

$$a^2yz+b^2zx+c^2xy = a^2 \cdot \frac{b}{c-a} \cdot \frac{c}{a-b} + b^2 \cdot \frac{c}{a-b} \cdot \frac{a}{b-c} + c^2 \cdot \frac{a}{b-c} \cdot \frac{b}{c-a} = 0.$$

and

$$\begin{aligned}
px+qy+rz &= p\cdot\frac{a}{b-c}+q\cdot\frac{b}{c-a}+r\cdot\frac{c}{a-b}\\
&= \frac{a(a+b+c)(b-c)^2}{(c+a)(a+b)}\cdot\frac{a}{b-c}+\frac{b(a+b+c)(c-a)^2}{(b+c)(b+a)}\cdot\frac{b}{c-a}\\
&\quad +\frac{c(a+b+c)(a-b)^2}{(c+a)(b+c)}\cdot\frac{c}{a-b}\\
&= 0.
\end{aligned}$$

Hence, the coordinates of $X(100)$ satisfy the equation of (Ω). This means that (Ω) passes through the anticomplement of the Feuerbach point. In other words (γ) passes through Feuerbach point. This completes our proof of Emelyanov's theorem.

References

[1] A. Bogomolny, Feuerbach's Theorem, Interactive Mathematics Miscellany and Puzzles, `http://www.cut-the-knot.org/Curriculum/Geometry/Feuerbach.shtml`.

[2] H. S. M. Coxeter, *Introduction to Geometry*, 2nd edition, John Wiley & Sons, Hoboken, N.J., 1969.

[3] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, The Math. Assoc. of America, 1967.

[4] L. Emelyanov and T. Emelyanova, A Note on the Feuerbach Point, *Forum Geom.*, 1 (2001) 121–124.

[5] C. Kimberling, Encyclopedia of Triangle Centers, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[6] J. S. MacKay, History of the nine point circle, *Proc. Edinb. Math. Soc.*, 11 (1892) 1961.

[7] H. M. Nguyen and D. P. Nguyen, Synthetic proofs of two theorems related to the Feuerbach point, *Forum Geom.*, 12 (2012) 39–46.

[8] K. J. Sanjana, An elementary proof of Feuerbach's theorem, *Mathematical Notes* 22 (1924) 11–12, Edinburgh Mathematical Society.

[9] M. J. G. Scheer, A simple vector proof of Feuerbach's theorem, *Forum Geom.*, 11 (2011) 205–210.

[10] E. W. Weisstein, *Feuerbach's Theorem*, from *MathWorld – A Wolfram Web Resource*, `http://mathworld.wolfram.com/FeuerbachsTheorem.html`.

[11] E. W. Weisstein, *Anticomplement*, from *MathWorld – A Wolfram Web Resource*, `http://mathworld.wolfram.com/Anticomplement.html`.

[12] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at `http://math.fau.edu/Yiu/Geometry.html`.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece.
E-mail address: `ndergiades@yahoo.gr`

Tran Quang Hung: High school for Gifted students, Hanoi University of Science, Hanoi National University, Hanoi, Vietnam.
E-mail address: `analgeomatica@gmail.com`

Forum Geometricorum
Volume 18 (2018) 361–365.

FORUM GEOM
ISSN 1534-1178

Pedals of the Poncelet Pencil and Fontené Points

Roger C. Alperin

Abstract. In this essay we describe special aspects of the Poncelet pencil, pedal circles and their relation to theorems of Fontené.

1. Review of the Poncelet Pencil, [1], [2], [3]

Given triangle $\triangle ABC$ we consider the pencil of lines at the circumcenter O. For each line $\mathcal{L}$ of this pencil we apply the isogonal transformation (denoted by $'$) to obtain the resulting conic $\mathcal{K} = \mathcal{L}'$ passing through $H = O'$, the orthocenter, and thus $\mathcal{K}$ is an equilateral hyperbola. The pencil of conics $\mathcal{K}$ (as $\mathcal{L}$ varies) is called the Poncelet pencil.

It is known that the locus of centers, $Z(\mathcal{K})$, of these conics is the Euler nine point circle $\mathcal{C}_9$. The circumcircle $\mathcal{C}$ meets each conic of the Poncelet pencil in the three vertices of the triangle and the circumcircle point $W(\mathcal{K})$. The midpoint of H and $W(\mathcal{K})$ is $Z(\mathcal{K})$.

2. Pedal Circles of the Poncelet Pencil

For each point P of the plane not on the circumcircle $\mathcal{C}$ we can form the pedal triangle and then its circumcircle $\mathcal{C}(P)$. Points on $\mathcal{C}$ have pedals which lie on the Simson line.

The history of the Theorem of Griffiths, sometimes known as Fontene's second theorem, is detailed in [7], $\S 403-6$, [9], [10] and asserts the following (see also [4]).

Theorem 1. *As P varies on $\mathcal{L}$ the pedal circles $\mathcal{C}(P)$ pass through $Z(\mathcal{K})$.*

The identification of the common point is in Johnson [7], see also [2]. For the points on $\mathcal{C} \cap L$ the associated Simson lines also pass through $Z(\mathcal{K})$ as indicated in [7].

3. Pedal Circles of Isogonal Points

According to theorems proven in Honsberger, [5], p. 67, [6], p. 56, we know the following (see Figure 1).

Theorem 2. *For P and its isogonal P' the pedal triangles of $\triangle$ lie on $\mathcal{C}(P)$ with center o at the midpoint of PP'.*

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

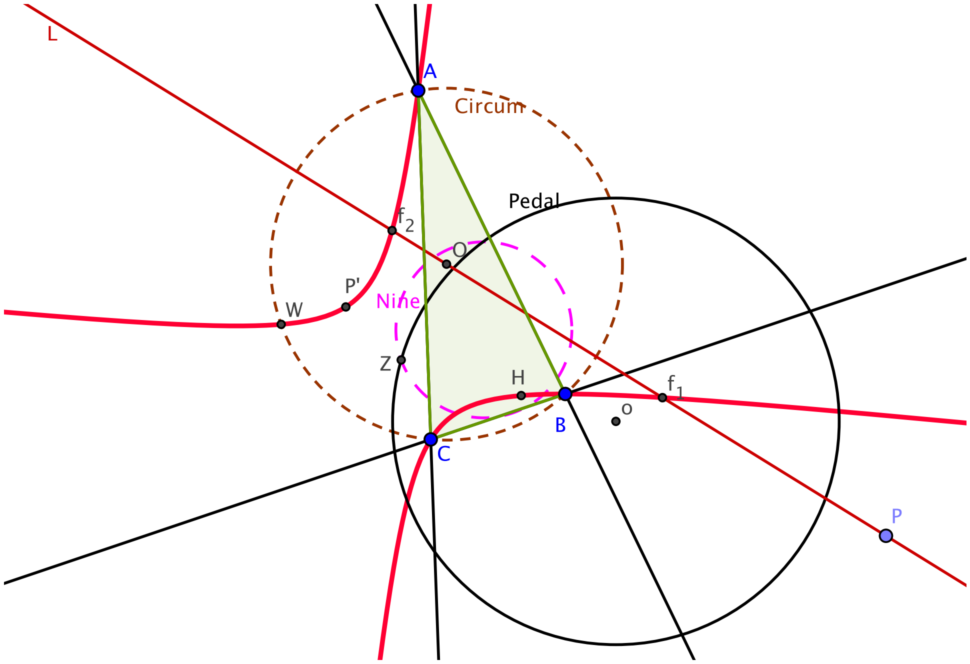


Figure 1. Pedal Circle from P

4. Fontene's Third Theorem and McCay's Cubic

Fontene's Third Theorem characterizes when the pedal circle and nine point circle, $\mathcal{C}_9$, are tangent: whenever P, O, P' are collinear.

Consider the intersections $\mathcal{L} \cap \mathcal{K}$ when $\mathcal{K}$ is irreducible. These two possible points will be called the Fontené points $\mathcal{K}$. A circle with these two Fontené points as diameter will be called the Fontené circle. By construction Fontené pairs are an isogonal pair.

Fontené's third theorem [10] can be expressed as follows (see Figure 2).

Theorem 3. *For $P \in \mathcal{L}$ the pedal circle $\mathcal{C}(P)$ and $\mathcal{C}_9$ are tangent at $Z(\mathcal{K})$ exactly when P is a Fontené point of $\mathcal{K}$.*

The next result follows immediately from the definition of the McCay cubic as the pivotal cubic determined from O and the isogonal transformation [8].

Theorem 4. *As $\mathcal{L}$ varies at O the locus of the Fontené points is the McCay cubic. The McCay cubic is self isogonal with isogonal pairs being the pairs of Fontené points of an irreducible conic of the Poncelet pencil. The reducible conics of the pencil give Fontené pairs consisting of a vertex and a point on the opposite side.*

5. Examples

5.1. *Isoceles Triangle.* In trilinear coordinates u, v, w the equation of McCay's cubic is $u(v^2 - w^2)\cos(A) + v(w^2 - u^2)\cos(B) + w(u^2 - v^2)\cos(C) = 0$. For

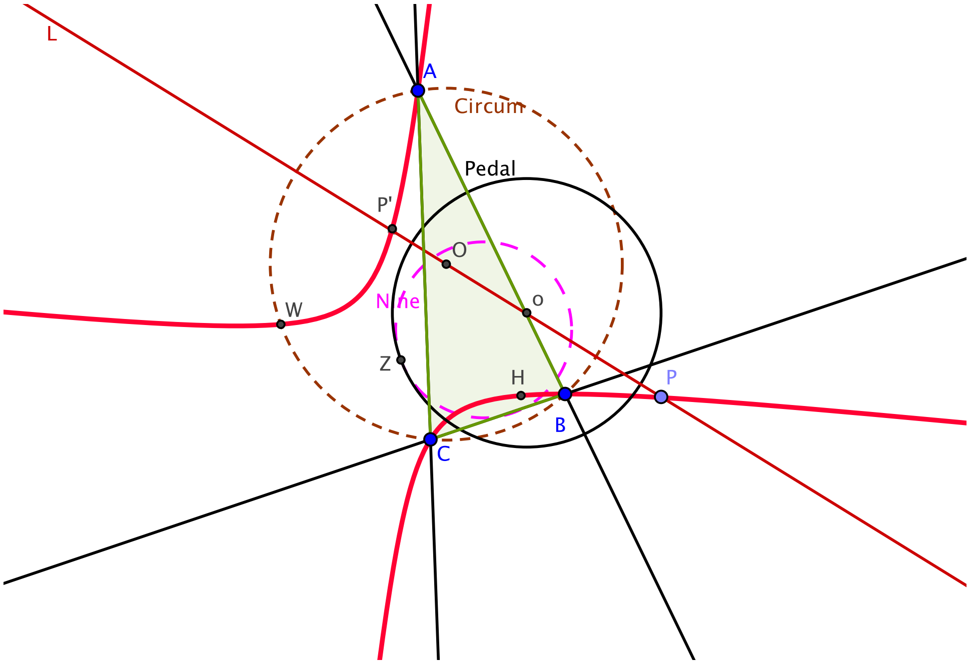


Figure 2. Tangent Pedal Circle

an isosceles triangle $A = B, u = v$ so the the equation factors up to a constant as $(u - v)(vu + w^2) = 0$. Hence McCay's cubic for an isosceles triangle consists of the perpendicular bisector $\mathcal{L}$ of the base and a hyperbola. For this line $\mathcal{L}$ there are infinitely many Fontené points on the reducible conic of the Poncelet pencil.

5.2. *Jerabek.* The Euler line $\mathcal{L}$ meets the Jerabek hyperbola $\mathcal{K} = \mathcal{L}'$ at O, H, its Fontené points. The pedal triangles from these points are the midpoint triangle and the orthic triangle. The pedal circle at these points is the Euler circle.

5.3. *Fuerbach.* $\mathcal{L} = OI$ is tangent to the Fuerbach hyperbola $\mathcal{K} = \mathcal{L}'$ at I; thus there is only one Fontené point. The pedal circle from I is the incircle tangent to the nine point circle at the Feuerbach point, the center of the Feuerbach hyperbola.

Similarly, for the other Feuerbach hyperbolas $\mathcal{K}_i = \mathcal{L}'_i$ using $\mathcal{L}_i = OI_j$ for the excenters $I_j,\ j = 1, 2, 3$ [1], we get the three ex-Feuerbach points on the nine point circle as centers of these hyperbolas. The excenter is the Fontené point.

It now follows that the McCay cubic passes through the nine points A, B, C, O, H, I, I_1, I_2, I_3 and the vertices P, Q, R of the anti-cevian triangle of the circumcenter.

5.4. *Kiepert,* $\mathcal{L} = OG'$,$\mathcal{K} = \mathcal{L}'$.

Theorem 5. *For a non-isosceles triangle the Fontené points of the Kiepert hyperbola are complex.*

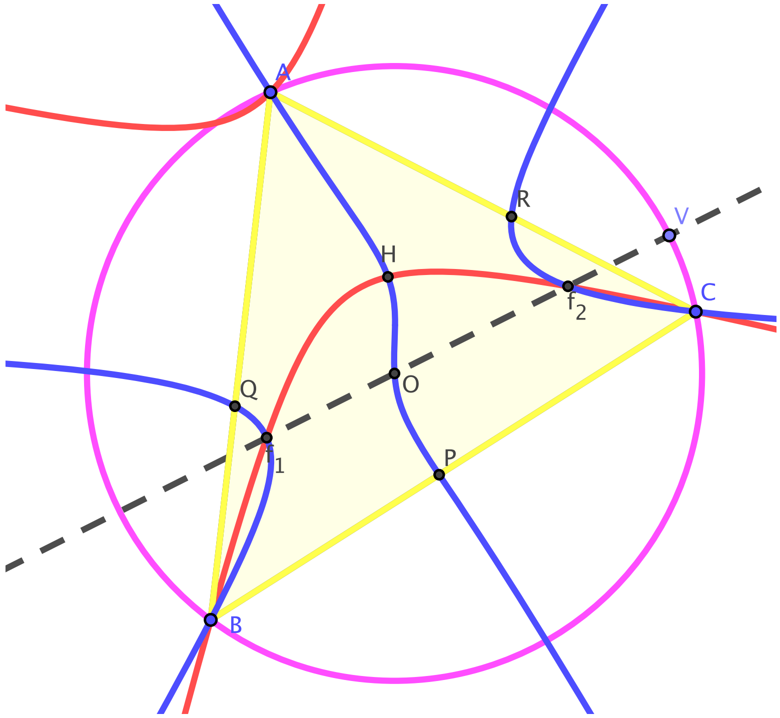


Figure 3. M'Cay Cubic

Proof. In trilinear coordinates u, v, w the McCay equation is $u(v^2 - w^2)\cos(A) + v(w^2 - u^2)\cos(B) + w(u^2 - v^2)\cos(C) = 0$ and the Kiepert's equation is $\sin(B - C)/u + \sin(C - A)/v + \sin(A - B)/w = 0$. Solving the Kiepert equation for w we can then eliminate w from the McCay equation giving a cubic equation in $x = u/v$. Since the orthocenter belongs to both curves we have a factor of $x - \cos(B)/\cos(A)$ and thus this reduces to a quadratic equation in x. Rewriting C in terms of A, B and simplifying gives the quadratic equation $x^2 - 2(\cos(A)\cos(B) + \sin(A)\sin(B)x + 1 = 0$. The discriminant is $-\sin(A - B)^2$ which is negative unless $A = B$; thus the Fontené points are complex for a non-isosceles triangle. □

References

[1] R. C. Alperin, The Poncelet pencil of rectangular hyperbolas, *Forum Geom.*, 10 (2010) 15–20.
[2] R. C. Alperin, Solving Euler's triangle problems with Poncelet's pencil, *Forum Geom.*, 11 (2011) 121–129.
[3] R. C. Alperin, Reflections on Poncelet's pencil, *Forum Geom.*, 15 (2015) 93–98.
[4] Z. Čerin, On Properties of Rectangular Hyperbolas, *Geom. Dedicata*, 84 (2001) 41–47.
[5] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library 37, MAA, 1995
[6] R. Honsberger, *More Mathematical Morsels*, Dolciani Mathematical Expositions 10, MAA, 1991
[7] R. A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960

[8] V. Prasolov, *Essays on Numbers and Figures*, Mathematical World 16, AMS, 2000
[9] J. Tabov, Four collinear Griffiths points, *Math. Mag.*, 68 (1995) 61–64.
[10] E. W. Weisstein, *Griffiths' Theorem*, *Fontené Theorems*, from *MathWorld – A Wolfram Web Resource*, http://mathworld.wolfram.com/GriffithsTheorem.html; http://mathworld.wolfram.com/FonteneTheorems.html

Roger C. Alperin: Department of Mathematics, San Jose State University, San Jose, CA 95192, USA
E-mail address: rcalperin@gmail.com

Forum Geometricorum
Volume 18 (2018) 367–370.

FORUM GEOM
ISSN 1534-1178

Circumconics with Asymptotes Making a Given Angle

Francisco Javier García Capitán

Abstract. Given a triangle ABC, we present a construction of the circumhyperbola whose two asymptotes are parallel to the sides of an isosceles triangle on a side of ABC.

Let ABC be a triangle and $0 < \theta < \frac{\pi}{2}$. Call U the point on the perpendicular bisector of BC such that $\angle BUC = 2\theta$ and $\mathcal{H}_{ab}$ the circumhyperbola of ABC with asymptotes parallel to UB and UM.

To construct this hyperbola by five ordinary points, we consider the points D and D' defined as follows:

- Let M be the midpoint BC and S the intersection point of the parallel to BU through C and the parallel to AB through M. Then the line AS intersects the perpendicular bisector of BC at D.
- Let S' be the intersections of the parallel through A to BC and the parallel through C to BU. If the parallel through S' to AB intersects BC at M', then the perpendicular through M' to BC intersects the parallel through A to BC at D'.

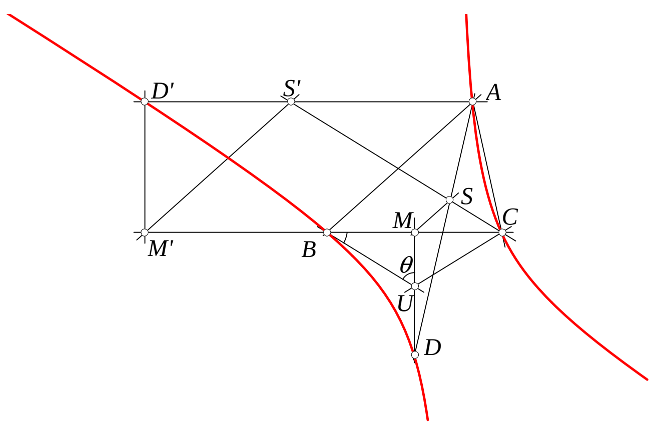


Figure 1

Proposition 1. *The points D, D' lie on $\mathcal{H}_{ab}$.*

Publication Date: November 19, 2018. Communicating Editor: Paul Yiu.

Proof. We have the coordinates

$$D = \left(a^4 : (S_B - S_C)(S_C - S\tan\theta) : -(S_B - S_C)(S_B + S\tan\theta)\right),$$
$$D' = \left(a^2 : S_B - S_C + S\tan\theta : -S_B + S_C - S\tan\theta\right).$$

On the other hand, the hyperbola $\mathcal{H}_{ab}$ has equation

$$a^4yz + S_C\left(S_C - S\tan\theta\right)zx + S_B\left(S_B + S\tan\theta\right)xy = 0.$$

□

Remark. Proposition 1 can also be proved by using Pascal theorem.

Let A_b the perspector of the hyperbola $\mathcal{H}_{ab}$, and A_c the perspector of the hyperbola $\mathcal{H}_{ac}$, defined in a similar way, that is, as the circumhyperbola with asymptotes parallel to UM and UC. Therefore we have the points

$$A_b = \left(a^4 : S_C\left(S_C - S\tan\theta\right) : S_B\left(S_B + S\tan\theta\right)\right),$$
$$A_c = \left(a^4 : S_C\left(S_C + S\tan\theta\right) : S_B\left(S_B - S\tan\theta\right)\right).$$

We also define B_c, B_a and C_a, C_b cyclically.

Proposition 2. *The six points A_b, A_c, B_c, B_a, C_a, C_b, lie on a same conic Γ_θ, whose equation is*

$$\tan^2\theta \cdot (S_Ax + S_By + S_Cz)^2 - S^2 \cdot \left(x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\right) = 0.$$

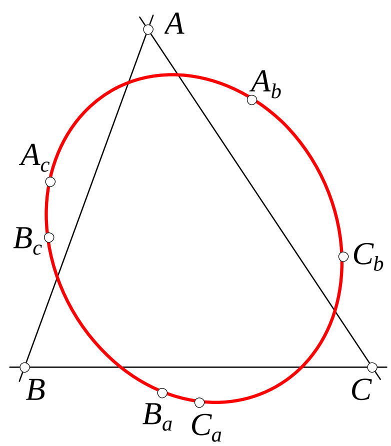


Figure 2

Proposition 3. *The center O_θ of Γ_θ lies on the line GK and the following ratio holds:*

$$\frac{GO_\theta}{O_\theta K} = -\frac{S_\omega^2 \cdot \sin^2\theta}{3S^2},$$

where G and K are the centroid and the symmedian point of ABC.

Proposition 4. *The conic Γ_θ is the locus of perspectors of all circumconics of ABC whose asymptotes make an angle θ.*

Now call XYZ the triangle bounded by lines A_bA_c, B_cB_a, C_aA_b. Surprisingly enough, XYZ does not depend on θ. More specifically, we have the following result.

Proposition 5. *The line A_bA_c trisects the $A-$altitude from the vertex and intersects BC at D', the harmonic conjugate of the feet D of the $A-$altitude with respect to BC. In other words, lines A_bA_c, BC and the sideline of the orthic triangle corresponding to vertex A are concurrent.*

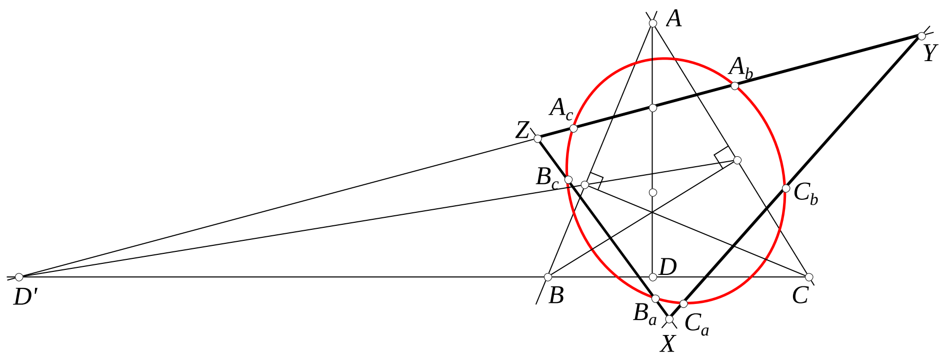


Figure 3

Proposition 6. *The triangle XYZ is perspective at K.*

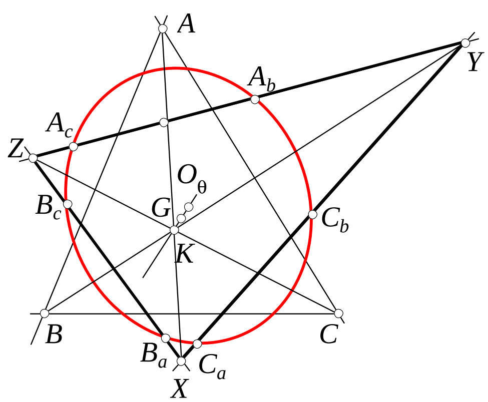


Figure 4

Proposition 7. *The discriminant of* Γ_θ *is given by the formula*

$$\Delta(\theta) = 2\left(\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2\right)\tan^2\theta - 12S^2.$$

The following figure shows a general view: The point U on the perpendicular bisector of BC has been used to construct the circumhyperbola with perspector A_b having asymptotes parallel to UB and perpendicular to BC. Similarly we construct A_c, B_c, B_a, C_a, C_b lying on the conic Γ_θ. The conic center O_θ of Γ_θ lies on line GK.

Now we take a point P on Γ_θ and construct the conic (hyperbola) with perspector P. The asymptotes of this hyperbola make an angle θ formed by UB and the perpendicular bisector of BC.

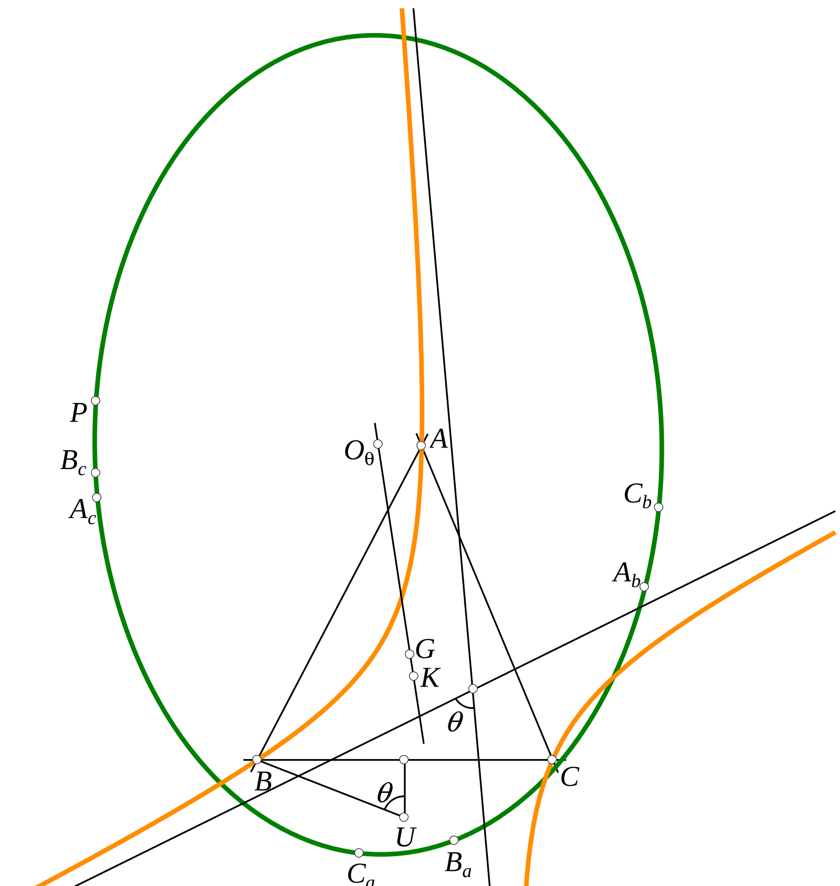


Figure 5

Francisco Javier García Capitán: Departamento de Matemáticas, I.E.S. Alvarez Cubero, Avda. Presidente Alcalá-Zamora, s/n, 14800 Priego de Córdoba, Córdoba, Spain

E-mail address: garciacapitan@gmail.com

Forum Geometricorum
Volume 18 (2018) 371–380.

FORUM GEOM
ISSN 1534-1178

Integral Triangles and Trapezoids Pairs with a Common Area and a Common Perimeter

Yong Zhang, Junyao Peng, and Jiamin Wang

Abstract. Using the theory of elliptic curves, we show that there are infinitely many integral right triangle and θ-integral right (resp. isosceles) trapezoid, integral isosceles triangle and θ-integral right (resp. isosceles) trapezoid, Heron triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.

1. Introduction

We say that a Heron (resp. rational) triangle is a triangle with integral (resp. rational) sides and integral (resp. rational) area, and a polygon (sides greater than 3) is integral (resp. rational) if the lengths of its sides are all integers (resp. rational numbers). We call a polygon θ-integral (resp. θ-rational) if the polygon has integral (resp. rational) side lengths, integral (resp. rational) area, and both $\sin\theta$ and $\cos\theta$ are rational numbers.

In 1995, R. K. Guy [6] introduced a problem of Bill Sands, that asked for examples of an integral right triangle and an integral rectangle with a common area and a common perimeter, but there are no non-degenerate such. In the same paper, R. K. Guy showed that there are infinitely many such integral isosceles triangle and integral rectangle pairs. Several authors studied other cases, such as two distinct Heron triangles by A. Bremner [1], Heron triangle and rectangle pairs by R. K. Guy and A. Bremner [2], integral right triangle and parallelogram pairs by Y. Zhang [8], integral right triangle and rhombus pairs by S. Chern [3], integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs by P. Das, A. Juyal and D. Moody [4], Heron triangle-rhombus and isosceles triangle-rhombus pairs by Y. Zhang and J. Peng [9], and rational (primitive) right triangle-isosceles triangle pairs by Y. Hirakawa and H. Matsumura [7].

Now we consider other pairs of geometric shapes having a common area and a common perimeter, such as triangles and trapezoids pairs (see figures 1 and 2). The sides of triangles and trapezoids are given in Table 1, and the area and perimeter of

---

Publication Date: December 6, 2018. Communicating Editor: Paul Yiu.

This research was supported by the National Natural Science Foundation of China (Grant No. 11501052) and Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology).

triangles and trapezoids are given in Table 2. For θ-integral trapezoid, we may set

$$\sin\theta = \frac{2t}{1+t^2},\ \cos\theta = \frac{1-t^2}{1+t^2},$$

where $0 < t \leq 1$ is a rational number. For $t = 1, \theta = \pi/2$, trapezoid reduces to rectangle, this is the case studied by R. K. Guy [6], thus we only need to consider the case $0 < t < 1$. Using the theory of elliptic curves, we prove

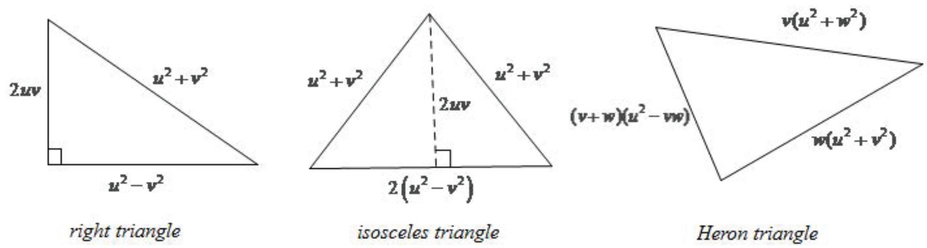


Figure 1. Right triangle, isosceles triangle and Heron triangle

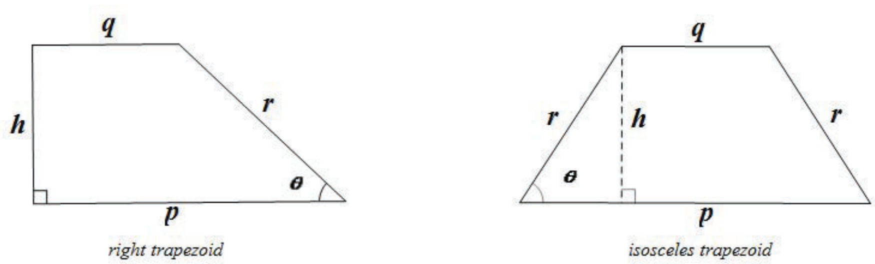


Figure 2. Right trapezoid and isosceles trapezoid

name	sides
right triangle	$(u^2 - v^2, 2uv, u^2 + v^2)\ (u > v)$
isosceles triangle	$(u^2 + v^2, u^2 + v^2, 2(u^2 - v^2))\ (u > v)$
Heron triangle	$((v + w)(u^2 - vw), v(u^2 + w^2), w(u^2 + v^2))\ (u^2 > vw)$
right trapezoid	(p, q, h, r)
isosceles trapezoid	(p, q, r, r)

Table 1. The sides of triangles and trapezoids

Theorem 1. *There are infinitely many integral right triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

name	area	perimeter
right triangle	$uv(u^2 - v^2)$	$2u(u + v)$
isosceles triangle	$2uv(u^2 - v^2)$	$4u^2$
Heron triangle	$uvw(v + w)(u^2 - vw)$	$2u^2(v + w)$
right trapezoid	$\frac{(p+q)h}{2}$	$p + q + h + r$
isosceles trapezoid	$\frac{(p+q)r}{2} \sin\theta$	$p + q + 2r$

Table 2. The area and perimeter of triangles and trapezoids

Theorem 2. *There are infinitely many integral isosceles triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

Theorem 3. *There are infinitely many Heron triangle and θ-integral right (resp. isosceles) trapezoid pairs with a common area and a common perimeter.*

2. Proofs of the theorems

Proof of Theorem 1. By the homogeneity of the sides of integral right triangle and θ-integral right trapezoid, we can set $u = 1$, and v, p, q, h, r to be positive rational numbers. Now we only need to study the rational right triangle and θ-rational right trapezoid pairs with a common area and a common perimeter, then we have

$$\begin{cases} v(1 - v^2) = \dfrac{1}{2}(p + q)h, \\ 2(1 + v) = p + q + r + h, \\ \dfrac{p - q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{1}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0 < t < 1$ a rational number.

Eliminating h, p, q in equation (1), we get

$$\frac{t(t+1)^2r^2 - 2t(t^2+1)(v+1)r - v(v-1)(v+1)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 2t(t^2+1)(v+1)r - v(v-1)(v+1)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_1(r)$ should be a rational perfect square. Let us consider the curve

$$\mathcal{C}_1 : \ s^2 = \Delta_1(r) = t(t+1)^2v^3 + t^2v^2 - t(t^2+1)v + t^2,$$

which is a cubic curve with a rational point $P_0 = (v, s) = (1, 2t)$. And the curve $\mathcal{C}_1$ is birationally equivalent to the elliptic curve $\mathcal{E}_1$ given by the equation in Weierstrass form

$$\mathcal{E}_1 : \ Y^2 = X^3 - 27t^2(3t^4+6t^3+7t^2+6t+3)X + 54t^4(3t^2+5t+3)(6t^2+11t+6).$$

The map $\varphi_1 : \mathcal{C}_1 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_1$ is given by

$$X = 3t(3t^2v + 6tv + t + 3v), \ Y = 27st(t+1)^2,$$

and the inverse map

$$\varphi_1^{-1} : \ v = \frac{-3t^2X}{9t(t+1)^2}, \ s = \frac{Y}{27t(t+1)^2}.$$

By the map φ_1, we get the point

$$P = \varphi_1(P_0) = (3t(3t^2 + 7t + 3), \ 54t^2(t+1)^2),$$

which lies on the elliptic curve $\mathcal{E}_1$. By the group law, we have

$$[2]P = \left(\frac{9t^4 - 6t^2 + 9}{4}, \ -\frac{27(t^2 - 4t + 1)(t+1)^4}{8}\right).$$

Through the inverse map φ_1^{-1}, we get

$$v = \varphi_1^{-1}(x([2]P)) = \frac{(t-1)^2}{4t},$$

then

$$\begin{aligned} r =& \frac{(t^2+1)(t-1)^2}{8t^2}, \\ p =& -\frac{(t+1)(t^2-4t-1)}{8t}, \\ q =& \frac{(t+1)(t^2+4t-1)}{8t^2}, \\ h =& \frac{(t-1)^2}{4t}. \end{aligned}$$

Hence, the rational right triangle has sides

$$(x, y, z) = \left(-\frac{(t^2-6t+1)(t+1)^2}{16t^2}, \ \frac{(t-1)^2}{2t}, \ \frac{t^4 - 4t^3 + 22t^2 - 4t + 1}{16t^2}\right),$$

and the θ-rational right trapezoid has sides

$$(p, q, h, r) = \left(-\frac{(t+1)(t^2-4t-1)}{8t}, \ \frac{(t+1)(t^2+4t-1)}{8t^2}, \ \frac{(t-1)^2}{4t}, \ \frac{(t^2+1)(t-1)^2}{8t^2}\right).$$

Since v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $v < 1$, we obtain the condition

$$0.236067977 < t < 1.$$

Then if $0.236067977 < t < 1$, there are infinitely many rational right triangle and θ-rational right trapezoid pairs with a common area and a common perimeter.

Therefore, there are infinitely many such integral right triangle and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such integral right triangle and θ-integral isosceles trapezoid pairs. □

Example 1. If $t = \frac{1}{2}$, we have a right triangle with sides $(x, y, z) = (63, 16, 65)$, and a right trapezoid with sides $(p, q, h, r) = (66, 60, 8, 10)$, which have a common area 504 and a common perimeter 144.

Proof of Theorem 2. As in Theorem 1, we only need to consider the θ-rational isosceles triangle and rational right trapezoid pairs with a common area and a common perimeter. We may set $u = 1$, and v, p, q, h, r to be positive rational numbers, then

$$\begin{cases} 2v(1-v^2) = \dfrac{1}{2}(p+q)h, \\ 4 = p+q+r+h, \\ \dfrac{p-q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{2}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0 < t < 1$ a rational number.

Eliminating h, p, q in equation (2), we have

$$\frac{t(t+1)^2r^2 - 4t(t^2+1)r - 2v(v-1)(v+1)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 4t(t^2+1)r - 2v(v-1)(v+1)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_2(r)$ should be a rational perfect square. Let us consider the curve

$$\mathcal{C}_2 : \ s^2 = \Delta_2(r) = 2t(t+1)^2v^3 - 2t(t+1)^2v + 4t^2,$$

which is a cubic curve with a rational point $Q_0 = (v, s) = (1, 2t)$. And the curve $\mathcal{C}_2$ is birationally equivalent with the elliptic curve $\mathcal{E}_2$ given by the equation in Weierstrass form

$$\mathcal{E}_2 : \ Y^2 = X^3 - 4t^2(t+1)^4X + 16t^4(t+1)^4.$$

The map $\varphi_2 : \mathcal{C}_2 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_2$ is given by

$$X = 2vt(t+1)^2, \ Y = 2st(t+1)^2,$$

and the inverse map

$$\varphi_2^{-1} : \ v = \frac{X}{2t(t+1)^2}, \ s = \frac{Y}{2t(t+1)^2}.$$

By the map φ_2, we get the point

$$Q = \varphi_2(Q_0) = (2t(t+1)^2, \ 4t^2(t+1)^2),$$

which lies on the elliptic curve $\mathcal{E}_2$. By the group law, we have

$$[2]Q = ((t-1)^2(t+1)^2,\ -(t^4-2t^3-2t^2-2t+1)(t+1)^2).$$

Through the inverse map φ_2^{-1}, we get

$$v = \varphi_2^{-1}(x([2]Q)) = \frac{(t-1)^2}{2t},$$

then

$$\begin{aligned} r =& \frac{(t-1)^2(t^2+1)^2}{2(t+1)^2t^2}, \\ p =& -\frac{t^4-2t^3-2t^2-6t+1}{t(t+1)}, \\ q =& -\frac{t^4-6t^3-2t^2-2t+1}{t^2(t+1)}, \\ h =& \frac{(t-1)^2(t^2+1)}{t(t+1)^2}. \end{aligned}$$

Hence, the rational isosceles triangle has sides

$$\begin{aligned} (x,y,z) = \Bigg(& \frac{t^4-4t^3+10t^2-4t+1}{4t^2},\ \frac{t^4-4t^3+10t^2-4t+1}{4t^2}, \\ & -\frac{t^4-4t^3-2t^2-4t+1}{4t^2}\Bigg), \end{aligned}$$

and the θ-rational right trapezoid has sides

$$\begin{aligned} (p,q,h,r) = \Bigg(& -\frac{t^4-2t^3-2t^2-6t+1}{t(t+1)},\ -\frac{t^4-6t^3-2t^2-2t+1}{t^2(t+1)}, \\ & \frac{(t-1)^2(t^2+1)}{t(t+1)^2},\ \frac{(t-1)^2(t^2+1)^2}{2(t+1)^2t^2}\Bigg). \end{aligned}$$

Since v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $v < 1$, we obtain the condition

$$0.3137501201 < t < 1.$$

Then if $0.3137501201 < t < 1$, there are infinitely many rational isosceles triangle and θ-rational right trapezoid pairs with a common area and a common perimeter. Therefore, there are infinitely many such integral isosceles triangle and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such integral isosceles triangle and θ-integral isosceles trapezoid pairs. □

Example 2. If $t = \frac{1}{2}$, we have an isosceles triangle with sides $(x,y,z) = (153, 153, 270)$ and a right trapezoid with sides $(p,q,h,r) = (258, 228, 40, 50)$, which have a common area 9720 and a common perimeter 576.

Proof of Theorem 3. As in Theorem 1, we only need to investigate the rational Heron triangle and θ-rational right trapezoid pairs with a common area and a common perimeter. We can set $w = 1$, and u, v, p, q, h, r to be positive rational numbers, then

$$\begin{cases} uv(v+1)(u^2-v) = \dfrac{1}{2}(p+q)h, \\ 2u^2(1+v) = p+q+r+h, \\ \dfrac{p-q}{r} = \cos\theta, \\ h = r\sin\theta, \end{cases} \tag{3}$$

where $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$, with $0 < t < 1$ a rational number.

Eliminating h, p, q in equation (3), we have

$$\frac{t(t+1)^2r^2 - 2tu^2(t^2+1)(v+1)r + uv(v+1)(u^2-v)(t^2+1)^2}{(t^2+1)^2} = 0.$$

It only needs to consider

$$t(t+1)^2r^2 - 2tu^2(t^2+1)(v+1)r + uv(v+1)(u^2-v)(t^2+1)^2 = 0.$$

If this quadratic equation has rational solutions r, then its discriminant $\Delta_3(r)$ should be a rational perfect square. Let us consider the curve

$$\begin{aligned} \mathcal{C}_3:\ s^2 = \Delta_3(r) =& (t+1)^2tuv^3 - (t^2u^2 - tu^3 + 2tu^2 - t^2 + u^2 - 2t - 1)tuv^2 \\ & - u^3(t^2 - 2tu + 2t + 1)tv + t^2u^4, \end{aligned}$$

which is a cubic curve with a rational point $R_0 = (v, s) = (u^2, tu^2(u^2+1))$. And the curve $\mathcal{C}_3$ is birationally equivalent to the elliptic curve $\mathcal{E}_3$ given by the equation in Weierstrass form

$$\mathcal{E}_3:\ Y^2 = X^3 + A_4X + A_6,$$

where

$$\begin{aligned} A_4 =& -27t^2u^2(t^4u^4 - 2t^3u^5 + t^2u^6 + 4t^3u^4 - 4t^2u^5 + t^4u^2 - 4t^3u^3 + 6t^2u^4 \\ & - 2tu^5 + 4t^3u^2 - 8t^2u^3 + 4tu^4 + t^4 + 6t^2u^2 - 4tu^3 + u^4 + 4t^3 + 4tu^2 \\ & + 6t^2 + u^2 + 4t + 1), \\ A_6 =& -27t^3u^3(t^2u^2 - tu^3 + 2tu^2 + 2t^2 + u^2 + 4t + 2)(2t^4u^4 - 4t^3u^5 + 2t^2u^6 \\ & + 8t^3u^4 - 8t^2u^5 - t^4u^2 - 8t^3u^3 + 12t^2u^4 - 4tu^5 - 4t^3u^2 - 16t^2u^3 + 8tu^4 \\ & - t^4 - 6t^2u^2 - 8tu^3 + 2u^4 - 4t^3 - 4tu^2 - 6t^2 - u^2 - 4t - 1). \end{aligned}$$

The map $\varphi_3 : \mathcal{C}_3 \ni (v, s) \mapsto (X, Y) \in \mathcal{E}_3$ is given by

$$\begin{aligned} X &= -3tu(t^2u^2 - tu^3 - 3t^2v + 2tu^2 - t^2 - 6tv + u^2 - 2t - 3v - 1), \\ Y &= 27sut(t+1)^2, \end{aligned}$$

and the inverse map φ_3^{-1} :

$$v = \frac{X + 3tu(t^2u^2 - tu^3 + 2tu^2 - t^2 + u^2 - 2t - 1)}{9ut(t+1)^2},$$
$$s = \frac{Y}{27ut(t+1)^2}.$$

By the map φ_3, we get the point

$$R = \varphi_3(R_0) = (3tu(2t^2u^2 + tu^3 + 4tu^2 + t^2 + 2u^2 + 2t + 1), \quad 27t^2u^3(u^2+1)(t+1)^2),$$

which lies on the elliptic curve $\mathcal{E}_3$. By the group law, we have

$$[2]R = \Big(\frac{3}{4}u(4t^2u^3 - 4t(t+1)^2u^2 + 3(t+1)^4u - 8t(t+1)^2), \quad -\frac{27}{8}u^2(t+1)^4(t^2u - 2tu^2 + 2tu - 4t + u)\Big).$$

Through the inverse map φ_3^{-1}, we get

$$v = \varphi_3^{-1}(x([2]R)) = \frac{t^2u - 4tu^2 + 2tu - 4t + u}{4t},$$

then

$$r = \frac{u(t^2+1)(t^2u + 2tu - 4t + u)}{8t^2},$$
$$p = -\frac{(t+1)(t^2u - 2tu^2 + 2tu - 2u^2 - 4t + u)u}{8t},$$
$$q = \frac{(t+1)(2t^2u^2 - t^2u + 2tu^2 - 2tu + 4t - u)u}{8t^2},$$
$$h = \frac{u(t^2u + 2tu - 4t + u)}{4t}.$$

To simplify the proof, we set $t = \frac{1}{2}$. Hence, the rational Heron triangle has sides

$$(x, y, z) = \left(\frac{9u(8u^2 - 9u + 8)}{64}, \frac{(9u-8)(u^2+1)}{8}, \frac{145u^2 - 144u + 64}{64}\right),$$

and the θ-rational right trapezoid has sides

$$(p, q, h, r) = \left(\frac{3u(12u^2 - 9u + 8)}{32}, \frac{3u(6u^2 - 9u + 8)}{16}, \frac{u(9u-8)}{8}, \frac{5u(9u-8)}{32}\right).$$

Since u, v, p, q, h, r are positive rational numbers, $0 < \sin\theta < 1$ and $u^2 > v$, we obtain the condition

$$u > \frac{8}{9}.$$

Then if $u > \frac{8}{9}$, there are infinitely many rational Heron triangles and θ-rational right trapezoid pairs with a common area and a common perimeter. Therefore, there are infinitely many such Heron triangles and θ-integral right trapezoid pairs.

Similarly, we can prove that there are infinitely many such Heron triangles and θ-integral isosceles trapezoid pairs. $\square$

Example 3. If $u = 2$, we have a Heron triangle with sides $(x, y, z) = (99, 100, 89)$ and a right trapezoid with sides $(p, q, h, r) = (114, 84, 40, 50)$, which have a common area 3960 and a common perimeter 288.

3. Some related questions

We have studied the integral triangles and θ-integral trapezoids pairs with a common area and a common perimeter. Noting that isosceles trapezoid is a cyclic quadrilateral, so it is also interesting to consider the following three questions.

Question 1. Are there infinitely many integral right triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

Question 2. Are there infinitely many integral isosceles triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

Question 3. Are there infinitely many Heron triangle and θ-integral cyclic (resp. rational) quadrilateral pairs with a common area and a common perimeter?

References

[1] A. Bremner, *On Heron triangles*, Ann. Math. Inform. **33** (2006), 15–21.
[2] A. Bremner and R. K. Guy, *Triangle-rectangle pairs with a common area and a common perimeter*, Int. J. Number Theory **2** (2006), 217–223.
[3] S. Chern, *Integral right triangle and rhombus pairs with a common area and a common perimeter*, Forum Geom. **16** (2016), 25–27.
[4] P. Das, A. Juyal and D. Moody, *Integral isosceles triangle-parallelogram and Heron triangle-rhombus pairs with a common area and common perimeter*, J. Number Theory **180** (2017), 208–218.
[5] L. E. Dickson, *History of the theory of numbers, Vol. II: Diophantine analysis*, Dover Publications, New York, 2005.
[6] R. K. Guy, *My favorite elliptic curve: a tale of two types of triangles*, Amer. Math. Monthly **102** (1995), 771–781.
[7] Y. Hirakawa and H. Matsumura, *A unique pair of triangles*, J. Number Theory **194** (2019), 297–302.
[8] Y. Zhang, *Right triangle and parallelogram paris with a common area and a common perimeter*, J. Number Theory **164** (2016), 179–190.
[9] Y. Zhang and J. Peng, *Heron triangle and integral rhombus paris with a common area and a common perimeter*, Forum Geom. **17** (2017), 419–423.

Yong Zhang: School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, People's Republic of China
E-mail address: zhangyongzju@163.com

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, People's Republic of China
E-mail address: junyaopeng906@163.com

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, People's Republic of China
E-mail address: 1281524597@qq.com

Forum Geometricorum
Volume 18 (2018) 381–400.

FORUM GEOM
ISSN 1534-1178

Pairs of Congruent-Like Quadrilaterals that are not Congruent

Giuseppina Anatriello, Francesco Laudano, and Giovanni Vincenzi

Abstract. In this article we will give a negative answer to a question referred to a congruence theorem for quadrilaterals that have been recently introduced in [5, section 4]. Precisely we will show that there exist pairs of quadrilaterals having 8 pieces (four sides and four angles) pairwise congruent, but that are not congruent. Computations and the use of geometric design by dynamical software will have a crucial role in the proof.

1. Introduction

One of the main topic in Mathematical Geometric Foundations is the concept of 'congruence' between polygons. Two convex polygons are said to be *congruent* if there is a one-to-one correspondence between their vertices such that consecutive vertices correspond to consecutive vertices, and all pairs of correspondent sides and all pairs of correspondent angles are congruent. Formally, we can say that two polygons are congruent if, and only if, one can be transformed into the other by an isometry.

Usually congruence theorems for triangles, for quadrilaterals and in general for polygons, are stated using sides and angles of pairs of such figures (see [4], [6], [11]); but for polygons certain remarks are required (see [4, Lesson 11] and [6, Chapter 8] for the definitions and the development of the basic properties related to polygons).

The general study of congruence between two polygons seems too hard, and different cases such as convex (see [4, Definition 9.7]), non convex or twisted polygons should be considered.

Extending a definition introduced in [5] we will say that two polygons P and P' are *congruent-like*, if there is a bijection between the sides of P and P', and a bijection between the angles of P and P', such that each side of P is congruent to a corresponding side of P' and each angle of P is congruent to a corresponding angle of P' (note that the two bijections are 'a priori' independent).

Publication Date: December 6, 2018. Communicating Editor: Paul Yiu.
This research has been supported by Ministry of Italian Research, INdAM - GNSAGA..

Clearly, two congruent n-gons are congruent-like. It is easy to check that the converse holds for pairs of triangles; but, as remarked in [5], it is not always true if $n > 4$ (see Figures 1 and 2).

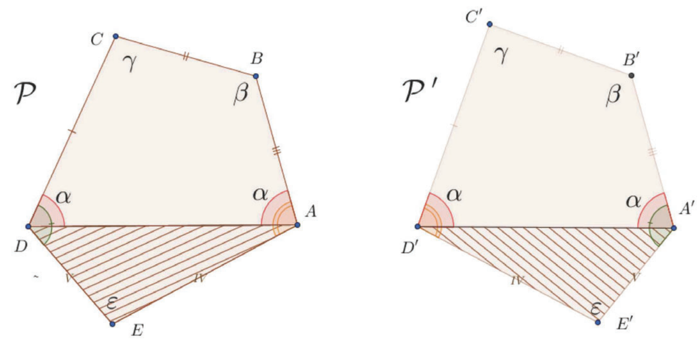


Figure 1. Starting from two congruent quadrilaterals (A, B, C, D) and (A', B', C', D') with two congruent consecutive angles of magnitude α, we may easily construct pairs of congruent-like polygons that are not congruent.

Following [5], in this paper we will denote by $P = (A_0, \ldots, A_{n-1})$ the polygon whose consecutive vertices are $A_0, \ldots, A_{n-1}$ (usually numbered counterclockwise), by a_i the length of the side A_iA_{i+1} (the index i is read mod n). The inner angle at the vertex A_i as well as its measure will be denoted by $\hat{A}_i$, $A_{i-1}\hat{A}_iA_{i+1}$ or simply by α_i. Moreover, we will denote by:

$$a_i \to a_{i+1} \to \ldots \to a_{i+4} \to a_{i+5} \to a_i, \quad \text{or}$$

$$A_iA_{i+1} \to A_{i+1}A_{i+2} \to \ldots \to A_{i+n-2}A_{i+n-1} \to A_{i+n-1}A_i \to A_iA_{i+1},$$

the ordered sequence of the 'sides' of P, starting from A_iA_{i+1} (see Figure 2).

We will say that two congruent-like polygons P and P' are *ordered congruent-like polygons*, if whichever ordered sequence of consecutive sides of P is equal to an ordered sequence of sides of P'. In [5, Theorem 3.5] it has been recently showed that convex ordered congruent-like quadrilaterals are indeed congruent (a complete list of other classical congruence theorems for quadrilaterals can be found in [4] and [6] for instance).

Theorem 1. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be ordered congruent-like convex quadrilaterals. Then Q and Q' are congruent.*

This result cannot be extended to hexagons (and any other n-gons, for $n \geq 6$), as Figure 2 shows, and it is still an open question for (ordered) pentagons.

Another natural question that arises is:

P) *Can we remove the hypothesis that Q and Q' are ordered?*

The aim of this article is to investigate this question. As we will see examples of (non-ordered) convex congruent-like quadrilaterals Q and Q' that are not congruent exist.

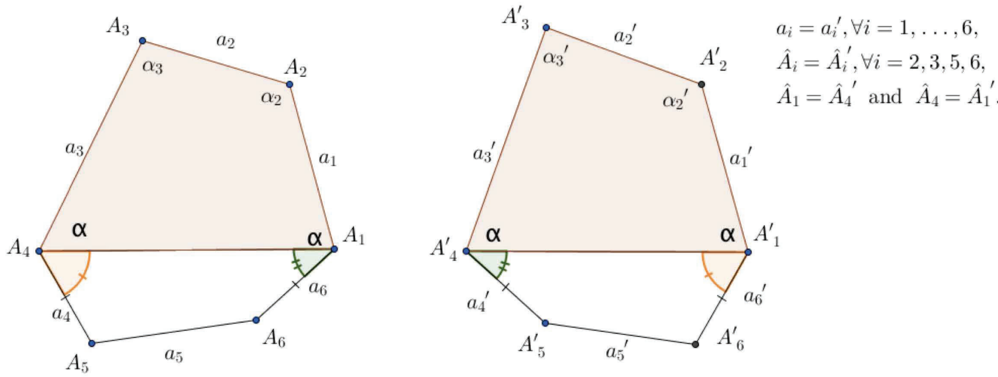


Figure 2. Starting from a quadrilateral (A, B, C, D) with two congruent consecutive angles, say α, we may easily construct pairs of ordered congruent-like n-gons that are not congruent, for every $n > 5$. Here any sequence $a_i \to a_{i+1} \to \ldots \to a_{i+4} \to a_{i+5} \to a_i$ of sides of the hexagon on the left is equal to $a_i' \to a_{i+1}' \to \ldots \to a_{i+4}' \to a_{i+5}' \to a_i'$ referred to the hexagon on the right.

As we will see the construction of such examples will be not so elementary: software for geometric designs, computation and combined arguments of synthetic geometry and analysis will be used. Here we only prove the existence, and on the other hand we will also provide for a method to obtain good approximations of such pairs of quadrilaterals that are not congruent with all sides and angles pairwise congruent (see Figure 8).

In section 2, we will start to define families $\mathcal{F}_{m,\alpha}$ of pairs of quadrilaterals $Q_{d,a}$ and $Q'_{d,a}$ (depending on two parameters, namely d and a that are also two consecutive sides of the $Q_{d,a}$), which have five elements pairwise congruent (three sides and two angles), and related 'functions' which describes the relations among sides and angles of $Q_{d,a}$ and $Q'_{d,a}$. In section 3, we will use dynamic software for restricting the study of 'functions' associated to pairs of quadrilaterals to smaller intervals.

In section 4 we will highlight the properties of those 'functions' introduced in section 2. In section 5 we will prove that convex congruent-like quadrilaterals Q and Q' that are not congruent exist!

The paper is suitable for a large audience of readers who are referred to [9], [10], [7] and reference therein, for deeper investigations on quadrilaterals and related topics.

The measure of angles that occur in this paper are expressed in degrees (instead of radians). This will make easier the reading of the figures for those angles close to the right one (see Lemma 4 for example); moreover in the computations, this will make more apparent the difference between the measure of angles and the measure of sides, as in Lemmas 9 and 10).

2. Families of pairs of quadrilaterals having three sides and two angles pairwise congruent

We begin presenting a technique for building pairs of quadrilaterals with at least five elements pairwise congruent, precisely three sides and two angles. Let's proceed by steps:

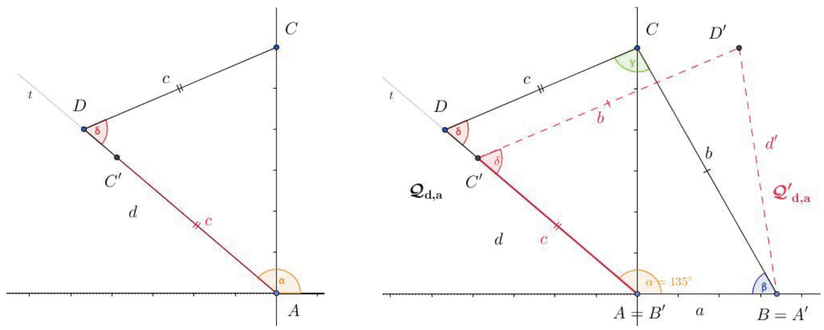


Figure 3. Two steps for constructing pairs of quadrilaterals with 5 pieces congruent. Note that $d = AD$, and $c = CD = AC'$.

1) Fix a point A at the origin of a coordinate system, and choose a point C on the (positive) y-axis at a distance say m from A. After, consider an obtuse angle α whose sides are the (positive) x-axis and t - which is a line lying in the second quadrant (see Figure 3 on the left). For every positive integer $d \in I = [m/\sqrt{2}, m\sqrt{2}]$, on the side t we consider a point D such that $d = AD$. Clearly $AD \geq DC$, and the smaller sides of the triangle (A, C, D) are $d = DA$ and $c = CD$. Now, let C' be the point lying on the line t such that the measure of $B'C'$ is c (see Figure 3).

2) Let now, a be a positive real number, and let B lie on the positive part of the x-axis, such that $AB = a$ (look at Figure 3 on the right). In this way two quadrilaterals are determined: $Q = Q_{d,a} = Q(A, B, C, D)$, where B has coordinates $(0, a)$; and $Q' = Q'_{d,a} = Q'(A', B', C', D')$, where $A' = B, B' = A$, and the vertex D', is obtained taking on the parallel line to DC trough C' a segment of length $b = BC$.

The pair $(Q_{d,a}, Q'_{d,a})$ obtained as above will be called m, α*-pair, determined by* (d, a).

In this way for every possible choice of m and α, we obtain the (m, α)*-family*

$$\mathcal{F}_{m,\alpha} = \{(Q = Q_{d,a}, Q' = Q'_{d,a})\}_{(d,a) \in I \times \mathbb{R}^+}$$

that is the collection of all (m, α)*-pairs of quadrilaterals* determined by all the possible choice of (d, a) in $I \times \mathbb{R}^+$. By construction, each pair o quadrilaterals have three sides and two angle pairwise congruent.

As our aim is just to find an example, in order to simplify the computations and to avoid formal complications, we will assume that $m = 3$, $\alpha = 135°$, thus we will focus our attention on the family of pairs $\mathcal{F}_{3,135°}$.

By construction, we have that the "pieces" of $Q = Q_{d,a}$ e $Q' = Q'_{d,a}$ can be considered as functions depending on $d \in [3/\sqrt{2}, 3\sqrt{2}]$ and a positive real number a. We will call these functions as *associated functions to the family* $\mathcal{F}_{3,135^\circ}$. Elementary geometry shows that:

$$c(d) = \sqrt{3^2 + d^2 - 2d\,3\ \cos 45^\circ}. \tag{1}$$

Arguing on the triangle (C, D, A) we have $3^2 = c(d)^2 + d^2 - 2dc(d)\ \cos(\delta(d))$

$$\delta(d) = \arccos\left(\frac{c(d)^2 + d^2 - 3^2}{2d\ c(d)}\right). \tag{2}$$

$$b(a) = \sqrt{a^2 + 3^2}. \tag{3}$$

Let now $u(d, a)$ and $v(d, a)$ be the coordinates of D'. Easy geometric considerations (see Figure 4) yield

$$\begin{aligned} u(d, a) =&\ -c(d)\sqrt{2}/2 + b(a)\ \cos(\alpha + \delta - 180^\circ); \text{and} \\ v(d, a) =&\ c(d)\sqrt{2}/2 + b(a)\ \sin(\alpha + \delta - 180^\circ); && (4) \\ d'(d, a) =&\ \sqrt{(u(d, a) - a)^2 + v(d, a)^2}. && (5) \end{aligned}$$

Moreover, arguing on the triangle (A, B, D'), we have $AD'^2 = a^2 + d'^2 - 2ad' \cos(\alpha')$, so that:

$$\begin{aligned} \alpha'(d, a) =&\ D'\hat{A}'B' = \arccos\left(\frac{a^2 + d'(d, a)^2 - u(d, a)^2 - v(d, a)^2}{2a\ d'(d, a)}\right), && (6) \\ \beta(d, a) =&\ \arctan(h/a), && (7) \\ \gamma(d, a) =&\ 360^\circ - 135^\circ - \beta(d, a) - \delta(d). && (8) \end{aligned}$$

Here we highlight that each of the above function associated to the family $\mathcal{F}_{3,135^\circ}$ is continuous.

We conclude this section with some remarks:

*R*emark 1. In the above construction we may think to fix d, and move a as variable. In this case, we may note that:

i) If a is close to 0, then $\gamma(d, a)$ is acute and $\alpha'(d, a) = D'\hat{B}A$ is obtuse. Indeed, as $b > c$, then one can check that D' lies in the I quadrant.
ii) If a is enough large, then $\gamma(d, a)$ is obtuse. Precisely, if we put $a = m = CA$, then $\beta = 45^\circ$, $B\hat{C}A = 45^\circ$. Moreover, as $c < d$ we have $D\hat{C}A > 45^\circ$. Therefore, $\gamma(d, a)$ is obtuse.

We also remark that when we fix d, as we will see, it is not trivial to establish the monotony of the functions $\alpha'(d, a)$, $\delta'(d, a)$, $\gamma(d, a)$, and $d'(d, a)$. On the other hand an investigation by dynamic software tools can give useful suggestions in this direction.

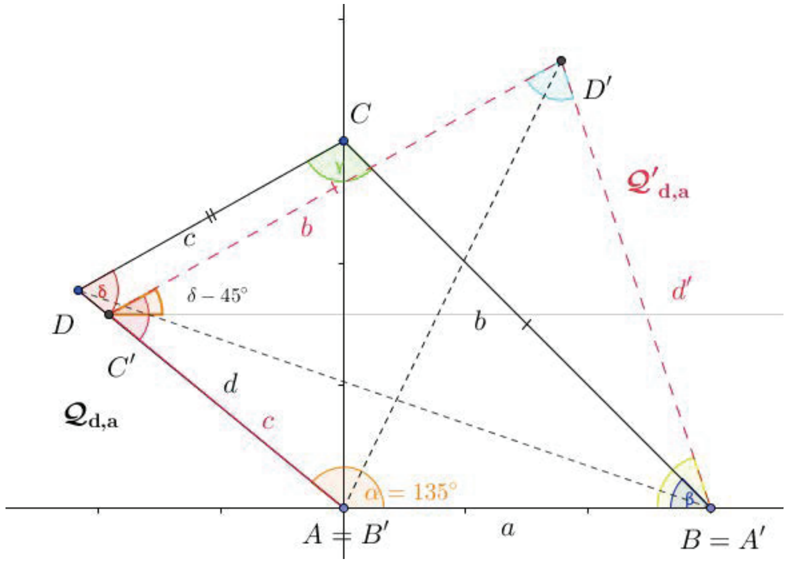


Figure 4. Geometric considerations to determine analytically the 'associated functions' to the family $\mathcal{F}_{3,135°}$.

3. The use of software for finding the approximation of the solution

In this section, we will study some properties related to the associated functions. As first step, we fix d, and in order to select pairs of quadrilaterals belonging to $\mathcal{F}_{3,135°}$ with all the angles pairwise congruent, we will investigate how much is the difference between the angles $\gamma(d,a)$ and $\alpha'(d,a)$ of the pairs $(Q_{d,a}, Q'_{d,a})$. Therefore, we draw the graphics of the following functions:

$$G_d : a \in \mathbb{R}^+ \to \alpha'(d,a) - \gamma(d,a) \tag{9}$$

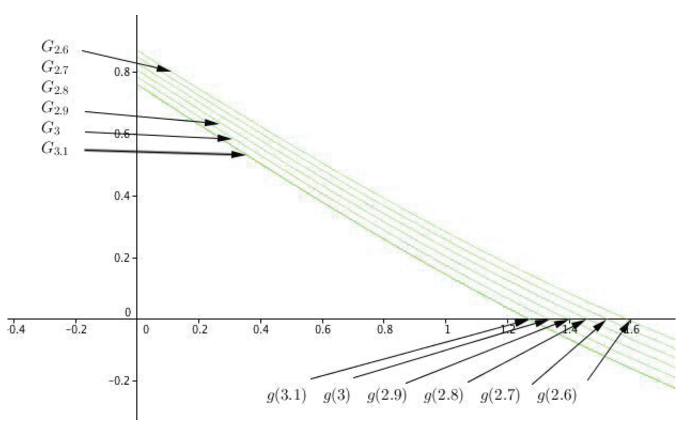


Figure 5. Some graphics related to G_d.

Looking at the Figure 5, we may think that for every $d = 2.6, \dots 3.1$, that G_d has a zero in $g(d)$ (we also note that $g(d)$ seems univocally determined), and therefore the angles $\gamma(d, g(d))$ and $\alpha'(d, g(d))$ of $(Q_{d,g(d)}, Q'_{d,g(d)})$ coincides. For these reasons (see also section 4) we will call g as *angular synchronization function*.

In this way we can 'determine' many pairs of quadrilaterals - $(Q_{d,g(d)}, Q'_{d,g(d)})$ - with seven elements (three sides and four angles) pairwise congruent.

Now, the problem is:
Is there among them one with all pieces pairwise congruent? or equivalently
Does there exist $\overline{d}$ *such that* $\overline{d} = d'(\overline{d}, g(\overline{d}))$*?* In other term:
P): *Can we determine* $\overline{d}$ *such that* $(Q_{\overline{d},g(\overline{d})}, Q'_{\overline{d},(\overline{d})})$ *are congruent-like?*

In order to reduce the interval in which to find a such $\overline{d}$, we combine the functions $\{G_d\}$ with the functions that describes the difference between $d'(d, a)$ and d, whenever d is fixed:

$$H_d(a) : a \in \mathbb{R} \to d'(d, a) - d. \tag{10}$$

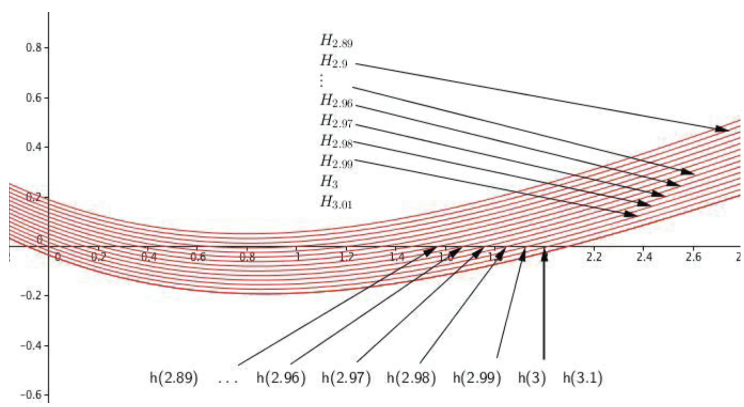


Figure 6. Some graphics related to H_d.

Clearly both G_d and H_d are continuous functions of the variable a.

We note that, if we consider small values of d, for example if we put $d = 2.9 \in [3/\sqrt{2}, 3\sqrt{2}]$, then $H_d(a) > 0, \quad \forall a \in (0, 2)$. It follows that $\forall a \in (0, 2)$ the side $A'D' = d'(2.9, a)$ of $Q'_{2.9,a}$ is greater than the side $AD = d$ of $Q_{2.9,a}$.

Let $h(d)$ such that $H_d(h(d)) = 0$, then $d = d'(d, h(d))$; but here it is not to say that the quadrilaterals $(Q_{d,h(d)}, Q'_{d,h(d)})$ are congruent-like. On the other hand if we find $\overline{d}$ such that $g(\overline{d}) = h(\overline{d})$, then $G_{\overline{d}}(g(\overline{d})) = H_{\overline{d}}(h(\overline{d})) = 0$, $\overline{d} = d'(\overline{d}, g(\overline{d})$, so that the quadrilaterals $(Q_{\overline{d},g(\overline{d})}, Q'_{\overline{d},(\overline{d})})$ are congruent like.

Remark 2. By comparing the graphics of $\{G_d\}$ and $\{H_d\}$ (see Figure 7) we see that a possible value $\overline{d}$ such that $g(\overline{d}) = h(\overline{d})$ lies in $U = (2.93, 2.97)$. Looking at the Figures 7 and 5, we also note that if d runs in U, then $g(d) < 2$. For these

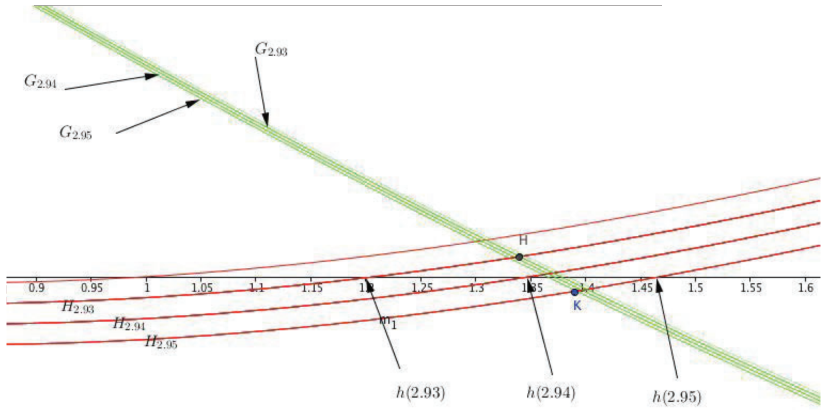


Figure 7. The intersection K between $G_{2.95}$ and $H_{2.95}$ and the intersection H between $G_{2.93}$ and $H_{2.93}$ are farer from the x-axis than the intersection between $G_{2.94}$ and $H_{2.94}$. Therefore, $\overline{d}$ is closer to 2.94 and $g(\overline{d})$ is about 1.37.

reasons we will restrict our investigation to pairs $\{(Q_{d,a}, Q'_{d,a})\}_{(d,a)\in U\times V}$, where $V := (0, 2)$.

Remark 3. If we refine the above graphics by using reduced scale, we may determine more and more precise values for $\overline{d}$ (see Figure 8 on the left), which allow to detect better approximations of other pairs of quadrilaterals we are looking for (see for example Figure 8 on the right).

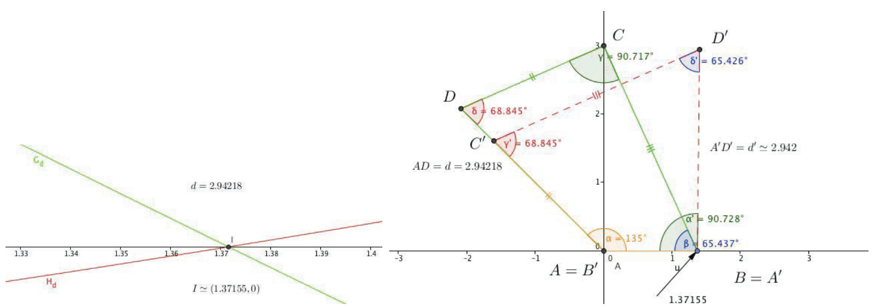


Figure 8. Using a large scale, we can establish that $\overline{d}$ is about 2.94218 and it follows that $g(\overline{d}) \simeq 1.37155$. Note that $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ are not congruent, but have 5 pieces (three sides and two angles) pairwise congruent by construction. Moreover it is also $AD \simeq A'D'$, $\gamma \simeq \alpha'$ and $\beta \simeq \delta'$.

Even if these 'qualitative' consideration do not prove the existence of $\overline{d}$, they will be crucial in order to study the associated functions in suitable intervals. In the next sections, in order to prove the main theorem, we will formalize analytically the behavior of the associated functions.

4. The analytic study of the associated functions to pairs of $\mathcal{F}_{3,135^\circ}$

In this section we will study some properties related to the associated functions to pairs of $\mathcal{F}_{3,135^\circ}$. In particular, we will prove that $\forall d \in [2.93, 2.97]$ the function

$$G_d : (0,2] \to \alpha'(d,a) - \gamma(d,a)$$

strictly increases, and has a zero.

This will permit to define analytically the *angular synchronization* function (that we have already mentioned at the beginning of section 3) as:

$$g : d \in (2.93, 2.97) \to g(d) \in (0,2),$$

where $g(d)$ is the unique real number such that $G_d(g(d)) = 0$.

The proof of the following Lemma is immediate.

Lemma 2. *Let (A, L, M) a triangle. On the line trough A and L let consider a segment $AS > AL$ and then consider an angle $\sigma > \lambda$ such that $\sigma + \alpha < 180^\circ$ (see Figure 9). Then the side s of σ intersects the line trough A and M in a point, say N, such that $LM < SN$ and $MA < NA$.*

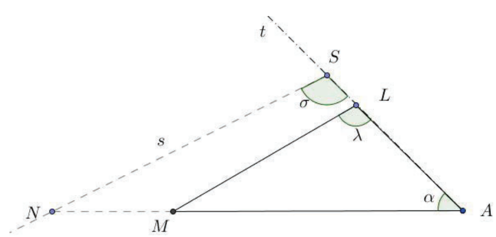


Figure 9. Inequalities between triangles.

The following two Lemmas give universal - that is $\forall d \in [2.93, 2.97]$ - inequalities referred to $\gamma(d,a), \alpha'(d,a), \delta'(d,a)$ when $a = 2$. We highlight that the proofs will use both analytic and synthetic arguments.

Lemma 3. *Let $a = 2$. Then $\forall d \in [2.93, 2.97]$ we have $\alpha'(d,2) > \delta'(d,2)$.*

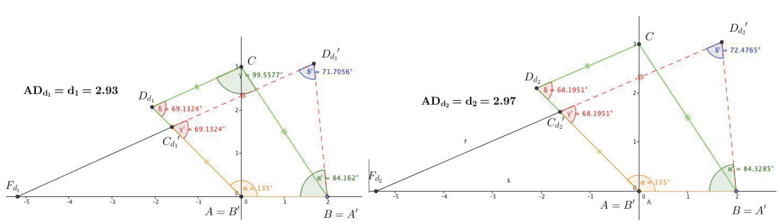


Figure 10. Note that the vertices F, C' e D' and D depend on d. Two extreme cases are related to pair ($\mathcal{Q}_{2.93,2}$, $\mathcal{Q}'_{2.93,2}$) and pair ($\mathcal{Q}_{2.97,2}$, $\mathcal{Q}'_{2.97,2}$).

Proof. 1) If $d_1 = 2.93$ (see Figure 10 on the left), then considering $\mathcal{Q}'_{d_1,2}$ we have: $3.925 < F_{d_1}C'_{d_1}$ and $5.188 < F_{d_1}A$, and in particular

$$F_{d_1}B = F_{d_1}A + a < 5.2 + 2 < 3.925 + \sqrt{13} \leq F_{d_1}C'_{d_1} + b = F_{d_1}D'_{d_1}.$$

2) If $d_2 = 2.97$ (see Figure 10 on the right), then considering $\mathcal{Q}'_{d_2,2}$ we have: $4.101 < F_{d_2}C'_{d_2} \simeq 4.102 < 4.103$ and $F_{d_2}A \simeq 5.386$, so that

$$F_{d_2}B = F_{d_2}A + a < 5.387 + 2 < 4.101 + \sqrt{13} \leq F_{d_2}C'_{d_2} + b = F_{d_2}D'_{d_2}.$$

In both the cases ($i = 1, 2$), looking at the triangle (F_{d_i}, D'_{d_i}, A'), the side $F_{d_i}D'_{d_i}$ is greater than $F_{d_i}B$, so that the same relation holds for the opposite angles, that is $\alpha'(d_i, 2) > \delta'(d_i, 2)$.

Let now $d \in (2.93, 2.97)$. Clearly $\delta(d_1, 2) > \delta(d, 2) > \delta(2.97, 2)$, and $c(2.93) < c(d, 2) < c(2.97)$. Then, if we consider the triangles (C'_{d_1}, A, F_{d_1}), (C'_d, A, F_d) and (C'_{d_2}, A, F_{d_2}), by Lemma 2 we have the following inequalities:

$$3.925 < F_{d_1}C'_{d_1} < F_dC'_d < F_{d_2}C'_{d_2} \simeq 4.102 < 4.103 \quad \text{and}$$
$$5.188 < F_{d_1}A \simeq 5.1885 < F_dA < F_{d_2}A \simeq 5.386 < 5.387.$$

It follows that $F_dA_d - F_dC'_d < 5.387 - 3.925 = 1.462 < b - a\sqrt{9+4} - 2 = b - a$.

Finally, looking at the triangle (F_d, D'_d, B) we have $F_dB = F_dA_d + a < F_dC'_d + b = F_dD'_d$, so that $\alpha'(d, 2) > \delta'(d, 2)$. □

Lemma 4. *Let* $a = 2$. *Then* $\forall d \in [2.93, 2.97]$ *the angle* $\gamma(d, 2)$ *is obtuse and* $\alpha'(d, 2)$ *is acute.*

Proof. Clearly, whenever we fix $a \in (0, 2]$, the function $\gamma^a : d \in [2.93, 2.97] \to \gamma(d, a)$ increases. On the other hand, a computation yields $99.55° < \gamma(2.93, 2)$, so that

$$\forall d \in (2.93, 2.97) \quad \text{we have} \quad 99.55° < \gamma(2.93, 2) < \gamma(d, 2),$$

and the first part of the statement holds.

To show that $\alpha'(d, 2)$ is acute we will bound all the associated functions that occur to determine it. Using Equation 6 we have:

$$\alpha'(d, 2) = D'\hat{A}'B' = \arccos\left(\frac{2^2 + d'(d, 2)^2 - u(d, 2)^2 - v(d, 2)^2}{2 \cdot 2\, d'(d, 2)}\right). \qquad (11)$$

We make the following preliminary remarks:

1) The associated function $c(d)$ (see Equation 1) increases -This can be checked by synthetic geometric way-.

2) By hypothesis $b = \sqrt{AC^2 + 2^2} = \sqrt{13}$ is fixed.

3)The following inequalities can be obtained by a direct computation by using Equations 1 – 8.

$$\begin{array}{ccc} & d=2.97 & \\ 68.195^\circ < & \delta(2.97,2) < & 68.196^\circ \\ 84.328^\circ < & \alpha'(2.97,2) < & 84.329^\circ \\ 100.495^\circ < & \gamma(2.97,2) < & 100.496^\circ \end{array} \quad \text{and} \quad \begin{array}{ccc} & d=2.93 & \\ 69.132^\circ < & \delta(2.93,2) < & 69.133^\circ \\ 84.161^\circ < & \alpha'(2.93,2) < & 84.162^\circ \\ 99.557^\circ < & \gamma(2.93,2) < & 99.558^\circ. \end{array} \tag{12}$$

In particular $\alpha'(d,2)$ is acute when $d = 2.93$ or $d = 2.97$.

Let now $2.93 < d < 2.97$. In the next four steps we will bound the functions $c(d), u(d,2), v(d,2)$ and $d'(d,2)$:

- We have just highlighted that $c(d)$ is an increasing function in $(2.93, 2.97)$, thus by Equation 1 we have the following bound in $(2.93, 2.97)$:

$$\begin{aligned} &\sqrt{3^2 + 2.93^2 - 2 \cdot 2.93 \cdot \sqrt{2}/2} \le c(d) \le \sqrt{3^2 + 2.97^2 - 2 \cdot 2.97 \cdot \sqrt{2}/2} \\ \iff &\sqrt{5.15396} = 2.27 \le c(d) \le \sqrt{5.2202} = 2.2848. \end{aligned} \tag{13}$$

- To bound $u(d,2)$ in $(2.93, 2.97)$, we recall that (see Equation 4)

$$u(d,a) = -c(d)\sqrt{2}/2 + b(a)\, \cos(\delta(d) - 45^\circ).$$

  As $(2.93, 2.97) \subset (3/\sqrt{2}, 3)$, we have that $\delta(d)$ lies in $(45^\circ, 90^\circ)$, $\forall d \in (2.93, 2.97)$ (see Figure 4). In particular $0^\circ < \delta(d) - 45^\circ < 45^\circ$.

  We have noted that $c(d)$ is an increasing function in $(2.93, 2.97)$, while the cosine function is decreasing in $(0^\circ, 90^\circ)$. Thus, by Relations (12) we have the following bounds for $u(d,2)$:

$$\begin{aligned} & - c(2.97)\sqrt{2}/2 + \sqrt{13}\, \cos(69.133^\circ - 45^\circ) \le u(d,2) \\ & \le -c(2.93)\sqrt{2}/2 + \sqrt{13}\, \cos(68.195^\circ - 45^\circ) \qquad (14) \\ \iff & - 2.2848\sqrt{2}/2 + \sqrt{13}\, \cos(24.133^\circ) \le u(d,2) \\ & \le -2.27\sqrt{2}/2 + \sqrt{13}\, \cos(23.195^\circ) \\ \iff & 1.6748 \le u(d,2) \le 1.7089. \end{aligned}$$

- Similarly, taking into account that the sine function is increasing in $(0^\circ, 90^\circ)$ we can bound $v(d,2)$. Indeed by Equation 4 we have

$$v(d,2) = c(d)\sqrt{2}/2 + b(a)\, \sin(\alpha + \delta(d) - \pi),$$

  so that

$$\begin{aligned} & c(2.93)\sqrt{2}/2 + \sqrt{13}\, \sin(23.195^\circ) \le v(d,2) \\ & \le c(2.97)\sqrt{2}/2 + \sqrt{13}\, \sin(24.133^\circ) \qquad (15) \\ \iff & 3.0252 \le v(d,2) \le 3.0897. \qquad (16) \end{aligned}$$

- To bound $d'(d,2)$ in $(2.93, 2.97)$ we can use Equation 5:

$$d'(d,2) = \sqrt{((u(d,2)-2)^2 + v(d,2)^2}. \tag{17}$$

By Equation 14, $1.6748 \le u(d,2) \le 1.7089$, then $(1.7089-2)^2 < (u(d,2)-2)^2 < (1.6748-2)^2$, and by relation 15 we have the following bound:

$$3.03917 < \sqrt{9.23658} < d'(d,2) < \sqrt{9.65201} < 3.1067.$$

Now we show that $\alpha'(d,2)$ is acute. Looking at relation 11, we note that the value "$4\ d'(d,2)$" is always a positive real number, so that it suffices to show that the numerator inside the argument of the trigonometric function $\arccos$ is positive. Thus we have to show that

$$2^2 + d'(d,2)^2 \ge u(d,2)^2 + v(d,2)^2.$$

By relations 14, 15, and 17 we have the following inequalities $\forall d \in (2.93, 2.97)$:

$$\begin{aligned} u(d,2)^2 + v(d,2)^2 &< 1.7089^2 + 3.0897^2 = 12.4665 \\ &\le 13.2365 \le 4^2 + 3.03917^2 \\ &\le 2^2 + d'(d,2)^2. \end{aligned}$$

The lemma is proved. □

We also need the following elementary properties, that in particular holds for rectangular triangles.

Lemma 5. *Let (A,B,C) a triangle, and let (A,B_1,C) a triangle as in Figure 11. Then $CB - CB_1 < BB_1$.*

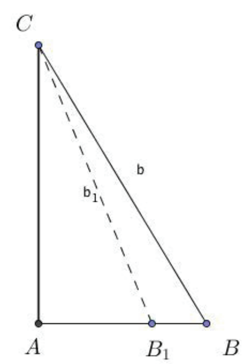


Figure 11. If $BA > B_1A$ and $BA - B_1A = \varepsilon$, then the hypotenuse of the smaller triangle AB_1C is shorter than the hypotenuse of the triangle ABC; moreover their difference $b - b_1$ is less than ε.

Proof. It is enough to apply the triangle inequality to the triangle (B, B_1, C). □

Now we are in a position to show that $\forall d \in (2.93, 2.97)$ the function

$$G_d|(0,2] : a \in (0,2] \to \alpha'(d,a) - \gamma(d,a)$$

strictly decreases (see also Definition 9 and Figure 6). Indeed the next Proposition shows that when we fix $d \in (2.93, 2.97)$, if a decreases, then $\alpha'(d,a)$ increases; while $\gamma(d,a)$ and $\delta'(d,a)$ decreases.

Proposition 6. *Let $d \in [2.93, 2.97]$, and let AB_1 and AB be segments whose length respectively a_1 and a, are such that $a_1 < a \leq 2$. Then the following relations referred to angles of $Q'_{d,AB} = (A', B', C', D')$, $Q'_{d,AB_1} = (A', B'_1, C', D'_1)$, $Q_{d,AB} = (A, B, C, D)$ and $Q_{d,AB_1} = (A, B_1, C, D)$ hold (see Figure 12).*

(i) $\gamma = \gamma(d,a) = D\hat{C}B > \gamma_1 = \gamma(d,a_1) = D\hat{C}B_1$.
(ii) $\alpha' = \alpha'(d,a) = D'\hat{A}'B' < \alpha'_1 = \alpha'(d,a_1) = D'_1\hat{B}_1B'$.
(iii) $\delta' = \delta'(d,a) > \delta'_1 = \delta'(d,a_1)$.

In particular $G_d|(0,2] : a \in (0,2] \to \alpha'(d,a) - \gamma(d,a)$ strictly decreases.

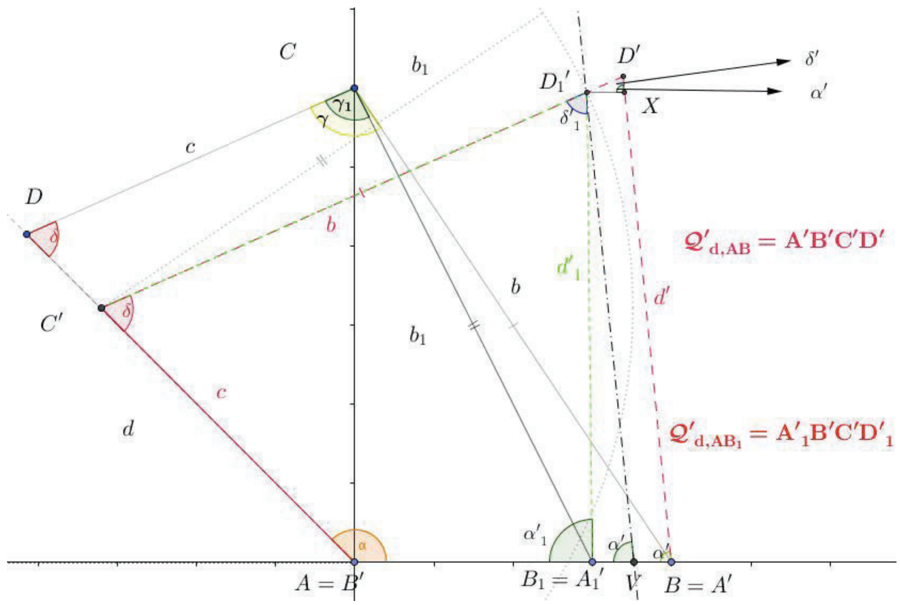


Figure 12. The monotony of the functions $\gamma(d,a)$, $\alpha'(d,a)$ and $\delta'(d,a)$, where d is fixed in $[2.93, 2.97]$.

Proof. (i) This condition follows by elementary geometry. If $a < a_1$ then $\gamma > \gamma_1$ (look at the Figure 12).

(ii) In order to show this condition we suppose first that $a = AB = 2$. By Lemma 5 the length $b - b_1$ of $D'D'_1$ is less than $A'A'_1$. Put V the intersection between the parallel line to $A'D'$ trough D'_1 and AB, and put X the intersection between $D'A'$ with the parallel line to AB trough D'_1. It turns out that (V, A', X, D'_1) is a parallelogram.

Note that V lies on the left of A'. By Lemma 3 we have $\alpha'(d,2) > \delta'(d,2) = C'\hat{D}'A'$, therefore looking at the triangle (D', D'_1, X) we have that $D'_1X = BV$ is less than D'_1D'.

By contradiction suppose that V doesn't belong to $A'A_1'$ (that is 'V lies on the left of A_1'') we have $A'V = D_1'X > A'A_1' > D'D_1'$, and this is a contradiction.

Therefore V belongs to the segment B_1B, and the external angle $\alpha_1' = D_1'\hat{B}_1B'$ is larger than $\alpha' = D_1'VB_1$.

Let now $AB < 2$, and put $z = 2$ e $z_1 := AB$. The first part of the proof shows in particular that $\alpha'(d, z_1) > \alpha'(d, z) > \delta'(d, z) > \delta'(d, z_1)$. This consideration yields that for every $a \leq 2$ the inequality $\alpha'(d, a) > \delta'(d, a)$- which extends Lemma 3 to every $a \in (0, 2]$ - holds. Then it is possible to run over the argument that we have used above. This complete the proof of (ii).

To show the condition (iii), it is enough to remember that by construction the quadrilaterals $Q'_{d,AB}$ and Q'_{d,AB_1} have a common angle of magnitude $135°$, and another one of magnitude $\delta(d, a)$. Thus the condition is proved applying (ii).

□

Remark 4. We highlight that the monotony of G_d is not guaranteed all over $\mathbb{R}^+$.

Proposition 7. *Let* $d \in (2.93, 2.97)$. *Then there exists (exactly one) real number* $g(d) \in (0, 2)$ *such that the angle* $\gamma(d, g(d))$ *of* $Q_{d,g(d)}$ *and the angle* $\alpha'(d, g(d))$ *di* $Q'_{d,g(d)}$ *coincides. In particular* $G_d(g(d)) = 0$, *for every* $d \in (2.93, 2.97)$.

Proof. For every $d \in (2.93, 2.97)$, we have seen (Remark 1) that if a is enough small, then $\gamma(d, a)$ is less than $\alpha'(d, a)$; moreover we have proved (Lemma 4) that if $a = 2$ then $\gamma(d, a) > 90° > \alpha'(d, 2)$.

It follows that for every $d \in (2.93, 2.97)$, the function

$$G_d|_{(0,2]} : a \in (0, 2] \to \alpha'(d, 2) - \gamma(d, a) \in \mathbb{R},$$

assumes both positive and negative values, so that, it has a zero in $(0, 2)$. Moreover by Proposition 6 G_d has only one zero. □

Remark 5. In virtue of Proposition 7 we can define

$$g : d \in (2.93, 2.97) \to g(d) \in (0, 2), \quad \text{such that } G_d(g(d)) = 0.$$

According to Section 3, this function shows the exact value $a = g(d)$, for which we have the 'synchronization' of all the angles of pairs of quadrilaterals of the type $(\mathcal{Q}_{d,a}, \mathcal{Q}'_{d,a})$ that have the same first parameter d. In fact, for every $d \in (2.93, 2.97)$ we have that $\mathcal{Q}_{d,g(d)}$ and $\mathcal{Q}'_{d,g(d)}$ have all the angles pairwise congruent. We may call such pairs *almost congruent quadrilateral*, extending a terminology used for triangles (see [3][1, section 2] for definition and recent results connected them).

In order to show that g is a continuous function, we point out some considerations.

Let f be the following function of two variables defined in an open interval of $\mathbb{R}^2$. Precisely:

$$f : (d, a) \in (2.93, 2.97) \times (0, 2) \to G_d(a) = \alpha'(d, a) - \gamma(d, a) \in \mathbb{R}^+.$$

Clearly, f is a continuous function in the variables d and a (see Equations 1–6). By Proposition 7, we can read the function g implicitly defined by the equation $f(d, g(d)) = 0$, but it is not easy to study the partial derived functions $\dfrac{\partial f}{\partial d}$ and $\dfrac{\partial f}{\partial a}$.

Lemma 8. *The function g is continuous.*

Proof. To show that g is a continuous function, we will fix a whichever element $\overline{x} \in U := (2.93, 2.97)$, and will prove that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $|x - \overline{x}| < \delta$, $x \in U$, then $|g(x) - g(\overline{x})| < \varepsilon$.

For every $x \in U$, the function $f^x : y \to f(x, y)$ coincides with G_x, so that it is decreasing by Proposition 6; moreover $f(x, g(x)) = 0$.

Now, let $\overline{x} \in U$ and $\varepsilon > 0$ such that $(\overline{x}, g(\overline{x}) - \varepsilon)$ and $(\overline{x}, g(\overline{x}) + \varepsilon)$ belongs to the dominio of f- $(2.93, 2.97) \times (0, 2)$, then:

$$f(\overline{x}, g(\overline{x}) + \varepsilon) < 0 < f(\overline{x}, g(\overline{x}) - \varepsilon).$$

It follows by the continuity of f (respect to x) and by sign permanence theorem, that there exists $\delta > 0$ such that

$$f(x, g(\overline{x}) + \varepsilon) < 0 < f(x, g(\overline{x}) - \varepsilon), \quad \forall |x - \overline{x}| < \delta, x \in U.$$

As $f(x, g(x)) = 0$ and $f^x(x, y)$ is a strictly decreasing function, it follows that

$$g(\overline{x}) - \varepsilon < g(x) < g(\overline{x}) + \varepsilon, \quad \forall |x - \overline{x}| < \delta, x \in U.$$

The proof is complete. □

5. The main theorem

Before showing the final theorem we need two technical results that give the approximations of $g(d)$ and $d'(d, g(d)$ for some values d close to the extremes 2.93 and 2.97 of their domain. They are obtained by computations. For completeness we will give the whole proofs.

Lemma 9. *Put* $d_1 = 2.9300001$. *Then* $a_1 = 1.3787 < g(d_1) < a_2 = 1.3794$. *Moreover,* $2.95528 < d'(d_1, g(d_1) < 2.95544$. *In particular* $d_1 < d'(d_1, g(d_1)$.

Proof. The following relations can be obtain directly by computing $\gamma(d_1, a_1)$ and $\alpha'(d_1, a_1)$ using the Formulas 6, 8.

$$\begin{array}{rcl} 90.5506^\circ < & \gamma(d_1, a_1) \simeq 90.55065^\circ < & 90.5507^\circ \\ 90.5630^\circ < & \alpha'(d_1, a_1) \simeq 90.5631^\circ < & 90.5632^\circ \end{array}$$

and

$$\begin{array}{rcl} 90.5601^\circ < & \gamma(d_1, a_2) \simeq 90.56011^\circ < & 90.5602^\circ \\ 90.5558^\circ < & \alpha'(d_1, a_2) \simeq 90.55588^\circ < & 90.5559^\circ \end{array}$$

By Proposition 6, the continuous functions $\gamma(d_1, a)$ and $\alpha'(d_1, a)$ are respectively (strictly) increasing and decreasing, then we have the synchronization of $(\mathcal{Q}_{d_1,a}, \mathcal{Q}'_{d_1,a})$ when a lies in $(1.3787, 1.3794)$. Therefore

$$1.3787 < g(d_1) < 1.3794. \tag{18}$$

We will show now the second part of the Lemma. Looking at the equations 1 and 2 we see that $c(d)$ and $\delta(d)$ do not depend on a. Computing $c(d_1)$ and $\delta(d_1)$ we have the following bounds.

$$\begin{array}{rcl} 2.27023 < & c(d_1) \simeq 2.270234 < & 2.27024 \\ 69.1323^\circ < & \delta(d_1) \simeq 69.13239^\circ < & 69.1324^\circ \end{array}. \tag{19}$$

Clearly, the function $b(a) = \sqrt{a^2 + 3^2}$ is strictly increasing (see 3), thus by Equation 18 we have

$$3.301638 < \sqrt{9 + 1.3787^2} < b(g(d_1)) < \sqrt{9 + 1.3794^2} < 3.30192 \tag{20}$$

By Equation 4 the coordinates of $D' \in \mathcal{Q}'_{d_1, g(d_1)}$ are

$$u(d_1, g(d_1)) = -c(d_1)\sqrt{2}/2 + b(g(d_1)) \cos(\delta(d_1) - 45^\circ), \quad \text{and}$$
$$v(d_1, g(d_1)) = c(d_1)\sqrt{2}/2 + b(g(d_1)) \sin(\delta(d_1) - 45^\circ).$$

We can bound them using the inequalities 19 and 20:

$$\begin{aligned} & -2.27024\sqrt{2}/2 + 3.301638 \cos(24.1324^\circ) \\ & < u(d_1, g(d_1)) < -2.27023\sqrt{2}/2 + 3.30192 \cos(24.1323^\circ), \end{aligned} \tag{21}$$

thus,

$$1.4077 < u(d_1, g(d_1)) < 1.4081.$$

$$\begin{aligned} & 2.27023 \sqrt{2}/2 + 3.301638 \sin(24.1323^\circ) \\ & < v(d_1, g(d_1)) < 2.27024 \sqrt{2}/2 + 3.30192 \sin(24.1324^\circ), \end{aligned} \tag{22}$$

thus,

$$2.95515 < v(d_1, g(d_1)) < 2.9553.$$

Now we can bound $d'(d_1, g(d_1)) = \sqrt{(u(d_1, g(d_1)) - g(d_1))^2 + v(d_1, g(d_1))^2}$.

For let, we will consider the minimum (resp. the maximum) difference between $u(d_1, g(d_1))$ and $g(d_1)$, and the minimum (resp. the maximum) of the bound for $v(d_1, g(d_1))$ obtained above (precisely 18, 21 and 22). We have:

$$\sqrt{(1.4077 - 1.3794)^2 + 2.95515^2} < d'(d_1, g(d_1)) < \sqrt{(1.4081 - 1.3787)^2 + 2.9553^2}.$$

Then

$$2.95528 < d'(d_1, g(d_1)) < 2.95544.$$

□

Lemma 10. *Let* $d_2 = 2.9699999$. *Then* $a_1 = 1.3538 < g(d_2) < a_2 = 1.3541$. *Moreover,*

$$2.9124404 < d'(d_2, g(d_2) < 2.912513.$$

In particular $d_2 > d'(d_2, g(d_2))$.

Proof. We will proceed as in the above Lemma. So that as first step we will compute $\gamma(d_2, a_1)$ and $\alpha'(d_2, a_1)$ using the Formulas 6, 8:

$$\begin{array}{rcl} 91.0929^\circ < & \gamma(d_2, a_1) \simeq 91.093^\circ < & 91.0931^\circ \\ 91.0995^\circ < & \alpha'(d_2, a_1) \simeq 91.09956^\circ < & 91.0996^\circ \end{array}$$

and

$$\begin{array}{rcl} 90.5601^\circ < & \gamma(d_2, a_2) \simeq 90.56011^\circ < & 90.5602^\circ \\ 91.0958^\circ < & \alpha'(d_2, a_2) \simeq 91.09587^\circ < & 91.0959^\circ \end{array}$$

By Proposition 6, the continuous functions $\gamma(d_2, a)$ and $\alpha'(d_2, a)$ are respectively (strictly) increasing and decreasing, then we have the synchronization of $(\mathcal{Q}_{d_2,a}, \mathcal{Q}'_{d_2,a})$ when a lies in $(1.3538, 1.3541)$. Therefore

$$1.3538 < g(d_2) < 1.3541. \tag{23}$$

We will show now the second part of the Lemma. Looking at the equations 1 and 2 we see that $c(d)$ and $\delta(d)$ do not depend on a. Computing $c(d_2)$ and $\delta(d_2)$ we have the following bounds.

$$\begin{array}{rcl} 2.28478 < & c(d_2) \simeq 2.28479 < & 2.2848 \\ 68.19505^\circ < & \delta(d_2) \simeq 68.19506^\circ < & 68.19507^\circ \end{array}. \tag{24}$$

Now using the Equation 23, and taking into account that $b(d)$ is an increasing function, we have

$$\begin{aligned} & 3.2913 < \sqrt{9 + 1.3538^2} \simeq 3.291318 \\ & < b(g(d_2)) < \sqrt{9 + 1.3541^2} \simeq 3.291441 < 3.2915. \end{aligned} \tag{25}$$

By Equations 4 and 5 the coordinates of $D' \in \mathcal{Q}'_{d_2, g(d_2)}$ are

$u(d_2, g(d_2)) = -c(d_2)\sqrt{2}/2 + b(g(d_2)) \cos(\delta(d_2) - 45^\circ)$ and
$v(d_2, g(d_2)) = c(d_2)\sqrt{2}/2 + b(g(d_2)) \sin(\delta(d_2) - 45^\circ)$.

We can bound them using the inequalities (24) and (25):

$$\begin{aligned} & -2.2848\sqrt{2}/2 + \sqrt{9 + 1.3538^2} \cos(23.19507^\circ) \\ & < u(d, g(d)) < -2.28478\sqrt{2}/2 + \sqrt{9 + 1.3541^2} \cos(23.19505^\circ), \end{aligned} \tag{26}$$

thus

$$1.409681 < u(d_2, g(d_2)) < 1.40981;$$

$$\begin{aligned} & 2.28478\sqrt{2}/2 + \sqrt{9 + 1.3538^2} \sin(23.19505^\circ) \\ & < v(d_2, g(d_2)) < 2.2848\sqrt{2}/2 + \sqrt{9 + 1.3541^2} \sin(23.19507^\circ) \end{aligned} \tag{27}$$

thus

$$2.91191 < v(d_2, g(d_2)) < 2.911974.$$

Now we can give a bound for $d'(d_2, g(d)) = \sqrt{(u(d_2, g(d_2)) - g(d_2))^2 + v(d_2, g(d_2))^2}$.

For let, we will consider the minimum (resp. the maximum) difference between $u(d_2, g(d_2))$ and $g(d_2)$, and the minimum (resp. the maximum) of the bound for $v(d_2, g(d_2))$ obtained above (precisely 23, 26 and 27). We have:

$$\sqrt{(1.409681 - 1.3541)^2 + 2.91191^2} < d'(d_2, g(d)) < \sqrt{(1.40981 - 1.3538)^2 + 2.911974^2}. \tag{28}$$

Then

$$2.9124404 < d'(d_2, g(d)) < 2.912513.$$

□

Theorem 11. *There exist* $\overline{d} \in (2.93, 2.97)$ *and* $\overline{a} \in (0, 2)$ *such that* $(\mathcal{Q}_{\overline{d},\overline{a}}, \mathcal{Q}'_{\overline{d},\overline{a}}) \in \mathcal{F}_{3,135^\circ}$ *is a pair of congruent-like quadrilaterals that are not congruent.*

Proof. Put $U = (2.93, 2.97)$ and $V = (0, 2)$. By Lemma 8 and by Relation 5 we have that:

$$g : d \in U \to g(d) \in V, \quad \text{and}$$
$$d' : (d, a) \in U \times V \to d'(d, a) \in \mathbb{R}$$

are continuous functions. It follows that

$$k : d \in U \to d'(d, g(d)) \in \mathbb{R}, \quad \text{and}$$
$$l : d \in U \to k(d) - d \in \mathbb{R}$$

are likewise continuous functions.

On the other hand by Lemmas 9 and 10, we have that there are two real numbers d_1 and d_2 such that:

$$d_1 < k(d_1) \quad \text{and} \quad d_2 > k(d_2).$$

It follows that l has a zero, and hence there exists $\overline{d}$ such that

$$k(\overline{d}) = \overline{d} = d'(\overline{d}, g(\overline{d})). \tag{29}$$

Now if we put $\overline{a} = g(\overline{d})$, then the quadrilaterals $\mathcal{Q}_{\overline{d},g(\overline{d})}$ e $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ have all angles pairwise congruent, and also the sides AD e $A'D'$ are congruent (the other three sides are congruent by construction), thus they are congruent-like quadrilaterals.

Finally, if we put $\mathcal{Q}_{\overline{d},g(\overline{d})} = (A, B, C, D)$ and $\mathcal{Q}'_{\overline{d},g(\overline{d})} = (A', B', C', D')$, then $AB = a < 2 < CD < 2.93 < DA < 2.97 < 3$ (see Equation 13 and Theorem 11). On the other hand trivial geometric consideration yields: $3 < CB$. In particular the sides of maximum size in $\mathcal{Q}_{\overline{d},g(\overline{d})}$, are BC and DA, that are not consecutive. On the other hand by construction $BC = C'D'$ and by Equation 29 $DA = D'A'$, thus the sides of maximum size in $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ are consecutive. Therefore $\mathcal{Q}_{\overline{d},g(\overline{d})}$ and $\mathcal{Q}'_{\overline{d},g(\overline{d})}$ are not congruent. The proof is complete. □

6. Conclusion

Congruence theorems for quadrilaterals (and more generally for polygons) could appear to be a difficult topic for many learners. The reasons for such difficulties relate to the complexities in learning to analyze the attributes of different quadrilaterals and to distinguish between critical and non-critical aspects (see [2] and [10]). We highlight that this kind of studies requires logical deduction, together with suitable interactions among concepts - of different areas of Mathematics - images and a certain skill in software computing.

We conclude by noting that the technique presented in section 2, is available also to define other pairs of congruent-like quadrilaterals that are not congruent, and hence the existence of other pairs (of congruent-like quadrilaterals that are not congruent) can be claimed. Finally we observe that Theorem 11 states the existence of pairs of non congruent quadrilaterals with all pieces pairwise congruent. Thus arise the following open questions:

1) Is it possible to characterize all pairs of quadrilaterals of such type?
2) Is it possible to find by elementary geometry (synthetic) a pair of non congruent quadrilaterals with all pieces pairwise congruent?

References

[1] G. Anatriello and G. Vincenzi, Logarithmic spirals and continue triangles, *J. Comp. App. Math.*, 296 (2016) 127–137.

[2] T. Fujita and K. Jones, Learners understanding of the definitions and hierarchical classification of quadrilaterals: towards a theoretical framing, *Research in Mathematics Education*, 9 no. 1 and 2 (2017) 3–20.

[3] R. T. Jones and B. B. Peterson, Almost congruent triangles, *Math Magazine*, 47 (1974) 180–189.

[4] M. Harvey. *Geometry illuminated. An illustrated introduction to Euclidean and hyperbolic plane geometry* . Washington, DC: MAA Textbooks. Mathematical Association of America, xvi+543 pp. 2015.

[5] F. Laudano and G. Vincenzi, Congruence Theorems for Quadrilaterals, *J. Geom. Graphics*, 21 (2017) 45–59.

[6] J. M. Lee. *Axiomatic geometry*. Providence, RI: Pure and Applied Undergraduate Texts, 21. American Mathematical Society, xviii+469 pp. 2013.

[7] H. Martini, Recent results in elementary geometry. Part II, *Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics*, (Munich, 1993), pp. 419–443, Sympos. Gaussiana, Eds. M. Behara, R. Fritsch, and R. G. Lintz, de Gruyter, Berlin, 1995.

[8] E. Moise. *Elementary Geometry From an Advanced Standpoint*, Reading, MA: Addison-WesleyPublishing Company, third edition, iv+502 pp; 1990.

[9] F. Pierro and G. Vincenzi, On a conjecture referring to orthic quadrilaterals, *Beitr Algebra Geom.*, 57 (2016) 441–451.

[10] Z. Usiskin, J. Griffin, D. Witonsky, and E. Willmore, *The Classification of Quadrilaterals: A Study of Definition*. Charlotte, NC: Information Age Pubblishing, 104 pp. 2008.

[11] I. E. Vance, Minimum Conditions for Congruence of Quadrilaterals, *School Science and Mathematics*, 82 (1982) 403–415.

Giuseppina Anatriello: Dipartimento di Architettura, Università di Napoli, via Via Monteoliveto, 3, I-80134 Napoli, Italy
E-mail address: anatriello@unina.it

Francesco Laudano: Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, I-84084 Fisciano (SA), Italy
E-mail address: flaudano@unisa.it

Giovanni Vincenzi: Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, I-84084 Fisciano (SA), Italy
E-mail address: vincenzi@unisa.it

Forum Geometricorum
Volume 18 (2018) 401–408.

FORUM GEOM
ISSN 1534-1178

A Remarkable Theorem on Eight Circles

Gábor Gévay

Abstract. For $i = 1, \dots, 6$, consider a closed chain of circles $\{\Gamma_i\}$ such that every two consecutive members Γ_i and Γ_{i+1} meet in the points (A_i, B_i), with indices modulo 6. We require that both sextuples $(A_1, \dots, A_6)$ and $(B_1, \dots, B_6)$ are cyclic. We prove the theorem that the three lines connecting the centers of the opposite circles of the chain concur. In the rest of the paper we present a slightly more general version of this theorem.

1. Introduction

We start from the following extension of Miquel's Six-Circles Theorem, which was given and proven in [9] (it is also mentioned in [10]).

Theorem 1 ([9]). *Let Γ_a and Γ_b be two circles. Let $n > 2$ be an even number, and take the points $A_1, \dots, A_n$ on Γ_a, and $B_1, \dots, B_n$ on Γ_b, such that each quadruple $(A_1, B_1, A_2, B_2), \dots, (A_{n-1}, B_{n-1}, A_n, B_n)$ is cyclic. Then the quadruple (A_n, B_n, A_1, B_1) is also cyclic.*

Note that this is an extension, indeed, since for $n = 4$ it returns Miquel's classical theorem. In the subsequent case, $n = 6$, the following remarkable concurrency occurs.

Theorem 2 (Szilassi). *For $i = 1, \dots, 6$, consider a closed chain of circles Γ_i in the plane such that two consecutive circles Γ_i and Γ_{i+1} intersect in the points (A_i, B_i) (with indices modulo 6). Denote the center of the ith circle of this chain by K_i. Suppose that the sextuple $(A_1, \dots, A_6)$ lies on a circle Γ_a, and likewise, the sextuple $(B_1, \dots, B_6)$ lies on a circle Γ_b different from Γ_a. Then the lines K_1K_4, K_2K_5 and K_3K_6 concur.*

This theorem was found by Lajos Szilassi when experimenting with the octuple of circles corresponding to Theorem 1 (cf. Figure 1). He communicated it to the author, but has given no proof [14]. In Section 2 we shall prove it. Section 3 is devoted to the background of this theorem, and in conclusion, we formulate a slightly generalized and extended version of it.

We note that there is a seemingly related theorem, due to Dao Thanh Oai, proved in several different ways [6, 2, 7]. It bears a formal resemblance to our theorem,

Publication Date: December 6, 2018. Communicating Editor: Paul Yiu.
This work is supported by the OTKA grant NN-114614, Hungary.

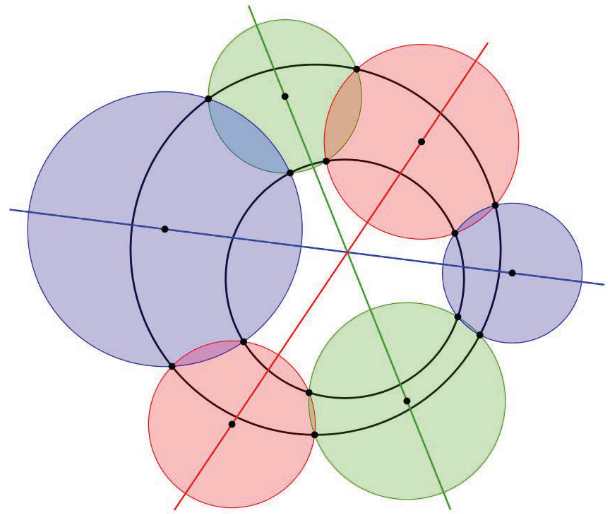


Figure 1. Illustration of Theorem 2 (drawn by Szilassi)

since it states the concurrence of lines connecting the opposite centers of a closed chain of six circles, like in our case (but with a different condition on these circles). Furthermore, a whole set of problems related to this theorem is posed in [8]. A careful scrutiny may reveal a closer connection between this theorem and ours; however, it is beyond the scope of this paper.

2. Proof of Theorem 2

In our proof, we shall use stereographic projection as well as duality in projective 3-space; so, to set the stage for this, we consider the following embeddings:

$$\mathbb{E}^2 \cup \{\infty\} \ \subset \ \mathbb{E}^2 \cup \{\ell_\infty\} \ \subset \ \mathbb{E}^3 \cup \{\pi_\infty\}. \tag{1}$$

In words, we use the Euclidean plane completed with the (unique) point at infinity, as the model of the inversive plane; it is embedded in the projective plane, whose model is the Euclidean plane completed with the line at infinity. This latter, in turn, is embedded in (actually, a restriction of) the projective 3-space, whose model is the Euclidean 3-space completed with the plane at infinity.

For the stereographic projection, we use the standard unit sphere $\mathbb{S}^2$, whose north pole N is the center of the projection, and the image plane is the tangent plane at the south pole S (for other details, see e.g. [4, 11, 13]). We also use an extension of this projection, from the same center and onto the same image plane, but with the whole projective 3-space as domain [13].

The dual transformation that we need is realized by polar reciprocity, whose reference sphere coincides with $\mathbb{S}^2$; in what follows, our term "duality" always refers to this reciprocity. Recall that in this case the dual of a point P is a plane P^* that passes through a point P' and is perpendicular to the radial line OP; here the point P' is the inverse of P, i.e. it lies on the ray from O to P at a distance $d(O, P') = d(O, P)^{-1}$.

The following statement is proved in [11], § 36.

Lemma 3 (Hilbert, Cohn-Vossen)**.** *Let Γ' be an arbitrary circle lying on the sphere and not passing through N. The planes tangent to the sphere at the points of Γ' envelop a circular cone, whose vertex we denote by V. Since Γ' does not pass through N, NV is not tangent to the sphere at N, and is therefore not parallel to the image plane; let M be the point at which NV intersects the image plane. The curve Γ that is the image of Γ' is a circle with M as center.*

Let (P, Q, R, S) be a cyclic quadruple of points lying on the sphere. Since it is cyclic, all the four points lie within a plane, which we denote by $\pi(P, Q, R, S)$. On the other hand, we denote by $V(P, Q, R, S)$ the vertex of the cone determined by the circumcircle of these points in the way described in the previous lemma. The following observation is a simple application of duality.

Proposition 4. *$\pi(P, Q, R, S)$ and $V(P, Q, R, S)$ are duals of each other.*

This implies, again by duality:

Proposition 5. *For two distinct cyclic quadruples of points (P_1, Q_1, R_1, S_1) and (P_2, Q_2, R_2, S_2) lying on the sphere, consider the line connecting $V(P_1, Q_1, R_1, S_1)$ and $V(P_2, Q_2, R_2, S_2)$. The dual of this line is the line of intersection of the planes $\pi(P_1, Q_1, R_1, S_1)$ and $\pi(P_2, Q_2, R_2, S_2)$.*

The following observation is implicit in the discussion of bundles of circles in [13] (but it can easily be seen as well).

Proposition 6. *Let Γ_1' and Γ_2' be two circles lying on the sphere, whose stereographic image are Γ_1 and Γ_2, respectively. The line of intersection of the planes of Γ_1' and Γ_2' projects from N precisely onto the radical line of Γ_1 and Γ_2.*

Note that this observation includes the case, too, when Γ_1' and Γ_2' are concentric; in this case the corresponding radical line coincides with the line at infinity.

The following lemma follows directly from Propositions 4 and 5, by a further application of duality.

Lemma 7. *Consider the following six quadruples of points lying on the sphere: $(P_1, Q_1, R_1, S_1), \ldots, (P_6, Q_6, R_6, S_6)$. For the point $V(P_i, Q_i, R_i, S_i)$ and plane $\pi(P_i, Q_i, R_i, S_i)$ $(i = 1, \ldots, 6)$ we use here the simplified notation V_i and π_i, respectively. Furthermore, we denote by ℓ_{14}, ℓ_{25}, ℓ_{36} the line connecting the pairs of points (V_1, V_4), (V_2, V_5), (V_3, V_6), respectively. Likewise, we denote by ℓ_{14}^*, ℓ_{25}^*, ℓ_{36}^* the line of intersection of the pairs of planes (π_1, π_4), (π_2, π_5) (π_3, π_6), respectively. Then, the lines $\ell_{14}, \ell_{25}, \ell_{36}$ are concurrent if and only if the lines $\ell_{14}^*, \ell_{25}^*, \ell_{36}^*$ lie within a common plane.*

Lemma 8. *Let Γ_1, Γ_2, Γ_4, Γ_5 be circles in the plane such that they are images under stereographic projection of the circles Γ_1', Γ_2', Γ_4', Γ_5', respectively, on the sphere. Suppose that the following conditions hold:*

(1) *Γ_1 and Γ_2 intersect in the points A_1, B_1;*
(2) *Γ_4 and Γ_5 intersect in the points A_4, B_4;*

(3) *the points* A_1, B_1, A_4, B_4 *form a cyclic quadruple.*

Denote by ℓ_{14} *the line of intersection of the planes* $\pi(\Gamma_1')$ *and* $\pi(\Gamma_4')$*, and by* ℓ_{25} *the line of intersection of the planes* $\pi(\Gamma_2')$ *and* $\pi(\Gamma_5')$ (*where* $\pi(\Gamma_i')$ *denotes the plane of the circle* Γ_i', $i = 1, 2, 4, 5$). *The lines* ℓ_{14} *and* ℓ_{25} *meet in a common point.*

Proof. Condition (1) implies that the radical line of Γ_1 and Γ_2 is A_1B_1. Hence we see by Proposition 6 that the planes $\pi(\Gamma_1')$ and $\pi(\Gamma_2')$ intersect in a line which meets the sphere of the stereographic projection in two points (actually, in the inverse image of A_1 and B_1). Denote this line by ℓ_{12}. Similarly, by Condition (2) we have a line, denoted by ℓ_{45}, which is the line of intersection of the planes $\pi(\Gamma_4')$ and $\pi(\Gamma_5')$.

Consider Condition (3). By the same reasoning as before, it implies that ℓ_{12} and ℓ_{45} lie within a common plane (this plane is precisely $\pi(A_1, B_1, A_4, B_4)$; cf. Figure 2). Hence ℓ_{12} and ℓ_{45} meet in a common point (recall that the plane in question is a projective plane, on account of (1)). This point is a common point of the planes $\pi(\Gamma_1')$, $\pi(\Gamma_2')$, $\pi(\Gamma_4')$ and $\pi(\Gamma_5')$. Thus it is the common point of the lines ℓ_{14} and ℓ_{25}, too. □

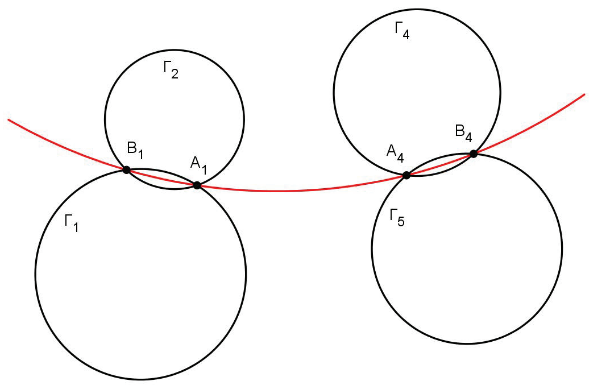


Figure 2. Arrangement of circles in Lemma 8

Lemma 9. *Consider the arrangement of points and circles given in Theorem 2. Each of the quadruples of points* (A_1, B_1, A_4, B_4), (A_2, B_2, A_5, B_5), (A_3, B_3, A_6, B_6) *is cyclic.*

Proof. Observe that the triple of circles $\Gamma_2, \Gamma_3, \Gamma_4$ closes to a four-membered chain through the circumcircle of the quadruple (A_1, B_1, A_4, B_4) such that these four circles, together with Γ_a and Γ_b, form a configuration corresponding to Miquel's classical six-circles theorem (cf. Figure 3). This means that Miquel's theorem implies the statement in this case. The other two cases can be verified analogously. □

Now we are ready to prove our theorem.

Proof of Theorem 2. Consider the following lines: $\ell_{14} = \pi(\Gamma_1') \cap \pi(\Gamma_4')$, $\ell_{25} = \pi(\Gamma_2') \cap \pi(\Gamma_5')$, $\ell_{36} = \pi(\Gamma_3') \cap \pi(\Gamma_6')$ (these lines exist, since we are in projective space). By Lemma 9, the quadruple (A_1, B_1, A_4, B_4) is cyclic. Thus the conditions

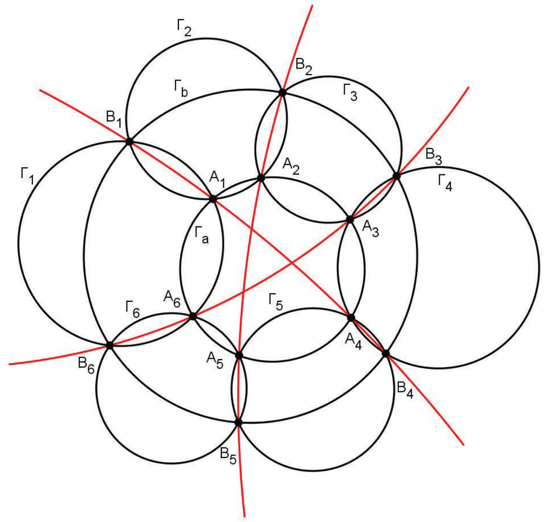


Figure 3. Completing the six-membered chain with three new circles

of Lemma 8 are fulfilled, hence the lines ℓ_{14} and ℓ_{25} meet in a common point (cf. Figures 2 and 3). By the same reasoning we obtain that each two of the lines ℓ_{14}, ℓ_{25} and ℓ_{36} have a common point. This means that they lie within the same plane. Denote their dual by ℓ^*_{14}, ℓ^*_{25}, and ℓ^*_{36}, respectively. By Lemma 7, these latter three lines connect the pairs of points (V_1, V_4), (V_2, V_5) and (V_3, V_6), respectively, and are concurrent. By Lemma 3, they project to the lines K_1K_4, K_2K_5 and K_3K_6, respectively. Thus it follows that those are also concurrent. $\square$

3. Generalization and extension

The circumcircles of the quadruples (A_1, B_1, A_4, B_4), (A_2, B_2, A_5, B_5) and (A_3, B_3, A_6, B_6) play an important role in our proof. In this section we explore what is this role. To simplify the reference, we introduce the following notation: we denote by $\mathcal{C}(P, Q, R, S)$ the circumcircle of a quadruple of points (P, Q, R, S). Furthermore, we denote the circles in question as follows:

$$\Gamma''_3 = \mathcal{C}(A_1, B_1, A_4, B_4), \quad \Gamma''_1 = \mathcal{C}(A_2, B_2, A_5, B_5), \quad \Gamma''_2 = \mathcal{C}(A_3, B_3, A_6, B_6).$$

Remark 10. (a) Observe that the role of the Miquel type condition of Theorem 2 (i.e. the existence of the circumcircles Γ_a and Γ_b) is merely to ensure the existence of the circles Γ''_1, Γ''_2 and Γ''_3 (cf. the proof Lemma 9). Consequently, in our theorem the former condition can be replaced by the latter, weaker condition. Thus, with this weaker condition, we obtain a slightly more general version of the theorem.

(b) The weaker condition provides the possibility to extend the conclusion of the theorem, too. Namely, it turns out that not only one concurrence occurs. This will be formulated in Theorem 15 below.

To this end, first we change our former notation for the circles of the six-membered chain in Theorem 2 as follows: $\Gamma_1, \Gamma_3, \Gamma_5$ will be denoted by $\Gamma_1, \Gamma_2, \Gamma_3$, and $\Gamma_2, \Gamma_4, \Gamma_6$ will be denoted $\Gamma'_3, \Gamma'_1, \Gamma'_2$ respectively. Also, the centers will be denoted by K_i and K'_i, with indices according to the new notation of the circles.

With these new notations, observe that we have three triples of circles whose mutual relationship can be characterized as follows:

$$\begin{aligned} \Gamma''_1 &= \mathcal{C}((\Gamma_2 \cap \Gamma'_3) \cup (\Gamma_3 \cap \Gamma'_2)); \\ \Gamma''_2 &= \mathcal{C}((\Gamma_3 \cap \Gamma'_1) \cup (\Gamma_1 \cap \Gamma'_3)); \\ \Gamma''_3 &= \mathcal{C}((\Gamma_1 \cap \Gamma'_2) \cup (\Gamma_2 \cap \Gamma'_1)). \end{aligned} \tag{2}$$

Recall that the centers of circles sharing a common chord lie on the perpendicular bisector of this chord. Hence, the centers of the circles in (2) can be grouped in triples given by the following relations:

$$K''_1 = K_2K'_3 \cap K_3K'_2; \quad K''_2 = K_3K'_1 \cap K_1K'_3; \quad K''_3 = K_1K'_2 \cap K_2K'_1, \tag{3}$$

where K''_i denotes the center of the circle Γ''_i $(i = 1, 2, 3)$.

Observe the formal analogy of relations (2) and (3). Using the term *cross triangle* introduced by van Lamoen [12], relations (3) can be formulated as follows:

Proposition 11. *Triangle $K''_1K''_2K''_3$ is the cross triangle of triangles $K_1K_2K_3$ and $K'_1K'_2K'_3$.*

These relationships are illustrated in Figure 4. The following property can easily be checked.

Proposition 12. *Both triples of relations (2) and (3) are invariant under the following cyclic permutation: $X \mapsto X' \mapsto X'' \mapsto X$, where X stands for the circles in relations (2) as well as for the corresponding centers in (3).*

As a consequence, Proposition 11 can be formulated in the following seemingly stronger, but in fact equivalent version.

Proposition $11'$. *Any of the triangles $K_1K_2K_3$, $K'_1K'_2K'_3$, $K''_1K''_2K''_3$ is the cross triangle of the other two triangles.*

We apply the following theorem of van Lamoen [12]:

Theorem 13 (van Lamoen)**.** *Let ABC, $A'B'C'$ and $A''B''C''$ be three triangles such that $A''B''C''$ is the cross triangle of ABC and $A'B'C'$. Suppose that ABC and $A'B'C'$ are centrally perspective. Then ABC and $A''B''C''$ are centrally perspective, and likewise, $A'B'C'$ and $A''B''C''$ are also centrally perspective. Moreover, the three perspectors are collinear.*

Note that here any two perspectivities imply the third, on account of Proposition 12. Observe that the three triangles in this theorem determine a configuration of type $(12_4, 16_3)$ which consists of

- the nine vertices of the triangles;
- the three perspectors;
- the six lines determining the cross triangle relationship;
- the nine projecting lines meeting by threes at the three perspectors;

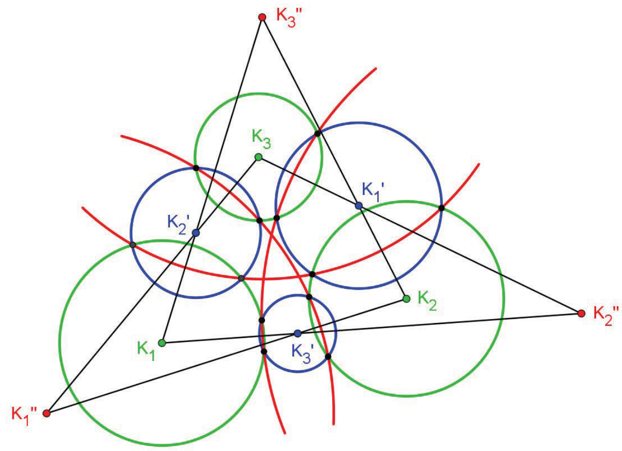


Figure 4. The desmic system of nine circles

- the line connecting these perspectors.

It is called the *desmic configuration*, and the triangles are called the *desmic mates* of each other [5]. This is a famous configuration which goes back to the work of Hesse and Salmon in the 19th century [3]. Moreover, as one can check in Figure 5, it is isomorphic with the Reye configuration (cf. [11], § 22). This latter, in turn, relates it with the celebrated *desmic tetrahedra* of Stephanos [1] (hence the term which we are using here).

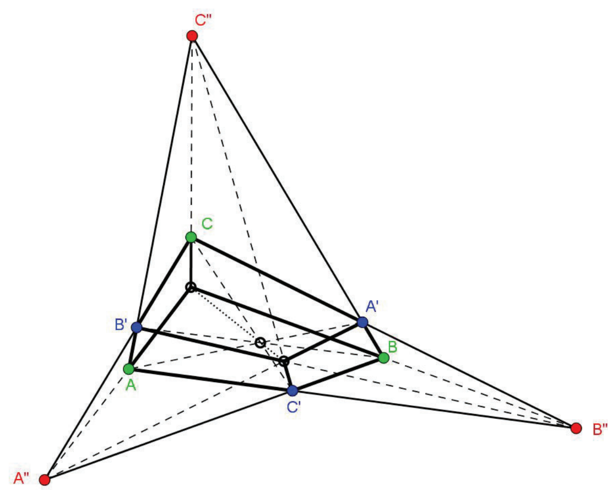


Figure 5. The desmic configuration of type $(12_4, 16_3)$. (A combinatorial cube is indicated showing the isomorphism with the Reye configuration)

Definition 14. *For* $i = 1, 2, 3$*, let* Γ_i*,* Γ_i'*,* Γ_i'' *be nine circles which satisfy the relations (2) above such that each of the intersections* $\Gamma_i \cap \Gamma_j'$ $(i \neq j)$ *consists* *of precisely two points. Suppose, furthermore, that the centers* K_i*,* K_i'*,* K_i'' *of these circles are the vertices of triangles* $K_1K_2K_3$*,* $K_1'K_2'K_3'$*,* $K_1''K_2''K_3''$ *which are desmic mates of each other. Then we call these circles a* desmic system of nine circles*.*

Using this definition, and taking into consideration Remark 10a as well as Theorem 13, we obtain the following slightly more general version of Theorem 2.

Theorem 15. *For* $i = 1, \dots, 6$*, consider a closed chain of circles* $\{\Gamma_i\}$ *such that every two consecutive members* Γ_i *and* Γ_{i+1} *meet in the points* (A_i, B_i)*, with indices modulo 6. Suppose that this chain can be completed with three circles which are circumscribed on the quadruples* (A_1, B_1, B_4, A_4)*,* (A_2, B_2, B_5, A_5) *and* (A_1, B_1, B_6, A_6)*. Then these nine circles form a desmic system.*

References

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea Publ. Co., New York, 1964.

[2] T. Cohl, A purely synthetic proof of Dao's theorem on six circumcenters associated with a cyclic hexagon, *Forum Geom.*, 14 (2014) 261–264.

[3] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* 56 (1950), 413–455. Reprinted in: H. S. M. Coxeter, *Twelve Geometric Essays*, Southern Illinois University Press, Carbondale, 1968, 106–149.

[4] H. S. M. Coxeter and S. L. Greitzer, *Az újra felfedezett geometria*, Gondolat, Budapest, 1977. Hungarian translation of: *Geometry Revisited*, Random House, New York, 1967.

[5] K. Dean and F. M. van Lamoen, Geometric construction of reciprocal conjugations, *Forum Geom.*, 1 (2001) 115–120.

[6] N. Dergiades, Dao's theorem on six circumcenters associated with a cyclic hexagon, *Forum Geom.*, 14 (2014) 243–246.

[7] N. T. Dung, A simple proof of Dao's theorem on six circumcircles, *Global Journal of Adv. Research on Classical and Modern Geometries*, 6 (2017) 58–61.

[8] N. Q. Duong, Some problems around the Dao's theorem on six circumcenters associated with a cyclic hexagon configuration, *International Journal of Computer Discovered Mathematics*, 1 (2016) 40–47.

[9] G. Gévay, Resolvable configurations. Submitted for publication.

[10] G. Gévay, An extension of Miquel's six-circles theorem, *Forum Geom.*, 18 (2018) 115–118.

[11] D. Hilbert and S. Cohn-Vossen, *Szemléletes geometria*, Gondolat, Budapest, 1982. Hungarian translation of: *Anschauliche Geometrie*, Springer, Berlin, 1932.

[12] F. M. van Lamoen, Bicentric triangles, *Nieuw Arch. Wisk.* 17 (1999) 363–372.

[13] H. Schwerdtfeger, *Geometry of Complex Numbers*, Dover Publications, Inc., New York, 1979.

[14] L. Szilassi, Personal communication, 2017.

Gábor Gévay: Bolyai Institute, University of Szeged, Szeged, Hungary

E-mail address: `gevay@math.u-szeged.hu`

Forum Geometricorum
Volume 18 (2018) 409–412.

FORUM GEOM
ISSN 1534-1178

An Analogue to Pappus Chain theorem with Division by Zero

Hiroshi Okumura

Abstract. Considering a line passing through the centers of two circles in a Pappus chain, we give a theorem analogue to Pappus chain theorem.

1. Introduction

Let α, β and γ be circles with diameters BC, CA and AB, respectively for a point C on the segment AB (see Figure 1). Pappus chain theorem says: if $\{\alpha = \delta_0, \delta_1, \delta_2, \cdots\}$ is a chain of circles whose members touch the circles β and γ, the distance between the center of the circle δ_n and the line AB equals $2nr_n$, where r_n is the radius of δ_n. In this article we show that if we consider a line passing through the centers of two circles in the chain instead of AB, a similar theorem still holds including the case in which the two circles are symmetric in the line AB.

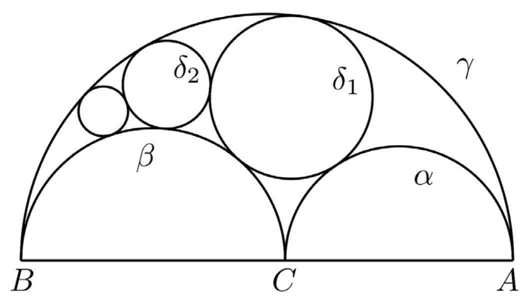


Figure 1.

2. Results

Let a and b be the radii of the circles α and β, respectively. We use a rectangular coordinate system with origin C such that A and B have coordinates $(-2b, 0)$ and $(2a, 0)$, respectively. Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\alpha, \beta, \gamma\}$. We consider the chain of circles $\mathcal{C} = \{\cdots, \delta_{-2}, \delta_{-1}, \varepsilon_1 = \delta_0, \delta_1, \delta_2, \cdots\}$ whose members touch the circles ε_2 and ε_3, where we assume that δ_i lies on the region $y > 0$ if $i > 0$ (see Figure 2).

If $\varepsilon_1 = \alpha$, the chain is explicitly denoted by $\mathcal{C}_\alpha$. The chains $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$ are defined similarly. Let $c = a + b$ and let (x_n, y_n) and r_n be the coordinates of the center and the radius of the circle δ_n. Pappus chain theorem also holds for the chains $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$, i.e., we have $y_n = 2nr_n$, and x_n and r_n are given in Table 1 [3, 4].

Publication Date: December 6, 2018. Communicating Editor: Paul Yiu.

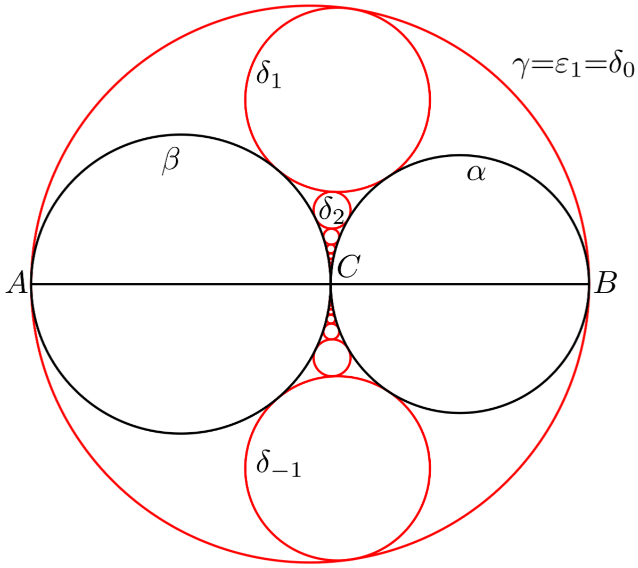


Figure 2: $\mathcal{C}_\gamma, \varepsilon_1 = \gamma,\ \{\varepsilon_2, \varepsilon_3\} = \{\alpha,\ \beta\}$

Chain	x_n	r_n
$\mathcal{C}_\alpha$	$-2b + \dfrac{bc(b+c)}{n^2a^2+bc}$	$\dfrac{abc}{n^2a^2+bc}$
$\mathcal{C}_\beta$	$2a - \dfrac{ca(c+a)}{n^2b^2+ca}$	$\dfrac{abc}{n^2b^2+ca}$
$\mathcal{C}_\gamma$	$\dfrac{ab(b-a)}{n^2c^2-ab}$	$\dfrac{abc}{n^2c^2-ab}$

Table 1: $y_n = 2nr_n$

Let $l_{i,j}$ $(i \neq j)$ be the line passing through the centers of the circles δ_i and δ_j for $\delta_i, \delta_j \in \mathcal{C}$. It is expressed by the equations

$$\begin{cases} 2(bc - a^2ij)x + a(b+c)(i+j)y - 2b(2a^2ij - c(b-c)) = 0, \\ 2(ca - b^2ij)x - b(c+a)(i+j)y + 2a(2b^2ij + c(c-a)) = 0, \\ 2(ab + c^2ij)x + c(a-b)(i+j)y - 2ab(a-b) = 0 \end{cases} \tag{1}$$

for $\mathcal{C}_\alpha$, $\mathcal{C}_\beta$, $\mathcal{C}_\gamma$, respectively.

Let $H_{i,j}(n)$ be the point of intersection of the lines $x = x_n$ and $l_{i,j}$ with y-coordinate $h_{i,j}(n)$. Let $d_{i,j}(n) = h_{i,j}(n) - y_n$, i.e., $d_{i,j}(n)$ is the signed distance between the center of δ_n and $H_{i,j}(n)$. The following theorem is an analogue to Pappus chain theorem. It is also a generalization of [1] (see Figure 3).

Theorem 1. *If $i + j \neq 0$, then $d_{i,j}(n) = f_{i,j}(n)r_n$ holds, where*

$$f_{i,j}(n) = \frac{2(n-i)(n-j)}{i+j} = 2\left(\frac{n^2}{i+j} - n + \frac{ij}{i+j}\right). \tag{2}$$

Proof. We consider the chain $\mathcal{C}_\alpha$. By Table 1 and (1), we get

$$h_{i,j}(n) = \frac{2(n^2+ij)abc}{(i+j)(n^2a^2+bc)} = 2\frac{(n^2+ij)}{(i+j)}r_n.$$

Therefore

$$d_{i,j}(n) = h_{i,j}(n) - y_n = 2\frac{(n^2+ij)}{(i+j)}r_n - 2nr_n = \frac{2(n-i)(n-j)}{i+j}r_n.$$

The rest of the theorem can be proved in a similar way. □

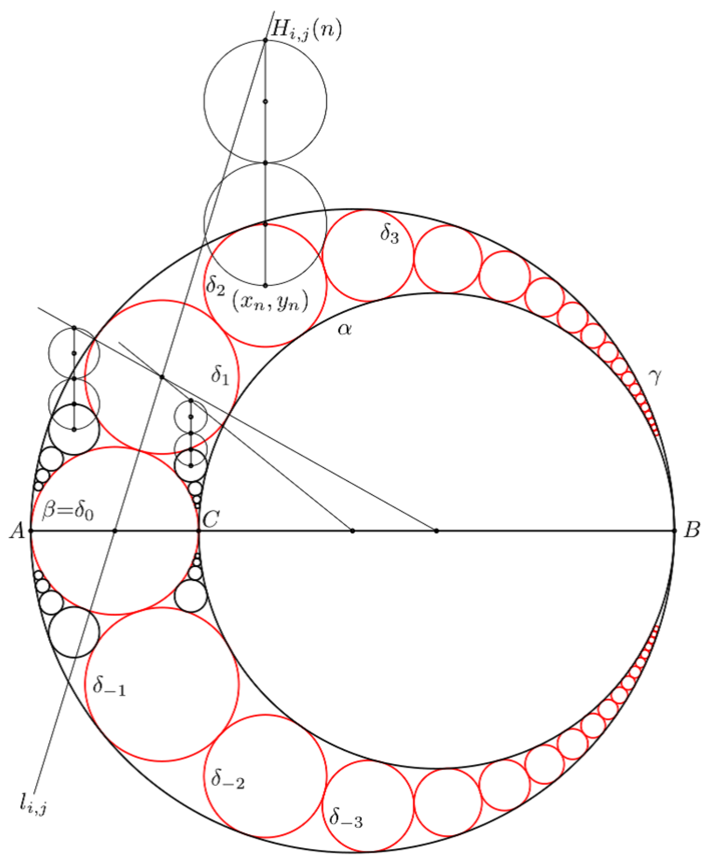


Figure 3: $\mathcal{C}_\beta,\ \{i,\ j\} = \{0,\ 1\},\ n = 2$

Corollary 2. *If $i = 0$ in Theorem 1, the following statements hold.*
(i) *If $j = \pm 1$, $d_{i,j}(n) = \pm 2n(n \mp 1)r_n$.*
(ii) *If $j = \pm 2$, $d_{i,j}(n) = \pm n(n \mp 2)r_n$.*

Corollary 3. $d_{i,j}(n) - d_{i,j}(-n) = -4nr_n$ *for any integers i, j, n with $i \neq \pm j$.*

3. The case $i + j = 0$

We consider Theorem 1 in the case $i + j = 0$ under the definition of division by zero: $z/0 = 0$ for any real number z [2]. In this case the line $l_{i,j}$ is perpendicular to AB. Since $i + j = 0$ implies $f_{i,j}(n) = -2n$ by (2), we get the same conclusion as to Pappus chain theorem for $l_{i,j}$.

Theorem 4. *If $i + j = 0$, $d(n) = -2nr_n$ holds.*

Since two parallel lines meet in the origin [5], $H_{i,j}(n)$ coincides with the origin, if $i + j = 0$, i.e., $h_{i,j}(n) = 0$. Hence Theorem 4 can also be derived from this fact with Pappus chain theorem. The theorem shows that *the distances from the center of the circle δ_n to the two lines AB and $x = x_i$ are the same for any integer i*. This is one of the unexpected phenomena for perpendicular lines derived from the definition of the division by zero. For another such example see [6]. Since Theorem 1 can be stated even in the case $i + j = 0$, we get :

Theorem 5. $d_{i,j}(n) = f_{i,j}(n)r_n$ *holds for any integers i, j, n, where $i \neq j$.*

Now Corollary 3 also holds even in the case $i + j = 0$.

Corollary 6. $d_{i,j}(n) - d_{i,j}(-n) = -4nr_n$ *for any integers i, j, n with $i \neq j$.*

4. Conclusion

The recent definition, $z/0 = 0$ for a real number z, yields several unexpected phenomena, which are especially significant for perpendicular lines. In this paper we get one more such result in section 3, for which we are still looking for a suitable interpretation.

References

[1] A. Altintas and H. Okumura, A note on Pappus chain and a collinear theorem, *Sangaku Journal of Mathematics*, 2 (2018) 11–12.
[2] M. Kuroda, H. Michiwaki, S. Saitoh, and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, *Int. J. Appl. Math.*, 27(2) (2014) 191–198.
[3] G. Lucca, Three Pappus chains inside the arbelos: some identities, *Forum Geom.*, 7 (2007) 107–109.
[4] G. Lucca, Some identities arising from inversion of Pappus chains in an arbelos, *Forum Geom.*, 8 (2008) 171–174.
[5] H. Michiwaki, H. Okumura, and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, *Int. J. Math. Computation*, 28(1) (2017) 1–16.
[6] H. Okumura, Wasan geometry with the division by 0, *Int. J. Geom.*, 7(1) (2018) 17–20.

Hiroshi Okumura: Takahanadai Maebashi Gunma 371-0123, Japan
E-mail address: hokmr@yandex.com

Forum Geometricorum
Volume 18 (2018) 413–418.

FORUM GEOM
ISSN 1534-1178

Polygons on the Sides of an Octagon

Rogério César dos Santos

Abstract. Van Aubel's theorem has some interesting generalizations. Some were dealt with by Krishna (2018a and 2018b). In this article, we intend, encouraged by the works cited, to prove a new generalization of Van Aubel's theorem, which consists of the construction of a parallelogram from an octagon surrounded by regular n-sided polygons.

The present article will demonstrate the following result about octagons, driven by the results of Krishna [1, 2].

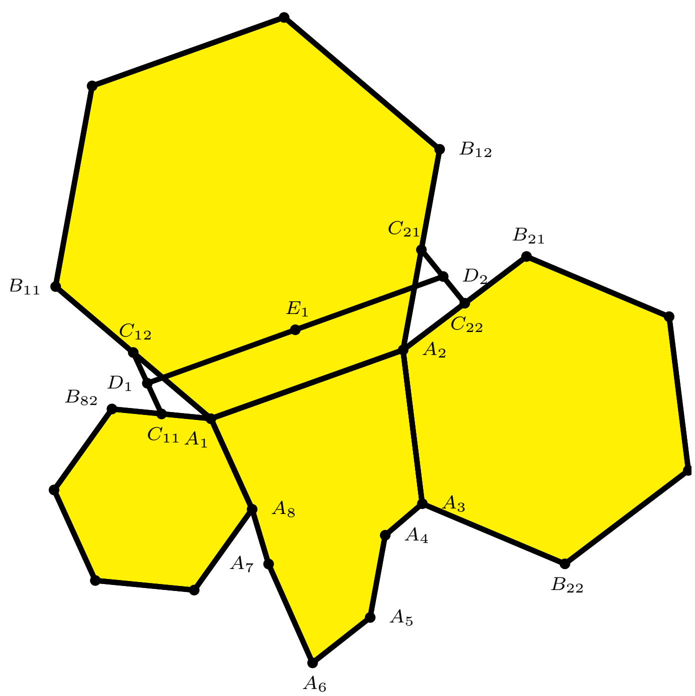


Figure 1

Theorem 1. *Take a random octagon* $O: \ A_1A_2\cdots A_8$*, either convex or concave. Having fixed an integer* $n \geq 3$*, consider the eight regular polygons of* n *sides constructed externally on the sides of* O*. Denote the sixteen sides of the eight*

Publication Date: December 14, 2018. Communicating Editor: Paul Yiu.

polygons that have the common vertex A_j *with the octagon,* $j = 1, \ldots, 8$, *by* A_1B_{82}, A_1B_{11}, A_2B_{12}, A_2B_{21}, $\ldots$, A_8B_{72}, *and* A_8B_{81}, *as illustrated in Figure 1. Denote the midpoints of these sides by* C_{11}, C_{12}, C_{21}, C_{22}, $\ldots$, C_{81}, *and* C_{82}, *respectively. The figure shows only three of the regular polygons so that the image is not overloaded.*

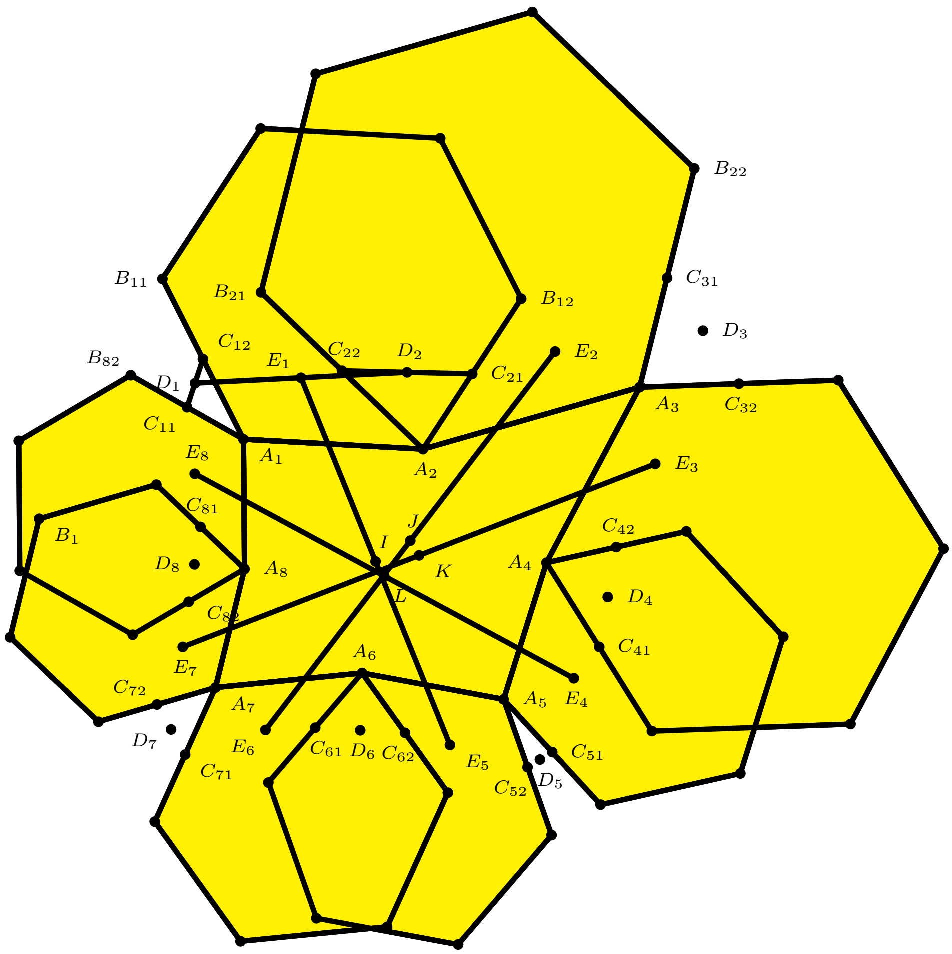


Figure 2

Also consider the midpoints of the segments $C_{j1}C_{j2}$*:*

$$D_j = \frac{C_{j1} + C_{j2}}{2}, \qquad j = 1, \ldots, 8 \pmod 8$$

and the midpoints of D_jD_{j+1}*:*

$$E_j = \frac{D_j + D_{j+1}}{2}.$$

.

Under these conditions, the midpoints of E_jE_{j+4}, $j = 1, \ldots, 8 \pmod 8$, *denoted by* I, J, K, *and* L, *respectively, form a parallelogram, as shown in Figure 2.*

Proof. Let us demonstrate by denoting the points I, J, K, and L as functions of the complex numbers corresponding to the vertices of the octagon and the internal angle of the polygon built on the side. Consider $A_j = (x_j,\ y_j)$, $j = 1, 2, \ldots, 8$ for each vertex of the octagon in the Cartesian Plane. Denote the complex number corresponding to each A_j by $a_j = x_j + iy_j$, where $i^2 = -1$. Similarly, denote the complex numbers corresponding to the points B_j, C_j, D_j, and E_j by b_j, c_j, d_j and e_j, respectively. Take M_j as the midpoint of the segment A_jA_{j+1}, so that $m_j = \frac{a_j+a_{j+1}}{2}$ is the complex number corresponding to M_j.

Considering $\hat{A} = \widehat{B_{11}A_1A_2}$ the internal angle of each of the eight regular polygons. Then, through the hypotheses, we can conclude that $c_{12} - a_1$ is equal to the complex number resulting from the rotation of $m_1 - a_1$ by $\hat{A}$ degrees counterclockwise. Namely,

$$c_{12} - a_1 = (m_1 - a_1)(\cos\hat{A} + i\sin\hat{A}).$$

From now on let us denote $\operatorname{cis}(\hat{A}) = \cos\hat{A} + i\sin\hat{A}$. Then,

$$c_{12} = a_1 + (m_1 - a_1)\operatorname{cis}(\hat{A}).$$

The complex number $c_{11} - a_1$ is equal to the complex number resulting from the rotation of $m_8 - a_1$ by $\hat{A}$ degrees clockwise, or $360° - \hat{A}$ counterclockwise, that is,

$$\begin{aligned}
& c_{11} - a_1 = (m_8 - a_1)\operatorname{cis}(360° - \hat{A}) \\
\Longrightarrow\ & c_{11} - a_1 = (m_8 - a_1)(\cos(360° - \hat{A}) + i\sin(360° - \hat{A})) \\
& \qquad\quad = (m_8 - a_1)(\cos\hat{A} - i\sin\hat{A}) \\
& \qquad\quad = (m_8 - a_1)(\cos(-\hat{A}) + i\sin(-\hat{A})) \\
& \qquad\quad = (m_8 - a_1)\operatorname{cis}(-\hat{A}) \\
\Longrightarrow\ & c_{11} = a_1 + (m_8 - a_1)\operatorname{cis}(-\hat{A}).
\end{aligned}$$

Therefore, we can find

$$\begin{aligned}
d_1 &= \frac{c_{11} + c_{12}}{2} \\
&= \frac{1}{2}\Big(a_1 + (m_8 - a_1)\operatorname{cis}(-\hat{A}) + a_1 + (m_1 - a_1)\operatorname{cis}(\hat{A})\Big) \\
&= 1/2\left(a_1 + \left(\frac{a_8 + a_1}{2} - a_1\right)\operatorname{cis}(-\hat{A}) + a_1 + \left(\frac{a_1 + a_2}{2} - a_1\right)\operatorname{cis}(\hat{A})\right) \\
&= 1/2\left(2a_1 + \frac{a_8 - a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_2 - a_1}{2}\operatorname{cis}(\hat{A})\right).
\end{aligned}$$

Similarly, we can find

$$\begin{aligned} d_2 = &\ \frac{c_{21}+c_{22}}{2} \\ = &\ \frac{1}{2}\left(2a_2 + \frac{a_1-a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3-a_2}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Then

$$\begin{aligned} e_1 = &\ \frac{d_1+d_2}{2} \\ = &\ \frac{1}{4}\left(2a_1 + \frac{a_8-a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_2-a_1}{2}\operatorname{cis}(\hat{A})\right. \\ &\qquad \left. +2a_2 + \frac{a_1-a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3-a_2}{2}\operatorname{cis}(\hat{A})\right) \\ = &\ \frac{1}{4}\left(2(a_1+a_2) + \frac{a_8-a_2}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3-a_1}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Similarly, we obtain

$$e_5 = \frac{1}{4}\left(2(a_5+a_6) + \frac{a_4-a_6}{2}\operatorname{cis}(-\hat{A}) + \frac{a_7-a_5}{2}\operatorname{cis}(\hat{A})\right).$$

Thus,

$$\begin{aligned} I = &\ \frac{e_1+e_5}{2} \\ = &\ \frac{1}{8}\left(2(a_1+a_2+a_5+a_6) + \frac{a_8+a_4-a_2-a_6}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. +\frac{a_3+a_7-a_1-a_5}{2}\operatorname{cis}(\hat{A})\right). \end{aligned}$$

Similarly, we can obtain:

$$\begin{aligned} J = &\ \frac{e_2+e_6}{2} \\ = &\ \frac{1}{8}\left(2(a_2+a_3+a_6+a_7) + \frac{a_1+a_5-a_3-a_7}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. +\frac{a_4+a_8-a_2-a_6}{2}\operatorname{cis}(\hat{A})\right), \\ K = &\ \frac{e_3+e_7}{2} \\ &\ \frac{1}{8}\left(2(a_3+a_4+a_7+a_8) + \frac{a_2+a_6-a_4-a_8}{2}\operatorname{cis}(-\hat{A})\right. \\ &\qquad \left. +\frac{a_5+a_1-a_3-a_7}{2}\operatorname{cis}(\hat{A})\right), \end{aligned}$$

and

$$\begin{aligned}
L = &\ \frac{e_4 + e_8}{2} \\
= &\ \frac{1}{8}\left(2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\operatorname{cis}(-\hat{A})\right. \\
&\left. + \frac{a_6 + a_2 - a_4 - a_8}{2}\operatorname{cis}(\hat{A})\right).
\end{aligned}$$

Anyway, let us calculate

$$\begin{aligned}
|J - K| = &\ \frac{1}{8}\left|\, 2(a_2 + a_6 - a_4 - a_8)\right. \\
&+ \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\operatorname{cis}(-\hat{A}) \\
&\left. + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\operatorname{cis}(\hat{A})\right|.
\end{aligned}$$

Remembering that

$$\begin{aligned}
I = &\ \frac{1}{8}\left(2(a_1 + a_2 + a_5 + a_6) + \frac{a_8 + a_4 - a_2 - a_6}{2}\operatorname{cis}(-\hat{A})\right. \\
&\left. + \frac{a_3 + a_7 - a_1 - a_5}{2}\operatorname{cis}(\hat{A})\right)
\end{aligned}$$

and

$$\begin{aligned}
L = &\ \frac{1}{8}\left(2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\operatorname{cis}(-\hat{A})\right. \\
&\left. + \frac{a_6 + a_2 - a_4 - a_8}{2}\operatorname{cis}(\hat{A})\right),
\end{aligned}$$

we have:

$$\begin{aligned}
|I - L| = &\ \frac{1}{8}\left|\, 2(a_2 + a_6 - a_4 - a_8)\right. \\
&+ \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\operatorname{cis}(-\hat{A}) \\
&\left. + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\operatorname{cis}(\hat{A})\right| \\
= &\ |J - K|.
\end{aligned}$$

In an analogous way, it is possible to prove that

$$|K - L| = |J - I|.$$

This concludes the demonstration. □

References

[1] D. N. V. Krishna, An equilateral triangle associated with a hexagon, *Math. Gazette*, 102 (555) (2018a) 515–516.
[2] D. N. V. Krishna, A note on special cases of Van Aubel's theorem, *International Jounal of Advances in Applied Mathematics and Mechanics*, 5(4) (2018b) 30–51.

Rogério César dos Santos: Quadra 04, Conjunto E, Bloco C, apt 102. Sobradinho, DF. Brazil, Zip code (CEP): 73025-040.
E-mail address: `professorrogeriocesar@gmail.com`

Forum Geometricorum
Volume 18 (2018) 419–430.

FORUM GEOM
ISSN 1534-1178

The Love for the Three Conics

Stefan Liebscher and Dierck-E. Liebscher

Abstract. Neville's theorem of the three focus-sharing conics finds a simplification and a new outreach in the context of projective geometry.

1. Introduction

In Euclidean geometry, let three points in the plane serve as three pairs of foci for three conics. The three pairs of conics define three lines through their intersections, and the three lines are concurrent, they pass a common point [4, 9, 10].

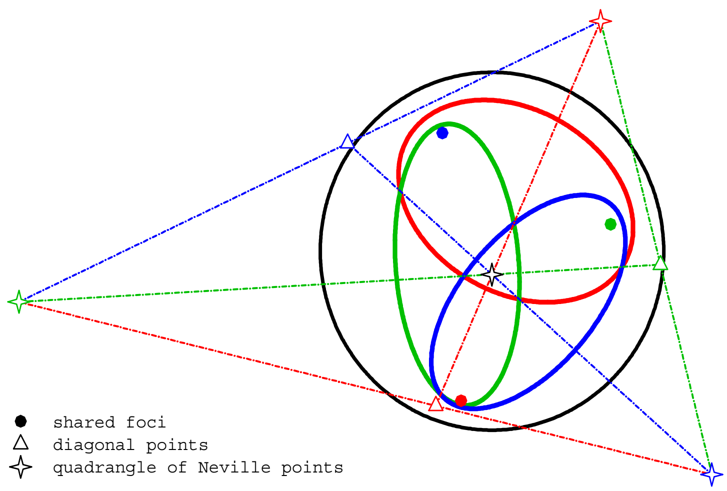


Figure 1. Three focus-sharing conics in Cayley-Klein geometries.

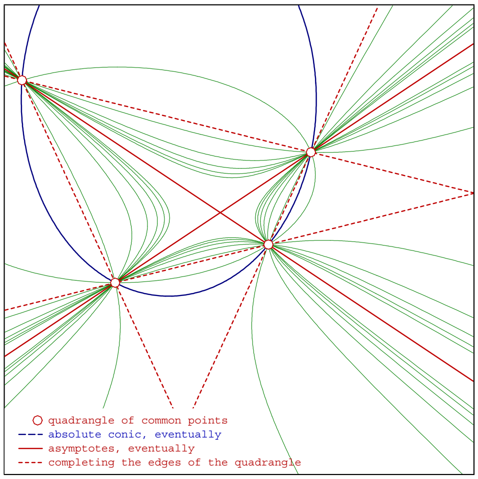
This theorem has found devotees and different proofs in Euclidean geometry [2, 3, 12]. It may be expanded to non-Euclidean geometries (Fig. 1) where it allows a generalising formulation in projective terms [7].

From the perspective of projective geometry, foci are defined for pairs of conics [11]. They are the vertices of the quadrilateral of common tangents and come in three pairs, which define the three diagonals of the quadrilateral [8]. Linear combination of the two conics yields the pencil of conics tangent to all four sides of the quadrilateral.

For a single conic, it is the metric-generating absolute conic of Cayley-Klein geometries which may serve as the second conic to define foci in the usual sense. This is the case in the Neville problem. Confocal conics conventionally form the pencil defined by two foci and the absolute conic in the background. Pencils exist

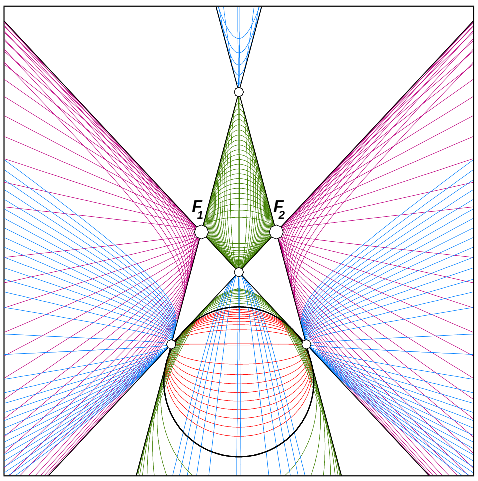
---

Publication Date: December 14, 2018. Communicating Editor: Paul Yiu.



The pencil can be seen as determined by the four common points, or as generated by two conics with their four intersections. Selecting any of the conics as absolute conic (strong-line ellipse), the pencil is determined by any other conic, in particular the singular conic given by a pair of focal lines.

Figure 2. Pencil of conics through four common points.



The dual pencil can be seen as determined by the four common tangents, or as generated by two conics with their four common tangents. Selecting any of the conics as absolute conic (strong-line ellipse), the pencil is determined by any other conic, in particular the singular conic given by a pair of foci. The plane contains points where two conics of the dual pencil intersect, and points which meet none of the conics.

Figure 3. Dual pencil of conics with four common tangents.

for each pair of the three focus-sharing conics, too. It is the consideration of these pencils which admits the short and snappy proofs (see ch. 9 in [7]).

2. Pairs of conics

To explore the relation between two conics, we consider the generated pencils of conics. Figure 2 shows the pencil with the four intersection points fixed. It contains three singular elements: the three pairs of common chords. Figure 3 shows the dual pencil with the four common tangents fixed. It also contains three singular elements: the three pairs of foci. To emphasize their duality, we call the common chords also focal lines. Whichever focus we choose, it determines two directrices, which intersect in a diagonal point common to both the quadrilateral of tangents and the quadrangle of common points (Fig. 4). This diagonal point carries one of the three pairs of focal lines (i.e. common chords), see Prop. 1 below.

For Neville's problem, we start with three points (Fig. 5). For any two of them we choose a conic for which the two points are a pair of foci with respect to the absolute conic. Any two of the conics share just one focus with one crossing of directrices and one singular conic consisting of a pair of common chords. We obtain such a pair of common chords for each of the three pairs of conics. The condition that the shared foci are also foci with respect to the absolute conic imposes linear dependence, and the common chords are the edges of a quadrangle of points in which the common chords intersect. This turns out to be simple algebra, and yields the desired proofs in section 3 below.

We use homogeneous coordinates and write "$\sim$" to denote equality up to a nonzero scalar factor. Conics can be represented by pairs $[\mathbf{c}, \mathbf{C}]$ of symmetric matrices. They provide a linear map of points onto lines: $Q \mapsto \mathbf{c}Q$, the polar of Q, and its dual, a linear map of lines onto points, $g \mapsto \mathbf{C}g$, the pole of g. The pair of both maps is also called polarity. The peripheral points of a conic are the zeros of the quadratic form, $P^\mathsf{T}\mathbf{c}P = 0$, its tangents are the zeros of the dual, $t^\mathsf{T}\mathbf{C}t = 0$.

For regular conics, the two matrices $\mathbf{C}$ and $\mathbf{c}$ are reciprocal, $\mathbf{Cc} \sim \mathbf{1}$. Singular conics satisfy $\mathbf{Cc} = 0 = \mathbf{cC}$. If one of $\mathbf{c}, \mathbf{C}$ has at least rank 2, the other one is (a scalar multiple of) its adjoint (or transpose cofactor) matrix.

For $\mathbf{c}$ of rank 2, we can write $\mathbf{c} = gh^\mathsf{T} + hg^\mathsf{T}$. This yields a singular conic consisting of two distinct lines g, h of peripheral points with a pencil of tangents trough the double point $g \times h$. For $\mathbf{C}$ of rank 2, we can write $\mathbf{C} = PQ^\mathsf{T} + QP^\mathsf{T}$ and find a singular conic consisting of a (double) line $P \times Q$ of peripheral points through the concurrency centers P, Q of the two pencils of tangents.

For two distinct regular conics $[\mathbf{k}_1, \mathbf{K}_1], [\mathbf{k}_2, \mathbf{K}_2]$, the pencil $\mathcal{P}[\mathbf{k}_1, \mathbf{k}_2]$ is given by the linear combinations $\{\alpha_1\mathbf{k}_1 + \alpha_2\mathbf{k}_2\}$ (Fig. 2), the dual pencil $\tilde{\mathcal{P}}[\mathbf{K}_1, \mathbf{K}_2]$ by $\{\beta_1\mathbf{K}_1 + \beta_2\mathbf{K}_2\}$ (Fig. 3).

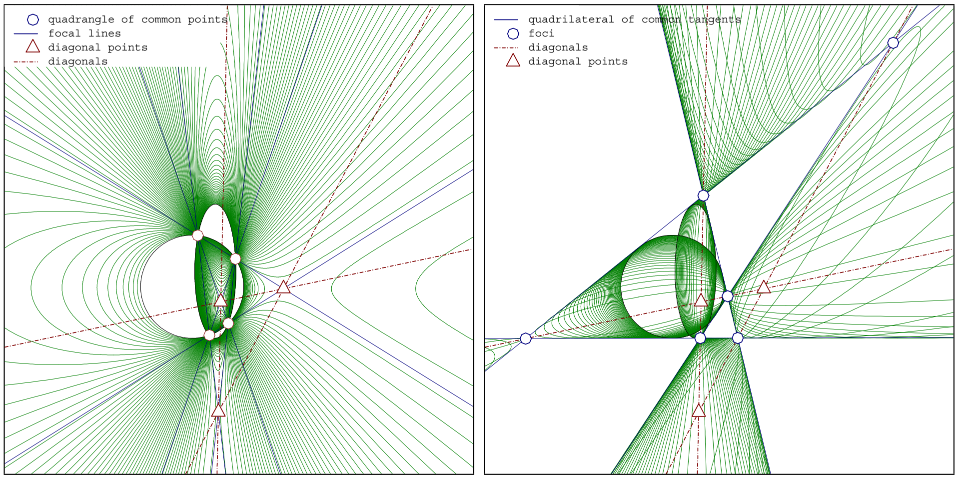
The pencil is uniquely determined by its quadrangle of common points, $\mathcal{P} = \mathcal{P}[Q_1, \ldots, Q_4]$. Its singular elements are the pairs of opposite chords, i.e. the pairs of opposite edges of the quadrangle $Q_1, \ldots, Q_4$ of common points, pairs which intersect in the diagonal points of the quadrangle. We note that for regular real conics the singular elements are real even in the case of only complex intersections of the conics.

Analogously, the dual pencil is uniquely determined by the quadrilateral of common tangents, $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}[t_1, \ldots, t_4]$. Its singular elements are the diagonal lines of the quadrilateral with tangent pencils in the vertices of the quadrilateral. The six intersections of the tangents t_i, the vertices of the quadrilateral, are the common foci of the dual pencil $\tilde{\mathcal{P}}$.

Proposition 1. *For any pair of conics, the triangle given by the diagonal points of the quadrangle of common points coincides with the triangle given by the diagonal lines of the quadrilateral of common tangents. This diagonal triangle is also self-polar: Each diagonal point is the pole of the opposite diagonal line with respect to all conics of the two pencils.*

Proof. See Fig. 4 and Thm. 7.6 of [7]. □

In other words: given a pair of conics, the diagonals of the tangent quadrilateral meet in the three intersection points of opposite common chords.



Two conics have 4 common tangents and 4 common points. On the left, part of the pencil with the common points is shown, with the six chords (focal lines) and the three diagonal points. On the right, part of the pencil with the common tangents is shown, with the six intersections (foci) and the three diagonal lines. The diagonal triangles coincide.

Figure 4. Two pencils for a pair of conics with their coinciding diagonal triangles.

3. Three focus-sharing conics

In Neville's problem, three conics $\mathbf{K}_k$, $k = 1,2,3$ share three foci F_k, $k = 1,2,3$ with respect to the absolute conic $\mathbf{C}$. The pair $[F_l, F_m]$ belongs to $\mathbf{K}_k$ (k, l, m cyclically), and F_k is focus in the pairs $[\mathbf{K}_l, \mathbf{K}_m]$, $[\mathbf{K}_l, \mathbf{C}]$, and $[\mathbf{K}_m, \mathbf{C}]$.

The pair $[\mathbf{K}_l, \mathbf{K}_m]$ itself has six foci (again in three pairs). F_k is one of them, and its adjoint partner on the diagonal passing F_k will be denoted by F_k^*. The line $d_k \sim F_k \times F_k^*$ is a diagonal line of the quadrilateral of tangents common to $\mathbf{K}_l$ and $\mathbf{K}_m$. The opposite diagonal point D_k can be represented as pole of d_k to both of the conics, $D_k \sim \mathbf{K}_l d_k \sim \mathbf{K}_m d_k$, and the four polars $\mathbf{k}_l F_k, \mathbf{k}_m F_k, \mathbf{k}_l F_k^*, \mathbf{k}_m F_k^*$ are incident with D_k.

$$D_k \sim \mathbf{K}_l d_k \sim \mathbf{K}_m d_k \sim \mathbf{k}_l F_k \times \mathbf{k}_m F_k \sim \mathbf{k}_l F_k^* \times \mathbf{k}_m F_k^* . \tag{1}$$

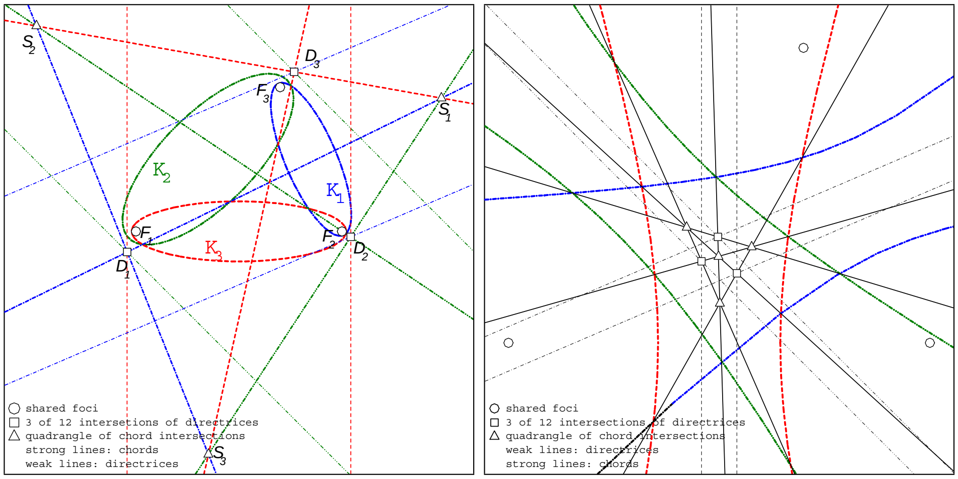
The focus F_k has a directrix (its polar) with both $\mathbf{K}_l$ and $\mathbf{K}_m$, and the intersection of the two directrices is the diagonal point D_k (Fig. 5).

We now use the identity of the diagonal triangles, Prop. 1. The diagonal point D_k is incident with two opposite chords of the pencil $\mathcal{P}(\mathbf{k}_l, \mathbf{k}_m)$. They form one of the singular elements of this pencil, $\mathbf{n}_k$. We can write

$$\mathbf{n}_k \sim (D_k^{\mathsf{T}} \mathbf{k}_l D_k)\, \mathbf{k}_m - (D_k^{\mathsf{T}} \mathbf{k}_m D_k)\, \mathbf{k}_l . \tag{2}$$

After substituting one D_k by $\mathbf{K}_l d_k$, the other by $\mathbf{K}_m d_k$, we obtain

$$\mathbf{n}_k \sim (d_k^{\mathsf{T}} \mathbf{K}_m \mathbf{k}_l \mathbf{K}_l d_k)\, \mathbf{k}_m - (d_k^{\mathsf{T}} \mathbf{K}_l \mathbf{k}_m \mathbf{K}_m d_k)\, \mathbf{k}_l \sim (d_k^{\mathsf{T}} \mathbf{K}_m d_k)\, \mathbf{k}_m - (d_k^{\mathsf{T}} \mathbf{K}_l d_k)\, \mathbf{k}_l \tag{3}$$



In the case of three ellipses, (at least) three of the common chords pass through complex intersection points. Nevertheless all six common chords are real and represent the edges of a quadrangle. In the case of three hyperbolas, we have chosen the eccentricities large enough to obtain only real intersections of the conics and real chords. For clearness, both drawings show an euclidean setting, for the general setting, see Fig. 1.

Figure 5. Three focus-sharing conics with the quadrangles of Neville centers.

This is a singular conic consisting of two lines, i.e. a pair of common chords, and a double point, the diagonal point D_k. For each pair$[\mathbf{K}_l, \mathbf{K}_m]$ of the three conics, we obtain such a singular conic consisting of that pair of opposite common chords which intersect in the diagonal point D_k.

Proposition 2. *The three singular conics $\mathbf{n}_k$ are linear dependent. Therefore, the common chords represented in these singular conics are the six sides of a quadrangle (Fig. 5).*

Proof. The conics $\mathbf{K}_m$ are elements of the dual pencils generated by $\mathbf{C}$ and the singular conics $F_k F_l^{\mathsf{T}} + F_l F_k^{\mathsf{T}}$. We fix representatives to remove the ambiguity of arbitrary nonzero scalar factors and write

$$\mathbf{K}_m = \omega_m \mathbf{C} + \lambda_m (F_k F_l^{\mathsf{T}} + F_l F_k^{\mathsf{T}}). \tag{4}$$

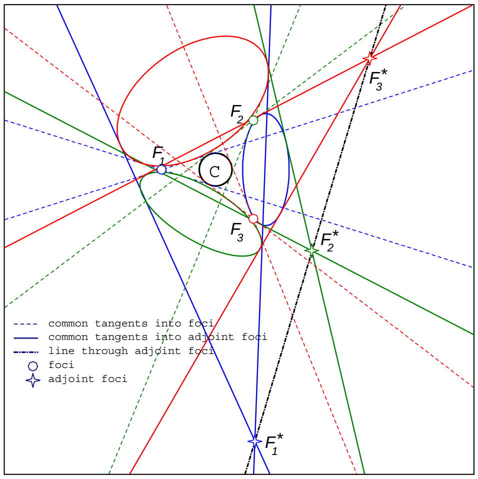
Since the diagonal d_m is incident with F_m, we obtain

$$(d_k^{\mathsf{T}} \mathbf{K}_m d_k) = \omega_m (d_k^{\mathsf{T}} \mathbf{C} d_k). \tag{5}$$

Equation (3) now yields $\mathbf{n}_{lm} = \omega_m \mathbf{k}_m - \omega_l \mathbf{k}_l$ (again choosing a suitable representative to remove the nonzero coefficient $d_k^{\mathsf{T}} \mathbf{C} d_k$) and linear dependency $\mathbf{n}_{12} + \mathbf{n}_{23} + \mathbf{n}_{31} = 0$ emerges. □

Starting with the absolute conic $\mathbf{C}$, we can reformulate Prop.2.

Corollary 3. *Given a conic with three pairs of tangents defining three intersections. Given one representative of each dual pencil generated by two of the three pairs of tangents. Then, the three intersections have two polars each (w.r.t. the*



The figure shows three conics $\mathbf{K}_m$ outside the absolute conic $\mathbf{C}$ in order to get real lines and points. The points F_m are the shared foci. The points F_m^* are opposite to the foci F_m in the quadrilateral of tangents common to $\mathbf{K}_k$ and $\mathbf{K}_l$. They are incident on a line.

Figure 6. Three focus-sharing conics with their collinear adjoint foci.

conics of the associated dual pencils). These polars intersect in points which carry two common chords. The six common chords are edges of a quadrangle.

We turn again to the adjoint foci. A pair $[\mathbf{K}_l, \mathbf{K}_m]$ defines six foci. They yield three diagonals which define a pairing of the foci, and the $[F_k, F_k^*]$ is such a pair. The focus F_k is the intersection of two tangents common to $[\mathbf{K}_l, \mathbf{K}_m]$, and F_k^* is the intersection of the two other common tangents. The focus F_k is a focus of the pair $[\mathbf{C}, \mathbf{K}_l]$ and of the pair $[\mathbf{C}, \mathbf{K}_m]$ as well. The focus adjoint to F_k in the pair $[\mathbf{K}_l, \mathbf{K}_m]$ is F_k^*, in the pair $[\mathbf{C}, \mathbf{K}_l]$ it is F_m, and in the pair $[\mathbf{C}, \mathbf{K}_m]$ it is F_l.

The following proposition was proven by Bogdanov [3] in the Euclidean case. Its generalization in Cayley-Klein geometries also generalizes the dual theorem to the four-conics-theorem (see Fig. 17 in [5]).

Proposition 4. *The three adjoint foci of the three focus-sharing conics are collinear. The six foci F_k, F_k^*, $i = 1 \ldots 3$, are the vertices of a quadrilateral (Fig. 6).*

Proof. We refer again to the singular members of dual pencils. The singular elements of the dual pencil $\mathbf{K}_{lm} = \alpha \mathbf{K}_l + \beta \mathbf{K}_m$ are given by the diagonal points itself:

$$\mathbf{N}_k = (d_k^\mathsf{T} \mathbf{K}_l d_k)\mathbf{K}_m - (d_k^\mathsf{T} \mathbf{K}_m d_k)\mathbf{K}_l. \tag{6}$$

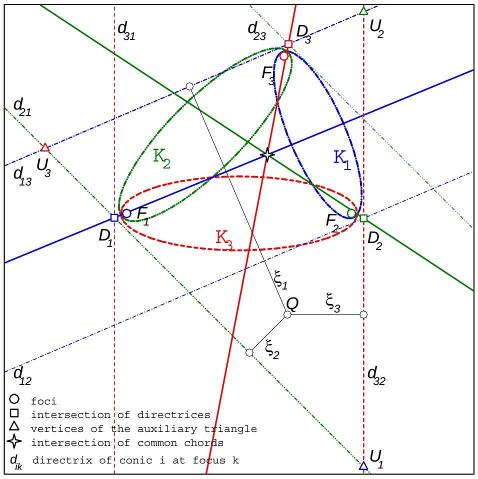
We take (5) and obtain

$$\sum_{k \in \{1,2,3\}} \omega_k (d_l^\mathsf{T} \mathbf{C} d_l)(d_m^\mathsf{T} \mathbf{C} d_m)\mathbf{N}_k = 0. \tag{7}$$

The three singular conics (all are pairs $F_k F_k^*$ of foci) are linear dependent, i.e. the foci F_k, F_k^*, $k = 1, 2, 3$ are the vertices of a complete quadrilateral, the three adjoint foci F_k^* are collinear. □

4. The limit of the Neville points

Any point can be found as a Neville point, if it is reached by the three pencils of conics generated by the three pairs of foci with the absolute conic. In particular,



Given three shared foci and three ellipses, one adds the six directrices first. We choose one of the two triangles contained in the convex hexagon (d_{13}, d_{21}, d_{32}) to establish trilinear coordinates ξ_k.
The distance of foci $(2f_i)$ is found in the distance of the directrices $(2\kappa_i = 2f_i/\varepsilon_i^2)$, so that intersections of two ellipses are given by comparing the distances from the focus (e.g. F_1) with the distances from the directrices (ξ_2 and $2\kappa_3 - \xi_3$ for the common chord of $\mathbf{K}_2$ and $\mathbf{K}_3$).

Figure 7. Three focus-sharing ellipses on the Euclidean plane.

any point in the inner part of the triangle can be the intersection of three ellipses of the set.

There is one particular point among the Neville centers. It is the limit for collapsing ellipses, for the singular members of the three dual pencils $\tilde{\mathcal{P}}[\mathbf{K}_m, \mathbf{C}]$. This limit is given by $\omega_m \to 0$ in eq. (4). We start with the Euclidean picture of Neville's proof (Fig 7). The triangle of foci is augmented with the six directrices. We identify two triangles of three directrices each. They are congruent and similar to the triangle of foci. We use distances (ξ_1, ξ_2, ξ_3) to the sides of one of them (d_{13}, d_{21}, d_{32}). After the calculation, $[\xi_1 : \xi_2 : \xi_3]$ can be reinterpreted as trilinear coordinates. We denote the eccentricity of the conic $\mathbf{K}_k$ by ε_k, the distances of the foci by $2f_k$, and the distance of the directrices by $2\kappa_k = 2f_k/\varepsilon_k^2$. The common chords of the three ellipses are

$$\varepsilon_1\xi_1 = \varepsilon_2(2\kappa_2 - \xi_2), \quad \varepsilon_2\xi_2 = \varepsilon_3(2\kappa_3 - \xi_3), \quad \varepsilon_3\xi_3 = \varepsilon_1(2\kappa_1 - \xi_1), \tag{8}$$

We obtain the intersection

$$\begin{aligned} \xi_1^S &= \kappa_1 + \varepsilon_1^{-1}(\kappa_2\varepsilon_2 - \kappa_3\varepsilon_3), \\ \xi_2^S &= \kappa_2 + \varepsilon_2^{-1}(\kappa_3\varepsilon_3 - \kappa_1\varepsilon_1), \\ \xi_3^S &= \kappa_3 + \varepsilon_3^{-1}(\kappa_1\varepsilon_1 - \kappa_2\varepsilon_2). \end{aligned} \tag{9}$$

The limit of collapsing ellipses is simply given by eccentricities equal one. In this case, the equations (8) for the chords reduce to the equations for the angular bisectors of the directrices. The common chords become the angular bisectors of the triangle $[D_1, D_2, D_3]$ approaching the focus triangle, too. The intersection of the chords approaches the incenter for eccentricities equal one.

We shall show, using the singular elements of pencils of conics, that in the general Cayley-Klein geometry, the limit of the intersection again approaches the incenter. More specifically, we shall show that the limit of the common chord of two conics sharing a focus is the angular bisector. We show the proof for the case of a regular absolute conic.

The shared foci are F_k, F_l of $[\mathbf{K}_m, \mathbf{C}]$, the diagonal points $D_m \sim \mathbf{k}_k F_m \times \mathbf{k}_l F_m \sim \mathbf{K}_k d_m \sim \mathbf{K}_l d_m$. The focal lines (i.e. the common chords) are the lines of the singular elements $\mathbf{n}_m \sim (D_m^\mathsf{T} \mathbf{k}_l D_m)\mathbf{k}_k - (D_m^\mathsf{T} \mathbf{k}_k D_m)\mathbf{k}_l$. As we are interested in the limit $\omega_m \to 0$, we fix $\lambda_m = 1$ in (4) and express the dual conics as

$$\mathbf{K}_m = \omega_m \mathbf{C} + F_k F_l^\mathsf{T} + F_l F_k^\mathsf{T}. \tag{10}$$

Proposition 5. *Given the dual conics (10). Then the the singular conics $\mathbf{n}_m$, given by (3), of the pencils $\mathcal{P}(\mathbf{k}_k, \mathbf{k}_l)$ are formed by the angular bisectors*

$$\begin{aligned} \mathbf{w}_m \quad \sim \quad & ((F_m \times F_k)^\mathsf{T} \mathbf{C} (F_m \times F_k))(F_m \times F_l)(F_m \times F_l)^\mathsf{T} \\ & -((F_m \times F_l)^\mathsf{T} \mathbf{C} (F_m \times F_l))(F_m \times F_k)(F_m \times F_k)^\mathsf{T}, \end{aligned} \tag{11}$$

i.e. $\mathbf{n}_m \sim \mathbf{w}_m$.

Proof. Step 1: $\mathbf{k}_m$ as inverse of $\mathbf{K}_m$ with suitable scaling. We normalize F_k such that $\gamma_k := F_k^\mathsf{T} \mathbf{c} F_k = \pm 1$ and write $p_k := \mathbf{c} F_k$, $\delta_m := F_k^\mathsf{T} \mathbf{c} F_l = F_k^\mathsf{T} p_l = F_l^\mathsf{T} p_k$. The coefficient ω_m in (10) is determined now. The limit of eccentricity 1 means $\omega_m \to 0$ here. The adjoint $\mathbf{k}_m$ to $\mathbf{K}_m$ is found as

$$\mathbf{k}_m = \omega_m \mathbf{K}_m^{-1} = \mathbf{c} + \frac{1}{(\mu_m^2 - \gamma_k \gamma_l)}(\gamma_l p_k p_k^\mathsf{T} + \gamma_k p_l p_l^\mathsf{T}) - \frac{\mu_m}{(\mu_m^2 - \gamma_k \gamma_l)}(p_l p_k^\mathsf{T} + p_k p_l^\mathsf{T}) \tag{12}$$

with $\mu_m = \omega_m + \delta_m$.

Step 2: Singular conics $\mathbf{n}_m$ of common chords. We use (3) and (5) from the proof of Prop. 2 to find

$$\mathbf{n}_m \sim \mathbf{k}_k - \mathbf{k}_l. \tag{13}$$

Step 3: $\mathbf{k}_m$ in the limit of eccentricity 1. We fix

$$\tilde{\mathbf{k}}_m = (F_k \times F_l)(F_k \times F_l)^\mathsf{T} \tag{14}$$

as representation of the singular conic given by the line through the foci F_k, F_l. Equation (12) yields

$$F_k^\mathsf{T} \mathbf{k}_m F_k = \frac{\gamma_k (\mu_m - \delta_m)^2}{\mu_m^2 - \gamma_k \gamma_l}, \qquad F_l^\mathsf{T} \mathbf{k}_m F_l \quad \text{the same,}$$

and finally

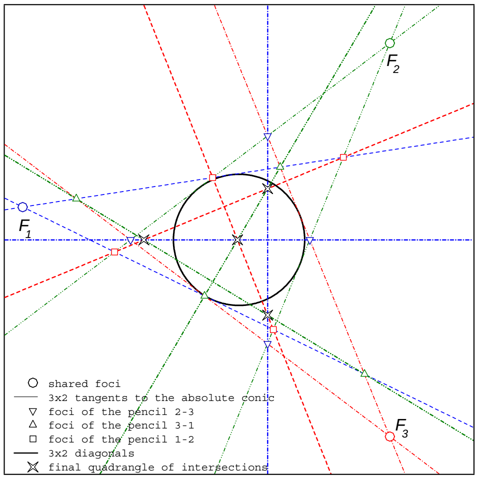
$$F_k^\mathsf{T} \mathbf{k}_m F_l = \frac{(\mu_m - \delta_m)(\mu_m \delta_m - \gamma_k \gamma_l)}{\mu_m^2 - \gamma_k \gamma_l}.$$

In the limit $\omega_m \to 0$, that is $\mu_m \to \delta_m$, these values vanish. Then both F_k and F_l are incident with the ($\mathbf{k}_m$)-polars of F_k and F_l. Consequently, the polar of any point on $F_k \times F_l$ is this line itself:

$$\lim_{\omega_m \to 0} \mathbf{k}_m \sim \tilde{\mathbf{k}}_m \tag{15}$$

Step 4: Comparing coefficients. At this point, we only have to check that the coefficients of $\mathbf{k}_k$ and $\mathbf{k}_l$ in (13) fit the coefficients in (11):

$$\lim_{\omega \to 0} (\mathbf{k}_k - \mathbf{k}_l) \overset{?}{\sim} ((F_m \times F_k)^\mathsf{T} \mathbf{C} (F_m \times F_k))\tilde{\mathbf{k}}_k - ((F_m \times F_l)^\mathsf{T} \mathbf{C} (F_m \times F_l))\tilde{\mathbf{k}}_l.$$



Each pair of foci generates a quadrilateral of tangents to the absolute conic. There are two other pairs of intersections, which are connected by the other two diagonals of the quadrilateral. These six diagonals are again edges of a quadrangle: It is the quadrangle of circumcenters of the focus triangle.

Figure 8. The quadrangle of diagonals of the set of the three pairs of foci.

Because of (15), it suffices to check one regular point. We choose $Q = \mathbf{C}(F_k \times F_l)$ and obtain

$$Q^\mathsf{T}\tilde{\mathbf{k}}_m Q = ((F_k \times F_l)^\mathsf{T} Q)^2, \qquad Q^\mathsf{T}\mathbf{k}_m Q = (F_k \times F_l)^\mathsf{T} Q$$

Therefore, indeed,

$$\begin{aligned} &\lim_{\omega\to 0}(\mathbf{k}_k - \mathbf{k}_l) \\ &= \frac{1}{(F_m \times F_l)^\mathsf{T}\mathbf{C}(F_m \times F_l)}\tilde{\mathbf{k}}_k - \frac{1}{(F_m \times F_k)^\mathsf{T}\mathbf{C}(F_m \times F_k)}\tilde{\mathbf{k}}_l \\ &\sim ((F_m \times F_k)^\mathsf{T}\mathbf{C}(F_m \times F_k))\tilde{\mathbf{k}}_k - ((F_m \times F_l)^\mathsf{T}\mathbf{C}(F_m \times F_l))\tilde{\mathbf{k}}_l. \end{aligned}$$

The singular conic of the diagonals is the singular conic of the angular bisectors, and the Neville center coincides with the incenter of the focus triangle. □

5. Trifles

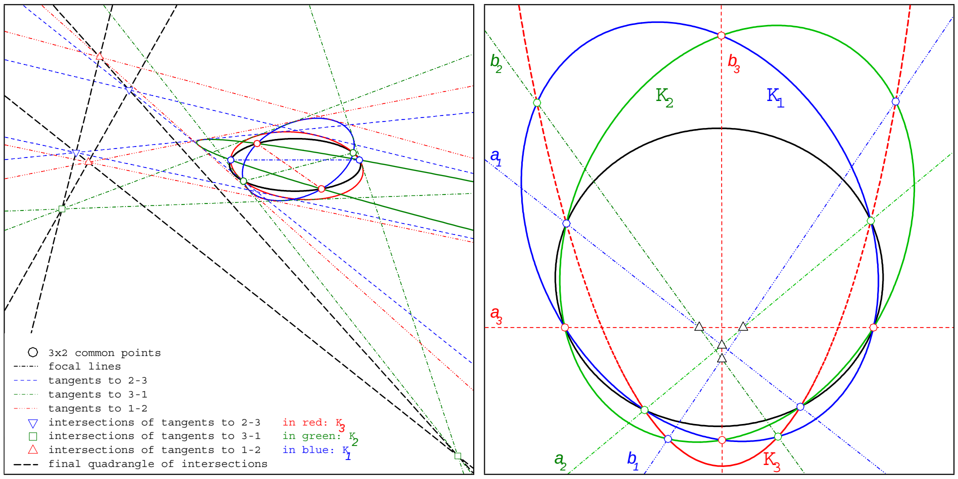
In the Euclidean setup, eq. (8), we obtain an incenter for the Neville point also in the case of equal eccentricities. The incenter is not that of the focus triangle, but of the directrix hexagon.

In the limit of infinite eccentricity, the Neville point approaches the circumcenter of the focus triangle. This circumcenter is one of the 60 Brianchon points of the tangent hexagon, which is generated by the three foci. More general: To each pair of foci belong two other pairs corresponding to the two other diagonals. These six diagonals are again edges of a quadrangle (Fig. 8).

The trilinear coordinates of the incenter of the triangle $[D_1, D_2, D_3]$ relative to the triangle $[U_1, U_2, U_3]$ in Fig. 7 are given by

$$N = [(a + b - c) : (b + c - a) : (c + a - b)]. \tag{16}$$

This seems to be a particular point [6] in the triangle $[U_1, U_2, U_3]$, but it comes in 32 different versions: The hexagon of directrices admits eight choices of directrix



Dual to the focus-sharing conics we obtain intersections of common tangents in a quadrilateral and connections of focal lines and their adjoints as edges of a quadrangle.

Figure 9. Three conics sharing focal lines.

triangles, and for each triangle a quadrangle of in- and excenters:

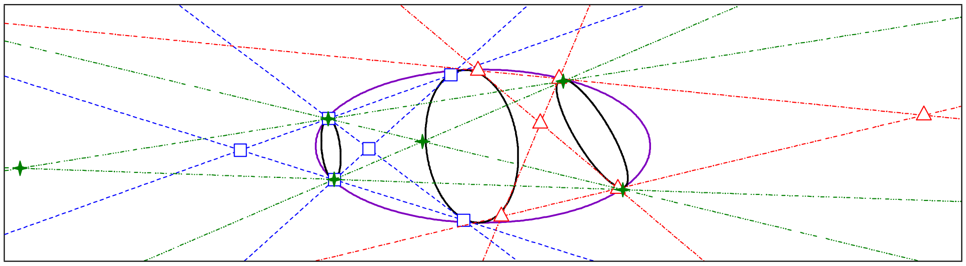
$$N = [(\pm a + s_\gamma b - s_\beta c) : (\pm b + s_\alpha c - s_\gamma a) : (\pm c + s_\beta a - s_\alpha b)] \qquad (17)$$

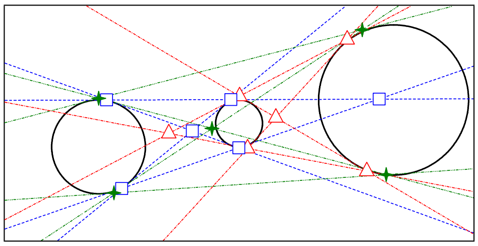
with $s_\alpha, s_\beta, s_\gamma = \pm 1$ and $s_\alpha s_\beta s_\gamma = 1$ for the quadrangle.

Since we calculate in projective spaces, the propositions of section 3 have dual counterparts. Instead of foci as intersections of tangents common with a fourth (absolute) conic, we consider the chords through intersections with this fourth (absolute) conic. We also call them focal lines to emphasize this duality. In Neville's problem, three pairs of tangents to the absolute conic generated three dual pencils from which three conics were taken. Now three pairs of points on the absolute conic generate three pencils from which three conics can be taken. In Neville's problem, each pair of the three conics had four intersections and three pairs of chords. Now each pair of three conics has four common tangents and three pairs of foci. In each pair of conics, one pair of chords intersected in the intersection of the polars of the shared focus. Now in each pair of conics, one pair of foci is collinear with the poles of the shared focal lines. In Neville's problem, the three chosen pairs of chords were the edges of a quadrangle. Now the three chosen pairs of foci are the vertices of a quadrilateral. This is the dual version of Prop. 2 (Fig. 9, left).

Each pair of three conics sharing a focal line defines an adjoint line (the partner in the pair to which the shared focal line belongs). Together with the shared focal lines we obtain six lines which are found to be the edges of a complete quadrangle. This is the dual version of Prop. 4 (Fig. 9, right), known in Euclidean notions as the four-conics theorem [5].

We conclude with a general remark on the relation between Euclidean and non-Euclidean constructions. Theorems about conics and lines only, without explicit





Three circles (in the upper part non-Euclidean, at left Euclidean) define three quadrilaterals of common tangents (differing in line style and point style). Choosing a vertex from two of these, some vertex of the third will be collinear with the two. The non-Euclidean view-point shows the connection of otherwise isolated theorems.

Figure 10. Collinearity between quadrilaterals of common tangents.

reference to symmetry and perpendiculars, can be interpreted as theorems in non-Euclidean geometry. Figure 6 is an example. One may state that if two common tangents of each pair of three conics touch a fourth conic, then the remaining common tangents of each pair intersect in three collinear points [5]. This formulation is purely Euclidean, but misses the non-Euclidean connection. In addition, Euclidean theorems about lines and circles should be expected to find simple non-Euclidean extensions by use of the non-Euclidean definitions of circles and perpendicularity. Figure 10 shows the connection between Monge's theorem, a dual three-conic theorem cited in [1, 5], and a Pascal-line construction cited in [12].

References

[1] A. Akopyan, On some classical constructions extended to hyperbolic geoometry, arXiv:1105.2153v1 math. MG, pages 1–12, 2011.

[2] A. V. Akopyan and A. A. Zaslawsky, *Geometry of conics*, volume 26 of Mathematical world. Providence, RI: American Mathematical Society (AMS), 2007.

[3] I. I. Bogdanov, Two theorems on the focus-sharing ellipses: a three-dimensional view, *J. Class. Geom.*, 1 (2012) 1–5.

[4] L. Crawford, Determination of the foci, directrices, axes and eccentricities of a conic whose equation is given with numerical coefficients, *Math. Gaz.*, 18 (1934) 43–46.

[5] C. J. A Evelyn, G. B. Money-Coutts, and J. A. Tyrrell, *The seven circles theorem and other new theorems*, London : Stacey, 1974.

[6] C. Kimberling, *Encyclopedia of triangle centers*, `http://faculty.evansville.edu/ck6/`.

[7] S. Liebscher, *Projektive Geometrie der Ebene. Ein klassischer Zugang mit interaktiver Visualisierung*, Heidelberg: Springer Spektrum, 2017.

[8] S. Liebscher and D.-E. Liebscher, The relativity of conics and circles, *Forum Geom.*, 18 (2018) 1–6.

[9] E. H. Neville, A focus-sharing set of three conics, *Math. Gaz.*, 20 (239) (1936) 182–183.

[10] E. H. Neville, The focus-sharing conics again, *Math. Gaz.*, 21 (242) (1937) 56.

[11] J. Richter-Gebert, *Perspectives on projective geometry. A guided tour through real and complex geometry*, Springer, Berlin, 2011.

[12] A. Skutin, Some new theorems in plane geometry, arXiv:1704.04923v3 math.MG, pages 1–19, 2017.

Stefan Liebscher: TNG Technology Consulting GmbH, Unterföhring, Germany
E-mail address: `geometry@stefan-liebscher.de`

Dierck-E. Liebscher: Leibniz-Institut für Astrophysik, Potsdam, Germany
E-mail address: `deliebscher@aip.de`

Forum Geometricorum
Volume 18 (2018) 431–433.

FORUM GEOM
ISSN 1534-1178

A Polynomial Approach to the "Bloom" of Thymaridas and the Apollonius' Circle

Gerasimos T. Soldatos

Abstract. A polynomial approach to the so-called "loom of Thymaridas" shows that it holds when it obeys a relationship that defines a circle of Apollonius.

Consider the following system of N linear equations in N unknowns:

$$\begin{aligned} x + x_1 + x_2 + \cdots + x_{N-1} &= C \\ x + x_1 &= C_1 \\ x + x_2 &= C_2 \\ &\vdots \\ x + x_{N-1} &= C_{N-1}. \end{aligned} \tag{1}$$

According to the "bloom or flower" of Thymaridas of Paros (c. 400-c. 350B.C.), the solution to this system is given by [1]:

$$x = \frac{C_1 + C_2 + \cdots + C_{N-1} - C}{N-2}. \tag{2}$$

In what follows, we show that this solution holds only if the coefficients of the quadratic polynomials whose roots enter the particular sums $(x + x_\nu)$, $\nu = 1,\ 2,\ \ldots,\ N-1$, obey a relationship that defines an Apollonius' circle as follows:

Let the left-hand side of (1) be an elementary symmetric function of degree 1 in terms of the roots of polynomial

$$\Pi(x) = a_N X^N + a_{N-1} X^{N-1} + \cdots + a_1 X + a_0$$

so that by Vieta's formulas:

$$C = -\frac{a_{N-1}}{a_N}. \tag{3}$$

Also, let any of the sums $x + x_\nu$ be the sum of the roots of quadratic polynomials $Q(X) = b_2^\nu X^2 + b_1^\nu X + b_0^\nu$ so that:

$$C_\nu = -\frac{b_1^\nu}{b_2^\nu}. \tag{4}$$

Publication Date: December 14, 2018. Communicating Editor: Paul Yiu.

Since x solves any two quadratic equations so that:

$$b_2^\nu x^2 + b_1^\nu x + b_0^\nu = 0 = b_2^m x^2 + b_1^m x + b_0^m$$

one obtains that:

$$(b_2^\nu - b_2^m)x^2 + (b_1^\nu - b_1^m)x + (b_0^\nu - b_0^m) = 0 \tag{5}$$

where $m = 1, 2, \ldots,\ N-1$, $m \neq \nu$. Inserting (3) and (4) in (2), and the result in (5), yields after some operations that:

$$\begin{aligned}(b_2^\nu - b_2^m)&\left[-\sum_{\nu=1}^{N-1}\frac{b_1^\nu}{b_2^\nu} - \left(-\frac{a_{N-1}}{a_N}\right)\right]^2\\ &+ (b_1^\nu - b_1^m)\left[-\sum_{\nu=1}^{N-1}\frac{b_1^\nu}{b_2^\nu} - \left(-\frac{a_{N-1}}{a_N}\right)\right](N-2)\\ &+ (b_0^\nu - b_0^m)(N-2)^2\\ &= 0.\end{aligned} \tag{6}$$

or, in shorthand notation:

$$B_2(A-S)^2 + B_1 S(N-2) + B_0(N-2)^2 = 0 \tag{6$'$}$$

where:

$$B_2 \equiv (b_2^\nu - b_2^m), \qquad B_1 \equiv (b_1^\nu - b_1^m), \qquad B_0 \equiv (b_0^\nu - b_0^m),$$
$$A \equiv \frac{a_{N-1}}{a_N} \quad \text{and} \quad S \equiv \sum_\nu \frac{b_1^\nu}{b_2^\nu}.$$

Treating (6′) as a quadratic equation in A:

$$B_2A^2 + [B_1(N-2) - 2B_2S]A + [B_2S^2 - B_1(N-2)S + B_0(N-2)^2] = 0$$

which has to have a unique solution and hence, a zero discriminant, one obtains after the necessary operations that:

$$A \equiv \frac{a_{N-1}}{a_N} = \frac{2B_2S - B_1(N-2)}{2B_2} = S - (N-2)\frac{B_1}{2B_2} \tag{7}$$

and

$$B_1^2 = 4B_2B_0. \tag{8}$$

When (8) is inserted in (7):

$$A \equiv \frac{a_{N-1}}{a_N} = S - (N-2)\sqrt{\frac{B_0}{B_2}} \tag{7$'$}$$

Given now that (2) may be rewritten in shorthand notation as follows:

$$x = \frac{A-S}{N-2} \tag{2$'$}$$

inserting (7) in (2'), yields:

$$x = \frac{S - (N-2)\frac{B_1}{2B_2} - S}{N-2} = -\frac{B_1}{2B_2} \equiv \frac{b_1^m - b_1^\nu}{2(b_2^\nu - b_2^m)} \tag{9}$$

or, inserting (7') in (2):

$$x = -\sqrt{\frac{B_0}{B_2}} \equiv \sqrt{\frac{b_0^m - b_0^\nu}{b_2^\nu - b_2^m}} \qquad (9')$$

The value of x has to be unique and hence, the ratio (8) or (8') between any two pairs of equations has to be the same. Geometrically, given any two points D and F along a straight line, we have a constant ratio x of varying distances from these points, distances like $(b_1^m - b_1^\nu) = PF$ and $2(b_2^\nu - b_2^m) = PD$, defining the set of points P in Figure 1, that is an Apollonius' circle centered at point O; and the set of quadratic equations that satisfy (8) or (8') is equal to the set of these points. Thymaridas' bloom holds only in this connection. And, when it does, one needs to consider the coefficients of any two quadratic equations to have a value for x.

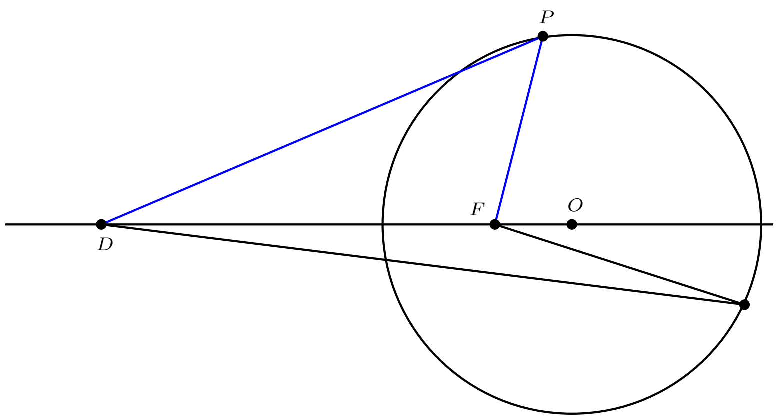


Figure 1. Thymaridas' bloom and Apollonius' circle

In sum, the bloom of Thymaridas is one way to put the algebra of the circle of Apollonius.

Reference.

[1] T. L. Heath, "The ('Bloom') of Thymaridas", in *A History of Greek Mathematics*, pp. 94–96, Clarendon Press, Oxford, 1921.

Gerasimos T. Soldatos: Taburishanska 73, 27505 Svitlovodsk, Ukraine
E-mail address: soldgera@yahoo.com

Forum Geometricorum
Volume 18 (2018) 435–437.

FORUM GEOM
ISSN 1534-1178

Author Index