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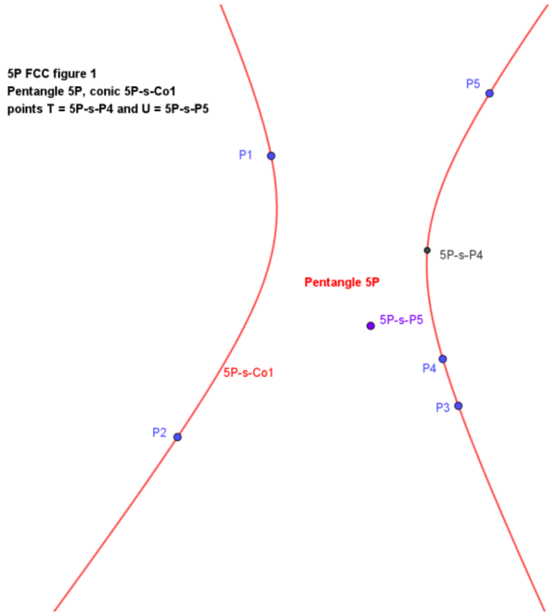
Pentangle's focal circular circumcubics

Let's consider a pentangle PA of 5 points Pi with it's circumconic 5P-s-Co1 and it's known points T = 5P-s-P4 and U = 5P-s-P5.



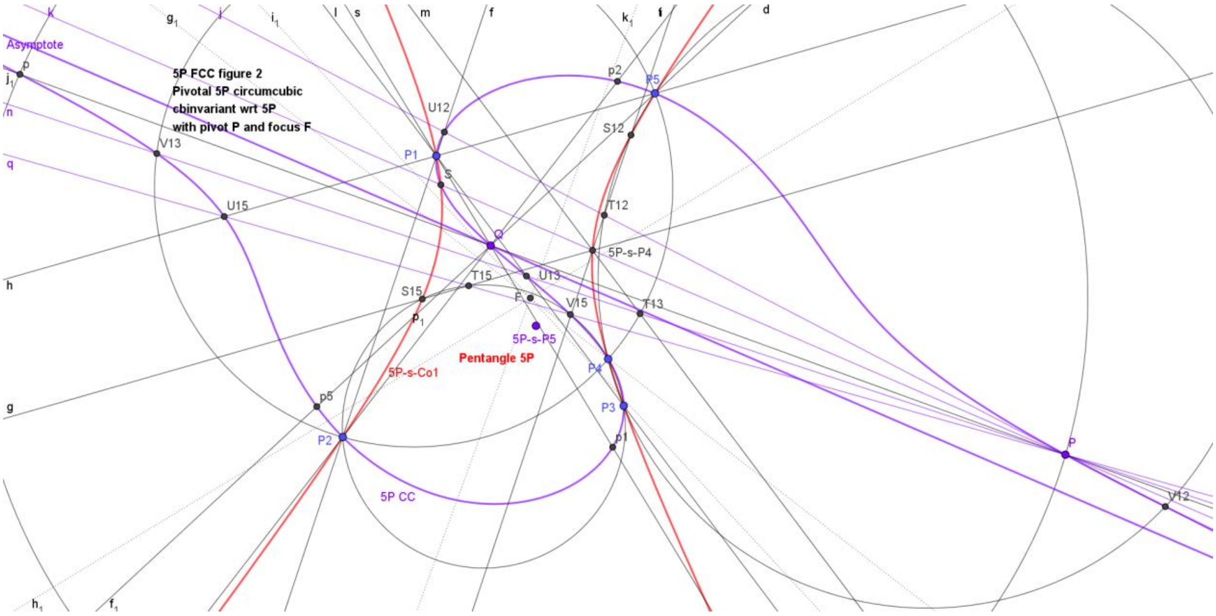
There are 11 degenerated circular circumcubics of the 5 points, 10 formed by a circle through 3 of the 5 points (and the circular points) and the line through the 2 other points and one 11th formed by the circumconic of the 5 points and the infinity line (through the circular points).

Each of these eleven cubics is a pivotal circumcubic of the 5 points and has a pivot Tij for the 10 1rst and T for the last one. Here is a method to determine the pivots Tij. The parallels through T to the lines PiPj intersect the circles through the 3 other points in 2 points ; one is Sij on the circumcubic, the other is the pivot Tij.

Having the 11 pivots, any pivotal circular circumcubic of the 5 points with pivot P is easy to draw. The line PT cuts the circumconic in a 2^{nd} point (other than T) S and the lines PTij intersect the lines PiPj in a point Uij and the circles through the 3 other points in a 2^{nd} point (other than Tij) Vij. Uij and Vij as well as S and the infinity point of PT are 11 copples of cb partners wrt the 5 points and the 2 circular points. This gives 21 points beside the 5 points and the pivot on the cubic.

For any point of the cubic, the circle Ci(P) pass through the 2 points U = 5P-sP5 and F, the focus of the cubic (for the point S on the circumconic, Ci(P) is the line UF), which allows to find F.

Using Eckart's construction allows to find the point Q where the cubic cuts it's asymptote and the asymptote as parallel to P5P-s-P4 through this point. (Any line through Q cuts the cubic in 2 points equidistant from the focus).



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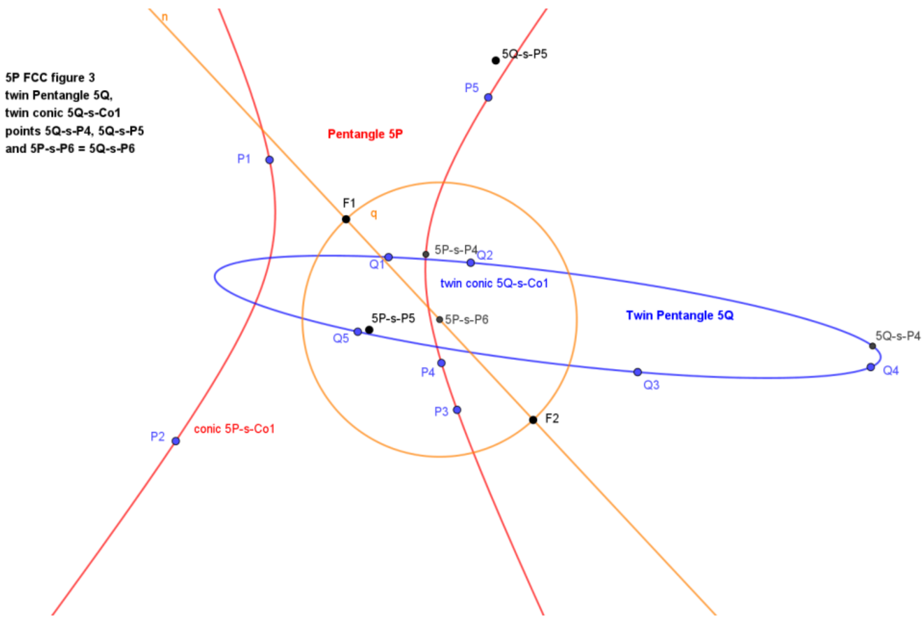
An interesting question is to find under which conditions this circular circumcubic of the 5 points is focal, id est it’s focus lies on the cubic. In this case, the 5P CC is a 5P FCC.

This item gives all the points, curves and constructions needed to answer the question.

1. the twin pentangle of 5 points Qi = Tf8(Pi) with the point M = 5P-s-P6, the twin conic 5Q-s-Co1 and the points 5Q-s-P4 and 5Q-s-P5 (note that this point is Tf8(5P-s-P5)).

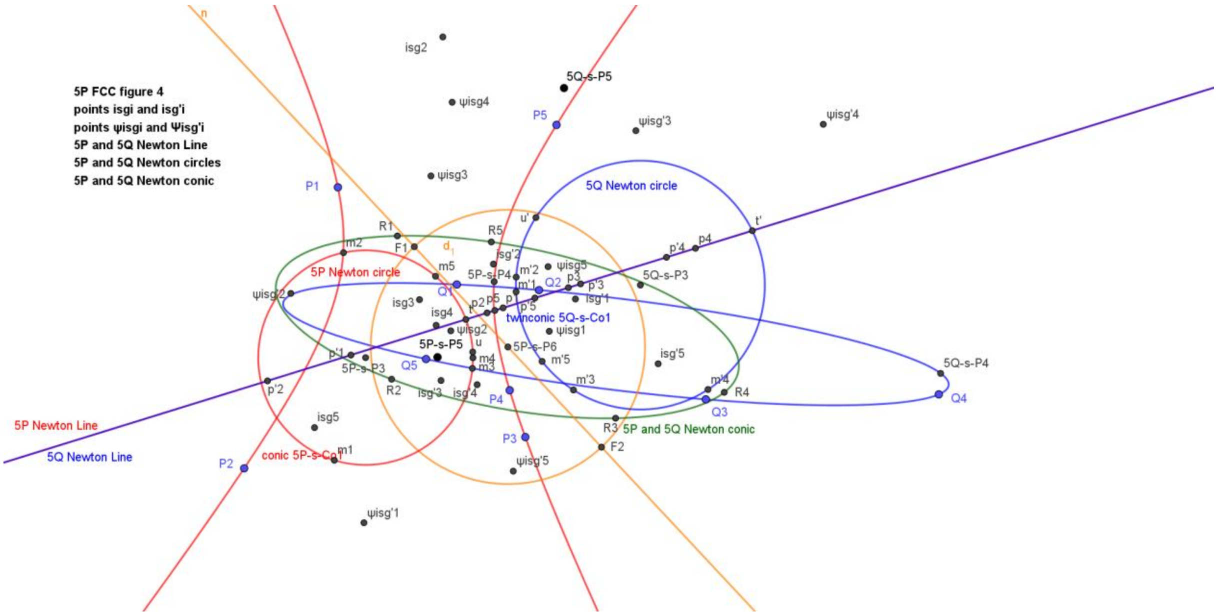


2. the 20 points isgi = QA-P4(5P less Pi), isg'i = QA-P4(5Q less Qi) and their Tf8 transformed ψisgi and ψisg'i.

3. the middles pi of Piψisg'i and p'i of Qiψisgi are aligned on a Newton Line through the middles t of TU and t' of T'U'.

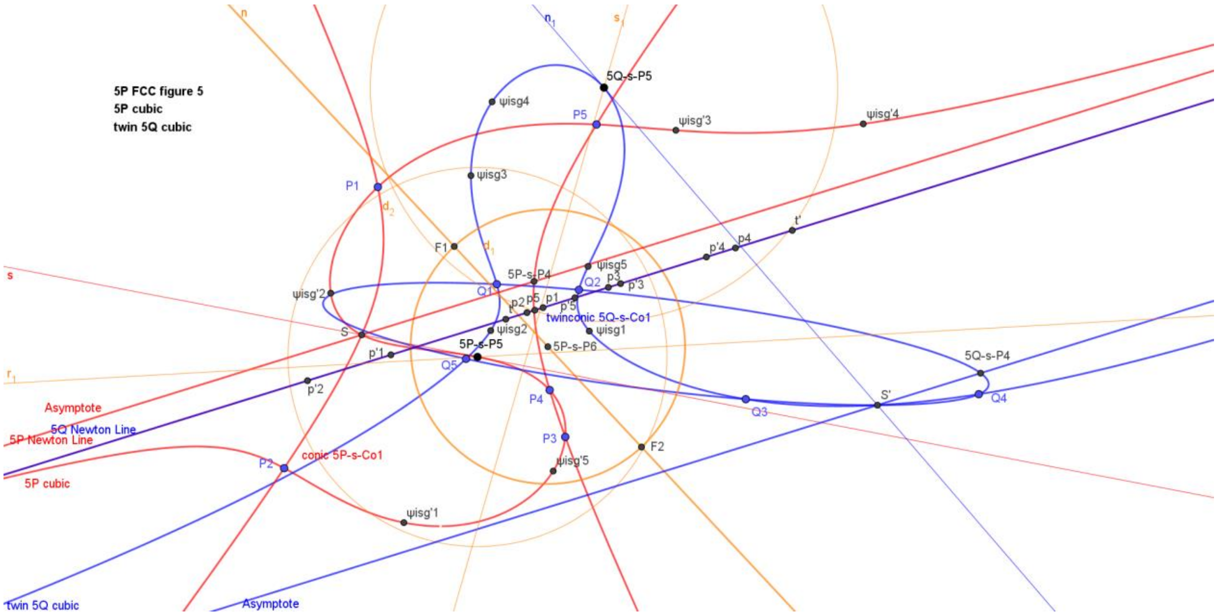
4. the middles mi of Piisgi and m'i of Qiisg'i are on the twin Newton circles, centered in ω = 5P-s-P3 and ω' = 5Q-s-P3, which pass also respectively through the middles m of MU and m' of MU'.

5. last, the middles Ri of PiQi are on the common Newton conic.



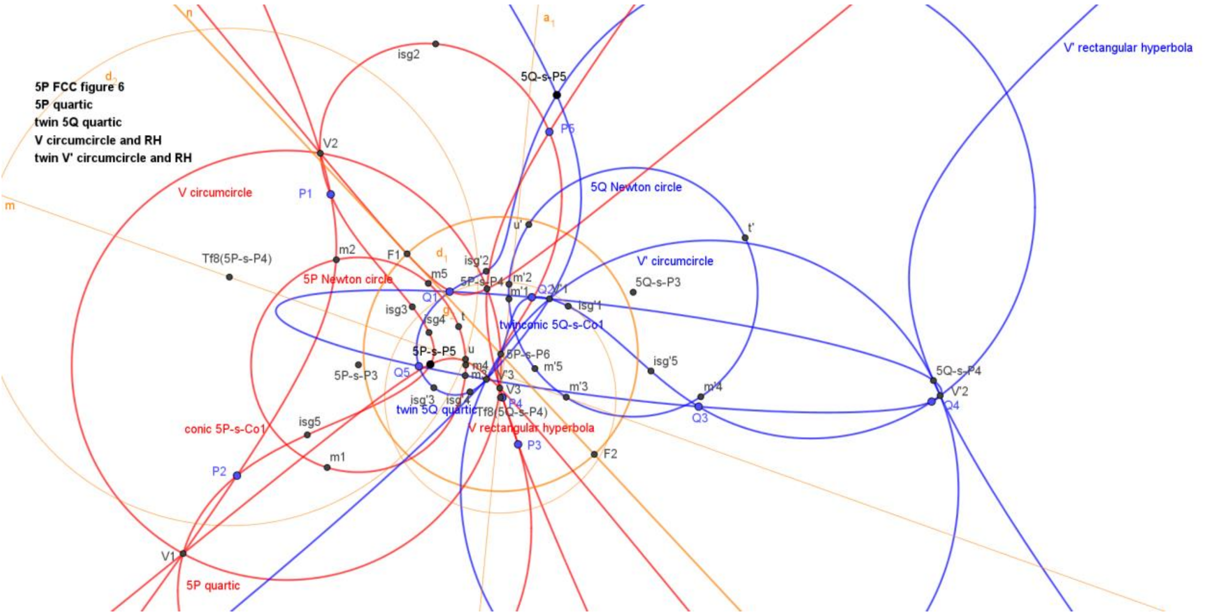
6. the 10 points Pi and ψisg'i are on the 5P cubic, which is a Van Rees focal circular circumcubic of the 5 points cb invariant with pivot in the infinity point of the Newton Line and focus in U = 5P-s-P5 ; it is invariant in the CSC2 centered in 5P-s-P5 and swapping the Pi and the ψisg'i and 5P-s-P6 and 5Q-s-P4.

7. the 10 points Qi and ψisgi are on the twin 5Q cubic, which is a Van Rees focal circular circumcubic of the 5 points cb invariant with pivot in the infinity point of the Newton Line and focus in U' = 5Q-s-P5 ; it is invariant in the twinCSC2 centered in 5Q-s-P5 and swapping the Qi and the ψisgi and 5P-s-P6 and 5P-s-P4.

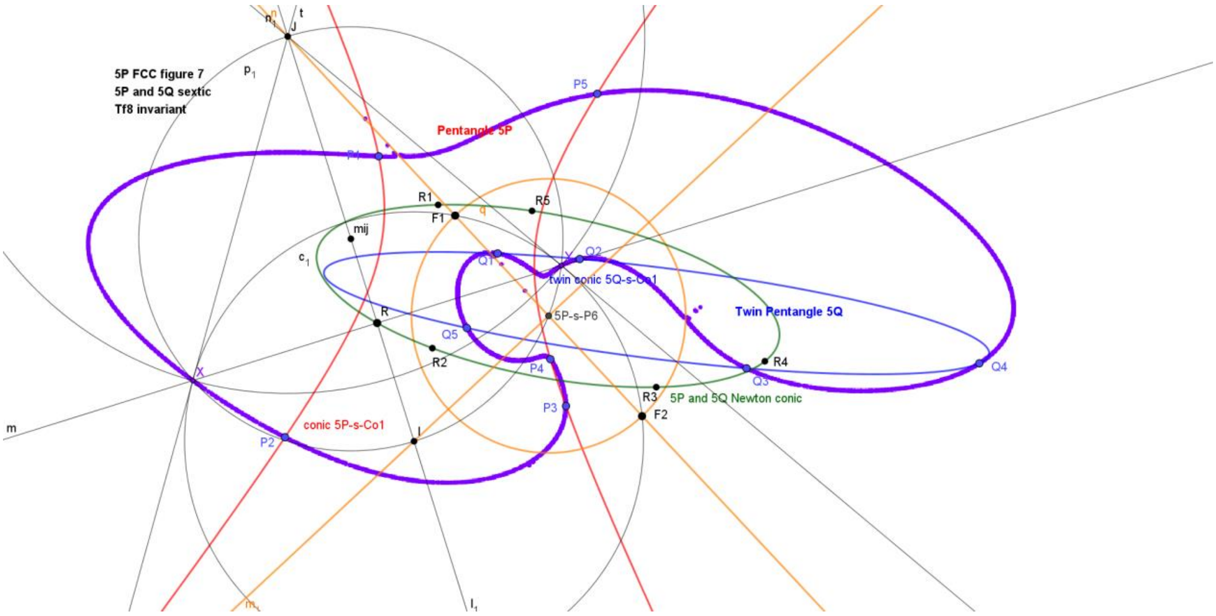


8. the 5P quartic is Tf8(5Q twin cubic), invariant in the CSC3 centered in Tf8(5P-s-P4), which swaps the Pi and the isgi and U = 5P-s-P5 and M = 5P-s-P6. CSC3 is a combination of CSC1 (or Tf8) and twinCSC2 ; CSC3 = CSC1*twinCSC2*CSC1.
   The 5P quartic and the 5P conic intersect beside the 5 points in 3 points V1, V2 and V3 forming the V triangle ; it's circumcircle passes through 5P-s-P4 and 5P-s-P6, it's Euler circle is the 5P Newton circle and it's orthocenter is 5P-s-P5.
   The V rectangular hyperbola passes through the vertices of the V triangle and through the points 5P-s-P5 and 5P-s-P4 and is centered in the middle of these 2 points. It's axes are are parallel to those of the 5P conic.

9. the 5Q quartic is the twin quartic, Tf8(5P cubic), invariant in the twinCSC3 centered in Tf8(5Q-s-P4), which swaps the Qi and the isg'i and U' = 5Q-s-P5 and M = 5P-s-P6. twinCSC3 is a combination of CSC1 (or Tf8) and CSC2 ; twinCSC3 = CSC1*CSC2*CSC1.
   The 5Q quartic and the 5Q conic intersect beside the 5 points in 3 points V'1, V'2 and V'3 forming the V' triangle ; it's circumcircle passes through 5Q-s-P4 and 5P-s-P6, it's Euler circle is the 5Q Newton circle and it's orthocenter is 5Q-s-P5 (Tf8(P-s-P5).
   The V' rectangular hyperbola passes through the vertices of the V' triangle and through the points 5Q-s-P5 and 5Q-s-P4 and is centered in the middle of these 2 points. It's axes are are parallel to those of the 5Q conic.



10. the 5P and 5Q sextic is invariant in Tf8 ; for a variable point R on the 5P and 5Q Newton conic, the 2 intersections between the bisector of the angle F1RF2 and it's Tf8 transformed give 2 Tf8 partners of the sextic.



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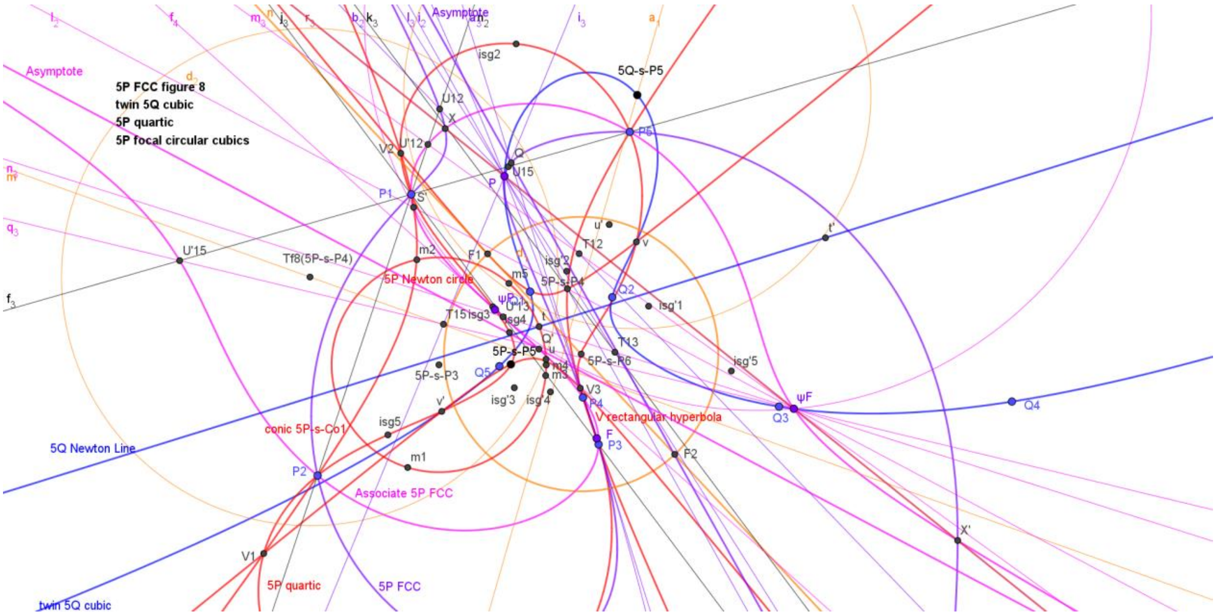
It now possible to answer the question above : a 5P circular circumcubic is focal if and only if it's pivot P lies on the twin 5Q cubic ; it's focus F is then CSC4(P), where CSC4 = twinCSC2*CSC1. As we have CSC3 = CSC1*twinCSC2*CSC1, we have also CSC4 = CSC1*CSC3 (Remember that CSC1 is Tf8).

This 5P FCC has an associate 5P FCC with pivot P' = twinCSC2(P) = Tf8(F) and focus F' = CSC3(F) = Tf8(P).



The 2 associated cubics intersect beside the 5 points Pi and the 2 circular points in 2 points X and X' cb partners wrt the 5 points on the pivot line PTf8(F).

There are many more or less interesting examples of such 5P FCC copples of associated cubics :

- pivot the infinity point of the Newton Line and focus 5P-s-P5 : this is our 1rst 5P cubic ; it's associate has pivot in 5Q-s-P5 and focus in 5P-s-P6.
- pivot in Qi and focus in isgi and associate with pivot in Tf8(isgi) and focus in Pi ; this gives 10 cubics.
- among the intersections of the twin 5Q cubic and the 5P quartic, 2 points v and v' are particularly interesting as they are at the same time CSC1 (or Tf8) partners and twin CSC2 partners, each point being therefore a fixed point for twin CSC2*CSC1.
  There are 2 associated cubics with pivot = focus in v or v'.

Perhaps a last interesting property, using the V rectangular hyperbola :

- a line through 5P-s-P4 intersect the 5Q twin cubic in 3 points, possible pivots
- the twinCSC2 transform of this line is a circle through the twinCSC2 transforms of the 3 points, which are 3 other possible pivots on the twin 5Q cubic, and through 5Q-s-P5 and 5P-s-P6 (as twinCSC2 transform of 5P-s-P4)
- the CSC1 (or Tf8) transform of this circle is a line through 5P-s-P5 (as Tf8(5Q-s-P5) and the 3 corresponding foci
- the 2 lines intersect in a point V on the V rectangular hyperbola

- conversely, for a point V on the RH, the 2 lines V5P-s-P4 and V5P-s-P5 are CSC4 partners ; the 1rst one intersects the twin 5Q cubic in 3 pivots P, Q and R and the 2nd intersects the 5P quartic in 3 foci, F, G and H giving 3 5P FCC
- the Tf8 of the 3 foci give in turn 3 other pivots ψF, ψG and ψH on a circle through 5Q-s-P5 on the twin 5Q cubic and the Tf8 of the 3 1rst pivots give 3 other foci ψP, ψQ and ψR on a circle through Tf8(5P-s-P4) on the 5P quartic, giving the 3 associate 5P FCC
- naturally, the lines ψF5P-s-P4 and ψP5P-s-P5 intersect in a point X, the lines ψG5P-s-P4 and ψQ5P-s-P5 intersect in a point Y and the lines ψH5P-s-P4 and ψR5P-s-P5 intersect in a point Z and the 3 points X, Y and Z lie on the RH