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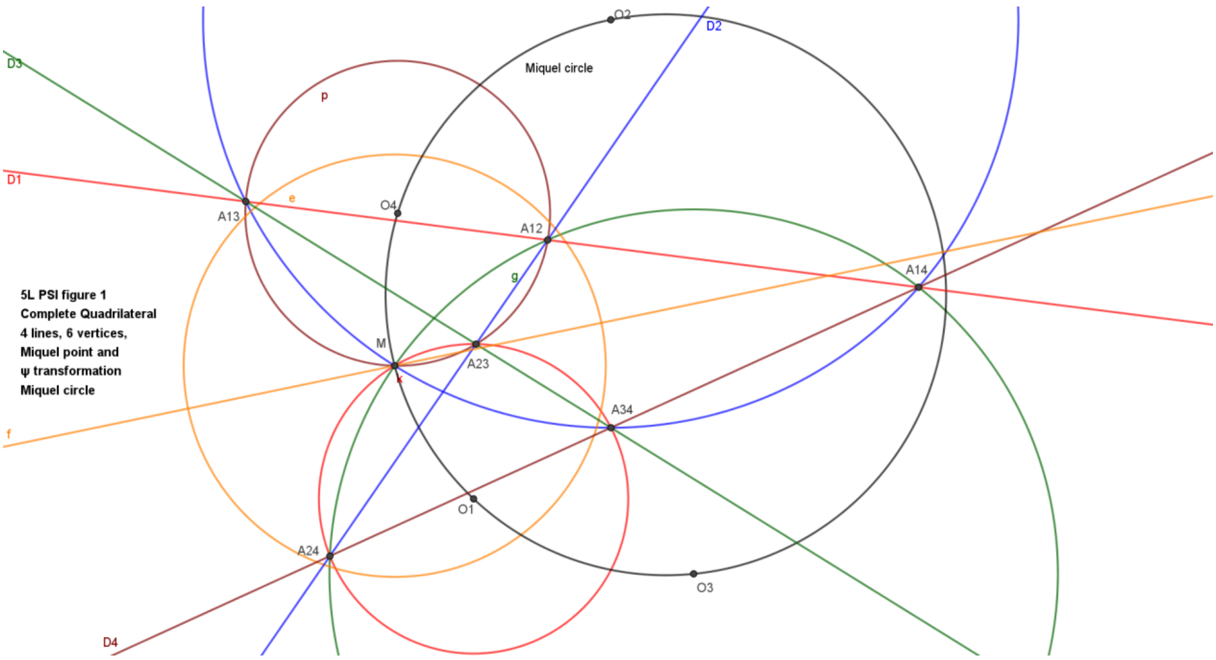
Pentalateral's transformation Ψ or Triple Moebius

A. Complete Quadrilateral

Let's first consider a quadrilateral 4L with it's 6 vertices Aij, it's point QL-P1 and it's Cl-S transformation QL-Tf1, which swaps 2 opposite vertices and the foci of all inscribed conics.

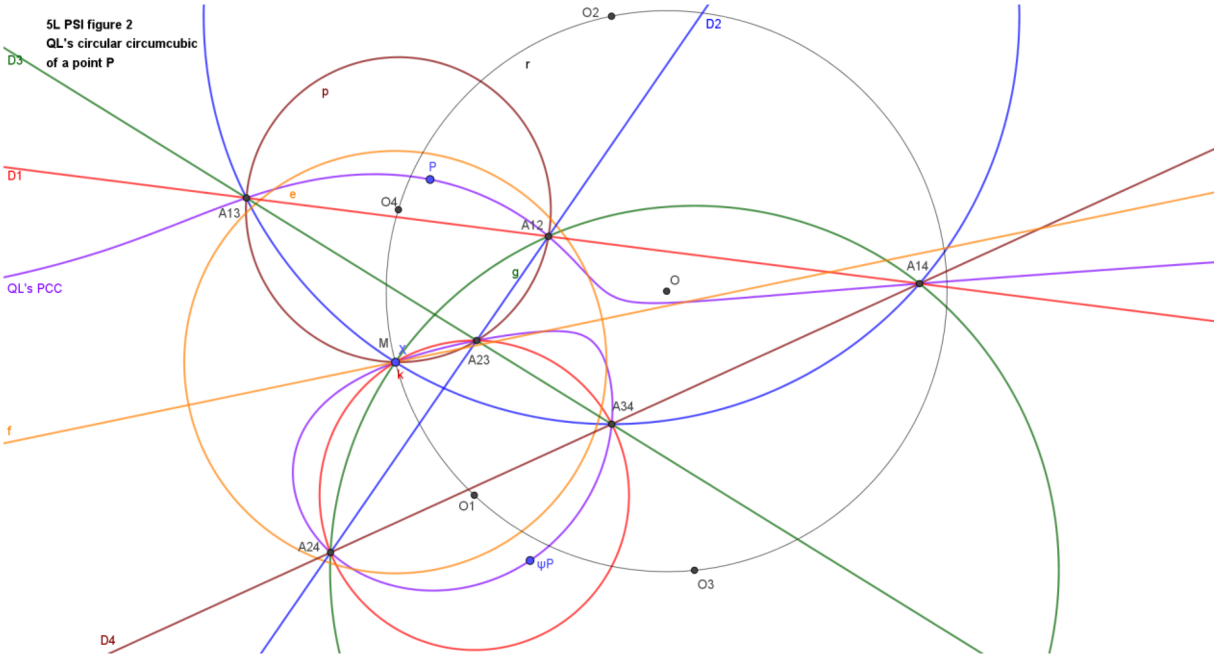
The 4 degenerated cubics formed by a line through 3 vertices and the circumcircle through the 3 other pass through the 6 vertices of the QL, through QL-P1 and through the circular points.

These 9 points form therefore a CB system. The Cl-S swaps each of the 4 lines with the corresponding circle and more generally a point with a point and a line not through QL-P1 with a circle through QL-P1.



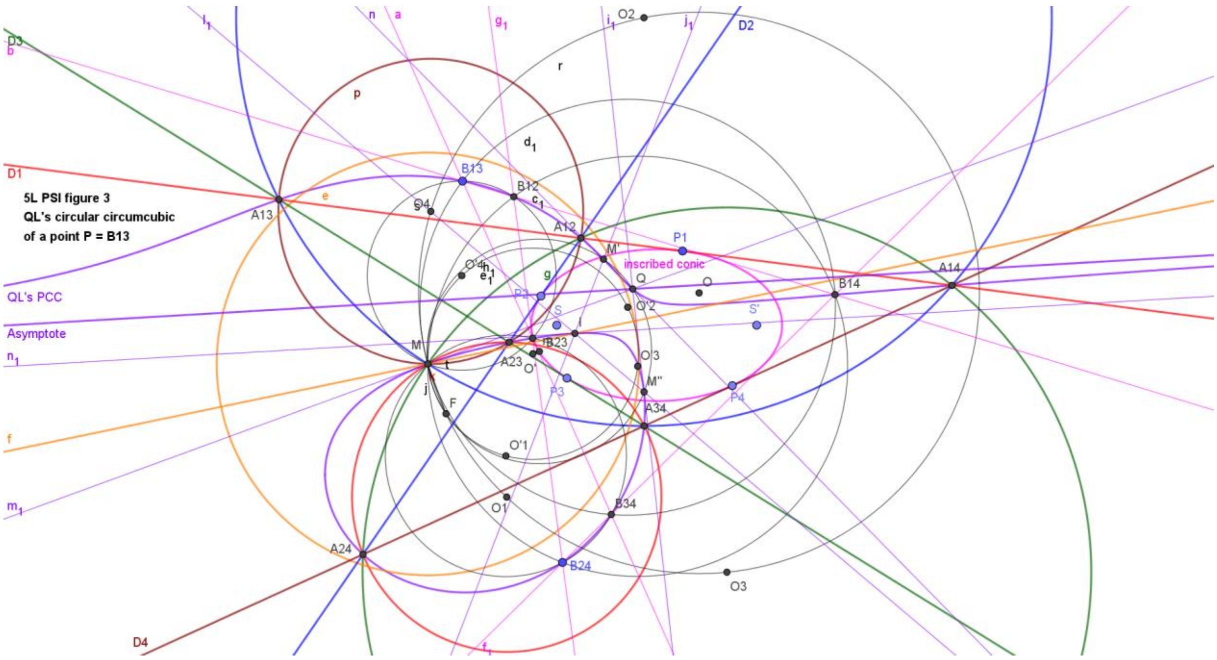
There are an infinity of cubics through these 9 points and we need another point P in order to identify a specific circular circumcubic of the 6 vertices of the QL and QL-P1.

The circular circumcubics of the 6 vertices and a 7th point P pass through QL-P1 and are all invariant in the Cl-S or QL-Tf1 transformation ψ and pass through ψP (CSC partner of P).

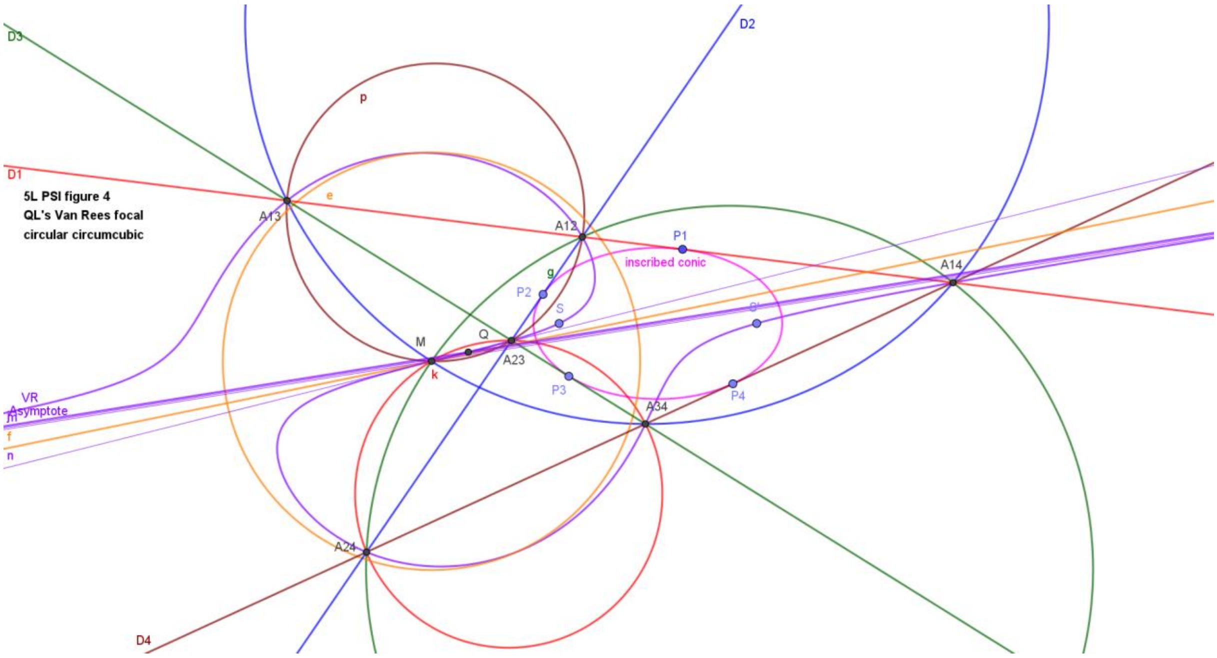


This QL's P circular circumcubic is a 7P-Cu1 and a QA-Cu1 and is generally not focal. It can be seen as the locus of the vertices of the QL's formed by the 2 tangents in P and ψP to all QL's inscribed conics (including the QL's parabola QL-Co1). These cubics are Poncelet-Darboux curves for any inscribed conic, meaning that for any point of the curve the tangents to the conic in this point and it's CSC partner form a 3rd QL inscribed in the curve and circumscribed to the conic.

The focus F of the cubic is the 2nd intersection (apart of QL-P1) of the Miquel circles of the 2 QL's. The perpendicular in M to MF gives the point Q, where the asymptote cuts the curve and the circle with diameter QF gives the 2 other vertices of the triangle QA-Tr2 of the Miquel points. The in- and excenters of this triangle are on the curve, with tangents parallel to the asymptote, which allows to draw it through Q. These 4 points are the centers of anallagmaty of the curve.



There is only one Van Rees focal circular circumcubic of the 6 vertices of a QL, it is QL-Cu1 with focus in QL-P1. This cubic is the locus of the foci of all QL's inscribed conics. For a given inscribed conic with foci S and S', all the QL's inscribed in the cubic and circumscribed to the conic have different VR's intersecting in QL-P1, S and S' and this curve is a particular Poncelet-Darboux curve wrt the conic.

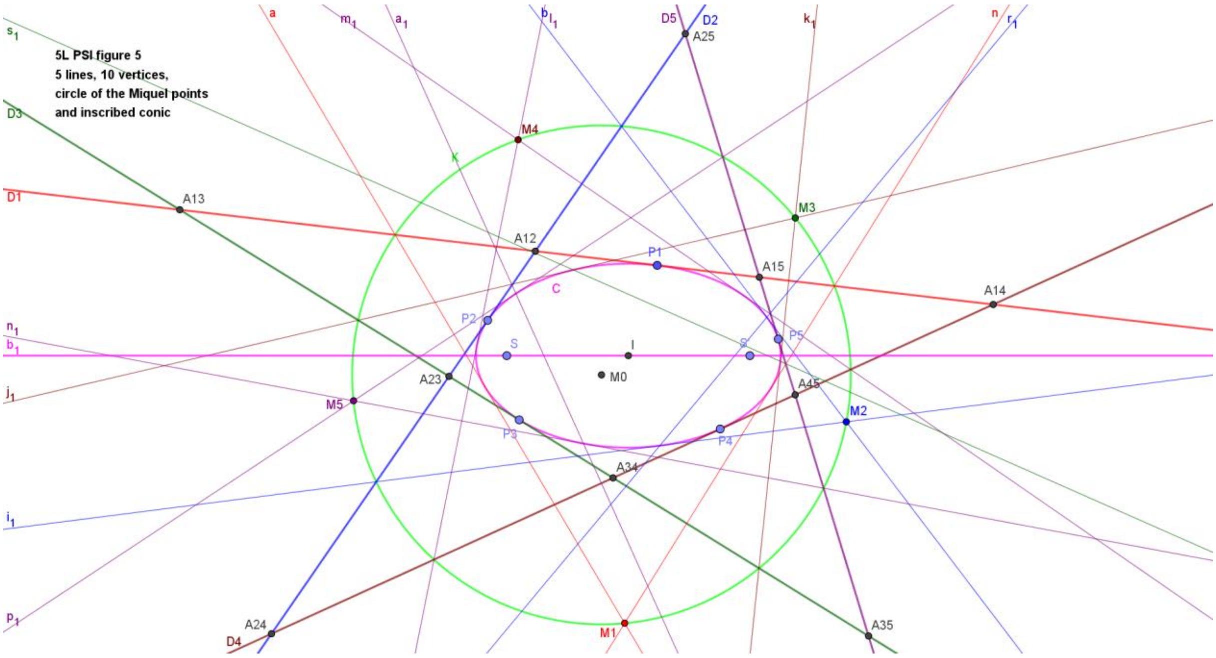


Let's remark in conclusion that the CB partner of the 6 vertices, QL-P1 and P is not ψP. The transformation ψ and the CB transformation of the 6 vertices and QL-P1 are different.

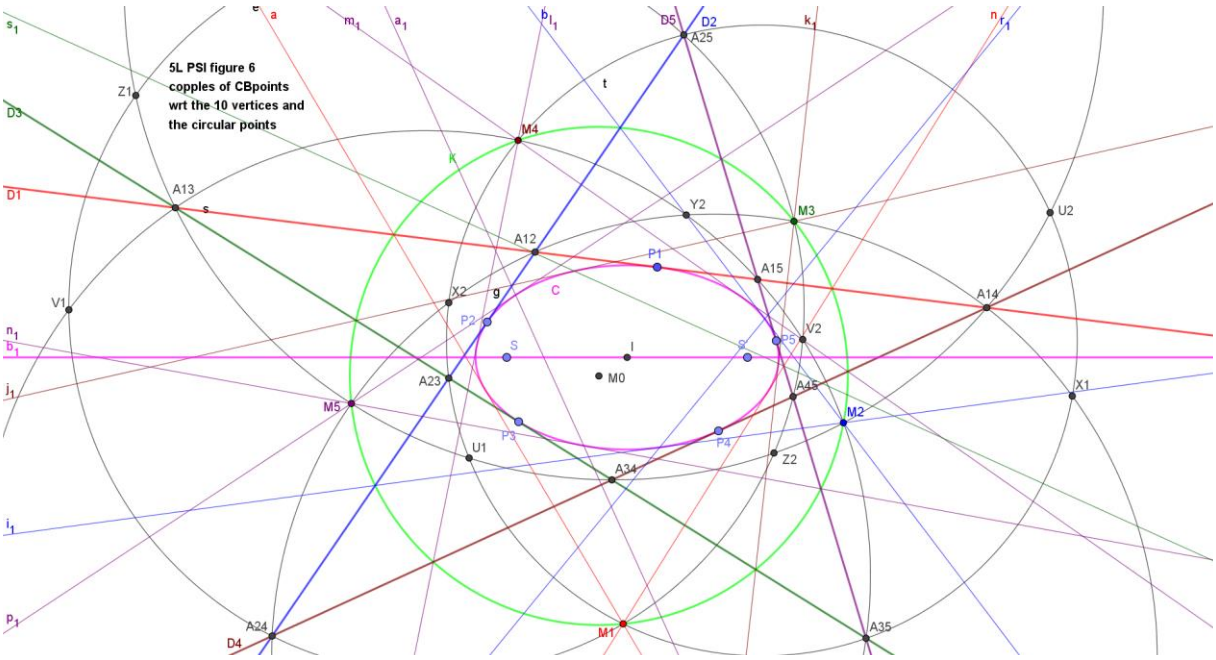
B. Pentalateral

Let's consider now a pentalateral 5L with it's 10 vertices Aij, it's inscribed conic 5L-s-Co1, hereafter inconic with foci S and S' and it's circle of Miquel points 5L-o-Ci1, hereafter Miquel circle.

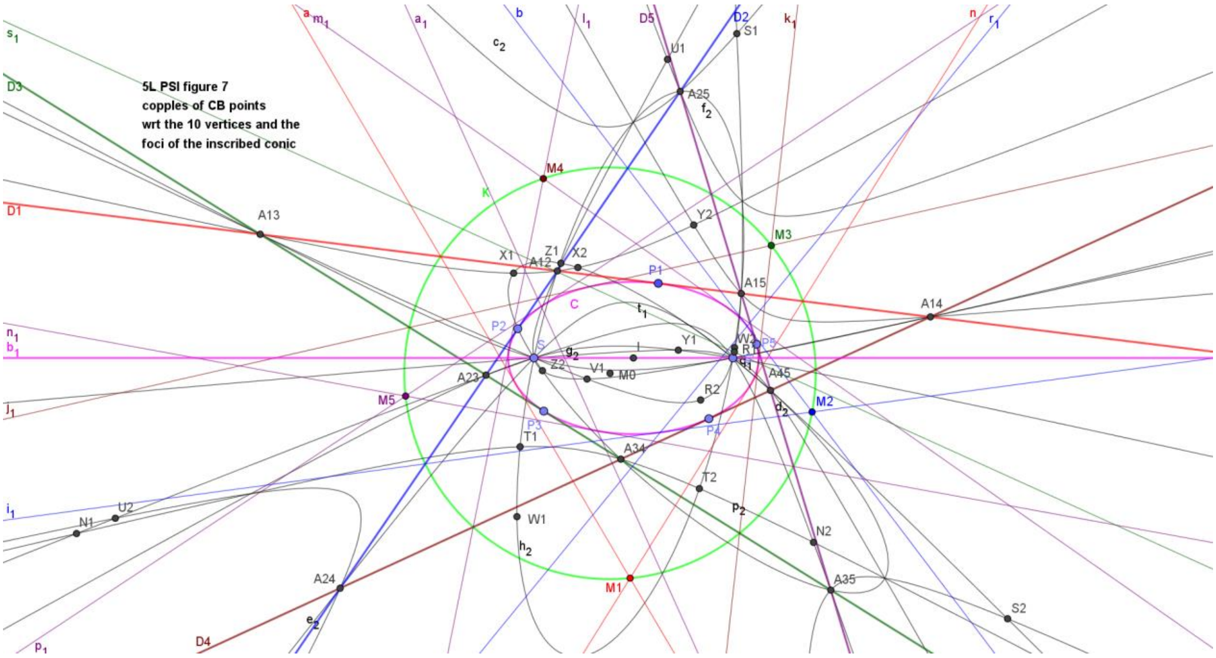
The Miquel circle is a Poncelet-Darboux curve for the insconic ; the 5 triangles formed by the lines Li and the tangents to the inscribed conic in the corresponding Miquel points Mi of the QL's formed by the 4 other lines are inscribed in the Miquel circle and circumscribed to the inconic.



The 10 degenerated quartics formed by 2 lines (through 7 points) and the circles through the 3 last points pass through the 10 vertices and through the circular points (but not through the foci S and S' of the inconic). 2 of these quartics intersect in 2 other points in the intersection of the 2 circles. All the copples of points are CB partners with the 10 vertices and the circular points.

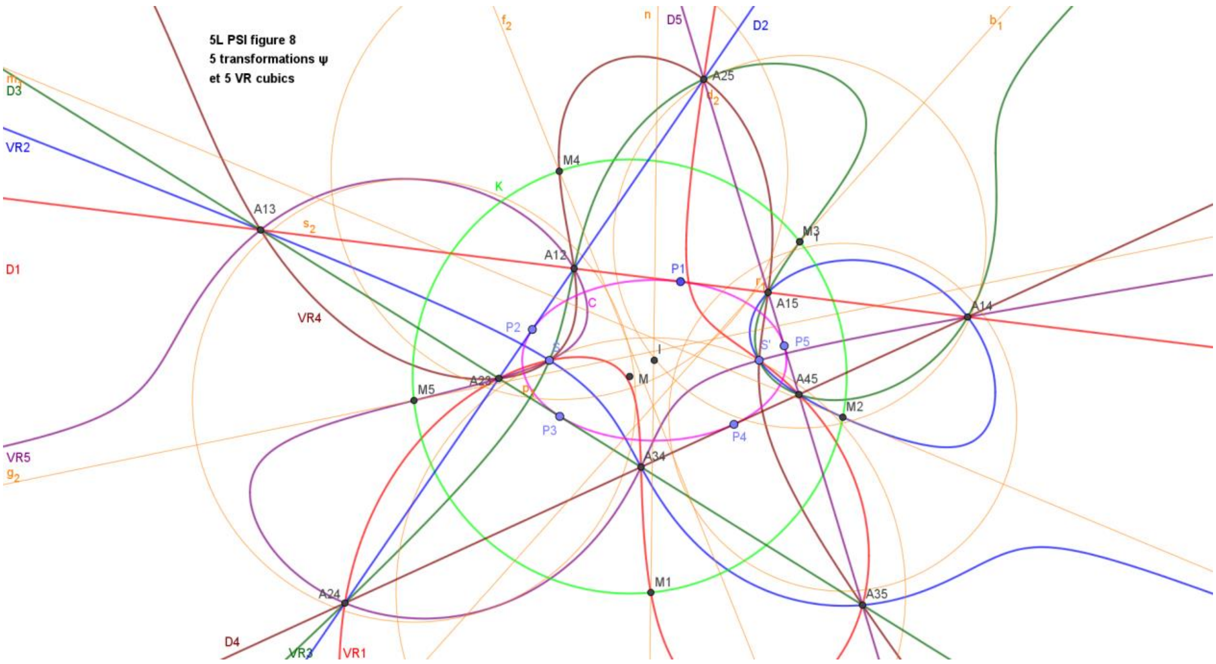


The same way, 10 degenerated quartics formed by 2 lines (through 7 points) and the conics through the 3 last points and the foci S and S' of the inconic pass through the 10 vertices and these foci S and S', but are not circular. 2 of these quartics intersect in 2 other points in the intersection of the 2 conics. All the copples of points are CB partners with the 10 vertices and the foci S and S'.



Despite all my efforts, I couldn't find interesting properties of these copples of points (anyway not all necessary real).

The 5 degenerated quartics formed by a line and the Van Rees focal circular cubic QL-Cu1 of the QL of the 4 other lines pass through the 10 vertices of the 5L, through the foci S and S' of the inconic and through the circular points. The foci of the VR's are the Miquel points.



These 14 points form therefore a CB system.

There are an infinity of quartics through these 14 points and we need another point P in order to identify a specific circular circumquartic of the 10 vertices of the PL and the foci S and S'.

This time, it is more promising. We are looking for a PL transformation Ψ swapping each of the 5 lines Li with the corresponding VRi and more generally a line tangent to the inconic with a VR and a point with a triple of points.

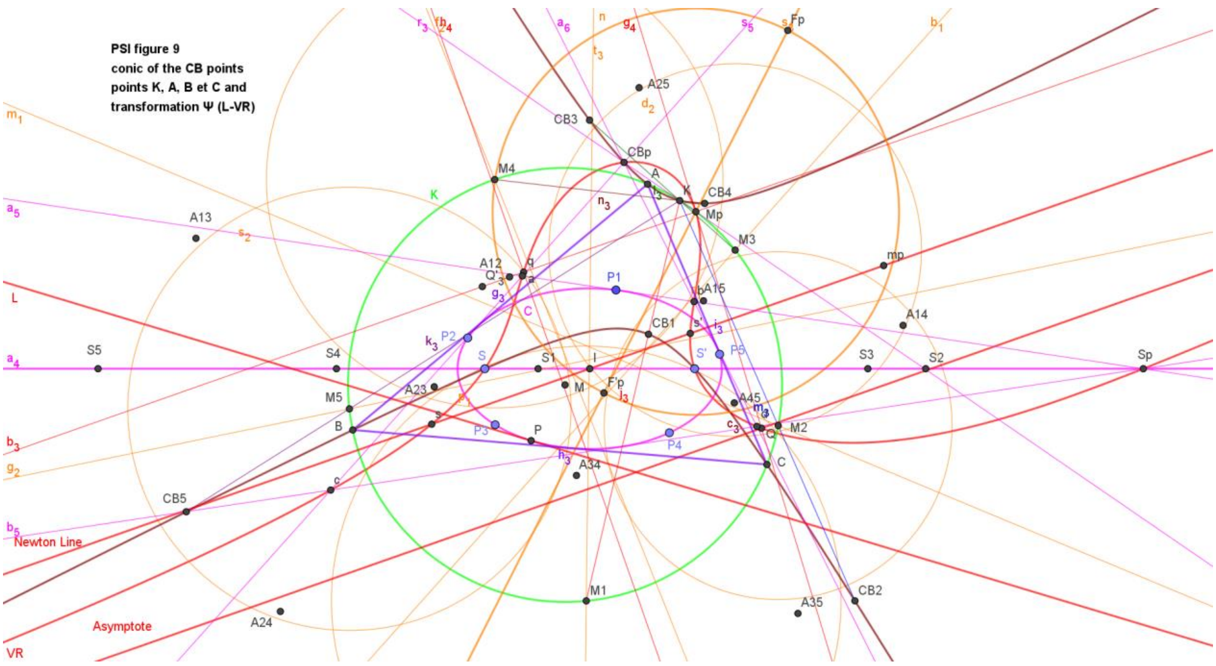
We will see that this transformation Ψ is a triple Moebius transformation, which associates to a vertice Aij of the PL the 3 vertices Akl, Akm and Alm, ψtransformed of Aij in the 3 Moebius transformations ψm, ψl and ψk centered in Mm, Ml and Mk swapping the foci S and S' of the inconic.

We will use the conic of the CB points for the VR's and the circle Ci(Q) for a point Q.

Let CBi be the CB of the 6 vertices of the QL of the 4 lines without Li and S and S' ; it is the 4th intersection with VRi of the circle through Mi, S and S' and the tangential of S and S' on this cubic. It is therefore also the CSCi of the 3rd intersection Si of SS' with VRi. The conic through the 5 CBi intersects the Miquel circle in 4 points K, A, B and C. ABC is circumscribed to the conic and K is aligned with MiCBi.

For any line Lp tangent to the inconic in a point P, we determine the point Mp on the Miquel circle as intersection of the 2nd tangents to the inconic in the points where Lp cuts the Miquel circle and the point CBp as 2nd intersection between the conic of the CB points and KMp. The CSCp centered in Mp swapping S and S' swaps CBp and a point Sp on SS'. Then VRp is the VR with focus Mp invariant in CSCp with Newton Line the line through I, the middle of SS' and mp, the middle of SpCBp, which gives the asymptote. This VRp passes through Mp, CBp, Sp, S and S' and is completely determined.

The transformation Ψ associates to the line Lp the Van Rees focal circular cubic VRp.



For any point Q, the circle Ci(Q) is the locus of the ψtransforms of Q in all ψtransformations centered in a point of the Miquel circle and swapping the foci S and S' of the inconic ; in particular, this circle Ci(Q) passes through the 5 ψtransforms of Q in the 5 ψtransformations CSCi of the 5 VRi.

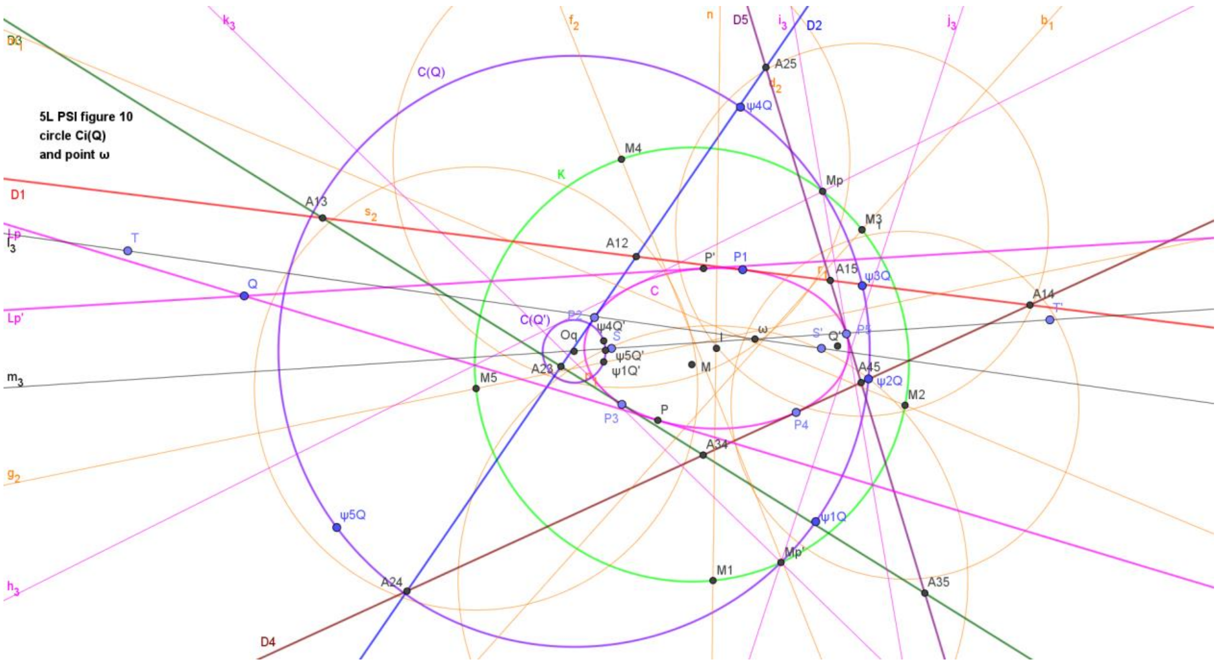
If the point Q is on the Miquel circle, Ci(Q) is a line tangent to the inconic.

If the point Q is on the line of infinity, Ci(Q) is the Miquel circle.

Let T and T' be inverses of S and S' wrt the Miquel circle ; ST' and S'T intersect in a point ω. The inversion with center ω which swaps S and T' and S' and T leaves the Miquel circle globally unchanged. The point Q' is the inverse of Q in this inversion. Then the circles Ci(Q) and Ci(Q') have the same center.

For Q on the Miquel circle, Q' is also on the Miquel circle, QQ' passes through ω and the lines Ci(Q) and Ci(Q') are parallel and symmetric wrt I, the middle of SS'.

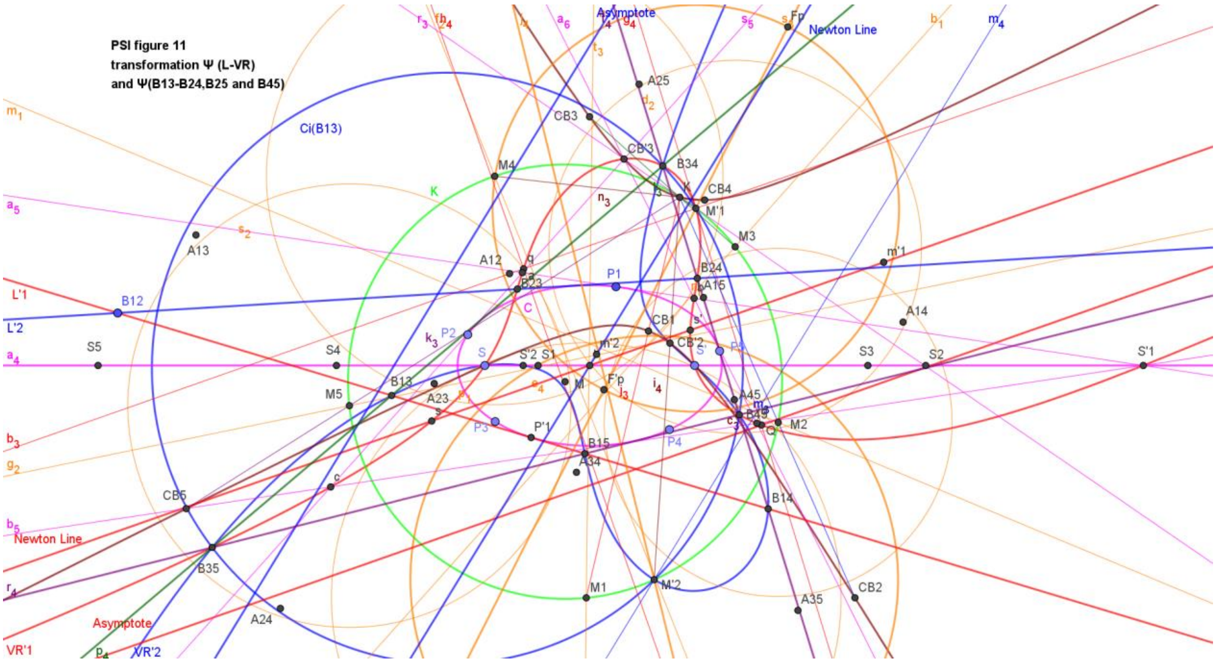
Let Lp and L'p be the tangents from Q to the inconic, the circle Ci(Q) intersects the Miquel circle in Mp and M'p, Ci(Q) is ψp(L'p) or ψ'p(Lp), where ψp and ψ'p are the ψtransformations centered in Mp and M'p and swapping S and S'. Ci(Mp) is the line Lp and Ci(M'p) is the line L'p.



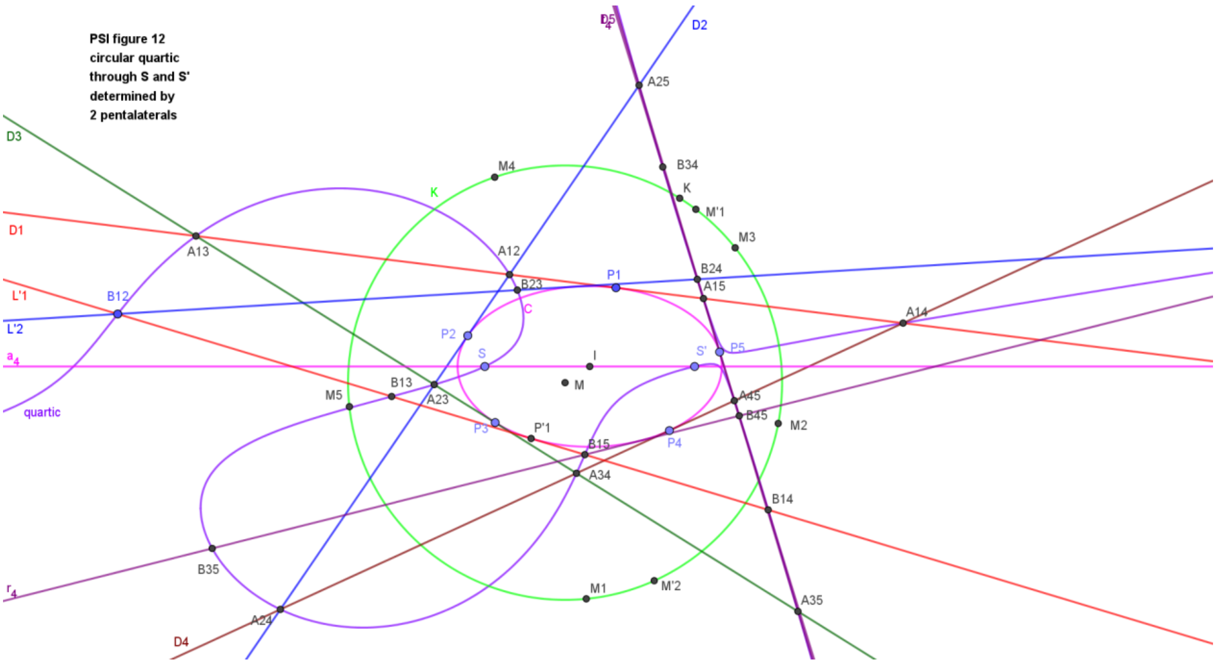
It's now possible to achieve the figur. For a point Q, named this time B12, the 2 tangents to the inscribed conic will be L'1 and L'2. The 2 Vr's associated to L'1 and L'2 are VR'1 and VR'2, which intersect in 3 points B34, B35 and B45 on the circle Ci(B12).

The triangle B34B35B45 is circumscribed to the inconic and it's sides form with the 2 lines L'1 and L'2 a 2nd pentalateral circumscribed to the inconic.

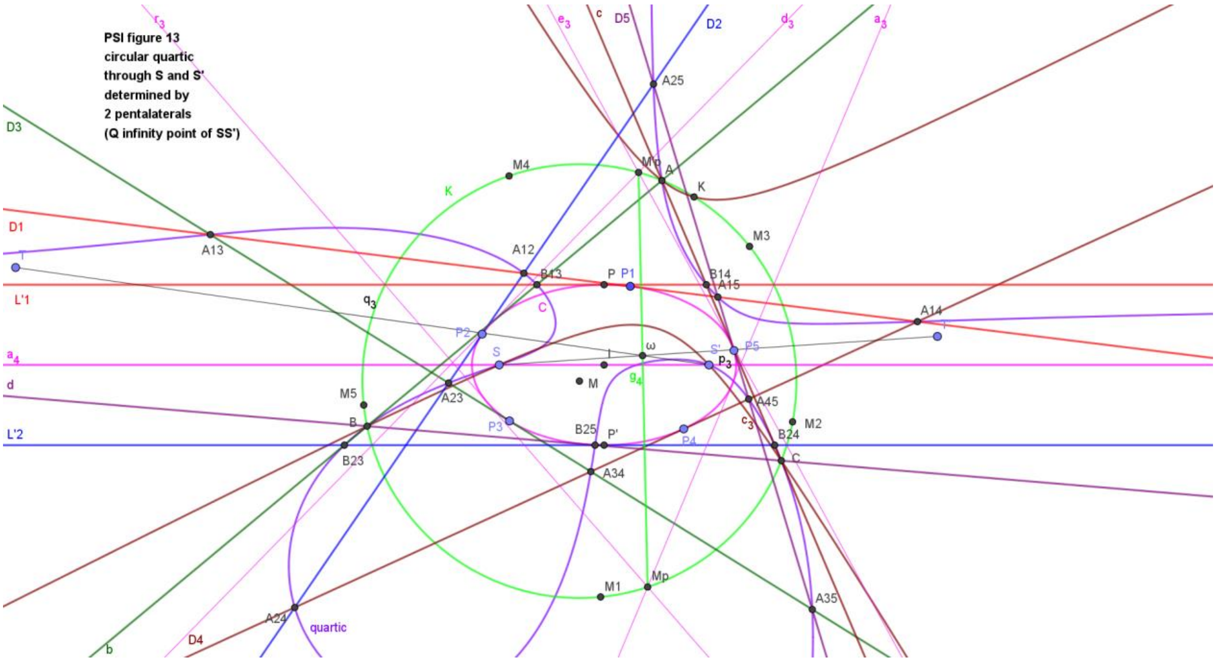
The transformation Ψ associates to the point B12 the triple of points B34, B35 and B45, intersections on the circle Ci(B12) of the 2 VR's associated to the tangents from this point to the inconic. If the VR's are invariant in the ψtransformations ψ1 and ψ2 centered in M'1 and M'2, it's interesting (and useful) to remark that B34, B35 and B45 are on VR'1 the ψ1 transforms of the 3 points where L'2 cuts VR'1 as well as on VR'2 the ψ2 transforms of the 3 points where L'1 cuts VR'2.



The 2 pentalaterals Aij and Bij determine a circular quartic through S and S'. This quartic is a Poncelet-Darboux curve for the inconic, as there are an infinity of pentalaterals inscribed in the quartic and circumscribed to the inconic.



If the point B12 is choosen as the infinity point of the line SS’, 2 lines are parallel to SS’ and the 3 others are the sides of the triangle ABC. The points Mp and Mp’ corresponding to the parallel line are aligned with the point ω, the 3 others are A, B and C.



Let’s finally remark that the Ψpartners of a point wrt a pentalateral are not the CB partners of the point wrt the 10 vertices of the pentalateral and the foci of the inconic. The Ψ transformation and the CB transformation are different.

As it involves many Moebius transformations ψ of quadrilaterals, I named this pentalateral’s transformation Ψ or triple Moebius.