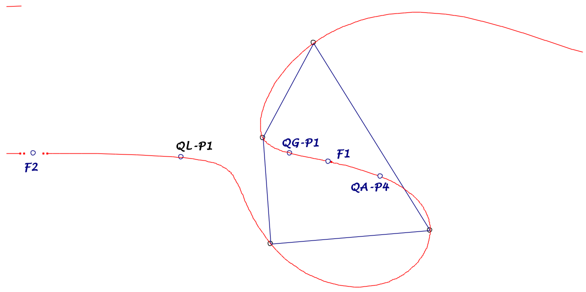
EQF-Note 2014-02-05

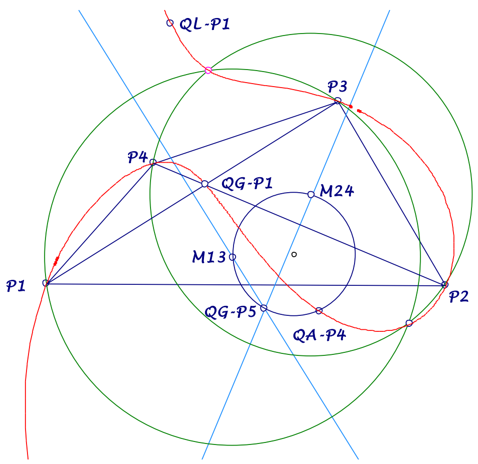
Background for these notes is:
Chris van Tienhoven: Encyclopedia of Quadri-Figures
http://chrisvantienhoven.nl/

QA-P4−*QL-P1*−*QG-P1*−Circumcubic

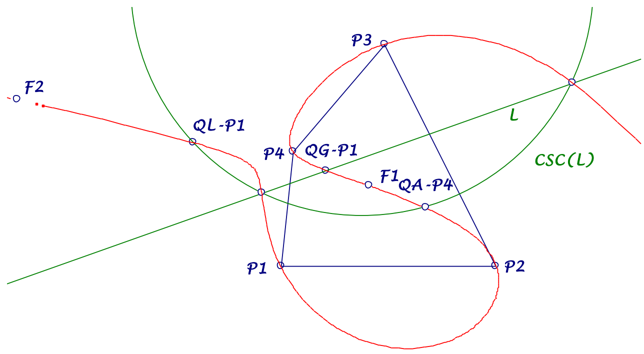
For lines L through the Diagonal Crosspoint QG-P1 of a quadrigon the intersections with their Clawson-Schmidt Conjugate (QL-Tf1) give a special QG-circumcubic, combining the geometry of quadrangle, quadrilateral and quadrigon.



We start with a construction for the cubic independent of the Clawson-Schmidt Conjugate. The Isogonal Center *QA-P4* is the second intersection of circles through the Miquel Point *QL-P1* and opposite vertices of the quadrigon (see *EQF*), . The *QG*-Quasi Circumcenter *QG-P5* is the intersection of the perpendicular bisectors of the diagonals (see EQF). Considering circles through *QA-P4* and *QG-P5* we get two points M_{13}, M_{24} on the perpendicular bisectors of the diagonals as centers of circles through opposite vertices. The intersections of these circles give points of the cubic.

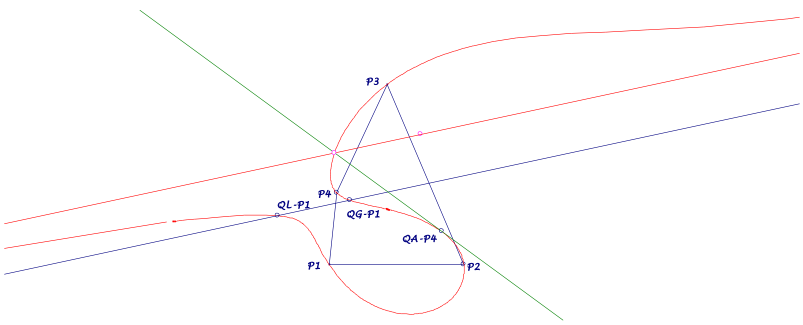


Using the Clawson-Schmidt Conjugate *QL-Tf1* – shortened *CSC* – we consider lines *L* through *QG-P1* and their intersections with *CSC(L)*, which are circles through *QL-P1* and *QA-P4* = *CSC(QG-P1)*.

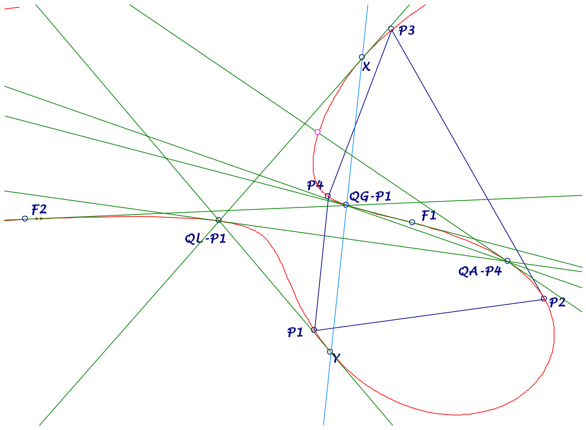


Properties:

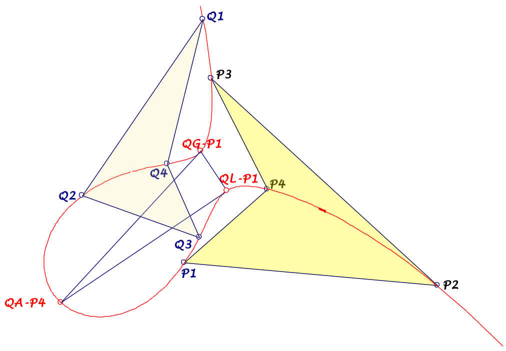
- The cubic contains the vertices of the quadrigon, the Miquel Point *QL-P1*, the Diagonal Crosspoint *QG-P1* and the Isogonal Center *QA-P4* = *CSC(QG-P1)* as well as the *CSC*-fixed points F_1, F_2 (see *QL-Tf1*).
- The cubic is *CSC*-invariant with pivot *QG-P1*.
- The asymptote is a parallel to *QG-P1.QL-P1* through the reflection of *QA-P4* in *QG-P1.QL-P1*.
- The asymptote cuts the cubic on the tangent at *QA-P4* to the cubic (see below).



- The tangent at *QL-P1* is *QA-P4.QG-P16*.
- The tangent at *QG-P1* contains *QA-P4*.
- The tangents in F_1, F_2 intersect in *QG-P1*.
- The tangent in *QA-P4* is the *CSC*-image of a circle through *QL-P1*, touching *QG-P1.QA-P4* in *QG-P1*.
- The angle bisector *L* of *QL-P1*, *QG-P1*, *QA-P4* cuts *CSC(L)* in two points *X* and *Y* on the cubic with tangents through *QL-P1*.



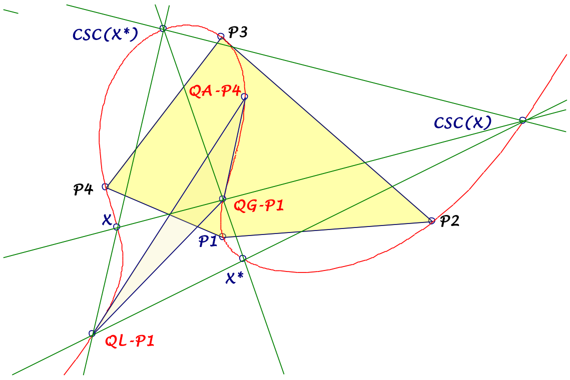
- The cubic is isogonal invariant wrt the triangle Δ = *QA-P4.QL-P1.QG-P1*. (This is a very curious property! For example: F_1 and F_2 are isogonal conjugate.)
- Wrt the triangle Δ the cubic contains the isogonal conjugates Q_i of the vertices P_i. The *P*- and *Q*-quadrigons are perspectiv wrt *QL-P1* and have the same Diagonal Crosspoint, Miquel Point and Isogonal Center.



- Finally the equation in *DT*-barycentric coordinates wrt *QL-DT*.

$$
\begin{aligned}
& a^2 (-\lambda\, SA + \mu\, SB + \nu\, SC)\, z\, (m^2 y^2 - n^2 z^2) - c^2 (\lambda\, SA + \mu\, SB - \nu\, SC)\, x\, (l^2 x^2 - m^2 y^2) \\
& + 2 m^2 b^2 (\lambda\, SA - \mu\, SB + \nu\, SC)\, x\, y\, z \\
& - (\lambda^2\, SA^2 + \mu\, (2 l^2 + n^2)\, SB^2 - \nu\, m^2\, SC^2 + (\mu\, m^2 + 2 \mu\, n^2 - \nu\, n^2)\, SA\, SB \\
& \qquad - (\lambda\, l^2 - 2 \mu\, m^2 + 2 \nu\, m^2)\, SA\, SC - (\lambda\, l^2 - 2 \mu\, m^2 + 2 \nu\, l^2)\, SB\, SC)\, x^2 z \\
& + (\lambda\, m^2\, SA^2 - \mu\, (l^2 + 2 n^2)\, SB^2 + \nu^2\, SC^2 + (\lambda\, n^2 - 2 \mu\, m^2 - \mu\, n^2)\, SA\, SB \\
& \qquad + (2 \lambda\, m^2 - 2 \mu\, m^2 + \nu\, n^2)\, SA\, SC + (\lambda\, l^2 - 2 \mu\, l^2 - \mu\, m^2)\, SB\, SC)\, x\, z^2 == 0 \\
& \qquad\qquad \text{with } \lambda := m^2 - n^2;\ \mu := n^2 - l^2;\ \nu := l^2 - m^2
\end{aligned}
$$

- Additional remark: For a triangle *ABC* we can consider a transformation, consisting of a reflection in the angle bisector at *B* and a reflection in a circle round *B*, so that *A* and *C* changes. This transformation shall be the “*ABC*-inversion wrt *B*”. For example the *CSC*-mapping is of this type wrt the Miquel Triangle.



The considered cubic is invariant under the Δ-inversions wrt *QA-P4*, *QL-P1* and *QG-P1*. Let *X* be a point on the cubic:
Δ-inversion of *X* wrt *QL-P1* is *CSC(X)*, collinear with *X* and *QG-P1*.
Δ-inversion of *X* wrt *QA-P4* is the Δ-isogonal conjugate *X** of *X*, collinear with *CSC(X)* and *QL-P1*.
Δ-inversion of *X* wrt *QG-P1* is *CSC(X*)*, collinear with *X* and *QL-P1*.

Further: *X.X** parallel *CSC(X).CSC(X*)*.

Final conclusion: The cubic depends only on the points *QA-P4*, *QL-P1*, *QG-P1*. If we take these points as *A*, *B*, *C* of a reference triangle *ABC* for barycentric coordinates, the *CSC*-mapping becomes

$$(x:y:z) \quad \rightarrow \quad \left(-a^2z : \frac{a^2yz+b^2zx+c^2xy}{x+y+z} : -c^2x\right)$$

and the cubic has the equation

$$x(a^2yz+b^2zx+c^2xy)+a^2yz(x+y+z)=0.$$

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