Dynamic Perspective Fields

The hidden structure in Triangle Geometry

Part 1 – Synthetic Approach

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Dynamic Perspective Fields

The hidden structure in Triangle Geometry

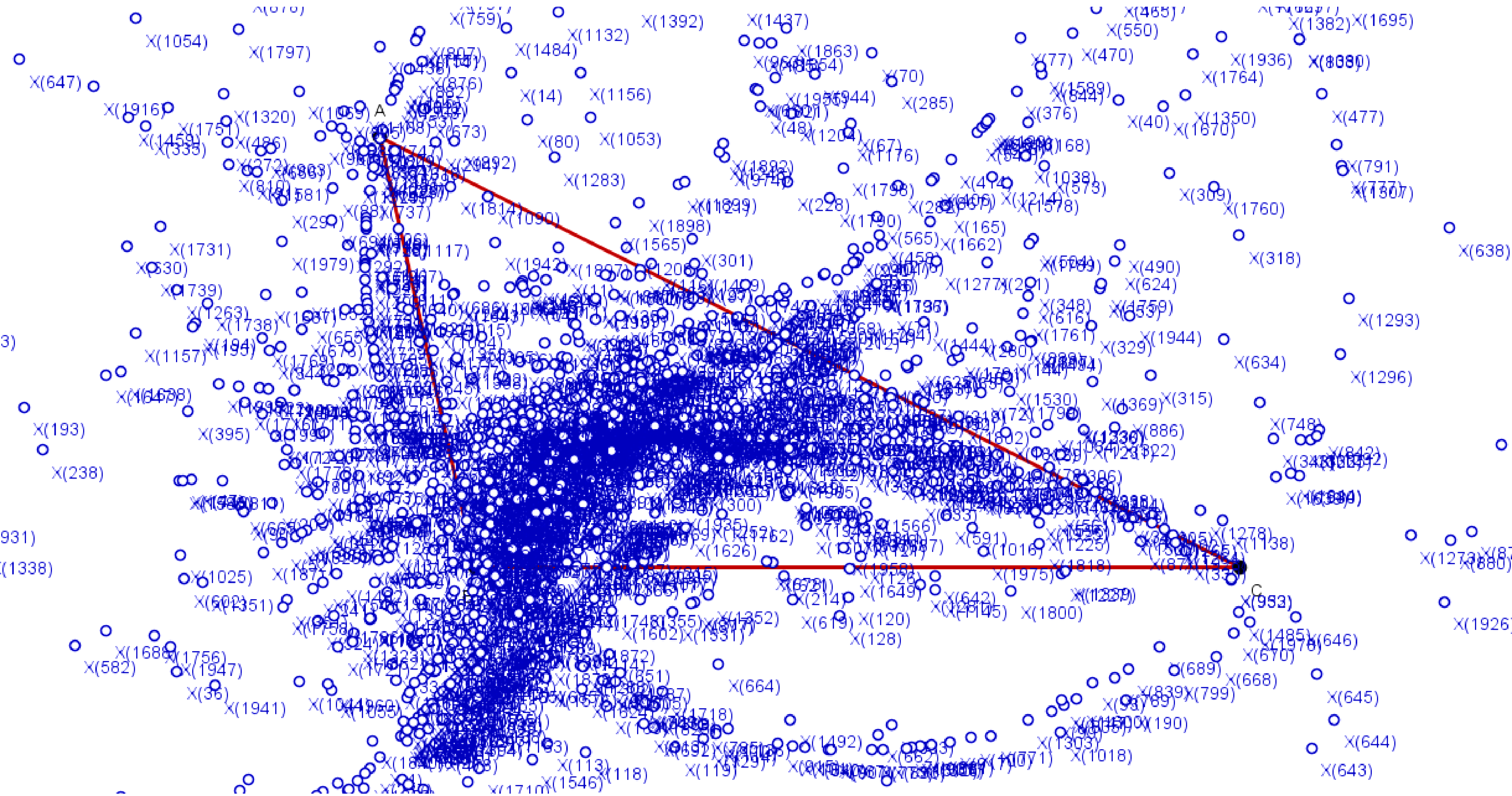
- This presentation is the report of a quest.
  A quest in which I have been searching for explicitness about the wonderful structure in Triangle Geometry.

- It reveals the cause for collinearities and many other regularities in Triangle Geometry.

- Also it unravels the composition and structure in trilinear and barycentric coordinates.

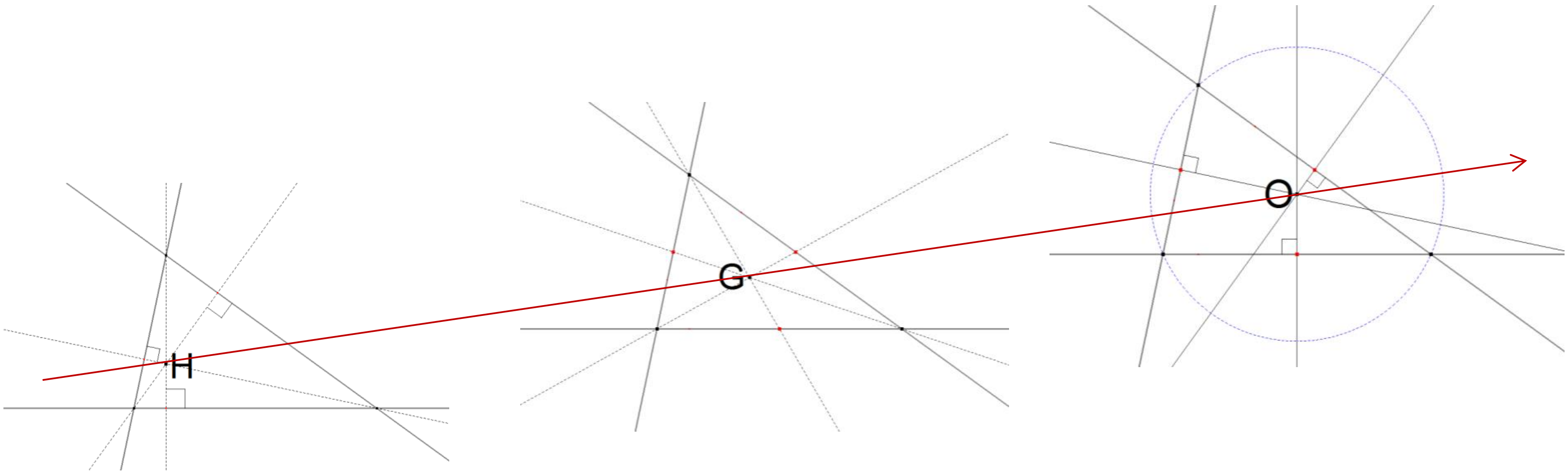
Let's get started . . . .

There are many many points known in Triangle Geometry



More than 3.500 points are registered.

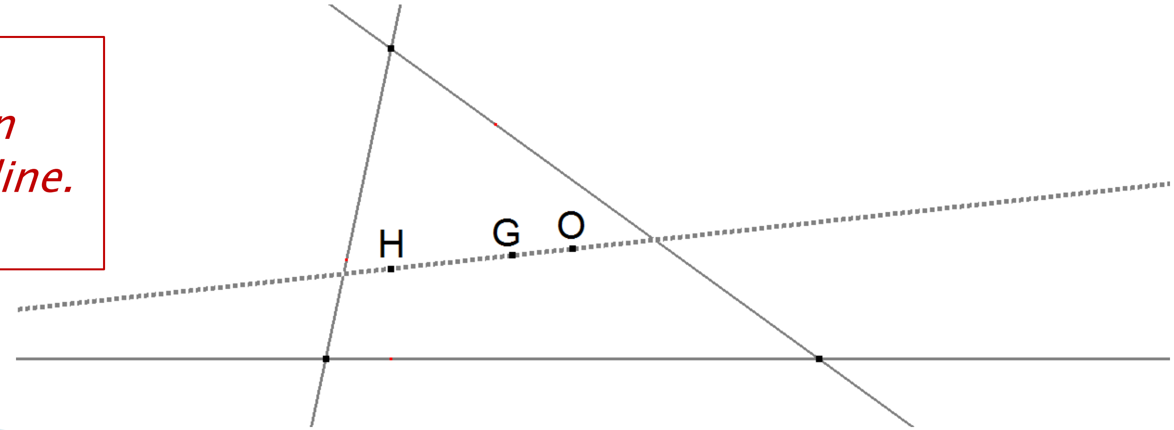
And there are many collinearities in Triangle Geometry.



For Example:

- Orthocenter H, Centroid G and Circumcenter O are always lined up.
- There is a proof for this well known property. But is it trivial ? Why ?

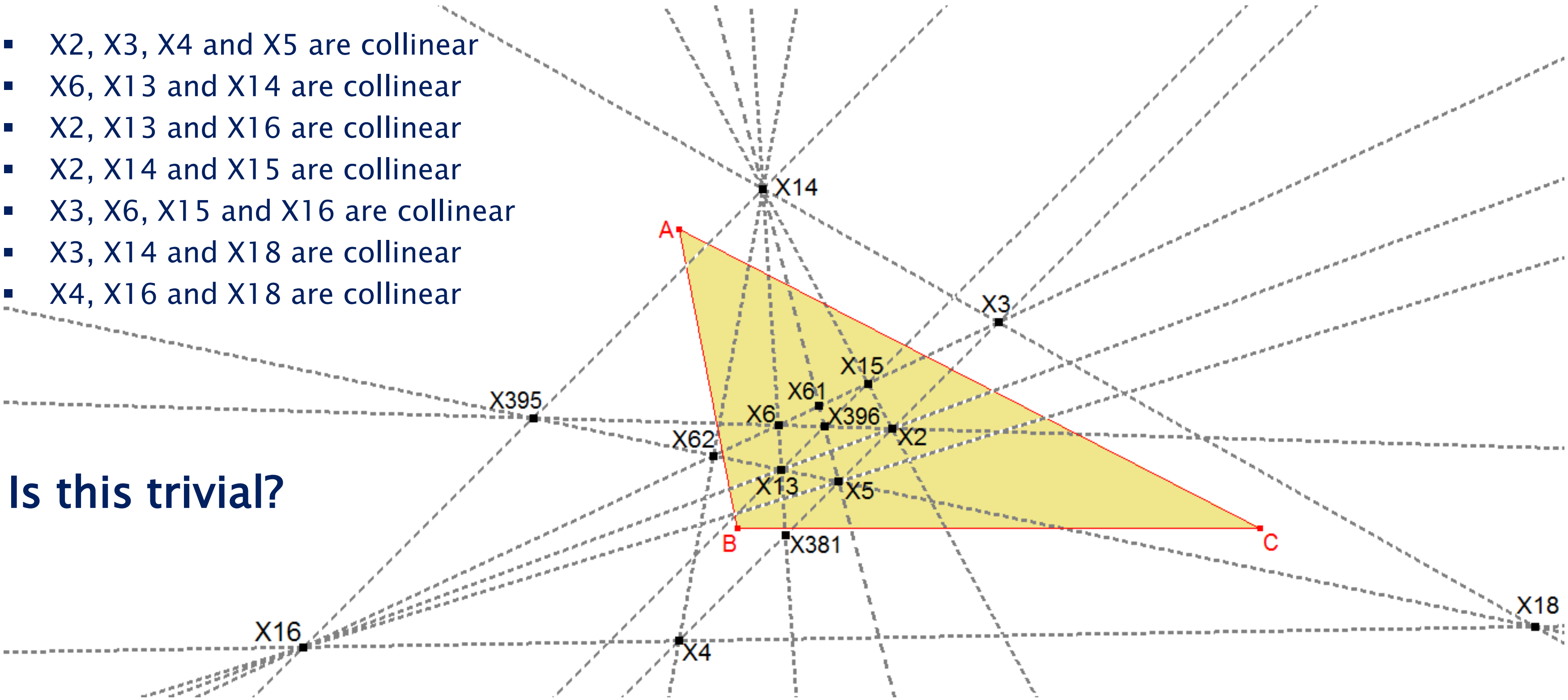
As a matter of fact
282 *points are known*
to occur on this special line.
How trivial is this ?



There are many more collinearities in Triangle Geometry. How come?

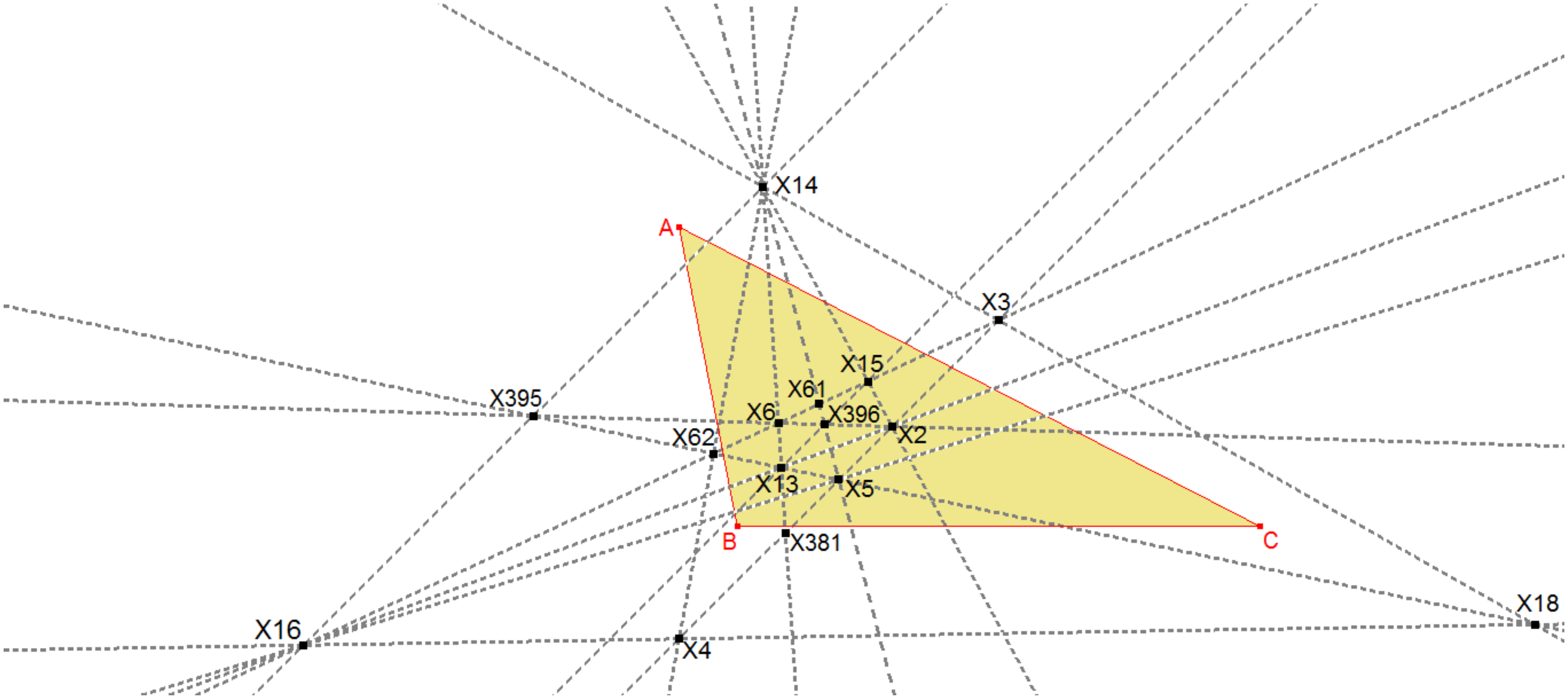
Some more examples of collinearities:

- X2, X3, X4 and X5 are collinear
- X6, X13 and X14 are collinear
- X2, X13 and X16 are collinear
- X2, X14 and X15 are collinear
- X3, X6, X15 and X16 are collinear
- X3, X14 and X18 are collinear
- X4, X16 and X18 are collinear



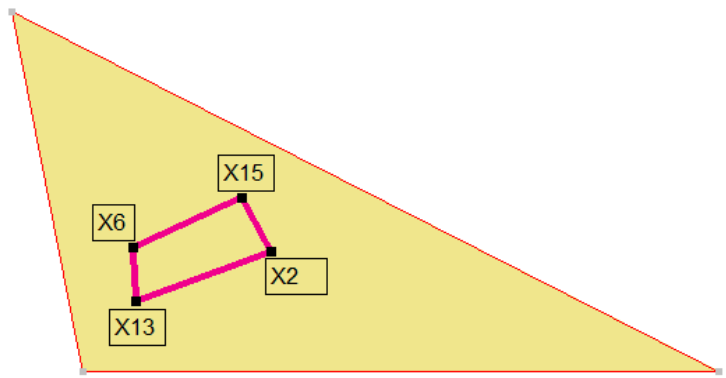
Is this trivial?

There are so many regularities in Triangle Geometry. How come?



- Is there a higher level structure in this?
- In this presentation we will search for this structure.

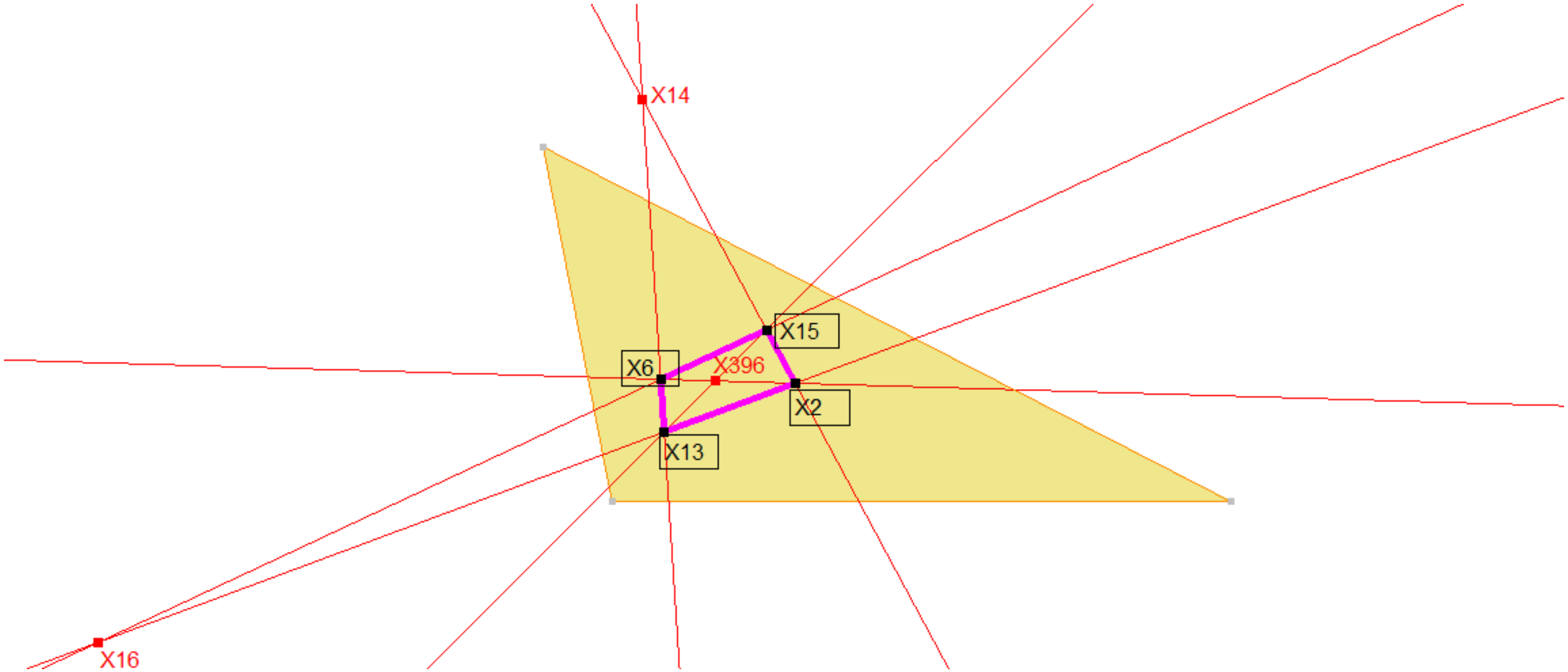
Analyses of collinearities
Fase 0



Let’s look at some basic points in a triangle.

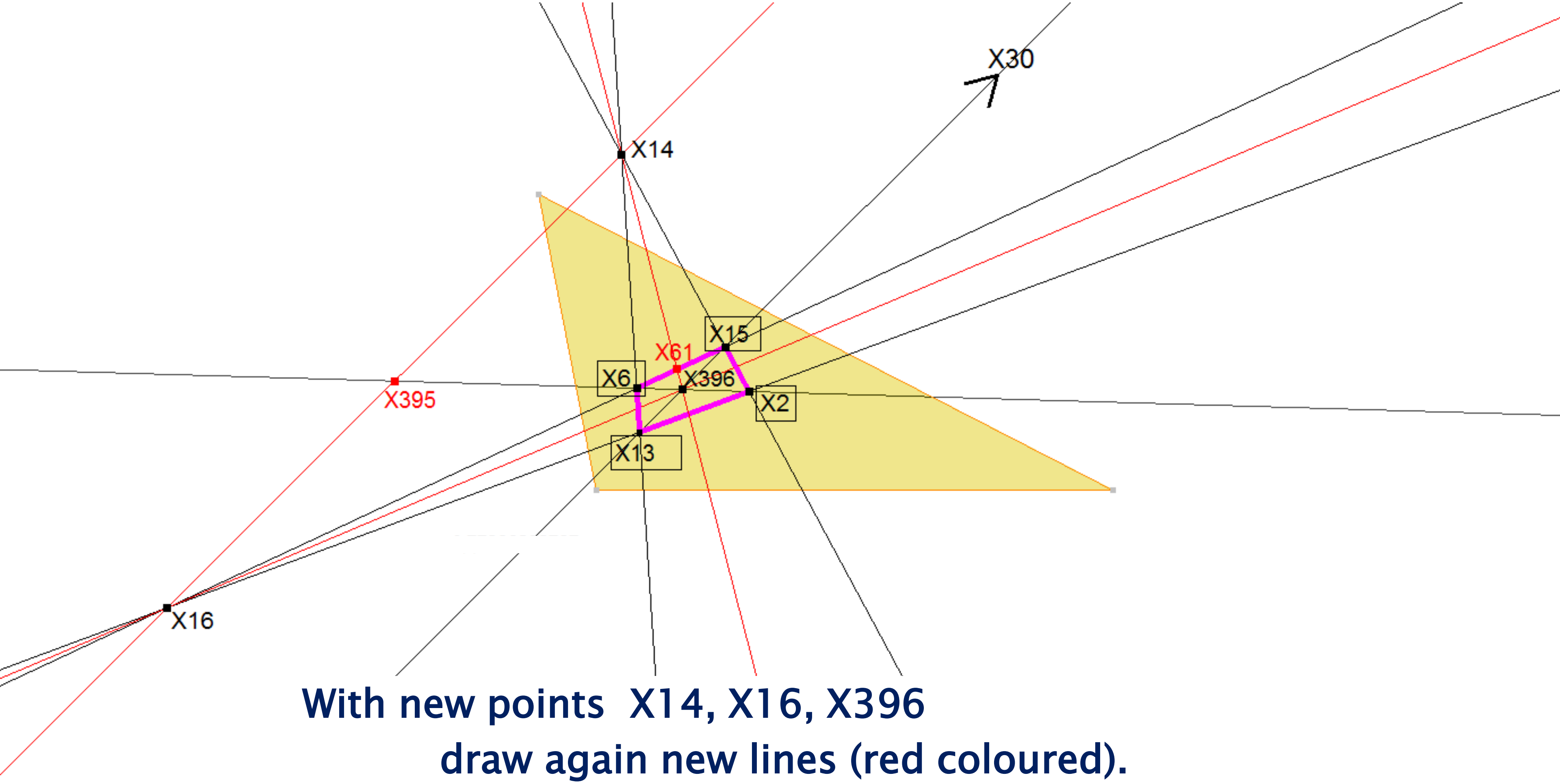
* X2 = Centroid
* X6 = Symmedian Point
* X13 = 1st Fermat Point
* X15 = 1st Isodynamic Center

Analyses of collinearities
Fase 1



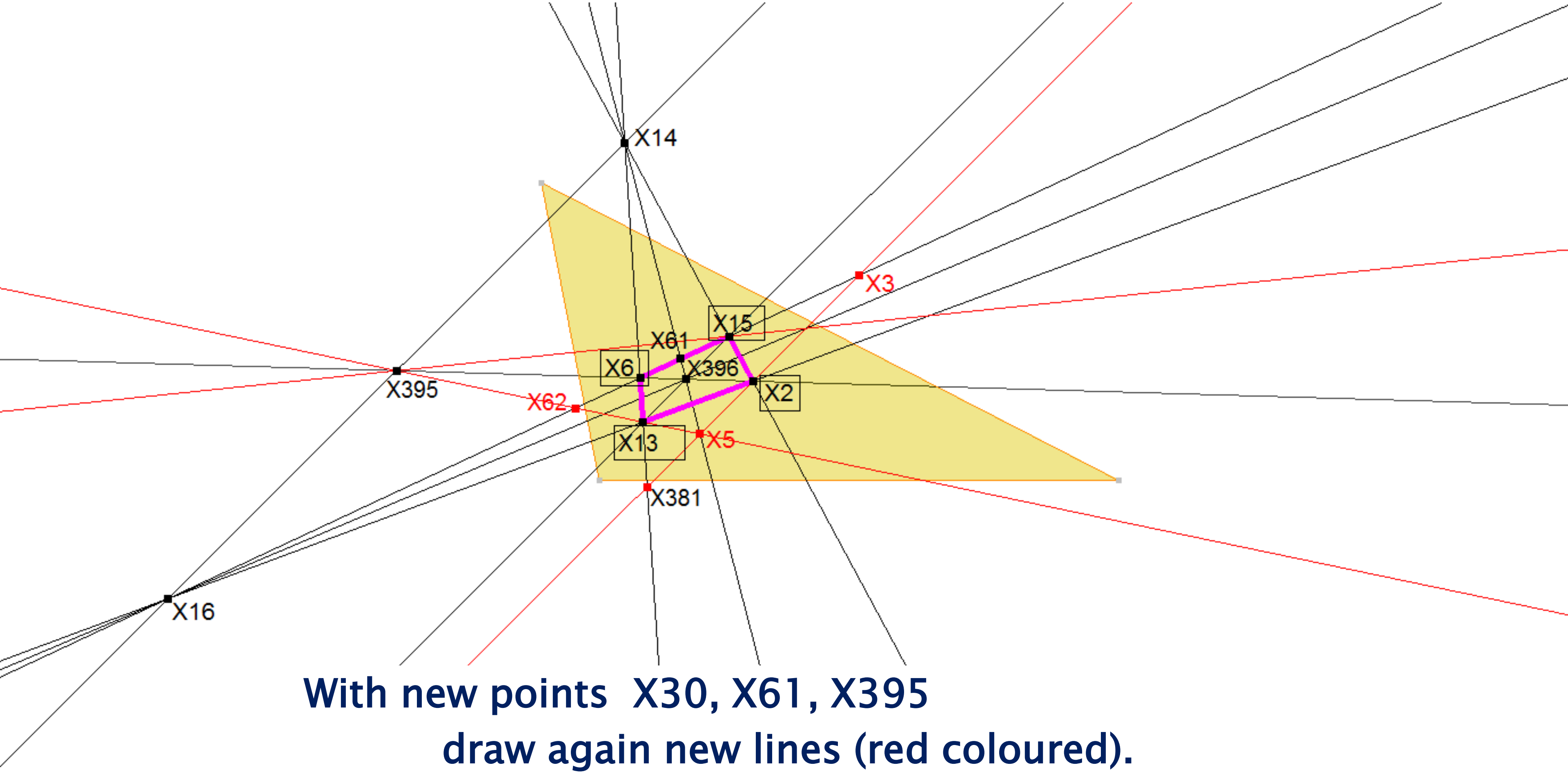
Draw all possible lines
from the points X2, X6, X13, X15
to the points X2, X6, X13, X15.
This results in 3 new intersectionpoints X14, X16, X396.

Analyses of collinearities
Fase 2



With new points X14, X16, X396
draw again new lines (red coloured).
This results in new intersectionpoints X30, X61, X395.

Analyses of collinearities
Fase 3

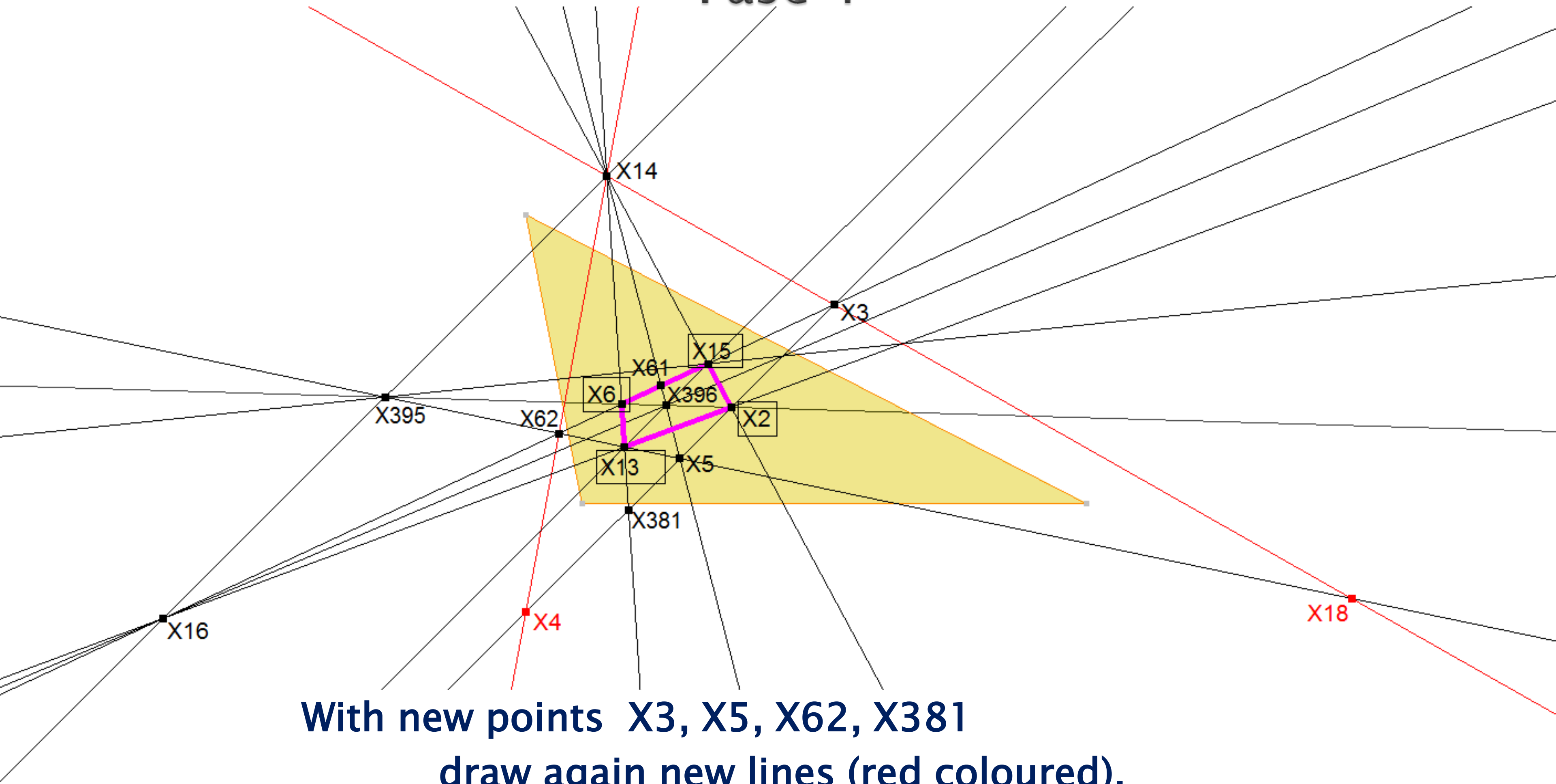


With new points X30, X61, X395

draw again new lines (red coloured).

This results in new intersectionpoints X3, X5, X62, X381.

Analyses of collinearities
Fase 4



With new points X3, X5, X62, X381
draw again new lines (red coloured).
This results in new intersectionpoints X4, X18.

Analyses of collinearities

CONCLUSION

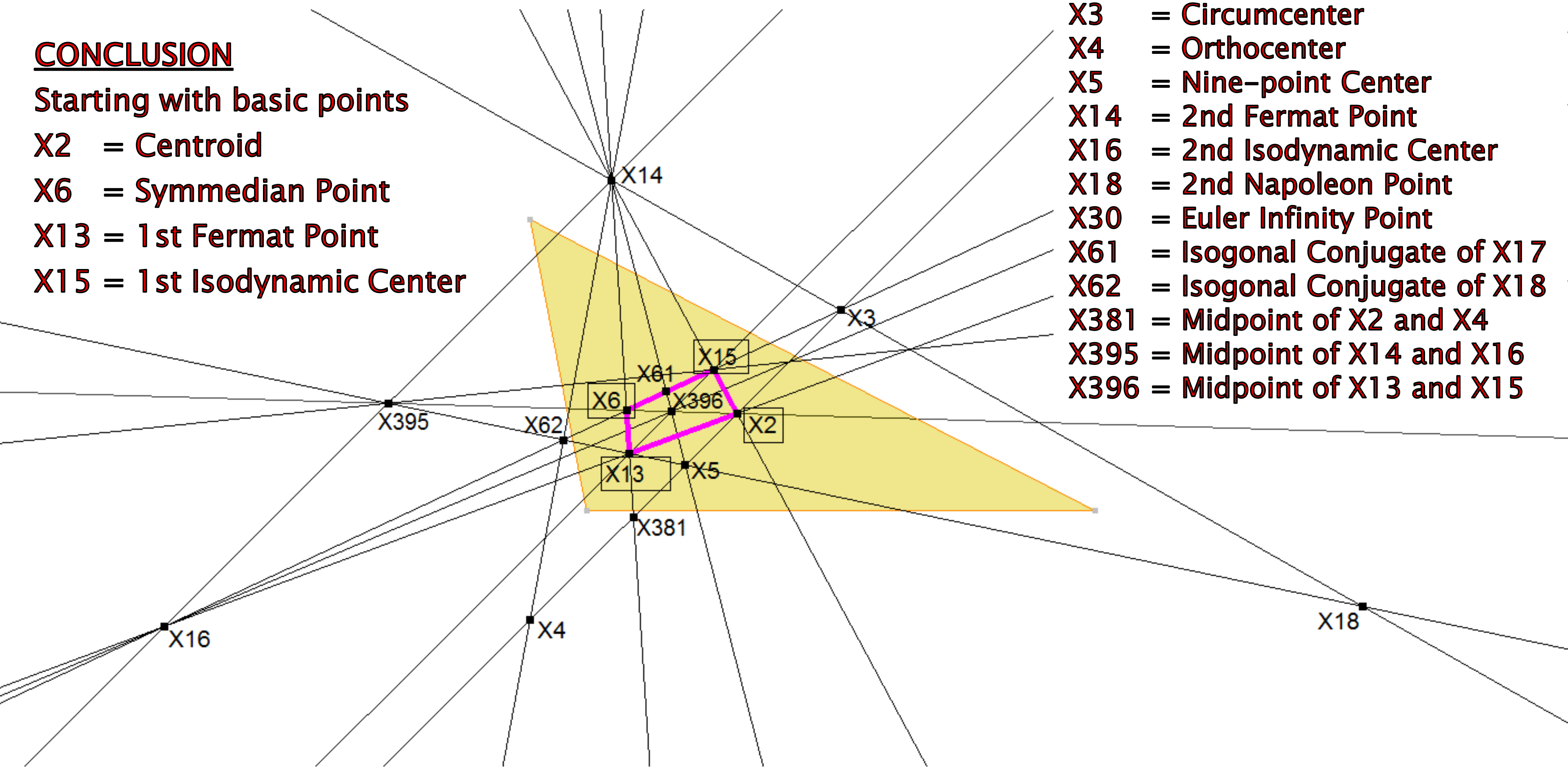
Starting with basic points

X2 = Centroid

X6 = Symmedian Point

X13 = 1st Fermat Point

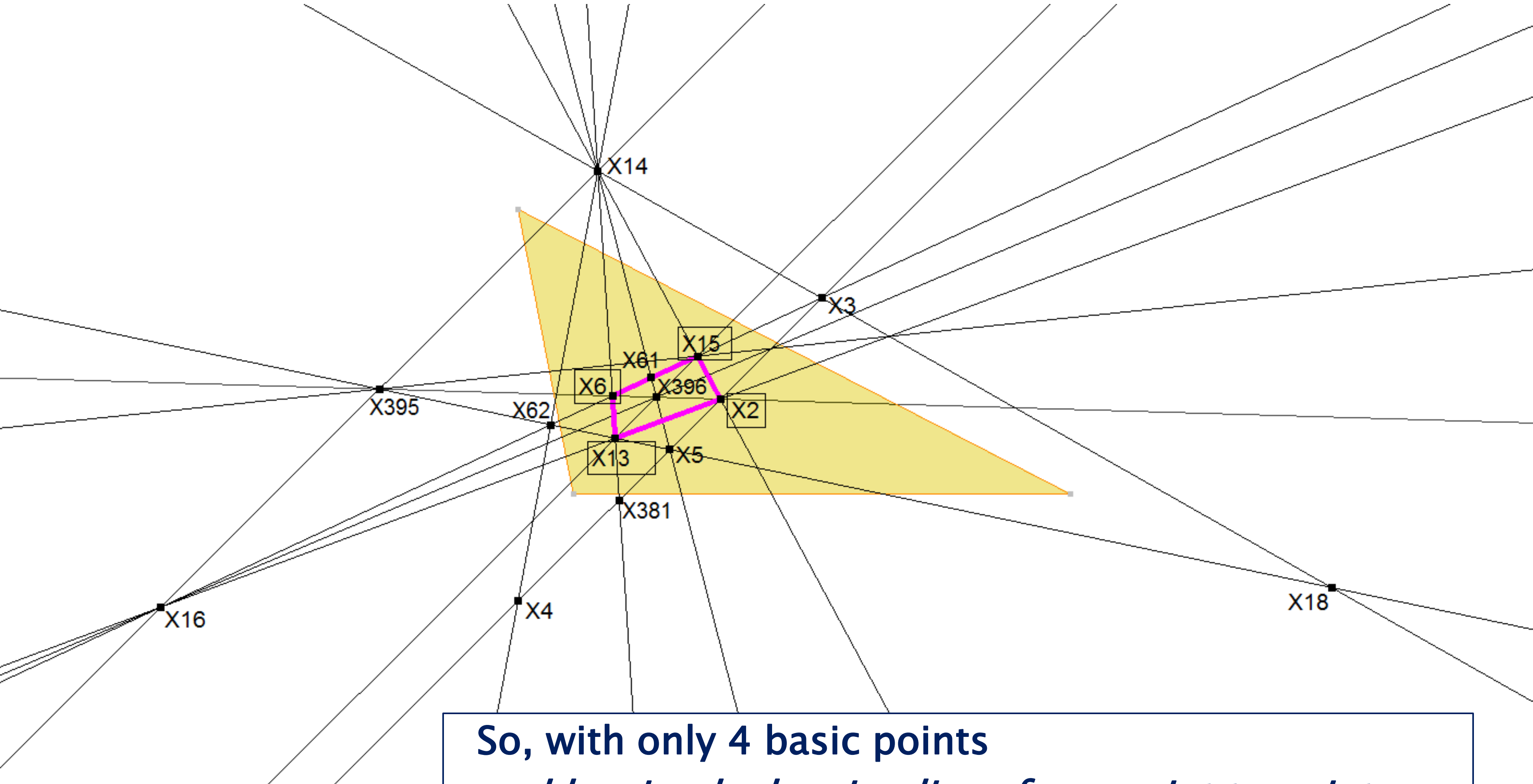
X15 = 1st Isodynamic Center



we constructed these points:

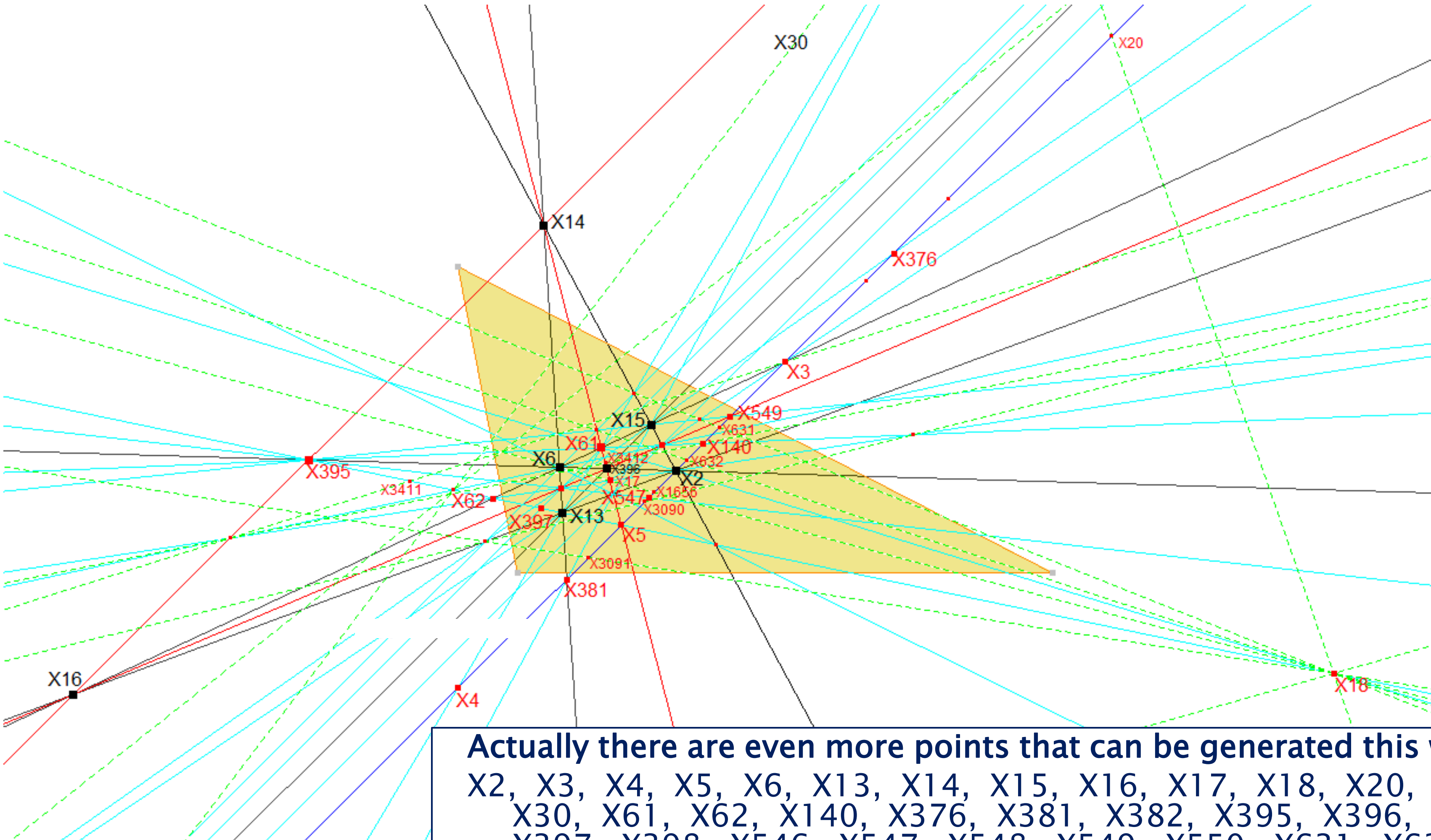
X3 = Circumcenter
X4 = Orthocenter
X5 = Nine-point Center
X14 = 2nd Fermat Point
X16 = 2nd Isodynamic Center
X18 = 2nd Napoleon Point
X30 = Euler Infinity Point
X61 = Isogonal Conjugate of X17
X62 = Isogonal Conjugate of X18
X381 = Midpoint of X2 and X4
X395 = Midpoint of X14 and X16
X396 = Midpoint of X13 and X15

Analyses of collinearities



So, with only 4 basic points
and by simply drawing lines from point to point,
we end up with 12 extra registered ETC-points.

Analyses of collinearities

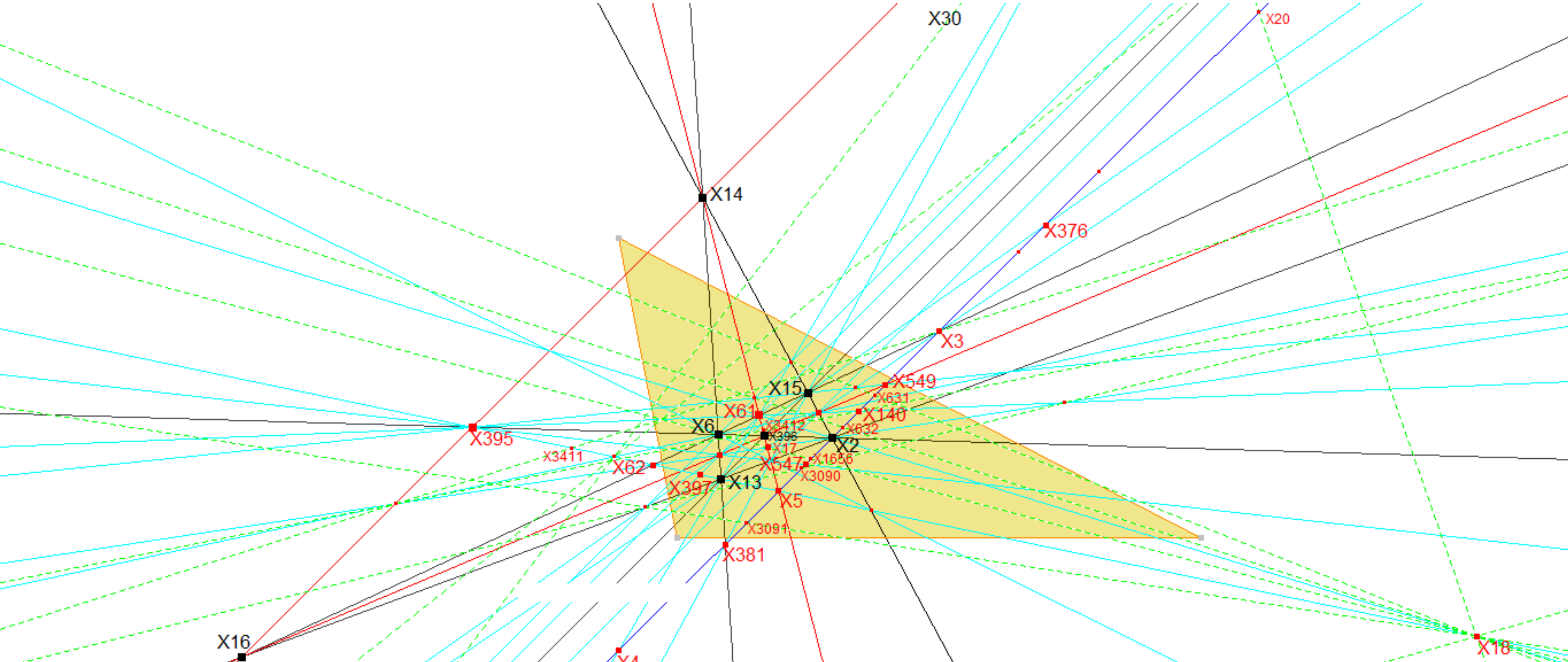


Actually there are even more points that can be generated this way:

X2, X3, X4, X5, X6, X13, X14, X15, X16, X17, X18, X20, X30, X61, X62, X140, X376, X381, X382, X395, X396, X397, X398, X546, X547, X548, X549, X550, X631, X632, X1656, X1657, X3090, X3091, X3146, X3411, X3412.

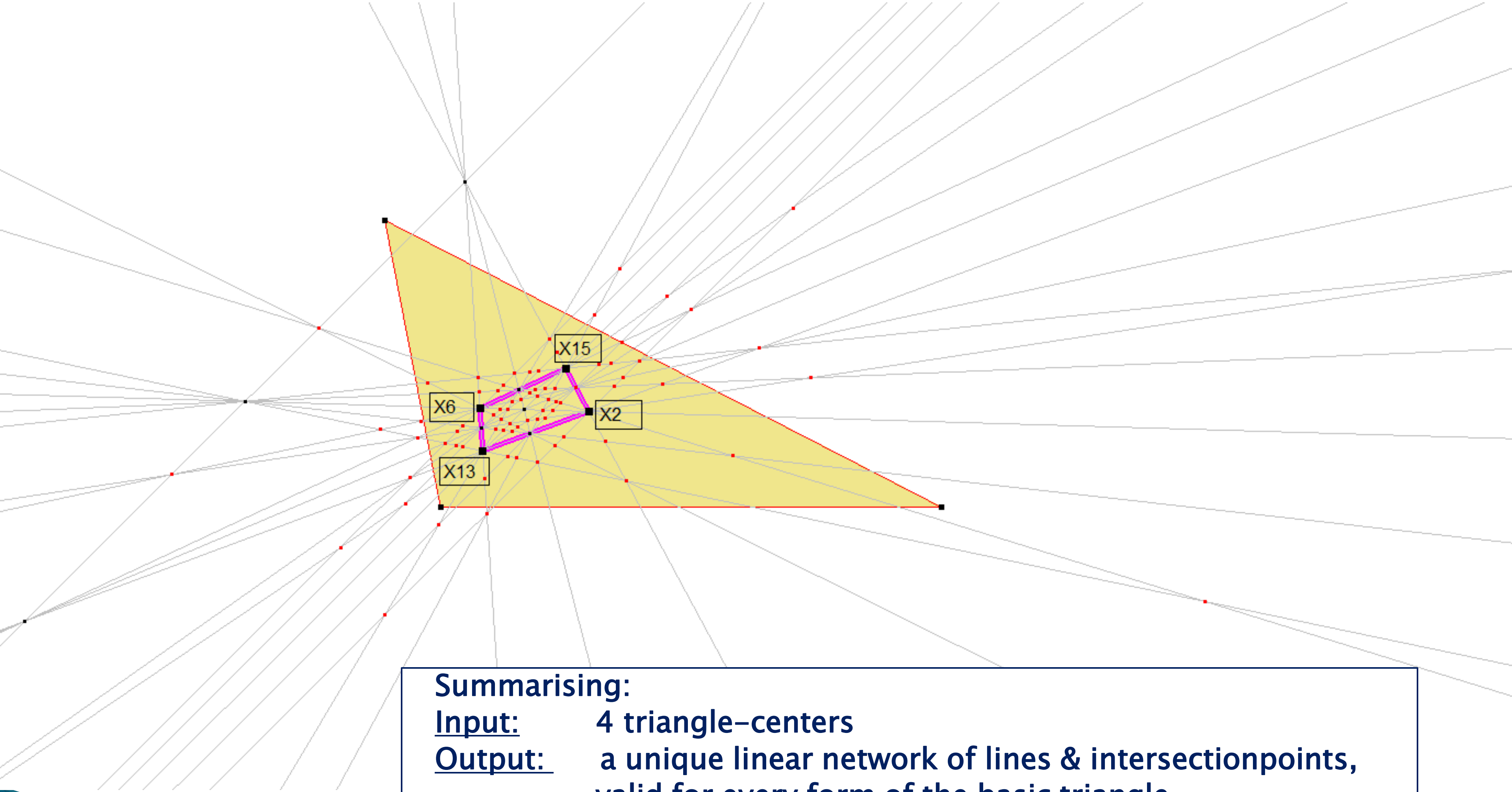
Alltogether 37 registered ETC-points in the range X1-X3500.

Analyses of collinearities



These points can be generated in steps.
Starting with: X2, X6, X13, X15
1st generation: X14, X16, X396
2nd generation: X30, X61, X395
3rd generation: X3, X4, X5, X17, X18, X62, X140, X376, X381, X397, X547, X549
4th generation: X20, X382, X398, X546, X548, X550, X631, X632, X1656, X1657, X3090, X3091, X3146, X3411, X3412

Analyses of collinearities



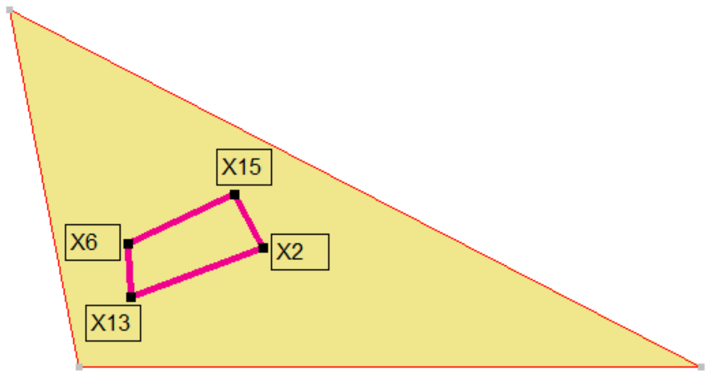
Summarising:

Input: 4 triangle-centers

Output: a unique linear network of lines & intersectionpoints, valid for every form of the basic triangle.

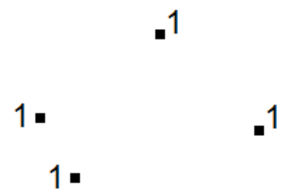
(not every point in this field is a registered ETC-Center, but at least a Center)

4-point-generated-network Theory



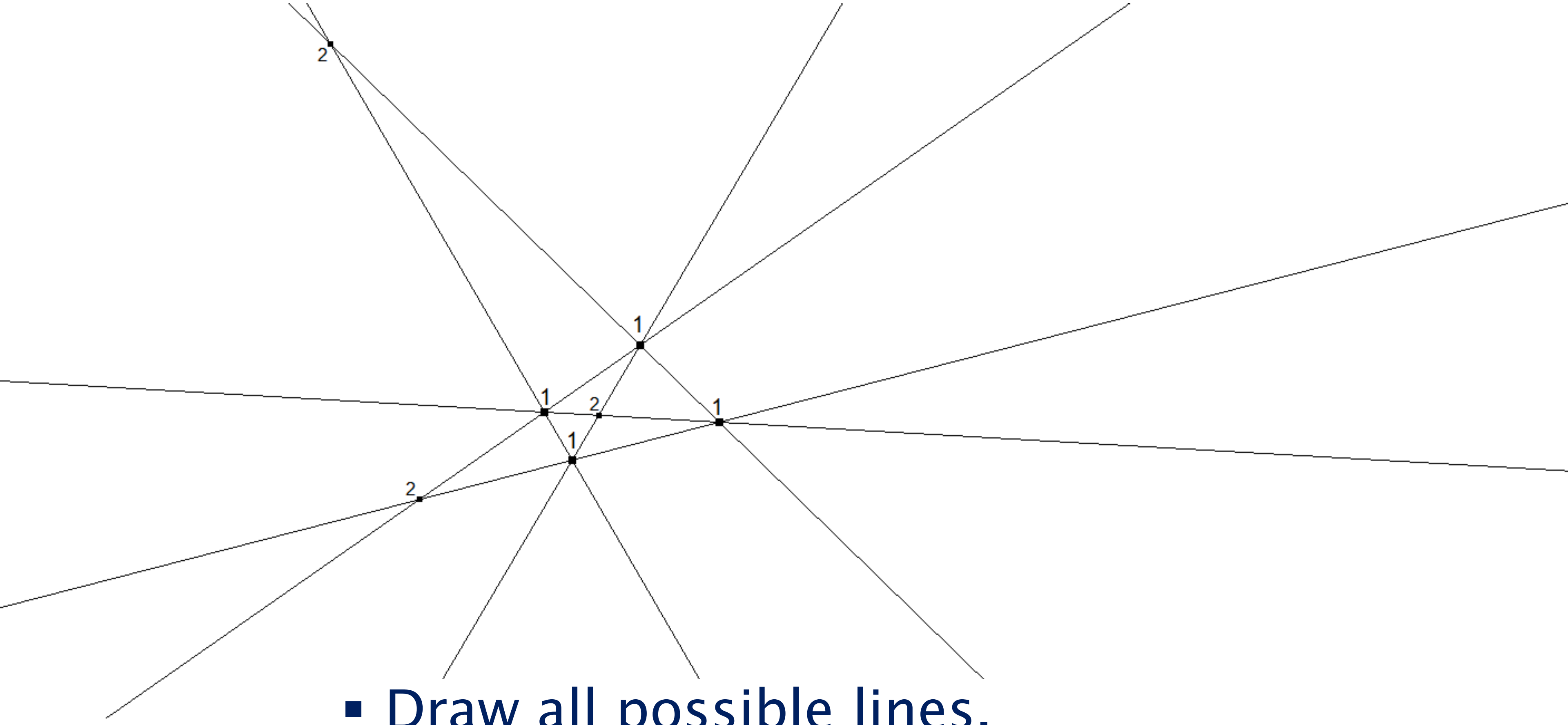
- Now what happened?
- Is this only one special case,
  or can this be generalised to other cases ?
- Is there a general theory behind this?

Theory 4-point-generated-network
Fase 1



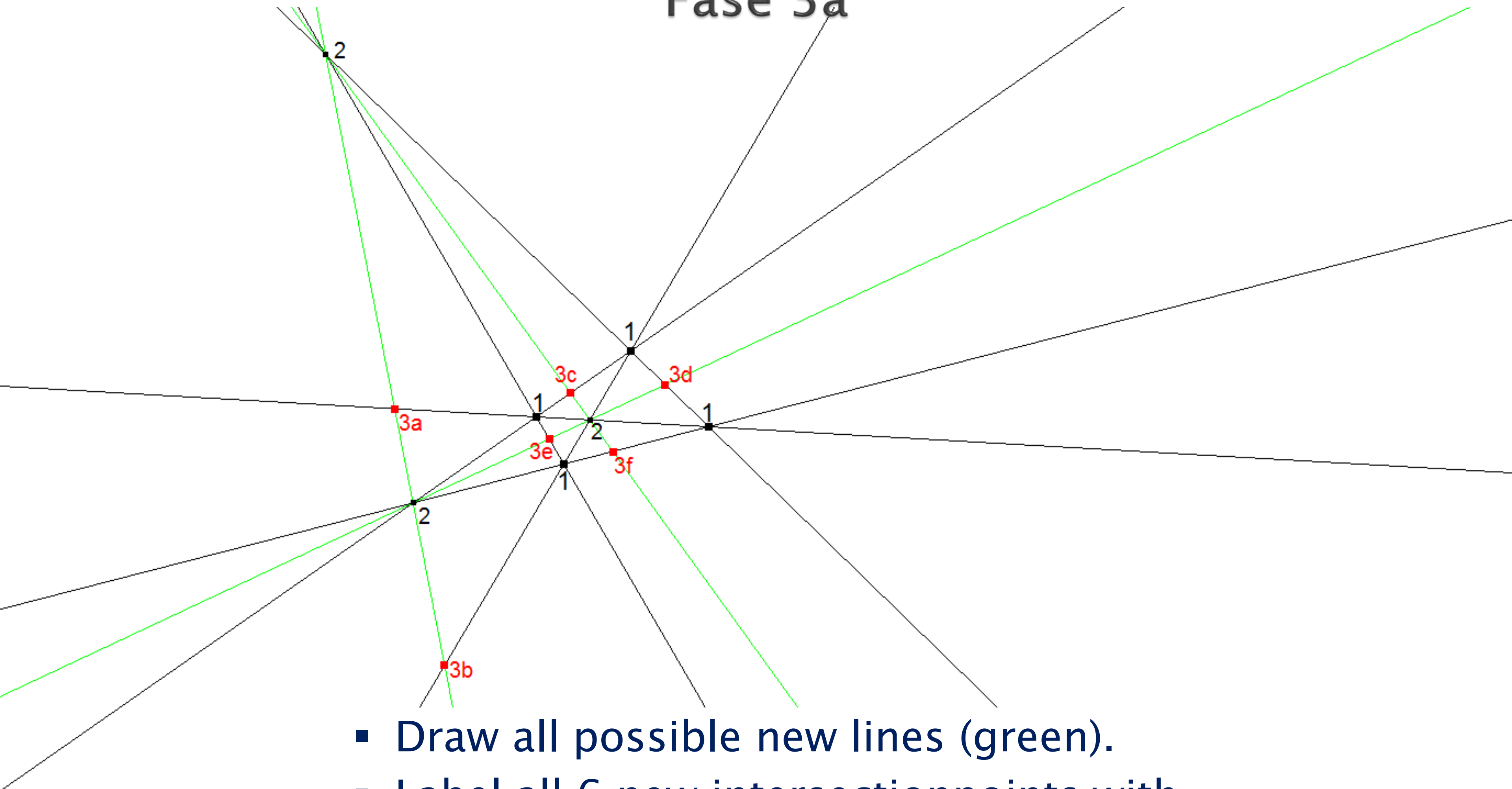
- Let's start with 4 random points.
- Label them with (fase) "1".

Theory 4-point-generated-network
Fase 2



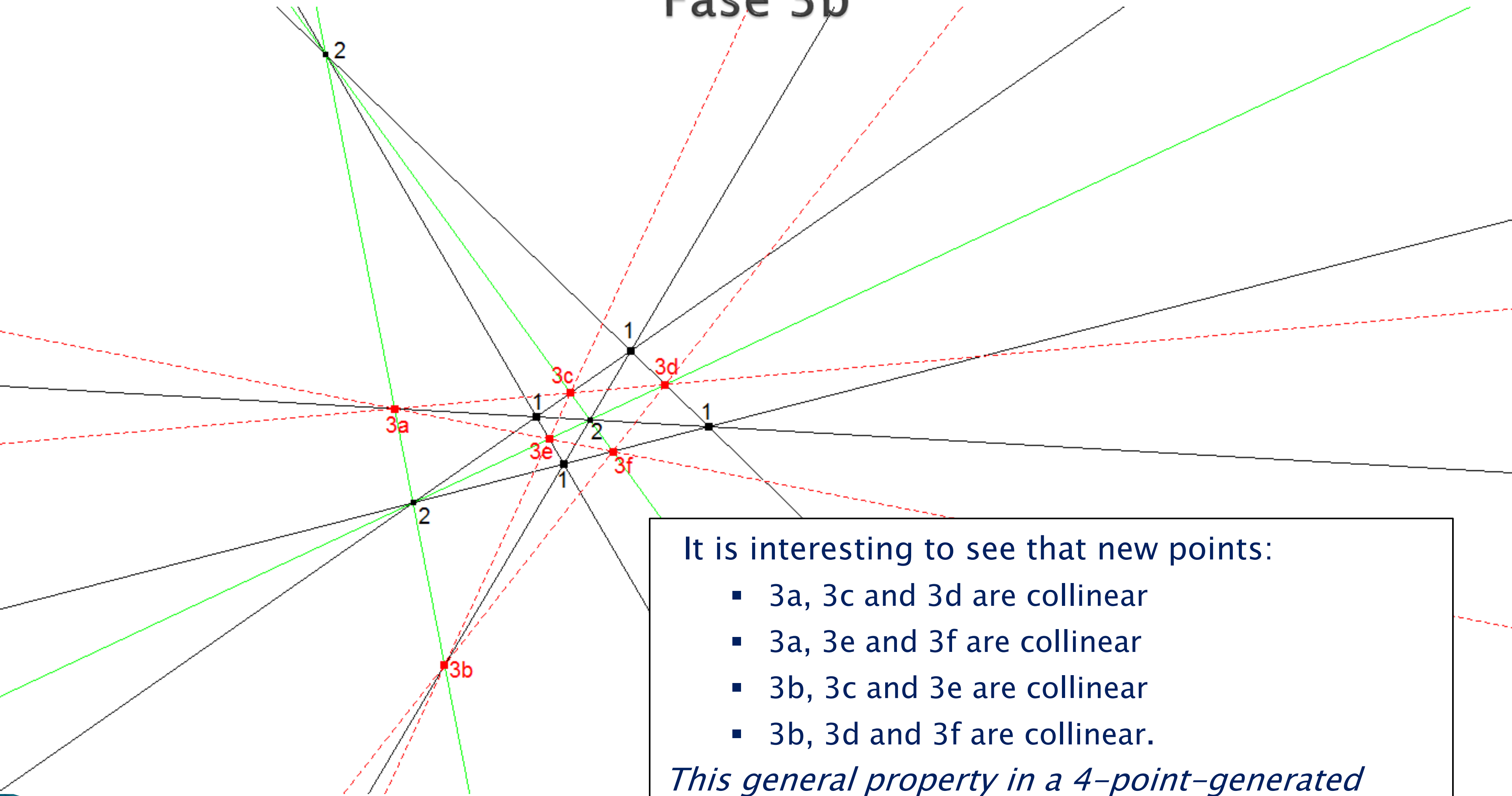
- Draw all possible lines.
- Label all new intersectionpoints with “2”.

Theory 4-point-generated-network
Fase 3a



- Draw all possible new lines (green).
- Label all 6 new intersectionpoints with "3a", "3b", "3c", "3d", "3e", "3f".

Theory 4-point-generated-network
Fase 3b

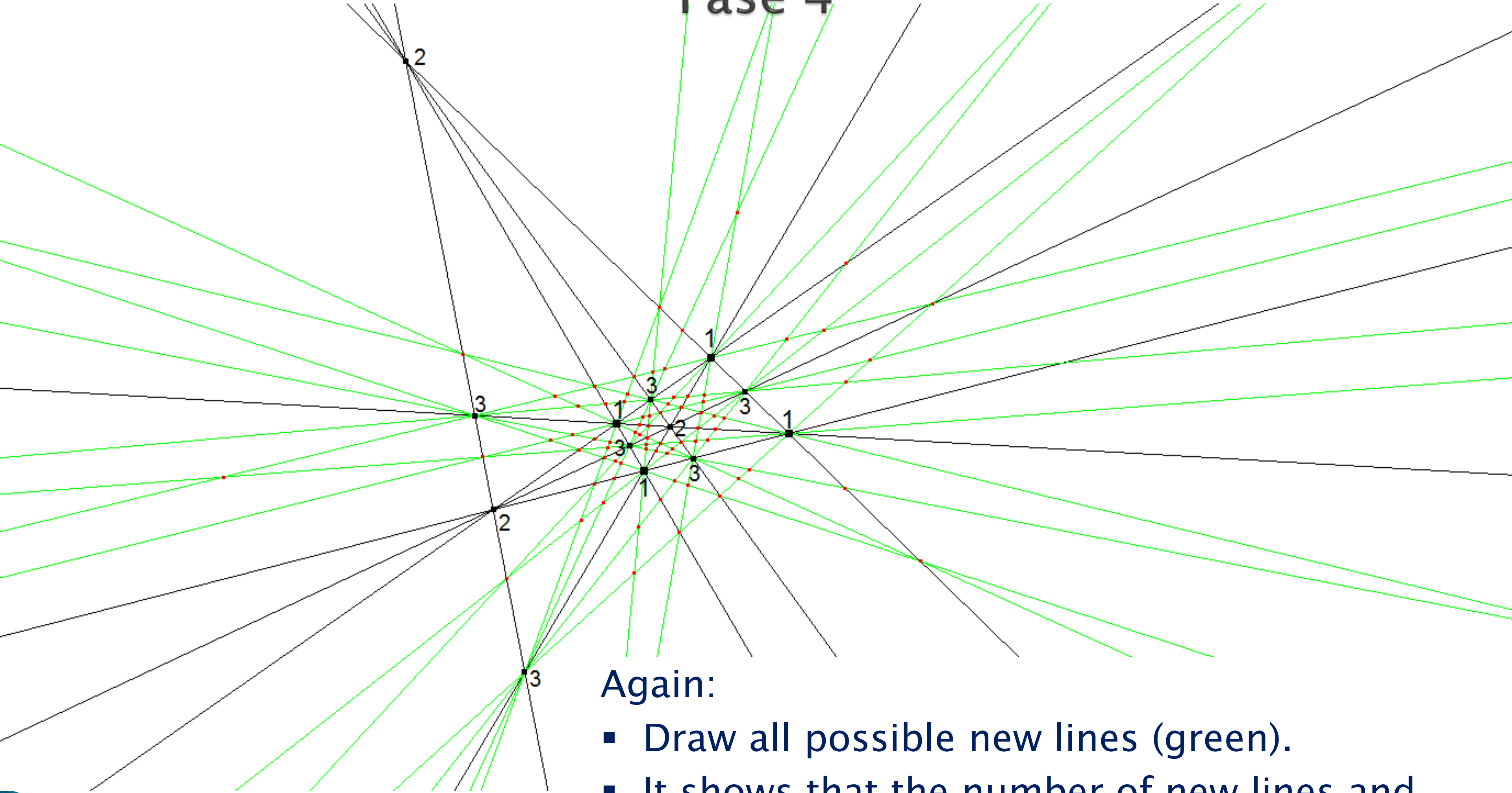


It is interesting to see that new points:

- 3a, 3c and 3d are collinear
- 3a, 3e and 3f are collinear
- 3b, 3c and 3e are collinear
- 3b, 3d and 3f are collinear.

This general property in a 4-point-generated network will appear to be one of the main reasons for collinearities within a triangular environment.

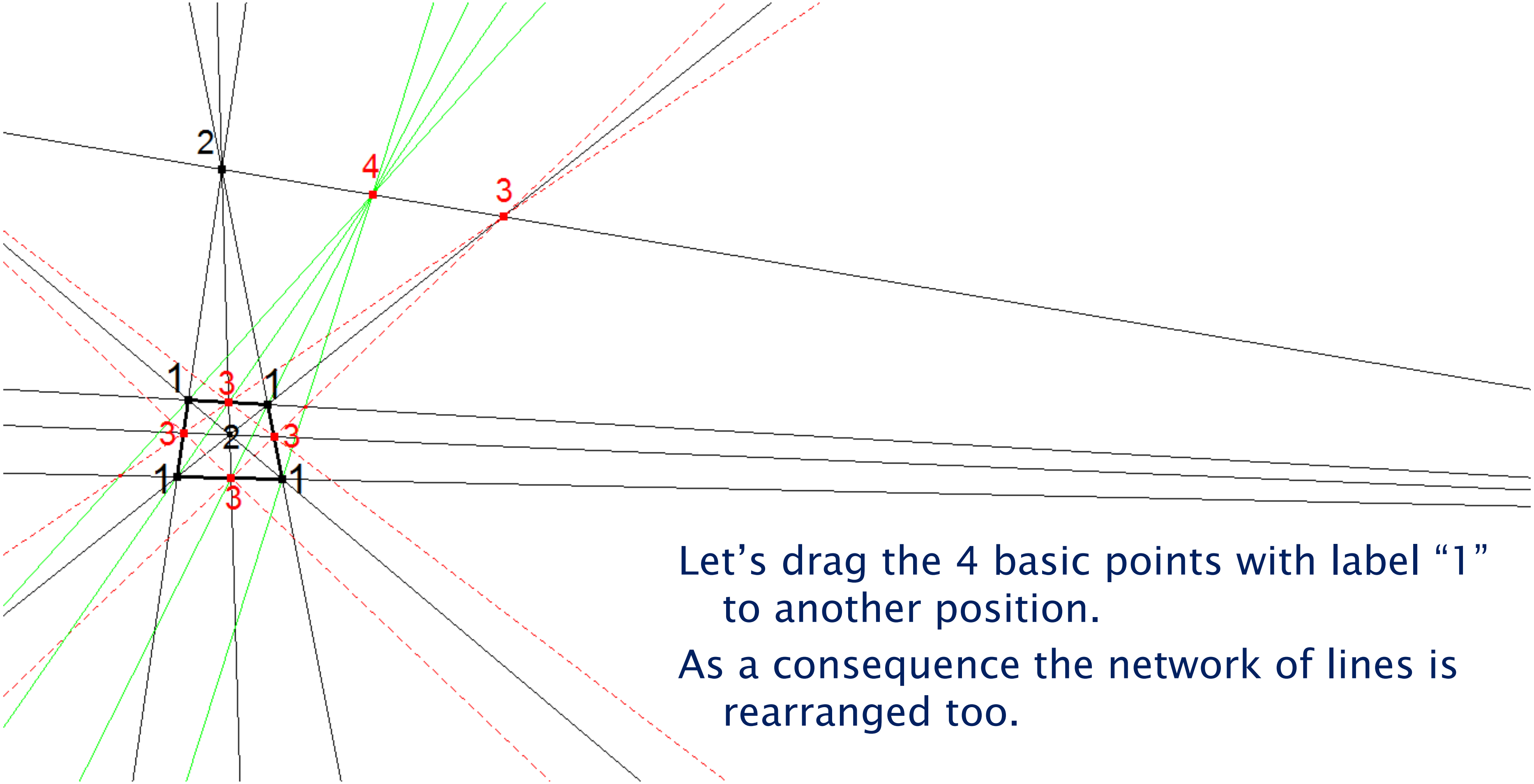
Theory 4-point-generated-network
Fase 4



Again:

- Draw all possible new lines (green).
- It shows that the number of new lines and new intersectionpoints is growing rapidly.
- New lines often pass through earlier points.

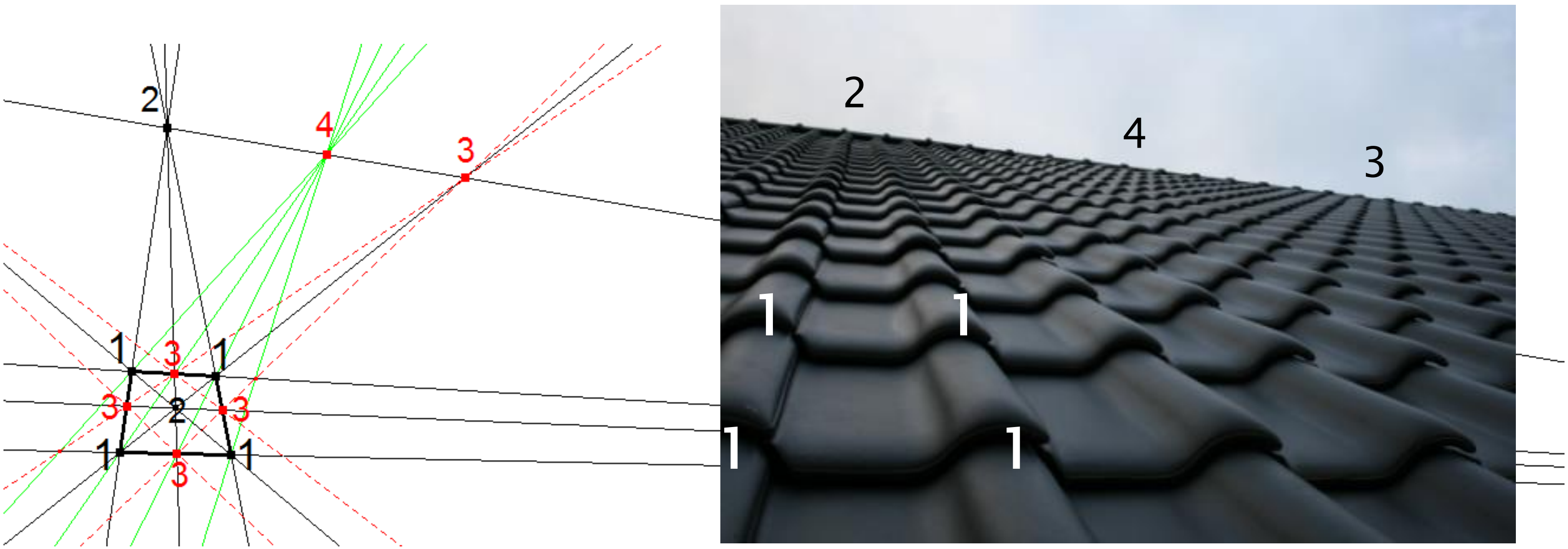
Theory 4-point-generated-network Reformat



Let's drag the 4 basic points with label "1" to another position.

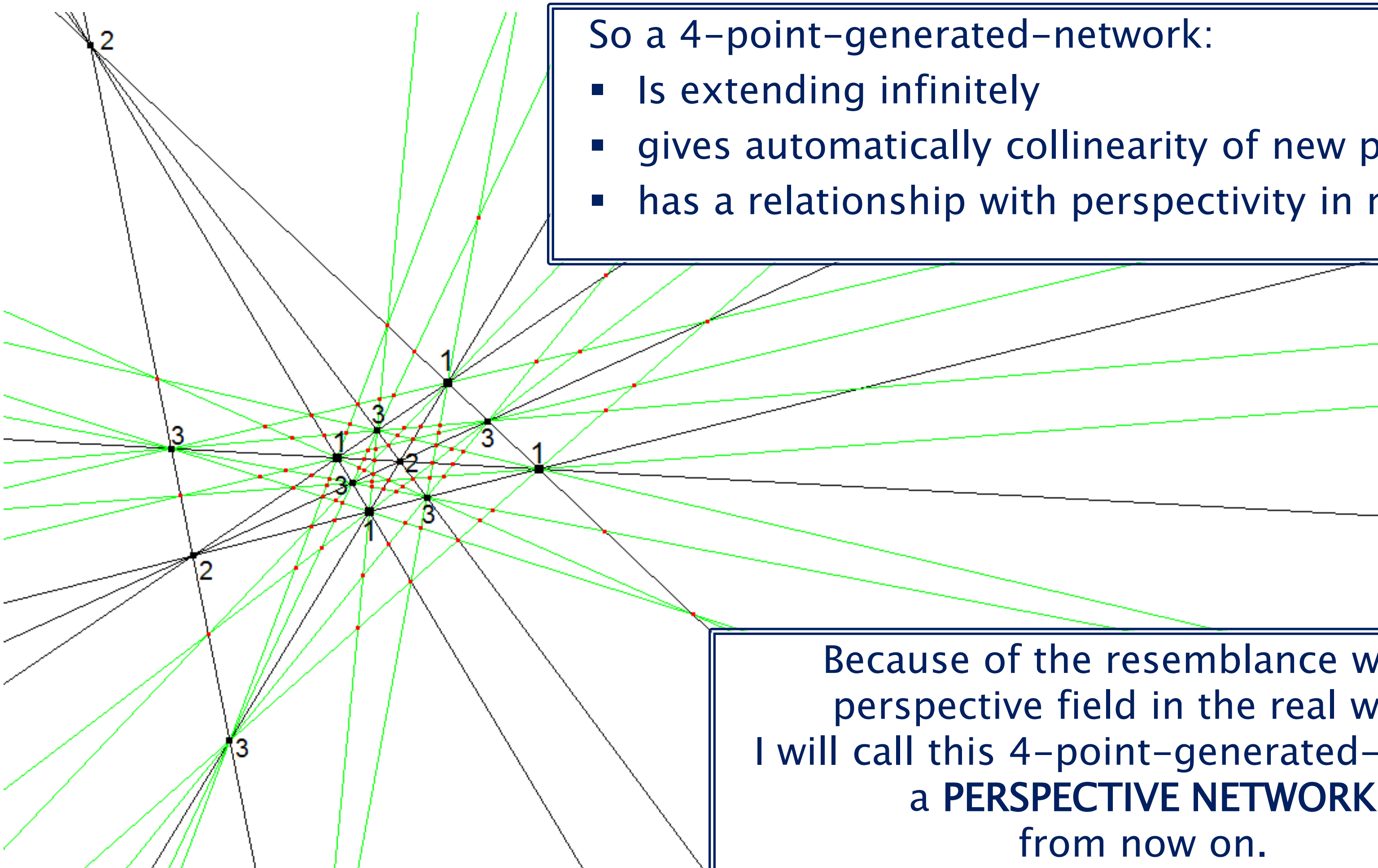
As a consequence the network of lines is rearranged too.

Theory 4-point-generated-network
Reformat



- There is a striking resemblance with a real life picture of perspectivity in roof-tiles.
- As a matter of fact when you take the vertices of one roof-tile and do the same exercise as we just did, then the same perspective structure comes forward like we constructed before.

Theory 4-point-generated-network
Summary

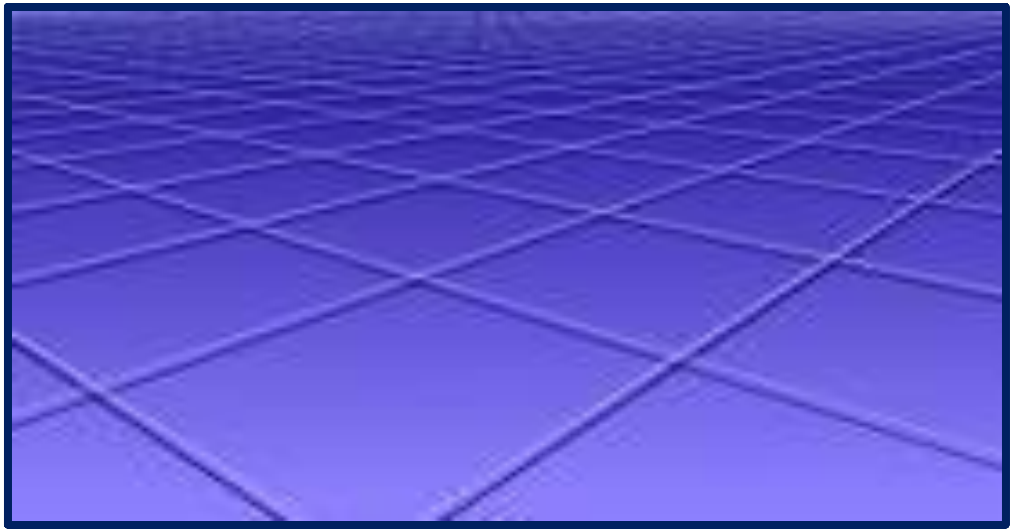


So a 4-point-generated-network:

- Is extending infinitely
- gives automatically collinearity of new points
- has a relationship with perspectivity in real world

Because of the resemblance with a
perspective field in the real world
I will call this 4-point-generated-network
a **PERSPECTIVE NETWORK**
from now on.

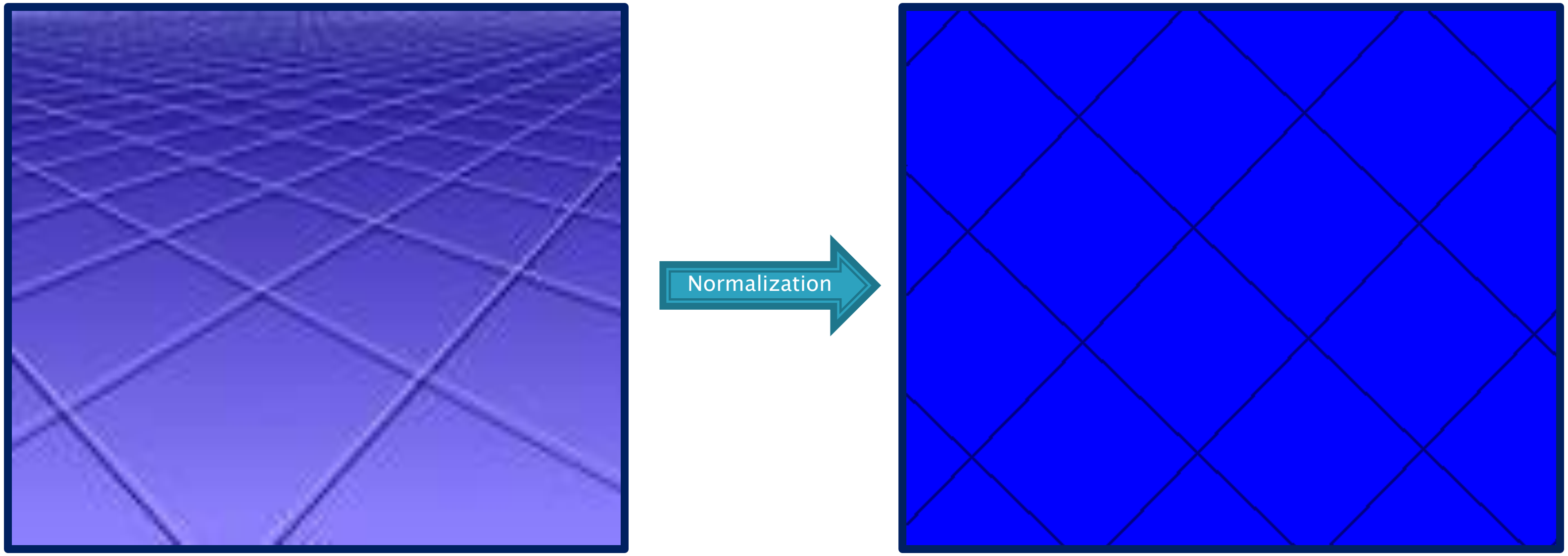
Perspective Network Theory



- A perspective network in real life arises when several items are placed in *a linear or rectangular structure*.
- Via the lens in a persons eye the real structure is transformed.
- This actually is a transformation as described in Projective Geometry.
- *A perspective network can be constructed in a projective way from a set of four points, no three of which are collinear.*
- Transformations in Projective Geometry preserve collinearity
  (i.e. all points lying on a line still lie on a line after transformation).

Perspective Network
Summary

- We first made a picture in a triangle consisting of 4 selected points.
- We saw this evolving into a 4-point-generated-network.
- We saw the resemblance with a perspective field in real life.



Now, a perspective field in real life is based on a rectangular structure.

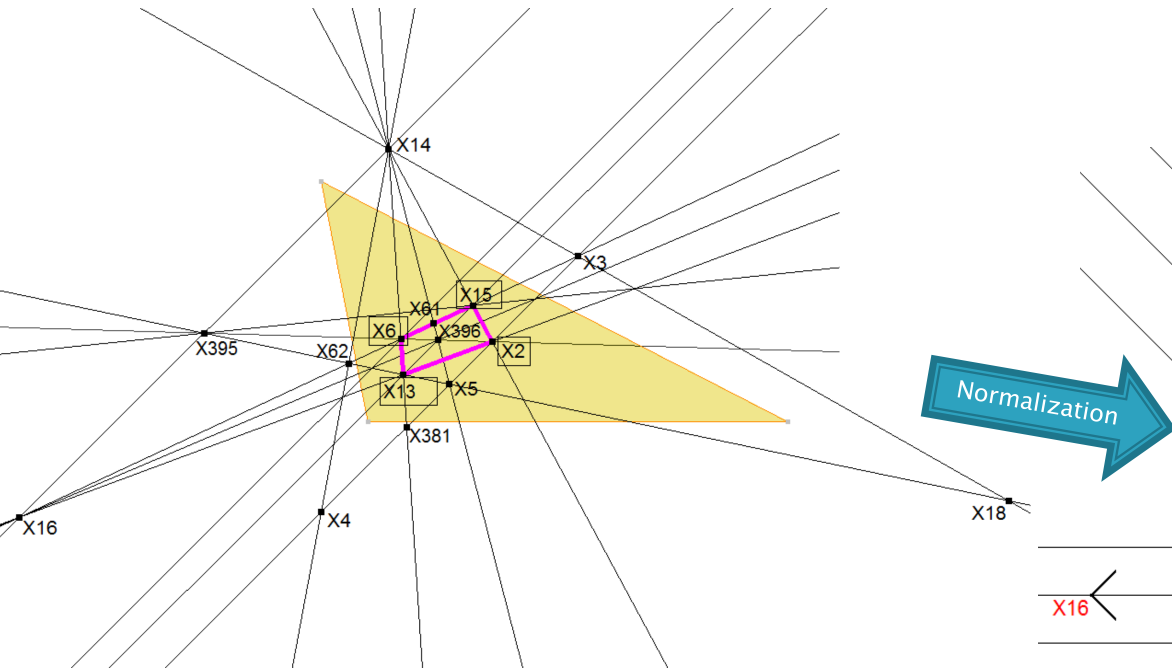
So why not transforming our 4-point-generated triangle network

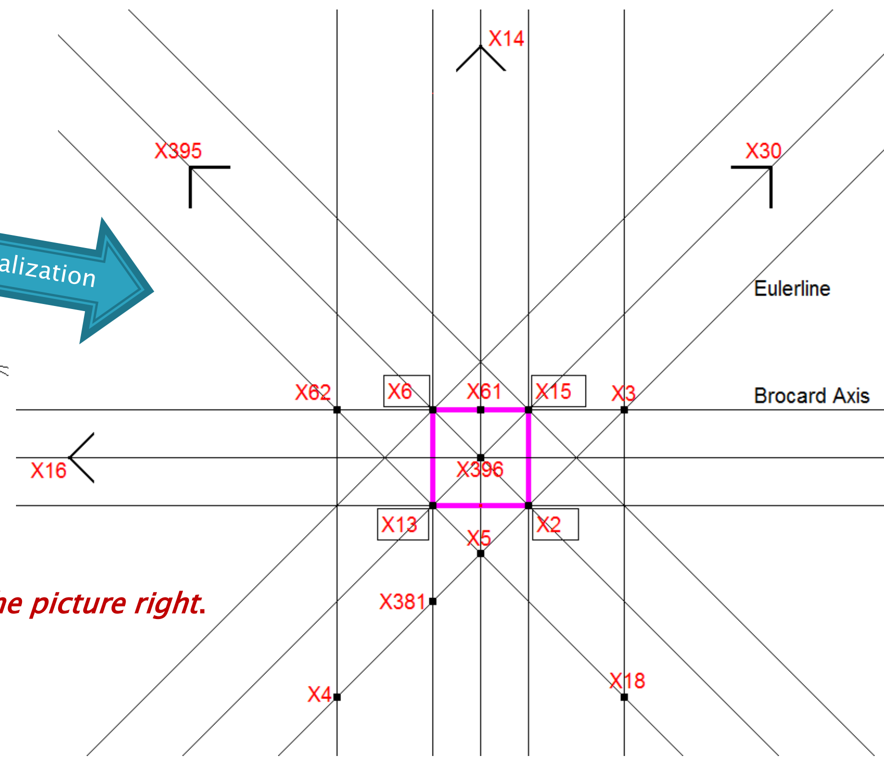
back to its "original" rectangular structure?

What appears than?

Normalized Perspective Network Transformation

Transforming the rectangular X13-X2-X15-X6-form into a square,
while obeying the rules of Projective Geometry
gives this special result:

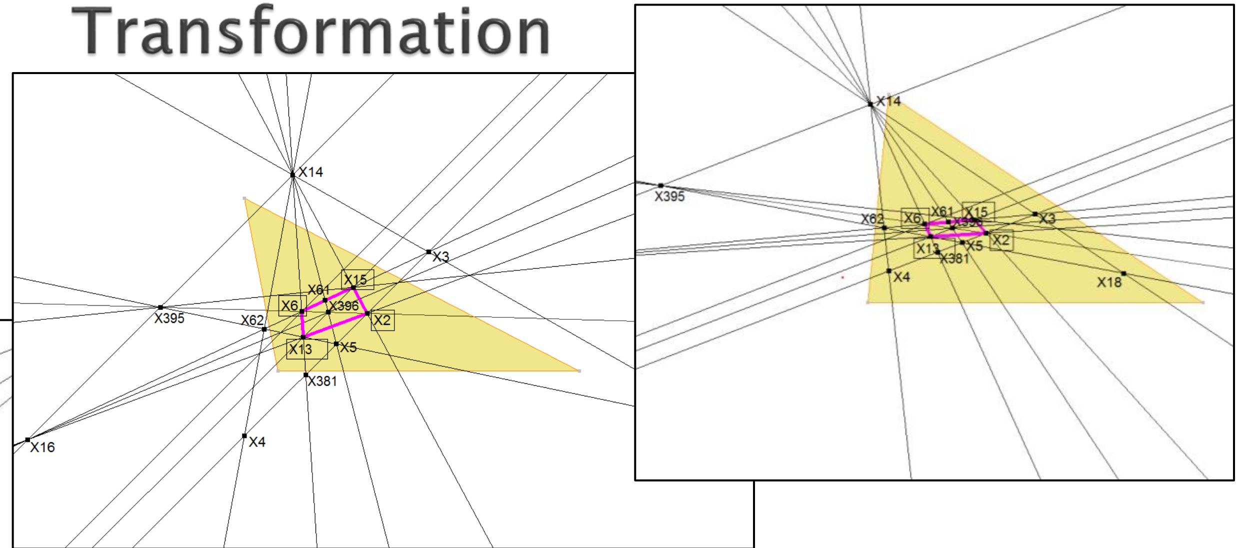


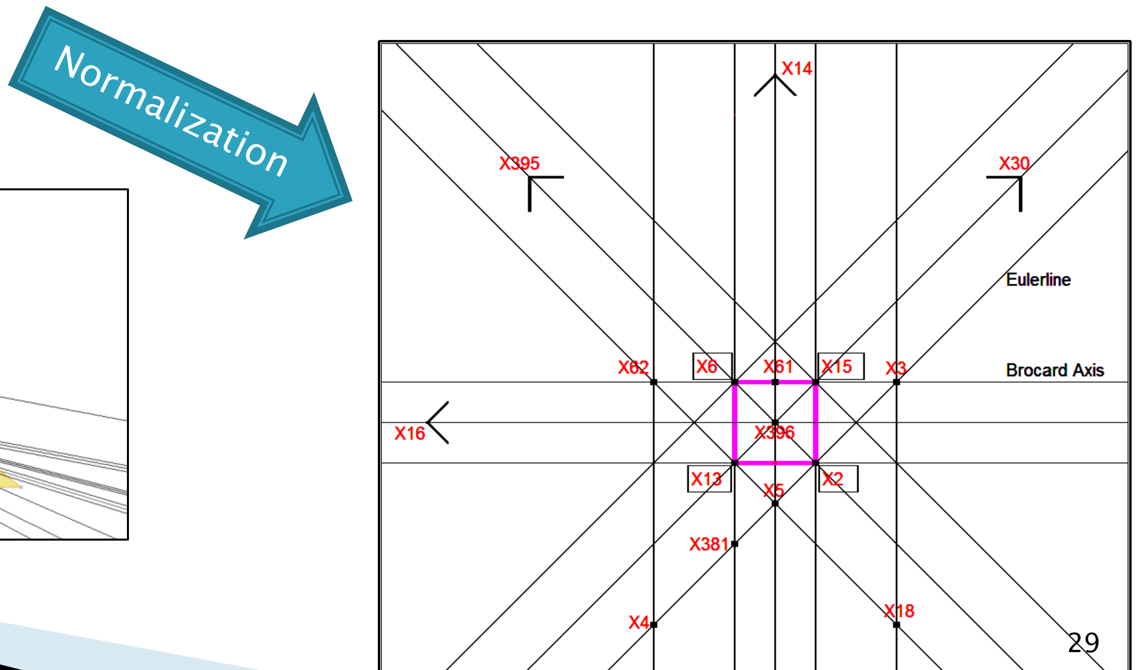


Note:
**** every line in the picture left has a corresponding line in the picture right.***
* collinearity has been preserved !
* centers X14, X16, X395 have become infinitypoints.

Normalized Perspective Network Transformation

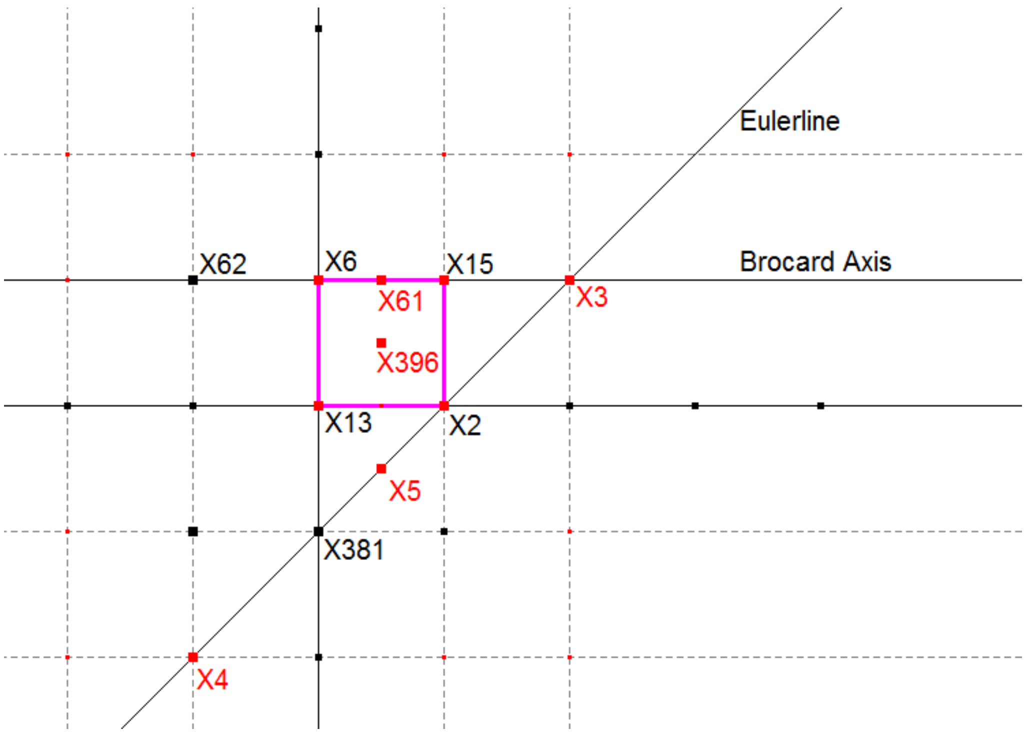
When the reference triangle changes, the 14 lines and 15 points change their position too.
However the normalized picture happens to be invariant !!!





Normalized Perspective Network

Examining the normalized picture



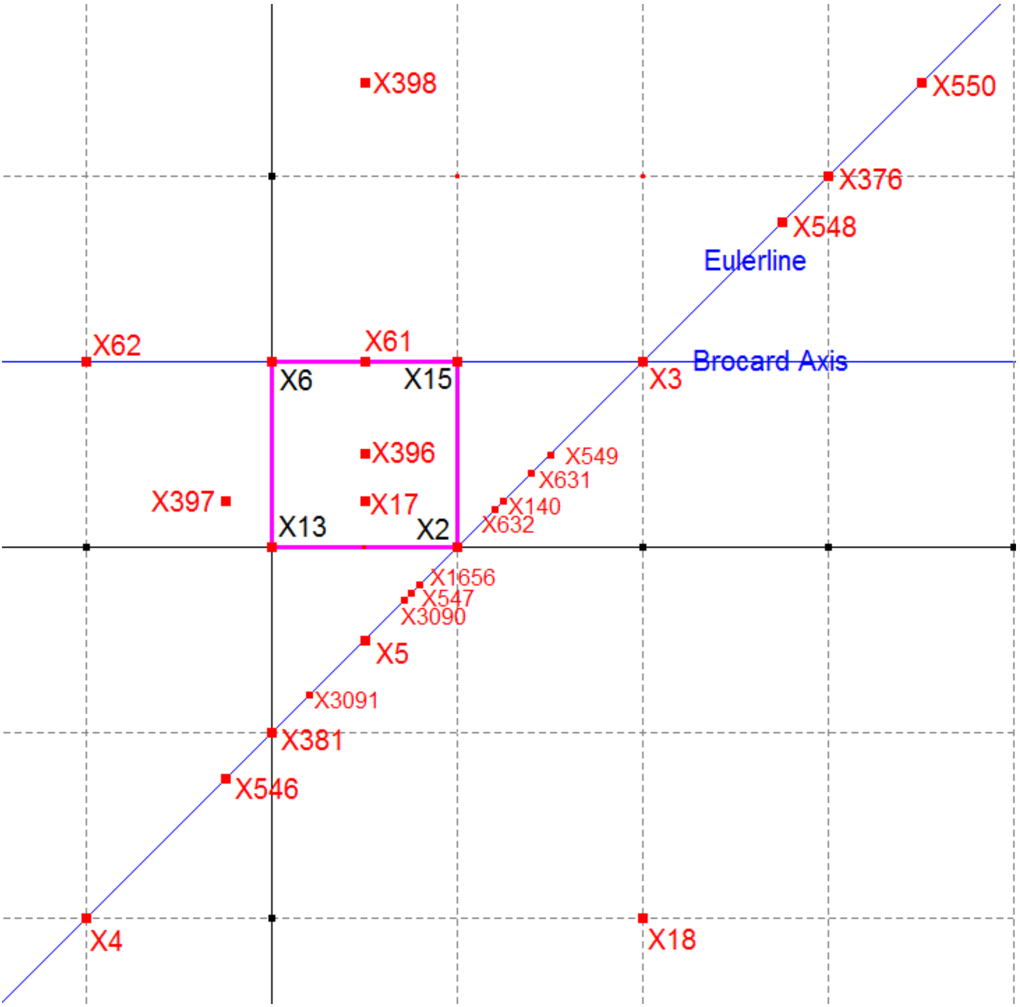
When we define X13 as origin and X13.X2 & X13.X6 as having lenght = 1 then all points can be given cartesian coordinates:

X2 = (1 , 0)
X3 = (2 , 1)
X4 = (-1 , -2)
X5 = (½ , -½)
X6 = (0 , 1)
X13 = (0 , 0)
X15 = (1 , 1)
X61 = (½ , 1)
X62 = (-1 , 1)
X381= (0 , -1)
X396= (½ , ½)

Note after transformation all points in this picture are invariant! That means whatever shape the reference triangle adopts, these coordinates are fixed in the normalized perspective network as seen from the X13-X2-X15-X6-perspectivitywindow!

In contrast to these ETC-points the vertices of the reference triangle are not invariant !

Normalized Perspective Network



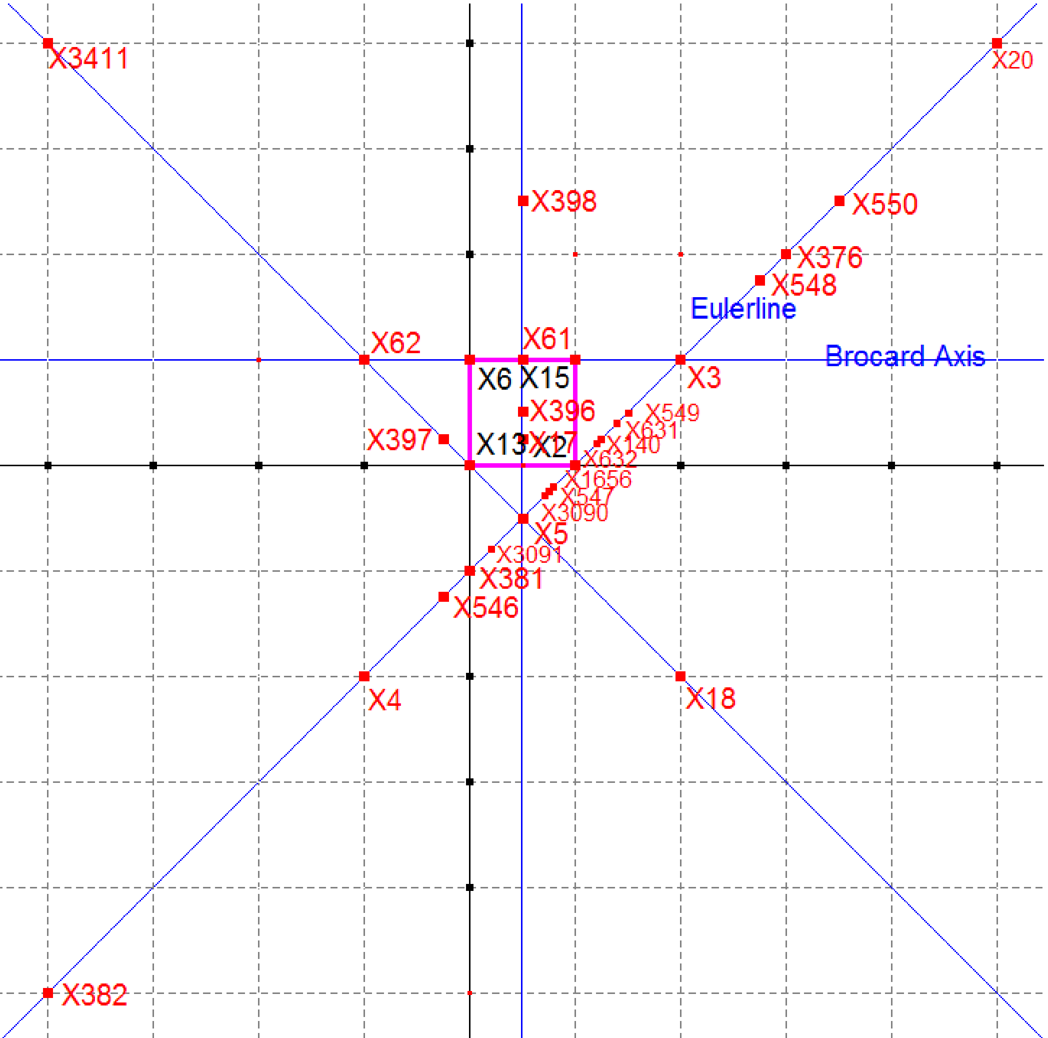
Note that all coordinates are sets of *rational* numbers.

More invariant points with their cartesian coordinates:

X2	(1 , 0)	
X3	(2 , 1)	
X4	(−1 , −2)	
X5	(½ , − ½)	
X6	(0 , 1)	
X13	(0 , 0)	
X14	(0 : 1 : 0)	= infinitypoint
X15	(1 , 1	
X16	(1 : 0 : 0)	= infinitypoint
X17	(½ , ¼)	
X18	(2 , −2)	
X20	(5 , 4)	
X30	(1 : 1 : 0)	= infinitypoint
X61	(½ , 1)	
X62	(−1 , 1)	
X140	(1 ¼ , ¼)	
X376	(3 , 2)	
X381	(0 , −1)	
X382	(−4 , −5)	
X395	(1 : −1 : 0)	= infinitypoint
X396	(½ , ½)	
X397	(− ¼ , ¼)	
X398	(½ , 2½)	
X546	(− ¼ , −1¼)	
X547	(¾ , − ¼)	
X548	(2 ¾ , 1¾)	
X549	(1½ , ½)	
X550	(3 ½ , 2 ½)	
X631	(7/5 , 2/5)	
X632	(11/10, 1/10)	
X1656	(4/5 , −1/5)	
X1657	(8 , 7)	
X3090	(5/7 , −2/7)	
X3091	(1/5 , −4/5)	
X3146	(−7 , −8)	
X3411	(−4 , 4)	
X3412	(4/8 , 5/8)	

Finite points are noted in cartesian coordinates.
Infinitypoints are noted in projective coordinates.

Normalized Perspective Network



Observations

A picture with a wider scope looks like this.

- The striking thing is that all these points have fixed coordinates like:
  * the CircumCenter X3 (2, 1)
  * the OrthoCenter X4 (1, -2).

- In this case all these points are on 4 lines:
  X2.X3, X3.X15, X5.X17, X5.X18.

It looks like this network is expanding from X5.

- This picture is in a way symmetric.

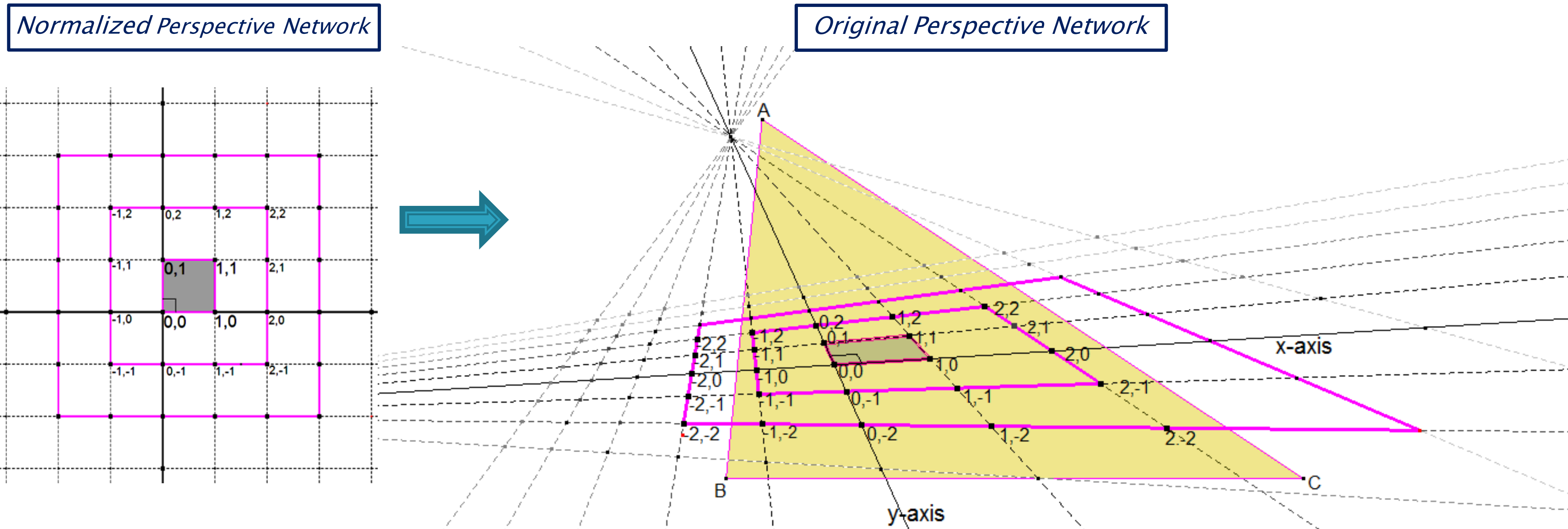
It also can be used to look for interesting "missing points".
For example (1, -1) and (5, -5).

- All coordinates are expressed in rational numbers. A rational number is a number that can be expressed as a fraction p/q where p and q are integers.

Examples:

X17 (1/2 , 1/4)
X631 (7/5 , 2/5)
X632 (11/10, 1/10)
X3090 (5/7 , -2/7)
X3091 (1/5 , -4/5)

Normalized Perspective Network



- In this picture it can be seen how rasterpoints from a Normalized Perspective Network are converted back to its Original Perspective Network.

- The raster of integer-coordinates are simply the intersectionpoints of several basic lines from the Original Perspective Network.

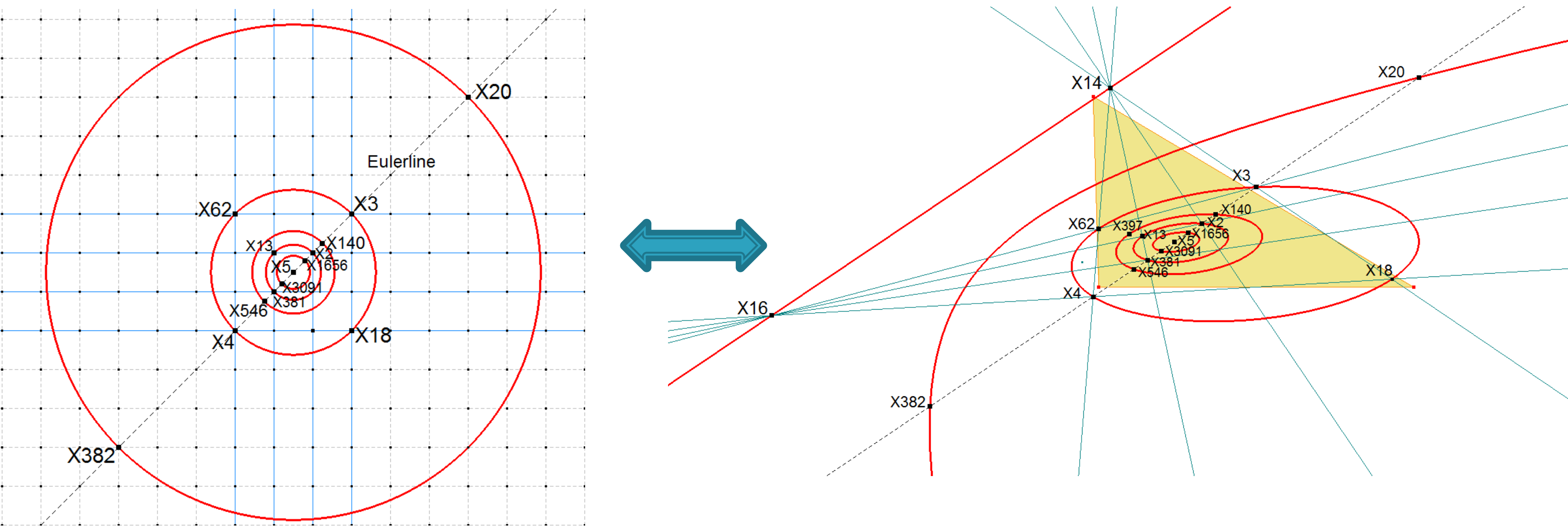
- Note: Only a limited number of these rasterpoints are registered in ETC.
  However they would be nice candidates because they occur on several familiar lines.

Normalized Perspective Network

Concentric Perspective Conics (CPC's)

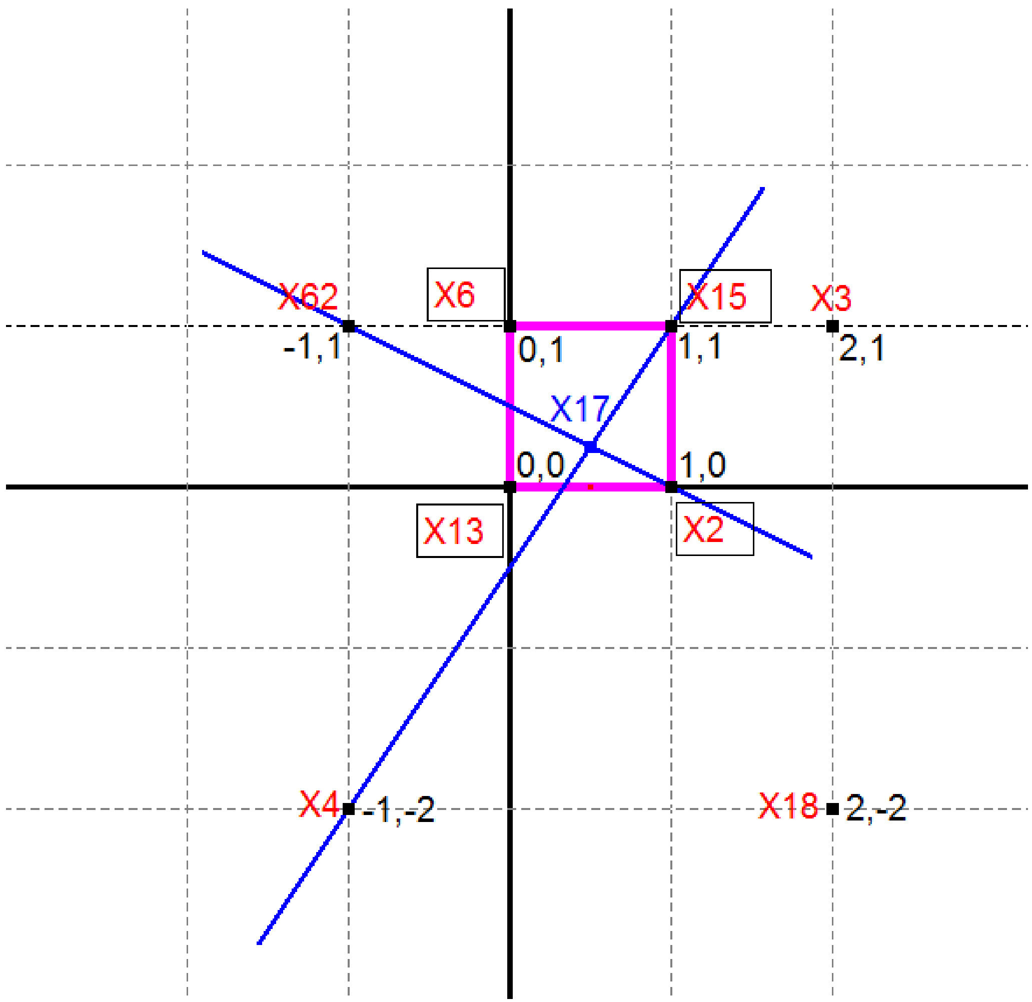
The spread in the field generated by a Perspective Network can be understood by observing the contourlines of Concentric Perspective Conics (CPC's).
All conics are widening from X5 because X5 has a centerfunction in this field.

Normalized Perspective Network | *Original Perspective Network*



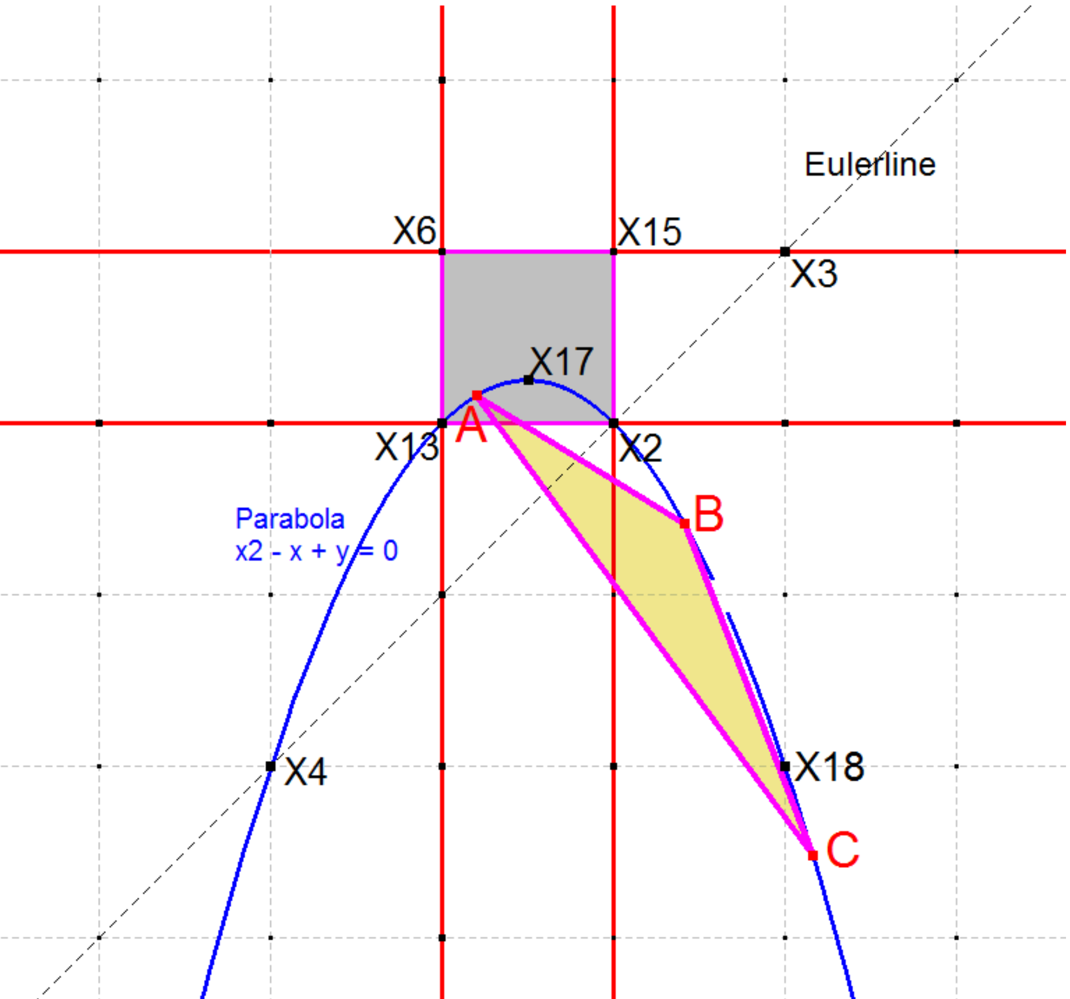
In this case the line X14.X16 is a degenerate conic.

Normalized Perspective Network



- In the Normalized Perspective Network the same collinearities exist as in its Original Perspective Network.
- It is well known that X17 is the intersectionpont of lines X2.X62 and X4.X15.
- When drawing these lines (blue coloured) in the Normalized Perspective Network picture it easily can be seen that the normalized coordinates should be rational.
- Some calculationwork shows that the coordinates of X17 are (½ , ¼). They are rational indeed.
- Because all points in a Normalized Perspective Network are intersectionpoints of lines through other points with rational coordinates the result again is a point with rational coordinates.
- So all points in a Normalized Perspective Network are rational !

Normalized Perspective Network



What happens to the vertices A, B and C in the Normalized Perspective Network?

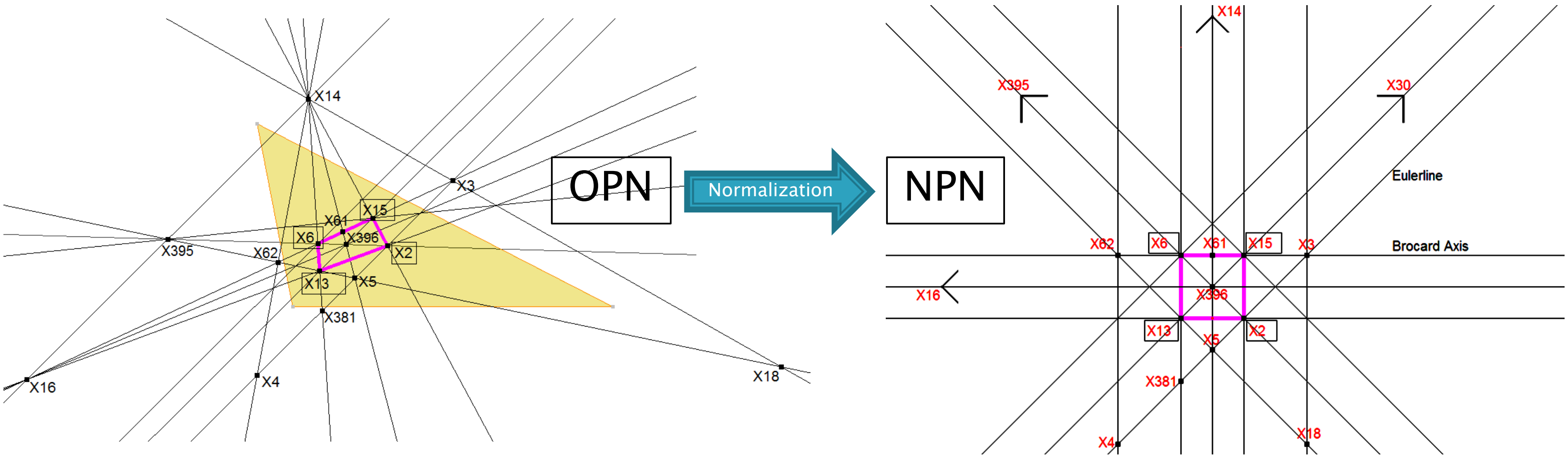
1. The vertices A, B and C are variant.

2. Dependant on the shape of ABC they appear on different places in the normalized picture.

3. The locus of A,B,C is a parabola with equation: $x^2 - x + y = 0$.

4. This parabola also happens to be the transform of the Kiepert Hyperbola into a normalized form.

Normalized Perspective Network



So an Original Perspective Network (OPN) is transformed into a Normalized PerspectiveNetwork (NPN).
The characteristics of an OPN are:

1. It can be constructed in a projective way from a set of four points, no three of which are collinear.
2. New points are generated *only* by making new lines from point to point and by making new points as an intersection of two lines (the projective way).

The characteristics of a NPN are:

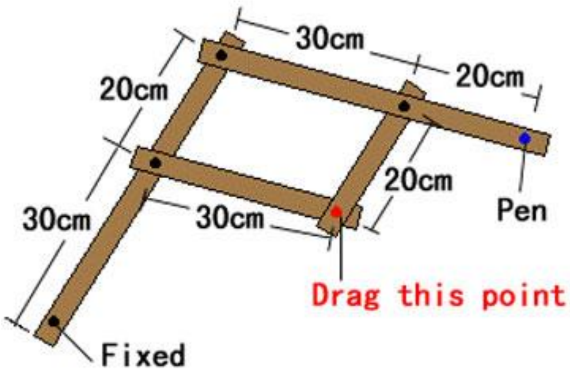
1. All points are invariant,
2. All points have rational cartesian coordinates.

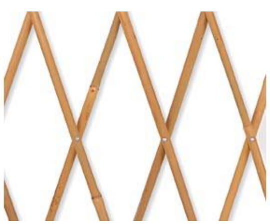
The characteristics of the OPN-NPN-transformation are:

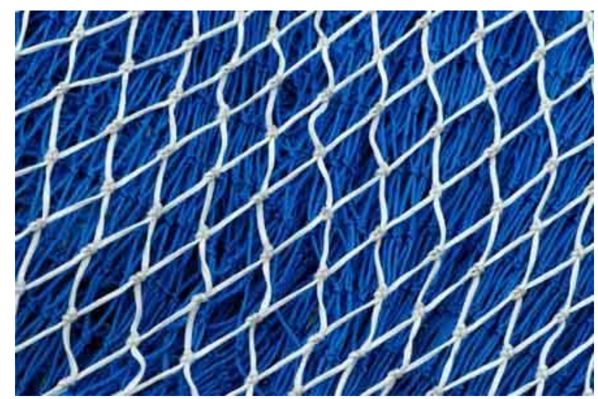
1. The four basicpoints of the OPN are transformed (stretched) in the form of a square.
2. The environment is transformed (stretched) accordingly in a projective way.
3. Collinearity is preserved.

Perspective Network

So we saw that a Perspective Network is a network of linked points and lines.
Nothing more and nothing less.
It is like a fishing-net with the possibility of endless small meshes.
It only hinges when the reference triangle changes.
Here are some practical applications from a Perspective Network:







Perspective Network

Application: Retro Transform

In a Perspective Network (as in Projective Geometry) 2 processes are admitted:

- Make lines
- Make points

Algebraically there are 2 calculations connected to these processes.

Let P1 and P2 be 2 points with homogeneous coordinates P1 (x1:y1:z1) and P2 (x2:y2:z2).
Let L be the line through P1 and P2 with equation p.x + q.y + r.z = 0.
Now p = y1.z2 - y2.z1 and q = z1.x2 - z2.x1 and r = x1.y2 - x2.y1.

Let L1 and L2 be 2 lines with equations p1.x + q1.y + r1.z = 0 and p2.x + q2.y + r2.z = 0.
Let P be the intersectionpoint of L1 and L2 with homogeneous coordinates (x:y:z).
Now x = q1.r2 - q2.r1 and y = r1.p2 - r2.p1 and z = p1.q2 - p2.q1.

As can be seen the algebraic calculations are just the same !
The calculation of a connecting line and an intersectionpoint use the same cross-ratio-formulas! This is a well known fact in Projective Geometry.

Perspective Network

There is a way of describing points with trilinear coordinates.
Trilinear coordinates describe the distance of a point within a triangle to the sides of this triangle.

These are homogeneous coordinates.
This means that no absolute coordinate values are needed for defining the point.
Just the ratio's of coordinates are enough for defining the point.
The 3 coefficients of a line can be considered as homogeneous too.
In notation homogeneous coordinates are separated with a semicolon in stead of a comma.

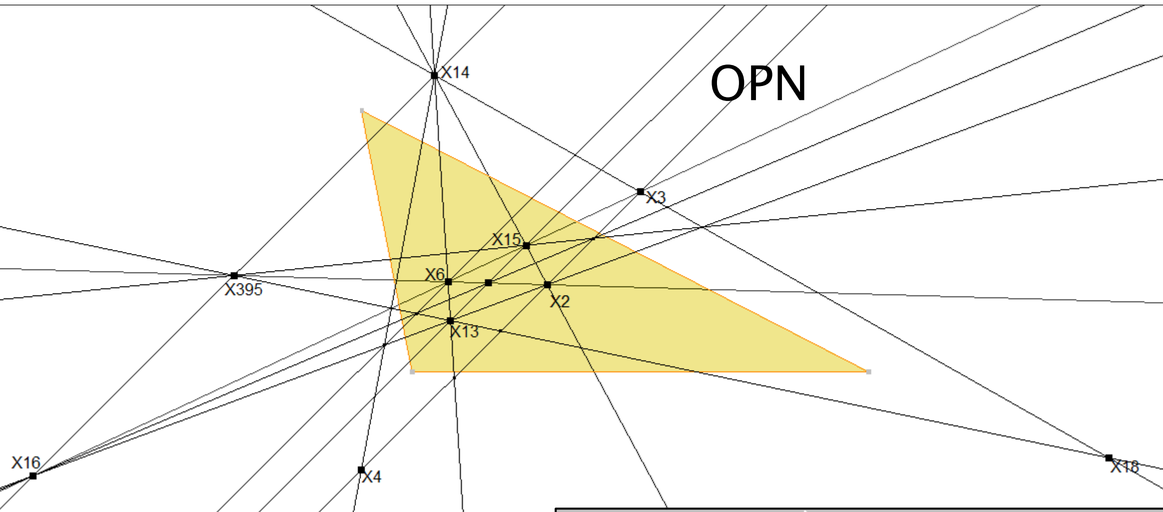
When using this coordinatesystem it is possible to *interchange points and lines*.
Let P be a point with trilinear coordinates (px : py : pz).
Let L be a line with trilinear equation lx.x + ly.y + lz.z = 0.
Now *point P* can be represented *as a line* with equation px.x + py.y + pz.z = 0.
And *line L* can be represented *as a point* with coordinates (lx : ly : lz).

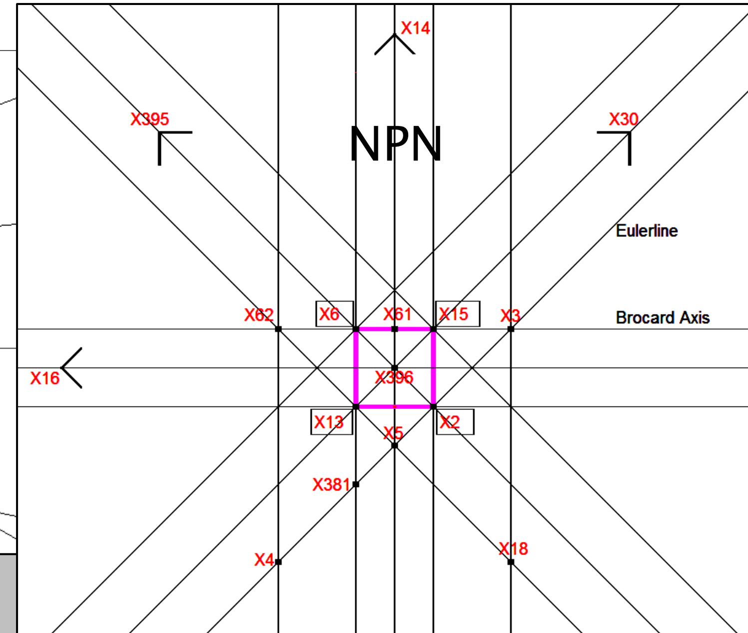
The whole world turns upside down. However the interesting thing is:

- Points that initially were *collinear* in the new setting become a *concurrent* pencil of lines.
- Lines that initially were *concurrent* in the new setting become a set of *collinear* points.

Look at next picture to see the result.

3 presentations of a Perspective Network



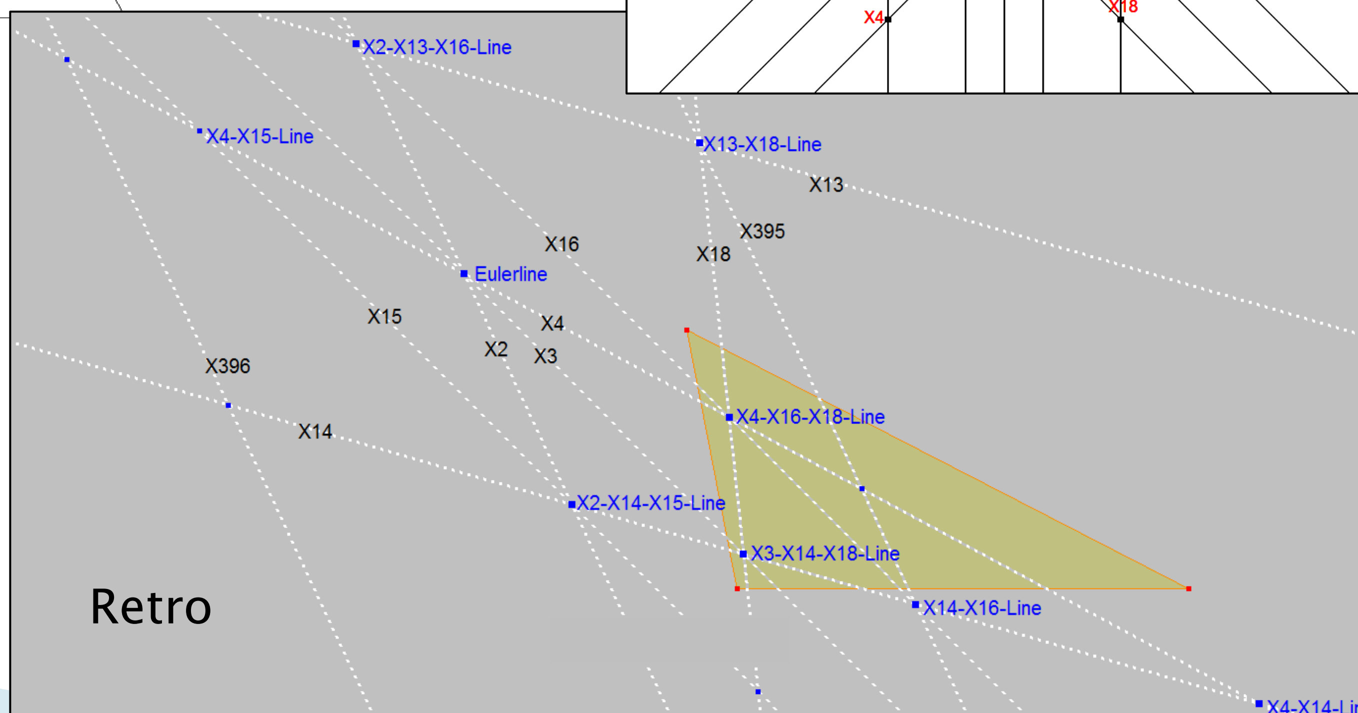


These 3 pictures show exactly the same points and lines however in different formats.

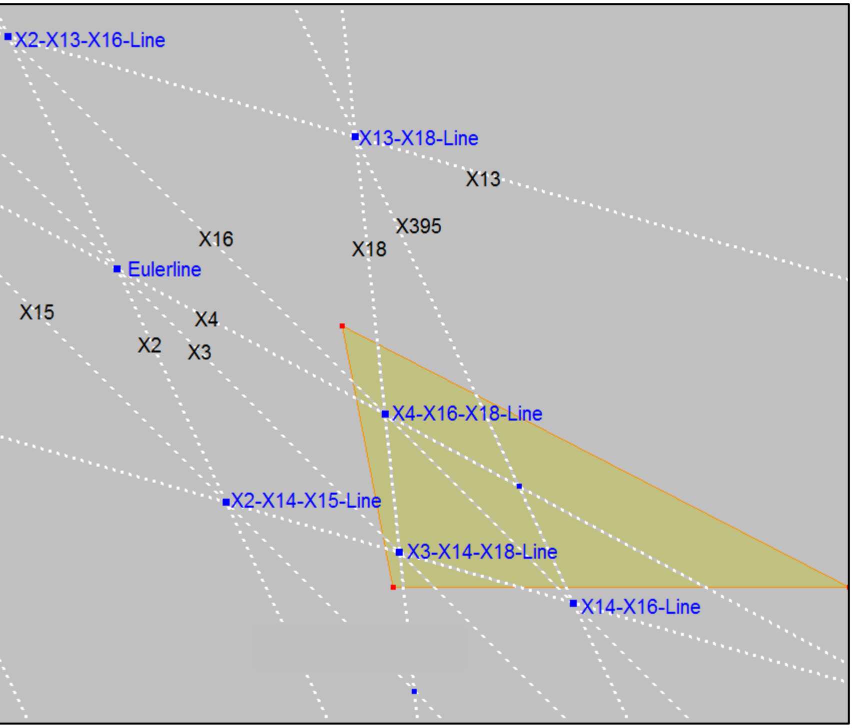
In the Retro picture:

** points have become lines*

** lines have become points*



RetroTransform of a Perspective Network



The transform of X6 is the line at infinity.
The transform of any line through X6 is an infinitypoint.
As a consequence: Two points collinear with X6 are transformed in homothetic retrolines.
Per definition the Retrotransform (just to mention a name) of any Line is the CrossDifference of any 2 points on this line.

THE WORLD UPSIDE DOWN
With the RetroTransform the whole plane gets upside down, because:

- a point is transformed into a line
- a line is transformed into a point
- ‘inside’ triangle is transformed into ‘outside’ triangle
- ‘outside’ triangle is transformed into ‘inside’ triangle
- collinear becomes concurrent
- concurrent becomes collinear.

Combined it gives this effect:

- an internal point is transformed into an external line
- an external point is transformed into an internal line
- an internal line is transformed into an external point
- an external line is transformed into an internal point.

An external line = a line that has no single point in common with the basic triangle.
An internal line = a line that has more than one point in common with the basic triangle.

A triangle consisting of 3 points is transformed into another triangle consisting of 3 lines (whose vertices are the transforms of the lines of the original triangle).

I estimate that more than 95% of the retro-transform-points are non-ETC-points.

Retrotransform of a Perspective Network

Special is too that:
When working with homogeneous coordinates
the normal network of points and lines
and the retro-network of points and lines
successively start overlapping each other per
generationstep after the 3rd generation.

Stated in another way: coordinates of points and coefficients start adopting corresponding values.
Overlap of identical coordinates/coefficients starts to occur after the 3rd generation-step.

Retrotransform of a Perspective Network

Why showing this Retro Picture?

- It is because the whole network of points and lines in a Perspective Network stay intact after this Retro Transformation.

- The development of a Perspective Network made it possible to visualize the Retro Transformation in its elementary structure.
  It's a merit of the phenomenon of a Perspective Network.

This is just one application of a Perspective Network.

Perspective Network

Now let us consider:

Could it be possible there are points :
that are invariant in the NPN
but
without rational coordinates?

The consequence in the Original Perspective Network (OPN) would be:

- These points keep in step with other networkpoints.
  When changing the shape of the basic triangle these points move in the same flow with all other points of the Perspective Network.
  They are “capsulated” in the network.
- But they are not constructible in a projective way.

They would be embedded in the perspective network,
like real numbers are embedded in the set of rational numbers.

Perspective Network

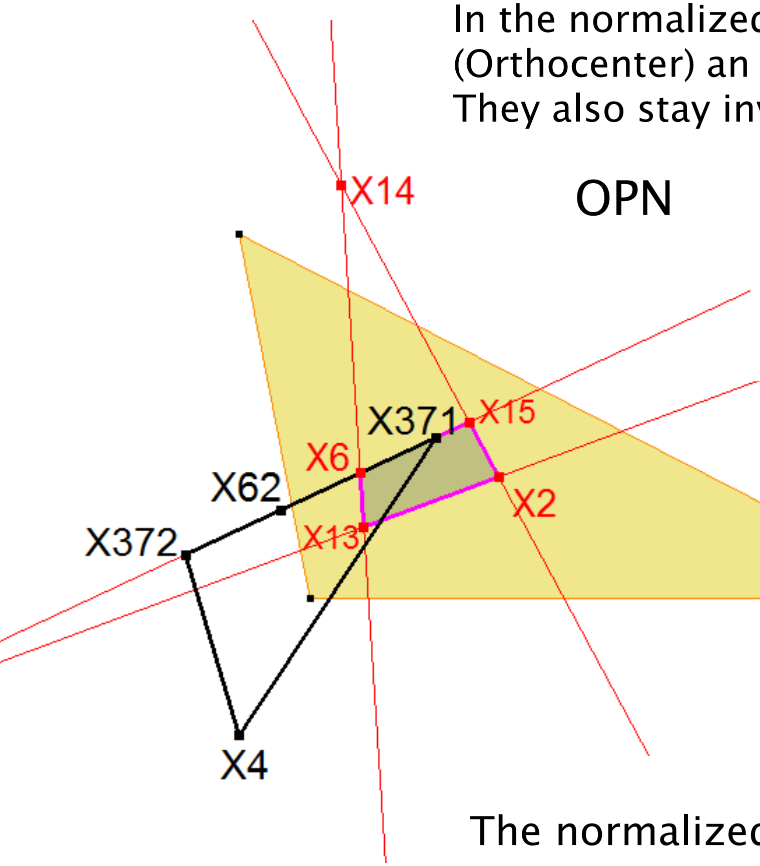
Could it be possible there are points :
that are invariant in the NPN
but
without rational coordinates?

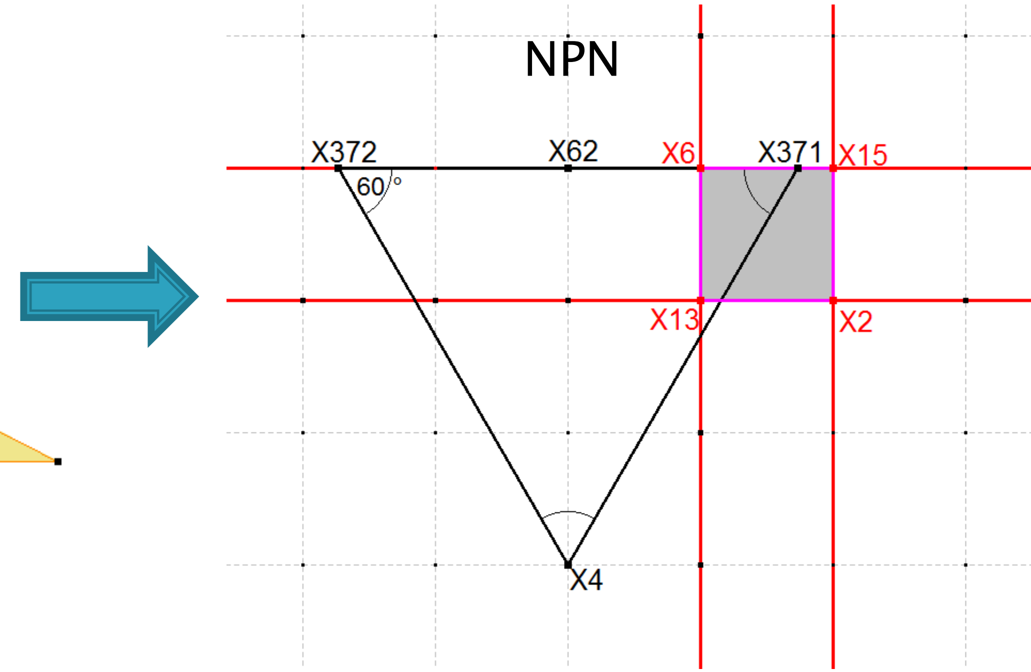
THE ANSWER IS:

YES

Perspective Network

Example 1: The normalized coordinates of X371 and X372 are not rational.
In the normalized picture X371 and X372 form together with X4 (Orthocenter) an equilateral triangle (quite a surprise).
They also stay invariant when changing the shape of the Basic Triangle.





The normalized cartesian coordinates of points in this picture are:

X2	(1 , 0)	X13	(0 , 0)
X4	(−1 , −2)	X15	(1 , 1)
X6	(0 , 1)	X62	(−1 , 1)

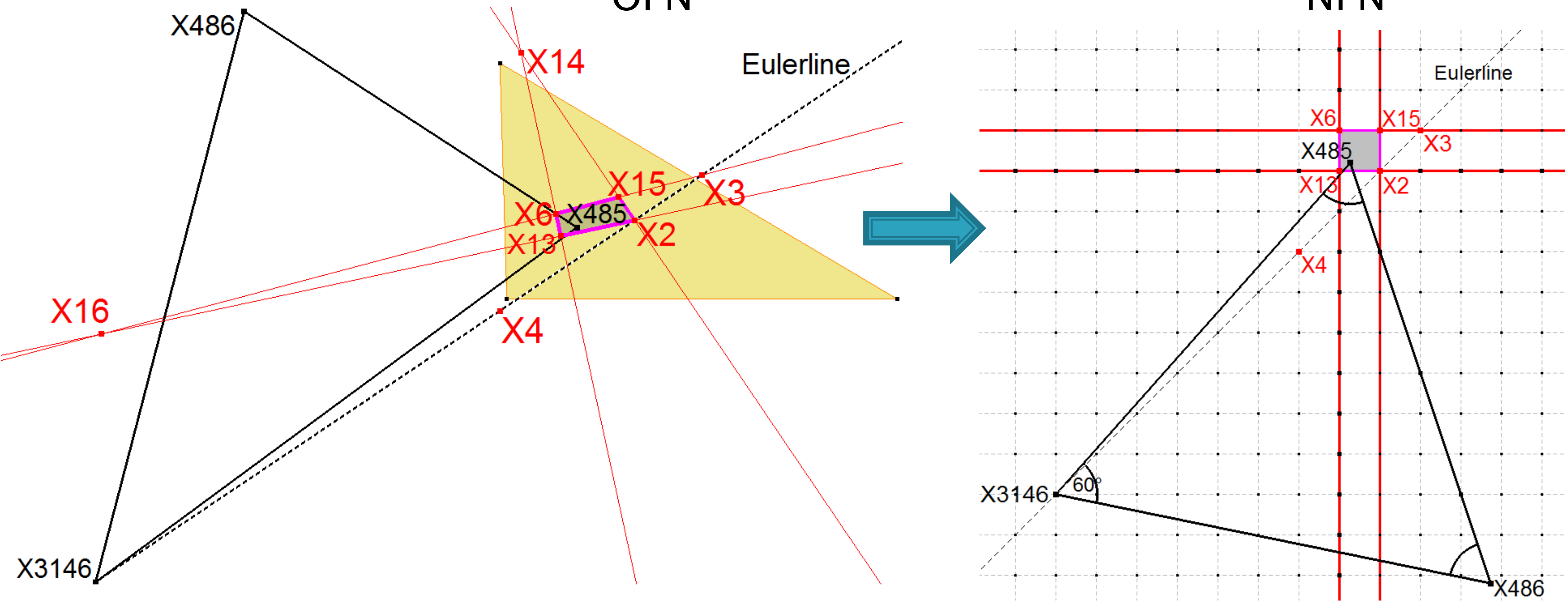
X371	$(-1+\sqrt{3}\ ,\ 1)$
X372	$(-1-\sqrt{3}\ ,\ 1)$

Because of the occurrence of a square root in the coordinates of X371 and X372 the coordinates are not rational and so these points are not constructible from the quadruple X2, X6, X13, X15 in a projective way.

Perspective Network

Example 2: The normalized coordinates of X485 and X486 aren't rational either .
In the normalized picture X485, X486 and X3146 form together an equilateral triangle.

OPN NPN



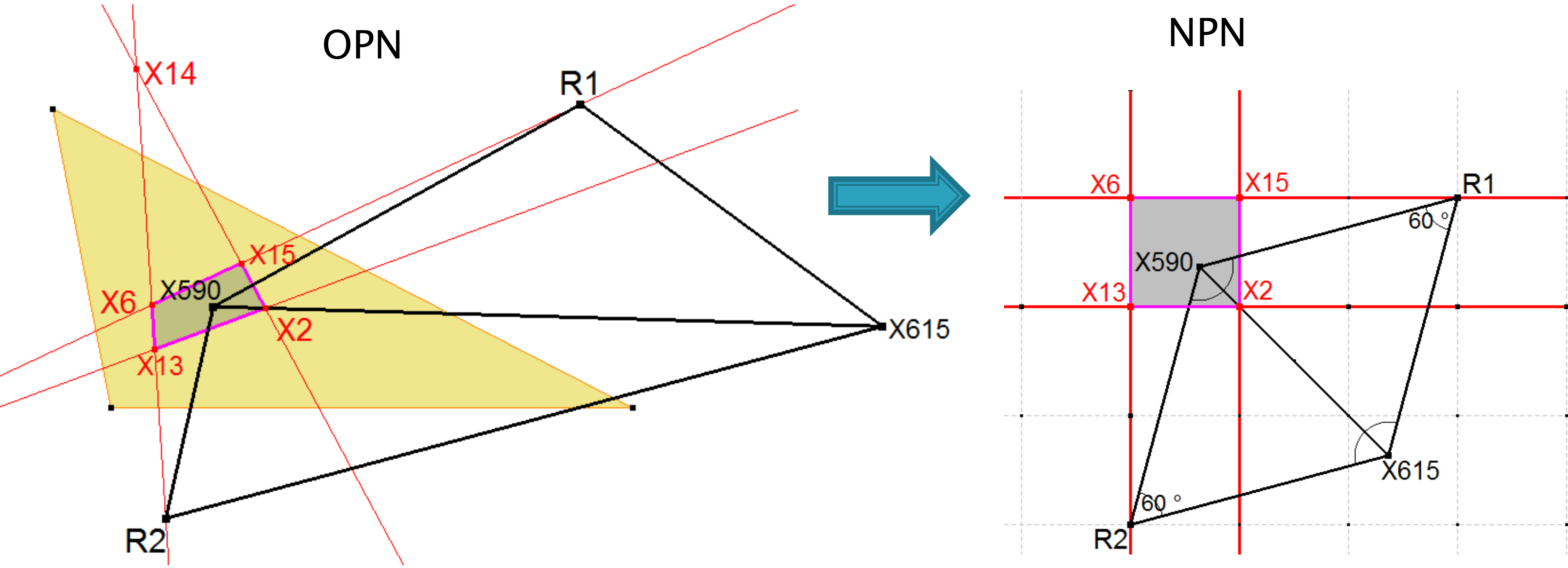
Normalized cartesian coordinates of X485 and X486 are

X485	$(\ 2-\sqrt{3}\ ,\ -5+3\sqrt{3})$
X486	$(\ 2+\sqrt{3}\ ,\ -5-3\sqrt{3})$

Note that all points in the OPN that are located "behind" the perspectivityline X14.X16 swap in the NPN "to the other side". That's because X14 and X16 have become infinitypoints in the NPN. In this case X486 has swapped "to the other side".

Perspective Network

<u>Example 3</u>: The normalized coordinates of X590 and X651 are not rational either.
In the normalized picture X590 and X615 form together with rasterpoints R1 and R2 equilateral triangles (R1 and R2 are non-ETC-centers).



Normalized cartesian coordinates of X590 and X615 are

X590	$(1\frac{1}{2} - \frac{1}{2}\sqrt{3}, -\frac{1}{2} + \frac{1}{2}\sqrt{3})$
X615	$(1\frac{1}{2} + \frac{1}{2}\sqrt{3}, -\frac{1}{2} - \frac{1}{2}\sqrt{3})$

Because of the occurrence of a square root in the coordinates of X590 and X615 the coordinates are not rational and so these points are not constructible from the quadruple X2, X6, X13, X15 in a projective way.

Perspective Network

- So now we found 3 pairs of points that are invariant in the X13–X2–X15–X6 Perspective Network though having no rational coordinates.

- When we run through ETC (range X1–X3500) we find many more of these points.

- As a matter of fact there are *64 extra points* that are invariant in the X13–X2–X15–X6 Perspective Network without having rational coordinates.

- However it is not sure that all of these points can be catagorized as having a square root in its coordinates.

- The final word has not been spoken about this.

Perspective Field – definition

- Now that we found points that are invariant in a Perspective Network though having no rational coordinates we can introduce a new term: *Perspective Field*.

- *A Perspective Field is a set of lines and points that stay invariant after a projective normalization transformation.*

- So now the condition from *being intersectionpoints in a network of lines* changes to *being invariant after a projective normalization transformation*.

- The same projective transformation that initially was set up for being used with points in a network now is used for every point that is available. The result of being invariant in the Normalized picture after projective transformation has become the only condition for belonging to this field.

- Now all 101 points will be listed of the Perspective Field as seen from the X13-X2-X15-X6-window.

Centers in the Normalized Perspective X13-X2-X15-X6-Field (1)

Here are all invariant points wrt X13, X2, X15, X6 (3 pages):

X(xxx) = Rational Coordinates

X(xxx) = Infinity Point

X(xxx) = Coordinates with √3

X(xxx) = Other Real Coordinates

X(2) = CENTROID	(1 , 0)	
X(3) = CIRCUMCENTER	(2 , 1)	
X(4) = ORTHOCENTER	(-1 , -2)	
X(5) = NINE-POINT CENTER	(1/2 , -1/2)	
X(6) = SYMMEDIAN POINT	(0 , 1)	
X(13) = 1st ISOGONIC CENTER (FERMAT POINT)	(0 , 0)	
X(14) = 2nd ISOGONIC CENTER	(0 : 1 : 0)	InfinityPoint
X(15) = 1st ISODYNAMIC POINT	(1 , 1)	
X(16) = 2nd ISODYNAMIC POINT	(1 : 0 : 0)	InfinityPoint
X(17) = 1st NAPOLEON POINT	(1/2 , 1/4)	
X(18) = 2nd NAPOLEON POINT	(2 , -2)	
X(20) = DE LONGCHAMPS POINT	(5 , 4)	
X(30) = EULER INFINITY POINT	(1 : 1 : 0)	InfinityPoint
X(61) = ISOGONAL CONJUGATE OF X(17)	(1/2 , 1)	
X(62) = ISOGONAL CONJUGATE OF X(18)	(-1 , 1)	
X(140) = MIDPOINT OF X(3) AND X(5)	(1 1/4 , 1/4)	
X(371) = KENMOTU POINT (CONGRUENT SQUARES POINT)	(-1+√3 , 1)	
X(372) = {X(3),X(6)}-HARMONIC CONJUGATE OF X(371)	(-1-√3 , 1)	
X(376) = CENTROID OF ANTIPEDAL TRIANGLE OF X(2)	(3 , 2)	
X(381) = MIDPOINT OF X(2) AND X(4)	(0 , -1)	
X(382) = REFLECTION OF CIRCUMCENTER IN ORTHOCENTER	(-4 , -5)	
X(395) = MIDPOINT OF X(14) AND X(16)	(1 : -1 : 0)	InfinityPoint
X(396) = MIDPOINT OF X(13) AND X(15)	(1/2 , 1/2)	
X(397) = CROSSPOINT OF X(4) AND X(17)	(-1/4 , 1/4)	
X(398) = CROSSPOINT OF X(4) AND X(18)	(1/2 , 2 1/2)	
X(485) = VECTEN POINT	(2-√3 , -5+3√3)	
X(486) = INNER VECTEN POINT	(2+√3 , -5-3√3)	
X(546) = MIDPOINT OF X(4) AND X(5)	(-1/4 , -1 1/4)	
X(547) = MIDPOINT OF X(2) AND X(5)	(3/4 , -1/4)	
X(548) = MIDPOINT OF X(5) AND X(20)	(2 3/4 , 1 3/4)	
X(549) = MIDPOINT OF X(2) AND X(3)	(1 1/2 , 1/2)	
X(550) = MIDPOINT OF X(3) AND X(20)	(3 1/2 , 2 1/2)	
X(590) = ISOGONAL CONJUGATE OF X(588)	(1½-½√3 , -½+½√3)	
X(615) = ISOGONAL CONJUGATE OF X(589)	(1½+½√3 , -½-½√3)	
X(631) = 3/5*OG	(1 2/5 , 2/5)	
X(632) = 9/10*OG	(1 1/10 , 1/10)	

Centers in the Normalized Perspective X13-X2-X15-X6-Field (2)

X(xxx) = Rational Coordinates

X(xxx) = Infinity Point

X(xxx) = Coordinates with √3

X(xxx) = Other Real Coordinates

X(1131) = ARCTAN(2) KIEPERT POINT	(-7+4√3 , -104+60√3)
X(1132) = ARCTAN(-2) KIEPERT POINT	(-7-4√3 , -104-60√3)
X(1139) = OUTER PENTAGON POINT	(-0.2798 , -0.358)
X(1140) = INNER PENTAGON POINT	(-3.5743 , -16.3502)
X(1151) = ISOGONAL CONJUGATE OF X(1131)	(8-4√3, 1)
X(1152) = ISOGONAL CONJUGATE OF X(1132)	(8+4√3, 1)
X(1327) = ARCTAN(3) KIEPERT POINT	(-0.2679 , -0.3397)
X(1328) = ARCTAN(-3) KIEPERT POINT	(-3.7321 , -17.6603)
X(1587) = POINT CASTOR I	((1-2√3)/11 , (14-6√3)/11)
X(1588) = POINT CASTOR II	((1+2√3)/11 , (14+6√3)/11)
X(1656) = INTERSECTION OF EULER LINE AND LINE X(17)X(18)	(4/5 , -1/5)
X(1657) = {X(3),X(4)}-HARMONIC CONJUGATE OF X(1656)	(8 , 7)
X(2041) = 1st EULER-VECTEN-GIBERT POINT	(-4-3√3 , -5-3√3)
X(2042) = 2nd EULER-VECTEN-GIBERT POINT	(-4+3√3 , -5+3√3)
X(2043) = 3rd EULER-VECTEN-GIBERT POINT	(2+√3 , 1+√3)
X(2044) = 4th EULER-VECTEN-GIBERT POINT	(2-√3 , 1-√3)
X(2045) = 5th EULER-VECTEN-GIBERT POINT	((14+3√3)/13 , (1+3√3)/13)
X(2046) = 6th EULER-VECTEN-GIBERT POINT	((14-3√3)/13 , (1-3√3)/13)
X(2671) = 1st GOLDEN ARBELOS POINT	(0.3879 , 0.2374)
X(2672) = 2nd GOLDEN ARBELOS POINT	(2.5782 , -4.0688)
X(2673) = ISOGONAL CONJUGATE OF X(2671)	(8 √ 3/(+9+4 √ 5+3 √ 5), 1)
X(2674) = ISOGONAL CONJUGATE OF X(2672)	(8 √ 3/(-9+4 √ 5-3 √ 5), 1)
X(2675) = INTERSECTION X(2671)X(2673) ^ X(2672)X(2674)	(0.034 , -0.966)
X(2676) = INTERSECTION OF EULER LINE AND X(2671)X(2672)	(0.6743 , -0.3257)
X(3068) = INTERSECTION OF X(2)X(6) AND X(4)X(371)	(-3+2√3 , 4-2√3)
X(3069) = INTERSECTION OF X(2)X(6) AND X(4)X(372)	(-3-2√3 , 4+2√3)
X(3070) = INTERSECTION OF X(4)X(6) AND X(5)X(372)	((1-√3)/2 , (5-3√3)/2)
X(3071) = INTERSECTION OF X(4)X(6) AND X(5)X(371)	((1+√3)/2 , (5+3√3)/2)
X(3090) = INTERSECTION OF LINES X(2)X(3) AND X(11)X(1058)	(5/7 , -2/7)
X(3091) = INTERSECTION OF LINES X(2)X(3) AND X(11)X(153)	(1/5 , -4/5)
X(3146) = X(253)-CEVA CONJUGATE OF X(2)	(-7 , -8)
X(3311) = INTERSECTION OF LINES X(3)X(6) AND X(4)X(1131)	((-2+4√3)/11 , 1)
X(3312) = INTERSECTION OF LINES X(3)X(6) AND X(4)X(1132)	((-2-4√3)/11 , 1)
X(3316) = ISOGONAL CONJUGATE OF X(3311)	((13-4√3)/11 , (-74+60√3)/121)
X(3317) = ISOGONAL CONJUGATE OF X(3312)	((13+4√3)/11 , (-74-60√3)/121)

Centers in the Normalized Perspective X13–X2–X15–X6–Field (3)

X(xxx) = Rational Coordinates

X(xxx) = Infinity Point

X(xxx) = Coordinates with √3

X(xxx) = Other Real Coordinates

X(3364) = COS (A – 5π/12) POINT	(2–√3 , 1)
X(3365) = SIN (A – 5π/12) POINT	(2+√3 , 1)
X(3366) = SEC (A – 5π/12) POINT	(–1+√3 , –5+3√3)
X(3367) = CSC (A – 5π/12) POINT	(–1–√3 , –5–3√3)
X(3368) = COS (A – 2π/5) POINT	(2 / (1 +√3 Tan (2π/5)) , 1)
X(3369) = SIN (A – 2π/5) POINT	(2 / (1 –√3 Cot (2π/5)) , 1)
X(3370) = SEC (A – 2π/5) POINT	((–1 + 2 Cos (4π/5)) / (–2 + Cos (4π/5) –√3 Sin (4π/5)) , (1 –√3 Sin (4π/5) + Cos (4π/5)) / (–2 + Cos (4π/5) –√3 Sin (4π/5))
X(3371) = COS (A – 3π/8) POINT	(2 / (1 +√3 +√6) , 1)
X(3372) = SIN (A – 3π/8) POINT	(2 / (1 +√3 –√6) , 1)
X(3373) = SEC (A – 3π/8) POINT	((2 +2√2)) / (4 +√2 +√6) , (–2 +√2 +√6)) / (4 +√2 +√6))
X(3374) = CSC (A – 3π/8) POINT	((2 –2√2)) / (4 –√2 –√6) , (–2 –√2 –√6)) / (4 –√2 –√6))
X(3379) = COS (A – π/5) POINT	(2 / (1 +√3 Tan (π/5)) , 1)
X(3380) = SIN (A – π/5) POINT	(2 / (1 –√3 Cot (π/5)) , 1)
X(3381) = SEC (A – π/5) POINT	((–1 + 2 Cos (2π/5)) / (–2 + Cos (2π/5) –√3 Sin (2π/5)) , (1 –√3 Sin (2π/5) + Cos (2π/5)) / (–2 + Cos (2π/5) –√3 Sin (2π/5))
X(3382) = CSC (A – π/5) POINT	((1 + 2 Cos(2π/5)) / (2 + Cos(2π/5) –√3 Sin(2π/5)) , (–1 –√3 Sin (2π/5) + Cos (2π/5)) / (2 + Cos (2π/5) –√3 Sin (2π/5))
X(3385) = COS (A – π/8) POINT	(2 / (1 +√3 –√6) , 1)
X(3386) = SIN (A – π/8) POINT	(2 / (1 –√3 –√6) , 1)
X(3387) = SEC (A – π/8) POINT	((2 – 2√2) / (4 –√2 +√6) , (–2 –√2 +√6)) / (4 –√2 +√6))
X(3388) = CSC (A – π/8) POINT	((2 +2√2) / (4 +√2 –√6) , (–2 +√2 –√6)) / (4 +√2 –√6))
X(3389) = COS (A – π/12) POINT	((1+√3) / 2 , 1)
X(3390) = SIN (A – π/12) POINT	((1–√3) / 2 , 1)
X(3391) = SEC (A – π/12) POINT	((1–√3) / 2 , –1/2)
X(3392) = CSC (A – π/12) POINT	((1+√3) / 2 , –1/2)
X(3393) = COS (A + π/5) POINT	(2 / (1 –√3 Tan (π/5)) , 1)
X(3394) = SIN (A + π/5) POINT	(2 / (1 +√3 Cot (π/5)) , 1)
X(3395) = COS (A + 2π/5) POINT	(2 / (1 –√3 Tan (2π/5)) , 1)
X(3396) = SIN (A + 2π/5) POINT	(2 / (1 +√3 Cot (2π/5)) , 1)
X(3397) = SEC (A + 2π/5) POINT	((–1 + 2 Cos (4π/5)) / (–2 + Cos (4π/5) +√3 Sin (4π/5)) , (1 +√3 Sin (4π/5) + Cos (4π/5)) / (–2 + Cos (4π/5) +√3 Sin (4π/5))
X(3411) = 1st BROCARD–KIEPERT–FERMAT CUSP	(–4 , 4)
X(3412) = 2nd BROCARD–KIEPERT–FERMAT CUSP	(1/2 , 5/8)

This makes a total of 101 Centers in the Perspective Field defined by quadruple X13–X2–X15–X6.

Perspective Fields

Thus far we only investigated a Perspective Network and Perspective Field originating from quadruple X13, X2, X15, X6.

A new question arises:

do other Perspective Networks or Perspective Fields exist,
originating from other quadruples,
with again many points involved ?

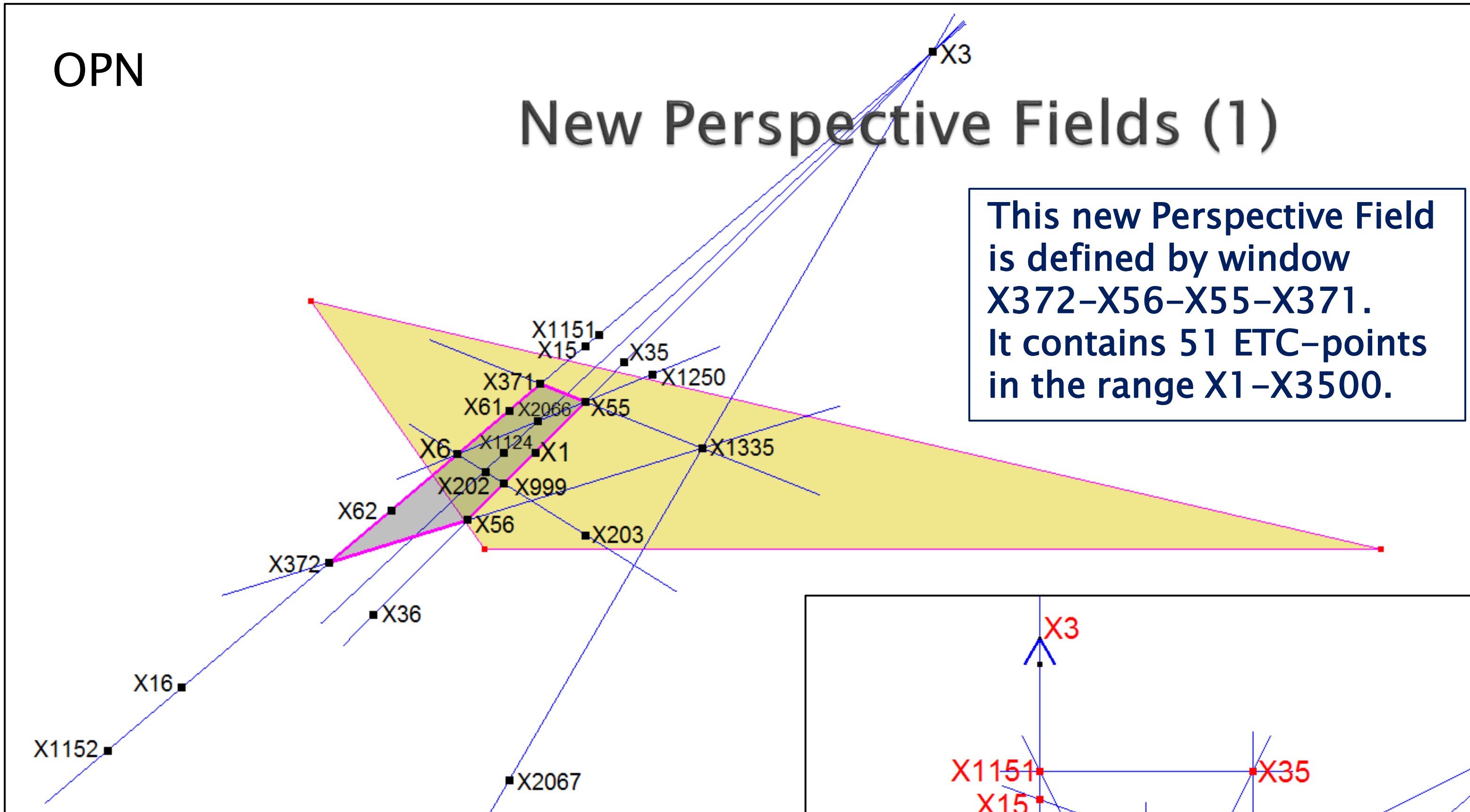
The answer is again:

YES

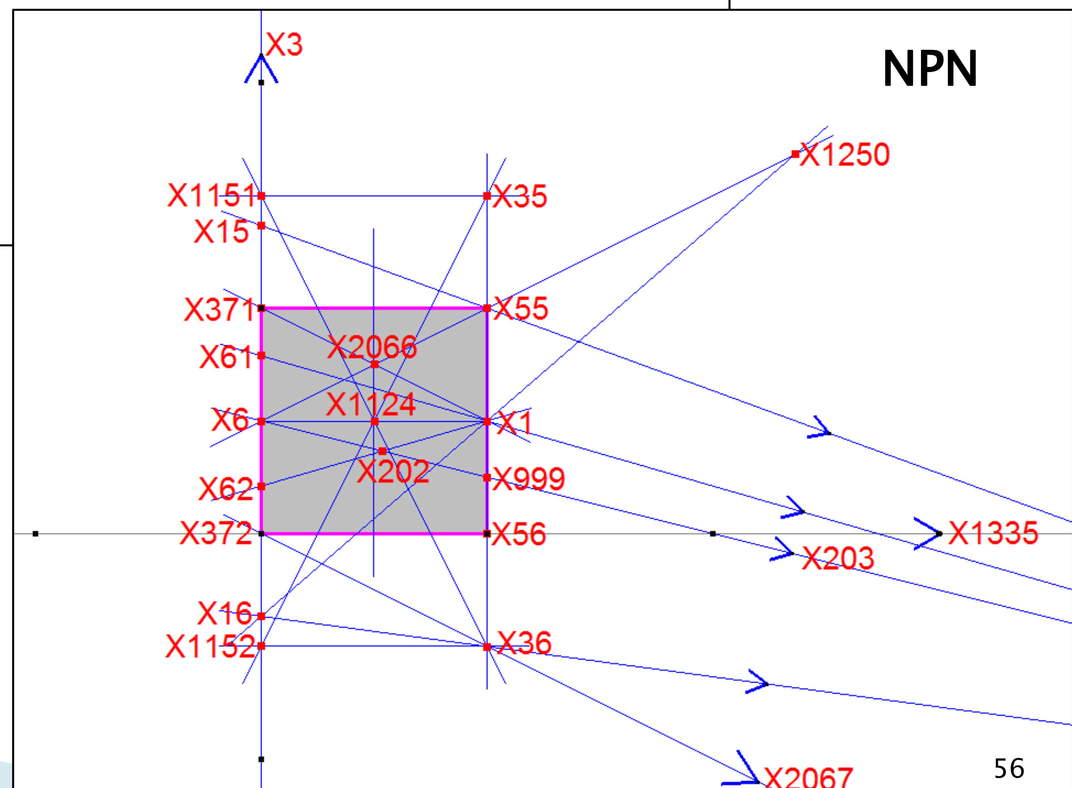
Here are some examples >

New Perspective Fields (1)

This new Perspective Field is defined by window X372–X56–X55–X371.
It contains 51 ETC–points in the range X1–X3500.



Only 23 of the 51 fieldpoints are displayed.



New Perspective Fields (1)

The 51 points of this new Perspective Field are:

X(xxx) = Rational Coordinates

X(xxx) = Infinity Point

X(xxx) = Coordinates with V3

X(xxx) = Other real Coordinates

		Cartesian Coordinates
X(1)	INCENTER	(1 , 1/2)
X(3)	CIRCUMCENTER	0 : 1 : 0
X(6)	SYMMEDIAN POINT (LEMOINE POINT, GREBE POINT)	(0 , 1/2)
X(15)	1st ISODYNAMIC POINT	(0 , (1 + V3)/2)
X(16)	2nd ISODYNAMIC POINT	(0 , (1 - V3)/2)
X(35)	{X(1),X(3)}-HARMONIC CONJUGATE OF X(36)	(1 , 3/2)
X(36)	INVERSE-IN-CIRCUMCIRCLE OF INCENTER	(1 , -1/2)
X(55)	INSIMILICENTER(CIRCUMCIRCLE, INCIRCLE)	(1 , 1)
X(56)	EXSIMILICENTER(CIRCUMCIRCLE, INCIRCLE)	(1 , 0)
X(61)	ISOGONAL CONJUGATE OF X(17)	(0 , 1/2 + 1/6 V3)
X(62)	ISOGONAL CONJUGATE OF X(18)	(0 , 1/2 - 1/6 V3)
X(202)	X(1)-CEVA CONJUGATE OF X(15)	(4 - 2V3 , (-1 + V3)/2)
X(203)	X(1)-CEVA CONJUGATE OF X(16)	(4 + 2V3 , (-1 - V3)/2)
X(371)	KENMOTU POINT (CONGRUENT SQUARES POINT)	(0 , 1)
X(372)	{X(3),X(6)}-HARMONIC CONJUGATE OF X(371)	(0 , 0)
X(999)	MIDPOINT OF X(1) AND X(57)	1 , 1/4)
X(1124)	ISOGONAL CONJUGATE OF X(1123)	(1/2 , 1/2)
X(1151)	ISOGONAL CONJUGATE OF X(1131)	(0 , 1 1/2)
X(1152)	ISOGONAL CONJUGATE OF X(1132)	(0 , -1/2)
X(1250)	ISOGONAL CONJUGATE OF X(1081)	((3 + V3)/2 , (5 + V3)/4)
X(1335)	{X(1),X(6)}-HARMONIC CONJUGATE OF X(1124)	1 : 0 : 0
X(2066)	POINT CAROLI I	(1/2 , 3/4)
X(2067)	POINT CAROLI II	1 : -1/2 : 0
X(2307)	X(2)-ISOCONJUGATE OF X(1251)	((1 - V3)/2, (-3 + V3)/4)

New Perspective Fields (1)

X(xxx) = Rational Coordinates

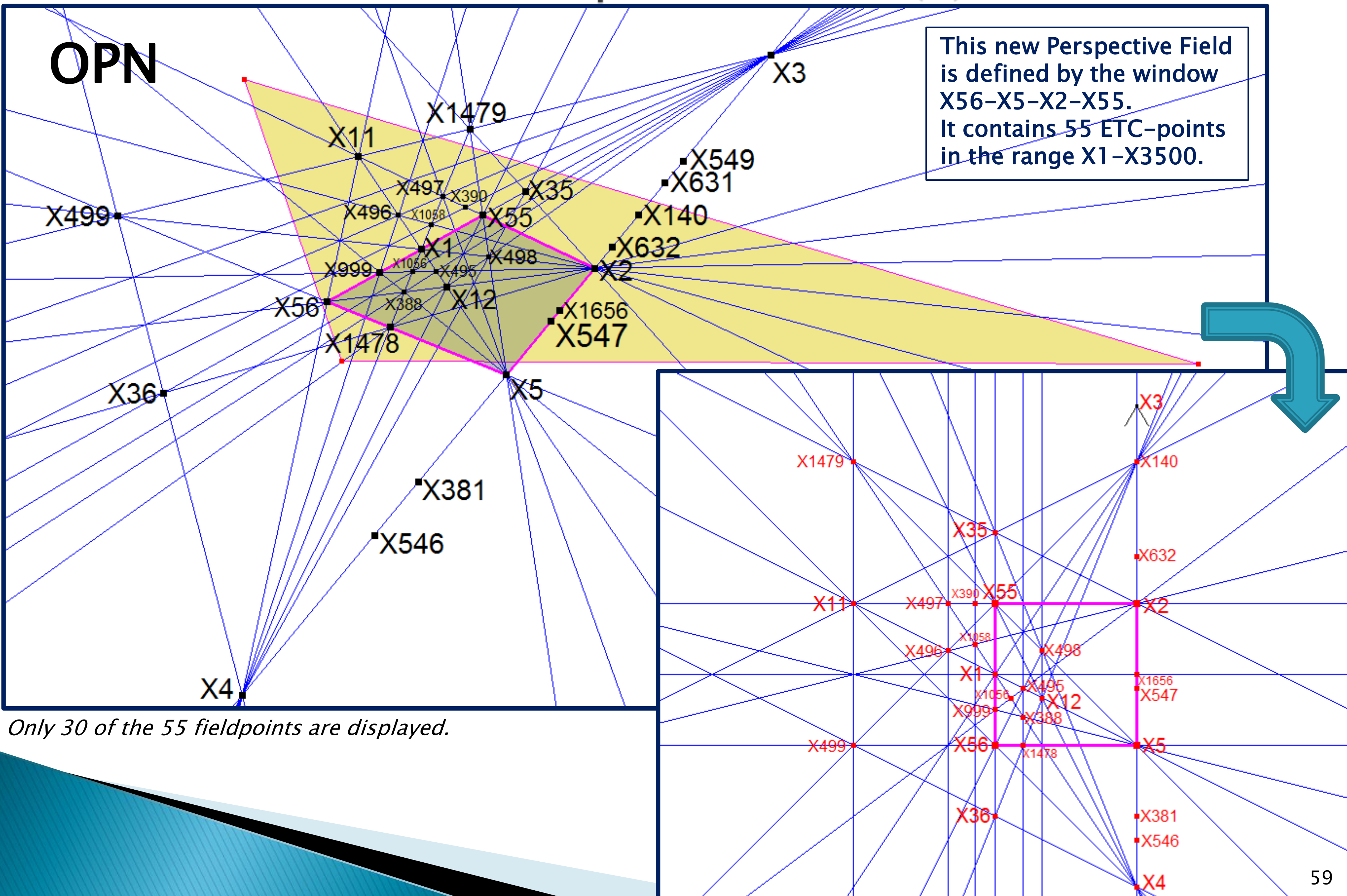
X(xxx) = Infinity Point

X(xxx) = Coordinates with V3

X(xxx) = Other real Coordinates

		Cartesian Coordinates
X(2673)	ISOGONAL CONJUGATE OF X(2671)	(0 , (4 - V5)/2)
X(2674)	ISOGONAL CONJUGATE OF X(2672)	(0 , (-2 + V5)/2)
X(3295)	INTERSECTION OF LINES X(1)X(3) AND X(4)X(390)	(1 , 3/4)
X(3297)	INTERSECTION OF LINES X(485)X(496) AND X(486)X(495)	(2/3 , 1/2)
X(3298)	INTERSECTION OF LINES X(485)X(495) AND X(486)X(496)	(2 , 1/2)
X(3299)	INTERSECTION OF LINES X(35)X(372) AND X(36)X(371)	(1/3 , 1/2)
X(3301)	INTERSECTION OF LINES X(35)X(371) AND X(36)X(372)	(-1 , 1/2)
X(3303)	INTERSECTION OF LINES X(35)X(371) AND X(36)X(372)	(1 , 2/3)
X(3304)	INTERSECTION OF LINES X(1)X(3) AND X(495)X(499)	(1 , 1/3)
X(3311)	INTERSECTION OF LINES X(3)X(6) AND X(4)X(1131)	(0 , 3/4)
X(3312)	INTERSECTION OF LINES X(3)X(6) AND X(4)X(1132)	(0 , 1/4)
X(3364)	COS(A - 5π/12) POINT	(0 , (1 + cot(5π/12)) / 2)
X(3365)	SIN(A - 5π/12) POINT	(0 , (1 - tan(5π/12)) / 2)
X(3368)	COS(A - 2π/5) POINT	(0 , (1 + cot(2π/5)) / 2)
X(3369)	SIN(A - 2π/5) POINT	(0 , (1 - tan(2π/5)) / 2)
X(3371)	COS(A - 3π/8) POINT	(0 , (1 + cot(3π/8)) / 2)
X(3372)	SIN(A - 3π/8) POINT	(0 , (1 - tan(3π/8)) / 2)
X(3379)	COS(A - π/5) POINT	(0 , (1 + cot(π/5)) / 2)
X(3380)	SIN(A - π/5) POINT	(0 , (1 - tan(π/5)) / 2)
X(3385)	COS(A - π/8) POINT	(0 , (1 + cot(π/8)) / 2)
X(3386)	SIN(A - π/8) POINT	(0 , (1 - tan(π/8)) / 2)
X(3389)	COS(A - π/12) POINT	(0 , (1 + cot(π/12)) / 2)
X(3390)	SIN(A - π/12) POINT	(0 , (1 - tan(π/12)) / 2)
X(3393)	COS(A + π/5) POINT	(0 , (1 - cot(π/5)) / 2)
X(3394)	SIN(A + π/5) POINT	(0 , (1 + tan(π/5)) / 2)
X(3395)	COS(A + 2π/5) POINT	(0 , (1 - cot(2π/5)) / 2)
X(3396)	SIN(A + 2π/5) POINT	(0 , (1 + tan(2π/5)) / 2)

New Perspective Fields (2)



This new Perspective Field is defined by the window X56–X5–X2–X55.
It contains 55 ETC-points in the range X1–X3500.

Only 30 of the 55 fieldpoints are displayed.

New Perspective Fields (2)

The 55 points of this new Perspective Field are:

X(xxx) = Rational Coordinates

X(xxx) = Infinity Point

X(xxx) = Coordinates with V3

X(xxx) = Other real Coordinates

		Cartesian Coordinates
X(1)	= INCENTER	(0 , 1/2)
X(2)	= CENTROID	(1 , 1)
X(3)	= CIRCUMCENTER	(0 : 1 : 0)
X(4)	= ORTHOCENTER	(1 , -1)
X(5)	= NINE-POINT CENTER	(1 , 0)
X(11)	= FEUERBACH POINT	(-1 , 1)
X(12)	= {X(1),X(5)}-HARMONIC CONJUGATE OF X(11)	(1/3 , 1/3)
X(20)	= DE LONGCHAMPS POINT	(1 , -3)
X(30)	= EULER INFINITY POINT	(1 , -2)
X(35)	= {X(1),X(3)}-HARMONIC CONJUGATE OF X(36)	(0 , 1 1/2)
X(36)	= INVERSE-IN-CIRCUMCIRCLE OF INCENTER	(0 , -1/2)
X(55)	= INSIMILICENTER(CIRCUMCIRCLE, INCIRCLE)	(0 , 1)
X(56)	= EXSIMILICENTER(CIRCUMCIRCLE, INCIRCLE)	(0 , 0)
X(140)	= MIDPOINT OF X(3) AND X(5)	(1 , 2)
X(376)	= CENTROID OF THE ANTIPEDAL TRIANGLE OF X(2)	(1 , -5)
X(381)	= MIDPOINT OF X(2) AND X(4)	(1 , -1/2)
X(382)	= REFLECTION OF CIRCUMCENTER IN ORTHOCENTER	(1 , -1 1/2)
X(388)	= INTERSECTION OF LINES X(1)X(4) and X(7)X(8)	(1/5 , 1/5)
X(390)	= REFLECTION OF GERGONNE POINT IN INCENTER	(-1/7 , 1)
X(495)	= JOHNSON MIDPOINT	(1/5 , 2/5)
X(496)	= {X(1),X(5)}-HARMONIC CONJUGATE OF X(495)	(-1/3 , 2/3)
X(497)	= CROSSPOINT OF GERGONNE POINT AND NAGEL POINT	(-1/3 , 1)
X(498)	= YFF CONCURRENT CONGRUENT CIRCLES POINT	(1/3 , 2/3)
X(499)	= {X(1),X(2)}-HARMONIC CONJUGATE OF X(498)	(-1 , 0)
X(546)	= MIDPOINT OF X(4) AND X(5)	(1 , -2/3)
X(547)	= MIDPOINT OF X(2) AND X(5)	(1 , 2/5)
X(548)	= MIDPOINT OF X(5) AND X(20)	(1 , -6)
X(549)	= MIDPOINT OF X(2) AND X(3)	(1 , 4)
X(550)	= MIDPOINT OF X(3) AND X(20)	(1 , -4)

New Perspective Fields (2)

The 55 points of this new Perspective Field are:

X(xxx) = Rational Coordinates

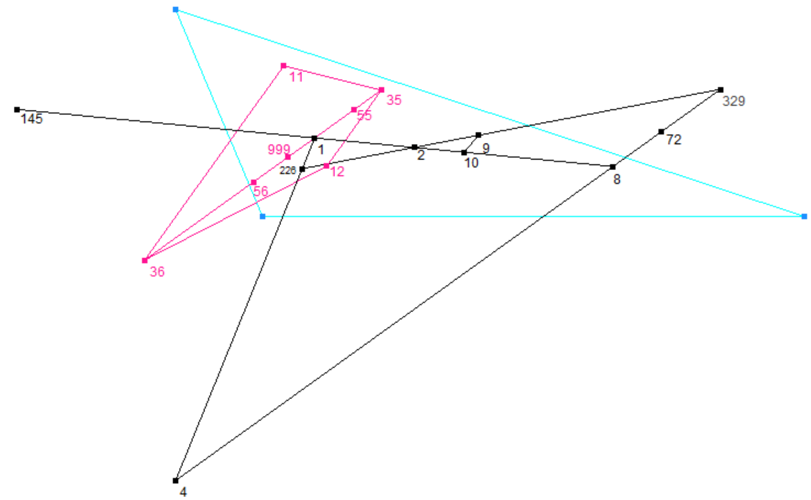
X(xxx) = Infinity Point

X(xxx) = Coordinates with V3

X(xxx) = Other real Coordinates

	Cartesian Coordinates
X(631) = 3/5*OG	(1 , 3)
X(632) = 9/10*OG	(1 , 1 1/3)
X(999) = MIDPOINT OF X(1) AND X(57)	(0 , 1/4)
X(1056) = POINT ALCYONE	(1/9 , 1/3)
X(1058) = POINT ALKALUROPS	(-1/7 , 5/7)
X(1478) = CENTER OF JOHNSON-YFF CIRCLE	(1/3 , 0)
X(1479) = {X(1),X(4)}-HARMONIC CONJUGATE OF X(1478)	(-1 , 2)
X(1656) = INTERSECTION OF EULER LINE AND LINE X(17)X(18)	(1 , 1/2)
X(1657) = {X(3),X(4)}-HARMONIC CONJUGATE OF X(1656)	(1 , -2 1/2)
X(2041) = 1st EULER-VECTEN-GIBERT POINT	(1 , -V3)
X(2042) = 2nd EULER-VECTEN-GIBERT POINT	(1 , +V3)
X(2043) = 3rd EULER-VECTEN-GIBERT POINT	(1 , -2 -V3)
X(2044) = 4th EULER-VECTEN-GIBERT POINT	(1 , -2+V3)
X(2045) = 5th EULER-VECTEN-GIBERT POINT	(1 , +2+V3)
X(2046) = 6th EULER-VECTEN-GIBERT POINT	(1 , +2 -V3)
X(2675) = INTERSECTION OF LINES X(2671)X(2673) and X(2672)X(2674)	(1 , (-25 + 8V5) / 15)
X(2676) = INTERSECTION OF EULER LINE AND LINE X(2671)X(2672)	(1 , (-5 + 4V5) / 15)
X(3058) = INTERSECTION OF LINES X(1)X(30) AND X(2)X(11)	(-1/5 , 1)
X(3085) = INTERSECTION OF LINES X(1)X(2) AND X(4)X(12)	(1/5 , 3/5)
X(3086) = INTERSECTION OF LINES X(1)X(2) AND X(4)X(11)	(-1/3 , 1/3)
X(3090) = INTERSECTION OF LINES X(2)X(3) AND X(11)X(1058)	(1 , 1/3)
X(3091) = INTERSECTION OF LINES X(2)X(3) AND X(11)X(153)	(1 , -1/3)
X(3146) = X(253)-CEVA CONJUGATE OF X(2)	(1 , -1 2/3)
X(3295) = INTERSECTION OF LINES X(1)X(3) AND X(4)X(390)	(0 , 3/4)
X(3303) = INTERSECTION OF LINES X(1)X(3) AND X(496)X(498)	(0 , 2/3)
X(3304) = INTERSECTION OF LINES X(1)X(3) AND X(495)X(499)	(0 , 1/3)

Perspective Fields



A Perspective Field looks like a constellation of stars at heaven.
All these points are like stars.
Each star moves in its own way.
Some stars adjourn together in a flow.
Just like in astronomy we just watch and sometimes we understand why.