Perspective Fields: Concepts and Calculations

Addition of Points

The homogeneous coordinates of points, expressed in barycentric coordinates can be added to yield a new point. The addition of points is defined as follows:

$$P_1(x_1, y_1, z_1) + P_2(x_2, y_2, z_2) = P(x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

Points can also be weighted by introducing coefficients, as shown below:

$$cf_1 \bullet P_1(x_1, y_1, z_1) + cf_2 \bullet P_2(x_2, y_2, z_2) = P(cf_1 \bullet x_1 + cf_2 \bullet x_2,\ cf_1 \bullet y_1 + cf_2 \bullet y_2,\ cf_1 \bullet z_1 + cf_2 \bullet z_2).$$

Compliance Factors

When adding three points (P_1, P_2, P_3) using coefficients to create a fourth point $P_4(x_4, y_4, z_4)$, the expression is:

$$cf_1 \bullet P_1(x_1, y_1, z_1) + cf_2 \bullet P_2(x_2, y_2, z_2) + cf_3 \bullet P_3(x_3, y_3, z_3) = P_4(x_4, y_4, z_4). \qquad (1)$$

This results in the following equations:

$$\begin{aligned} cf_1 \bullet x_1 + cf_2 \bullet x_2 + cf_3 \bullet x_3 &= x_4 \\ cf_1 \bullet y_1 + cf_2 \bullet y_2 + cf_3 \bullet y_3 &= y_4 \\ cf_1 \bullet z_1 + cf_2 \bullet z_2 + cf_3 \bullet z_3 &= z_4. \end{aligned} \qquad (2)$$

These are three equations with three unknowns.
Using Cramer's rule, the coefficients (cf_1, cf_2, cf_3) can be solved as follows:

$$\begin{aligned} cf_1 &= \frac{\mathrm{Det}[P_4, P_2, P_3]}{\mathrm{Det}[P_1, P_2, P_3]} \\ cf_2 &= \frac{\mathrm{Det}[P_1, P_4, P_3]}{\mathrm{Det}[P_1, P_2, P_3]} \\ \mathrm{cf}_3 &= \frac{\mathrm{Det}[\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_4]}{\mathrm{Det}[P_1, P_2, P_3]}. \end{aligned} \qquad (3)$$

Here, $\mathrm{Det}[P_1, P_2, P_3]$ is the determinant of a 3x3 matrix whose rows are the barycentric coordinates of $P_1, P_2, and\ P_3$.

The coefficients (cf_1, cf_2, cf_3) depend on the specific representations of the points and are therefore called *compliance factors*.

Substituting (3) in (1) yields the following expression:

$$\mathrm{Det}[P_1, P_2, P_3] \bullet P_4 = \mathrm{Det}[P_4, P_2, P_3] \bullet P_1 + \mathrm{Det}[P_1, P_4, P_3] \bullet P_2 + \mathrm{Det}[P_1, P_2, P_4] \cdot P_3. \qquad (4)$$

Since homogeneous coordinates are used, the factor for target point P_4 can be omitted, simplifying the expression to:

$$P_4 = cf1 \bullet P_1 + cf2 \bullet P_2 + cf3 \bullet P_3, \qquad (5)$$

where

$$cf_1 = \mathrm{Det}[P_4, P_2, P_3],\ \ cf_2 = \mathrm{Det}[P_1, P_4, P_3],\ \ cf_3 = \mathrm{Det}[P_1, P_2, P_4]. \qquad (6)$$

Conclusion: P4 can be expressed as the sum of P1, P2, and P3 using compliance factors cf1, cf2, and cf3.

Starting Point
Given: $P_4 = cf1 \bullet P_1 + cf2 \bullet P_2 + cf3 \bullet P_3$.
Let Px be a given point.
Can it be expressed in the form: $P_x = n1 \bullet cf1 \bullet P_1 + n2 \bullet cf2 \bullet P_2 + n3 \bullet cf3 \bullet P_3$?
And how can the coefficients (n1,n2,n3) be determined?

Solution
This can again be solved using *Cramer's rule* resulting in:

$n1 \bullet cf1 = Det[Px,P2,P3]$
$n2 \bullet cf2 = Det[P1,Px,P3]$
$n3 \bullet cf3 = Det[P1,P2,Px]$.

Substituting the expressions for the compliance factors cf1, cf2, and cf3 from (6) yields:

$$n1 = Det[Px,P2,P3] \,/\, Det[P4,P2,P3]$$
$$n2 = Det[P1,Px,P3] \,/\, Det[P1,P4,P3] \qquad (7)$$
$$n3 = Det[P1,P2,Px] \,/\, Det[P1,P2,P4].$$

Result
Given that $P_4 = cf1 \bullet P_1 + cf2 \bullet P_2 + cf3 \bullet P_3$, other points Px can be expressed as the weighted sum of three points (P1,P2,P3) as follows:

$$P_x = n1 \bullet cf1 \bullet P_1 + n2 \bullet cf2 \bullet P_2 + n3 \bullet cf3 \bullet P_3.$$

The elements (cf1,cf2,cf3) and (n1,n2,n3) can be computed from the coordinates of P1, P2, P3, P4, and Px using the formulas (6) and (7).

Definition: Perspective Field
A Perspective Field is a set of points Px, that satisfy:

$P_x = n1 \bullet cf1 \bullet P_1 + n2 \bullet cf2 \bullet P_2 + n3 \bullet cf3 \bullet P_3$, where:

cf1=Det[P4,P2,P3], cf2=Det[P1,P4,P3], cf3=Det[P1,P2,P4]
and (n1, n2, n3) is a set of three numerical values.

Thus, a Perspective Field is defined by 3 **primary points** (P1, P2, P3) and a 4th **secondary point** P4, used to calculate the compliance factors (cf1, cf2, cf3). It is denoted as: **PF[P1, P2, P3; P4].**
The semicolon indicates that P4 is not part of the set (P1, P2, P3), but is essential for defining the field.

Note: The values (n1, n2, n3) are not always numerical. When they are not numerical, but expressions of (a,b,c), then they can be used to convert coordinates relating to triangle ABC to coordinates relating to triangle P1P2P3 – thus establishing a *reference change*.

Main Transformations used in Perspective Fields

Barycentric to Perspective Coordinates
To calculate the perspective coordinates, we use the transformation *BCtoPC*, meaning Barycentric Coordinates to Perspective Coordinates (Barycentric wrt ABC, Perspective wrt P1P2P3).
Let P1, P2, P3, P4, and Px be points given in barycentric coordinates. Then:

$$\boldsymbol{BCtoPC}[\boldsymbol{P1},\boldsymbol{P2},\boldsymbol{P3},\boldsymbol{P4},\boldsymbol{Px}] = \left(\frac{\det(P_x,P_2,P_3)}{\det(P_4,P_2,P_3)} : \frac{\det(P_1,P_x,P_3)}{\det(P_1,P_4,P_3)} : \frac{\det(P_1,P_2,P_x)}{\det(P_1,P_2,P_4)}\right)$$

This gives the perspective coordinates of P_x with respect to the Perspective Field $[P1,P2,P3;P4]$.

A particularly useful case of this transformation occurs when $P4$ is the centroid of the triangle $(P1,P2,P3)$. In this case, **BCtoPC** becomes:

$$\boldsymbol{BCtoPC4G}[\boldsymbol{P1},\boldsymbol{P2},\boldsymbol{P3},\boldsymbol{Px}] = \left(\det(P_x,P_2,P_3)\bullet(p1_x+p1_y+p1_z) : \det(P_1,P_x,P_3)\bullet(p2_x+p2_y+p2_z) : \det(P_1,P_2,P_x)\bullet(p3_x+p3_y+p3_z)\right)$$

assuming $P1=(p1_x : p1_y : p1_z),\ P2=(p2_x : p2_y : p2_z),\ and\ P3=(p3_x : p3_y : p3_z)$.

This provides the perspective coordinates of P_x with respect to the Perspective Field $[P1,P2,P3;Centroid(P1,P2,P3)]$.
It represents the designation of an ETC point expressed in coordinates relative to △ABC as it functions with respect to △P1P2P3.

Note: The coordinates are usually expressed in terms of (a, b, c), representing the lengths of the sides of the reference triangle ABC. However, since perspective coordinates involve expressions with the side lengths of triangle P1P2P3, it is generally also necessary to convert (a, b, c) into (a1,b1,c1).

Perspective to Barycentric Coordinates
To calculate barycentric coordinates from perspective coordinates, we use the transformation **PCtoBC** (Perspective Coordinates to Barycentric Coordinates):

$$\boldsymbol{PCtoBC}[\boldsymbol{P1},\boldsymbol{P2},\boldsymbol{P3},\boldsymbol{P4},\{\boldsymbol{n1},\boldsymbol{n2},\boldsymbol{n3}\}] =$$

$$n1\bullet\det(P4,P2,P3)\bullet P1 + n2\bullet\det(P1,P4,P3)\bullet P2 + n3\bullet\det(P1,P2,P4)\bullet P3$$

where {n1,n2,n3} represents the point in perspective coordinates to be transformed.
For the special case where $P4$ is the centroid, **PCtoBC** simplifies to:

$$\boldsymbol{PCtoBC4G}[\boldsymbol{P1},\boldsymbol{P2},\boldsymbol{P3},\boldsymbol{P4},\{\boldsymbol{n1},\boldsymbol{n2},\boldsymbol{n3}\}] =$$

$$\frac{n1\bullet P1}{(p1x+p1y+p1z)}+\frac{n2\bullet P2}{(p2x+p2y+p2z)}+\frac{n3\bullet P3}{(p3y+p3y+p3z)}$$

assuming $P1=(p1_x{:}p1_y{:}p1_z),\ P2=(p2_x{:}p2_y{:}p2_z), and\ P3=(p3_x{:}p3_y{:}p3_z)$ are points in barycentric coordinates wrt △ABC and {n1,n2,n3} is the point in perspective coordinates to be transformed.
This represents the designation of an ETC point expressed in coordinates relative to △P1P2P3 as it functions with respect to △ABC.

Note: The coordinates of the outcome are typically expressed in terms of (a1, b1, c1), which represent the lengths of the sides of the triangle P1P2P3. However, since barycentric coordinates involve expressions with the side lengths of reference triangle ABC, it is generally also necessary to convert (a1, b1, c1) into (a,b,c).

Main Transformations used in Perspective Linear Sets

In Perspective Fields often several points are found on lines. They also have a perspective setting with transformations and parameters that share similar structures.

The similar Transformations are as follows:

1. **BCtoPCpls Transformation:**

   BCtoPCpls[P1,P2,P3,Px] = BCtoPC[P1,P2,NCP,P3,Px],
   where NCP = the Non-Collinear Point, meaning that it is *not collinear* with P1 and P2.
   For simpler calculations, this can be chosen as a point in its simplest form, such as {1,1,1}, unless this point happens to be collinear with P1 and P2. In such cases, an alternative like {1,2,1} should be selected.
   - The outcome will be the perspective coordinates of Px in the format {n1,n2,0}.
   - The coordinates relating to P1 and P2 will be {n1,n2}.

2. **BCtoPC4Gpls Transformation:**

   BCtoPC4Gpls[P1,P2,Px] = BCtoPC4G[P1,P2,NCP,Px],

3. **PCtoBCpls Transformation:**

   PCtoBCpls[P1,P2,P3,{n1,n2,0}] = PCtoBC[P1,P2,NCP,P3,(n1,n2,0}],

4. **PCtoBC4Gpls Transformation:**

   PCtoBC4Gpls[P1,P2,{n1,n2,0}] = BCtoPC4G[P1,P2,NCP,{n1,n2,0}]

In all cases, the use of NCP (Non-Collinear Point) is essential to ensure the transformations are well-defined and consistent.

Examples

Example 1

The vertices of the Incentral Triangle IT are ITa = $(0, b, c)$, ITb = (a, 0, c), ITc = (a, b, 0).

X3 = $(a^2\ SA,\ \ b^2 SB,\ \ c^2 SC)$

$X8143 =$

$(a(a^5b + a^4b^2 - 2a^3b^3 - 2a^2b^4 + ab^5 + b^6 + a^5c - 3a^3b^2c - 2a^2b^3c + 2ab^4c + 2b^5c + a^4c^2 - 3a^3bc^2 - 2a^2b^2c^2 - 3ab^3c^2 - b^4c^2 - 2a^3c^3 - 2a^2bc^3 - 3ab^2c^3 - 4b^3c^3 - 2a^2c^4 + 2abc^4 - b^2c^4 + ac^5 + 2bc^5 + c^6),$
$b(a^6 + a^5b - 2a^4b^2 - 2a^3b^3 + a^2b^4 + ab^5 + 2a^5c + 2a^4bc - 2a^3b^2c - 3a^2b^3c + b^5c - a^4c^2 - 3a^3bc^2 - 2a^2b^2c^2 - 3ab^3c^2 + b^4c^2 - 4a^3c^3 - 3a^2bc^3 - 2ab^2c^3 - 2b^3c^3 - a^2c^4 + 2abc^4 - 2b^2c^4 + 2ac^5 + bc^5 + c^6),$
$c(a^6 + 2a^5b - a^4b^2 - 4a^3b^3 - a^2b^4 + 2ab^5 + b^6 + a^5c + 2a^4bc - 3a^3b^2c - 3a^2b^3c + 2ab^4c + b^5c - 2a^4c^2 - 2a^3bc^2 - 2a^2b^2c^2 - 2ab^3c^2 - 2b^4c^2 - 2a^3c^3 - 3a^2bc^3 - 3ab^2c^3 - 2b^3c^3 + a^2c^4 + b^2c^4 + ac^5 + bc^5))$

***Question*: Which point in △ABC, corresponds to the circumcenter X3 of △IT?**
Solution:

1. In the expression for X3, the side lengths (a,b,c) from △ABC must be converted to side lengths (ai,bi,ci) from △IT, denoted as X3 -> X3t.
2. **PCtoBC4G** [ITa, ITb, ITc, X3t] yields X8143.

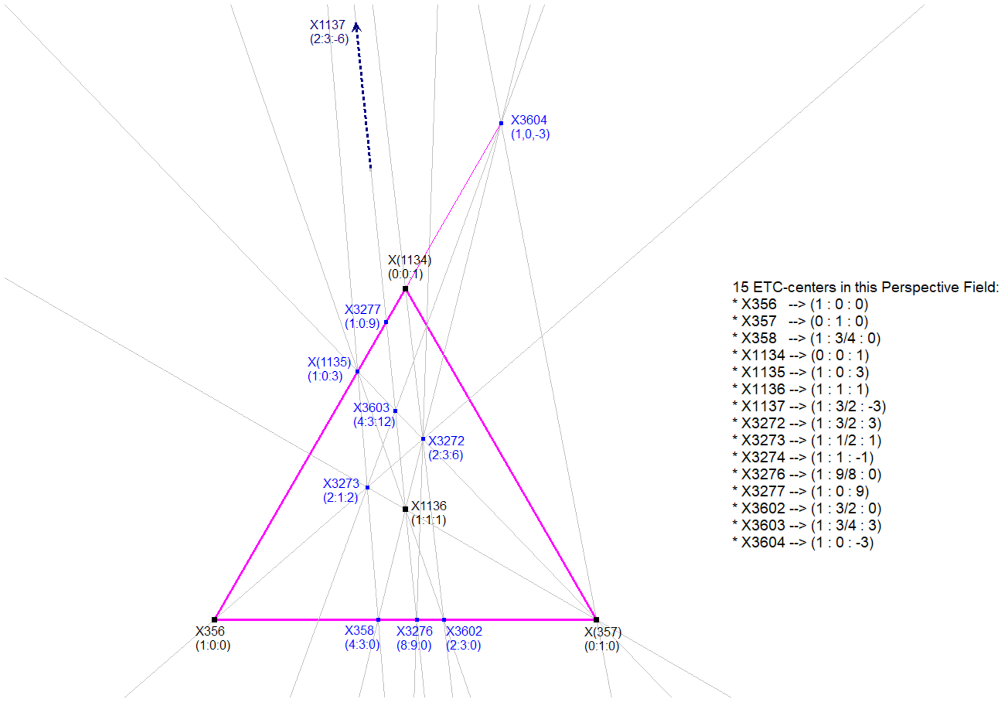
***Question*: The point X8143 of △ABC, corresponds to which point relative to △IT?**
Solution:

1. **BCtoPC4G** [ITa, ITb, ITc, X8143] yields X3t.
2. X3 = X3t, where side lengths (ai,bi,ci) are converted from triangle IT to side lengths (a,b,c) of triangle ABC.

Here, in both cases, ITa, ITb, ITc, X3, and X8143 are all expressed in barycentric coordinates with respect to the reference triangle △ABC.

In summary: X8143 wrt △ABC = X3 wrt △IT.

Example 2



Above picture illustrates the collinearities of some Morley-related points, expressed in a normalized perspective.
The Perspective Field shown is defined as [X356,X357,X1134; X1136], meaning:

X1136 = cf1.X356 + cf2.X357 + cf3.X1134

Any point Xi in this field can be expressed as:

Xi = n1.cf1.X356 + n2.cf2.X357 + n3.cf3.X1134,

where (n1,n2,n3) are the perspective coordinates.
The compliance factors (cf1,cf2,cf3) are calculated as:

cf1 = -12
cf2 = 32 Cos[A/3] Cos[B/3] Cos[C/3]
cf3 = 1 - 2 Cos[2A/3] - 2 Cos[2B/3] - 2 Cos[2C/3]}

PF-Transformations

BCtoPC [X356, X357, X1134, X1136, X3272] = {2,3,6}
PCtoBC [X356, X357, X1134, X1136, {2,3,6}] = X3271

Example 3

Let P1=X(13), P2=X(2) and P3=X(6).

The choice of P4 determines the values of the compliance factors (cf1, cf2, cf3). For P4=X(15), the calculations are as follows:

$cf1 = 1$

$cf2 = (a-b-c)(a+b-c)(a-b+c)(a+b+c)$

$cf3 = -\sqrt{3}\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}$

Thus, we know that

X(15) = cf1.X(13) + cf2.X(2) + cf3.X(6).

To calculate other points in the Perspective Field, we search for points that satisfy the equation:

Xi = n1.cf1.X(13) + n2.cf2.X(2) + n3.cf3.X(6),

where (n1,n2,n3) are the numeric perspective coordinates.

PF-Transformations

Using the transformation **BCtoPC**, we calculate perspective coordinates for various points as follows:

BCtoPC [X13, X2, X6, X15, X2] = {0,1,0}
BCtoPC [X13, X2, X6, X15, X3] = {2,2,1}
BCtoPC [X13, X2, X6, X15, X4] = {4,1,2}
BCtoPC [X13, X2, X6, X15, X5] = {2,-1,1}
BCtoPC [X13, X2, X6, X15, X6] = {0,0,1}
BCtoPC [X13, X2, X6, X15, X13] = {1,0,0}
BCtoPC [X13, X2, X6, X15, X14] = {1,0,1}
BCtoPC [X13, X2, X6, X15, X15] = {1,1,1}
BCtoPC [X13, X2, X6, X15, X16] = {1,1,0}
BCtoPC [X13, X2, X6, X15, X17] = {-1,2,1}
BCtoPC [X13, X2, X6, X15, X18] = {1,-2,2}
BCtoPC [X13, X2, X6, X15, X20] = {8,5,4}
BCtoPC [X13, X2, X6, X15, X30] = {2,1,1}
BCtoPC [X13, X2, X6, X15, X61] = {1,1,2}
BCtoPC [X13, X2, X6, X15, X62] = {1,1,-1}
BCtoPC [X13, X2, X6, X15, X140] = {2,5,1}
BCtoPC [X13, X2, X6, X15, X371] = {6,6, $\sqrt{3}(3+\sqrt{3})$}
BCtoPC [X13, X2, X6, X15, X372] = {6,6, $\sqrt{3}(3-\sqrt{3})$}
BCtoPC [X13, X2, X6, X15, X376] = {4,3,2}
and many more.

Points that belong to the Perspective Field have fixed numeric coordinates.

Points that do not be belong to the Perspective Field have coordinates in terms of (a,b,c), indicating that they are not fixed points in the field.

For example:

BCtoPC [X13, X2, X6, X15, X21] =

$\{ 2(a-b-c)(a+b-c)(a-b+c),$
$(2a^3-2a^2b-2ab^2+2b^3-2a^2c+abc-2b^2c-2ac^2-2bc^2+2c^3),$
$(a-b-c)(a+b-c)(a-b+c) \}$

To find the point in the field with perspective coordinates (1:2:3) we apply:

PCtoBC [X13, X2, X6, X15, {1,2,3}] =

$\{\sqrt{3}\,(5a^4-7a^2b^2+2b^4-7a^2c^2-4b^2c^2+2c^4)-60\,a^2\Delta,$
$\sqrt{3}\,(2a^4-7a^2b^2+5b^4-4a^2c^2-7b^2c^2+2c^4)-60\,b^2\Delta,$
$\sqrt{3}\,(2a^4-4a^2b^2+2b^4-7a^2c^2-7b^2c^2+5c^4)-60\,c^2\Delta\}$

Example 4
Perspective Fields also have their implications for sets of points on lines.
Just as the Triangular Plane contains many Perspective Fields, each line in the Triangular Plane contains many **Perspective Linear Sets**.
A Perspective Field is defined by [Ax, Bx, Cx; Px], whereas a Perspective Linear Set is defined by [Ax, Bx; Px].

Main Perspective Linear Set on the Euler Line:

0. **PERSPECTIVE LINEAR SET** [X2, X4; X3]
   Compliance Factors: {cf1, cf2} = {(a - b - c)(a + b - c)(a - b + c)(a + b + c), 4}
   Perspective Coordinates can be calculated using **BCtoPCpls** [X2, X4, X3, Xi].
   Points and their perspective linear coordinates:
   - X2 → {1, 0}
   - X3 → {1, 1}
   - X4 → {0, 1}
   - X5 → {1, -1}
   - X20 → {1, 2}
   - X140 → {3, 1}
   - X376 → {2, 3}
   - X381 → {1, -3}
   - X382 → {1, 5}
   - X546 → {1, -5}
   - X547 → {7, -3}
   - X548 → {5, 7}
   - X549 → {5, 3}
   - X550 → {3, 5}
   - X631 → {2, 1}
   - X632 → {7, 1}
   - X1656 → {3, -1}
   - X1657 → {3, 7}
   - X2041 → {$-1 + \sqrt{3}$, $1 + \sqrt{3}$}
   - X2042 → {$1 + \sqrt{3}$, $-1 + \sqrt{3}$}
   - X2043 → {$1 + \sqrt{3}$, $3 + \sqrt{3}$}
   - X2044 → {$-1 + \sqrt{3}$, $-3 + \sqrt{3}$}
   - X2045 → {$3 - \sqrt{3}$, $-1 + \sqrt{3}$}
   - X2046 → {$3 + \sqrt{3}$, $-1 - \sqrt{3}$}
   - X2675 → {$-4(-1 + \sqrt{5})(7 + \sqrt{5})$, 128}
   - X2676 → {$(1 + \sqrt{5})^4$, -64}
   - X3090 → {-2, 1}
   - X3091 → {1, -2}
   - X3146 → {1, 4}

Other Perspective Linear Sets on the Euler Line:

1. **PERSPECTIVE LINEAR SET** [X4, X5; X235]
   Compliance Factors: {cf1, cf2} = {$8a^2b^2c^2$, $(a^2 - b^2 - c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)$}
   Perspective Coordinates can be calculated using **BCtoPCpls** [X4, X5, X235, Xi].
   Points and their perspective linear coordinates:
   - X4 → {1, 0}
   - X5 → {0, 1}
   - X235 → {1, 1}
   - X403 → {1, 2}
   - X427 → {-1, 1}
   - X1594 → {-1, 2}
   - X1595 → {-2, 1}
   - X1596 → {2, 1}
   - X1906 → {3, 1}
   - X1907 → {3, -1}

2. **PERSPECTIVE LINEAR SET** [X3, X4; X24]
   Compliance Factors: {cf1, cf2} = $\{(a^2 - b^2 - c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2), 4a^2b^2c^2\}$
   Perspective Coordinates can be calculated using **BCtoPCpls** [X3, X4, X24, Xi].
   Points and their perspective linear coordinates:
   - X3 → {1, 0}
   - X4 → {0, 1}
   - X24 → {1, 1}
   - X25 → {1, 2}
   - X186 → {2, 1}
   - X378 → {-1, 1}
   - X1593 → {1, -2}
   - X1597 → {1, -4}
   - X1598 → {1, 4}

3. **PERSPECTIVE LINEAR SET** [X2, X4; X470]
   Compliance Factors:
   {cf1, cf2} = $\{\sqrt{3}(a^2 - b^2 - c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2), -4\sqrt{(a + b - c)(a - b + c)(-a + b + c)(a + b + c)}\}$
   Perspective Coordinates can be calculated using **BCtoPCpls** [X2, X4, X470, Xi].
   Points and their perspective linear coordinates:
   - X2 → {1, 0}
   - X4 → {0, 1}
   - X470 → {1, 1}
   - X471 → {1, -1}
   - X472 → {-1, 3}
   - X473 → {1, 3}
   - X1585 → {1, $\sqrt{3}$}
   - X1586 → {1, -$\sqrt{3}$}

There are several more Perspective Linear Sets on the Euler Line.
In fact, lines other than the Euler Line also often contain multiple Perspective Linear Sets.

Example 5
Perspective Linear Sets and Soddy Circles
Next Perspective Linear Sets shows that sets of points in a Perspective Field or Linear Set often a general theme through their corresponding coordinates address. In this case, all points are related to the *Soddy Circles*.

PERSPECTIVE LINEAR SET [X1, X1371; X176]
Compliance Factors:
{cf1, cf2} = {1, 3}
Perspective Coordinates can be calculated using: **BCtoPCpls** [X1, X1371, X176, Xi].
Points and their Perspective Linear Coordinates:

- X1 → {1, 0},
- X176 → {1, 1},
- X1371 → {0, 1},
- X482 → {1, -1},
- X1373 → {2, -1},
- X7 → {3, -1},
- X1374 → {4, -1}
- X481 → {5, -1},
- X1372 → {6, -1},
- X175 → {7, -1}.

In next picture it can be seen how this works out: